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Solution homework exercise 3.3: (Bazaraa-Shetty function)

Statement:

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = (x-2)^4 + (x-2y)^2$$

has a strict global minimum at $(2, 1)^T$ *and no other minima or maxima.*

Proof: Assume that f has a minimum or maximum $(\hat{x}, \hat{y})^T \in \mathbb{R}^2$. Due to the necessary first order optimality condition,

$$\nabla f(\hat{x}, \hat{y}) = 0.$$

For all $(x, y)^T \in \mathbb{R}^2$,

$$\nabla f(x, y) = \begin{pmatrix} 4(x-2)^3 + 2(x-2y) \\ 2(x-2y)(-2) \end{pmatrix}.$$

Thus it follows

$$\hat{x} = 2\hat{y}$$

from $2(\hat{x} - 2\hat{y})(-2) = 0$ and thus

$$\hat{x} = 2$$

from $4(\hat{x} - 2)^3 + 2(\hat{x} - 2\hat{y}) = 0$. Hence it

$$\hat{x} = 2$$

and

$$\hat{y} = 1.$$

So if f has a minimum or maximum, it must be $(2, 1)^T$. For all $(x, y)^T \in \mathbb{R}^2$,

$$\nabla^2 f(x, y) = \begin{pmatrix} 12(x-2)^2 + 2 & -4 \\ -4 & 8 \end{pmatrix}.$$

Since

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 12(x-2)^2 + 2 \geq 0,$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = 8 \geq 0$$

and

$$\det\left(\frac{\partial^2 f}{\partial x^2}(\hat{x}, \hat{y})\right) = (12(x-2)^2 + 2)8 - (-4)^2 = 96(x-2)^2 \geq 0$$

for all $(x, y)^T \in \mathbb{R}^2$, it follows that $\nabla^2 f(x, y)^T$ is positive semidefinite for all $(x, y)^T \in \mathbb{R}^2$ due to Silvester's criterion. Thus $(2, 1)^T$ is a minimum of f due to the second order sufficient optimality condition. Since

$$f(2, 1) = 0 \leq (x-2)^4 + (x-2y)^2 = f(x, y)$$

for all $(x, y)^T \in \mathbb{R}^2$, $(2, 1)^T$ is also a global minimum. It is also strict since it is the only (global) minimum. $\square$