SoSe 2020 Kiel University
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Solution homework exercise 4.1:

Statement:

$$\min_{x\in\mathbb{R}^2} \|Ax-z\|_2^2 \quad \textit{where} \quad A := \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \textit{and} \quad z := \begin{pmatrix} 6 \\ 12 \\ 6 \end{pmatrix}$$

has

$$x^* := \begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

as unique solution.

Proof: Assume that $x \in R^2$ is a solution of the above linear regression problem. Due to the normal equation,

$$A^T A x = A^T z.$$

We calculate some intermediate results

$$\begin{aligned} A^T A &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A^T z &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 12 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 24 \\ 24 \end{pmatrix}. \end{aligned}$$

Now we can check if the solution is unique. For this we can check if the matrix $A^T A$ is invertible. Since

$$\begin{aligned} \det(A^T A) &= \det\left(\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}\right) \\ &= 5 \cdot 3 - 3 \cdot 3 \\ &= 6 \\ &\neq 0, \end{aligned}$$

$A^T A$ is invertible and thus x does exist and is unique. Moreover

$$\begin{aligned} (A^T A)^{-1} &= \left(\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}\right)^{-1} \\ &= \frac{1}{6} \begin{pmatrix} 3 & -3 \\ -3 & 5 \end{pmatrix} \end{aligned}$$

and thus

$$\begin{aligned} x &= (A^T A)^{-1} A^T z \\ &= \frac{1}{6} \begin{pmatrix} 3 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 24 \\ 24 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 8 \end{pmatrix}. \end{aligned}$$

$\square$