SoSe 2020 Kiel University
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Solution homework exercise 5.1:

Statement: Consider the general descent method applied to a continuous differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. Let $(x_k)_{k\in\mathbb{N}_0}$ be the resulting sequence of iterates.

a) If x^* and $\hat{x}$ are two accumulation points of $(x_k)_{k\in\mathbb{N}_0}$, then $f(x^*) = f(\hat{x})$.

b) If x^* is an accumulation points of $(x_k)_{k\in\mathbb{N}_0}$, then x^* is not a strict maximum.

c) For $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto -x^2$ ant $x_0 = 0$ the general descent method stops at a strict maximum.

Proof: a) Assume that

$$f(x^*) \neq f(\hat{x}).$$

Without restriction of generality assume that $f(x^*) > f(\hat{x})$. Then for

$$\epsilon := \tfrac{1}{2}(f(x^*) - f(\hat{x}))$$

holds $\epsilon > 0$. Since x^* and $\hat{x}$ are accumulation points of $(x_k)_{k\in\mathbb{N}_0}$, $(x_{n_k^*})_{k\in\mathbb{N}_0}$ and $(x_{\hat{n}_k})_{k\in\mathbb{N}_0}$ exist such that

$$\lim_{k\to\infty} x_{n_k^*} = x^* \text{ and } \lim_{k\to\infty} x_{\hat{n}_k} = \hat{x}.$$

Since f is continuous,

$$\lim_{k\to\infty} f(x_{n_k^*}) = f(x^*) \text{ and } \lim_{k\to\infty} f(x_{\hat{n}_k}) = f(\hat{x}).$$

Thus $k_0 \in \mathbb{N}_0$ exists such that

$$|f(x_{n_k^*}) - f(x^*)| < \epsilon \text{ and } |f(x_{\hat{n}_k}) - f(\hat{x})| < \epsilon$$

for all $k \in \mathbb{N}_0$ with $k \geq k_0$. Hence $f(x^*) \neq f(\hat{x})$ and $\epsilon = \frac{1}{2}(f(x^*) - f(\hat{x}))$ imply

$$f(x_{n_k^*}) > f(x_{\hat{n}_k})$$

for all $k \in \mathbb{N}_0$ with $k \geq k_0$. Thus $i, j \in \mathbb{N}_0$ with $i < j$ exist such that

$$f(x_j) > f(x_i).$$

But this is a contradiction to

$$f(x_{k+1}) < f(x_k)$$

for all $k \in \mathbb{N}_0$ which is true due to the definition of the general descent method. Thus

$$f(x^*) = f(\hat{x})$$

must be true.

b) Since x^* is an accumulation points of $(x_k)_{k\in\mathbb{N}_0}$, $(x_{n_k^*})_{k\in\mathbb{N}_0}$ exists such that

$$\lim_{k\to\infty} x_{n_k^*} = x^*.$$

The continuity of f implies then

$$\lim_{k\to\infty} f(x_{n_k^*}) = f(\lim_{k\to\infty} x_{n_k^*}) = f(x^*).$$

Assume that x^* is a strict maximum. Then $\epsilon > 0$ exists such that

$$f(x) < f(x^*) \text{ for all } x \in \mathbb{R}^n \text{ with } \|x - x^*\|_2 \le \epsilon.$$

Since x^* is an accumulation points of $(x_k)_{k\in\mathbb{N}_0}$ a k_0 exists such that

$$\|x_{k_0} - x^*\|_2 \le \epsilon \text{ for all } k \in \mathbb{N}_0 \text{ with } k \ge k_0.$$

Thus

$$f(x_{k_0}) < f(x^*) \text{ for all } k \in \mathbb{N}_0 \text{ with } k \ge k_0.$$

Furthermore

$$f(x_{k+1}) < f(x_k) \text{ for all } k \in \mathbb{N}_0$$

due to the definition of the general descent method. This implies

$$f(x_k) < f(x_{k_0}) < f(x^*) \text{ for all } k \in \mathbb{N}_0 \text{ with } k \ge k_0$$

and thus

$$\lim_{k\to\infty} f(x_{n_k^*}) \le f(x_{k_0}) \ne f(x^*).$$

This is a contradiction to

$$\lim_{k\to\infty} f(x_{n_k^*}) = f(x^*).$$

Thus x^* can not be a strict maximum.

c) Use $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto -x^2$ and $x_0 = 0$. x_0 a strict global maximum of f. Furthermore $\nabla f(x_0) = 0$ and thus no descent direction exists and the general descent method must stop at x_0.

$\square$