SoSe 2020 Kiel University
Dr. Joscha Reimer Department of Computer Science

Solution homework exercise 3.2: (Roosenbrock function)

Statement:

$$f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto 100(y - x^2)^2 + (1 - x)^2$$

has a strict global minimum at $(1, 1)^T$ *and no other minima or maxima.*

Proof: Assume that f has a minimum or maximum $(\hat{x}, \hat{y})^T \in \mathbb{R}^2$. Due to the necessary first order optimality condition,

$$\nabla f(\hat{x}, \hat{y}) = 0.$$

For all $(x, y)^T \in \mathbb{R}^2$,

$$\nabla f(x, y) = \begin{pmatrix} 200(y - x^2)(-2x) + 2(1 - x)(-1) \\ 200(y - x^2) \end{pmatrix} = \begin{pmatrix} -400x(y - x^2) + 2x - 2 \\ 200(y - x^2) \end{pmatrix}.$$

Thus it follows

$$\hat{x}^2 = \hat{y}$$

from $200(\hat{y} - \hat{x}^2) = 0$ and thus

$$2\hat{x} = 2$$

from $-400\hat{x}(\hat{y} - \hat{x}^2) + 2\hat{x} - 2 = 0$. Hence,

$$\hat{x} = 1$$

and

$$\hat{y} = 1.$$

So if f has a minimum or a maximum, it must be $(1, 1)^T$. For all $(x, y)^T \in \mathbb{R}^2$,

$$\nabla^2 f(x, y) = \begin{pmatrix} -400(y - 3x^2) + 2 & -400x \\ -400x & 200 \end{pmatrix}.$$

Hence

$$\nabla^2 f(\hat{x}, \hat{y}) = \nabla^2 f(1, 1) = \begin{pmatrix} 802 & -400 \\ -400 & 200 \end{pmatrix}.$$

Since $802 > 0$ and $\det(\nabla^2 f(\hat{x}, \hat{y})) = 802 * 200 - (-400)^2 = 400 > 0$, $\nabla^2 f(\hat{x}, \hat{y})$ is positive definite due to Silvester's criterion. Due to the second order sufficient optimality condition, $(1, 1)^T$ is a strict minimum. It is also global since

$$f(1, 1) = 0 \leq 100(y - x^2)^2 + (1 - x)^2 = f(x, y)$$

for all $(x, y)^T \in \mathbb{R}^2$. $\square$