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Solution homework exercise 3.4:

Statement: For

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x,y) = x^2 - 2xy + y^2$$

are all $(x,x)^T$ with $x \in \mathbb{R}$ nonstrict global minima. Other minima or maxima do not exist.

Proof: Assume that f has a minimum or a maximum $(\hat{x},\hat{y})^T \in \mathbb{R}^2$. Due to the necessary first order optimality condition,

$$\nabla f(\hat{x},\hat{y}) = 0.$$

For all $(x,y)^T \in \mathbb{R}^2$,

$$\nabla f(x,y) = \begin{pmatrix} 2x - 2y \\ -2x + 2y \end{pmatrix}.$$

Thus it follows

$$\hat{x} = \hat{y}.$$

Furthermore

$$\nabla^2 f(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

for all $(x,y)^T \in \mathbb{R}^2$. Since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 \geq 0,$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = 2 \geq 0$$

and

$$\det\left(\frac{\partial^2 f}{\partial x^2}(\hat{x},\hat{y})\right) = 2^2 - (-2)^2 = 0 \geq 0$$

for all $(x,y)^T \in \mathbb{R}^2$, it follows that $\nabla^2 f(x,y)^T$ is positive semidefinite for all $(x,y)^T \in \mathbb{R}^2$ due to Silvester's criterion. Thus $(x,x)^T$ is a minimum of f for every $x \in \mathbb{R}$ due to the second order sufficient optimality condition. Since

$$f(x,x) = x^2 - 2xx + x^2 = 0 \leq (x-y)^2 = f(x,y)$$

for all $(x,y)^T \in \mathbb{R}^2$, all these minima are nonstrict global minima. $\square$