Factoring Part 2: Il Prodotto Speciale

More advanced techniques!

Before we start...

Pair up with somebody and try to simplify the expression (*hint: look for common factors!*)

$$\frac{36x^2 + 24xy + 18x^3}{6xyz + 48x^2}$$

We have a common factor of $6x$ on both the top and bottom!

$$\frac{36x^2 + 24xy + 18x^3}{6xyz + 48x^2} = \frac{6x(6x + 4y + 3x^2)}{6x(yz + 8x)} = \boxed{\frac{6x + 4y + 3x^2}{yz + 8x}}$$

What *is* factoring?

- Last week:

> ***breaking down*** *a number a number into the* ***product*** *of* ***smaller numbers****.*

- But what is a *number*?

1. The number 5 is a number
2. The expression $5x$ represents a number, for some x
3. But.... what about the expression $(x^2 + 6x + 9)$
    - There's no common factor we can take out 😟

The Special Product

Let us consider the following expression

$$(x + a)(x + a) = (x + a)^2$$

here is what happens when we multiply everything out

$$(x + a)(x + a) = x^2 + ax + ax + a^2 = x^2 + 2ax + a^2$$

Look what happens when we let $a = 3$!

$$(x + 3)(x + 3) = x^2 + 6x + 9$$

😮 😮 😮

What to look for

Let's study the equation

$$x^2 + 2ax + a^2$$

There are four things to look for

1. All terms are positive (they have a $+$)
2. The first term has x^2
3. The second term is $2x$ times some number a
4. The third term is the ***square*** of that number a

If we see this form, we know we can factor into $(x + a)^2$! 😃

An example

The formula is

$$x^2 + 2ax + a^2 = (x + a)^2$$

Let us try to factor

$$x^2 + 6x + 9$$

$$\underbrace{x^2 + 2(3)x + 3(3)}_{a=3}$$

$$(x + 3)(x + 3)$$

Let's practice!

$$x^2 + 2ax + a^2 = (x + a)^2$$

With a partner, try to factor these expressions

$$x^2 + 4x + 4 = (x + \ ?)^2$$
$$x^2 + 12x + 36 = (x + \ ?)^2$$
$$x^2 + 24x + 144 = (x + \ ?)$$
$$2x^2 + 16x + 32 = \ ?$$

Let's practice!

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Let's practice!

$$x^2 + 2ax + a^2 = (x + a)^2$$

With a partner, try to factor these expressions

$$\begin{aligned} x^2 + 4x + 4 &= (x + 2)^2 \\ x^2 + 12x + 36 &= (x + 6)^2 \\ x^2 + 24x + 144 &= (x + \ ?) \\ 2x^2 + 16x + 32 &= \ ? \end{aligned}$$

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$$x^2 + 24x + 144 = (x + 12)$$
$$2x^2 + 16x + 32 = 2(x^2 + 8x + 16) = 2(x + 4)^2$$

A harder example

Recall the formula is

$$x^2 + 2ax + a^2 = (x + a)^2$$

Let us take a look at

$$4x^2 + 12x + 9$$

True or False: this is also the case where $a = 3$

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True or False: this is also the case where $a = 3$

True! We can always look at the last term

A harder example

Recall the formula is

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We can write this as

$$(2x)^2 + 2(3)(2x) + 3^2$$

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Let us take a look at

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We can write this as

$$(2x)^2 + 2(3)(2x) + 3^2 = (2x + 3)^2$$

Let's Practice

With a partner, factor the following expression (*Hint: what is* a?)

$$9x^2 + 12x + 4$$

Let’s Practice

Recall the formula is

$$x^2 + 2ax + a^2 = (x + a)^2$$

Let us re-write our expression after seeing $a = 2$

$$9x^2 + 12x + 4$$
$$(3x)^2 + 2(2)(3x) + 2^2$$
$$(3x + 2)^2$$

Some more special products

Work together with the ***same*** partner to try and factor the following two equations

$$x^2 + 18x + 81$$

$$x^2 - 18x + 81 \qquad \text{Hint: how can we change our answer from above?}$$

Some more special products

Work together with the ***same*** partner to try and factor the following two equations

$$x^2 + 18x + 81 = (x + 9)^2$$
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$$x^2 - 18x + 81 = (x-9)^2$$

Let's review this second example

A different special form

What if we expand

$$(x-a)(x-a)$$
$$x^2-ax-ax+a^2$$
$$x^2-2ax+a^2$$

This is almost the same as before! We get

$$x^2-2ax+a^2=(x-a)^2$$

This is why $x^2-18x+81=(x-9)^2$

A very special form

Let's look at what happens if signs are opposite now!

$$(x+a)(x-a)$$
$$x^2 - ax + ax - a^2$$
$$x^2 - a^2$$

We're left with only $x^2 - a^2$ 😮😮😮!

This means that we can factor

$$x^2 - a^2 = (x+a)(x-a)$$

$x^2 - a^2$ is known as the ***difference of two perfect squares***

An example

We can factor

$$x^2 - 9 = (x + 3)(x - 3)$$

Let's Practice

Recall the formula is

$$x^2 - a^2 = (x + a)(x - 1)$$

With a partner, factor the following equations

$$x^2 - 4$$
$$x^2 - 81$$
$$2x^2 - 32$$
$$4x^2 - 9$$
$$x^4y^8 - 16$$
$$x^2y^4 - 4z^6$$

Let's Practice

Recall the formula is

$$x^2 - a^2 = (x + a)(x - 1)$$

With a partner, factor the following equations

$$x^2 - 4 = (x + 2)(x - 2)$$

$$x^2 - 81$$

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$$x^4y^8 - 16 = \left((x^2y^4)^2 - 4^2\right) = (x^2y^4 + 4)(x^2y^4 - 4)$$
$$x^2y^4 - 4z^6 = \left((xy^2)^2 - (2z^3)^2\right) = (xy^2 + 2z^3)(xy^2 - 2z^3)$$

Wrapping Up

We've seen some really useful equations for factoring today!

$$\begin{aligned} x^2 + 2ax + a^2 &= (x+a)^2 \\ x^2 - 2ax + a^2 &= (x-a)^2 \\ x^2 - a^2 &= (x+a)(x-a) \end{aligned}$$

Don't forget what we saw last time: ***factoring out a common factor***.