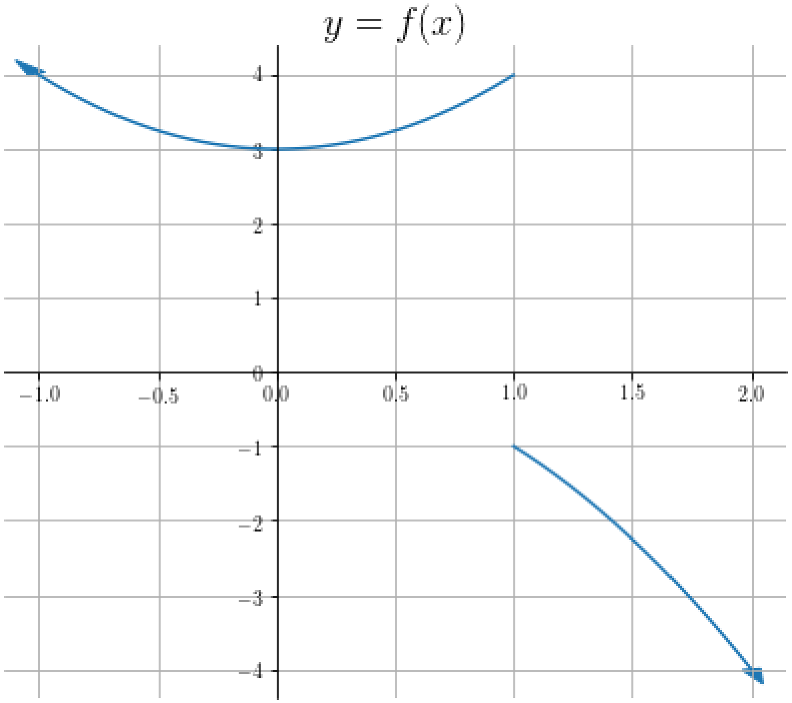
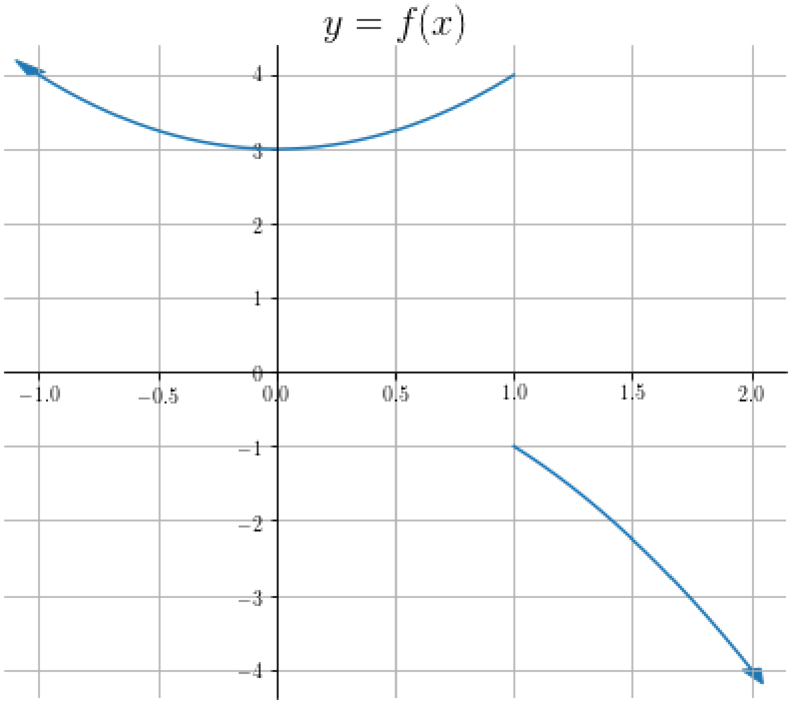
Limits Day 2

Warm-Up Questions!



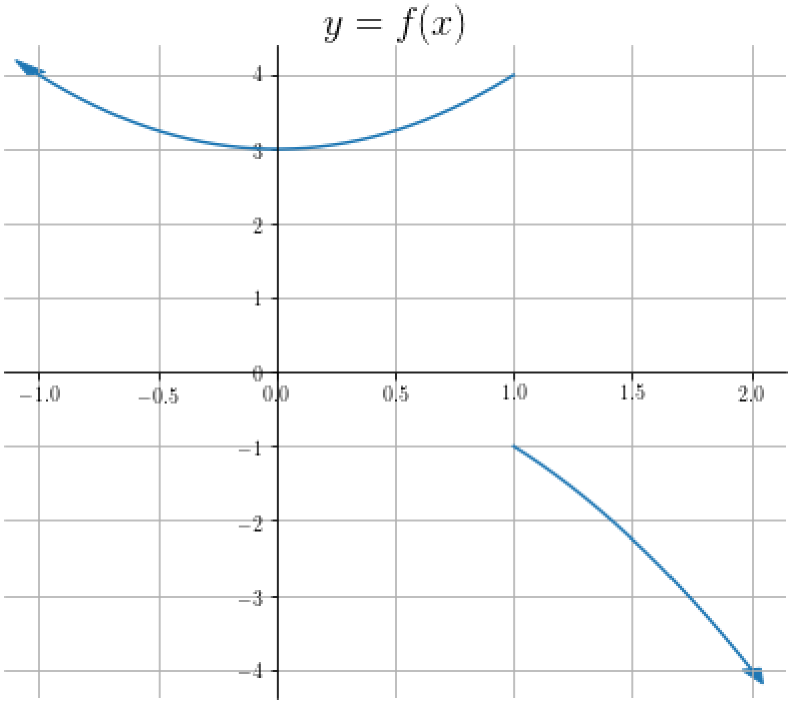
What is $\lim_{x \to 1^-} f(x)$?

Warm-Up Questions!



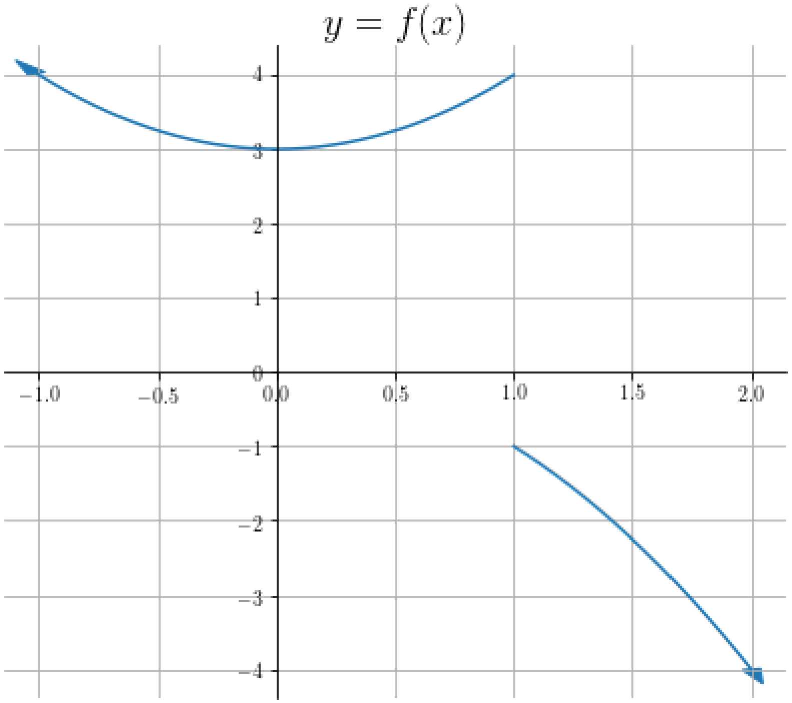
$$\lim_{x \to 1^-} f(x) = 4$$

Warm-Up Questions!



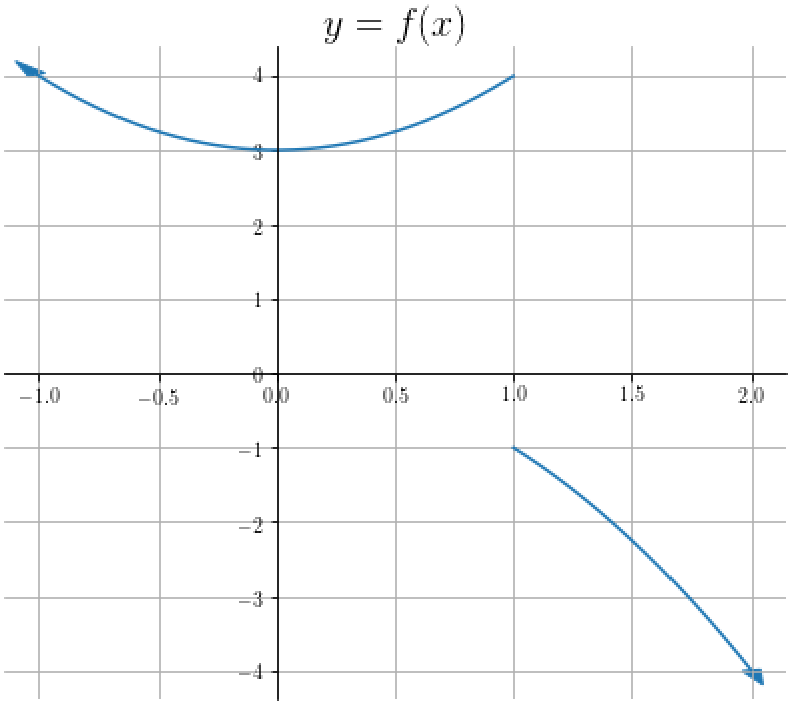
What is $\lim_{x \to 1^+} f(x)$?

Warm-Up Questions!



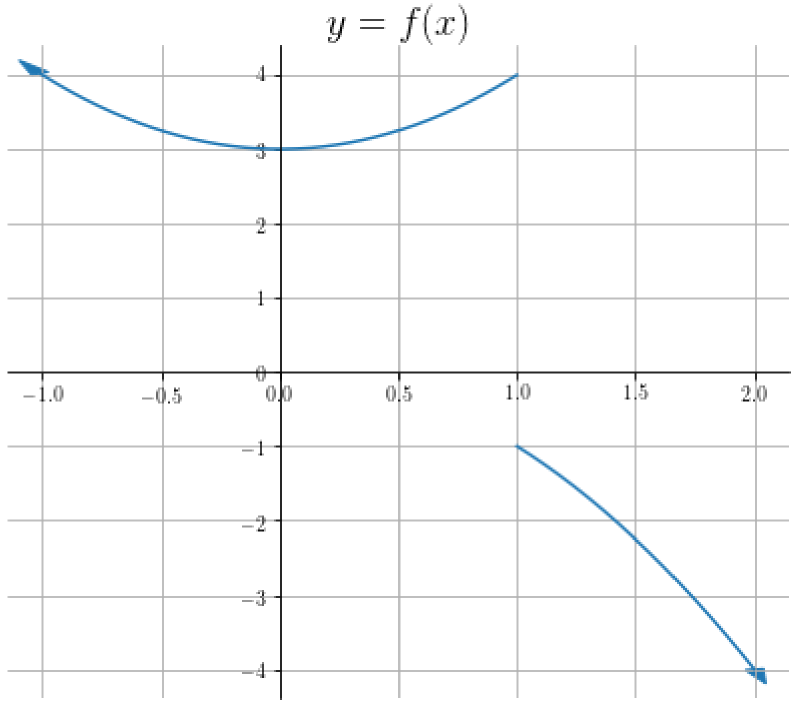
$$\lim_{x \to 1^+} f(x) = -1$$

Warm-Up Questions!



What is $\lim_{x \to 1} f(x)$?

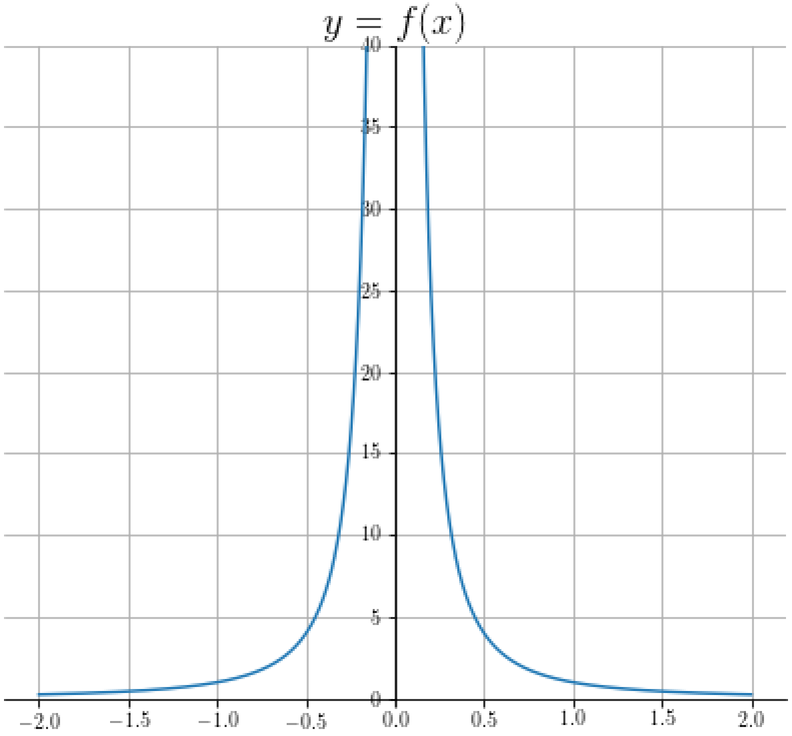
Warm-Up Questions!



$\lim_{x \to 1} f(x)$ is undefined, because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

A new problem: asymptotes

Consider the following function: $f(x) = \frac{1}{x^2}$



At $x = 0$, the function is undefined. But, there is not a clear value that it "approaches" there; how do we take the limit?

A new problem: asymptotes

Consider the following function: $f(x) = \frac{1}{x^2}$

We say that $x = 0$ is a vertical ***asymptote***. A vertical asymptote is a value at which the function "blows up" around.

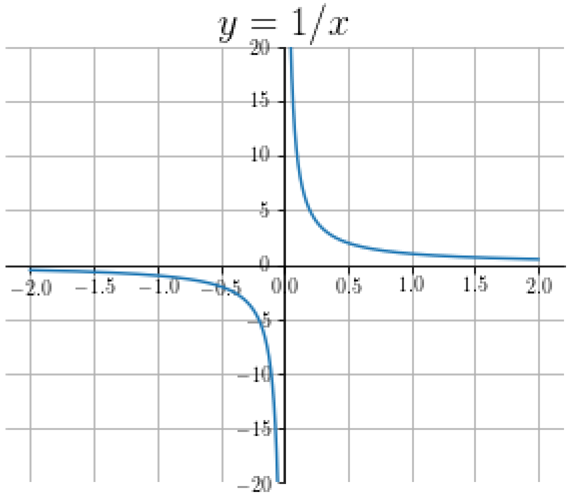
$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Concept Question

True or False: $\lim_{x \to 0} \frac{1}{x} = \infty$

False! $\lim_{x \to 0} \frac{1}{x}$ = undefined because

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty$$



Determining Asymptotes

If we are trying to evaluate

$$\lim_{x \to a} f(x)$$

and

$$f(a) = \frac{1}{0}$$

then we say that a is a *vertical asymptote.*

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x) = \pm\infty$.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \infty \to \lim_{x \to a} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = -\infty \to \lim_{x \to a} f(x) = -\infty$$

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \to \lim_{x \to a} f(x) = \text{undefined}$$

Note that $\frac{1}{0}$ is NOT an indeterminant form!

Constructing Graphs

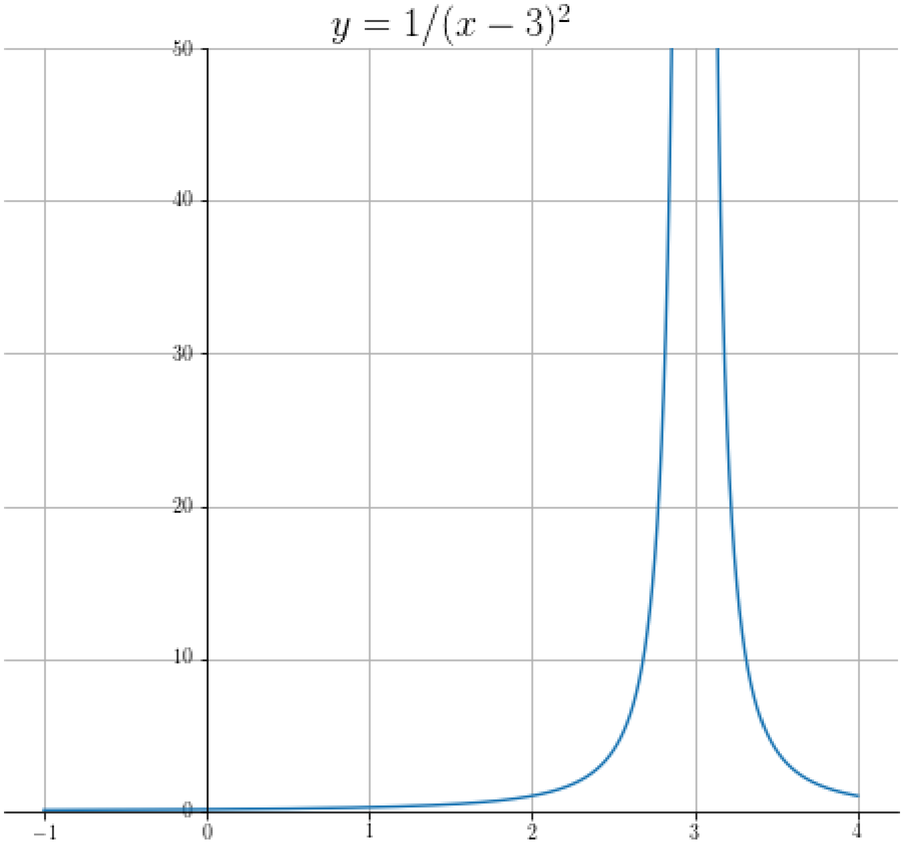
In ***groups of 3***, create 3 functions that have the desired limits

1. $\lim_{x \to 3} f(x) = \infty$
2. $\lim_{x \to 3^-} g(x) = -\infty$ and $\lim_{x \to 3^+} g(x) = \infty$
3. $\lim_{x \to 0^+} h(x) = -\infty$ and $h(x)$ is not a fraction

Constructing Graphs Example 1

$$\lim_{x \to 3} f(x) = \infty$$

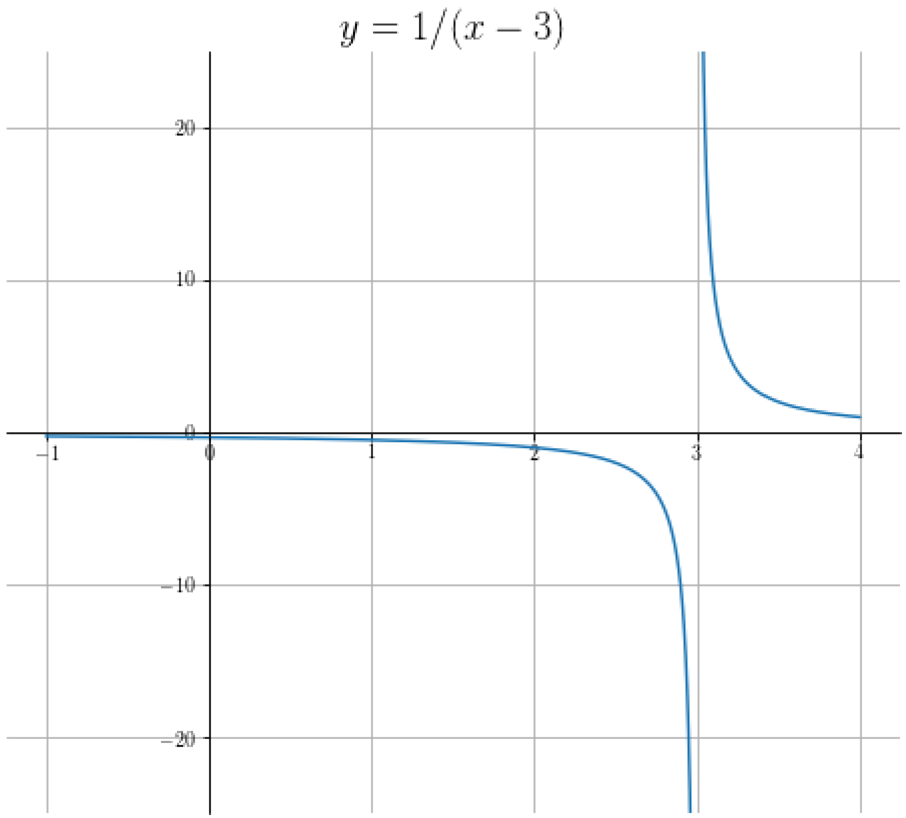
$$f(x) = \frac{1}{(x-3)^2}$$



Constructing Graphs Example 2

$$\lim_{x \to 3^-} g(x) = -\infty \qquad \lim_{x \to 3^+} g(x) = \infty$$

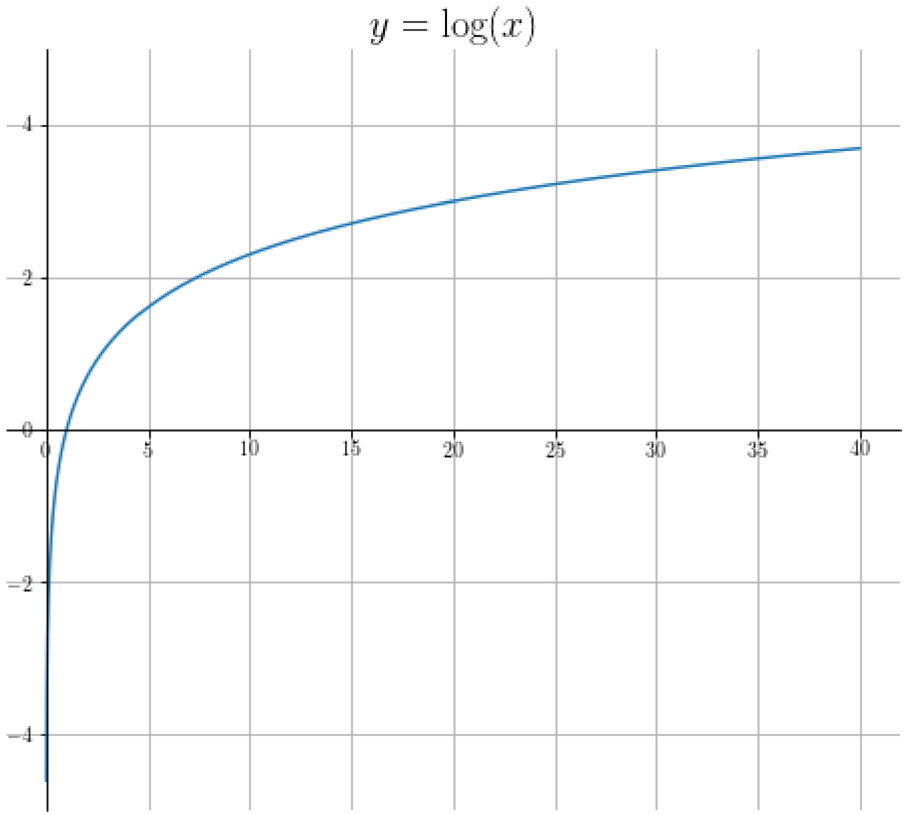
$$g(x) = \frac{1}{(x-3)}$$



Constructing Graphs

$$\lim_{x \to 0^+} h(x) = -\infty$$

$$h(x) = \log(x)$$



A New Idea: What if $x \to \infty$?

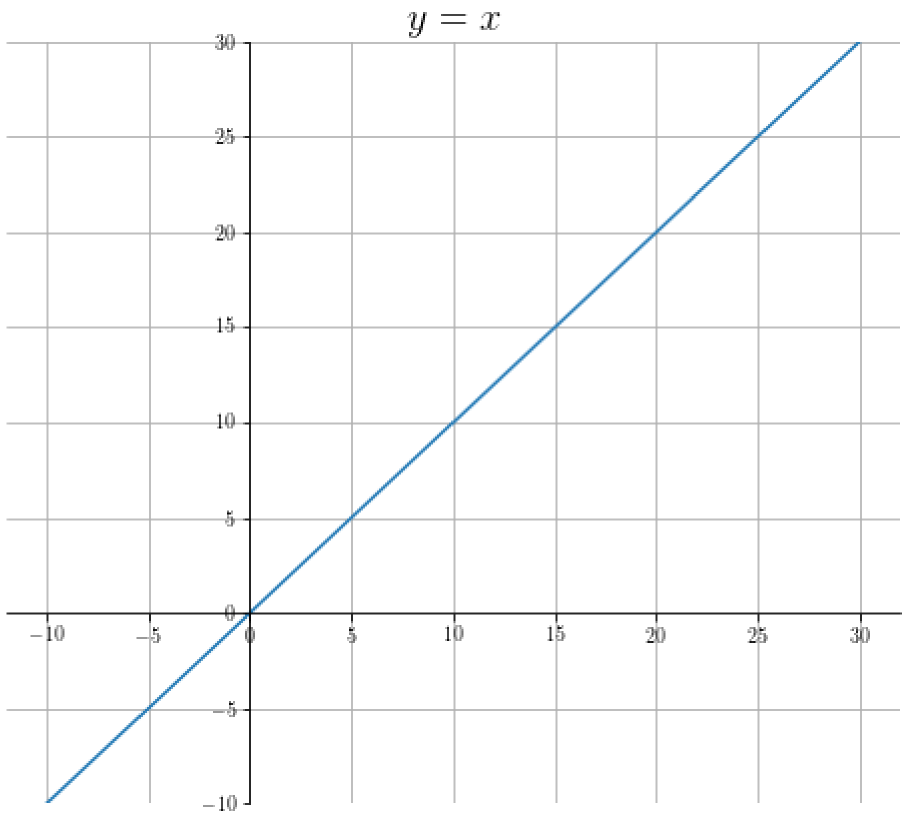
Sometimes we care about understanding what happens to the value of a function for ***arbitrarily large inputs***. Generally, one of two things will happen:

1. The function ***diverges to*** $\pm\infty$
2. The function ***converges to some exact value***.

We'll look at some examples now.

A simple diverging graph

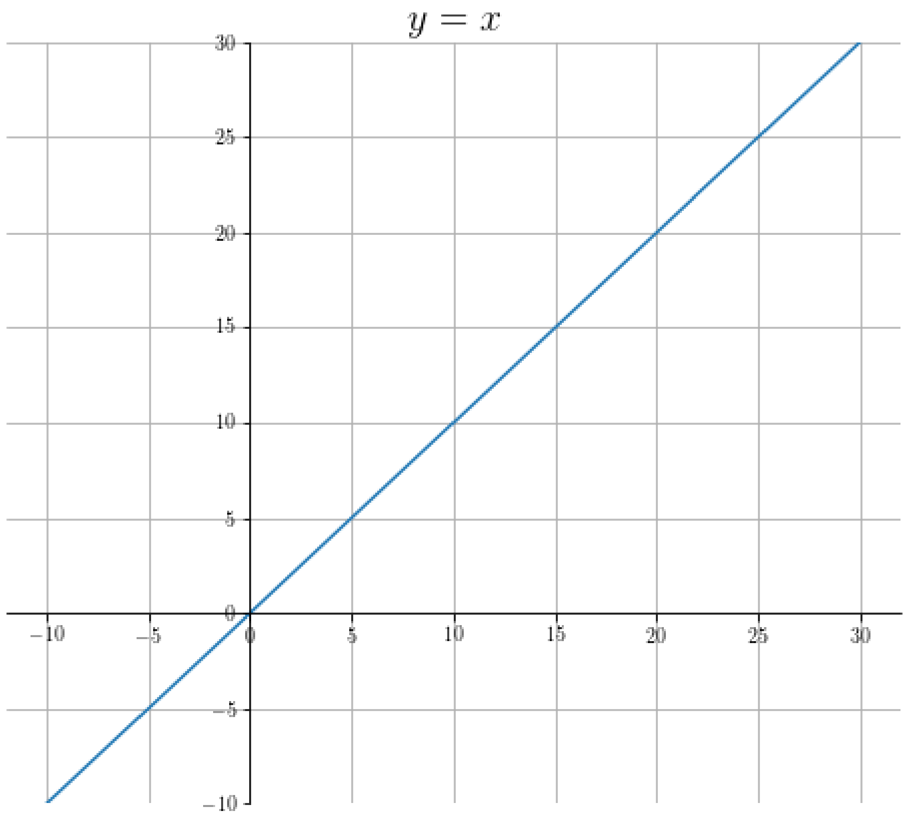
$$f(x) = x$$



What happens as $x \to \infty$?

A simple diverging graph

$$f(x) = x$$



What happens as $x \to \infty$? $\lim_{x \to \infty} f(x) = \infty$!

Some more diverging functions

What are some more functions that diverge as $x \to \infty$?

- $f(x) = x^2$
- $f(x) = \log(x)$
- $f(x) = e^x$

Concept Question

True or False: If $\lim_{x \to \infty} f(x) = \infty$, then $\lim_{x \to \infty} c \cdot f(x) = \infty$ for any constant number c

True!: constants do not change the *behavior* of a function, only its specific values.

This makes it easy to evaluate limits with a bunch of constants in the function by ignoring them. I.e.

$$\lim_{x \to \infty} c \cdot f(x) = c \cdot \lim_{x \to \infty} f(x)$$

Rational Expressions

Consider the following limit

$$\lim_{x\to\infty} \frac{x^3 + 3x - 10}{2x^2}$$

How do we think about evaluating this?

On top, as x gets arbitrarily large, really only the x^3 term matters since it's much larger than the other two. So this limit should really be the same as

$$\lim_{x\to\infty} \frac{x^3}{2x^2}$$

Rational Expressions

$$\lim_{x\to\infty} \frac{x^3}{2x^2} = \frac{1}{2} \lim_{x\to\infty} \frac{x^3}{x^2}$$

As x grows arbitrarily large, x^3 grows ***much, much, much faster*** than x^2.

Since the numerator of the fraction is getter larger than the denominator, we get that

$$\lim_{x\to\infty} \frac{x^3}{2x^2} = \infty$$

Thus,

$$\lim_{x\to\infty} \frac{x^3 + 3x - 10}{2x^2} = \infty$$

What does *grows faster* mean?

Remember that we are letting x get arbitrarily large. Let's look at a table of values

x	x^3	x^2	$x^3 - x^2$
...	...	...	...
10	1000	100	900
11	1331	121	1210
12	1728	144	1584
13	2197	169	2028
14	2744	196	2548
...	...	...	...

The difference between the numerator and denominator is increasing (at an increasing rate!), meaning the fraction gets larger and larger.

Diverges?

Determine if each of the limits below diverges (does the limit $= \infty$?)

$$\lim_{x\to\infty} \frac{x^3}{x}$$

$$\lim_{x\to\infty} \frac{\log(x)}{x}$$

$$\lim_{x\to\infty} \frac{e^x}{x}$$

$$\lim_{x\to\infty} \frac{x^3}{\log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x^5 \log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + \log(x)}{x \log(x)}$$

Diverges?

Determine if each of the limits below diverges (does the limit $= \infty$?)

$$\lim_{x\to\infty} \frac{x^3}{x} = \infty$$

$$\lim_{x\to\infty} \frac{\log(x)}{x}$$

$$\lim_{x\to\infty} \frac{e^x}{x}$$

$$\lim_{x\to\infty} \frac{x^3}{\log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x^5 \log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + \log(x)}{x \log(x)}$$

Diverges?

Determine if each of the limits below diverges (does the limit $= \infty$?)

$$\lim_{x\to\infty} \frac{x^3}{x} = \infty$$

$$\lim_{x\to\infty} \frac{\log(x)}{x} \neq \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x}$$

$$\lim_{x\to\infty} \frac{x^3}{\log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x^5 \log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + \log(x)}{x \log(x)}$$

Diverges?

Determine if each of the limits below diverges (does the limit $= \infty$?)

$$\lim_{x\to\infty} \frac{x^3}{x} = \infty$$

$$\lim_{x\to\infty} \frac{\log(x)}{x} \neq \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty$$

$$\lim_{x\to\infty} \frac{x^3}{\log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x^5 \log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + \log(x)}{x \log(x)}$$

Diverges?

Determine if each of the limits below diverges (does the limit $= \infty$?)

$$\lim_{x\to\infty} \frac{x^3}{x} = \infty$$

$$\lim_{x\to\infty} \frac{\log(x)}{x} \neq \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty$$

$$\lim_{x\to\infty} \frac{x^3}{\log(x)} = \infty$$

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x^5 \log(x)}$$

$$\lim_{x\to\infty} \frac{x^2 + \log(x)}{x \log(x)}$$

Diverges?

Determine if each of the limits below diverges (does the limit $= \infty$?)

$$\lim_{x\to\infty} \frac{x^3}{x} = \infty$$

$$\lim_{x\to\infty} \frac{\log(x)}{x} \neq \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty$$

$$\lim_{x\to\infty} \frac{x^3}{\log(x)} = \infty$$

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x^5 \log(x)} \neq \infty$$

$$\lim_{x\to\infty} \frac{x^2 + \log(x)}{x \log(x)}$$

Diverges?

Determine if each of the limits below diverges (does the limit $= \infty$?)

$$\lim_{x\to\infty} \frac{x^3}{x} = \infty$$

$$\lim_{x\to\infty} \frac{\log(x)}{x} \neq \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty$$

$$\lim_{x\to\infty} \frac{x^3}{\log(x)} = \infty$$

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x^5 \log(x)} \neq \infty$$

$$\lim_{x\to\infty} \frac{x^2 + \log(x)}{x \log(x)} = \infty$$

Wrapping Up

- If we are taking a limit of $\frac{p(x)}{q(x)}$, if $p(x)$ ***grows faster*** than $q(x)$, then the function diverges (the limit as $x \to \infty$ is ∞).
    - Some more examples include $\frac{x}{\log(x)}$ and $\frac{e^x}{x^c}$ for any value of c!
- Next time we will explore graphs that ***converge*** to a specific value as $x \to \infty$!