Second Degree Equations Part 2

Factoring

Remember that we have seen before that we can factor a quadratic expression in a couple of ways

$$\begin{aligned} x^2 - 4 &= (x+2)(x-2) && \text{special product} \\ x^2 + 2x - 24 &= (x-4)(x+6) && \text{factor directly} \end{aligned}$$

We have also seen that there are some equations that we cannot factor with direct methods, such as

$$x^2 - 3x - 7$$

But maybe the *quadratic formula* can help factor our expression.

Remember our goal is to factor $x^2 - 3x - 7$ into something that looks like $(x + c_1)(x + c_2)$. This means we want to find c_1 and c_2 such that

$$x^2 - 3x - 7 = (x + c_1)(x + c_2)$$

Factoring

$$x^2 - 3x - 7 = (x + c_1)(x + c_2)$$

The *quadratic equation* gives us all x such that $x^2 - 3x - 7 = 0$. This means that we know an x such that

$$x^2 - 3x - 7 = (x + c_1)(x + c_2) = 0$$

In order for $(x + c_1)(x + c_2) = 0$, it ***must*** be true that eiter $(x + c_1) = 0$ or $(x + c_2) = 0$.

Hmmmmmm.....

Let’s try

Let's use the *quadratic equation* on $x^2 - 3x - 7 = 0$. We get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot -7}}{2 \cdot 1}$$

$$x = \frac{3 \pm \sqrt{9 + 28}}{2}$$

$$x = \frac{3 \pm \sqrt{37}}{2}$$

$$x = \frac{3 + \sqrt{37}}{2}, \frac{3 - \sqrt{37}}{2} = x_+, x_-$$

Let's try

$$x_+, x_- = \frac{3+\sqrt{37}}{2}, \frac{3-\sqrt{37}}{2}$$

We want to find c_1 and c_2 so that $(x+c_1)(x+c_2) = 0$ only when $x = x_+$ and $x = x_-$

Idea: let $c_1 = -x_+$ and $c_2 = -x_-$!

We get $x^2 - 3x - 7 = (x - x_+)(x - x_-)$, which is zero *only when* $x = x_+$ or $x = x_-$!

This gives us

$$x^2 - 3x - 7 = \left(x - \frac{3+\sqrt{37}}{2}\right)\left(x - \frac{3-\sqrt{37}}{2}\right)$$

(which if you expand can see are equal)!

Last Comments on Factoring

We factored the expression $x^2 - 3x - 7$ using the quadratic formula. But what happens if $a \neq 1$?

All we have to do is add a factor of a into our factorization!

To factor an expression $ax^2 + bx + c$ for any values of a, b, c

Calculate

$$x_+, x_- = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Factor into $a(x - x_+)(x - x_-)$

(Optionally) distribute out a to make our factors cleaner. For example,

$$6\left(x - \frac{2}{3}\right)\left(x + \frac{3}{2}\right) = 3\left(x - \frac{2}{3}\right)2\left(x + \frac{3}{2}\right) = (3x - 2)(2x + 3)$$

Practice

In *groups of 3*, factor the following expressions with the quadratic formula

$$x^2 + x - 6$$
$$3x^2 - 9x + 6$$
$$5x^2 + 6x + 1$$
$$6x^2 - 24x - 30$$

Practice

In *groups of 3*, factor the following expressions with the quadratic formula

$$x^2 + x - 6 = (x - 2)(x + 3)$$
$$3x^2 - 9x + 6$$
$$5x^2 + 6x + 1$$
$$6x^2 - 24x - 30$$

Practice

In *groups of 3*, factor the following expressions with the quadratic formula

$$
\begin{aligned}
x^2 + x - 6 &= (x-2)(x+3) \\
3x^2 - 9x + 6 &= 3(x-2)(x-1) = (3x-6)(x-1) \\
5x^2 + 6x + 1 & \\
6x^2 - 24x - 30 &
\end{aligned}
$$

Practice

In *groups of 3*, factor the following expressions with the quadratic formula

$$\begin{aligned} x^2 + x - 6 &= (x-2)(x+3) \\ 3x^2 - 9x + 6 &= 3(x-2)(x-1) = (3x-6)(x-1) \\ 5x^2 + 6x + 1 &= 5(x+1)\left(x + \frac{2}{10}\right) = (x+1)\left(5x+1\right) \end{aligned}$$

$$6x^2 - 24x - 30$$

Practice

In *groups of 3*, factor the following expressions with the quadratic formula

$$
\begin{aligned}
x^2 + x - 6 &= (x-2)(x+3) \\
3x^2 - 9x + 6 &= 3(x-2)(x-1) = (3x-6)(x-1) \\
5x^2 + 6x + 1 &= 5(x+1)\left(x + \frac{2}{10}\right) = (x+1)(5x+1) \\
6x^2 - 24x - 30 &= 6(x+1)(x-5) = (6x+6)(x-5)
\end{aligned}
$$

Graphing Parabolas

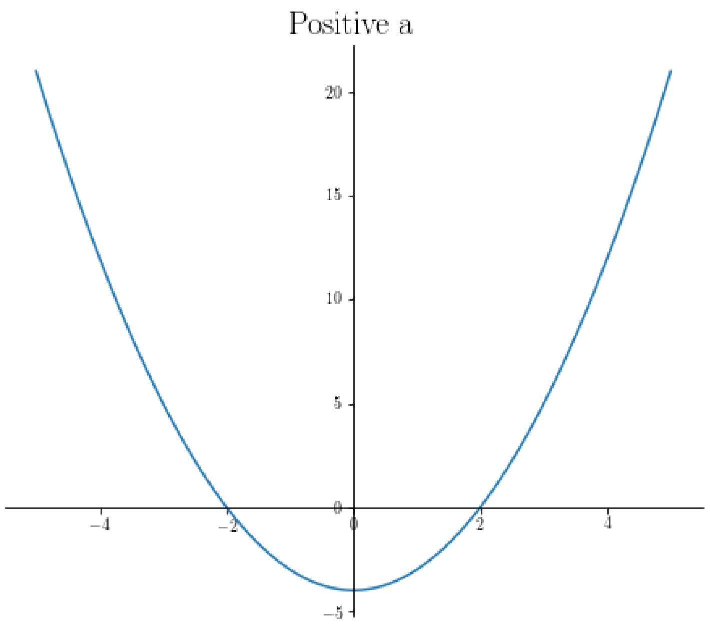
Before we start computing points to graph, let us understand the parts of a second degree equation. Remember we are looking at the graph

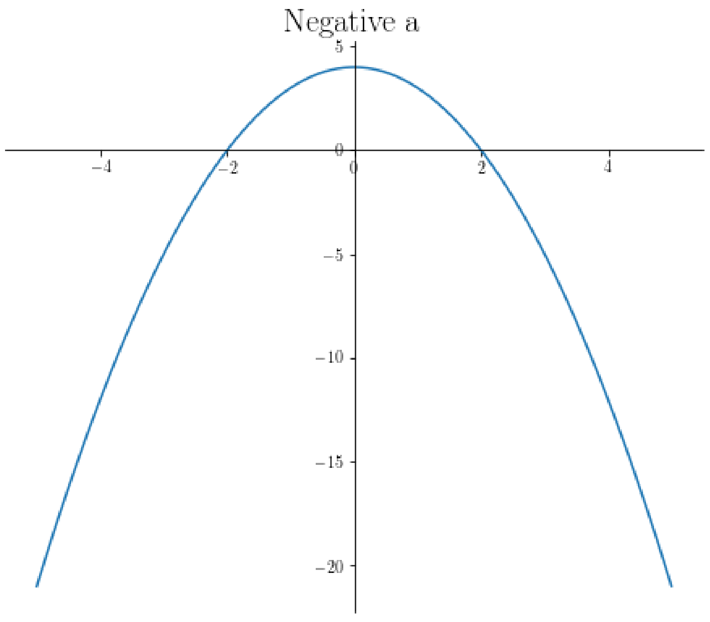
$$y = ax^2 + bx + c$$

True or False: If a is positive, the parabola always "opens upwards" (looks like a ∪), and if a is negative, the parabola always "opens downwards" (looks luke an ∩).

Graphing Parabolas

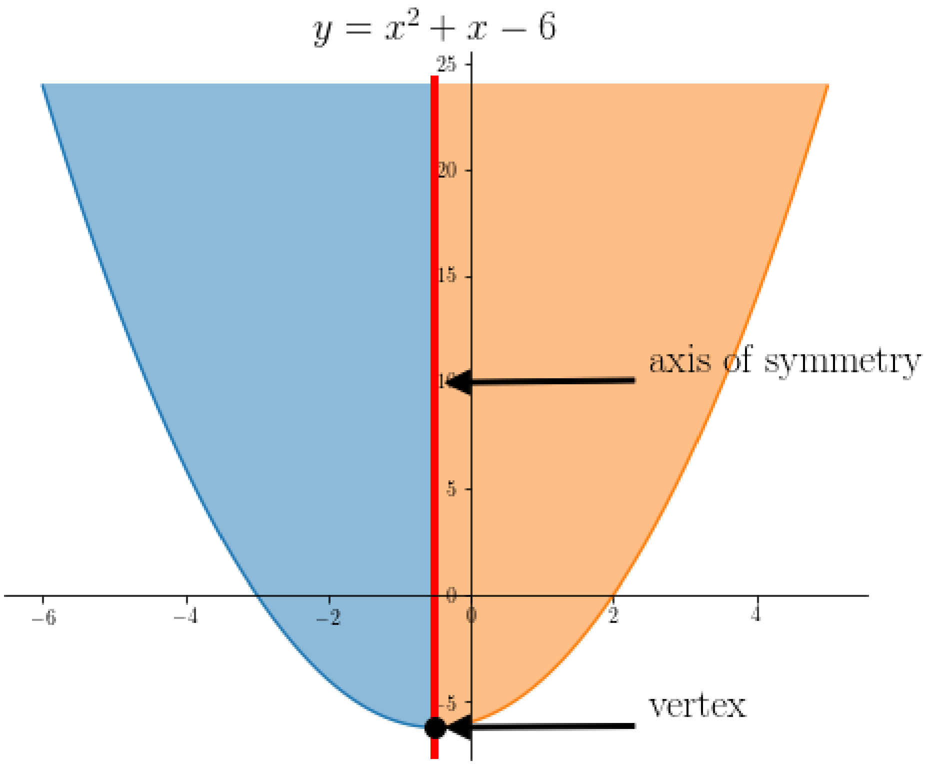
True! If a is positive, we will have some x value that is a minimum, and then get more positive as we move away from it. If a is negative, we will have some x value that is a maximum, and then get more negative as we move away from it.





Axis of Symmetry

Notice how a parabola is *symmetric* about its minimum/maximum point (called its vertex).



Axis of Symmetry

For a parabola given as

$$y = ax^2 + bx + c$$

We can calculalte the ***x value*** for the *axis of symmetry* (the red line from before!) as

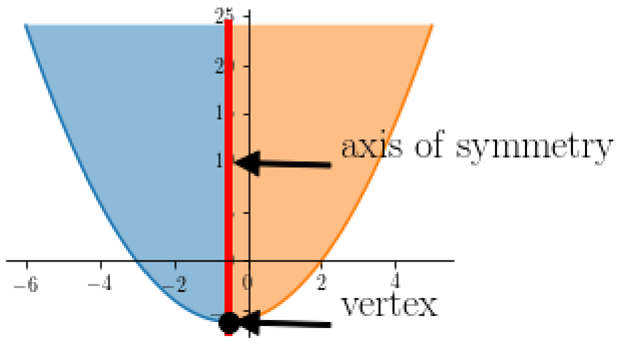
$$x_{axis} = \frac{-b}{2a}$$

True or False: The vertex of a parabola is always on the parabola itself

True! The vertex is defined to be the minimum/maximum value *of the parabola*, so it must be on it.

Maybe we can use this fact...

Finding the Vertex



If we know the equation of the parabola $y = f(x) = ax^2 + bx + c$ and we calculated the axis of symmetry

$$x_{axis} = \frac{-b}{2a}$$

then the vertex is located at

$$(x_{vertex}, y_{vertex}) = (x_{axis}, f(x_{axis}))$$

Why is $x_{vertex} = x_{axis}$?

Practice

In *groups of 3*, find the vertex of each of the following parabolas

$$\begin{aligned} x^2 + x - 6 \\ 3x^2 - 9x + 6 \\ 5x^2 + 6x + 1 \\ 6x^2 - 24x - 30 \end{aligned}$$

Practice Example 1

$$f(x) = x^2 + x - 6$$

$$x_{axis} = \frac{-b}{2a}$$

$$x_{axis} = \frac{-1}{2} = -0.5$$

$$y_{vertex} = f(-0.5) = (-0.5)^2 - 0.5 - 6 = -6.5$$

$$(x_{vertex}, y_{vertex}) = (-0.5, -6.5)$$

Practice Example 2

$$f(x) = 3x^2 - 9x + 6$$
$$x_{axis} = \frac{-b}{2a}$$
$$x_{axis} = \frac{9}{6} = \frac{3}{2}$$
$$y_{vertex} = f(3/2) = 3(3/2)^2 - 9(3/2) + 6 = \frac{-3}{4}$$
$$(x_{vertex}, y_{vertex}) = (3/2, -3/4)$$

Practice Example 3

$$f(x) = 5x^2 + 6x + 1$$

$$x_{axis} = \frac{-b}{2a}$$

$$x_{axis} = \frac{-6}{10} = \frac{-3}{5}$$

$$y_{vertex} = f(-3/5) = 5(-3/5)^2 + 6(-3/5) + 1 = \frac{-4}{5}$$

$$(x_{vertex}, y_{vertex}) = (-3/5, -4/5)$$

Practice Example 3

$$f(x) = 6x^2 - 24x - 30$$

$$x_{axis} = \frac{-b}{2a}$$

$$x_{axis} = \frac{24}{12} = 2$$

$$y_{vertex} = f(2) = 6(2)^2 - 24(2) - 30 = -54$$

$$(x_{vertex}, y_{vertex}) = (2, -54)$$

Tomorrow

Tomorrow we will finish learning how to graph parabolas!