Second Degree Equations Part 2

Factoring

Remember that we have seen before that we can factor a quadratic expression in a couple of ways

$$\begin{aligned} x^2 - 4 &= (x+2)(x-2) \qquad && \text{special product} \\ x^2 + 2x - 24 &= (x-4)(x+6) \qquad && \text{factor directly} \end{aligned}$$

We have also seen that there are some equations that we cannot factor with direct methods, such as

$$x^2 - 3x - 7$$

But maybe the *quadratic formula* can help factor our expression.

Some Review

Let's use the *quadratic equation* on $x^2 - x - 6 = 0$. We get

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{1 \pm \sqrt{1 - 4(1)(-6)}}{2 \cdot 1} \\
x &= \frac{1 \pm \sqrt{1 + 24}}{2} \\
x &= \frac{1 \pm \sqrt{25}}{2} \\
x &= \frac{1 \pm 5}{2} \\
x &= 3, -2 = x_+, x_-
\end{aligned}
$$

Factoring

Let us try to factor

$$x^2 - x - 6$$

using a new method.

Our goal is to find c_1 and c_2 such that

$$x^2 - x - 6 = (x + c_1)(x + c_2)$$

The *quadratic equation* gives us all x such that $x^2 - x - 6 = 0$. This means that we know an x such that

$$x^2 - x - 6 = (x + c_1)(x + c_2) = 0$$

In order for $(x + c_1)(x + c_2) = 0$, it ***must*** be true that eiter $(x + c_1) = 0$ or $(x + c_2) = 0$.

Hmmmmmm.....

Let's try

$$x_+, x_- = 3, -2$$

We want to find c_1 and c_2 so that $(x + c_1)(x + c_2) = 0$ only when $x = x_+$ and $x = x_-$

Idea: let $c_1 = -x_+$ and $c_2 = -x_-$!

We get $x^2 - 3x - 7 = (x - x_+)(x - x_-)$, which is zero *only when* $x = x_+$ or $x = x_-$!

This gives us

$$x^2 - x - 6 = (x - 3)\,(x + 2)$$

Last Comments on Factoring

What happens if $a \neq 1$?

All we have to do is add a factor of a into our factorization!

To factor an expression $ax^2 + bx + c$ for any values of a, b, c

Calculate

$$x_+, x_- = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Factor

$$ax^2 + bx + c = a(x - x_+)(x - x_-)$$

Practice

In *groups of 3*, factor the following expressions with the quadratic formula

$$x^2 + x - 6$$
$$3x^2 - 9x + 6$$
$$5x^2 + 6x + 1$$

Practice Example 1

$$x^2 + x - 6$$

$$\begin{aligned}
x_+, x_- &= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-6)}}{2(1)} \\
&= \frac{-1 \pm \sqrt{1 + 24}}{2} \\
&= \frac{-1 \pm 5}{2} \\
&= 2, -3
\end{aligned}$$

$$x^2 + x - 6 = (1)(x - x_+)(x - x_-)$$

$$x^2 + x - 6 = (x - 2)(x + 3)$$

Practice Example 2

$$3x^2 - 9x + 6$$

$$\begin{aligned}
x_+, x_- &= \frac{9 \pm \sqrt{(-9)^2 - 4(3)(6)}}{2(3)} \\
&= \frac{9 \pm \sqrt{81 - 72}}{6} \\
&= \frac{9 \pm \sqrt{9}}{6} \\
&= \frac{9 \pm 3}{6} \\
&= 2, 1
\end{aligned}$$

$$3x^2 - 9x + 6 = (3)(x - x_+)(x - x_-)$$

$$3x^2 - 9x + 6 = 3(x - 2)(x - 1)$$

Practice Example 3

$$5x^2 + 6x + 1$$

$$\begin{aligned}
x_+, x_- &= \frac{-6 \pm \sqrt{(6)^2 - 4(5)(1)}}{2(5)} \\
&= \frac{-6 \pm \sqrt{36 - 20}}{10} \\
&= \frac{-6 \pm \sqrt{16}}{10} \\
&= \frac{-6 \pm 4}{10} \\
&= \frac{-2}{10}, -1
\end{aligned}$$

$$5x^2 + 6x + 1 = (5)(x - x_+)(x - x_-)$$

$$5x^2 + 6x + 1 = 5\left(x + \frac{2}{10}\right)(x + 1) = (5x + 1)(x + 1)$$

Graphing Parabolas

Before we start computing points to graph, let us understand the parts of a second degree equation. Remember we are looking at the graph

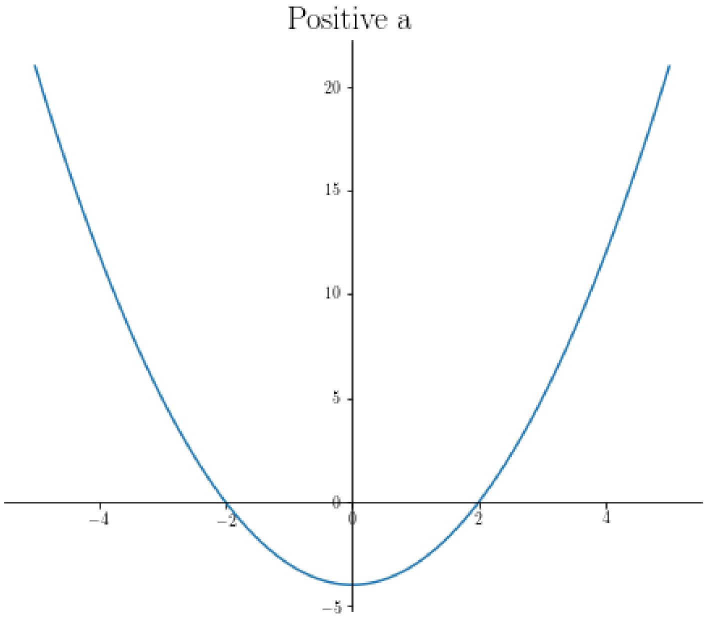
$$y = ax^2 + bx + c$$

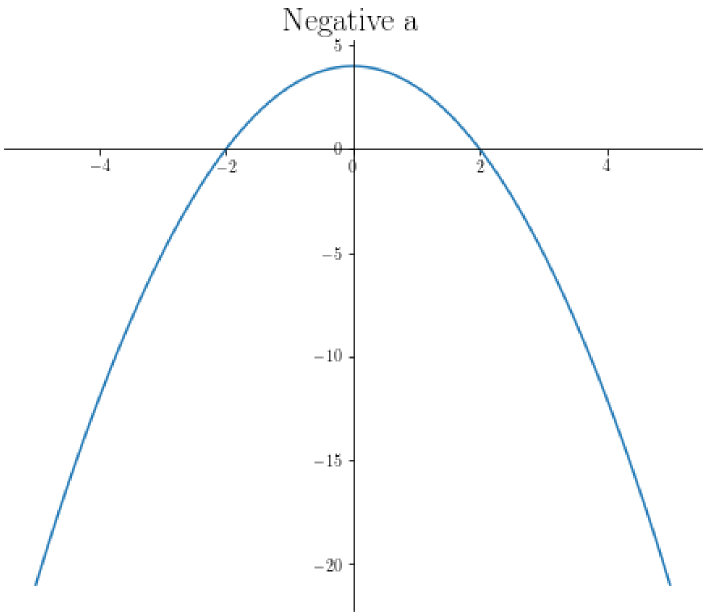
True or False: If a is positive, the parabola always looks like a ∪, and if a is negative, the parabola always looks like a ∩.

Graphing Parabolas

True!

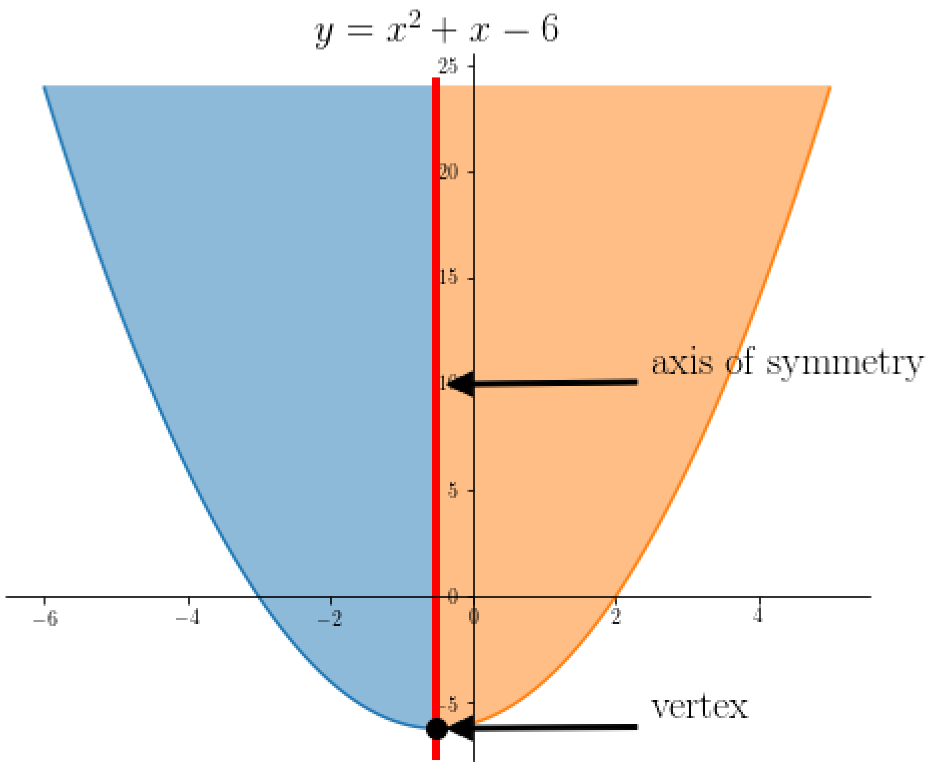
Positive: 😊, Negative: 😩





Axis of Symmetry

Notice how a parabola is *symmetric* about its minimum/maximum point (called its vertex).



Axis of Symmetry

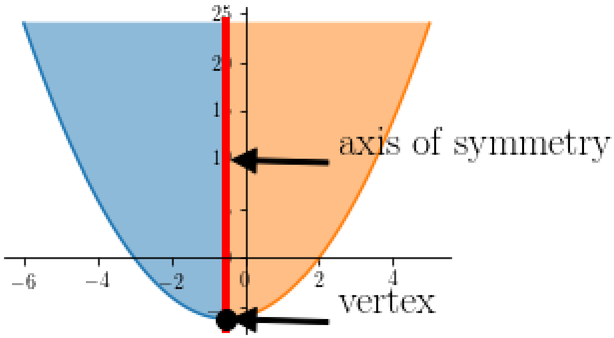
For a parabola given as

$$y = ax^2 + bx + c$$

We can calculalte the ***x value*** for the *axis of symmetry* (the red line from before!) as

$$x_{axis} = \frac{-b}{2a}$$

Finding the Vertex



If we know the equation of the parabola $y = ax^2 + bx + c$ and we calculated the axis of symmetry

$$x_{axis} = \frac{-b}{2a}$$

then the vertex is located at

$$(x_{vertex}, y_{vertex}) = \left(\frac{-b}{2a}, \frac{-\left(b^2 - 4ac\right)}{4a} \right)$$

Practice

$$(x_{vertex}, y_{vertex}) = \left(\frac{-b}{2a}, \frac{-\left(b^2 - 4ac\right)}{4a} \right)$$

In *groups of 3*, find the vertex of each of the following parabolas

$$\begin{aligned} x^2 + x - 6 \\ 3x^2 - 9x + 6 \\ 5x^2 + 6x + 1 \\ 6x^2 - 24x - 30 \end{aligned}$$

Practice Example 1

Parabola: $y = x^2 + x - 6$

$$x_{vertex} = \frac{-b}{2a}$$
$$x_{vertex} = \frac{-1}{2} = -0.5$$
$$y_{vertex} = \frac{-\left(b^2 - 4ac\right)}{4a}$$
$$y_{vertex} = \frac{-\left(1^2 - 4(1)(-6)\right)}{4} = \frac{-(1+24)}{4} = \frac{-25}{4} = -6.25$$
$$(x_{vertex}, y_{vertex}) = (-0.5, -6.25)$$

Practice Example 2

Parabola: $y = 3x^2 - 9x + 6$

$$x_{vertex} = \frac{-b}{2a}$$

$$x_{vertex} = \frac{9}{6} = \frac{3}{2}$$

$$y_{vertex} = \frac{-\left(b^2 - 4ac\right)}{4a}$$

$$y_{vertex} = \frac{-\left((-9)^2 - 4(3)(6)\right)}{4(3)} = \frac{-(81 - 72)}{12} = \frac{-9}{12} = \frac{-3}{4}$$

$$(x_{vertex}, y_{vertex}) = (3/2, -3/4)$$

Practice Example 3

Parabola: $y = 5x^2 + 6x + 1$

$$x_{vertex} = \frac{-b}{2a}$$

$$x_{vertex} = \frac{-6}{10} = \frac{-3}{5}$$

$$y_{vertex} = \frac{-\left(b^2 - 4ac\right)}{4a}$$

$$y_{vertex} = \frac{-\left(6^2 - 4(5)(1)\right)}{4(5)} = \frac{-(36 - 20)}{20} = \frac{-16}{20} = \frac{-4}{5}$$

$$(x_{vertex}, y_{vertex}) = (-3/5, -4/5)$$

Practice Example 4

Parabola: $y = 6x^2 - 24x - 30$

$$x_{vertex} = \frac{-b}{2a}$$
$$x_{vertex} = \frac{24}{12} = 2$$
$$y_{vertex} = \frac{-\left(b^2 - 4ac\right)}{4a}$$
$$y_{vertex} = \frac{-\left(24^2 - 4(6)(-30)\right)}{4(6)} = \frac{-(576 + 720)}{24} = \frac{-1296}{24} = -54$$
$$(x_{vertex}, y_{vertex}) = (2, -54)$$

We can also use a trick to solve this question. We know that $x_{vertex} = x_{axis}$, so we can just evaluate the parabola at x_{axis} to find y_{axis}! This means

$$\begin{aligned} y_{vertex} &= 6(x_{vertex})^2 - 24(x_{vertex}) - 30 \\ &= 6(2)^2 - 24(2) - 30 \\ &= 6(4) - 48 - 30 \\ &= 24 - 48 - 30 \\ &= -54 \end{aligned}$$

Tomorrow

Tomorrow we will finish learning how to graph parabolas!