SGN – Assignment #1

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1 Periodic orbit

Exercise 1

Consider the 3D Earth–Moon Circular Restricted Three-Body Problem with $\mu = 0.012150$.

1) Find the x-coordinate of the Lagrange point L_1 in the rotating, adimensional reference frame with at least 10-digit accuracy.

Solutions to the 3D CRTBP satisfy the symmetry

$$\mathcal{S} : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t).$$

Thus, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.

2) Given the initial guess $\mathbf{x}_0 = (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0})$, with

$$\begin{aligned} x_0 &= 1.08892819445324 \\ y_0 &= 0 \\ z_0 &= 0.0591799623455459 \\ v_{x0} &= 0 \\ v_{y0} &= 0.257888699435051 \\ v_{z0} &= 0 \end{aligned}$$

Find the periodic halo orbit that passes through z_0; that is, develop the theoretical framework and implement a differential correction scheme that uses the STM either approximated through finite differences or achieved by integrating the variational equation.

The periodic orbits in the CRTBP exist in families. These can be computed by continuing the orbits along one coordinate, e.g., z_0. This is an iterative process in which one component of the state is varied, while the other components are taken from the solution of the previous iteration.

3) By gradually decreasing z_0 and using numerical continuation, compute the families of halo orbits until $z_0 = 0.034$.

(8 points)

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1) By definition, the Lagrange point L_1 is collinear with the two main bodies.
Using adimensional Synodic Rotating Frame coordinates, the CRTBP potential:

$$U(x, y, z) = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x+1-\mu)^2 + y^2 + z^2}}$$

The x-coordinate can be therefore found as one of the stationary points of the potential U, or, in other words, one of the zeros of its derivative along x, given the aforementioned assumption $y_{L_1} = z_{L_1} = 0$:

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \mu\frac{x+\mu-1}{|x+mu-1|^3} = 0$$

By appropriately selecting the search region so that $x \in (-\mu, 1-\mu)$, the Lagrange point L_1 is found using the `MatLab` function `fzero`. To achieve the required accuracy the algorithm tolerance was set to $TolX = 10^{-11}$. It should be noted that $TolX$ is a relative tolerance:

$$|x_i - x_{i+1}| < TolX\,|x_{i+1}|$$

Given the fact that the expected solution $|x_{L_1}| < 1$, such choice is satisfactory, and therefore:

$$x_{L_1} = 0.8369180073$$

2) Given the state initial guess $\tilde{\mathbf{x}}_0$, the state is propagated forwards in time according to CRTBP dynamics:

$$\begin{cases} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \\ \dot{v}_x = \dfrac{\partial U}{\partial x} + 2\dot{y} \\ \dot{v}_y = \dfrac{\partial U}{\partial y} - 2\dot{x} \\ \dot{v}_z = \dfrac{\partial U}{\partial z} \end{cases}$$

The numerical propagation is done using `ode113` with $AbsTol = 10^{-9}$ and $RelTol = 10^{-12}$. The event function `plane_intersection` stops the integration at $t = T_{1/2} = t_f$, where the y component of the state approaches $y = 0$ from above.

This process allows for the computation of half of the orbit. Exploiting the symmetry property $\mathcal{S}$ it is then possible to compute the complete orbit, granting continuity in position, but not in velocity.

The State Transition Matrix $\Phi(\tilde{\mathbf{x}}_0, t_0, t) = \Phi(t_0, t)$ is computed through variational equations:

$$\dot{\Phi}(t_0, t) = A(t)\,\Phi(t_0, t) \quad \text{with} \quad A(t) = \frac{\partial f(t)}{\partial \mathbf{x}}$$

$$A(t) = \begin{bmatrix} I & 0 \\ U_{\mathbf{xx}}(t) & B \end{bmatrix} \quad \text{with} \quad (U_{\mathbf{xx}})_{i,j} = \frac{\partial^2 f(t)}{\partial x_i \partial x_j} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is propagated along with the state $\mathbf{x}$.

It is used to update $\tilde{x}_0$ and $\tilde{v}_{y0}$ via differential corrections in function of $\delta v_x(t_f)$ and $\delta v_z(t_f)$:

$$\begin{bmatrix} \delta x(t_0) \\ \delta v_y(t_0) \end{bmatrix} = \begin{bmatrix} \tilde{x}(t_0) - x(t_0) \\ \tilde{v}_y(t_0) - v_y(t_0) \end{bmatrix} = \hat{\Phi}^{-1}(t_0, t_f) \begin{bmatrix} \delta v_x(t_f) \\ \delta v_z(t_f) \end{bmatrix} = \hat{\Phi}^{-1}(t_0, t_f) \begin{bmatrix} v_x(t_f) \\ v_z(t_f) \end{bmatrix}$$

$$\hat{\Phi}(t_0, t_f) = \begin{bmatrix} \Phi_{4,1}(t_0, t_f) & \Phi_{4,5}(t_0, t_f) \\ \Phi_{6,1}(t_0, t_f) & \Phi_{6,5}(t_0, t_f) \end{bmatrix}$$

The use of variational equations yields a more precise computation of $\Phi(t_0, t)$ with respect to the one obtainable using finite differences. The variational equations are however based on linear approximations, and the non-linear residual error builds up over integration time. The differential corrections are therefore applied over multiple iterations, each based on updated $\tilde{\mathbf{x}}_0$, until $v_x(t_f) \approx v_z(t_f) \approx 0$.

In particular, the stopping criterion of the iterative algorithm is set as $err \leq Tol$, with:

$$err = |v_x(t_f)| + |v_z(t_f)|$$

$$Tol = \frac{TU}{LU} 10^{-5} \quad \text{where} \quad \begin{cases} LU = 384000\,km \\ TU = 27.3\,days \end{cases}$$

Such tolerance grants final dimensional error less or equal than $10^{-2}\,m/s$.
The algorithm converges in 7 iterations with error $err = 0.00669995\,m/s$.

The corrected initial conditions:

$$\begin{aligned} x_0 &= 1.0902928805382091 \\ z_0 &= 0.0591799623455459 \\ v_{y0} &= 0.2602901632984244 \end{aligned}$$

The remaining components: $y_0 = v_{x0} = v_{z0} = 0$.

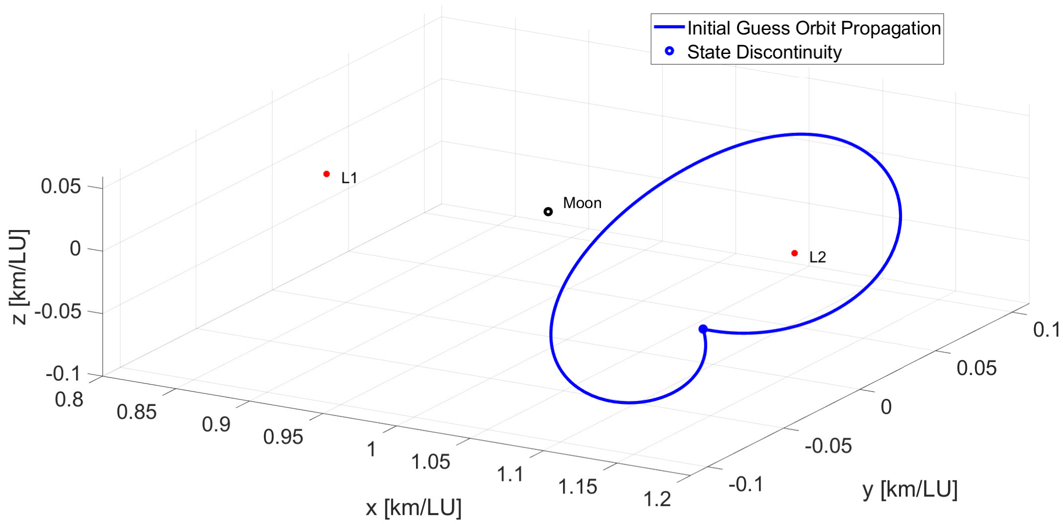

Figure 1: Halo orbit propagated with given initial condition $\tilde{\mathbf{x}}_0$

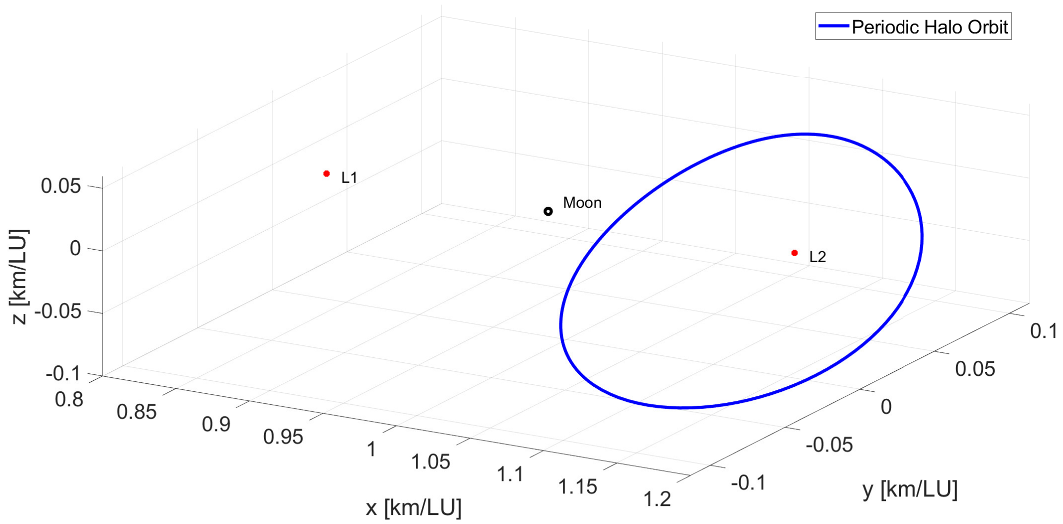

Figure 2: Periodic Halo orbit propagated with corrected initial conditions $\mathbf{x}_0$

3) By exploiting numerical continuation, it is possible to identify new periodic Halo orbits in function of z_0. The initial state guess $\tilde{\mathbf{x}}_0$ is iteratively taken as the corrected initial state $\mathbf{x}_0$ of the previous iteration with updated z_0, the latter taken from a set of N_H evenly spaced points in $[0.034, 0.0591799623455459]$. In the same fashion as per point 2, at each iteration the state initial guess is corrected until $err \leq Tol$.

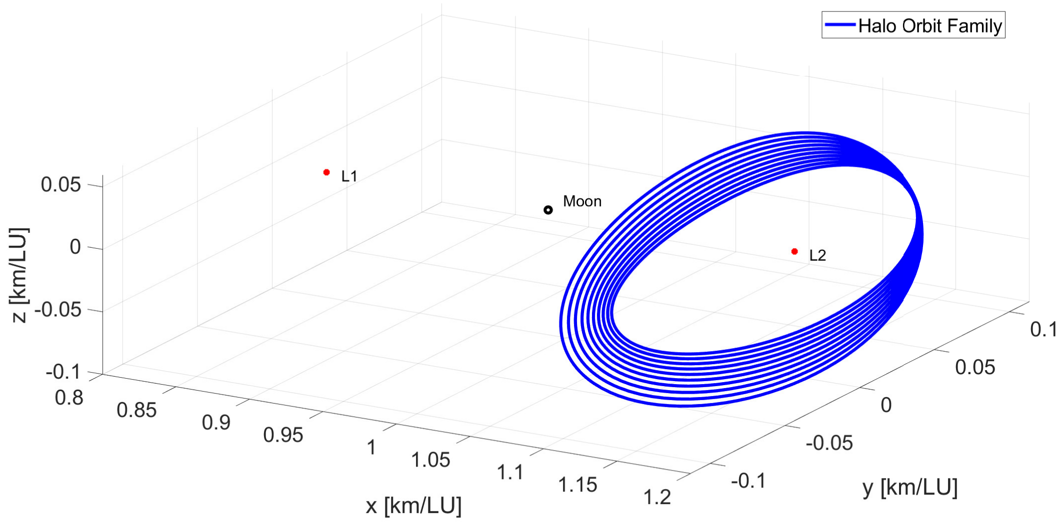

Figure 3: Periodic Halo family of orbits for $z_0 \in [0.034, 0.0591799623455459]$, $N_H = 10$

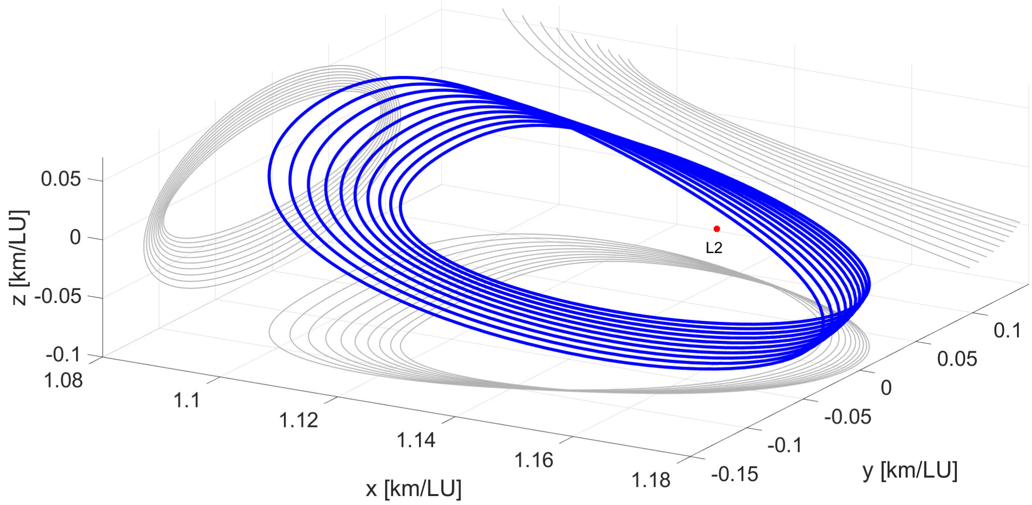

Figure 4: Periodic Halo family of orbits and projections

Again, at each iteration the algorithm converges in ≤ 10 iterations with $err \leq 10^{-2}\, m/s$.

2 Impulsive guidance

Exercise 2

The Aphophis close encounter with Earth will occur on April 2029. You shall design a planetary protection guidance solution aimed at reducing the risk of impact with the Earth.

The mission shall be performed with an impactor spacecraft, capable of imparting a $\Delta\mathbf{v} = 0.00005\,\mathbf{v}(t_{\text{imp}})$, where $\mathbf{v}$ is the spacecraft velocity and t_{imp} is the impact time. The spacecraft is equipped with a chemical propulsion system that can perform impulsive manoeuvres up to a total Δv of 5 km/s.

The objective of the mission is to maximize the distance from the Earth at the time of the closest approach. The launch shall be performed between 2024-10-01 (LWO, Launch Window Open) and 2025-02-01 (LWC, Launch Window Close), while the impact with Apophis shall occur between 2028-08-01 and 2029-02-28. An additional Deep-Space Maneuver (DSM) can be performed between LWO+6 and LWC+18 months.

1) Analyse the close encounter conditions reading the SPK kernel and plotting in the time window [2029-01-01; 2029-07-31] the following quantities:
   a) The distance between Apophis and the Sun, the Moon and the Earth respectively.
   b) The evolution of the angle Earth-Apophis-Sun
   c) The ground-track of Apophis for a time-window of 12 hours centered around the time of closest approach (TCA).
2) Formalize an unambiguous statement of the problem specifying the considered optimization variables, objective function, the linear and non-linear equality and inequality constraints, starting from the description provided above. Consider a multiple-shooting problem with $N = 3$ points (or equivalently 2 segments) from t_0 to t_{imp}.
3) Solve the problem with multiple shooting. Propagate the dynamics of the spacecraft considering only the gravitational attraction of the Sun; propagate the post-impact orbit of Apophis using a full n-body integrator. Use an event function to stop the integration at TCA to compute the objective function; read the position of the Earth at t_0 and that of Apophis at t_{imp} from the SPK kernels. Provide the optimization solution, that is, the optimized departure date, DSM execution epoch and the corresponding $\Delta\mathbf{v}$'s, the spacecraft impact epoch, and time and Distance of Closest Approach (DCA) in Earth radii. Suggestion: try different initial conditions.

(11 points)

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1) The relative positions of Aphophis with respect to the Sun $\mathbf{r}_{Aph-\odot}$, Earth $\mathbf{r}_{Aph-\oplus}$ and Moon $\mathbf{r}_{Aph-☽}$ are extracted from `CSPICE` SPK kernels at equally spaced time instants across the Close Encounter time window.
   The distance of Aphophis with respect to the aforementioned bodies is simply computed as the magnitude of the respective relative position, while the $\beta_{\odot-Aph-\oplus}$ angle is computed exploiting trigonometric properties:

$$\beta_{\odot-Aph-\oplus} = tan^{-1}\left(\frac{s}{c}\right) \quad \text{with} \quad c = \frac{\mathbf{r}_{Aph-\odot}\cdot\mathbf{r}_{Aph-\oplus}}{r_{Aph-\odot}r_{Aph-\oplus}}, \quad s = \frac{\mathbf{r}_{Aph-\odot}\times\mathbf{r}_{Aph-\oplus}}{r_{Aph-\odot}r_{Aph-\oplus}}$$

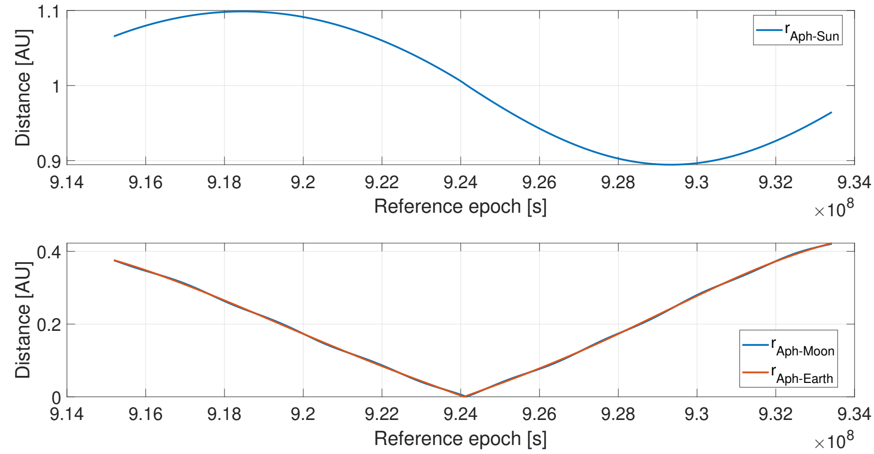

Figure 5: Aphophis relative distance with respect to the Sun (top), Earth and Moon (bottom), [2029-01-01, 2029-07-31]

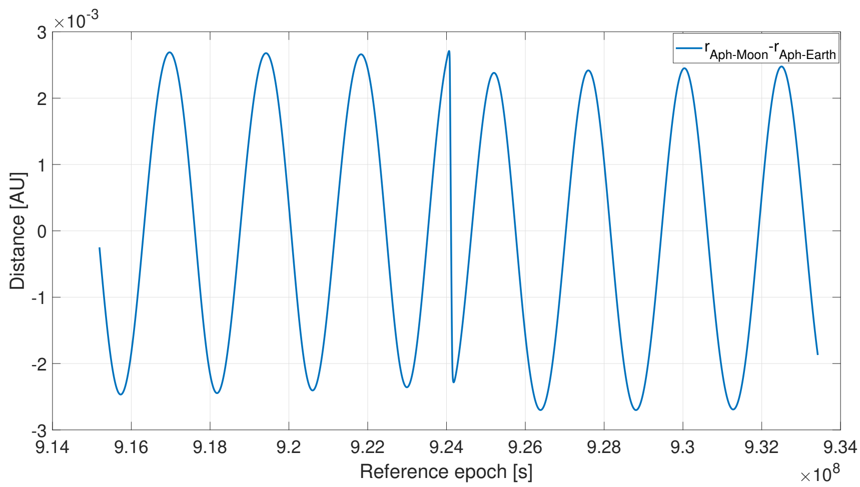

Figure 6: $r_{Aph-☽}$ in relation to $r_{Aph-\oplus}$, [2029-01-01, 2029-07-31]

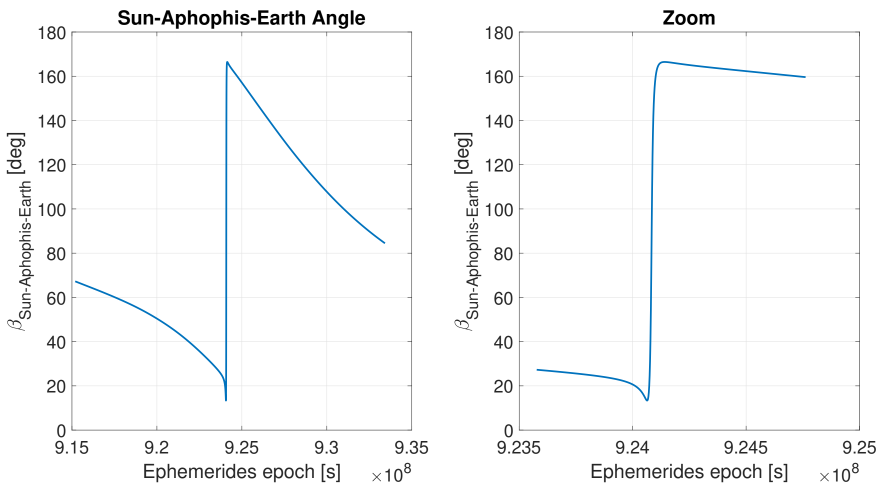

Figure 7: $\beta_{\odot-Aph-\oplus}$ angle, [2029-01-01, 2029-07-31].
On the left a zoom around TCA.

For the purpose of the computation of the Ground Track, TCA prior maneuver is computed as:

$$TCA = arg \min_{t \in CE} r_{Aph-\oplus}(t) = \text{2029-04-13 21:44:56.669}$$

where CE is the Close Encounter window [2029-01-01, 2029-07-31].

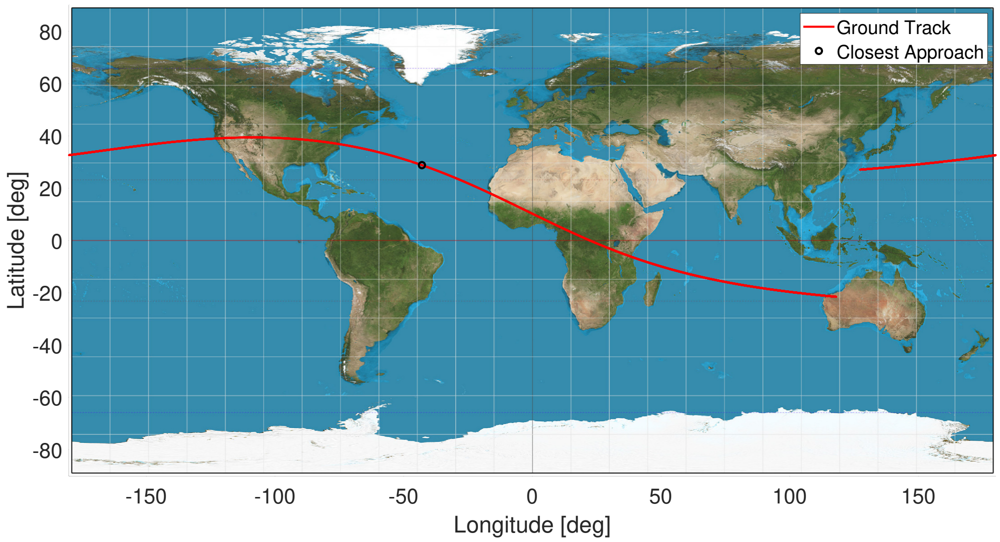

Figure 8: TCA centered 24 hour Aphophis Ground Track

2) The problem is formulated as the constrained minimization of the multiple shooting problem:

$$\min_{\mathbf{y}\in\mathbf{Y}} -\left\|\mathbf{r}_{\oplus}(TCA) - \mathbf{r}_{Aph}(TCA)\right\| \text{ s.t. } \begin{cases} A\mathbf{y} \le B \\ C(\mathbf{y}) \le 0 \\ \mathbf{C}_{eq}(\mathbf{y}) = \mathbf{0} \end{cases}$$

The optimization variable $\mathbf{y}$ collects the states of $N = 3$ nodes $\mathbf{x}_i$, with corresponding time instants t_i:

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}_1^\intercal & \mathbf{x}_2^\intercal & \mathbf{x}_3^\intercal & t_1 & t_2 & t_3 \end{bmatrix}^\intercal \quad \text{where} \quad \mathbf{x}_i = \begin{bmatrix} \mathbf{r}_i \\ \mathbf{v}_i \end{bmatrix} \quad \text{and} \quad \begin{cases} t_1 = t_0 \\ t_2 = t_{DSM} \\ t_3 = t_{imp} \end{cases}$$

Only the time instants t_i are bounded by corresponding Opening and Closing windows. For the purpose of numerical analysis however, each component of the states $\mathbf{x}_i$ is also bounded to an arbitrarily large value:

$$\mathbf{Y} = \left\{\mathbf{y} \in \mathbb{R}^{21} \middle| \quad y_i \in [-1, +1] \cdot 10^{12} \quad i = 1:18; \quad y_i \in \left[t_i^{LB}, t_i^{UB}\right] \quad i = 19:21\right\}$$

The linear inequality constraint is designed to ensure the maneuvers happen in the correct order $t_0 \le t_{DSM} \le t_{imp}$:

$$A = \begin{bmatrix} 0 & \dots & 0 & 1 & -1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 21} \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It ought to be noted that the time windows bounding t_i are not overlapping, therefore such constraint is redundant. It is added for generality of the problem formulation, but omitted in the code.

Given the Two Body Problem equations of motion:

$$\ddot{\mathbf{r}} = -\mu_{\odot}\frac{\mathbf{r}}{r^3} \quad \text{with} \quad \mathbf{r} = \mathbf{r}_{SC} - \mathbf{r}_{\odot}$$

The state of S/C is propagated twice: firstly from t_1 to t_2 given initial condition $\mathbf{x}_1$, secondly from t_2 to t_3 given initial condition $\mathbf{x}_2$. The integration is performed using `ode113` with $AbsTol = 10^{-9}$ and $RelTol = 10^{-12}$.

The non-linear equality constraint is designed to ensure continuity of specific S/C state components at each node. In particular $\mathbf{C}_{eq,1}(\mathbf{y})$ and $\mathbf{C}_{eq,4}(\mathbf{y})$ constrain S/C initial and final position to those of Earth and Aphophis respectively. $\mathbf{C}_{eq,2}(\mathbf{y})$ ensures positional continuity in the flow from the first node to the second. $\mathbf{C}_{eq,3}(\mathbf{y})$ ensures continuity in the flow from the second node to the third. $\boldsymbol{\varphi}$ refers to the flow of S/C dynamics:

$$\mathbf{C}_{eq}(\mathbf{y}) = \begin{pmatrix} \mathbf{C}_{eq,1}(\mathbf{y}) \\ \mathbf{C}_{eq,2}(\mathbf{y}) \\ \mathbf{C}_{eq,3}(\mathbf{y}) \\ \mathbf{C}_{eq,4}(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \mathbf{r}_{\oplus}(t_1) - \mathbf{r}_1 \\ \boldsymbol{\varphi}_{\mathbf{r}}(\mathbf{x}_1, t_1, t_2) - \mathbf{r}_2 \\ \boldsymbol{\varphi}(\mathbf{x}_2, t_2, t_3) - \mathbf{x}_3 \\ \mathbf{r}_3 - \mathbf{r}_{Aph}(t_3) \end{pmatrix}$$

The non-linear inequality constraint is designed to ensure the total Δv imparted by S/C, sum of Δv_L and Δv_{DSM}, is below the maximum allowed:

$$C(\mathbf{y}) = \|\mathbf{v}_{\oplus}(t_1) - \mathbf{v}_1\| + \|\boldsymbol{\varphi}_{\mathbf{v}}(\mathbf{x}_1, t_1, t_2) - \mathbf{v}_2\| - 5$$

3) The problem is solved using the `MatLab` function `fmincon` which computes the constrained minimum of `avoidance`, given an initial feasible guess $\mathbf{y}_0$.
Inside `avoidance` the state of Aphophis at $t_3 = t_{imp}$ is retrived from `CSPICE` SPK kernels, and perturbed with the $\Delta\mathbf{v}$ imparted by S/C at impact. The perturbed state is taken as Aphophis initial condition for numerical propagation, according to non-inertial Sun-centered n-body equations of motion:

$$\ddot{\mathbf{r}} = -\mu_{\odot}\frac{\mathbf{r}}{r^3} - \sum_j \mu_j \frac{\mathbf{r} + \boldsymbol{\rho}_i f(q_j)}{d_j^3}$$

$$f(q_j) = \frac{q_j\left(3 + 3q_j + q_j^2\right)}{1 + (1+q_j)^{1.5}} \qquad q_j = \frac{\mathbf{r}\cdot(\mathbf{r} - 2\boldsymbol{\rho}_j)}{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j} \qquad \begin{gathered} \mathbf{r} = \mathbf{r}_{Aph} - \mathbf{r}_{\odot} \\ \mathbf{d}_j = \mathbf{r}_{Aph} - \mathbf{r}_j \\ \boldsymbol{\rho}_j = \mathbf{r}_j - \mathbf{r}_{\odot} \end{gathered}$$

The subscript j identifies the j^{th} body amongst all the planets and the Moon.

Again, the integration is performed using `ode113` with absolute and relative tolerances set to $AbsTol = RelTol = 10^{-12}$.
The event function `close_encounter` halts the solver propagation at $t = TCA$, once Aphophis reaches DCA:

$$\rho(t = TCA) = \|\mathbf{r}_{\oplus}(t) - \mathbf{r}_{Aph}(t)\|\Big|_{t=TCA} = DCA$$

Such condition is identified as a stationary minimum of $\rho(t)$:

$$\frac{d\rho(t)}{dt} = \frac{\boldsymbol{\rho}(t)\cdot\dot{\boldsymbol{\rho}}(t)}{\rho(t)} = 0$$

Analogously, it is possible to identify the same minimum using:

$$\frac{d\rho^2(t)}{dt} = 2\boldsymbol{\rho}(t)\cdot\dot{\boldsymbol{\rho}}(t) = 0$$

The latter option is preferable with respect to the first one, as $\rho^2(t)$ is more likely to correctly capture the behaviour of $\rho(t)$ around it's minimum, therefore providing a more accurate evaluation.

At each iteration `fmincon` computes TCA, and updates the variables $\mathbf{y}$ starting from the initial condition $\mathbf{y}_0$ until the solution is deemed satisfactory in terms of both optimality and feasibility. The Optimality Tolerance is set to its default value 10^{-6}. Constraint Tolerance is instead set to $ConTol = 10^{-10}$. In fact, given the general non-linear constraint:

$$\mathbf{C}(\mathbf{y}) = \begin{bmatrix} \mathbf{C}_{eq}(\mathbf{y}) \\ C(\mathbf{y}) \end{bmatrix} \qquad \frac{\|\mathbf{C}(\mathbf{y})\|_\infty}{1 + \|\mathbf{C}(\mathbf{y_0})\|_\infty} \leq ConTol$$

The chosen $\mathbf{y}_0$ yields $\|\mathbf{C}(\mathbf{y_0})\|_\infty$ in the order of 10^8. Such choice of $ConTol$ allows for maximum constraint violation to be at most in the order of $\approx 10^{-2}\,km$, $10\,m/s$.
This choice over constrains the positional requirements, which could be relaxed up to three orders of magnitude. It however ensures that S/C total Δv requirement is satisfied with high precision.

It ought to be noted that the problem has high sensitivity with respect to the initial condition $\mathbf{y}_0$. Different initial conditions might lead to the identification of different final feasible solutions satisfying the previously defined constraints.
The initial condition relative to the provided results was generated as follows:

$$t_i \quad \text{randomly picked s.t.} \quad t_i \in \left[t_i^{LB}, t_i^{UB}\right] \quad i = 1:3$$

$$\mathbf{x}_1 = \mathbf{x}_{\oplus}(t_1) \quad \mathbf{x}_2 = \boldsymbol{\varphi}(\mathbf{x}_1, t_1, t_2) \quad \mathbf{x}_3 = \boldsymbol{\varphi}(\mathbf{x}_2, t_2, t_3)$$

Given the long computational time required to solve the minimization problem, the code runs with a fixed initial condition $\mathbf{y}_0$. The initial condition generation process is present in the code as a comment.

Launch	`2024-11-30-06:53:25.912 UTC`		
DSM	`2025-11-12-23:43:38.107 UTC`		
Impact	`2028-11-22-12:35:01.164 UTC`		
TCA	`2029-04-13-22:49:24.897 UTC`		
$\Delta\mathbf{v}_L$ [km/s]	−2.0279	+1.6628	+0.2714
$\Delta\mathbf{v}_{DSM}$ [km/s]	−0.3242	−0.1155	−2.3384
DCA [Re]	10.5452		

Table 1: Guidance solution for the impactor mission

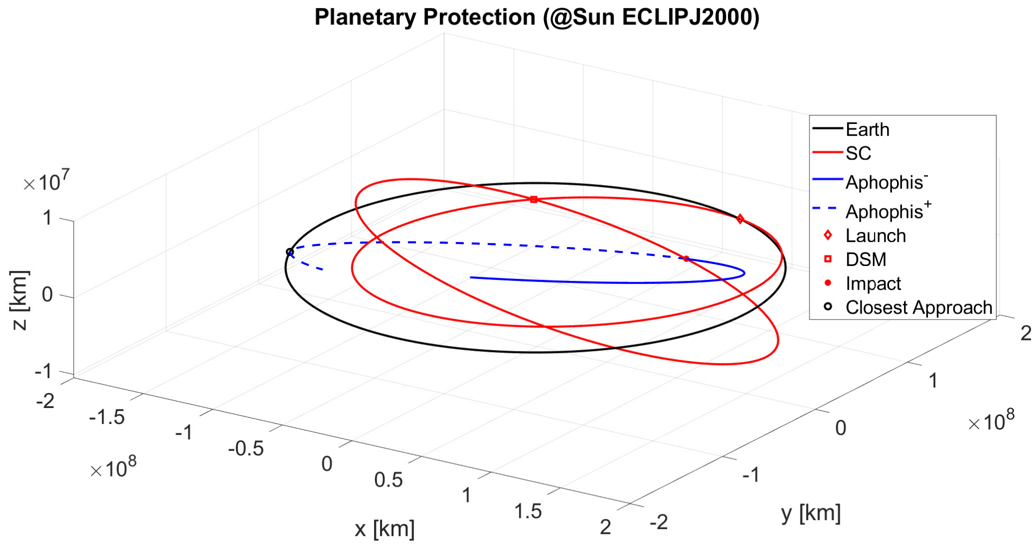

Figure 9: Planetary Protection optimal maneuver. Aphophis$^-$ indicates prior impact trajectory. Aphophis$^+$ indicates post impact trajectory

3 Continuous guidance

Exercise 3

A low-thrust option is being considered for an Earth-Venus transfer. Provide a *time-optimal* solution under the following assumptions: the spacecraft moves in the heliocentric two-body problem, Venus instantaneous acceleration is determined only by the Sun gravitational attraction, the departure date is fixed, and the spacecraft initial and final states are coincident with those of the Earth and Venus, respectively.

1) Using the PMP, write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$ with the appropriate transversality condition.

2) Adimensionalize the problem using as reference length $LU = 1$ AU and reference mass $MU = m_0$, imposing that $\mu = 1$. Report all the adimensionalized parameters.

3) Solve the problem considering the following data:

   - Launch date: `2023-05-28-14:13:09.000 UTC`
   - Spacecraft mass: $m_0 = 1000$ kg
   - Electric propulsion properties: $T_{\max} = 800$ mN, $I_{sp} = 3120$ s

   To obtain an initial guess for the costate, generate random numbers such that $\lambda_{0,i} \in [-20; +20]$, while $t_f < 2\pi$. Report the obtained solution in terms of $\{\boldsymbol{\lambda}_0, t_f\}$ and the error with respect to the target. Assess your results exploiting the properties of the Hamiltonian in problems that are not time-dependent and time-optimal solutions.

4) Solve the problem for a lower thrust level $T_{\max} = [500]$ mN. Tip: exploit numerical continuation.

(11 points)

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1) Given generic equations of motion $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$, cost function $l = l(\mathbf{x}, \mathbf{u}, t)$ the Hamiltonian $H = l(\mathbf{x}, \mathbf{u}, t) + \boldsymbol{\lambda}^\intercal \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ yields the Lagrange-Euler equations, together with PMP optimal control:

$$\begin{cases} \dot{\mathbf{x}} = \dfrac{\partial H}{\partial \boldsymbol{\lambda}}, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \dot{\boldsymbol{\lambda}} = -\dfrac{\partial H}{\partial \mathbf{x}}, \quad x_i(t_f) = x_{i,f} \\ \mathbf{u} = arg \min\limits_{\mathbf{u} \in U} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \lambda_i(t_f) = 0 \quad i \neq j \end{cases}$$

Given that the time-optimal nature of the problem, using Two Body Problem system dynamics the equations simplify as follows:

$$H = 1 + \boldsymbol{\lambda}^\intercal \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad \text{with} \quad \mathbf{u} = u\hat{\boldsymbol{\alpha}} = -\frac{\boldsymbol{\lambda}_\mathbf{v}}{\lambda_\mathbf{v}}$$

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\dfrac{\mu_\odot}{r^3}\mathbf{r} - \dfrac{T_{max}}{m}\dfrac{\boldsymbol{\lambda}_\mathbf{v}}{\lambda_\mathbf{v}} \\ \dot{m} = -\dfrac{T_{max}}{I_{sp} g_0} \end{cases} \qquad \begin{cases} \dot{\boldsymbol{\lambda}}_\mathbf{r} = -3\dfrac{\mu_\odot}{r^5}\left(\mathbf{r}^\intercal \boldsymbol{\lambda}_\mathbf{v}\right)\mathbf{r} + \dfrac{\mu_\odot}{r^3}\boldsymbol{\lambda}_\mathbf{v} \\ \dot{\boldsymbol{\lambda}}_\mathbf{v} = -\boldsymbol{\lambda}_\mathbf{r} \\ \dot{\lambda}_m = -\lambda_\mathbf{v}\dfrac{T_{max}}{m^2} \end{cases}$$

The state of S/C is presented in $ECLIPJ2000$ coordinates, and subject to initial conditions:

$$\begin{cases} \mathbf{r}(t_0) = \mathbf{r}_{\oplus}(t_0) \\ \mathbf{v}(t_0) = \mathbf{v}_{\oplus}(t_0) \\ m(t_0) = m_0 \end{cases}$$

The equations are also subjected to final boundary conditions:

$$\begin{cases} \mathbf{r}(t_f) = \mathbf{r}_{♀}(t_f) \\ \mathbf{v}(t_f) = \mathbf{v}_{♀}(t_f) \\ \lambda_m(t_f) = 0 \end{cases}$$

Combined with the transversality equation:

$$H(t_f) = \boldsymbol{\lambda}^\intercal \dot{\boldsymbol{\psi}}(t_f) \quad \text{with} \quad \boldsymbol{\psi}(t_f) = \begin{bmatrix} \mathbf{r}_{♀}(t_f) \\ \mathbf{v}_{♀}(t_f) \end{bmatrix}$$

The final boundary constraints and the transversality equation provide 8 constraints to the 8 unknowns of the problem $\boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$, t_f.

2) To fully adimensionalize the problem only three reference fundamental quantities are required: reference length LU, reference mass MU and reference time TU. The first two are given, while TU is computed by imposing the adimensional Sun gravitational parameter $\mu = 1$:

$$TU = \sqrt{\mu \frac{LU^3}{\mu_\odot}} \quad \text{so that} \quad \mu_\odot \frac{TU^2}{LU^3} = \mu$$

The following derived quantities are required:

- Reference velocity: $VU = LU/TU$
- Reference acceleration: $AU = VU/TU$
- Reference force: $FU = MU\,AU$

All quantities but time are adimensionalized by simply dividing the dimensional quantity by the appropriate reference unit.
Given the launch date reference epoch t_0, time is adimesionalised as follows:

$$t^{AD} = \frac{t - t_0}{TU}$$

This ensures that $t_0^{AD} = 0$.

From this point onwards, the adimensional subscript is omitted.

$\mathbf{r}_0$	-4.0093×10^{-1}	-9.3063×10^{-1}	4.8795×10^{-5}
$\mathbf{v}_0$	9.0233×10^{-1}	-3.9917×10^{-1}	4.2073×10^{-5}
m_0	1.0000		
I_{sp}	6.2119×10^{-4}		
$T_{\max}$	1.3491×10^{-1}		
g_0	1.6537×10^{3}		
GM	1.0000		

Table 2: Adimensionalized quantities ($T_{\max} = 800$ mN; @Sun ECLIPJ2000).

3) The problem is set up as a zero-finding problem: an appropriate set of $\{\boldsymbol{\lambda}_0,\, t_f\}$ is to be used to propagate S/C dynamics so that the final constraints are met, or, in other words, the residuals Δ_f are zero.

The function `timeopt_conv` computes such residuals in function of $\{\boldsymbol{\lambda}_0,\, t_f\}$: starting from $\{\mathbf{x}_0,\, \boldsymbol{\lambda}_0\}$ the state and costate are propagated in time up to t_f according to the previously defined dynamics. The integration is performed using `ode113` with absolute and relative tolerances respectively $AbsTol = 10^{-9}$ and $RelTol = 10^{-12}$.

At time t_f, the residual are computed as follows:

$$\Delta_f = \begin{bmatrix} \begin{bmatrix} \mathbf{r}\,(t_f) \\ \mathbf{v}\,(t_f) \end{bmatrix} - \boldsymbol{\psi}\,(t_f) \\ \lambda_m\,(t_f) \\ H\,(t_f) - \boldsymbol{\lambda}^\intercal \dot{\boldsymbol{\psi}}\,(t_f) \end{bmatrix}$$

For the transversality condition:

$$\dot{\boldsymbol{\psi}}\,(t_f) = \begin{bmatrix} \mathbf{v}_{♀}\,(t_f) \\ \mathbf{a}_{♀}\,(t_f) \end{bmatrix} \quad \text{with} \quad \mathbf{a}_{♀}\,(t_f) = -\mu \frac{\mathbf{r}_{♀}\,(t_f)}{r^3_{♀}\,(t_f)}$$

`fsolve` perturbs an initial guess $\{\boldsymbol{\lambda}_0,\, t_f\}_0$ until the the relative norm of the gradient of the sum of the squared elements of Δ_f is less than the Optimality Tolerance $OptTol$.
It was found that $OptTol = 10^{-4}$ yields satisfactory results.

The initial guess $\{\boldsymbol{\lambda}_0,\, t_f\}_0$ is generated by randomly picking each element in the appropriate interval:

$$\lambda_{0,i} \in [-20,\, +20] \quad i = 1:6; \quad \lambda_{0,7} \in [0,\, +20]\,; \quad t_f \in [0,\, 2\pi]$$

Given the aleatory nature of the initial guess, the convegence of `fsolve` is not always ensured. The whole zero-finding process is therefore iterated until an initial guess $\{\boldsymbol{\lambda}_0,\, t_f\}_0$ granting convergence is generated.

The results reported were obtained starting from a fixed chosen initial condition. The random initial condition generation process is present in the code as a comment.

$\boldsymbol{\lambda}_{0,r}$	4.9785×10^{-1}	-1.3815×10^{1}	1.0817×10^{-1}
$\boldsymbol{\lambda}_{0,v}$	5.1823	-1.0397×10^{1}	1.4848
$\lambda_{0,m}$	1.7511		
t_f	`2023-10-17-03:57:58.043 UTC`		
TOF [days]	141.5728		

Table 3: Time-optimal Earth-Venus transfer solution ($T_{\max} = 800$ mN).

$\lvert\lvert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\rvert\rvert$	[km]	4.1427×10^{-2}
$\lvert\lvert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\rvert\rvert$	[m/s]	1.1808×10^{-5}

Table 4: Final state error with respect to Venus' center ($T_{\max} = 800$ mN).

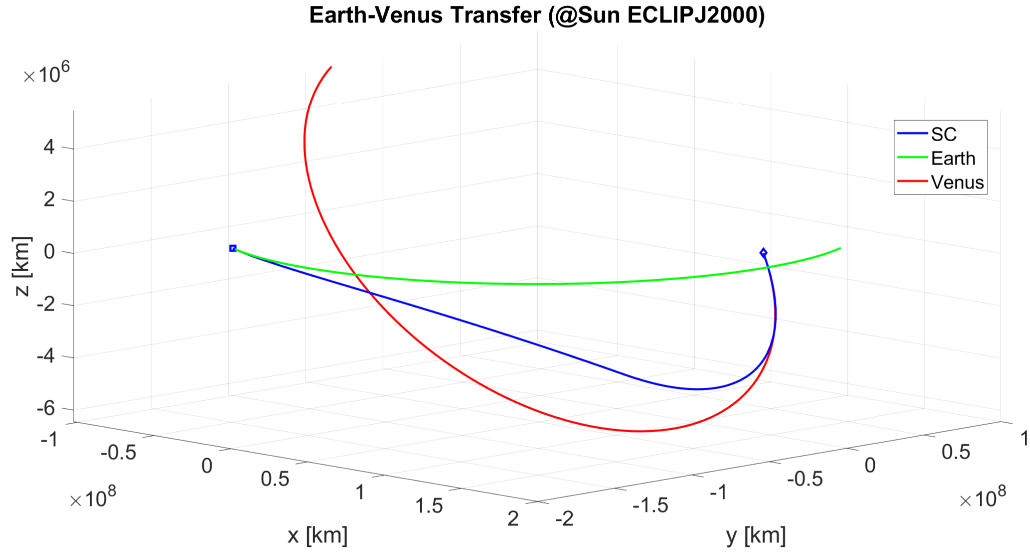

Figure 10: Time-optimal Earth-Venus transfer solution ($T_{\max} = 800$ mN)

It is possible to assess the veracity of the results exploiting Hamiltonian properties. Since the system dynamics are autonomous (non time dependant):

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \quad \text{meaning} \quad H\left(t\right) = const$$

It is possible to observe that H is in fact approximately constant, exhibiting maximum relative variation of $\approx 10^{-6}\,\%$. The fluctuations are attributable to numerical error.

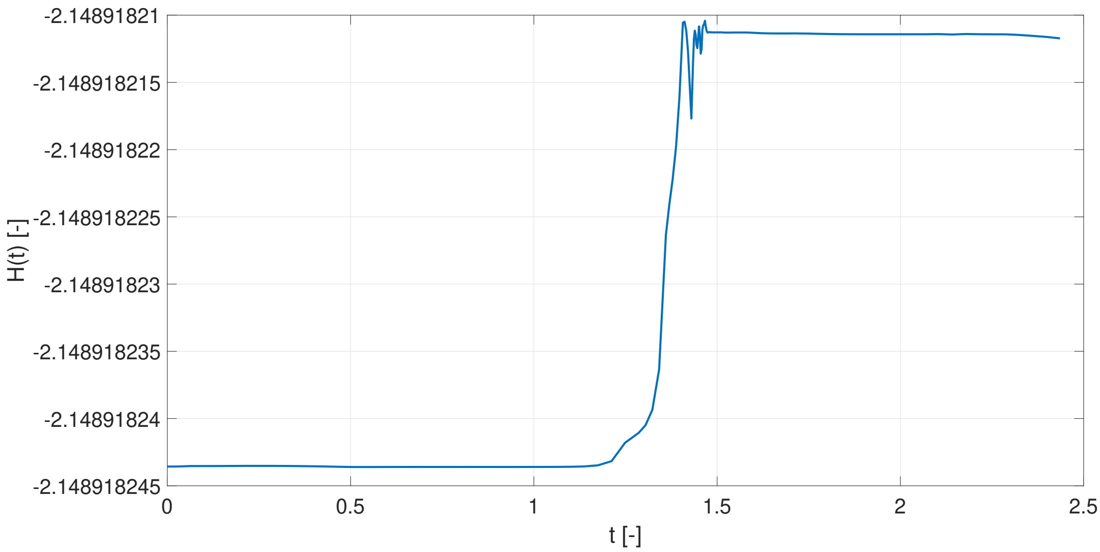

Figure 11: Hamiltonian ($T_{\max} = 800$ mN)

Another way to assess the result is to observe the behaviour of the switching function:

$$S_t = -\frac{\lambda_v \, I_{sp} \, g_0}{m} - \lambda_m$$

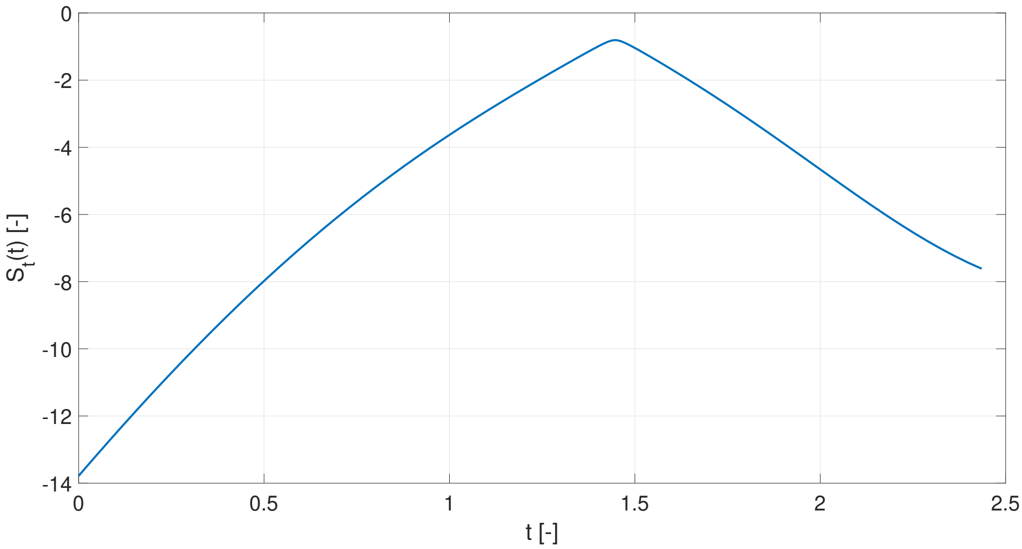

Figure 12: Switching function ($T_{\max} = 800$ mN)

As expected S_t is negative and shows no change in sign, therefore validating the assumptions made for time-optimal problems on $\mathbf{u}$.

4) Exploiting numerical continuation, the problem is solved for gradually decreasing thrust, until $T_{max} = 500\,mN$.
Iteratively, each iteration initial guess $\{\boldsymbol{\lambda}_0,\, t_f\}_0$ is taken as the previous iteration solution, the dimensional thrust is lowered in steps of $25mN$ and the zero-finding problem solved in the same fashion as the previous point.

$\boldsymbol{\lambda}_{0,r}$	-5.6343×10^{-1}	-2.4535×10^{1}	-4.7502×10^{-1}
$\boldsymbol{\lambda}_{0,v}$	1.4180×10^{1}	-1.8908×10^{1}	1.9967
$\lambda_{0,m}$	2.7282		
t_f	`2024-01-01-04:10:55.651 UTC`		
TOF [days]	217.5818		

Table 5: Time-optimal Earth-Venus transfer solution ($T_{\max} = 500$ mN).

$\vert\vert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\vert\vert$	[km]	1.6261
$\vert\vert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\vert\vert$	[m/s]	5.5088×10^{-4}

Table 6: Final state error with respect to Venus' center ($T_{\max} = 500$ mN).

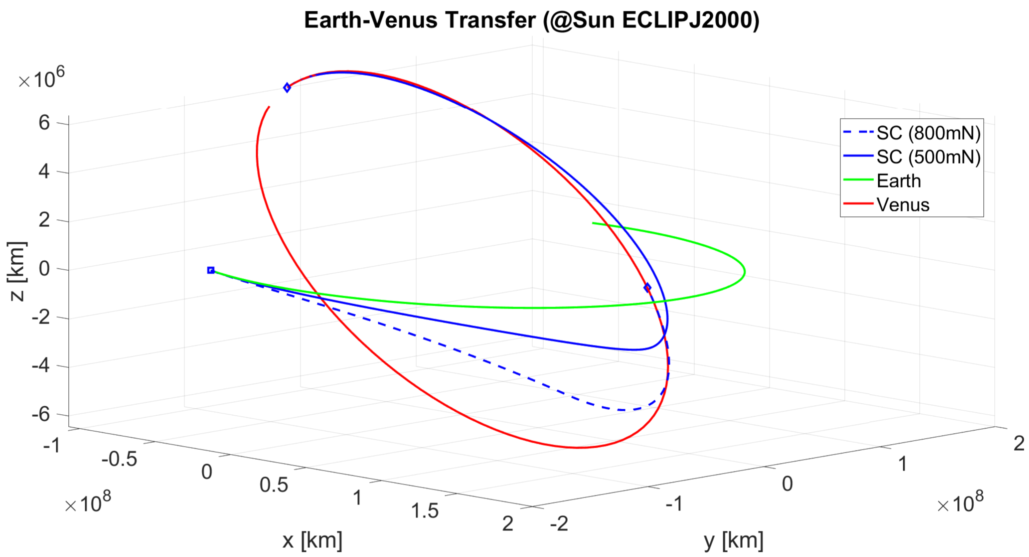

Figure 13: Time-optimal Earth-Venus transfer solution ($T_{\max} = 500$ mN)

Again, the Hamiltonian H exhibits maximum relative variation of $\approx 10^{-6}\,\%$, while the switching function S_t is negative with no change in sign.

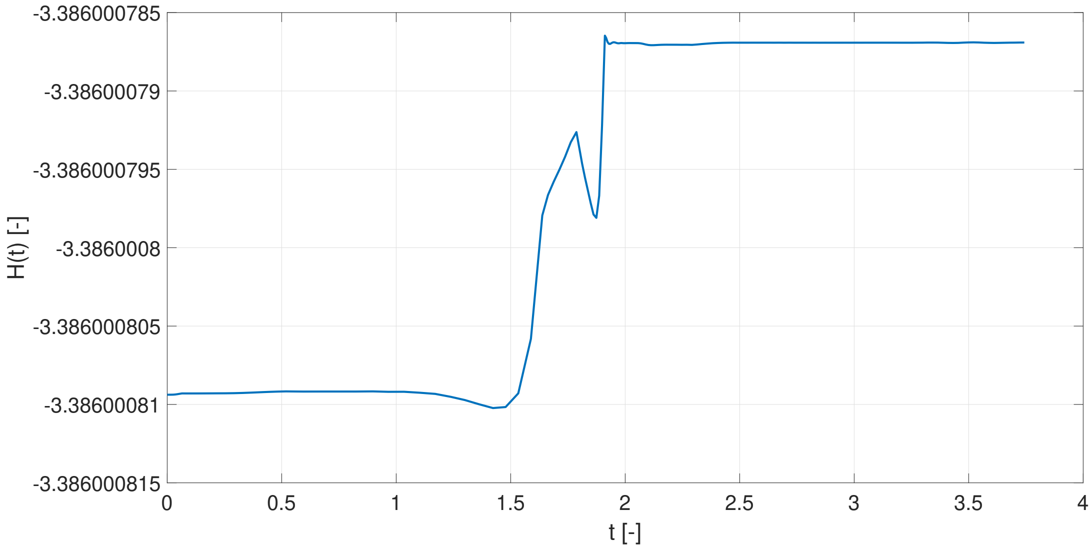

Figure 14: Hamiltonian ($T_{\max} = 500$ mN)

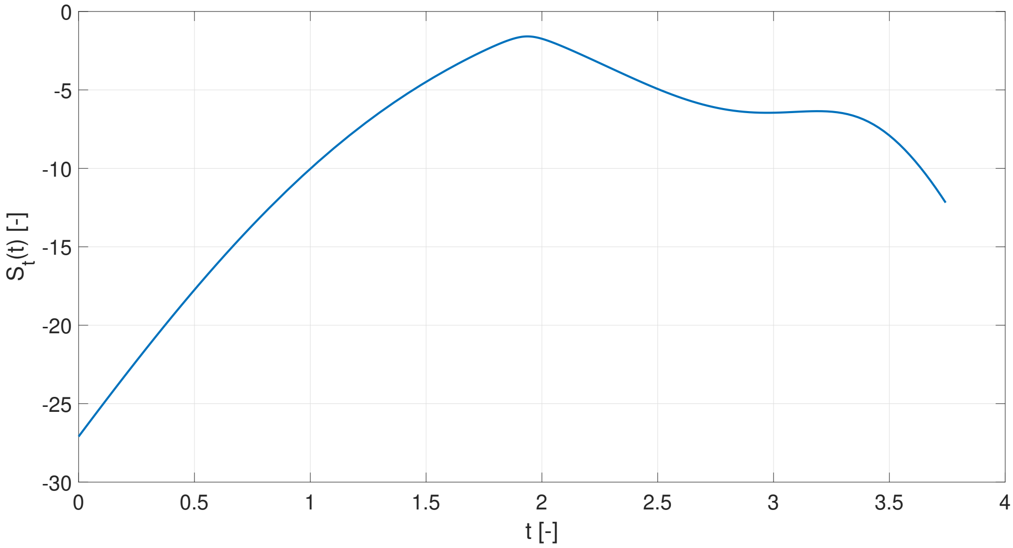

Figure 15: Switching function ($T_{\text{max}} = 500$ mN)