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MFRA - Computational Finance CAT 2

Assignment 2

SIMS

2) Underlying security

$S \sim dS_t = \underbrace{\mu S_t dt}_{drift} + \sigma_t S_t dZ_t$; Z - Std Brownian Motion

$Var(V(t)) = \sigma_t^2$

$V(t) \sim dV_t = \underbrace{\kappa(\theta - v(t))}_{drift} dt + \gamma \sqrt{V_t}\, dW_t$.

Correlation ρ

$cov(dZ_t, dW_t) = \rho dt$.

Show that the generalized Black Scholes equation is

$$\frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \gamma v S \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}\gamma^2 v \frac{\partial U}{\partial v^2} + rS\frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} =$$

drift ; $S_t = rS_t dt$

; $V_t = \kappa(\theta - V_t) - \lambda V_t$ by Girsanov's Theorem

SDE ; $dS_t = rS_t dt + \sqrt{V_t} S_t dZ_t^{Q}$

$dV_t = [\kappa(\theta - V_t) - \lambda V_t] dt + \gamma \sqrt{V_t}\, dW_t^{Q}$

$cov(dZ_t^{Q}, dW_t^{Q}) = \rho dt$.

Multivariate Ito's lemma

$$dU = \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial v}dv + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}(dv)^2 + \frac{\partial^2 U}{\partial S \partial v} dS dv$$

$S = S_t$, $V = V_t$

$$dS = rS\,dt + \sqrt{v}\,S\,dz$$

$$dv = [\kappa(\theta - v) - \lambda v]\,dt + \gamma\sqrt{v}\,dW$$

$(dS)^2 = vS^2 dt$; $dS = \mu S dt + \sqrt{v} S dz$; $(dz)^2 = dt$; $(dt)^2 = 0$; $(\mu S dt + \sqrt{v} S dz)^2$
$= (\mu S)^2 (dt^2) + 2\mu S \cdot d\, vS dt dz + vS^2 (dz)^2 = vS^2 dt.$

$(dv)^2 = \gamma^2 v\,dt$; $(dv)^2 = ((\kappa(\theta - v)dt) + \gamma\sqrt{v}\,dW)^2 = (\gamma\sqrt{v})^2 (dW)^2 = \gamma^2 v\,dt.$

$dS\,dv = \sqrt{v} S \cdot \gamma\sqrt{v} \cdot \rho dt = \rho\gamma v S dt.$; $dS = \mu S dt + \sqrt{v} S dz$; $dv = \kappa(\theta - v)dt + \gamma\sqrt{v}\,dW$
$= dS\,dv = (\sqrt{v} S dz)(\gamma\sqrt{v}\,dW) = \gamma v S \cdot dz\,dW$
$= \gamma v S \rho dt.$

$$dU = \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial S}(rS dt + \sqrt{v} S dz) + \frac{\partial U}{\partial v}\left([\kappa(\theta - v) - \lambda v] dt + \gamma\sqrt{v}\,dW\right) + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} vS^2 dt$$

$$+ \frac{1}{2}\frac{\partial^2 U}{\partial v^2} vS^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\gamma^2 v\,dt + \frac{\partial^2 U}{\partial S \partial v}\rho\gamma v S dt$$

$$= \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial S} rS dt + \frac{\partial U}{\partial S}\sqrt{v} S dz \} + \frac{\partial U}{\partial v}[\kappa(\theta - v) - \lambda v] dt + \frac{\partial U}{\partial v}\gamma\sqrt{v}\,dW$$

$$+ \frac{1}{2}\frac{\partial^2 U}{\partial S^2} vS^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2} vS^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\gamma^2 v\,dt + \frac{\partial^2 U}{\partial S dv}\rho\gamma v S dt.$$

$$= \left[\frac{\partial U}{\partial t} + rS\frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} + \frac{1}{2} vS^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2} vS^2 \frac{\partial^2 U}{\partial v^2} + \rho\gamma v S \frac{\partial^2 U}{\partial S \partial v}\right] dt$$

under risk neutral measure; discounted derivative price must be a martingale.

$$\Longrightarrow E^{Q}[dU - rU dt] = 0$$

$$= \frac{\partial U}{\partial t} + rS\frac{\partial U}{\partial S} + (\kappa(\theta - v) - \lambda v)\frac{\partial U}{\partial v} + \frac{1}{2} vS^2 \frac{\partial^2 U}{\partial S^2} + \rho\gamma v S\frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2 U}{\partial v^2} - rU = 0$$

which forms the Generalized Black Scholes model.