Lecture 2

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2.8 Random Variable

R.V.

A random variable is a function that assigns a proprobability to each outcome in the sample space.

Discrete Random Variable:

The range of values is finite or countable infinite.

Continuous Random Variable:

The range is an interval or uncountable infinite.

Example: two dice. Let X denote the sum of the two dice.

$$X = \begin{cases} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \\ 12 \end{cases}$$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

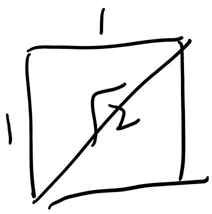
$$p(X=2) = \frac{1}{36} = f(X=2)$$

$$p(X=3) = \frac{2}{36} = f(X=3)$$

$\frac{1}{2}$ $\frac{2}{3}$ $\sqrt{2}$

$A = [1, 2]$

$B =$ All integer. 1 2 3 ....



$\sqrt{2}$

X. A. B. C.

Definition: (Discrete)

For a discrete R.V. X with possible values $x_1 \cdots x_n$, a probability mass function (p.m.f) is a function satisfies:

a probability mass function (p.m.f) is a function satisfies:

(1) $f(X_i) \geq 0$

(2) $\sum f(X_i) = 1$

$\Rightarrow$ (3) $f(X_i) = P(X = x_i)$

Example: 3 Coins, Coin 1 has $P(H) = \frac{1}{2}$

coin 2 has $P(H) = \frac{1}{3}$

Coin 3 has $P(H) = \frac{1}{4}$

Find the p.m.f of X. X denots # of heads, draw 3 coins.

X	0	1	2	3
f(X)	$\frac{1}{4}$	$\frac{11}{24}$	$\frac{1}{4}$	$\frac{1}{24}$

p.m.f = f(X)

$f(X=0) = P(X=0) = (1-\frac{1}{2}) \cdot (1-\frac{1}{3}) \cdot (1-\frac{1}{4}) = \frac{1}{4}$

1. HTT 2. THT 3. TTH

$f(X=1) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{11}{24}$

$f(X=2) = \frac{1}{4}$

$f(X=3) = \frac{1}{24}$

Example: 10 balls in an urn, 5 white, 5 Black.

(a) balls are drawn each time, without replacement until a black one is get. what's the probability we get the black at 4th draw?

Let denote X as # of draw for the first black ball.

$x = 1, 2, 3, 4, 5, 6$

$P(X=1) = f(X=1) = \frac{5}{10} = \frac{1}{2}$

$P(X=2) = f(X=2) = \frac{5}{10} \cdot \frac{5}{9} = \frac{5}{18}$

$$p(X=2) = f(X=2) = \underset{\text{white}}{\frac{5}{10}} \cdot \underset{\text{black}}{\frac{5}{9}} = \frac{5}{18}$$

$$p(X=4) = f(X=4) = \underset{\text{white}}{\frac{5}{10}} \quad \underset{\text{white}}{\frac{4}{9}} \quad \underset{\text{white}}{\frac{3}{8}} \cdot \underset{\text{black}}{\frac{5}{7}} \approx 0.0595$$

(b) what if we draw until two <u>consecutive black</u> balls are obtained. $P(X=5)$?

$\square \quad \square \quad \boxed{W} \quad \underbrace{\boxed{B} \quad \boxed{B}}$

① W W $\qquad \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{4}{6}$

② B W $\qquad \frac{5}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{4}{7} \cdot \frac{3}{6}$

③ W B $\qquad \frac{5}{10} \quad \frac{5}{9} \quad \frac{4}{8} \quad \frac{4}{7} \cdot \frac{3}{6}$

~~B B~~

$$P(X=5) = 0.11905.$$

Example: 200 students, 40 from 3J, 160 from 3Y.

randomly choose 3 students.

X denotes the # of 3J students.

Find p.m.f of X:

$$X = \{0, 1, 2, 3\}$$

$$p(X=0) = \boxed{\frac{{}_{40}C_0 \cdot {}_{160}C_3}{{}_{200}C_3}} = \frac{160}{200} \cdot \frac{159}{199} \cdot \frac{158}{198}$$

$$= 0.5101$$

$$p(X=1) = \frac{{}_{40}C_1 \cdot {}_{160}C_2}{{}_{200}C_3} = 0.3874$$

$$p(X=2) = \frac{{}_{40}C_2 \cdot {}_{160}C_1}{{}_{200}C_3} = 0.0950$$

$$P(X=2) = \frac{\quad}{200C_3} = 0.0950$$

$$P(X=3) = \frac{40C_3 \cdot 160C_0}{200C_3} = 0.0075$$

Cumulative Distribution Function. C.D.F.

$$F(x) = P(X \le x) = \sum_{X \le x} f(x)$$

X: r.v. x: specific number

Example:

$$P(X \le x) = \begin{cases} 0.5105 & x=0 \\ 0.5105 + 0.3874 & x=1 \\ 0.5105 + 0.3874 + 0.095 & x=2 \\ \underline{1} & x=3 \end{cases}$$

Property:

(1) $0 \le F(x) \le 1$

(2) If $x < y$, then $F(x) \le F(y)$

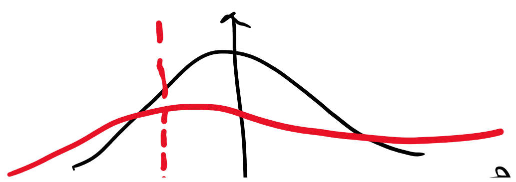
Mean or Expected Value of R.V.

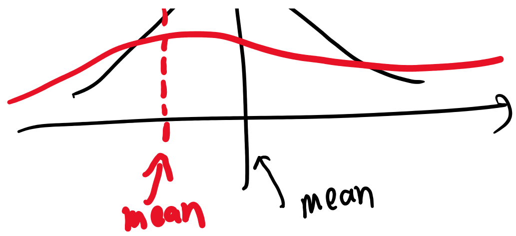
$$\mu = E(X) = \sum_{x} x \, \underline{f(x)}$$

$\sum_x$: all x in the range.

Variance of R.V.

$$\sigma^2 = V(X) = Var(X) = E([X-\mu]^2) = \sum_{x} (x-\mu)^2 f(x)$$





Example: P.M.F $p(X=x) = \begin{cases} \frac{1}{6} & 1 \\ \vdots & 2 \\ \vdots & 3 \\ & 4 \\ & 5 \\ \frac{1}{6} & 6 \end{cases}$

Mean: $\mu = E(X) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6$

$f(1)\cdot 1 + f(2)\cdot 2 \quad - - - - \quad f(6)\cdot 6$

$= 3.5$

Variance:

$\sigma^2 = V(X) = E[(X-\mu)^2]$

$f(1)\cdot(1-\mu)^2 + f(2)\cdot(2-\mu)^2 + \cdots$

$= \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \frac{1}{6}(3-3.5)^2 + \cdots \frac{1}{6}(6-3.5)^2$

$= 2.91667$

Standard Deviation:

$\sigma = \sqrt{Var(X)}$

$$\boxed{\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2}$$

$E(X^2) = \frac{1}{6}\cdot 1^2 + \frac{1}{6}\cdot 2^2 + \frac{1}{6}\cdot 3^2 + \cdots \frac{1}{6}\cdot 6^2$

Binomial Distribution:

Bernoulli Trial: a trial with two possible outcomes

Example: Flip a coin.

Example: Roll a dice and see whether getting a 6.

Binomal Distribution:

A random variable consist of n Bernoulli Trial.

(1) Trials are independent

(2) Each trial results in only two possible outcomes, labeled as success or failure

(3) The probability of success is same p, $0<p<1$

P.M.F.: $\quad {}_nC_x = \binom{n}{x}$

$$\boxed{P(X=x)=f(x)=\binom{n}{x}p^x(1-p)^{n-x}, \quad x=1,\dots n.}$$

Example:

Suppose 10% people are left handed. A sample of 5. Let X denotes the # of left handed person among 5.

$p=0.1 \qquad n=5.$

p.m.f of X:

$$P(X=0)=\underbrace{\binom{5}{0}}_{5\text{ choose }0}\underbrace{0.1^0}_{\text{left}^0}\cdot \underbrace{0.9^5}_{\text{right}^5}$$

$$P(X=1)=\binom{5}{1}0.1^1\ 0.9^4$$

$$P(X=2)=\binom{5}{2}0.1^2\ 0.9^3$$

$$P(X=3)=\binom{5}{3}0.1^3\ 0.9^2$$

$$P(X=4)=\binom{5}{4}0.1^4\ 0.9^1 \qquad n=5$$

$$P(X=5)=\binom{5}{5}0.1^5\ 0.9^0 \qquad p=0.1$$

Mean and Variance of X.

$$\mu=E(X)=0\cdot P(X=0)+1\cdot P(X=1)+\dots \quad 5\,P(X=5)$$

$\mu = E(X) = 0 \cdot p(X=0) + 1 \cdot p(X=1) + \cdots \quad 3\,p(X=3)$

$= 0.5$

$E(X) = n \cdot p$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = 0.7$

$Var(X) = 0.45$

$Var(X) = np(1-p)$

Binomial Expansion:

Flip a coin n times. what's the total # of outcomes.

$n=1 \quad 2$

$n=2 \quad 4$

$n=3 \quad 8$

$\vdots$

$n=n \quad 2^n$

0 tail, all heads

1 tail $n-1$ heads

$\vdots$

n tail 0 heads

$${}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{n-1} + {}_nC_n$$

$2^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots \quad {}_nC_{n-1} + {}_nC_n$

$\Rightarrow (a+b)^n = {}_nC_0\, a^0 b^n + {}_nC_1\, a^1 b^{n-1} + \cdots \quad {}_nC_r\, a^r b^{n-r} + \cdots \quad {}_nC_n\, a^n b^0$

Biomial Expansion.

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$

Example:

$(a+b)^{10}$ what's the coefficient of term ab^9.

Example:

$(a+5b)^{10}$ what's the coefficient of term ab^9.

$${}_{10}C_0\, a^0 (5b)^{10} + {}_{10}C_1\, a^1 (5b)^9 + \cdots\cdots$$

$$= {}_{10}C_1\, 5^9\, ab^9$$

Geometric Distribution:

In a series of Bernoulli Trials. (Independent, p)

The R.V. X equals # of trials until the first success.

p.m.f:

$$p(X=x) = f(x) = (1-p)^{x-1} p, \qquad x = 1, 2, \cdots \infty$$

Example:

Roll a dice, until the first '6' apper.

X = # of Rolls.

$$p = \frac{1}{6}$$

$$p(X=x) = \begin{cases} \frac{1}{6} & X=1 \\ \frac{5}{6} \cdot \frac{1}{6} & X=2 \\ \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} & X=3 \\ \vdots & \vdots \\ \left(\frac{5}{6}\right)^{r-1} \frac{1}{6} & X=r \end{cases}$$

(a) Probability that I got a '6' within 2 to 4 rolls inclusively.

$$p(X=2) + p(X=3) + p(X=4)$$

$$= \frac{5}{6} \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6}.$$

$$= \frac{5}{6}\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^3\frac{1}{6}.$$

(b) Mean Variance.

$$\mu = E(X) = \frac{1}{6}\cdot 1 + \left(\frac{5}{6}\right)\cdot\frac{1}{6}\cdot 2 + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6}\cdot 3 + \cdots\cdots$$

$$① \quad A = \frac{1}{6}\cdot 1 + \left(\frac{5}{6}\right)\cdot\frac{1}{6}\cdot 2 + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6}\cdot 3 + \left(\frac{5}{6}\right)^3\cdot\frac{1}{6}\cdot 4 + \cdots\cdots$$

$$② \quad \frac{5}{6}A = \left(\frac{5}{6}\right)\frac{1}{6}\cdot 1 + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6}\cdot 2 + \left(\frac{5}{6}\right)^3\cdot\frac{1}{6}\cdot 3 + \left(\frac{5}{6}\right)^4\frac{1}{6}\cdot 4$$

① − ②

$$\frac{1}{6}A = \frac{1}{6} + \left(\frac{5}{6}\right)\frac{1}{6} + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^3\cdot\frac{1}{6}$$

$$A = 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots\cdots$$

$$= 6$$

$$\boxed{\mu = E(X) = \frac{1}{p}}$$

$$\boxed{Var(X) = \frac{1-p}{p^2}}$$

Negative Binomial Distribution:

In a series of Bernoulli trials, Independent, with constant p

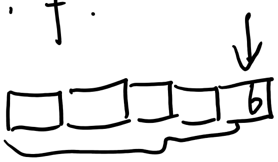
X denotes # of trials until r success happens.

Example:

Roll a dice until 3 '6' occurs.

X = # of trials until we get 3 '6'.

p.m.f.



last success

$$f(x=5) = C^{5-1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2\cdot \frac{1}{6}$$

$$p(X=5) = \underbrace{\binom{5-1}{3-1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2}_{\text{2 success out of 4}} \cdot \boxed{\frac{1}{6}}$$

p.m.f.:

$$p(X=x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r, \quad x=r, r+1, \ldots +\infty$$

Mean:

$$\mu = E(X) = \frac{r}{p}$$

Variance:

$$\sigma^2 = Var(X) = \frac{r(1-p)}{p^2}$$

Remark:

Binomial, Geometric, Negative Binomial has Bernoulli trial with constant p, with replacement.

Hypergeometric Distribution: (without replacement).

A set of N objects, contains K objects classified as success, $N-K$ objects as failure.

A sample of size n objects randomly chosen without replacement from the N objects.

when $K \le N$, $n \le N$

X = # of success.

p.m.f

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Example: students 40 from 3J, 160 from 3Y

Example: 200 students, 40 from 3J, 160 from 3Y

with replacement, I pick 3 randomly.

X = # of 3J students

$p(X=1)$ $n=3$ $x=1$

$p=\frac{40}{200} \Rightarrow$ Binomial

$$p(X=1)=\binom{n}{x}p^x(1-p)^{n-x}$$

$$=\binom{3}{1}\left(\frac{40}{200}\right)^1\left(1-\frac{40}{200}\right)^2$$

with out replacement " " 3.

X = # of 3J students.

$p(X=1)$ $N=200$ $K=40$

$N-K=160$ $n=3$

p is not constant $\Rightarrow$ Hypergeometric

$$p(X=1)=\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$=\frac{\binom{40}{1}\binom{160}{2}}{\binom{200}{3}}$$

Mean:

$$\mu=E(X)=np$$

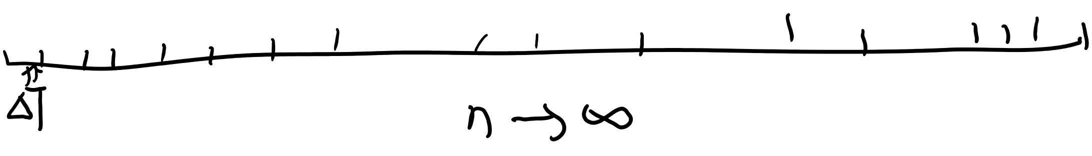
Variance:

$$\sigma^2=Var(X)=np(1-p)\cdot\frac{N-n}{N-1}$$

Poisson Distribution:

Example: The average of emails I get per day is 10.

whats the probability that I get 20 emails tomorrow. A Day



$p = \frac{10}{n} \Leftarrow \Delta T$

$p(X=x) = \binom{n}{x} p^{n}(1-p)^{n-x}$

If I know x, I know $nx \Leftarrow$ emails for a specificday

$= \binom{n}{x}\left(\frac{10}{n}\right)^{x}\left(1-\frac{10}{n}\right)^{n-x}$

$n \to \infty$

limits $= \boxed{\frac{10^{x} e^{-10}}{x!}}$

$\left(\lim_{n\to\infty}\left(1+\frac{10}{n}\right)^{n} = e^{10}\right)$

$p(X=20) = \frac{10^{20} e^{-10}}{20!}$

p.m.f of Poisson:

$p(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$ where λ is the average of X

$x = 0, \cdots +\infty$

Mean :

$\mu = E(X) = \lambda$

Variance :

$\sigma^2 = Var(X) = \lambda$

Example:

# of people in the hospital emergency on average is 5. per hour.

(a) Find the probability that at least 1 patient is there?

$p(X \geq 1) = p(X=1) + p(X=2) \cdots\cdots + p(X=100000) + \cdots$

$= 1 - p(X=0)$

$= 1 - \frac{e^{-5}5^{0}}{1}$

$\boxed{0! = 1}$

$\boxed{0! = 1}$

$= 1 - p(\ldots)$

$= 1 - \frac{e^{-5} 5^0}{0!}$

$= 1 - e^{-5}$

(b) Mean and Variace.

$\mu = \sigma^2 = \lambda = 5$

(c) How long would you wait on averge for (hours) where no people are addmitted.

$p(X=0) = e^{-5} = p \Rightarrow$ for an hour. no people's probability.

Geometric distribution with $p = e^{-5}$

Y = # of hours on averge to wait.

Y follows a geometric.

$\mu_Y = E(Y) = \frac{1}{p} = e^5$

Mutually Exclusive

$p(A \cap B) = 0$

Independent:

$p(A \cap B) = p(A) \cdot p(B)$

$\Rightarrow p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Example: In classroom, 70% people drive here $\Rightarrow A$

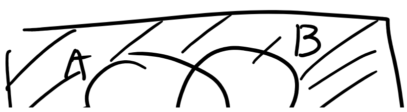
80% people are from 3Y $\Rightarrow B$.

6% people are neither A nor B.

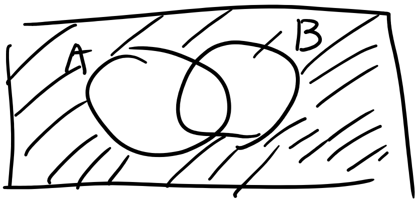
Are A and B Independent?

$(A \cup B)' = 0.06$

$p(A) = 0.7$ $p(B) = 0.8$



$p(A) = 0.7 \qquad p(B) = 0.8$



$p[(A \cup B)'] = 0.06$

$p(A \cup B) = 0.94$

$\therefore p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$0.94 = 0.7 + 0.8 - p(A \cap B)$

$\therefore p(A \cap B) = 0.56$

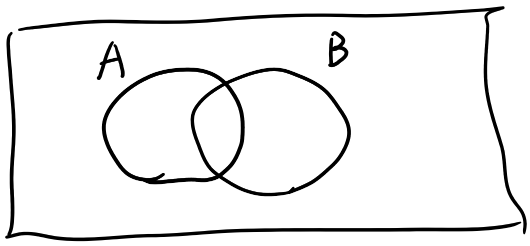
$\therefore p(A \cap B) = p(A) \cdot p(B) \iff$ Indepeadent

Example:
Assume the passing rate for a Eng course is 90%. If I randomly choose 2 students. whats the probability at least one of pass.

Method 1:

$$1 - 0.1^2 = 0.99$$

Method 2:



$p(A) = 0.9 \qquad p(A \cap B) = p(A) \cdot p(B) = 0.81$

$p(B) = 0.9$

$\therefore p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.99$