Lecture 8

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Note:

1. 95% C.I. for people's height is (165, 185), how to interpret this interval.

We are 95% confident / certain that the true mean of people's height is between 165, 185.

2. Independence: X and Y

① $Cov(X,Y) = 0 = E(XY) - E(X)\cdot E(Y)$

② $p(X \text{ and } Y) = p(X)\cdot p(Y)$

③ $p(X|Y) = p(X)$

④ $p(Y|X) = p(Y)$

Mutually Exclusive

$p(X \cap Y) = 0$

9.1 Hypothesis Testing

Statistical hypothesis:

is a state about the parameter of one or more poplucations.

— Null Hypothesis (H_0) is that the population parameter is equal to some value.

H_0: $\mu = 65$

— Alternative Hypothesis (H_a/H_1) is that the population parameter is $>$, $<$, $\neq$ to some value.

H_1: $\mu > 65$

or

H_1: $\mu < 65$

or

H_1: $\mu \neq 65$

Critical region:

consists of values of the test statistic that would result in rejecting H_0.

Test statistic:

is a statistic where value is used to make a decision about hypothesis from a sample. $(\bar{X})$

Example: H_0: $\mu = 70$

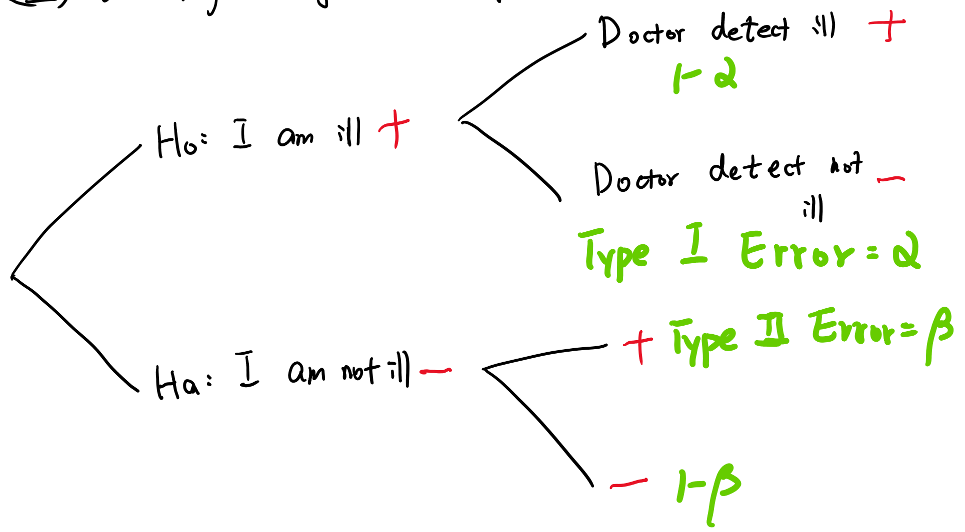
H_1: $\mu \neq 70$

Critical Region: $\bar{X} \geq 72$ or $\bar{X} \leq 68$

reject H_0 if $\bar{X} \geq 72$ or $\bar{X} \leq 68$

Critical Region: $\bar{X} \geq 72$ or $\bar{X} \leq 68$

$\Leftrightarrow$ We might reject H_0 if $\bar{X} \geq 72$ or $\bar{X} \leq 68$



Type I Error:

A Type I Error occurs if we reject H_0 but H_0 is actually true. α = significance level.

Type II Error:

A Type II Error occurs if we fail to H_0 but H_1/H_0 is true. β

$1-\beta$ = power of the test.

Example: In testing, $H_0: \mu = 70$, $H_1: \mu \neq 70$ based on a sample of size 64 from Normal. Suppose that we reject H_0 if $\bar{X} \geq 72$ or $\bar{X} \leq 68$

Find α when $\sigma = 16$.

$\alpha = P(\text{Type I Error})$ (Assume H_0 is True $\Leftrightarrow \mu = 70$)

$= P(\bar{X} \geq 72) + P(\bar{X} \leq 68)$

$\left(Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)$ $= P\left(Z \geq \frac{72-70}{16/\sqrt{64}}\right) + P\left(Z \leq \frac{68-70}{16/\sqrt{64}}\right)$

$= P(Z \geq 1) + P(Z \leq -1)$

$= 0.3173$

Note: Increasing sample size will always decrease α.

$\beta = P(\text{Type II Error})$

$= P(\text{failing to reject } H_0 \text{ when } H_1 \text{ is true})$

$= P(\text{do not reject } H_0 \mid \underline{H_1 \text{ True}})$

$= p(\text{do not reject } H_0 \mid \underline{H_1 \text{ True}})$

Find β if $\boxed{\mu = 74}$ true mean

$= p(68 \le \bar{X} \le 72 \mid \mu = 74)$

$= p\left(\frac{68-74}{16/\sqrt{64}} \le Z \le \frac{72-74}{16/\sqrt{64}}\right)$

$= p(1 \le Z \le 3)$

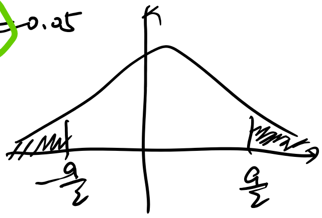
$= 0.1573$

Example: Find the boundary of the critical region if we want $\alpha = 0.05$.

$p(\bar{X} \ge \boxed{70+a}) + p(\bar{X} \le \boxed{70-a}) = 0.05$

$p\left(Z \ge \frac{70+a-70}{16/\sqrt{64}}\right) + p\left(Z \le \frac{70-a-70}{16/\sqrt{64}}\right) = 0.05$

$p(Z \ge \frac{a}{2}) + p(Z \le \frac{-a}{2}) = 0.05$



$\Leftrightarrow 2p(Z \ge \frac{a}{2}) = 0.05$

$\frac{a}{2} = 1.96$

$a = 3.92$

Critical Region: $\bar{X} \ge \underline{73.92}$ or $\bar{X} \le \underline{66.08}$.

Note: 1. For a fixed sample size, decreasing α, increases β.

2. For a fixed α, increase sample size, n will decrease β.

P value:

The p-value is the probability of obtaining a value of the test statistic at least as etreme as the observed value when H_0 is true

(Same direction with H_1)

Example: H_0: $\mu = 70$

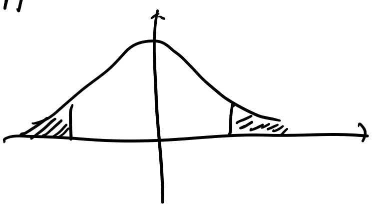
H_1: $\mu \ne 70$ two tailed

Example: $H_0: \mu = 70$

$H_1: \mu \neq 70$ two tailed

$n = 64$, $\sigma = 16$. Suppose $\bar{x} = 74$

Find p-value. alternative '$\neq$'



$$p\text{-value} = \boxed{2} \cdot P(\bar{x} > 74)$$

$$= 2 \cdot P\left(Z > \frac{74-70}{16/\sqrt{64}}\right)$$

$$= 2 \cdot P(Z > 2)$$

$$= 0.0455$$

(the probability of mistakes to reject H_0 when $\bar{x} = 74$.)

If for a given α for a test.

p-value $< \alpha$, then reject H_0.

Hypothesis Test:

① Hypothesis: H_0: $\mu = 0$ always equal

H_1: $>$

$<$

$\neq$

② Test statistic:

③ Critical Region / Critical Value

p-value

④ Conclusion:

Test Stat is in the critical region

$\Leftrightarrow$ p-value $< \alpha$

$\Leftrightarrow$ reject H_0.

9.2 Test on the mean of Normal (σ known)

① H_0: $\mu = \mu_0$

H_1: ① $\mu \neq \mu_0$

H_1: ① $\mu \neq \mu_0$

② $\mu > \mu_0$

③ $\mu < \mu_0$

② Test statistic

$$Z_0 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

③

	①	②	③
Critical Region:	$Z_0 > Z^*_{\frac{\alpha}{2}}$ or $Z < -Z^*_{\frac{\alpha}{2}}$	$Z_0 > Z^*_{\alpha}$	$Z_0 < -Z^*_{\alpha}$
p-value:	$2p(Z > \lvert Z_0 \rvert)$	$p(Z > Z_0)$	$p(Z < Z_0)$

④ Conclusion: p-value < α, reject H_0.

or |Test stat| > |Criticital value|, reject H_0

For all tests

Example:

The average house price in Canada in 2010 was $\boxed{\$339100}$ (μ_0) with standard deviation (σ) $\boxed{\$194300}$. according to CREA.

Today, a consumer randomly select $\boxed{54}$ $n=54$ house sample. find an average of $\$\boxed{418200}$. $\bar{x}$

State a hypothesis to test whether the price has increased significantly, $\alpha = 0.1$

① Hypothesis: $H_0: \mu = 339100$

$H_a: \mu > 339100$

② Test Stat:

$$Z_0 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$= \frac{418200 - 339100}{194300/\sqrt{54}}$$

$$= 2.99$$

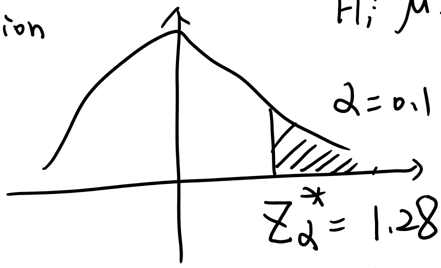
$= 2.99$

$\approx 3.$

③ Critical Region

(a)

$H_1: \mu > 339100$



$\alpha = 0.1$

$Z_\alpha^* = 1.28$

$\Leftrightarrow p(Z < Z_\alpha^*) = 0.9$

(b) p-value $= p(Z > Z_0)$

$= p(Z > 3)$

$= 0.001$

④ Conclusion:

(a) $|\text{Test Stat}| = 3 > |\text{Critical Value}| = |Z_\alpha^*|$

$= 1.28$

$\Leftrightarrow$

(b) p-value $= 0.001 < \alpha = 0.01$

$\Leftrightarrow$ reject H_0.

9.3 Tests on the Mean of a Normal Distribution (σ unknown)

② Test Stat:

$$t_{n-1} = \frac{\bar{x} - \mu}{s/\sqrt{n}}.$$

All other steps same 9.1

Example: A company claims that its batteries last at least 35 hours.

Here is sample of their products:

35 34 32 31 34 34 32 33 35 55 32

31

$n = 12$

Do a hypothesis test for battery does not last longer than 35 hours. $\alpha = 0.05$

H_1: $>$, $<$ } one tail

$\neq$ two tail

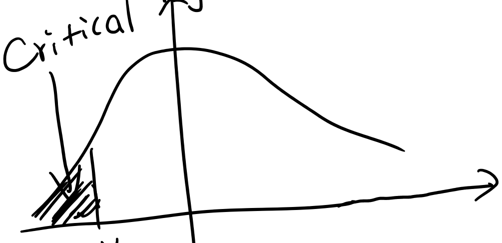
① $H_0: \mu = 35$

$H_1: \mu < 35$ → one tail

$\bar{X} = 34.8333$ $\boxed{S} = 6.506$

② Test Stat: $t_{n-1} = \dfrac{\bar{X} - \mu}{S/\sqrt{n}} = -0.089$

Critical region



$t^*_{n-1,\alpha}$

③ $H_1: \mu < 35$

$-t^*_{n-1,\alpha}$ = Critical value

$= -t^*_{11,0.05} \Leftrightarrow$ one tail t, $\alpha = 0.05$, $df = 11$

$= -1.796$

④ |Test Stat| < |Critical Value|

∴ do not reject H_0.

9.5 Tests on a Population Proportion.

① $H_0: p = p_0$

$H_1: p > p_0$

$p < p_0$

$p \neq p_0$

where p_0 is a given number.

② Test Stat:

$$Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)$$

($\hat{p}$ sample statistic)

③
④ > same with 9.1

Example: In 2006, 34% of students had not been absent from school even once. $p_0 = 0.34$

In 2011 survey responses from 8302 students $n = 8302$

showed the rate is 33%. $\hat{p} = 0.33$

Officials would be concerned if student attendence were declining. Do a hypothesis test $\alpha = 0.05$.

① $H_0: p = 0.34$

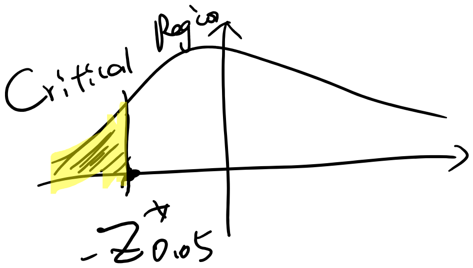
$H_1: p < 0.34$

② Test Stat

$$Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.33 - 0.34}{\sqrt{\frac{0.34(1-0.34)}{8302}}} = -1.923$$

③

Critical value $|Z^*_{0.05}| = 1.645$



④ |Test Stat| > |Critical Value|

⟺ reject H_0.

⟺ p-value $= P(Z < -1.923)$

$= 0.0272 < 0.05$

'ς calculation for the exams.

There will be no β's calculation for the exams.

10.1 - 10.2 Inference for difference of two means

Case I:

Two independent samples from population (σ_1, σ_2 unknown)

① Hypothesis: $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 > \mu_2$

$\mu_1 < \mu_2$

$\mu_1 \neq \mu_2$

② Test Statistic:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

③ ④ same with 9.1

Case II

Two independent sample (σ_1 σ_2 unknown, assuming equal variance for two population)

①

② Test Stat:

$$\boxed{t_{df} = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}}$$

where $\boxed{df = n_1 + n_2 - 2}$

$$\boxed{S_p^2 = \frac{(n_1 - 1)\sigma_1^2 + (n_2 - 1)\sigma_2^2}{n_1 + n_2 - 2}}$$

$$\boxed{Sp^2 = \frac{(n_1-1)\sigma_1^2 + (n_2-1)\sigma_2^2}{n_1+n_2-2}}$$

pooled variance

③, ④ same with q1

Example: A developer wants to know if the houses in two different neighbourhoods were built at roughly the same time. She took random samples

Neighbour 1

$n_1 = 30$

$\bar{x}_1 = 57.2$

$s_1 = 7.51$

Neighbour 2

$n_2 = 35$

$\bar{x}_1 = 47.6$

$s_2 = 7.85$

Does N1 has significantly larger age than N2? $\alpha = 0.05$

Assume the population variances are equal. $(\sigma_1^2 = \sigma_2^2)$

① $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 > \mu_2$ one tail

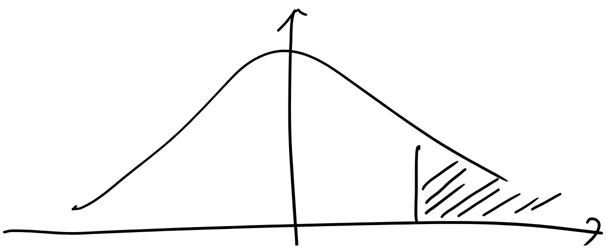
② Test Stat:

$$t_{df} = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{sp^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$sp^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = 59.218$$

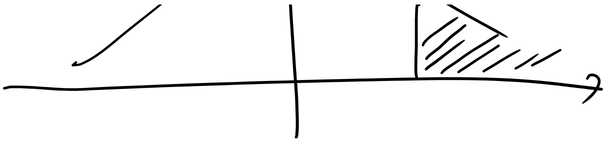
$$df = n_1 + n_2 = 63$$

$$t_{df} = 5.01$$



③ Critical value

③ Critical vale



$t^{*}_{63,\,0.05} = 1.671$

④ $|\text{Test Stat}| > |\text{Critical Value}|$

$\therefore$ reject H_0.

CI for $(\mu_1 - \mu_2)$: $(\sigma_1^2 = \sigma_2^2)$

$$\bar{X}_1 - \bar{X}_2 \pm t^{*}_{n_1+n_2-2,\,\frac{\alpha}{2},\,\text{two tail}} \cdot \sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Case III: two independent sample $(\sigma_1^2 \neq \sigma_2^2)$

② Test Stat:

$$t_{df} = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$\text{where } df = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

CI for $\mu_1 - \mu_2$ $(\sigma_1^2 \neq \sigma_2^2)$

$$\bar{X}_1 - \bar{X}_2 \pm t_{df,\,\frac{\alpha}{2}\ \text{two tail}} \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}.$$

Example: Two suppliers manufature a plastic gear used in a laser printer.

Supplier 1: $\bar{X}_1 = 296$ $\quad S_1 = 12$ $\quad n_1 = 10$

" 2: $\bar{X}_2 = 321$ $\quad S_2 = 22.$ $\quad n_2 = 16.$

Test Ho hypothesis that the suppliers provide gears with

Test the hypothesis that the suppliers provide gears with different mean. Assume variances are not equal, $\alpha = 0.05$.

① Hypothesis: $H_0: \mu_1 = \mu_2$

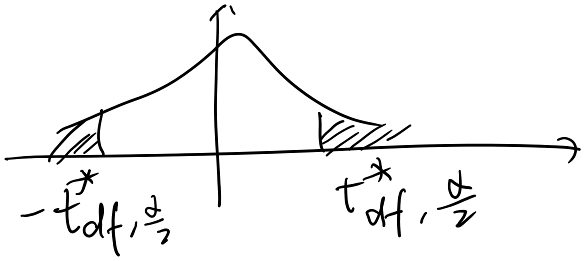
$H_1: \mu_1 \neq \mu_2$ two tail.

② Test Statistic:

$$t_{df} = \frac{290 - 321 - (0)}{\sqrt{\frac{12^2}{10} + \frac{22^2}{16}}} = -4.64$$

$df = 23.72 \approx 23$ round down

③ Critical Value



$t^*_{23, \frac{0.05}{2}}$, two tail

$= 1.714$

④ $|\text{Test Stat}| > |\text{Critical Value}|$

$\Longleftrightarrow$ reject H_0.