Lecture 4

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5.1 Two or More Random Variables:

Joint probability mass function (p.m.f) → Discrete R.V.

X and Y are both discrete R.V.

Then $f_{X,Y}(x,y) = P(X=x, Y=y)$

(1) $f_{X,Y}(x,y) \geq 0$

(2) $\sum_x \sum_y f_{X,Y}(x,y) = 1$

Example: Suppose X, Y two dice, X is fair, Y is not

$$f_Y(y) = P(Y=y) = \begin{cases} \frac{1}{6} & 1 \\ \frac{1}{6} & 2 \\ \frac{1}{6} & 3 \\ \frac{1}{5} & 4 \\ \frac{1}{5} & 5 \\ \frac{1}{10} & 6 \end{cases} \Rightarrow \text{p.m.f of } Y.$$

Then, X and Y independent. What's the joint p.m.f of X and Y.

$$f_{X,Y}(x,y) = P(X=x, Y=y) = \left.\begin{cases} \frac{1}{6}\cdot\frac{1}{6} & x=1,\ y=1 \\ \frac{1}{6}\cdot\frac{1}{6} & x=1,\ y=2 \\ \vdots & \\ \frac{1}{6}\cdot\frac{1}{5} & x=2,\ y=4 \\ \vdots & \\ \frac{1}{6}\cdot\frac{1}{10} & x=6,\ y=6 \end{cases}\right\} \text{36 pairs}$$

The joint probability density function: p.d.f. Continuous.

X, Y Continuous R.V.

Then $f_{X,Y}(x,y)$ follows:

(1) $f_{XY}(x,y) \geq 0$

(2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

(3) for any probability in a certain region B.

$P((X,Y) \in B) = \iint f_{XY}(x,y)\,dx\,dy$

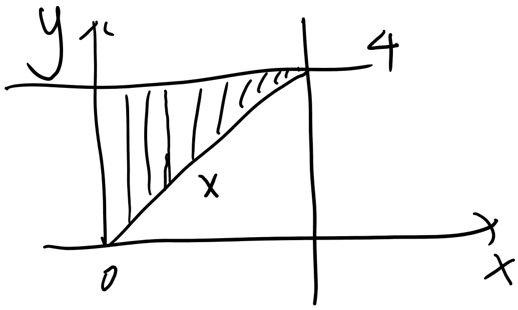
(3) for any ...

$$P(X,Y \in B) = \iint_B f_{XY}(x,y)\,dx\,dy$$

Example: $f(x,y)$

Let $f_{X,Y}(x,y) = c(x+y)$, $\boxed{0<x<y<4}$

(a) Find $c$.




$$\int_0^4 \int_x^4 c(x+y)\,dy\,dx = 1$$

$$\int_0^4 \left\{ c\left[xy+\frac{y^2}{2}\right]\Big|_x^4 \right\} dx = 1$$

$$c\int_0^4 \left(-\frac{3}{2}x^2+4x+8\right) dx = 1$$

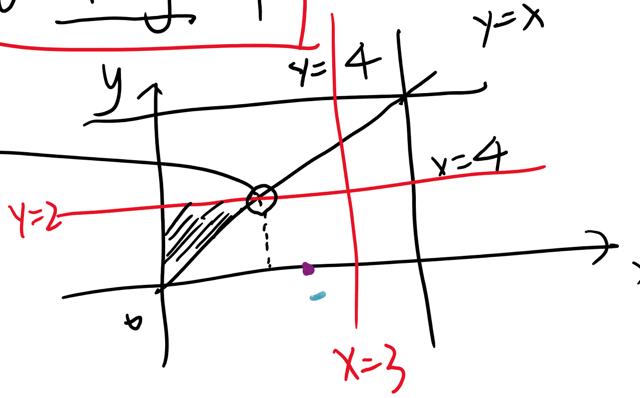
$$\therefore c\cdot\left[-\frac{3}{6}x^3+2x^2+8x\right]\Big|_0^4 = 1$$

$$\therefore c = \frac{1}{32}$$

$$\therefore f(x,y) = \frac{1}{32}(x+y), \quad 0<x<y<4.$$

range in the question

(b) $P(X<3, Y<2)$ $\quad \boxed{0<x<y<4}$




$$\begin{cases} y=x \\ y=2 \end{cases}$$

$\therefore$ $x=2$, $y=2$

$$P(X<3, Y<2) = \int_0^2 \int_x^2 \frac{1}{32}(x+y)\,dy\,dx$$

$$= \int_0^2 \frac{1}{32}\left(xy+\frac{y^2}{2}\right)\Big|_x^2 dx$$

$$= \int_0^2 \frac{1}{32}\left[(2x+2)-\left(x^2+\frac{x^2}{2}\right)\right] dx$$

$$= \frac{1}{32}\left[-\frac{3}{6}x^3+x^2+2x\right]_0^2$$

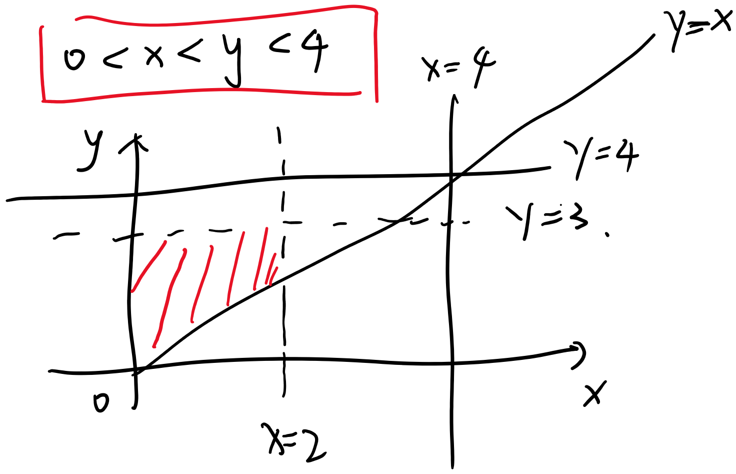
$$= \frac{1}{8}$$

$= \frac{1}{8}$

(3) $P(X<2,\ Y<3)$

$$\int_0^2 \int_x^3 \frac{1}{32}(x+y)\, dy\, dx$$

$\boxed{0<x<y<4}$




## Marginal Probability Distrition:

$f(x,y)$

$\Rightarrow f(x)$ ?

$\Rightarrow f(y)$ ?

Discrete:

$$f_X(x) = \sum_y f_{XY}(x,y)$$

$$f_Y(y) = \sum_x f_{XY}(x,y)$$

Continuous

$$f_X(x) = \int_{\text{all } y} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{\text{all } x} f_{X,Y}(x,y)\, dx$$

$f_X(x)$ marginal distribution of $X$.

$f_Y(y)$ marginal distribution of $Y$.

Example:

$f(x,y) = \frac{1}{32}(x+y)$ $\boxed{0<x<y<4}$

Find $f_X(x)$, $f_Y(y)$.

$f(x)$ $f(y)$

$\boxed{x<y<4}$

$$f_X(x) = \int_x^4 \frac{1}{32}(x+y)\, dy$$

$$= \frac{1}{32}\left(xy + \frac{y^2}{2}\right)\Big|_x^4$$

$$= \frac{1}{32}\left[-\frac{3}{2}x^2 + 4x + 8\right], \quad 0<x<4$$

$$= \frac{1}{32}\left[-\frac{3}{2}x^2 + 4x + 8\right], \quad 0<x<4$$

$$f_Y(y) = \int_0^y \frac{1}{32}(x+y)\,dx \qquad \boxed{0<x<y}$$

## Conditional Distribution and Independence.

$f_{XY}(x,y)$

**Conditional Distribution**

Discrete: $P_{Y|X}(y|x) = f_{Y|X}(y|x) = \dfrac{P(X=x, Y=y)}{P(X=x)} \quad \left(P(A|B) = \dfrac{P(A\cap B)}{P(B)}\right)$

Continuous $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

follows property of probability distribution

**Independence.**

Discrete: $P(X=x, Y=y) = f(x,y) = f(x)\cdot f(y) = P(X=x)\cdot P(Y=y)$

$\Longleftrightarrow \left(P(A\cap B) = P(A)\cdot P(B)\right)$

Continuous: $f(x,y) = f(x)\cdot f(y)$

Example: R.V. $X, Y$ with joint p.m.f.

| | $Y=2$ | $Y=4$ | $Y=5$ | |
|---|---|---|---|---|
| $X=1$ | $\frac{1}{12}$ | $\frac{1}{24}$ | $\frac{1}{24}$ | $\frac{1}{6}$ |
| $X=2$ | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{3}{8}$ |
| $X=3$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{12}$ | $\frac{11}{24}$ |
| | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | |

Discrete.

(a) Find $P(X\le 2, Y\le 4)$ $\qquad \frac{3}{8}$

(b) Find Marginal distribution for $X, Y$.

$P(Y=y) = \begin{cases} \frac{1}{2} & Y=2 \\ \ldots \end{cases}$

(b) Find Marginal

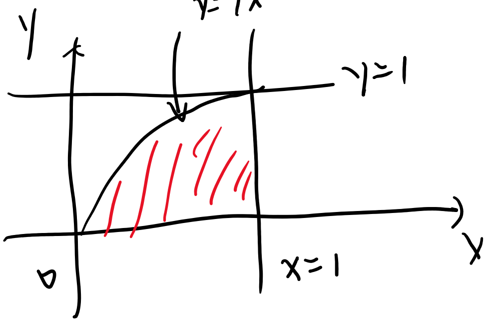
$$p(Y=y)=\begin{cases}\frac{1}{2} & Y=2\\ \frac{1}{4} & Y=4\\ \frac{1}{4} & Y=5\end{cases}$$

$$P(X=x)=\begin{cases}\frac{1}{6} & X=1\\ \frac{3}{8} & X=2\\ \frac{11}{24} & X=3\end{cases}$$

Example:

$f(x,y)=6xy, \quad 0\le x\le 1, \quad 0\le y\le \sqrt{x}$.

(a) find $f_X(x)$, $f_Y(y)$.




$$f_X(x)=\int_0^{\sqrt{x}} 6xy\,dy$$

$$=3x^2, \qquad 0\le x\le 1$$

$$f_Y(y)=\int_{y^2}^{1} 6xy\,dx \qquad \begin{cases} y\le\sqrt{x}\\ 0\le x\le 1\end{cases} \Rightarrow y^2\le x\le 1$$

$\cancel{0}$

$$=3y(1-y^4)$$

(b) Find $f_{X|Y}(x|y)$ $\qquad x|y$ x given that y occurs

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f(y)}$$

$$=\frac{6xy}{3y(1-y^4)}$$ Condition p.d.f.

$$=\frac{2x}{1-y^4}, \qquad y^2\le x\le 1$$

(c) Are X and Y Independent?

either way ① $f(x,y)\ne f(x)f(y)$ ② $f(x|y)\ne f(x)$ $\Big\}$ not independent

5.2 Covariance and Correlation.

## 5.2 Covariance and Correlation.

**Covariance:** (Continuous only)

For $h(x,y)$, the expected value of $h(x,y)$ is defined as:

$$E(h(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dy\, dx$$

The covariance:

$$Cov(X,Y) = \sigma_{xy} = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$= E(XY) - E(X)\cdot E(Y)$$

**Correlation:**

$$\rho_{xy} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

Property:

(1) $-1 \le \rho_{xy} \le 1$ (linearly)

(2) when $cov(X,Y) = \rho_{xy} = 0 \iff X, Y$ are independent

(3) $0 < \rho_{xy} \le 1$ positive correlated $X\uparrow$ $Y\uparrow$

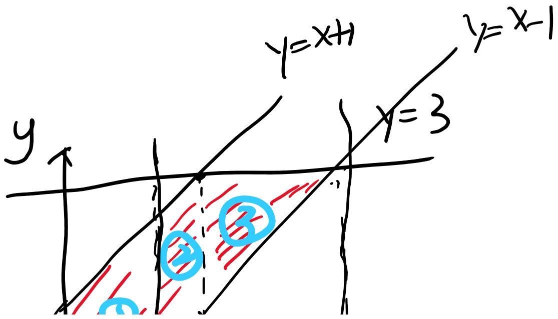
$-1 \le \rho_{xy} < 0$ negative correlated $X\uparrow$ $Y\downarrow$

Example:

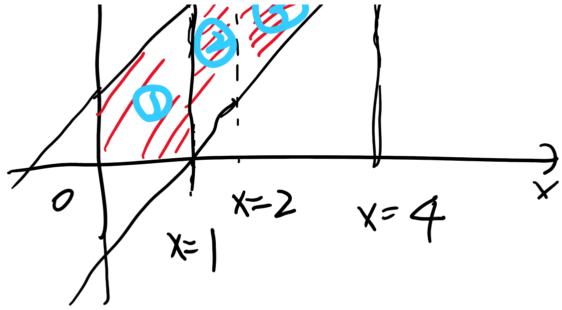
$$f(x,y) = \frac{2}{11}, \quad \begin{cases} 0 < y < 3 \\ y-1 < x < y+1 \\ x > 0 \end{cases} \Rightarrow \begin{matrix} Y = X+1 \\ Y = X-1 \end{matrix}$$

Find $Cov(X,Y)$

$$Cov(X,Y) = \underbrace{E(XY)}_{a} - \underbrace{E(X)}_{b} \cdot \underbrace{E(Y)}_{c}$$




a. $E(XY) = \int\int x\cdot y\, f(x,y)\, dy\, dx$




a. $E(XY) = \int\int x \cdot y \, f(x,y) \, dy \, dx$

$= 1 \int_0^1 \int_0^{x+1} xy \cdot \frac{2}{11} \, dy \, dx$

$+ 2 \int_1^2 \int_{x-1}^{x+1} xy \cdot \frac{2}{11} \, dy \, dx$

$+ 3 \int_2^4 \int_{x-1}^{3} xy \cdot \frac{2}{11} \, dy \, dx$

$= \frac{433}{132}$

b. $E(X) = \frac{5}{3}$ (also 3 parts)

c. $E(Y) = \frac{53}{33}$

$cov(X,Y) = \frac{433}{132} - \frac{5}{3} \cdot \frac{53}{33}$

## 5.4 Linear Functions of R.V.

Given R.V. $X_1, X_2 \cdots X_p$ and Contants $C_1, C_2 \cdots C_p$.

Denote $Y = C_1X_1 + C_2X_2 + C_3X_3 + \cdots C_pX_p$.

$E(Y) = C_1E(X_1) + C_2E(X_2) + \cdots C_pE(X_p)$

$Var(Y) = C_1^2 Var(X_1) + C_2^2 Var(X_2) + \cdots C_p^2 Var(X_p)$

$+ \sum_{i<j} \sum C_i C_j Cov(X_i, Y_i)$ if independent then equal to 0.

Example: $X_1, X_2, X_3$

$Y = 2X_1 + 3X_2 + 4X_3$.

$E(Y) = 2E(X_1) + 3E(X_2) + 4E(X_3)$

$Var(Y) = 4Var(X_1) + 9Var(X_2) + 16Var(X_3)$

$Var(Y) = 4Var(X_1) + 9Var(X_2) + 16Var(X_3)$

$+ 6Cov(X_1, X_2) + 8Cov(X_1, X_3) + 12Cov(X_2, X_3)$

Suppose $X_1 \cdots X_p$ are independent and identical distributions. with mean $\mu$ and Variance $\sigma^2$. $\boxed{i.i.d.}$

Denote $\bar{X}$ is the average of $X_1 \cdots X_p$.

$$E(\bar{X}) = E\left(\frac{X_1 + X_2 + \cdots X_p}{p}\right)$$

$$= \frac{1}{p}E(X_1) + \frac{1}{p}E(X_2) + \cdots \frac{1}{p}E(X_p)$$

$$= \underbrace{\frac{1}{p}\mu + \frac{1}{p}\mu + \cdots \frac{1}{p}\mu}_{p}$$

$$= \mu.$$

$$Var(\bar{X}) = Var\left(\frac{X_1 + \cdots X_p}{p}\right)$$

$$= \frac{1}{p^2}Var(X_1) + \frac{1}{p^2}Var(X_2) + \cdots \frac{1}{p^2}Var(X_p)$$

$$= \frac{p}{p^2}\sigma^2$$

$$= \frac{\sigma^2}{p}$$

Standize Normal. (an individual)

$$Z = \frac{X - \mu}{\sigma}$$

Standize Normal ( Sample or a group with n )

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Example:

The mean fill volume for an automatic filling machine for soft drinks is mean 12.1 oz with S.D 0.1 oz

$\mu$ $\sigma$

If the fill volume are IID Normal.

(a) whats the probability the average volume of 10 is less than 12 oz?

$\bar{X}$ of 10

$$p(\bar{X} < 12) = p\left(Z < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right) = p\left(Z < \frac{12 - 12.1}{0.1/\sqrt{10}}\right) = p(Z < -\sqrt{10})$$

(b) whats the probability for one volume is less than 12 o.z?

X of 1

$$p(X < 12) = p\left(Z < \frac{X - \mu}{\sigma}\right) = p\left(Z < \frac{12 - 12.1}{0.1}\right) = p(Z < -1)$$

Example:

Let $X_1$ and $X_2$ be length and width respectively of a manufactured part.

Assume that $X_1$ is $N(2, 0.1^2)$

$X_2$ is $N(5, 0.2^2)$

$X_1$ and $X_2$ are independent.

Find the probability that the perimeter exceeds 14.5.

Denote $Y = 2X_1 + 2X_2$

$P(Y > 14.5)$

$E(Y) = E(2X_1 + 2X_2) = 2 \cdot E(X_1) + 2E(X_2) = 2 \cdot 2 + 2 \cdot 5 = 14$

$Var(Y) = Var(2X_1 + 2X_2) = 4Var(X_1) + 4Var(X_2) = 0.2$

$$\therefore \quad p(Y > 14.5) = p\left(Z > \frac{14.5 - 14}{\sqrt{0.2}}\right)$$

$$p(Z > \frac{0.5}{\ }$$

$$\therefore \quad p(Y > 4.5) = p\left(Z > \cdots \;\; \sqrt{0.2}\right)$$

$$= p\left(Z > \frac{0.5}{\sqrt{0.2}}\right)$$

$$= p(Z > 1.12)$$

$$= p(Z < -1.12) = 0.1314$$