Lecture 7

May 29, 2019 6:51 PM

CLT: $X_1, \dots X_n$, i.i.d. (R.S.)

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$\bar{X}$	S	$\hat{p}$	....	$\hat{\theta}$	Statistic	point estimation
$\Downarrow$	$\Downarrow$	$\Downarrow$		$\Downarrow$	$\Downarrow$	
μ	σ	p		θ	parameter	

Interval Estimation:

$\bar{X} = 120$

I am 95% confident that the gas price would be within 110 and 130.

mid point $\bar{X} = 120$

$(110, 130)$

$(110, 130)$ 95% $\boxed{\text{confidence Interval}}$

A $100(1-\alpha)\%$ confidence interval for the population mean μ is a random interval (L, U) with the property that:

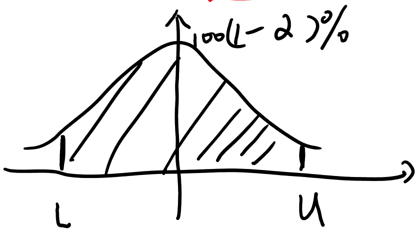
$\alpha = 5\%$ $\alpha = 10\%$

$1-\alpha = 90\%$

$$p(L \le \mu \le U) = 1 - \alpha \simeq 95\%$$

$$\begin{cases} 1-\alpha = \text{confidence coefficient.} \\ \alpha = \text{significance level.} \end{cases}$$

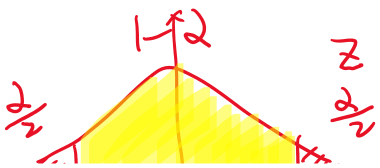
$$\boxed{Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}} \sim N(0,1)$$

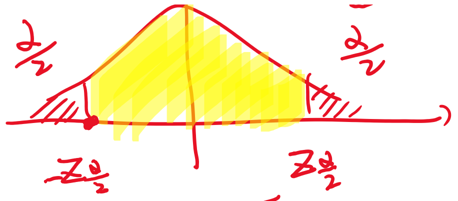


$$Pr(L \le \boxed{\bar{X}} \le U) = 1 - \alpha$$

$$\Leftrightarrow Pr\left(\frac{L-\mu}{\sigma/\sqrt{n}} \le \boxed{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}} \le \frac{U-\mu}{\sigma/\sqrt{n}}\right) = 1 - \alpha$$

$$\Leftrightarrow Pr\left(\frac{L-\mu}{\sigma/\sqrt{n}} \le \boxed{Z} \le \frac{U-\mu}{\sigma/\sqrt{n}}\right) = 1 - \alpha$$





critical value. (for z only).

$$\Longleftrightarrow \Pr\left(-Z_{\frac{\alpha}{2}} \le \boxed{Z} \le Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$\boxed{100(1-\alpha)\%\ \text{CI for } \mu}$: $\left(\mu - Z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{n}},\ \mu + Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$.

$\Uparrow \bar{x}$ $\quad \Uparrow \bar{x}$

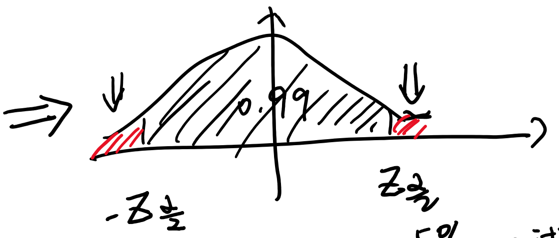
$$\left(\bar{x} - Z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{n}},\ \bar{x} + Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$\Downarrow$ σ known

Example:

A sample of 106 healthy adults gave a mean body temperature of $\bar{x} = 98.2$ F. Suppose that σ is known to be 0.62 F. Estimate the mean body temperature of them using a 99% C.I. .

$\Rightarrow$



$$\begin{cases} \alpha = 0.01 \\ -Z_{\frac{0.01}{2}} = -Z_{0.005} \\ Z_{\frac{0.01}{2}} = Z_{0.005} \end{cases}$$

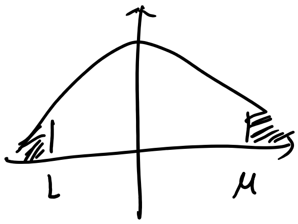
critical value

$Z_{0.005} = ?\ 2.575$

$\boxed{P(Z \le -Z_{0.005}) = 0.005}$

Two sides Z - C.I.

CI	Critical Value	α
90%	$Z_{0.05} = 1.645$	10%
95%	$Z_{0.025} = 1.96$	5%
99%.	$Z_{0.005} = 2.575$	1%



$$\mu \in \left(98.2 - 2.575\cdot\frac{0.6}{\sqrt{106}},\ 98.2 + 2.575\cdot\frac{0.6}{\sqrt{106}}\right) = 99\%\ \text{CI}$$

$$\mu \in \left(98.2 - 2.575 \cdot \frac{0.6}{\sqrt{106}},\ 98.2 + 2.575 \cdot \frac{\ }{\sqrt{106}}\right) = 99\%\ \text{CI}$$

$$= (98.04,\ 98.36)$$

Determine the sample size:

μ is around 170

Marginal Error

$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$. $\quad ME = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

8.4 Large Sample C.I. for proportion.

$0 < p < 1$

Recall x is the number of "success" in independent Bernoulli trials with size n.

p is the probability of success.

Then a $100(1-\alpha)\%$ CI for p is $\hat{p} = \frac{x}{n}$

$$\hat{p} \pm \underbrace{z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}_{ME}$$

Example: A random sample of 80 Canadians. 25 are left-handed. Find a 95% CI for the proportion of Canadians that are left-handed.

$n = 80$, $\quad x = 25$, $\quad \hat{p} = \frac{25}{80}$

95% $\quad z_{\frac{0.05}{2}} = 1.96$

$$\hat{p} \pm z_{\frac{0.05}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= \frac{25}{80} \pm 1.96 \cdot \sqrt{\frac{25/80\,(1-\frac{25}{80})}{80}}$$

$= (0.2109, 0.4107)$

Example: How many people would have to be sampled to estimate this proportion accurate within 3% with a 99% CI.

± 0.03

$Z_{\frac{0.01}{2}} = 2.575$

$ME = 0.03 = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$\hat{p} = \frac{25}{80}$

$\Rightarrow n = \frac{2.575^2 \cdot \hat{p}(1-\hat{p})}{0.03^2}$

$ME = Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Case ① $\boxed{n = \frac{Z_{\frac{\alpha}{2}}^2 \hat{p}(1-\hat{p})}{ME^2}}$

What if $\hat{p}$ is unknown.

$\boxed{\hat{p}(1-\hat{p})} \le \left[\frac{\hat{p} + (1-\hat{p})}{2}\right]^2 = \boxed{\frac{1}{4}}$

$\boxed{ab \le \left(\frac{a+b}{2}\right)^2} \iff 4ab \le a^2 + b^2 + 2ab$

$\iff a^2 + b^2 - 2ab \geq 0$

Case ② $\boxed{n \le \frac{1}{4} \frac{Z_{\frac{\alpha}{2}}^2}{ME^2}}$

Blackboard notes:

Example: How many students should be sampled if we want to estimate the class average μ accurate to within 3 marks with a 98% CI. Assume $\sigma = 16.17$

$\therefore n = \frac{Z_{\frac{\alpha}{2}}^2 \sigma^2}{ME^2} = 157.72 \approx 158$

Example: A researcher collects a sample of size 57 and produces the following C.I. for μ: $(68.5858, 74.2505)$. The population standard deviation is known to be 10.4122. What's the level of confidence?

$$ME = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = \frac{74.2505 - 68.5858}{2}$$

$$\therefore z_{\frac{\alpha}{2}} = 2.05$$

$$\therefore \Phi(2.05) = 1 - \frac{\alpha}{2} \Rightarrow \alpha = 0.0404$$

$$\Rightarrow 100(1-\alpha) = 95.96\%$$

Large Sample CI for μ:

when σ is unknown but n is Large.

$\frac{\bar{X} - \mu}{s/\sqrt{n}}$ is approximately $N(0,1)$

$\therefore$ The CI becomes

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

8.2 C.I. for the mean of a normal distribution.

σ unknown and small sample size.

Result: Let $X_1, \dots X_n$ be a r.s. from a normal distribution with unknown μ and σ^2.

Then

$\frac{\bar{X} - \mu}{S/\sqrt{n}}$ has a t distribution with $n-1$ degree of freedom.

$100(1-\alpha)\%$ CI for μ is:

$$\bar{X} \pm t_{n-1,\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

Example:

10 students with mean score $\bar{x} = 78$ and $S = 12.63$. Find a 99% CI for the class average.

$$78 \pm \underbrace{t_{9,0.05}}_{3.250} \frac{12.63}{\sqrt{10}}$$

Test 2