THE INDEFINITE INTEGRAL

ANTIDERIVATIVES:

4.2.1 DEFINITION A function F is called an ***antiderivative*** of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

Example 3

$$\int (3x^6 - 2x^2 + 7x + 1)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

Sometimes it is useful to rewrite an integrand in a different form before performing the integration. This is illustrated in the following example.

Example 4 Evaluate

$$\text{(a)} \int \frac{\cos x}{\sin^2 x}\,dx \qquad \text{(b)} \int \frac{t^2 - 2t^4}{t^4}\,dt$$

Solution (a).

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx = -\csc x + C$$

Formula 8 in Table 4.2.1

Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt = \int (t^{-2} - 2)\,dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

PRINCIPLES OF INTEGRAL EVALUATION
INTEGRATION by U-SUBSTITUTION

Example 5

$$\int \left(\frac{1}{x} + \sec^2 \pi x\right) dx = \int \frac{dx}{x} + \int \sec^2 \pi x \, dx$$

$$= \ln|x| + \int \sec^2 \pi x \, dx$$

$$= \ln|x| + \frac{1}{\pi} \int \sec^2 u \, du$$

$$\boxed{\begin{array}{c} u = \pi x \\ du = \pi \, dx \text{ or } dx = \frac{1}{\pi} du \end{array}}$$

$$= \ln|x| + \frac{1}{\pi} \tan u + C = \ln|x| + \frac{1}{\pi} \tan \pi x + C$$

The next four examples illustrate a substitution $u = g(x)$ where $g(x)$ is a nonlinear function.

Example 6 Evaluate $\int \sin^2 x \cos x \, dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x \, dx$$

Thus,

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

Example 7 Evaluate $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$.

Solution. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}} dx \quad \text{or} \quad 2\,du = \frac{1}{\sqrt{x}} dx$$

Thus,

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int 2\cos u\,du = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C \blacktriangleleft$$

▶ Example 8 Evaluate $\int t^4 \sqrt[3]{3-5t^5}\,dt$.

Solution.

$$\int t^4 \sqrt[3]{3-5t^5}\,dt = -\frac{1}{25}\int \sqrt[3]{u}\,du = -\frac{1}{25}\int u^{1/3}\,du$$

$$u = 3 - 5t^5$$
$$du = -25t^4\,dt \text{ or } -\tfrac{1}{25}\,du = t^4\,dt$$

$$= -\frac{1}{25}\frac{u^{4/3}}{4/3} + C = -\frac{3}{100}(3-5t^5)^{4/3} + C \blacktriangleleft$$

EXERCISE SET 7.1

Evaluate the integrals by making appropriate u-substitutions and applying the formulas reviewed in this section.

1. $\int (4-2x)^3\,dx$

2. $\int 3\sqrt{4+2x}\,dx$

3. $\int x\sec^2(x^2)\,dx$

4. $\int 4x\tan(x^2)\,dx$

5. $\int \frac{\sin 3x}{2+\cos 3x}\,dx$

6. $\int \frac{1}{9+4x^2}\,dx$

7. $\int e^x \sinh(e^x)\,dx$

8. $\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$

9. $\int e^{\tan x}\sec^2 x\,dx$

10. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

11. $\displaystyle\int \cos^5 5x \sin 5x \, dx$

12. $\displaystyle\int \frac{\cos x}{\sin x \sqrt{\sin^2 x + 1}} \, dx$

13. $\displaystyle\int \frac{e^x}{\sqrt{4 + e^{2x}}} \, dx$

14. $\displaystyle\int \frac{e^{\tan^{-1} x}}{1 + x^2} \, dx$

15. $\displaystyle\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}} \, dx$

16. $\displaystyle\int (x+1)\cot(x^2 + 2x) \, dx$

17. $\displaystyle\int \frac{\cosh \sqrt{x}}{\sqrt{x}} \, dx$

18. $\displaystyle\int \frac{dx}{x(\ln x)^2}$

19. $\displaystyle\int \frac{dx}{\sqrt{x}\, 3^{\sqrt{x}}}$

20. $\displaystyle\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta \, d\theta$

21. $\displaystyle\int \frac{\operatorname{csch}^2(2/x)}{x^2} \, dx$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4}}$

23. $\displaystyle\int \frac{e^{-x}}{4 - e^{-2x}} \, dx$

24. $\displaystyle\int \frac{\cos(\ln x)}{x} \, dx$

25. $\displaystyle\int \frac{e^x}{\sqrt{1 - e^{2x}}} \, dx$

26. $\displaystyle\int \frac{\sinh(x^{-1/2})}{x^{3/2}} \, dx$

27. $\displaystyle\int \frac{x}{\csc(x^2)} \, dx$

28. $\displaystyle\int \frac{e^x}{\sqrt{4 - e^{2x}}} \, dx$

29. $\displaystyle\int x 4^{-x^2} \, dx$

30. $\displaystyle\int 2^{\pi x} \, dx$

SOLUTION SET

1. $u = 4 - 2x$, $du = -2dx$, $\quad -\frac{1}{2}\int u^3\,du = -\frac{1}{8}u^4 + C = -\frac{1}{8}(4-2x)^4 + C.$

3. $u = x^2$, $du = 2xdx$, $\quad \frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan(x^2) + C.$

5. $u = 2 + \cos 3x$, $du = -3\sin 3x dx$, $\quad -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln(2+\cos 3x) + C.$

7. $u = e^x$, $du = e^x dx$, $\quad \int \sinh u\,du = \cosh u + C = \cosh e^x + C.$

9. $u = \tan x$, $du = \sec^2 x dx$, $\quad \int e^u\,du = e^u + C = e^{\tan x} + C.$

11. $u = \cos 5x$, $du = -5\sin 5x dx$, $\quad -\frac{1}{5}\int u^5\,du = -\frac{1}{30}u^6 + C = -\frac{1}{30}\cos^6 5x + C.$

13. $u = e^x$, $du = e^x dx$, $\quad \int \frac{du}{\sqrt{4+u^2}} = \ln\left(u + \sqrt{u^2+4}\right) + C = \ln\left(e^x + \sqrt{e^{2x}+4}\right) + C.$

15. $u = \sqrt{x-1}$, $du = \frac{1}{2\sqrt{x-1}}\,dx$, $\quad 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x-1}} + C.$

17. $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $\quad \int 2\cosh u\,du = 2\sinh u + C = 2\sinh\sqrt{x} + C.$

19. $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $\quad \int \frac{2\,du}{3^u} = 2\int e^{-u\ln 3}\,du = -\frac{2}{\ln 3}e^{-u\ln 3} + C = -\frac{2}{\ln 3}3^{-\sqrt{x}} + C.$

21. $u = \frac{2}{x}$, $du = -\frac{2}{x^2}\,dx$, $\quad -\frac{1}{2}\int \operatorname{csch}^2 u\,du = \frac{1}{2}\coth u + C = \frac{1}{2}\coth\frac{2}{x} + C.$

23. $u = e^{-x}$, $du = -e^{-x}dx$, $\quad -\int \frac{du}{4-u^2} = -\frac{1}{4}\ln\left|\frac{2+u}{2-u}\right| + C = -\frac{1}{4}\ln\left|\frac{2+e^{-x}}{2-e^{-x}}\right| + C.$

25. $u = e^x$, $du = e^x dx$, $\quad \int \frac{e^x\,dx}{\sqrt{1-e^{2x}}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} e^x + C.$

27. $u = x^2$, $du = 2xdx$, $\quad \frac{1}{2}\int \frac{du}{\csc u} = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2) + C.$

29. $4^{-x^2} = e^{-x^2\ln 4}$, $u = -x^2\ln 4$, $du = -2x\ln 4\,dx = -x\ln 16\,dx$, $-\frac{1}{\ln 16}\int e^u\,du = -\frac{1}{\ln 16}e^u + C = -\frac{1}{\ln 16}e^{-x^2\ln 4} + C = -\frac{1}{\ln 16}4^{-x^2} + C.$