Vectors

Mechanics

Lecture Content

- Vector definition and types
- Vector representation
- Vector algebra

Vectors

What are vectors? How many are there? What do they want???

Vectors

- Vectors are mathematical entities that help define a system, its actions and/or its evolution.
- Vectors are essentially just the combination of elements.
- These elements can be numbers (scalars; not vectors) or vectors.
- Vectors are used is all scientific fields such as physics, chemistry, biology, and of course mathematics. Sometimes it can also be used in literature.
- Numbers help define amount or quantities — Scalars.
- Vectors help define amount or quantities of multiple objects — Vectors.

Types of vectors in physics

Vectors in physics are identified by their magnitude (length) and directions.

- **Null vectors**: vectors with zero magnitude
- **Position vectors**: vectors that starts from origin
- **Free vectors**: vectors that start from any point in space
- **Unit vectors**: vectors of magnitude 1
- **Equal vectors**: same in magnitude and direction
- **Opposite vectors**: same in magnitude but opposite in direction

What do they want?

They just want to help you define:

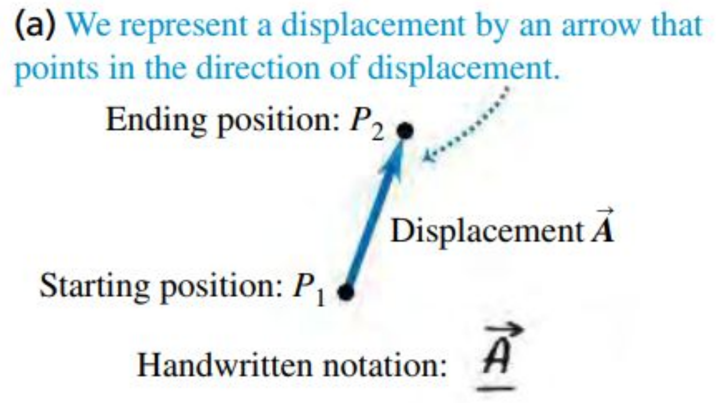
- Physical systems
- Motions of isolated particles
- Interactions of isolated particles

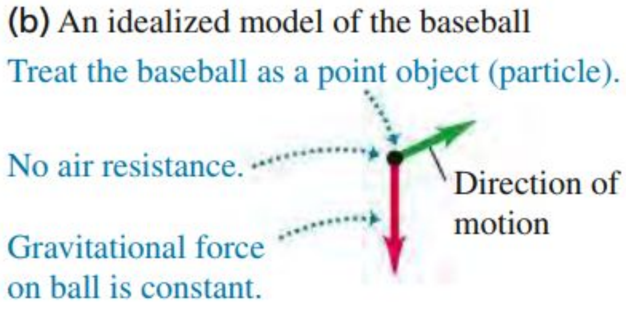
Vector Representation

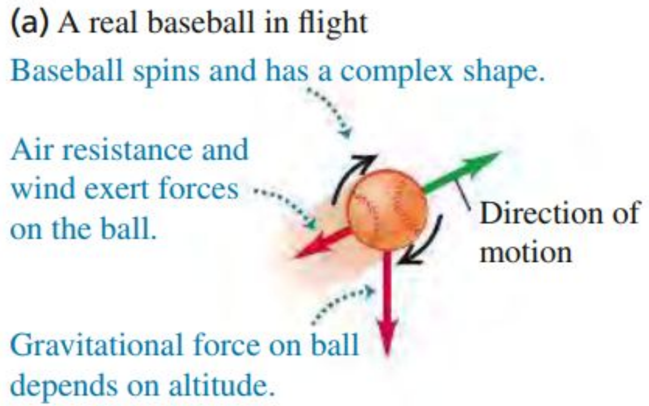
How do they look depends on how you want them to look…

Representations

- Representation are essential in physics
- Vectors can be represented by arrows that define the motion or positions.
- The distinct features of these arrows are their tail, head, length and orientations

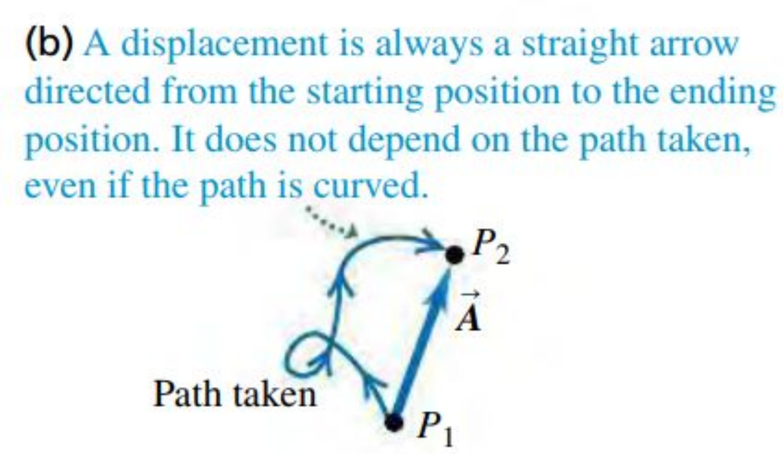


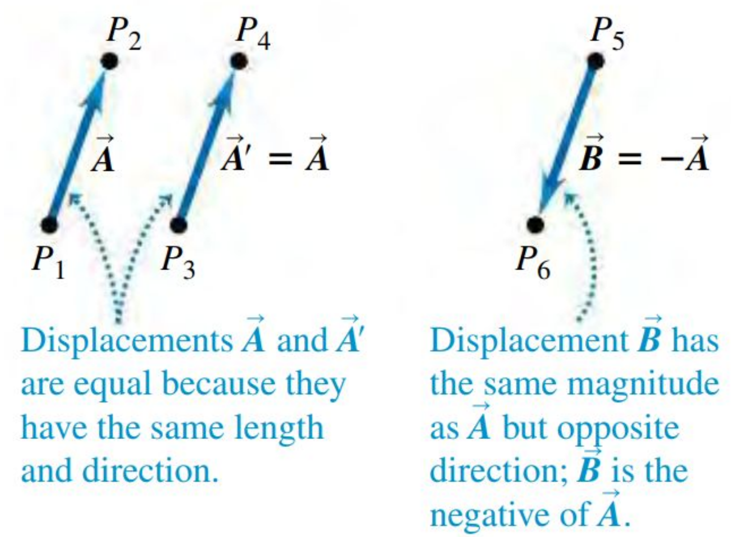




Representations

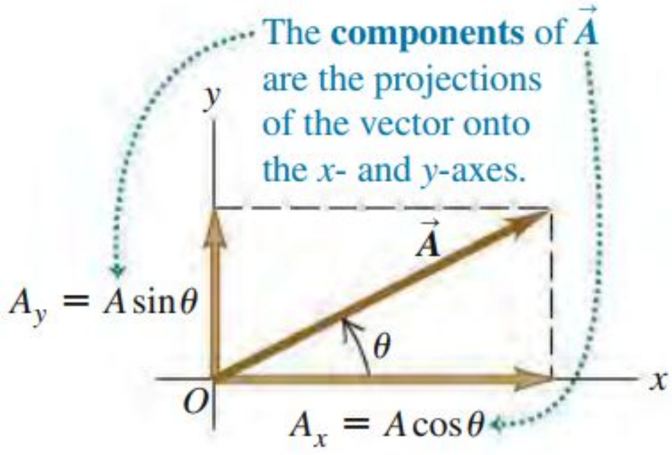
For example, a physical quantity of displacement is explained by using vectors as follows:

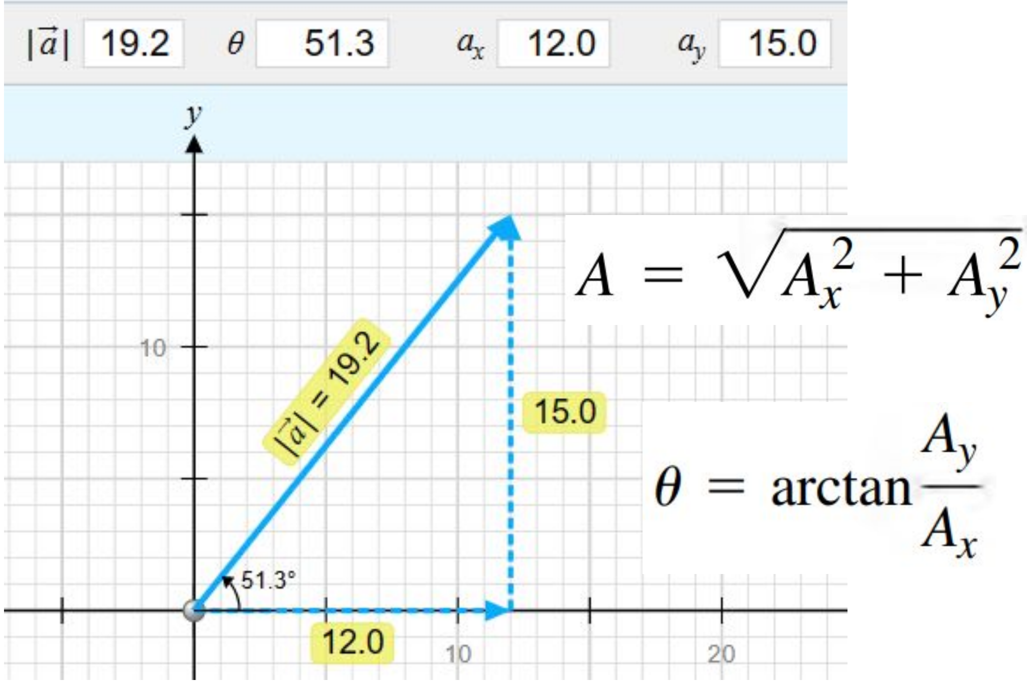




Representations

As a geometric object, we can plot the vector as follows





Representations

Vector Notations is a representations in algebraic form.

Component of each direction is written with their identifier unit vector

Since different directions cannot be summed together, they remain in a combination of addition.

$$\vec{A} = 12\,unit\ \vec{i} + 19.2\,unit\ \vec{j}$$

Secondary/Standard representation is also in the following form

$$\vec{A} = (12\,,\ 19.2)$$

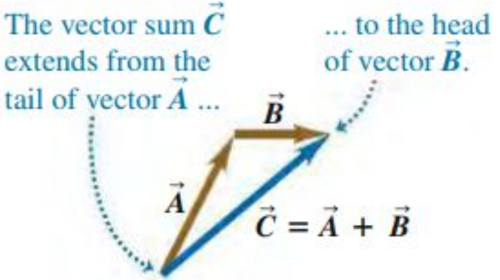
Vector Algebra

Now let’s get all these vectors together

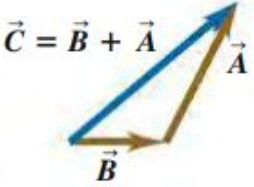
Addition

$$\vec{C} = \vec{B} + \vec{A} \quad \text{and} \quad \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

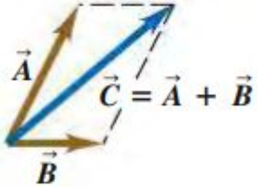
(a) We can add two vectors by placing them head to tail.



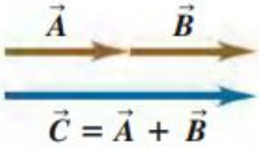
(b) Adding them in reverse order gives the same result: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. The order doesn't matter in vector addition.



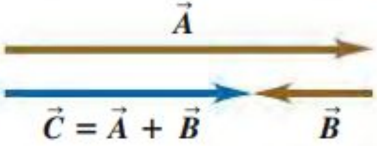
(c) We can also add two vectors by placing them tail to tail and constructing a parallelogram.



(a) Only when vectors $\vec{A}$ and $\vec{B}$ are parallel does the magnitude of their vector sum $\vec{C}$ equal the sum of their magnitudes: $C = A + B$.



(b) When $\vec{A}$ and $\vec{B}$ are antiparallel, the magnitude of their vector sum $\vec{C}$ equals the *difference* of their magnitudes: $C = |A - B|$.



Addition

$$\vec{R} = \vec{A} + (\vec{B} + \vec{C}) = \vec{A} + \vec{E}$$

$$\vec{R} = (\vec{A} + \vec{B}) + \vec{C} = \vec{D} + \vec{C}$$

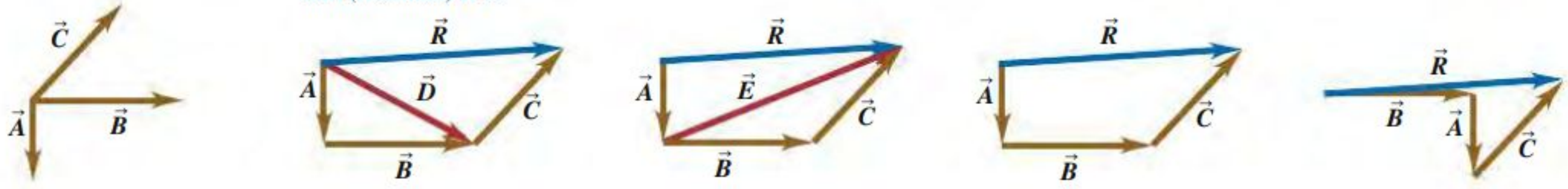
(a) To find the sum of these three vectors ...

(b) ... add $\vec{A}$ and $\vec{B}$ to get $\vec{D}$ and then add $\vec{C}$ to $\vec{D}$ to get the final sum (resultant) $\vec{R}$ ...

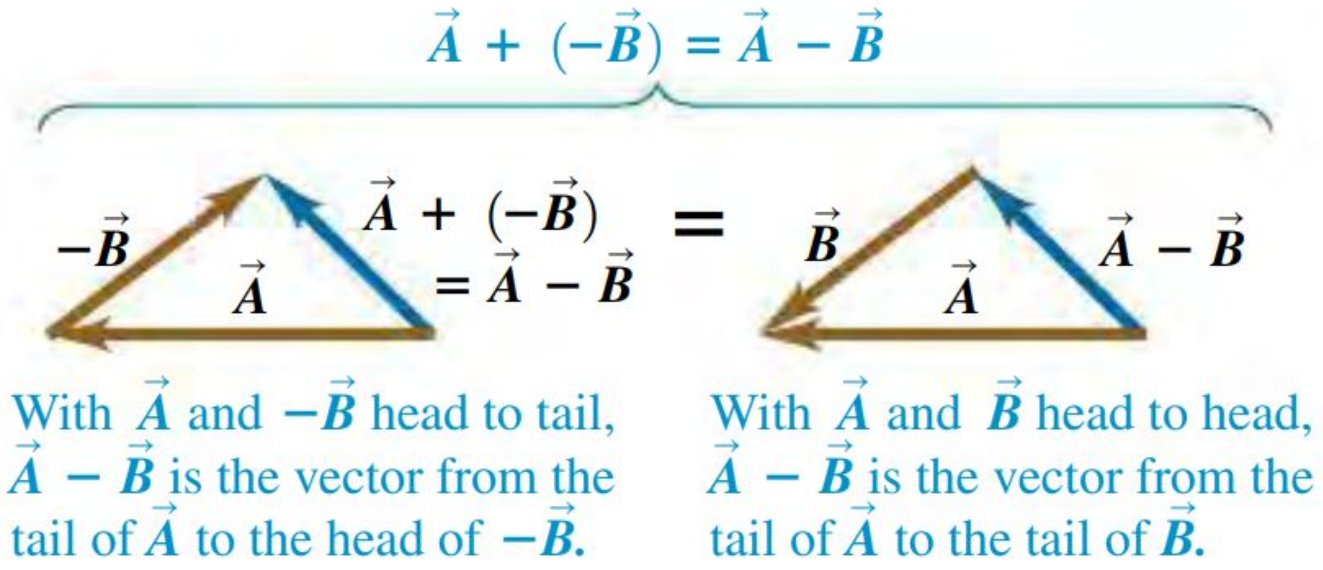
(c) ... or add $\vec{B}$ and $\vec{C}$ to get $\vec{E}$ and then add $\vec{A}$ to $\vec{E}$ to get $\vec{R}$ ...

(d) ... or add $\vec{A}$, $\vec{B}$, and $\vec{C}$ to get $\vec{R}$ directly ...

(e) ... or add $\vec{A}$, $\vec{B}$, and $\vec{C}$ in any other order and still get $\vec{R}$.

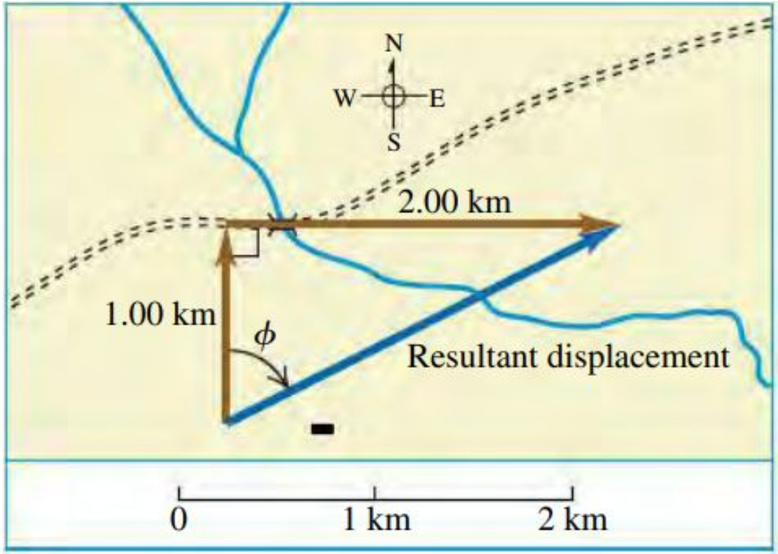


Addition


$\vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$
$-\vec{B}$
$\vec{A}$
$\vec{A} + (-\vec{B})$
$= \vec{A} - \vec{B}$
$=$
$\vec{B}$
$\vec{A}$
$\vec{A} - \vec{B}$
With $\vec{A}$ and $-\vec{B}$ head to tail, $\vec{A} - \vec{B}$ is the vector from the tail of $\vec{A}$ to the head of $-\vec{B}$.
With $\vec{A}$ and $\vec{B}$ head to head, $\vec{A} - \vec{B}$ is the vector from the tail of $\vec{A}$ to the tail of $\vec{B}$.

Numerical Example

A cross-country skier skis 1.00 km north and then 2.00 km east on a horizontal snowfield. How far and in what direction is she from the starting point?



$$\sqrt{(1.00\text{ km})^2 + (2.00\text{ km})^2} = 2.24\text{ km}$$

$$\tan\phi = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{2.00\text{ km}}{1.00\text{ km}} = 2.00$$

$$\phi = \arctan 2.00 = 63.4^\circ$$

We can describe the direction as 63.4° east of north or $90^\circ - 63.4^\circ = 26.6^\circ$ north of east.

Multiplication

Scalar Product

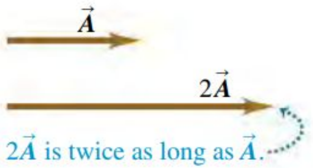
- The product of a scalar s and a vector $\vec{v}$ is a new vector whose magnitude is sv and whose direction is the same as that of $\vec{v}$ if s is positive, and opposite that of $\vec{v}$ if s is negative. To divide $\vec{v}$ by s, multiply $\vec{v}$ by $1/s$.

scaling up → $100\vec{V} = \vec{\omega}$, direction remains the same

$|\vec{\omega}| = 100|\vec{V}|$

scaling down → $0.5\vec{V} = \vec{u}$

(a) Multiplying a vector by a positive scalar changes the magnitude (length) of the vector but not its direction.



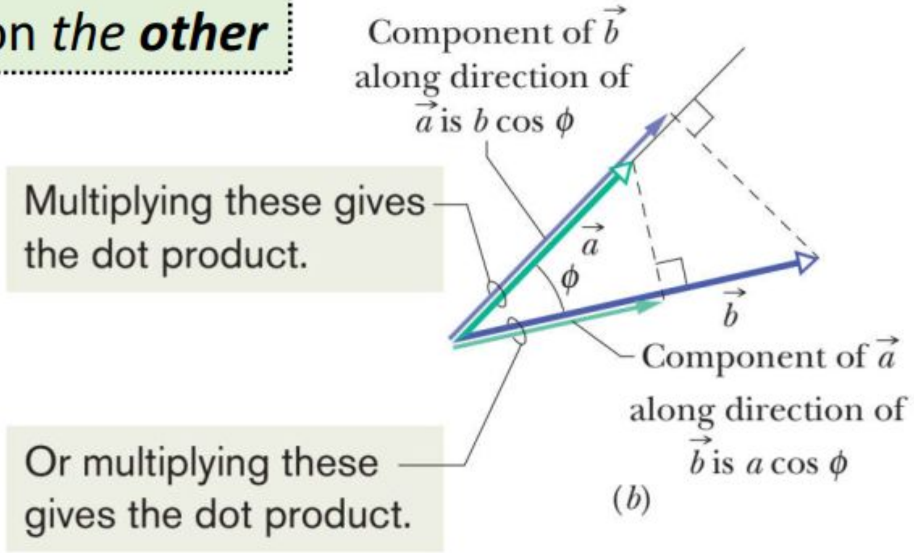
(b) Multiplying a vector by a negative scalar changes its magnitude and reverses its direction.

Multiplication

Dot Product

The Projection of one ***vector*** on *the* ***other***

How much does two vector point in the same direction



Multiplication

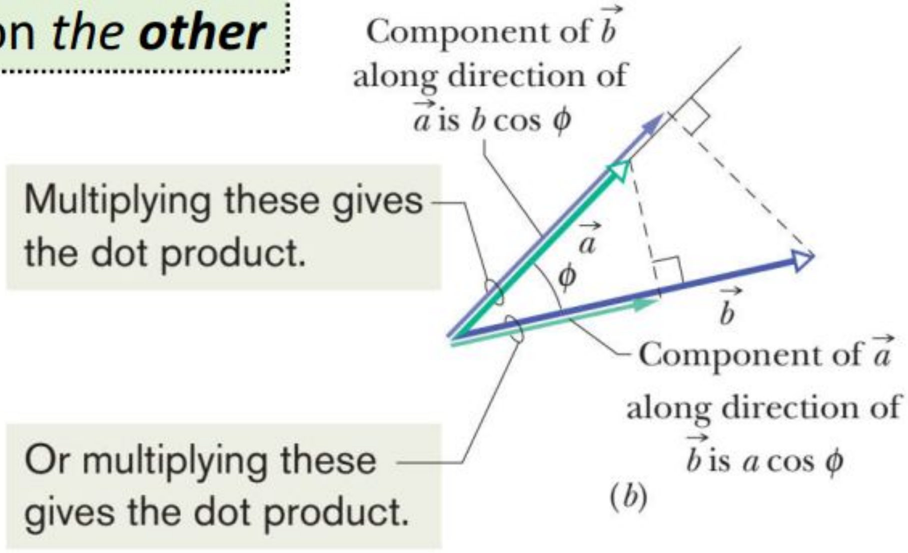
Dot Product

The Projection of one ***vector*** on *the* ***other***

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \emptyset$$

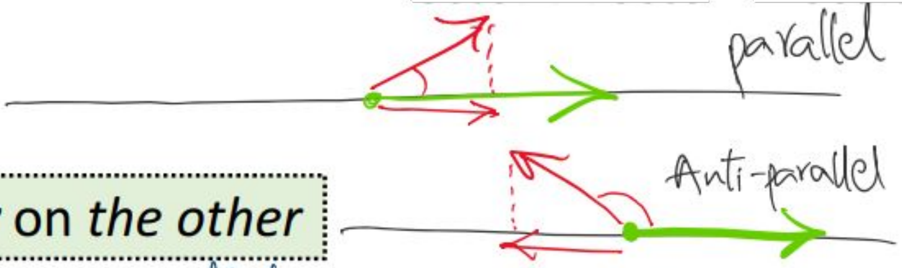
$$\vec{b} \cdot \vec{a} = |\vec{b}||\vec{a}| \cos \emptyset$$

$$\boxed{\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}}$$



Multiplication

Dot Product



The Projection of one ***vector*** on *the other*

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \emptyset$$

parallel $\longrightarrow$ Dot product is +ve

Anti parallel $\longrightarrow$ Dot product is -ve

If the angle ϕ between two vectors is $0°$, the component of one vector along the other is maximum, and so also is the dot product of the vectors. If, instead, ϕ is $90°$, the component of one vector along the other is zero, and so is the dot product.

Multiplication

$$\sum_{m=0}^{4}(a_m b_m) = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

Dot Product

$$(a_x \hat{i} + a_y \hat{j}) \cdot (b_x \hat{i} + b_y \hat{j})$$

Or Sum of (**Element wise multiplication**)

summation upper limit

repeated index

$$\vec{A} \cdot \vec{B} = \sum_{u=1}^{2} (a_u b_u)$$

Summation

sum starts from

(x, y)

$(1, 2)$

just multiply and rember

$$\boxed{\begin{aligned} \hat{i} \cdot \hat{i} &= 1 \\ \hat{j} \cdot \hat{j} &= 1 \end{aligned}}$$

$$= a_1 b_1 + a_2 b_2 \Rightarrow a_x b_x + a_y b_y$$

Multiplication

Dot Product

Or Sum of (Element wise multiplication)

$$\vec{A} \cdot \vec{B} = \sum_{u=1}^{2} (a_u b_u)$$

$$\vec{A} \cdot \vec{B} = a_x b_x + a_y b_y$$

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = (1)(1) \cos 0^\circ = 1$$

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = (1)(1) \cos 90^\circ = 0$$

Multiplication

Cross Product

Rotational Information

Important points

The system must be in three dimensions or more.

The resultant vector is always perpendicular to the two vectors multiplied.

Right-hand rule

rotate your hand towards crossed vector

Thumb gives the direction of resultant vector

$\vec{a} \times \vec{b}$

$\vec{a}$ and $\vec{b}$ are necessarily on a plane.

and $\vec{c}$ is perpendicular to this plane.

Multiplication

Cross Product

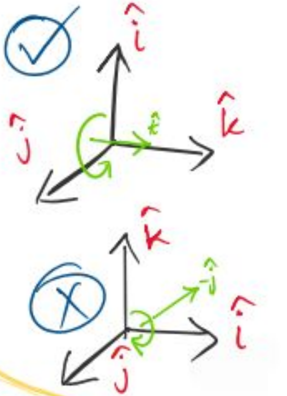
Rotational Information

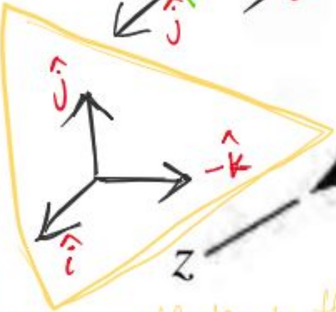
using the right-hand rule

$$\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$$

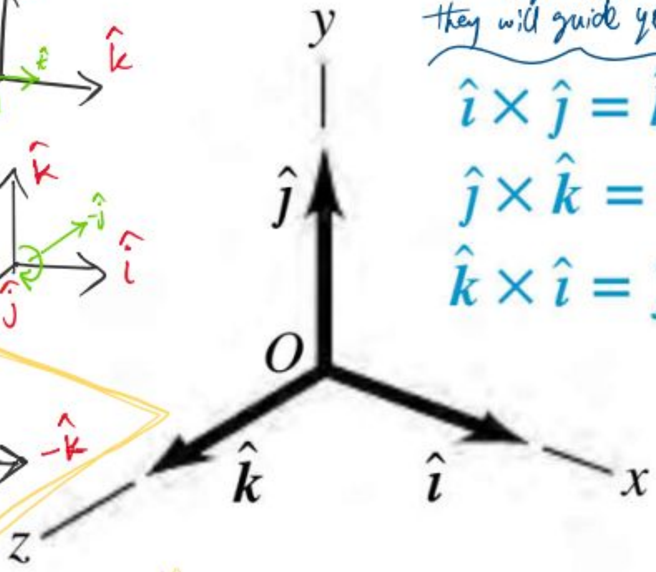
$$\hat{\jmath} \times \hat{k} = -\hat{k} \times \hat{\jmath} = \hat{\imath}$$

$$\hat{k} \times \hat{\imath} = -\hat{\imath} \times \hat{k} = \hat{\jmath}$$





What do you think, is this correct ?



Hold tight to these
they will guide you

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$

$$\hat{\jmath} \times \hat{k} = \hat{\imath}$$

$$\hat{k} \times \hat{\imath} = \hat{\jmath}$$

Multiplication

Cross Product

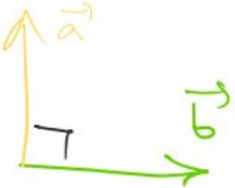
Rotational Information

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \emptyset$$

parallel
(No room for rotation)

anti-parallel
(will be understood by next explanation)

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a} \times \vec{b} = 0$. The magnitude of $\vec{a} \times \vec{b}$, which can be written as $|\vec{a} \times \vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

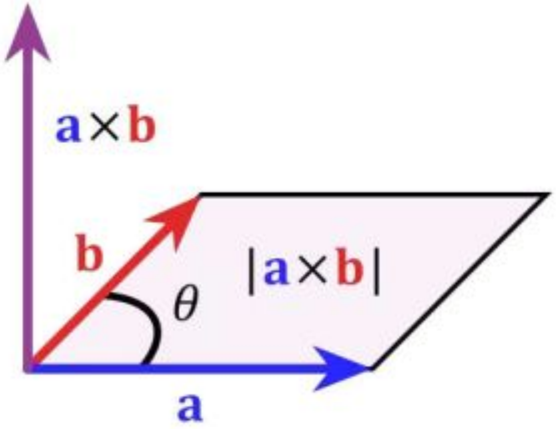


Multiplication

Cross Product

Determinant (because determinants show how area is stretched and rotated)

$$\vec{a} \times \vec{b} = det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}\right)$$



Multiplication

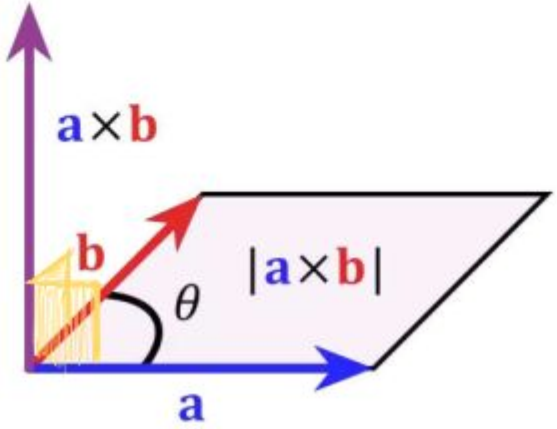
Cross Product

Determinant (because determinants show how area is stretched and rotated)

→ Length of $\vec{a} \times \vec{b}$ is the same as *area of parallelogram*.

→ $\vec{a} \times \vec{b}$ is perpendicular to the $\vec{a}$ and $\vec{b}$

for parallel and antiparallel vectors, the area of parallelogram will remain zero.



Home work:

Write the summary of this lecture. Noting down important formulas and drawing appropriate diagrams and sketches to help explain the ideas.

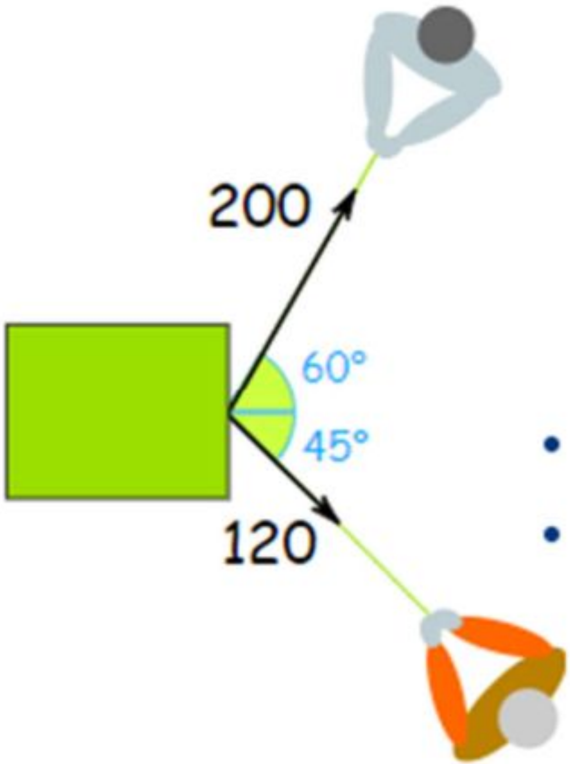
End of lecture.

Now wonder...

1. Is the vector **i** + **j** + **k** a unit vector? Is the vector (3.0 **i** − 2.0 **j**) a unit vector? Justify your answers.

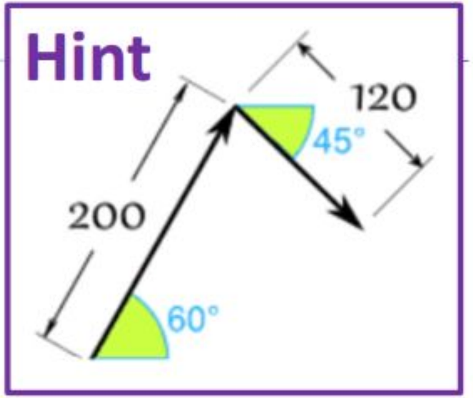
2. Can you find two vectors with different lengths that have a vector sum of zero? What length restrictions are required for three vectors to have a vector sum of zero? Explain

Study the following practice numerical problems on next slides...



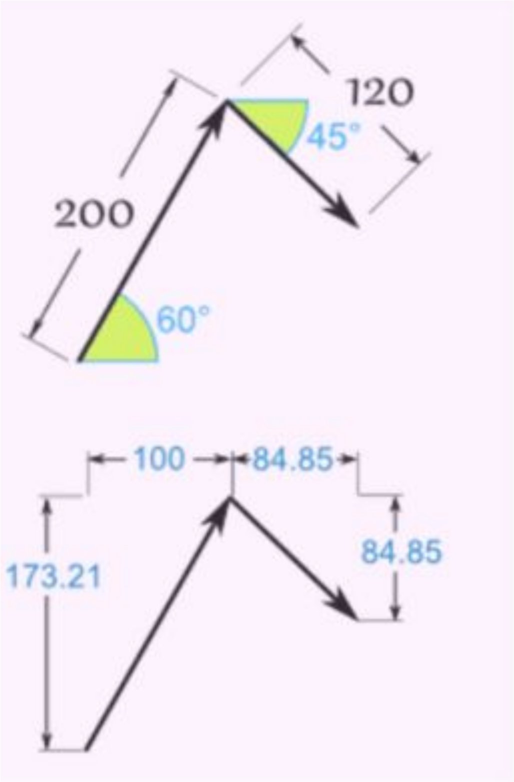
An Example

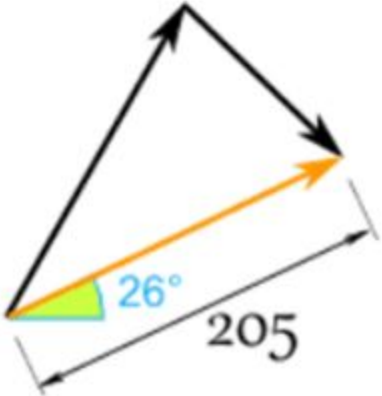
Sam and Alex are pulling a box.

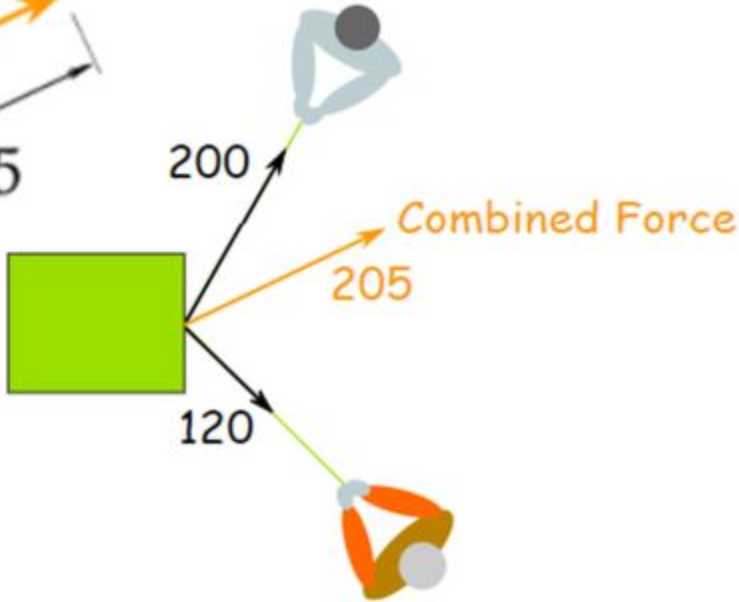


- Sam pulls with 200 Newtons of force at 60°
- Alex pulls with 120 Newtons of force at 45° as shown

What is the combined force, and its direction?


120
45°
200
60°
100
84.85
173.21
84.85


26°
205

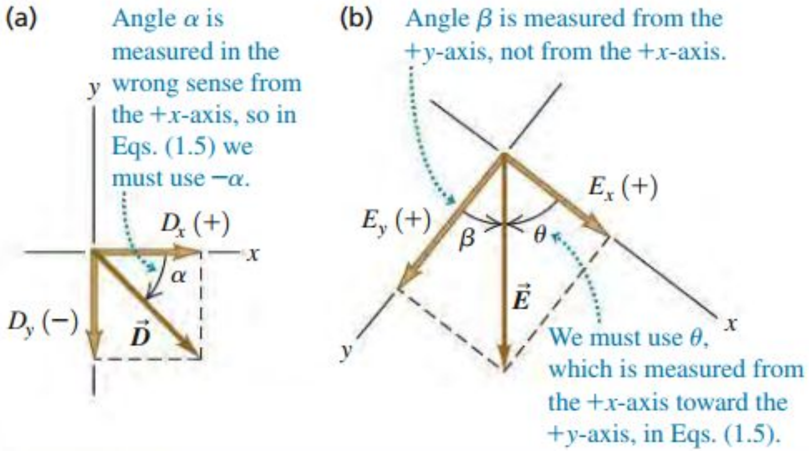

200
Combined Force
205
120

(a) What are the x- and y-components of vector $\vec{D}$ in **Fig. 1.19a**? The magnitude of the vector is $D = 3.00$ m, and angle $\alpha = 45°$. (b) What are the x- and y-components of vector $\vec{E}$ in Fig. 1.19b? The magnitude of the vector is $E = 4.50$ m, and angle $\beta = 37.0°$.

SOLUTION

IDENTIFY and SET UP: We can use Eqs. (1.5) to find the components of these vectors, but we must be careful: Neither angle α nor β in Fig. 1.19 is measured from the $+x$-axis toward the $+y$-axis. We estimate from the figure that the lengths of both components in part (a) are roughly 2 m, and that those in part (b) are 3 m and 4 m. The figure indicates the signs of the components.

1.19 Calculating the x- and y-components of vectors.



EXECUTE: (a) The angle α (the Greek letter alpha) between the positive x-axis and $\vec{D}$ is measured toward the *negative* y-axis. The angle we must use in Eqs. (1.5) is $\theta = -\alpha = -45°$. We then find

$$D_x = D\cos\theta = (3.00\text{ m})(\cos(-45°)) = +2.1\text{ m}$$

$$D_y = D\sin\theta = (3.00\text{ m})(\sin(-45°)) = -2.1\text{ m}$$

Had we carelessly substituted $+45°$ for θ in Eqs. (1.5), our result for D_y would have had the wrong sign.

(b) The x- and y-axes in Fig. 1.19b are at right angles, so it doesn't matter that they aren't horizontal and vertical, respectively. But we can't use the angle β (the Greek letter beta) in Eqs. (1.5), because β is measured from the $+y$-axis. Instead, we must use the angle $\theta = 90.0° - \beta = 90.0° - 37.0° = 53.0°$. Then we find

$$E_x = E\cos 53.0° = (4.50\text{ m})(\cos 53.0°) = +2.71\text{ m}$$

$$E_y = E\sin 53.0° = (4.50\text{ m})(\sin 53.0°) = +3.59\text{ m}$$

EVALUATE: Our answers to both parts are close to our predictions. But why do the answers in part (a) correctly have only two significant figures?

Given the two displacements

$$\vec{D} = (6.00\,\hat{\imath} + 3.00\,\hat{\jmath} - 1.00\hat{k})\ \text{m} \quad \text{and}$$

$$\vec{E} = (4.00\,\hat{\imath} - 5.00\,\hat{\jmath} + 8.00\hat{k})\ \text{m}$$

find the magnitude of the displacement $2\vec{D} - \vec{E}$.

SOLUTION

IDENTIFY and SET UP: We are to multiply vector $\vec{D}$ by 2 (a scalar) and subtract vector $\vec{E}$ from the result, so as to obtain the vector $\vec{F} = 2\vec{D} - \vec{E}$. Equation (1.8) says that to multiply $\vec{D}$ by 2, we multiply each of its components by 2. We can use Eq. (1.15) to do the subtraction; recall from Section 1.7 that subtracting a vector is the same as adding the negative of that vector.

EXECUTE: We have

$$\begin{aligned}\vec{F} &= 2(6.00\hat{\imath} + 3.00\,\hat{\jmath} - 1.00\hat{k})\ \text{m} - (4.00\hat{\imath} - 5.00\,\hat{\jmath} + 8.00\hat{k})\ \text{m}\\ &= [(12.00 - 4.00)\hat{\imath} + (6.00 + 5.00)\,\hat{\jmath} + (-2.00 - 8.00)\hat{k}]\ \text{m}\\ &= (8.00\hat{\imath} + 11.00\,\hat{\jmath} - 10.00\hat{k})\ \text{m}\end{aligned}$$

From Eq. (1.11) the magnitude of $\vec{F}$ is

$$\begin{aligned}F &= \sqrt{F_x^2 + F_y^2 + F_z^2}\\ &= \sqrt{(8.00\ \text{m})^2 + (11.00\ \text{m})^2 + (-10.00\ \text{m})^2}\\ &= 16.9\ \text{m}\end{aligned}$$

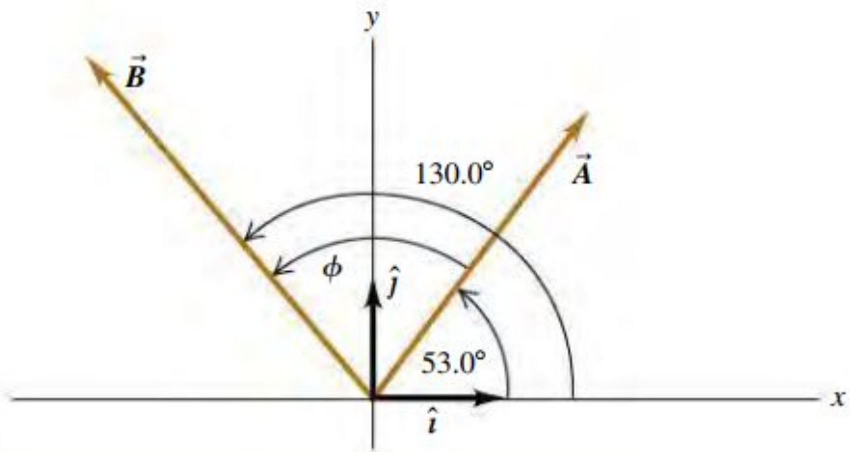
EVALUATE: Our answer is of the same order of magnitude as the larger components that appear in the sum. We wouldn't expect our answer to be much larger than this, but it could be much smaller.

Find the scalar product $\vec{A} \cdot \vec{B}$ of the two vectors in **Fig. 1.28.** The magnitudes of the vectors are $A = 4.00$ and $B = 5.00$.

SOLUTION

IDENTIFY and SET UP: We can calculate the scalar product in two ways: using the magnitudes of the vectors and the angle between them (Eq. 1.16), and using the components of the vectors (Eq. 1.19). We'll do it both ways, and the results will check each other.

1.28 Two vectors $\vec{A}$ and $\vec{B}$ in two dimensions.



EXECUTE: The angle between the two vectors $\vec{A}$ and $\vec{B}$ is $\phi = 130.0° - 53.0° = 77.0°$, so Eq. (1.16) gives us

$$\vec{A} \cdot \vec{B} = AB \cos \phi = (4.00)(5.00) \cos 77.0° = 4.50$$

To use Eq. (1.19), we must first find the components of the vectors. The angles of $\vec{A}$ and $\vec{B}$ are given with respect to the $+x$-axis and are measured in the sense from the $+x$-axis to the $+y$-axis, so we can use Eqs. (1.5):

$$A_x = (4.00) \cos 53.0° = 2.407$$
$$A_y = (4.00) \sin 53.0° = 3.195$$
$$B_x = (5.00) \cos 130.0° = -3.214$$
$$B_y = (5.00) \sin 130.0° = 3.830$$

As in Example 1.7, we keep an extra significant figure in the components and round at the end. Equation (1.19) now gives us

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$
$$= (2.407)(-3.214) + (3.195)(3.830) + (0)(0) = 4.50$$

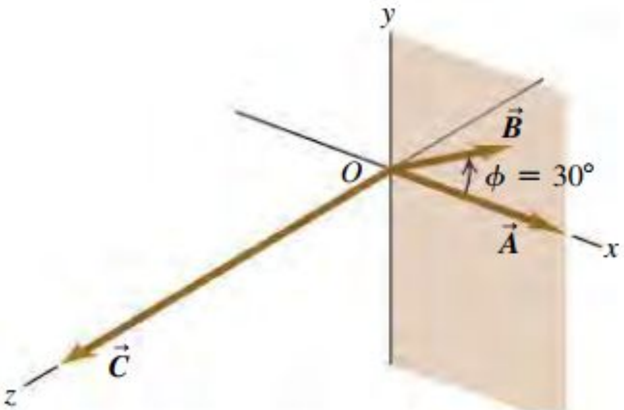
EVALUATE: Both methods give the same result, as they should.

Vector $\vec{A}$ has magnitude 6 units and is in the direction of the $+x$-axis. Vector $\vec{B}$ has magnitude 4 units and lies in the xy-plane, making an angle of 30° with the $+x$-axis (**Fig. 1.33**). Find the vector product $\vec{C} = \vec{A} \times \vec{B}$.

SOLUTION

IDENTIFY and SET UP: We'll find the vector product in two ways, which will provide a check of our calculations. First we'll use Eq. (1.20) and the right-hand rule; then we'll use Eqs. (1.25) to find the vector product by using components.

1.33 Vectors $\vec{A}$ and $\vec{B}$ and their vector product $\vec{C} = \vec{A} \times \vec{B}$. Vector $\vec{B}$ lies in the xy-plane.



EXECUTE: From Eq. (1.20) the magnitude of the vector product is

$$AB \sin \phi = (6)(4)(\sin 30°) = 12$$

By the right-hand rule, the direction of $\vec{A} \times \vec{B}$ is along the $+z$-axis (the direction of the unit vector $\hat{k}$), so $\vec{C} = \vec{A} \times \vec{B} = 12\hat{k}$.

To use Eqs. (1.25), we first determine the components of $\vec{A}$ and $\vec{B}$. Note that $\vec{A}$ points along the x-axis, so its only nonzero component is A_x. For $\vec{B}$, Fig. 1.33 shows that $\phi = 30°$ is measured from the $+x$-axis toward the $+y$-axis, so we can use Eqs. (1.5):

$$A_x = 6 \qquad A_y = 0 \qquad A_z = 0$$

$$B_x = 4\cos 30° = 2\sqrt{3} \qquad B_y = 4 \sin 30° = 2 \qquad B_z = 0$$

Then Eqs. (1.25) yield

$$C_x = (0)(0) - (0)(2) = 0$$

$$C_y = (0)(2\sqrt{3}) - (6)(0) = 0$$

$$C_z = (6)(2) - (0)(2\sqrt{3}) = 12$$

Thus again we have $\vec{C} = 12\hat{k}$.

EVALUATE: Both methods give the same result. Depending on the situation, one or the other of the two approaches may be the more convenient one to use.