CONVERGENCE TESTS

The Ratio Test

9.5.5 THEOREM (*The Ratio Test*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}$$

(a) *If $\rho < 1$, the series converges.*

(b) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(c) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

▶ **Example 3** Each of the following series has positive terms, so the ratio test applies. In each part, use the ratio test to determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$ (d) $\displaystyle\sum_{k=3}^{\infty} \frac{(2k)!}{4^k}$ (e) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-1}$

Solution (a). The series converges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k\to+\infty} \frac{k!}{(k+1)!} = \lim_{k\to+\infty} \frac{1}{k+1} = 0 < 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \frac{1}{2} \lim_{k\to+\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

Solution (c). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k\to+\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to+\infty} \left(1 + \frac{1}{k}\right)^k = e > 1$$

See Formula (4) of Section 6.1

Solution (d). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \lim_{k\to+\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4} \right)$$

$$= \lim_{k\to+\infty} \left(\frac{(2k+2)(2k+1)(2k)!}{(2k)!} \cdot \frac{1}{4} \right) = \frac{1}{4} \lim_{k\to+\infty} (2k+2)(2k+1) = +\infty$$

Solution (e). The ratio test is of no help since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} = \lim_{k\to+\infty} \frac{2k-1}{2k+1} = 1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty} \frac{dx}{2x-1} = \lim_{b\to+\infty} \int_1^b \frac{dx}{2x-1} = \lim_{b\to+\infty} \frac{1}{2} \ln(2x-1) \Big]_1^b = +\infty$$

Both the comparison test and the limit comparison test would also have worked here (verify). ◀

The Root Test

9.5.6 THEOREM (*The Root Test*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \sqrt[k]{u_k} = \lim_{k \to +\infty} (u_k)^{1/k}$$

(*a*) *If $\rho < 1$, the series converges.*

(*b*) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(*c*) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

▶ Example 4 Use the root test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

Solution (a). The series diverges, since

$$\rho = \lim_{k \to +\infty} (u_k)^{1/k} = \lim_{k \to +\infty} \frac{4k-5}{2k+1} = 2 > 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k \to +\infty} (u_k)^{1/k} = \lim_{k \to +\infty} \frac{1}{\ln(k+1)} = 0 < 1 \quad ◀$$

The Comparison Test

9.5.1 THEOREM (*The Comparison Test*) *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonnegative terms and suppose that*

$$a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3, \ldots, a_k \leq b_k, \ldots$$

(*a*) *If the "bigger series" Σb_k converges, then the "smaller series" Σa_k also converges.*

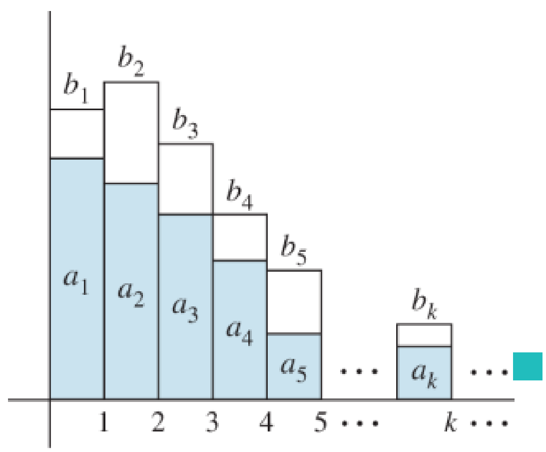
(*b*) *If the "smaller series" Σa_k diverges, then the "bigger series" Σb_k also diverges.*

Using the Comparison Test

There are two steps required for using the comparison test to determine whether a series $\sum u_k$ with positive terms converges:

Step 1. Guess at whether the series $\sum u_k$ converges or diverges.

Step 2. Find a series that proves the guess to be correct. That is, if we guess that $\sum u_k$ diverges, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if we guess that $\sum u_k$ converges, we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.



For each rectangle, a_k denotes the area of the blue portion and b_k denotes the combined area of the white and blue portions.

▲ **Figure 9.5.1**

▶ **Example 1** Use the comparison test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{2k^2+k}$$

Solution (a). According to Principle 9.5.2, we should be able to drop the constant in the denominator without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \tag{1}$$

which is a divergent p-series $\left(p=\frac{1}{2}\right)$. Thus, we will guess that the given series diverges and try to prove this by finding a divergent series that is "smaller" than the given series. However, series (1) does the trick since

$$\frac{1}{\sqrt{k}-\frac{1}{2}} > \frac{1}{\sqrt{k}} \quad \text{for } k=1,2,\ldots$$

Thus, we have proved that the given series diverges.

Solution (b). According to Principle 9.5.3, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} \tag{2}$$

which converges since it is a constant times a convergent p-series ($p=2$). Thus, we will guess that the given series converges and try to prove this by finding a convergent series that is "bigger" than the given series. However, series (2) does the trick since

$$\frac{1}{2k^2+k} < \frac{1}{2k^2} \quad \text{for } k=1,2,\ldots$$

Thus, we have proved that the given series converges. ◀

The Limit Comparison Test

9.5.4 THEOREM (*The Limit Comparison Test*) *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

▶ **Example 2** Use the limit comparison test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{2k^2+k} \qquad \text{(c)} \sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$$

Solution (a). As in Example 1, Principle 9.5.2 suggests that the series is likely to behave like the divergent p-series (1). To prove that the given series diverges, we will apply the limit comparison test with

$$a_k = \frac{1}{\sqrt{k}+1} \quad \text{and} \quad b_k = \frac{1}{\sqrt{k}}$$

We obtain

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{\sqrt{k}}{\sqrt{k}+1} = \lim_{k\to+\infty} \frac{1}{1+\dfrac{1}{\sqrt{k}}} = 1$$

Since ρ is finite and positive, it follows from Theorem 9.5.4 that the given series diverges.

Solution (b). As in Example 1, Principle 9.5.3 suggests that the series is likely to behave like the convergent series (2). To prove that the given series converges, we will apply the limit comparison test with

$$a_k = \frac{1}{2k^2+k} \quad \text{and} \quad b_k = \frac{1}{2k^2}$$

We obtain

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{2k^2}{2k^2+k} = \lim_{k\to+\infty} \frac{2}{2+\dfrac{1}{k}} = 1$$

Since ρ is finite and positive, it follows from Theorem 9.5.4 that the given series converges, which agrees with the conclusion reached in Example 1 using the comparison test.

Solution (c). From Principle 9.5.3, the series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{3k^3}{k^7} = \sum_{k=1}^{\infty} \frac{3}{k^4} \tag{3}$$

which converges since it is a constant times a convergent p-series. Thus, the given series is likely to converge. To prove this, we will apply the limit comparison test to series (3) and the given series. We obtain

$$\rho = \lim_{k \to +\infty} \frac{\dfrac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}}{\dfrac{3}{k^4}} = \lim_{k \to +\infty} \frac{3k^7 - 2k^6 + 4k^4}{3k^7 - 3k^3 + 6} = 1$$

Since ρ is finite and nonzero, it follows from Theorem 9.5.4 that the given series converges, since (3) converges. ◀

EXERCISE SET 9.5

25–49 Use any method to determine whether the series converges.

25. $\sum_{k=0}^{\infty} \frac{7^k}{k!}$ **26.** $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ **27.** $\sum_{k=1}^{\infty} \frac{k^2}{5^k}$

28. $\sum_{k=1}^{\infty} \frac{k!\,10^k}{3^k}$ **29.** $\sum_{k=1}^{\infty} k^{50}e^{-k}$ **30.** $\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$

31. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$ **32.** $\sum_{k=1}^{\infty} \frac{4}{2+3^k k}$

33. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$ **34.** $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{5^k}$

35. $\sum_{k=1}^{\infty} \frac{2+\sqrt{k}}{(k+1)^3-1}$ **36.** $\sum_{k=1}^{\infty} \frac{4+|\cos x|}{k^3}$

37. $\sum_{k=1}^{\infty} \frac{1}{1+\sqrt{k}}$ **38.** $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ **39.** $\sum_{k=1}^{\infty} \frac{\ln k}{e^k}$

40. $\sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$ **41.** $\sum_{k=0}^{\infty} \frac{(k+4)!}{4!\,k!\,4^k}$ **42.** $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

43. $\sum_{k=1}^{\infty} \frac{1}{4+2^{-k}}$ **44.** $\sum_{k=1}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1}$ **45.** $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$

46. $\sum_{k=1}^{\infty} \frac{5^k+k}{k!+3}$ **47.** $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$ **48.** $\sum_{k=1}^{\infty} \frac{[\pi(k+1)]^k}{k^{k+1}}$

49. $\sum_{k=1}^{\infty} \frac{\ln k}{3^k}$

SOLUTION SET

25. Ratio Test, $\rho = \lim_{k\to+\infty} 7/(k+1) = 0$, converges.

27. Ratio Test, $\rho = \lim_{k\to+\infty} \dfrac{(k+1)^2}{5k^2} = 1/5 < 1$, converges.

29. Ratio Test, $\rho = \lim_{k\to+\infty} e^{-1}(k+1)^{50}/k^{50} = e^{-1} < 1$, converges.

31. Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} 1/k^{5/2}$, $\rho = \lim_{k\to+\infty} \dfrac{k^3}{k^3+1} = 1$, converges.

33. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$, $\rho = \lim_{k\to+\infty} \dfrac{k}{\sqrt{k^2+k}} = 1$, diverges.

35. Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} \dfrac{1}{k^{5/2}}$, $\rho = \lim_{k\to+\infty} \dfrac{k^3+2k^{5/2}}{k^3+3k^2+3k} = 1$, converges.

37. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/\sqrt{k}$.

39. Ratio Test, $\rho = \lim_{k\to+\infty} \dfrac{\ln(k+1)}{e\ln k} = \lim_{k\to+\infty} \dfrac{k}{e(k+1)} = 1/e < 1$, converges.

41. Ratio Test, $\rho = \lim_{k\to+\infty} \dfrac{k+5}{4(k+1)} = 1/4$, converges.

43. Diverges by the Divergence Test, because $\lim_{k\to+\infty} \dfrac{1}{4+2^{-k}} = 1/4 \neq 0$.

45. $\dfrac{\tan^{-1} k}{k^2} < \dfrac{\pi/2}{k^2}$, $\sum_{k=1}^{\infty} \dfrac{\pi/2}{k^2}$ converges so $\sum_{k=1}^{\infty} \dfrac{\tan^{-1} k}{k^2}$ converges.

47. Ratio Test, $\rho = \lim_{k\to+\infty} \dfrac{(k+1)^2}{(2k+2)(2k+1)} = 1/4$, converges.

49. $a_k = \dfrac{\ln k}{3^k}$, $\dfrac{a_{k+1}}{a_k} = \dfrac{\ln(k+1)}{\ln k}\dfrac{3^k}{3^{k+1}} \to \dfrac{1}{3}$, converges.