CONTINUITY

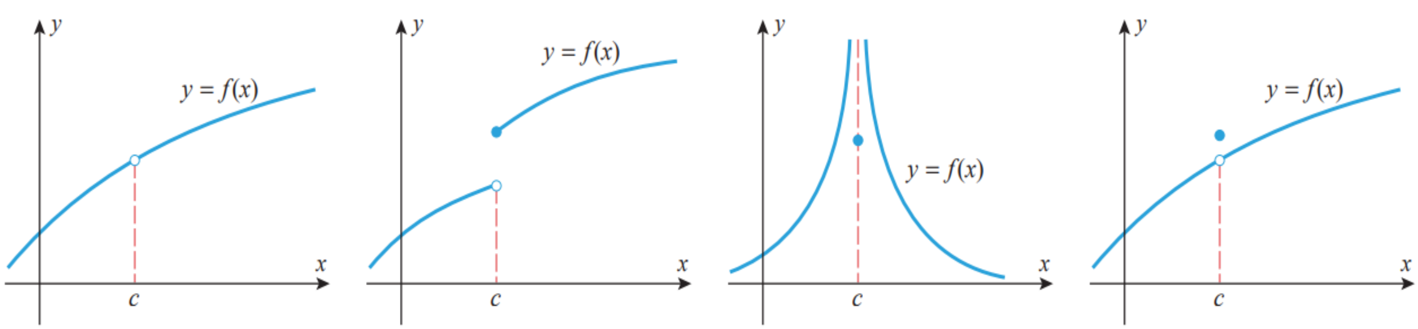
DEFINITION OF CONTINUITY:

Intuitively, the graph of a function can be described as a "continuous curve" if it has no breaks or holes. To make this idea more precise we need to understand what properties of a function can cause breaks or holes. Referring to Figure 1.5.1, we see that the graph of a function has a break or hole if any of the following conditions occur:

- The function f is undefined at c (Figure 1.5.1*a*).
- The limit of $f(x)$ does not exist as x approaches c (Figures 1.5.1*b*, 1.5.1*c*).
- The value of the function and the value of the limit at c are different (Figure 1.5.1*d*).

1.5.1 DEFINITION A function f is said to be ***continuous at $x = c$*** provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.



(a) (b) (c) (d)

1.5.2 DEFINITION A function f is said to be ***continuous on a closed interval [a, b]*** if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(a) *$f + g$ is continuous at c.*

(b) *$f - g$ is continuous at c.*

(c) *fg is continuous at c.*

(d) *f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.*

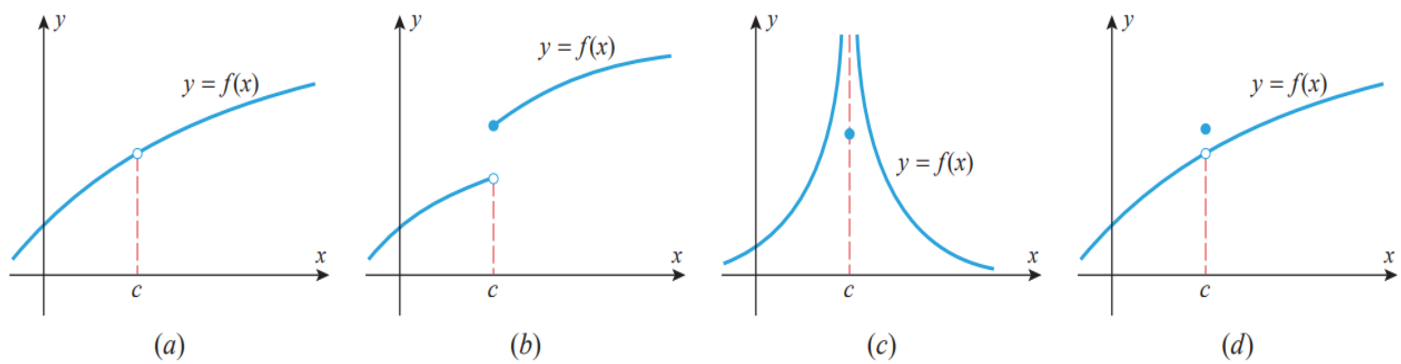
CONTINUITY OF POLYNOMIALS AND RATIONAL FUNCTIONS

1.5.4 THEOREM

(a) *A polynomial is continuous everywhere.*

(b) *A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.*

TYPES OF DISCONTINUITY



Each function drawn in Figure 1.5.1 illustrates a discontinuity at $x = c$. In Figure 1.5.1*a*, the function is not defined at c, violating the first condition of Definition 1.5.1. In Figure 1.5.1*b*, the one-sided limits of $f(x)$ as x approaches c both exist but are not equal. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1*b* has a ***jump discontinuity*** at c. In Figure 1.5.1*c*, the one-sided limits of $f(x)$ as x approaches c are infinite. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1*c* has an ***infinite discontinuity*** at c. In Figure 1.5.1*d*, the function is defined at c and $\lim_{x\to c} f(x)$ exists, but these two values are not equal, violating the third condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1*d* has a ***removable discontinuity*** at c. Exercises 33 and 34 help to explain why discontinuities of this type are given this name.

Example 1 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, namely,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$ (Figure 1.5.2*a*). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as x approaches 2; hence, g is also not continuous at $x = 2$ (Figure 1.5.2*b*). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit as x approaches 2; hence, h is continuous at $x = 2$ (Figure 1.5.2*c*). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀

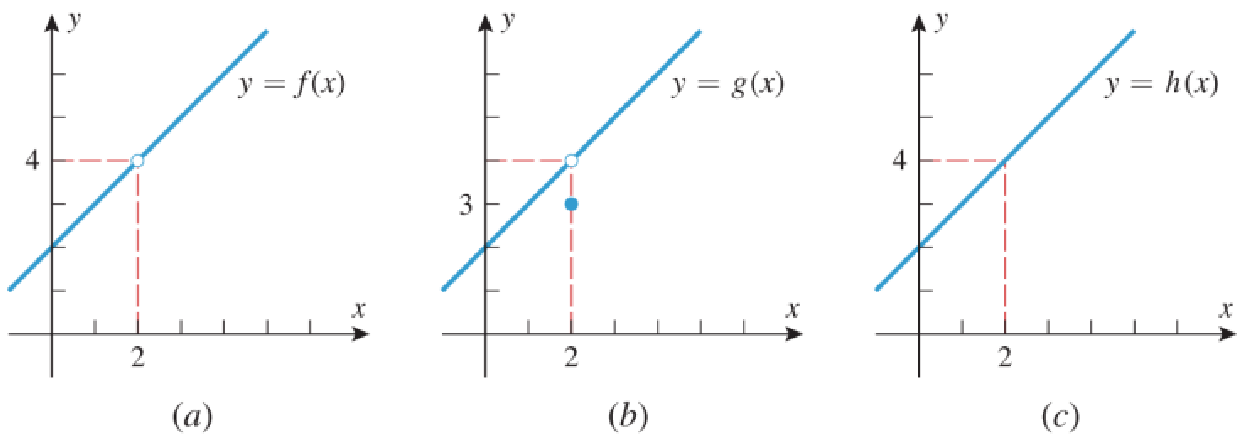


Figure 1.5.2

▶ Example 2 What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution. Because the natural domain of this function is the closed interval $[-3, 3]$, we will need to investigate the continuity of f on the open interval $(-3, 3)$ and at the two endpoints. If c is any point in the interval $(-3, 3)$, then it follows from Theorem 1.2.2(e) that

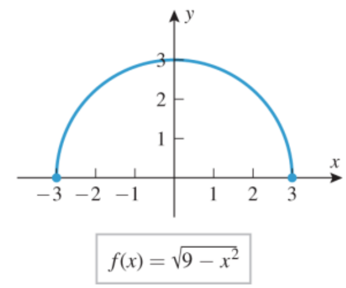
$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{9 - x^2} = \sqrt{\lim_{x\to c}(9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

which proves f is continuous at each point in the interval $(-3, 3)$. The function f is also continuous at the endpoints since

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x\to 3^-}(9 - x^2)} = 0 = f(3)$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x\to -3^+}(9 - x^2)} = 0 = f(-3)$$

Thus, f is continuous on the closed interval $[-3, 3]$ (Figure 1.5.5).



▲ **Figure 1.5.5**

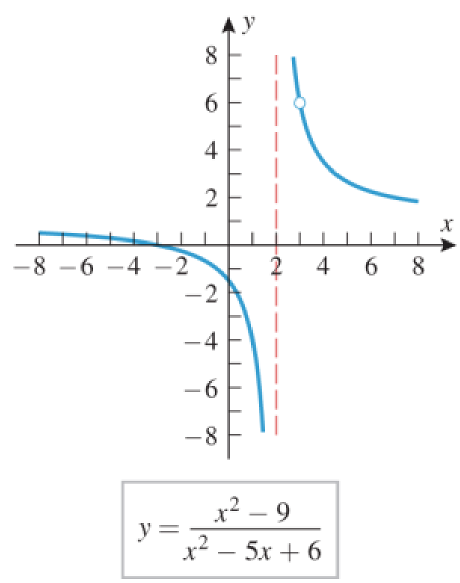
▶ Example 3 For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

$$x^2 - 5x + 6 = 0$$

yields discontinuities at $x = 2$ and at $x = 3$ (Figure 1.5.6).



▲ **Figure 1.5.6**

Example 4 Show that $|x|$ is continuous everywhere.

Solution. We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

so $|x|$ is the same as the polynomial x on the interval $(0, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, 0)$. But polynomials are continuous everywhere, so $x = 0$ is the only possible discontinuity for $|x|$. Since $|0| = 0$, to prove the continuity at $x = 0$ we must show that

$$\lim_{x \to 0} |x| = 0 \tag{2}$$

Because the piecewise formula for $|x|$ changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Thus, (2) holds and $|x|$ is continuous at $x = 0$ (Figure 1.5.7).

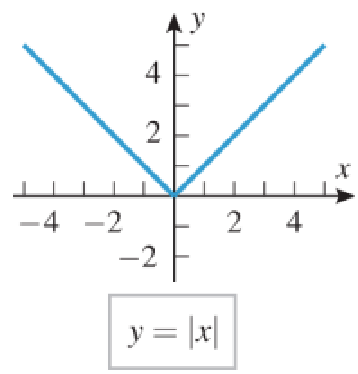


Figure 1.5.7

EXERCISE SET 1.2

Question:

5. Consider the functions

$$f(x) = \begin{cases} 1, & x \neq 4 \\ -1, & x = 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 4x - 10, & x \neq 4 \\ -6, & x = 4 \end{cases}$$

In each part, is the given function continuous at $x = 4$?

(a) $f(x)$ (b) $g(x)$ (c) $-g(x)$ (d) $|f(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $g(x) - 6f(x)$

Answer:

(a) No. **(b)** No. **(c)** No. **(d)** Yes. **(e)** Yes. **(f)** No. **(g)** Yes.

Question:

Find values of x, if any, at which f is not continuous.

17. $f(x) = \dfrac{3}{x} + \dfrac{x-1}{x^2-1}$

Answer:

The function is not continuous at $x = 0$, $x = 1$ and $x = -1$.

Question:

Find values of x, if any, at which f is not continuous.

22. $f(x) = \begin{cases} \dfrac{3}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

Answer:

The function is not continuous at $x = 1$, as $\lim\limits_{x \to 1} f(x)$ does not exist.

Question:

29–30 Find a value of the constant k, if possible, that will make the function continuous everywhere.

29. (a) $f(x) = \begin{cases} 7x - 2, & x \leq 1 \\ kx^2, & x > 1 \end{cases}$

Answer:

(a) f is continuous for $x < 1$, and for $x > 1$; $\lim_{x \to 1^-} f(x) = 5$, $\lim_{x \to 1^+} f(x) = k$, so if $k = 5$ then f is continuous for all x.

Question:

35–36 Find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity.

35. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$

Answer:

(a) $x = 0$, $\lim_{x \to 0^-} f(x) = -1 \neq +1 = \lim_{x \to 0^+} f(x)$ so the discontinuity is not removable.

(b) $x = -3$; define $f(-3) = -3 = \lim_{x \to -3} f(x)$, then the discontinuity is removable.