Assignment : 02

23k-2001

BCS 1-J

Question#1

$A = ?$, $f(x) = 4x - x^2$, $[0,4]$

x_k^* as right endpoint

$\because A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$

For right endpoint,

$x_k^* = a + k\Delta x$, $\Delta x = \frac{b-a}{n}$

$x_k^* = 0 + k\left(\frac{4}{n}\right)$, $\Delta x = \frac{4}{n}$

$x_k^* = \frac{4k}{n}$

$f(x_k^*) = 4\left(\frac{4k}{n}\right) - \left(\frac{4k}{n}\right)^2$

$f(x_k^*) = \frac{16k}{n} - \frac{16k^2}{n^2}$

$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{16k}{n} - \frac{16k^2}{n^2}\right)\frac{4}{n}$

$A = \lim_{n\to\infty} \left\{ \frac{4\times 16}{n^2} \sum_{k=1}^{n} k - \frac{4\times 16}{n^3} \sum_{k=1}^{n} k^2 \right\}$

$A = \lim_{n\to\infty} \left\{ \frac{64}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \right\}$

$A = 32 \lim_{n\to\infty} \frac{n.n(1+1/n)}{n^2} - \frac{32}{3} \lim_{n\to\infty} \frac{n.n.n(1+1/n)(2+1/n)}{n^3}$

$A = 32 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) - \frac{32}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$

Applying limits,

$A = 32(1) - \frac{32}{3}(1)(2)$

$A = \frac{32}{3}$ units^2 Ans.

Question #2

$$\int 8x^4 \cos 2x \, dx \qquad \text{(Tabular I.B.P)}$$

$$I = 8 \int x^4 \cos 2x \, dx$$

(Derivative) U		V (Integral)
x^4	$+$	$\cos 2x$
$4x^3$	$-$	$\frac{\sin 2x}{2}$
$12x^2$	$+$	$-\frac{\cos 2x}{4}$
$24x$	$-$	$-\frac{\sin 2x}{8}$
24	$+$	$\frac{\cos 2x}{16}$
0		$\frac{\sin 2x}{32}$

$$I = 8\left(\frac{x^4 \sin 2x}{2}\right) + 8\left(4x^3 \frac{\cos 2x}{4}\right) - 8\left(12x^2 \frac{\sin 2x}{8}\right) - 8\left(24x \frac{\cos 2x}{16}\right) + 8\left(\frac{24 \sin 2x}{32}\right) + C$$

$$I = 4x^4 \sin 2x + 8x^3 \cos 2x - 12x^2 \sin 2x - 12x \cos 2x + 6 \sin 2x + C$$

Ans.

Question #3

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$$

Integration by parts

$$I = \int_0^1 x^2 \cdot \frac{x}{\sqrt{x^2+1}}\,dx$$

$$I = \left[x^2\left[\frac{1}{2}\frac{(x^2+1)^{1/2}}{1/2}\right]\right]_0^1 - \int_0^1 \frac{1}{2}\frac{(x^2+1)^{1/2}}{1/2} \cdot 2x\,dx$$

$$I = \left[x^2\sqrt{x^2+1}\right]_0^1 - \int_0^1 (x^2+1)^{1/2}\,2x\,dx$$

$$I = \left[x^2\sqrt{x^2+1}\right]_0^1 - \left[\frac{(x^2+1)^{3/2}}{3/2}\right]_0^1$$

$$I = \left[x^2\sqrt{x^2+1}\right]_0^1 - \frac{2}{3}\left[(x^2+1)^{3/2}\right]_0^1$$

$$I = \left[1^2\sqrt{1^2+1} - 0^2\sqrt{0^2+1}\right] - \frac{2}{3}(1^2+1)^{3/2} + \frac{2}{3}(0^2+1)^{3/2}$$

$$I = (\sqrt{2} - 0) - \frac{2}{3}(2)^{3/2} + \frac{2}{3}$$

$$I = \sqrt{2} - \frac{2\sqrt{8}}{3} + \frac{2}{3}$$

$$I = \frac{3\sqrt{2} - 4\sqrt{2} + 2}{3}$$

$$I = \frac{-\sqrt{2}+2}{3}$$ Ans.

By substibution

let, $u = \sqrt{x^2+1}$

$\frac{du}{dx} = \frac{1}{2}(x^2+1)^{-1/2} 2x \Rightarrow$ $u^2 = x^2+1$

$x^2 = u^2-1$

$du = \frac{x}{\sqrt{x^2+1}} dx$

For $x=1$, For $x=0$

$u = \sqrt{2}$ $u = 1$

$$I = \int_0^1 x^2 . \frac{x dx}{\sqrt{x^2+1}}$$

$$I = \int_1^{\sqrt{2}} (u^2-1) du.$$

$$I = \int_1^{\sqrt{2}} u^2 du - \int_1^{\sqrt{2}} du$$

$$I = \left[\frac{u^3}{3}\right]_1^{\sqrt{2}} - \left[u\right]_1^{\sqrt{2}}$$

$$I = \frac{(\sqrt{2})^3 - 1^3}{3} - (\sqrt{2}-1)$$

$$I = \frac{2\sqrt{2} - 1 - 3\sqrt{2} + 3}{3}$$

$$I = \frac{-\sqrt{2}+2}{3}$$ Ans.

Question # 4

a. $\displaystyle\int_{-\infty}^{2} \frac{dx}{x^2+4}$

$$\Rightarrow I = \lim_{a\to-\infty} \int_a^2 \frac{dx}{x^2+4}$$

$$I = \lim_{a\to-\infty} \left| \frac{1}{2}\tan^{-1}\frac{x}{2} \right|_a^2$$

$$I = \frac{1}{2} \lim_{a\to-\infty} \left(\tan^{-1}\frac{2}{2} - \tan^{-1}\frac{a}{2} \right)$$

$$I = \frac{1}{2} \lim_{a\to-\infty} \left(\tan^{-1}1 - \tan^{-1}\frac{a}{2} \right)$$

$$I = \frac{1}{2} \lim_{a\to-\infty} \left(\frac{\pi}{4} - \tan^{-1}\frac{(a)}{2} \right)$$

$$I = \frac{1}{2} \left(\frac{\pi}{4} - \tan^{-1}\frac{(-\infty)}{2} \right)$$

$$I = \frac{1}{2} \left(\frac{\pi}{4} - \tan^{-1}(-\infty) \right)$$

$$I = \frac{1}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{2} \right) \right]$$

$$I = \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{2} \right)$$

$$I = \frac{3\pi}{8}$$ Ans. (converges)

b. $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ discontinuity at $x = \frac{1}{2}$

$$I = \int_0^{1/2} \frac{dx}{2x-1} + \int_{1/2}^{1} \frac{dx}{2x-1}$$

Consider,

$$\lim_{b \to (1/2)^-} \int_0^b \frac{dx}{2x-1}$$

$$= \lim_{b \to (1/2)^-} \frac{1}{2} \int_0^b \frac{2dx}{2x-1}$$

$$= \frac{1}{2} \lim_{b \to (1/2)^-} \Big[\ln(2x-1) \Big]_0^b$$

$$= \frac{1}{2} \lim_{b \to (1/2)^-} \big(\ln(2b-1) - \ln(0-1) \big)$$

$$= \frac{1}{2} \left[\ln\left(2 \cdot \frac{1}{2} - 1\right) - \ln(-1) \right]$$

$$= \frac{1}{2} (0 - \infty)$$

$$= -\infty$$

Hence

$$\int_0^1 \frac{1}{2x-1}\,dx \rightarrow \text{diverges}$$

Ans.

Question #5:

1. $\displaystyle\int_0^3 \frac{x}{\sqrt{3+2x}}\,dx$

let, $u = \sqrt{3+2x}$ $\Rightarrow x = \frac{u^2-3}{2}$

$du = \frac{1}{2}(3+2x)^{-1/2} \cdot 2\,dx$

$$du = \frac{dx}{(3+2x)^{1/2}}$$

For $x = 3$ For $x = 0$

$u = 3$ $u = \sqrt{3}$

$$I = \int_{\sqrt{3}}^{3} \frac{u^2-3}{2} \cdot du$$

$$= \frac{1}{2}\int_{\sqrt{3}}^{3} u^2\,du - \frac{3}{2}\int_{\sqrt{3}}^{3} du$$

$$= \frac{1}{2}\left[\frac{u^3}{3}\right]_{\sqrt{3}}^{3} - \frac{3}{2}\left[u\right]_{\sqrt{3}}^{3}$$

$$= \frac{1}{6}\left[3^3 - (\sqrt{3})^3\right] - \frac{3}{2}\left(3 - \sqrt{3}\right)$$

$$= \frac{1}{6}\left(27 - 3\sqrt{3}\right) - \frac{3}{2}\left(3 - \sqrt{3}\right)$$

$$= \frac{9}{2} - \frac{\sqrt{3}}{2} - \frac{9}{2} + \frac{3\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3} - \sqrt{3}}{2}$$

$I = \sqrt{3}$ Ans.

2. $\int \frac{2^y}{2^y+5}\,dy$

let , $x = 2^y$

$dx = 2^y \ln 2\, dy$

$\frac{dx}{\ln 2} = 2^y dy$

$I = \frac{1}{\ln 2}\int \frac{dx}{x+5}$

$I = \frac{1}{\ln 2}\ln(x+5) + C$

$I = \frac{\ln(2^y+5)}{\ln 2} + C$ Ans.

3. $\int \frac{1}{\sqrt{1+x^2}\,\sinh^{-1}x}\,dx$

$\because \frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$

$I = \int \frac{1}{\sinh^{-1}x}\left(\frac{1}{\sqrt{1+x^2}}dx\right)$

$I = \ln(\sinh^{-1}x) + C$ Ans.

4. $\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta} d\theta$, let $u = \tan\theta$
$du = \sec^2\theta d\theta$

$I = \int \frac{du}{u^3 - u^2}$

$I = \int \frac{du}{u^2(u-1)}$

Consider,

$\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1}$

Multiply, divide by l.cm

$1 = A(u)(u-1) + B(u-1) + C(u^2)$

put $x = 1$

$1 = 0 + 0 + C$

$C = 1$

put $x = 0$

$1 = 0 + B(-1) + 0$

$B = -1$

compare u^2 coefficients

$0 = A + C$

$0 = A + 1$

$A = -1$

$I = \int \left(\frac{-1}{u} - \frac{1}{u^2} + \frac{1}{u-1} \right) du$

$I = -\int \frac{du}{u} - \int \frac{du}{u^2} + \int \frac{du}{u-1}$

$I = -\int \frac{du}{u} - \int u^{-2} du + \int \frac{du}{u-1}$

$I = -\ln u - \frac{u^{-1}}{-1} + \ln(u-1) + C$

$I = \frac{1}{u} - \ln u + \ln(u-1) + C$

$I = \frac{1}{\tan\theta} - \ln\tan\theta + \ln(\tan\theta - 1) + C$

Ans.

5. $\int \frac{x^3+8}{(x^2-1)(x-2)} dx$

$$I = \int \frac{x^3+8}{x^3-2x^2-x+2} dx$$

$$\begin{array}{r|l} & 1 \\ x^3-2x^2-x+2 & x^3+8 \\ & x^3-2x^2+2-x \\ \hline & 2x^2+x+6 \end{array}$$

$$I = \int \left(1 + \frac{2x^2+x+6}{x^3-2x^2-x+2}\right) dx$$

Consider,

$$\frac{2x^2+x+6}{x^3-2x^2-x+2} \Rightarrow \frac{2x^2+x+6}{(x-1)(x+1)(x-2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2}$$

Mulbiply, divide by L.C.M

$$2x^2+x+6 = A(x+1)(x-2) + B(x-1)(x-2) + C(x-1)(x+1)$$

Put $x = 1$

$$2(1)^2+1+6 = A(1+1)(1-2) + 0 + 0$$

$$9 = -2A$$

$$A = -\frac{9}{2}$$

Put $x = -1$

$$2-1+6 = 0 + B(-2)(-3) + 0$$

$$\frac{7}{6} = B$$

Put $x = 2$

$$8 + 2 + 6 = 0 + 0 + C(1)(3)$$

$$16 = 3C$$

$$C = \frac{16}{3}$$

$$I = \int \left[1 + \left(\frac{-9}{2(x-1)} + \frac{7}{6(x+1)} + \frac{16}{3(x-2)} \right) \right] dx$$

$$I = \int dx - \frac{9}{2} \int \frac{dx}{x-1} + \frac{7}{6} \int \frac{dx}{x+1} + \frac{16}{3} \int \frac{dx}{x-2}$$

$$I = x - \frac{9}{2} \ln(x-1) + \frac{7}{6} \ln(x+1) + \frac{16}{3} \ln(x-2) + C$$

Ans.

Q6. $\int x^5 e^{x^3} dx$

let, $y = x^3$

$dy = 3x^2 dx$

$\frac{dy}{3} = x^2 dx$

$I = \int x^3 . e^{x^3} . x^2 dx$

$I = \frac{1}{3} \int y . e^y dy$

$I = \frac{1}{3} y e^y - \frac{1}{3} \int e^y . dy$

$I = \frac{1}{3} y . e^y - \frac{1}{3} e^y + C$

$I = \frac{1}{3} x^3 . e^{x^3} - \frac{1}{3} e^{x^3} + C$

Ans.

$I = \frac{(x^3 - 1) e^{x^3}}{3} + C$

Q7. $\int \frac{1}{\sqrt{x^2-4x}}\,dx$

let,

$x-2 = 2\sec\theta$

$dx = 2\tan\theta\sec\theta\,d\theta$

$I = \int \frac{1}{\sqrt{x^2-4x+4-4}}\,dx$

$\Rightarrow \sec\theta = \frac{x-2}{2} \;;\; \frac{H}{B}$

$I = \int \frac{1}{\sqrt{(x-2)^2-2^2}}\,dx$

$P = \sqrt{(x-2)^2+2^2}$

$P = \sqrt{x^2-4x}$

$I = \int \frac{2\tan\theta\sec\theta\,d\theta}{\sqrt{(2\sec\theta)^2-2^2}}$

$\tan\theta = \frac{\sqrt{x^2-4x}}{2}$

$I = \int \frac{2\tan\theta\sec\theta\,d\theta}{2\sqrt{\tan^2\theta}}$

$I = \int \frac{\tan\theta\sec\theta\,d\theta}{\tan\theta}$

$I = \int \sec\theta\,d\theta$

$I = \ln(\sec\theta+\tan\theta)+C$

$I = \ln\left(\frac{x-2}{2}+\frac{\sqrt{x^2-4x}}{2}\right)+C$ Ans.

Q8. $\int \frac{x}{\sqrt{x^2+4x+5}} dx$

$$I = \int \frac{x}{\sqrt{x^2+4x+4+1}} dx$$

$$I = \int \frac{x \, dx}{\sqrt{(x+2)^2+1^2}}$$, let

$y = x+2$, $x = y-2$

$dy = dx$

$$I = \int \frac{(y-2)dy}{\sqrt{y^2+1^2}}$$

$$I = \int \frac{y\,dy}{\sqrt{y^2+1}} - 2\int \frac{dy}{\sqrt{y^2+1}}$$

$$I = \int (y^2+1)^{-1/2} y\,dy - 2\int \frac{dy}{\sqrt{y^2+1}}$$

$$I = \frac{1}{2}\int (y^2+1)^{-1/2} 2y\,dy - 2\int \frac{dy}{\sqrt{y^2+1}}$$

$$I = \frac{1}{2}\frac{(y^2+1)^{1/2}}{1/2} - 2\ln\left(y+\sqrt{y^2+1}\right) + C$$

$$I = \sqrt{(x+2)^2+1} - 2\ln\left(x+2+\sqrt{(x+2)^2+1}\right) + C$$

$$I = \sqrt{x^2+4x+5} - 2\ln\left(x+2+\sqrt{x^2+4x+5}\right) + C$$

Ans.

Q9. $\int \ln(2x+3)\,dx$

let, $y = 2x+3$

$dy = 2dx$

$\frac{dy}{2} = dx$

$$I = \frac{1}{2}\int \ln y\,dy$$

$$I = \frac{1}{2}\int \ln y \cdot y^{\circ} dy$$

$$I = \frac{1}{2}\ln y \cdot y - \frac{1}{2}\int y \cdot \frac{1}{y}\,dy$$

$$I = \frac{y\ln y}{2} - \frac{1}{2}y + C$$

$$I = \frac{(2x+3)\ln(2x+3)}{2} - \frac{(2x+3)}{2} + C$$

Ans.

10. $\int \sin^4 2x\, dx$, let $u = 2x$

$\frac{du}{2} = dx$

$$I = \frac{1}{2}\int \sin^4 u\, du$$

$$I = \frac{1}{2}\left[\frac{-1}{4}\sin^3 u\cos u + \frac{3}{4}\int \sin^2 u\, du\right]$$

$$I = \frac{1}{2}\left[\frac{-1}{4}\sin^3 u\cos u + \frac{3}{4}\int \frac{1-\cos 2u}{2}\, du\right]$$

$$I = \frac{1}{2}\left[\frac{-1}{4}\sin^3 u\cos u + \frac{3}{8}\int du - \frac{3}{8}\int \cos 2u\, du\right]$$

$$I = \frac{1}{2}\left[\frac{-1}{4}\sin^3 u\cos u + \frac{3}{8}u - \frac{3}{8}\frac{\sin 2u}{2}\right] + C$$

$$I = \frac{-1}{8}\sin^3 u\cos u + \frac{3}{16}u - \frac{3}{32}\sin 2u + C$$

$$I = \frac{-1}{8}\sin^3 2x\cos 2x + \frac{3}{16}\cdot 2x - \frac{3}{32}\sin 4x + C$$

$$I = \frac{-1}{8}\sin^3 2x\cos 2x + \frac{3x}{8} - \frac{3}{32}\sin 4x + C$$

Ans.

11. $\int \frac{\sqrt{1+4x^2}}{x}\,dx$ let $x = \frac{1}{2}\tan\theta$

$I = \int \frac{\sqrt{4(x^2 + 1/4)}}{x}\,dx$ $dx = \frac{1}{2}\sec^2\theta\,d\theta$

$I = 2\int \frac{\sqrt{(x^2 + 1/2^2)}}{x}\,dx$

$I = 2\int \frac{\sqrt{\frac{1}{4}\tan^2\theta + \frac{1}{4}}}{\frac{1}{2}\tan\theta} \cdot \frac{1}{2}\sec^2\theta\,d\theta$

$I = 2 \cdot \frac{1}{2}\int \frac{\sqrt{\tan^2\theta + 1}}{\tan\theta}\sec^2\theta\,d\theta$

$I = \int \frac{\sec\theta}{\tan\theta} \cdot \sec^2\theta\,d\theta$

$I = \int \frac{\sec^3\theta}{\tan\theta}\,d\theta$

$I = \int \frac{1}{\cos^3\theta} \cdot \frac{\cos\theta}{\sin\theta}\,d\theta$

$I = \int \sec^2\theta \operatorname{cosec}\theta\,d\theta$

$I = \int (1+\tan^2\theta)\operatorname{cosec}\theta\,d\theta$

$I = \int \operatorname{cosec}\theta\,d\theta + \int \tan^2\theta \operatorname{cosec}\theta\,d\theta$

$I = \int \operatorname{cosec}\theta\,d\theta + \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\sin\theta}\,d\theta$

$$I = \int \operatorname{cosec}\theta \, d\theta + \int \frac{\sin\theta}{\cos^2\theta} d\theta$$

$$I = \int \operatorname{cosec}\theta \, d\theta + \int \cos^{-2}\theta \sin\theta \, d\theta$$

$$I = \ln(\operatorname{cosec}\theta - \cot\theta) - \frac{\cos^{-1}\theta}{-1} + C$$

$$I = \ln(\operatorname{cosec}\theta - \cot\theta) + \sec\theta + C$$

$\because$ $x = \frac{1}{2}\tan\theta$ $\qquad H = (\sqrt{4x^2+1})$

$\tan\theta = 2x$ $\qquad \cos\theta = \frac{1}{\sqrt{4x^2+1}}$, $\sec\theta = \sqrt{4x^2+1}$

$\cot\theta = \frac{1}{2x}$

$\sin\theta = \frac{2x}{\sqrt{4x^2+1}}$, $\operatorname{cosec}\theta = \frac{\sqrt{4x^2+1}}{2x}$

$$I = \ln\left(\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right) + \sqrt{4x^2+1} + C$$

$$I = \ln\left(\frac{\sqrt{4x^2+1} - 1}{2x}\right) + \sqrt{4x^2+1} + C$$

Ans.

Q12. $\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

let,

$$y = \sqrt{e^x - 1}$$

$$y^2 = e^x - 1 \quad , \quad e^x = y^2 + 1$$

$$2y\,dy = e^x dx$$

$$\frac{2y\,dy}{y^2+1} = dx$$

at $x = \ln 2$

$y = \sqrt{e^{\ln 2} - 1}$

$y = 1$

at $x = 0$

$y = \sqrt{e^0 - 1}$

$y = 0$

$$I = \int_0^1 y \cdot \frac{2y\,dy}{y^2+1}$$

$$\begin{array}{r|l} & 1 \\ \hline y^2+1 & y^2 \\ & y^2+1 \\ \hline & -1 \end{array}$$

$$I = 2\int_0^1 \frac{y^2\,dy}{y^2+1}$$

$$I = 2\int_0^1 \left(1 - \frac{1}{y^2+1}\right) dy$$

$$I = 2\int_0^1 dy - 2\int_0^1 \frac{dy}{y^2+1}$$

$$I = 2\left|y\right|_0^1 - 2\left|\tan^{-1} y\right|_0^1$$

$$I = 2(1-0) - 2\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$I = 2 - 2\left(\frac{\pi}{4} - 0\right)$$

$$I = \frac{4-\pi}{2} \quad \text{Ans.}$$