Limits

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \tag{14}$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \tag{15}$$

Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

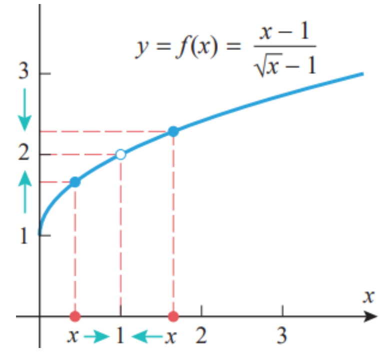
denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$

Use numerical evidence to make a conjecture about the value of

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$$

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

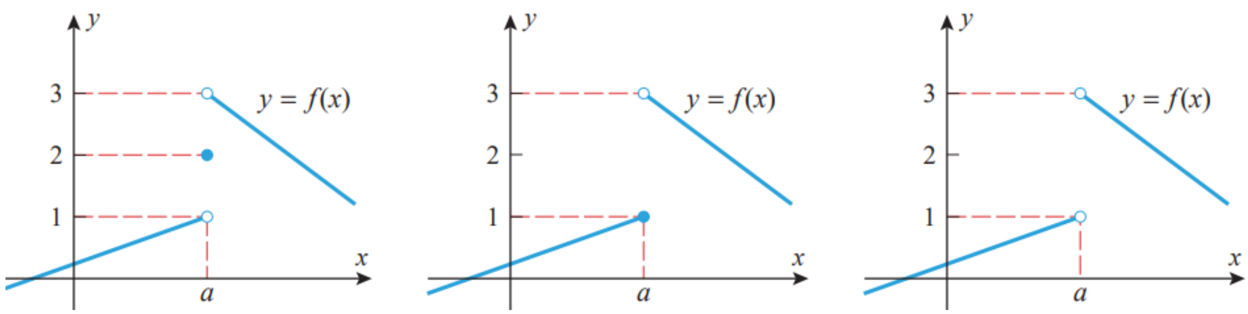


x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

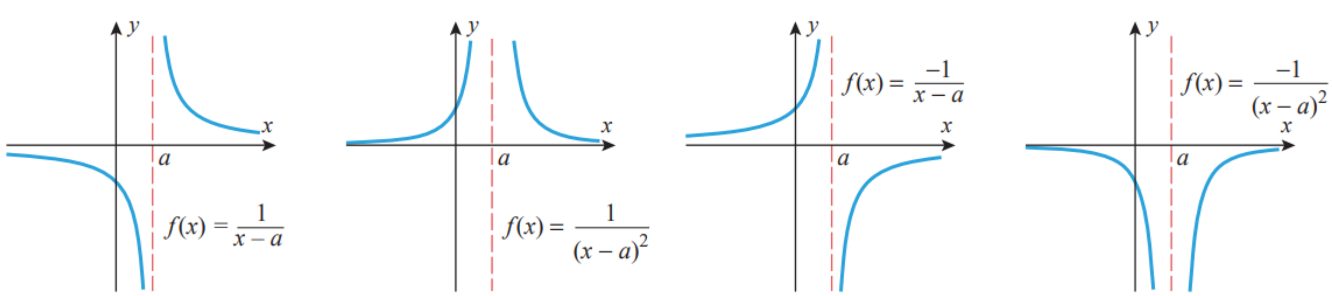
Left side → ← Right side

Find one sided and two sided limit of the function at $x = a$, if it Exists

(a)



(b)



(a) (b) (c) (d)

Problem Set:

For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$

(c) $\lim_{x \to 2} f(x)$ (d) $f(2)$.

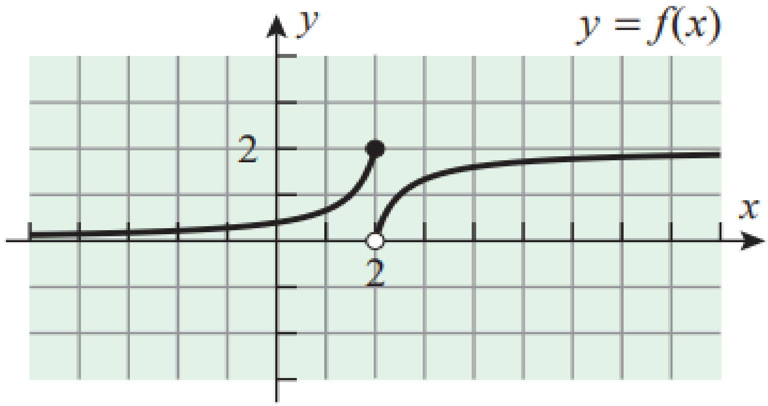


Figure Ex-4

For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$

(c) $\lim_{x \to 3} f(x)$ (d) $f(3)$.

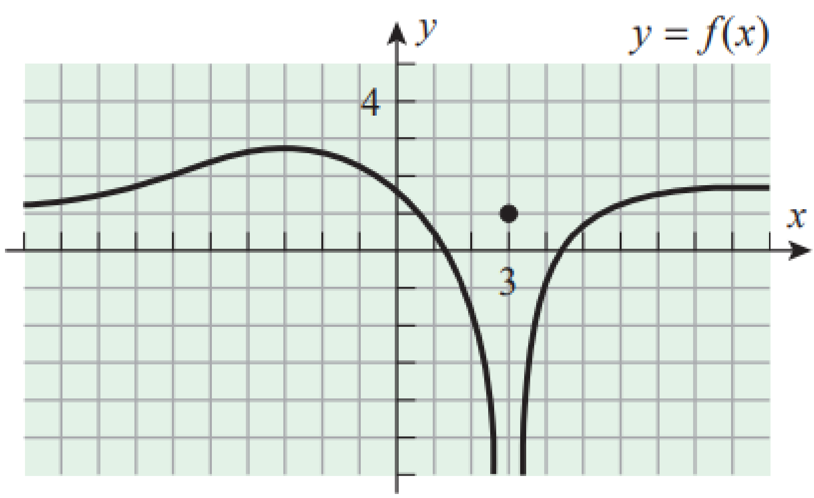
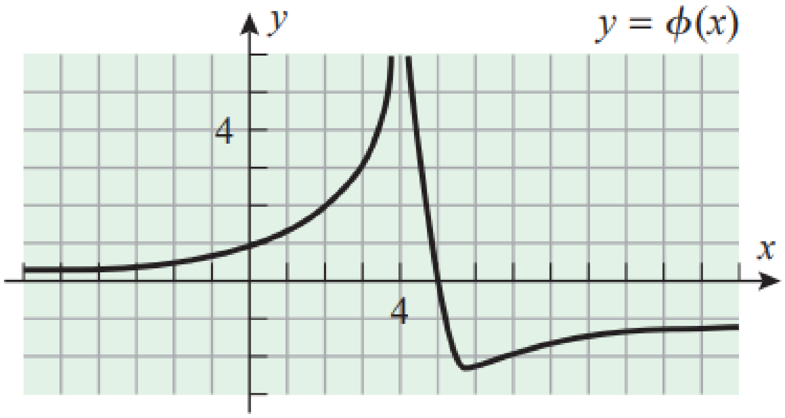


Figure Ex-7

For the function ϕ graphed in the accompanying figure, find

(a) $\lim_{x \to 4^-} \phi(x)$ (b) $\lim_{x \to 4^+} \phi(x)$

(c) $\lim_{x \to 4} \phi(x)$ (d) $\phi(4)$.



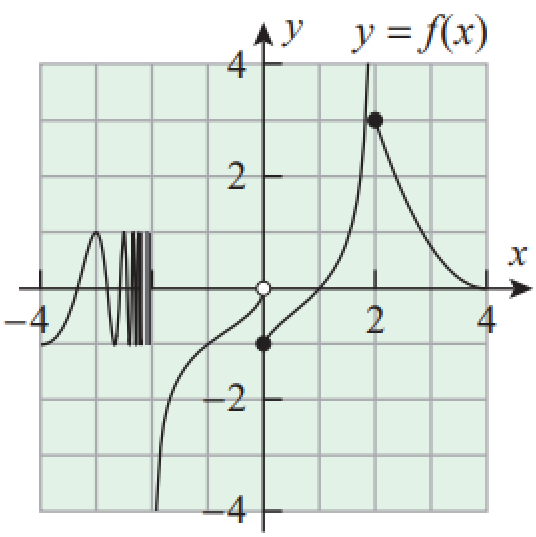
◀ **Figure Ex-8**

For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$

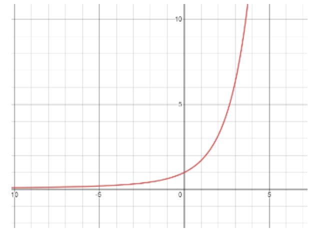
(d) $\lim_{x \to 0^+} f(x)$ (e) $\lim_{x \to 2^-} f(x)$ (f) $\lim_{x \to 2^+} f(x)$

(g) the vertical asymptotes of the graph of f.



◀ **Figure Ex-10**

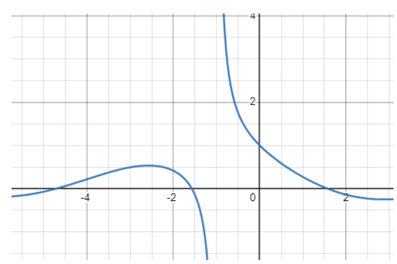
11. $f(x) = \dfrac{e^x - 1}{x}$; $\lim\limits_{x \to 0} f(x)$



x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$						

13–16 (i) Make a guess at the limit (if it exists) by evaluating the function at the specified x-values. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.]

$$\lim_{x \to -1} \frac{\cos x}{x+1};\ x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$$



$$\lim_{x \to 0} \frac{\sin(5x)}{\sin(2x)};\ x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$$