National University of Computer & Emerging Sciences
MT-1003 Calculus and Analytical Geometry

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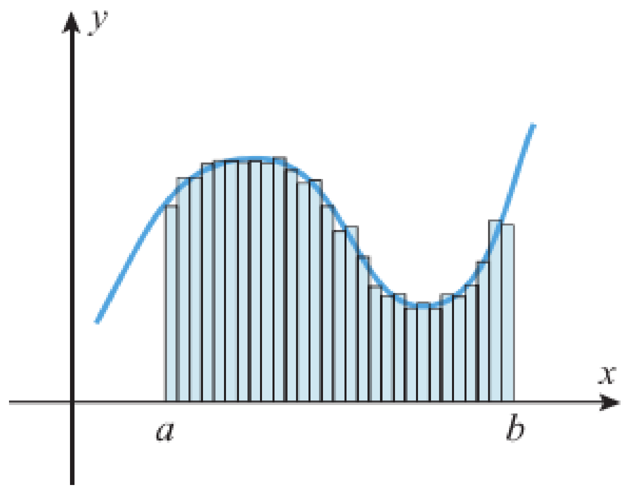
5.4.3 DEFINITION ***(Area Under a Curve)*** If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{2}$$

The limit in (2) is interpreted to mean that given any number $\epsilon > 0$ the inequality

$$\left| A - \sum_{k=1}^{n} f(x_k^*)\Delta x \right| < \epsilon$$

holds when n is sufficiently large, no matter how the points x_k^* are selected.



Thus, the left endpoint, right endpoint, and midpoint choices for $x_1^*, x_2^*, \ldots, x_n^*$ are given by

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \boxed{\text{Left endpoint}} \tag{3}$$

$$x_k^* = x_k = a + k\Delta x \qquad \boxed{\text{Right endpoint}} \tag{4}$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \qquad \boxed{\text{Midpoint}} \tag{5}$$

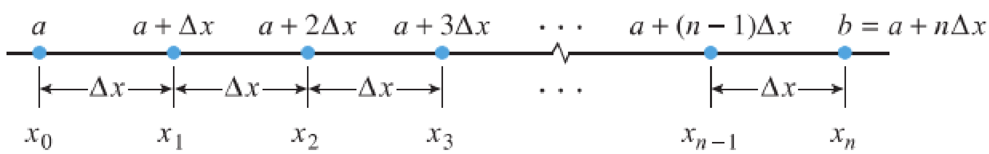


Figure 5.4.6

5.4.2 THEOREM

(a) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(b) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(c) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

5.4.4 THEOREM

(a) $$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1$$ (b) $$\lim_{n \to +\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$$

(c) $$\lim_{n \to +\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{3}$$ (d) $$\lim_{n \to +\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{1}{4}$$

▶ **Example 4** Use Definition 4.4.3 with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$.

Solution. The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so it follows from (4) that

$$x_k^* = a + k\Delta x = \frac{k}{n}$$

Thus,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (x_k^*)^2 \Delta x = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$ Part (*b*) of Theorem 4.4.2

$$= \frac{1}{6}\left(\frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}\right) = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

from which it follows that

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n \to +\infty} \left[\frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] = \frac{1}{3}$$

▶ Example 5 Use Definition 4.4.3 with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = f(x) = 9 - x^2$ and over the interval $[0, 3]$.

Solution. Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so it follows from (5) that

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$\begin{aligned}
f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x &= \left[9 - \left(k - \frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) \\
&= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right) \\
&= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}
\end{aligned}$$

from which it follows that

$$\begin{aligned}
A &= \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \lim_{n \to +\infty} \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}\right) \\
&= \lim_{n \to +\infty} 27\left[\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right] \\
&= 27\left[1 - \frac{1}{3} + 0 \cdot \frac{1}{2} - 0 \cdot 1\right] = 18 \qquad \boxed{\text{Theorem 4.4.4}}
\end{aligned}$$

◀

Question:

41–44 Use Definition 5.4.3 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval.

44. $f(x) = 4 - \frac{1}{4}x^2$; $[0, 3]$

Solution:

$\Delta x = \frac{3}{n}$, $x_k^* = (k-1)\frac{3}{n}$; $f(x_k^*)\Delta x = \left[4 - \frac{1}{4}(x_k^*)^2\right]\Delta x = \left[4 - \frac{1}{4}\frac{9(k-1)^2}{n^2}\right]\frac{3}{n} = \frac{12}{n} - \frac{27k^2}{4n^3} + \frac{27k}{2n^3} - \frac{27}{4n^3}$,

$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\frac{12}{n} - \frac{27}{4n^3}\sum_{k=1}^{n}k^2 + \frac{27}{2n^3}\sum_{k=1}^{n}k - \frac{27}{4n^3}\sum_{k=1}^{n}1 = 12 - \frac{27}{4n^3}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{27}{2n^3}\frac{n(n+1)}{2} - \frac{27}{4n^2} =$

$= 12 - \frac{9}{8}\frac{(n+1)(2n+1)}{n^2} + \frac{27}{4n} + \frac{27}{4n^2} - \frac{27}{4n^2}$,

$A = \lim_{n\to+\infty}\left[12 - \frac{9}{8}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] + 0 + 0 - 0 = 12 - \frac{9}{8}(1)(2) = 39/4.$

Question:

45–48 Use Definition 5.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y = f(x)$ over the specified interval.

46. $f(x) = 6 - x$; $[1, 5]$

Solution:

Endpoints $1, 1+\frac{4}{n}, 1+\frac{8}{n}, \ldots, 1+\frac{4(n-1)}{n}, 1+4 = 5$, and midpoints $1+\frac{2}{n}, 1+\frac{6}{n}, 1+\frac{10}{n}, \ldots, 1+\frac{4(n-1)-2}{n}, \frac{4n-2}{n}$. Approximate the area with the sum $\sum_{k=1}^{n}\left(6 - \left(1 + \frac{4k-2}{n}\right)\right)\frac{4}{n} = \sum_{k=1}^{n}\left(5\frac{4}{n} - \frac{16}{n^2}k + \frac{8}{n^2}\right) = 20 - \frac{16}{n^2}\frac{n(n+1)}{2} + \frac{8}{n} = 20 - 8 = 12$, which is exact, because f is linear.

EXERCISE SET 4.4

35–40 Use Definition 4.4.3 with x_k^* as the *right* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

35. $f(x) = x/2;\ [1, 4]$

36. $f(x) = 5 - x;\ [0, 5]$

37. $f(x) = 9 - x^2;\ [0, 3]$

38. $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

39. $f(x) = x^3;\ [2, 6]$

40. $f(x) = 1 - x^3;\ [-3, -1]$

41–44 Use Definition 4.4.3 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

41. $f(x) = x/2;\ [1, 4]$

42. $f(x) = 5 - x;\ [0, 5]$

43. $f(x) = 9 - x^2;\ [0, 3]$

44. $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

45–48 Use Definition 4.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

45. $f(x) = 2x;\ [0, 4]$

46. $f(x) = 6 - x;\ [1, 5]$

47. $f(x) = x^2;\ [0, 1]$

48. $f(x) = x^2;\ [-1, 1]$

SOLUTION SET

35. $\Delta x = \frac{3}{n}$, $x_k^* = 1 + \frac{3}{n}k$; $f(x_k^*)\Delta x = \frac{1}{2}x_k^*\Delta x = \frac{1}{2}\left(1 + \frac{3}{n}k\right)\frac{3}{n} = \frac{3}{2}\left[\frac{1}{n} + \frac{3}{n^2}k\right]$,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{3}{2}\left[\sum_{k=1}^{n}\frac{1}{n} + \sum_{k=1}^{n}\frac{3}{n^2}k\right] = \frac{3}{2}\left[1 + \frac{3}{n^2}\cdot\frac{1}{2}n(n+1)\right] = \frac{3}{2}\left[1 + \frac{3}{2}\frac{n+1}{n}\right],$$

$$A = \lim_{n\to+\infty}\frac{3}{2}\left[1 + \frac{3}{2}\left(1 + \frac{1}{n}\right)\right] = \frac{3}{2}\left(1 + \frac{3}{2}\right) = \frac{15}{4}.$$

37. $\Delta x = \frac{3}{n}$, $x_k^* = 0 + k\frac{3}{n}$; $f(x_k^*)\Delta x = \left(9 - 9\frac{k^2}{n^2}\right)\frac{3}{n}$,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(9 - 9\frac{k^2}{n^2}\right)\frac{3}{n} = \frac{27}{n}\sum_{k=1}^{n}\left(1 - \frac{k^2}{n^2}\right) = 27 - \frac{27}{n^3}\sum_{k=1}^{n}k^2,$$

$$A = \lim_{n\to+\infty}\left[27 - \frac{27}{n^3}\sum_{k=1}^{n}k^2\right] = 27 - 27\left(\frac{1}{3}\right) = 18.$$

39. $\Delta x = \frac{4}{n}$, $x_k^* = 2 + k\frac{4}{n}$; $f(x_k^*)\Delta x = (x_k^*)^3\Delta x = \left[2 + \frac{4}{n}k\right]^3\frac{4}{n} = \frac{32}{n}\left[1 + \frac{2}{n}k\right]^3 = \frac{32}{n}\left[1 + \frac{6}{n}k + \frac{12}{n^2}k^2 + \frac{8}{n^3}k^3\right]$,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{32}{n}\left[\sum_{k=1}^{n}1 + \frac{6}{n}\sum_{k=1}^{n}k + \frac{12}{n^2}\sum_{k=1}^{n}k^2 + \frac{8}{n^3}\sum_{k=1}^{n}k^3\right] =$$

$$= \frac{32}{n}\left[n + \frac{6}{n}\cdot\frac{1}{2}n(n+1) + \frac{12}{n^2}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{8}{n^3}\cdot\frac{1}{4}n^2(n+1)^2\right] =$$

$$= 32\left[1 + 3\frac{n+1}{n} + 2\frac{(n+1)(2n+1)}{n^2} + 2\frac{(n+1)^2}{n^2}\right],$$

$$A = \lim_{n\to+\infty}32\left[1 + 3\left(1 + \frac{1}{n}\right) + 2\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 2\left(1 + \frac{1}{n}\right)^2\right] = 32[1 + 3(1) + 2(1)(2) + 2(1)^2] = 320.$$

41. $\Delta x = \frac{3}{n}$, $x_k^* = 1 + (k-1)\frac{3}{n}$; $f(x_k^*)\Delta x = \frac{1}{2}x_k^*\Delta x = \frac{1}{2}\left[1 + (k-1)\frac{3}{n}\right]\frac{3}{n} = \frac{1}{2}\left[\frac{3}{n} + (k-1)\frac{9}{n^2}\right]$,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{2}\left[\sum_{k=1}^{n}\frac{3}{n} + \frac{9}{n^2}\sum_{k=1}^{n}(k-1)\right] = \frac{1}{2}\left[3 + \frac{9}{n^2}\cdot\frac{1}{2}(n-1)n\right] = \frac{3}{2} + \frac{9}{4}\frac{n-1}{n},$$

$$A = \lim_{n\to+\infty}\left[\frac{3}{2} + \frac{9}{4}\left(1 - \frac{1}{n}\right)\right] = \frac{3}{2} + \frac{9}{4} = \frac{15}{4}.$$

43. $\Delta x = \dfrac{3}{n}, x_k^* = 0 + (k-1)\dfrac{3}{n}; f(x_k^*)\Delta x = \left[9 - 9\dfrac{(k-1)^2}{n^2}\right]\dfrac{3}{n}$,

$\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[9 - 9\frac{(k-1)^2}{n^2}\right]\frac{3}{n} = \frac{27}{n}\sum_{k=1}^{n}\left(1 - \frac{(k-1)^2}{n^2}\right) = 27 - \frac{27}{n^3}\sum_{k=1}^{n} k^2 + \frac{54}{n^3}\sum_{k=1}^{n} k - \frac{27}{n^2}$,

$A = \displaystyle\lim_{n\to+\infty} = 27 - 27\left(\frac{1}{3}\right) + 0 + 0 = 18.$

45. Endpoints $0, \dfrac{4}{n}, \dfrac{8}{n}, \ldots, \dfrac{4(n-1)}{n}, \dfrac{4n}{n} = 4$, and midpoints $\dfrac{2}{n}, \dfrac{6}{n}, \dfrac{10}{n}, \ldots, \dfrac{4n-6}{n}, \dfrac{4n-2}{n}$. Approximate the area with the sum $\displaystyle\sum_{k=1}^{n} 2\left(\frac{4k-2}{n}\right)\frac{4}{n} = \frac{16}{n^2}\left[2\frac{n(n+1)}{2} - n\right] \to 16$ (exact) as $n \to +\infty$.

47. $\Delta x = \dfrac{1}{n}, x_k^* = \dfrac{2k-1}{2n}; f(x_k^*)\Delta x = \dfrac{(2k-1)^2}{(2n)^2}\dfrac{1}{n} = \dfrac{k^2}{n^3} - \dfrac{k}{n^3} + \dfrac{1}{4n^3}, \displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{n^3}\sum_{k=1}^{n} k^2 - \frac{1}{n^3}\sum_{k=1}^{n} k + \frac{1}{4n^3}\sum_{k=1}^{n} 1.$

Using Theorem 5.4.4, $A = \displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{3} + 0 + 0 = \frac{1}{3}.$