Assignment : 01

23K-2001

Question #1:

$$f(x) = \begin{cases} x^3+4, & x<1 \\ 7, & x=1 \\ x+6, & x>1 \end{cases}$$

i) $\lim\limits_{x\to1^+} : ?$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x+6)$$
$$= (1)+6$$
$$= 7 \qquad \text{Ans.}$$

ii) $\lim\limits_{x\to1^-} : ?$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^3+4)$$
$$= 1^3+4$$
$$= 5 \qquad \text{Ans.}$$

iii) $\lim\limits_{x\to1} : ?$

Since, $\lim\limits_{x\to1^+} f(x) \neq \lim\limits_{x\to1^-} f(x)$

$$7 \neq 5$$

$\lim\limits_{x\to1} f(x) = \text{DNE}$

Ans.

iv) $f(1)$:

$f(x) = 7$; $x = 1$

$\therefore f(1) = 7$ Ans.

Question #2:

$$f(x) = \begin{cases} -x^2 , & x < 2 \\ -x-1 , & x \geq 2 \end{cases}$$

i) $\lim_{x \to -2} f(x)$: ?

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} (-x^2)$$
$$= -(-2)^2$$
$$= -4$$ Ans.

ii) $\lim_{x \to 2} f(x)$: ?

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$
$$\lim_{x \to 2^-} (-x^2) = \lim_{x \to 2^+} (-x-1)$$
$$(-2^2) \neq (-2-1)$$
$$-4 \neq -3$$

Hence,

$$\lim_{x \to 2} f(x) = DNE$$ Ans.

iii) $\lim_{x \to 4} f(x)$: ?

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} (-x-1)$$
$$= -(4)-1$$
$$= -5$$ Ans.

iv) $f(-2)$: ? ; $x = -2$

$f(x) = -x^2$

$f(-2) = -(-2)^2$

$f(-2) = -4$ Ans.

v) $f(2)$: ? ; $x = 2$

$f(x) = -x-1$

$f(+2) = -2-1$

$f(2) = -3$ Ans.

vi) $f(4)$: ? ; $x = 4$

$f(x) = -x-1$

$f(4) = -4-1$

$f(4) = -5$ Ans.

Question # 3:

1. $\lim_{x\to 0^-} \frac{3x+4}{x^2}$; $\left(\frac{4}{0}\right)$

$\Rightarrow = \lim_{x\to 0^-} \frac{3x+4}{x^2}$

$= \frac{3(0)+4}{0^2}$

$= \frac{4}{0}$

$= +\infty$ Ans.

2. $\lim_{x\to 3^+} \frac{x^2-9}{\sqrt{x-3}}$

$\Rightarrow = \lim_{x\to 3^+} \frac{x^2-9}{\sqrt{x-3}} \times \frac{\sqrt{x+3}}{\sqrt{x+3}}$

$= \lim_{x\to 3^+} \frac{(x^2-9)\sqrt{x+3}}{(x^2-3^2)^{1/2}}$

$= \lim_{x\to 3^+} \sqrt{x^2-9}\sqrt{x+3}$

$= \sqrt{3^2-9}\sqrt{3+3}$

$= 0(\sqrt{6})$

$= 0$ Ans.

3. $\lim_{x\to -2} \frac{x^2-4}{x^2-x-6}$

$\Rightarrow = \lim_{x\to -2} \frac{(x-2)(x+2)}{(x^2-3x+2x-6)}$

$= \lim_{x\to -2} \frac{(x-2)(x+2)}{x(x-3)+2(x-3)}$

$= \lim_{x\to -2} \frac{(x-2)(x+2)}{(x+2)(x-3)}$

$= \lim_{x\to -2} \frac{(x-2)}{(x-3)}$

$= \frac{-2-2}{-2-3}$

$= \frac{-4}{-5}$

$= \frac{4}{5}$ Ans.

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4. $\lim_{t \to 1} \frac{t^3+t^2-5t+3}{t^3-3t+2}$

$\Rightarrow$ By synthetic division

$= \lim_{t \to 1} \frac{(t-1)(t^2+2t-3)}{(t-1)\cdot(t^2+t-2)}$

$= \lim_{t \to 1} \frac{t^2+2t-3}{t^2+t-2}$

$= \lim_{t \to 1} \frac{(t-1)(t+3)}{t^2+2t-t-2}$

$= \lim_{t \to 1} \frac{(t-1)(t+3)}{(t-1)(t+2)}$

$= \lim_{t \to 1} \frac{(t+3)}{(t+2)}$

$= \frac{1+3}{1+2}$

$= \frac{4}{3}$ Ans.

5. $\lim_{x \to 0} \frac{\sqrt{x+64}-8}{x}$

$\Rightarrow = \lim_{x \to 0} \frac{\sqrt{x+64}-8}{x} \times \frac{\sqrt{x+64}+8}{\sqrt{x+64}+8}$

$= \lim_{x \to 0} \frac{(x+64)-8^2}{x[(\sqrt{x+64})+8]}$

$= \lim_{x \to 0} \frac{x}{x(\sqrt{x+64}+8)}$

$= \lim_{x \to 0} \frac{1}{\sqrt{x+64}+8}$

$= \frac{1}{\sqrt{0+64}+8}$

$= \frac{1}{8+8}$

$= \frac{1}{16}$ Ans.

6. $\lim_{y \to 0} \dfrac{5y^3 + 8y^2}{3y^4 - 16y^2}$

$\Rightarrow = \lim_{y \to 0} \dfrac{y^2(5y+8)}{y^2(3y^2-16)}$

$= \lim_{y \to 0} \dfrac{5y+8}{3y^2-16}$

$= \dfrac{5(0)+8}{3(0)^2-16}$

$= \dfrac{-8}{16}$

$= -\dfrac{1}{2}$ Ans.

Question # 4:

1. $\lim_{x \to -4^-} f(x) = \text{DNE}$ Ans.
2. $\lim_{x \to -2^+} f(x) = 2$ Ans.
3. $\lim_{x \to -1} f(x) = \text{DNE}$ Ans.
4. $\lim_{x \to 3} f(x) = \text{DNE}$ Ans.

Question #5

1. $g(x) = \dfrac{\tan 3x}{(x+7)^4}$

$g(x) = \tan 3x \cdot (x+7)^{-4}$

$\dfrac{d}{dx} g(x) = \dfrac{d}{dx} (\tan 3x)(x+7)^{-4}$

$g'(x) = (x+7)^{-4} \dfrac{d}{dx} \tan 3x + \tan 3x \dfrac{d}{dx} (x+7)^{-4}$

$= (x+7)^{-4} \cdot \sec^2 3x \cdot (3) + \tan 3x \dfrac{(-4)}{(x+7)^5}$

$= \dfrac{3\sec^2 3x}{(x+7)^4} - \dfrac{4\tan 3x}{(x+7)^5}$

$g'(x) = \dfrac{3\sec^2 3x\,(x+7) - 4\tan 3x}{(x+7)^5}$ Ans.

2. $r = \tan\sqrt{\theta} \cdot \sec\left(\frac{1}{\theta}\right)$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\tan\sqrt{\theta}.\sec\frac{1}{\theta}.$$

$$= \tan\sqrt{\theta}\frac{d}{d\theta}\sec\frac{1}{\theta} + \sec\frac{1}{\theta}\frac{d}{d\theta}\tan\sqrt{\theta}$$

$$= \tan\sqrt{\theta}.\sec\frac{1}{\theta}\tan\frac{1}{\theta}\left(\frac{-1}{\theta^2}\right) + \sec\frac{1}{\theta}.\sec^2\sqrt{\theta}.\frac{1}{2\sqrt{\theta}}$$

$$= -\frac{1}{\theta^2}\tan\sqrt{\theta}\sec\frac{1}{\theta}.\tan\frac{1}{\theta} + \frac{1}{2\sqrt{\theta}}\sec\frac{1}{\theta}\sec^2\sqrt{\theta} \quad \text{Ans.}$$

3. $y = \left[\frac{t^2}{t^{3-4t}}\right]^3$

$$y = \left[t^{4t-1}\right]^3$$

$$y = t^{12t-3}$$

Take ln b/s

$$\ln y = \ln(t^{12t-3})$$

Differentiate:

$$\frac{1}{y}\frac{dy}{dt} = \frac{d}{dt}(12t-3)\ln t$$

$$\frac{1}{y}\frac{dy}{dt} = (12t-3)\frac{1}{t} + \ln t.(12)$$

$$\frac{dy}{dt} = \left[\frac{(12t-3)}{t} + 12\ln t\right]y$$

$$\frac{dy}{dt} = \frac{(12t-3+12t.\ln t)}{t}\, t^{12t-3}$$

$$= 3(4t-1+4t.\ln t).\, t^{12t-3-1}$$

$$= 3(4t+4t\ln t-1).\, t^{12t-4}$$

$$\frac{dy}{dt} = 3t^{12t-4}(4t+4t\ln t-1) \quad \text{Ans.}$$

4. $q = \tan\left(\frac{\cos t}{t}\right)$

$$\frac{d}{dt} q = \frac{d}{dt} \tan\left(\frac{\cos t}{t}\right)$$

$$= \sec^2\left(\frac{\cos t}{t}\right) \cdot \frac{d}{dt} \frac{\cos t}{t}$$

$$= \sec^2\left(\frac{\cos t}{t}\right) \cdot \left[\frac{t(-\sin t) - \cos t}{\cancel{1s}\ t^2}\right]$$

$$q' = -\sec^2\left(\frac{\cos t}{t}\right)\left[\frac{t \sin t + \cos t}{t^2}\right] \quad \text{Ans.}$$

5. $y = (t^{-3/4} \sin t)^{4/3}$

$$y = t^{-1} \cdot \sin^{4/3} t$$

$$y' = t^{-1} \cdot \frac{4}{3} \sin^{1/3} t \cdot (\cos t) + \sin^{4/3} t \cdot \left(\frac{-1}{t^2}\right)$$

$$= \frac{4}{3t} \sin^{1/3} t \cdot \cos t - \frac{1}{t^2} \sin^{4/3} t$$

$$y' = \frac{4t \cos t \sin^{1/3} t - 3 \sin^{4/3} t}{3t^2} \quad \text{Ans.}$$

6. $f(x) = [x^4 - \sec(4x^2-2)]^{-4}$

Differentiate b/s:

$$\frac{d}{dx} f(x) = \frac{d}{dx} [x^4 - \sec(4x^2-2)]^{-4}$$

$$f'(x) = -4[x^4 - \sec(4x^2-2)]^{-5} \cdot \frac{d}{dx}[x^4 - \sec(4x^2-2)]$$

$$f'(x) = \frac{-4}{(x^4 - \sec 4x^2 - \cancel{3}2)^5}\left(4x^3 - \sec(4x^2-2)\tan(4x^2-2) \cdot 8x\right)$$

$$= \frac{-4}{[x^4 - \sec(4x^2-2)]^5}\left(4x^3 - 8x\sec(4x^2-2)\tan(4x^2-2)\right)$$

$$f'(x) = \frac{-16x[x^2 - 2\sec(4x^2-2)\tan(4x^2-2)]}{[x^4 - \sec(4x^2-2)]^5} \quad \text{Ans}$$

Question #6

$$y''' + xy'' - 2y' = 0$$

Prove:

Given: $y = x^3 + 3x + 1$

$\Rightarrow \frac{d}{dx} y = \frac{d}{dx}(x^3 + 3x + 1)$

$\frac{dy}{dx} = 3x^2 + 3$

$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}(3x^2 + 3)$

$\frac{d^2y}{dx^2} = 6x$

$\frac{d}{dx}\frac{d^2y}{dx} = \frac{d}{dx} 6x$

$\frac{d^3y}{dx^3} = 6$

$\Rightarrow y''' + xy'' - 2y' = 0$

$(6) + (x)(6x) - 2(3x^2 + 3) = 0$

$6 + 6x^2 - 6x^2 - 6 = 0$

$0 = 0$

L.H.S = R.H.S

Hence Proved

Question #7:

1. $5y^2 = x^2y + \frac{2}{xy^2}$

Differentiate b/s

$5\frac{d}{dx} y^2 = \frac{d}{dx} x^2y + 2\frac{d}{dx}\frac{1}{xy^2}$

$10y\frac{dy}{dx} = \left(x^2\frac{dy}{dx} + 2xy\right) + 2\frac{d}{dx} x^{-1}y^{-2}$

$10y\frac{dy}{dx} = x^2\frac{dy}{dx} + 2xy + 2\left[x^{-1}\frac{(-2)}{y^3}\frac{dy}{dx} + y^{-2}\left(\frac{-1}{x^2}\right)\right]$

$10y\frac{dy}{dx} = x^2\frac{dy}{dx} + 2xy - \frac{4}{xy^3}\frac{dy}{dx} - \frac{2}{x^2y^2}$

$\frac{dy}{dx}\left(10y + \frac{4}{xy^3} - x^2\right) = 2xy - \frac{2}{x^2y^2}$

$$= \frac{2xy - \frac{2}{x^2y^2}}{10y + \frac{4}{xy^3} - x^2}$$

$$= \frac{\frac{2x^3y^3 - 2}{x^2y^2}}{\frac{10xy^4 + 4 - x^3y^3}{xy^3}}$$

$$\frac{dy}{dx} = \frac{y(2x^3y^3 - 2)}{x(10xy^4 + 4 - x^3y^3)}$$ Ans.

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2. $x^{3/2} + y^{3/2} = 2$

$$\frac{d}{dx}\left(x^{3/2} + y^{3/2}\right) = \frac{d}{dx} 2$$

$$\frac{3}{2} x^{1/2} + \frac{3}{2} y^{1/2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3}{2} x^{1/2} \times \frac{2}{3y^{1/2}}$$

$$\frac{dy}{dx} = -\frac{x^{1/2}}{y^{1/2}}$$

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{1/2} \quad \text{Ans.}$$

3. $x^5 + 3x^2y^3 + 3x^3y^2 + y^5 = 8$

$$\frac{d}{dx}\left(x^5 + 3x^2y^3 + 3x^3y^2 + y^5\right) = \frac{d}{dx} 8$$

$$5x^4 + 3\left(x^2 . 3y^2\frac{dy}{dx} + y^3 . 2x\right) + 3\left(x^3 . 2y\frac{dy}{dx} + y^2 . 3x^2\right) + 5y^4\frac{dy}{dx} = 0$$

$$5x^4 + 9x^2y^2\frac{dy}{dx} + 6xy^3 + 6x^3y\frac{dy}{dx} + 9x^2y^2 + 5y^4\frac{dy}{dx} = 0$$

$$(9x^2y^2 + 6x^3y + 5y^4)\frac{dy}{dx} = -5x^4 - 6xy^3 - 9x^2y^2$$

$$\frac{dy}{dx} = \frac{-5x^4 - 6xy^3 - 9x^2y^2}{9x^2y^2 + 6x^3y + 5y^4}$$

$$\frac{dy}{dx} = \frac{-x\,(5x^3 + 6y^3 + 9xy^2)}{y\,(9x^2y + 6x^3 + 5y^3)} \quad \text{Ans.}$$

Question #8:

1. $y = x^3$, $x = \frac{1}{\sqrt{t^2+5}}$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 \quad , \quad \frac{dx}{dt} = \frac{d}{dt}\frac{1}{\sqrt{t^2+5}}$$

$$\frac{dy}{dx} = 3x^2 \qquad = \frac{-1\,(2t)}{2\,(t^2+5)^{+3/2}}$$

$$\frac{dx}{dt} = \frac{-t}{(t^2+5)^{3/2}}$$

$$\because \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)).g'(x)$$

$$= 3x^2 . \frac{-t}{(t^2+5)^{3/2}}$$

$$= -3\left(\frac{1}{\sqrt{t^2+5}}\right)^2 \frac{t}{(t^2+5)^{3/2}}$$

$$\cancel{= \frac{-3t}{t^2+5t}}$$

$$f'(x) = \frac{-3t}{(t^2+5)^{5/2}} \quad \text{Ans.}$$

2. $y = x^2 + 3x + 2$, $x = \frac{t-1}{t+1}$

$\frac{dy}{dx} = 2x + 3$

$$, \frac{dx}{dt} = \frac{(t+1)(1) - (t-1)(1)}{(t+1)^2}$$

$$\frac{dx}{dt} = \frac{t+1-t+1}{(t+1)^2}$$

$$\frac{dx}{dt} = \frac{2}{(t+1)^2}$$

$$f'(x) = (2x+3) . \frac{2}{(t+1)^2}$$

$$= \left[2\left(\frac{t-1}{t+1}\right) + 3\right] \frac{2}{(t+1)^2}$$

$$= \left[4 \frac{(t-1)}{(t+1)} + 6\right] \times \frac{1}{(t+1)^2}$$

$$= \frac{4t - 4 + 6(t+1)}{(t+1)^3}$$

$$= \frac{4t - 4 + 6t + 6}{(t+1)^3}$$

$$= \frac{10t + 2}{(t+1)^3}$$

$$= \frac{2(5t+1)}{(t+1)^3}$$ Ans.

3. $f(u) = \frac{5}{\left(u + \frac{1}{\sqrt{u}}\right)^4}$

$g = \frac{5}{x^4}$, $K = u + \frac{1}{\sqrt{u}}$

$g' = 5 \frac{d}{dx} x^{-4}$ $\qquad \frac{dx}{du} = 1 + \left(\frac{-1}{2} u^{-3/2}\right)$

$g' = \frac{-20}{x^5}$ $\qquad \frac{dx}{du} = 1 - \frac{1}{2u^{3/2}}$

$$f'(x) = \frac{-20}{x^5}\left(1 - \frac{1}{2u^{3/2}}\right)$$

$$= \frac{-20}{x^5} + \frac{20}{2u^{3/2} . x^5}$$

$$= \frac{-20}{\left(u + \frac{1}{\sqrt{u}}\right)^5} + \frac{20}{2u^{3/2}\left(u + \frac{1}{\sqrt{u}}\right)^5}$$

$$f'(u) = \frac{-20u^{3/2} + 10}{\left(u + \frac{1}{\sqrt{u}}\right)^5}$$ Ans.