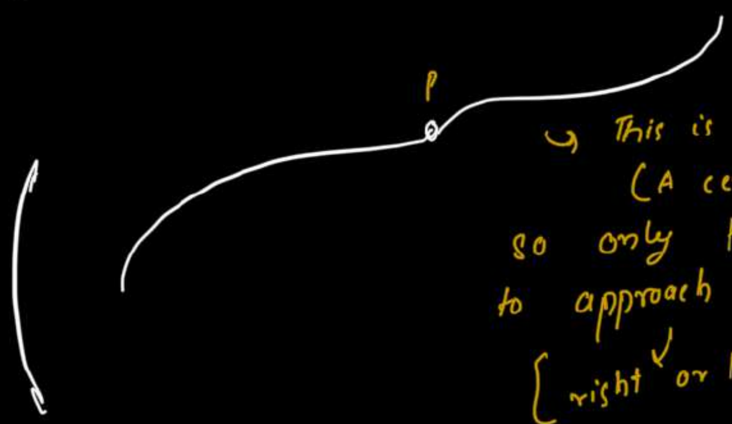
Limits & continuity (MVC)

$\lim_{x\to a} f(x) = L$ → 1-var: Limit



↳ This is a curve (A certain path) so only two directions to approach to point [P] [right or left]

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

But.....

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

→ For 2 var: we have a surface : There are ∞ paths along the surface that approach our point.

→ To prove a limit exist, we must prove that along all paths we approach to a same point

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< Notes

a same p

↳ can use "SQUEEZE THEORM"

↳ we can prove A limit [DNE] by showing that along a paths,
↳ we get a different value as approach the same point

eg $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{2x^2+y^2}$

Sol:

Along $x=0$ ← Travelling along The surface Directly over/under "Y-Axis"

$\lim\limits_{y\to 0} \dfrac{0^2-y^2}{2(0)^2+y^2}$

$\lim\limits_{y\to 0} \dfrac{-\cancel{y^2}}{\cancel{y^2}}$

$= -1$

Along $y=0$.

$\lim\limits_{x\to 0} \dfrac{x^2-(0)^2}{2x^2+0^2}$

$\lim\limits_{x\to 0} \dfrac{\cancel{x^2}}{2\cancel{x^2}}$

$= \frac{1}{2}$

$\boxed{-1 \neq \frac{1}{2}}$ → $\boxed{\text{Limit DNE}}$

$\frac{\#N}{=}$

⇒ when solving limits
first i) Try $x=0$, $y=0$ paths 1st.
2) If don't work, choose other paths

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Notes

a) Be certain about the point (a, b) is actually on your path.

b) Try to substitute, so Degrees of Numerator & Denominator becomes equal.

c) Always use either $x=0$ or $y=0$ as 1 path

Eg. $\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2}$ ⟹ prove DNE

Sol

along $x=0$

$$\lim_{y\to 0} \frac{0}{y^2} = 0$$

along $y=0$

$$\lim_{x\to 0} \frac{0}{3x^2} = 0$$

along $y=x$.

$$\lim_{x\to 0} \frac{3x(x)}{3x^2+(x)^2}$$

$$\lim_{x\to 0} \frac{3\cancel{x^2}}{4\cancel{x^2}} = 3/4$$

Since $0 \neq 3/4$ → Hence Limit DNE

Eg. $\lim_{(x,y)\to(0,0)} \frac{xy^3\cos x}{2x^2+y^6}$ → Prove DNE

Eg. $\lim_{(x,y)\to(0,0)} \frac{xy^3\cos x}{2x^2+y^6}$ → Prove DNE

Sol

along $x=0$

$\lim_{y\to 0} \frac{0}{y^6} = 0$

along $y=0$

$\lim_{x\to 0} \frac{0}{2x^2} = 0$

along $x=y^3$

$$\lim_{y\to 0} \frac{(y^3)(y^3)\cos y^3}{2\cancel{y}^6+y^6}$$

$$\lim_{y\to 0} \frac{\cancel{y^6}\cos y^3}{3\cancel{y^6}}$$

$$\lim_{y\to 0} \frac{\cos y^3}{3} = \frac{1}{3} \neq 0$$

A·L

Hence Limit DNE

Eg. $\lim_{(x,y)\to(1,0)} \frac{2xy-2y}{x^2+y^2-2x+1}$

— along $\boxed{x=0}$ Not possible b/c it doesn't pass through our point (a,b)

Sol

along $y=0$

↳ $\lim_{x\to 1} \frac{0}{x^2-2x+1} = 0$

along $x=1$

$\lim_{y\to 0} \frac{2y-2y}{y^2} = 0$

⇒ a simple trick: Try to form factor & decide the substitution

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Notes

along $\boxed{x=0}$ b/c it doesn't pass through our point (a,b)

Eg. $\lim_{(x,y)\to(1,0)} \dfrac{2xy-2y}{x^2+y^2-2x+1}$

Sol:

along $y=0$

$\hookrightarrow \lim_{x\to 1} \dfrac{0}{x^2-2x+1} = 0$

along $x=1$

$\lim_{y\to 0} \dfrac{2y-2y}{y^2} = 0$

$\Rightarrow$ a simple trick: Try to form factor & decide the substitution

$\hookrightarrow$

$$\lim_{(x,y)\to(1,0)} \frac{2y(x-1)}{y^2+(x-1)^2}$$

let $y = x-1$

$$\lim_{y\to 0} \frac{2y(y)}{y^2+(y)^2}$$

$$\lim_{y\to 0} \frac{2y^2}{y^2+y^2}$$

$$\lim_{y\to 0} \frac{2\cancel{y^2}}{2\cancel{y^2}}$$

$$= \boxed{1}$$

$\boxed{0 \neq 1}$

Hence Limit DNE

⑥ we define curves in 3D with the help of parametric (t)

... 2 variable ... Paths are now

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< Notes

⇒ for 3-variable, Paths are now parametric "t"

⇒ always choose an "Axis" is 1st path

Eg Show $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$

Sol

Along x-axis (y=0, z=0)

$\lim_{x\to 0} \frac{0}{x^2} = 0$

Set all of your variables equal to a function of t [parameter] such that your degrees match up

Along C → a curve → a line through space.

C: $x=t,\ y=t,\ z=t$

$\lim_{t\to 0} \frac{t\cdot t + t\cdot t + t\cdot t}{t^2+t^2+t^2}$

$\lim_{t\to 0} \frac{\cancel{t^2+t^2+t^2}}{\cancel{t^2+t^2+t^2}}$

$\boxed{= 1}$

letting $t\to 0$ means that all parameters of x, y, z all tends to zero

$\boxed{0 \neq 1}$ → Hence limit DNE.

Eg: $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xz^2+zy^2}{x^2+2y^2+z^4}$

Sol

along x-axis

$\lim\limits_{x\to 0} \dfrac{0}{x^2} = 0$

Along C

$x = t^2, \quad y = t^2, \quad z = t$

$$\lim_{t\to 0} \frac{t^2\cdot(t)^2 + t\cdot(t^2)^2}{(t^2)^2 + 2(t^2)^2 + (t)^4}$$

$$\lim_{t\to 0} \frac{t^4 + t^5}{t^4 + 2t^4 + t^4} = \lim_{t\to 0} \frac{t^4+t^5}{4t^4}$$

$$\lim_{t\to 0} \frac{\cancel{t^4}(1+t)}{4\cancel{t^4}} \Rightarrow \text{A.C} \Rightarrow \frac{1+0}{4} = 1/4$$

$\boxed{0 \neq 1/4} \Rightarrow$ limit DNE

∴ try the highest power variable equalling to t & match the other variable to nth power

Eg. $\lim\limits_{(x,y)\to(1,-2)} \dfrac{3xy}{2x^2-y^2}$

Sol
1st plug in

$\lim\limits_{(x,y)\to(1,-2)} \dfrac{3xy}{2x^2-y^2} = 3$

Q) $\lim\limits_{(x,y,z)\to(0,3,1)} \left[e^{\sin(\pi x)} + \ln\left[\cos(\pi(y-z))\right] \right]$

Sol

$= e^0 + \ln\left[\cos(\pi(3-1))\right]$

$e^0 + \ln\left[\cos(2\pi)\right]$

$1 + \ln(1)$

$1+0 = 1$

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Notes

Eg $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

$\Rightarrow$ $x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2+y^2$

Sol

$$\lim_{r\to 0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2}$$

$$\lim_{r\to 0^+} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2}$$

$$\lim_{r\to 0^+} r(\cos^3\theta + \sin^3\theta)$$

$$\boxed{= 0}$$

$\therefore$ we approach to $r\to 0^+$ b/c from $x^2+y^2 \to$ there always will be a +ve answer

Hence limit exist

b/c we just plug the number ie: 0 which results ina constant number, Hence

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< Notes

Q) $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = (0\cdot(-\infty)) = I.f$

$x^2+y^2=r^2$

Sol:

$\lim_{r\to 0^+} r^2 \cdot \ln(r^2)$

$\lim_{r\to 0^+} \dfrac{\ln r^2}{1/r^2} \Rightarrow$ [Try : L.HoP]

$\lim_{r\to 0^+} \dfrac{2\ln r}{1/r^2}$

Exist

$= \dfrac{2}{r} \div \dfrac{-1}{r^3}$

$= -\dfrac{2}{\cancel{r}} \times r^{\cancel{3}\,2}$

$= -2r^2$ A.L = $\boxed{0}$

Continuity.

↳ A function is continuous at any point on the region for which it is $\underbrace{\text{defined}}_{\text{(Domain)}}$

⇒ polynomial $f(x)$ → Cont. Everywhere

⇒ Rational func: → " " → Denom $\neq 0$

⇒ Continuity Holds for compositions.

eg.

$$F(x, y) = \frac{x^3 + xy + y^3}{x^2 + y^2 + 1}$$

Sol:

$f(x)$ will be continuous on all ordered pairs (x, y)

Q) $$f(x, y, z) = \frac{xyz}{x^2 + y^2 + z^2 - 4}$$

Sol:

$x^2 + y^2 + z^2 - 4 \neq 0$

$x^2 + y^2 + z^2 \neq 4$

Cont : ON

$\{(x, y, z) \mid x^2 + y^2 + z^2 \neq 4\}$

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Notes

Q] $f(x,y) = x^2 + xy + y^2$, $g(t) = t\cos t + \sin t$

Sol

↳ cont on all (x,y)

↳ cont on all "t"

$h(x,y) = g[f(x,y)]$

↳ cont on all (x,y)

Q) $f(x,y) = x - 2y + 3$, $g(t) = \sqrt{t} + \frac{1}{\sqrt{t}}$

Sol

↳ cont on all x

↳ cont on $\{t \mid t > 0\}$

$h(x,y) = g(f(x,y))$

if $0 < t$

& $t = x - 2y + 3$

so

$0 < x - 2y + 3 \to x - 2y > -3$

cont on all $\{(x,y) \mid x - 2y > 3$

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Notes

Q) $f(x,y) = x \tan y$, $g(t) = \cos t$

↳ cont on all "t"

↳ const on

$\{(x,y) \mid y \neq (\pi/2 + k\pi)\}$

$h(x,y) = g(f(x,y))$

↳ cont ON $\{(x,y) \mid y \neq (\pi/2 + k\pi)\}$

Q) $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$ ⟹ we can prove limit exist by using polar method or by squeeze theorm

Sol

for squeeze theorm we take absolute value of our function & set the lower & upper boundry if possible and then if the limits of our upper boundry function and lower boundry function is same then the the limit of that actuall function will be that too .... and exists.

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Notes

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$$

sol

$$\frac{5x^2y}{x^2+y^2} \Rightarrow \left|\frac{5x^2y}{x^2+y^2}\right| \Rightarrow \frac{5x^2|y|}{x^2+y^2} \leq 5|y|$$

$$0 \leq \frac{5x^2|y|}{x^2+y^2} \leq 5|y|$$

$$\lim_{(x,y)\to(0,0)} 0 = 0 \qquad \lim_{(x,y)\to(0,0)} 5|y| = 0$$

$$\boxed{\text{limit} = 0}$$

Exist