National University of Computer & Emerging Sciences
Karachi Campus

Multivariable Calculus (MT2008)

Sessional-II Exam

Date: April 5th, 2024
Time: 8:30 am - 9:30 am
Course Instructor(s)
Dr. Fahad Riaz, Dr. Nazish Kanwal
Ms. Fareeha Sultan, MS. Alishba, & Ms. Uzma.

Total Time: 1 Hour
Total Marks: 30
Total Questions: 04

Student Name | **Roll No** | **Section** | **Student Signature**

Attempt all questions. There are 4 questions and 1 pages.

CLO #1: Understand the basic concepts and know the basic techniques of differential and integral calculusof functions of several variables.

Question 1 **[9 marks]**

(a) Let $f(x, y) = y\cos(2x) - \sin(2x)$.

i. $\boxed{\text{3 points}}$ Find the direction derivative of f at $(0, 0)$ in the direction $i - j$.

ii. $\boxed{\text{2 points}}$ What is the value of the largest directional derivative of f at $(0, 0)$.

Solution:

(a) We have

$$\nabla f = \langle -2y\sin(2x) - 2\cos(2x), \cos(2x)\rangle$$

so

$$\nabla f(0,0) = \langle -2, 1\rangle .$$

The directional derivative is then given by

$$D_{\langle 1,-1\rangle} f(0,0) = \frac{\langle -2,1\rangle \cdot \langle 1,-1\rangle}{|\langle 1,-1\rangle|} = \frac{-1}{\sqrt{2}}$$

(b) The value of the largest directional derivative is

$$|\nabla f(0,0)| = \sqrt{5}.$$

(b) $\boxed{\text{4 points}}$ Find an equation for the tangent plane and parametric equations for the normal line to the surface $x^2y^3z^4 + xyz = 2$ at the point $(2, 1, -1)$.

Solution:

b) $x^2y^3z^4 + xyz = 2$; $P_0(2,1,-1)$.

$F_x(P_0)(x-x_0) + F_y(P_0)(y-y_0) + F_z(P_0)(z-z_0) = 0$

$F_x = 2xy^3z^4 + yz$, $F_x(P_0) = 3$

$F_y = 3x^2y^2z^4 + xz$, $F_y(P_0) = 10$

$F_z = 4x^2y^3z^3 + xy$, $F_z(P_0) = -14$

$3(x-2) + 10(y-1) - 14(z+1) = 0$

$3x + 10y - 14z = 30.$

normal line : $x(t) = 2 + 3t$

$y(t) = 1 + 10t$

$z(t) = -1 - 14t.$

CLO #2: Apply the theory to calculate the gradients, directional derivatives, arc length of curves, area of surfaces, and volume of solids. .

Question 2 **[6 marks]**

(a) 4 points Evaluate the double integral over the rectangular region R.

$$\int\int_R \frac{xy}{x^2+1}\,dA; \quad R = \{(x,y) : 0 \le x \le 1,\ -3 \le y \le 3\}.$$

Solution:

$$\iint_R \frac{xy^2}{x+1}\,dA = \int_{-3}^{3}\int_0^1 y^2\left(\frac{x}{x^2+1}\right)dx\,dy$$

$$= \int_{-3}^{3} y^2\left[\frac{1}{2}\ln(x^2+1)\right]_0^1 dy$$

$$= \int_{-3}^{3} \frac{1}{2}(\ln 2 - \ln 1)\, y^2\, dy$$

$$= \frac{\ln 2}{2}\left[\frac{y^3}{3}\right]_{-3}^{3}$$

$$= \frac{\ln 2}{2}\left(\frac{27+27}{3}\right)$$

$$= 9\ln 2$$

(b) 2 points Write a formula to find the volume of the solid enclosed between the surface $z = \frac{x}{y}$ and the rectangular region $R : 0 \le x \le 2,\ 1 \le y \le e^2$.

Solution:

$$V = \int_0^2\int_1^{e^2} \frac{x}{y}\,dydx \quad \text{or } V = \int_1^{e^2}\int_1^2 \frac{x}{y}\,dxdy$$

CLO #3: Solve problems involving maxima and minima, line integral and surface integral, and vector calculus.

Question 3 **[5 marks]**

Compute the local minima of the given function by using **gradient descent algorithm** by taking **step size as 0.15** and initial point as **(2,2)**. Perform **three** iterations.

$$f(x, y) = 4x^2 + 2.5y^2 + 3xy - 5.5x - 4.1y$$

Solution:

$f(x,y) = 4x^2 + 2.5y^2 + 3xy - 5.5x - 4.1y$					fx=8x1+3x2-5.5		
x0=(2.2)		alpha=0.15			fy=5x2+3x1-4		
S.No.	x1	x2	fx	fy	x1	x2	f(x1,x2)
					2	2	19
1	2	2	16.5	12	-0.475	0.2	2.53
2	-0.475	0.2	-8.7	-4.425	0.83	0.86375	-1.248502344
3	0.83	0.86375	3.73125	2.80875	0.2703125	0.4424375	-2.116026846
4	0.2703125	0.4424375	-2.0101875	-0.976875	0.57184062	0.58896875	-2.31539239
5	0.57184062	0.58896875	0.84163125	0.66036562	0.44559593	0.48991390	-2.361260295
6	0.44559593	0.48991390	-0.46549078	-0.21364265	0.51541955	0.52196030	-2.371827458
7	0.51541955	0.52196030	0.18923735	0.15606018	0.48703395	0.49855127	-2.374265927
8	0.48703395	0.49855127	-0.10807455	-0.04614176	0.50324513	0.50547254	-2.374829727
9	0.50324513	0.50547254	0.04237870	0.03709810	0.49688832	0.49990782	-2.374960388
10	0.49688832	0.49990782	-0.02516988	-0.00979588	0.50066381	0.50137720	-2.374990753
11	0.50066381	0.50137720	0.00944212	0.00887747	0.49924749	0.50004558	-2.374997833
12	0.49924749	0.50004558	-0.00588329	-0.00202958	0.50012998	0.50035002	-2.37499949

CLO #3: Solve problems involving maxima and minima, line integral and surface integral, and vector calculus.

Question 4 **[10 marks]**

(a) $\boxed{\text{5 points}}$ Given the three points $P_1(1, 4)$, $P_2(5, 2)$, and $P_3(3, -2)$. Let

$$G(x, y) = (x - 1)^2 + (y - 4)^2 + (x - 5)^2 + (y - 2)^2 + (x - 3)^2 + (y + 2)^2$$

is the sum of the squares of the distances from point $P(x, y)$ to the three points $(P_1, P_2, \&P_3)$. Find the values of x and y so that this $G(x, y)$ is minimized.

Solution:

$$G_x = 2(x - 1) + 2(x - 5) + 2(x - 3) = 6x - 18 = 0 \Rightarrow x = 3$$

$$G_y = 2(y - 4) + 2(y - 2) + 2(y + 2) = 6y - 8 = 0 \Rightarrow y = \frac{4}{3}$$

so critical point is $(3, \frac{4}{3})$.

$$G_{xx} = 6,\ G_{yy} = 6,\ G_{xy} = 0 \Rightarrow D = G_{xx}G_{yy} - \{G_{xy}\}^2 = 36 > 0$$

and $G_{xx} = 6 > 0 \Rightarrow (3, \frac{4}{3})$ is minimum point and $G(3, \frac{4}{3}) = \frac{80}{3}$ is the minimum value of G.

(b) $\boxed{\text{5 points}}$ Use **Lagrange multipliers** to find the maximum and minimum values of the function subject to the given constraint. Also find the points at which these values occurs.

$$f(x, y) = x^2 + y^2; \quad xy = 1.$$

Solution: We have $\bigtriangledown f = \lambda \bigtriangledown g \Rightarrow 2xi + 2yj = \lambda(yi + xj)$
$\Rightarrow 2x = \lambda y \Rightarrow \lambda = \frac{2x}{y}$ and $2y = \lambda x \Rightarrow \lambda = \frac{2y}{x}$. On equating these two equations, we get $x^2 = y^2 \Rightarrow x = \pm y$. Then $g(x, y)$ gives $y^2 = 1 \Rightarrow y = \pm 1$ and $x = \pm 1$, and $f(1, 1) = 2 = f(-1, -1)$ is the minimum value. No maximum value as function is the sum of the squares of x and y.