Partial Derivatives

Exercise Set 13.1

1. **(a)** $f(2,1) = (2)^2(1) + 1 = 5.$ **(b)** $f(1,2) = (1)^2(2) + 1 = 3.$ **(c)** $f(0,0) = (0)^2(0) + 1 = 1.$

(d) $f(1,-3) = (1)^2(-3) + 1 = -2.$ **(e)** $f(3a,a) = (3a)^2(a) + 1 = 9a^3 + 1.$

(f) $f(ab,\, a-b) = (ab)^2(a-b) + 1 = a^3b^2 - a^2b^3 + 1.$

2. **(a)** $2t$ **(b)** $2x$ **(c)** $2y^2 + 2y$

3. **(a)** $f(x+y, x-y) = (x+y)(x-y) + 3 = x^2 - y^2 + 3.$ **(b)** $f\left(xy, 3x^2y^3\right) = (xy)\left(3x^2y^3\right) + 3 = 3x^3y^4 + 3.$

4. **(a)** $(x/y)\sin(x/y)$ **(b)** $xy\sin(xy)$ **(c)** $(x-y)\sin(x-y)$

5. $F(g(x), h(y)) = F\left(x^3, 3y+1\right) = x^3 e^{x^3(3y+1)}.$

6. $g(u(x,y), v(x,y)) = g\left(x^2y^3, \pi xy\right) = \pi xy \sin\left[\left(x^2y^3\right)^2(\pi xy)\right] = \pi xy \sin\left(\pi x^5 y^7\right).$

7. **(a)** $t^2 + 3t^{10}$ **(b)** 0 **(c)** 3076

8. $\sqrt{t}e^{-3\ln\left(t^2+1\right)} = \dfrac{\sqrt{t}}{\left(t^2+1\right)^3}.$

9. **(a)** 2.50 mg/L. **(b)** $C(100,t) = 20(e^{-0.2t} - e^{-t}).$ **(c)** $C(x,1) = 0.2x(e^{-0.2} - e^{-1}).$

10. **(a)** $e^{-0.2t} - e^{-t} = e^{-0.1} - e^{-0.5}$ at $t \approx 6.007$, the medication remains effective for 5 and a half hours longer.

(b) The maximum concentration is about 10.6998 mg/L, at time $t \approx 2.0118$ hours.

11. **(a)** $v = 7$ lies between $v = 5$ and $v = 15$, and $7 = 5 + 2 = 5 + \dfrac{2}{10}(15-5)$, so $WCI \approx 19 + \dfrac{2}{10}(13-19) = 19 - 1.2 = 17.8°\text{F}$.

(b) $T = 28$ lies between $T = 25$ and $T = 30$, and $28 = 25 + \dfrac{3}{5}(30-25)$, so $WCI \approx 19 + \dfrac{3}{5}(25-19) = 19 + 3.6 = 22.6°\text{F}$.

12. **(a)** At $T = 35$, $14 = 5 + 9 = 5 + \dfrac{9}{10}(15-5)$, so $WCI \approx 31 + \dfrac{9}{10}(25-31) = 25.6°\text{F}$.

(b) At $v = 15$, $32 = 30 + \dfrac{2}{5}(35-30)$, so $WCI \approx 19 + \dfrac{2}{5}(25-19) = 21.4°\text{F}$.

13. **(a)** At $v = 25$, $WCI = 16$, so $T = 30°\text{F}$.

(b) At $v = 25$, $WCI = 6 = 3 + \dfrac{1}{2}(9-3)$, so $T \approx 20 + \dfrac{1}{2}(25-20) = 22.5°\text{F}$.

14. (a) At $T = 25$, $WCI = 7$, so $v = 35$ mi/h.

(b) At $T = 30$, $WCI = 15 = 16 + \dfrac{1}{2}(14-16)$, so $v \approx 25 + \dfrac{1}{2}(35-25) = 30$ mi/h.

15. (a) The depression is $20 - 16 = 4$, so the relative humidity is 66%.

(b) The relative humidity $\approx 77 - (1/2)7 = 73.5\%$.

(c) The relative humidity $\approx 59 + (2/5)4 = 60.6\%$.

16. (a) $4°$ C.

(b) The relative humidity $\approx 62 - (1/4)9 = 59.75\%$.

(c) The relative humidity $\approx 77 + (1/5)(79-77) = 77.4\%$.

17. (a) 19 **(b)** -9 **(c)** 3 **(d)** $a^6 + 3$ **(e)** $-t^8 + 3$ **(f)** $(a+b)(a-b)^2 b^3 + 3$

18. (a) $x^2(x+y)(x-y) + (x+y) = x^2\left(x^2 - y^2\right) + (x+y) = x^4 - x^2y^2 + x + y$.

(b) $(xz)(xy)(y/x) + xy = xy^2z + xy$.

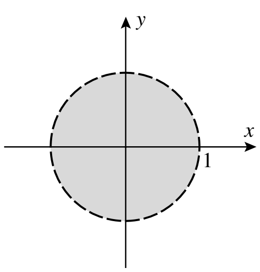
19. $F\left(x^2, y+1, z^2\right) = (y+1)e^{x^2(y+1)z^2}$.

20. $g\left(x^2z^3, \pi xyz, xy/z\right) = (xy/z)\sin\left(\pi x^3yz^4\right)$.

21. (a) $f(\sqrt{5}, 2, \pi, -3\pi) = 80\sqrt{\pi}$. **(b)** $f(1,1,\ldots,1) = \displaystyle\sum_{k=1}^{n} k = n(n+1)/2$.

22. (a) $f(-2,2,0,\pi/4) = 1$. **(b)** $f(1,2,\ldots,n) = n(n+1)(2n+1)/6$, see Theorem 5.4.2(b), Section 5.4.

23.



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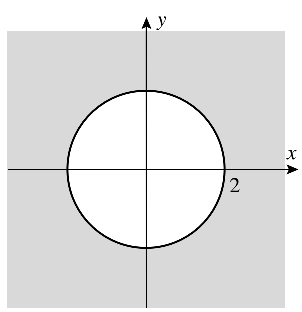

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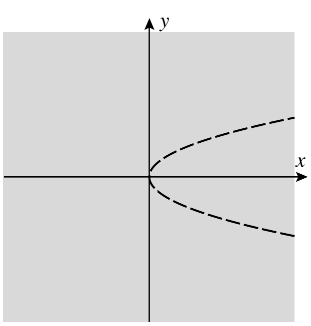

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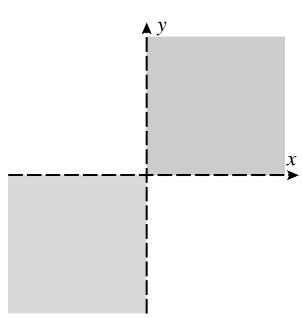


27. **(a)** All points in 2-space above or on the line $y = -2$.

(b) All points in 3-space on or within the sphere $x^2 + y^2 + z^2 = 25$.

(c) All points in 3-space.

28. **(a)** All points in 2-space on or between the vertical lines $x = \pm 2$.

(b) All points in 2-space above the line $y = 2x$.

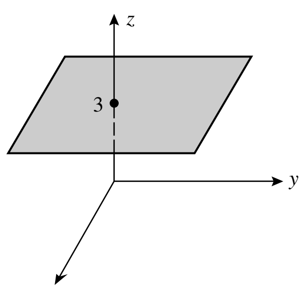
(c) All points in 3-space not on the plane $x + y + z = 0$.

29. True; it is the intersection of the domain $[-1, 1]$ of $\sin^{-1} t$ and the domain $[0, +\infty)$ of $\sqrt{t}$.

30. False, the origin is not in the domain of the function.

31. False; z has no constraints so the domain is an infinite solid circular cylinder.

32. True; $f(x, y, z) = D$ yields the plane with normal vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and which passes through $(D, 0, 0)$.



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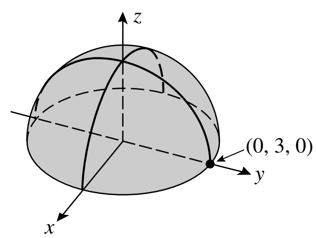

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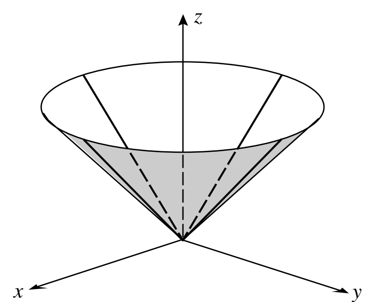

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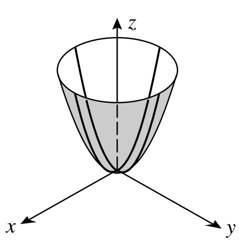

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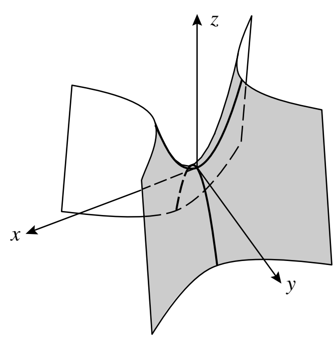

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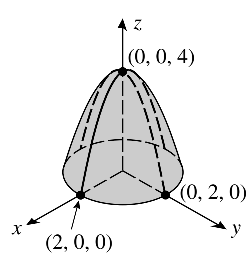

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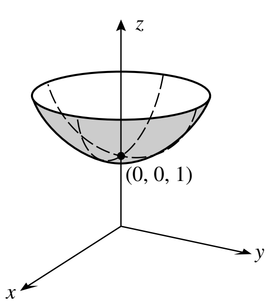

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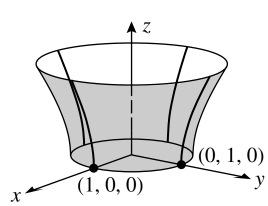

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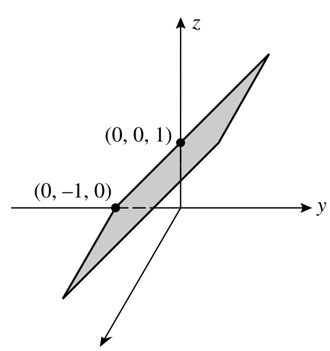

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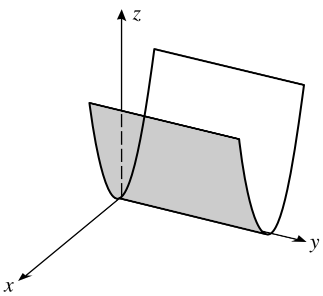
42. 

43. **(a)** Hyperbolas. **(b)** Parabolas. **(c)** Noncircular ellipses. **(d)** Lines.

44. **(a)** Lines. **(b)** Circles. **(c)** Hyperbolas. **(d)** Parabolas.

45. **(a)** $\approx \$130$. **(b)** $\approx \$275$ more.

46. **(a)** $\approx \$55$. **(b)** $\approx \$250$ less.

47. **(a)** $f(x, y) = 1 - x^2 - y^2$, because $f = c$ is a circle of radius $\sqrt{1-c}$ (provided $c \leq 1$), and the radii in (a) decrease as c increases.

(b) $f(x, y) = \sqrt{x^2 + y^2}$ because $f = c$ is a circle of radius c, and the radii increase uniformly.

(c) $f(x, y) = x^2 + y^2$ because $f = c$ is a circle of radius $\sqrt{c}$ and the radii in the plot grow like the square root function.

48. **(a)** III, because the surface has 9 peaks along the edges, three peaks to each edge.

(b) I, because in the first quadrant of the xy-plane, $z \geq 0$ for $x \geq y$, and $z \leq 0$ for $x \leq y$.

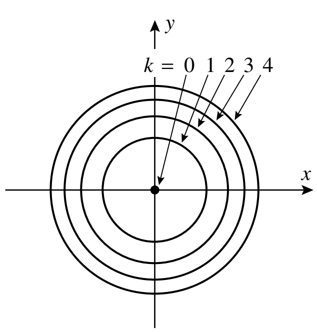
(c) IV, because in the first quadrant of the xy-plane, $z \leq 0$ for $x \geq y$, and $z \geq 0$ for $x \leq y$.

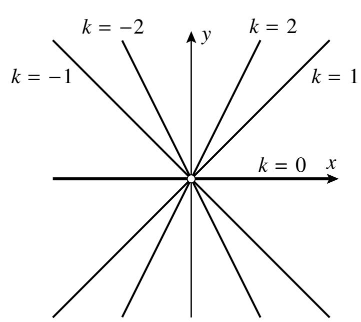
(d) II, because the surface has four peaks.

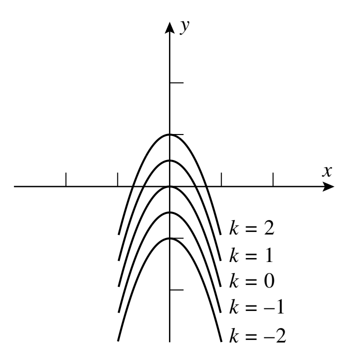
49. **(a)** A **(b)** B **(c)** Increase. **(d)** Decrease. **(e)** Increase. **(f)** Decrease.

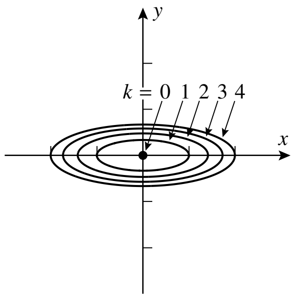
50. **(a)** Medicine Hat, since the contour lines are closer together near Medicine Hat than they are near Chicago.

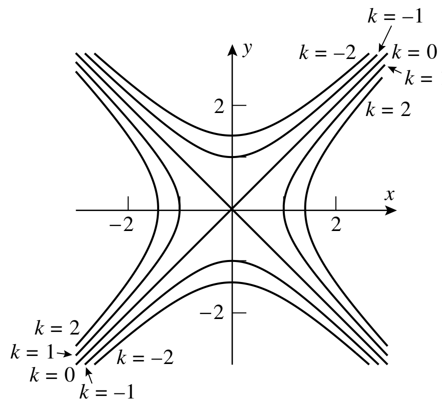
(b) The change in atmospheric pressure is about $\Delta p \approx 999 - 1010 = -11$, so the average rate of change is $\Delta p/1400 \approx -0.0079$.

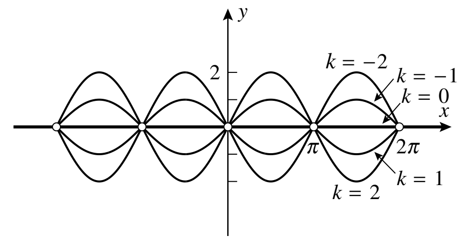
51. 

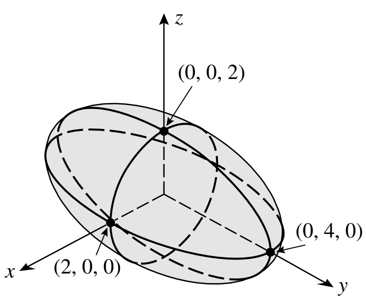
52. 

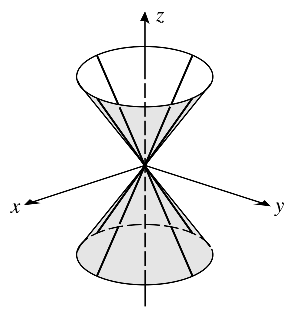
53. 

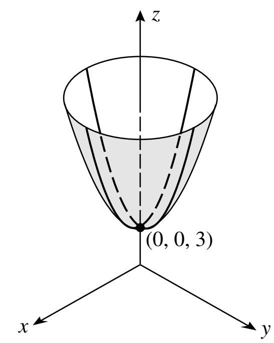
54. 

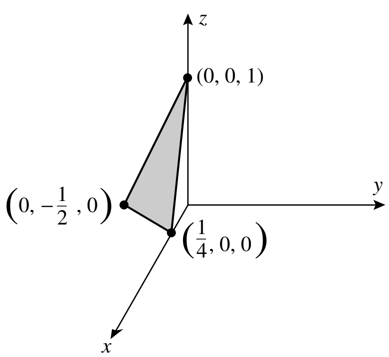
55. 

56. 

57. 

58. 

59. 

60. 

61. Concentric spheres, common center at (2,0,0).

62. Parallel planes, common normal $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

63. Concentric cylinders, common axis the y-axis.

64. Circular paraboloids, common axis the z-axis, all the same shape but with different vertices along z-axis.

65. **(a)** $f(-1,1) = 0$; $x^2 - 2x^3 + 3xy = 0$. **(b)** $f(0,0) = 0$; $x^2 - 2x^3 + 3xy = 0$.

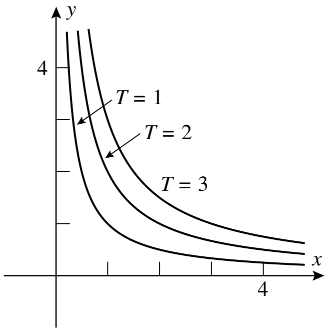
(c) $f(2,-1) = -18$; $x^2 - 2x^3 + 3xy = -18$.

66. **(a)** $f(\ln 2,\, 1) = 2$; $ye^x = 2$. **(b)** $f(0,3) = 3$; $ye^x = 3$. **(c)** $f(1,-2) = -2e$; $ye^x = -2e$.

67. **(a)** $f(1,-2,0) = 5$; $x^2 + y^2 - z = 5$. **(b)** $f(1,0,3) = -2$; $x^2 + y^2 - z = -2$. **(c)** $f(0,0,0) = 0$; $x^2 + y^2 - z = 0$.

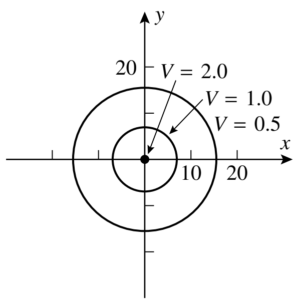
68. **(a)** $f(1,0,2) = 3$; $xyz + 3 = 3$, $xyz = 0$. **(b)** $f(-2,4,1) = -5$; $xyz + 3 = -5$, $xyz = -8$.

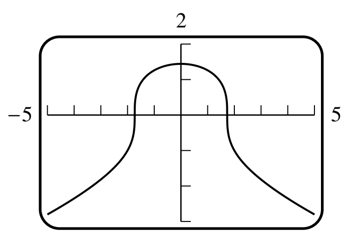
(c) $f(0,0,0) = 3$; $xyz = 0$.

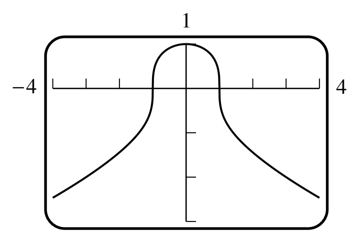
69. **(a)** 

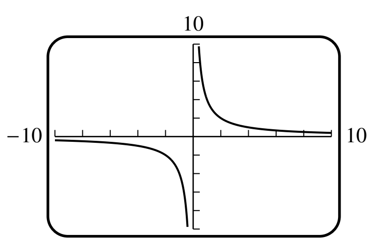
(b) At $(1,4)$ the temperature is $T(1,4) = 4$ so the temperature will remain constant along the path $xy = 4$.

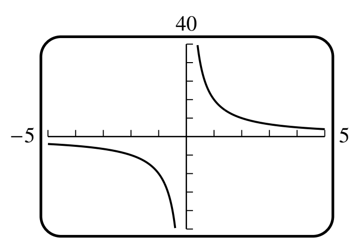
70. $V = \dfrac{8}{\sqrt{16 + x^2 + y^2}}$, $x^2 + y^2 = \dfrac{64}{V^2} - 16$, the equipotential curves are circles.

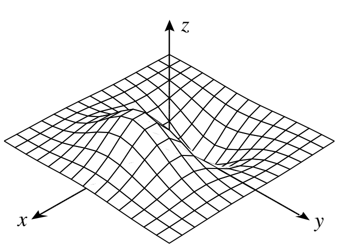


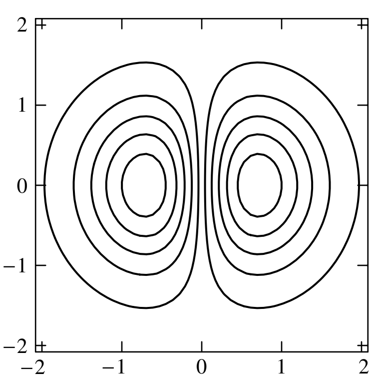
71. **(a)** 

(b) 

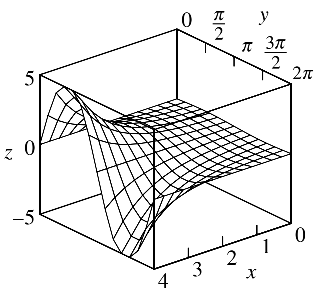
72. **(a)** 

(b) 

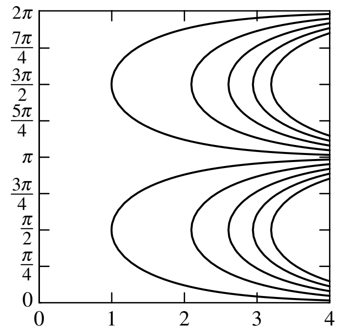
73. **(a)** 

(b) 

74. (a)



(b)

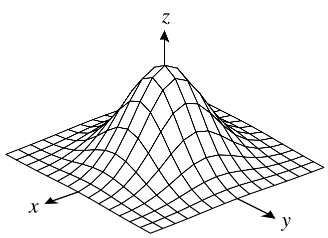


75. (a) The graph of g is the graph of f shifted one unit in the positive x-direction.

(b) The graph of g is the graph of f shifted one unit up the z-axis.

(c) The graph of g is the graph of f shifted one unit down the y-axis and then inverted with respect to the plane $z = 0$.

76. (a)



(b) If a is positive and increasing then the graph of g is more pointed, and in the limit as $a \to +\infty$ the graph approaches a 'spike' on the z-axis of height 1. As a decreases to zero the graph of g gets flatter until it finally approaches the plane $z = 1$.

Exercise Set 13.2

1. $\displaystyle\lim_{(x,y)\to(1,3)} (4xy^2 - x) = 4 \cdot 1 \cdot 3^2 - 1 = 35.$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x - y}{\sin y - 1} = \frac{4 \cdot 0 - 0}{\sin 0 - 1} = 0.$

3. $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x + y} = \frac{-1 \cdot 2^3}{-1 + 2} = -8.$

4. $\displaystyle\lim_{(x,y)\to(1,-3)} e^{2x-y^2} = e^{2\cdot 1-(-3)^2} = e^{-7}.$

5. $\displaystyle\lim_{(x,y)\to(0,0)} \ln(1 + x^2y^3) = \ln(1 + 0^2 \cdot 0^3) = 0.$

6. $\displaystyle\lim_{(x,y)\to(4,-2)} x\sqrt[3]{y^3 + 2x} = (-2) \cdot \sqrt[3]{(-2)^3 + 2 \cdot 4} = 0.$

7. (a) Along $x = 0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3}{x^2 + 2y^2} = \lim_{y\to 0} \frac{3}{2y^2}$ does not exist.

(b) Along $x = 0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + y}{2x^2 + y^2} = \lim_{y\to 0} \frac{1}{y}$ does not exist.

8. **(a)** Along $y=0$: $\lim\limits_{x\to 0}\dfrac{x}{x^2}=\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist, so the original limit does not exist.

(b) Along $y=0$: $\lim\limits_{x\to 0}\dfrac{1}{x^2}$ does not exist, so the original limit does not exist.

9. Let $z=x^2+y^2$, then $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}=\lim\limits_{z\to 0^+}\dfrac{\sin z}{z}=1$.

10. Let $z=x^2+y^2$, then $\lim\limits_{(x,y)\to(0,0)}\dfrac{1-\cos\left(x^2+y^2\right)}{x^2+y^2}=\lim\limits_{z\to 0^+}\dfrac{1-\cos z}{z}=\lim\limits_{z\to 0^+}\dfrac{\sin z}{1}=0$.

11. Let $z=x^2+y^2$, then $\lim\limits_{(x,y)\to(0,0)}e^{-1/\left(x^2+y^2\right)}=\lim\limits_{z\to 0^+}e^{-1/z}=0$.

12. With $z=x^2+y^2$, $\lim\limits_{z\to 0}\dfrac{1}{\sqrt{z}}e^{-1/\sqrt{z}}$; let $w=\dfrac{1}{\sqrt{z}}$, $\lim\limits_{w\to+\infty}\dfrac{w}{e^w}=0$.

13. $\lim\limits_{(x,y)\to(0,0)}\dfrac{\left(x^2+y^2\right)\left(x^2-y^2\right)}{x^2+y^2}=\lim\limits_{(x,y)\to(0,0)}\left(x^2-y^2\right)=0$.

14. $\lim\limits_{(x,y)\to(0,0)}\dfrac{\left(x^2+4y^2\right)\left(x^2-4y^2\right)}{x^2+4y^2}=\lim\limits_{(x,y)\to(0,0)}\left(x^2-4y^2\right)=0$.

15. Along $y=0$: $\lim\limits_{x\to 0}\dfrac{0}{3x^2}=\lim\limits_{x\to 0}0=0$; along $y=x$: $\lim\limits_{x\to 0}\dfrac{x^2}{5x^2}=\lim\limits_{x\to 0}1/5=1/5$, so the limit does not exist.

16. Let $z=x^2+y^2$, then $\lim\limits_{(x,y)\to(0,0)}\dfrac{1-x^2-y^2}{x^2+y^2}=\lim\limits_{z\to 0^+}\dfrac{1-z}{z}=+\infty$ so the limit does not exist.

17. $\lim\limits_{(x,y,z)\to(2,-1,2)}\dfrac{xz^2}{\sqrt{x^2+y^2+z^2}}=\dfrac{2\cdot 2^2}{\sqrt{2^2+(-1)^2+2^2}}=\dfrac{8}{3}$.

18. $\lim\limits_{(x,y,z)\to(2,0,-1)}\ln(2x+y-z)=\ln(2\cdot 2+0-(-1))=\ln 5$.

19. Let $t=\sqrt{x^2+y^2+z^2}$, then $\lim\limits_{(x,y,z)\to(0,0,0)}\dfrac{\sin\left(x^2+y^2+z^2\right)}{\sqrt{x^2+y^2+z^2}}=\lim\limits_{t\to 0^+}\dfrac{\sin\left(t^2\right)}{t}=0$.

20. With $t=\sqrt{x^2+y^2+z^2}$, $\lim\limits_{t\to 0^+}\dfrac{\sin t}{t^2}=\lim\limits_{t\to 0^+}\dfrac{\cos t}{2t}=+\infty$ so the limit does not exist.

21. $\dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}=\dfrac{e^\rho}{\rho}$, so $\lim\limits_{(x,y,z)\to(0,0,0)}\dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}=\lim\limits_{\rho\to 0^+}\dfrac{e^\rho}{\rho}$ does not exist.

22. $\lim\limits_{(x,y,z)\to(0,0,0)}\tan^{-1}\left[\dfrac{1}{x^2+y^2+z^2}\right]=\lim\limits_{\rho\to 0^+}\tan^{-1}\dfrac{1}{\rho^2}=\dfrac{\pi}{2}$.

23. $\lim\limits_{r\to 0}r\ln r^2=\lim\limits_{r\to 0}(2\ln r)/(1/r)=\lim\limits_{r\to 0}(2/r)/(-1/r^2)=\lim\limits_{r\to 0}(-2r)=0$.

24. $y\ln(x^2+y^2)=r\sin\theta\ln r^2=2r(\ln r)\sin\theta$, so $\lim\limits_{(x,y)\to(0,0)}y\ln(x^2+y^2)=\lim\limits_{r\to 0^+}2r(\ln r)\sin\theta=0$.

25. $\dfrac{x^2y^2}{\sqrt{x^2+y^2}}=\dfrac{(r^2\cos^2\theta)(r^2\sin^2\theta)}{r}=r^3\cos^2\theta\sin^2\theta$, so $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^2}{\sqrt{x^2+y^2}}=0$.

26. $\left|\dfrac{r^2\cos\theta\sin\theta}{\sqrt{r^2+2r^2\sin^2\theta}}\right| \le \dfrac{r^2}{\sqrt{r^2}} = r$ so $\displaystyle\lim_{(x,y)\to(0,0)}\left|\frac{xy}{x^2+2y^2}\right| = 0$.

27. $\left|\dfrac{\rho^3\sin^2\phi\cos\phi\sin\theta\cos\theta}{\rho^2}\right| \le \rho$, so $\displaystyle\lim_{(x,y,z)\to(0,0,0)}\frac{xyz}{x^2+y^2+z^2} = 0$.

28. $\left|\dfrac{\sin x\sin y}{\sqrt{x^2+2y^2+3z^2}}\right| \le \left|\dfrac{xy}{\sqrt{x^2+y^2+z^2}}\right| = \left|\dfrac{\rho^2\sin^2\phi\cos\theta\sin\theta}{\rho}\right| \le \rho$, so $\displaystyle\lim_{(x,y,z)\to(0,0,0)}\frac{\sin x\sin y}{\sqrt{x^2+2y^2+3z^2}} = 0$.

29. True: contains no boundary points, therefore each point of D is an interior point.

30. False: $f(x,y) = xy/(x^2+y^2)$ has limit zero along $x=0$ as well as along $y=0$, but not, if $m \ne 0$, along the line $y = mx$.

31. False: let $f(x,y) = -1$ for $x < 0$ and $f(x,y) = 1$ for $x \ge 0$ and let $g(x.y) = -f(x,y)$.

32. True; there is a $\delta > 0$ such that $|f(x)| > |L|/2$ if $0 < x < \delta$, so $\dfrac{x^2+y^2}{|f(x^2+y^2)|} \le \dfrac{x^2+y^2}{|L|/2} < \epsilon$ if $x^2+y^2 < \delta$ and $x^2+y^2 < |L|\epsilon/2$.

33. **(a)** No, since there seem to be points near $(0,0)$ with $z=0$ and other points near $(0,0)$ with $z \approx 1/2$.

(b) $\displaystyle\lim_{x\to0}\frac{mx^3}{x^4+m^2x^2} = \lim_{x\to0}\frac{mx}{x^2+m^2} = 0$.

(c) $\displaystyle\lim_{x\to0}\frac{x^4}{2x^4} = \lim_{x\to0}1/2 = 1/2$.

(d) A limit must be unique if it exists, so $f(x,y)$ cannot have a limit as $(x,y)\to(0,0)$.

34. **(a)** Along $y = mx$: $\displaystyle\lim_{x\to0}\frac{mx^4}{2x^6+m^2x^2} = \lim_{x\to0}\frac{mx^2}{2x^4+m^2} = 0$; along $y = kx^2$: $\displaystyle\lim_{x\to0}\frac{kx^5}{2x^6+k^2x^4} = \lim_{x\to0}\frac{kx}{2x^2+k^2} = 0$.

(b) $\displaystyle\lim_{x\to0}\frac{x^6}{2x^6+x^6} = \lim_{x\to0}\frac{1}{3} = \frac{1}{3} \ne 0$.

35. **(a)** We may assume that $a^2+b^2+c^2 > 0$, since we are dealing with a line (not just the point $(0,0,0)$). Assume first that $a \ne 0$. Then $\displaystyle\lim_{t\to0}\frac{abct^3}{a^2t^2+b^4t^4+c^4t^4} = \lim_{t\to0}\frac{abct}{a^2+b^4t^2+c^4t^2} = 0$. If, on the other hand, $a = 0$, the result is trivial, as the quotient is then zero.

(b) $\displaystyle\lim_{t\to0}\frac{t^4}{t^4+t^4+t^4} = \lim_{t\to0}1/3 = 1/3$.

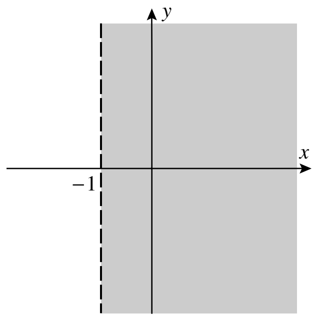
36. $\pi/2$ because $\dfrac{x^2+1}{x^2+(y-1)^2} \to +\infty$ as $(x,y)\to(0,1)$.

37. $-\pi/2$ because $\dfrac{x^2-1}{x^2+(y-1)^2} \to -\infty$ as $(x,y)\to(0,1)$.

38. With $z = x^2+y^2$, $\displaystyle\lim_{z\to0^+}\frac{\sin z}{z} = 1 = f(0,0)$.

39. The required limit does not exist, so the singularity is not removable.

40. $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ so the limit exists, and $f(0,0) = -4 \neq 0$, thus the singularity is removable.

41.


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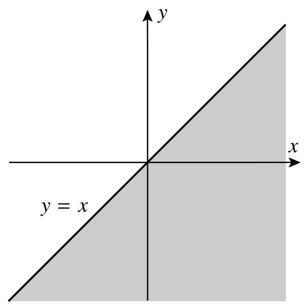

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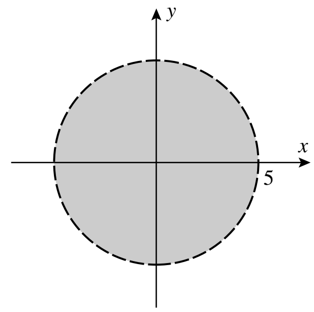

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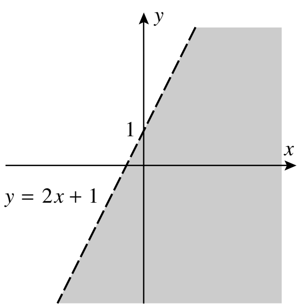

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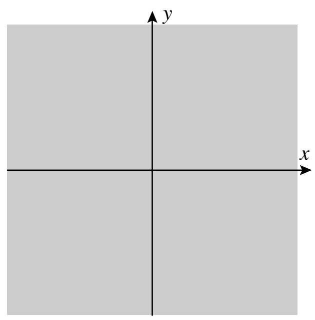

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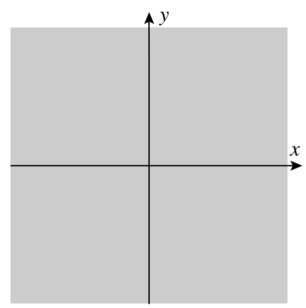

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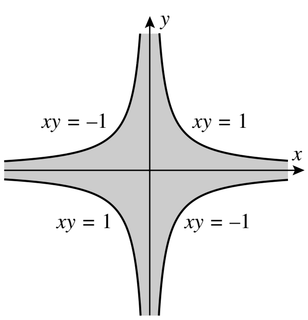
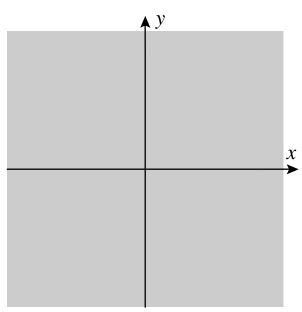


48.


49. All of 3-space.

50. All points inside the sphere with radius 2 and center at the origin.

51. All points not on the cylinder $x^2 + z^2 = 1$.

52. All of 3-space.

Exercise Set 13.3

1. **(a)** $9x^2y^2$ **(b)** $6x^3y$ **(c)** $9y^2$ **(d)** $9x^2$ **(e)** $6y$ **(f)** $6x^3$ **(g)** 36 **(h)** 12

2. **(a)** $2e^{2x}\sin y$ **(b)** $e^{2x}\cos y$ **(c)** $2\sin y$ **(d)** 0 **(e)** $\cos y$ **(f)** e^{2x} **(g)** 0 **(h)** 4

3. $\dfrac{\partial z}{\partial x} = 18xy - 15x^4y$, $\dfrac{\partial z}{\partial y} = 9x^2 - 3x^5$.

4. $f_x(x,y) = 20xy^4 - 6y^2 + 20x$, $f_y(x,y) = 40x^2y^3 - 12xy$.

5. $\dfrac{\partial z}{\partial x} = 8(x^2 + 5x - 2y)^7(2x + 5)$, $\dfrac{\partial z}{\partial y} = -16(x^2 + 5x - 2y)^7$.

6. $f_x(x,y) = (-1)(xy^2 - x^2y)^{-2}(y^2 - 2xy)$, $f_y(x,y) = (-1)(xy^2 - x^2y)^{-2}(2xy - x^2)$.

7. $\dfrac{\partial}{\partial p}(e^{-7p/q}) = -7e^{-7p/q}/q$, $\dfrac{\partial}{\partial q}(e^{-7p/q}) = 7pe^{-7p/q}/q^2$.

8. $\dfrac{\partial}{\partial x}(xe^{\sqrt{15xy}}) = e^{\sqrt{15xy}} + xe^{\sqrt{15xy}}\dfrac{1}{2}\dfrac{1}{\sqrt{15xy}}15y$, $\dfrac{\partial}{\partial y}(xe^{\sqrt{15xy}}) = xe^{\sqrt{15xy}}\dfrac{1}{2}\dfrac{1}{\sqrt{15xy}}15x$.

9. $\dfrac{\partial z}{\partial x} = (15x^2y + 7y^2)\cos(5x^3y + 7xy^2)$, $\dfrac{\partial z}{\partial y} = (5x^3 + 14xy)\cos(5x^3y + 7xy^2)$.

10. $f_x(x,y) = -(2y^2 - 6xy^2)\sin(2xy^2 - 3x^2y^2)$, $f_y(x,y) = -(4xy - 6x^2y)\sin(2xy^2 - 3x^2y^2)$.

11. **(a)** $\dfrac{\partial z}{\partial x} = \dfrac{3}{2\sqrt{3x+2y}}$; slope $= \dfrac{3}{8}$. **(b)** $\dfrac{\partial z}{\partial y} = \dfrac{1}{\sqrt{3x+2y}}$; slope $= \dfrac{1}{4}$.

12. **(a)** $\dfrac{\partial z}{\partial x} = e^{-y}$; slope $= 1$. **(b)** $\dfrac{\partial z}{\partial y} = -xe^{-y} + 5$; slope $= 2$.

13. **(a)** $\dfrac{\partial z}{\partial x} = -4\cos(y^2 - 4x)$; rate of change $= -4\cos 7$. **(b)** $\dfrac{\partial z}{\partial y} = 2y\cos(y^2 - 4x)$; rate of change $= 2\cos 7$.

14. **(a)** $\dfrac{\partial z}{\partial x} = -\dfrac{1}{(x+y)^2}$; rate of change $= -\dfrac{1}{4}$. **(b)** $\dfrac{\partial z}{\partial y} = -\dfrac{1}{(x+y)^2}$; rate of change $= -\dfrac{1}{4}$.

15. $\partial z/\partial x =$ slope of line parallel to xz-plane $= -4$; $\partial z/\partial y =$ slope of line parallel to yz-plane $= 1/2$.

16. Moving to the right from (x_0, y_0) decreases $f(x,y)$, so $f_x < 0$; moving up increases f, so $f_y > 0$.

17. **(a)** The right-hand estimate is $\partial r/\partial v \approx (222 - 197)/(85 - 80) = 5$; the left-hand estimate is $\partial r/\partial v \approx (197 - 173)/(80 - 75) = 4.8$; the average is $\partial r/\partial v \approx 4.9$.

(b) The right-hand estimate is $\partial r/\partial \theta \approx (200 - 197)/(45 - 40) = 0.6$; the left-hand estimate is $\partial r/\partial \theta \approx (197 - 188)/(40 - 35) = 1.8$; the average is $\partial r/\partial \theta \approx 1.2$.

18. **(a)** The right-hand estimate is $\partial r/\partial v \approx (253-226)/(90-85) = 5.4$; the left-hand estimate is $(226-200)/(85-80) = 5.2$; the average is $\partial r/\partial v \approx 5.3$.

(b) The right-hand estimate is $\partial r/\partial \theta \approx (222 - 226)/(50 - 45) = -0.8$; the left-hand estimate is $(226 - 222)/(45 - 40) = 0.8$; the average is $\partial r/\partial v \approx 0$.

19. III is a plane, and its partial derivatives are constants, so III cannot be $f(x,y)$. If I is the graph of $z = f(x,y)$ then (by inspection) f_y is constant as y varies, but neither II nor III is constant as y varies. Hence $z = f(x,y)$ has II as its graph, and as II seems to be an odd function of x and an even function of y, f_x has I as its graph and f_y has III as its graph.

20. The slope at P in the positive x-direction is negative, the slope in the positive y-direction is negative, thus $\partial z/\partial x < 0, \partial z/\partial y < 0$; the curve through P which is parallel to the x-axis is concave down, so $\partial^2 z/\partial x^2 < 0$; the curve parallel to the y-axis is concave down, so $\partial^2 z/\partial y^2 < 0$.

21. True: f is constant along the line $y = 2$ so $f_x(4,2) = 0$.

22. True, $f(3,y) = y^2$, so $f_y(3,4) = 8$.

23. True; z is a linear function of both x and y.

24. False; if so then $2y+2=\dfrac{\partial f_x}{\partial y}=\dfrac{\partial f_y}{\partial x}=2y$, a contradiction.

25. $\partial z/\partial x=8xy^3e^{x^2y^3}$, $\partial z/\partial y=12x^2y^2e^{x^2y^3}$.

26. $\partial z/\partial x=-5x^4y^4\sin(x^5y^4)$, $\partial z/\partial y=-4x^5y^3\sin\left(x^5y^4\right)$.

27. $\partial z/\partial x=x^3/(y^{3/5}+x)+3x^2\ln(1+xy^{-3/5})$, $\partial z/\partial y=-(3/5)x^4/(y^{8/5}+xy)$.

28. $\partial z/\partial x=ye^{xy}\sin(4y^2)$, $\partial z/\partial y=8ye^{xy}\cos(4y^2)+xe^{xy}\sin(4y^2)$.

29. $\dfrac{\partial z}{\partial x}=-\dfrac{y(x^2-y^2)}{(x^2+y^2)^2}$, $\dfrac{\partial z}{\partial y}=\dfrac{x(x^2-y^2)}{(x^2+y^2)^2}$.

30. $\dfrac{\partial z}{\partial x}=\dfrac{xy^3(3x+4y)}{2(x+y)^{3/2}}$, $\dfrac{\partial z}{\partial y}=\dfrac{x^2y^2(6x+5y)}{2(x+y)^{3/2}}$.

31. $f_x(x,y)=(3/2)x^2y\left(5x^2-7\right)\left(3x^5y-7x^3y\right)^{-1/2}$, $f_y(x,y)=(1/2)x^3\left(3x^2-7\right)\left(3x^5y-7x^3y\right)^{-1/2}$.

32. $f_x(x,y)=-2y/(x-y)^2$, $f_y(x,y)=2x/(x-y)^2$.

33. $f_x(x,y)=\dfrac{y^{-1/2}}{y^2+x^2}$, $f_y(x,y)=-\dfrac{xy^{-3/2}}{y^2+x^2}-\dfrac{3}{2}y^{-5/2}\tan^{-1}(x/y)$.

34. $f_x(x,y)=3x^2e^{-y}+(1/2)x^{-1/2}y^3\sec\sqrt{x}\tan\sqrt{x}$, $f_y(x,y)=-x^3e^{-y}+3y^2\sec\sqrt{x}$.

35. $f_x(x,y)=-(4/3)y^2\sec^2x\left(y^2\tan x\right)^{-7/3}$, $f_y(x,y)=-(8/3)y\tan x\left(y^2\tan x\right)^{-7/3}$.

36. $f_x(x,y)=2y^2\cosh\sqrt{x}\sinh\left(xy^2\right)\cosh\left(xy^2\right)+\dfrac{1}{2}x^{-1/2}\sinh\sqrt{x}\sinh^2\left(xy^2\right)$,
$f_y(x,y)=4xy\cosh\sqrt{x}\sinh\left(xy^2\right)\cosh\left(xy^2\right)$.

37. $f_x(x,y)=-2x$, $f_x(3,1)=-6$; $f_y(x,y)=-21y^2$, $f_y(3,1)=-21$.

38. $\partial f/\partial x=x^2y^2e^{xy}+2xye^{xy}$, $\partial f/\partial x\big|_{(1,1)}=3e$; $\partial f/\partial y=x^3ye^{xy}+x^2e^{xy}$, $\partial f/\partial y\big|_{(1,1)}=2e$.

39. $\partial z/\partial x=x(x^2+4y^2)^{-1/2}$, $\partial z/\partial x\big|_{(1,2)}=1/\sqrt{17}$; $\partial z/\partial y=4y(x^2+4y^2)^{-1/2}$, $\partial z/\partial y\big|_{(1,2)}=8/\sqrt{17}$.

40. $\partial w/\partial x=-x^2y\sin xy+2x\cos xy$, $\dfrac{\partial w}{\partial x}(1/2,\pi)=-\pi/4$; $\partial w/\partial y=-x^3\sin xy$, $\dfrac{\partial w}{\partial y}(1/2,\pi)=-1/8$.

41. **(a)** $2xy^4z^3+y$ **(b)** $4x^2y^3z^3+x$ **(c)** $3x^2y^4z^2+2z$ **(d)** $2y^4z^3+y$ **(e)** $32z^3+1$ **(f)** 438

42. **(a)** $2xy\cos z$ **(b)** $x^2\cos z$ **(c)** $-x^2y\sin z$ **(d)** $4y\cos z$ **(e)** $4\cos z$ **(f)** 0

43. $f_x=2z/x$, $f_y=z/y$, $f_z=\ln(x^2y\cos z)-z\tan z$.

44. $f_x=y^{-5/2}z\sec(xz/y)\tan(xz/y)$, $f_y=-xy^{-7/2}z\sec(xz/y)\tan(xz/y)-(3/2)y^{-5/2}\sec(xz/y)$,
$f_z=xy^{-5/2}\sec(xz/y)\tan(xz/y)$.

45. $f_x=-y^2z^3/\left(1+x^2y^4z^6\right)$, $f_y=-2xyz^3/\left(1+x^2y^4z^6\right)$, $f_z=-3xy^2z^2/\left(1+x^2y^4z^6\right)$.

46. $f_x=4xyz\cosh\sqrt{z}\sinh\left(x^2yz\right)\cosh\left(x^2yz\right)$, $f_y=2x^2z\cosh\sqrt{z}\sinh\left(x^2yz\right)\cosh\left(x^2yz\right)$,
$f_z=2x^2y\cosh\sqrt{z}\sinh\left(x^2yz\right)\cosh\left(x^2yz\right)+(1/2)z^{-1/2}\sinh\sqrt{z}\sinh^2\left(x^2yz\right)$.

47. $\partial w/\partial x = yze^z \cos xz$, $\partial w/\partial y = e^z \sin xz$, $\partial w/\partial z = ye^z(\sin xz + x\cos xz)$.

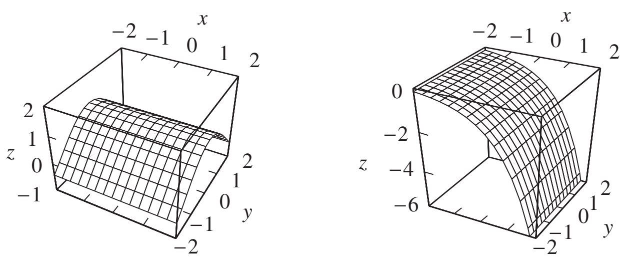
48. $\partial w/\partial x = 2x/\left(y^2+z^2\right)$, $\partial w/\partial y = -2y\left(x^2+z^2\right)/\left(y^2+z^2\right)^2$, $\partial w/\partial z = 2z\left(y^2-x^2\right)/\left(y^2+z^2\right)^2$.

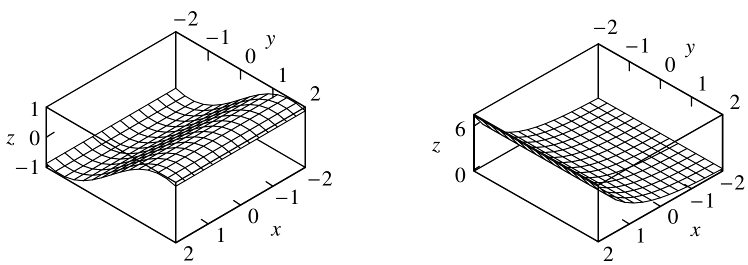
49. $\partial w/\partial x = x/\sqrt{x^2+y^2+z^2}$, $\partial w/\partial y = y/\sqrt{x^2+y^2+z^2}$, $\partial w/\partial z = z/\sqrt{x^2+y^2+z^2}$.

50. $\partial w/\partial x = 2y^3e^{2x+3z}$, $\partial w/\partial y = 3y^2e^{2x+3z}$, $\partial w/\partial z = 3y^3e^{2x+3z}$.

51. **(a)** e **(b)** $2e$ **(c)** e

52. **(a)** $2/\sqrt{7}$ **(b)** $4/\sqrt{7}$ **(c)** $1/\sqrt{7}$

53. 

54. 

55. $\partial z/\partial x = 2x + 6y(\partial y/\partial x) = 2x$, $\partial z/\partial x\big|_{(2,1)} = 4$.

56. $\partial z/\partial y = 6y$, $\partial z/\partial y\,\big|_{(2,1)} = 6$.

57. $\partial z/\partial x = -x\left(29-x^2-y^2\right)^{-1/2}$, $\partial z/\partial x|_{(4,3)} = -2$.

58. **(a)** $\partial z/\partial y = 8y,\ \partial z/\partial y|_{(-1,1)} = 8$. **(b)** $\partial z/\partial x = 2x,\ \partial z/\partial x|_{(-1,1)} = -2$.

59. **(a)** $\partial V/\partial r = 2\pi rh$. **(b)** $\partial V/\partial h = \pi r^2$. **(c)** $\partial V/\partial r|_{r=6,\ h=4} = 48\pi$. **(d)** $\partial V/\partial h|_{r=8,\ h=10} = 64\pi$.

60. **(a)** $\partial V/\partial s = \dfrac{\pi s d^2}{6\sqrt{4s^2-d^2}}$. **(b)** $\partial V/\partial d = \dfrac{\pi d(8s^2-3d^2)}{24\sqrt{4s^2-d^2}}$. **(c)** $\partial V/\partial s|_{s=10,\ d=16} = 320\pi/9$.

(d) $\partial V/\partial d|_{s=10,\ d=16} = 16\pi/9$.

61. **(a)** $P = 10T/V$, $\partial P/\partial T = 10/V$, $\partial P/\partial T|_{T=80,\ V=50} = 1/5$ lb/(in^{2}K).

(b) $V = 10T/P, \partial V/\partial P = -10T/P^2$, if $V = 50$ and $T = 80$, then $P = 10(80)/(50) = 16$, $\partial V/\partial P|_{T=80,\ P=16} = -25/8(\text{in}^5/\text{lb})$.

62. **(a)** $\partial T/\partial x = 3x^2+1,\ \partial T/\partial x|_{(1,2)} = 4\dfrac{^\circ\text{C}}{\text{cm}}$. **(b)** $\partial T/\partial y = 4y,\ \partial T/\partial y|_{(1,2)} = 8\dfrac{^\circ\text{C}}{\text{cm}}$.

63. **(a)** $V = lwh, \partial V/\partial l = wh = 6$. **(b)** $\partial V/\partial w = lh = 15$. **(c)** $\partial V/\partial h = lw = 10$.

64. **(a)** $\partial A/\partial a = (1/2)b\sin\theta = (1/2)(10)\left(\sqrt{3}/2\right) = 5\sqrt{3}/2$.

(b) $\partial A/\partial\theta = (1/2)ab\cos\theta = (1/2)(5)(10)(1/2) = 25/2$.

(c) $b = (2A\csc\theta)/a$, $\partial b/\partial a = -(2A\csc\theta)/a^2 = -b/a = -2$.

65. $\partial V/\partial r = \dfrac{2}{3}\pi rh = \dfrac{2}{r}(\dfrac{1}{3}\pi r^2h) = 2V/r$.

66. **(a)** $\partial z/\partial y = x^2,\ \partial z/\partial y|_{(1,3)} = 1$, $\mathbf{j}+\mathbf{k}$ is parallel to the tangent line so $x = 1,\ y = 3+t,\ z = 3+t$.

(b) $\partial z/\partial x = 2xy,\ \partial z/\partial x|_{(1,3)} = 6$, $\mathbf{i}+6\mathbf{k}$ is parallel to the tangent line so $x = 1+t,\ y = 3,\ z = 3+6t$.

67. **(a)** $2x - 2z(\partial z/\partial x) = 0,\ \partial z/\partial x = x/z = \pm 3/(2\sqrt{6}) = \pm\sqrt{6}/4$.

(b) $z = \pm\sqrt{x^2+y^2-1},\ \partial z/\partial x = \pm x/\sqrt{x^2+y^2-1} = \pm\sqrt{6}/4$.

68. **(a)** $2y - 2z(\partial z/\partial y) = 0,\ \partial z/\partial y = y/z = \pm 4/(2\sqrt{6}) = \pm\sqrt{6}/3$.

(b) $z = \pm\sqrt{x^2+y^2-1},\ \partial z/\partial y = \pm y/\sqrt{x^2+y^2-1} = \pm\sqrt{6}/3$.

69. $\dfrac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}\left(2x + 2z\dfrac{\partial z}{\partial x}\right) = 0$, $\partial z/\partial x = -x/z$; similarly, $\partial z/\partial y = -y/z$.

70. $\dfrac{4x - 3z^2(\partial z/\partial x)}{2x^2+y-z^3} = 1,\ \dfrac{\partial z}{\partial x} = \dfrac{4x - 2x^2 - y + z^3}{3z^2};\ \dfrac{1 - 3z^2(\partial z/\partial y)}{2x^2+y-z^3} = 0,\ \dfrac{\partial z}{\partial y} = \dfrac{1}{3z^2}$.

71. $2x + z\left(xy\dfrac{\partial z}{\partial x} + yz\right)\cos xyz + \dfrac{\partial z}{\partial x}\sin xyz = 0,\ \dfrac{\partial z}{\partial x} = -\dfrac{2x + yz^2\cos xyz}{xyz\cos xyz + \sin xyz}$;
$z\left(xy\dfrac{\partial z}{\partial y} + xz\right)\cos xyz + \dfrac{\partial z}{\partial y}\sin xyz = 0,\ \dfrac{\partial z}{\partial y} = -\dfrac{xz^2\cos xyz}{xyz\cos xyz + \sin xyz}$.

72. $e^{xy}(\cosh z)\dfrac{\partial z}{\partial x} + ye^{xy}\sinh z - z^2 - 2xz\dfrac{\partial z}{\partial x} = 0,\ \dfrac{\partial z}{\partial x} = \dfrac{z^2 - ye^{xy}\sinh z}{e^{xy}\cosh z - 2xz}$;
$e^{xy}(\cosh z)\dfrac{\partial z}{\partial y} + xe^{xy}\sinh z - 2xz\dfrac{\partial z}{\partial y} = 0,\ \dfrac{\partial z}{\partial y} = -\dfrac{xe^{xy}\sinh z}{e^{xy}\cosh z - 2xz}$.

73. $(3/2)\left(x^2+y^2+z^2+w^2\right)^{1/2}\left(2x + 2w\dfrac{\partial w}{\partial x}\right) = 0$, $\partial w/\partial x = -x/w$; similarly, $\partial w/\partial y = -y/w$ and $\partial w/\partial z = -z/w$.

74. $\partial w/\partial x = -4x/3$, $\partial w/\partial y = -1/3$, $\partial w/\partial z = (2x^2 + y - z^3 + 3z^2 + 3w)/3$.

75. $\dfrac{\partial w}{\partial x} = -\dfrac{yzw\cos xyz}{2w + \sin xyz},\ \dfrac{\partial w}{\partial y} = -\dfrac{xzw\cos xyz}{2w + \sin xyz},\ \dfrac{\partial w}{\partial z} = -\dfrac{xyw\cos xyz}{2w + \sin xyz}$.

76. $\dfrac{\partial w}{\partial x} = \dfrac{ye^{xy}\sinh w}{z^2 - e^{xy}\cosh w},\ \dfrac{\partial w}{\partial y} = \dfrac{xe^{xy}\sinh w}{z^2 - e^{xy}\cosh w},\ \dfrac{\partial w}{\partial z} = \dfrac{2zw}{e^{xy}\cosh w - z^2}$.

77. $f_x = e^{x^2},\ f_y = -e^{y^2}$.

78. $f_x = ye^{x^2y^2},\ f_y = xe^{x^2y^2}$.

79. $f_x = 2xy^3\sin x^6y^9, f_y = 3x^2y^2\sin x^6y^9$.

80. $f_x = \sin(x-y)^3 - \sin(x+y)^3,\ f_y = -\sin(x-y)^3 - \sin(x+y)^3.$

81. **(a)** $-\dfrac{1}{4x^{3/2}}\cos y$ **(b)** $-\sqrt{x}\cos y$ **(c)** $-\dfrac{\sin y}{2\sqrt{x}}$ **(d)** $-\dfrac{\sin y}{2\sqrt{x}}$

82. **(a)** $8 + 84x^2y^5$ **(b)** $140x^4y^3$ **(c)** $140x^3y^4$ **(d)** $140x^3y^4$

83. **(a)** $6\cos(3x^2+6y^2) - 36x^2\sin(3x^2+6y^2)$ **(b)** $12\cos(3x^2+6y^2) - 144y^2\sin(3x^2+6y^2)$

(c) $-72xy\sin(3x^2+6y^2)$ **(d)** $-72xy\sin(3x^2+6y^2)$

84. **(a)** 0 **(b)** $4xe^{2y}$ **(c)** $2e^{2y}$ **(d)** $2e^{2y}$

85. $f_x = 8x - 8y^4,\ f_y = -32xy^3 + 35y^4,\ f_{xy} = f_{yx} = -32y^3.$

86. $f_x = x/\sqrt{x^2+y^2},\ f_y = y/\sqrt{x^2+y^2},\ f_{xy} = f_{yx} = -xy(x^2+y^2)^{-3/2}.$

87. $f_x = e^x\cos y,\ f_y = -e^x\sin y,\ f_{xy} = f_{yx} = -e^x\sin y.$

88. $f_x = e^{x-y^2},\ f_y = -2ye^{x-y^2},\ f_{xy} = f_{yx} = -2ye^{x-y^2}.$

89. $f_x = 4/(4x-5y),\ f_y = -5/(4x-5y),\ f_{xy} = f_{yx} = 20/(4x-5y)^2.$

90. $f_x = 2x/(x^2+y^2), f_y = 2y/(x^2+y^2),\ f_{xy} = -4xy/(x^2+y^2)^2.$

91. $f_x = 2y/(x+y)^2, f_y = -2x/(x+y)^2, f_{xy} = f_{yx} = 2(x-y)/(x+y)^3.$

92. $f_x = 4xy^2/(x^2+y^2)^2,\ f_y = -4x^2y/(x^2+y^2)^2,\ f_{xy} = f_{yx} = 8xy(x^2-y^2)/(x^2+y^2)^3.$

93. **(a)** $\dfrac{\partial^3 f}{\partial x^3}$ **(b)** $\dfrac{\partial^3 f}{\partial y^2\partial x}$ **(c)** $\dfrac{\partial^4 f}{\partial x^2\partial y^2}$ **(d)** $\dfrac{\partial^4 f}{\partial y^3\partial x}$

94. **(a)** f_{xyy} **(b)** f_{xxxx} **(c)** f_{xxyy} **(d)** f_{yyyxx}

95. **(a)** $30xy^4 - 4$ **(b)** $60x^2y^3$ **(c)** $60x^3y^2$

96. **(a)** $120(2x-y)^2$ **(b)** $-240(2x-y)^2$ **(c)** $480(2x-y)$

97. **(a)** $f_{xyy}(0,1) = -30$ **(b)** $f_{xxx}(0,1) = -125$ **(c)** $f_{yyxx}(0,1) = 150$

98. **(a)** $\dfrac{\partial^3 w}{\partial y^2\partial x} = -e^y\sin\ x,\ \left.\dfrac{\partial^3 w}{\partial y^2\partial x}\right|_{(\pi/4,0)} = -1/\sqrt{2}.$ **(b)** $\dfrac{\partial^3 w}{\partial x^2\partial y} = -e^y\cos\ x,\ \left.\dfrac{\partial^3 w}{\partial x^2\partial y}\right|_{(\pi/4,0)} = -1/\sqrt{2}.$

99. **(a)** $f_{xy} = 15x^2y^4z^7 + 2y.$ **(b)** $f_{yz} = 35x^3y^4z^6 + 3y^2.$ **(c)** $f_{xz} = 21x^2y^5z^6.$

(d) $f_{zz} = 42x^3y^5z^5.$ **(e)** $f_{zyy} = 140x^3y^3z^6 + 6y.$ **(f)** $f_{xxy} = 30xy^4z^7.$

(g) $f_{zyx} = 105x^2y^4z^6.$ **(h)** $f_{xxyz} = 210xy^4z^6.$

100. **(a)** $160(4x-3y+2z)^3$ **(b)** $-1440(4x-3y+2z)^2$ **(c)** $-5760(4x-3y+2z)$

101. **(a)** $z_x = 2x+2y,\ z_{xx} = 2,\ z_y = -2y+2x,\ z_{yy} = -2;\ z_{xx}+z_{yy} = 2-2 = 0.$

(b) $z_x = e^x\sin y - e^y\sin x,\ z_{xx} = e^x\sin y - e^y\cos x,\ z_y = e^x\cos y + e^y\cos x,\ z_{yy} = -e^x\sin y + e^y\cos x;\ z_{xx}+z_{yy} = e^x\sin y - e^y\cos x - e^x\sin y + e^y\cos x = 0.$

(c) $z_x = \dfrac{2x}{x^2+y^2} - 2\dfrac{y}{x^2}\dfrac{1}{1+(y/x)^2} = \dfrac{2x-2y}{x^2+y^2}$, $z_{xx} = -2\dfrac{x^2-y^2-2xy}{(x^2+y^2)^2}$, $z_y = \dfrac{2y}{x^2+y^2} + 2\dfrac{1}{x}\dfrac{1}{1+(y/x)^2} = \dfrac{2y+2x}{x^2+y^2}$, $z_{yy} = -2\dfrac{y^2-x^2+2xy}{(x^2+y^2)^2}$; $z_{xx}+z_{yy} = -2\dfrac{x^2-y^2-2xy}{(x^2+y^2)^2} - 2\dfrac{y^2-x^2+2xy}{(x^2+y^2)^2} = 0$.

102. **(a)** $z_t = -e^{-t}\sin(x/c)$, $z_x = (1/c)e^{-t}\cos(x/c)$, $z_{xx} = -(1/c^2)e^{-t}\sin(x/c)$; $z_t - c^2 z_{xx} = -e^{-t}\sin(x/c) - c^2(-(1/c^2)e^{-t}\sin(x/c)) = 0$.

(b) $z_t = -e^{-t}\cos(x/c)$, $z_x = -(1/c)e^{-t}\sin(x/c)$, $z_{xx} = -(1/c^2)e^{-t}\cos(x/c)$; $z_t - c^2 z_{xx} = -e^{-t}\cos(x/c) - c^2(-(1/c^2)e^{-t}\cos(x/c)) = 0$.

103. $u_x = \omega\sin c\omega t\cos\omega x$, $u_{xx} = -\omega^2\sin c\omega t\sin\omega x$, $u_t = c\omega\cos c\omega t\sin\omega x$, $u_{tt} = -c^2\omega^2\sin c\omega t\sin\omega x$; $u_{xx} - \dfrac{1}{c^2}u_{tt} = -\omega^2\sin c\omega t\sin\omega x - \dfrac{1}{c^2}(-c^2)\omega^2\sin c\omega t\sin\omega x = 0$.

104. **(a)** $\partial u/\partial x = \partial v/\partial y = 2x$, $\partial u/\partial y = -\partial v/\partial x = -2y$.

(b) $\partial u/\partial x = \partial v/\partial y = e^x\cos y$, $\partial u/\partial y = -\partial v/\partial x = -e^x\sin y$.

(c) $\partial u/\partial x = \partial v/\partial y = 2x/(x^2+y^2)$, $\partial u/\partial y = -\partial v/\partial x = 2y/(x^2+y^2)$.

105. $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ so $\partial^2 u/\partial x^2 = \partial^2 v/\partial x\partial y$, and $\partial^2 u/\partial y^2 = -\partial^2 v/\partial y\partial x$, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \partial^2 v/\partial x\partial y - \partial^2 v/\partial y\partial x$, if $\partial^2 v/\partial x\partial y = \partial^2 v/\partial y\partial x$ then $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$; thus u satisfies Laplace's equation. The proof that v satisfies Laplace's equation is similar. Adding Laplace's equations for u and v gives Laplaces' equation for $u+v$.

106. $\partial^2 R/\partial R_1^2 = -2R_2^2/(R_1+R_2)^3$, $\partial^2 R/\partial R_2^2 = -2R_1^2/(R_1+R_2)^3$, $\left(\partial^2 R/\partial R_1^2\right)\left(\partial^2 R/\partial R_2^2\right) = 4R_1^2R_2^2/\left(R_1+R_2\right)^6 = \left[4/\left(R_1+R_2\right)^4\right]\left[R_1R_2/\left(R_1+R_2\right)\right]^2 = 4R^2/\left(R_1+R_2\right)^4$.

107. $\partial f/\partial v = 8vw^3x^4y^5$, $\partial f/\partial w = 12v^2w^2x^4y^5$, $\partial f/\partial x = 16v^2w^3x^3y^5$, $\partial f/\partial y = 20v^2w^3x^4y^4$.

108. $\partial w/\partial r = \cos st + ue^u\cos ur$, $\partial w/\partial s = -rt\sin st$, $\partial w/\partial t = -rs\sin st$, $\partial w/\partial u = re^u\cos ur + e^u\sin ur$.

109. $\partial f/\partial v_1 = 2v_1/\left(v_3^2+v_4^2\right)$, $\partial f/\partial v_2 = -2v_2/\left(v_3^2+v_4^2\right)$, $\partial f/\partial v_3 = -2v_3\left(v_1^2-v_2^2\right)/\left(v_3^2+v_4^2\right)^2$,
$\partial f/\partial v_4 = -2v_4\left(v_1^2-v_2^2\right)/\left(v_3^2+v_4^2\right)^2$.

110. $\dfrac{\partial V}{\partial x} = 2xe^{2x-y} + e^{2x-y}$, $\dfrac{\partial V}{\partial y} = -xe^{2x-y} + w$, $\dfrac{\partial V}{\partial z} = w^2e^{zw}$, $\dfrac{\partial V}{\partial w} = wze^{zw} + e^{zw} + y$.

111. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 0 **(e)** $2(1+yw)e^{yw}\sin z\cos z$ **(f)** $2xw(2+yw)e^{yw}\sin z\cos z$

112. 128, -512, 32, 64/3.

113. $\partial w/\partial x_i = -i\sin(x_1 + 2x_2 + \ldots + nx_n)$.

114. $\partial w/\partial x_i = \dfrac{1}{n}\left(\displaystyle\sum_{k=1}^{n} x_k\right)^{(1/n)-1}$.

115. **(a)** xy-plane, $f_x = 12x^2y + 6xy$, $f_y = 4x^3 + 3x^2$, $f_{xy} = f_{yx} = 12x^2 + 6x$.

(b) $y \neq 0, f_x = 3x^2/y$, $f_y = -x^3/y^2$, $f_{xy} = f_{yx} = -3x^2/y^2$.

116. (a) $x^2 + y^2 > 1$, (the exterior of the circle of radius 1 about the origin); $f_x = x/\sqrt{x^2+y^2-1}$, $f_y = y/\sqrt{x^2+y^2-1}$, $f_{xy} = f_{yx} = -xy\left(x^2+y^2-1\right)^{-3/2}$.

(b) xy-plane, $f_x = 2x\cos(x^2+y^3)$, $f_y = 3y^2\cos(x^2+y^3)$, $f_{xy} = f_{yx} = -6xy^2\sin\left(x^2+y^3\right)$.

117. $f_x(2,-1) = \lim_{x\to 2}\dfrac{f(x,-1)-f(2,-1)}{x-2} = \lim_{x\to 2}\dfrac{2x^2+3x+1-15}{x-2} = \lim_{x\to 2}(2x+7) = 11$ and

$f_y(2,-1) = \lim_{y\to -1}\dfrac{f(2,y)-f(2,-1)}{y+1} = \lim_{y\to -1}\dfrac{8-6y+y^2-15}{y+1} = \lim_{y\to -1} y-7 = -8.$

118. $f_x(x,y) = \dfrac{2}{3}(x^2+y^2)^{-1/3}(2x) = \dfrac{4x}{3(x^2+y^2)^{1/3}}$, $(x,y)\neq(0,0)$; and by definition, $f_x(0,0) = \lim_{h\to 0}\dfrac{((h)^2)^{2/3}-0}{h} = 0.$

119. (a) $f_y(0,0) = \dfrac{d}{dy}[f(0,y)]\Big|_{y=0} = \dfrac{d}{dy}[y]\Big|_{y=0} = 1.$

(b) If $(x,y)\neq(0,0)$, then $f_y(x,y) = \dfrac{1}{3}(x^3+y^3)^{-2/3}(3y^2) = \dfrac{y^2}{(x^3+y^3)^{2/3}}$; $f_y(x,y)$ does not exist when $y\neq 0$ and $y=-x$.

Exercise Set 13.4

1. $f(x,y) \approx f(3,4) + f_x(x-3) + f_y(y-4) = 5 + 2(x-3) - (y-4)$ and $f(3.01, 3.98) \approx 5 + 2(0.01) - (-0.02) = 5.04$.

2. $f(x,y) \approx f(-1,2) + f_x(x+1) + f_y(y-2) = 2 + (x+1) + 3(y-2)$ and $f(-0.99, 2.02) \approx 2 + 0.01 + 3(0.02) = 2.07$.

3. $L(x,y,z) = f(1,2,3) + (x-1) + 2(y-2) + 3(z-3)$, $f(1.01, 2.02, 3.03) \approx 4 + 0.01 + 2(0.02) + 3(0.03) = 4.14$.

4. $L(x,y,z) = f(2,1,-2) - (x-2) + (y-1) - 2(z+2)$, $f(1.98, 0.99, -1.97) \approx 0.02 - 0.01 - 2(0.03) = -0.05$.

5. Suppose $f(x,y) = c$ for all (x,y). Then at (x_0,y_0) we have $\dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} = 0$ and hence $f_x(x_0,y_0)$ exists and is equal to 0 (Definition 13.3.1). A similar result holds for f_y. From equation (2), it follows that $\Delta f = 0$, and then by Definition 13.4.1 we see that f is differentiable at (x_0,y_0). An analogous result holds for functions $f(x,y,z)$ of three variables.

6. Let $f(x,y) = ax + by + c$. Then $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = ax_0 + by_0 + c + a(x-x_0) + b(y-y_0) = ax + by + c$, so $L = f$ and thus E is zero. For three variables the proof is analogous.

7. $f_x = 2x, f_y = 2y, f_z = 2z$ so $L(x,y,z) = 0$, $E = f - L = x^2+y^2+z^2$, and $\lim_{(x,y,z)\to(0,0,0)}\dfrac{E(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \lim_{(x,y,z)\to(0,0,0)}\sqrt{x^2+y^2+z^2} = 0$, so f is differentiable at $(0,0,0)$.

8. $f_x = 2xr(x^2+y^2+z^2)^{r-1}, f_y = 2yr(x^2+y^2+z^2)^{r-1}, f_z = 2zr(x^2+y^2+z^2)^{r-1}$, so the partials of f exist only if $r \geq 1$. If so then $L(x,y,z) = 0, E(x,y,z) = f(x,y,z)$ and $\dfrac{E(x,y,z)}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{r-1/2}$, so f is differentiable at $(0,0,0)$ if and only if $r \geq \max(1/2, 1) = 1$.

9. $dz = 7dx - 2dy$.

10. $dz = ye^{xy}dx + xe^{xy}dy$.

11. $dz = 3x^2y^2dx + 2x^3ydy$.

12. $dz = (10xy^5 - 2)dx + (25x^2y^4 + 4)dy$.

13. $dz = \left[y/\left(1+x^2y^2\right)\right]dx + \left[x/\left(1+x^2y^2\right)\right]dy$.

14. $dz = -3e^{-3x}\cos 6y dx - 6e^{-3x}\sin 6y dy$.

15. $dw = 8dx - 3dy + 4dz$.

16. $dw = yze^{xyz}dx + xze^{xyz}dy + xye^{xyz}dz$.

17. $dw = 3x^2y^2zdx + 2x^3yzdy + x^3y^2dz$.

18. $dw = \left(8xy^3z^7 - 3y\right)dx + \left(12x^2y^2z^7 - 3x\right)dy + \left(28x^2y^3z^6 + 1\right)dz$.

19. $dw = \dfrac{yz}{1+x^2y^2z^2}dx + \dfrac{xz}{1+x^2y^2z^2}dy + \dfrac{xy}{1+x^2y^2z^2}dz$.

20. $dw = \dfrac{1}{2\sqrt{x}}dx + \dfrac{1}{2\sqrt{y}}dy + \dfrac{1}{2\sqrt{z}}dz$.

21. $df = (2x + 2y - 4)dx + 2xdy$; $x = 1$, $y = 2$, $dx = 0.01$, $dy = 0.04$ so $df = 0.10$ and $\Delta f = 0.1009$.

22. $df = (1/3)x^{-2/3}y^{1/2}dx + (1/2)x^{1/3}y^{-1/2}dy$; $x = 8$, $y = 9$, $dx = -0.22$, $dy = 0.03$ so $df = -0.045$ and $\Delta f \approx -0.045613$.

23. $df = -x^{-2}dx - y^{-2}dy$; $x = -1$, $y = -2$, $dx = -0.02$, $dy = -0.04$ so $df = 0.03$ and $\Delta f \approx 0.029412$.

24. $df = \dfrac{y}{2(1+xy)}dx + \dfrac{x}{2(1+xy)}dy$; $x = 0$, $y = 2$, $dx = -0.09$, $dy = -0.02$ so $df = -0.09$ and $\Delta f \approx -0.098129$.

25. $df = 2y^2z^3dx + 4xyz^3dy + 6xy^2z^2dz, x = 1, y = -1, z = 2, dx = -0.01, dy = -0.02, dz = 0.02$ so $df = 0.96$ and $\Delta f \approx 0.97929$.

26. $df = \dfrac{yz(y+z)}{(x+y+z)^2}dx + \dfrac{xz(x+z)}{(x+y+z)^2}dy + \dfrac{xy(x+y)}{(x+y+z)^2}dz, x = -1, y = -2, z = 4, dx = -0.04, dy = 0.02, dz = -0.03$ so $df = 0.58$ and $\Delta f \approx 0.60529$.

27. False: Example 9, Section 13.3 gives such a function which is not even continuous at (x_0, y_0), let alone differentiable.

28. False; only where f is continuous, since by Theorem 13.2.3 the condition given is equivalent to continuity.

29. True; indeed, by Theorem 13.4.4, f is differentiable.

30. True; from (9), it has normal vector $f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$ and passes through $(x_0, y_0, f(x_0, y_0))$.

31. Label the four smaller rectangles A, B, C, D starting with the lower left and going clockwise. Then the increase in the area of the rectangle is represented by B, C and D; and the portions B and D represent the approximation of the increase in area given by the total differential.

32. $V + \Delta V = (\pi/3)4.05^2(19.95) \approx 109.0766250\pi, V = 320\pi/3, \Delta V \approx 2.40996\pi$; $dV = (2/3)\pi rhdr + (1/3)\pi r^2dh$; $r = 4$, $h = 20$, $dr = 0.05$, $dh = -0.05$ so $dV = 2.4\pi$, and $\Delta V/dV \approx 1.00415$.

33. **(a)** $f(P) = 1/5, f_x(P) = -x/(x^2+y^2)^{-3/2}\Big|_{(x,y)=(4,3)} = -4/125$, $f_y(P) = -y/(x^2+y^2)^{-3/2}\Big|_{(x,y)=(4,3)} = -3/125, L(x,y) = \dfrac{1}{5} - \dfrac{4}{125}(x-4) - \dfrac{3}{125}(y-3)$.

(b) $L(Q) - f(Q) = \dfrac{1}{5} - \dfrac{4}{125}(-0.08) - \dfrac{3}{125}(0.01) - 0.2023342382 \approx -0.0000142382$, $|PQ| = \sqrt{0.08^2 + 0.01^2} \approx 0.08062257748$, $|L(Q) - f(Q)|/|PQ| \approx 0.000176603$.

34. (a) $f(P) = 1, f_x(P) = 0.5, f_y(P) = 0.3, L(x,y) = 1 + 0.5(x-1) + 0.3(y-1)$.

(b) $L(Q) - f(Q) = 1 + 0.5(0.05) + 0.3(-0.03) - 1.05^{0.5}0.97^{0.3} \approx 0.00063$, $|PQ| = \sqrt{0.05^2 + 0.03^2} \approx 0.05831$, $|L(Q) - f(Q)|/|PQ| \approx 0.0107$.

35. (a) $f(P) = 0, f_x(P) = 0, f_y(P) = 0, L(x,y) = 0$.

(b) $L(Q) - f(Q) = -0.003\sin(0.004) \approx -0.000012, |PQ| = \sqrt{0.003^2 + 0.004^2} = 0.005$, $|L(Q) - f(Q)|/|PQ| \approx 0.0024$.

36. (a) $f(P) = \ln 2, f_x(P) = 1, f_y(P) = 1/2, L(x,y) = \ln 2 + (x-1) + \dfrac{1}{2}(y-2)$.

(b) $L(Q) - f(Q) = \ln 2 + 0.01 + (1/2)(0.02) - \ln 2.0402 \approx 0.0000993383$, $|PQ| = \sqrt{0.01^2 + 0.02^2} \approx 0.02236067978$, $|L(Q) - f(Q)|/|PQ| \approx 0.0044425$.

37. (a) $f(P) = 6, f_x(P) = 6, f_y(P) = 3, f_z(P) = 2, L(x,y) = 6 + 6(x-1) + 3(y-2) + 2(z-3)$.

(b) $L(Q) - f(Q) = 6 + 6(0.001) + 3(0.002) + 2(0.003) - 6.018018006 = -.000018006$,
$|PQ| = \sqrt{0.001^2 + 0.002^2 + 0.003^2} \approx .0003741657387$; $|L(Q) - f(Q)|/|PQ| \approx -0.000481$.

38. (a) $f(P) = 0, f_x(P) = 1/2, f_y(P) = 1/2, f_z(P) = 0, L(x,y) = \dfrac{1}{2}(x+1) + \dfrac{1}{2}(y-1)$.

(b) $L(Q) - f(Q) = 0, |L(Q) - f(Q)|/|PQ| = 0$.

39. (a) $f(P) = e, f_x(P) = e, f_y(P) = -e, f_z(P) = -e, L(x,y) = e + e(x-1) - e(y+1) - e(z+1)$.

(b) $L(Q) - f(Q) = e - 0.01e + 0.01e - 0.01e - 0.99e^{0.9999} = 0.99(e - e^{0.9999})$, $|PQ| = \sqrt{0.01^2 + 0.01^2 + 0.01^2} \approx 0.01732, |L(Q) - f(Q)|/|PQ| \approx 0.01554$.

40. (a) $f(P) = 0, f_x(P) = 1, f_y(P) = -1, f_z(P) = 1,\ L(x,y,z) = (x-2) - (y-1) + (z+1)$.

(b) $L(Q) - f(Q) = 0.02 + 0.03 - 0.01 - \ln 1.0403 \approx 0.00049086691$, $|PQ| = \sqrt{0.02^2 + 0.03^2 + 0.01^2} \approx 0.03742$, $|L(Q) - f(Q)|/|PQ| \approx 0.01312$.

41. (a) Let $f(x,y) = e^x \sin y$; $f(0,0) = 0, f_x(0,0) = 0, f_y(0,0) = 1$, so $e^x \sin y \approx y$.

(b) Let $f(x,y) = \dfrac{2x+1}{y+1}$; $f(0,0) = 1, f_x(0,0) = 2, f_y(0,0) = -1$, so $\dfrac{2x+1}{y+1} \approx 1 + 2x - y$.

42. $f(1,1) = 1, f_x(x,y) = \alpha x^{\alpha-1}y^\beta, f_x(1,1) = \alpha, f_y(x,y) = \beta x^\alpha y^{\beta-1}, f_y(1,1) = \beta$, so $x^\alpha y^\beta \approx 1 + \alpha(x-1) + \beta(y-1)$.

43. (a) Let $f(x,y,z) = xyz + 2$, then $f_x = f_y = f_z = 1$ at $x = y = z = 1$, and $L(x,y,z) = f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) = 3 + x - 1 + y - 1 + z - 1 = x + y + z$.

(b) Let $f(x,y,z) = \dfrac{4x}{y+z}$, then $f_x = 2, f_y = -1, f_z = -1$ at $x = y = z = 1$, and $L(x,y,z) = f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) = 2 + 2(x-1) - (y-1) - (z-1) = 2x - y - z + 2$.

44. Let $f(x,y,z) = x^\alpha y^\beta z^\gamma$, then $f_x = \alpha, f_y = \beta, f_z = \gamma$ at $x = y = z = 1$, and $f(x,y,z) \approx f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) = 1 + \alpha(x-1) + \beta(y-1) + \gamma(z-1)$.

45. $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$ and $L(1.1, 0.9) = 3.15 = 3 + 2(0.1) + f_y(1,1)(-0.1)$ so $f_y(1,1) = -0.05/(-0.1) = 0.5$.

46. $L(x,y) = 3 + f_x(0,-1)x - 2(y+1)$, $3.3 = 3 + f_x(0,-1)(0.1) - 2(-0.1)$, so $f_x(0,-1) = 0.1/0.1 = 1$.

47. $x - y + 2z - 2 = L(x,y,z) = f(3,2,1) + f_x(3,2,1)(x-3) + f_y(3,2,1)(y-2) + f_z(3,2,1)(z-1)$, so $f_x(3,2,1) = 1, f_y(3,2,1) = -1, f_z(3,2,1) = 2$ and $f(3,2,1) = L(3,2,1) = 1$.

48. $L(x,y,z) = x + 2y + 3z + 4 = (x-0) + 2(y+1) + 3(z+2) - 4, f(0,-1,-2) = -4, f_x(0,-1,-2) = 1, f_y(0,-1,-2) = 2, f_z(0,-1,-2) = 3$.

49. $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0), 2y - 2x - 2 = x_0^2 + y_0^2 + 2x_0(x-x_0) + 2y_0(y-y_0)$, from which it follows that $x_0 = -1, y_0 = 1$.

50. $f(x,y) = x^2y$, so $f_x(x_0,y_0) = 2x_0y_0,\ f_y(x_0,y_0) = x_0^2$, and $L(x,y) = f(x_0,y_0) + 2x_0y_0(x-x_0) + x_0^2(y-y_0)$. But $L(x,y) = 8 - 4x + 4y$, hence $-4 = 2x_0y_0,\ 4 = x_0^2$ and $8 = f(x_0,y_0) - 2x_0^2y_0 - x_0^2y_0 = -2x_0^2y_0$. Thus either $x_0 = -2, y_0 = 1$ from which it follows that $8 = -8$, a contradiction, or $x_0 = 2, y_0 = -1$, which is a solution since then $8 = -2x_0^2y_0 = 8$ is true.

51. $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0),\ y + 2z - 1 = x_0y_0 + z_0^2 + y_0(x-x_0) + x_0(y-y_0) + 2z_0(z-z_0)$, so that $x_0 = 1, y_0 = 0, z_0 = 1$.

52. $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$. Then $x - y - z - 2 = x_0y_0z_0 + y_0z_0(x-x_0) + x_0z_0(y-y_0) + x_0y_0(z-z_0)$, hence $y_0z_0 = 1, x_0z_0 = -1, x_0y_0 = -1$, and $-2 = x_0y_0z_0 - 3x_0y_0z_0$, or $x_0y_0z_0 = 1$. Since now $x_0 = -y_0 = -z_0$, we must have $|x_0| = |y_0| = |z_0| = 1$ or else $|x_0y_0z_0| \neq 1$, impossible. Thus $x_0 = 1, y_0 = z_0 = -1$ (note that $(-1,1,1)$ is not a solution).

53. $A = xy,\ dA = ydx + xdy,\ dA/A = dx/x + dy/y,\ |dx/x| \le 0.03$ and $|dy/y| \le 0.05,\ |dA/A| \le |dx/x| + |dy/y| \le 0.08 = 8\%$.

54. $V = (1/3)\pi r^2 h,\ dV = (2/3)\pi rh dr + (1/3)\pi r^2 dh,\ dV/V = 2(dr/r) + dh/h,\ |dr/r| \le 0.01$ and $|dh/h| \le 0.04$, $|dV/V| \le 2|dr/r| + |dh/h| \le 0.06 = 6\%$.

55. $dT = \dfrac{\pi}{g\sqrt{L/g}}dL - \dfrac{\pi L}{g^2\sqrt{L/g}}dg,\ \dfrac{dT}{T} = \dfrac{1}{2}\dfrac{dL}{L} - \dfrac{1}{2}\dfrac{dg}{g}$; $|dL/L| \le 0.005$ and $|dg/g| \le 0.001$ so $|dT/T| \le (1/2)(0.005) + (1/2)(0.001) = 0.003 = 0.3\%$.

56. $d\nu = \dfrac{1}{2}B^{-1/2}\rho^{-1/2}dB - \dfrac{1}{2}B^{1/2}\rho^{-3/2}d\rho$, thus $\dfrac{d\nu}{\nu} = \dfrac{1}{2}\dfrac{dB}{B} - \dfrac{1}{2}\dfrac{d\rho}{\rho}$. We are given that $|dB/B| \le 0.007$ and $|d\rho/\rho| \le 0.003$, so $|d\nu/\nu| \le (1/2)(0.007) + (1/2)(0.003) = 0.005 = 0.5\%$.

57. $E = kq/r^2$, thus $dE = kr^{-2}dq - 2kqr^{-3}dr$, and then $dE/E = dq/q - 2dr/r$. We are given that $|dq/q| \le 0.002$ and $|dr/r| \le 0.005$, so $|dE/E| \le 0.002 + 2(0.005) = 0.012 = 1.2\%$.

58. $dP = (k/V)dT - (kT/V^2)dV,\ dP/P = dT/T - dV/V$; if $dT/T = 0.03$ and $dV/V = 0.05$ then $dP/P = -0.02$ so there is about a 2% decrease in pressure.

59. **(a)** $\left|\dfrac{d(xy)}{xy}\right| = \left|\dfrac{y\,dx + x\,dy}{xy}\right| = \left|\dfrac{dx}{x} + \dfrac{dy}{y}\right| \le \left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \le \dfrac{r}{100} + \dfrac{s}{100}$; $(r+s)\%$.

(b) $\left|\dfrac{d(x/y)}{x/y}\right| = \left|\dfrac{y\,dx - x\,dy}{xy}\right| = \left|\dfrac{dx}{x} - \dfrac{dy}{y}\right| \le \left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \le \dfrac{r}{100} + \dfrac{s}{100}$; $(r+s)\%$.

(c) $\left|\dfrac{d(x^2y^3)}{x^2y^3}\right| = \left|\dfrac{2xy^3\,dx + 3x^2y^2\,dy}{x^2y^3}\right| = \left|2\dfrac{dx}{x} + 3\dfrac{dy}{y}\right| \le 2\left|\dfrac{dx}{x}\right| + 3\left|\dfrac{dy}{y}\right| \le 2\dfrac{r}{100} + 3\dfrac{s}{100}$; $(2r+3s)\%$.

(d) $\left|\dfrac{d(x^3y^{1/2})}{x^3y^{1/2}}\right| = \left|\dfrac{3x^2y^{1/2}\,dx + (1/2)x^3y^{-1/2}\,dy}{x^3y^{1/2}}\right| = \left|3\dfrac{dx}{x} + \dfrac{1}{2}\dfrac{dy}{y}\right| \le 3\left|\dfrac{dx}{x}\right| + \dfrac{1}{2}\left|\dfrac{dy}{y}\right| \le 3\dfrac{r}{100} + \dfrac{1}{2}\dfrac{s}{100}$; $(3r + \dfrac{1}{2}s)\%$.

60. $R = 1/(1/R_1 + 1/R_2 + 1/R_3)$, $\partial R/\partial R_1 = \dfrac{1}{R_1^2(1/R_1 + 1/R_2 + 1/R_3)^2} = R^2/R_1^2$, similarly $\partial R/\partial R_2 = R^2/R_2^2$ and $\partial R/\partial R_3 = R^2/R_3^2$ so $\dfrac{dR}{R} = (R/R_1)\dfrac{dR_1}{R_1} + (R/R_2)\dfrac{dR_2}{R_2} + (R/R_3)\dfrac{dR_3}{R_3}$, $\left|\dfrac{dR}{R}\right| \le (R/R_1)\left|\dfrac{dR_1}{R_1}\right| + (R/R_2)\left|\dfrac{dR_2}{R_2}\right| + (R/R_3)\left|\dfrac{dR_3}{R_3}\right| \le (R/R_1)(0.10) + (R/R_2)(0.10) + (R/R_3)(0.10) = R(1/R_1 + 1/R_2 + 1/R_3)(0.10) = (1)(0.10) = 0.10 = 10\%$.

61. $dA = \dfrac{1}{2}b\sin\theta da + \dfrac{1}{2}a\sin\theta db + \dfrac{1}{2}ab\cos\theta d\theta$, $|dA| \le \dfrac{1}{2}b\sin\theta|da| + \dfrac{1}{2}a\sin\theta|db| + \dfrac{1}{2}ab\cos\theta|d\theta| \le \dfrac{1}{2}(50)(1/2)(1/2) + \dfrac{1}{2}(40)(1/2)(1/4) + \dfrac{1}{2}(40)(50)\left(\sqrt{3}/2\right)(\pi/90) = 35/4 + 50\pi\sqrt{3}/9 \approx 39\text{ ft}^2$.

62. $V = \ell wh$, $dV = whd\ell + \ell hdw + \ell wdh$, $|dV/V| \le |d\ell/\ell| + |dw/w| + |dh/h| \le 3(r/100) = 3r\%$.

63. $f_x = 2x\sin y, f_y = x^2\cos y$ are both continuous everywhere, so f is differentiable everywhere.

64. $f_x = y\sin z, f_y = x\sin z, f_z = xy\cos z$ are all continuous everywhere, so f is differentiable everywhere.

65. That f is differentiable means that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{E_f(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$, where $E_f(x,y) = f(x,y) - L_f(x,y)$; here $L_f(x,y)$ is the linear approximation to f at (x_0,y_0). Let f_x and f_y denote $f_x(x_0,y_0), f_y(x_0,y_0)$ respectively. Then $g(x,y,z) = z - f(x,y), L_f(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0), L_g(x,y,z) = g(x_0,y_0,z_0) + g_x(x-x_0) + g_y(y-y_0) + g_z(z-z_0) = 0 - f_x(x-x_0) - f_y(y-y_0) + (z-z_0)$, and $E_g(x,y,z) = g(x,y,z) - L_g(x,y,z) = (z-f(x,y)) + f_x(x-x_0) + f_y(y-y_0) - (z-z_0) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - f(x,y) = -E_f(x,y)$. Thus $\dfrac{|E_g(x,y,z)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \le \dfrac{|E_f(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$, so
$\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E_g(x,y,z)}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} = 0$ and g is differentiable at (x_0,y_0,z_0).

66. The condition $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$ is equivalent to $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \epsilon(\Delta x,\Delta y) = 0$ which is equivalent to ϵ being continuous at $(0,0)$ with $\epsilon(0,0) = 0$. Since ϵ is continuous, f is differentiable.

Exercise Set 13.5

1. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 42t^{13}$.

2. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \dfrac{2(3 + t^{-1/3})}{3(2t + t^{2/3})}$.

3. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 3t^{-2}\sin(1/t)$.

4. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \dfrac{1 - 2t^4 - 8t^4\ln t}{2t\sqrt{1 + \ln t - 2t^4\ln t}}$.

5. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = -\dfrac{10}{3}t^{7/3}e^{1-t^{10/3}}$.

6. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (1+t)e^t\cosh\left(te^t/2\right)\sinh\left(te^t/2\right)$.

7. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 165t^{32}$.

8. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = \dfrac{3-(4/3)t^{-1/3}-24t^{-7}}{3t-2t^{2/3}+4t^{-6}}$.

9. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = -2t\cos\left(t^2\right)$.

10. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = \dfrac{1-512t^5-2560t^5\ln t}{2t\sqrt{1+\ln t-512t^5\ln t}}$.

11. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 3264$.

12. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 0$.

13. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 3(2t)_{t=2} - (3t^2)_{t=2} = 12-12 = 0$.

14. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 1 + 2(\pi\cos\pi t)_{t=1} + 3(2t)_{t=1} = 1-2\pi+6 = 7-2\pi$.

15. Let $z = xy$, and let $x = f(t)$ and $y = g(t)$. Then $z = f(t)g(t)$ and $(f(t)g(t))' = \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = y\dfrac{dx}{dt} + x\dfrac{dy}{dt} = g(t)f'(t) + f(t)g'(t)$.

16. Let $z = x^y$, and let $x = t$ and $y = t$. Then $z = t^t$ and $(t^t)' = \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = yx^{y-1}\dfrac{dx}{dt} + (\ln x)\,x^y\dfrac{dy}{dt} = t\cdot t^{t-1} + (\ln t)\,t^t = t^t + (\ln t)\,t^t$.

17. $\partial z/\partial u = 24u^2v^2 - 16uv^3 - 2v + 3,\ \partial z/\partial v = 16u^3v - 24u^2v^2 - 2u - 3$.

18. $\partial z/\partial u = 2u/v^2 - u^2v\sec^2(u/v) - 2uv^2\tan(u/v)$, $\partial z/\partial v = -2u^2/v^3 + u^3\sec^2(u/v) - 2u^2v\tan(u/v)$.

19. $\partial z/\partial u = -\dfrac{2\sin u}{3\sin v},\ \partial z/\partial v = -\dfrac{2\cos u\cos v}{3\sin^2 v}$.

20. $\partial z/\partial u = 3 + 3v/u - 4u,\ \partial z/\partial v = 2 + 3\ln u + 2\ln v$.

21. $\partial z/\partial u = e^u$, $\partial z/\partial v = 0$.

22. $\partial z/\partial u = -\sin(u-v)\sin\left(u^2+v^2\right) + 2u\cos(u-v)\cos\left(u^2+v^2\right)$,
$\partial z/\partial v = \sin(u-v)\sin\left(u^2+v^2\right) + 2v\cos(u-v)\cos\left(u^2+v^2\right)$.

23. $\partial T/\partial r = 3r^2\sin\theta\cos^2\theta - 4r^3\sin^3\theta\cos\theta$, $\partial T/\partial\theta = -2r^3\sin^2\theta\cos\theta + r^4\sin^4\theta + r^3\cos^3\theta - 3r^4\sin^2\theta\cos^2\theta$.

24. $dR/d\phi = 5e^{5\phi}$.

25. $\partial t/\partial x = \left(x^2+y^2\right)/\left(4x^2y^3\right)$, $\partial t/\partial y = \left(y^2-3x^2\right)/\left(4xy^4\right)$.

26. $\partial w/\partial u = \dfrac{2v^2\left[u^2v^2-(u-2v)^2\right]}{\left[u^2v^2+(u-2v)^2\right]^2}$, $\partial w/\partial v = \dfrac{u^2\left[(u-2v)^2-u^2v^2\right]}{\left[u^2v^2+(u-2v)^2\right]^2}$.

27. $\partial z/\partial r = (dz/dx)(\partial x/\partial r) = 2r\cos^2\theta/\left(r^2\cos^2\theta+1\right)$, $\partial z/\partial\theta = (dz/dx)(\partial x/\partial\theta) = -2r^2\sin\theta\cos\theta/\left(r^2\cos^2\theta+1\right)$.

28. $\partial u/\partial x = (\partial u/\partial r)(dr/dx) + (\partial u/\partial t)(\partial t/\partial x) = \left(s^2\ln t\right)(2x) + \left(rs^2/t\right)\left(y^3\right) = x(4y+1)^2\left(1+2\ln xy^3\right)$, $\partial u/\partial y = (\partial u/\partial s)(ds/dy) + (\partial u/\partial t)(\partial t/\partial y) = (2rs\ln t)(4) + \left(rs^2/t\right)\left(3xy^2\right) = 8x^2(4y+1)\ln xy^3 + 3x^2(4y+1)^2/y$.

29. $\partial w/\partial\rho = 2\rho\left(4\sin^2\phi+\cos^2\phi\right)$, $\partial w/\partial\phi = 6\rho^2\sin\phi\cos\phi$, $\partial w/\partial\theta = 0$.

30. $\dfrac{dw}{dx} = \dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dx} = 3y^2z^3 + (6xyz^3)(6x) + 9xy^2z^2\dfrac{1}{2\sqrt{x-1}} = 3(3x^2+2)^2(x-1)^{3/2}+$
$+36x^2(3x^2+2)(x-1)^{3/2} + \dfrac{9}{2}x(3x^2+2)^2\sqrt{x-1} = \dfrac{3}{2}(3x^2+2)(39x^3-30x^2+10x-4)\sqrt{x-1}$.

31. $-\pi$.

32. $351/2$, -168.

33. $\sqrt{3}e^{\sqrt{3}}$, $\left(2-4\sqrt{3}\right)e^{\sqrt{3}}$.

34. 1161.

35. $A = \dfrac{1}{2}ab\sin\theta$, so $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial a}\dfrac{da}{dt} + \dfrac{\partial A}{\partial b}\dfrac{db}{dt} + \dfrac{\partial A}{\partial\theta}\dfrac{d\theta}{dt}$. This gives us $0 = \dfrac{dA}{dt} = \dfrac{1}{2}b\sin\theta\dfrac{da}{dt} + \dfrac{1}{2}a\sin\theta\dfrac{db}{dt} + \dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}$. From here, $\dfrac{d\theta}{dt} = -(b\sin\theta\dfrac{da}{dt} + a\sin\theta\dfrac{db}{dt})/(ab\cos\theta)$, and with the given values, $\dfrac{d\theta}{dt} = -\dfrac{9\sqrt{3}}{20} \approx -0.779423$ rad/s.

36. $V = IR$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial I}\dfrac{dI}{dt} + \dfrac{\partial V}{\partial R}\dfrac{dR}{dt} = R\dfrac{dI}{dt} + I\dfrac{dR}{dt}$. We also know that $R = \dfrac{R_1R_2}{R_1+R_2}$, which gives us $\dfrac{dR}{dt} = \dfrac{\partial R}{\partial R_1}\dfrac{dR_1}{dt} + \dfrac{\partial R}{\partial R_2}\dfrac{dR_2}{dt} = \dfrac{R_2^2}{(R_1+R_2)^2}\dfrac{dR_1}{dt} + \dfrac{R_1^2}{(R_1+R_2)^2}\dfrac{dR_2}{dt}$. With the given values, we get $\dfrac{dV}{dt} \approx 0.455$ V/s.

37. False; by themselves they have no meaning.

38. True; this is the chain rule.

39. False; consider $z = xy, x = t, y = t$; then $z = t^2$.

40. True; use the chain rule to differentiate both sides of the equation $f(t,t) = c$.

41. $F(x,y) = x^2y^3 + \cos y$, $\dfrac{dy}{dx} = -\dfrac{2xy^3}{3x^2y^2 - \sin y}$.

42. $F(x,y) = x^3 - 3xy^2 + y^3 - 5$, $\dfrac{dy}{dx} = -\dfrac{3x^2-3y^2}{-6xy+3y^2} = \dfrac{x^2-y^2}{2xy-y^2}$.

43. $F(x,y) = e^{xy} + ye^y - 1$, $\dfrac{dy}{dx} = -\dfrac{ye^{xy}}{xe^{xy}+ye^y+e^y}$.

44. $F(x,y) = x - (xy)^{1/2} + 3y - 4$, $\dfrac{dy}{dx} = -\dfrac{1-(1/2)(xy)^{-1/2}y}{-(1/2)(xy)^{-1/2}x+3} = \dfrac{2\sqrt{xy}-y}{x-6\sqrt{xy}}$.

45. $\dfrac{\partial z}{\partial x} = \dfrac{2x+yz}{6yz-xy}$, $\dfrac{\partial z}{\partial y} = \dfrac{xz-3z^2}{6yz-xy}$.

46. $\ln(1+z) + xy^2 + z - 1 = 0$; $\dfrac{\partial z}{\partial x} = -\dfrac{y^2(1+z)}{2+z}$, $\dfrac{\partial z}{\partial y} = -\dfrac{2xy(1+z)}{2+z}$.

47. $ye^x - 5\sin 3z - 3z = 0$; $\dfrac{\partial z}{\partial x} = -\dfrac{ye^x}{-15\cos 3z - 3} = \dfrac{ye^x}{15\cos 3z + 3}$, $\dfrac{\partial z}{\partial y} = \dfrac{e^x}{15\cos 3z + 3}$.

48. $\dfrac{\partial z}{\partial x} = -\dfrac{ze^{yz}\cos xz - ye^{xy}\cos yz}{ye^{xy}\sin yz + xe^{yz}\cos xz + ye^{yz}\sin xz}$, $\dfrac{\partial z}{\partial y} = -\dfrac{ze^{xy}\sin yz - xe^{xy}\cos yz + ze^{yz}\sin xz}{ye^{xy}\sin yz + xe^{yz}\cos xz + ye^{yz}\sin xz}$.

49. **(a)** $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x}$, $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\dfrac{\partial u}{\partial y}$.

(b) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}\left(\dfrac{dz}{du}\right)\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{d^2 z}{du^2}\left(\dfrac{\partial u}{\partial x}\right)^2$;

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial}{\partial y}\left(\dfrac{dz}{du}\right)\dfrac{\partial u}{\partial y} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y^2} + \dfrac{d^2 z}{du^2}\left(\dfrac{\partial u}{\partial y}\right)^2$; $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{\partial}{\partial y}\left(\dfrac{dz}{du}\right)\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{d^2 z}{du^2}\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}$.

50. **(a)** $z = f(u)$, $u = x^2 - y^2$; $\partial z/\partial x = (dz/du)(\partial u/\partial x) = 2x dz/du$; $\partial z/\partial y = (dz/du)(\partial u/\partial y) = -2y dz/du$, $y\partial z/\partial x + x\partial z/\partial y = 2xy dz/du - 2xy dz/du = 0$.

(b) $z = f(u)$, $u = xy$; $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x} = y\dfrac{dz}{du}$, $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\dfrac{\partial u}{\partial y} = x\dfrac{dz}{du}$, $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = xy\dfrac{dz}{du} - xy\dfrac{dz}{du} = 0$.

(c) $yz_x + xz_y = y(2x\cos(x^2 - y^2)) - x(2y\cos(x^2 - y^2)) = 0$.

(d) $xz_x - yz_y = xye^{xy} - yxe^{xy} = 0$.

51. Let $z = f(u)$ where $u = x + 2y$; then $\partial z/\partial x = (dz/du)(\partial u/\partial x) = dz/du$, $\partial z/\partial y = (dz/du)(\partial u/\partial y) = 2dz/du$ so $2\partial z/\partial x - \partial z/\partial y = 2dz/du - 2dz/du = 0$.

52. Let $z = f(u)$ where $u = x^2 + y^2$; then $\partial z/\partial x = (dz/du)(\partial u/\partial x) = 2x\ dz/du$, $\partial z/\partial y = (dz/du)(\partial u/\partial y) = 2y dz/du$ so $y\ \partial z/\partial x - x\partial z/\partial y = 2xy dz/du - 2xy dz/du = 0$.

53. $\dfrac{\partial w}{\partial x} = \dfrac{dw}{du}\dfrac{\partial u}{\partial x} = \dfrac{dw}{du}$, $\dfrac{\partial w}{\partial y} = \dfrac{dw}{du}\dfrac{\partial u}{\partial y} = 2\dfrac{dw}{du}$, $\dfrac{\partial w}{\partial z} = \dfrac{dw}{du}\dfrac{\partial u}{\partial z} = 3\dfrac{dw}{du}$, so $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = 6\dfrac{dw}{du}$.

54. $\partial w/\partial x = (dw/d\rho)(\partial\rho/\partial x) = (x/\rho)dw/d\rho$, similarly $\partial w/\partial y = (y/\rho)dw/d\rho$ and $\partial w/\partial z = (z/\rho)dw/d\rho$ so $(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + (\partial w/\partial z)^2 = (dw/d\rho)^2$.

55. $z = f(u, v)$ where $u = x - y$ and $v = y - x$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$ and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = -\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$ so $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

56. Let $w = f(r, s, t)$ where $r = x - y$, $s = y - z$, $t = z - x$; $\partial w/\partial x = (\partial w/\partial r)(\partial r/\partial x) + (\partial w/\partial t)(\partial t/\partial x) = \partial w/\partial r - \partial w/\partial t$, similarly $\partial w/\partial y = -\partial w/\partial r + \partial w/\partial s$ and $\partial w/\partial z = -\partial w/\partial s + \partial w/\partial t$ so $\partial w/\partial x + \partial w/\partial y + \partial w/\partial z = 0$.

57. **(a)** $1 = -r\sin\theta\dfrac{\partial\theta}{\partial x} + \cos\theta\dfrac{\partial r}{\partial x}$ and $0 = r\cos\theta\dfrac{\partial\theta}{\partial x} + \sin\theta\dfrac{\partial r}{\partial x}$; solve for $\partial r/\partial x$ and $\partial\theta/\partial x$.

(b) $0 = -r\sin\theta\dfrac{\partial\theta}{\partial y} + \cos\theta\dfrac{\partial r}{\partial y}$ and $1 = r\cos\theta\dfrac{\partial\theta}{\partial y} + \sin\theta\dfrac{\partial r}{\partial y}$; solve for $\partial r/\partial y$ and $\partial\theta/\partial y$.

(c) $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial x} = \dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\sin\theta$, $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial y} = \dfrac{\partial z}{\partial r}\sin\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\cos\theta$.

(d) Square and add the results of parts (a) and (b).

(e) From part (c), $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial r}\left(\dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial \theta}\sin\theta\right)\dfrac{\partial r}{\partial x} + \dfrac{\partial}{\partial \theta}\left(\dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial \theta}\sin\theta\right)\dfrac{\partial \theta}{\partial x} =$

$= \left(\dfrac{\partial^2 z}{\partial r^2}\cos\theta + \dfrac{1}{r^2}\dfrac{\partial z}{\partial \theta}\sin\theta - \dfrac{1}{r}\dfrac{\partial^2 z}{\partial r \partial \theta}\sin\theta\right)\cos\theta + \left(\dfrac{\partial^2 z}{\partial \theta \partial r}\cos\theta - \dfrac{\partial z}{\partial r}\sin\theta - \dfrac{1}{r}\dfrac{\partial^2 z}{\partial \theta^2}\sin\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial \theta}\cos\theta\right)\left(-\dfrac{\sin\theta}{r}\right) =$
$\dfrac{\partial^2 z}{\partial r^2}\cos^2\theta + \dfrac{2}{r^2}\dfrac{\partial z}{\partial \theta}\sin\theta\cos\theta - \dfrac{2}{r}\dfrac{\partial^2 z}{\partial \theta \partial r}\sin\theta\cos\theta + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2}\sin^2\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial r}\sin^2\theta.$

Similarly, from part (c), $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2}\sin^2\theta - \dfrac{2}{r^2}\dfrac{\partial z}{\partial \theta}\sin\theta\cos\theta + \dfrac{2}{r}\dfrac{\partial^2 z}{\partial \theta \partial r}\sin\theta\cos\theta + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2}\cos^2\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial r}\cos^2\theta.$

Add these to get $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r}.$

58. $z_x = \dfrac{-2y}{x^2+y^2}, z_{xx} = \dfrac{4xy}{(x^2+y^2)^2}, z_y = \dfrac{2x}{x^2+y^2}, z_{yy} = -\dfrac{4xy}{(x^2+y^2)^2}, z_{xx} + z_{yy} = 0$; $z = \tan^{-1}\dfrac{2r^2\cos\theta\sin\theta}{r^2(\cos^2\theta - \sin^2\theta)} =$ $\tan^{-1}\tan 2\theta = 2\theta + k\pi$ for some fixed k; $z_r = 0, z_{\theta\theta} = 0$.

59. (a) By the chain rule, $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and $\dfrac{\partial v}{\partial \theta} = -\dfrac{\partial v}{\partial x}r\sin\theta + \dfrac{\partial v}{\partial y}r\cos\theta$, use the Cauchy-Riemann conditions $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ in the equation for $\dfrac{\partial u}{\partial r}$ to get $\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial y}\cos\theta - \dfrac{\partial v}{\partial x}\sin\theta$ and compare to $\dfrac{\partial v}{\partial \theta}$ to see that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$. The result $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ can be obtained by considering $\dfrac{\partial v}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$.

(b) $u_x = \dfrac{2x}{x^2+y^2}$, $v_y = 2\dfrac{1}{x}\dfrac{1}{1+(y/x)^2} = \dfrac{2x}{x^2+y^2} = u_x$; $u_y = \dfrac{2y}{x^2+y^2}$, $v_x = -2\dfrac{y}{x^2}\dfrac{1}{1+(y/x)^2} = -\dfrac{2y}{x^2+y^2} =$ $-u_y$; $u = \ln r^2, v = 2\theta, u_r = 2/r, v_\theta = 2$, so $u_r = \dfrac{1}{r}v_\theta$, $u_\theta = 0, v_r = 0$, so $v_r = -\dfrac{1}{r}u_\theta$.

60. (a) $u_x = f'(x+ct)$, $u_{xx} = f''(x+ct)$, $u_t = cf'(x+ct)$, $u_{tt} = c^2 f''(x+ct)$; $u_{tt} = c^2 u_{xx}$.

(b) Substitute g for f and $-c$ for c in part (a).

(c) Since the sum of derivatives equals the derivative of the sum, the result follows from parts (a) and (b).

(d) $\sin t \sin x = \dfrac{1}{2}(-\cos(x+t) + \cos(x-t))$.

61. $\partial w/\partial \rho = (\sin\phi\cos\theta)\partial w/\partial x + (\sin\phi\sin\theta)\partial w/\partial y + (\cos\phi)\,\partial w/\partial z$,
$\partial w/\partial \phi = (\rho\cos\phi\cos\theta)\partial w/\partial x + (\rho\cos\phi\sin\theta)\partial w/\partial y - (\rho\sin\phi)\partial w/\partial z$,
$\partial w/\partial \theta = -(\rho\sin\phi\sin\theta)\partial w/\partial x + (\rho\sin\phi\cos\theta)\partial w/\partial y$.

62. (a) $\dfrac{\partial w}{\partial x} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial x}$. **(b)** $\dfrac{\partial w}{\partial y} = \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial y}$.

63. $w_r = e^r/(e^r+e^s+e^t+e^u)$, $w_{rs} = -e^re^s/(e^r+e^s+e^t+e^u)^2$, $w_{rst} = 2e^re^se^t/(e^r+e^s+e^t+e^u)^3$, $w_{rstu} =$ $-6e^re^se^te^u/(e^r+e^s+e^t+e^u)^4 = -6e^{r+s+t+u}/e^{4w} = -6e^{r+s+t+u-4w}$.

64. $\partial w/\partial y_1 = a_1\partial w/\partial x_1 + a_2\partial w/\partial x_2 + a_3\partial w/\partial x_3$, $\partial w/\partial y_2 = b_1\partial w/\partial x_1 + b_2\partial w/\partial x_2 + b_3\partial w/\partial x_3$.

65. (a) $dw/dt = \displaystyle\sum_{i=1}^{4}(\partial w/\partial x_i)(dx_i/dt)$. **(b)** $\partial w/\partial v_j = \displaystyle\sum_{i=1}^{4}(\partial w/\partial x_i)(\partial x_i/\partial v_j)$ for $j = 1, 2, 3$.

66. Let $u = x_1^2 + x_2^2 + ... + x_n^2$; then $w = u^k$, $\partial w/\partial x_i = ku^{k-1}(2x_i) = 2k\,x_iu^{k-1}$, $\partial^2 w/\partial x_i^2 = 2k(k-1)x_iu^{k-2}(2x_i) +$ $2ku^{k-1} = 4k(k-1)x_i^2u^{k-2} + 2ku^{k-1}$ for $i = 1, 2, \ldots, n$, so $\displaystyle\sum_{i=1}^{n}\partial^2 w/\partial x_i^2 = 4k(k-1)u^{k-2}\sum_{i=1}^{n}x_i^2 + 2kn\,u^{k-1} =$

$4k(k-1)u^{k-2}u + 2kn\,u^{k-1} = 2ku^{k-1}[2(k-1)+n]$, which is 0 if $k=0$ or if $2(k-1)+n=0$, $k=1-n/2$.

67. $dF/dx = (\partial F/\partial u)(du/dx) + (\partial F/\partial v)(dv/dx) = f(u)g'(x) - f(v)h'(x) = f(g(x))g'(x) - f(h(x))h'(x)$.

68. Represent the line segment C that joins A and B by $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$ for $0 \le t \le 1$. Let $F(t) = f(x_0 + (x_1 - x_0)t, y_0 + (y_1 - y_0)t)$ for $0 \le t \le 1$; then $f(x_1, y_1) - f(x_0, y_0) = F(1) - F(0)$. Apply the Mean Value Theorem to $F(t)$ on the interval [0,1] to get $[F(1) - F(0)]/(1-0) = F'(t^*)$, $F(1) - F(0) = F'(t^*)$ for some t^* in (0,1) so $f(x_1, y_1) - f(x_0, y_0) = F'(t^*)$. By the chain rule, $F'(t) = f_x(x,y)(dx/dt) + f_y(x,y)(dy/dt) = f_x(x,y)(x_1 - x_0) + f_y(x,y)(y_1 - y_0)$. Let (x^*, y^*) be the point on C for $t = t^*$ then $f(x_1, y_1) - f(x_0, y_0) = F'(t^*) = f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0)$.

69. Let (a,b) be any point in the region, if (x,y) is in the region then by the result of Exercise 74 $f(x,y) - f(a,b) = f_x(x^*, y^*)(x-a) + f_y(x^*, y^*)(y-b)$, where (x^*, y^*) is on the line segment joining (a,b) and (x,y). If $f_x(x,y) = f_y(x,y) = 0$ throughout the region then $f(x,y) - f(a,b) = (0)(x-a) + (0)(y-b) = 0$, $f(x,y) = f(a,b)$ so $f(x,y)$ is constant on the region.

Exercise Set 13.6

1. $\nabla f(x,y) = (3y/2)(1+xy)^{1/2}\mathbf{i} + (3x/2)(1+xy)^{1/2}\mathbf{j}$, $\nabla f(3,1) = 3\mathbf{i} + 9\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 12/\sqrt{2} = 6\sqrt{2}$.

2. $\nabla f(x,y) = 5\cos(5x-3y)\mathbf{i} - 3\cos(5x-3y)\mathbf{j}$, $\nabla f(3,5) = 5\mathbf{i} - 3\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -27/5$.

3. $\nabla f(x,y) = \left[2x/\left(1+x^2+y\right)\right]\mathbf{i} + \left[1/\left(1+x^2+y\right)\right]\mathbf{j}$, $\nabla f(0,0) = \mathbf{j}$, $D_{\mathbf{u}}f = -3/\sqrt{10}$.

4. $\nabla f(x,y) = -\left[(c+d)y/(x-y)^2\right]\mathbf{i} + \left[(c+d)x/(x-y)^2\right]\mathbf{j}$, $\nabla f(3,4) = -4(c+d)\mathbf{i} + 3(c+d)\mathbf{j}$, $D_{\mathbf{u}}f = -(7/5)(c+d)$.

5. $\nabla f(x,y,z) = 20x^4y^2z^3\mathbf{i} + 8x^5yz^3\mathbf{j} + 12x^5y^2z^2\mathbf{k}$, $\nabla f(2,-1,1) = 320\mathbf{i} - 256\mathbf{j} + 384\mathbf{k}$, $D_{\mathbf{u}}f = -320$.

6. $\nabla f(x,y,z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + (xye^{xz} + 2z)\mathbf{k}$, $\nabla f(0,2,3) = 6\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, $D_{\mathbf{u}}f = 45/7$.

7. $\nabla f(x,y,z) = \dfrac{2x}{x^2+2y^2+3z^2}\mathbf{i} + \dfrac{4y}{x^2+2y^2+3z^2}\mathbf{j} + \dfrac{6z}{x^2+2y^2+3z^2}\mathbf{k}$, $\nabla f(-1,2,4) = (-2/57)\mathbf{i} + (8/57)\mathbf{j} + (24/57)\mathbf{k}$, $D_{\mathbf{u}}f = -314/741$.

8. $\nabla f(x,y,z) = yz\cos xyz\mathbf{i} + xz\cos xyz\mathbf{j} + xy\cos xyz\mathbf{k}$, $\nabla f(1/2, 1/3, \pi) = (\pi\sqrt{3}/6)\mathbf{i} + (\pi\sqrt{3}/4)\mathbf{j} + (\sqrt{3}/12)\mathbf{k}$, $D_{\mathbf{u}}f = (1-\pi)/12$.

9. $\nabla f(x,y) = 12x^2y^2\mathbf{i} + 8x^3y\mathbf{j}$, $\nabla f(2,1) = 48\mathbf{i} + 64\mathbf{j}$, $\mathbf{u} = (4/5)\mathbf{i} - (3/5)\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 0$.

10. $\nabla f(x,y) = 27x^2\mathbf{i} - 6y^2\mathbf{j}$, $\nabla f(1,0) = 27\mathbf{i}$, $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 27/\sqrt{2}$.

11. $\nabla f(x,y) = \left(y^2/x\right)\mathbf{i} + 2y\ln x\mathbf{j}$, $\nabla f(1,4) = 16\mathbf{i}$, $\mathbf{u} = (-\mathbf{i} + \mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = -8\sqrt{2}$.

12. $\nabla f(x,y) = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$, $\nabla f(0,\pi/4) = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, $\mathbf{u} = (5\mathbf{i} - 2\mathbf{j})/\sqrt{29}$, $D_{\mathbf{u}}f = 7/\sqrt{58}$.

13. $\nabla f(x,y) = -\left[y/\left(x^2+y^2\right)\right]\mathbf{i} + \left[x/\left(x^2+y^2\right)\right]\mathbf{j}$, $\nabla f(-2,2) = -(\mathbf{i} + \mathbf{j})/4$, $\mathbf{u} = -(\mathbf{i} + \mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = \sqrt{2}/4$.

14. $\nabla f(x,y) = (e^y - ye^x)\mathbf{i} + (xe^y - e^x)\mathbf{j}$, $\nabla f(0,0) = \mathbf{i} - \mathbf{j}$, $\mathbf{u} = (5\mathbf{i} - 2\mathbf{j})/\sqrt{29}$, $D_{\mathbf{u}}f = 7/\sqrt{29}$.

15. $\nabla f(x,y,z) = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla f(-3,0,4) = -3\mathbf{j} + 8\mathbf{k}$, $\mathbf{u} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$, $D_{\mathbf{u}}f = 5/\sqrt{3}$.

16. $\nabla f(x,y,z) = -x\left(x^2+z^2\right)^{-1/2}\mathbf{i} + \mathbf{j} - z\left(x^2+z^2\right)^{-1/2}\mathbf{k}$, $\nabla f(-3,1,4) = (3/5)\mathbf{i} + \mathbf{j} - (4/5)\mathbf{k}$, $\mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k})/3$, $D_{\mathbf{u}}f = 0$.

17. $\nabla f(x,y,z) = -\dfrac{1}{z+y}\mathbf{i} - \dfrac{z-x}{(z+y)^2}\mathbf{j} + \dfrac{y+x}{(z+y)^2}\mathbf{k}$, $\nabla f(1,0,-3) = (1/3)\mathbf{i} + (4/9)\mathbf{j} + (1/9)\mathbf{k}$, $\mathbf{u} = (-6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})/7$, $D_{\mathbf{u}}f = -8/63$.

18. $\nabla f(x,y,z) = e^{x+y+3z}(\mathbf{i}+\mathbf{j}+3\mathbf{k})$, $\nabla f(-2,2,-1) = e^{-3}(\mathbf{i}+\mathbf{j}+3\mathbf{k})$, $\mathbf{u} = (20\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})/21$, $D_{\mathbf{u}}f = (31/21)e^{-3}$.

19. $\nabla f(x,y) = (y/2)(xy)^{-1/2}\mathbf{i} + (x/2)(xy)^{-1/2}\mathbf{j}$, $\nabla f(1,4) = \mathbf{i} + (1/4)\mathbf{j}$, $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} = (1/2)\mathbf{i} + \left(\sqrt{3}/2\right)\mathbf{j}$, $D_{\mathbf{u}}f = 1/2 + \sqrt{3}/8$.

20. $\nabla f(x,y) = [2y/(x+y)^2]\mathbf{i} - [2x/(x+y)^2]\mathbf{j}$, $\nabla f(-1,-2) = -(4/9)\mathbf{i} + (2/9)\mathbf{j}$, $\mathbf{u} = \mathbf{j}$, $D_{\mathbf{u}}f = 2/9$.

21. $\nabla f(x,y) = 2\sec^2(2x+y)\mathbf{i} + \sec^2(2x+y)\mathbf{j}$, $\nabla f(\pi/6, \pi/3) = 8\mathbf{i} + 4\mathbf{j}$, $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = 2\sqrt{2}$.

22. $\nabla f(x,y) = \cosh x \cosh y\mathbf{i} + \sinh x \sinh y\mathbf{j}$, $\nabla f(0,0) = \mathbf{i}$, $\mathbf{u} = -\mathbf{i}$, $D_{\mathbf{u}}f = -1$.

23. $\nabla f(x,y) = y(x+y)^{-2}\mathbf{i} - x(x+y)^{-2}\mathbf{j}$, $\nabla f(1,0) = -\mathbf{j}$, $\overrightarrow{PQ} = -2\mathbf{i} - \mathbf{j}$, $\mathbf{u} = (-2\mathbf{i} - \mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = 1/\sqrt{5}$.

24. $\nabla f(x,y) = -e^{-x}\sec y\mathbf{i} + e^{-x}\sec y \tan y\mathbf{j}, \nabla f(0,\pi/4) = \sqrt{2}(-\mathbf{i} + \mathbf{j})$, $\overrightarrow{PO} = -(\pi/4)\mathbf{j}$, $\mathbf{u} = -\mathbf{j}$, $D_{\mathbf{u}}f = -\sqrt{2}$.

25. $\nabla f(x,y) = \dfrac{ye^y}{2\sqrt{xy}}\mathbf{i} + \left(\sqrt{xy}e^y + \dfrac{xe^y}{2\sqrt{xy}}\right)\mathbf{j}$, $\nabla f(1,1) = (e/2)(\mathbf{i} + 3\mathbf{j})$, $\mathbf{u} = -\mathbf{j}$, $D_{\mathbf{u}}f = -3e/2$.

26. $\nabla f(x,y) = -y(x+y)^{-2}\mathbf{i} + x(x+y)^{-2}\mathbf{j}$, $\nabla f(2,3) = (-3\mathbf{i} + 2\mathbf{j})/25$, if $D_{\mathbf{u}}f = 0$ then $\mathbf{u}$ and ∇f are orthogonal, by inspection $2\mathbf{i} + 3\mathbf{j}$ is orthogonal to $\nabla f(2,3)$ so $\mathbf{u} = \pm(2\mathbf{i} + 3\mathbf{j})/\sqrt{13}$.

27. $\nabla f(2,1,-1) = -\mathbf{i} + \mathbf{j} - \mathbf{k}$. $\overrightarrow{PQ} = -3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{u} = (-3\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{11}$, $D_{\mathbf{u}}f = 3/\sqrt{11}$.

28. $\nabla f(-1,-2,1) = 13\mathbf{i} + 5\mathbf{j} - 20\mathbf{k}$, $\mathbf{u} = -\mathbf{k}$, $D_{\mathbf{u}}f = 20$.

29. Solve the system $(3/5)f_x(1,2) - (4/5)f_y(1,2) = -5$, $(4/5)f_x(1,2) + (3/5)f_y(1,2) = 10$ for

(a) $f_x(1,2) = 5$. **(b)** $f_y(1,2) = 10$. **(c)** $\nabla f(1,2) = 5\mathbf{i} + 10\mathbf{j}$, $\mathbf{u} = (-\mathbf{i} - 2\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = -5\sqrt{5}$.

30. $\nabla f(-5,1) = -3\mathbf{i} + 2\mathbf{j}$, $\overrightarrow{PQ} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{u} = (\mathbf{i} + 2\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = 1/\sqrt{5}$.

31. f increases the most in the direction of III.

32. The contour lines are closer at P, so the function is increasing more rapidly there, hence ∇f is larger at P.

33. $\nabla z = -7y\cos(7y^2 - 7xy)\mathbf{i} + (14y - 7x)\cos(7y^2 - 7xy)\mathbf{j}$.

34. $\nabla z = (42/y)\cos(6x/y)\mathbf{i} - (42x/y^2)\cos(6x/y)\mathbf{j}$.

35. $\nabla z = -\dfrac{84y}{(6x-7y)^2}\mathbf{i} + \dfrac{84x}{(6x-7y)^2}\mathbf{j}$.

36. $\nabla z = \dfrac{48ye^{3y}}{(x+8y)^2}\mathbf{i} + \dfrac{6xe^{3y}(3x+24y-8)}{(x+8y)^2}\mathbf{j}$.

37. $\nabla w = -9x^8\mathbf{i} - 3y^2\mathbf{j} + 12z^{11}\mathbf{k}$.

38. $\nabla w = e^{8y}\sin 6z\mathbf{i} + 8xe^{8y}\sin 6z\mathbf{j} + 6xe^{8y}\cos 6z\mathbf{k}$.

39. $\nabla w = \dfrac{x}{x^2+y^2+z^2}\mathbf{i} + \dfrac{y}{x^2+y^2+z^2}\mathbf{j} + \dfrac{z}{x^2+y^2+z^2}\mathbf{k}$.

40. $\nabla w = e^{-5x}\sec(x^2yz)\left[\left(2xyz\tan(x^2yz)-5\right)\mathbf{i} + x^2z\tan(x^2yz)\,\mathbf{j} + x^2y\tan(x^2yz)\,\mathbf{k}\right]$.

41. $\nabla f(x,y) = 10x\mathbf{i} + 4y^3\mathbf{j}$, $\nabla f(4,2) = 40\mathbf{i} + 32\mathbf{j}$.

42. $\nabla f(x,y) = 10x\cos(x^2)\mathbf{i} - 3\sin 3y\mathbf{j}$, $\nabla f(\sqrt{\pi}/2, 0) = 5\sqrt{\pi/2}\mathbf{i}$.

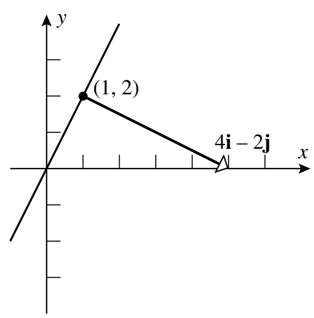
43. $\nabla f(x,y) = 3(2x+y)\left(x^2+xy\right)^2\mathbf{i} + 3x\left(x^2+xy\right)^2\mathbf{j}$, $\nabla f(-1,-1) = -36\mathbf{i} - 12\mathbf{j}$.

44. $\nabla f(x,y) = -x\left(x^2+y^2\right)^{-3/2}\mathbf{i} - y\left(x^2+y^2\right)^{-3/2}\mathbf{j}$, $\nabla f(3,4) = -(3/125)\mathbf{i} - (4/125)\mathbf{j}$.

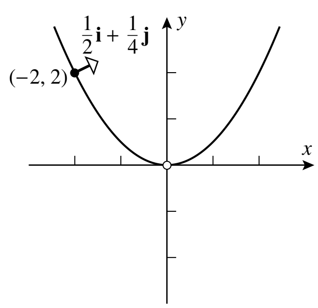
45. $\nabla f(x,y,z) = [y/(x+y+z)]\mathbf{i} + [y/(x+y+z) + \ln(x+y+z)]\mathbf{j} + [y/(x+y+z)]\mathbf{k}$, $\nabla f(-3,4,0) = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$.

46. $\nabla f(x,y,z) = 3y^2z\tan^2 x\sec^2 x\,\mathbf{i} + 2yz\tan^3 x\,\mathbf{j} + y^2\tan^3 x\,\mathbf{k}$, $\nabla f(\pi/4,-3) = 54\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k}$.

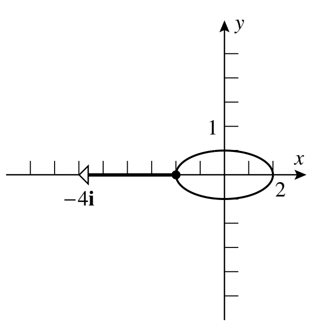
47. $f(1,2) = 3$, level curve $4x - 2y + 3 = 3$, $2x - y = 0$; $\nabla f(x,y) = 4\mathbf{i} - 2\mathbf{j}$, $\nabla f(1,2) = 4\mathbf{i} - 2\mathbf{j}$.



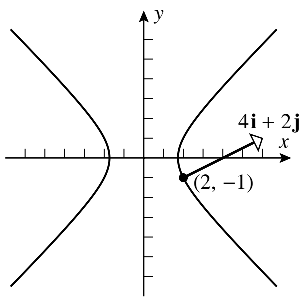
48. $f(-2,2) = 1/2$, level curve $y/x^2 = 1/2$, $y = x^2/2$ for $x \neq 0$. $\nabla f(x,y) = -\left(2y/x^3\right)\mathbf{i} + \left(1/x^2\right)\mathbf{j}$, $\nabla f(-2,2) = (1/2)\mathbf{i} + (1/4)\mathbf{j}$.



49. $f(-2,0) = 4$, level curve $x^2 + 4y^2 = 4$, $x^2/4 + y^2 = 1$. $\nabla f(x,y) = 2x\mathbf{i} + 8y\mathbf{j}$, $\nabla f(-2,0) = -4\mathbf{i}$.



50. $f(2,-1) = 3$, level curve $x^2 - y^2 = 3$. $\nabla f(x,y) = 2x\mathbf{i} - 2y\mathbf{j}$, $\nabla f(2,-1) = 4\mathbf{i} + 2\mathbf{j}$.



51. $\nabla f(x,y) = 8xy\mathbf{i} + 4x^2\mathbf{j}$, $\nabla f(1,-2) = -16\mathbf{i} + 4\mathbf{j}$ is normal to the level curve through P so $\mathbf{u} = \pm(-4\mathbf{i} + \mathbf{j})/\sqrt{17}$.

52. $\nabla f(x,y) = (6xy - y)\mathbf{i} + \left(3x^2 - x\right)\mathbf{j}$, $\nabla f(2,-3) = -33\mathbf{i} + 10\mathbf{j}$ is normal to the level curve through P so $\mathbf{u} = \pm(-33\mathbf{i} + 10\mathbf{j})/\sqrt{1189}$.

53. $\nabla f(x,y) = 12x^2y^2\mathbf{i} + 8x^3y\mathbf{j}$, $\nabla f(-1,1) = 12\mathbf{i} - 8\mathbf{j}$, $\mathbf{u} = (3\mathbf{i} - 2\mathbf{j})/\sqrt{13}$, $\|\nabla f(-1,1)\| = 4\sqrt{13}$.

54. $\nabla f(x,y) = 3\mathbf{i} - (1/y)\mathbf{j}$, $\nabla f(2,4) = 3\mathbf{i} - (1/4)\mathbf{j}$, $\mathbf{u} = (12\mathbf{i} - \mathbf{j})/\sqrt{145}$, $\|\nabla f(2,4)\| = \sqrt{145}/4$.

55. $\nabla f(x,y) = x\left(x^2 + y^2\right)^{-1/2}\mathbf{i} + y\left(x^2 + y^2\right)^{-1/2}\mathbf{j}, \nabla f(4,-3) = (4\mathbf{i} - 3\mathbf{j})/5$, $\mathbf{u} = (4\mathbf{i} - 3\mathbf{j})/5$, $\|\nabla f(4,-3)\| = 1$.

56. $\nabla f(x,y) = y(x+y)^{-2}\mathbf{i} - x(x+y)^{-2}\mathbf{j}$, $\nabla f(0,2) = (1/2)\mathbf{i}$, $\mathbf{u} = \mathbf{i}$, $\|\nabla f(0,2)\| = 1/2$.

57. $\nabla f(1,1,-1) = 3\mathbf{i} - 3\mathbf{j}$, $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, $\|\nabla f(1,1,-1)\| = 3\sqrt{2}$.

58. $\nabla f(0,-3,0) = (\mathbf{i}-3\mathbf{j}+4\mathbf{k})/6$, $\mathbf{u} = (\mathbf{i}-3\mathbf{j}+4\mathbf{k})/\sqrt{26}$, $\|\nabla f(0,-3,0)\| = \sqrt{26}/6$.

59. $\nabla f(1,2,-2) = (-\mathbf{i} + \mathbf{j})/2$, $\mathbf{u} = (-\mathbf{i} + \mathbf{j})/\sqrt{2}$, $\|\nabla f(1,2,-2)\| = 1/\sqrt{2}$.

60. $\nabla f(4,2,2) = (\mathbf{i} - \mathbf{j} - \mathbf{k})/8$, $\mathbf{u} = (\mathbf{i} - \mathbf{j} - \mathbf{k})/\sqrt{3}$, $\|\nabla f(4,2,2)\| = \sqrt{3}/8$.

61. $\nabla f(x,y) = -2x\mathbf{i} - 2y\mathbf{j}$, $\nabla f(-1,-3) = 2\mathbf{i} + 6\mathbf{j}$, $\mathbf{u} = -(\mathbf{i} + 3\mathbf{j})/\sqrt{10}$, $-\|\nabla f(-1,-3)\| = -2\sqrt{10}$.

62. $\nabla f(x,y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$; $\nabla f(2,3) = e^6(3\mathbf{i} + 2\mathbf{j})$, $\mathbf{u} = -(3\mathbf{i} + 2\mathbf{j})/\sqrt{13}$, $-\|\nabla f(2,3)\| = -\sqrt{13}e^6$.

63. $\nabla f(x,y) = -3\sin(3x - y)\mathbf{i} + \sin(3x - y)\mathbf{j}$, $\nabla f(\pi/6, \pi/4) = (-3\mathbf{i} + \mathbf{j})/\sqrt{2}$, $\mathbf{u} = (3\mathbf{i} - \mathbf{j})/\sqrt{10}$, $-\|\nabla f(\pi/6, \pi/4)\| = -\sqrt{5}$.

64. $\nabla f(x,y) = \dfrac{y}{(x+y)^2}\sqrt{\dfrac{x+y}{x-y}}\mathbf{i} - \dfrac{x}{(x+y)^2}\sqrt{\dfrac{x+y}{x-y}}\mathbf{j}$, $\nabla f(3,1) = (\sqrt{2}/16)(\mathbf{i} - 3\mathbf{j})$, $\mathbf{u} = -(\mathbf{i} - 3\mathbf{j})/\sqrt{10}$, $-\|\nabla f(3,1)\| = -\sqrt{5}/8$.

65. $\nabla f(5,7,6) = -\mathbf{i} + 11\mathbf{j} - 12\mathbf{k}$, $\mathbf{u} = (\mathbf{i} - 11\mathbf{j} + 12\mathbf{k})/\sqrt{266}$, $-\|\nabla f(5,7,6)\| = -\sqrt{266}$.

66. $\nabla f(0,1,\pi/4) = 2\sqrt{2}(\mathbf{i} - \mathbf{k})$, $\mathbf{u} = -(\mathbf{i} - \mathbf{k})/\sqrt{2}$, $-\|\nabla f(0,1,\pi/4)\| = -4$.

67. False; actually they are equal: $D_{\mathbf{v}}(f) = \nabla f \cdot \mathbf{v}/\|\mathbf{v}\| = \nabla f \cdot 2\|\mathbf{u}\|/2 = D_{\mathbf{u}}(f)$.

68. True: let $\mathbf{u} = (x, x^2)$. Then $0 = Df_{\mathbf{u}} = f_x(0,0)\cdot 1 + f_y(0,0)\cdot 0 = f_x(0,0)$.

69. False; $f(x,y) = x$ and $\mathbf{u} = \mathbf{j}$.

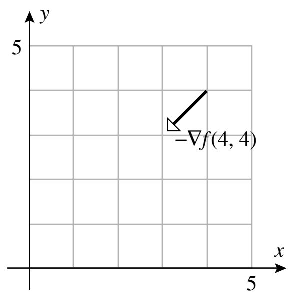
70. False, for example $f(x,y) = \sin x, (x_0, y_0) = (0,0)$ and $(x_1, y_1) = (3\pi/2, 0)$.

71. $\nabla f(4,-5) = 2\mathbf{i} - \mathbf{j}$, $\mathbf{u} = (5\mathbf{i} + 2\mathbf{j})/\sqrt{29}$, $D_{\mathbf{u}}f = 8/\sqrt{29}$.

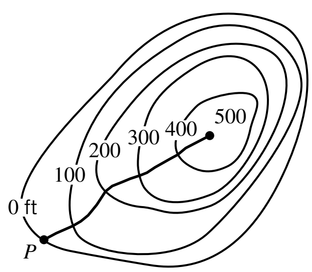
72. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ where $u_1^2 + u_2^2 = 1$, but $D_\mathbf{u}f = \nabla f \cdot \mathbf{u} = u_1 - 2u_2 = -2$ so $u_1 = 2u_2 - 2$, $(2u_2 - 2)^2 + u_2^2 = 1$, $5u_2^2 - 8u_2 + 3 = 0$, $u_2 = 1$ or $u_2 = 3/5$ thus $u_1 = 0$ or $u_1 = -4/5$; $\mathbf{u} = \mathbf{j}$ or $\mathbf{u} = -\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$.

73. (a) At $(1, 2)$ the steepest ascent seems to be in the direction $\mathbf{i} + \mathbf{j}$ and the slope in that direction seems to be $0.5/(\sqrt{2}/2) = 1/\sqrt{2}$, so $\nabla f \approx \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$, which has the required direction and magnitude.

(b) The direction of $-\nabla f(4, 4)$ appears to be $-\mathbf{i} - \mathbf{j}$ and its magnitude appears to be $1/0.8 = 5/4$.



74.



Depart from each contour line in a direction orthogonal to that contour line, as an approximation to the optimal path.

75. $\nabla z = 6x\mathbf{i} - 2y\mathbf{j}$, $\|\nabla z\| = \sqrt{36x^2 + 4y^2} = 6$ if $36x^2 + 4y^2 = 36$; all points on the ellipse $9x^2 + y^2 = 9$.

76. $\nabla z = 3\mathbf{i} + 2y\mathbf{j}$, $\|\nabla z\| = \sqrt{9 + 4y^2}$, so $\nabla\|\nabla z\| = \dfrac{4y}{\sqrt{9 + 4y^2}}\mathbf{j}$, and $\nabla\|\nabla z\|\Big|_{(x,y)=(5,2)} = \dfrac{8}{5}\mathbf{j}$.

77. $\mathbf{r} = t\mathbf{i} - t^2\mathbf{j}$, $d\mathbf{r}/dt = \mathbf{i} - 2t\mathbf{j} = \mathbf{i} - 4\mathbf{j}$ at the point $(2, -4)$, $\mathbf{u} = (\mathbf{i} - 4\mathbf{j})/\sqrt{17}$; $\nabla z = 2x\mathbf{i} + 2y\mathbf{j} = 4\mathbf{i} - 8\mathbf{j}$ at $(2, -4)$, hence $dz/ds = D_\mathbf{u}z = \nabla z \cdot \mathbf{u} = 36/\sqrt{17}$.

78. (a) $\nabla T(x, y) = \dfrac{y\left(1 - x^2 + y^2\right)}{\left(1 + x^2 + y^2\right)^2}\mathbf{i} + \dfrac{x\left(1 + x^2 - y^2\right)}{\left(1 + x^2 + y^2\right)^2}\mathbf{j}$, $\nabla T(1, 1) = (\mathbf{i} + \mathbf{j})/9$, $\mathbf{u} = (2\mathbf{i} - \mathbf{j})/\sqrt{5}$, $D_\mathbf{u}T = 1/\left(9\sqrt{5}\right)$.

(b) $\mathbf{u} = -(\mathbf{i} + \mathbf{j})/\sqrt{2}$, opposite to $\nabla T(1, 1)$.

79. (a) $\nabla V(x, y) = -2e^{-2x}\cos 2y\mathbf{i} - 2e^{-2x}\sin 2y\mathbf{j}$, $\mathbf{E} = -\nabla V(\pi/4, 0) = 2e^{-\pi/2}\mathbf{i}$.

(b) $V(x, y)$ decreases most rapidly in the direction of $-\nabla V(x, y)$ which is $\mathbf{E}$.

80. $\nabla z = -0.04x\mathbf{i} - 0.08y\mathbf{j}$, if $x = -20$ and $y = 5$ then $\nabla z = 0.8\mathbf{i} - 0.4\mathbf{j}$.

(a) $\mathbf{u} = -\mathbf{i}$ points due west, $D_\mathbf{u}z = -0.8$, the climber will descend because z is decreasing.

(b) $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$ points northeast, $D_\mathbf{u}z = 0.2\sqrt{2}$, the climber will ascend at the rate of $0.2\sqrt{2}$ m per m of travel in the $xy-$plane.

(c) The climber will travel a level path in a direction perpendicular to $\nabla z = 0.8\mathbf{i} - 0.4\mathbf{j}$, by inspection $\pm(\mathbf{i}+2\mathbf{j})/\sqrt{5}$ are unit vectors in these directions; $(\mathbf{i}+2\mathbf{j})/\sqrt{5}$ makes an angle of $\tan^{-1}(1/2) \approx 27^\circ$ with the positive y-axis so $-(\mathbf{i}+2\mathbf{j})/\sqrt{5}$ makes the same angle with the negative y-axis. The compass direction should be N 27° E or S 27° W.

81. Let $\mathbf{u}$ be the unit vector in the direction of $\mathbf{a}$, then $D_{\mathbf{u}}f(3,-2,1) = \nabla f(3,-2,1) \cdot \mathbf{u} = \|\nabla f(3,-2,1)\| \cos\theta = 5\cos\theta = -5$, $\cos\theta = -1$, $\theta = \pi$ so $\nabla f(3,-2,1)$ is oppositely directed to $\mathbf{u}$; $\nabla f(3,-2,1) = -5\mathbf{u} = -10/3\mathbf{i} + 5/3\mathbf{j} + 10/3\mathbf{k}$.

82. **(a)** $\nabla T(1,1,1) = (\mathbf{i}+\mathbf{j}+\mathbf{k})/8$, $\mathbf{u} = -(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$, $D_{\mathbf{u}}T = -\sqrt{3}/8$.

(b) $(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$. **(c)** $\sqrt{3}/8$.

83. **(a)** $\nabla r = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j} = \mathbf{r}/r$.

(b) $\nabla f(r) = \dfrac{\partial f(r)}{\partial x}\mathbf{i} + \dfrac{\partial f(r)}{\partial y}\mathbf{j} = f'(r)\dfrac{\partial r}{\partial x}\mathbf{i} + f'(r)\dfrac{\partial r}{\partial y}\mathbf{j} = f'(r)\nabla r$.

84. **(a)** $\nabla\left(re^{-3r}\right) = \dfrac{(1-3r)}{r}e^{-3r}\mathbf{r}$.

(b) $3r^2\mathbf{r} = \dfrac{f'(r)}{r}\mathbf{r}$ so $f'(r) = 3r^3$, $f(r) = \dfrac{3}{4}r^4 + C$, $f(2) = 12 + C = 1$, $C = -11$; $f(r) = \dfrac{3}{4}r^4 - 11$.

85. $\mathbf{u}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $\mathbf{u}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, $\nabla z = \dfrac{\partial z}{\partial x}\mathbf{i} + \dfrac{\partial z}{\partial y}\mathbf{j} = \left(\dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\sin\theta\right)\mathbf{i} + \left(\dfrac{\partial z}{\partial r}\sin\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\cos\theta\right)\mathbf{j} = \dfrac{\partial z}{\partial r}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dfrac{\partial z}{\partial r}\mathbf{u}_r + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\mathbf{u}_\theta$.

86. **(a)** $\nabla(f+g) = (f_x + g_x)\mathbf{i} + (f_y + g_y)\mathbf{j} = (f_x\mathbf{i} + f_y\mathbf{j}) + (g_x\mathbf{i} + g_y\mathbf{j}) = \nabla f + \nabla g$.

(b) $\nabla(cf) = (cf_x)\mathbf{i} + (cf_y)\mathbf{j} = c(f_x\mathbf{i} + f_y\mathbf{j}) = c\nabla f$.

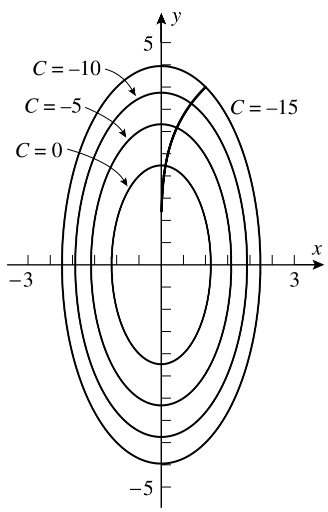
(c) $\nabla(fg) = (fg_x + gf_x)\mathbf{i} + (fg_y + gf_y)\mathbf{j} = f(g_x\mathbf{i} + g_y\mathbf{j}) + g(f_x\mathbf{i} + f_y\mathbf{j}) = f\nabla g + g\nabla f$.

(d) $\nabla(f/g) = \dfrac{gf_x - fg_x}{g^2}\mathbf{i} + \dfrac{gf_y - fg_y}{g^2}\mathbf{j} = \dfrac{g(f_x\mathbf{i} + f_y\mathbf{j}) - f(g_x\mathbf{i} + g_y\mathbf{j})}{g^2} = \dfrac{g\nabla f - f\nabla g}{g^2}$.

(e) $\nabla(f^n) = \left(nf^{n-1}f_x\right)\mathbf{i} + \left(nf^{n-1}f_y\right)\mathbf{j} = nf^{n-1}(f_x\mathbf{i} + f_y\mathbf{j}) = nf^{n-1}\nabla f$.

87. $\mathbf{r}'(t) = \mathbf{v}(t) = k(x,y)\nabla\mathbf{T} = -8k(x,y)x\mathbf{i} - 2k(x,y)y\mathbf{j}$; $\dfrac{dx}{dt} = -8kx$, $\dfrac{dy}{dt} = -2ky$. Divide and solve to get $y^4 = 256x$; one parametrization is $x(t) = e^{-8t}$, $y(t) = 4e^{-2t}$.

88. $\mathbf{r}'(t) = \mathbf{v}(t) = k\nabla\mathbf{T} = -2k(x,y)x\mathbf{i} - 4k(x,y)y\mathbf{j}$. Divide and solve to get $y = \dfrac{3}{25}x^2$; one parametrization is $x(t) = 5e^{-2t}$, $y(t) = 3e^{-4t}$.



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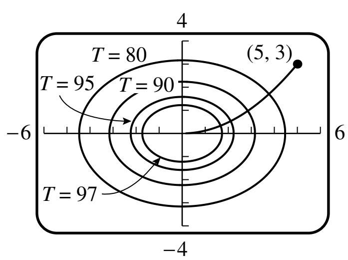
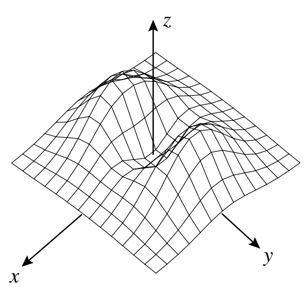


90.



91. (a)

(c) $\nabla f = [2x - 2x(x^2 + 3y^2)]e^{-(x^2+y^2)}\mathbf{i} + [6y - 2y(x^2 + 3y^2)]e^{-(x^2+y^2)}\mathbf{j}$.

(d) $\nabla f = \mathbf{0}$ if $x = y = 0$ or $x = 0, y = \pm 1$ or $x = \pm 1, y = 0$.

92. $dz/dt = (\partial z/\partial x)(dx/dt) + (\partial z/\partial y)(dy/dt) = (\partial z/\partial x\mathbf{i} + \partial z/\partial y\mathbf{j}) \cdot (dx/dt\mathbf{i} + dy/dt\mathbf{j}) = \nabla z \cdot \mathbf{r}'(t)$.

93. $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$, if $\nabla f(x, y) = 0$ throughout the region then $f_x(x, y) = f_y(x, y) = 0$ throughout the region, the result follows from Exercise 69, Section 13.5.

94. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be nonparallel unit vectors for which the directional derivative is zero. Let $\mathbf{u}$ be any other unit vector, then $\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ for some choice of scalars c_1 and c_2, $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = c_1\nabla f(x, y) \cdot \mathbf{u}_1 + c_2\nabla f(x, y) \cdot \mathbf{u}_2 = c_1 D_{\mathbf{u}_1} f(x, y) + c_2 D_{\mathbf{u}_2} f(x, y) = 0$.

95. $$\nabla f(u, v, w) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}\right)\mathbf{j} +$$
$$+ \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}\right)\mathbf{k} = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w.$$

Exercise Set 13.7

1. (a) $f(x, y, z) = x^2 + y^2 + 4z^2, \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}, \nabla f(2, 2, 1) = 4\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}, \mathbf{n} = \mathbf{i} + \mathbf{j} + 2\mathbf{k},\ x + y + 2z = 6$.

(b) $\mathbf{r}(t) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + 2\mathbf{k}), x(t) = 2 + t, y(t) = 2 + t, z(t) = 1 + 2t$.

(c) $\cos\theta = \dfrac{\mathbf{n}\cdot\mathbf{k}}{\|\mathbf{n}\|} = \dfrac{\sqrt{2}}{\sqrt{3}}, \theta \approx 35.26°$.

2. (a) $f(x,y,z) = xz - yz^3 + yz^2$, $\mathbf{n} = \nabla f(2,-1,1) = \mathbf{i} + 3\mathbf{k}$; tangent plane $x + 3z = 5$.

(b) Normal line $x = 2 + t$, $y = -1$, $z = 1 + 3t$.

(c) $\cos\theta = \dfrac{\mathbf{n}\cdot\mathbf{k}}{\|\mathbf{n}\|} = \dfrac{3}{\sqrt{10}}, \theta \approx 18.43°$.

3. $\nabla F = \langle 2x, 2y, 2z\rangle$, so $\mathbf{n} = \langle -6, 0, 8\rangle$, so the tangent plane is given by $-6(x+3) + 8(z-4) = 0$ or $3x - 4z = -25$, normal line $x = -3 - 6t$, $y = 0$, $z = 4 + 8t$.

4. $\nabla F = \langle 2xy, x^2, -8z\rangle$, so $\mathbf{n} = \langle -6, 9, 16\rangle$, so the tangent plane is given by $-6x + 9y + 16z = -5$, normal line $x = -3 - 6t$, $y = 1 + 9t$, $z = -2 + 16t$.

5. $\nabla F = \langle 2x - yz, -xz, -xy\rangle$, so $\mathbf{n} = \langle -18, 8, 20\rangle$, so the tangent plane is given by $-18x + 8y + 20z = 152$, normal line $x = -4 - 18t$, $y = 5 + 8t$, $z = 2 + 20t$.

6. At P, $\partial z/\partial x = 4$ and $\partial z/\partial y = -6$, tangent plane $4x - 6y - z = 13$, normal line $x = 2 + 4t$, $y = -3 - 6t$, $z = 13 - t$.

7. At P, $\partial z/\partial x = 48$ and $\partial z/\partial y = -14$, tangent plane $48x - 14y - z = 64$, normal line $x = 1 + 48t$, $y = -2 - 14t$, $z = 12 - t$.

8. At P, $\partial z/\partial x = 14$ and $\partial z/\partial y = -2$, tangent plane $14x - 2y - z = 16$, normal line $x = 2 + 14t$, $y = 4 - 2t$, $z = 4 - t$.

9. At P, $\partial z/\partial x = 1$ and $\partial z/\partial y = -1$, tangent plane $x - y - z = 0$, normal line $x = 1 + t$, $y = -t$, $z = 1 - t$.

10. At P, $\partial z/\partial x = -1$ and $\partial z/\partial y = 0$, tangent plane $x + z = -1$, normal line $x = -1 - t$, $y = 0$, $z = -t$.

11. At P, $\partial z/\partial x = 0$ and $\partial z/\partial y = 3$, tangent plane $3y - z = -1$, normal line $x = \pi/6$, $y = 3t$, $z = 1 - t$.

12. At P, $\partial z/\partial x = 1/4$ and $\partial z/\partial y = 1/6$, tangent plane $3x + 2y - 12z = -30$, normal line $x = 4 + t/4$, $y = 9 + t/6$, $z = 5 - t$.

13. The tangent plane is horizontal if the normal $\partial z/\partial x\mathbf{i} + \partial z/\partial y\mathbf{j} - \mathbf{k}$ is parallel to $\mathbf{k}$ which occurs when $\partial z/\partial x = \partial z/\partial y = 0$.

(a) $\partial z/\partial x = 3x^2y^2$, $\partial z/\partial y = 2x^3y$; $3x^2y^2 = 0$ and $2x^3y = 0$ for all (x,y) on the x-axis or y-axis, and $z = 0$ for these points, the tangent plane is horizontal at all points on the x-axis or y-axis.

(b) $\partial z/\partial x = 2x - y - 2$, $\partial z/\partial y = -x + 2y + 4$; solve the system $2x - y - 2 = 0$, $-x + 2y + 4 = 0$, to get $x = 0$, $y = -2$. $z = -4$ at $(0,-2)$, the tangent plane is horizontal at $(0,-2,-4)$.

14. $\partial z/\partial x = 6x$, $\partial z/\partial y = -2y$, so $6x_0\mathbf{i} - 2y_0\mathbf{j} - \mathbf{k}$ is normal to the surface at a point (x_0, y_0, z_0) on the surface. $6\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ is normal to the given plane. The tangent plane and the given plane are parallel if their normals are parallel so $6x_0 = 6$, $x_0 = 1$ and $-2y_0 = 4$, $y_0 = -2$. $z = -1$ at $(1,-2)$, the point on the surface is $(1,-2,-1)$.

15. $\partial z/\partial x = -6x$, $\partial z/\partial y = -4y$ so $-6x_0\mathbf{i} - 4y_0\mathbf{j} - \mathbf{k}$ is normal to the surface at a point (x_0, y_0, z_0) on the surface. This normal must be parallel to the given line and hence to the vector $-3\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ which is parallel to the line so $-6x_0 = -3$, $x_0 = 1/2$ and $-4y_0 = 8$, $y_0 = -2$. $z = -3/4$ at $(1/2,-2)$. The point on the surface is $(1/2,-2,-3/4)$.

16. $(3,4,5)$ is a point of intersection because it satisfies both equations. Both surfaces have $(3/5)\mathbf{i} + (4/5)\mathbf{j} - \mathbf{k}$ as a normal so they have a common tangent plane at $(3,4,5)$.

17. **(a)** $2t+7=(-1+t)^2+(2+t)^2$, $t^2=1$, $t=\pm 1$ so the points of intersection are $(-2,1,5)$ and $(0,3,9)$.

(b) $\partial z/\partial x = 2x$, $\partial z/\partial y = 2y$ so at $(-2,1,5)$ the vector $\mathbf{n}=-4\mathbf{i}+2\mathbf{j}-\mathbf{k}$ is normal to the surface. $\mathbf{v}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ is parallel to the line; $\mathbf{n}\cdot\mathbf{v}=-4$ so the cosine of the acute angle is $[\mathbf{n}\cdot(-\mathbf{v})]/(\|\mathbf{n}\|\,\|-\mathbf{v}\|)=4/\left(\sqrt{21}\sqrt{6}\right)=4/\left(3\sqrt{14}\right)$. Similarly, at (0,3,9) the vector $\mathbf{n}=6\mathbf{j}-\mathbf{k}$ is normal to the surface, $\mathbf{n}\cdot\mathbf{v}=4$ so the cosine of the acute angle is $4/\left(\sqrt{37}\sqrt{6}\right)=4/\sqrt{222}$.

18. $z=xf(u)$ where $u=x/y$, $\partial z/\partial x = xf'(u)\partial u/\partial x + f(u) = (x/y)f'(u)+f(u)=uf'(u)+f(u)$, $\partial z/\partial y = xf'(u)\partial u/\partial y = -(x^2/y^2)f'(u) = -u^2 f'(u)$. If (x_0,y_0,z_0) is on the surface then, with $u_0=x_0/y_0$, $[u_0 f'(u_0)+f(u_0)]\,\mathbf{i}-u_0^2 f'(u_0)\,\mathbf{j}-\mathbf{k}$ is normal to the surface so the tangent plane is $[u_0 f'(u_0)+f(u_0)]\,x - u_0^2 f'(u_0)y - z = [u_0 f'(u_0)+f(u_0)]x_0 - u_0^2 f'(u_0)y_0 - z_0 = \left[\dfrac{x_0}{y_0}f'(u_0)+f(u_0)\right]x_0 - \dfrac{x_0^2}{y_0^2}f'(u_0)\,y_0 - z_0 = x_0 f(u_0) - z_0 = 0$, so all tangent planes pass through the origin.

19. False, they only need to be parallel.

20. False, $f_x(1,1)=-1/2, f_y(1,1)=1/2$.

21. True, see Section 13.4 equation (15).

22. True, see equation (5) in Theorem 13.7.2.

23. Set $f(x,y,z)=z+x-z^4(y-1)$, then $f(x,y,z)=0, \mathbf{n}=\pm\nabla f(3,5,1)=\pm(\mathbf{i}-\mathbf{j}-15\mathbf{k})$, unit vectors $\pm\dfrac{1}{\sqrt{227}}(\mathbf{i}-\mathbf{j}-15\mathbf{k})$.

24. $f(x,y,z)=\sin xz - 4\cos yz$, $\nabla f(\pi,\pi,1)=-\mathbf{i}-\pi\mathbf{k}$; unit vectors $\pm\dfrac{1}{\sqrt{1+\pi^2}}(\mathbf{i}+\pi\mathbf{k})$.

25. $f(x,y,z)=x^2+y^2+z^2$, if (x_0,y_0,z_0) is on the sphere then $\nabla f(x_0,y_0,z_0)=2\,(x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k})$ is normal to the sphere at (x_0,y_0,z_0), the normal line is $x=x_0+x_0t$, $y=y_0+y_0t$, $z=z_0+z_0t$ which passes through the origin when $t=-1$.

26. $f(x,y,z)=2x^2+3y^2+4z^2$, if (x_0,y_0,z_0) is on the ellipsoid then $\nabla f(x_0,y_0,z_0)=2\,(2x_0\mathbf{i}+3y_0\mathbf{j}+4z_0\mathbf{k})$ is normal there and hence so is $\mathbf{n}_1=2x_0\mathbf{i}+3y_0\mathbf{j}+4z_0\mathbf{k}$; $\mathbf{n}_1$ must be parallel to $\mathbf{n}_2=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ which is normal to the given plane so $\mathbf{n}_1=c\mathbf{n}_2$ for some constant c. Equate corresponding components to get $x_0=c/2$, $y_0=-2c/3$, and $z_0=3c/4$; substitute into the equation of the ellipsoid yields $2\left(c^2/4\right)+3\left(4c^2/9\right)+4\left(9c^2/16\right)=9$, $c^2=108/49$, $c=\pm 6\sqrt{3}/7$. The points on the ellipsoid are $\left(3\sqrt{3}/7,-4\sqrt{3}/7,9\sqrt{3}/14\right)$ and $\left(-3\sqrt{3}/7,4\sqrt{3}/7,-9\sqrt{3}/14\right)$.

27. $f(x,y,z)=x^2+y^2-z^2$, if (x_0,y_0,z_0) is on the surface then $\nabla f(x_0,y_0,z_0)=2\,(x_0\mathbf{i}+y_0\mathbf{j}-z_0\mathbf{k})$ is normal there and hence so is $\mathbf{n}_1=x_0\mathbf{i}+y_0\mathbf{j}-z_0\mathbf{k}$; $\mathbf{n}_1$ must be parallel to $\overrightarrow{PQ}=3\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ so $\mathbf{n}_1=c\,\overrightarrow{PQ}$ for some constant c. Equate components to get $x_0=3c$, $y_0=2c$ and $z_0=2c$ which when substituted into the equation of the surface yields $9c^2+4c^2-4c^2=1$, $c^2=1/9$, $c=\pm 1/3$ so the points are $(1,2/3,2/3)$ and $(-1,-2/3,-2/3)$.

28. $f_1(x,y,z)=2x^2+3y^2+z^2$, $f_2(x,y,z)=x^2+y^2+z^2-6x-8y-8z+24$, $\mathbf{n}_1=\nabla f_1(1,1,2)=4\mathbf{i}+6\mathbf{j}+4\mathbf{k}$, $\mathbf{n}_2=\nabla f_2(1,1,2)=-4\mathbf{i}-6\mathbf{j}-4\mathbf{k}$, $\mathbf{n}_1=-\mathbf{n}_2$ so $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. Note that $(1,1,2)$ lies on each of the two surfaces.

29. $\mathbf{n}_1=2\mathbf{i}-2\mathbf{j}-\mathbf{k}, \mathbf{n}_2=2\mathbf{i}-8\mathbf{j}+4\mathbf{k}, \mathbf{n}_1\times\mathbf{n}_2=-16\mathbf{i}-10\mathbf{j}-12\mathbf{k}$ is tangent to the line, so $x(t)=1+8t, y(t)=-1+5t, z(t)=2+6t$.

30. $f(x,y,z)=\sqrt{x^2+y^2}-z, \mathbf{n}_1=\nabla f(4,3,5)=\dfrac{4}{5}\mathbf{i}+\dfrac{3}{5}\mathbf{j}-\mathbf{k}, \mathbf{n}_2=\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \mathbf{n}_1\times\mathbf{n}_2=(16\mathbf{i}-13\mathbf{j}+5\mathbf{k})/5$ is tangent to the line, $x(t)=4+16t, y(t)=3-13t, z(t)=5+5t$. The point $(4,3,5)$ lies on both surfaces.

31. $f(x,y,z) = x^2+z^2-25$, $g(x,y,z) = y^2+z^2-25$, $\mathbf{n}_1 = \nabla f(3,-3,4) = 6\mathbf{i}+8\mathbf{k}$, $\mathbf{n}_2 = \nabla g(3,-3,4) = -6\mathbf{j}+8\mathbf{k}$, $\mathbf{n}_1 \times \mathbf{n}_2 = 48\mathbf{i}-48\mathbf{j}-36\mathbf{k}$ is tangent to the line, $x(t) = 3+4t, y(t) = -3-4t, z(t) = 4-3t$. The point $(3,-3,4)$ lies on both surfaces.

32. (a) $f(x,y,z) = z-8+x^2+y^2, g(x,y,z) = 4x+2y-z, \mathbf{n}_1 = 4\mathbf{j}+\mathbf{k}, \mathbf{n}_2 = 4\mathbf{i}+2\mathbf{j}-\mathbf{k}, \mathbf{n}_1 \times \mathbf{n}_2 = -6\mathbf{i}+4\mathbf{j}-16\mathbf{k}$ is tangent to the line, $x(t) = 3t, y(t) = 2-2t, z(t) = 4+8t$.

33. Use implicit differentiation to get $\partial z/\partial x = -c^2x/\left(a^2z\right)$, $\partial z/\partial y = -c^2y/\left(b^2z\right)$. At (x_0,y_0,z_0), $z_0 \neq 0$, a normal to the surface is $-\left[c^2x_0/\left(a^2z_0\right)\right]\mathbf{i} - \left[c^2y_0/\left(b^2z_0\right)\right]\mathbf{j} - \mathbf{k}$ so the tangent plane is $-\dfrac{c^2x_0}{a^2z_0}x - \dfrac{c^2y_0}{b^2z_0}y - z = -\dfrac{c^2x_0^2}{a^2z_0} - \dfrac{c^2y_0^2}{b^2z_0} - z_0$, $\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}+\dfrac{z_0z}{c^2} = \dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{z_0^2}{c^2} = 1$.

34. $\partial z/\partial x = 2x/a^2$, $\partial z/\partial y = 2y/b^2$. At (x_0,y_0,z_0) the vector $\left(2x_0/a^2\right)\mathbf{i}+\left(2y_0/b^2\right)\mathbf{j}-\mathbf{k}$ is normal to the surface so the tangent plane is $\left(2x_0/a^2\right)x+\left(2y_0/b^2\right)y-z = 2x_0^2/a^2+2y_0^2/b^2-z_0$, but $z_0 = x_0^2/a^2+y_0^2/b^2$ so $\left(2x_0/a^2\right)x+\left(2y_0/b^2\right)y-z = 2z_0-z_0 = z_0$, $2x_0x/a^2+2y_0y/b^2 = z+z_0$.

35. $\mathbf{n}_1 = f_x(x_0,y_0)\mathbf{i}+f_y(x_0,y_0)\mathbf{j}-\mathbf{k}$ and $\mathbf{n}_2 = g_x(x_0,y_0)\mathbf{i}+g_y(x_0,y_0)\mathbf{j}-\mathbf{k}$ are normal, respectively, to $z = f(x,y)$ and $z = g(x,y)$ at P; $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$, $f_x(x_0,y_0)\,g_x(x_0,y_0)+f_y(x_0,y_0)\,g_y(x_0,y_0)+1 = 0$, $f_x(x_0,y_0)\,g_x(x_0,y_0)+f_y(x_0,y_0)\,g_y(x_0,y_0) = -1$.

36. $f_x = x/\sqrt{x^2+y^2}, f_y = y/\sqrt{x^2+y^2}, g_x = -x/\sqrt{x^2+y^2}, g_y = -y/\sqrt{x^2+y^2}, f_xg_x+f_yg_y = -(x^2+y^2)/(x^2+y^2) = -1$, so by Exercise 35 the normal lines are perpendicular.

37. $\nabla f = f_x\mathbf{i}+f_y\mathbf{j}+f_z\mathbf{k}$ and $\nabla g = g_x\mathbf{i}+g_y\mathbf{j}+g_z\mathbf{k}$ evaluated at (x_0,y_0,z_0) are normal, respectively, to the surfaces $f(x,y,z) = 0$ and $g(x,y,z) = 0$ at (x_0,y_0,z_0). The surfaces are orthogonal at (x_0,y_0,z_0) if and only if $\nabla f \cdot \nabla g = 0$ so $f_xg_x+f_yg_y+f_zg_z = 0$.

38. $f(x,y,z) = x^2+y^2+z^2-a^2 = 0, g(x,y,z) = z^2-x^2-y^2 = 0$, $f_xg_x+f_yg_y+f_zg_z = -4x^2-4y^2+4z^2 = 4g(x,y,z) = 0$.

39. $z = \dfrac{k}{xy}$; at a point $\left(a,b,\dfrac{k}{ab}\right)$ on the surface, $\left\langle -\dfrac{k}{a^2b}, -\dfrac{k}{ab^2}, -1\right\rangle$ and hence $\langle bk, ak, a^2b^2\rangle$ is normal to the surface so the tangent plane is $bkx+aky+a^2b^2z = 3abk$. The plane cuts the x, y, and z-axes at the points $3a$, $3b$, and $\dfrac{3k}{ab}$, respectively, so the volume of the tetrahedron that is formed is $V = \dfrac{1}{3}\left(\dfrac{3k}{ab}\right)\left[\dfrac{1}{2}(3a)(3b)\right] = \dfrac{9}{2}k$, which does not depend on a and b.

Exercise Set 13.8

1. (a) Minimum at $(2,-1)$, no maxima. **(b)** Maximum at $(0,0)$, no minima. **(c)** No maxima or minima.

2. (a) Maximum at $(-1,5)$, no minima. **(b)** No maxima or minima. **(c)** No maxima or minima.

3. $f(x,y) = (x-3)^2+(y+2)^2$, minimum at $(3,-2)$, no maxima.

4. $f(x,y) = -(x+1)^2-2(y-1)^2+4$, maximum at $(-1,1)$, no minima.

5. $f_x = 6x+2y = 0$, $f_y = 2x+2y = 0$; critical point (0,0); $D = 8 > 0$ and $f_{xx} = 6 > 0$ at (0,0), relative minimum.

6. $f_x = 3x^2-3y = 0$, $f_y = -3x-3y^2 = 0$; critical points (0,0) and $(-1,1)$; $D = -9 < 0$ at (0,0), saddle point; $D = 27 > 0$ and $f_{xx} = -6 < 0$ at $(-1,1)$, relative maximum.

7. $f_x = 2x-2xy = 0$, $f_y = 4y-x^2 = 0$; critical points (0,0) and $(\pm 2,1)$; $D = 8 > 0$ and $f_{xx} = 2 > 0$ at (0,0), relative minimum; $D = -16 < 0$ at $(\pm 2,1)$, saddle points.

8. $f_x = 3x^2 - 3 = 0$, $f_y = 3y^2 - 3 = 0$; critical points $(-1, \pm 1)$ and $(1, \pm 1)$; $D = -36 < 0$ at $(-1, 1)$ and $(1, -1)$, saddle points; $D = 36 > 0$ and $f_{xx} = 6 > 0$ at (1,1), relative minimum; $D = 36 > 0$ and $f_{xx} = -36 < 0$ at $(-1, -1)$, relative maximum.

9. $f_x = y + 2 = 0$, $f_y = 2y + x + 3 = 0$; critical point $(1, -2)$; $D = -1 < 0$ at $(1, -2)$, saddle point.

10. $f_x = 2x + y - 2 = 0$, $f_y = x - 2 = 0$; critical point $(2, -2)$; $D = -1 < 0$ at $(2, -2)$, saddle point.

11. $f_x = 2x + y - 3 = 0$, $f_y = x + 2y = 0$; critical point $(2, -1)$; $D = 3 > 0$ and $f_{xx} = 2 > 0$ at $(2, -1)$, relative minimum.

12. $f_x = y - 3x^2 = 0$, $f_y = x - 2y = 0$; critical points (0,0) and $(1/6, 1/12)$; $D = -1 < 0$ at (0,0), saddle point; $D = 1 > 0$ and $f_{xx} = -1 < 0$ at $(1/6, 1/12)$, relative maximum.

13. $f_x = 2x - 2/\left(x^2 y\right) = 0$, $f_y = 2y - 2/\left(xy^2\right) = 0$; critical points $(-1, -1)$ and $(1, 1)$; $D = 32 > 0$ and $f_{xx} = 6 > 0$ at $(-1, -1)$ and $(1, 1)$, relative minima.

14. $f_x = e^y = 0$ is impossible, no critical points.

15. $f_x = 2x = 0$, $f_y = 1 - e^y = 0$; critical point $(0, 0)$; $D = -2 < 0$ at $(0, 0)$, saddle point.

16. $f_x = y - 2/x^2 = 0$, $f_y = x - 4/y^2 = 0$; critical point (1,2); $D = 3 > 0$ and $f_{xx} = 4 > 0$ at $(1, 2)$, relative minimum.

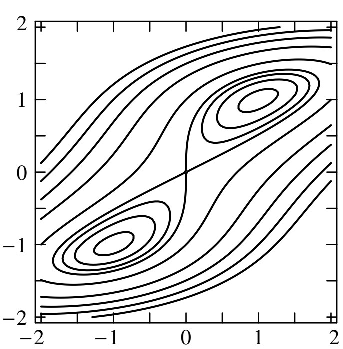
17. $f_x = e^x \sin y = 0$, $f_y = e^x \cos y = 0$, $\sin y = \cos y = 0$ is impossible, no critical points.

18. $f_x = y \cos x = 0$, $f_y = \sin x = 0$; $\sin x = 0$ if $x = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$ and $\cos x \neq 0$ for these values of x so $y = 0$; critical points $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$; $D = -1 < 0$ at $(n\pi, 0)$, saddle points.

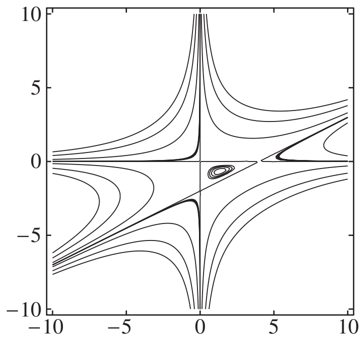
19. $f_x = -2(x+1)e^{-\left(x^2+y^2+2x\right)} = 0$, $f_y = -2ye^{-\left(x^2+y^2+2x\right)} = 0$; critical point $(-1, 0)$; $D = 4e^2 > 0$ and $f_{xx} = -2e < 0$ at $(-1, 0)$, relative maximum.

20. $f_x = y - a^3/x^2 = 0$, $f_y = x - b^3/y^2 = 0$; critical point $\left(a^2/b, b^2/a\right)$; if $ab > 0$ then $D = 3 > 0$ and $f_{xx} = 2b^3/a^3 > 0$ at $\left(a^2/b, b^2/a\right)$, relative minimum; if $ab < 0$ then $D = 3 > 0$ and $f_{xx} = 2b^3/a^3 < 0$ at $\left(a^2/b, b^2/a\right)$, relative maximum.

21. $\nabla f = (4x - 4y)\mathbf{i} - (4x - 4y^3)\mathbf{j} = \mathbf{0}$ when $x = y, x = y^3$, so $x = y = 0$ or $x = y = \pm 1$. At $(0,0), D = -16$, a saddle point; at $(1,1)$ and $(-1,-1), D = 32 > 0, f_{xx} = 4$, a relative minimum.



22. $\nabla f = (2y^2 - 2xy + 4y)\mathbf{i} + (4xy - x^2 + 4x)\mathbf{j} = \mathbf{0}$ when $2y^2 - 2xy + 4y = 0, 4xy - x^2 + 4x = 0$, with solutions $(0,0), (0,-2), (4,0), (4/3, -2/3)$. At $(0,0)$, $D = -16$, a saddle point. At $(0,-2)$, $D = -16$, a saddle point. At $(4,0), D = -16$, a saddle point. At $(4/3, -2/3), D = 16/3$, $f_{xx} = 4/3 > 0$, a relative minimum.



23. False, e.g. $f(x, y) = x$.

24. False; $f(x, y) = (x^2 + y^2 - 1/4)^2$, every point on $x^2 + y^2 = 1/4$ is a critical point of f.

25. True, Theorem 13.8.6.

26. True, Theorem 13.8.6.

27. **(a)** Critical point (0,0); $D = 0$.

(b) $f(0, 0) = 0$, $x^4 + y^4 \geq 0$ so $f(x, y) \geq f(0, 0)$, relative minimum.

28. **(a)** $f_x(x, y) = 4x^3$, $f_y(x, y) = -4y^3$, both equal zero only at $(0, 0)$ where $D = 0$.

(b) The trace of the surface $z = x^4 - y^4$ in the xz-plane has a relative minimum at the origin, whereas the trace in the yz-plane has a relative maximum there. Therefore, f has a saddle point at $(0, 0)$.

29. **(a)** $f_x = 3e^y - 3x^2 = 3\left(e^y - x^2\right) = 0$, $f_y = 3xe^y - 3e^{3y} = 3e^y\left(x - e^{2y}\right) = 0$, $e^y = x^2$ and $e^{2y} = x$, $x^4 = x$, $x\left(x^3 - 1\right) = 0$ so $x = 0, 1$; critical point $(1, 0)$; $D = 27 > 0$ and $f_{xx} = -6 < 0$ at $(1, 0)$, relative maximum.

(b) $\lim\limits_{x \to -\infty} f(x, 0) = \lim\limits_{x \to -\infty} \left(3x - x^3 - 1\right) = +\infty$ so no absolute maximum.

30. $f_x = 8xe^y - 8x^3 = 8x(e^y - x^2) = 0$, $f_y = 4x^2e^y - 4e^{4y} = 4e^y(x^2 - e^{3y}) = 0$, $x^2 = e^y$ and $x^2 = e^{3y}$, $e^{3y} = e^y$, $e^{2y} = 1$, so $y = 0$ and $x = \pm 1$; critical points (1,0) and $(-1, 0)$. $D = 128 > 0$ and $f_{xx} = -16 < 0$ at both points so a relative maximum occurs at each one.

31. $f_x = y - 1 = 0$, $f_y = x - 3 = 0$; critical point $(3, 1)$. Along $y = 0$: $u(x) = -x$; no critical points, along $x = 0$: $v(y) = -3y$; no critical points, along $y = -\dfrac{4}{5}x + 4$: $w(x) = -\dfrac{4}{5}x^2 + \dfrac{27}{5}x - 12$; critical point $(27/8, 13/10)$.

(x, y)	$(3, 1)$	$(0, 0)$	$(5, 0)$	$(0, 4)$	$(27/8, 13/10)$
$f(x, y)$	-3	0	-5	-12	$-231/80$

Absolute maximum value is 0, absolute minimum value is -12.

32. $f_x = y - 2 = 0$, $f_y = x = 0$; critical point $(0, 2)$, but $(0, 2)$ is not in the interior of R. Along $y = 0$: $u(x) = -2x$; no critical points, along $x = 0$: $v(y) = 0$; along $y = 4 - x$: $w(x) = 2x - x^2$; critical point $(1, 3)$.

(x, y)	$(0, 0)$	$(0, 4)$	$(4, 0)$	$(1, 3)$
$f(x, y)$	0	0	-8	1

Absolute maximum value is 1, absolute minimum value is -8.

33. $f_x = 2x - 2 = 0$, $f_y = -6y + 6 = 0$; critical point $(1, 1)$. Along $y = 0$: $u_1(x) = x^2 - 2x$; critical point $(1, 0)$, along $y = 2$: $u_2(x) = x^2 - 2x$; critical point $(1, 2)$, along $x = 0$: $v_1(y) = -3y^2 + 6y$; critical point $(0, 1)$, along $x = 2$: $v_2(y) = -3y^2 + 6y$; critical point $(2, 1)$.

(x, y)	$(1, 1)$	$(1, 0)$	$(1, 2)$	$(0, 1)$	$(2, 1)$	$(0, 0)$	$(0, 2)$	$(2, 0)$	$(2, 2)$
$f(x, y)$	2	-1	-1	3	3	0	0	0	0

Absolute maximum value is 3, absolute minimum value is -1.

34. $f_x = e^y - 2x = 0$, $f_y = xe^y - e^y = e^y(x - 1) = 0$; critical point $(1, \ln 2)$. Along $y = 0$: $u_1(x) = x - x^2 - 1$; critical point $(1/2, 0)$, along $y = 1$: $u_2(x) = ex - x^2 - e$; critical point $(e/2, 1)$, along $x = 0$: $v_1(y) = -e^y$; no critical points, along $x = 2$: $v_2(y) = e^y - 4$; no critical points.

(x, y)	$(0, 0)$	$(0, 1)$	$(2, 1)$	$(2, 0)$	$(1, \ln 2)$	$(1/2, 0)$	$(e/2, 1)$
$f(x, y)$	-1	$-e$	$e - 4$	-3	-1	$-3/4$	$e(e - 4)/4 \approx -0.87$

Absolute maximum value is $-3/4$, absolute minimum value is -3.

35. $f_x = 2x - 1 = 0$, $f_y = 4y = 0$; critical point $(1/2, 0)$. Along $x^2 + y^2 = 4$: $y^2 = 4 - x^2$, $u(x) = 8 - x - x^2$ for $-2 \le x \le 2$; critical points $(-1/2, \pm\sqrt{15}/2)$.

(x, y)	$(1/2, 0)$	$\left(-1/2, \sqrt{15}/2\right)$	$\left(-1/2, -\sqrt{15}/2\right)$	$(-2, 0)$	$(2, 0)$
$f(x, y)$	$-1/4$	$33/4$	$33/4$	6	2

Absolute maximum value is $33/4$, absolute minimum value is $-1/4$.

36. $f_x = y^2 = 0$, $f_y = 2xy = 0$; no critical points in the interior of R. Along $y = 0$: $u(x) = 0$; along $x = 0$: $v(y) = 0$; along $x^2 + y^2 = 1$: $w(x) = x - x^3$ for $0 \le x \le 1$; critical point $\left(1/\sqrt{3}, \sqrt{2/3}\right)$.

(x, y)	$(0, 0)$	$(0, 1)$	$(1, 0)$	$\left(1/\sqrt{3}, \sqrt{2/3}\right)$
$f(x, y)$	0	0	0	$2\sqrt{3}/9$

Absolute maximum value is $\dfrac{2}{9}\sqrt{3}$, absolute minimum value is 0.

37. Maximize $P = xyz$ subject to $x + y + z = 48$, $x > 0$, $y > 0$, $z > 0$. $z = 48 - x - y$ so $P = xy(48 - x - y) = 48xy - x^2y - xy^2$, $P_x = 48y - 2xy - y^2 = 0$, $P_y = 48x - x^2 - 2xy = 0$. But $x \neq 0$ and $y \neq 0$ so $48 - 2x - y = 0$ and $48 - x - 2y = 0$; critical point (16,16). $P_{xx}P_{yy} - P_{xy}^2 > 0$ and $P_{xx} < 0$ at $(16, 16)$, relative maximum. $z = 16$ when $x = y = 16$, the product is maximum for the numbers $16, 16, 16$.

38. Minimize $S = x^2+y^2+z^2$ subject to $x+y+z = 27$, $x > 0$, $y > 0$, $z > 0$. $z = 27-x-y$ so $S = x^2+y^2+(27-x-y)^2$, $S_x = 4x + 2y - 54 = 0$, $S_y = 2x + 4y - 54 = 0$; critical point $(9, 9)$; $S_{xx}S_{yy} - S_{xy}^2 = 12 > 0$ and $S_{xx} = 4 > 0$ at $(9, 9)$, relative minimum. $z = 9$ when $x = y = 9$, the sum of the squares is minimum for the numbers $9, 9, 9$.

39. Maximize $w = xy^2z^2$ subject to $x + y + z = 5$, $x > 0$, $y > 0$, $z > 0$. $x = 5 - y - z$ so $w = (5 - y - z)y^2z^2 = 5y^2z^2 - y^3z^2 - y^2z^3$, $w_y = 10yz^2 - 3y^2z^2 - 2yz^3 = yz^2(10 - 3y - 2z) = 0$, $w_z = 10y^2z - 2y^3z - 3y^2z^2 = y^2z(10 - 2y - 3z) = 0$, $10 - 3y - 2z = 0$ and $10 - 2y - 3z = 0$; critical point when $y = z = 2$; $w_{yy}w_{zz} - w_{yz}^2 = 320 > 0$ and $w_{yy} = -24 < 0$ when $y = z = 2$, relative maximum. $x = 1$ when $y = z = 2$, xy^2z^2 is maximum at $(1, 2, 2)$.

40. Minimize $w = D^2 = x^2 + y^2 + z^2$ subject to $x^2 - yz = 5$. $x^2 = 5 + yz$ so $w = 5 + yz + y^2 + z^2$, $w_y = z + 2y = 0$, $w_z = y + 2z = 0$; critical point when $y = z = 0$; $w_{yy}w_{zz} - w_{yz}^2 = 3 > 0$ and $w_{yy} = 2 > 0$ when $y = z = 0$, relative minimum. $x^2 = 5$, $x = \pm\sqrt{5}$ when $y = z = 0$. The points $\left(\pm\sqrt{5}, 0, 0\right)$ are closest to the origin.

41. The diagonal of the box must equal the diameter of the sphere, thus we maximize $V = xyz$ or, for convenience, $w = V^2 = x^2y^2z^2$ subject to $x^2 + y^2 + z^2 = 4a^2$, $x > 0$, $y > 0$, $z > 0$; $z^2 = 4a^2 - x^2 - y^2$ hence $w = 4a^2x^2y^2 - x^4y^2 - x^2y^4$, $w_x = 2xy^2(4a^2 - 2x^2 - y^2) = 0, w_y = 2x^2y\left(4a^2 - x^2 - 2y^2\right) = 0, 4a^2 - 2x^2 - y^2 = 0$ and $4a^2 - x^2 - 2y^2 = 0$; critical point $\left(2a/\sqrt{3}, 2a/\sqrt{3}\right)$; $w_{xx}w_{yy} - w_{xy}^2 = \dfrac{4096}{27}a^8 > 0$ and $w_{xx} = -\dfrac{128}{9}a^4 < 0$ at $\left(2a/\sqrt{3}, 2a/\sqrt{3}\right)$, relative maximum. $z = 2a/\sqrt{3}$ when $x = y = 2a/\sqrt{3}$, the dimensions of the box of maximum volume are $2a/\sqrt{3}, 2a/\sqrt{3}, 2a/\sqrt{3}$.

42. Maximize $V = xyz$ subject to $x + y + z = 129$, $x > 0$, $y > 0$, $z > 0$. $z = 129 - x - y$ so $V = 129xy - x^2y - xy^2$, $V_x = y(129 - 2x - y) = 0$, $V_y = x(129 - x - 2y) = 0$, $129 - 2x - y = 0$ and $129 - x - 2y = 0$; critical point $(43, 43)$; $V_{xx}V_{yy} - V_{xy}^2 = 7439 > 0$ and $V_{xx} = -86 < 0$ at $(43, 43)$, relative maximum. The maximum volume is $V = (43)(43)(43) = 79{,}507$ cm^3.

43. Let x, y, and z be, respectively, the length, width, and height of the box. Minimize $C = 10(2xy) + 5(2xz + 2yz) = 10(2xy + xz + yz)$ subject to $xyz = 16$. $z = 16/(xy)$, so $C = 20(xy + 8/y + 8/x)$, $C_x = 20(y - 8/x^2) = 0$, $C_y = 20(x - 8/y^2) = 0$; critical point (2,2); $C_{xx}C_{yy} - C_{xy}^2 = 1200 > 0$ and $C_{xx} = 40 > 0$ at (2,2), relative minimum. $z = 4$ when $x = y = 2$. The cost of materials is minimum if the length and width are 2 ft and the height is 4 ft.

44. Maximize the profit $P = 500(y-x)(x-40)+[45,000+500(x-2y)](y-60) = 500(-x^2-2y^2+2xy-20x+170y-5400)$. $P_x = 1000(-x + y - 10) = 0$, $P_y = 1000(-2y + x + 85) = 0$; critical point (65,75); $P_{xx}P_{yy} - P_{xy}^2 = 1{,}000{,}000 > 0$ and $P_{xx} = -1000 < 0$ at (65,75), relative maximum. The profit will be maximum when $x = 65$ and $y = 75$.

45. (a) $x = 0 : f(0, y) = -3y^2$, minimum -3, maximum $0; x = 1, f(1, y) = 4 - 3y^2 + 2y, \dfrac{\partial f}{\partial y}(1, y) = -6y + 2 = 0$ at $y = 1/3$, minimum 3, maximum 13/3; $y = 0, f(x, 0) = 4x^2$, minimum 0, maximum 4; $y = 1, f(x, 1) = 4x^2 + 2x - 3, \dfrac{\partial f}{\partial x}(x, 1) = 8x + 2 \neq 0$ for $0 < x < 1$, minimum -3, maximum 3.

(b) $f(x, x) = 3x^2$, minimum 0, maximum 3; $f(x, 1-x) = -x^2+8x-3, \dfrac{d}{dx}f(x, 1-x) = -2x+8 \neq 0$ for $0 < x < 1$, maximum 4, minimum -3.

(c) $f_x(x, y) = 8x + 2y = 0, f_y(x, y) = -6y + 2x = 0$, solution is $(0, 0)$, which is not an interior point of the square, so check the sides: minimum -3, maximum 13/3.

46. Maximize $A = ab \sin\alpha$ subject to $2a + 2b = \ell$, $a > 0$, $b > 0$, $0 < \alpha < \pi$. $b = (\ell - 2a)/2$ so $A = (1/2)(\ell a - 2a^2)\sin\alpha$, $A_a = (1/2)(\ell - 4a)\sin\alpha$, $A_\alpha = (a/2)(\ell - 2a)\cos\alpha$; $\sin\alpha \neq 0$ so from $A_a = 0$ we get $a = \ell/4$ and then from $A_\alpha = 0$ we get $\cos\alpha = 0$, $\alpha = \pi/2$. $A_{aa}A_{\alpha\alpha} - A_{a\alpha}^2 = \ell^2/8 > 0$ and $A_{aa} = -2 < 0$ when $a = \ell/4$ and $\alpha = \pi/2$, the area is maximum.

47. Minimize $S = xy+2xz+2yz$ subject to $xyz = V$, $x > 0$, $y > 0$, $z > 0$ where x, y, and z are, respectively, the length, width, and height of the box. $z = V/(xy)$ so $S = xy + 2V/y + 2V/x$, $S_x = y - 2V/x^2 = 0$, $S_y = x - 2V/y^2 = 0$; critical point $(\sqrt[3]{2V}, \sqrt[3]{2V})$; $S_{xx}S_{yy} - S_{xy}^2 = 3 > 0$ and $S_{xx} = 2 > 0$ at this point so there is a relative minimum there. The length and width are each $\sqrt[3]{2V}$, the height is $z = \sqrt[3]{2V}/2$.

48. The altitude of the trapezoid is $x \sin\phi$ and the lengths of the lower and upper bases are, respectively, $27 - 2x$ and $27 - 2x + 2x\cos\phi$ so we want to maximize $A = (1/2)(x\sin\phi)[(27 - 2x) + (27 - 2x + 2x\cos\phi)] = 27x\sin\phi - 2x^2\sin\phi + x^2\sin\phi\cos\phi$. $A_x = \sin\phi(27 - 4x + 2x\cos\phi)$, $A_\phi = x(27\cos\phi - 2x\cos\phi - x\sin^2\phi + x\cos^2\phi) = x(27\cos\phi - 2x\cos\phi + 2x\cos^2\phi - x)$. $\sin\phi \neq 0$ so from $A_x = 0$ we get $\cos\phi = (4x - 27)/(2x)$, $x \neq 0$ so from $A_\phi = 0$ we get $(27 - 2x + 2x\cos\phi)\cos\phi - x = 0$ which, for $\cos\phi = (4x - 27)/(2x)$, yields $4x - 27 - x = 0$, $x = 9$. If $x = 9$ then $\cos\phi = 1/2$, $\phi = \pi/3$. The critical point occurs when $x = 9$ and $\phi = \pi/3$; $A_{xx}A_{\phi\phi} - A_{x\phi}^2 = 729/2 > 0$ and $A_{xx} = -3\sqrt{3}/2 < 0$ there, the area is maximum when $x = 9$ and $\phi = \pi/3$.

49. (a) $\dfrac{\partial g}{\partial m} = \displaystyle\sum_{i=1}^{n} 2\left(mx_i + b - y_i\right)x_i = 2\left(m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_iy_i\right) = 0$ if $\left(\displaystyle\sum_{i=1}^{n} x_i^2\right)m + \left(\displaystyle\sum_{i=1}^{n} x_i\right)b =$

$\sum_{i=1}^{n} x_i y_i, \frac{\partial g}{\partial b} = \sum_{i=1}^{n} 2(mx_i + b - y_i) = 2\left(m\sum_{i=1}^{n} x_i + bn - \sum_{i=1}^{n} y_i\right) = 0$ if $\left(\sum_{i=1}^{n} x_i\right) m + nb = \sum_{i=1}^{n} y_i$.

(b) $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}\left(x_i^2 - 2\bar{x}x_i + \bar{x}^2\right) = \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - \frac{2}{n}\left(\sum_{i=1}^{n} x_i\right)^2 + \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 =$ $\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 \geq 0$ so $n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 \geq 0$. This is an equality if and only if $\sum_{i=1}^{n}(x_i - \bar{x})^2 = 0$, which means $x_i = \bar{x}$ for each i.

(c) The system of equations $Am + Bb = C, Dm + Eb = F$ in the unknowns m and b has a unique solution provided $AE \neq BD$, and if so the solution is $m = \frac{CE - BF}{AE - BD}, b = \frac{F - Dm}{E}$, which after the appropriate substitution yields the desired result.

50. (a) $g_{mm} = 2\sum_{i=1}^{n} x_i^2,\ g_{bb} = 2n,\ g_{mb} = 2\sum_{i=1}^{n} x_i,\ D = g_{mm}g_{bb} - g_{mb}^2 = 4\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right] > 0$ and $g_{mm} > 0$.

(b) $g(m, b)$ is of the second-degree in m and b so the graph of $z = g(m, b)$ is a quadric surface.

(c) The function $z = g(m, b)$, as a function of m and b, has only one critical point, found in Exercise 49, and tends to $+\infty$ as either $|m|$ or $|b|$ tends to infinity, since g_{mm} and g_{bb} are both positive. Thus the only critical point must be a minimum.

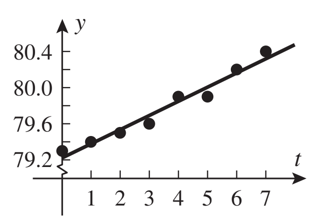
51. $n = 3, \sum_{i=1}^{3} x_i = 3, \sum_{i=1}^{3} y_i = 7, \sum_{i=1}^{3} x_i y_i = 13, \sum_{i=1}^{3} x_i^2 = 11, y = \frac{3}{4}x + \frac{19}{12}$.

52. $n = 4, \sum_{i=1}^{4} x_i = 7, \sum_{i=1}^{4} y_i = 4, \sum_{i=1}^{4} x_i^2 = 21, \sum_{i=1}^{4} x_i y_i = -2, y = -\frac{36}{35}x + \frac{14}{5}$.

53. $\sum_{i=1}^{4} x_i = 10, \sum_{i=1}^{4} y_i = 8.2, \sum_{i=1}^{4} x_i^2 = 30, \sum_{i=1}^{4} x_i y_i = 23, n = 4;\ m = 0.5,\ b = 0.8,\ y = 0.5x + 0.8$.

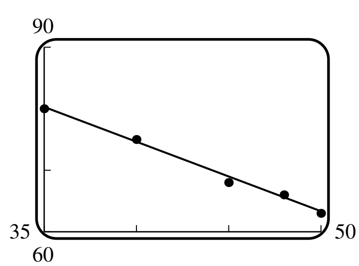
54. $\sum_{i=1}^{5} x_i = 15, \sum_{i=1}^{5} y_i = 15.1, \sum_{i=1}^{5} x_i^2 = 55, \sum_{i=1}^{5} x_i y_i = 39.8, n = 5; m = -0.55, b = 4.67, y = 4.67 - 0.55x$.

55. (a) $y \approx 79.225 + 0.1571t$. **(b)**



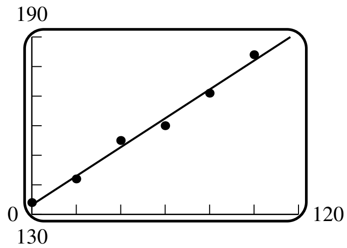
(c) $y \approx 81.6$.

56. (a) $y \approx 119.84 - 1.13x$. **(b)**



(c) About 52 units.

57. **(a)** $P = \dfrac{2798}{21} + \dfrac{171}{350}T$. **(b)**



(c) $T \approx -272.7096°$ C.

58. **(a)** For example, $z = y$.

(b) For example, on $0 \le x \le 1, 0 \le y \le 1$ let $z = \begin{cases} y & \text{if} \quad 0 < x < 1, 0 < y < 1; \\ 1/2 & \text{if} \quad x = 0, 1 \text{ or } y = 0, 1. \end{cases}$

59. $f(x_0, y_0) \ge f(x, y)$ for all (x, y) inside a circle centered at (x_0, y_0) by virtue of Definition 14.8.1. If r is the radius of the circle, then in particular $f(x_0, y_0) \ge f(x, y_0)$ for all x satisfying $|x - x_0| < r$ so $f(x, y_0)$ has a relative maximum at x_0. The proof is similar for the function $f(x_0, y)$.

Exercise Set 13.9

1. **(a)** $xy = 4$ is tangent to the line, so the maximum value of f is 4.

(b) $xy = 2$ intersects the curve and so gives a smaller value of f.

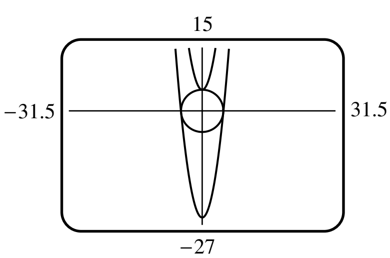
(c) Maximize $f(x, y) = xy$ subject to the constraint $g(x, y) = x + y - 4 = 0, \nabla f = \lambda \nabla g, y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$, so solve the equations $y = \lambda, x = \lambda$ with solution $x = y = \lambda$, but $x + y = 4$, so $x = y = 2$, and the maximum value of f is $f = xy = 4$.

2. **(a)** $x^2 + y^2 = 25$ is tangent to the line at $(3, 4)$, so the minimum value of f is 25.

(b) A larger value of f yields a circle of a larger radius, and hence intersects the line.

(c) Minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 3x + 4y - 25 = 0, \nabla f = \lambda \nabla g, 2x\mathbf{i} + 2y\mathbf{j} = 3\lambda\mathbf{i} + 4\lambda\mathbf{j}$, so solve $2x = 3\lambda, 2y = 4\lambda$ and $3x + 4y - 25 = 0$; solution is $x = 3, y = 4$, minimum $= 25$.

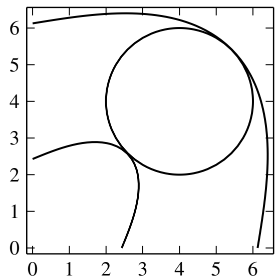
3. **(a)**



(b) One extremum at $(0, 5)$ and one at approximately $(\pm 5, 0)$, so minimum value -5, maximum value ≈ 25.

(c) Find the minimum and maximum values of $f(x, y) = x^2 - y$ subject to the constraint $g(x, y) = x^2 + y^2 - 25 = 0, \nabla f = \lambda \nabla g, 2x\mathbf{i} - \mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, so solve $2x = 2\lambda x, -1 = 2\lambda y, x^2 + y^2 - 25 = 0$. If $x = 0$ then $y = \pm 5, f = \mp 5$, and if $x \ne 0$ then $\lambda = 1, y = -1/2, x^2 = 25 - 1/4 = 99/4, f = 99/4 + 1/2 = 101/4$, so the maximum value of f is $101/4$ at $(\pm 3\sqrt{11}/2, -1/2)$ and the minimum value of f is -5 at $(0, 5)$.

4. (a)



(b) $f \approx 15$.

(d) Set $f(x,y) = x^3+y^3-3xy$, $g(x,y) = (x-4)^2+(y-4)^2-4$; minimize f subject to the constraint $g = 0:\ \nabla f = \lambda g, (3x^2-3y)\mathbf{i}+(3y^2-3x)\mathbf{j} = 2\lambda(x-4)\mathbf{i}+2\lambda(y-4)\mathbf{j}$, so solve (use a CAS) $3x^2-3y = 2\lambda(x-4), 3y^2-3x = 2\lambda(y-4)$ and $(x-4)^2+(y-4)^2-4 = 0$; minimum value $f = 14.52$ at $(2.5858, 2.5858)$.

5. $y = 8x\lambda$, $x = 16y\lambda$; $y/(8x) = x/(16y)$, $x^2 = 2y^2$ so $4\left(2y^2\right)+8y^2 = 16$, $y^2 = 1$, $y = \pm 1$. Test $\left(\pm\sqrt{2},-1\right)$ and $(\pm\sqrt{2},1)$. $f\left(-\sqrt{2},-1\right) = f\left(\sqrt{2},1\right) = \sqrt{2}$, $f\left(-\sqrt{2},1\right) = f\left(\sqrt{2},-1\right) = -\sqrt{2}$. Maximum $\sqrt{2}$ at $\left(-\sqrt{2},-1\right)$ and $\left(\sqrt{2},1\right)$, minimum $-\sqrt{2}$ at $\left(-\sqrt{2},1\right)$ and $\left(\sqrt{2},-1\right)$.

6. $2x = 2x\lambda, -2y = 2y\lambda, x^2+y^2 = 25$. If $x \neq 0$ then $\lambda = 1$ and $y = 0$ so $x^2+0^2 = 25$, $x = \pm 5$. If $x = 0$ then $0^2+y^2 = 25$, $y = \pm 5$. Test $(\pm 5,0)$ and $(0,\pm 5)$: $f(\pm 5,0) = 25$, $f(0,\pm 5) = -25$, maximum 25 at $(\pm 5,0)$, minimum -25 at $(0,\pm 5)$.

7. $12x^2 = 4x\lambda$, $2y = 2y\lambda$. If $y \neq 0$ then $\lambda = 1$ and $12x^2 = 4x$, $12x(x-1/3) = 0$, $x = 0$ or $x = 1/3$ so from $2x^2+y^2 = 1$ we find that $y = \pm 1$ when $x = 0$, $y = \pm\sqrt{7}/3$ when $x = 1/3$. If $y = 0$ then $2x^2+(0)^2 = 1$, $x = \pm 1/\sqrt{2}$. Test $(0,\pm 1)$, $\left(1/3,\pm\sqrt{7}/3\right)$, and $\left(\pm 1/\sqrt{2},0\right)$. $f(0,\pm 1) = 1$, $f\left(1/3,\pm\sqrt{7}/3\right) = 25/27$, $f\left(1/\sqrt{2},0\right) = \sqrt{2}$, $f\left(-1/\sqrt{2},0\right) = -\sqrt{2}$. Maximum $\sqrt{2}$ at $\left(1/\sqrt{2},0\right)$, minimum $-\sqrt{2}$ at $\left(-1/\sqrt{2},0\right)$.

8. $1 = 2x\lambda$, $-3 = 6y\lambda$; $1/(2x) = -1/(2y)$, $y = -x$ so $x^2+3(-x)^2 = 16$, $x = \pm 2$. Test $(-2,2)$ and $(2,-2)$. $f(-2,2) = -9$, $f(2,-2) = 7$. Maximum 7 at $(2,-2)$, minimum -9 at $(-2,2)$.

9. $2 = 2x\lambda, 1 = 2y\lambda, -2 = 2z\lambda$; $1/x = 1/(2y) = -1/z$ thus $x = 2y$, $z = -2y$ so $(2y)^2+y^2+(-2y)^2 = 4$, $y^2 = 4/9$, $y = \pm 2/3$. Test $(-4/3,-2/3,4/3)$ and $(4/3,2/3,-4/3)$. $f(-4/3,-2/3,4/3) = -6$, $f(4/3,2/3,-4/3) = 6$. Maximum 6 at $(4/3,2/3,-4/3)$, minimum -6 at $(-4/3,-2/3,4/3)$.

10. $3 = 4x\lambda$, $6 = 8y\lambda$, $2 = 2z\lambda$; $3/(4x) = 3/(4y) = 1/z$ thus $y = x$, $z = 4x/3$, so $2x^2+4x^2+(4x/3)^2 = 70$, $x^2 = 9$, $x = \pm 3$. Test $(-3,-3,-4)$ and $(3,3,4)$. $f(-3,-3,-4) = -35$, $f(3,3,4) = 35$. Maximum 35 at (3, 3, 4), minimum -35 at $(-3,-3,-4)$.

11. $yz = 2x\lambda$, $xz = 2y\lambda$, $xy = 2z\lambda$; $yz/(2x) = xz/(2y) = xy/(2z)$ thus $y^2 = x^2$, $z^2 = x^2$ so $x^2+x^2+x^2 = 1$, $x = \pm 1/\sqrt{3}$. Test the eight possibilities with $x = \pm 1/\sqrt{3}$, $y = \pm 1/\sqrt{3}$, and $z = \pm 1/\sqrt{3}$ to find the maximum is $1/\left(3\sqrt{3}\right)$ at $\left(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}\right)$, $\left(1/\sqrt{3},-1/\sqrt{3},-1/\sqrt{3}\right)$, $\left(-1/\sqrt{3},1/\sqrt{3},-1/\sqrt{3}\right)$, and $\left(-1/\sqrt{3},-1/\sqrt{3},1/\sqrt{3}\right)$; the minimum is $-1/\left(3\sqrt{3}\right)$ at $\left(1/\sqrt{3},1/\sqrt{3},-1/\sqrt{3}\right)$, $\left(1/\sqrt{3},-1/\sqrt{3},1/\sqrt{3}\right)$, $\left(-1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}\right)$, and $\left(-1/\sqrt{3},-1/\sqrt{3},-1/\sqrt{3}\right)$.

12. $4x^3 = 2\lambda x, 4y^3 = 2\lambda y, 4z^3 = 2\lambda z$; if x (or y or z) $\neq 0$ then $\lambda = 2x^2$ (or $2y^2$ or $2z^2$). Assume for the moment that $|x| \le |y| \le |z|$. Then:

Case I: $x,y,z \neq 0$ so $\lambda = 2x^2 = 2y^2 = 2z^2, x = \pm y = \pm z$, $3x^2 = 1, x = \pm 1/\sqrt{3}$, $f(x,y,z) = 3/9 = 1/3$.

Case II: $x = 0, y,z \neq 0$; then $y = \pm z, 2y^2 = 1, y = \pm z = \pm 1/\sqrt{2}, f(x,y,z) = 2/4 = 1/2$.

Case III: $x = y = 0, z \neq 0$; then $z^2 = 1, z = \pm 1, f(x,y,z) = 1$.

Case IV: all other cases follow by symmetry.

Thus f has a maximum value of 1 at $(0,0,\pm 1), (0,\pm 1,0)$, and $(\pm 1,0,0)$ and a minimum value of $1/3$ at $(\pm 1/\sqrt{3},\pm 1/\sqrt{3},\pm 1/\sqrt{3})$.

13. False, it is a scalar.

14. False, they must be parallel, not necessarily equal.

15. False, there are three equations in three unknowns.

16. True, see the discussion before equation (3).

17. $f(x,y) = x^2 + y^2$; $2x = 2\lambda$, $2y = -4\lambda$; $y = -2x$ so $2x - 4(-2x) = 3$, $x = 3/10$. The point is $(3/10, -3/5)$.

18. $f(x,y) = (x-4)^2 + (y-2)^2$, $g(x,y) = y - 2x - 3$; $2(x-4) = -2\lambda$, $2(y-2) = \lambda$; $x - 4 = -2(y-2)$, $x = -2y + 8$ so $y = 2(-2y+8) + 3$, $y = 19/5$. The point is $(2/5, 19/5)$.

19. $f(x,y,z) = x^2 + y^2 + z^2$; $2x = \lambda$, $2y = 2\lambda$, $2z = \lambda$; $y = 2x$, $z = x$ so $x + 2(2x) + x = 1$, $x = 1/6$. The point is $(1/6, 1/3, 1/6)$.

20. $f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2$; $2(x-1) = 4\lambda$, $2(y+1) = 3\lambda$, $2(z-1) = \lambda$; $x = 4z - 3$, $y = 3z - 4$ so $4(4z-3) + 3(3z-4) + z = 2$, $z = 1$. The point is $(1, -1, 1)$.

21. $f(x,y) = (x-1)^2 + (y-2)^2$; $2(x-1) = 2x\lambda$, $2(y-2) = 2y\lambda$; $(x-1)/x = (y-2)/y$, $y = 2x$ so $x^2 + (2x)^2 = 45$, $x = \pm 3$. $f(-3,-6) = 80$ and $f(3,6) = 20$ so $(3,6)$ is closest and $(-3,-6)$ is farthest.

22. $f(x,y,z) = x^2+y^2+z^2$; $2x = y\lambda$, $2y = x\lambda$, $2z = -2z\lambda$. If $z \neq 0$ then $\lambda = -1$ so $2x = -y$ and $2y = -x$, $x = y = 0$; substitute into $xy - z^2 = 1$ to get $z^2 = -1$ which has no real solution. If $z = 0$ then $xy - (0)^2 = 1$, $y = 1/x$, and also (from $2x = y\lambda$ and $2y = x\lambda$), $2x/y = 2y/x$, $y^2 = x^2$ so $(1/x)^2 = x^2$, $x^4 = 1$, $x = \pm 1$. Test $(1,1,0)$ and $(-1,-1,0)$ to see that they are both closest to the origin.

23. $f(x,y,z) = x+y+z$, $x^2+y^2+z^2 = 25$ where x, y, and z are the components of the vector; $1 = 2x\lambda$, $1 = 2y\lambda$, $1 = 2z\lambda$; $1/(2x) = 1/(2y) = 1/(2z)$; $y = x$, $z = x$ so $x^2 + x^2 + x^2 = 25$, $x = \pm 5/\sqrt{3}$. $f\left(-5/\sqrt{3}, -5/\sqrt{3}, -5/\sqrt{3}\right) = -5\sqrt{3}$ and $f\left(5/\sqrt{3}, 5/\sqrt{3}, 5/\sqrt{3}\right) = 5\sqrt{3}$ so the vector is $5(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$.

24. $x^2 + y^2 = 25$ is the constraint; solve $8x - 4y = 2x\lambda$, $-4x + 2y = 2y\lambda$. If $x = 0$ then $y = 0$ and conversely; but $x^2+y^2 = 25$, so x and y are nonzero. Thus $\lambda = (4x-2y)/x = (-2x+y)/y$, so $0 = 2x^2+3xy-2y^2 = (2x-y)(x+2y)$, hence $y = 2x$ or $x = -2y$. If $y = 2x$ then $x^2 + (2x)^2 = 25$, $x = \pm\sqrt{5}$. If $x = -2y$ then $\left(-2y^2\right) + y^2 = 25$, $y = \pm\sqrt{5}$. $T\left(-\sqrt{5}, -2\sqrt{5}\right) = T\left(\sqrt{5}, 2\sqrt{5}\right) = 0$ and $T\left(2\sqrt{5}, -\sqrt{5}\right) = T\left(-2\sqrt{5}, \sqrt{5}\right) = 125$. The highest temperature is 125 and the lowest is 0.

25. Minimize $f = x^2 + y^2 + z^2$ subject to $g(x,y,z) = x + y + z - 27 = 0$. $\nabla f = \lambda \nabla g$, $2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda\mathbf{i} + \lambda\mathbf{j} + \lambda\mathbf{k}$, solution $x = y = z = 9$, minimum value 243.

26. Maximize $f(x,y,z) = xy^2z^2$ subject to $g(x,y,z) = x + y + z - 5 = 0, \nabla f = \lambda\nabla g = \lambda(\mathbf{i}+\mathbf{j}+\mathbf{k}), \lambda = y^2z^2 = 2xyz^2 = 2xy^2z$, $\lambda = 0$ is impossible, hence $x,y,z \neq 0$, and $z = y = 2x, 5x - 5 = 0$, $x = 1, y = z = 2$, maximum value 16 at $(1,2,2)$.

27. Minimize $f = x^2 + y^2 + z^2$ subject to $x^2 - yz = 5, \nabla f = \lambda\nabla g, 2x = 2x\lambda, 2y = -z\lambda, 2z = -y\lambda$. If $\lambda \neq \pm 2$, then $y = z = 0, x = \pm\sqrt{5}, f = 5$; if $\lambda = \pm 2$ then $x = 0$, and since $-yz = 5$, $y = -z = \pm\sqrt{5}, f = 10$, thus the minimum value is 5 at $(\pm\sqrt{5}, 0, 0)$.

28. The diagonal of the box must equal the diameter of the sphere so maximize $V = xyz$ or, for convenience, maximize $f = V^2 = x^2y^2z^2$ subject to $g(x,y,z) = x^2+y^2+z^2-4a^2 = 0, \nabla f = \lambda\nabla g$, $2xy^2z^2 = 2\lambda x, 2x^2yz^2 = 2\lambda y, 2x^2y^2z = 2\lambda z$. Since $V \neq 0$ it follows that $x,y,z \neq 0$, hence $x = y = z, 3x^2 = 4a^2, x = 2a/\sqrt{3}$, maximum volume $8a^3/(3\sqrt{3})$.

29. Let x, y, and z be, respectively, the length, width, and height of the box. Minimize $f(x,y,z) = 10(2xy) + 5(2xz + 2yz) = 10(2xy + xz + yz)$ subject to $g(x,y,z) = xyz - 16 = 0$, $\nabla f = \lambda\nabla g, 20y + 10z = \lambda yz, 20x + 10z = \lambda xz, 10x + 10y = \lambda xy$. Since $V = xyz = 16, x,y,z \neq 0$, thus $\lambda z = 20 + 10(z/y) = 20 + 10(z/x)$, so $x = y$.

From this and $10x + 10y = \lambda xy$ it follows that $20 = \lambda x$, so $10z = 20x, z = 2x = 2y, V = 2x^3 = 16$ and thus $x = y = 2$ ft, $z = 4$ ft, $f(2, 2, 4) = 240$ cents.

30. **(a)** If $g(x, y) = x = 0$ then $8x+2y = \lambda, -6y+2x = 0$; but $x = 0$, so $y = \lambda = 0$, $f(0, 0) = 0$ maximum, $f(0, 1) = -3$, minimum. If $g(x, y) = x - 1 = 0$ then $8x + 2y = \lambda, -6y + 2x = 0$; but $x = 1$, so $y = 1/3$, $f(1, 1/3) = 13/3$ maximum, $f(1, 0) = 4$, $f(1, 1) = 3$ minimum. If $g(x, y) = y = 0$ then $8x + 2y = 0, -6y + 2x = \lambda$; but $y = 0$ so $x = \lambda = 0$, $f(0, 0) = 0$ minimum, $f(1, 0) = 4$, maximum. If $g(x, y) = y - 1 = 0$ then $8x + 2y = 0, -6y + 2x = \lambda$; but $y = 1$ so $x = -1/4$, no solution, $f(0, 1) = -3$ minimum, $f(1, 1) = 3$ maximum.

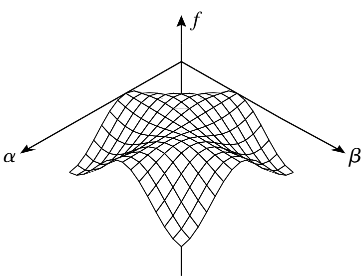
(b) If $g(x, y) = x - y = 0$ then $8x + 2y = \lambda, -6y + 2x = -\lambda$; but $x = y$ so solution $x = y = \lambda = 0, f(0, 0) = 0$ minimum, $f(1, 1) = 3$ maximum. If $g(x, y) = 1 - x - y = 0$ then $8x + 2y = -1, -6y + 2x = -1$; but $x + y = 1$ so solution is $x = -2/13, y = 3/2$ which is not on diagonal, $f(0, 1) = -3$ minimum, $f(1, 0) = 4$ maximum.

31. Maximize $A(a, b, \alpha) = ab \sin\alpha$ subject to $g(a, b, \alpha) = 2a + 2b - \ell = 0, \nabla_{(a,b,\alpha)}A = \lambda\nabla_{(a,b,\alpha)}g$, $b \sin\alpha = 2\lambda, a \sin\alpha = 2\lambda, ab \cos\alpha = 0$ with solution $a = b\ (= \ell/4), \alpha = \pi/2$ maximum value if parallelogram is a square.

32. Minimize $f(x, y, z) = xy + 2xz + 2yz$ subject to $g(x, y, z) = xyz - V = 0, \nabla f = \lambda\nabla g$, $y + 2z = \lambda yz, x + 2z = \lambda xz, 2x + 2y = \lambda xy; \lambda = 0$ leads to $x = y = z = 0$, impossible, so solve for $\lambda = 1/z + 2/x = 1/z + 2/y = 2/y + 2/x$, so $x = y = 2z, x^3 = 2V$, minimum value $3(2V)^{2/3}$.

33. **(a)** Maximize $f(\alpha, \beta, \gamma) = \cos\alpha\cos\beta\cos\gamma$ subject to $g(\alpha, \beta, \gamma) = \alpha+\beta+\gamma-\pi = 0, \nabla f = \lambda\nabla g, -\sin\alpha\cos\beta\cos\gamma = \lambda, -\cos\alpha\sin\beta\cos\gamma = \lambda, -\cos\alpha\cos\beta\sin\gamma = \lambda$ with solution $\alpha = \beta = \gamma = \pi/3$, maximum value $1/8$.

(b) For example, $f(\alpha, \beta) = \cos\alpha\cos\beta\cos(\pi - \alpha - \beta)$.



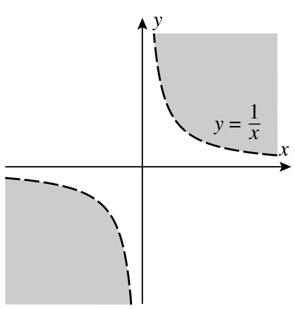
34. Find maxima and minima $z = x^2 + 4y^2$ subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$, $\nabla z = \lambda\nabla g, 2x\mathbf{i} + 8y\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, solve $2x = 2\lambda x, 8y = 2\lambda y$. If $y \neq 0$ then $\lambda = 4, x = 0, y^2 = 1$ and $z = x^2 + 4y^2 = 4$. If $y = 0$ then $x^2 = 1$ and $z = 1$, so the maximum height is obtained for $(x, y) = (0, \pm1), z = 4$ and the minimum height is $z = 1$ at $(\pm1, 0)$.

Chapter 13 Review Exercises

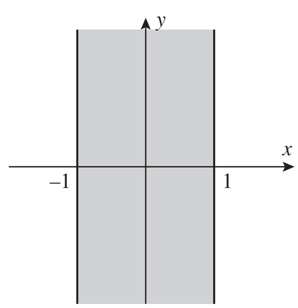
1. **(a)** $f(\ln y, e^x) = e^{\ln y} \ln e^x = xy$.

(b) $f(r + s, rs) = e^{r+s} \ln(rs)$.

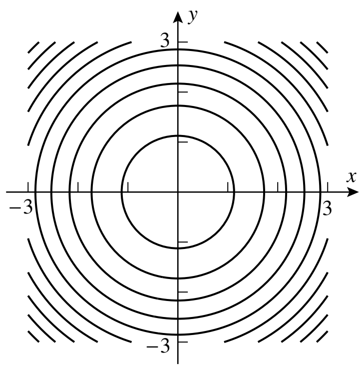
2. **(a)**



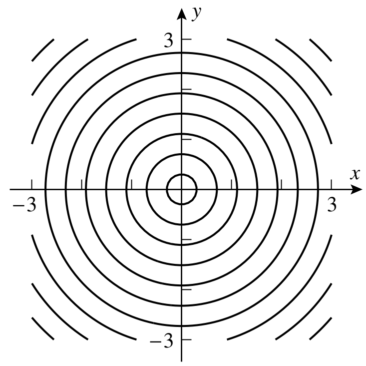
(b)



3. $z = \sqrt{x^2 + y^2} = c$ implies $x^2 + y^2 = c^2$, which is the equation of a circle; $x^2 + y^2 = c$ is also the equation of a circle (for $c > 0$).



$z = x^2 + y^2$



$z = \sqrt{x^2 + y^2}$

4. **(b)** $f(x, y, z) = z - x^2 - y^2$.

5. $x^4 - x + y - x^3y = (x^3 - 1)(x - y)$, limit $= -1$, not defined on the line $y = x$ so not continuous at $(0, 0)$.

6. If $(x, y) \neq (0, 0)$, then $\dfrac{x^4 - y^4}{x^2 + y^2} = x^2 - y^2$, limit: $\displaystyle\lim_{(x,y)\to(0,0)} (x^2 - y^2) = 0$, continuous.

7. **(a)** They approximate the profit per unit of any additional sales of the standard or high-resolution monitors, respectively.

(b) The rates of change with respect to the two directions x and y, and with respect to time.

9. **(a)** $P = \dfrac{10T}{V}$, $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial P}{\partial V}\dfrac{dV}{dt} = \dfrac{10}{V} \cdot 3 - \dfrac{10T}{V^2} \cdot 0 = \dfrac{30}{V} = \dfrac{30}{2.5} = 12\ \mathrm{N/(m^2min)} = 12\ \mathrm{Pa/min}$.

(b) $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial P}{\partial V}\dfrac{dV}{dt} = \dfrac{10}{V} \cdot 0 - \dfrac{10T}{V^2} \cdot (-3) = \dfrac{30T}{V^2} = \dfrac{30 \cdot 50}{(2.5)^2} = 240\ \mathrm{Pa/min}$.

10. **(a)** $z = 1 - y^2$, slope $= \dfrac{\partial z}{\partial y} = -2y = 4$. **(b)** $z = 1 - 4x^2$, $\dfrac{\partial z}{\partial x} = -8x = -8$.

11. $w_x = 2x\sec^2(x^2+y^2)+\sqrt{y}$, $w_{xy} = 8xy\sec^2(x^2+y^2)\tan(x^2+y^2)+\dfrac{1}{2}y^{-1/2}$, $w_y = 2y\sec^2(x^2+y^2)+\dfrac{1}{2}xy^{-1/2}$, $w_{yx} = 8xy\sec^2(x^2 + y^2)\tan(x^2 + y^2) + \dfrac{1}{2}y^{-1/2}$.

12. $\partial w/\partial x = \dfrac{1}{x - y} - \sin(x + y)$, $\partial^2 w/\partial x^2 = -\dfrac{1}{(x - y)^2} - \cos(x + y)$, $\partial w/\partial y = -\dfrac{1}{x - y} - \sin(x + y)$, $\partial^2 w/\partial y^2 = -\dfrac{1}{(x - y)^2} - \cos(x + y) = \partial^2 w/\partial x^2$.

13. $F_x = -6xz, F_{xx} = -6z, F_y = -6yz, F_{yy} = -6z, F_z = 6z^2-3x^2-3y^2$, $F_{zz} = 12z, F_{xx}+F_{yy}+F_{zz} = -6z-6z+12z = 0$.

14. $f_x = yz + 2x, f_{xy} = z, f_{xyz} = 1, f_{xyzx} = 0$; $f_z = xy - (1/z), f_{zx} = y, f_{zxx} = 0, f_{zxxy} = 0$.

16. $\Delta w = (1.1)^2(-0.1) - 2(1.1)(-0.1) + (-0.1)^2(1.1) - 0 = 0.11$, $dw = (2xy - 2y + y^2)dx + (x^2 - 2x + 2yx)dy = -(-0.1) = 0.1$.

17. $dV = \dfrac{2}{3}xhdx + \dfrac{1}{3}x^2dh = \dfrac{2}{3}2(-0.1) + \dfrac{1}{3}(0.2) = -0.06667\ \mathrm{m}^3$; $\Delta V = -0.07267\ \mathrm{m}^3$.

18. $f_x\left(\dfrac{1}{3}, \pi\right) = \pi\cos\dfrac{\pi}{3} = \dfrac{\pi}{2}$, $f_y\left(\dfrac{1}{3}, \pi\right) = \dfrac{1}{3}\cos\dfrac{\pi}{3} = \dfrac{1}{6}$, so $L(x, y) = \dfrac{\sqrt{3}}{2} + \dfrac{\pi}{2}\left(x - \dfrac{1}{3}\right) + \dfrac{1}{6}(y - \pi)$.

19. $\dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$, so when $t=0$, $4\left(-\dfrac{1}{2}\right)+2\dfrac{dy}{dt}=2$. Solve to obtain $\left.\dfrac{dy}{dt}\right|_{t=0}=2$.

20. **(a)** $\dfrac{dy}{dx}=-\dfrac{6x-5y+y\sec^2 xy}{-5x+x\sec^2 xy}$. **(b)** $\dfrac{dy}{dx}=-\dfrac{\ln y+\cos(x-y)}{x/y-\cos(x-y)}$.

21. $\dfrac{dy}{dx}=-\dfrac{f_x}{f_y}$, $\dfrac{d^2y}{dx^2}=-\dfrac{f_y(d/dx)f_x-f_x(d/dx)f_y}{f_y^2}=-\dfrac{f_y(f_{xx}+f_{xy}(dy/dx))-f_x(f_{xy}+f_{yy}(dy/dx))}{f_y^2}=$
$=-\dfrac{f_y(f_{xx}+f_{xy}(-f_x/f_y))-f_x(f_{xy}+f_{yy}(-f_x/f_y))}{f_y^2}=\dfrac{-f_y^2f_{xx}+2f_xf_yf_{xy}-f_x^2f_{yy}}{f_y^3}$.

22. **(a)** $\dfrac{d}{dt}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)\dfrac{dx}{dt}+\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)\dfrac{dy}{dt}=\dfrac{\partial^2 z}{\partial x^2}\dfrac{dx}{dt}+\dfrac{\partial^2 z}{\partial y\partial x}\dfrac{dy}{dt}$ by the Chain Rule, and $\dfrac{d}{dt}\left(\dfrac{\partial z}{\partial y}\right)=$
$\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)\dfrac{dx}{dt}+\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)\dfrac{dy}{dt}=\dfrac{\partial^2 z}{\partial x\partial y}\dfrac{dx}{dt}+\dfrac{\partial^2 z}{\partial y^2}\dfrac{dy}{dt}$.

(b) $\dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$, $\dfrac{d^2z}{dt^2}=\dfrac{dx}{dt}\left(\dfrac{\partial^2 z}{\partial x^2}\dfrac{dx}{dt}+\dfrac{\partial^2 z}{\partial y\partial x}\dfrac{dy}{dt}\right)+\dfrac{\partial z}{\partial x}\dfrac{d^2x}{dt^2}+\dfrac{dy}{dt}\left(\dfrac{\partial^2 z}{\partial x\partial y}\dfrac{dx}{dt}+\dfrac{\partial^2 z}{\partial y^2}\dfrac{dy}{dt}\right)+\dfrac{\partial z}{\partial y}\dfrac{d^2y}{dt^2}$.

25. $\nabla f=\dfrac{y}{x+y}\mathbf{i}+\left(\ln(x+y)+\dfrac{y}{x+y}\right)\mathbf{j}$, so when $(x,y)=(-3,5)$, $\dfrac{\partial f}{\partial u}=\nabla f\cdot\mathbf{u}=\left[\dfrac{5}{2}\mathbf{i}+\left(\ln 2+\dfrac{5}{2}\right)\mathbf{j}\right]\cdot\left[\dfrac{3}{5}\mathbf{i}+\dfrac{4}{5}\mathbf{j}\right]=$
$\dfrac{3}{2}+2+\dfrac{4}{5}\ln 2=\dfrac{7}{2}+\dfrac{4}{5}\ln 2$.

26. **(a)** $\mathbf{u}$ is a unit vector parallel to the gradient, so $\mathbf{u}=\dfrac{2}{5}\left(2\mathbf{i}+\dfrac{3}{2}\mathbf{j}\right)=\dfrac{4}{5}\mathbf{i}+\dfrac{3}{5}\mathbf{j}$. The maximum value is $\nabla f(0,0)\cdot\mathbf{u}=$
$\dfrac{8}{5}+\dfrac{9}{10}=\dfrac{5}{2}$.

(b) The unit vector to give the minimum has the opposite sense of the vector in Part(a), so $\mathbf{u}=-\dfrac{4}{5}\mathbf{i}-\dfrac{3}{5}\mathbf{j}$ and $\nabla f(0,0)\cdot\mathbf{u}=-\dfrac{5}{2}$.

27. Use the unit vectors $\mathbf{u}=\langle\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\rangle,\mathbf{v}=\langle 0,-1\rangle,\mathbf{w}=\langle-\dfrac{1}{\sqrt{5}},-\dfrac{2}{\sqrt{5}}\rangle=-\dfrac{\sqrt{2}}{\sqrt{5}}\mathbf{u}+\dfrac{1}{\sqrt{5}}\mathbf{v}$, so that $D_{\mathbf{w}}f=-\dfrac{\sqrt{2}}{\sqrt{5}}D\mathbf{u}f+$
$\dfrac{1}{\sqrt{5}}D\mathbf{v}f=-\dfrac{\sqrt{2}}{\sqrt{5}}2\sqrt{2}+\dfrac{1}{\sqrt{5}}(-3)=-\dfrac{7}{\sqrt{5}}$.

28. **(a)** $\mathbf{n}=z_x\mathbf{i}+z_y\mathbf{j}-\mathbf{k}=8\mathbf{i}+8\mathbf{j}-\mathbf{k}$, tangent plane $8x+8y-z=4+8\ln 2$, normal line $x(t)=1+8t,y(t)=\ln 2+8t,z(t)=4-t$.

(b) $\mathbf{n}=3\mathbf{i}+10\mathbf{j}-14\mathbf{k}$, tangent plane $3x+10y-14z=30$, normal line $x(t)=2+3t,y(t)=1+10t,z(t)=-1-14t$.

29. The origin is not such a point, so assume that the normal line at $(x_0,y_0,z_0)\neq(0,0,0)$ passes through the origin, then $\mathbf{n}=z_x\mathbf{i}+z_y\mathbf{j}-\mathbf{k}=-y_0\mathbf{i}-x_0\mathbf{j}-\mathbf{k}$; the line passes through the origin and is normal to the surface if it has the form $\mathbf{r}(t)=-y_0t\mathbf{i}-x_0t\mathbf{j}-t\mathbf{k}$ and $(x_0,y_0,z_0)=(x_0,y_0,2-x_0y_0)$ lies on the line if $-y_0t=x_0,-x_0t=y_0,-t=2-x_0y_0$, with solutions $x_0=y_0=-1$, $x_0=y_0=1,x_0=y_0=0$; thus the points are $(0,0,2),(1,1,1),(-1,-1,1)$.

30. $\mathbf{n}=\dfrac{2}{3}x_0^{-1/3}\mathbf{i}+\dfrac{2}{3}y_0^{-1/3}\mathbf{j}+\dfrac{2}{3}z_0^{-1/3}\mathbf{k}$, tangent plane $x_0^{-1/3}x+y_0^{-1/3}y+z_0^{-1/3}z=x_0^{2/3}+y_0^{2/3}+z_0^{2/3}=1$; intercepts are $x=x_0^{1/3},y=y_0^{1/3},z=z_0^{1/3}$, sum of squares of intercepts is $x_0^{2/3}+y_0^{2/3}+z_0^{2/3}=1$.

31. The line is tangent to $6\mathbf{i}+4\mathbf{j}+\mathbf{k}$, a normal to the surface is $\mathbf{n}=18x\mathbf{i}+8y\mathbf{j}-\mathbf{k}$, so solve $18x=6k,8y=4k,-1=k$; $k=-1,x=-1/3,y=-1/2,z=2$.

32. Solve $(t-1)^2/4 + 16e^{-2t} + (2-\sqrt{t})^2 = 1$ for t to get $t = 1.833223, 2.839844$; the particle strikes the surface at the points $P_1(0.83322, 0.639589, 0.646034)$, $P_2(1.83984, 0.233739, 0.314816)$. The velocity vectors are given by $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \mathbf{i} - 4e^{-t}\mathbf{j} - 1/(2\sqrt{t})\mathbf{k}$, and a normal to the surface is $\mathbf{n} = \nabla(x^2/4 + y^2 + z^2) = x/2\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. At the points P_i these are $\mathbf{v}_1 = \mathbf{i} - 0.639589\mathbf{j} - 0.369286\mathbf{k}$, $\mathbf{v}_2 = \mathbf{i} - 0.233739\mathbf{j} - 0.296704\mathbf{k}$; $\mathbf{n}_1 = 0.41661\mathbf{i} + 1.27918\mathbf{j} + 1.29207\mathbf{k}$ and $\mathbf{n}_2 = 0.91992\mathbf{i} + 0.46748\mathbf{j} + 0.62963\mathbf{k}$ so $\cos^{-1}[(\mathbf{v}_i \cdot \mathbf{n}_i)/(\|\mathbf{v}_i\|\,\|\mathbf{n}_i\|)] = 112.3°, 61.1°$; the acute angles are $67.7°, 61.1°$.

33. $\nabla f = (2x + 3y - 6)\mathbf{i} + (3x + 6y + 3)\mathbf{j} = \mathbf{0}$ if $2x + 3y = 6, x + 2y = -1, x = 15, y = -8, D = 3 > 0, f_{xx} = 2 > 0$, so f has a relative minimum at $(15, -8)$.

34. $\nabla f = (2xy - 6x)\mathbf{i} + (x^2 - 12y)\mathbf{j} = \mathbf{0}$ if $2xy - 6x = 0, x^2 - 12y = 0$; if $x = 0$ then $y = 0$, and if $x \neq 0$ then $y = 3, x = \pm 6$, thus the gradient vanishes at $(0,0), (-6,3), (6,3)$; $f_{xx} = 2y - 6$, $f_{yy} = -12$, $f_{xy} = 2x$, so $D = -24y + 72 - 4x^2$, so $(\pm 6, 3)$ are saddle points, and $(0,0)$ is a relative maximum.

35. $\nabla f = (3x^2 - 3y)\mathbf{i} - (3x - y)\mathbf{j} = \mathbf{0}$ if $y = x^2, 3x = y$, so $x = y = 0$ or $x = 3, y = 9$; at $x = y = 0, D = -9$, saddle point; at $x = 3, y = 9, D = 9, f_{xx} = 18 > 0$, relative minimum.

36. $\nabla f = (8x - 12y)\mathbf{i} + (-12x + 18y)\mathbf{j} = \mathbf{0}$ if $y = \frac{2}{3}x$; $f_{xx} = 8, f_{xy} = -12, f_{yy} = 18, D = 0$, from which we can draw no conclusion. Upon inspection, however, $f(x,y) = (2x - 3y)^2$, so f has a relative (and an absolute) minimum of 0 at every point on the line $y = \frac{2}{3}x$, no relative maximum.

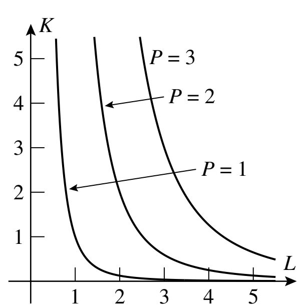
37. (a) $y^2 = 8 - 4x^2$, find extrema of $f(x) = x^2(8 - 4x^2) = -4x^4 + 8x^2$ defined for $-\sqrt{2} \le x \le \sqrt{2}$. Then $f'(x) = -16x^3 + 16x = 0$ when $x = 0, \pm 1, f''(x) = -48x^2 + 16$, so f has a relative maximum at $x = \pm 1, y = \pm 2$ and a relative minimum at $x = 0, y = \pm 2\sqrt{2}$. At the endpoints $x = \pm\sqrt{2}, y = 0$ we obtain the minimum $f = 0$ again.

(b) $f(x,y) = x^2y^2, g(x,y) = 4x^2 + y^2 - 8 = 0, \nabla f = 2xy^2\mathbf{i} + 2x^2y\mathbf{j} = \lambda\nabla g = 8\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, so solve $2xy^2 = \lambda 8x, 2x^2y = \lambda 2y$. If $x = 0$ then $y = \pm 2\sqrt{2}$, and if $y = 0$ then $x = \pm\sqrt{2}$. In either case f has a relative and absolute minimum. Assume $x, y \neq 0$, then $y^2 = 4\lambda, x^2 = \lambda$, use $g = 0$ to obtain $x^2 = 1, x = \pm 1, y = \pm 2$, and $f = 4$ is a relative and absolute maximum at $(\pm 1, \pm 2)$.

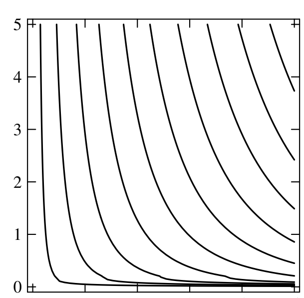
38. Let the first octant corner of the box be (x,y,z), so that $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. Maximize $V = 8xyz$ subject to $g(x,y,z) = (x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, solve $\nabla V = \lambda\nabla g$, or $8(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = (2\lambda x/a^2)\mathbf{i} + (2\lambda y/b^2)\mathbf{j} + (2\lambda z/c^2)\mathbf{k}, 8a^2yz = 2\lambda x, 8b^2xz = 2\lambda y, 8c^2xy = 2\lambda z$. For the maximum volume, $x, y, z \neq 0$; divide the first equation by the second to obtain $a^2y^2 = b^2x^2$; the first by the third to obtain $a^2z^2 = c^2x^2$, and finally $b^2z^2 = c^2y^2$. From $g = 1$ get $3(x/a)^2 = 1, x = \pm a/\sqrt{3}$, and then $y = \pm b/\sqrt{3}, z = \pm c/\sqrt{3}$. The dimensions of the box are $\frac{2a}{\sqrt{3}} \times \frac{2b}{\sqrt{3}} \times \frac{2c}{\sqrt{3}}$, and the maximum volume is $8abc/(3\sqrt{3})$.

39. Denote the currents I_1, I_2, I_3 by x, y, z respectively. Then minimize $F(x,y,z) = x^2R_1 + y^2R_2 + z^2R_3$ subject to $g(x,y,z) = x + y + z - I = 0$, so solve $\nabla F = \lambda\nabla g, 2xR_1\mathbf{i} + 2yR_2\mathbf{j} + 2zR_3\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\lambda = 2xR_1 = 2yR_2 = 2zR_3$, so the minimum value of F occurs when $I_1 : I_2 : I_3 = \frac{1}{R_1} : \frac{1}{R_2} : \frac{1}{R_3}$.

40. (a)



(b)



41. **(a)** $\partial P/\partial L = c\alpha L^{\alpha-1}K^{\beta}, \partial P/\partial K = c\beta L^{\alpha}K^{\beta-1}$.

(b) The rates of change of output with respect to labor and capital equipment, respectively.

(c) $K(\partial P/\partial K) + L(\partial P/\partial L) = c\beta L^{\alpha}K^{\beta} + c\alpha L^{\alpha}K^{\beta} = (\alpha+\beta)P = P$.

42. **(a)** Maximize $P = 1000L^{0.6}K^{0.4}$ subject to $50L + 100K = 200{,}000$ or $L + 2K = 4000$. $600\left(\dfrac{K}{L}\right)^{0.4} = \lambda$, $400\left(\dfrac{L}{K}\right)^{0.6} = 2\lambda$, $L + 2K = 4000$; so $\dfrac{2}{3}\left(\dfrac{L}{K}\right) = 2$, thus $L = 3K$, $L = 2400, K = 800, P(2400, 800) = 1000 \cdot 2400^{0.6} \cdot 800^{0.4} = 1000 \cdot 3^{0.6} \cdot 800 = 800{,}000 \cdot 3^{0.6} \approx \$1{,}546{,}545.64$.

(b) The value of labor is $50L = 120{,}000$ and the value of capital is $100K = 80{,}000$.

Chapter 13 Making Connections

1. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = \cos\theta\dfrac{\partial z}{\partial x} + \sin\theta\dfrac{\partial z}{\partial y}$, multiply by r to get the first equation. $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta} = -r\sin\theta\dfrac{\partial z}{\partial x} + r\cos\theta\dfrac{\partial z}{\partial y} = -y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y}$.

2. **(a)** $f(tx, ty) = 3t^2x^2 + t^2y^2 = t^2 f(x, y)$; $n = 2$.

(b) $f(tx, ty) = \sqrt{t^2x^2 + t^2y^2} = tf(x, y)$; $n = 1$.

(c) $f(tx, ty) = t^3x^2y - 2t^3y^3 = t^3 f(x, y)$; $n = 3$.

(d) $f(tx, ty) = 5/\left(t^2x^2 + 2t^2y^2\right)^2 = t^{-4}f(x, y)$; $n = -4$.

3. Suppose $g(\theta)$ exists such that $f(x, y) = r^n g(\theta)$ is homogeneous of degree n. Then $f(tx, ty) = (tr)^n g(\theta) = t^n[r^n g(\theta)] = t^n f(x, y)$. Conversely if $f(x, y)$ is homogeneous of degree n then let $g(\theta) = f(\cos\theta, \sin\theta)$. Then $f(x, y) = f(r\cos\theta, r\sin\theta) = r^n f(\cos\theta, \sin\theta) = r^n g(\theta)$; moreover, $g(\theta)$ has period 2π.

4. **(a)** If $f(u, v) = t^n f(x, y)$, then $\dfrac{\partial f}{\partial u}\dfrac{du}{dt} + \dfrac{\partial f}{\partial v}\dfrac{dv}{dt} = nt^{n-1}f(x, y)$, $x\dfrac{\partial f}{\partial u} + y\dfrac{\partial f}{\partial v} = nt^{n-1}f(x, y)$; let $t = 1$ to get $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$.

(b) Let $f(x, y)$ be homogeneous of degree n, so that $f(x, y) = r^n g(\theta)$, where g has period 2π. Then $f_x = nr^{n-1}g(\theta)\dfrac{x}{r} - r^n g'\theta\dfrac{y}{r^2}$, $f_y = nr^{n-1}g(\theta)\dfrac{y}{r} + r^n g'(\theta)\dfrac{x}{r^2}$, so
$xf_x + yf_y = r^n\left(ng(\theta)\cos^2\theta + ng(\theta)\sin^2\theta - g'(\theta)[\cos\theta\sin\theta - \sin\theta\cos\theta]\right) = nr^n g(\theta) = nf(x, y)$.

(c) If $f(x, y) = 3x^2 + y^2$ then $xf_x + yf_y = 6x^2 + 2y^2 = 2f(x, y)$; if $f(x, y) = \sqrt{x^2 + y^2}$ then $xf_x + yf_y = x^2/\sqrt{x^2+y^2} + y^2/\sqrt{x^2+y^2} = \sqrt{x^2+y^2} = f(x, y)$; if $f(x, y) = x^2y - 2y^3$ then $xf_x + yf_y = 3x^2y - 6y^3 = 3f(x, y)$; if $f(x, y) = \dfrac{5}{(x^2+2y^2)^2}$ then $xf_x + yf_y = x\dfrac{5(-2)2x}{(x^2+2y^2)^3} + y\dfrac{5(-2)4y}{(x^2+2y^2)^3} = -4f(x, y)$.

5. Write $f(x, y) = z(r, \theta)$ in polar form. From the hypotheses and Exercise 1 of this section we see that $r\dfrac{\partial z}{\partial r} - nz = 0$. Divide by r^{n+1} to obtain $r^{-n}\dfrac{\partial z}{\partial r} - nr^{-n-1}z = 0$, $\dfrac{\partial}{\partial r}(r^{-n}z) = 0$. Thus $r^{-n}z$ is independent of r, say $r^{-n}z = g(\theta), z = r^n g(\theta)$. From Exercise 3 it follows that f is homogeneous of degree n provided that g is 2π periodic; but this follows from the fact that z is defined in terms of sines and cosines.