Ex # 13.9:

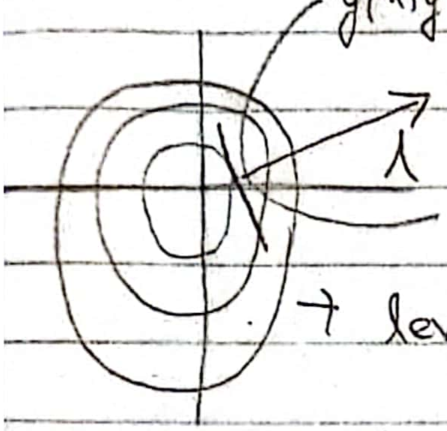
Langrange Multipliers:

Idea: $f(x,y) = z \rightarrow$ A Surface

Constraint $\rightarrow g(x,y) = c$ Another surface

A level Curve to a surface $g(x,y) = z$

$g(x,y) = c$ ☆ The Intersection gives constraint maxima and minima



$\rightarrow$ level Curves of $f(x,y) = c_1, c_2, c_3 \ldots$

Normals of level Curves are scalar multiplier.

normal to a level curve $\leftrightarrow$ gives gradient

$f(x,y) = k \cdot g(x,y)$

$\nabla f(x,y) = k \cdot \nabla g(x,y)$

$$\boxed{\nabla f(x,y) = \lambda \cdot \nabla g(x,y)}$$

$\therefore \lambda =$ langrange multiplier

Ex # $f(x,y) = xy$ with a constraint of

$2x + 3y = 6$

$\rightarrow$ ignore constant

Sol: $f(x,y) = xy$, $g(x,y) = 2x + 3y$

$\Rightarrow \nabla f(x,y) = \lambda . \nabla g(x,y)$ — (A)

$\nabla f(x,y) = y\hat{i} + x\hat{j}$, $\nabla g(x,y) = 2\hat{i} + 3\hat{j}$

$y\hat{i} + x\hat{j} = \lambda . (2\hat{i} + 3\hat{j})$

$y\hat{i} + x\hat{j} = 2\lambda\hat{i} + 3\lambda\hat{j}$

Compare

$y = 2\lambda$, $x = 3\lambda$

Put x & y in Constraint:

$2(3\lambda) + 3(2\lambda) = 6$

$6\lambda + 6\lambda = 6 \Rightarrow \lambda = 1/2$

$y = 2\left(\frac{1}{2}\right) = 1$, $x = 3/2$

Critical Point $(3/2, 1)$

Ex: $f(x,y,z) = x+2y-2z \quad , \quad x^2+2y^2+4z^2=1$

$g(x,y,z) = x^2+2y^2+4z^2$

$\nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z)$

$\nabla f(x,y,z) = 1\hat{i}+2\hat{j}-2\hat{k} \quad , \quad \nabla g(x,y,z) = 2x\hat{i}+4y\hat{j}+8z\hat{k}$

$\hat{i}+2\hat{j}-2\hat{k} = \lambda \cdot (2x\hat{i}+4y\hat{j}+8z\hat{k})$

$1 = 2x\lambda \quad , \quad 2 = 4y\lambda \quad , \quad -2 = 8z\lambda$

$x = \frac{1}{2\lambda} \quad , \quad y = \frac{1}{2\lambda} \quad , \quad z = \frac{-1}{4\lambda}$

Put x, y, z in Constraint

$\left(\frac{1}{2\lambda}\right)^2 + 2\left(\frac{1}{2\lambda}\right)^2 + 4\left(\frac{-1}{4\lambda}\right)^2 = 1$

$\frac{1}{4\lambda^2} + \frac{2}{4\lambda^2} + \frac{4}{16\lambda^2} = 1$

$\frac{1}{4\lambda^2} + \frac{1}{2\lambda^2} + \frac{1}{4\lambda^2} = 1$

$\frac{1+2+1}{4\lambda^2} = 1 \quad , \quad \frac{4}{4\lambda^2} = 1$

$\lambda^2 = 1 \quad , \quad \lambda = \pm 1$

When $\lambda = 1$: $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = -1/4$

When $\lambda = -1$: $x = -\frac{1}{2}$, $y = -\frac{1}{2}$, $z = \frac{1}{4}$

Now Put in $f(x,y,z) = x+2y-2z$ to find max and min.

$f(1/2, 1/2, -1/4) = 2 \rightarrow \boxed{MAX}$

$f(-1/2, -1/2, 1/4) = -2 \rightarrow \boxed{MIN}$