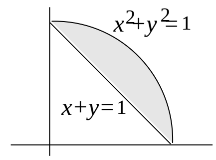
Limits in Iterated Integrals

For most students, the trickiest part of evaluating multiple integrals by iteration is to put in the limits of integration. Fortunately, a fairly uniform procedure is available which works in any coordinate system. *You must always begin by sketching the region; in what follows we'll assume you've done this.*

1. Double integrals in rectangular coordinates.



Let's illustrate this procedure on the first case that's usually taken up: double integrals in rectangular coordinates. Suppose we want to evaluate over the region R pictured the integral

$$\iint_R f(x,y)\,dy\,dx\ , \qquad\qquad R = \ \text{ region between } x^2+y^2=1 \quad \text{and} \quad x+y=1\ ;$$

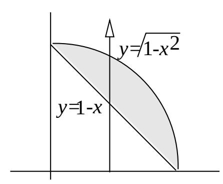
we are integrating first with respect to y. Then to put in the limits,

1. Hold x fixed, and let y increase (since we are integrating with respect to y). As the point (x,y) moves, it traces out a vertical line.

2. Integrate from the y-value where this vertical line enters the region R, to the y-value where it leaves R.

3. Then let x increase, integrating from the lowest x-value for which the vertical line intersects R, to the highest such x-value.

Carrying out this program for the region R pictured, the vertical line enters R where $y = 1-x$, and leaves where $y = \sqrt{1-x^2}$.



The vertical lines which intersect R are those between $x=0$ and $x=1$. Thus we get for the limits:

$$\iint_R f(x,y)\,dy\,dx \;=\; \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

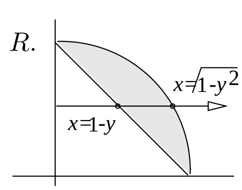
To calculate the double integral, integrating in the reverse order $\iint_R f(x,y)\,dx\,dy$,

1. Hold y fixed, let x increase (since we are integrating first with respect to x). This traces out a horizontal line.

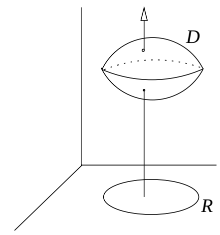
2. Integrate from the x-value where the horizontal line enters R to the x-value where it leaves.

3. Choose the y-limits to include all of the horizontal lines which intersect R.

Following this prescription with our integral we get:



$$\iint_R f(x,y)\,dx\,dy \;=\; \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$



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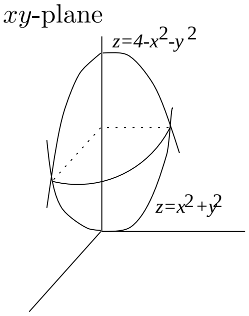
3. Triple integrals in rectangular and cylindrical coordinates.

You do these the same way, basically. To supply limits for $\iiint_D dz\,dy\,dx$ over the region D, we integrate first with respect to z. Therefore we

1. Hold x and y fixed, and let z increase. This gives us a vertical line.

2. Integrate from the z-value where the vertical line enters the region D to the z-value where it leaves D.

3. Supply the remaining limits (in either xy-coordinates or polar coordinates) so that you include all vertical lines which intersect D. This means that you will be integrating the remaining double integral over the region R in the xy-plane which D projects onto.



For example, if D is the region lying between the two paraboloids

$$z = x^2 + y^2 \qquad\qquad z = 4 - x^2 - y^2,$$

we get by following steps 1 and 2,

$$\iiint_D dz\,dy\,dx \;=\; \iint_R \int_{x^2+y^2}^{4-x^2-y^2} dz\,dA$$

where R is the projection of D onto the xy-plane. To finish the job, we have to determine what this projection is. From the picture, what we should determine is the xy-curve over which the two surfaces intersect. We find this curve by eliminating z from the two equations, getting

$$\begin{aligned} x^2 + y^2 &= 4 - x^2 - y^2, \qquad \text{which implies} \\ x^2 + y^2 &= 2. \end{aligned}$$

Thus the xy-curve bounding R is the circle in the xy-plane with center at the origin and radius $\sqrt{2}$.

This makes it natural to finish the integral in polar coordinates. We get

$$\iiint_D dz\,dy\,dx \;=\; \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,r\,dr\,d\theta \;;$$

the limits on z will be replaced by r^2 and $4 - r^2$ when the integration is carried out.