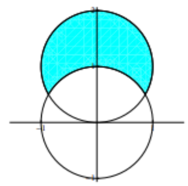
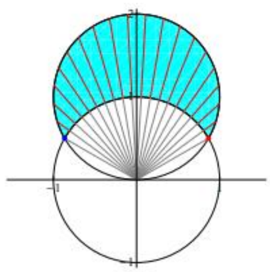
9. *Find the area of the region which lies inside the circle $x^2+(y-1)^2 = 1$ but outside the circle $x^2+y^2 = 1$.*



First, let's write the equations of the two circles in polar coordinates. The circle $x^2 + y^2 = 1$ is just $r = 1$. The circle $x^2 + (y-1)^2 = 1$ is more complicated:

$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ (r\cos\theta)^2 + (r\sin\theta - 1)^2 &= 1 \\ r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + \cancel{1} &= \cancel{1} \\ r^2(\cos^2\theta + \sin^2\theta) - 2r\sin\theta &= 0 \\ r^2 &= 2r\sin\theta \\ r &= 2\sin\theta \end{aligned}$$



We are slicing from the θ of the red point to the θ of the blue point. Let's find these values. The red point and blue point are points where the curves $r = 1$ and $r = 2\sin\theta$ intersect, so let's solve $1 = 2\sin\theta$. This happens when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. So, the red point has $\theta = \frac{\pi}{6}$, the blue point has $\theta = \frac{5\pi}{6}$, and our outer integral will have θ going from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$.

Along each slice, r goes from the lower circle ($r = 1$) to the upper circle ($r = 2\sin\theta$), so the inner integral will have r going from 1 to $2\sin\theta$. So, we can rewrite our double integral as

$$\begin{aligned} \boxed{\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} 1\cdot r\; dr\; d\theta} &= \int_{\pi/6}^{5\pi/6}\left(\frac{1}{2}r^2\Big|_{r=1}^{r=2\sin\theta}\right)\,d\theta \\ &= \int_{\pi/6}^{5\pi/6}\left(2\sin^2\theta - \frac{1}{2}\right)\,d\theta \\ &= \int_{\pi/6}^{5\pi/6}\left(\frac{1}{2} - \cos 2\theta\right)\,d\theta \\ &\quad \text{by the identity } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\ &= \frac{\theta}{2} - \frac{1}{2}\sin 2\theta\Big|_{\theta=\pi/6}^{\theta=5\pi/6} \\ &= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}} \end{aligned}$$

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$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{4\sin\theta} r\sin 2\theta \, dr\, d\theta$$

$r = 0$ &

$r = 4\sin\theta$

$r^2 = 4r\sin\theta$

$x^2+y^2 = 4x$

$x^2 - 4x + y^2 = 0$

$(x)^2 - 2(x)(2) + (2)^2 + y^2 =$

$(x-2)^2 + y^2 = 2^2$

$\therefore y = \pm\sqrt{4-(x-2)^2}$

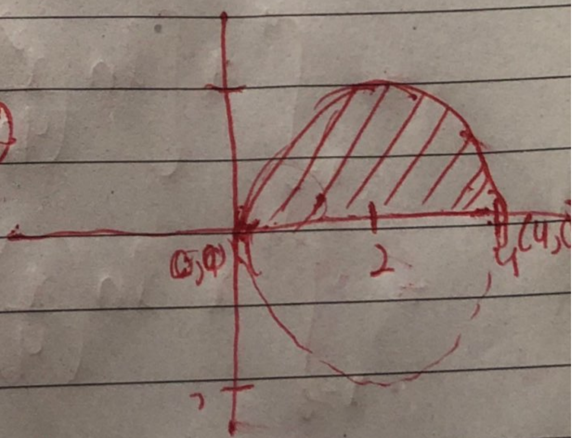
$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$\therefore \sin\theta = \frac{x}{r}$ & $\cos\theta = \frac{y}{r}$

$\sin 2\theta = \frac{2xy}{r^2}$

$\sin 2\theta = \frac{2xy}{x^2+y^2}$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{4\sin\theta} \sin 2\theta \, r\, dr\, d\theta$$



$$\int_{x=0}^{4}\int_{y=0}^{y=\sqrt{4-(x-2)^2}} \frac{2xy}{x^2+y^2}\, dy\, dx$$

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$$= \int_{x=0}^{4} \left| x \ln(x^2+y^2) \right|_{y=0}^{y=\sqrt{4-(x-2)^2}} dx$$

$$= \int_{x=0}^{4} x \left[\ln(x^2) - \ln(x^2+4-(x-2)^2) \right] dx$$

$$= \int_{x=0}^{4} x \left[\ln(x^2) - \ln(\cancel{x^2}+\cancel{4}-\cancel{x^2}+4x-\cancel{4}) \right] dx$$

$\cancel{x \ln(x^2) - x \ln}$

$$= \int_{x=0}^{4} x \left[\ln(x^2) - \ln(4x) \right] dx$$

$$= \int_{x=0}^{4} x \left[\ln\left(\frac{x^2}{4x}\right) \right] dx$$

$$= \int_{x=0}^{4} x \ln\left(\frac{x}{4}\right) dx$$

$$= \left| \frac{x^2}{2} \ln\left(\frac{x}{4}\right) - \frac{x^2}{4} \right|_{x=0}^{4}$$

$$\boxed{= -4}$$ Ans