Ex. 18.4 (3 - 10)

* Green Theorm:

$$\int f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Q3) $\oint_c \underbrace{3xy}_{f}\,dx + \underbrace{2xy}_{g}\,dy$, rectangle $[x=-2,\ x=4,\ y=2,\ y=1]$

↳ Fubini's Theorm (order don't matte

- $\frac{\partial f}{\partial y} = 3x(1)$
- $\frac{\partial g}{\partial x} = 2y(1)$

$$\int_1^2 \int_{-2}^4 2y - 3x \cdot dx\,dy$$

- $\int_1^2 \left(2yx - \frac{3x^2}{2}\right)\Big|_{-2}^{4} dy \Rightarrow 2y(4-(-2)) - \frac{3}{2}(4^2-(-2)^2)$

$= 12y - 2\ \frac{3}{2}(12) \Rightarrow 12y - 18$

- $12\int_1^2 y \to \frac{12y^2}{2} \to 6y^2\Big|_1^2 \to 6(2^2-1^2) \Rightarrow 6(3) = 18$

-18, $-18y$, $-18y$

$18(2-1) \Rightarrow 18 - 18$

$= 0$

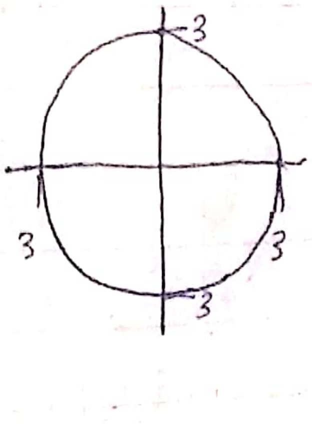
Q4) $\oint_c \underbrace{(x^2-y^2)}_{f}\,dx + \underbrace{x}_{g}\,dy$, Circle with $x^2+y^2=9$

↳ Most easist with polar — Circle with $r=3$

- $\frac{\partial f}{\partial y} = 0 - 2y$
- $\frac{\partial g}{\partial x} = 1$

$$= \int_0^{2\pi}\int_0^3 1 + 2y \cdot r\,dr\,d\theta$$

↳ $r\sin\theta$



- $\int_0^{2\pi}\int_0^3 r + 2r^2\sin\theta\,dr\,d\theta$

- $\int_0^3 r \cdot dr + \cancel{\int_0^{3}} 2\sin\theta \int r^2 \cdot dr \Rightarrow \left(\frac{r^2}{2} + \frac{2\sin\theta\, r^3}{3}\right)\Big|_0^3$

$$\left(\frac{r^2 8}{2} + \frac{2}{3}\sin\theta r^3\right)\Bigg|_0^3 \Rightarrow \frac{9\cdot 8}{2} + \frac{2}{3}\sin\theta(3)^3 - 0$$

$$= \frac{9\cdot 8}{2} + 2\sin(9) \Rightarrow \frac{98}{2} + 18\sin\theta$$

$$\cdot\ \frac{98}{2}\int_0^{2\pi} d\theta + 18\int \sin\theta \cdot d\theta$$

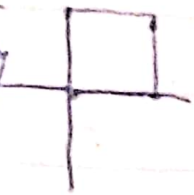
$$= \frac{98\theta}{2} + 18(-\cos\theta)\Big|_0^{2\pi} \Rightarrow \cancel{\frac{9}{2}}(2\pi - 0) + -18\cos(2\pi)$$

$$\frac{98\pi}{\cancel{8}} - \cancel{18(1)} + 18\cos(0)$$

$$= 9\pi$$

Q5) $\int \underset{f}{\underbrace{x\cos y}}\, dx - \underset{g}{\underbrace{y\sin(x)}}\, dy$, square $[(0,0), (\frac{\pi}{2}, 0), (\frac{\pi}{2}, \frac{\pi}{2}), (0, \frac{\pi}{2})]$

$$\left.\begin{aligned} &\cdot\ \frac{\partial f}{\partial y} = -x\sin y \\ &\cdot\ \frac{\partial g}{\partial x} = -y\cos x \end{aligned}\right\} \int^{\pi/2}\int_0^{\pi/2} +x\sin y - y\cos x\, \cancel{dxdy}\ dydx$$



$$\cdot\ x\int_0^{\pi/2} \sin y \cdot dy - \cancel{g}\cos x \int y \cdot dy$$

$$\left[-x\cdot\cos y - \cos(x)\frac{y^2}{2}\right]\Bigg|_0^{\pi/2} \Rightarrow -\cancel{x\cos\left(\frac{\pi}{2}\right)} - x\cos(x)\left(\frac{\pi^2/4}{2}\right)$$

$$+ x\cos(0) + \cancel{\cos(x)^2(0)}$$

$$= +\frac{\pi^2}{8}\cos(x) - x$$

$$\cdot\ +\frac{\pi^2}{8}\int_0^{\pi/2}\cos(x)\cdot dx - \int x\cdot dx \rightarrow +\frac{\pi^2}{8}\sin x - \frac{x^2}{2}\Bigg|_0^{\pi/2}$$

$$= +\frac{\pi^2}{8}\left[\sin\left(\frac{\pi}{2}\right) - \cancel{\sin(0)}\right] - \left[\frac{\pi^2/4}{2} - 0\right]$$

$$\Rightarrow +\frac{\pi^2}{8} - \frac{\pi^2}{8} = 0$$

Q6) $\oint_c \underbrace{y\tan^2 x}_{f}\, dx + \underbrace{\tan x}_{g}\, dy$, $\underbrace{C}_{\text{Polar}}$ is circle $\underbrace{x^2+(y+1)^2=1}_{\text{translate with } r=1}$

$$\left.\begin{aligned} &\cdot\ \frac{\partial f}{\partial y} = \tan^2(x)(1) \\ &\cdot\ \frac{\partial g}{\partial x} = \sec^2(x) \end{aligned}\right\} \int_0^{2\pi}\int_0^{1} \cancelto{1}{(\sec^2 x - \tan^2 x)}\, r\,dr\,d\theta$$

- $\int_0^{2\pi}\int_0^{1} r\,dr\,d\theta \to \frac{r^2}{2}\Big|_0^1 \Rightarrow 1$
- $\frac{1}{2}\int_0^{2\pi} d\theta \to \frac{1}{2}\theta\Big|_0^{2\pi} \to \frac{1}{2}(2\pi) - 0 \to \pi$

Q7) $\oint_c \underbrace{(x^2-y)}_{f}\, dx + \underbrace{x}_{g}\, dy$, C is circle $x^2+y^2=4$

$$\left.\begin{aligned} &\cdot\ \frac{\partial f}{\partial y} = 0-1 \\ &\cdot\ \frac{\partial g}{\partial x} = 1 \end{aligned}\right\} \int_0^{2\pi}\int_0^{2} 1+1\ r\,dr\,d\theta \Rightarrow 2\int_0^{2\pi}\int_0^{2} 1\ r\,dr\,d\theta$$

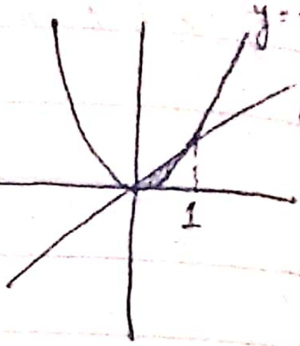
- $2r\Big|_0^2 \Rightarrow 2(2-0) \Rightarrow 4$
- $4\int d\theta \to 4\theta\Big|_0^{2\pi} \Rightarrow 4(2\pi - 0) = 8\pi$

Q8) $\oint_c \underbrace{(e^x + y^2)}_{f}\, dx + \underbrace{(e^y + x^2)}_{g}\, dy$, $y = x^2$ & $y = x$

$\frac{\partial f}{\partial y} = 0 + 2y$

$\frac{\partial g}{\partial x} = 0 + 2x$

$$\int_0^1\int_{x^2}^{x}(2x-2y)\,dy\,dx$$



$x^2 - x = 0$
$x(x-1) = 0$
$x = 0, x = 1$

$$2\int_{x^2}^{x}(x-y)\,dy \to 2xy - \frac{y^2}{2}\Big|_{x^2}^{x} \Rightarrow 2x(x-x^2) - \frac{1}{2}(x^2-x^4)$$

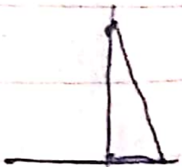
$$= 2x^2 - 2x^3 - \frac{x^2}{2} + \frac{x^4}{2} \Rightarrow \frac{3}{2}x^2 - 2x^3 + \frac{x^4}{2}$$

$$\frac{3}{2}\int_0^1 x^2\,dx - 2\int_0^1 x^3\,dx + \frac{1}{2}\int_0^1 x^4\,dx$$

$$\frac{3}{2}\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{1}{2}\frac{x^5}{5}\Big|_0^1 \Rightarrow \frac{3(1)}{2(3)} - \frac{2}{4}(1) + \frac{1}{2(5)}(1)$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{10} = \frac{1}{30}$$

Q9) $\oint_C \underbrace{\ln(1+y)}_{f}\,dx - \underbrace{\frac{xy}{1+y}}_{g}\,dy$, vertices $(0,0)$, $(2,0)$, $(0,4)$



$\frac{\partial f}{\partial y} = \frac{1(1)}{1+y}$

$\frac{\partial g}{\partial x} = \frac{-y}{1+y}(1)$

$$\int_0^2\int_0^{-2x+4} \frac{-y}{1+y} - \frac{1}{1+y}\,dy\,dx$$

$m = \frac{4-0}{0-2} = -2$

$y - 0 = -2(x-2)$

$y = -2x+4$

$$-\int_0^2\int_0^{-2x+4} \cancel{\frac{y+1}{y+1}}\,dy\,dx$$

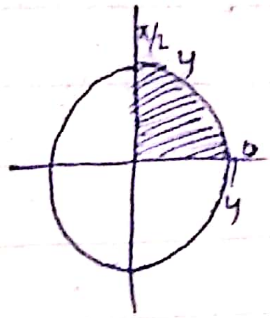
$$-y\Big|_0^{-2x+4} \to +2x - 4 - 0$$

$$2\int x\,dx - 4\int dx \to 2\frac{x^2}{2} - 4x\Big|_0^2$$

$$= \frac{1}{4}x^2 - 4x \Big|_0^2 \Rightarrow \frac{1}{4}(2)^2 - 4(2) - 0$$

$$\frac{1}{4}(4) - 8 \Rightarrow -4$$

(10) $\int \underbrace{x^2y}_{f}\,dx - \underbrace{y^2x}_{g}\,dy$, first quad, $x^2+y^2=16$



$$\left.\begin{array}{l} \frac{\partial f}{\partial y} = x^2 \\ \frac{\partial g}{\partial x} = -y^2 \end{array}\right\} \int_0^{\pi/2}\int_0^4 \underbrace{-y^2 - x^2}_{-(y^2+x^2) = r^2} \cdot r\,dr\,d\theta$$

$$\int_0^{\pi/2}\int_0^4 -r^3\,dr\,d\theta$$

$$-\frac{r^4}{4}\Big|_0^4 \Rightarrow -\frac{1}{4}(4)^4 - 0 \Rightarrow -\frac{256}{4}$$

$$-\frac{256}{4}\theta\Big|_0^{\pi/2} \Rightarrow -\frac{256}{4}\left(\frac{\pi}{2}\right) - 0 \Rightarrow \frac{256\pi}{8} \Rightarrow 32\pi$$