Line Integrals

$$\int_C f(x,y)\, ds$$

arc length

curve

use parametric eq: to define it.

$$f(x,y) = f\big(x(t),\, y(t)\big)$$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\; dt$$

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\vec{r}\,'(t) = x'(t)\,\hat{i} + y'(t)\,\hat{j}$$

$$\|\vec{r}\,'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

$\therefore \; \|\vec{r}\,'(t)\| = \sqrt{(\ldots)}$

$$\boxed{ds = \|\vec{r}\,'(t)\| \, dt}$$

$$\int_C f(x,y)\, ds = \int_{t=a}^{t=b} f\big(x(t), y(t)\big) \cdot \|\vec{r}\,'(t)\| \, dt$$

$$= \int_{t=a}^{t=b} f\big(x(t), y(t)\big) \cdot \sqrt{[x'(t)]^2 + (y'(t))^2}\, dt$$

Q: $\int_C y \, ds$, $C: \vec{r}(t) = 2t\,\hat{i} + t^3\,\hat{j}$, $0 \leq t \leq 1$

$x = 2t$, $y = t^3$

$\therefore$

1) $\vec{r}'(t) = 2\hat{i} + 3t^2\hat{j}$

2) $\|\vec{r}'(t)\| = \sqrt{(2)^2 + (3t^2)^2}$

$= \sqrt{4+9t^4}$

or, $\sqrt{fx^2 + fy^2} = \sqrt{4+9t^4}$

$:\int_a^b f(x(t), y(t)) \cdot \|\vec{r}'(t)\| \, dt$

$\int_0^1 t^3 \cdot \left(\sqrt{4+9t^4}\right) dt$

$\frac{1}{36}\int_0^1 36t^3 \sqrt{4+9t^4} \, dt$

$\frac{1}{36} \times \frac{2}{3} \left[(4+9t^4)^{3/2}\right]_0^1$

$\cdot \left[(13)^{3/2} - (4)^{3/2}\right]$

$\frac{1}{54} \left((13) \right.$

$$\boxed{\frac{1}{54} \cdot \left(13\sqrt{13} - 8 \right)}$$

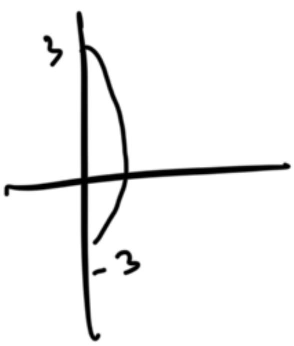
$r^2 = 9 \to r = 3$

Eg $\int_C x^2 + y^2 \, ds$; C: The Right $\frac{1}{2}$ circle $x^2+y^2=9$

for circle:

$x = r\cos t$, $y = r\sin t$

$x = 3\cos t$, $y = 3\sin t$



$$\int_{-\pi/2}^{\pi/2} \left(9\cos^2 t + 9\sin^2 t\right) \sqrt{(-3\sin t)^2 + (3\cos t)^2} \, dt$$

$$\int^{\pi/2} 9\left(\cos^2 t + \sin^2 t\right) \sqrt{9} \quad dt$$

$$\int_{-\pi/2}$$

$$\int_{-\pi/2}^{\pi/2} 9 \cdot 3 \, dt$$

$$\Rightarrow 27 \int_{-\pi/2}^{\pi/2} dt = 27\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$27(\pi)$$

$$27\pi$$

(x) for segments your "t" should go from $0 \to 1 \to 0 \le t \le 1$ (always)

ie: constant

Eg $\int_C 2xy \, ds$; C: The segment $(-2,-1)$ to $(1,3)$

t=0 t=1

you can choose anyone.

Note: For segments, we want $0 \le t \le 1$

$\ldots = c_2 + k_2 t$

$x = c_1 + k_1 t$ $\qquad y =$

at $t = 0 : (-2, -1)$, at $t = 1 : (1, 3)$

$-2 = c_1 + k_1(0)$

$\boxed{c_1 = -2}$

$x = c_1 + k_1 t$

$1 = -2 + k_1(1)$

$k_1 = 3$

now combine both values to form equation of x : remember we use both points for value of x & y

$$\boxed{x = -2 + 3t}$$

for y

at $t = 0 : (-2, -1)$

$-1 = c_2 + k_2(0)$

$\boxed{-1 = c_2}$

at $t = 1 : (1, 3)$

$y = c_2 + k_2 t$

$3 = -1 + k_2(1)$

$k_2 = 4$

$$\boxed{y = -1 + 4t}$$

(2) $$\int_0^1 2(-2+3t)(-1+4t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt$$

$$\int_0^1 2\left(2 - 8t - 3t + 12t^2\right)\sqrt{9+16}\;dt$$

$$\int_0^1 2\left(12t^2 - 11t + 2\right)\sqrt{25}\;dt$$

$$\int_0^1 10\left(12t^2 - 11t + 2\right)\;dt$$

$$\int_0^1 120t^2 - 110t + 20\;dt$$

$$120\frac{t^3}{3} - \frac{110t^2}{2} + 20t \Big|_0^1$$

$$\frac{\cancel{120}^{40}}{\cancel{3}} - \frac{110}{2} + 20$$

$$60 - 55$$

$$5$$

Eg $\int_C xyz^2\, ds$, C: The segment $(1,1,0) \to (2,3,1)$

$(1,1,0) \to (2,3,1)$

$t=0$ $t=1$

$x = c_1 + k_1 t$ $y = c_2 + k_2 t$ $z = c_3 + k_3 t$

$t=0$: $(1,1,0)$

$c_1 = 1$ $c_2 = 1$ $c_3 = 0$

$(2,3,1)$

$t = 1 \quad (2, \ldots$

$x = c_1 + k_1 t$

$2 = 1 + k_1(1)$

$\boxed{k_1 = 1}$

$y = c_2 + k_2 t$

$3 = 1 + k_2(1)$

$\boxed{k_2 = 2}$

$z = c_3 + k_3 t$

$1 = 0 + k_3(1)$

$\boxed{k_3 = 1}$

$\boxed{x = 1 + t}$ $\quad \boxed{y = 1 + 2t}$ $\quad \boxed{z = t}$

$$\int_0^1 (1+t)(1+2t)(t)^2 \cdot \sqrt{(1)^2 + (2)^2 + (1)^2} \quad dt$$

$$\int_0^1 (1+3t+2t^2)(t^2) \sqrt{1+4+1} \quad dt$$

$$\sqrt{6} \int_0^1 2t^4 + 3t^3 + t^2 \quad dt$$

$$\sqrt{6} \left[\frac{2t^5}{5} + \frac{3t^4}{4} + \frac{t^3}{3} \right]_0^1$$

$$\sqrt{6}\left[\frac{2}{5}+\frac{3}{4}+\frac{1}{3}\right]$$

$$\sqrt{6}\left[\frac{24+45+20}{(5)(4)(3)}\right]$$

$$\frac{89\sqrt{6}}{60}$$

Ans

④ if two segments are given then the starting point should be taken as the $t=0$

Eg $\int_C xy\,dx + (x+y)\,dy$

C: Two segment

C_1: $(1,2) \to (3,4)$

C_2: $(3,4) \to (4,0)$

① first find x & y for C_1, integrate
② then find x & y for C_2, integrate] ⊕

$$C_1^1 : \underset{t=0}{(1,2)} \to \underset{t=1}{(3,4)}$$

$y = c_2 + k_2 t$

$x = c_1 + k_1 t$

$c_2 = 2$

$c_1 = 1$

at $t=1$: $(3,4)$

$4 = 2 + k_2(1)$

$3 = 1 + k_1(1)$

$\boxed{k_2 = 2}$

$k_1 = 2$

$\boxed{x = 1 + 2t}$ $\boxed{y = 2 + 2t}$

$\boxed{dy = 2\,dt}$

$\boxed{dx = 2\,dt}$

No: need to do the $\boxed{\sqrt{f_x^2 + f_y^2}\,dt}$

$$C_1^1 : \int^1 (1+2t)(2+2t)\,2dt + (1+2t+2+2t)\,2dt$$

$$C_1' : \int_0^1 \left[(1+2t)(2+2t)2 + (1+2t+2+2t)2 \right] dt$$

$$\boxed{C_1' : \frac{68}{3}}$$

$t=0$ $\quad$ $t=1$

Now $\quad C_2' : (3,4) \longrightarrow (4,0)$

$x = c_1 + k_1 t$ $\qquad$ $\cancel{y} = c_2 + k_2 t$

at $t=0$

$c_1 = 3$ $\qquad$ $c_2 = 4$

at $t=1$

$4 = 3 + k_1(1)$ $\qquad$ $0 = 4 + k_2(t)$

$\boxed{k_1 = 1}$ $\qquad$ $\boxed{k_2 = -4}$

C_2^1: $\boxed{x = 3+t}$ $\quad$ $\boxed{y = 4-4t}$

$\boxed{dx = dt}$ $\quad$ $\boxed{dy = -4dt}$

C_2^1: $\int_0^1 (3+t)(4-4t)\,dt + (3+t+4-4t)-4\,dt$

C_2^1: $\int_0^1 \left[(3+t)(4-4t) + [(3+t+4-4t)]-4\right] dt$

C_2: $-46/3$

Total $= C_1 + C_2$

$= \frac{68}{3} - \frac{46}{3}$

$= 22/3$

l_3

Eg $\int_C (x+y)\,dx + (xy)\,dy + y\,dz$,

$C: \vec{r}(t) = e^{t}\hat{i} + e^{-t}\hat{j} + 2e^{2t}\hat{k}$

sol

$x = e^{t} \qquad y = e^{-t} \qquad z = 2e^{2t}$

$dx = e^{t}\,dt \qquad dy = -e^{-t}\,dt \qquad dz = 4e^{2t}\,dt$

$$\int_0^1 (e^{t}+e^{-t})\,e^{t}\,dt + (e^{t}\cdot e^{-t})\,-e^{-t}\,dt + e^{-t}\,(4e^{2t})\,dt$$

$$\int_0^1 \left(e^{t}+1 - e^{-t} + 4e^{t}\right) dt$$

$$= \frac{1}{2}e^{2} + 4e + \frac{1}{e} - 9/2$$

ANS

(*) Line integrals then a vector field.

1\. for $F(x,y,z) = F\big(x(t), y(t), z(t)\big)$

& $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

The Line $\int$ along "c", then the vector field "F" is

$$W = \int_{t=a}^{t=b} F\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\, dt \qquad - \quad W = \int F \cdot dr$$

Qu: Find work done by moving along the parabola $y = x^2$ from $(-1,1) \to (2,4)$ then

V. F : $F(x,y) = xe$ i + j ,

should satisfy the points

for Non-Segments

Do trivial parametric eq:

$x = t, \quad y = t^2$

$-1 \leq t \leq 2$

$t = -1$	$t = 2$
$x = 1$	$x = 4$

$F(x,y) = xe^{y}\hat{i} + y\hat{j}$

$F = te^{t^2}\hat{i} + t^2\hat{j}$

$\vec{r}(t) = x\hat{i} + y\hat{j}$

(: compare

$\vec{r}(t) = t\hat{i} + t^2\hat{j}$

derivative

$F \cdot \vec{r}\,'(t)$

$(te^{t^2}\hat{i} + t^2\hat{j}) \cdot (\hat{i} + 2t\hat{j})$

$$= \frac{te^{t^2} + 2t^3}{f \cdot dr}$$

$$\int_{-1}^{2} te^{t^2} + 2t^3 \; dt = \frac{e^4 - e + 15}{2}$$

Ans

Q) $F(x,y,z) = (x+2y)\hat{i} + 2z\hat{j} + (x-y)\hat{k}$

$C: \quad (-1,3,2) \rightarrow (1,-2,4)$

$t=0$ $\quad$ $t=1$

segment : $0 \le t \le 1$

at $t=0$: $(-1,3,2)$

$x = c_1 + k_1 t$

$y = c_2 + k_2 t$

$z = c_3 + k_3 t$

at $t=0$

$c_1 = -1$

$c_2 = 3$

$c_3 = 2$

at $t=1$ $(1,-2,4)$

$x = -1 + 2t$

$y = 3 - 5t$

$z = 2 + 2t$

compare

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= (-1+2t)\hat{i} + (3-3t)\hat{j} + (2+2t)\hat{k}$$

$$F = \left[(-1+2t) + 2(3-3t)\right]\hat{i} + \left[2(2+2t)\right]\hat{j} + \left[(-1+2t) - (3-3t)\right]\hat{k}$$

$$F_{(t)} = (5-8t)\hat{i} + (4+4t)\hat{j} + (-4+7t)k$$

$$\vec{r}'(t) = 2\hat{i} - 5\hat{j} + 2\hat{k}$$

answer = 0 in conservative v.f

for non-conservative v.f

(*) Green's theorem:

Format:

$\oint_C P\,dx + Q\,dy$ has a curve "C" that is enclosed a region on a plane &

C is a simple closed curved that is travelled in the counter clockwise direction

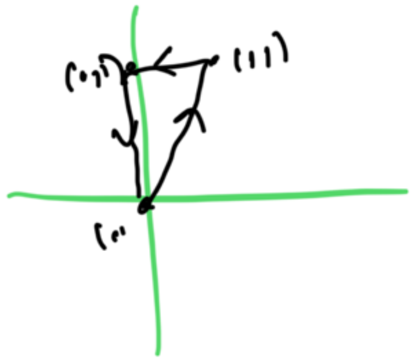
then $$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA}$$

↳ give a line $\int$ for a simple closed curve = $\iint$ area a region that a curve contain.

Eg 2

$\oint x^3\,dx + xy\,dy$; C : Δ : (0,0) (1,1)

$\hat{i}$ $(0,1)$



$$\oint P\,dx + Q\,dy$$

Sol

$$F(x,y) = P\hat{i} + Q\hat{j}$$

$$F(x,y) = x^3\hat{i} + xy\hat{j}$$

$$\oint_C x^3\,dx + xy\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

$$= \iint_R \left(\frac{\partial\,[xy]}{\partial x} - \frac{\partial\,[x^3]}{dy}\right] dA$$

$$= \iint_R [y - 0]\,dA$$

dA

$$= \iint_R y$$

$$\int_0^1 \int_0^y y \; dx \; dy$$

$$\int_0^1 y \, x \Big]_0^y \; dy$$

$$\int_0^1 y^2 \; dy$$

$$\frac{y^3}{3} \Big]_0^1 = \frac{1}{3}$$

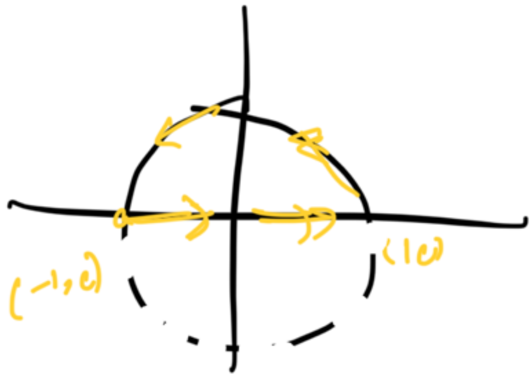
ANS

Ex

$$\oint_C x^2 y \, dx + y^3 dy$$

C; $(-1, 0) \to (1, 0)$

R $x^2 + y^2 = 1$



$$\iint_R \left(\frac{d(y^3)}{dx} - \frac{d(x^2 y)}{dy} \right) dA$$

$$\iint_R 0 - x^2 \quad dA$$

$$\iint_R -x^2 \; dA$$

$$\int_0^{\pi} - \int_0^{1} r^2 \cos^2\theta \cdot r \, dr \, d\theta$$

$= +\frac{1}{8}$ ANS