$$-\frac{\sqrt{5}}{4}\left[\frac{1}{e^{2\pi}}-\frac{1}{e^{0}}\right] \Rightarrow -\frac{\sqrt{5}}{4e^{2\pi}}$$

Ex # 15.4 :

$$\oint_C f(x,y)\,dx + g(x,y)\,dy$$

$$\Rightarrow \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

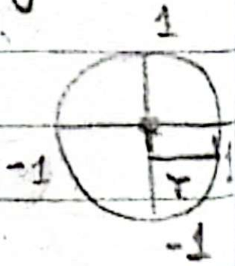
Ex #2 :

$$F(x,y) = (e^{x} - y^{3})\,\hat{i} + (\cos y + x^{3})\,\hat{j}$$

$x^2+y^2=1$ in counterclockwise

$f(x,y) = e^{x} - y^{3}$ , $g(x,y) = \cos y + x^{3}$

$\frac{\partial f}{\partial y} = -3y^{2}$ , $\frac{\partial g}{\partial x} = 3x^{2}$




$$\iint_R 3x^2 + 3y^2 \, dA$$

put $x = r\cos\theta$

$y = r\sin\theta$

$$3\int_0^{2\pi}\int_0^{1} (x^2+y^2)\, r\, dr\, d\theta$$

in circle

$x^2+y^2 = r^2$

$(r\cos\theta)^2 + (r\sin\theta)^2$

$r^2\cos^2\theta + r^2\sin^2\theta$

$\boxed{r^2 = r^2}$

$$3\int_0^{2\pi}\int_0^{1} r^3\, dr\, d\theta$$

$$\frac{3}{4}\int_0^{2\pi} [r^2]_0^1 \, d\theta \Rightarrow \frac{3}{4}\int_0^{2\pi} d\theta$$

$$\Rightarrow \frac{3}{4}[\theta]_0^{2\pi} \Rightarrow \frac{3}{\cancel{4}_2}[\cancel{2}\pi - 0] = \boxed{\frac{3\pi}{2}}$$

(3) $\oint 3xy \, dx + 2xy \, dy$,


$f(x,y) = 3xy, \; g(x,y) = 2xy$

$\frac{\partial f}{\partial y} = 3x, \; \frac{\partial g}{\partial x} = 2y$

$$\iint_R (2y - 3x)\, dA \Rightarrow \int_{-2}^{4}\int_{1}^{2}[(2y-3x)\,dy]\,dx$$

$$\int_{-2}^{4} I \, dx \Rightarrow I = \int_1^2 (2y - 3x)\,dy$$

$$\Rightarrow 2\int_1^2 y\,dy - 3\int_1^2 x\,dy$$

$$\Rightarrow \frac{2}{2}\left[y^2\right]_1^2 - 3x[1]$$

$$\Rightarrow 4 - 1 - 3x \Rightarrow 3 - 3x$$

$$\int_{-2}^{4}(3 - 3x)\,dx \Rightarrow 3\int_{-2}^{4} dx - 3\int_{-2}^{4} x\,dx$$

$$\Rightarrow 3[4+2] - \frac{3}{2}[16 - (4)]$$

$$\Rightarrow 18 - \frac{3}{\cancel{2}}\left[\cancel{12}_{6}\right] = \boxed{0}$$

④ $\oint_C (x^2-y^2)dx + x\,dy \;,\; C \Rightarrow x^2+y^2=9$

$f(x,y) = x^2-y^2 \;,\; g(x,y) = x$

$\frac{\partial f}{\partial y} = -2y \;,\; \frac{\partial g}{\partial x} = 1 \qquad r = \pm 3$

$$\iint_R (1+2y)\,dA \Rightarrow \int_0^{2\pi}\int_0^{3} (1+2y)\,r\,dr\,d\theta$$

$$\Rightarrow \int_0^{2\pi} I\,d\theta \Rightarrow I = \int_0^{3} (1+2r\sin\theta)\,r\,dr$$

$$I = \int_0^3 r\,dr + 2\int_0^3 r^2\sin\theta\,dr$$

$$I = \frac{1}{2}\left|r^2\right|_0^3 + 2\sin\theta \int_0^3 r^2\,dr$$

$$\Rightarrow \frac{1}{2}\left|9-0\right| + \frac{2\sin\theta}{3}\left|r^3\right|_0^3$$

$$\Rightarrow \frac{9}{2} + \frac{2\sin\theta}{\cancel{3}_1}\left[\cancel{27}_9\right] \Rightarrow \frac{9}{2} + 18\sin\theta$$

$$\int_0^{2\pi}\left(\frac{9}{2} + 18\sin\theta\right)d\theta$$

$$\frac{9}{2}\int_0^{2\pi} d\theta + 18\int_0^{2\pi} \sin\theta\,d\theta$$

$$\frac{9}{2}[2\pi] - 18\left[\cos(2\pi) - \cos(0)\right]$$

$$9\pi - 18[0] \Rightarrow \boxed{9\pi}$$

(51) $\oint x\cos y\,dx - y\sin x\,dy$

square $(0,0), (\pi/2, 0), (\pi/2, \pi/2), (0, \pi/2)$

$f(x,y) = x\cos y \quad , \quad g(x,y) = -y\sin x$

$\frac{\partial f}{\partial y} = -x\sin(y) \quad , \quad \frac{\partial g}{\partial x} = -y\cos(x)$

$$\iint_R [-y\cos(x) + x\sin(y)]\,dA$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/2} [-y\cos(x) + x\sin(y)]\,dx\,dy \quad - (1)$$

$$I = \int_0^{\pi/2} [-y\cos(x) + x\sin(y)]\,dx$$

$$\Rightarrow -y\int_0^{\pi/2} \cos(x)\,dx + \sin(y)\int_0^{\pi/2} x\,dx$$

$$\Rightarrow -y\left[\sin(x)\right]_0^{\pi/2} + \frac{\sin(y)}{2}\left[x^2\right]_0^{\pi/2}$$

$$\Rightarrow -y\left[\sin(\pi/2) - \sin(0)\right] + \frac{\sin(y)}{2}\left[\left(\frac{\pi}{2}\right)^2 - (0)^2\right]$$

$$\Rightarrow -y + \frac{\sin(y)}{2}\left[\frac{\pi^2}{4}\right] \Rightarrow -y + \frac{\pi^2}{8}\sin(y) \quad - (2)$$

$$\int_0^{\pi/2}\left[-y + \frac{\pi^2}{8}\sin(y)\right]dy \text{ in } (1)$$

$$-\int_0^{\pi/2} y\,dy + \frac{\pi^2}{8}\int_0^{\pi/2}\sin(y)\,dy$$

$$\frac{1}{2}\left[y^2\right]_0^{\pi/2} - \frac{\pi^2}{8}\left[\cos(y)\right]_0^{\pi/2}$$

$$\Rightarrow \frac{1}{2}\left[\frac{\pi^2}{4}\right] - \frac{\pi^2}{8}\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$$

$$\frac{\pi^2}{8} - \frac{\pi^2}{8}\left[-0-1\right] = \boxed{0}$$

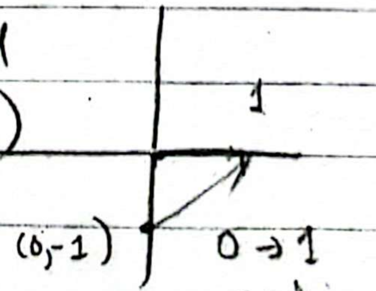
⑥ $\oint_C y\text{-}\tan^2 x\,dx + \tan x\,dy \;,\; x^2+(y+1)^2=1$

$$(x-h)^2+(y-k)^2 = 1$$

$$(h,k) = (0,-1) \;,\; r = \pm 1$$

$f(x,y) = y\tan^2 x$, $g(x,y) = \tan x$

$\frac{\partial f}{\partial y} = \tan^2 x$, $\frac{\partial g}{\partial x} = \sec^2(x)$




$$\iint_R \left[\sec^2(x) - \tan^2(x)\right] dA$$

$$\int_0^{2\pi}\int_0^1 r\,dr\,d\theta \Rightarrow \frac{1}{2}\left[r^2\right]_0^1$$

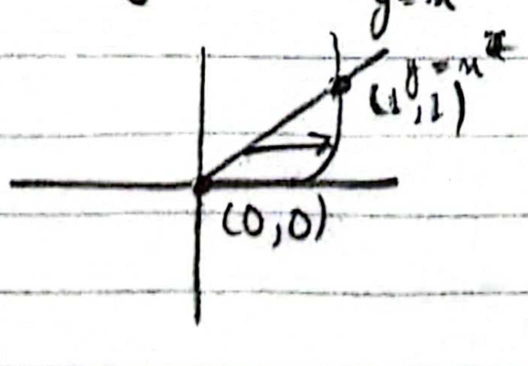
$$\frac{1}{2}\left[1\right]$$

$$\frac{1}{2}\int_0^{2\pi} d\theta$$

$$\Rightarrow \frac{1}{2}\left[2\pi - 0\right] = \boxed{\pi}$$

Answer.

⑧ $\oint_C (e^x+y^2)\,dx + (e^y+x^2)\,dy$




$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$\Rightarrow f(x,y) = e^x + y^2 \, , \; g(x,y) = e^y + x^2$

$$\frac{\partial f}{\partial y} = 2y \, , \; \frac{\partial g}{\partial x} = 2x$$

$$\iint_R (2x - 2y)\, dA \Rightarrow 2 \iint_R (x-y)\, dA$$

$$2\int_0^1 \int_{x^2}^{x} (x-y)\, dy\, dx \Rightarrow 2\int_0^1 I \, dx \quad - (1)$$

$$I = \int_{x^2}^{x} (x-y)\, dy \Rightarrow x\int_{x^2}^{x} dy - \int_{x^2}^{x} y\, dy$$

$$\Rightarrow x\left[x - x^2\right] - \frac{1}{2}\left[x^2 - x^4\right]$$

$$\Rightarrow x^2 - x^3 - \frac{1}{2}x^2 + \frac{x^4}{2}$$

$$2\int_0^1 \left(x^2 - x^3 - \frac{1}{2}x^2 + \frac{x^4}{2}\right) dx$$

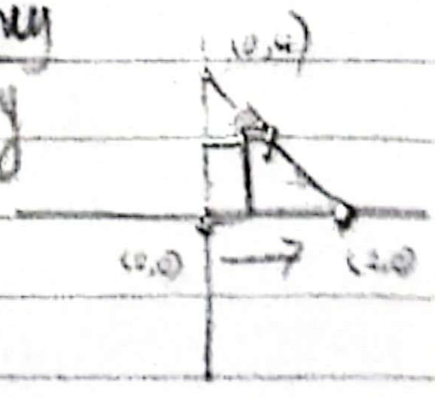
$$\Rightarrow \frac{2}{3}\left[x^3\right]_0^1 - \frac{2}{4}\left[x^4\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1 + \frac{1}{5}\left[x^5\right]_0^1$$

$$\Rightarrow \frac{2}{3} - \frac{2}{4} - \frac{1}{3} + \frac{1}{5} = 1/30 \text{ Answer}$$

(9) $$\oint_C \ln(1+y)\, dx - \frac{xy}{1+y}\, dy$$

$$f(x,y) = \ln(1+y), \; g(x,y) = \frac{-xy}{1+y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{(1+y)} \, , \; \frac{\partial g}{\partial x} = \frac{-y}{1+y}$$



$$\iint_R \left[\frac{-y}{1+y} - \frac{1}{(1+y)}\right] dA$$

$(0,4)\ (2,0) \Rightarrow y = mx + c$

$$m = \frac{0-4}{2-0} = \frac{-4}{2} = \boxed{-2}$$

$y = -2x + c$ — ① $4 = -2(0) + c$

$c = 4$

$$\boxed{y = -2x + 4}$$

$$\int_0^2 \int_0^{-2x+4} \left(\frac{-y}{1+y} - \frac{1}{1+y}\right) dy\, dx$$

$\Rightarrow$ solve