MVC

Assignment : 01

23k-2001

BCS-2J

Question#1:

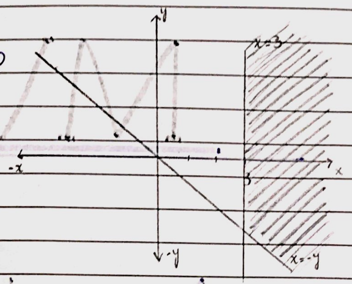
a. i. $f(x,y) = \sqrt{x+y} - \sqrt{x-3}$

Sol:

$\sqrt{x+y} \geq 0$ & $\sqrt{x-3} \geq 0$

$x+y \geq 0$, $x-3 \geq 0$

$x \geq 3$



ii. $f(x,y) = \sqrt{\frac{1}{x^2} - \frac{1}{y^2}}$

Sol:

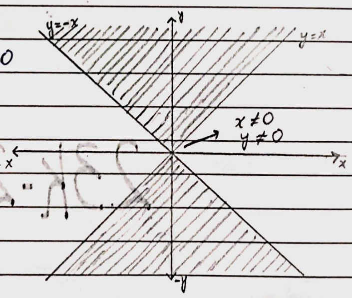
$\sqrt{\frac{1}{x^2} - \frac{1}{y^2}} \geq 0$ OR $\frac{y^2 - x^2}{x^2y^2} \geq 0$

$\rightarrow y^2 - x^2 \geq 0$, $x^2y^2 \neq 0$

$(y-x)(y+x) \geq 0$

$y - x \geq 0$, $y + x \geq 0$

$y \geq x$, $y \geq -x$

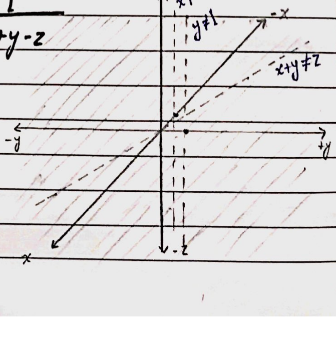


iii. $f(x,y,z) = \frac{1}{x+1} + \frac{1}{y-1} + \frac{1}{x+y-z}$

Sol:

$x+1 \neq 0$, $y-1 \neq 0$, $x+y-z \neq 0$

$x \neq -1$ $y \neq 1$



iv. $f(x,y) = \ln(x^2 - 8y)$

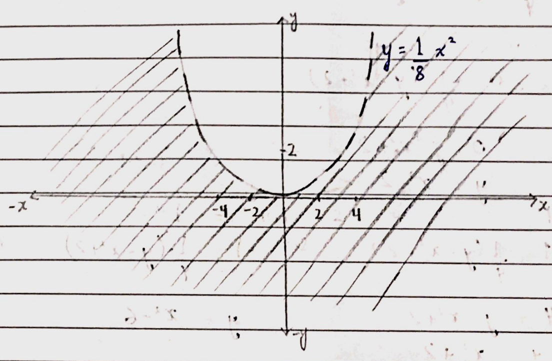
Sol:

$x^2 - 8y > 0$

$x^2 > 8y$ $\rightarrow$ parabola:

$y < \frac{1}{8}x^2$ $a = 2$, endpoints:

$(-2a, a), (2a, a) \rightarrow (-4, 2), (4, 2)$



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b.

i. $x^2 - 4z - y = 2$

Sol:

Divide eqn by 4:

$$z = \frac{y}{4} + \frac{1}{2} - \frac{x^2}{4}$$

$$z = \frac{1}{4}(y - x^2 + 2)$$

For $z = 0$

$z = 0$, $0 = \frac{1}{4}(y - x^2 + 2)$

$0 = y - x^2 + 2$

$y = x^2 - 2$

For $z = 1$

$1 = \frac{1}{4}(y - x^2 + 2)$

$4 = y - x^2 + 2$

$y = x^2 + 2$

For $z = -1$

$-1 = \frac{1}{4}(y - x^2 + 2)$

$y = x^2 - 6$

For $z = 2$

$2 = \frac{1}{4}(y - x^2 + 2)$

$8 = y - x^2 + 2$

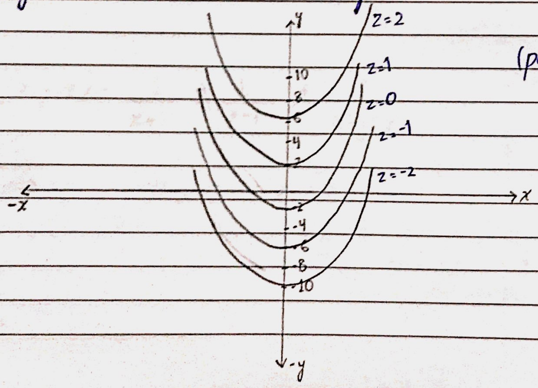
$y = x^2 + 6$

For $z = -2$

$-2 = \frac{1}{4}(y - x^2 + 2)$

$-8 = y - x^2 + 2$

$y = x^2 - 10$



(parabolas)

Date

ii. $z^2 + 4x^2 = 1 - y^2$

Sol:

$z^2 = 1 - y^2 - 4x^2$

$z^2 = 1 - (y^2 + 4x^2)$

For $z = 0$

$\rightarrow 0 = 1 - (y^2 + 4x^2)$

$4x^2 + y^2 = 1$ (ellipse)

For $z = 1$

$\rightarrow 1 = 1 - (y^2 + 4x^2)$

$4x^2 + y^2 = 0$

(origin)

For $z = -1$

$-1 = 1 - (y^2 + 4x^2)$

$-2 = -(y^2 + 4x^2)$

$4x^2 + y^2 = 2$ (ellipse)

For $z = 2$

$\rightarrow 2 = 1 - (y^2 + 4x^2)$

$-1 = y^2 + 4x^2$

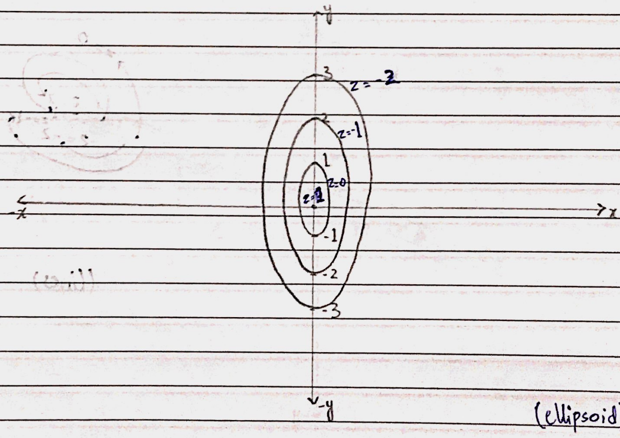
(Not possible)

For $z = -2$

$\rightarrow -2 = 1 - (y^2 + 4x^2)$

$y^2 + 4x^2 = 3$

(ellipse)



(ellipsoid)

Date

III. $2x - 6y + z = -2$

Sol:

$$z = -2 - 2x + 6y$$
$$z = -2(1 + x + 3y)$$

For $z = 0$

$$\rightarrow 0 = -2(1 + x + 3y)$$
$$x + 3y = -1$$
$$3y - x = 1$$

For $z = 1$

$$\rightarrow 1 = -2(1 + x - 3y)$$
$$-1/2 = 1 + x - 3y$$
$$3y - x = 3/2$$

For $z = -1$

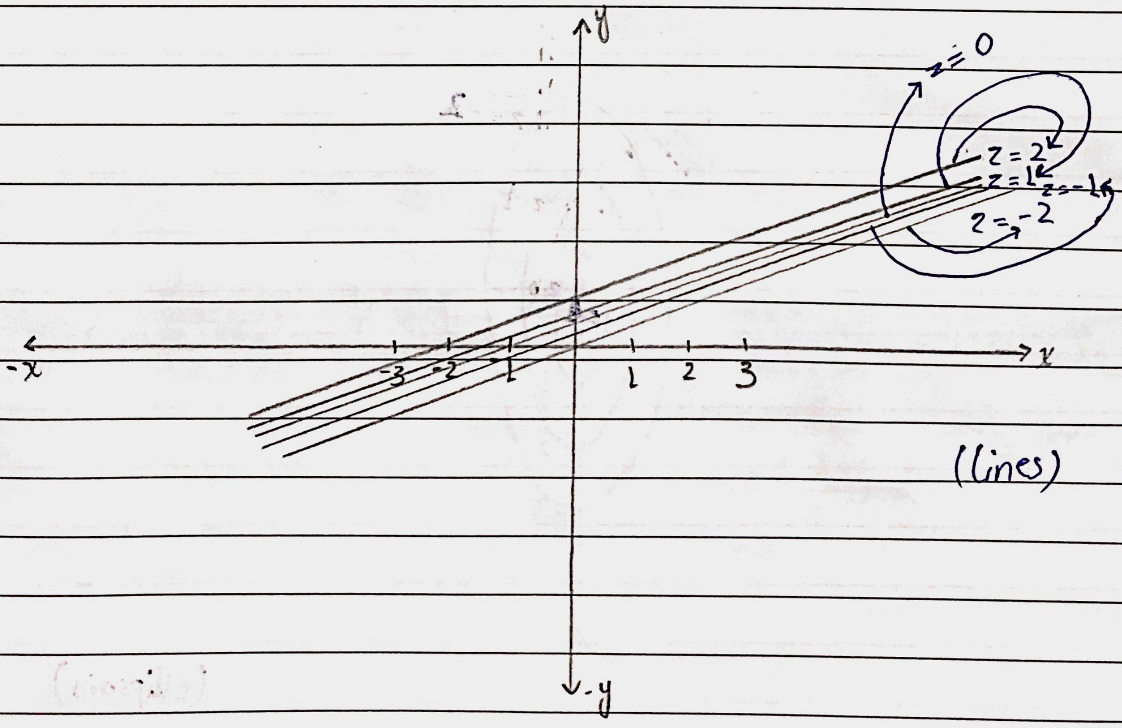
$$\rightarrow -1 = -2(1 + x - 3y)$$
$$1/2 = 1 + x - 3y$$
$$3y - x = \frac{1}{2}$$

For $z = 2$

$$\rightarrow 2 = -2(1 + x - 3y)$$
$$-1 = 1 + x - 3y$$
$$3y - x = 2$$

For $z = -2$

$$\rightarrow -2 = -2(1 + x - 3y)$$
$$1 = 1 + x - 3y$$
$$3y - x = 0$$



(lines)

c.

i. $f(x,y,z) = x^2 - y^2 - z^2 + 1$; $k = -3$

$\Rightarrow k = x^2 - y^2 - z^2 + 1$

$-3 = x - y^2 - z^2 + 1$

$-4 = x - y^2 - z^2 \Rightarrow y^2 + z^2 = 4 + x$

By traces:

xy-plane:

$-4 = x - y^2$

$x = y^2 - 4$ (parabola)

xz-plane:

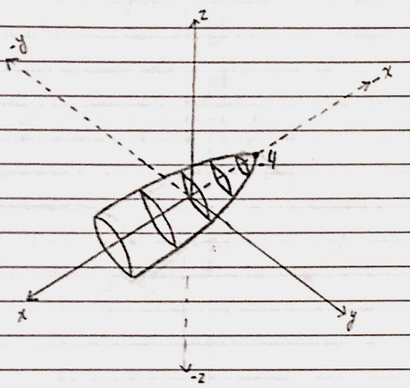
$-4 = x - z^2$

$x = z^2 - 4$ (parabola)

yz-plane:

$-4 = -y^2 - z^2$

$y^2 + z^2 = 4$ (ellipse)



elliptic paraboloid along x-axis

ii. $f(x,y,z) = \frac{3x^2 + y^2}{z^2}$; $k = 9$

Sol:

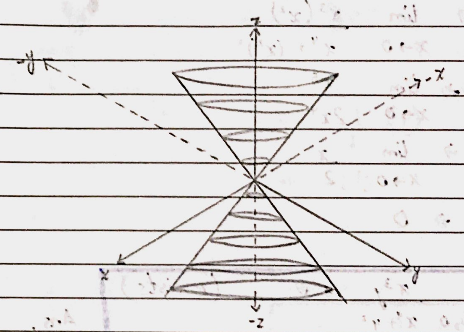
$\Rightarrow k = \frac{3x^2 + y^2}{z^2}$

$kz^2 = 3x^2 + y^2$

$9z^2 = 3x^2 + y^2$ elliptic

$z^2 = \frac{x^2}{3} + \frac{y^2}{9}$ (cone)

$c = \pm 1$, $a = \pm\sqrt{3}$, $b = \pm 3$



iii. $f(x,y,z) = 9x^2 + 4y^2 + z^2$; $k = 4$

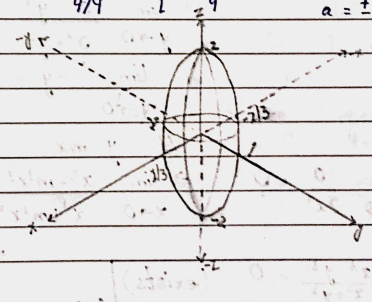
Sol:

$\Rightarrow k = 9x^2 + 4y^2 + z^2$

$4 = 9x^2 + 4y^2 + z^2$

$1 = \frac{x^2}{4/9} + \frac{y^2}{1} + \frac{z^2}{4}$ (ellipsoid)

$a = \pm\frac{2}{3}$, $b = \pm 1$, $c = \pm 2$



Date

Question#2:

a.

i. $$\lim_{(x,y)\to 0,0} \frac{x^3y}{x^4+y^2}$$

Sol:

At x-axis, $y=0$

$$\Rightarrow \lim_{x\to 0} \frac{x^3(0)}{x^4+(0)^2} = 0$$

At y-axis, $x=0$

$$\Rightarrow \lim_{y\to 0} \frac{0^3(y)}{0^4+y^2} = 0$$

At $y=x^2$:

$$\Rightarrow \lim_{x\to 0} \frac{x^3(x^2)}{x^4+(x^2)^2}$$

$$\Rightarrow \lim_{x\to 0} \frac{x^5}{2x^4}$$

$$\Rightarrow \lim_{x\to 0} \frac{x}{2}$$

$$\Rightarrow 0$$

Hence,

$$\boxed{\lim_{(x,y)\to 0,0} \frac{x^3y}{x^4+y^2} = 0 \quad \text{(exists)}}$$ Ans.

ii. $$\lim_{(x,y)\to 0,0} \frac{x^3-y^3}{x^2+y^2}$$

Sol:

At $y=0$

$$\Rightarrow \lim_{x\to 0} \frac{x^3-0^3}{x^2+0^2}$$

$$\Rightarrow \lim_{x\to 0} x = 0$$

At $x=0$

$$\Rightarrow \lim_{y\to 0} \frac{0^3-y^3}{0^2+y^2}$$

$$\Rightarrow \lim_{y\to 0} -y = 0$$

At $y=x$

$$\Rightarrow \lim_{x\to 0} \frac{x^3-x^3}{x^2+x^2} = \lim_{x\to 0} \frac{0}{2x^2} = 0$$

At $y=mx$

$$\Rightarrow \lim_{x\to 0} \frac{x^3-m^3x^3}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x(1-m^3)}{1+m^2}$$

$$= 0$$

Hence,

$$\boxed{\lim_{(x,y)\to 0,0} \frac{x^3-y^3}{x^2+y^2} = 0 \quad \text{(exists)}}$$ Ans.

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III. $\lim_{(x,y)\to 0,0} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$

Sol:

At $y=0$

$\Rightarrow \lim_{x\to 0} \frac{1-\cos(x^2+0)}{(x^2+0)^2}$

$\Rightarrow \lim_{x\to 0} \frac{1-\cos x^2}{x^4}$

Apply L-Hopital Rule

$\Rightarrow \lim_{x\to 0} \frac{0+2x\sin x^2}{4x^3}$

$\Rightarrow \lim_{x\to 0} \frac{1.\sin x^2}{2.\ x^2}$

$\Rightarrow \frac{1}{2}(1)$

$\Rightarrow 1/2$

At $x=0$

$\Rightarrow \lim_{y\to 0} \frac{1-\cos(0+y^2)}{(0+y^2)^2}$

$\Rightarrow \lim_{y\to 0} \frac{1-\cos y^2}{y^4}$

Apply L-Hopital Rule

$\Rightarrow \lim_{y\to 0} \frac{0+2y\sin y^2}{4y^3}$

$\Rightarrow \lim_{y\to 0} \frac{1}{2}\frac{\sin y^2}{y^2}$

$\Rightarrow \frac{1}{2}(1)$

$\Rightarrow 1/2$

At $y=x$

$\Rightarrow \lim_{x\to 0} \frac{1-\cos(x^2+x^2)}{(x^2+x^2)^2}$

$\Rightarrow \lim_{x\to 0} \frac{1-\cos 2x^2}{4x^4}$

Apply L-Hopital Rule

$\Rightarrow \lim_{x\to 0} \frac{0+4x\sin 2x^2}{16x^3}$

$\Rightarrow \lim_{x\to 0} \frac{1}{2}\frac{\sin 2x^2}{2x^2}$

$\Rightarrow \frac{1}{2}(1)$

$\Rightarrow 1/2$

Hence,

$$\boxed{\lim_{(x,y)\to 0,0} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} = \frac{1}{2}}$$

(exists)

Ans.

→ OR by polar coordinates: (r,θ)

let $r^2 = x^2+y^2$, at $r\to 0^+$

$\Rightarrow \lim_{r\to 0} \frac{1-\cos r^2}{(r^2)^2}$, Apply L-hopital

$\Rightarrow \lim_{r\to 0} \frac{+2r\sin r^2}{4r^3}$

$\Rightarrow \lim_{r\to 0} \frac{1}{2}\frac{\sin r^2}{r^2}$

$\Rightarrow \frac{1}{2}(1) = \frac{1}{2}$ (limit exists)

Date

b.

i. $\frac{xy}{\sqrt{x^2+y^2}}$

Sol:

at $y=0$

$\Rightarrow \lim_{x\to 0} \frac{x(0)}{\sqrt{x^2+0^2}} = 0$

at $x=0$

$\Rightarrow \lim_{y\to 0} \frac{0(y)}{\sqrt{0^2+y^2}} = 0$

- at $y=mx$

$\Rightarrow \lim_{x\to 0} \frac{x(mx)}{\sqrt{x^2+(mx)^2}}$

$\Rightarrow \lim_{x\to 0} \frac{x^2m}{x\sqrt{1+m^2}}$

$\Rightarrow 0$

Since,

$\lim_{(x,y)\to 0,0} \frac{xy}{\sqrt{x^2+y^2}} = 0$

&

$\lim_{(x,y)\to 0,0} \frac{xy}{\sqrt{x^2+y^2}} = f(0,0) = 0$

The function is continuous.

Ans.

ii. $\frac{xy}{x^2+y^2}$

Sol:

at $y=0$

$\Rightarrow \lim_{x\to 0} \frac{x(0)}{x^2+0^2} = 0$

at $x=0$

$\Rightarrow \lim_{y\to 0} \frac{0(y)}{0^2+y^2} = 0$

at $y=mx$

$\Rightarrow \lim_{x\to 0} \frac{x(mx)}{x^2+m^2x^2}$

$\Rightarrow \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$

$\Rightarrow \frac{m}{1+m^2}$

$\because \lim_{(x,y)\to 0,0} \frac{xy}{x^2+y^2} \Rightarrow$ Does not exist

Hence,

The function is not continuous

Ans.

Date

iii. $\dfrac{x^4 - y^2}{x^4 + y^2}$

Sol:

At $y = 0$

$\Rightarrow \lim_{x \to 0} \dfrac{x^4 - 0^2}{x^4 + 0^2} = 1$

At $x = 0$

$\Rightarrow \lim_{y \to 0} \dfrac{0^4 - y^2}{0^4 + y^2} = -1$

$\because \lim_{(x,y) \to 0,0} \dfrac{x^4 - y^2}{x^4 + y^2} =$ Does not exist

Hence,

The function is not continuous.

Ans.

iv. $\dfrac{x^2 y}{x^4 + y^2}$

Sol:

At $y = 0$

$\Rightarrow \lim_{x \to 0} \dfrac{x^2 (0)}{x^4 + 0^2} = 0$

At $x = 0$

$\Rightarrow \lim_{y \to 0} \dfrac{0^2 (y)}{x^4 + y^2} = 0$

At $y = x^2$

$\Rightarrow \lim_{x \to 0} \dfrac{x^2 . (x^2)}{x^4 + (x^2)^2}$

$\Rightarrow \lim_{x \to 0} \dfrac{x^4}{2x^4}$

$\Rightarrow \dfrac{1}{2}$

$\because \lim_{(x,y) \to 0,0} \dfrac{x^2 y}{x^4 + y^2} =$ Does not exist

Hence,

The function is not continuous.

Ans.

Date ________

Question #3:

Part I:

a. $f_x(0,y) = -y$ Prove when:
$f_y(x,0) = x$

$$f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}$$

$$\Rightarrow f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$$

$$f_x(0,y) = \frac{(x^2+y^2)(3x^2y-y^3)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2}$$

$$f_x(0,y) = \frac{3x^4y+3x^2y^3-x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2}$$

$$f_x(0,y) = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

$$f_x(0,y) = \frac{(0)y+4(0)y^3-y^5}{(0^2+y^2)^2}$$

$$f_x(0,y) = \frac{-y^5}{y^4}$$

$f_x(0,y) = -y$ Proved.

$$\Rightarrow f_y(x,y) = \frac{(x^2+y^2)(x^3-3xy^2)-(x^3y-xy^3)(2y)}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{x^5-3x^3y^2+x^3y-3xy^3-2x^3y^2+2xy^4}{(x^2+y^2)^2}$$

$$f_y(x,0) = \frac{x^5-0+0-0-0+0}{(x^2+0^2)^2}$$

$$f_y(x,0) = \frac{x^5}{x^4}$$

$f_y(x,0) = x$ Proved.

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b. $f_{xy}(0,0) = -1$

By part(a):
$f_x(0,y) = -y$
$f_{xy}(0,y) = -1$
$f_{xy}(0,0) = -1$ Proved.

$f_{yx}(0,0) = 1$

By f_y:
$f_y(x,0) = x$
$f_{yx}(x,0) = 1$
$f_{yx}(0,0) = 1$ Proved

c. Prove: $f(x,y)$ is differentiable at $(0,0)$:

By polar coordinates: $r \to 0^+$

$x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$

$$\Rightarrow \lim_{r\to 0^+} (r\cos\theta)(r\sin\theta)\left(\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}\right)$$

$$\Rightarrow \lim_{r\to 0^+} r^2\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta)$$

$$\Rightarrow \lim_{r\to 0^+} r^2(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$$

Apply limit

$\Rightarrow = 0$ Ans.

Hence limit exists and since all first-order partial derivatives of f exist, $f(x,y)$ is differentiable at $(0,0)$

Part II:

$f_x = 0$ implies that f does not change whenever x changes similarily $f_y = 0$ implies that f does not change whenever y changes so, f does not depend on x,y implying that is a constant.

We can prove by integrating the partial derivatives:

$$\int f_x(x,y)\,dx = \int 0\,dx \quad \& \quad \int f_y(x,y)\,dy = \int 0\,dy$$

$f(x,y) = 0 + C$ $\quad$ $f(x,y) = 0 + C$

Hence $f(x,y) = c \to$ (constant) Proved!

Part III:

Date

Question #4:

a.

$$D_u f(x_0,y_0) = \nabla f(x_0,y_0)\hat{u}$$

$$D_u f(0,0) = \nabla f(0,0)(1,2) = 1$$

$$D_u f(0,0) = \nabla f(0,0)(2,1) = 2$$

$$f_x(0,0) = ? \quad , \quad f_y(0,0) = ?$$

Sol:

∵ First we have to make both vectors to unit vectors

$$u_1 = \frac{i+2j}{\sqrt{1+4}} \quad , \quad u_2 = \frac{2i+j}{\sqrt{1+4}}$$

$$u_1 = \frac{i}{\sqrt{5}} + \frac{2}{\sqrt{5}}j \quad , \quad u_2 = \frac{2i}{\sqrt{5}} + \frac{j}{\sqrt{5}}$$

$$\Rightarrow f_x\hat{i} + f_y\hat{j} = 1 \quad , \quad \Rightarrow f_x i + f_y j = 2$$

$$f_x\left(\frac{1}{\sqrt{5}}\right) + f_y\left(\frac{2}{\sqrt{5}}\right) = 1 \qquad f_x\left(\frac{2}{\sqrt{5}}\right) + f_y\left(\frac{1}{\sqrt{5}}\right) = 2$$

$$f_x + 2f_y = \sqrt{5} \quad \text{(i)} \qquad 2f_x + f_y = 2\sqrt{5} \quad \text{(ii)}$$

Multiply whole eqn by (2)

Subtracting:

$$2f_x + 4f_y = 2\sqrt{5}$$

$$\underline{-2f_x - f_y = -2\sqrt{5}}$$

$$\Rightarrow 3f_y = 0$$

$$f_y = 0$$

$$\text{(i)} \Rightarrow f_x + 2(0) = \sqrt{5}$$

$$f_x = \sqrt{5}$$

Hence

$$\boxed{\begin{aligned} f_x(0,0) &= \sqrt{5} \\ f_y(0,0) &= 0 \end{aligned}}$$

Ans.

b. $z = f(x,y)$

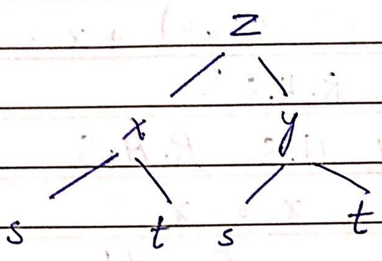
$x = g(s,t)$, $y = h(s,t)$

$g(1,2) = 3$ $\qquad$ $h(1,2) = 6$

$g_s(1,2) = -1$, $g_t(1,2) = 4$ $\qquad$ $h_s(1,2) = -5$, $h_t(1,2) = 10$

$$f_x(3,6) = 7 \quad \& \quad f_y(3,6) = 8$$

$\left.\dfrac{\partial z}{\partial s}\right|_{\substack{s=1\\t=2}} = ?$, $\left.\dfrac{\partial z}{\partial t}\right|_{\substack{s=1\\t=2}} = ?$



$\because \dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y} \dfrac{\partial y}{\partial s}$

$\dfrac{\partial z}{\partial s}_{(1,2)} = f_x(3,6) g_s(1,2) + f_y(3,6) h_s(1,2)$

$\dfrac{\partial z}{\partial s}_{(1,2)} = (7)(-1) + (8)(-5)$

$\dfrac{\partial z}{\partial s}_{(1,2)} = -7 - 40$

$\boxed{\dfrac{\partial z}{\partial s}_{(1,2)} = -47}$ Ans.

$\because \dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x} \dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y} \dfrac{\partial y}{\partial t}$

$\dfrac{\partial z}{\partial t}_{(1,2)} = f_x(3,6) g_t(1,2) + f_y(3,6) h_t(1,2)$

$\dfrac{\partial z}{\partial t}_{(1,2)} = 7(4) + 8(10)$

$\boxed{\dfrac{\partial z}{\partial t}_{(1,2)} = 108}$ Ans.

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c. $z = y + f(x^2 - y^2)$

Prove:

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$$

L.H.S:

$$\Rightarrow y\left[0 + f'(x^2-y^2).2x\right] + x\left[1 + f'(x^2-y^2)(-2y)\right]$$

$$\Rightarrow 2xy\, f'(x^2-y^2) + x - 2xy\, f'(x^2-y^2)$$

$$\Rightarrow x = R.H.S$$

$\because$ L.H.S = R.H.S

$x = x$

Hence Proved!

Question # 5:

a. $f(x,y,z) = x^3\sqrt{y^2+z^2}$ at $(2,3,4)$

Find linear approximation

estimate:

$$(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$$

Sol:

$$\because L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$\Rightarrow = 2^3\sqrt{3^2+4^2} + 3x_0^2\sqrt{y_0^2+z_0^2}(x-2) + \frac{x_0^3\, 2y_0}{2\sqrt{y_0^2+z_0^2}}(-3+y) + \frac{x_0^3\, 2z_0}{2\sqrt{y_0^2+z_0^2}}(z-4)$$

$$\Rightarrow = 8\sqrt{25} + 3(2^2)\sqrt{3^2+4^2}(x-2) + \frac{8(3)}{\sqrt{3^2+4^2}}(y-3) + \frac{8(4)}{\sqrt{3^2+4^2}}(z-4)$$

at $(x,y,z) = (1.98, 3.01, 3.97)$

$$\Rightarrow = 40 + 60(1.98-2) + \frac{24}{5}(3.01-3) + \frac{32}{5}(3.97-4)$$

$\Rightarrow \boxed{L = 38.656}$ Ans.

$f(1.98, 3.01, 3.97) = 38.672 \approx L = 38.656$ Ans.

Date

b. $0 \le x \le 10$ $\quad$ x, y : meters

$0 \le y \le 8$ $\quad$ $T(x,y)$: celsius

i. $T_x(6,4)$: ?

$T_y(6,4)$: ?

$$\therefore T_x(6,4) = \lim_{\Delta v \to 0} \frac{T(6+\Delta v, 4) - T(6,4)}{\Delta v}$$

For $\Delta v = +2$

$$T_x(6,4) \approx \frac{T(6+2,4) - T(6,4)}{2}$$

$$T_x(6,4) \approx \frac{86-80}{2}$$

$$T_x(6,4) \approx 3$$

For $\Delta v = -2$

$$T_x(6,4) \approx \frac{T(6-2,4) - T(6,4)}{-2}$$

$$T_x(6,4) \approx \frac{72-80}{-2}$$

$$T_x(6,4) \approx 4$$

$$\therefore \boxed{T_x(6,4) \approx \frac{3+4}{2} = 3.5 \ ^\circ C/m}$$ Ans.

$$\therefore T_y(6,4) = \lim_{\Delta v \to 0} \frac{T(6, 4+\Delta v) - T(6,4)}{\Delta v}$$

$$\Rightarrow T_y(6,4) \approx \frac{T(6,4+2) - T(6,4)}{2}$$

$$T_y(6,4) \approx \frac{75-80}{2}$$

$$T_y(6,4) \approx -2.5$$

$$\Rightarrow T_y(6,4) \approx \frac{T(6,4-2) - T(6,4)}{-2}$$

$$T_y(6,4) \approx \frac{87-80}{-2}$$

$$T_y(6,4) \approx -3.5$$

$$\boxed{T_y(6,4) \approx \frac{-2.5-3.5}{2} = -3 \ ^\circ C/m}$$ Ans.

Date

ii. $D_u T(6,4)$; $u = (i+j)/\sqrt{2}$

$\because D_u T(6,4) = f_x(6,4)u_i + f_y(6,4)u_j$

From previous part: we get f_x & f_y at $(6,4)$

$\Rightarrow D_u T(6,4) = 3.5(1/\sqrt{2}) + (-3)(1/\sqrt{2})$

$D_u T(6,4) = \frac{1}{2\sqrt{2}}$

$\boxed{D_u T(6,4) \approx 0.353\ °C/m}$ Ans.

We conclude that if we move a small distance from the point (6,4) in the direction of u, the function T(x,y), (i.e temperature) will increase at a rate of 0.353 °C/m

iii. $T_{xy}(6,4)$

From part(i) we have,

$T_x(6,4) \approx 3.5\ °C/m$

$\because T_{xy}(6,4) = \lim_{\Delta v \to 0} \frac{T_x(6, 4+\Delta v) - T_x(6,4)}{\Delta v}$

at $\Delta v = 2$

$T_{xy}(6,4) \approx \frac{T_x(6,6) - T_x(6,4)}{\Delta v}$ — (A)

at $\Delta v = -2$

$T_{xy}(6,4) \approx \frac{T_x(6,2) - T_x(6,4)}{-2}$ — (B)

For $T_x(6,6)$:

[Right-hand]

$\Rightarrow T_x(6,6) \approx \frac{T(6+2,6) - T(6,6)}{2}$

$T_x(6,6) \approx \frac{80-75}{2}$

$T_x(6,6) \approx 2.5$

[Left-Hand]

$\Rightarrow T_x(6,6) \approx \frac{T(6-2,6) - T(6,6)}{2}$

$T_x(6,6) \approx \frac{68-75}{2}$

$T_x(6,6) \approx 3.5$

Date

$\Rightarrow T_x(6,6) \approx \frac{2.5+3.5}{2} = 3\ °C/m$

For $T_x(6,2)$:

[Right-hand]

$\Rightarrow T_x(6,2) \approx \frac{T(6+2,2) - T(6,2)}{2}$

$T_x(6,2) \approx \frac{90-87}{2}$

$T_x(6,2) \approx 1.5$

[Left-Hand]

$\Rightarrow T_x(6,2) \approx \frac{T(6-2,2) - T(6,2)}{-2}$

$T_x(6,2) \approx \frac{74-87}{-2}$

$T_x(6,2) \approx 6.5$

$\Rightarrow T_x(6,2) \approx \frac{1.5+6.5}{2} = 4\ °C/m$

Now,

(A) $\Rightarrow T_{xy}(6,4) \approx \frac{3-3.5}{2}$

$T_{xy}(6,4) \approx -0.25$

(B) $\Rightarrow T_{xy}(6,4) \approx \frac{4-3.5}{-2}$

$T_{xy}(6,4) \approx -0.25$

$\Rightarrow \boxed{T_{xy}(6,4) \approx \frac{-0.25-0.25}{2} = -0.25\ °C/m^2 \text{ Ans.}}$

iv. linear approximation of $T(x,y)$ near $(6,4)$:!

Sol:

$\because L = T(x_0,y_0) + T_x(x_0,y_0)(x-x_0) + T_y(x_0,y_0)(y-y_0)$

From previous parts,

$L = T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4)$

$L = 80 + (3.5)(5-6) + (-3)(3.8-4)$; $\because (x,y) = (5, 3.8)$

$L = 80 + 3.5(5-6) - 3(3.8-4)$

$\boxed{L = 77.1\ °C}$ Ans.

Date

Question #6:

a. $x = 5m$ $\quad \Delta x = 0.002\ m$

$y = 12m$ $\quad \Delta y = 0.002\ m$

i. Max error in area:

$$\because A = \frac{1}{2}xy \longrightarrow dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$

$$\Rightarrow dA = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$$

$$dA = \frac{1}{2}(12)(0.002) + \frac{1}{2}(5)(0.002)$$

$$\boxed{dA = 0.017\ m^2}$$ Ans.

ii. Max error in hypotenus:

By pythagorus theorem:

$$H = \sqrt{x^2+y^2}$$

$$\boxed{dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy}$$

$$dH = \frac{2x\,dx}{2\sqrt{x^2+y^2}} + \frac{2y\,dy}{2\sqrt{x^2+y^2}}$$

$$dH = \frac{5(0.002)}{\sqrt{5^2+12^2}} + \frac{12(0.002)}{\sqrt{5^2+12^2}}$$

$$dH = 2.615\times10^{-3}$$

$$\boxed{dH = 0.002615\ m}$$ Ans.

Date ____________

b. $\frac{dx}{dt} = +3 \text{ in/s}$, $\frac{dy}{dt} = -2 \text{ in/s}$, $\frac{d\theta}{dt} = 0.05 \text{ rad/s}$

$\left[\frac{dA}{dt}\right]_{\substack{x=40 \\ y=50 \\ \theta=\pi/6}} : ?$ $\quad \because A = \frac{1}{2} xy\sin\theta$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} y\sin\theta(3) + \frac{1}{2} x\sin\theta(-2) + \frac{1}{2} xy\cos\theta\,(0.05)$$

$$\frac{dA}{dt} = \frac{3}{2}(50)\sin\frac{\pi}{6} - 40\sin\frac{\pi}{6} + \frac{(40)(50)}{2}\cos\frac{\pi}{6}(0.05)$$

$$\frac{dA}{dt} = 3.25\left(\frac{1}{2}\right) - 40\left(\frac{1}{2}\right) + 50\left(\frac{\sqrt{3}}{2}\right)$$

$$\boxed{\left[\frac{dA}{dt}\right]_{\substack{x=40 \\ y=50 \\ \theta=\pi/6}} = 60.801 \text{ in}^2/\text{sec}}$$ Ans.

Question #7:

a. $f(x,y,z) = ze^{xy}$; $P(0,1,2)$

direction : ? $\quad |\nabla F| : ?$

Sol:

$$\nabla F = f_x(x,y,z)\hat{i} + f_y(x,y,z)\hat{j} + f_z(x,y,z)\hat{k}$$

$$\nabla F = (ze^{xy}.y)\hat{i} + (ze^{xy}.x)\hat{j} + e^{xy}\hat{k}$$

$$\nabla F(0,1,2) = 2e^{0}(1)\hat{i} + 2e^{0}(0)\hat{j} + e^{0}\hat{k}$$

$$\nabla F(0,1,2) = 2\hat{i} + \hat{k}$$

Direction:

$2\hat{i} + \hat{k}$ OR $(2,0,1)$ Ans.

$$|\nabla F| = \sqrt{2^2+0^2+1^2}$$

$$\boxed{|\nabla F| = \sqrt{5}}$$ max rate of increase. Ans.

Date

6. $f(x,y,z) = |r|^{-n}$; $r = xi + yj + zk$

Prove:

$$\nabla f = \frac{-nr}{|r|^{n+2}}$$

Sol:

$\because |r| = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow f(x,y,z) = \left(\sqrt{x^2+y^2+z^2}\right)^{-n}$$

$$f(x,y,z) = (x^2+y^2+z^2)^{-n/2}$$

$$\because \nabla F = f_x(x,y,z)\hat{i} + f_y(x,y,z)\hat{j} + f_z(x,y,z)\hat{k}$$

$$\rightarrow f_x(x,y,z) = \frac{-n}{2}(x^2+y^2+z^2)^{-n/2-1}\cdot(2x)$$

$$f_x(x,y,z) = \frac{-nx}{(x^2+y^2+z^2)^{(n+2)/2}}$$

$$\rightarrow f_y(x,y,z) = \frac{-n}{2}(x^2+y^2+z^2)^{-n/2-1}(2y)$$

$$f_y(x,y,z) = \frac{-ny}{(x^2+y^2+z^2)^{(n+2)/2}}$$

$$\rightarrow f_z(x,y,z) = \frac{-n}{2}(x^2+y^2+z^2)^{-n/2-1}(2z)$$

$$f_z(x,y,z) = \frac{-nz}{(x^2+y^2+z^2)^{(n+2)/2}}$$

$$\rightarrow \nabla F = \frac{-nx\hat{i}}{(x^2+y^2+z^2)^{(n+2)/2}} - \frac{ny\hat{j}}{(x^2+y^2+z^2)^{(n+2)/2}} - \frac{nz\hat{k}}{(x^2+y^2+z^2)^{(n+2)/2}}$$

$$\nabla F = \frac{-n(x\hat{i}+y\hat{j}+z\hat{k})}{\left[(x^2+y^2+z^2)^{1/2}\right]^{(n+2)}}$$

$\because r = xi + yj + zk$

$|r| = \sqrt{x^2+y^2+z^2}$

$$\boxed{\nabla F = \frac{-nr}{|r|^{n+2}}}$$

Hence proved!

Date

c. $f(x,y,z) = x^2 e^{yz^2}$; $\nabla F = ?$

Sol:

$\Rightarrow \nabla F = f_x(x,y,z)\hat{i} + f_y(x,y,z)\hat{j} + f_z(x,y,z)\hat{k}$

$\rightarrow f_x(x,y,z) = 2x e^{yz^2}$

$\rightarrow f_y(x,y,z) = x^2 e^{yz^2} . z^2$

$\rightarrow f_z(x,y,z) = x^2 e^{yz^2} . 2yz$

Now,

$\Rightarrow \boxed{\nabla f = 2xe^{yz^2}\hat{i} + x^2z^2e^{yz^2}\hat{j} + 2x^2yze^{yz^2}\hat{k}}$ Ans.

$\nabla f = < 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2} >$

i. When is the directional derivative maximum:

Ans:

$D_uf(x,y,z)$ is maximum when ∇f and u are in same direction i.e $\theta = 0°$

$D_uf(x,y,z) = \nabla f . u$

$D_uf(x,y,z) = |\nabla f||u| \cos\theta$

$D_uf(x,y,z) = |\nabla f|(1) \cos 0°$

$D_uf(x,y,z) = |\nabla f|$ (max)

ii. When is D_uf minimum:

Ans:

$D_uf(x,y,z)$ is minimum when ∇f and u are antiparallel i.e $\theta = 180°$

$D_uf(x,y,z) = |\nabla f||u| \cos 180°$

$D_uf(x,y,z) = -|\nabla f|$ (min)

iii. When is D_uf equal to zero:

Ans: $D_uf(x,y,z)$ is zero when ∇f and u are perpendicular i.e $\theta = 90°$

$D_uf(x,y,z) = |\nabla f||u| \cos 90°$

$D_uf(x,y,z) = 0$ (zero)

Date

iv. When is $D_uf(x,y,z)$ half of its maximum value:

Ans:

$D_u'f(x,y,z) = \frac{1}{2} D_uf_{max}$

$|\nabla f||u| \cos\theta = \frac{1}{2} |\nabla f|$

$(1) \cos\theta = \frac{1}{2}$

$\cos\theta = 1/2$

$\theta = \cos^{-1}(1/2)$

$\boxed{\theta = \frac{\pi}{3}}$ or $\theta = 60°$

$D_uf(x,y,z)$ is half of its maximum value when θ between ∇f and u is $60°$.