MVC (Multi - Variable Calculus)

13.1

Multi - Variables

$f(x) = x^2 + 2x + 3$

$f(0.5) = 0.5^2 + 2(0.5) + 3$

$f(x,y) = x^2 + y^2 + xy^2$

$f(1,e) = (1)^2 + (e)^2 + (1)(e)^2$

$\boxed{f(1,e) = 1 + 2e^2}$

$f(t,t^2) = t^2 + t^4 + t\,(t^2)^2$

$\boxed{f(t,t^2) = t^2 + t^4 + t^5}$

Q5 $F(g(x), h(y)) = ?$

$F(x,y) = x\,e^{xy}$, $g(x) = x^3$, $h(y) = 3y+1$

$F(g(x), h(y)) = F(x^3, 3y+1)$

$= (x^3)\,e^{(x^3)(3y+1)} = \boxed{x^3\,e^{3x^3y + x^3}}$ Ans.

Q3 $f(x,y) = xy + 3$

$f(\overbrace{x+y}^{x}, \overbrace{x-y}^{y}) = ?$

$f(x+y, x-y) = (x+y)(x-y) + 3 = x^2 + \cancel{xy} - \cancel{xy} - y^2 + 3$

$= \boxed{x^2 - y^2 + 3}$ Ans

Q2 $f(x,y) = x + \sqrt[3]{xy}$

$f(2y^2, 4y) = 2y^2 + \sqrt[3]{(2y^2)(4y)} = 2y^2 + \sqrt[3]{8y^3}$

$= \boxed{2y^2 + 2y}$ Ans.

Q6 $F(\overbrace{u(x,y)}^{x}, \overbrace{v(x,y)}^{y}) = ?$

$F(x,y) = y \sin(x^2 y)$

$u(x,y) = x^2 y^3 \qquad v(x,y) = \pi x y$

$\rightarrow F(x^2y^3, \pi x y) = \pi x y \sin((x^2y^3)^2(\pi x y))$

$= \pi x y \sin(x^4 y^6 \cdot (\pi x y)) = \boxed{\pi x y \sin(\pi x^5 y^7)}$ Ans

Q8 $g(x,y) = y e^{-3x}$

$x(t) = t^2, \quad y(t) = t^3$

$g(x(t), y(t)) = g(t^2, t^3) = t^3 e^{-3(t^2)} = \boxed{e^{-3t^2} t^3}$ Ans.

Q17 $f(x,y,z) = x y^2 z^3 + 3$

1) $f(t, t^2, -t) = t \cdot (t^2)^2 (-t)^3 + 3 = \boxed{-t^8 + 3}$ Ans.

2) $f(a+b, a-b, b) = (a+b)(a-b)^2 (b)^3 + 3$

$= \boxed{(a+b)(a^2 - 2ab + b^2)(b^3) + 3 \ \vdots}$ Ans

Q19 $F(f(x), g(y), h(z)) = ?$

$F(x,y,z) = y e^{xyz}, \quad f(x) = x^2, \quad g(y) = y+1, \quad h(y) = z^2$

$F(x^2, y+1, z^2) = \boxed{(y+1) e^{(x^2)(y+1)(z^2)}}$ Ans.

Q20 $g(u(x,y,z), v(x,y,z), w(x,y,z)) = ?$

Q20 $g(u(x,y,z),\ v(x,y,z),\ w(x,y,z)) = ?$

$g(x,y,z) = z \sin xy$, $u(x,y,z) = x^2 z^3$

$v(x,y,z) = \pi xyz$, $w(x,y,z) = xy/z$

$$g(x^2z^3,\ \pi xyz,\ xy/z) = \frac{xy}{z} \sin\left((x^2z^3)(\pi xyz)\right)$$

$$= \boxed{\frac{xy}{z} \cdot \sin(\pi x^3 y z^4)} \text{ Ans.}$$

Q18 $f(x) = xyz + x$

a) $f(x+y,\ x-y,\ x^2) = ?$

$$f((x+y)(x-y)(x^2) + (x+y)) = x^4 - \cancel{x^3y} + \cancel{x^3y} - x^2y + x + y$$

$$= \boxed{x^4 - x^2y + x + y} \text{ Ans}$$

13.1

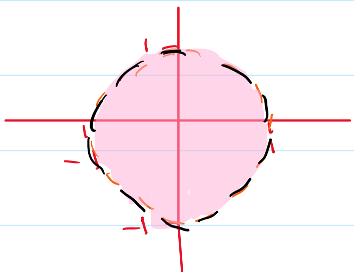
Domain

Q23 $f(x,y) = \ln(1 - x^2 - y^2)$

$$1 - x^2 - y^2 > 0$$

$$x^2 + y^2 < 1^2$$

so dotted line because of '<' sign.

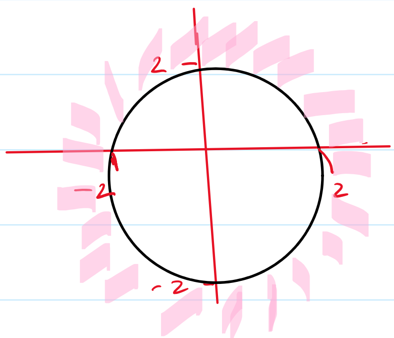


Q24 $f(x,y) = \sqrt{-4 + x^2 + y^2}$

$$x^2 + y^2 - 4 \geq 0$$

$$x^2 + y^2 \geq 2^2$$

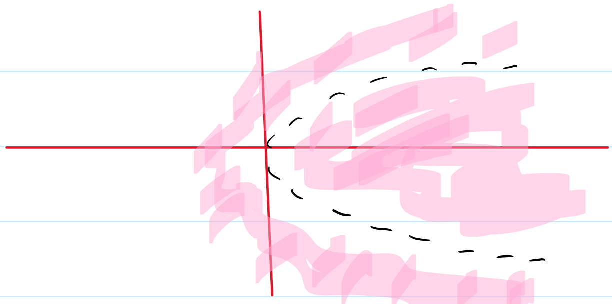
so solid line '≥'



Q25 $f(x,y) = \dfrac{1}{x - y^2}$

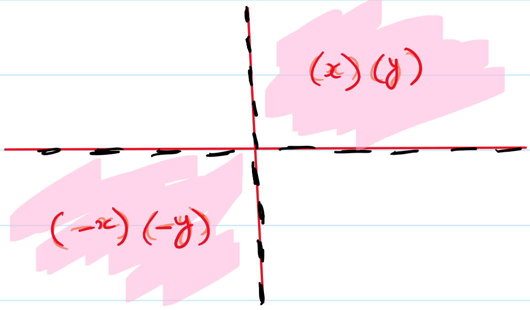
$$x - y^2 \neq 0$$

$x \neq y^2 \rightarrow$ so a parabola on x-axis.



Q 26 $f(x,y) = \ln(xy)$

$xy > 0$



- In donon cases main product greater than zero rahega.
- Dotted line b/c zero is not included.

Q 27 a) $f(x,y) = x\,e^{-\sqrt{y+2}}$

$\rightarrow \sqrt{y+2} \geq 0$

$y + 2 \geq 0$

$\boxed{y \geq -2}$

In words

All values in 2-space that lies on or above the line $\boxed{y = -2}$. Ans

b) $f(x,y,z) = \sqrt{25 - x^2 - y^2 - z^2}$

$$25 - x^2 - y^2 - z^2 \geq 0$$
$$-x^2 - y^2 - z^2 \geq -25$$
$$\boxed{x^2 + y^2 + z^2 \leq 25}$$

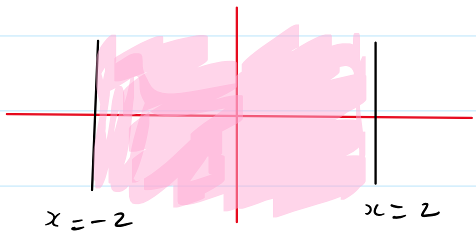
All values inside or on the boundary of sphere in 3-space.

c) $f(x,y,z) = e^{xyz}$

- All values in 3-space

Q28 a) $f(x,y) = \dfrac{\sqrt{4-x^2}}{y^2+3}$

$$\sqrt{4-x^2} \geq 0$$
$$4 - x^2 \geq 0$$
$$-x^2 \geq -4$$
$$x^2 \leq 4$$
$$x \leq \pm 2$$



All values in 2-space which lies on or b/w $x=2$ and $x=-2$

$\boxed{x \leq 2}$ $\boxed{x \geq -2}$

b) $f(x,y) = \ln(y - 2x)$

$$y - 2x > 0$$
$$\boxed{y > 2x}$$

straight line

All values in 2-space that lies above line $y = 2x$.

c) $f(x,y,z) = \dfrac{xyz}{x+y+z}$

$$x + y + z \neq 0$$

→ All values in 3-space that does not lies on the boundary of the plane.

Parabola → $y^2 = 4ax$ / $x^2 = 4ay$

Ellipse → $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbola → $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$

Circle → $x^2 + y^2 = a^2$

43–44 In each part, select the term that best describes the level curves of the function f. Choose from the terms lines, circles, noncircular ellipses, parabolas, or hyperbolas.

43. (a) $f(x, y) = 5x^2 - 5y^2$ (b) $f(x, y) = y - 4x^2$
(c) $f(x, y) = x^2 + 3y^2$ (d) $f(x, y) = 3x^2$

44. (a) $f(x, y) = x^2 - 2xy + y^2$ (b) $f(x, y) = 2x^2 + 2y^2$
(c) $f(x, y) = x^2 - 2x - y^2$ (d) $f(x, y) = 2y^2 - x$

43) a) Hyperbola b) Parabola

c) Ellipse d) suppose → $3x^2 = 3$, $x^2 = 1$, $x = \pm 1$ so lines.

44) a) $(x - y)^2 = 4$

$x - y = \pm 2$

so lines

b) Circle → $2x^2 + 2y^2 = 2$

$x^2 + y^2 = 1$

c) $x^2 - 2x - y^2$

$x^2 - 2(1)(x) + 1 - 1 - y^2$

$(x - 1)^2 - y^2 = 5$

$\frac{(x-1)^2}{5} - \frac{y^2}{5} = 1$

so hyperbola

d) $2y^2 - x = 2$

$2y^2 = 2 + x$

Parabola

2 dimensions $\rightarrow$ level curve

3 dimensions $\rightarrow$ level surface.

51–56 Sketch the level curve $z = k$ for the specified values of k.

51. $\underbrace{z = x^2 + y^2}_{\text{circle}}$; $k = 0, 1, 2, 3, 4$

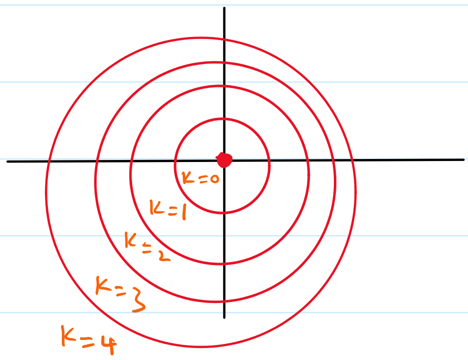
$k=0 \rightarrow$ point

$k=1 \rightarrow$ circle with $r=1$

$k=2 \rightarrow$ circle with $r=\sqrt{2}$

$k=3 \rightarrow$ " " $r=\sqrt{3}$

$k=4 \rightarrow$ circle with $r=2$



52. $z = y/x$; $k = -2, -1, 0, 1, 2$

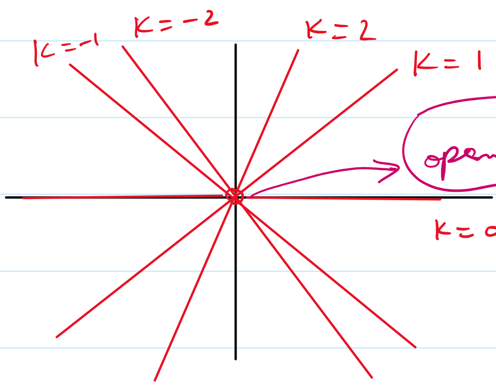
$k=-2 \rightarrow -2 = \frac{y}{x} \rightarrow y = -2x$

$k=-1 \rightarrow y = -x$

$k=0 \rightarrow y = 0$

$k=1 \rightarrow y = x$

$k=2 \rightarrow y = 2x$



open circle b/c $\frac{y}{x}$ so x can't be zero.

53. $z = x^2 + y$; $k = -2, -1, 0, 1, 2$

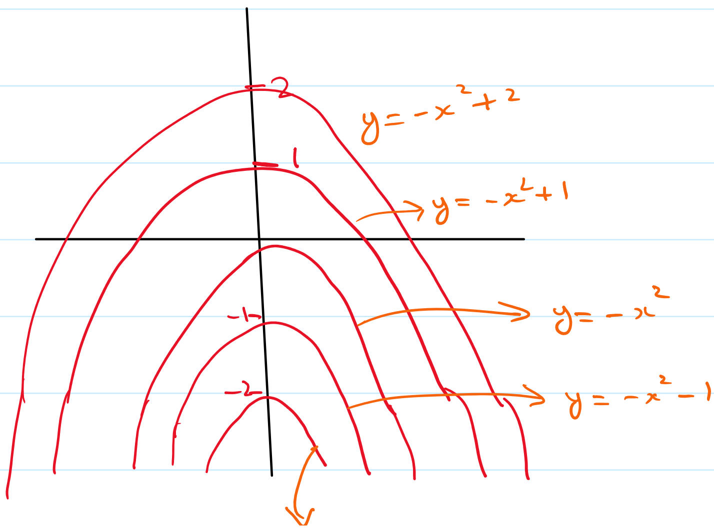
$k=-2 \longrightarrow y = -2 - x^2$

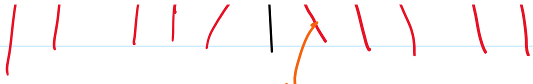
$k=-1 \longrightarrow y = -1 - x^2$

$k=0 \longrightarrow y = -x^2$

$k=1 \longrightarrow y = 1 - x^2 = -x^2+1$

$k=2 \longrightarrow y = 2 - x^2$





$y = -x^2 - 2$

54. $z = x^2 + 9y^2;\ k = 0, 1, 2, 3, 4$

Ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

For $a > b$, upon x-axis

For $b > a$, upon y-axis.

$k = 0 \rightarrow x^2 + 9y^2 = 0$

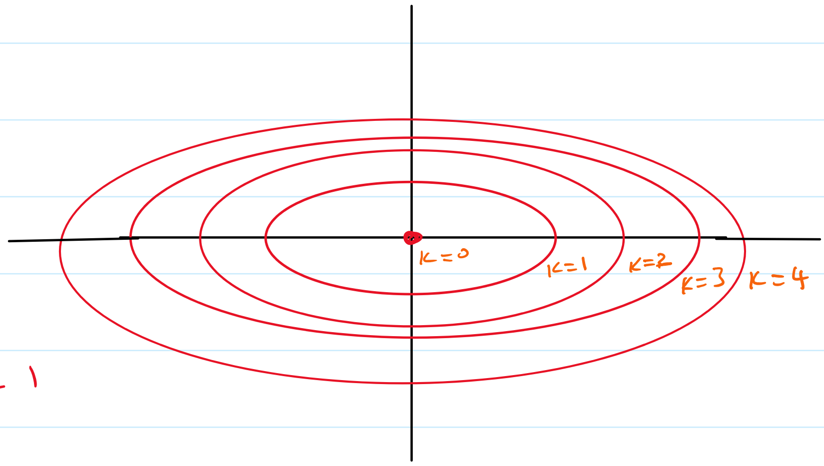
$k = 0 \rightarrow \dfrac{x^2}{0} + \dfrac{y^2}{\left(\frac{1}{3}\right)^2_0} = 0$

$k = 1 \rightarrow \dfrac{x^2}{1^2} + \dfrac{y^2}{\left(\frac{1}{3}\right)^2} = 1$

$k = 2 \rightarrow \dfrac{x^2}{2} + \dfrac{9y^2}{2} = \dfrac{2}{2}$

$\dfrac{x^2}{(\sqrt{2})^2} + \dfrac{y^2}{\left(\frac{\sqrt{2}}{3}\right)^2} = 1$

$k = 3 \rightarrow \dfrac{x^2}{(\sqrt{3})^2} + \dfrac{y^2}{\left(\frac{\sqrt{3}}{3}\right)^2} = 1$



55. $z = x^2 - y^2;\ k = -2, -1, 0, 1, 2$

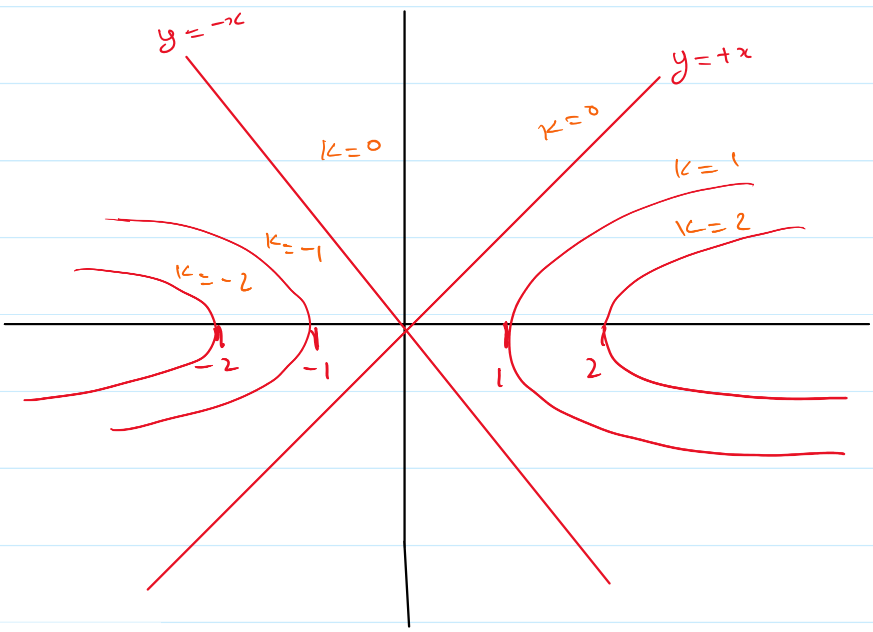
$k = 0 \rightarrow x^2 - y^2 = 0$

$x^2 = y^2$

$y = \pm x$

$k = 1 \rightarrow x^2 - y^2 = 1$

$k = 2 \rightarrow x^2 - y^2 = 2$



56. $z = y \csc x;\ k = -2, -1, 0, 1, 2$

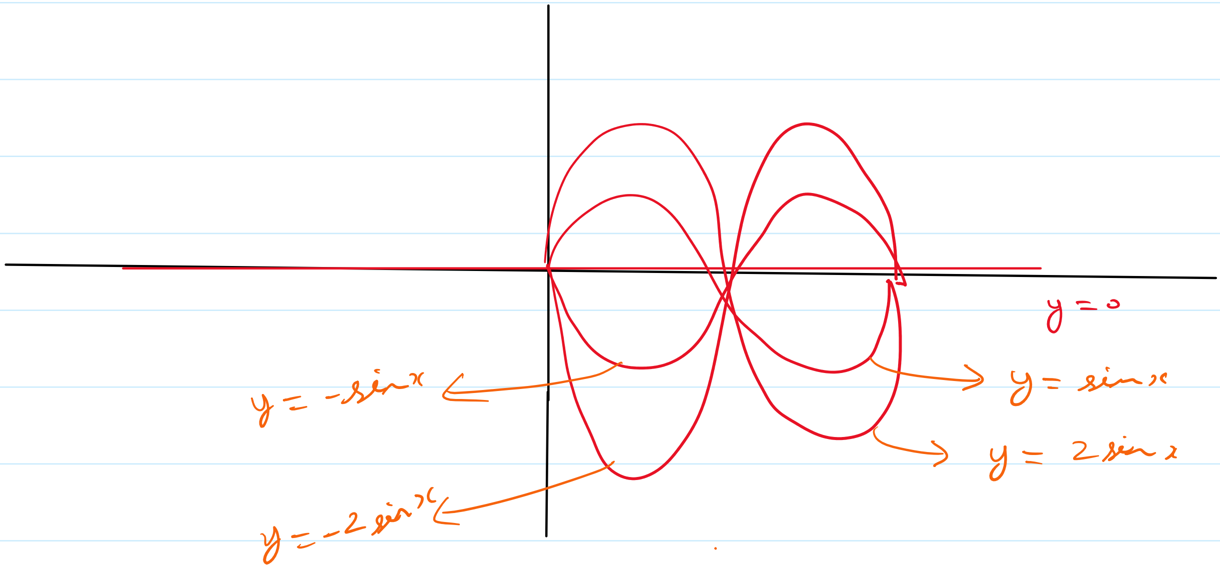
$k=0 \rightarrow 0 = \frac{y}{\sin x} \rightarrow y = 0$

$k=1 \rightarrow 1 = \frac{y}{\sin x} \rightarrow y = \sin x$

$k=2 \rightarrow 2 = \frac{y}{\sin x} \rightarrow y = 2\sin x$

$k=-1 \rightarrow y = -\sin x$

$k=-2 \rightarrow y = -2\sin x$



Monday, 29 January 2024 10:07 am

Level Surfaces

57–60 Sketch the level surface $f(x, y, z) = k$.

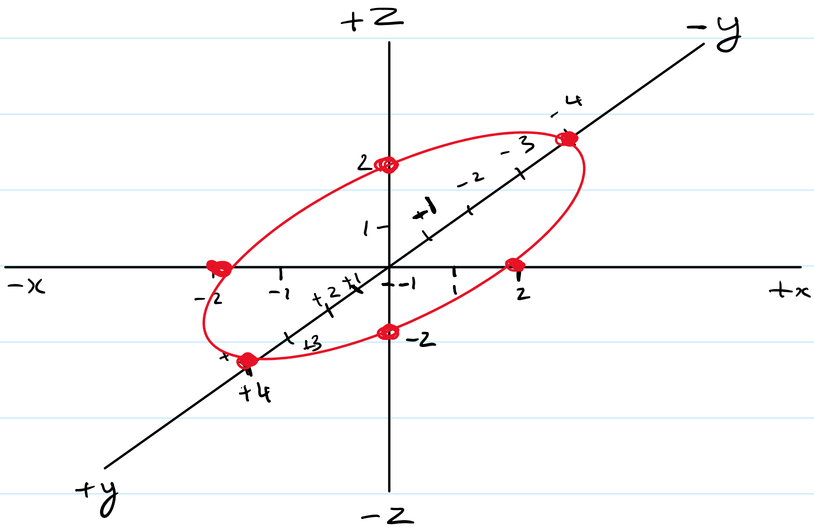
57. $f(x, y, z) = 4x^2 + y^2 + 4z^2$; $k = 16$

$$\frac{16}{16} = \frac{4x^2}{16} + \frac{y^2}{16} + \frac{4z^2}{16}$$

$$1 = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{4}$$

$$\frac{x^2}{(2)^2} + \frac{y^2}{(4)^2} + \frac{z^2}{(2)^2} = 1$$

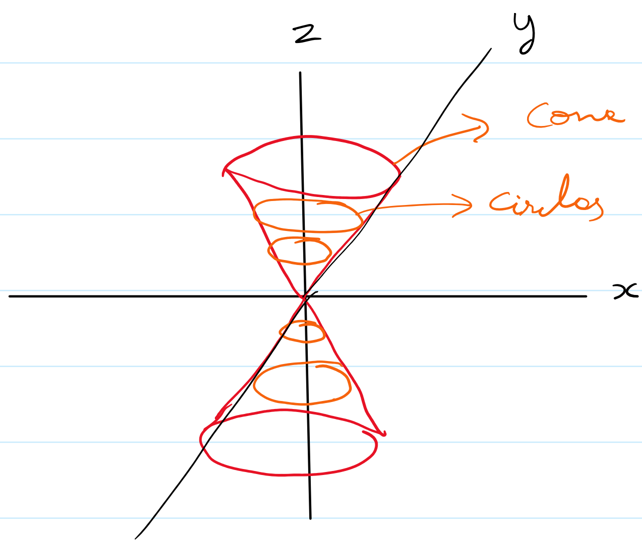
Ellipsoid



58. $f(x, y, z) = x^2 + y^2 - z^2$; $k = 0$

$x^2 + y^2 - z^2 = 0 \longrightarrow$ cone

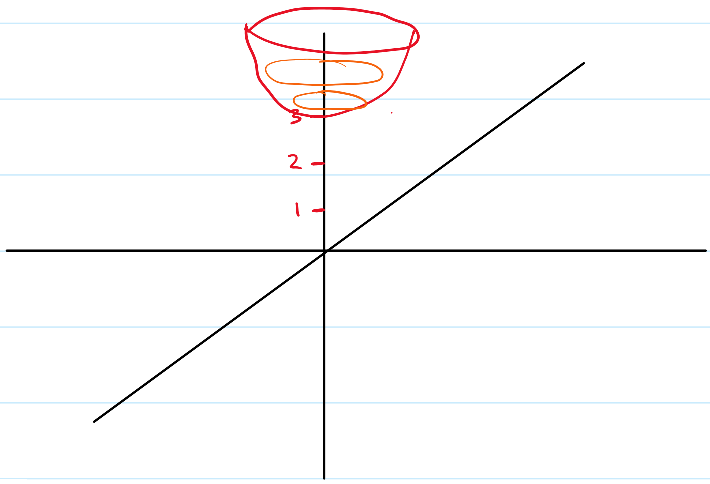
$x^2 + y^2 = z^2 \longrightarrow$ inside circle when $k = 0$



59. $f(x, y, z) = \underbrace{z - x^2 - y^2}_{\text{Paraboloid.}} + 4$; $k = 7$

$7 = z - x^2 - y^2 + 4$

$z - \underbrace{3}_{\text{translation.}} = \underbrace{x^2 + y^2}_{\text{Circle inside.}}$



61–64 Describe the level surfaces in words.

61–64 Describe the level surfaces in words.

61. $f(x, y, z) = (x - 2)^2 + y^2 + z^2$

Sphere in 3-D

62. $f(x, y, z) = 3x - y + 2z$

All planes in 3-D

63. $f(x, y, z) = x^2 + z^2$

The level surfaces for $f(x, y, z)$ result

$$f(x, y, z) = k, \quad k \in \mathcal{R}$$

Since $f(x, y, z) = x^2 + z^2$ result the level surface:

$$x^2 + z^2 = k$$

We can note that $x^2 + z^2 \geq 0$ for any reals x, y, z, then $k \geq 0$. Since $k \geq 0$ and,

$$x^2 + z^2 = k$$

do not depend on y, the surface represent cylinders whith simmetric axis $axis\ y$

64. $f(x, y, z) = z - x^2 - y^2$

$z - x^2 - y^2 = K$

$z - K = x^2 + y^2$

A circular parabolid.

13.2 ~ Limits & Continuity

1–6 Use limit laws and continuity properties to evaluate the limit.

1. $\lim_{(x,y)\to(1,3)} (4xy^2 - x)$ **2.** $\lim_{(x,y)\to(0,0)} \dfrac{4x - y}{\sin y - 1}$

1) $\lim_{(x,y)\to(1,3)} (4(1)(3)^2 - (1))$

$36 - 1 = \boxed{35}$ Ans.

2) $\lim_{(x,y)\to(0,0)} \dfrac{4(0) - 0}{\sin 0 - 1} = \dfrac{0}{-1}$

$\boxed{0}$ Ans

3. $\lim \dfrac{xy^3}{\ldots}$ **4.** $\lim e^{2x - y^2}$

3. $\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y}$

4. $\lim_{(x,y)\to(1,-3)} e^{2x-y^2}$

3) $\lim_{(x,y)\to(-1,2)} \frac{(-1)(2)^3}{(-1)+2}$

$\frac{-8}{1} = \boxed{-8}$ Ans

4) $\lim_{(x,y)\to(1,-3)} e^{2(1)-(-3)^2}$

e^{2-9}

$\boxed{\frac{1}{e^7}}$

5. $\lim_{(x,y)\to(0,0)} \ln(1+x^2y^3)$

6. $\lim_{(x,y)\to(4,-2)} x\sqrt[3]{y^3+2x}$

5) $\lim_{(x,y)\to(0,0)} \ln(1+(0)(0))$

$\ln(1) = \boxed{0}$ Ans

6) $\lim_{(x,y)\to(4,-2)} (4)\sqrt[3]{(-2)^3+2(4)}$

$(4)\sqrt[3]{-8+8}$

$\boxed{0}$ Ans.

7–8 Show that the limit does not exist by considering the limits as $(x, y)\to(0, 0)$ along the coordinate axes.

7. (a) $\lim_{(x,y)\to(0,0)} \frac{3}{x^2+2y^2}$ (b) $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x^2+y^2}$

a) Assume $x=0$

$\lim_{y\to 0} \frac{3}{(0)^2+2y^2}$

$\lim_{y\to 0} \frac{3}{2y^2} = \frac{3}{0} = \boxed{\infty}$

so limit D.N.E

b) Assume $y=0$

$\lim_{x\to 0} \frac{x+0}{2x^2+0} = \frac{x}{2x^2}$

$\lim_{x\to 0} \frac{1}{2x} = \frac{1}{0} = \boxed{\infty}$

Limit D.N.E

8. (a) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$ (b) $\lim_{(x,y)\to(0,0)} \frac{\cos xy}{x^2+y^2}$

a) Assume $x=0$

$\lim \frac{0-y}{} = \frac{-\cancel{y}}{} = \frac{-1}{}$

b) $\lim_{y\to 0} \frac{\cos(0)(y)}{0^2+y^2} = \frac{1}{y^2}$

$$\lim_{y \to 0} \frac{0 - y}{0 + y^2} = \frac{-\cancel{y}}{y^{\cancel{2}}} = \frac{-1}{y}$$

$$\lim_{y \to 0} \frac{-1}{0} = \boxed{\infty}$$ Sign Analysis then D.N.E

$$\lim_{y \to 0} \frac{\cos(\ldots)}{0^2 + y^2} = \frac{\ldots}{y^2}$$

$$\lim_{y \to 0} \frac{1}{0} = \boxed{\infty}$$

9–12 Evaluate the limit using the substitution $z = x^2 + y^2$ and observing that $z \to 0^+$ if and only if $(x, y) \to (0, 0)$.

9. $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Assume $z = x^2 + y^2 \rightarrow z = 0^2 + 0^2 = 0$

$$\lim_{z\to 0} \frac{\sin z}{z} = \frac{\cos z}{1} = \frac{\cos 0}{1} = \boxed{1} \text{ Ans.}$$

10. $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

Assume $z = x^2 + y^2$

$$\lim_{z\to 0} \frac{1-\cos z}{z} = \frac{1-\cos 0}{0} = \frac{0}{0}$$

L-Hopital

$$\lim_{z\to 0} \frac{0+\sin z}{1} = \frac{\sin 0}{1} = \boxed{0} \text{ Ans.}$$

13–22 Determine whether the limit exists. If so, find its value.

15. $\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+2y^2} = \frac{0}{0}$

L-hopital can't be applied b/c of two variables

- If indeterminant form, follow below steps.
- First, check on either coordinate
- Secondly, check on line
- Compare the answers.
- If not same, limit does not exist.

Assume $x = 0$

$$\lim_{x\to 0} \frac{(0)\,y}{3(0)^2+2y^2} = \frac{0}{2y^2} = \boxed{0}$$

On line

$y = x$

$$\lim_{y\to 0} \frac{(y)(y)}{3(y)^2+2y^2} = \frac{\cancel{y^2}}{5\cancel{y^2}} = \boxed{\frac{1}{5}}$$

$\boxed{0 \neq 1/5}$ Limit D.N.E.

19. $\displaystyle \lim_{(x,y,z)\to(0,0,0)} \frac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$

Assume $u = x^2+y^2+z^2$

$$\lim_{u\to 0} \frac{\sin u}{\sqrt{u}} = \boxed{\frac{0}{0}}$$

L-Hopital

$$\lim_{u\to 0} \frac{\cos u}{\frac{1}{2\sqrt{u}}} = 2\sqrt{u}\,\cos u = 0 \cos 0 = \boxed{0} \text{ Ans.}$$

21. $\displaystyle \lim_{(x,y,z)\to(0,0,0)} \frac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$

$u = \sqrt{x^2+y^2+z^2}$

$$\lim_{u\to 0} \frac{e^u}{u} = \frac{e^0}{0} = \frac{1}{0} = \boxed{\infty} \text{ Ans.}$$

13. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\,\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = 0 - 0 = \boxed{0} \text{ Ans.}$$

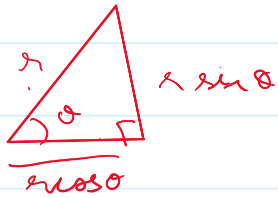
23–26 Evaluate the limits by converting to polar coordinates, as in Example 7.

23. $\displaystyle \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\ln(x^2+y^2)$

$(x, y) \longrightarrow (r, \theta)$

$(x, y) \longrightarrow (r, \theta)$

Cartesian coordinates — Polar Cordinates.



$x = r\cos\theta$

$y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\ln(x^2+y^2)$$

$$\lim_{(x,y)\to(0,0)} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\;\ln\left(r^2\cos^2\theta + r^2\sin^2\theta\right)$$

$$\sqrt{\cancel{r^2} \times 1}\quad \ln(r^2)$$

$$\lim_{r\to 0} r\ln r^2 = \boxed{0\times\infty} \text{ Indeterminat.}$$

L-Hopital

$$\lim_{r\to 0} \frac{\ln r^2}{\frac{1}{r}} = \frac{\frac{1}{\cancel{r^2}}\times 2r}{\frac{-1}{\cancel{r^2}}} = -2r \qquad r^{-1} \quad -1\,r^{-2} \quad \frac{-1}{r^2}$$

$$\lim_{r\to 0} -2r = \boxed{0} \text{ Ans.}$$

24. $\lim\limits_{(x,y)\to(0,0)} y\ln(x^2+y^2)$

$r^2 = x^2+y^2$

$r^2 = 0 + 0$

$r = 0$

$$\lim_{r\to 0} r\sin\theta\,\ln(r^2\cos^2\theta + r^2\sin^2\theta)$$

$$\lim_{r\to 0} r\sin\theta\,\ln(r^2) = 0\times\infty$$

L-hopital

$$\lim_{r\to 0} \frac{\sin\theta\cdot\ln r^2}{\frac{1}{r}} = \frac{\sin\theta\cdot\frac{1}{\cancel{r^2}}\times 2r}{\frac{-1}{\cancel{r^2}}} = -\sin\theta\times 2r = \boxed{0} \text{ Ans.}$$

34. (a) Show that as $(x, y) \to (0, 0)$ along any straight line $y = mx$, or along any parabola $y = kx^2$, the value of

$$\frac{x^3 y}{2x^6 + y^2}$$

approaches 0.

$\rightarrow$ $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x^3 \cdot mx}{2x^6 + (mx)^2} = \frac{mx^4}{2x^6 + m^2x^2} = \frac{0}{0}$$

L-Hopital

$$\frac{4mx^3}{12x^5 + 2m^2x} = \frac{0}{0}$$

$$\frac{12mx^2}{60x^4 + 2m^2} = \frac{0}{0 + 2m^2} = \boxed{0}$$ Ans

$y = Kx^2$

$$\lim_{x\to 0} \frac{x^3 Kx^2}{2x^6 + K^2x^4} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{5Kx^4}{12x^5 + 4K^2x^3} = \frac{5Kx^4}{x^3(12x^2 + 4K^2)}$$

$$\frac{0}{0 + 4K^2} = \frac{0}{4K^2} = \boxed{0}$$

Ans.

(b) Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6 + y^2}$$

does not exist by letting $(x, y) \to (0, 0)$ along the curve $y = x^3$.

$y = x^3$

$$\lim_{x\to 0} \frac{x^3 \cdot x^3}{2x^6 + x^6} = \frac{\cancel{x^6}}{3\cancel{x^6}} = \boxed{\frac{1}{3}}$$ $\rightarrow$ For curve

Now for coordinate

$y = 0$

$$\lim_{x\to 0} \frac{x^3 \cdot 0}{2x^6 + 0} = \frac{0}{2x^6} = \boxed{0}$$

$$\boxed{0 \neq \frac{1}{3}}$$

Limit D.N.E.

Friday, 2 February 2024 10:35 am

13.3 ~ Partial Derivatives

$$z = f(x,y) = x^2 + 2xy + y^2$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x = 2x + 2y + 0$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y = 0 + 2x + 2y$$

1. Let $f(x, y) = 3x^3y^2$. Find
(a) $f_x(x, y)$ (b) $f_y(x, y)$

(g) $f_x(1, 2)$ (h) $f_y($

2. Let $z = e^{2x} \sin y$. Find
(a) $\partial z/\partial x$ (b) ∂z
(d) $\partial z/\partial x|_{(x,0)}$ (e) ∂z

a) $f_x(x,y) = 9x^2y^2$

d) $\partial z/\partial x \big|_{(x,0)} = 2e^{2x} \sin y = 2e^{2x} \sin 0 = \boxed{0}$ Ans

9. $z = \sin(5x^3y + 7xy^2)$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = \cos(5x^3y + 7xy^2) \cdot (15x^2y + 7y^2)$$

$$\frac{\partial z}{\partial y} = \cos(5x^3y + 7xy^2) \cdot (5x^3 + 7x)$$

8. $\dfrac{\partial}{\partial x}(xe^{\sqrt{15xy}}), \dfrac{\partial}{\partial y}(xe^{\sqrt{15xy}})$

$$\frac{\partial z}{\partial x} = e^{\sqrt{15xy}} + xe^{\sqrt{15xy}} \cdot \frac{1}{2\sqrt{15xy}} \times 15y$$

11. Let $f(x, y) = \sqrt{3x + 2y}$.
(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(4, 2)$.
(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(4, 2)$.

a)

$$\frac{\partial z}{\partial x}\bigg|_{(4,2)} = \frac{1}{2\sqrt{3x+2y}} \times 3 + 0 = \frac{3+0}{2\sqrt{12+4}}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{3}{8}}$$ Ans

b) $$\frac{\partial z}{\partial y} = \frac{1}{\cancel{2}\sqrt{3x+2y}} \times 0 + \cancel{2} = \frac{1}{\sqrt{3(4)+2(2)}} = \boxed{\frac{1}{4}}$$ Ans

13. Let $z = \sin(y^2 - 4x)$.

(a) Find the rate of change of z with respect to x at the point (2, 1) with y held fixed.

(b) Find the rate of change of z with respect to y at the point (2, 1) with x held fixed.

a)
$$\frac{\partial z}{\partial x} = \cos(y^2 - 4x)\cdot(0-4) = -4\cos(1^2 - 4(2))$$
$$= -4\cos(-7)$$
$$= \boxed{-3.97}$$

b)
$$\frac{\partial z}{\partial y} = \cos(y^2 - 4x)\cdot(2y) = 2\cos(-7) = \boxed{1.98}$$

36. $f(x, y) = \cosh(\sqrt{x})\sinh^2(xy^2)$

$$f_x = \left(\sinh(\sqrt{x})\,\sinh^2(xy^2)\cdot\frac{1}{2\sqrt{x}}\right) + \left(\cosh(\sqrt{x})\cdot 2\sinh(xy)\cosh(xy^2)\cdot y^2\right)$$

17. Suppose that Nolan throws a baseball to Ryan and that the baseball leaves Nolan's hand at the same height at which it is caught by Ryan. It we ignore air resistance, the horizontal range r of the baseball is a function of the initial speed v of the ball when it leaves Nolan's hand and the angle θ above the horizontal at which it is thrown. Use the accompanying table and the method of Example 6 to estimate

(a) the partial derivative of r with respect to v when $v = 80$ ft/s and $\theta = 40°$

(b) the partial derivative of r with respect to θ when $v = 80$ ft/s and $\theta = 40°$.

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0}\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$f(x, y)$

$$f'(x) = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$f'(y) = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0}\frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}$$

SPEED v (ft/s) → +5 Δv

5 $\Delta\theta$

ANGLE θ (degrees)	75	80	85	90
35	165	188	212	238
40	173	197	222	249
45	176	200	226	253
50	173	197	222	249

Table Ex-17

$\Delta v = +5 \qquad \Delta\theta = 5$

a)
$$\left.\frac{\partial r}{\partial v}\right|_{(80,40)} = \lim_{\Delta v \to 0}\frac{r(v_0+\Delta v, \theta_0) - r(v_0, \theta_0)}{\Delta v}$$

$(80, 40)$: $80 \to v_0$, $40 \to \theta_0$

$$= \lim_{\Delta v \to 0}\frac{r(80+5, 40) - r(80, 40)}{5}$$

$$= \lim_{\Delta v \to 0}\frac{r(85, 40) - r(80, 40)}{5}$$

$$= \lim_{\Delta v \to 0}\frac{222 - 197}{5} = \boxed{5}$$

Now for $r(v_0 - \Delta v)$

$$\frac{\partial r}{\partial v} = \lim_{\Delta v \to 0}\frac{r(v_0 - \Delta v, \theta_0) - r(v_0, \theta_0)}{\Delta v}$$

$$\frac{\partial r}{\partial v} = \lim_{\Delta v \to 0} \frac{r(v_0 - \Delta v, \theta_0) - r(v_0, \theta_0)}{\Delta v}$$

$$= \lim_{\Delta v \to 0} \frac{r(75, 40) - r(80, 40)}{\Delta v} = \frac{173 - 197}{5}$$

$$= \boxed{-4.8}$$

$$\frac{5 + 4.8}{2} = \boxed{4.9} \text{ Ans.}$$

57. A point moves along the intersection of the plane $y = 3$ and the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is z changing with respect to x when the point is at $(4, 3, 2)$?

$z = \sqrt{29 - x^2 - y^2} \qquad y = 3$

$\dfrac{\partial z}{\partial x}(4,3,2) = \dfrac{1}{\cancel{2}\sqrt{29 - x^2 - y^2}} \times 0 - \cancel{2}x - 0$

$= \dfrac{-x}{\sqrt{29 - x^2 - y^2}} = \dfrac{-4}{\sqrt{29 - 4^2 - 3^2}} = \boxed{-2}$ Ans.

58. Find the slope of the tangent line at $(-1, 1, 5)$ to the curve of intersection of the surface $z = x^2 + 4y^2$ and
(a) the plane $x = -1$ (b) the plane $y = 1$.

b) $\dfrac{\partial z}{\partial x} = 2x = 2(-1) = \boxed{-2}$

a) $\dfrac{\partial z}{\partial y} = 8y = 8(1) = \boxed{8}$

59. The volume V of a right circular cylinder is given by the formula $V = \pi r^2 h$, where r is the radius and h is the height.

(a) Find a formula for the instantaneous rate of change of V with respect to r if r changes and h remains constant.

a) $\dfrac{\partial V}{\partial r} = \boxed{2\pi r h}$

(b) Find a formula for the instantaneous rate of change of V with respect to h if h changes and r remains constant.

b) $\dfrac{\partial V}{\partial h} = \pi r^2$

(c) Suppose that h has a constant value of 4 in, but r varies. Find the rate of change of V with respect to r at the point where $r = 6$ in.

c) $\dfrac{\partial V}{\partial r} = 2\pi(6)(4) = 48\pi$

(d) Suppose that r has a constant value of 8 in, but h varies. Find the instantaneous rate of change of V with respect to h at the point where $h = 10$ in.

d) $\dfrac{\partial V}{\partial h} = \pi(8)^2 = 64\pi$

60. The volume V of a right circular cone is given by

$$V = \frac{\pi}{24} d^2 \sqrt{4s^2 - d^2}$$

where s is the slant height and d is the diameter of the base.

(a) Find a formula for the instantaneous rate of change of V with respect to s if d remains constant.

(b) Find a formula for the instantaneous rate of change of V with respect to d if s remains constant.

(c) Suppose that d has a constant value of 16 cm, but s varies. Find the rate of change of V with respect to s when $s = 10$ cm.

(d) Suppose that s has a constant value of 10 cm, but d varies. Find the rate of change of V with respect to d when $d = 16$ cm.

a) $\frac{\partial V}{\partial s} = \frac{\pi}{\cancel{24}\,6} d^2 \times \frac{1}{\cancel{2}\sqrt{4s^2-d^2}} \times \overset{4s}{8s}$

$= \frac{\pi d^2 s}{6\sqrt{4^2 s - d^2}}$

c) $\frac{\partial V}{\partial s} = \frac{\pi (16)^2 (10)}{6\sqrt{4(10)^2 - 16^2}}$

$= \boxed{\frac{320\pi}{9}}$

b) $\frac{\pi}{24} d^2 \sqrt{4s^2 - d^2}$

$\frac{\partial V}{\partial d} = \left(\frac{\pi}{\cancel{24}\,12} \times \cancel{2}d\right)\sqrt{4s^2 - d^2} + \frac{\pi}{24} d^2 \times \frac{1}{\cancel{2}\sqrt{4s^2-d^2}} \times -\cancel{2}d$

$= \frac{\pi d \sqrt{4s^2 - d^2}}{12} - \frac{\pi d^3}{24\sqrt{4s^2 - d^2}}$

d) $\frac{\pi (16)\sqrt{4(10)^2 - 16^2}}{12} - \frac{\pi (16)^3}{24\sqrt{4(10)^2 - 16^2}}$

$16\pi - \frac{128}{9}\pi = \boxed{\frac{16}{9}\pi}$ Ans.

61. According to the ideal gas law, the pressure, temperature, and volume of a gas are related by $P = kT/V$, where k is a constant of proportionality. Suppose that V is measured in cubic inches (in^3), T is measured in kelvins (K), and that for a certain gas the constant of proportionality is $k = 10$ in·lb/K.

(a) Find the instantaneous rate of change of pressure with respect to temperature if the temperature is 80 K and the volume remains fixed at 50 in^3.

(b) Find the instantaneous rate of change of volume with respect to pressure if the volume is 50 in^3 and the temperature remains fixed at 80 K.

a)

$P = \frac{kT}{V}$

$\partial P = \frac{k}{V} \times 1 = \frac{10}{} = 0.2$

b) $V = \frac{kT}{P}$

$\frac{\partial V}{\partial P} = kTP^{-1}$

$= kT \times \frac{-1}{P^2}$

$= \frac{-\cancel{kT}}{k^2T^{\cancel{2}}} \times V^2$

$= -V^2 = -(50)^2$

$$\frac{\partial P}{\partial T} = \frac{K}{V} \times 1 = \frac{1\cancel{0}}{5\cancel{0}} = 0.2$$

$$= \frac{-V^2}{KT} = \frac{-(50)^2}{80 \times 10} = -3.125$$

62. The temperature at a point (x, y) on a metal plate in the xy-plane is $T(x, y) = x^3 + 2y^2 + x$ degrees Celsius. Assume that distance is measured in centimeters and find the rate at which temperature changes with respect to distance if we start at the point $(1, 2)$ and move

(a) to the right and parallel to the x-axis

(b) upward and parallel to the y-axis.

b) $\frac{\partial T}{\partial y} = 0 + 4y + 0 = 4y = 4(2) = 8$

a) $\frac{\partial T}{\partial x} = 3x^2 + 1 = 3(1)^2 + 1 = 4$

63. The length, width, and height of a rectangular box are $l = 5$, $w = 2$, and $h = 3$, respectively.

(a) Find the instantaneous rate of change of the volume of the box with respect to the length if w and h are held constant.

(b) Find the instantaneous rate of change of the volume of the box with respect to the width if l and h are held constant.

(c) Find the instantaneous rate of change of the volume of the box with respect to the height if l and w are held constant.

$V = l \times w \times h$

a) $\frac{\partial V}{\partial l} = w \times h = 6$

c) $\frac{\partial V}{\partial h} = l \times w = 5 \times 2 = 10$

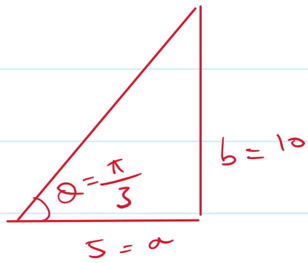
b) $\frac{\partial V}{\partial w} = l \times h = 15$

64. The area A of a triangle is given by $A = \frac{1}{2}ab\sin\theta$, where a and b are the lengths of two sides and θ is the angle between these sides. Suppose that $a = 5$, $b = 10$, and $\theta = \pi/3$.

(a) Find the rate at which A changes with respect to a if b and θ are held constant.

(b) Find the rate at which A changes with respect to θ if a and b are held constant.

(c) Find the rate at which b changes with respect to a if A and θ are held constant.



a) $\frac{\partial A}{\partial a} = \frac{1}{2} b \sin\theta = \frac{1}{\cancel{2}_1} \times \cancel{10}^5 \times \sin\frac{\pi}{3} = \frac{5\sqrt{3}}{2}$

b) $\frac{\partial A}{\partial \theta} = \frac{1}{2} a b \times \cos\theta = \frac{1}{\cancel{2}_1} \times 5 \times \cancel{10}^5 \times \cos\frac{\pi}{3} = 25 \times \frac{1}{2} = 12.5$

$\therefore \frac{\partial A}{\partial b} = \frac{1}{2} a \sin\theta = \frac{1}{2} \times 5 \times \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \times \frac{5}{2} = \frac{5\sqrt{3}}{4}$

$\frac{1}{2} a b \sin\theta$

$\therefore \frac{\partial A}{\partial b} = \frac{1}{2} a \sin\theta$ ... $\frac{3}{2}$ ... 2 ... 2 ...

c) $\frac{\partial b}{\partial a}$

$$A = \frac{1}{2} a b \sin\theta$$

$$b = \frac{2A}{a \sin\theta}$$

$$\frac{\partial b}{\partial a} = \frac{2A}{\sin\theta} \times \frac{-1}{a^2} = -\frac{\cancel{2}\left(\frac{1}{\cancel{2}} \cancel{a} b \cancel{\sin\theta}\right)}{a^{\cancel{2}} \cancel{\sin\theta}} = \frac{-b}{a} = -2$$

65. The volume of a right circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Show that if the height remains constant while the radius changes, then the volume satisfies

$$\frac{\partial V}{\partial r} = \frac{2V}{r}$$

$$h = \frac{3V}{\pi r^2}$$

$$\frac{\partial V}{\partial r} = \frac{1}{3}\pi h \times 2r = \frac{2\pi r h}{3} = \frac{2\cancel{\pi}\cancel{r} \times \frac{3V}{\cancel{\pi} r^{\cancel{2}}}}{\cancel{3}}$$

$$= \frac{2V}{r}$$ Ans

69–72 Calculate $\partial z/\partial x$ and $\partial z/\partial y$ using implicit differentiation. Leave your answers in terms of x, y, and z.

69. $(x^2 + y^2 + z^2)^{3/2} = 1$

$$\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}} \times \left(2x + 0 + 2z\frac{\partial z}{\partial x}\right) = 0$$

$$2x + 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{-\cancel{2}x}{\cancel{2}z} = \frac{\partial z}{\partial x}$$

$$\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}} \times \left(0 + 2y + 2z\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial z}{\partial y} = \frac{-y}{z}$$

70. $\ln(2x^2 + y - z^3) = x$

$$\frac{1}{2x^2+y-z^3} \times 4x + 0 - 3z^2 \times \frac{\partial z}{\partial x} = 1$$

$$4x - 3z^2\frac{\partial z}{\partial x} = 2x^2 + y - z^3$$

$$\partial z = 2x^2 + y - z^3 - 4x$$

∂x

$$\frac{\partial z}{\partial x} = \frac{2x^2 + y - z^3 - 4x}{-3z^2}$$

$$\frac{1}{2x^2 + y - z^3} \times \left(1 - 3z^2 \frac{\partial z}{\partial y}\right) = 0$$

$$3z^2 \frac{\partial z}{\partial y} = 1$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{1}{3z^2}}$$

71. $x^2 + z \sin xyz = 0$

$$2x + \left(\frac{\partial z}{\partial x}\right) \sin xyz + z \cdot \cos(xyz) \cdot xy = 0$$

$$2xy \cos(xyz) = -2x - \frac{\partial z}{\partial x} \sin xyz$$

$$\frac{-(2xy \cos(xyz) + 2x)}{\sin xyz} = \frac{\partial z}{\partial x}$$

13.4

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$z = f(x, y)$$

Total differential

$$dz = df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

$$dz \approx \Delta z \qquad dx \approx \Delta x \qquad dy \approx \Delta y \qquad \Delta y = y_2 - y_1$$

$$\frac{dz}{dx} = f_x \qquad \frac{dz}{dy} = f_y$$

$$dz = f_x\, dx \qquad dz = f_y\, dy$$

9–20 Compute the differential dz or dw of the function.

11. $z = x^3y^2$

$$dz = f_x\, dx + f_y\, dy$$

$$\boxed{dz = (3x^2y^2)\,dx + (2x^3y)\,dy}$$

18. $w = 4x^2y^3z^7 - 3xy + z + 5$

$$dw = f_x\, dx + f_y\, dy + f_z\, dz$$

$$\boxed{dw = (8xy^3z^7 - 3y)\,dx + (12x^2y^2z^7 - 3x)\,dy + (28x^2y^3z^6 + 1)\,dz} \text{ Ans}$$

21. $f(x, y) = x^2 + 2xy - 4x$; $P(1, 2), Q(1.01, 2.04)$

P = Actual Point
Q = After change

$$dz = (2x + 2y - 4)\,dx + (2x)\,dy \qquad dx = 1.01 - 1 = 0.01$$

$dz = (2x + 2y - 4)\,dx + (2x)\,dy$

$dx = 1.01 - 1 = 0.01$

$dy = 2.04 - 2 = 0.04$

at P

$dz = (2(1) + 2(2) - 4)(0.01) + (2(1))\,0.04$

$\boxed{dz = 0.1}$ ✓ → Actual

Now put P & Q in $f(x,y)$.

$z_1 = (1)^2 + 2(1)(2) - 4(1) = \boxed{1}$

$z_2 = (1.01)^2 + 2(1.01)(2.04) - 4(1.01) = \boxed{1.1009}$

$\Delta z = z_2 - z_1 = \boxed{0.1009}$ ✓ Approximately Equal.

→ Approximated

26. $f(x, y, z) = \dfrac{xyz}{x + y + z}$; $P(-1, -2, 4)$, $Q(-1.04, -1.98, 3.97)$

$df = f_x\,dx + f_y\,dy + f_z\,dz$

$$df = \left(\frac{(yz)(x+y+z) - (xyz)(1)}{(x+y+z)^2}\right)dx + \left(\frac{(xz)(x+y+z) - (xyz)(1)}{(x+y+z)^2}\right)dy + \left(\frac{(xy)(x+y+z) - (xyz)(1)}{(x+y+z)^2}\right)dz$$

$dx = -1.04 - (-1) = -0.04$

$dy = -1.98 - (-2) = 0.02$

$dz = 3.97 - (4) = -0.03$

$$df = \left(\frac{(-2\times 4)(-1-2+4) - (-1\times -2\times 4)}{(-1-2+4)^2}\right)\times -0.04 + \left(\frac{(-1\times 4)(-1+2+4) - (-1\times -2\times 4)}{(-1-2+4)^2}\right)\times 0.02 +$$

$$\left(\frac{(-1\times -2)(-1+2+4) - (-1\times -2\times 4)}{(-1-2+4)^2}\right)\times -0.03$$

$\boxed{df = 0.58}$ → Actual value

$f_1 = \dfrac{(-1\times -2\times 4)}{-1-2+4} = \boxed{8}$

$$f_1 = \frac{(-1 \times -2 \times 4)}{-1-2+4} = \boxed{8}$$

$$f_2 = \frac{(-1.04 \times -1.98 \times 3.97)}{-1.04-1.98+3.97} = \boxed{8.605}$$

$$\Delta f = 8.605 - 8 = \boxed{0.605} \longrightarrow \text{Approximated value.}$$

14. $z = e^{-3x}\cos 6y$

$$dz = f_x\,dx + f_y\,dy$$

$$dz = (-3e^{-3x}\cos 6y)\,dx + (-e^{-3x}\sin 6y \cdot 6)\,dy$$

25. $f(x, y, z) = 2xy^2z^3$; $P(1, -1, 2)$, $Q(0.99, -1.02, 2.02)$

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

$$df = (2y^2z^3)\,dx + (4xyz^3)\,dy + (6xy^2z^2)\,dz$$

$dx = 0.99 - 1 = -0.01$ $\quad\quad dz = 0.02$

$dy = -1.02 + 1 = -0.02$

$$df = (2(-1)^2(2)^3)(-0.01) + (4(1)(-1)(2)^3)(-0.02) + (6(1)(-1)^2(2)^2)(0.02)$$

$$df = -0.16 + 0.64 + 0.48 = \boxed{0.96} \longrightarrow \text{Approximate value}$$

$$f_1 = 2(1)(-1)^2(2)^3 = 16$$

$$f_2 = 2(0.99)(-1.02)^2(2.02)^3 = 16.979$$

$$\Delta f = 16.979 - 16 = \boxed{0.979} \longrightarrow \text{Actual value}$$

10

Wednesday, 14 February 2024 11:58 pm

13.4 — Local Linear Approximation

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$y = y_0 + y_0'(x - x_0)$$

$f(x,y)$ $\quad P(x_0, y_0)$

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$$\boxed{L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)}$$

Q37 $f(x,y,z) = xyz$

$P(1, 2, 3)$, $Q(1.001, 2.002, 3.003)$

$x_0 \; y_0 \; z_0$

$$L(x,y,z) = 6 + yz(x-1) + xz(y-2) + xy(z-3)$$
$$\boxed{L(x,y,z) = 6 + 6(x-1) + 3(y-2) + 2(z-3)}$$
$$L(Q) = 6 + 6(1.001-1) + 3(2.002-2) + 2(3.003-3)$$
$$\boxed{L(Q) = 6.018}$$
$$f(Q) = (1.001)(2.002)(3.003) = 6.018018$$
$$\boxed{f(Q) - L(Q) = 0.000018}$$

$$|PQ| = \sqrt{(1.001-1)^2 + (2.002-2)^2 + (3.003-3)^2}$$
$$\boxed{|PQ| = 0.00374}$$

Ratio of change and distance

$$\frac{|f(Q) - L(Q)|}{|PQ|} = \frac{0.000018}{0.00374} = \boxed{0.00481} \text{ Ans.}$$

Q38 $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ $\quad P(4,3)$ $\quad Q(3.92, 3.01)$

$$\cancel{L(x,y) = \frac{1}{5} + \frac{1}{2\sqrt{x^2+y^2}}}$$

$$L(x,y) = \frac{1}{5} - \frac{1}{\cancel{2}(x^2+y^2)^{3/2}} \times \cancel{2}x\,(x-4) + \frac{-1}{\cancel{2}(x^2+y^2)^{3/2}} \times \cancel{2}y\,(y-3)$$

$$L(x,y) = \frac{1}{5} - \frac{x(x-4)}{(x^2+y^2)^{3/2}} - \frac{y(y-3)}{(x^2+y^2)^{3/2}}$$

$$L(Q) = \frac{1}{5} - \frac{3.92(3.92-4)}{(3.92^2+3.01^2)^{3/2}} - \frac{3.01(3.01-3)}{(3.92^2+3.01^2)^{3/2}}$$

$$\boxed{L(Q) = 0.2023483}$$

$$f(Q) = \frac{1}{\sqrt{3.92^2+3.01^2}} = \boxed{0.202334}$$

$$|PQ| = \sqrt{(3.92-4)^2 + (3.01-3)^2}$$
$$\boxed{|PQ| = 0.08062}$$

$$\frac{|f(Q) - L(Q)|}{|PQ|} = \frac{|0.202334 - 0.2023483|}{0.08062}$$

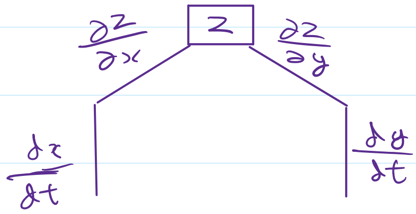
$$\boxed{0.000177} \text{ Ans.}$$

13.5 ~ Chain Rule (for derivatives)

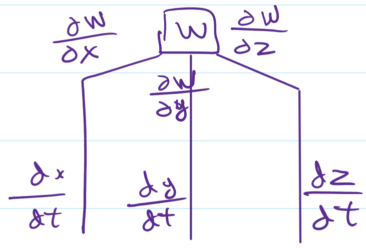
$z = f(x, y)$

$x = x(t)$

$y = y(t)$



$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$



$w = f(x, y, z)$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}}$$

1–6 Use an appropriate form of the chain rule to find dz/dt.

2. $z = \ln(2x^2 + y)$; $x = \sqrt{t}$, $y = t^{2/3}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{4x}{(2x^2 + y)} \cdot \frac{1}{2\sqrt{t}} + \frac{1}{(2x^2 + y)} \cdot \frac{2}{3} t^{-1/3}$$

Now substitute with t b/c we have to find $\frac{dz}{dt}$

$x = \sqrt{t}$ $\quad$ $y = t^{2/3}$

$$\frac{dz}{dt} = \frac{\cancel{4}^{2} \cancel{\sqrt{t}}}{2t + t^{2/3}} \cdot \frac{1}{\cancel{2} \cancel{\sqrt{t}}} + \frac{1}{2t + t^{2/3}} \cdot \frac{2}{3} t^{-\frac{1}{3}}$$

$$\boxed{\frac{dz}{dt} = \frac{2}{2t + t^{2/3}} + \frac{2}{3(2t^{4/3} + t)}}$$ Ans.

Chain Rule for Partial Derivatives

$z = f(x, y)$

$x = x(u, v)$

$y = y(u, v)$

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}}$$

Q $w = x^2 + y \sin z$.

$x = u^2 + v^2$

$y = \sin(uv)$

$z = \sqrt{uv}$

$$\rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial u} = (2x) \cdot (2u) + (\sin z) \cdot (\cos(uv) \cdot v) + (y \cos z) \cdot \left(\frac{1 \cdot v}{2\sqrt{uv}}\right)$$

$$\boxed{\frac{\partial w}{\partial u} = 4u(u^2+v^2) + \sin\sqrt{uv} \cdot \cos(uv) \cdot v + \sin uv \cdot \cos\sqrt{u} \cdot \frac{v}{2\sqrt{uv}}}$$

$$\rightarrow \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial v} = (2x) \cdot (2v) + \sin z \cdot u \cos(uv) + y \cos z \cdot \left(\frac{u}{2\sqrt{uv}}\right)$$

$$\boxed{\frac{\partial w}{\partial v} = 4v(u^2+v^2) + u \cos(uv) \cdot \sin\sqrt{uv} + \sin(uv) \cdot \cos\sqrt{uv} \cdot \frac{u}{2\sqrt{uv}}}$$

13. Suppose that $z = f(x, y)$ is differentiable at the point $(4, 8)$ with $f_x(4, 8) = 3$ and $f_y(4, 8) = -1$. If $x = t^2$ and $y = t^3$, find dz/dt when $t = 2$.

$z = f(x, y)$

$x = t^2$

$y = t^3$

$$\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = 3 \cdot (2t) + -1(3t^2) = 6t - 3t^2$$

$$= 6(2) - 3(2)^2$$

$$= \boxed{0} \text{ Ans.}$$

14. Suppose that ...

14. Suppose that $w = f(x, y, z)$ is differentiable at the point $(1, 0, 2)$ with $f_x(1, 0, 2) = 1$, $f_y(1, 0, 2) = 2$, and $f_z(1, 0, 2) = 3$. If $x = t$, $y = \sin(\pi t)$, and $z = t^2 + 1$, find dw/dt when $t = 1$.

$w = f(x, y, z)$

$x = t$

$y = \sin(\pi t)$

$z = t^2 + 1$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} + f_z \cdot \frac{dz}{dt}$$

$$\frac{dw}{dt} = (1)(1) + 2 \cdot (\pi \cos \pi t) + 3 \cdot (2t)$$

$$\frac{dw}{dt} = 1 + 2\pi \cos(\pi t) + 6t = 1 + 2\pi \cos \pi + 6$$

$$= 1 - 2\pi + 6 = \boxed{7 - 2\pi} \text{ Ans}$$

33. Use a chain rule to find the values of

$$\left.\frac{\partial z}{\partial r}\right|_{r=2,\theta=\pi/6} \quad \text{and} \quad \left.\frac{\partial z}{\partial \theta}\right|_{r=2,\theta=\pi/6}$$

if $z = xye^{x/y}$; $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \left((1)\, y e^{x/y} + (x)(y e^{x/y})\left(\frac{1}{y}\right)\right) \cdot (\cos\theta) + \left((1)(x e^{x/y}) + x y e^{x/y} \cdot \left(\frac{-x}{y^2}\right)\right)(\sin\theta)$$

$$\frac{\partial z}{\partial r} = \left(r\sin\theta\, e^{\cot\theta} + r^2 \cancel{\sin\theta} \cos\theta\, e^{\cot\theta} \cdot \frac{1}{\cancel{r\sin\theta}}\right)\cos\theta + \left(r\cos\theta\, e^{\cot\theta} + r^2 \cancel{\sin\theta} \cos\theta\, e^{\cot\theta} \cdot -\frac{\cot\theta}{\cancel{r\sin\theta}}\right)\sin\theta$$

$$\frac{\partial z}{\partial r} = \left(2\sin\frac{\pi}{6}\, e^{\frac{1}{\tan\pi/6}} + 2\cos\theta\, e^{\frac{1}{\tan\pi/6}}\right) \cdot \cos\frac{\pi}{6} + \left(2\cos\frac{\pi}{6}\, e^{\frac{1}{\tan\pi/6}} - 2\cos\theta\, e^{\pi/6} \times \frac{1}{\tan\pi/6}\right)\sin\frac{\pi}{6}$$

13.5 ~ Implicit Differentiation

First Method

Theorem 13.5.2 $\longrightarrow$ $\boxed{\dfrac{dy}{dx} = \dfrac{-\partial f/\partial x}{\partial f/\partial y}}$

Second Method

Calculus wala jo tha

36. The voltage, V (in volts), across a circuit is given by Ohm's law: $V = IR$, where I is the current (in amperes) flowing through the circuit and R is the resistance (in ohms). If two circuits with resistances R_1 and R_2 are connected in parallel, then their combined resistance, R, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that the current is 3 amperes and is increasing at 10^{-2} ampere/s, R_1 is 2 ohms and is increasing at 0.4 ohm/s, and R_2 is 5 ohms and is decreasing at 0.7 ohm/s. Estimate the rate at which the voltage is changing.

$V = IR$ $R_1 = 2$ ohms , $\frac{dR_1}{dt} = 0.4$ ohms

$I = 3A$ $R_2 = 5$ ohms $\frac{dR_2}{dt} = -0.7$ ohm.

$\frac{dI}{dt} = 10^{-2}$ A

$\frac{dV}{dt} = ?$

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\cdot\frac{dI}{dt} + \frac{\partial V}{\partial R}\cdot\frac{dR}{dt}$$

$$\frac{dV}{dt} = R\cdot\frac{dI}{dt} + I\cdot\frac{dR}{dt}$$

not given so we must find breaking it further.

$$\frac{dR}{dt} = \frac{\partial R}{\partial R_1}\cdot\frac{dR_1}{dt} + \frac{\partial R}{\partial R_2}\cdot\frac{dR_2}{dt}$$

$\because \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$R = \frac{R_1 R_2}{R_1 + R_2} = \boxed{\frac{10}{7}}$$

$$\frac{\partial R}{\partial R_1} = \frac{(R_2)(R_1+R_2) - (R_1R_2)}{(R_1+R_2)^2} = \frac{R_2(\cancel{R_1} + R_2 - \cancel{R_1})}{(R_1+R_2)^2}$$

$$\boxed{\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1+R_2)^2}} = \boxed{\frac{25}{49}}$$

$$\boxed{\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1+R_2)^2}} = \boxed{\frac{4}{49}}$$

$$\frac{dR}{dt} = \frac{25}{49}\times(0.4) + \frac{4}{49}\times(-0.7) = \boxed{0.1469}$$

$$\frac{dV}{dt} = R\cdot\frac{dI}{dt} + I\cdot\frac{dR}{dt}$$

$$\frac{dV}{dt} = \frac{10}{7}\times(10^{-2}) + (3)(0.1469) = \boxed{0.455 \text{ V/s}}$$ Ans.

43. $e^{xy} + ye^y = 1$

$$e^{xy}\cdot\left((1)(y) + (x)\left(\frac{dy}{dx}\right)\right) + \left(\left(\frac{dy}{dx}\right)e^y + y\cdot e^y\cdot\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} + e^y\frac{dy}{dx} + ye^y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} + e^y + ye^y) = -ye^{xy}$$

$$\boxed{\frac{dy}{dx} = \frac{-e^{xy}\cdot y}{xe^{xy} + e^y + ye^y}}$$ Ans.

Shown!

$$\boxed{\frac{dy}{dx} = \frac{-ye^{xy}}{xe^{xy} + ye^y + e^y}}$$

47. $ye^x - 5\sin 3z = 3z$

$$ye^x - 15\cos 3z\cdot\frac{\partial z}{\partial x} = 3\frac{\partial z}{\partial x}$$

$$ye^x = \frac{\partial z}{\partial x}(3 + 15\cos 3z)$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{ye^x}{3 + 15\cos 3z}}$$ Ans.

$$\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z}$$

$$\frac{\partial z}{\partial x} = \frac{-(ye^x)}{-15\cos 3z - 3}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\cancel{-}ye^x}{\cancel{-}(15\cos 3z + 3)}}$$

Shown!

$$e^x - 15\cos 3z\cdot\frac{\partial z}{\partial y} = 3\cdot\frac{\partial z}{\partial y}$$

$$e^x = \frac{\partial z}{\partial y}(3 + 15\cos 3z)$$

$$e^{x} - 15\cos 3z \cdot \frac{\partial z}{\partial y} = 3 \cdot \frac{\partial z}{\partial y}$$

$$e^{x} = \frac{\partial z}{\partial y}\left(3 + 15\cos 3z\right)$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{e^{x}}{3 + 15\cos 3z}}$$ Ans.

48. $e^{xy}\cos yz - e^{yz}\sin xz + 2 = 0$

$$\frac{\partial z}{\partial x} = ?$$

$$ye^{xy}\cos yz - e^{xy}\cdot y\sin yz \cdot \frac{\partial z}{\partial x} - \left[ye^{yz}\cdot\frac{\partial z}{\partial x}\cdot(\sin xz) + e^{yz}\cdot\cos xz\cdot\left((1)(z) + (x)\left(\frac{\partial z}{\partial x}\right)\right)\right] = 0$$

$$ye^{xy}\cos yz - y\sin yz\; e^{xy}\cdot\frac{\partial z}{\partial x} - y\sin xz\; e^{yz}\cdot\frac{\partial z}{\partial x} \quad z\cos xz\; e^{yz} - x\cos xz\; e^{yz}\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(-ye^{xy}\sin yz - ye^{yz}\sin xz - xe^{yz}\cos xz\right. = ze^{yz}\cos xz - ye^{xy}\cos yz$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\cancel{-}\left(ye^{xy}\cos yz - ze^{yz}\cos xz\right)}{\cancel{-}\left(ye^{xy}\sin yz + ye^{yz}\sin xz + xe^{yz}\cos xz\right)}}$$ Ans.

48. $\underbrace{e^{xy}\cos yz}_{\text{Product rule}} - \underbrace{e^{yz}\sin xz}_{\text{Product rule}} + 2 = 0$

$$\frac{\partial z}{\partial y} = ?$$

$$xe^{xy}\cdot\cos yz + e^{xy}\cdot(-\sin yz)\cdot\left((1)(z) + (y)\left(\frac{\partial z}{\partial y}\right)\right) - \left[e^{yz}\left((1)(z) + (y)\left(\frac{\partial z}{\partial y}\right)\right)\cdot\sin xz + e^{yz}\cdot\cos xz\cdot x\right] = 0$$

$$xe^{xy}\cos yz - ze^{xy}\sin yz - ye^{xy}\sin yz\left(\frac{\partial z}{\partial y}\right) - ze^{yz}\sin xz - ye^{yz}\left(\frac{\partial z}{\partial y}\right)\cdot\sin xz - xe^{yz}\cdot\cos xz\cdot\left(\frac{\partial z}{\partial y}\right) = 0$$

$$xe^{xy}\cos yz - ze^{xy}\sin yz - e^{yz}\sin xz = \frac{\partial z}{\partial y}\left(ye^{xy}\sin yz + ye^{yz}\sin xz + xe^{yz}\cos xz\right)$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{xe^{xy}\cos yz - ze^{xy}\sin yz - ze^{yz}\sin xz}{ye^{xy}\sin yz + ye^{yz}\sin xz + xe^{yz}\cos xz}}$$ Ans.

13.6 ~ Directional Derivatives

at point P

$D_u f(x, y, z) = \nabla f \cdot u$

$\nabla f(x, y) = f_x i + f_y j$

$\nabla f(x, y, z) = f_x i + f_y j + f_z k$

$u = u_1 i + u_2 j$

$u = u_1 i + u_2 j + u_3 k$

$$\boxed{D_u f(x_0, y_0, z_0) = f_x u_1 + f_y u_2 + f_z u_3}$$

1–8 Find $D_\mathbf{u} f$ at P.

3. $f(x, y) = \ln(1 + x^2 + y)$; $P(0, 0)$;
$\mathbf{u} = -\frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$

$$D_u f(0,0) = f_x u_1 + f_y u_2$$

$$= \left(\frac{2x}{1 + x^2 + y^2}\right)\left(\frac{-1}{\sqrt{10}}\right) + \left(\frac{1}{1 + x^2 + y^2}\right)\left(\frac{-3}{\sqrt{10}}\right)$$

Putting $P(0,0)$

$$D_u f(0,0) = 0 + \frac{1}{1 + 0^2 + 0^2}\left(\frac{-3}{\sqrt{10}}\right)$$

$$\boxed{D_u f(0,0) = -\frac{3}{\sqrt{10}}}$$ Ans.

9–18 Find the directional derivative of f at P in the direction of $\mathbf{a}$.

13. $f(x, y) = \tan^{-1}(y/x)$; $P(-2, 2)$; $\mathbf{a} = -\mathbf{i} - \mathbf{j}$

First find unit vector of it.

$$u = \frac{a}{|a|} = \frac{-i-j}{\sqrt{(-1)^2+(-1)^2}} = \frac{-1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

$$D_u f(-2,2) = f_x u_1 + f_y u_2$$

$$= \left(\frac{-y/x^2}{1+(y/x)^2}\right)\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{1/x}{1+(y/x)^2}\right)\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \left(\frac{-2/(-2)^2}{1+(2/-2)^2}\right)\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{1/-2}{1+(2/-2)^2}\right)\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \boxed{\frac{1}{2\sqrt{2}}}$$ Ans.

19–22 Find the directional derivative of f at P in the direction of a vector making the counterclockwise angle θ with the positive x-axis.

21. $f(x, y) = \tan(2x + y)$; $P(\pi/6, \pi/3)$; $\theta = 7\pi/4$

$$\boxed{u = \cos\theta \, i + \sin\theta \, j}$$

$$u = \cos\left(\frac{7\pi}{4}\right)\cdot i + \sin\left(\frac{7\pi}{4}\right)\cdot j$$

$$\boxed{u = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j}$$

$$D_u f(\pi/6, \pi/3) = \left(2\sec^2(2x+y)\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\sec^2(2x+y)\right)\left(\frac{-1}{\sqrt{2}}\right)$$

$$\boxed{D_u f(\pi/6, \pi/3) = 2\sqrt{2}}$$ Ans.

23. Find the directional derivative of
$$f(x,y) = \frac{x}{x+y}$$
at $P(1,0)$ in the direction of $Q(-1,-1)$.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{-2i - j}$$

unit vector $= u = \dfrac{-2i - j}{\sqrt{(-2)^2 + (-1)^2}} = \boxed{\dfrac{-2}{\sqrt{5}} i - \dfrac{1}{\sqrt{5}} j}$ Ans.

$$D_u f(1,0) = \left(\frac{(x+y) - x}{(x+y)^2}\right)\left(\frac{-2}{\sqrt{5}}\right) + \left(\frac{-x}{(x+y)^2}\right)\left(\frac{-1}{\sqrt{5}}\right)$$

$\boxed{D_u f(1,0) = \dfrac{1}{\sqrt{5}}}$ Ans.

25. Find the directional derivative of $f(x,y) = \sqrt{xy}e^y$ at $P(1,1)$ in the direction of the negative y-axis.

so the direction vector is $Q(1,-1)$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{0i - 2j}$$

$$u = \frac{0i - 2j}{\sqrt{0^2 + (-2)^2}} = \boxed{0i - 1j}$$

$$D_u f(1,1) = \left(\frac{ye^y}{2\sqrt{xy}}\right)(0) + \left(\left(\frac{x}{2\sqrt{xy}}\right)(e^y) + (\sqrt{xy})(e^y)\right)(-1)$$

$$D_u f(1,1) = \left(\frac{e}{2} + e\right)(-1) = \frac{3e}{2} = \boxed{-\frac{3e}{2}}$$ Ans.

29. Suppose that $D_{\mathbf{u}}f(1,2) = -5$ and $D_{\mathbf{v}}f(1,2) = 10$, where $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$ and $\mathbf{v} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$. Find
(a) $f_x(1,2)$ (b) $f_y(1,2)$
(c) the directional derivative of f at $(1,2)$ in the direction of the origin.

$$D_u f(1,2) = f_x \cdot u_1 + f_y \cdot u_2$$
$$-5 = f_x \cdot \left(\frac{3}{5}\right) + f_y \left(\frac{-4}{5}\right)$$

$$D_v f(1,2) = f_x \cdot v_1 + f_y \cdot v_2$$
$$10 = f_x \cdot \left(\frac{4}{5}\right) + f_y \left(\frac{3}{5}\right)$$

a)
$$\left(\frac{4}{5} f_x + \frac{3}{5} f_y = 10\right) \times 4$$
$$\left(\frac{3}{5} f_x - \frac{4}{5} f_y = -5\right) \times 3$$
$$\frac{16}{5} f_x + \cancel{\frac{12}{5} f_y} = 40$$
$$\frac{9}{5} f_x - \cancel{\frac{12}{5} f_y} = -15$$
$$5 f_x = 25$$
$$\boxed{f_x = 5} \text{ Ans}$$

b)
$$\frac{9}{5}(5) - \frac{12}{5} f_y = -15$$
$$9 - \frac{12}{5} f_y = -15$$
$$\boxed{f_y = 10} \text{ Ans.}$$

$$\vec{PO} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1i - 2j \qquad u = \frac{-1i - 2j}{\sqrt{1^2+2^2}} = -\frac{1}{\sqrt{5}} i - \frac{2}{\sqrt{5}} j$$

$$D_p f(1,2) = (5)\left(\frac{-1}{\sqrt{5}}\right) + (10)\left(\frac{-2}{\sqrt{5}}\right) = \boxed{-5\sqrt{5}} \text{ Ans.}$$

30. Given that $f_x(-5,1) = -3$ and $f_y(-5,1) = 2$, find the directional derivative of f at $P(-5,1)$ in the direction of the vector from P to $Q(-4,3)$.

$$\vec{PQ} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} - \begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$u = \frac{1i + 2j}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j$$

$$D_u f(-5,1) = f_x u_1 + f_y u_2$$

$$= (-3)\left(\frac{1}{\sqrt{5}}\right) + (2)\left(\frac{2}{\sqrt{5}}\right)$$

$$= \frac{-3}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}}}$$ Ans.

If f decreasing, $u = \frac{-\Delta \vec{f}}{|\Delta \vec{f}|}$

53–60 Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f at P in that direction.

53. $f(x, y) = 4x^3y^2$; $P(-1, 1)$

$\nabla f = f_x i + f_y j$

$\nabla f = 12x^2y^2\,\hat{i} + 8x^3y\,\hat{j}$

$a = \nabla f(-1, 1) = 12i - 8j$

Since increasing, +ve

$\therefore u = \frac{\Delta f}{|\Delta f|} = \frac{12i - 8j}{\sqrt{(12)^2 + (8)^2}} = \boxed{\frac{3}{\sqrt{13}} i - \frac{2}{\sqrt{13}} j}$

$D_u f(-1, 1) = f_x u_1 + f_y u_2$

$D_u f(-1, 1) = (12)\left(\frac{3}{\sqrt{13}}\right) + (-8)\left(-\frac{2}{\sqrt{13}}\right)$

$\boxed{D_u f(-1, 1) = 4\sqrt{13}}$ Ans.

61–66 Find a unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f at P in that direction.

66. $f(x, y, z) = 4e^{xy} \cos z$; $P(0, 1, \pi/4)$

$\nabla f = f_x i + f_y j + f_z k$

$\nabla f = (4y e^{xy} \cos z) i + (4x e^{xy} \cos z) j - (4 e^{xy} \sin z) k$

$$\nabla f = \left(4y e^{xy} \cos z\right) i + \left(4x e^{xy} \cos z\right) j - \left(4 e^{xy} \sin z\right) k$$

$$\nabla f(0, 1, \pi/4) = 2\sqrt{2}\, i + 0 j - 2\sqrt{2}\, k$$

Since decreasing, -ve sign

$$u = -\frac{\nabla f}{|\nabla f|} = -\left(\frac{2\sqrt{2}\, i + 0 j - 2\sqrt{2}\, k}{\sqrt{(2\sqrt{2})^2 + (-2\sqrt{2})^2}}\right)$$

$$\boxed{u = -\frac{\sqrt{2}}{2} i + 0 j + \frac{\sqrt{2}}{2} k}$$

$$D_u f(-1, 1) = f_x i + f_y j + f_z k$$

$$D_u f(-1, 1) = (2\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right) + 0 + (-2\sqrt{2})\left(\frac{\sqrt{2}}{2}\right)$$

$$\boxed{D_u f(-1, 1) = -4}$$ Ans

13.7 ≈ Tangent Planes & Normal Vectors

Standard Equation

$$ax + by + cz + d = 0$$

$$n = \langle \underbrace{a, b, c}_{\text{Components of vector.}} \rangle$$ → Vector representation

$$P(x_0, y_0, z_0)$$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$f(x,y,z)$, $P(x_0, y_0, z_0)$

$n = \nabla f(x_0, y_0, z_0) = f_x\, i + f_y\, j + f_z\, k$ → Normal vector

$n = < f_x, f_y, f_z >$

Eq of plane :- $\underbrace{f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0}$

This plane is tangent to the curve.

Parametric equations of line → Parameter is t

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \rightarrow \begin{aligned} x &= x_0 + f_x t \\ y &= y_0 + f_y t \\ z &= z_0 + f_z t \end{aligned}$$

3–12 Find an equation for the tangent plane and parametric equations for the normal line to the surface at the point P.

3. $x^2 + y^2 + z^2 = 25$; $P(-3, 0, 4)$

$\nabla f = f_x\, i + f_y\, j + f_z\, k$

$$\boxed{\nabla f = 2x\, i + 2y\, j + 2z\, k}$$

$\nabla f(-3,0,4) = 2(-3)\, i + 2(0)\, j + 2(4)\, k$

$\nabla f(-3,0,4) = 2(-3)\,i + 2(0)\,j + 2(4)\,K$

$$\boxed{\nabla f(-3,0,4) = -6\,i + 0\,j + 8\,K}$$

Eq of plane :- $f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$

$-6(x-(-3)) + 0(y-0) + 8(z-4) = 0$

$-6(x+3) + 8(z-4) = 0$

$-6x - 18 + 8z - 32 = 0$

$$\boxed{8z + 6x + 50 = 0}$$ Ans.

Parameteric

$$\boxed{\begin{aligned} x &= -3 - 6t \\ y &= 0 + 0t \\ z &= 4 + 8t \end{aligned}}$$

Ans.

10. $z = \ln\sqrt{x^2+y^2}$; $P(-1,0,0)$ $\rightarrow$ $\ln\sqrt{x^2+y^2} - z = 0$

$\nabla f = f_x\, i + f_y\, j + f_z\, K$

$\nabla f = \left(\dfrac{1}{\sqrt{x^2+y^2}} \times \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\right) i + \left(\dfrac{1}{\sqrt{x^2+y^2}} \times \dfrac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\right) j - 1\,K$

$\nabla f = \left(\dfrac{x}{x^2+y^2}\right) i + \left(\dfrac{y}{x^2+y^2}\right) j - K$

$$\boxed{\nabla f(-1,0,0) = -i + 0\,j - K}$$

Plane $\rightarrow$ $-1(x-(-1)) + 0(y-0) - 1(z-0) = 0$

$-1(x+1) + 0 - 1(z) = 0$

$-x - 1 - z = 0$

$\boxed{x + z + 1 = 0}$ Ans.

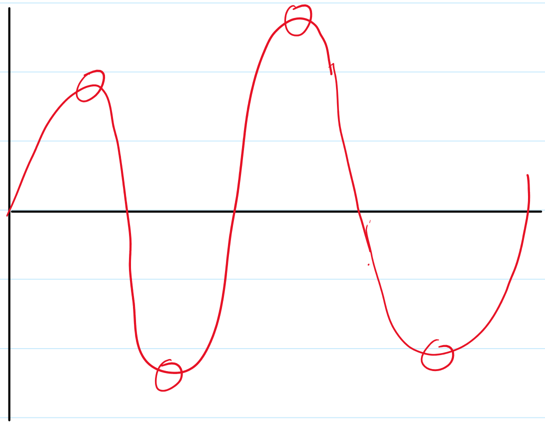
Parametric

$x = -1 - 1(t) = \boxed{-1 - t}$

$y = 0 + 0t$

$z = 0 - 1t = \boxed{0 - t}$

13.8 ~ Relative Extrema



$z = f(x, y)$

$$\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\} \rightarrow \text{critical Points}$$

Now Second derivatives

$$D = f_{xx} f_{yy} - \underbrace{(f_{xy})^2}_{}$$

$\rightarrow$ This is actually $f_{xy} \cdot f_{yx}$

① If $D > 0$, check $f_{xx}(x_0, y_0) > 0 \rightarrow$ a min at (x_0, y_0)

② If $D > 0$, check $f_{xx}(x_0, y_0) < 0 \rightarrow$ a max at (x_0, y_0)

③ $D < 0$ Saddle Point

④ $D = 0$ No conclusion.

Q $f(x, y) = 4xy - x^4 - y^4$

$\rightarrow f_x = 4y - 4x^3 \longrightarrow 4y - 4x^3 = 0 \longrightarrow y = x^3$ —— ①

$f_y = 4x - 4y^3 \longrightarrow 4x - 4y^3 = 0 \longrightarrow x = y^3$ —— ②

$\rightarrow f_x = 4y - 4x^3 \longrightarrow 4y - 4x^3 = 0 \longrightarrow y = x^3$ — ①

$f_y = 4x - 4y^3 \longrightarrow 4x - 4y^3 = 0 \longrightarrow x = y^3$ — ②

Now equate

$y = y^9$

$y^9 - y = 0$

$y(y^8 - 1) = 0$

$\boxed{y = 0}$ $\boxed{y = \pm 1}$

$\boxed{x = 0}$ $\boxed{x = 1}$

$(0,0)$ $(1,1)$ $(1,-1)$

$f_{xx} = -12x^3$

$f_{yy} = -12y^2$

$f_{xy} = 4$

$\longrightarrow D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	result
$(0,0)$	0	0	4	$-16 < 0$	Saddle Point
$(1,1)$	$-12 < 0$	-12	4	128 (+ve)	Relative Maxima
$(1,-1)$	$-12 < 0$	-12	4	$128 > 0$	Relative Maxima.

12. $f(x, y) = xy - x^3 - y^2$

$f_x = y - 3x^2 \longrightarrow y - 3x^2 = 0 \longrightarrow y = 3x^2$ — ①

$f_y = x - 2y \longrightarrow x - 2y = 0 \longrightarrow x = 2y$ — ②

$y = 3(2y)^2$

$y = 12y^2$

$12y^2 - y = 0$

$\boxed{y = 0}$ $\boxed{y = 1/12}$

$(0,0)$ $\left(\frac{1}{12}, \frac{1}{16}\right)$

$\boxed{x = 0}$ $\boxed{x = 1/16}$

$$\left.\begin{aligned} f_{xx} &= -6x \\ f_{yy} &= -2 \\ f_{xy} &= 1 \end{aligned}\right\} \longrightarrow D = f_{xx} f_{yy} - (f_{xy})^2$$

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	result
$(0, 0)$	0	-2	1	-1	Saddle Point
$\left(\frac{1}{12}, \frac{1}{16}\right)$	$-1 < 0$	-2	1	$1 > 0$	Relative Maxima

14. $f(x, y) = xe^y$

$f(x,y) = x e^y$

$f_x = e^y \longrightarrow e^y = 0 \longrightarrow y = \ln(0) \longrightarrow$ D.N.E

$f_y = x e^y \longrightarrow x e^y = 0 \longrightarrow y = \ln(0) \longrightarrow$ D.N.E

$\boxed{\text{No Critical Points}}$ Ans.

17. $f(x, y) = e^x \sin y$

$f_x = e^x \sin y \longrightarrow e^x = 0, \sin y = 0$ —— ①

$f_y = e^x \cos y \longrightarrow e^x = 0, \cos y = 0$ —— ②

Contradictory Equations as sin y and cos y cannot be 0 at same value of y.

15. $f(x, y) = x^2 + y - e^y$

$f_x = 2x \longrightarrow 2x = 0 \longrightarrow \boxed{x = 0}$

$f_y = 1 - e^y \longrightarrow 1 - e^y = 0 \longrightarrow e^y = 1 \longrightarrow \boxed{y = 0}$

$f_x = 2x \longrightarrow 2x = 0 \longrightarrow \boxed{x = 0}$

$f_y = 1 - e^y \longrightarrow 1 - e^y = 0 \longrightarrow e^y = 1 \longrightarrow \boxed{y = 0}$

$$\left.\begin{aligned} f_{xx} &= 2 \\ f_{yy} &= -e^y \\ f_{xy} &= 0 \end{aligned}\right\} \longrightarrow D = f_{xx} f_{yy} - (f_{xy})^2$$

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	Result
$(0,0)$	2	-1	0	$-2 < 0$	Saddle Point

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

$f_x = y - \frac{2}{x^2} \longrightarrow y - \frac{2}{x^2} = 0 \longrightarrow y = \frac{2}{x^2}$ —— ①

$f_y = x - \frac{4}{y^2} \longrightarrow x - \frac{4}{y^2} = 0 \longrightarrow x = \frac{4}{y^2}$ —— ②

$$y = \frac{2}{\left(\frac{4}{y^2}\right)^2} = \frac{2}{\frac{16}{y^4}} = \frac{\cancel{2}^1 y^4}{\cancel{16}8} = \frac{y^4}{8}$$

$$y - \frac{y^4}{8} = 0$$

$$8y - y^4 = 0$$

$y^4 - 8y = 0$ $\qquad\qquad (1, 2)$

$$y(y^3 - 8) = 0$$

$\boxed{y = 0}$ $\boxed{y = 2}$

$\boxed{x = \infty}$ $\boxed{x = 1}$

$$f_{xx} = f_x\left(y - 2x^{-2}\right) = \frac{4}{x^3}$$

$$f_{yy} = f_y\left(x - 4y^{-2}\right) = \frac{8}{y^3}$$

$$f_{xy} = 1$$

$$\rightarrow D = f_{xx} f_{yy} - (f_{xy})^2$$

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	Result
(1, 2)	$4 > 0$	1	1	$3 > 0$	Relative Minima.

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

18

Wednesday, 13 March 2024 10:53 am

1–2 Locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

1. (a) $f(x, y) = (x-2)^2 + (y+1)^2$

(b) $f(x, y) = 1 - x^2 - y^2$

$f(x,y) = 1-(x^2+y^2) \rightarrow$ So 1 is the maximum value at $(0,0)$ and minimum value will be approaching towards the negative infinity.

9–20 Locate all relative maxima, relative minima, and saddle points, if any.

13. $f(x, y) = x^2 + y^2 + \dfrac{2}{xy} \rightarrow x^2+y^2+2x^{-1}y^{-1}$

$f_x = 2x - 2x^{-2}y^{-1} \rightarrow 2x = \dfrac{2}{x^2y} \rightarrow \cancel{2}x^3y = \cancel{2} \rightarrow x^3 = y$

$f_y = 2y - 2x^{-1}y^{-2} \rightarrow 2y = \dfrac{2}{xy^2} \rightarrow \cancel{2}xy^3 = \cancel{2} \rightarrow x = y^3$

Equating

$(y^3)^3 = y$ $\qquad (0,0)$

$y^9 - y = 0$ $\qquad (1,1)$

$y(y^8-1) = 0$ $\qquad (-1,-1)$

$\boxed{y=0}$ $\qquad y^8 = 1$

$\boxed{y = 0}$ $\quad y^8 = 1$

$\boxed{y = \pm 1}$

$$f_{xx} = f_x\left(2x - 2x^{-2}y^{-1}\right) = 2 + 4x^{-3}y^{-1}$$

$$f_{yy} = f_y\left(2y - 2x^{-1}y^{-2}\right) = 2 + 4x^{-1}y^{-3}$$

$$D = f_{xx} \cdot f_{yy} - \left(f_{xy}\right)^2$$

$$f_{xy} = 2x^{-2}y^{-2}$$

(x_0, y_0)	f_x	f_y	f_{xy}	D	Result
(0, 0)	2	2	∞		
(1, 1)					
(-1, -1)					

Lagrange Multiplier

5–12 Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint. Also, find the points at which these extreme values occur.

5. $f(x, y) = xy;\ 4x^2 + 8y^2 = 16$

$$\nabla f = \lambda \nabla g$$

$$\nabla f = y\,i + x\,j$$

$$y\,i + x\,j = \lambda(8x\,i + 16y\,j)$$

$y = \lambda 8x$ $\quad\quad$ $x = \lambda 16y$

$\lambda = \dfrac{y}{8x}$ — (1) $\quad\quad$ $\lambda = \dfrac{x}{16y}$ — (2)

$$\frac{y}{8x} = \frac{x}{16y}$$

$$\cancel{16}^{2} y^2 = {}^{1}\cancel{8} x^2$$

$$\boxed{x = \pm\sqrt{2}\,y}$$

Put in g

$$4x^2 + 8y^2 = 16$$

$$4(\sqrt{2}y)^2 + 8y^2 = 16$$

$$8y^2 + 8y^2 = 16$$

$$y^2 = 1$$

$$\boxed{y = \pm 1}$$

(x_0, y_0)	$f(x,y) = xy$
$(\sqrt{2}, 1)$	$\sqrt{2} \longrightarrow$ Max
$(\sqrt{2}, -1)$	$-\sqrt{2} \longrightarrow$ Min
$(-\sqrt{2}, 1)$	$-\sqrt{2} \longrightarrow$ Min
$(-\sqrt{2}, -1)$	$\sqrt{2} \longrightarrow$ Max

$y^2 = 1$

$\boxed{y = \pm 1}$

7. $f(x, y) = 4x^3 + y^2;\ 2x^2 + y^2 = 1$

$\nabla f = \lambda \nabla g$

$12x^2 i + 2y j = \lambda(4x i + 2y j)$

$12x^2 = \lambda 4x$

$12x^2 - 4\lambda x = 0$

$4x(3x - \lambda) = 0$

$\boxed{x = 0}$ $\quad 3x = \lambda$

$\boxed{x = \frac{\lambda}{3}}$

$2y = \lambda 2y$

$\boxed{\lambda = 1}$

$x = \frac{\lambda}{3} = \boxed{\frac{1}{3}}$

$2(0)^2 + y^2 = 1$

$\boxed{y = \pm 1}$

$2x^2 + y^2 = 1$

$2\left(\frac{1}{3}\right)^2 + y^2 = 1$

$\boxed{y = \pm \frac{\sqrt{7}}{3}}$

(x_0, y_0)	$f(x,y) = 4x^3 + y^2$
$(0, 1)$	1
$(0, -1)$	1
$(\frac{1}{3}, \sqrt{7}/3)$	25/27
$(\frac{1}{3}, -\sqrt{7}/3)$	25/27

Max (for the values 1, 1)

Min (for the values 25/27, 25/27)

6. $f(x, y) = x^2 - y^2;\; x^2 + y^2 = 25$

$$\nabla f = \lambda \nabla g$$

$$2x\,i - 2y\,j = \lambda\,(2x\,i + 2y\,j)$$

$$2x = 2x\lambda$$
$$2x\lambda - 2x = 0$$
$$2x(\lambda - 1) = 0$$
$$\boxed{x = 0} \quad \boxed{\lambda = 1}$$

Put in g

$$x^2 + y^2 = 25$$
$$0 + y^2 = 25$$
$$\boxed{y = \pm 5}$$
$$(0, 5) \quad (0, -5)$$

$$-2y = 2y\lambda$$
$$2y\lambda + 2y = 0$$
$$2y(\lambda + 1) = 0$$
$$\boxed{y = 0} \quad \boxed{\lambda = -1}$$

Put in g

$$x^2 + 0 = 25$$
$$\boxed{x = \pm 5}$$
$$(5, 0) \quad (-5, 0)$$

(x_0, y_0)	$x^2 - y^2$	
$(0, 5)$	-25	Min
$(0, -5)$	-25	
$(5, 0)$	25	Max
$(-5, 0)$	25	

8. $f(x, y) = x - 3y - 1;\; x^2 + 3y^2 = 16$

$$\nabla f = \lambda \nabla g$$

$$i - 3j = \lambda\,(2x + 6y)$$

$$2\lambda x = 1$$
$$\boxed{\lambda = \frac{1}{2x}} \rightarrow (1)$$

$$6y\lambda = -3$$
$$\boxed{\lambda = \frac{-1}{2y}} \rightarrow (2)$$

$$\frac{1}{2x} = \frac{-1}{2y}$$

$$\cancel{2}y = -\cancel{2}x$$

$$\boxed{y = -x} \rightarrow \text{Put in g}$$

(x_0, y_0)	$x - 3y - 1$

$\boxed{y = -x} \to$ Put in g

$x^2 + 3y^2 = 16$

$x^2 + 3(-x)^2 = 16$

$x^2 + 3x^2 = 16$

$x^2 = 4$

$\boxed{x = \pm 2}$

$\boxed{y = -2}$ $\boxed{y = 2}$

$(2, -2)$ $(-2, 2)$

(x_0, y_0)	$x - 3y - 1$	
$(2, -2)$	7	$\to$ Max
$(-2, 2)$	-9	$\to$ Min

11. $f(x, y, z) = xyz;\ x^2 + y^2 + z^2 = 1$

$\nabla f = \lambda \nabla g$

$yz\,i + xz\,j + xy\,k = \lambda(2x\,i + 2y\,j + 2z\,k)$

$yz = 2x\lambda$ | $xz = 2y\lambda$ | $xy = \lambda 2z$

$\boxed{\lambda = \frac{yz}{2x}}$ — (1) | $\boxed{\lambda = \frac{xz}{2y}}$ — (2) | $\boxed{\lambda = \frac{xy}{2z}}$ — (3)

$\frac{y\cancel{z}}{\cancel{2}x} = \frac{x\cancel{z}}{\cancel{2}y}$ $\qquad$ $\frac{\cancel{x}z}{\cancel{2}y} = \frac{\cancel{x}y}{\cancel{2}z}$

$\boxed{y^2 = x^2}$ $\qquad$ $\boxed{z^2 = y^2}$

$x^2 + y^2 + z^2 = 1$

$y^2 + y^2 + y^2 = 1$

$3y^2 = 1$

$\boxed{y = \pm\sqrt{\frac{1}{3}}}$

$\left(\sqrt{\frac{1}{3}}\right)^2 = z^2$

$\boxed{z = \pm\sqrt{\frac{1}{3}}}$

$y^2 = x^2$

$\left(\sqrt{\frac{1}{3}}\right)^2 = x^2$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$$\boxed{x = \pm\sqrt{\frac{1}{3}}}$$

(x_0, y_0, z_0)	xyz	
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\sqrt{3}/9$	$\longrightarrow$ Max
$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$-\sqrt{3}/9$	$\longrightarrow$ Min.

14.1 ≈ Double Integrals

$f(x,y) \quad a \le x \le b \;,\quad c \le y \le d$

$$\underbrace{\int_c^d \left[\int_a^b f(x,y)\, dx\right] dy}_{\text{Type I}} = \underbrace{\int_a^b \int_c^d f(x,y)\, dy\, dx}_{\text{Type II}}$$

Q1 $\displaystyle\int_0^1 \int_0^2 (x+1)\, dy\, dx$

$$\int_0^1 \left| (x+1)\cdot y \right|_0^2 dx$$

$$\int_0^1 2(x+1)\, dx$$

$$\left| 2\left(\frac{x^2}{2} + x\right) \right|_0^1$$

$$\left| x^2 + 2x \right|_0^1$$

$$(1+2) - (0+0) = \boxed{3} \text{ Ans.}$$

Shortcut to integrate directly, below scenario.

$$\int (f(x))^n \cdot f'(x)\, dx$$

$$(f(x))^{n+1}$$

$$\frac{(f(x))^{n+1}}{n+1}$$

9. $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$

$$\int_0^1\int_0^1 x(xy+1)^{-2}\, dy\, dx$$

$$\int_0^1 \left| \frac{(xy+1)^{-1}}{-1} \right|_0^1 dx$$

$$\int_0^1 \left(\frac{(x+1)^{-1}}{-1} - \frac{(0+1)^{-1}}{-1} \right) dx$$

$$\int_0^1 \frac{-1}{x+1} + 1\, dx$$

$$\left(-\ln(x+1) + x\right)_0^1$$

$$[-\ln(1+1)+1] - [-\ln(0+1)+0]$$

$$-\ln 2 + 1 + \cancel{\ln 1} - \cancel{0}$$

$$\boxed{1-\ln 2} \text{ Ans.}$$

11. $\int_0^{\ln 2} \int_0^1 xye^{y^2x}\, dy\, dx$

$$u = xy^2$$

$$\frac{du}{dy} = 2xy$$

$$\frac{du}{2} = xy\, dy$$

$$\int_0^{\ln 2}\int_0^1 e^u \cdot \frac{du}{2} \cdot dx$$

$$\int_0^{\ln 2} \left| \frac{e^u}{2} \right|_0^1 dx$$

$$\frac{1}{2}\int_0^{\ln 2} \left| e^{xy^2} \right|_0^1 dx$$

$$\frac{1}{2}\int_0^{\ln 2} \left[e^x - e^0 \right] dx$$

$$\frac{1}{2}\int_0^{\ln 2} (e^x - 1)\, dx$$

$$\frac{1}{2} \left| e^x - x \right|_0^{\ln 2}$$

$$\frac{1}{2}\left[e^{\ln 2} - \ln 2 \right] - \frac{1}{2}\left[e^0 - 0 \right]$$

$$\frac{1}{2}\left[2 - \ln 2 \right] - \frac{1}{2}\left[1 \right]$$

$$1 - \frac{\ln 2}{2} - \frac{1}{2}$$

$$\frac{1}{2} - \frac{\ln 2}{2} = \boxed{\frac{1}{2}(1 - \ln 2)} \text{ Ans.}$$

Gradient Decent Algorithm.

$$f(X) = f(x_1, x_2) = 4x_1^2 + 3x_1x_2 + 2.5x_2^2 - 5.5x_1 - 4x_2$$

$$X_{n+1} = X_n - \alpha f'(X_n)$$

$n = 0$

$$X_1 = X_0 - \alpha f'(X_0)$$

Initial value OR Initial guess

Initial guess can be any value from the domain of function except where the slope is zero.

$$X_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{matrix} \to x_1 \\ \to x_2 \end{matrix} \quad , \quad \alpha = 0.135$$

- We have assumed these values, keeping in mind that $\frac{\partial f}{\partial x_1}$ or $\frac{\partial f}{\partial x_2}$ does not become 0.
- Values of x_1 and x_2 can be different.

$$f'(x) = \begin{bmatrix} 8x_1 + 3x_2 - 5.5 \\ 3x_1 + 5x_2 - 4 \end{bmatrix} \begin{matrix} \to \frac{\partial f}{\partial x_1} = 16.5 \\ \to \frac{\partial f}{\partial x_2} = 12 \end{matrix}$$

$$x_1 = x_0 - \alpha f'(x_0)$$

$$x_1 = x_0 - \alpha f'(x_0)$$

$$x_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 0.135 \begin{bmatrix} 16.5 \\ 12 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} -0.2275 \\ 0.38 \end{bmatrix}$$

$\downarrow$

Now put these values in $f(x_1)$

$$f(x_1) = 4(-0.2275)^2 + 3(-0.2275)(0.38) + 2.5(0.38)^2 - 5.5(-0.2275) - 4(0.38)$$

$$\boxed{f(x_1) = 0.0399}$$

$$f'(x_1) = \begin{bmatrix} 8(-0.2275) + 3(0.38) - 5.5 \\ 3(-0.2275) + 5(0.38) - 4 \end{bmatrix} = \begin{bmatrix} -6.18 \\ -2.7825 \end{bmatrix}$$

$$x_2 = x_1 - \alpha f'(x_1)$$

$$x_2 = \begin{bmatrix} -0.2275 \\ 0.38 \end{bmatrix} - 0.135 \begin{bmatrix} -6.18 \\ -2.7825 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0.6068 \\ 0.7556 \end{bmatrix} \rightarrow f(x_2) = -2.0841$$

$$f'(x_2) = \begin{bmatrix} 8(0.6068) + 3(0.7556) - 5.5 \\ 3(0.6068) + 5(0.7556) - 4 \end{bmatrix} = \begin{bmatrix} 1.6212 \\ 1.5984 \end{bmatrix}$$

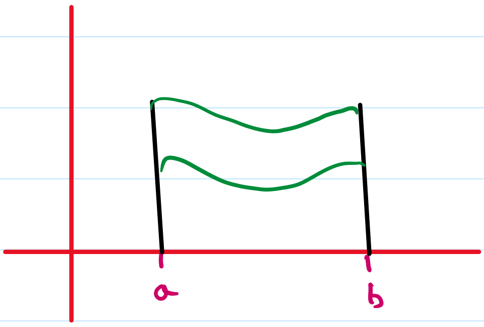
$$x_3 = x_2 - \alpha f'(x_2)$$

$$x_3 = \begin{bmatrix} 0.6068 \\ 0.7556 \end{bmatrix} - 0.135 \begin{bmatrix} 1.6212 \\ 1.5984 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 0.3879 \\ 0.5398 \end{bmatrix}$$

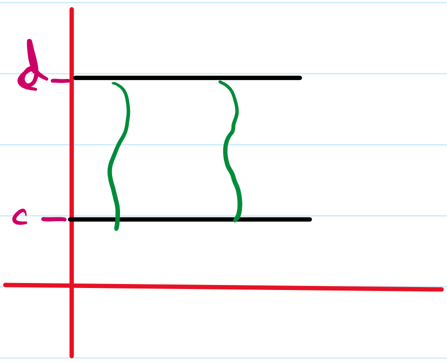
14.2 ~ Double Integrals over non-Rectangular regions.

Type I Region



$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Type II Region



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

1–8 Evaluate the iterated integral.

1. $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2\, dy\, dx$

$$\int_0^1 \left| x \frac{y^3}{3} \right|_{x^2}^{x} dx$$

$$\int_0^1 \left[\left(\frac{x(x)^3}{3} \right) - \left(\frac{x\,(x^2)^3}{3} \right) \right] dx$$

$$\int_0^1 \left(\frac{x^4}{3} - \frac{x^7}{3} \right) dx$$

$$\frac{1}{3}\left|\frac{x^5}{5}-\frac{x^8}{8}\right|_0^1 = \frac{1}{3}\left[\frac{1}{5}-\frac{1}{8}\right] = \boxed{\frac{1}{40}} \text{ Ans.}$$

3. $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}} y\,dx\,dy$

$$\int_0^3 \left| x\,y \right|_0^{\sqrt{9-y^2}} dy$$

$$\int_0^3 \left(y\sqrt{9-y^2}\right) dy$$

$$\int_0^3 -\frac{1}{2}\times\left(-2y\,(9-y^2)^{\frac{1}{2}}\right) dy$$

$$-\frac{1}{\cancel{2}}\left|\frac{(9-y^2)^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}\right|_0^3$$

$$-\frac{1}{3}\left[\left(\left(9-(3)^2\right)^{\frac{3}{2}}\right)-\left(9-0\right)^{\frac{3}{2}}\right] = -\frac{1}{3}\left[0-27\right] = \boxed{9} \text{ Ans.}$$

5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}}\int_0^{x^3} \sin\frac{y}{x}\,dy\,dx$

$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \left|\frac{-\cos y/x}{\frac{1}{x}}\right|_0^{x^3} dx$$

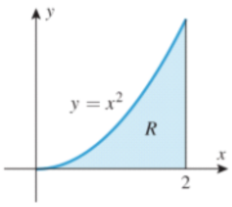
$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \left(-x\cos x^2 + x\right) dx$$

$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \left[\left(-\frac{1}{2}\times -2x\cos x^2\right) + x\right] dx$$

$$\left|-\frac{1}{2}\sin x^2 + \frac{x^2}{2}\right|_{\sqrt{\pi}}^{\sqrt{2\pi}} = \left(-\frac{1}{2}\sin 2\pi + \frac{2\pi}{2}\right) - \left(-\frac{1}{2}\sin\pi + \frac{\pi}{2}\right)$$

$$= \left(0 + \pi\right) - \left(0 + \frac{\pi}{2}\right) = \pi - \frac{\pi}{2} = \boxed{\frac{\pi}{2}} \text{ Ans.}$$

9. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.



▲ **Figure Ex-9**

Type I

$$\boxed{\int_0^2 \int_0^{x^2} f(x,y)\; dy\; dx}$$

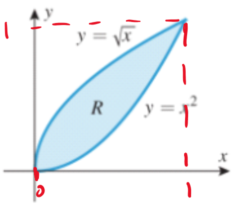
Type II

$y = x^2$

$y = 2^2 = 4$

$$\boxed{\int_0^4 \int_{\sqrt{y}}^{2} f(x,y)\; dx\; dy.}$$

10. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.



▲ **Figure Ex-10**

$x^2 = \sqrt{x}$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$\boxed{x = 0}$ $\boxed{x = 1}$

Type I

$$\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\; dy\; dx$$

Type II

$y = \sqrt{x}$ $y = x^2$

$x = y^2$ $x = \sqrt{y}$

$y^2 = \sqrt{y}$

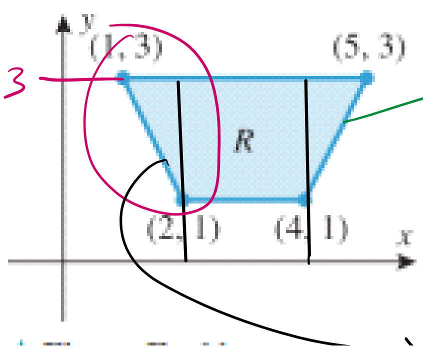
$y^4 = y$

$y(y^3 - 1) = 0$

$\boxed{y = 0}$ $\boxed{y = 1}$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} f(x,y)\; dx\; dy$$

11. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

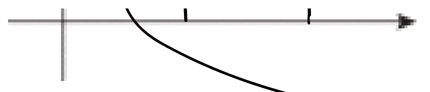


$y = mx + c$ $m = \frac{3-1}{5-4} = \boxed{2}$

$3 = 2(5) + c$

$\boxed{c = -7}$ $\boxed{y = 2x - 7}$ 2

$m = \frac{3-1}{1-2} = \boxed{-2}$



▲ **Figure Ex-11**

$1 - 2$

$y = mx + c$

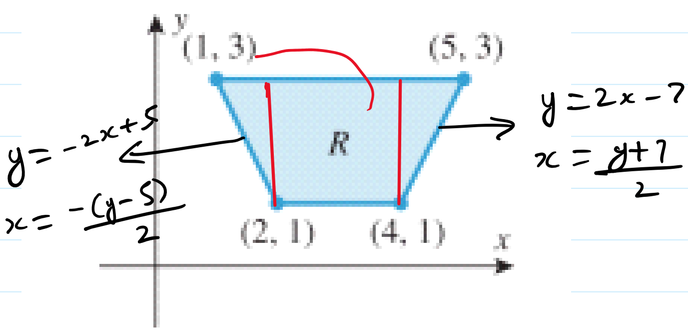
$3 = -2(1) + c$ $\longrightarrow$ $\boxed{y = -2x + 5}$ — 1

$\boxed{c = 5}$

Type I

$$\int_1^2 \int_{-2x+5}^{3} f(x,y)\, dy\, dx + \int_2^4 \int_1^3 f(x,y)\, dy\, dx + \int_4^5 \int_{2x-7}^{3} f(x,y)\, dy\, dx$$

Type II



▲ **Figure Ex-11**

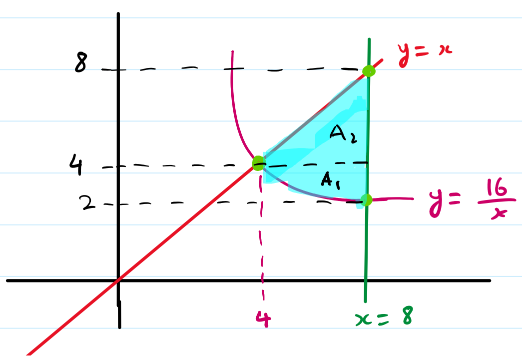
$$\boxed{\int_1^3 \int_{\frac{-(y+5)}{2}}^{\frac{y+7}{2}} f(x,y)\, dx\, dy}$$

Ans.

15–18 Evaluate the double integral in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region.

15. $\iint_R x^2\, dA$; R is the region bounded by $y = 16/x$, $y = x$, and $x = 8$.

$y = 16/x$, $y = x$, $x = 8$

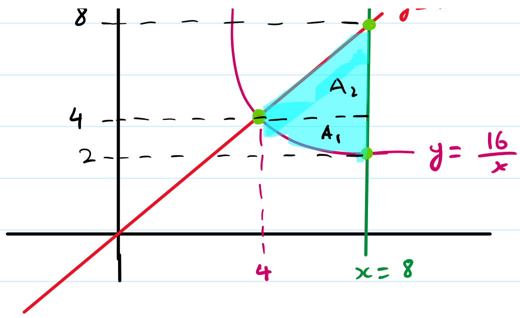


$y = x$, $y = \frac{16}{x}$

$x = \frac{16}{x}$

$16 = x^2$

$\boxed{x = \pm 4}$



$y = x \quad , \quad y = \frac{16}{x}$

$x = \frac{16}{x}$

$16 = x^2$

$\boxed{x = \pm 4}$

Type I

$$A = \int_4^8 \int_{16/x}^{x} f(x, y)\, dy\, dx$$

Type II

$x = 8 \quad , \quad y = \frac{16}{x}$

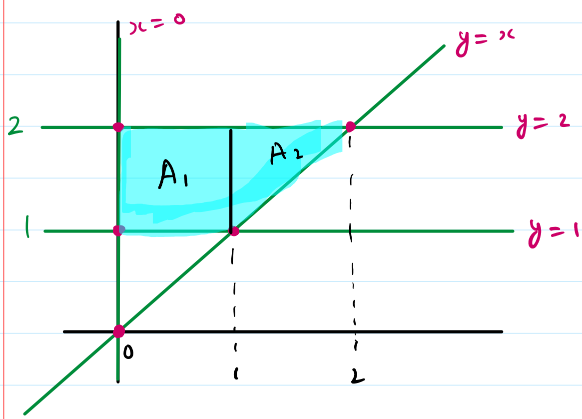
$8 = \frac{16}{x}$

$\boxed{x = 2}$

$A = A_1 + A_2$

$$\boxed{A = \int_2^4 \int_{16/y}^{8} f(x,y)\, dx\, dy + \int_4^8 \int_y^8 f(x,y)\, dx\, dy .}$$ Ans.

16. $\iint_R xy^2\, dA$; R is the region enclosed by $y = 1$, $y = 2$, $x = 0$, and $y = x$.



Type I

$A = A_1 + A_2$

$$A = \int_0^1 \int_1^2 (xy^2)\, dy\, dx + \int_1^2 \int_x^2 (xy^2)\, dy\, dx$$

Type II

$$A = \int_1^2 \int_0^y (xy^2)\, dx\, dy .$$

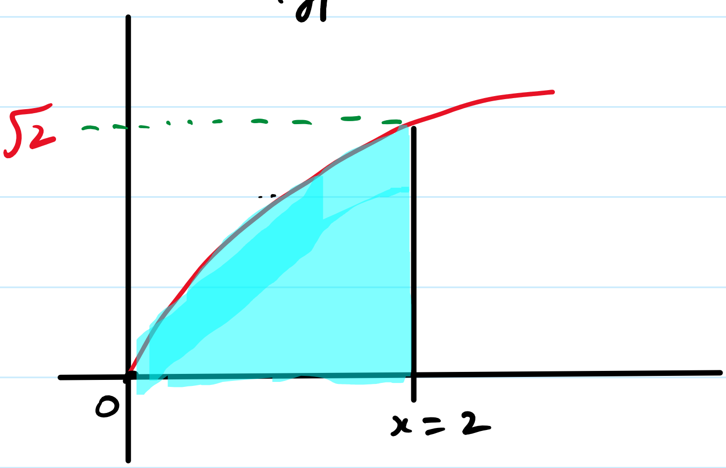
47–52 Express the integral as an equivalent integral with the order of integration reversed.

47. $\int_0^2 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$

$\downarrow$ Type I

$y = \sqrt{x}$

$y^2 = x$



Type II

$y^2 = 2$

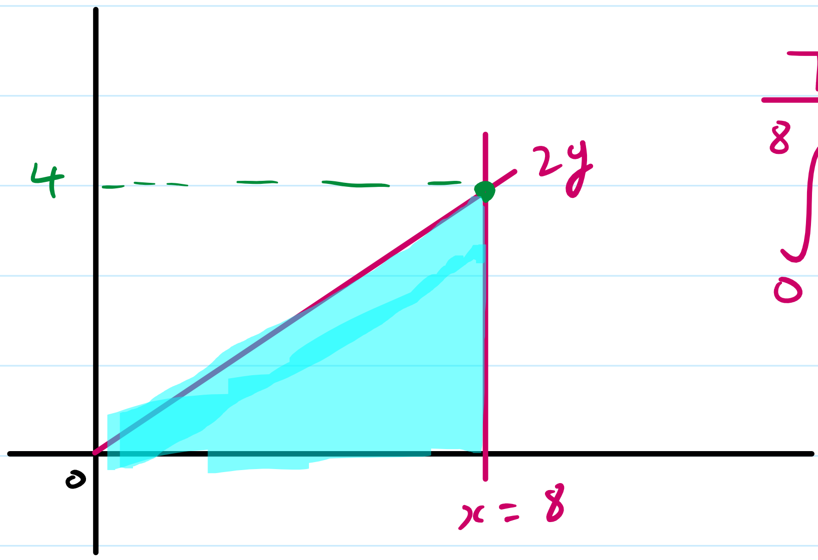
$\boxed{y = \sqrt{2}}$

$$\int_0^{\sqrt{2}} \int_{y^2}^{2} (\quad)\, dx\, dy$$

48. $\int_0^4 \int_{2y}^8 f(x, y)\, dx\, dy$

$2y = x$

$y = \frac{x}{2}$



Type I

$$\int_0^8 \int_0^{x/2} f(x,y)\, dy\, dx$$

49. $\int_0^2 \int_1^{e^y} f(x, y)\, dx\, dy$

$x = e^y$

49. $\int_0 \int_1 f(x, y)\,dx\,dy$

$x = e^y$

$\boxed{\ln x = y}$

$\ln x = 2$

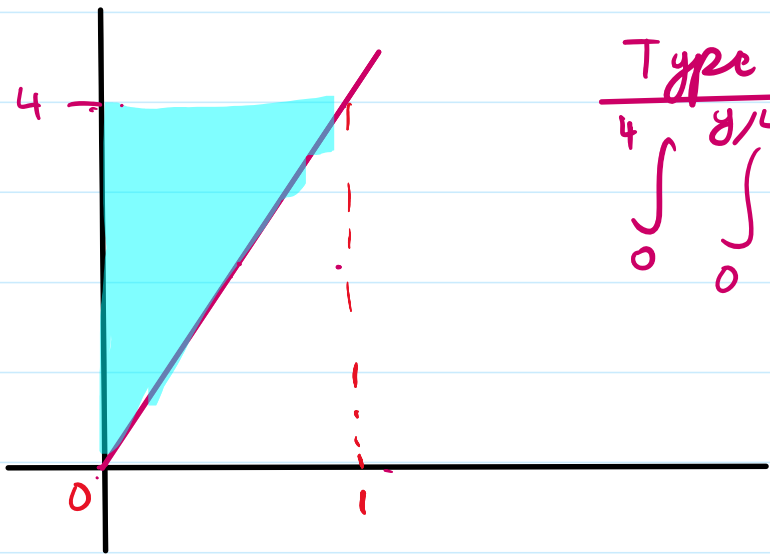
$\boxed{x = e^2}$

Type I

$$\int_1^{e^2}\int_{\ln x}^{2} f(x,y)\,dy\,dx$$

53–56 Evaluate the integral by first reversing the order of integration.

53. $\displaystyle\int_0^1\int_{4x}^4 e^{-y^2}\,dy\,dx$



Type II

$$\int_0^4\int_0^{y/4} e^{-y^2}\,dx\,dy$$

$$\int_0^4 \left| x \cdot e^{-y^2} \right|_0^{y/4} dy$$

$$\int_0^4 \left(\frac{y}{4} e^{-y^2} - 0\right) dy$$

$$\int_0^4 -\frac{1}{2} \times \left(-\frac{2y}{4} e^{-y^2}\right) dy$$

$$-\frac{1}{2} \left| \frac{e^{-y^2}}{4} \right|_0^4$$

$$\left(\frac{-e^{-16}}{8}\right) - \left(\frac{-e^{0}}{8}\right)$$

$$\frac{-e^{-16}}{8} + \frac{1}{8}$$

$$\boxed{\frac{1}{8}\left(1 - e^{-16}\right)}$$ Ans.

14.3 ~ Double Integrals in Polar Coordinates

(r, θ)

$x = r\cos\theta \qquad y = r\sin\theta$

$r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$

$A = \iint f(r,\theta)\, dA$

$\downarrow$

$dA = r\, dr\, d\theta$

$a \le r \le b$

$r_1(\theta) \le \theta \le r_2\theta$

$$\int_a^b \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

1–6 Evaluate the iterated integral.

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\, dr\, d\theta$

$$\int_0^{\frac{\pi}{2}} \left| \frac{r^2}{2}\cos\theta \right|_0^{\sin\theta} d\theta$$

$$\int_0^{\frac{\pi}{2}} \left(\left(\frac{\sin^2\theta}{2}\cos\theta\right) - (0)\right) d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{2}\sin^2\theta\, d\theta$$

$$\left| \frac{1}{2} \times \frac{(\sin\theta)^3}{3} \right|_0^{\pi/2}$$

$$\left(\frac{1}{6}\right) - (0) = \boxed{\frac{1}{6}} \text{ Ans.}$$

2. $\displaystyle\int_0^{\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta$

$$\int_0^{\pi} \left|\frac{r^2}{2}\right|_0^{1+\cos\theta} d\theta$$

$$\int_0^{\pi} \frac{(1+\cos\theta)^2}{2}\, d\theta$$

$$\frac{1}{2}\int_0^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$\frac{1}{2}\int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2}\left(1+\cos 2\theta\right)\right) d\theta$$

$$\frac{1}{2}\int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$\frac{1}{2}\left|\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right|_0^{\pi}$$

$$\frac{1}{2}\left[\left(\frac{3}{2}\pi\right) - (0)\right] = \boxed{\frac{3}{4}\pi} \text{ Ans.}$$

4. $\displaystyle\int_0^{\pi/6} \int_0^{\cos 3\theta} r\,dr\,d\theta$

$$\int_0^{\pi/6} \left|\frac{r^2}{2}\right|_0^{\cos 3\theta} d\theta$$

$$\frac{1}{2}\int_0^{\pi/6} (\cos 3\theta)^2\, d\theta \qquad \because \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\frac{1}{2}\int_0 (\ldots)\, d\theta \quad \bullet \quad \cos^2 \ldots = \frac{1}{2}(1+\cos\ldots)$$

$$\frac{1}{2}\int_0^{\pi/6} \frac{1}{2}\left(1+\cos 6\theta\right)\, d\theta)$$

$$\frac{1}{4}\left| \theta + \frac{\sin 6\theta}{6} \right|_0^{\pi/6}$$

$$\frac{1}{4}\left[\left(\frac{\pi}{6}\right) - (0+0)\right] = \boxed{\frac{\pi}{24}} \text{ Ans.}$$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$

$$\int_0^{\pi} \left| \frac{r^3}{3}\cos\theta \right|_0^{1-\sin\theta} d\theta$$

$$\frac{1}{3}\int_0^{\pi} \left((1-\sin\theta)^3 \cos\theta\right)\, d\theta$$

$$\frac{1}{3}\left| \frac{(1-\sin\theta)^4}{4} \right|_0^{\pi} = \frac{1}{12}\left[(1-\sin\pi)^4 - (1-\sin 0)^4\right]$$

$$= \frac{1}{12}\left[1^4 - 1^4\right] = \boxed{0}$$

6. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^3\, dr\, d\theta$

$$\int_0^{\pi/2} \left| \frac{r^4}{4} \right|_0^{\cos\theta} d\theta$$

$$\frac{1}{4}\int_0^{\pi/2} (\cos\theta)^4\, d\theta \approx \frac{1}{4}\int_0^{\pi/2} (\cos^2\theta)^2\, d\theta$$

$$\frac{1}{4}\int \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

$$\frac{1}{4}\int\left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

$$\frac{1}{16}\int_0^{\pi/2}(1+\cos 2\theta)^2\, d\theta$$

$$\frac{1}{16}\int\left(1 + 2\cos 2\theta + (\cos 2\theta)^2\right) d\theta$$

$$\frac{1}{16}\int\left(1 + 2\cos 2\theta + \left(\frac{1}{2}(1+\cos 4\theta)\right)\right) d\theta$$

$$\frac{1}{16}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right) d\theta$$

$$\frac{1}{16}\left|\frac{3}{2}\theta + \frac{\cancel{2}\sin 2\theta}{\cancel{2}} + \frac{\sin 4\theta}{8}\right|_0^{\pi/2}$$

$$\frac{1}{16}\left[\left(\frac{3\pi}{4}\right) - (0)\right] = \boxed{\frac{3\pi}{64}}$$ Ans.

7–10 Use a double integral in polar coordinates to find the area of the region described.

7. The region enclosed by the cardioid $r = 1 - \cos\theta$.

27–34 Evaluate the iterated integral by converting to polar coordinates.

27. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

$\downarrow$

dA

$r\, dr\, d\theta$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

+ve x-axis

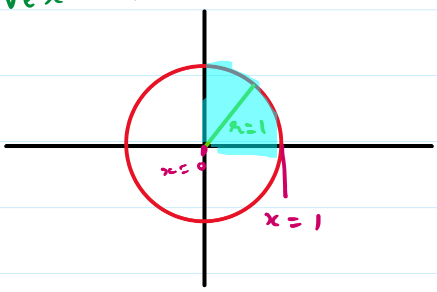
+ve y-axis.

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$\boxed{x^2 + y^2 = 1}$ → Circle eq

radius.



• $\frac{\pi}{2}$ b/c shaded region is 1st quadrant.

• 0 to 1 b/c of the origin to radius.

$$\int_0^{\pi/2}\int_0^1 r^2 \cdot r\, dr\, d\theta = \boxed{\int_0^{\pi/2}\int_0^1 r^3\, dr\, d\theta}$$

Further solve.

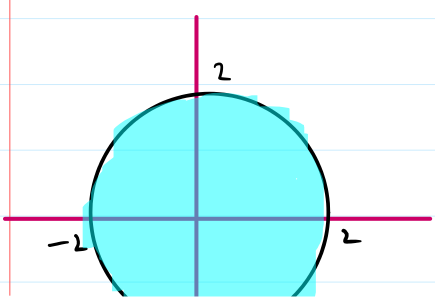
28. $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\, dx\, dy$

$x = \sqrt{4-y^2}$ → $r\cos\theta = \sqrt{4 - r^2\sin^2\theta}$

$x^2 = 4 - y^2$ $\quad r^2\cos^2\theta = 4 - r^2\sin^2\theta$

$x^2 + y^2 = 2^2$ $\quad r^2 = 4$

$\boxed{r = \pm 2}$

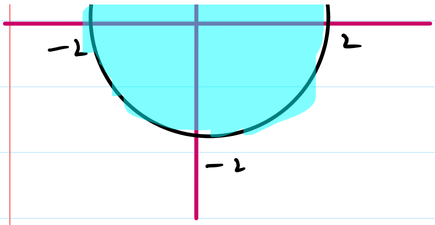


$$\int_0^{2\pi} \frac{1}{2}\int_0^{2} e^{-r^2}\, 2r\, dr\, d\theta$$

$$\frac{1}{2}\left| (e^{-4}-1)\,\theta \right|_0^{2\pi}$$

$$\frac{1}{\cancel{2}}\left[(e^{-4}-1)\,\cancel{2}\pi\right]$$

2π



$$\frac{1}{2}\int_0^{2\pi}\left|\, e^{-r^2}\right|_0^2 \, d\theta$$

$$\frac{1}{2}\int_0^{2\pi}\left((e^{-4}) - 1\right) d\theta$$

$\frac{1}{2}\left[(e^{-4}-1)\, 2\pi\right]$

$\boxed{\pi(e^{-4}-1)}$ Ans.

29. $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

$x=0,\ x=2$

$y=0$, $(y)^2 = \left(\sqrt{2x-x^2}\right)^2$

$y^2 = 2x - x^2$

$r^2\sin^2\theta = 2(r\cos\theta) - (r\cos\theta)^2$

$r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$

$r^2(1) = 2r\cos\theta$

$\boxed{r = 2\cos\theta}$

$y = 0$

$r\sin\theta = 0$

$\sin\theta = 0$

$\boxed{r = 0}$

$x = 0$

$r\cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\boxed{\theta = \frac{\pi}{2}}$

, $x = 2$

$r\cos\theta = 2$

$(2\cos\theta)\cos\theta = 2$

$2\cos^2\theta = 2$

$\cos\theta = 1$

$\theta = \cos^{-1}(1)$

$\boxed{\theta = 0}$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} r \cdot r\, dr\, d\theta$$

$$\boxed{\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\, dr\, d\theta}$$ Ans.

32. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2+y^2}\, dx\, dy$

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$y = 0$

$r \sin\theta = 0$

$\sin\theta = 0$

$\boxed{\theta = 0}$

$x = y$

$\cancel{r}\cos\theta = \cancel{r}\sin\theta$

$\tan\theta = 1$

$\boxed{\theta = \frac{\pi}{4}}$

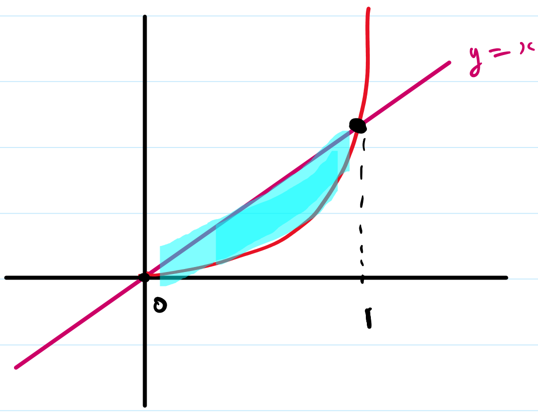
, $x = \sqrt{y}$

$x^2 = y$

$r^{\cancel{2}}\cos^2\theta = \cancel{r}\sin\theta$

$r = \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}$

$\boxed{r = \tan\theta \cdot \sec\theta}$



$$\int_0^{\pi/4}\int_0^{\sec\theta\tan\theta} r \cdot r \, dr \, d\theta = \int_0^{\pi/4}\int_0^{\sec\theta\tan\theta} r^2 \, dr \, d\theta$$

$$\int_0^{\pi/4} \left| \frac{r^3}{3} \right|_0^{\sec\theta\tan\theta}$$

$$\frac{1}{3}\int_0^{\pi/4} \sec^3\theta \tan^3\theta \, d\theta$$

$u = \sec\theta$

$\frac{du}{d\theta} = \sec\theta\tan\theta$

$d\theta = \frac{du}{\sec\theta\tan\theta}$

$$\frac{1}{3}\int_0^{\pi/4} \sec^{\cancel{3}\,2}\theta \tan^{\cancel{3}\,2}\theta \times \frac{du}{\cancel{\sec\theta\tan\theta}}$$

$$\frac{1}{3}\int_0^{\pi/4} \sec^2\theta \,(\sec^2\theta - 1)\, du$$

$$\frac{1}{3}\int^{\pi/4} u^2(u^2 - 1)\, du = \frac{1}{3}\int^{\pi/4} (u^4 - u^2)\, du$$

$$\frac{1}{3}\int_0^{\pi/4} u^2(u^2-1)\,du = \frac{1}{3}\int_0^{\pi/4}(u^4-u^2)\,du$$

$$\frac{1}{3}\left|\frac{u^5}{5}-\frac{u^3}{3}\right|_1^{2/\sqrt{2}}$$

$u = \sec\theta$

$u = \sec(0)$

$u = \frac{1}{\cos 0}$

$\boxed{u = 1}$

$u = \frac{1}{\cos\left(\frac{\pi}{4}\right)}$

$u = \frac{1}{\frac{\sqrt{2}}{2}}$

$\boxed{u = \frac{2}{\sqrt{2}}} = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

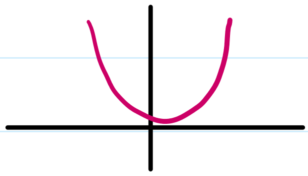
$u = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \boxed{\sqrt{2}}$

$$\frac{1}{3}\left[\left(\frac{(\sqrt{2})^5}{5}-\frac{(\sqrt{2})^3}{3}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)\right]$$

$$\frac{1}{3}\left[\frac{3(\sqrt{2})^5-5(\sqrt{2})^3}{15}-\frac{3-5}{15}\right]$$

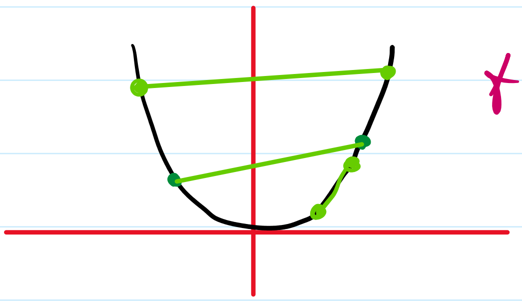
$$\frac{1}{3}\left[\frac{3(\sqrt{2})^5-5(\sqrt{2})^3+2}{15}\right] = \boxed{0.107}\ \text{Ans.}$$

Convex Optimization



Convex function

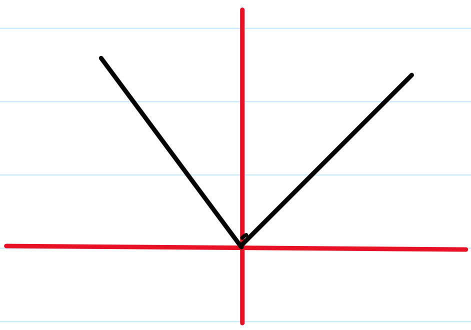
- If we draw line on two points lying on this curve, the line would always be above the curve or on the curve. (NEVER BELOW).



$f(x) = x^2$

$\longrightarrow$ Convex

$f(x) = |x|$

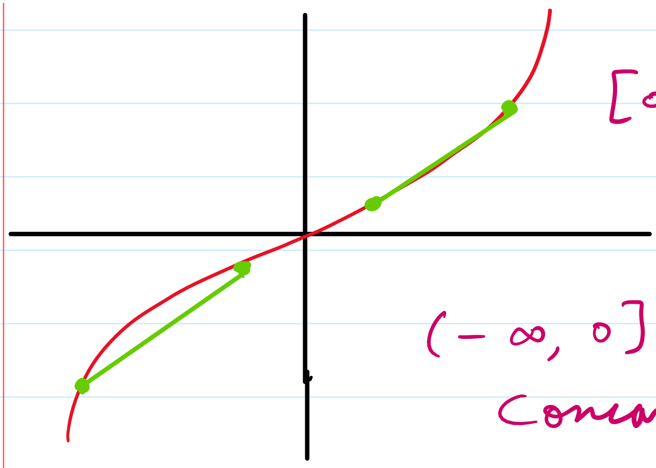


$\longrightarrow$ Convex

Concave function

- If we draw line on two points lying on this curve, the line would always be below the curve or on the curve. (NEVER ABOVE).

$f(x) = x^3$



$$f(t x_1 + (1-t) x_2) \leq t f(x_1) + (1-t) f(x_2)$$

→ We will not use the above technique.

Hessian Matrix

1. Positive definite → Convex so relative minimum point
2. Negative definite → Concave so relative maximum point
3. Indefinite → Saddle Point.

$$f(x_1, x_2, x_3) = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{vmatrix}$$

1. +ve definite → Convex (relative minima)

Conditions

$D_1 \quad 1 \times 1 > 0$

D_2 $\quad 2 \times 2 \quad > 0$

D_3 $\quad 3 \times 3 \quad > 0$

② −ve definite → Concave (relative maxima)

Conditions:

D_1 $\quad 1 \times 1 \quad < 0$

D_2 $\quad 2 \times 2 \quad > 0$

D_3 $\quad 3 \times 3 \quad < 0$

③ Indefinite → Saddle Point.

→ For this, any other order except above two.

Q $f(x, y, z) = x^4 + y^4 + z^4 + x^2 + y^2 + z^2$

$f_x = 4x^3 + 2x \rightarrow 4x^3 + 2x = 0 \rightarrow x(4x^2 + 2) = 0$

$\boxed{x = 0}$

$f_y = 4y^3 + 2y \rightarrow 4y^3 + 2y = 0 \rightarrow y(4y^2 + 2) = 0$

$\boxed{y = 0}$

$f_z = 4z^3 + 2z \rightarrow 4z^3 + 2z = 0 \rightarrow z(4z^2 + 2) = 0$

$\boxed{z = 0}$

$(0, 0, 0)$

$$\begin{vmatrix} |12x^2+2| & 0 & 0 \\ 0 & |12y^2+2| & 0 \\ 0 & 0 & |12z^2+2| \end{vmatrix}$$

$f(x,y,z) = (0,0,0)$

$D_1 = 12x^2+2 = 12(0)+2 = 2 > 0$

$D_2 = (12x^2+2)(12y^2+2) = 4 > 0$

$D_3 = (12x^2+2)(12y^2+2)(12z^2+2) = 8 > 0$

Hence, +ve definite $\rightarrow$ Convex $\rightarrow$ relative Minima

14.5 ≈ Triple Integrals

1–8 Evaluate the iterated integral.

1. $\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz$

$$\int_{-1}^{1}\int_{0}^{2}\left|\frac{x^3}{3}+y^2x+z^2x\right|_0^1 dy\,dz$$

$$\int_{-1}^{1}\int_{0}^{2}\left(\frac{1}{3}+y^2+z^2\right)dy\,dz$$

$$\int_{-1}^{1}\left|\frac{1}{3}y+\frac{y^3}{3}+z^2y\right)_0^2 dz$$

$$\int_{-1}^{1}\left(\frac{2}{3}+\frac{8}{3}+2z^2\right)dz$$

$$\left|\frac{10}{3}z+\frac{2z^3}{3}\right|_{-1}^{1}=\left[\frac{10}{3}+\frac{2}{3}\right]-\left[\frac{-10}{3}-\frac{2}{3}\right]$$

$$\frac{12}{3}+\frac{12}{3}=\boxed{8}\text{ Ans.}$$

5. $\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{0}^{x}xy\,dy\,dx\,dz$

$$\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\left|\frac{xy^2}{2}\right|_0^x dx\,dz$$

$$\frac{1}{}\int^{3}\int^{\sqrt{9-z^2}}x^3\,dx\,dz=\frac{1}{}\int^{3}\left(\left|\frac{x^4}{}\right)^{\sqrt{9-z^2}}dz$$

$$\frac{1}{2}\int_0^3\int_0^{\sqrt{9-z^2}} x^3 \, dx \, dz = \frac{1}{2}\int_0^3 \left| \frac{x^4}{4} \right|_0^{\sqrt{9-z^2}} dz$$

$$\frac{1}{8}\int_0^3 (9-z^2)^2 \, dz = \frac{1}{8}\int^3 (81 - 18z + z^4) \, dz$$

$$\frac{1}{8}\left| 81z - 9z^2 + \frac{z^5}{5} \right|_0^3 = \frac{1}{8}\left[243 - 81 + \frac{243}{5} \right] = \boxed{\frac{1053}{40}}$$

Ans

14.4 ≈ Surface Area

$z = f(x,y)$

$$A = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

1–4 Express the area of the given surface as an iterated double integral, and then find the surface area.

1. The portion of the cylinder $y^2 + z^2 = 9$ that is above the rectangle $R = \{(x, y) : 0 \leq x \leq 2, -3 \leq y \leq 3\}$.

• First make z subject b/c x & y limits are given.

$y^2 + z^2 = 9$

$z^2 = 9 - y^2$

$z = \sqrt{9 - y^2}$

$$\frac{\partial z}{\partial x} = 0 \quad , \quad \frac{\partial z}{\partial y} = \frac{1}{\cancel{2}\sqrt{9-y^2}} \times -\cancel{2}y$$

$$\iint \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$\iint \sqrt{0^2 + \left(\frac{-y}{\sqrt{9-y^2}}\right)^2 + 1}\, dA = \iint \sqrt{\left(\frac{y^2}{9-y^2} + 1\right)}\, dA$$

$$\begin{array}{r} -1 \\ 9-y^2 \overline{) \; y^2 \qquad} \end{array}$$

$$9-y^2 \overline{\big) \cancel{y^2}} \quad -1$$
$$\cancel{y^2} - 9$$
$$- \quad +$$

$$\rightarrow -1 + \frac{9}{9-y^2}$$

$$\iint \sqrt{\cancel{-1} + \frac{9}{9-y^2} + \cancel{1}} \; dA$$

$$\int_0^2 \int_{-3}^{3} \frac{3}{\sqrt{(3)^2-y^2}} \, dy \, dx$$

$$3\int_0^2 \left| \sin^{-1} \frac{y}{3} \right|_{-3}^{3} dx$$

$$3\int_0^2 \left[\sin^{-1}(1) - \sin^{-1}(-1) \right] dx$$

$$3\int_0^2 \left(\pi - \frac{3\pi}{2} \right) dx$$

$$-\frac{3}{2} \int_0^2 \pi \, dx$$

$$-\frac{3}{2} \left| \pi x \right|_0^2$$

$$-\frac{3}{2} \left[2\pi - 0 \right] = \boxed{-3\pi} \text{ Ans.}$$

2. The portion of the plane $2x + 2y + z = 8$ in the first octant.

$2x + 2y + z = 8$

$z = 8 - 2x - 2y$

$0 = 8 - 2x - 2y$

$0 = 2(4 - x - y)$

$\boxed{y = 4 - x}$

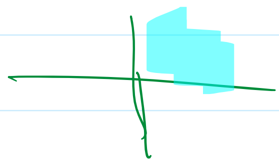
$0 = 4 - x$

$\boxed{x = 4}$

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{array}\right\} \rightarrow \begin{array}{l} x = 0 \\ y = 0 \\ z = 0 \end{array}$

• Lower limit will always be 0 in first octant

$z = 8 - 2x - 2y$



$\dfrac{\partial z}{\partial x} = -2 \qquad \dfrac{\partial z}{\partial y} = -2$

$$\int_0^4 \int_0^{4-x} \sqrt{(-2)^2 + (-2)^2 + 1}\; dy\, dx = \int_0^4 \int_0^{4-x} 3 \; dy\, dx$$

$$\int_0^4 \left| 3y \right|_0^{4-x} dx$$

$$3 \int_0^4 (4 - x)\; dx$$

$$3 \left| 4x - \frac{x^2}{2} \right|_0^4 = 3\left[\left(16 - \frac{16}{2}\right) - \left(0 - 0^2\right)\right] = \boxed{24} \text{ Ans.}$$

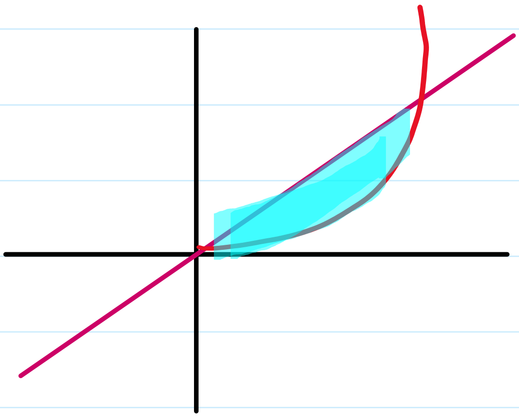
3. The portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

$y = x \qquad y = x^2$

$z^2 = 4x^2 + 4y^2$

$z = 4x + 4y^2$

$z = \sqrt{4x^2 + 4y^2}$



$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1$

$$\frac{\partial z}{\partial x} = \frac{1}{\cancel{2}\sqrt{4x^2+4y^2}} \times \overset{4}{\cancel{8}}x = \frac{\overset{2}{\cancel{4}}x}{\cancel{2}\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}}$$

$$\int_0^1 \int_x^{x^2} \sqrt{\frac{4x^2}{x^2+y^2} + \frac{4y^2}{x^2+y^2} + 1} \, dy \, dx$$

$$\int_0^1 \int_x^{x^2} \sqrt{\frac{5x^2+5y^2}{x^2+y^2}} \, dy \, dx = \int_0^1 \left| \sqrt{5}\, y \right|_x^{x^2} dx$$

$$\int_0^1 \left(\sqrt{5}\, x^2 - \sqrt{5}\, x\right) dx = \sqrt{5} \left| \frac{x^3}{3} - \frac{x^2}{2} \right|_0^1$$

$$\sqrt{5}\left[\left(\frac{1}{3} - \frac{1}{2}\right)\right] = \boxed{\frac{-\sqrt{5}}{6}} \text{ Ans.}$$

5. The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 2x$.

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

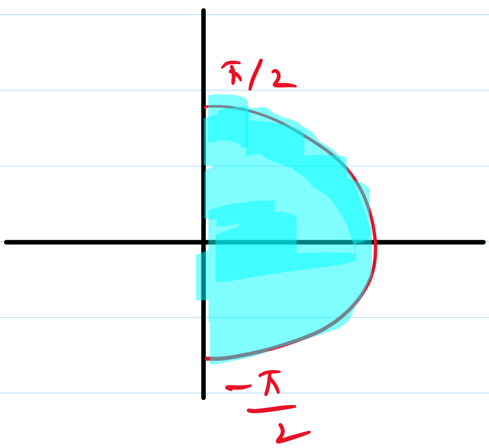
$$x^2 + y^2 = 2x$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$$

$$r^2 - 2r\cos\theta = 0$$

$$r(r - 2\cos\theta) = 0$$

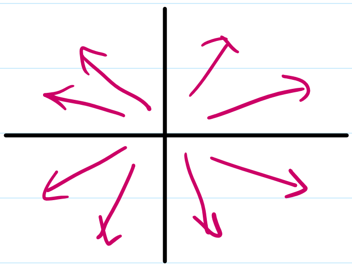
$$\boxed{r = 0} \quad \boxed{r = 2\cos\theta}$$



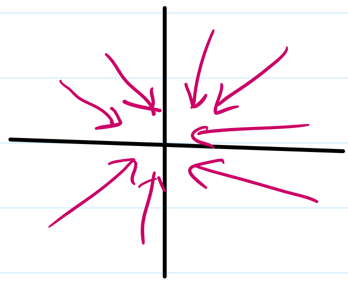
$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \sqrt{2}\, r\, dr\, d\theta$$

$$\int_{-\pi/2}^{\pi/2} \left| \frac{\sqrt{2}}{2} r^2 \right|_0^{2\cos\theta}$$

15.1 ≈ Vector Fields



+ve divergence



-ve divergence

Divergence

$\mathbf{F}(x,y,z) = f(x,y,z)\,i + g(x,y,z)\,k + h(x,y,z)\,k$

represents a vector

$\mathrm{Div}(F) = \nabla \cdot F$ → Dot Product

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$\mathrm{Curl}\, F = \nabla \times F$ → Cross Product.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$$

$$\nabla \times F = i\left(\partial/\partial y \cdot h - \partial/\partial z \cdot g\right) - j\left(\partial/\partial x \cdot g - \partial/\partial z \cdot f\right)$$

$$+ k\left(\partial/\partial x \cdot g - \partial/\partial y \cdot f\right)$$

17–22 Find div **F** and curl **F**.

22. $\mathbf{F}(x, y, z) = \ln x\mathbf{i} + e^{xyz}\mathbf{j} + \tan^{-1}(z/x)\mathbf{k}$

$$\operatorname{Div}(F) = \nabla \cdot F = \frac{1}{x} + e^{xyz} \cdot (xz) + \frac{1}{1+\left(\frac{z}{x}\right)^2} \cdot \left(\frac{1}{x}\right)$$

$$\operatorname{Curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln x & e^{xyz} & \tan^{-1}\left(\frac{z}{x}\right) \end{vmatrix}$$

$$\nabla \times F = i\left(\partial/\partial y \cdot \tan^{-1}\left(\frac{z}{x}\right) - \partial/\partial z \cdot e^{xyz}\right) - j\left(\partial/\partial x \cdot \tan^{-1}\left(\frac{z}{x}\right) - \partial/\partial z \cdot \ln x\right)$$
$$+ k\left(\partial/\partial x \cdot e^{xyz} - \partial/\partial y \cdot \ln x\right)$$

$$\nabla \times F = i\left(0 - xye^{xyz}\right) - j\left(\frac{\cancel{x^2}}{x^2+z^2} \cdot \frac{-z}{\cancel{x^2}} - 0\right) + k\left(yze^{xyz} - 0\right)$$

$$\boxed{\nabla \times F = -xye^{xyz}\,i + \frac{z}{x^2+z^2}\,j + yze^{xyz}\,k}$$ Ans.

27–28 Find $\nabla \times (\nabla \times \mathbf{F})$.

28. $\mathbf{F}(x, y, z) = y^2x\mathbf{i} - 3yz\mathbf{j} + xy\mathbf{k}$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2x & -3yz & xy \end{vmatrix}$$

$$\nabla \times F = i\,(x+3y) - j\,(y-0) + k\,(0-2yx)$$

$$\boxed{\nabla \times F = (x+3y)\,i - y\,j - 2y\,k}$$

$$\nabla \times (\nabla \times F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+3y) & -y & -2y \end{vmatrix}$$

$$\nabla \times (\nabla \times F) = i(-2x - 0) - j(-2y - 0) + k(0 - 3)$$
$$= \boxed{-2x\,i + 2y\,j - 3\,k} \text{ Ans.}$$

23–24 Find $\nabla \cdot (\mathbf{F} \times \mathbf{G})$.

24. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
$\mathbf{G}(x, y, z) = xy\mathbf{j} + xyz\mathbf{k}$

$$F \times G = \begin{vmatrix} i & j & k \\ yz & xz & xy \\ 0 & xy & xyz \end{vmatrix}$$

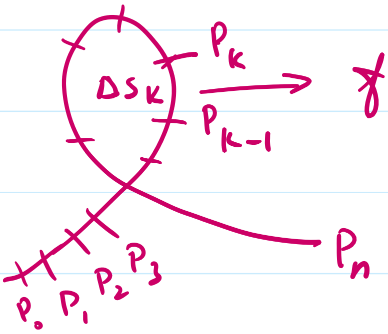
$$F \times G = i(x^2yz^2 - x^2y^2) - j(xy^2z^2 - 0) + k(xy^2z - 0)$$

$$\boxed{F \times G = (x^2yz^2 - x^2y^2)i - (xy^2z^2)j + (xy^2z)k}$$

$$\nabla \cdot (F \times G) = 2x\cancel{yz^2} - 2xy^2 - 2x\cancel{yz^2} + xy^2$$
$$= \boxed{-xy^2} \text{ Ans.}$$

15.2 ≈ Line Integral

Area b/w curves not meeting at some point.



$$\rightarrow f(x_k^*, y_k^*)$$

$$\Delta M_k = f(x_k^*, y_k^*)\, \Delta s_k$$

$$M = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta s_k$$

$$\Rightarrow \int_C f(x,y)\, ds = \int_a^b f(r(t)) \cdot |r'(t)|\, dt$$

$$\Rightarrow \int_C \vec{F} \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

$$r(t) = x(t)\, i + y(t)\, j + z(t)\, k$$

$$f(r(t)) = f(x(t), y(t), z(t))$$

$$F(r(t)) = F(x(t), y(t), z(t))$$

$$r(t) = x(t)\, i + y(t)\, j$$

$r(t) = x(t)\,i + y(t)\,j$

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

7–10 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the line segment C from P to Q.

7. $\mathbf{F}(x, y) = 8\mathbf{i} + 8\mathbf{j}$; $\underbrace{P(-4, 4)}_{P(x_0,\, y_0)}$, $Q(-4, 5)$

$v = Q - P = \langle (-4-4), (5-4) \rangle = \langle \underset{a}{0}, \underset{b}{1} \rangle$

$x(t) = x_0 + at = -4 + 0(t) = -4$

$y(t) = y_0 + bt = 4 + 1(t) = 4 + t$

$r(t) = x(t)\,i + y(t)\,j$

$\boxed{r(t) = -4\,i + (4+t)\,j}$

$\boxed{r'(t) = j}$

$F(r(t)) = F(x(t), y(t)) = 8i + 8j$

$\int_a^b \vec{F} \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$

$\int^1 (8i + 8j) \cdot j\, dt = 8 \int^1 dt = 8\, |t|_0^1$

$$\int_0^1 (8i + 8j) \cdot j \, dt = 8\int_0^1 dt = 8\,|t|_0^1$$

$$8\left[(1) - (0)\right] = \boxed{8} \text{ Ans.}$$

11. Let C be the curve represented by the equations

$$x = 2t, \quad y = t^2 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\displaystyle\int_C \overbrace{(x - \sqrt{y})}^{f(x,y)}\, ds$

$$r(t) = x(t)\, i + y(t)\, j$$

$$r(t) = 2t\, i + t^2\, j$$

$$r'(t) = 2\, i + 2t\, j$$

$$|r'(t)| = \sqrt{(2)^2 + (2t)^2} = \sqrt{4 + 4t^2} = \boxed{2\sqrt{1+t^2}}$$

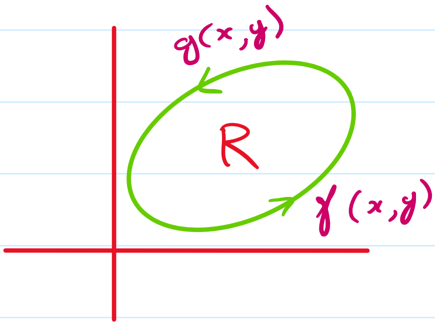
$$f(r(t)) = f(x(t), y(t)) = \int_0^1 (2t - t) \cdot (2\sqrt{1+t^2})\, dt$$

$$\int_0^1 2t\sqrt{1+t^2}\, dt = \left| \frac{2(1+t^2)^{\frac{3}{2}}}{3} \right|_0^1$$

$$\boxed{\frac{2}{3}\left((2)^{3/2} - (1)^{3/2}\right)} \text{ Ans.}$$

15.4 ~ Green's Theorem

→ Area b/w curves meeting at the same point.



$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

3–13 Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise.

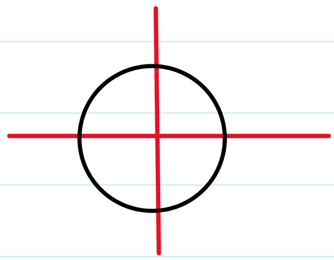
3. $\oint_C 3xy\,dx + 2xy\,dy$, where C is the rectangle bounded by $x = -2$, $x = 4$, $y = 1$, and $y = 2$.

($f(x,y)$, $g(x,y)$)

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \boxed{\int_1^2\int_{-2}^4 (2y - 3x)\,dx\,dy}$$

4. $\oint_C (x^2 - y^2)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 9$.

$x^2 + y^2 = r^2$



$$\iint (1 - 2y)\,dA$$

$$\int_0^{2\pi}\int_0^3 (1-2y)\cdot r\,dr\,d\theta$$

$$\int^{2\pi}\int^{3} (1 - 2r\sin\theta)\cdot r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{3} (1 - 2r\sin\theta)\cdot r \, dr\, d\theta$$

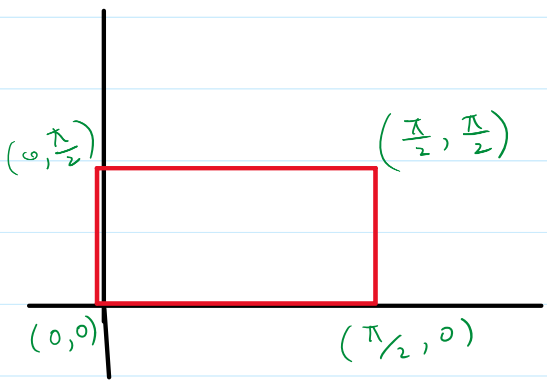
$$\int_0^{2\pi}\int_0^{3} (r - 2r^2\sin\theta)\, dr\, d\theta = \int_0^{2\pi} \left| \frac{r^2}{2} - \frac{2r^3}{3}\sin\theta \right|_0^3$$

$$\int_0^{2\pi} \left(\frac{9}{2} - 18\sin\theta\right) d\theta$$

$$\left| \frac{9}{2}\theta + 18\cos\theta \right|_0^{2\pi} = (9\pi - 1) - (0 + 18)$$

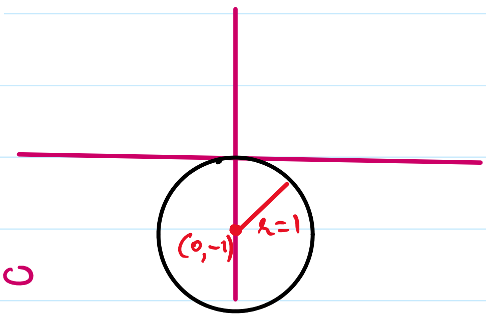
$$= 9\pi - 1 - 18 = \boxed{9\pi - 19} \text{ Ans.}$$

5. $\oint_C x\cos y\, dx - y\sin x\, dy$, where C is the square with vertices $(0, 0)$, $(\pi/2, 0)$, $(\pi/2, \pi/2)$, and $(0, \pi/2)$.



$$\int_0^{\pi/2}\int_0^{\pi/2} (y\cos x + x\sin y)\, dx\, dy$$

6. $\oint_C y\tan^2 x\, dx + \tan x\, dy$, where C is the circle $x^2 + (y + 1)^2 = 1$.



$$x^2 + y^2 + 2y + 1 = 1$$

$$x^2 + y^2 + 2y = 0$$

$$r^2\cos^2\theta + r^2\sin^2\theta + 2r\sin\theta = 0$$

$$r^2 + 2r\sin\theta = 0$$

$r\,(r + 2\sin\theta) = 0$

$\boxed{r = 0}$ $\boxed{r = -2\sin\theta}$

$$\iint \left(\sec^2 x - \tan^2 x\right) dA$$

$\because \sec^2 x - \tan^2 x = 1$

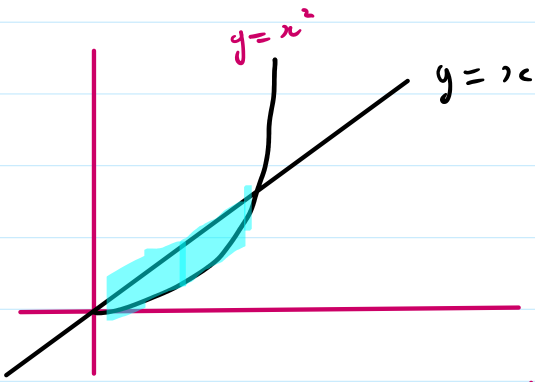
$$\int_{\pi}^{2\pi}\int_{0}^{-2\sin\theta} 1 \cdot r\,dr\,d\theta = \int_{0}^{2\pi} \left|\frac{r^2}{2}\right|_{0}^{-2\sin\theta} d\theta$$

$$\int_{0}^{2\pi} \frac{\cancel{4}^{2}\sin^2\theta}{\cancel{2}}\,d\theta = 2\int_{0}^{2\pi} \sin^2\theta\,d\theta = \cancel{2}\int \frac{1}{\cancel{2}}\left(1 - \cos 2\theta\right) d\theta$$

$$\int_{0}^{2\pi}\left(1 - \cos 2\theta\right) d\theta = \left|\theta - \frac{\sin 2\theta}{2}\right|_{0}^{2\pi} = \left[2\pi - \frac{\sin 4\pi}{2}\right]$$

$= \boxed{2\pi}$ Ans.

8. $\oint_C (e^x + y^2)\,dx + (e^y + x^2)\,dy$, where C is the boundary of the region between $y = x^2$ and $y = x$.



$$\boxed{\int_{0}^{1}\int_{x^2}^{x} (2x - 2y)\,dy\,dx}$$

Ans.

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$\boxed{x = 0}$ $\boxed{x = 1}$

$y = x$

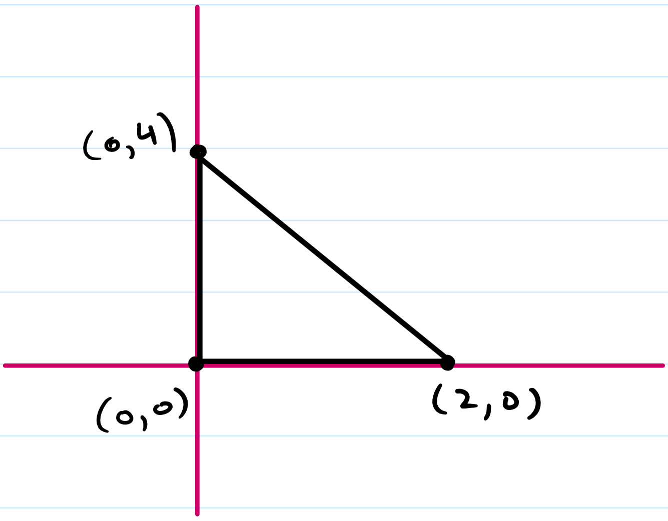
$\boxed{y = 0}$

$\boxed{y = 1}$

$x(x-1) = 0$

$x = 0$ $\quad$ $x = 1$ $\quad$ | $\quad$ $y = 1$

9. $\oint_C \ln(1+y)\,dx - \dfrac{xy}{1+y}\,dy$, where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,4)$.



$y - y_1 = m(x - x_1)$

$m = \dfrac{0-4}{2-0} = -\dfrac{4}{2} = \boxed{-2}$

$y - 4 = -2(x - 0)$

$y - 4 = -2x$

$\boxed{y = -2x + 4}$

$$\boxed{\int_0^2 \int_0^{-2x+4} \left(\frac{-y}{1+y} - \frac{1}{1+y} \right) dy\, dx}$$ Ans.