SOLUTION - KEY | MVC - Mid 1

Question #01

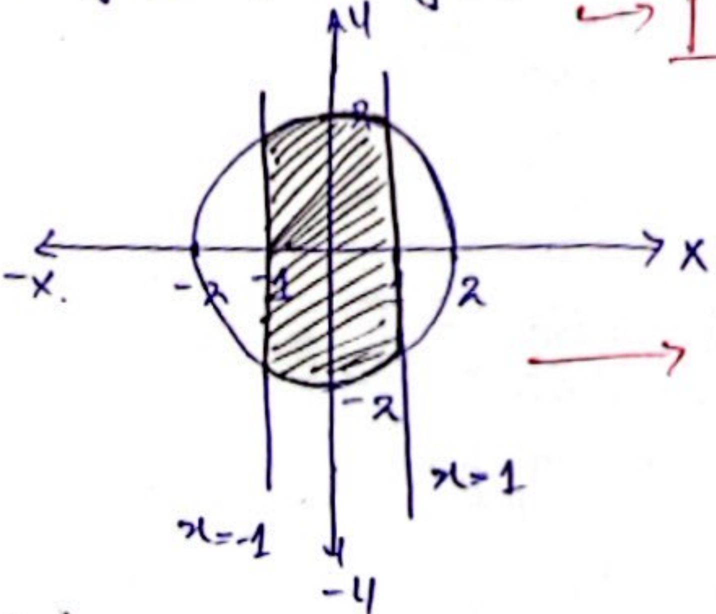
Part a

$f(x,y) = \sqrt{4-x^2-y^2} + \sqrt{1-x^2}$

$1-x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$

$4-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 4$ circle with boundary

→ $\boxed{1 \text{ mark}}$ for finding



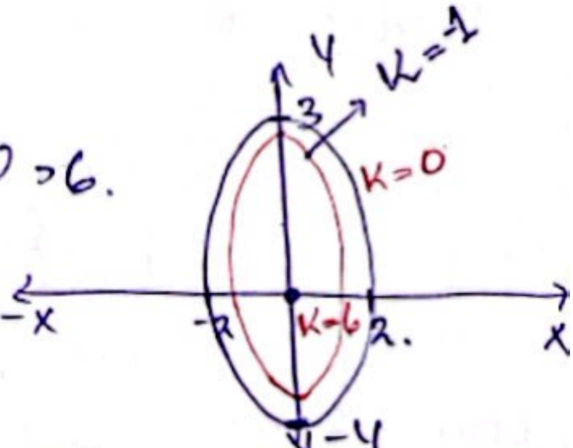
→ $\boxed{2\text{-mark}}$ for sketch.

Part b

$z = \sqrt{36-9x^2-4y^2}$, $K = 1, 0, 6$.

Put $z = K$

$K = \sqrt{36-9x^2-4y^2}$



When $K = -1$ $\boxed{1 \text{ mark}}$

$1 = 36-9x^2-4y^2 \Rightarrow 9x^2+4y^2 = 35 \rightarrow$ ellipse.

When $K = 0$ $\boxed{1 \text{ mark}}$

$9x^2+4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \rightarrow$ ellipse.

When $K = 6$ $\boxed{1 \text{ mark}}$

$36 = 36-9x^2-4y^2 \Rightarrow 9x^2+4y^2 = 0 \Rightarrow$ Point $(0,0)$

Question #02

Part a

$$\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{y^2+x}.$$

Along x-axis → $\boxed{1 \text{ mark}}$

Put $y=0$.

$$\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{y^2+x} = \lim_{x\to 0} \frac{0}{x} = 0$$

Along y axis → $\boxed{1 \text{ mark}}$

Put $x=0$.

$$\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{y^2+x} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

Limit is different along two different path

So limit does not exist.

Part b we have $\boxed{f(0,0)=0}$ → $\boxed{1 \text{ mark}}$

$$f(x,y) = \frac{2xy}{x^2+y^2} \quad \text{when } (x,y) \neq (0,0)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \quad \boxed{1.5 \text{ mark}}$$

along $y = mx$., put $x=t$ & $y=mt$.

$$\lim_{t\to 0} \frac{2m\cancel{t^2}}{\cancel{t^2}(1+m^2)} = \lim_{t\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}.$$

limit is different for different values of m

=> limit does not exist.

Function is not continuous at $(0,0)$ $\boxed{0.5 \text{ mark}}$

Scanned with CamScanner

Question #03

Part a

$$z = \ln(e^x + e^y)$$

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x+e^y} \Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{1}{(e^x+e^y)^2}\{(e^x+e^y)e^x - e^x(e^x)\}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{e^{x+y}}{(e^x+e^y)^2}} \text{ — } (1) \quad \boxed{1 \text{ mark}}$$

Also,

$$\frac{\partial z}{\partial y} = \frac{e^y}{e^x+e^y} \Rightarrow \cancel{\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(e^x+e^y)^2}\{(e^x+e^y)e^x - e^y(e^y)\}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(e^x+e^y)^2}\{(e^x+e^y)0 - e^y(e^x)\}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = -\frac{e^{x+y}}{(e^x+e^y)^2}} \text{ — } (2) \quad \boxed{1 \text{ mark}}$$

Adding (1) & (2) gives

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 0} \rightarrow \boxed{1 \text{ mark}}$$

Hence proved.

Part b

$$f_{xy}(3,2) \approx \frac{f_x(3,2.2) - f_x(3,2)}{0.2} = \frac{16.8-12.2}{0.2} = 23. \quad \boxed{1 \text{ mark}}$$

$$f_{xy}(3,2) \approx \frac{f_x(3,1.8) - f_x(3,2)}{-0.2} = \frac{7.5-12.2}{-0.2} = 23.5 \quad \boxed{1 \text{ mark}}$$

$$\boxed{f_{xy}(3,2) = \frac{1}{2}(23+23.5) = 23.25} \rightarrow \boxed{1 \text{ mark}}$$

Ans

Part C

using chain rule

$$\frac{dA}{dt} = \left(\frac{\partial A}{\partial a} \times \frac{da}{dt}\right) + \left(\frac{\partial A}{\partial b} \times \frac{db}{dt}\right) + \left(\frac{\partial A}{\partial \theta} \times \frac{d\theta}{dt}\right) \quad (1)$$

1 mark

$$A = \frac{1}{2} ab \sin\theta$$

$$\frac{\partial A}{\partial a} = \frac{1}{2} b \sin\theta \Rightarrow \boxed{\frac{\partial A}{\partial a}(40, 50, \pi/6) = \cancel{500}\ 25/2}$$

$$\frac{\partial A}{\partial b} = \frac{1}{2} a \sin\theta \Rightarrow \boxed{\frac{\partial A}{\partial b}(40, 50, \pi/6) = 10}$$

$$\frac{\partial \theta}{\partial b} = \frac{1}{2} ab \cos\theta \Rightarrow \boxed{\frac{\partial \theta}{\partial b}(40, 50, \pi/6) = 500\sqrt{3}}$$

1 mark

we have

$$\boxed{\frac{da}{dt} = 3}, \boxed{\frac{db}{dt} = -2}, \boxed{\frac{d\theta}{dt} = 0.05} \rightarrow \text{1 mark.}$$

$$(1) \Rightarrow \left.\frac{dA}{dt}\right|_{(40,50,\pi/6)} = \left(\frac{25}{2} \times 3\right) + (10 \times -2) + (500\sqrt{3} \times 0.05)$$

$$= \frac{75}{2} - 20 + 25\sqrt{3}.$$

$$\boxed{\left.\frac{dA}{dt}\right|_{(40,50,\pi/6)} = 60.801} \rightarrow \text{1 mark}$$

Ans.

Question #04

Part a

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$L(x,y,z) = f_x(3,2,6)\Delta x + f_y(3,2,6)\Delta y + f_z(3,2,6)\Delta z + f(3,2,6)$

$f_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow \boxed{f_x(3,2,6) = \dfrac{3}{7}}$, $\boxed{f(3,2,6) = 7.}$

$f_y = \dfrac{y}{\sqrt{x^2+y^2+z^2}} \rightarrow \boxed{f_y(3,2,6) = \dfrac{2}{7}}$

$f_z = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \rightarrow \boxed{f_z(3,2,6) = \dfrac{6}{7}}$

$\boxed{\Delta x = 0.26}$, $\boxed{\Delta y = -0.03}$, $\boxed{\Delta z = -0.01}$

2 mark

$L(x,y,z) = \dfrac{3}{7}(x-3) + \dfrac{2}{7}(y-2) + \dfrac{6}{7}(z-6) + 7 \rightarrow$

$L(3.26, 1.97, 5.99) = 7 + \dfrac{3}{7}(0.26) + \dfrac{2}{7}(-0.03) + \dfrac{6}{7}(-0.01)$

1 mark

$\boxed{L(3.26, 1.97, 5.99) = \cancel{7.11914285\overline{7}}}$ 7.0942857

$f(3.26, 1.97, 5.99) = 7.09849279$

1 mark

$\Delta f = |7.09849279 - 7.0942857| = 0.00420709$

$\overline{PQ} = \sqrt{(0.26)^2 + (-0.03)^2 + (-0.01)^2}$

$\boxed{\overline{PQ} = 0.2619160}$ $\rightarrow$ 1 mark

Error in f is $\frac{1}{100}$ times of $\overline{PQ}$

Part b

$x e^{y} + y e^{z} + 2\ln x - 2 - 3\ln 2 = 0.$

$\frac{\partial}{\partial x}(x e^{y} + y e^{z} + 2\ln x - 2 - 3\ln 2) = 0.$

$e^{y} + y e^{z}\frac{\partial z}{\partial x} + \frac{2}{x} = 0. \rightarrow$ 2 marks

$\boxed{\frac{\partial z}{\partial x} = -\frac{(e^{y}+2/x)}{y e^{z}}} \rightarrow \boxed{\left.\frac{\partial z}{\partial x}\right|_{(1,\ln 2,\ln 3)} = \frac{-4}{3\ln 2}} \rightarrow$ (1 mark)

Alternatively

$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}$, $f(x,y,z) = x e^{y} + y e^{z} + 2\ln x + 2 - 3\ln 2$

$\frac{\partial f}{\partial x} = e^{y} + 2/x.$

$\frac{\partial f}{\partial z} = y e^{z}$

So $\boxed{\frac{\partial z}{\partial x} = -\frac{(e^{y}+2/x)}{y e^{z}}} \Rightarrow \boxed{\left.\frac{\partial z}{\partial x}\right|_{(1,\ln 2,\ln 3)} = -\frac{4}{3\ln 2}}.$

Part c

True $\rightarrow$ 0.5 mark

equation of plane is. $\rightarrow$ 0.5 mark.

$ax + by + cz + d = 0$

$\Rightarrow z = \frac{-d - ax - by}{c} = f(x,y)$

$f_x = -a/c$ & $f_y = -\frac{b}{c}$ are constant functions.