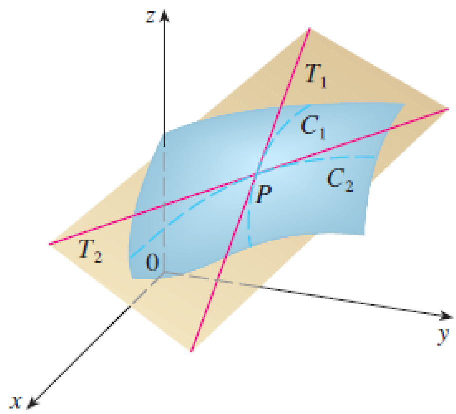
Section 14.4: Tangent Planes and Linear Approximations

Let S be the surface defined by $z = f(x, y)$. For each point $P(x_0, y_0, z_0)$ on the surface, the vertical planes $y = y_0$ and $x = x_0$ intersect S in curves C_1 and C_2. Let T_1 and T_2 denote the tangent lines to C_1 and C_2 at the point P. Then the plane that contains both tangent lines T_1 and T_2 is called the **tangent plane** to the surface S at the point P.



Equation of Tangent Plane: An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note how this is similar to the equation of a tangent line.

Example 1: Find the tangent plane to the surface $f(x, y) = 3x^2 - 2y^2$ at the point $(1, 2, -5)$.

Example 2: Find the tangent plane to the surface $f(x, y) = e^x \cos(y)$ at the point $(0, 0, 1)$.

Linearization:

The **linearization** of $f(x, y)$ at (x_0, y_0) is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **linear approximation** (or **tangent plane approximation**) of $f(x, y)$ at (x_0, y_0).

Note: The linearization $L(x, y)$ is a good approximation of $f(x, y)$ for points (x, y) near (x_0, y_0).

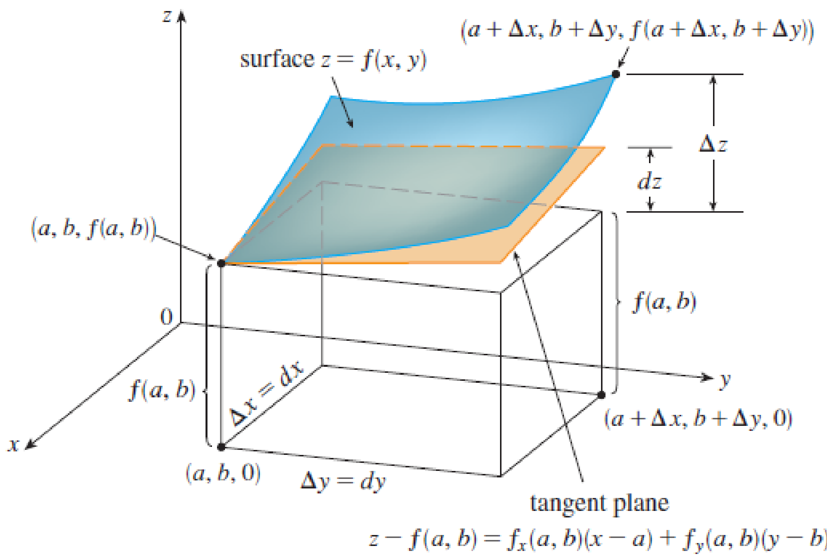
Example 3: Find the linearization of $f(x, y) = \sin(x - 2y)$ at $(2, 1)$ and use it to approximate $f(2.1, 0.95)$.

Differentials:

If $z = f(x, y)$, then the **differentials** dx and dy are independent variables and the **differential** dz is defined by

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy$$

Thus, dz is a dependent variable that depends on x, y, dx, and dy. Graphically, suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. If $dx = \Delta x$ and $dy = \Delta y$, then $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ represents the change in height of the surface, while $dz = f_x(a, b)\, dx + f_y(a, b)\, dy$ represents the change in height of the tangent plane. Therefore, dz is an approximation of Δz.



Example 4: Find the differential of $z = y \sin(xy)$.

Note: We can naturally generalize differentials to functions of three (or more) variables.

Example 5: Find the differential of $w = xye^z$.

Example 6: The dimensions of a closed rectangular box are measured as 81 cm, 53 cm, and 18 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

The **linearization** of $f(x, y, z)$ at (x_0, y_0, z_0) is

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

We have $f(x, y, z) \approx L(x, y, z)$ for points (x, y, z) near (x_0, y_0, z_0).

Example 7: Use differentials (or linear approximations) to approximate $\sqrt{4.05^2 + 3.97^2 + 7.04^2}$.