National University of Computer & Emerging Sciences, Karachi
Multivariable Calculus Assignment - 01 (CLO2)
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Submission Date - February 20, 2024 Total Marks: 40

Question 1.[2+3+3=08 marks]

(a) Find and sketch the domain of the given functions.

(i) $f(x,y)=\sqrt{x+y}-\sqrt{x-3}$ (ii) $f(x,y)=\sqrt{\frac{1}{x^2}-\frac{1}{y^2}}$

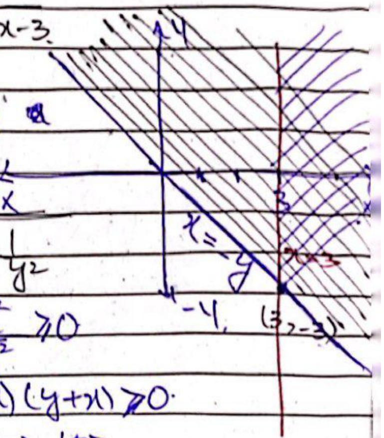
(iii) $f(x,y,z)=\frac{1}{x+1}+\frac{1}{y-1}+\frac{1}{x+y-z}$ (iv) $f(x,y)=ln(x^2-8y)$

Q#1 (Part a)

(Part i) $f(x,y)=\sqrt{x+y}-\sqrt{x-3}$

$\sqrt{x+y}\geq 0 \Rightarrow x+y\geq 0 \Rightarrow$

& $x-3\geq 0 \Rightarrow x\geq 3$

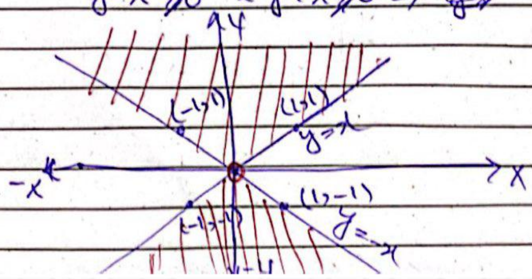


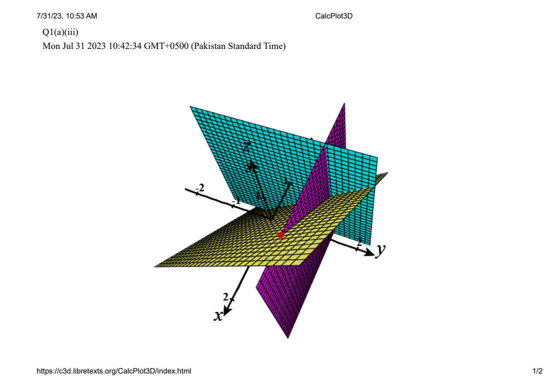
Part (ii) $f(x,y)=\sqrt{\frac{1}{x^2}-\frac{1}{y^2}}$

$\frac{1}{x^2}-\frac{1}{y^2}\geq 0 \Rightarrow \frac{y^2-x^2}{x^2y^2}\geq 0$

$\Rightarrow y^2-x^2\geq 0 \Rightarrow (y-x)(y+x)\geq 0$

$y-x\geq 0$ & $y+x\geq 0 \Rightarrow y\geq$



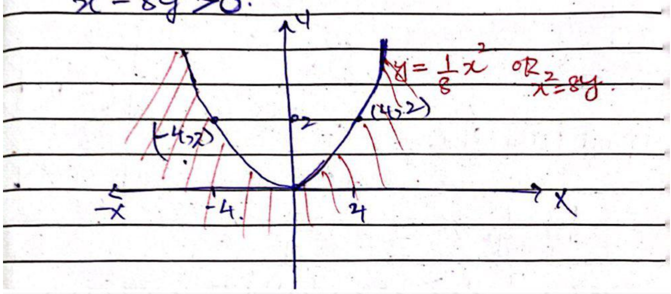


Q#01 (Part a)

Part (iv)

$f(x,y)=\ln(x^2-8y)$

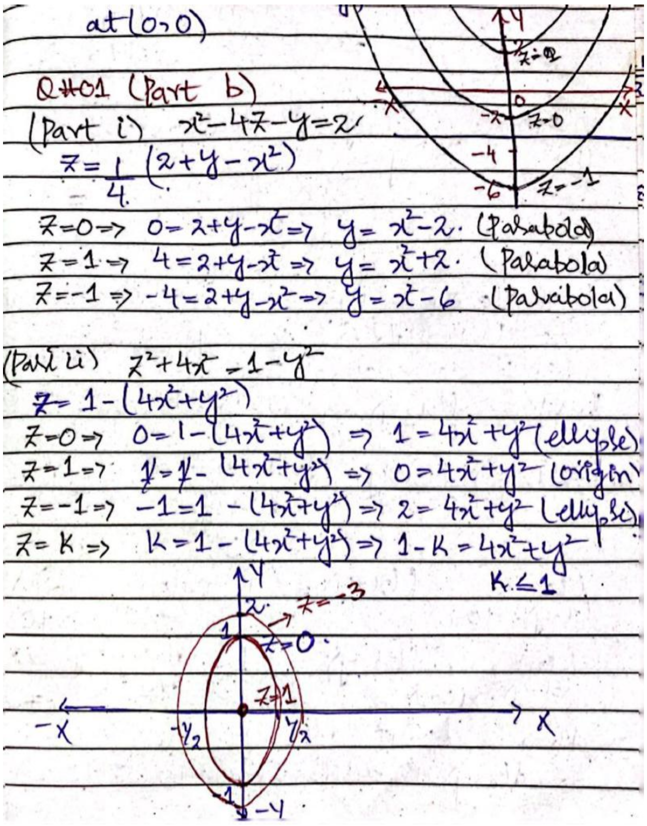
$x^2-8y>0.$



(b) Identify and sketch the level curves (or contours) for the given function.

(i) $x^2-4z-y=2$ (ii) $z^2+4x^2=1-y^2$ (iii) $2x-6y+z=-2$

Solution:


at (0,0)

Q#01 (Part b)

(Part i) $x^2-4z-y=2$

$z=\frac{1}{4}(2+y-x^2)$

$z=0 \Rightarrow 0=2+y-x^2 \Rightarrow y=x^2-2$ (Parabola)

$z=1 \Rightarrow 4=2+y-x^2 \Rightarrow y=x^2+2$ (Parabola)

$z=-1 \Rightarrow -4=2+y-x^2 \Rightarrow y=x^2-6$ (Parabola)

(Part ii) $z^2+4x^2=1-y^2$

$z=1-(4x^2+y^2)$

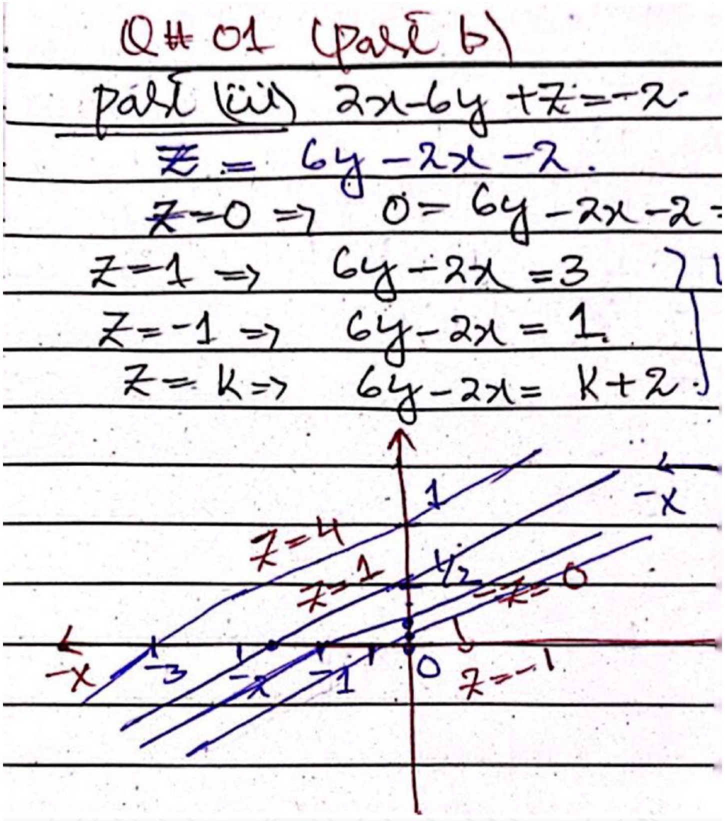
$z=0 \Rightarrow 0=1-(4x^2+y^2) \Rightarrow 1=4x^2+y^2$ (ellipse)

$z=1 \Rightarrow \cancel{1}=\cancel{1}-(4x^2+y^2) \Rightarrow 0=4x^2+y^2$ (origin)

$z=-1 \Rightarrow -1=1-(4x^2+y^2) \Rightarrow 2=4x^2+y^2$ (ellipse)

$z=k \Rightarrow k=1-(4x^2+y^2) \Rightarrow 1-k=4x^2+y^2$

$k \le 1$


Q# 01 (Part b)

Part (iii) $2x-6y+z=-2$

$z=6y-2x-2$.

$z=0 \Rightarrow 0=6y-2x-2$

$z=1 \Rightarrow 6y-2x=3$

$z=-1 \Rightarrow 6y-2x=1$

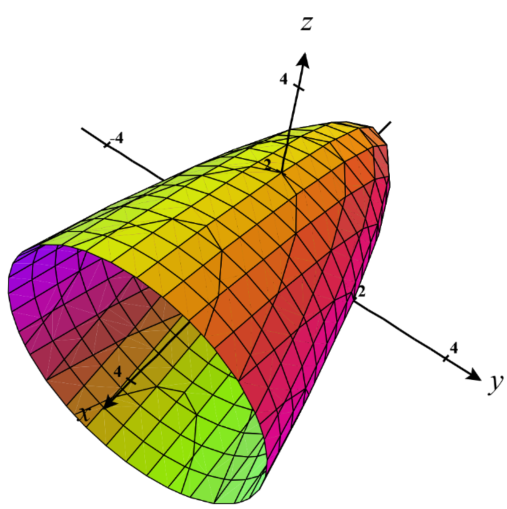
$z=k \Rightarrow 6y-2x=k+2$

(c) Identify and sketch the level surfaces (or contours) for the given functions at the specified value of k.

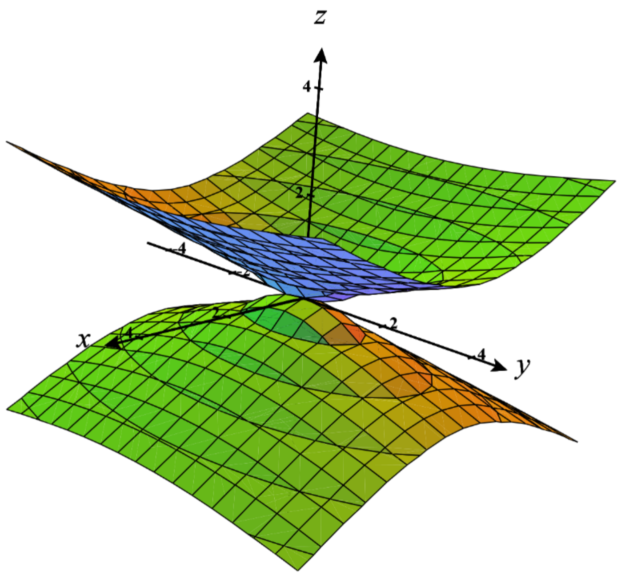
$$(i)\ f(x,y,z)=x-y^2-z^2+1,\ k=-3, \quad (ii)\ f(x,y,z)=\frac{3x^2+y^2}{z^2},\ k=9$$
$$(iii)\ f(x,y,z)=9x^2+4y^2+z^2,\ k=4.$$

Solution:

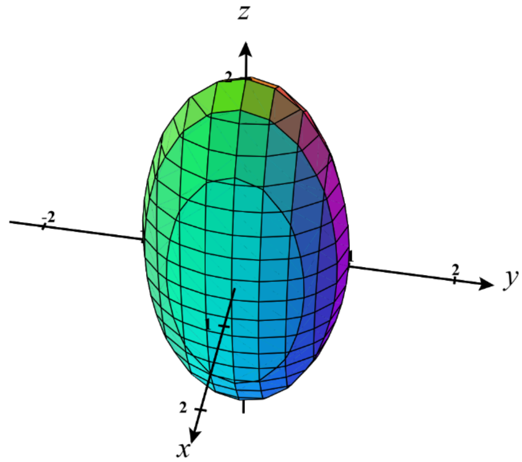
(i) Elliptical Paraboloid along x-axis



(ii) Elliptical cone along z-axis



(iii) Ellipsoid



Question 2.[3+2=5 marks]

(a) Examine whether the following limits exist and find their values if they exist.

(i) $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$ (ii) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (iii) $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$)

Solution:

Q#02 (part a)

Part (i) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$

Along x-axis

i.e $y=0$ but $x\neq 0$.

$\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$.

Along y-axis

i.e $x=0$ but $y\neq 0$.

$\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$

Along $y=x^2$

when $x\to 0 \Rightarrow y\to 0$.

$\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$

$= \lim\limits_{(x,x^2)\to(0,0)} \frac{x^5}{2x^4} = 0$

It seems that limit exist & equal to zero since

$x^3 \leq x^4+y^2$.

$\Rightarrow 0 \leq \left|\frac{x^3}{x^4+y^2}\right| \leq 1$

& $0 \leq \left|\frac{x^3y}{x^4+y^2}\right| \leq |y|$

and $\lim\limits_{y\to 0} |y| = 0$.

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$.

Part (ii) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

Along x-axis $y=0$ but $x\neq 0$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{x\to 0} x$

$= 0$.

Along y-axis $x=0$ but $y\neq 0$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{y\to 0} -y$.

$= 0$

Along $y=x$ $x\to 0 \Rightarrow y\to 0$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$.

Along $y=mx$ $x\to 0 \Rightarrow y\to 0$

$\lim\limits_{(x,mx)\to(0,0)} \frac{x^3-m^3x^3}{x^2+m^2x^2}$

$= \lim\limits_{x\to 0} \frac{x^3(1-m^3)}{x^2(1+m^2)} = 0$.

It seems that limit exist & equal to zero since

$x^2 \leq x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \leq 1$

$y^2 \leq x^2+y^2 \Rightarrow \frac{y^2}{x^2+y^2} \leq 1$

$\Rightarrow \left|\frac{x^3-y^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right|$

$\Rightarrow \left|\frac{x^3-y^3}{x^2+y^2}\right| \leq |x|+|y|$

≤ 0

Q#02 (part a)

Part (iii)

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$$

Along x-axis $y=0$ but $x\neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} = \lim_{x\to 0} \frac{1-\cos(x^2)}{x^4} = \frac{0}{0}.$$

by L-Hopital Rule

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{x^4} = \lim_{x\to 0} \frac{\sin(x^2)\times 2x}{4x^3} = \lim_{x\to 0} \frac{\sin(x^2)}{2x^2} =$$

$$\Rightarrow \lim_{x\to 0} \frac{\cos(x^2)\times 2x}{2\cdot 2x} = \frac{1}{2}.$$

Along y-axis $x=0$ but $y\neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^4} = \lim_{y\to 0} \frac{1-\cos(y^2)}{y^4} = \frac{1}{2}.$$

Q#02 (part a) (part iii) continue

Continuation Sheet ______ of ______

Along $y=x$ when $x\to 0 \Rightarrow y\to 0$.

$$\lim_{(x,x)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} = \lim_{x\to 0} \frac{1-\cos(2x^2)}{4x^4}$$

$$= \lim_{x\to 0} \frac{\sin(2x^2)\times 4x}{16x^3} = \lim_{x\to 0} \frac{\sin(2x^2)}{4x^2}$$

$$= \lim_{x\to 0} \frac{\cos(2x^2)\times 4x}{2\cdot 8x} = \frac{1}{2}$$

It seems that limit exist and equal to 1.

If $0 < \sqrt{x^2+y^2} < 8$ then $\left|\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} - \frac{1}{2}\right| < \varepsilon$.

That is.

If $0 < \sqrt{x^2+y^2} < 8$ then $\left|\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} + \frac{1}{2}\right| < \varepsilon$

But $\left|\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}\right| \leq \frac{1}{(x^2+y^2)^2} + \frac{|\cos(x^2+y^2)|}{(x^2+y^2)^2}$

also $|\cos(x^2+y^2)| = 1 \quad \forall (x,y)$

therefore

$$\left|\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}\right| \leq \frac{2}{(x^2+y^2)^2} = \frac{2}{8^4} = \varepsilon$$

Hence

$$\left|\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} + \frac{1}{2}\right| \leq \frac{2}{8^4} + \frac{1}{2} = \frac{4+8^4}{2\cdot 8^4} = \varepsilon$$

Hence $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} = \frac{1}{2}$ Ans.

(b) Examine the following functions for continuity at the point $(0,0)$, where $f(0,0)=0$ and $f(x,y)$ for $(x,y)\neq(0,0)$ is given by

$i)\ \frac{xy}{\sqrt{x^2+y^2}} \qquad ii)\ \frac{xy}{x^2+y^2} \qquad iii)\ \frac{x^4-y^2}{x^4+y^2} \qquad iv)\ \frac{x^2y}{x^4+y^2}.$

Solution:

Q#02 (Part b)

(part i) $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=\lim_{(x,mx)\to(0,0)}\frac{mx^2}{\sqrt{x^2(1+m^2)}}$$
$$=\lim_{x\to 0}\frac{mx^2}{\sqrt{x^2}(1+m^2)}$$
$$=\lim_{x\to 0}\frac{mx^2}{|x|(1+m^2)}=0.$$

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=0=f(0,0)$$

Hence $f(x,y)$ is continuous.

(Part ii) $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 & \end{cases}$

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}=\lim_{(x,mx)\to(0,0)}\frac{mx^2}{x^2(1+m^2)}=\frac{m}{1+m^2}$$

limit is different for different m so

limit does not exist

Therefore $f(x,y)$ is dis continuous at $(0,0)$

(Part iii) $f(x,y)=\begin{cases}\frac{x^4-y^2}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$$\lim_{(x,y)\to(0,0)}f(x,y)=1 \quad \left(\text{along } x\text{-axis} \atop y=0 \text{ but } x\neq 0\right)$$

$$\lim_{(x,y)\to(0,0)}f(x,y)=-1 \quad \left(\text{along } y\text{-axis} \atop x=0 \text{ but } y\neq 0\right)$$

Hence limit does not exist

$\Rightarrow f(x,y)$ is dis continuous at $(0,0)$.

Part (iv) $f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Along $y=mx^2$

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,mx^2)\to(0,0)}\frac{mx^4}{x^4(1+m^2)}=\frac{m}{1+m^2}$$

For different value of m limit is different. Hence limit does not exist. and $f(x,y)$ is dis continuous at $(0,0)$

Question 3.[2+1+2=5 marks]

Let $f(x,y)=xy\frac{x^2-y^2}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and 0, otherwise. Prove that

(a) $f_x(0,y)=-y$ and $f_y(x,0)=x$ for all x and y;

(b) $f_{xy}(0,0)=-1$ and $f_{yx}(0,0)=1$ and (c) $f(x,y)$ is differentiable at $(0,0)$.

(I)

Q# 03 (Part I)

$$f(x,y)=\begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0\ ; & (x,y)=(0,0)\end{cases}$$

(Part a) $f_x(0,y)=-y$

$$f_x(a,b)=\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

$$f_x(0,y)=\lim_{h\to 0}\frac{f(0+h,y)-f(0,y)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{hy\,\frac{h^2-y^2}{h^2+y^2}-0\right\}$$

$$=\lim_{h\to 0} y\,\frac{h^2-y^2}{h^2+y^2}.$$

$$\boxed{f_x(0,y)=\frac{y(-y^2)}{y^2}=-y}$$

(Part b) $f_y(x,0)=x$.

$$f_y(x,0)=\lim_{k\to 0}\frac{f(x,0+k)-f(x,0)}{k}$$

$$=\lim_{k\to 0}\frac{1}{k}\left[xk\,\frac{x^2-k^2}{x^2+k^2}-0\right]$$

$$=\lim_{k\to 0} x\,\frac{x^2-k^2}{x^2+k^2}$$

$$\boxed{f_y(x,0)=\frac{x(x^2)}{x^2}=x}$$

(Part c) $f_{xy}(0,0)=-1$

we have from Part (a)

$f_x(0,y)=-y.$

Diff b.s w.r.t y

$f_{xy}(0,y)=-1 \ \forall y \Rightarrow f_{xy}(0,0)=-1$

(Part d) $f_{yx}(0,0)=1$

from part (b) we have

$f_y(x,0)=x.$

Diff b.s w.r.t x

$f_{yx}=1 \quad \forall$ all x.

$$\Rightarrow \boxed{f_{yx}(0,0)=1}$$

Since both first order partial derivatives are exist and continuous at $(0,0)$ therefore f is differentiable at $(0,0)$.

(II) Suppose f is a function with $f_x(x,y)=f_y(x,y)=0$ for all (x,y). Then show that $f(x,y)=c$, a constant.

Q# 03 (part II)

$f_x(x,y)=f_y(x,y)=0. \quad \forall (x,y).$

Let $f_x(x,y)=0.$

Integrate b.s w.r.t 'x'

$$\Rightarrow \int\frac{\partial f}{\partial x}\,dx=\int 0\,dx.$$

$\Rightarrow$ ~~$f(x,y)=0+c \Rightarrow f(x,y)=c$~~

$f(x,y)=0+h(y).$ —— (I)

Diff w.r.t y.

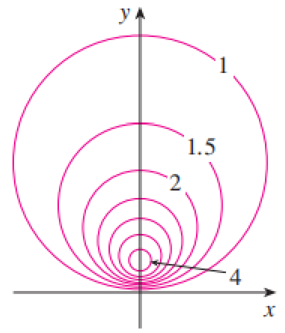
$\Rightarrow f_y(x,y)=h'(y)$

$\Rightarrow 0=h'(y) \quad$ b/c $f_y(x,y)=0.$

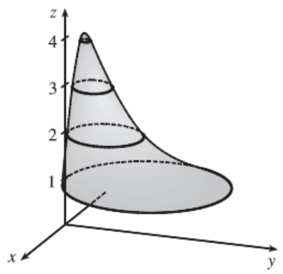
$\Rightarrow \boxed{h(y)=c}$

(I) $\Rightarrow \boxed{f(x,y)=c}$ Hence proved.

(III) A contour map of a function f is shown. Use it to make a rough sketch of the graph of f.



Sketch of graph whose contour map is given:



Question 4.[2+2+2=6 marks]

(a) The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions $(1,2)$ and $(2,1)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$.

Q#4(part a)

$D_u f(a,b) = u \cdot \nabla f(a,b)$

$D_{(1,2)} f(0,0) = (1,2) \cdot \nabla f(0,0) = 1$

$D_{(2,1)} f(0,0) = (2,1) \cdot \nabla f(0,0) = 2.$

$\langle 1,2 \rangle \langle f_x(0,0), f_y(0,0) \rangle = 1$

$\langle 2,1 \rangle \langle f_x(0,0), f_y(0,0) \rangle = 2.$

$$\left.\begin{aligned} f_x + 2f_y &= 1 \\ 2f_x + f_y &= 2 \end{aligned}\right\} \Rightarrow \boxed{f_y = 0}$$

$\Rightarrow \boxed{f_x = 1}$

(b) Suppose $z = f(x, y)$, where $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $g_s(1, 2) = -1$, $g_t(1, 2) = 4$, $h(1, 2) = 6$, $h_s(1, 2) = -5$, $h_t(1, 2) = 10$, $f_x(3, 6) = 7$, and $f_y(3, 6) = 8$. Find $\partial z/\partial s$ and $\partial z/\partial t$ when $s = 1$ and $t = 2$.

By the Chain Rule, $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so

$\frac{\partial z}{\partial s} = f_x(3,6)g_s(1,2) + f_y(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$, so

$\frac{\partial z}{\partial t} = f_x(3,6)g_t(1,2) + f_y(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108$.

(c) If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$$

$\frac{\partial z}{\partial x} = 2xf'(x^2-y^2)$, $\frac{\partial z}{\partial y} = 1 - 2yf'(x^2-y^2)$ [where

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 2xyf'(x^2-y^2) + x - 2xyf'(x^2-y^2) =$

Question 5.[2+2+2=6 marks]

(a) Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point (2, 3, 4) and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

$f(x,y,z) = x^3\sqrt{y^2+z^2} \quad\Rightarrow\quad f_x(x,y,z) = 3x^2\sqrt{y^2+z^2}$, $f_y(x,y,z) = \frac{yx^3}{\sqrt{y^2+z^2}}$, $f_z(x,y,z) = \frac{zx^3}{\sqrt{y^2+z^2}}$,

so $f(2,3,4) = 8(5) = 40$, $f_x(2,3,4) = 3(4)\sqrt{25} = 60$, $f_y(2,3,4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2,3,4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the

linear approximation of f at $(2,3,4)$ is

$$\begin{aligned} f(x,y,z) &\approx f(2,3,4) + f_x(2,3,4)(x-2) + f_y(2,3,4)(y-3) + f_z(2,3,4)(z-4) \\ &= 40 + 60(x-2) + \tfrac{24}{5}(y-3) + \tfrac{32}{5}(z-4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2} = f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 = 38.656$.

(b) A metal plate is situated in the xy-plane and occupies the rectangle $0 \leqslant x \leqslant 10$, $0 \leqslant y \leqslant 8$, where x and y are measured in meters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperatures at equally spaced points were measured and recorded in the table.

(a) Estimate the values of the partial derivatives $T_x(6, 4)$ and $T_y(6, 4)$. What are the units?

(b) Estimate the value of $D_{\mathbf{u}}T(6, 4)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. Interpret your result.

(c) Estimate the value of $T_{xy}(6, 4)$.

y \ x	0	2	4	6	8
0	30	38	45	51	55
2	52	56	60	62	61
4	78	74	72	68	66
6	98	87	80	75	71
8	96	90	86	80	75
10	92	92	91	87	78

(d) Find a linear approximation to the temperature function $T(x, y)$ near the point $(6, 4)$. Then use
it to estimate the temperature at the point $(5, 3.8)$.

Solution:

. (a) $T_x(6,4) = \lim_{h\to 0} \frac{T(6+h,4) - T(6,4)}{h}$, so we can approximate $T_x(6,4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6,4) \approx \frac{T(8,4) - T(6,4)}{2} = \frac{86-80}{2} = 3$, $T_x(6,4) \approx \frac{T(4,4) - T(6,4)}{-2} = \frac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately $3.5°\mathrm{C/m}$. Similarly, $T_y(6,4) = \lim_{h\to 0} \frac{T(6,4+h) - T(6,4)}{h}$, which we can approximate with $h = \pm 2$: $T_y(6,4) \approx \frac{T(6,6) - T(6,4)}{2} = \frac{75-80}{2} = -2.5$, $T_y(6,4) \approx \frac{T(6,2) - T(6,4)}{-2} = \frac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately $-3.0°\mathrm{C/m}$.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 14.6.9, $D_{\mathbf{u}}T(6,4) = \nabla T(6,4) \cdot \mathbf{u} = T_x(6,4)\frac{1}{\sqrt{2}} + T_y(6,4)\frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}T(6,4) \approx (3.5)\frac{1}{\sqrt{2}} + (-3.0)\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35°\mathrm{C/m}$.

Alternatively, we can use Definition 14.6.2: $D_{\mathbf{u}}T(6,4) = \lim_{h\to 0} \frac{T\left(6 + h\frac{1}{\sqrt{2}}, 4 + h\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$, which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_{\mathbf{u}}T(6,4) \approx \frac{T(8,6) - T(6,4)}{2\sqrt{2}} = \frac{80-80}{2\sqrt{2}} = 0$, $D_{\mathbf{u}}T(6,4) \approx \frac{T(4,2) - T(6,4)}{-2\sqrt{2}} = \frac{74-80}{-2\sqrt{2}} = \frac{3}{\sqrt{2}}$. Averaging these values, we have $D_{\mathbf{u}}T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1°\mathrm{C/m}$.

(c) $T_{xy}(x,y) = \frac{\partial}{\partial y}[T_x(x,y)] = \lim_{h\to 0} \frac{T_x(x,y+h) - T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim_{h\to 0} \frac{T_x(6,4+h) - T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \frac{T(8,6) - T(6,6)}{2} = \frac{80 - 75}{2} = 2.5$, $T_x(6,6) \approx \frac{T(4,6) - T(6,6)}{-2} = \frac{68 - 75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$T_x(6,2) \approx \frac{T(8,2) - T_x(6,2)}{2} = \frac{90 - 87}{2} = 1.5$, $T_x(6,2) \approx \frac{T(4,2) - T(6,2)}{-2} = \frac{74 - 87}{-2} = 6.5$.

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$T_{xy}(6,4) \approx \frac{T_x(6,6) - T_x(6,4)}{2} = \frac{3.0 - 3.5}{2} = -0.25$, $T_{xy}(6,4) \approx \frac{T_x(6,2) - T_x(6,4)}{-2} = \frac{4.0 - 3.5}{-2} = -0.25$.

Averaging these values, we have $T_{xy}(6,4) \approx -0.25$.

From the table, $T(6,4) = 80$, and from Exercise 11 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$T(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) = 3.5x - 3y + 71$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1°\mathrm{C}$.

Question 6.[2+2=4 marks]

(a) The two legs of a right triangle are measured as 5 m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of (a) the area of the triangle and (b) the length of the hypotenuse.

(a) $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017\ \mathrm{m}^2$ or $170\ \mathrm{cm}^2$.

(b) $z = \sqrt{x^2+y^2}$, $dz = \dfrac{x}{\sqrt{x^2+y^2}}\,dx + \dfrac{y}{\sqrt{x^2+y^2}}\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026$ m or 0.26 cm.

(b) The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/6$?

$A = \frac{1}{2}xy\sin\theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and $\dfrac{dA}{dt} = \dfrac{1}{2}\left[(y\sin\theta)\dfrac{dx}{dt} + (x\sin\theta)\dfrac{dy}{dt} + (xy\cos\theta)\dfrac{d\theta}{dt}\right]$.

So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$, $\dfrac{dA}{dt} = \dfrac{1}{2}\left[(25)(3) + (20)(-2) + (1000\sqrt{3})(0.05)\right] = \dfrac{35+50\sqrt{3}}{2} \approx 60.8\ \mathrm{in}^2/\mathrm{s}$.

Question 7.[2+2+2=6 marks]

(a) Find the direction in which $f(x,y,z) = ze^{xy}$ increases most rapidly at the point $(0,1,2)$. What is the maximum rate of increase?
Solution:
$\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy}\rangle$, $\nabla f(0,1,2) = \langle 2,0,1\rangle$ is the direction of most rapid increase while the rate is $|\langle 2,0,1\rangle| = \sqrt{5}$.

(b) Let $f(x,y,z) = |r|^{-n}$ where $r = xi + yj + zk$. Show that

$$\nabla \mathbf{f} = \frac{-\mathbf{n}\,\mathbf{r}}{|\mathbf{r}|^{\mathbf{n}+2}}$$

.

(c) Find the gradient of the function $f(x,y,z) = x^2\, e^{yz^2}$.

(i) When is the directional derivative of f a maximum?
(ii) When it is minimum?
(iii) When it is zero?
(iv) When is it half of its maximum value?

Solution:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle 2xe^{yz^2}, x^2e^{yz^2} \cdot z^2, x^2e^{yz^2} \cdot 2yz \right\rangle = \left\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2} \right\rangle$$

(a) By Theorem 14.6.15, the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}} f = |\nabla f| \cos\theta$ (see the proof of Theorem 14.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}} f = |\nabla f| \cos\theta = \frac{1}{2}|\nabla f| \quad \Leftrightarrow \quad \cos\theta = \frac{1}{2} \quad \Leftrightarrow \quad \theta = \frac{\pi}{3}$.