Multiple Integrals

Exercise Set 14.1

1. $\displaystyle\int_0^1\int_0^2 (x+3)\,dy\,dx = \int_0^1 (2x+6)\,dx = 7.$

2. $\displaystyle\int_1^3\int_{-1}^1 (2x-4y)\,dy\,dx = \int_1^3 4x\,dx = 16.$

3. $\displaystyle\int_2^4\int_0^1 x^2y\,dx\,dy = \int_2^4 \frac{1}{3}y\,dy = 2.$

4. $\displaystyle\int_{-2}^0\int_{-1}^2 (x^2+y^2)\,dx\,dy = \int_{-2}^0 (3+3y^2)\,dy = 14.$

5. $\displaystyle\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dy\,dx = \int_0^{\ln 3} e^x\,dx = 2.$

6. $\displaystyle\int_0^2\int_0^1 y\sin x\,dy\,dx = \int_0^2 \frac{1}{2}\sin x\,dx = \frac{1-\cos 2}{2}.$

7. $\displaystyle\int_{-1}^0\int_2^5 dx\,dy = \int_{-1}^0 3\,dy = 3.$

8. $\displaystyle\int_4^6\int_{-3}^7 dy\,dx = \int_4^6 10\,dx = 20.$

9. $\displaystyle\int_0^1\int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx = \int_0^1 \left(1-\frac{1}{x+1}\right)dx = 1-\ln 2.$

10. $\displaystyle\int_{\pi/2}^{\pi}\int_1^2 x\cos xy\,dy\,dx = \int_{\pi/2}^{\pi} (\sin 2x - \sin x)\,dx = -2.$

11. $\displaystyle\int_0^{\ln 2}\int_0^1 xy\,e^{y^2x}\,dy\,dx = \int_0^{\ln 2} \frac{1}{2}(e^x-1)\,dx = \frac{1-\ln 2}{2}.$

12. $\displaystyle\int_3^4\int_1^2 \frac{1}{(x+y)^2}\,dy\,dx = \int_3^4 \left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx = \ln(25/24).$

13. $\displaystyle\int_{-1}^1\int_{-2}^2 4xy^3\,dy\,dx = \int_{-1}^1 0\,dx = 0.$

14. $\displaystyle\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx = \int_0^1 [x(x^2+2)^{1/2} - x(x^2+1)^{1/2}]\,dx = \frac{1}{3}(3\sqrt{3}-4\sqrt{2}+1).$

15. $\int_0^1 \int_2^3 x\sqrt{1-x^2}\,dy\,dx = \int_0^1 x(1-x^2)^{1/2}\,dx = \frac{1}{3}$.

16. $\int_0^{\pi/2} \int_0^{\pi/3} (x \sin y - y \sin x) dy\,dx = \int_0^{\pi/2} \left(\frac{x}{2} - \frac{\pi^2}{18}\sin x\right) dx = \frac{\pi^2}{144}$.

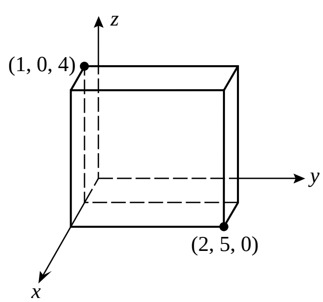
17. **(a)** $x_k^* = k/2 - 1/4, k = 1, 2, 3, 4; y_l^* = l/2 - 1/4, l = 1, 2, 3, 4, \iint\limits_R f(x, y)\,dx\,dy \approx \sum_{k=1}^{4}\sum_{l=1}^{4} f(x_k^*, y_l^*)\Delta A_{kl} = \sum_{k=1}^{4}\sum_{l=1}^{4}\left[\left(\frac{k}{2}-\frac{1}{4}\right)^2 + \left(\frac{l}{2}-\frac{1}{4}\right)\right]\left(\frac{1}{2}\right)^2 = \frac{37}{4}$.

(b) $\int_0^2 \int_0^2 (x^2 + y)\,dx\,dy = \frac{28}{3}$; the error is $\left|\frac{37}{4} - \frac{28}{3}\right| = \frac{1}{12}$.

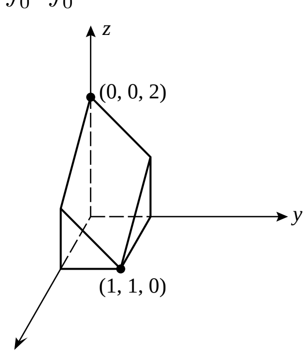
18. **(a)** $x_k^* = k/2 - 1/4, k = 1, 2, 3, 4; y_l^* = l/2 - 1/4, l = 1, 2, 3, 4, \iint\limits_R f(x, y)\,dx\,dy \approx \sum_{k=1}^{4}\sum_{l=1}^{4} f(x_k^*, y_l^*)\Delta A_{kl} = \sum_{k=1}^{4}\sum_{l=1}^{4}\left[\left(\frac{k}{2}-\frac{1}{4}\right) - 2\left(\frac{l}{2}-\frac{1}{4}\right)\right]\left(\frac{1}{2}\right)^2 = -4$.

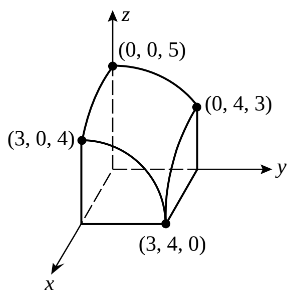
(b) $\int_0^2 \int_0^2 (x - 2y)\,dx\,dy = -4$; the error is zero.

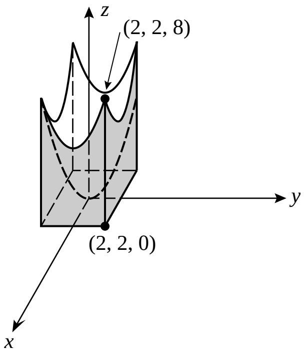
19. The solid is a rectangular box with sides of length 1, 5, and 4, so its volume is $1 \cdot 5 \cdot 4 = 20$;
$\int_0^5 \int_1^2 4\,dx\,dy = \int_0^5 4x\Big]_{x=1}^{2} dy = \int_0^5 4\,dy = 20$.



20. Two copies of the solid will fit together to form a rectangular box whose base is a square of side 1 and whose height is 2, so the solid's volume is $(1^2 \cdot 2)/2 = 1$;
$\int_0^1 \int_0^1 (2 - x - y)\,dx\,dy = \int_0^1 \left[2x - \frac{1}{2}x^2 - xy\right]_{x=0}^{1} dy = \int_0^1 \left(\frac{3}{2} - y\right) dy = \left[\frac{3}{2}y - \frac{1}{2}y^2\right]_0^1 = 1$.



21. 

22. 

23. False. ΔA_k represents the <u>area</u> of such a region.

24. True. $\displaystyle\iint_R f(x,y)\,dA = \int_1^4\int_0^3 f(x,y)\,dy\,dx = \int_1^4 2x\,dx = x^2\Big]_1^4 = 15.$

25. False. $\displaystyle\iint_R f(x,y)\,dA = \int_1^5\int_2^4 f(x,y)\,dy\,dx.$

26. True, by equation (12).

27. $\displaystyle\iint_R f(x,y)\,dA = \int_a^b\left[\int_c^d g(x)h(y)\,dy\right]dx = \int_a^b g(x)\left[\int_c^d h(y)\,dy\right]dx = \left[\int_a^b g(x)\,dx\right]\left[\int_c^d h(y)\,dy\right].$

28. The integral of $\tan x$ (an odd function) over the interval $[-1,1]$ is zero, so the iterated integral is also zero.

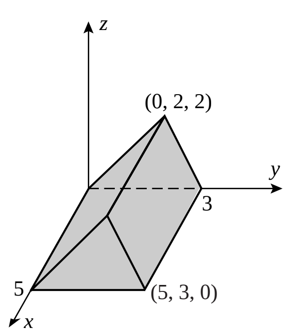
29. $\displaystyle V = \int_3^5\int_1^2 (2x+y)\,dy\,dx = \int_3^5\left(2x+\frac{3}{2}\right)dx = 19.$

30. $\displaystyle V = \int_1^3\int_0^2 (3x^3+3x^2y)\,dy\,dx = \int_1^3 (6x^3+6x^2)\,dx = 172.$

31. $\displaystyle V = \int_0^2\int_0^3 x^2\,dy\,dx = \int_0^2 3x^2\,dx = 8.$

32. $\displaystyle V = \int_0^3\int_0^4 5\left(1-\frac{x}{3}\right)dy\,dx = \int_0^3 5\left(4-\frac{4x}{3}\right)dx = 30.$

33. $\displaystyle \int_0^{1/2}\int_0^{\pi} x\cos(xy)\cos^2\pi x\,dy\,dx = \int_0^{1/2}\cos^2\pi x\sin(xy)\Big]_0^{\pi}dx = \int_0^{1/2}\cos^2\pi x\sin\pi x\,dx = -\frac{1}{3\pi}\cos^3\pi x\Big]_0^{1/2} = \frac{1}{3\pi}.$

34. **(a)** 

(b) $V = \int_0^5 \int_0^2 y\, dy\, dx + \int_0^5 \int_2^3 (-2y+6)\, dy\, dx = 10 + 5 = 15.$

35. $f_{\text{ave}} = \dfrac{1}{48}\int_0^6 \int_0^8 xy^2\, dx\, dy = \dfrac{1}{48}\int_0^6 \left(\dfrac{1}{2}x^2y^2\Big]_{x=0}^{x=8}\right) dy = \dfrac{1}{48}\int_0^6 32y^2\, dy = 48.$

36. $f_{\text{ave}} = \dfrac{1}{18}\int_0^6 \int_0^3 x^2 + 7y\, dx\, dy = \dfrac{1}{18}\int_0^6 \left(\dfrac{1}{3}x^3 + 7yx\Big]_{x=0}^{x=3}\right) dy = \dfrac{1}{18}\int_0^6 9 + 21y\, dy = 24.$

37. $f_{\text{ave}} = \dfrac{2}{\pi}\int_0^{\pi/2} \int_0^1 y \sin xy\, dx\, dy = \dfrac{2}{\pi}\int_0^{\pi/2} \left(-\cos xy\Big]_{x=0}^{x=1}\right) dy = \dfrac{2}{\pi}\int_0^{\pi/2} (1-\cos y)\, dy = 1 - \dfrac{2}{\pi}.$

38. $f_{\text{ave}} = \dfrac{1}{3}\int_0^3 \int_0^1 x(x^2+y)^{1/2}\, dx\, dy = \int_0^3 \dfrac{1}{9}[(1+y)^{3/2} - y^{3/2}]\, dy = \dfrac{2}{45}(31 - 9\sqrt{3}).$

39. $T_{\text{ave}} = \dfrac{1}{2}\int_0^1 \int_0^2 \left(10 - 8x^2 - 2y^2\right) dy\, dx = \dfrac{1}{2}\int_0^1 \left(\dfrac{44}{3} - 16x^2\right) dx = \left(\dfrac{14}{3}\right)^\circ \mathrm{C}.$

40. $f_{\text{ave}} = \dfrac{1}{A(R)}\int_a^b \int_c^d k\, dy\, dx = \dfrac{1}{A(R)}(b-a)(d-c)k = k.$

41. 1.381737122

42. 2.230985141

43. The first integral equals 1/2, the second equals −1/2. This does not contradict Theorem 14.1.3 because the integrand is not continuous at $(x,y) = (0,0)$; if $f(x,y) = \dfrac{y-x}{(x+y)^3}$, then $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \dfrac{-1}{x^2} \to -\infty$.

44. $V = \int_0^1 \int_0^\pi xy^3 \sin(xy)\, dx\, dy = \int_0^1 \Big[y\sin(xy) - xy^2\cos(xy)\Big]_{x=0}^{\pi} dy = \int_0^1 [y\sin(\pi y) - \pi y^2 \cos(\pi y)]\, dy =$

$= \left[\dfrac{3}{\pi^2}\sin(\pi y) - \dfrac{3}{\pi} y\cos(\pi y) - y^2 \sin(\pi y)\right]_0^1 = \dfrac{3}{\pi}.$

45. If R is a rectangular region defined by $a \le x \le b$, $c \le y \le d$, then the volume given in equation (5) can be written as an iterated integral: $V = \iint_R f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy\right) dx$. The inner integral, $\int_c^d f(x,y)\, dy$, is the area $A(x)$ of the cross-section with x-coordinate x of the solid enclosed between R and the surface $z = f(x,y)$. So $V = \int_a^b A(x)\, dx$, as found in Section 6.2.

Exercise Set 14.2

1. $\int_0^1 \int_{x^2}^x xy^2\, dy\, dx = \int_0^1 \dfrac{1}{3}(x^4 - x^7)\, dx = \dfrac{1}{40}.$

2. $\int_1^{3/2} \int_y^{3-y} y\, dx\, dy = \int_1^{3/2} (3y - 2y^2) dy = \dfrac{7}{24}.$

3. $\int_0^3 \int_0^{\sqrt{9-y^2}} y\, dx\, dy = \int_0^3 y\sqrt{9-y^2}\, dy = 9.$

4. $\displaystyle\int_{1/4}^{1}\int_{x^2}^{x}\sqrt{x/y}\,dy\,dx=\int_{1/4}^{1}\int_{x^2}^{x}x^{1/2}y^{-1/2}\,dy\,dx=\int_{1/4}^{1}2(x-x^{3/2})\,dx=\frac{13}{80}.$

5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}}\int_{0}^{x^3}\sin(y/x)\,dy\,dx=\int_{\sqrt{\pi}}^{\sqrt{2\pi}}[-x\cos(x^2)+x]\,dx=\frac{\pi}{2}.$

6. $\displaystyle\int_{-1}^{1}\int_{-x^2}^{x^2}(x^2-y)\,dy\,dx=\int_{-1}^{1}2x^4\,dx=\frac{4}{5}.$

7. $\displaystyle\int_{0}^{1}\int_{0}^{x}y\sqrt{x^2-y^2}\,dy\,dx=\int_{0}^{1}\frac{1}{3}x^3\,dx=\frac{1}{12}.$

8. $\displaystyle\int_{1}^{2}\int_{0}^{y^2}e^{x/y^2}\,dx\,dy=\int_{1}^{2}(e-1)y^2\,dy=\frac{7(e-1)}{3}.$

9. **(a)** $\displaystyle\int_{0}^{2}\int_{0}^{x^2}f(x,y)\,dy\,dx.$ **(b)** $\displaystyle\int_{0}^{4}\int_{\sqrt{y}}^{2}f(x,y)\,dx\,dy.$

10. **(a)** $\displaystyle\int_{0}^{1}\int_{x^2}^{\sqrt{x}}f(x,y)\,dy\,dx.$ **(b)** $\displaystyle\int_{0}^{1}\int_{y^2}^{\sqrt{y}}f(x,y)\,dx\,dy.$

11. **(a)** $\displaystyle\int_{1}^{2}\int_{-2x+5}^{3}f(x,y)\,dy\,dx+\int_{2}^{4}\int_{1}^{3}f(x,y)\,dy\,dx+\int_{4}^{5}\int_{2x-7}^{3}f(x,y)\,dy\,dx.$

(b) $\displaystyle\int_{1}^{3}\int_{(5-y)/2}^{(y+7)/2}f(x,y)\,dx\,dy.$

12. **(a)** $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}f(x,y)\,dy\,dx.$ **(b)** $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}f(x,y)\,dx\,dy.$

13. **(a)** $\displaystyle\int_{0}^{2}\int_{0}^{x^2}xy\,dy\,dx=\int_{0}^{2}\frac{1}{2}x^5\,dx=\frac{16}{3}.$

(b) $\displaystyle\int_{1}^{3}\int_{(5-y)/2}^{(y+7)/2}xy\,dx\,dy=\int_{1}^{3}(3y^2+3y)\,dy=38.$

14. **(a)** $\displaystyle\int_{0}^{1}\int_{x^2}^{\sqrt{x}}(x+y)\,dy\,dx=\int_{0}^{1}\left(x^{3/2}+\frac{x}{2}-x^3-\frac{x^4}{2}\right)dx=\frac{3}{10}.$

(b) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}x\,dy\,dx+\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}y\,dy\,dx=\int_{-1}^{1}2x\sqrt{1-x^2}\,dx+0=0.$

15. **(a)** $\displaystyle\int_{4}^{8}\int_{16/x}^{x}x^2dy\,dx=\int_{4}^{8}(x^3-16x)\,dx=576.$

(b) $\displaystyle\int_{2}^{4}\int_{16/y}^{8}x^2\,dx\,dy+\int_{4}^{8}\int_{y}^{8}x^2\,dx\,dy=\int_{4}^{8}\left[\frac{512}{3}-\frac{4096}{3y^3}\right]dy+\int_{4}^{8}\frac{512-y^3}{3}\,dy=\frac{640}{3}+\frac{1088}{3}=576.$

16. **(a)** $\displaystyle\int_{0}^{1}\int_{1}^{2}xy^2\,dy\,dx+\int_{1}^{2}\int_{x}^{2}xy^2\,dy\,dx=\int_{0}^{1}7x/3\,dx+\int_{1}^{2}\frac{8x-x^4}{3}\,dx=\frac{7}{6}+\frac{29}{15}=\frac{31}{10}.$

(b) $\displaystyle\int_1^2 \int_0^y xy^2\,dx\,dy = \int_1^2 \frac{1}{2}y^4\,dy = \frac{31}{10}.$

17. (a) $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3x-2y)\,dy\,dx = \int_{-1}^1 6x\sqrt{1-x^2}\,dx = 0.$

(b) $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (3x-2y)\,dx\,dy = \int_{-1}^1 -4y\sqrt{1-y^2}\,dy = 0.$

18. (a) $\displaystyle\int_0^5 \int_{5-x}^{\sqrt{25-x^2}} y\,dy\,dx = \int_0^5 (5x-x^2)\,dx = \frac{125}{6}.$

(b) $\displaystyle\int_0^5 \int_{5-y}^{\sqrt{25-y^2}} y\,dx\,dy = \int_0^5 y\left(\sqrt{25-y^2}-5+y\right)\,dy = \frac{125}{6}.$

19. $\displaystyle\int_0^4 \int_0^{\sqrt{y}} x(1+y^2)^{-1/2}\,dx\,dy = \int_0^4 \frac{1}{2}y(1+y^2)^{-1/2}\,dy = \frac{\sqrt{17}-1}{2}.$

20. $\displaystyle\int_0^\pi \int_0^x x\cos y\,dy\,dx = \int_0^\pi x\sin x\,dx = \pi.$

21. $\displaystyle\int_0^2 \int_{y^2}^{6-y} xy\,dx\,dy = \int_0^2 \frac{1}{2}(36y-12y^2+y^3-y^5)\,dy = \frac{50}{3}.$

22. $\displaystyle\int_0^{\pi/4} \int_{\sin y}^{1/\sqrt{2}} x\,dx\,dy = \int_0^{\pi/4} \frac{1}{4}\cos 2y\,dy = \frac{1}{8}.$

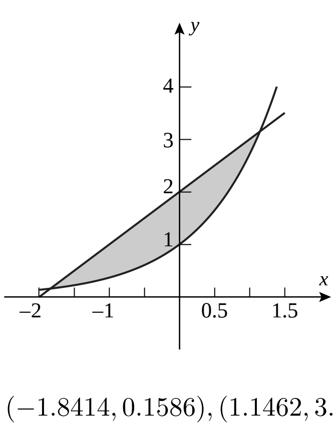
23. $\displaystyle\int_0^1 \int_{x^3}^{x} (x-1)\,dy\,dx = \int_0^1 (-x^4+x^3+x^2-x)\,dx = -\frac{7}{60}.$

24. $\displaystyle\int_0^{1/\sqrt{2}} \int_x^{2x} x^2\,dy\,dx + \int_{1/\sqrt{2}}^1 \int_x^{1/x} x^2\,dy\,dx = \int_0^{1/\sqrt{2}} x^3\,dx + \int_{1/\sqrt{2}}^1 (x-x^3)dx = \frac{1}{8}.$

25. $\displaystyle\int_0^2 \int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 y^2\sin(y^3)\,dy = \frac{1-\cos 8}{3}.$

26. $\displaystyle\int_0^1 \int_{e^x}^{e} x\,dy\,dx = \int_0^1 (ex-xe^x)\,dx = \frac{e}{2}-1.$

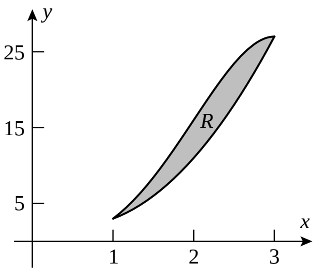
27. (a)



(b) $(-1.8414, 0.1586), (1.1462, 3.1462).$

(c) $\displaystyle\iint_R x\,dA \approx \int_{-1.8414}^{1.1462}\int_{e^x}^{x+2} x\,dy\,dx = \int_{-1.8414}^{1.1462} x(x+2-e^x)\,dx \approx -0.4044.$

(d) $\displaystyle\iint_R x\,dA \approx \int_{0.1586}^{3.1462}\int_{y-2}^{\ln y} x\,dx\,dy = \int_{0.1586}^{3.1462}\left[\frac{\ln^2 y}{2} - \frac{(y-2)^2}{2}\right] dy \approx -0.4044.$

28. **(a)** 

(b) $(1,3),(3,27)$.

(c) $\displaystyle\int_1^3\int_{3-4x+4x^2}^{4x^3-x^4} x\,dy\,dx = \int_1^3 x[(4x^3-x^4)-(3-4x+4x^2)]\,dx = \frac{224}{15}.$

29. $\displaystyle A = \int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx = \int_0^{\pi/4}(\cos x - \sin x)\,dx = \sqrt{2}-1.$

30. $\displaystyle A = \int_{-4}^{1}\int_{3y-4}^{-y^2} dx\,dy = \int_{-4}^{1}(-y^2-3y+4)\,dy = \frac{125}{6}.$

31. $\displaystyle A = \int_{-3}^{3}\int_{1-y^2/9}^{9-y^2} dx\,dy = \int_{-3}^{3} 8\left(1-\frac{y^2}{9}\right) dy = 32.$

32. $\displaystyle A = \int_0^1\int_{\sinh x}^{\cosh x} dy\,dx = \int_0^1(\cosh x - \sinh x)\,dx = 1-e^{-1}.$

33. False. The expression on the right side doesn't make sense. To evaluate an integral of the form $\displaystyle\int_{x^2}^{2x} g(y)\,dy$, x must have a fixed value. But then we can't use x as a variable in defining $\displaystyle g(y) = \int_0^1 f(x,y)\,dx$.

34. True. This is Theorem 14.2.2(a).

35. False. For example, if $f(x,y)=x$ then $\displaystyle\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{x^2}^{1} x\,dy\,dx = \int_{-1}^{1} xy\Big]_{y=x^2}^{1} dx = \int_{-1}^{1} x(1-x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{-1}^{1} = 0$, but $\displaystyle 2\int_0^1\int_{x^2}^{1} x\,dy\,dx = \int_0^1 xy\Big]_{y=x^2}^{1} dx = \int_0^1 x(1-x^2)\,dx = \left[\frac{1}{2}x^2-\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}.$

36. False. For example, if R is the square $0 \le x \le 1$, $0 \le y \le 1$, then the area of R is 1, but $\displaystyle\iint_R xy\,dA = \int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 \frac{1}{2}xy^2\Big]_{y=0}^{1} dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{4}x^2\Big]_0^1 = \frac{1}{4}.$

37. $\displaystyle\int_0^4\int_0^{6-3x/2}\left(3-\frac{3x}{4}-\frac{y}{2}\right) dy\,dx = \int_0^4\left[\left(3-\frac{3x}{4}\right)\left(6-\frac{3x}{2}\right) - \frac{1}{4}\left(6-\frac{3x}{2}\right)^2\right] dx = 12.$

38. $\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx = \int_0^2 (4-x^2)\,dx = \frac{16}{3}$.

39. $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (3-x)\,dy\,dx = \int_{-3}^{3} \left(6\sqrt{9-x^2} - 2x\sqrt{9-x^2}\right) dx = 27\pi$.

40. $V = \int_0^1 \int_{x^2}^{x} (x^2+3y^2)\,dy\,dx = \int_0^1 (2x^3 - x^4 - x^6)\,dx = \frac{11}{70}$.

41. $V = \int_0^3 \int_0^2 (9x^2+y^2)\,dy\,dx = \int_0^3 \left(18x^2 + \frac{8}{3}\right) dx = 170$.

42. $V = \int_{-1}^{1} \int_{y^2}^{1} (1-x)\,dx\,dy = \int_{-1}^{1} \left(\frac{1}{2} - y^2 + \frac{y^4}{2}\right) dy = \frac{8}{15}$.

43. $V = \int_{-3/2}^{3/2} \int_{-\sqrt{9-4x^2}}^{\sqrt{9-4x^2}} (y+3)\,dy\,dx = \int_{-3/2}^{3/2} 6\sqrt{9-4x^2}\,dx = \frac{27\pi}{2}$.

44. $V = \int_0^3 \int_{y^2/3}^{3} (9-x^2)\,dx\,dy = \int_0^3 \left(18 - 3y^2 + \frac{y^6}{81}\right) dy = \frac{216}{7}$.

45. $V = 4\int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx = \frac{8}{3}\int_0^1 (1-x^2)^{3/2}\,dx = \frac{\pi}{2}$.

46. $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx = \int_0^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2}\right] dx = 2\pi$.

47. $\int_0^{\sqrt{2}} \int_{y^2}^{2} f(x,y)\,dx\,dy$.

48. $\int_0^8 \int_0^{x/2} f(x,y)\,dy\,dx$.

49. $\int_1^{e^2} \int_{\ln x}^{2} f(x,y)\,dy\,dx$.

50. $\int_0^1 \int_{e^y}^{e} f(x,y)\,dx\,dy$.

51. $\int_0^{\pi/2} \int_0^{\sin x} f(x,y)\,dy\,dx$.

52. $\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\,dy\,dx$.

53. $\int_0^4 \int_0^{y/4} e^{-y^2}\,dx\,dy = \int_0^4 \frac{1}{4} y e^{-y^2}\,dy = \frac{1-e^{-16}}{8}$.

54. $\int_0^1 \int_0^{2x} \cos(x^2)\,dy\,dx = \int_0^1 2x\cos(x^2)\,dx = \sin 1$.

55. $\displaystyle\int_0^2\int_0^{x^2} e^{x^3}\,dy\,dx = \int_0^2 x^2e^{x^3}\,dx = \frac{e^8-1}{3}.$

56. $\displaystyle\int_0^{\ln 3}\int_{e^y}^3 x\,dx\,dy = \frac{1}{2}\int_0^{\ln 3}(9-e^{2y})\,dy = \frac{9\ln 3-4}{2}.$

57. **(a)** $\displaystyle\int_0^4\int_{\sqrt{x}}^2 \sin\left(\pi y^3\right)dy\,dx$; the inner integral is non-elementary.

$$\int_0^2\int_0^{y^2}\sin\left(\pi y^3\right)dx\,dy = \int_0^2 y^2\sin\left(\pi y^3\right)dy = -\frac{1}{3\pi}\cos\left(\pi y^3\right)\Big]_0^2 = 0.$$

(b) $\displaystyle\int_0^1\int_{\sin^{-1}y}^{\pi/2}\sec^2(\cos x)\,dx\,dy$; the inner integral is non-elementary.

$$\int_0^{\pi/2}\int_0^{\sin x}\sec^2(\cos x)\,dy\,dx = \int_0^{\pi/2}\sec^2(\cos x)\sin x\,dx = \tan 1.$$

58. $\displaystyle V = 4\int_0^2\int_0^{\sqrt{4-x^2}}(x^2+y^2)\,dy\,dx = 4\int_0^2\left(x^2\sqrt{4-x^2}+\frac{1}{3}(4-x^2)^{3/2}\right)dx =$
$\displaystyle = \int_0^{\pi/2}\left(\frac{64}{3}+\frac{64}{3}\sin^2\theta-\frac{128}{3}\sin^4\theta\right)d\theta = \frac{64}{3}\frac{\pi}{2}+\frac{64}{3}\frac{\pi}{4}-\frac{128}{3}\frac{\pi}{2}\frac{1\cdot 3}{2\cdot 4} = 8\pi$ (by substituting $x = 2\sin\theta$).

59. The region is symmetric with respect to the y-axis, and the integrand is an odd function of x, hence the answer is zero.

60. This is the volume in the first octant under the surface $z = \sqrt{1-x^2-y^2}$, so 1/8 of the volume of the sphere of radius 1, thus $\dfrac{\pi}{6}$.

61. Area of triangle is 1/2, so $\displaystyle f_{\text{ave}} = 2\int_0^1\int_x^1\frac{1}{1+x^2}\,dy\,dx = 2\int_0^1\left[\frac{1}{1+x^2}-\frac{x}{1+x^2}\right]dx = \frac{\pi}{2}-\ln 2.$

62. Area $\displaystyle = \int_0^2(3x-x^2-x)\,dx = \frac{4}{3}$, so $\displaystyle f_{\text{ave}} = \frac{3}{4}\int_0^2\int_x^{3x-x^2}(x^2-xy)\,dy\,dx = \frac{3}{4}\int_0^2\left(-2x^3+2x^4-\frac{x^5}{2}\right)dx = -\frac{3}{4}\frac{8}{15} =$ $-\dfrac{2}{5}.$

63. $\displaystyle T_{\text{ave}} = \frac{1}{A(R)}\iint_R(5xy+x^2)\,dA$. The diamond has corners $(\pm 2,0),(0,\pm 4)$ and thus has area $A(R) = 4\frac{1}{2}2(4) =$ 16m^2. Since $5xy$ is an odd function of x (as well as y), $\displaystyle\iint_R 5xy\,dA = 0$. Since x^2 is an even function of both x and y, $\displaystyle T_{\text{ave}} = \frac{4}{16}\iint_{\substack{R\\x,y>0}}x^2\,dA = \frac{1}{4}\int_0^2\int_0^{4-2x}x^2\,dy\,dx = \frac{1}{4}\int_0^2(4-2x)x^2\,dx = \frac{1}{4}\left[\frac{4}{3}x^3-\frac{1}{2}x^4\right]_0^2 = \left(\frac{2}{3}\right)^\circ\text{C}.$

64. The area of the lens is $\pi R^2 = 4\pi$ and the average thickness T_{ave} is $\displaystyle T_{\text{ave}} = \frac{4}{4\pi}\int_0^2\int_0^{\sqrt{4-x^2}}\left(1-\frac{x^2+y^2}{4}\right)dy\,dx =$ $\displaystyle\frac{1}{\pi}\int_0^2\frac{1}{6}(4-x^2)^{3/2}\,dx = \frac{8}{3\pi}\int_0^{\pi/2}\sin^4\theta\,d\theta = \frac{8}{3\pi}\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2} = \frac{1}{2}$ in (by substituting $x = 2\cos\theta$).

65. $y = \sin x$ and $y = x/2$ intersect at $x = 0$ and $x = a \approx 1.895494$, so $\displaystyle V = \int_0^a\int_{x/2}^{\sin x}\sqrt{1+x+y}\,dy\,dx \approx 0.676089.$

67. See Example 7. Given an iterated integral $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, draw the type II region R defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$. If R is also a type I region, try to determine the numbers a and b and functions $g_1(x)$ and $g_2(x)$ such that R is also described by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$. Then $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$. This isn't always possible: R may not be a type I region. Even if it is, it may not be possible to find formulas for $g_1(x)$ and $g_2(x)$.

Exercise Set 14.3

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}\sin^2\theta\cos\theta\,d\theta = \frac{1}{6}.$

2. $\displaystyle\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi}\frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{3\pi}{4}.$

3. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\,dr\,d\theta = \int_0^{\pi/2}\frac{a^3}{3}\sin^3\theta\,d\theta = \frac{2}{9}a^3.$

4. $\displaystyle\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_0^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = \frac{\pi}{24}.$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\,dr\,d\theta = \int_0^{\pi}\frac{1}{3}(1-\sin\theta)^3\cos\theta\,d\theta = 0.$

6. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^3\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{4}\cos^4\theta\,d\theta = \frac{3\pi}{64}.$

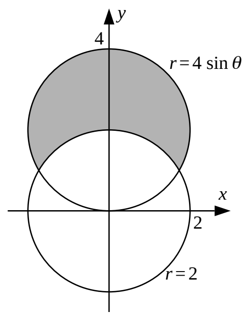
7. $\displaystyle A = \int_0^{2\pi}\int_0^{1-\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}.$

8. $\displaystyle A = 4\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{\pi}{2}.$

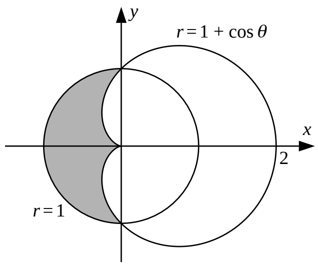
9. $\displaystyle A = \int_{\pi/4}^{\pi/2}\int_{\sin 2\theta}^{1} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\frac{1}{2}(1-\sin^2 2\theta)\,d\theta = \frac{\pi}{16}.$

10. $\displaystyle A = 2\int_0^{\pi/3}\int_{\sec\theta}^{2} r\,dr\,d\theta = \int_0^{\pi/3}(4-\sec^2\theta)\,d\theta = \frac{4\pi}{3} - \sqrt{3}.$

11. $\displaystyle A = \int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} f(r,\theta)\,r\,dr\,d\theta.$



12. $A = \int_{\pi/2}^{3\pi/2} \int_{1+\cos\theta}^{1} f(r,\theta) r \; dr \, d\theta.$



13. $V = 8\int_0^{\pi/2}\int_1^3 r\sqrt{9-r^2}\, dr\, d\theta.$

14. $V = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^2 dr\, d\theta.$

15. $V = 2\int_0^{\pi/2}\int_0^{\cos\theta} (1-r^2) r\, dr\, d\theta.$

16. $V = 4\int_0^{\pi/2}\int_1^3 dr\, d\theta.$

17. $V = 8\int_0^{\pi/2}\int_1^3 r\sqrt{9-r^2}\, dr\, d\theta = \frac{128}{3}\sqrt{2}\int_0^{\pi/2} d\theta = \frac{64}{3}\sqrt{2}\pi.$

18. $V = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^2 dr\, d\theta = \frac{16}{3}\int_0^{\pi/2} \sin^3\theta\, d\theta = \frac{32}{9}.$

19. $V = 2\int_0^{\pi/2}\int_0^{\cos\theta} (1-r^2) r\, dr\, d\theta = \frac{1}{2}\int_0^{\pi/2} (2\cos^2\theta - \cos^4\theta)\, d\theta = \frac{5\pi}{32}.$

20. $V = 4\int_0^{\pi/2}\int_1^3 dr\, d\theta = 8\int_0^{\pi/2} d\theta = 4\pi.$

21. $V = \int_0^{\pi/2}\int_0^{3\sin\theta} r^2 \sin\theta\, dr\, d\theta = 9\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{27\pi}{16}.$

22. $V = 4\int_0^{\pi/2}\int_{2\cos\theta}^{2} \sqrt{4-r^2}\, r\, dr\, d\theta + 4\int_{\pi/2}^{\pi}\int_0^2 \sqrt{4-r^2}\, r\, dr\, d\theta = \int_0^{\pi/2} \frac{32}{3}(1-\cos^2\theta)^{3/2}\theta\, d\theta + \int_{\pi/2}^{\pi} \frac{32}{3}\, d\theta = \frac{64}{9} + \frac{16\pi}{3}.$

23. $\displaystyle\int_0^{2\pi}\int_0^3 \sin(r^2)r\,dr\,d\theta = \frac{1}{2}(1-\cos 9)\int_0^{2\pi} d\theta = \pi(1-\cos 9).$

24. $\displaystyle\int_0^{\pi/2}\int_0^3 r\sqrt{9-r^2}\,dr\,d\theta = 9\int_0^{\pi/2} d\theta = \frac{9\pi}{2}.$

25. $\displaystyle\int_0^{\pi/4}\int_0^2 \frac{1}{1+r^2}r\,dr\,d\theta = \frac{1}{2}\ln 5\int_0^{\pi/4} d\theta = \frac{\pi}{8}\ln 5.$

26. $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} 2r^2\sin\theta\,dr\,d\theta = \frac{16}{3}\int_{\pi/4}^{\pi/2}\cos^3\theta\sin\theta\,d\theta = \frac{1}{3}.$

27. $\displaystyle\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{\pi}{8}.$

28. $\displaystyle\int_0^{2\pi}\int_0^2 e^{-r^2}r\,dr\,d\theta = \frac{1}{2}(1-e^{-4})\int_0^{2\pi} d\theta = (1-e^{-4})\pi.$

29. $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 dr\,d\theta = \frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta = \frac{16}{9}.$

30. $\displaystyle\int_0^{\pi/2}\int_0^1 \cos(r^2)r\,dr\,d\theta = \frac{1}{2}\sin 1\int_0^{\pi/2} d\theta = \frac{\pi}{4}\sin 1.$

31. $\displaystyle\int_0^{\pi/2}\int_0^a \frac{r}{(1+r^2)^{3/2}}\,dr\,d\theta = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{1+a^2}}\right).$

32. $\displaystyle\int_0^{\pi/4}\int_0^{\sec\theta\tan\theta} r^2\,dr\,d\theta = \frac{1}{3}\int_0^{\pi/4}\sec^3\theta\tan^3\theta\,d\theta = \frac{2(\sqrt{2}+1)}{45}.$

33. $\displaystyle\int_0^{\pi/4}\int_0^2 \frac{r}{\sqrt{1+r^2}}\,dr\,d\theta = \frac{\pi}{4}(\sqrt{5}-1).$

34. $\displaystyle\int_{\pi/2}^{3\pi/2}\int_0^4 3r^2\cos\theta\,dr\,d\theta = \int_{\pi/2}^{3\pi/2} 64\cos\theta\,d\theta = -128.$

35. True. It can be defined by the inequalities $0 \le \theta \le 2\pi$, $0 \le r \le 2$.

36. False. The volume is $\displaystyle\iint_R f(r,\theta)\,dA$. The extra factor r isn't introduced until we write this as an iterated integral as in Theorem 14.3.3.

37. False. The integrand in the iterated integral should be multiplied by r: $\displaystyle\iint_R f(r,\theta)\,dA = \int_0^{\pi/2}\int_1^2 f(r,\theta)\,r\,dr\,d\theta.$

38. False. The region is described by $0 \le \theta \le \pi$, $0 \le r \le \sin\theta$, so $A = \displaystyle\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta.$

39. $V = \displaystyle\int_0^{2\pi}\int_0^a hr\,dr\,d\theta = \int_0^{2\pi} h\frac{a^2}{2}\,d\theta = \pi a^2 h.$

40. $V = \displaystyle\int_0^{2\pi}\int_0^R D(r)r\,dr\,d\theta = \int_0^{2\pi}\int_0^R ke^{-r}r\,dr\,d\theta = -2\pi k(1+r)e^{-r}\Big]_0^R = 2\pi k[1-(R+1)e^{-R}].$

41. $\displaystyle\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}\int_0^2 r^3\cos^2\theta\,dr\,d\theta = 4\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}\cos^2\theta\,d\theta = 2\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}(1+\cos(2\theta))\,d\theta = \Big[2\theta+2\cos\theta\sin\theta\Big]_{\tan^{-1}(1/3)}^{\tan^{-1}(2)} = 2\tan^{-1}(2)+2\cdot\frac{1}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}}-2\tan^{-1}(1/3)-2\cdot\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}} = 2\left(\tan^{-1}(2)-\tan^{-1}(1/3)\right)+\frac{1}{5} = 2\tan^{-1}(1)+\frac{1}{5} = \frac{\pi}{2}+\frac{1}{5}.$

42. $\displaystyle A = \int_0^{\phi}\int_0^{2a\sin\theta} r\,dr\,d\theta = 2a^2\int_0^{\phi}\sin^2\theta\,d\theta = a^2\phi-\frac{1}{2}a^2\sin 2\phi.$

43. **(a)** $\displaystyle V = 8\int_0^{\pi/2}\int_0^a \frac{c}{a}(a^2-r^2)^{1/2}\,r\,dr\,d\theta = -\frac{4c}{3a}\pi(a^2-r^2)^{3/2}\Big]_0^a = \frac{4}{3}\pi a^2 c.$

(b) $\displaystyle V \approx \frac{4}{3}\pi(6378.1370)^2 6356.5231\ \text{km}^3 \approx 1.0831682\cdot 10^{12}\ \text{km}^3 = 1.0831682\cdot 10^{21}\ \text{m}^3.$

44. $\displaystyle V = 2\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{c}{a}(a^2-r^2)^{1/2} r\,dr\,d\theta = \frac{2}{3}a^2c\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta = \frac{(3\pi-4)a^2c}{9}.$

45. $\displaystyle A = 4\int_0^{\pi/4}\int_0^{a\sqrt{2\cos 2\theta}} r\,dr\,d\theta = 4a^2\int_0^{\pi/4}\cos 2\theta\,d\theta = 2a^2.$

46. $\displaystyle A = \int_{\pi/6}^{\pi/4}\int_{\sqrt{8\cos 2\theta}}^{4\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{\pi/4}(8\sin^2\theta-4\cos 2\theta)\,d\theta + \int_{\pi/4}^{\pi/2} 8\sin^2\theta\,d\theta = \frac{4\pi}{3}+2\sqrt{3}-2.$

Exercise Set 14.4

1. $z=\sqrt{9-y^2}$, $z_x=0$, $z_y=-y/\sqrt{9-y^2}$, $z_x^2+z_y^2+1 = 9/(9-y^2)$, $\displaystyle S = \int_0^2\int_{-3}^3 \frac{3}{\sqrt{9-y^2}}\,dy\,dx = \int_0^2 3\pi\,dx = 6\pi.$

2. $z = 8-2x-2y$, $z_x^2+z_y^2+1 = 4+4+1 = 9$, $\displaystyle S = \int_0^4\int_0^{4-x} 3\,dy\,dx = \int_0^4 3(4-x)dx = 24.$

3. $z^2 = 4x^2+4y^2$, $2zz_x = 8x$ so $z_x = 4x/z$; similarly $z_y = 4y/z$ so $z_x^2+z_y^2+1 = (16x^2+16y^2)/z^2+1 = 5$, $\displaystyle S = \int_0^1\int_{x^2}^x \sqrt{5}\,dy\,dx = \sqrt{5}\int_0^1 (x-x^2)\,dx = \frac{\sqrt{5}}{6}.$

4. $z_x = 2$, $z_y = 2y$, $z_x^2+z_y^2+1 = 5+4y^2$, $\displaystyle S = \int_0^1\int_0^y \sqrt{5+4y^2}\,dx\,dy = \int_0^1 y\sqrt{5+4y^2}\,dy = \frac{27-5\sqrt{5}}{12}.$

5. $z^2 = x^2+y^2$, $z_x = x/z$, $z_y = y/z$, $z_x^2+z_y^2+1 = (x^2+y^2)/z^2+1 = 2$, $\displaystyle S = \iint_R \sqrt{2}\,dA = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\sqrt{2}\,r\,dr\,d\theta = 4\sqrt{2}\int_0^{\pi/2}\cos^2\theta\,d\theta = \sqrt{2}\pi.$

6. $z_x = -2x$, $z_y = -2y$, $z_x^2+z_y^2+1 = 4x^2+4y^2+1$, $\displaystyle S = \iint_R \sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^1 r\sqrt{4r^2+1}\,dr\,d\theta = \frac{1}{12}(5\sqrt{5}-1)\int_0^{2\pi} d\theta = \frac{\pi}{6}(5\sqrt{5}-1).$

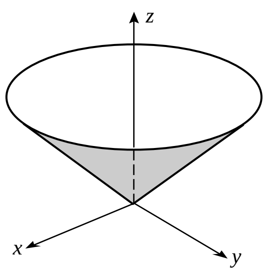
7. $z_x = y$, $z_y = x$, $z_x^2+z_y^2+1 = x^2+y^2+1$, $\displaystyle S = \iint_R \sqrt{x^2+y^2+1}\,dA = \int_0^{\pi/6}\int_0^3 r\sqrt{r^2+1}\,dr\,d\theta = \frac{1}{3}(10\sqrt{10}-1)\int_0^{\pi/6} d\theta = \frac{\pi}{18}(10\sqrt{10}-1).$

8. $z_x = x$, $z_y = y$, $z_x^2+z_y^2+1 = x^2+y^2+1$, $S = \iint\limits_R \sqrt{x^2+y^2+1}\,dA = \int_0^{2\pi}\int_0^{\sqrt{8}} r\sqrt{r^2+1}\,dr\,d\theta = \dfrac{26}{3}\int_0^{2\pi} d\theta = \dfrac{52\pi}{3}$.

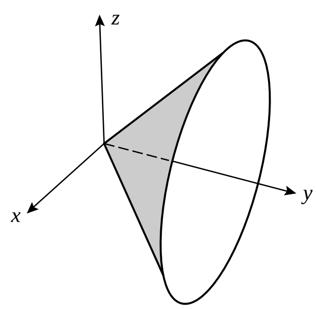
9. On the sphere, $z_x = -x/z$ and $z_y = -y/z$ so $z_x^2+z_y^2+1 = (x^2+y^2+z^2)/z^2 = 16/(16-x^2-y^2)$. The planes $z=1$ and $z=2$ intersect the sphere along the circles $x^2+y^2=15$ and $x^2+y^2=12$, so $S = \iint\limits_R \dfrac{4}{\sqrt{16-x^2-y^2}}dA = \int_0^{2\pi}\int_{\sqrt{12}}^{\sqrt{15}} \dfrac{4r}{\sqrt{16-r^2}}\,dr\,d\theta = 4\int_0^{2\pi} d\theta = 8\pi$.

10. On the sphere, $z_x = -x/z$ and $z_y = -y/z$ so $z_x^2+z_y^2+1 = (x^2+y^2+z^2)/z^2 = 8/(8-x^2-y^2)$; the cone cuts the sphere in the circle $x^2+y^2=4$; $S = \int_0^{2\pi}\int_0^2 \dfrac{2\sqrt{2}r}{\sqrt{8-r^2}}dr\,d\theta = (8-4\sqrt{2})\int_0^{2\pi} d\theta = 8(2-\sqrt{2})\pi$.

11. **(a)**



(b)



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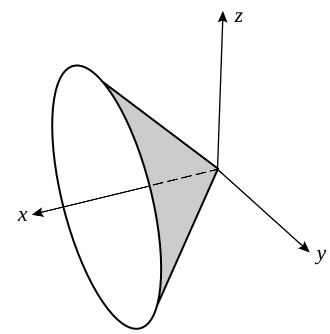
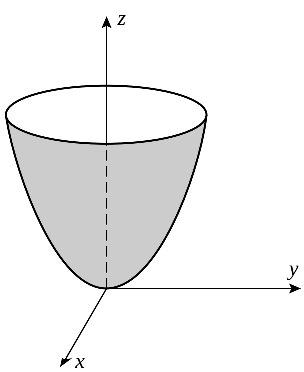
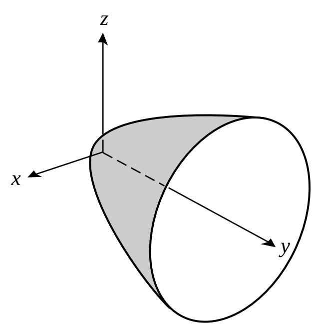


12. **(a)**



(b)



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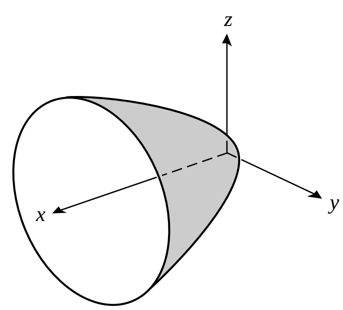


13. **(a)** $x = u, y = v, z = \dfrac{5}{2} + \dfrac{3}{2}u - 2v$. **(b)** $x = u, y = v, z = u^2$.

14. **(a)** $x = u, y = v, z = \dfrac{v}{1+u^2}$. **(b)** $x = u, y = v, z = \dfrac{1}{3}v^2 - \dfrac{5}{3}$.

15. **(a)** $x = \sqrt{5}\cos u, y = \sqrt{5}\sin u, z = v; 0 \le u \le 2\pi, 0 \le v \le 1$.

(b) $x = 2\cos u, y = v, z = 2\sin u; 0 \le u \le 2\pi, 1 \le v \le 3$.

16. **(a)** $x = u, y = 1-u, z = v; -1 \le v \le 1$ **(b)** $x = u, y = 5+2v, z = v; 0 \le u \le 3$.

17. $x = u, y = \sin u\cos v, z = \sin u \sin v$.

18. $x = u, y = e^u\cos v, z = e^u \sin v$.

19. $x = r\cos\theta, y = r\sin\theta, z = \dfrac{1}{1+r^2}$.

20. $x = r\cos\theta, y = r\sin\theta, z = e^{-r^2}$.

21. $x = r\cos\theta, y = r\sin\theta, z = 2r^2\cos\theta\sin\theta$.

22. $x = r\cos\theta, y = r\sin\theta, z = r^2(\cos^2\theta - \sin^2\theta)$.

23. $x = r\cos\theta, y = r\sin\theta, z = \sqrt{9 - r^2}; r \le \sqrt{5}$.

24. $x = r\cos\theta, y = r\sin\theta, z = r; r \le 3$.

25. $x = \dfrac{1}{2}\rho\cos\theta, y = \dfrac{1}{2}\rho\sin\theta, z = \dfrac{\sqrt{3}}{2}\rho$.

26. $x = 3\cos\theta, y = 3\sin\theta, z = 3\cot\phi$.

27. $z = x - 2y$; a plane.

28. $y = x^2 + z^2, 0 \le y \le 4$; part of a circular paraboloid.

29. $(x/3)^2 + (y/2)^2 = 1; 2 \le z \le 4$; part of an elliptic cylinder.

30. $z = x^2 + y^2; 0 \le z \le 4$; part of a circular paraboloid.

31. $(x/3)^2 + (y/4)^2 = z^2; 0 \le z \le 1$; part of an elliptic cone.

32. $x^2 + (y/2)^2 + (z/3)^2 = 1$; an ellipsoid.

33. (a) I: $x = r\cos\theta,\ y = r\sin\theta,\ z = r,\ 0 \le r \le 2$; II: $x = u,\ y = v,\ z = \sqrt{u^2 + v^2};\ 0 \le u^2 + v^2 \le 4$.

34. (a) I: $x = r\cos\theta,\ y = r\sin\theta,\ z = r^2,\ 0 \le r \le \sqrt{2}$; II: $x = u,\ y = v,\ z = u^2 + v^2;\ u^2 + v^2 \le 2$.

35. (a) $0 \le u \le 3,\ 0 \le v \le \pi$. **(b)** $0 \le u \le 4,\ -\pi/2 \le v \le \pi/2$.

36. (a) $0 \le u \le 6,\ -\pi \le v \le 0$. **(b)** $0 \le u \le 5,\ \pi/2 \le v \le 3\pi/2$.

37. (a) $0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi$. **(b)** $0 \le \phi \le \pi,\ 0 \le \theta \le \pi$.

38. (a) $\pi/2 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$. **(b)** $0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2$.

39. $u = 1,\ v = 2,\ \mathbf{r}_u \times \mathbf{r}_v = -2\mathbf{i} - 4\mathbf{j} + \mathbf{k};\ 2x + 4y - z = 5$.

40. $u = 1,\ v = 2,\ \mathbf{r}_u \times \mathbf{r}_v = -4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k};\ 2x + y - 4z = -6$.

41. $u = 0,\ v = 1,\ \mathbf{r}_u \times \mathbf{r}_v = 6\mathbf{k};\ z = 0$.

42. $\mathbf{r}_u \times \mathbf{r}_v = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k};\ 2x - y - 3z = -4$.

43. $\mathbf{r}_u \times \mathbf{r}_v = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j} + \dfrac{1}{2}\mathbf{k};\ x - y + \dfrac{1}{\sqrt{2}}z = \dfrac{\pi\sqrt{2}}{8}$.

44. $\mathbf{r}_u \times \mathbf{r}_v = 2\mathbf{i} - \ln 2\,\mathbf{k};\ 2x - (\ln 2)z = 0$.

45. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + 2u\,\mathbf{k},\ \mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j},\ \|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{4u^2 + 1};\ S = \displaystyle\int_0^{2\pi}\int_1^2 u\sqrt{4u^2 + 1}\,du\,dv = \dfrac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$.

46. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$, $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2}u$; $S = \int_0^{\pi/2}\int_0^{2v} \sqrt{2}\,u\,du\,dv = \dfrac{\sqrt{2}}{12}\pi^3$.

47. False. For example, if $f(x,y) = 1$ then the surface has the same area as R, $\iint\limits_R dA$, not $\iint\limits_R \sqrt{2}\,dA$.

48. True. $\mathbf{q}\times\mathbf{r} = \left\langle -\dfrac{\partial z}{\partial x}, -\dfrac{\partial z}{\partial y}, 1\right\rangle$, so $\iint\limits_R \|\mathbf{q}\times\mathbf{r}\|\,dA = \iint\limits_R \sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}\,dA = S$, by equation (2).

49. True, as explained before Definition 14.4.1.

50. True. $\|\langle 1,0,a\rangle \times \langle 0,1,b\rangle\| = \|\langle -a,-b,1\rangle\| = \sqrt{a^2+b^2+1} = \sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}$, so the area of the surface is $\iint\limits_R \sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}\,dA = \iint\limits_R \|\langle 1,0,a\rangle \times \langle 0,1,b\rangle\|\,dA = \|\langle 1,0,a\rangle \times \langle 0,1,b\rangle\| \cdot \iint\limits_R dA =$
$= \|\langle 1,0,a\rangle \times \langle 0,1,b\rangle\|\cdot$(area of R).

51. $\mathbf{r}(u,v) = a\cos u\sin v\mathbf{i} + a\sin u\sin v\mathbf{j} + a\cos v\mathbf{k}$, $\|\mathbf{r}_u\times\mathbf{r}_v\| = a^2\sin v$, $S = \int_0^{\pi}\int_0^{2\pi} a^2\sin v\,du\,dv = 2\pi a^2\int_0^{\pi}\sin v\,dv =$ $4\pi a^2$.

52. $\mathbf{r} = r\cos u\mathbf{i} + r\sin u\mathbf{j} + v\mathbf{k}$, $\|\mathbf{r}_u\times\mathbf{r}_v\| = r$; $S = \int_0^h\int_0^{2\pi} r\,du\,dv = 2\pi rh$.

53. $z_x = \dfrac{h}{a}\dfrac{x}{\sqrt{x^2+y^2}}$, $z_y = \dfrac{h}{a}\dfrac{y}{\sqrt{x^2+y^2}}$, $z_x^2 + z_y^2 + 1 = \dfrac{h^2x^2+h^2y^2}{a^2(x^2+y^2)} + 1 = \dfrac{a^2+h^2}{a^2}$, $S = \int_0^{2\pi}\int_0^a \dfrac{\sqrt{a^2+h^2}}{a}\,r\,dr\,d\theta =$ $\dfrac{1}{2}a\sqrt{a^2+h^2}\int_0^{2\pi} d\theta = \pi a\sqrt{a^2+h^2}$.

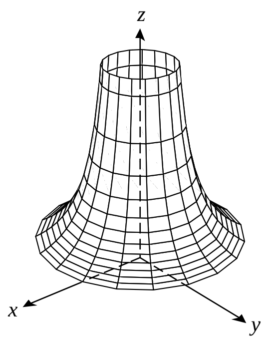
54. **(a)** Revolving a point $(a_0, 0, b_0)$ of the xz-plane around the z-axis generates a circle, an equation of which is $\mathbf{r} = a_0\cos u\mathbf{i} + a_0\sin u\mathbf{j} + b_0\mathbf{k}, 0 \le u \le 2\pi$. A point on the circle $(x-a)^2 + z^2 = b^2$ which generates the torus can be written $\mathbf{r} = (a + b\cos v)\mathbf{i} + b\sin v\mathbf{k}, 0 \le v \le 2\pi$. Set $a_0 = a + b\cos v$ and $b_0 = a + b\sin v$ and use the first result: any point on the torus can thus be written in the form $\mathbf{r} = (a + b\cos v)\cos u\mathbf{i} + (a + b\cos v)\sin u\mathbf{j} + b\sin v\mathbf{k}$, which yields the result.

55. $\mathbf{r}_u = -(a+b\cos v)\sin u\,\mathbf{i} + (a+b\cos v)\cos u\,\mathbf{j}$, $\mathbf{r}_v = -b\sin v\cos u\,\mathbf{i} - b\sin v\sin u\,\mathbf{j} + b\cos v\,\mathbf{k}$, $\|\mathbf{r}_u\times\mathbf{r}_v\| = b(a+b\cos v)$; $S = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos v)\,du\,dv = 4\pi^2 ab$.

56. $\|\mathbf{r}_u\times\mathbf{r}_v\| = \sqrt{u^2+1}$; $S = \int_0^{4\pi}\int_0^5 \sqrt{u^2+1}\,du\,dv = 4\pi\int_0^5 \sqrt{u^2+1}\,du \approx 174.7199011$.

57. $z = -1$ when $v \approx 0.27955$, $z = 1$ when $v \approx 2.86204$, $\|\mathbf{r}_u\times\mathbf{r}_v\| = |\cos v|$; $S \approx \int_0^{2\pi}\int_{0.27955}^{2.86204} |\cos v|\,dv\,du \approx 9.099$.

58. **(a)** $x = v\cos u, y = v\sin u, z = f(v)$, for example. **(b)** $x = v\cos u, y = v\sin u, z = 1/v^2$.



(c)

59. $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$, ellipsoid.

60. $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2=1$, hyperboloid of one sheet.

61. $-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$, hyperboloid of two sheets.

Exercise Set 14.5

1. $\displaystyle\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz=\int_{-1}^{1}\int_{0}^{2}(1/3+y^2+z^2)\,dy\,dz=\int_{-1}^{1}(10/3+2z^2)\,dz=8.$

2. $\displaystyle\int_{1/3}^{1/2}\int_{0}^{\pi}\int_{0}^{1}zx\sin xy\,dz\,dy\,dx=\int_{1/3}^{1/2}\int_{0}^{\pi}\frac{1}{2}x\sin xy\,dy\,dx=\int_{1/3}^{1/2}\frac{1}{2}(1-\cos\pi x)\,dx=\frac{1}{12}+\frac{\sqrt{3}-2}{4\pi}.$

3. $\displaystyle\int_{0}^{2}\int_{-1}^{y^2}\int_{-1}^{z}yz\,dx\,dz\,dy=\int_{0}^{2}\int_{-1}^{y^2}(yz^2+yz)\,dz\,dy=\int_{0}^{2}\left(\frac{1}{3}y^7+\frac{1}{2}y^5-\frac{1}{6}y\right)dy=\frac{47}{3}.$

4. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{1}\int_{0}^{x^2}x\cos y\,dz\,dx\,dy=\int_{0}^{\pi/4}\int_{0}^{1}x^3\cos y\,dx\,dy=\int_{0}^{\pi/4}\frac{1}{4}\cos y\,dy=\frac{\sqrt{2}}{8}.$

5. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{0}^{x}xy\,dy\,dx\,dz=\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\frac{1}{2}x^3dx\,dz=\int_{0}^{3}\frac{1}{8}(81-18z^2+z^4)\,dz=\frac{81}{5}.$

6. $\displaystyle\int_{1}^{3}\int_{x}^{x^2}\int_{0}^{\ln z}xe^y\,dy\,dz\,dx=\int_{1}^{3}\int_{x}^{x^2}(xz-x)\,dz\,dx=\int_{1}^{3}\left(\frac{1}{2}x^5-\frac{3}{2}x^3+x^2\right)dx=\frac{118}{3}.$

7. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2}x\,dz\,dy\,dx=\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}[2x(4-x^2)-2xy^2]\,dy\,dx=\int_{0}^{2}\frac{4}{3}x(4-x^2)^{3/2}\,dx=\frac{128}{15}.$

8. $\displaystyle\int_{1}^{2}\int_{z}^{2}\int_{0}^{\sqrt{3}y}\frac{y}{x^2+y^2}dx\,dy\,dz=\int_{1}^{2}\int_{z}^{2}\frac{\pi}{3}dy\,dz=\int_{1}^{2}\frac{\pi}{3}(2-z)\,dz=\frac{\pi}{6}.$

9. $\displaystyle\int_{0}^{\pi}\int_{0}^{1}\int_{0}^{\pi/6}xy\sin yz\,dz\,dy\,dx=\int_{0}^{\pi}\int_{0}^{1}x[1-\cos(\pi y/6)]\,dy\,dx=\int_{0}^{\pi}(1-3/\pi)x\,dx=\frac{\pi(\pi-3)}{2}.$

10. $\displaystyle\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y}y\,dz\,dy\,dx=\int_{-1}^{1}\int_{0}^{1-x^2}y^2\,dy\,dx=\int_{-1}^{1}\frac{1}{3}(1-x^2)^3\,dx=\frac{32}{105}.$

11. $\int_0^{\sqrt{2}}\int_0^x\int_0^{2-x^2} xyz\,dz\,dy\,dx = \int_0^{\sqrt{2}}\int_0^x \frac{1}{2}xy(2-x^2)^2 dy\,dx = \int_0^{\sqrt{2}} \frac{1}{4}x^3(2-x^2)^2\,dx = \frac{1}{6}.$

12. $\int_{\pi/6}^{\pi/2}\int_y^{\pi/2}\int_0^{xy} \cos(z/y)\,dz\,dx\,dy = \int_{\pi/6}^{\pi/2}\int_y^{\pi/2} y\sin x\,dx\,dy = \int_{\pi/6}^{\pi/2} y\cos y\,dy = \frac{5\pi - 6\sqrt{3}}{12}.$

13. $\int_0^3\int_1^2\int_{-2}^1 \frac{\sqrt{x+z^2}}{y}\,dz\,dy\,dx \approx 9.425.$

14. $8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} e^{-x^2-y^2-z^2}\,dz\,dy\,dx \approx 2.381.$

15. $V = \int_0^4\int_0^{(4-x)/2}\int_0^{(12-3x-6y)/4} dz\,dy\,dx = \int_0^4\int_0^{(4-x)/2} \frac{1}{4}(12-3x-6y)\,dy\,dx = \int_0^4 \frac{3}{16}(4-x)^2\,dx = 4.$

16. $V = \int_0^1\int_0^{1-x}\int_0^{\sqrt{y}} dz\,dy\,dx = \int_0^1\int_0^{1-x} \sqrt{y}\,dy\,dx = \int_0^1 \frac{2}{3}(1-x)^{3/2}\,dx = \frac{4}{15}.$

17. $V = 2\int_0^2\int_{x^2}^4\int_0^{4-y} dz\,dy\,dx = 2\int_0^2\int_{x^2}^4 (4-y)\,dy\,dx = 2\int_0^2 \left(8 - 4x^2 + \frac{1}{2}x^4\right) dx = \frac{256}{15}.$

18. $V = \int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} dz\,dx\,dy = \int_0^1\int_0^y \sqrt{1-y^2}\,dx\,dy = \int_0^1 y\sqrt{1-y^2}\,dy = \frac{1}{3}.$

19. The projection of the curve of intersection onto the xy-plane is $x^2+y^2=1$,

(a) $\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\,dz\,dy\,dx.$ **(b)** $\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\,dz\,dx\,dy.$

20. The projection of the curve of intersection onto the xy-plane is $2x^2+y^2=4$,

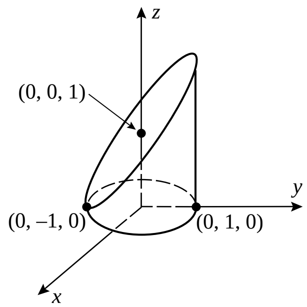
(a) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-2x^2}}^{\sqrt{4-2x^2}}\int_{3x^2+y^2}^{8-x^2-y^2} f(x,y,z)\,dz\,dy\,dx.$ **(b)** $\int_{-2}^2\int_{-\sqrt{(4-y^2)/2}}^{\sqrt{(4-y^2)/2}}\int_{3x^2+y^2}^{8-x^2-y^2} f(x,y,z)\,dz\,dx\,dy.$

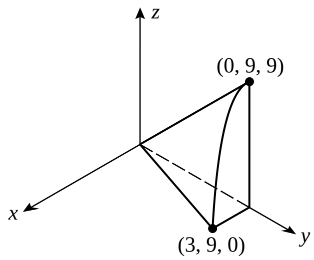
21. Let $f(x,y,z)=1$ in Exercise 19(a). $V = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} dz\,dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} dz\,dy\,dx.$

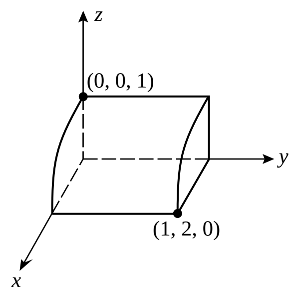
22. Let $f(x,y,z)=1$ in Exercise 20(a). $V = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-2x^2}}^{\sqrt{4-2x^2}}\int_{3x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{4-2x^2}}\int_{3x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx.$

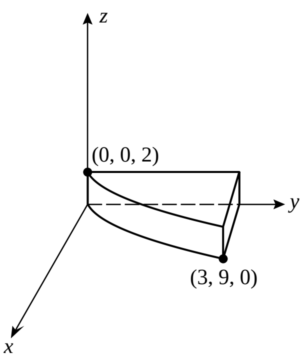
23. $V = 2\int_{-3}^3\int_0^{\sqrt{9-x^2}/3}\int_0^{x+3} dz\,dy\,dx.$

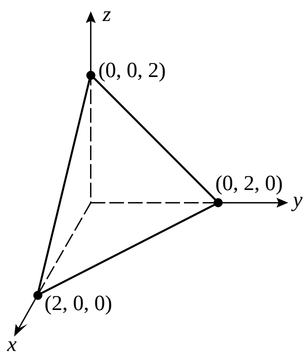
24. $V = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx.$

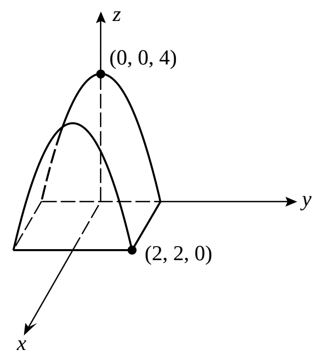
25. **(a)** 

(b) 

(c) 

26. **(a)** 

(b) 

(c) 

27. True, by changing the order of integration in Theorem 14.5.1.

28. False. For example, consider the simple xy-solid G defined by $-1 \le x \le 1$, $-1 \le y \le 1$, $0 \le z \le x^2 + y^2$. Cross-sections of G parallel to the xy-plane with $z > 0$ are neither type I nor type II regions, so the triple integral over G can't be expressed as an integral whose outermost integration is performed with respect to z. (As shown in Theorem 14.5.2, the triple integral can be expressed as an iterated integral whose innermost integration is performed with respect to z.)

29. False. The middle integral (with respect to y) should be $\displaystyle\int_0^{\sqrt{1-x^2}}$.

30. False. For example, let G be described by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and let $f(x, y, z) = 2x$. Then
$$\iiint_G 2x\,dV = \int_0^1 \int_0^1 \int_0^1 2x\,dz\,dy\,dx = \int_0^1 \int_0^1 2x\,dy\,dx = \int_0^1 2x\,dx = x^2\Big]_0^1 = 1 = \text{volume of } G.$$

31. $$\int_a^b \int_c^d \int_k^\ell f(x)g(y)h(z)dz\,dy\,dx = \int_a^b \int_c^d f(x)g(y)\left[\int_k^\ell h(z)\,dz\right]dy\,dx = \left[\int_a^b f(x)\left[\int_c^d g(y)\,dy\right]dx\right]\left[\int_k^\ell h(z)\,dz\right] = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)dy\right]\left[\int_k^\ell h(z)\,dz\right].$$

32. **(a)** $\displaystyle\left[\int_{-1}^1 x\,dx\right]\left[\int_0^1 y^2\,dy\right]\left[\int_0^{\pi/2} \sin z\,dz\right] = (0)(1/3)(1) = 0.$

(b) $\displaystyle\left[\int_0^1 e^{2x}\,dx\right]\left[\int_0^{\ln 3} e^y\,dy\right]\left[\int_0^{\ln 2} e^{-z}dz\right] = [(e^2-1)/2](2)(1/2) = (e^2-1)/2.$

33. $\displaystyle V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx = 1/6,\ f_{\text{ave}} = 6\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\,dz\,dy\,dx = \frac{3}{4}.$

34. The integrand is an odd function of each of x, y, and z, so the average is zero.

35. The volume $V = \dfrac{3\pi}{\sqrt{2}}$, and thus

$$r_{\text{ave}} = \frac{\sqrt{2}}{3\pi}\iiint\limits_G \sqrt{x^2+y^2+z^2}\,dV = \frac{\sqrt{2}}{3\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{5x^2+5y^2}^{6-7x^2-y^2}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \approx 3.291.$$

36. $V = 1, d_{\text{ave}} = \dfrac{1}{V}\displaystyle\int_0^1\int_0^1\int_0^1 \sqrt{(x-z)^2+(y-z)^2+z^2}\,dx\,dy\,dz \approx 0.771.$

37. **(a)** $\displaystyle\int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)} dz\,dy\,dx, \int_0^b\int_0^{a(1-y/b)}\int_0^{c(1-x/a-y/b)} dz\,dx\,dy,$

$\displaystyle\int_0^c\int_0^{a(1-z/c)}\int_0^{b(1-x/a-z/c)} dy\,dx\,dz, \int_0^a\int_0^{c(1-x/a)}\int_0^{b(1-x/a-z/c)} dy\,dz\,dx, \int_0^c\int_0^{b(1-z/c)}\int_0^{a(1-y/b-z/c)} dx\,dy\,dz,$

$\displaystyle\int_0^b\int_0^{c(1-y/b)}\int_0^{a(1-y/b-z/c)} dx\,dz\,dy.$

(b) Use the first integral in part (a) to get $\displaystyle\int_0^a\int_0^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx = \int_0^a \frac{1}{2}bc\left(1-\frac{x}{a}\right)^2 dx = \frac{1}{6}abc.$

38. $\displaystyle V = 8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx = 8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx =$

$\displaystyle = \frac{8c}{b}\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\,dy\,dx =$

$\displaystyle = \frac{8c}{b}\int_0^a\left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}+\frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\sin^{-1}\frac{y}{\sqrt{b^2(1-x^2/a^2)}}\right]_{y=0}^{b\sqrt{1-x^2/a^2}} dx =$

$\displaystyle = \frac{8c}{b}\int_0^a \frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}\,dx = 2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a = \frac{4\pi abc}{3}$, by Endpaper Integral Table Formula 74.

39. **(a)** $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^5 f(x,y,z)\,dz\,dy\,dx$ **(b)** $\displaystyle\int_0^9\int_0^{3-\sqrt{x}}\int_y^{3-\sqrt{x}} f(x,y,z)\,dz\,dy\,dx$

(c) $\displaystyle\int_0^2\int_0^{4-x^2}\int_y^{8-y} f(x,y,z)\,dz\,dy\,dx$

40. **(a)** $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$ **(b)** $\displaystyle\int_0^4\int_0^{x/2}\int_0^2 f(x,y,z)\,dz\,dy\,dx$

(c) $\displaystyle\int_0^2\int_0^{4-x^2}\int_{x^2}^{4-y} f(x,y,z)\,dz\,dy\,dx$

41. See discussion after Theorem 14.5.2.

Exercise Set 14.6

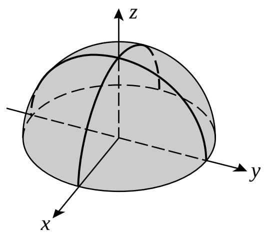
1. $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}(1-r^2)r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{8}\,d\theta = \frac{\pi}{4}.$

2. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{r^2} r\sin\theta\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{\cos\theta} r^3\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{4}\cos^4\theta\sin\theta\,d\theta = \frac{1}{20}.$

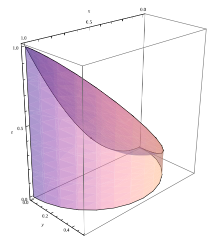
3. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{4}\sin\phi\cos\phi\,d\phi\,d\theta = \int_0^{\pi/2}\frac{1}{8}\,d\theta = \frac{\pi}{16}.$

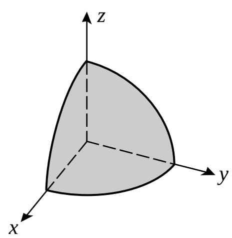
4. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{3}a^3\sec^3\phi\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\frac{1}{6}a^3\,d\theta = \frac{\pi a^3}{3}.$

5. $f(r,\theta,z) = z$

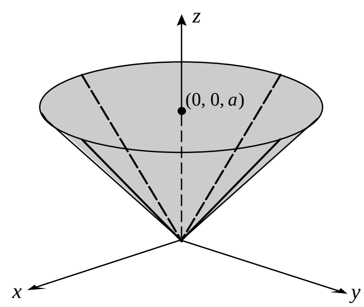


6. $f(r,\theta,z) = \sin\theta$





7. $f(\rho,\theta,\phi) = \rho\cos\phi$



8. $f(\rho,\theta,\phi) = 1$

9. $\displaystyle V = \int_0^{2\pi}\int_0^3\int_{r^2}^9 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta = \int_0^{2\pi}\frac{81}{4}d\theta = \frac{81\pi}{2}.$

10. $\displaystyle V = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_r^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1/\sqrt{2}} r\sqrt{1-r^2}-r^2\,dr\,d\theta = \int_0^{2\pi}\frac{1}{6}(2-\sqrt{2})d\theta = \frac{\pi}{3}(2-\sqrt{2}).$

11. $r^2+z^2=20$ intersects $z=r^2$ in a circle of radius 2; the volume consists of two portions, one inside the cylinder $r=2$ and one outside that cylinder: $\displaystyle V = \int_0^{2\pi}\int_0^2\int_{-\sqrt{20-r^2}}^{r^2} r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_2^{\sqrt{20}}\int_{-\sqrt{20-r^2}}^{\sqrt{20-r^2}} r\,dz\,dr\,d\theta =$ $\displaystyle\int_0^{2\pi}\int_0^2 r\left(r^2+\sqrt{20-r^2}\right)dr\,d\theta + \int_0^{2\pi}\int_2^{\sqrt{20}} 2r\sqrt{20-r^2}\,dr\,d\theta = \frac{4}{3}(10\sqrt{5}-13)\int_0^{2\pi}d\theta + \frac{128}{3}\int_0^{2\pi}d\theta = \frac{152}{3}\pi + \frac{80}{3}\pi\sqrt{5}.$

12. $z = hr/a$ intersects $z = h$ in a circle of radius a, $\displaystyle V = \int_0^{2\pi}\int_0^a\int_{hr/a}^h r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a\frac{h}{a}(ar-r^2)\,dr\,d\theta =$ $\displaystyle\int_0^{2\pi}\frac{1}{6}a^2h\,d\theta = \frac{\pi a^2 h}{3}.$

13. $\displaystyle V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^4\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{64}{3}\sin\phi\,d\phi\,d\theta = \frac{32}{3}\int_0^{2\pi}d\theta = \frac{64\pi}{3}.$

14. $\displaystyle V = \int_0^{2\pi}\int_0^{\pi/4}\int_1^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{7}{3}\sin\phi\,d\phi\,d\theta = \frac{7}{6}(2-\sqrt{2})\int_0^{2\pi}d\theta = \frac{7\pi}{3}(2-\sqrt{2}).$

15. In spherical coordinates the sphere and the plane $z=a$ are $\rho=2a$ and $\rho=a\sec\phi$, respectively. They intersect at $\phi=\pi/3$, $\displaystyle V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{a\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2a}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta =$

$$= \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}a^3\sec^3\phi\sin\phi\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\frac{8}{3}a^3\sin\phi\,d\phi\,d\theta = \frac{1}{2}a^3\int_0^{2\pi}d\theta + \frac{4}{3}a^3\int_0^{2\pi}d\theta = \frac{11\pi a^3}{3}.$$

16. $V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^3 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_{\pi/4}^{\pi/2} 9\sin\phi\,d\phi\,d\theta = \frac{9\sqrt{2}}{2}\int_0^{2\pi}d\theta = 9\sqrt{2}\pi.$

17. $\int_0^{\pi/2}\int_0^a\int_0^{a^2-r^2} r^3\cos^2\theta\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^a (a^2r^3 - r^5)\cos^2\theta\,dr\,d\theta = \frac{1}{12}a^6\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{\pi a^6}{48}.$

18. $\int_0^{\pi}\int_0^{\pi/2}\int_0^1 e^{-\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}(1-e^{-1})\int_0^{\pi}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta = \frac{\pi}{3}(1-e^{-1}).$

19. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta = \frac{32\pi}{15}(2\sqrt{2}-1).$

20. $\int_0^{2\pi}\int_0^{\pi}\int_0^3 \rho^3\sin\phi\,d\rho\,d\phi\,d\theta = 81\pi.$

21. False. The factor r^2 should be just r.

22. True. If G is the spherical wedge then the volume of G is $\iiint\limits_G 1\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, by equation (9).

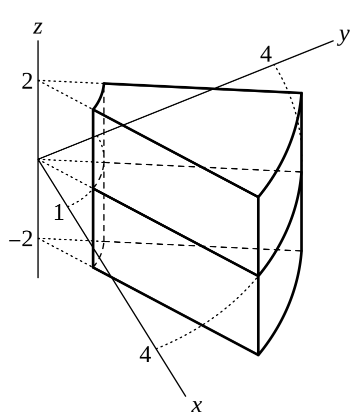
23. True. The region is described by $0 \le \phi \le \pi/4$, $0 \le \theta \le 2\pi$, $1 \le \rho \le 3$, so the volume is $\iiint\limits_G 1\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_1^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$

24. False. The "$\sin\theta$" and "$\cos\theta$" in the iterated integral are reversed.

25. **(a)** $\int_{-2}^{2}\int_1^4\int_{\pi/6}^{\pi/3}\frac{r\tan^3\theta}{\sqrt{1+z^2}}\,d\theta\,dr\,dz = \left(\int_{-2}^{2}\frac{1}{\sqrt{1+z^2}}\,dz\right)\left(\int_1^4 r\,dr\right)\left(\int_{\pi/6}^{\pi/3}\tan^3\theta\,d\theta\right) =$

$$= 2\ln(2+\sqrt{5})\cdot\frac{15}{2}\cdot\left(\frac{4}{3}-\frac{1}{2}\ln 3\right) = \frac{5}{2}(8-3\ln 3)\ln(2+\sqrt{5}) \approx 16.97774195.$$

(b) G is the cylindrical wedge $\pi/6 \le \theta \le \pi/3$, $1 \le r \le 4$, $-2 \le z \le 2$. Since $dx\,dy\,dz = dV = r\,d\theta\,dr\,dz$, the integrand in rectangular coordinates is $\frac{1}{r}\cdot\frac{r\tan^3\theta}{\sqrt{1+z^2}} = \frac{(y/x)^3}{\sqrt{1+z^2}}$, so $f(x,y,z) = \frac{y^3}{x^3\sqrt{1+z^2}}$.



26. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/4}\frac{1}{18}\cos^{37}\theta\cos\phi\,d\phi\,d\theta=\frac{\sqrt{2}}{36}\int_0^{\pi/2}\cos^{37}\theta\,d\theta=\frac{4{,}294{,}967{,}296}{755{,}505{,}013{,}725}\sqrt{2}\approx 0.008040.$

27. **(a)** $\displaystyle V=2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}}r\,dz\,dr\,d\theta=\frac{4\pi a^3}{3}.$ **(b)** $\displaystyle V=\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{4\pi a^3}{3}.$

28. **(a)** $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}xyz\,dz\,dy\,dx=\int_0^2\int_0^{\sqrt{4-x^2}}\frac{1}{2}xy(4-x^2-y^2)\,dy\,dx=\frac{1}{8}\int_0^2 x(4-x^2)^2\,dx=\frac{4}{3}.$

(b) $\displaystyle\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}}r^3z\sin\theta\cos\theta\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^2\frac{1}{2}(4r^3-r^5)\sin\theta\cos\theta\,dr\,d\theta=\frac{8}{3}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta=\frac{4}{3}.$

(c) $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2\rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\phi\,d\theta=\int_0^{\pi/2}\int_0^{\pi/2}\frac{32}{3}\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\phi\,d\theta=$
$\displaystyle=\frac{8}{3}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta=\frac{4}{3}.$

29. $\displaystyle V=\int_0^{\pi/2}\int_{\pi/6}^{\pi/3}\int_0^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{\pi/2}\int_{\pi/6}^{\pi/3}\frac{8}{3}\sin\phi\,d\phi\,d\theta=\frac{4}{3}(\sqrt{3}-1)\int_0^{\pi/2}d\theta=\frac{2\pi}{3}(\sqrt{3}-1).$

30. **(a)** The sphere and cone intersect in a circle of radius $\rho_0\sin\phi_0$, $\displaystyle V=\int_{\theta_1}^{\theta_2}\int_0^{\rho_0\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{\rho_0^2-r^2}}r\,dz\,dr\,d\theta=$
$\displaystyle\int_{\theta_1}^{\theta_2}\int_0^{\rho_0\sin\phi_0}\left(r\sqrt{\rho_0^2-r^2}-r^2\cot\phi_0\right)dr\,d\theta=\int_{\theta_1}^{\theta_2}\frac{1}{3}\rho_0^3(1-\cos^3\phi_0-\sin^3\phi_0\cot\phi_0)\,d\theta=$
$\displaystyle=\frac{1}{3}\rho_0^3(1-\cos^3\phi_0-\sin^2\phi_0\cos\phi_0)(\theta_2-\theta_1)=\frac{1}{3}\rho_0^3(1-\cos\phi_0)(\theta_2-\theta_1).$

(b) From part (a), the volume of the solid bounded by $\theta=\theta_1$, $\theta=\theta_2$, $\phi=\phi_1$, $\phi=\phi_2$, and $\rho=\rho_0$ is $\frac{1}{3}\rho_0^3(1-\cos\phi_2)(\theta_2-\theta_1)-\frac{1}{3}\rho_0^3(1-\cos\phi_1)(\theta_2-\theta_1)=\frac{1}{3}\rho_0^3(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1)$, so the volume of the spherical wedge between $\rho=\rho_1$ and $\rho=\rho_2$ is $\Delta V=\frac{1}{3}\rho_2^3(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1)-\frac{1}{3}\rho_1^3(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1)=\frac{1}{3}(\rho_2^3-\rho_1^3)(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1)$.

(c) $\frac{d}{d\phi}\cos\phi=-\sin\phi$ so from the Mean-Value Theorem $\cos\phi_2-\cos\phi_1=-(\phi_2-\phi_1)\sin\phi^*$ where ϕ^* is between ϕ_1 and ϕ_2. Similarly $\frac{d}{d\rho}\rho^3=3\rho^2$ so $\rho_2^3-\rho_1^3=3\rho^{*2}(\rho_2-\rho_1)$ where ρ^* is between ρ_1 and ρ_2. Thus $\cos\phi_1-\cos\phi_2=\sin\phi^*\Delta\phi$ and $\rho_2^3-\rho_1^3=3\rho^{*2}\Delta\rho$ so $\Delta V=\rho^{*2}\sin\phi^*\Delta\rho\Delta\phi\Delta\theta$.

31. The fact that none of the limits involves θ means that the solid is obtained by rotating a region in the xz-plane about the z-axis, between two angles θ_1 and θ_2. If the integral is expressed in cylindrical coordinates, then the plane region must be either a type I region or a type II region (with the role of y replaced by z); see Definition 14.2.1. If the integral is expressed in spherical coordinates, then the plane region may be a simple polar region (with the roles of θ and r replaced by ϕ and ρ); see Definition 14.3.1. Or it may be described by inequalities of the form $\rho_1\le\rho\le\rho_2$, $\phi_1(\rho)\le\phi\le\phi_2(\rho)$ for some numbers $\rho_1\le\rho_2$ and functions $\phi_1(\rho)\le\phi_2(\rho)$.

Exercise Set 14.7

1. $\displaystyle\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & 4\\ 3 & -5\end{vmatrix}=-17.$

2. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 4v \\ 4u & -1 \end{vmatrix} = -1 - 16uv.$

3. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos u & -\sin v \\ \sin u & \cos v \end{vmatrix} = \cos u \cos v + \sin u \sin v = \cos(u-v).$

4. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{2(v^2-u^2)}{(u^2+v^2)^2} & -\dfrac{4uv}{(u^2+v^2)^2} \\ \dfrac{4uv}{(u^2+v^2)^2} & \dfrac{2(v^2-u^2)}{(u^2+v^2)^2} \end{vmatrix} = 4/(u^2+v^2)^2.$

5. $x = \dfrac{2}{9}u + \dfrac{5}{9}v,\ y = -\dfrac{1}{9}u + \dfrac{2}{9}v;\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2/9 & 5/9 \\ -1/9 & 2/9 \end{vmatrix} = \dfrac{1}{9}.$

6. $x = \ln u,\ y = uv;\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/u & 0 \\ v & u \end{vmatrix} = 1.$

7. $x = \dfrac{\sqrt{u+v}}{\sqrt{2}},\ y = \dfrac{\sqrt{v-u}}{\sqrt{2}};\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{1}{2\sqrt{2}\sqrt{u+v}} & \dfrac{1}{2\sqrt{2}\sqrt{u+v}} \\ -\dfrac{1}{2\sqrt{2}\sqrt{v-u}} & \dfrac{1}{2\sqrt{2}\sqrt{v-u}} \end{vmatrix} = \dfrac{1}{4\sqrt{v^2-u^2}}.$

8. $x = \dfrac{u^{3/2}}{v^{1/2}},\ y = \dfrac{v^{1/2}}{u^{1/2}};\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{3u^{1/2}}{2v^{1/2}} & -\dfrac{u^{3/2}}{2v^{3/2}} \\ -\dfrac{v^{1/2}}{2u^{3/2}} & \dfrac{1}{2u^{1/2}v^{1/2}} \end{vmatrix} = \dfrac{1}{2v}.$

9. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 5.$

10. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix} = u^2v.$

11. $y = v,\ x = \dfrac{u}{y} = \dfrac{u}{v},\ z = w - x = w - \dfrac{u}{v};\ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1 & 0 \\ -1/v & u/v^2 & 1 \end{vmatrix} = \dfrac{1}{v}.$

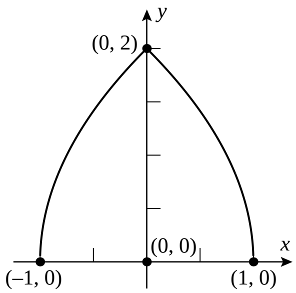
12. $x = \dfrac{v+w}{2},\ y = \dfrac{u-w}{2},\ z = \dfrac{u-v}{2};\ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & -1/2 & 0 \end{vmatrix} = -\dfrac{1}{4}.$

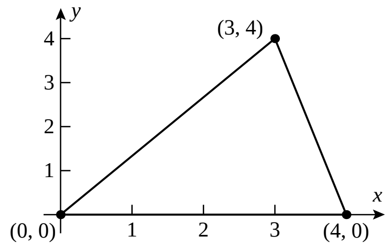
13. False. It is the <u>area</u> of the parallelogram.

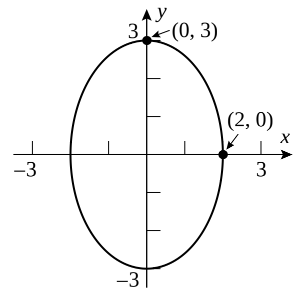
14. False. If the mapping is not one-to-one, then the integral may be larger than the area. For example, let $x = u$, $y = (v-3)^2$. Then R is the rectangle $0 \le x \le 2$, $0 \le y \le 4$, with area 8, but $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 2(v-3) \end{vmatrix} = 2(v-3)$, so $\displaystyle\int_1^5 \int_0^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_1^5 \int_0^2 2|v-3|\,du\,dv = \int_1^5 4|v-3|\,dv = \int_1^3 4(3-v)\,dv + \int_3^5 4(v-3)\,dv = (12v - 2v^2)\Big]_1^3 + (2v^2-12v)\Big]_3^5 = 8 + 8 = 16.$

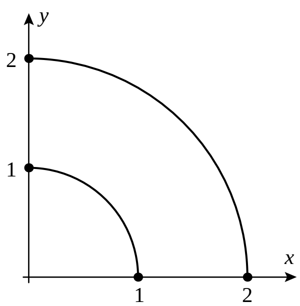
15. False. The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$

16. True. See the solution of Exercise 14.7.48(b).

17. 

18. 

19. 

20. 

21. $x = \dfrac{1}{5}u + \dfrac{2}{5}v,\ y = -\dfrac{2}{5}u + \dfrac{1}{5}v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{5};\ \dfrac{1}{5}\displaystyle\iint\limits_S \dfrac{u}{v}\, dA_{uv} = \dfrac{1}{5}\int_1^3 \int_1^4 \dfrac{u}{v}\, du\, dv = \dfrac{3}{2}\ln 3.$

22. $x = \dfrac{1}{2}u + \dfrac{1}{2}v,\ y = \dfrac{1}{2}u - \dfrac{1}{2}v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{2};\ \dfrac{1}{2}\displaystyle\iint\limits_S ve^{uv}\, dA_{uv} = \dfrac{1}{2}\int_1^4 \int_0^1 ve^{uv}\, du\, dv = \dfrac{1}{2}(e^4 - e - 3).$

23. $x = u + v,\ y = u - v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = -2$; the boundary curves of the region S in the uv-plane are $v = 0, v = u$, and $u = 1$ so $2\displaystyle\iint\limits_S \sin u \cos v\, dA_{uv} = 2\int_0^1 \int_0^u \sin u \cos v\, dv\, du = 1 - \dfrac{1}{2}\sin 2.$

24. $x = \sqrt{v/u},\ y = \sqrt{uv}$ so, from Example 3, $\dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{2u}$; the boundary curves of the region S in the uv-plane are $u = 1, u = 3, v = 1$, and $v = 4$ so $\displaystyle\iint\limits_S uv^2\left(\dfrac{1}{2u}\right) dA_{uv} = \dfrac{1}{2}\int_1^4 \int_1^3 v^2 du\, dv = 21.$

25. $x = 3u,\ y = 4v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = 12$; S is the region in the uv-plane enclosed by the circle $u^2 + v^2 = 1$. Use polar coordinates to obtain $\displaystyle\iint\limits_S 12\sqrt{u^2+v^2}(12)\, dA_{uv} = 144\int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta = 96\pi.$

26. $x = 2u, y = v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = 2$; S is the region in the uv-plane enclosed by the circle $u^2+v^2 = 1$. Use polar coordinates to obtain $\displaystyle\iint\limits_S e^{-(4u^2+4v^2)}(2)\, dA_{uv} = 2\int_0^{2\pi}\int_0^1 re^{-4r^2} dr\, d\theta = \dfrac{\pi}{2}(1 - e^{-4}).$

27. Let S be the region in the uv-plane bounded by $u^2 + v^2 = 1$, so $u = 2x, v = 3y, x = u/2, y = v/3,\ \dfrac{\partial(x,y)}{\partial(u,v)} =$

$\begin{vmatrix} 1/2 & 0 \\ 0 & 1/3 \end{vmatrix} = 1/6$, use polar coordinates to get $\dfrac{1}{6}\iint\limits_S \sin(u^2+v^2)\,dA_{uv} = \dfrac{1}{6}\int_0^{\pi/2}\int_0^1 r\sin r^2\,dr\,d\theta =$

$= \dfrac{\pi}{24}(-\cos r^2)\Big]_0^1 = \dfrac{\pi}{24}(1-\cos 1)$.

28. $u = x/a, v = y/b, x = au, y = bv;\ \dfrac{\partial(x,y)}{\partial(u,v)} = ab;\ A = ab\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi ab.$

29. $x = u/3, y = v/2, z = w,\ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 1/6$; S is the region in uvw-space enclosed by the sphere $u^2+v^2+w^2 = 36$, so $\iiint\limits_S \dfrac{u^2}{9}\dfrac{1}{6}dV_{uvw} = \dfrac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^6 (\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^6 \rho^4\sin^3\phi\cos^2\theta\,d\rho\,d\phi\,d\theta = \dfrac{192\pi}{5}$.

30. Let G_1 be the region $u^2+v^2+w^2 \le 1$, with $x = au, y = bv, z = cw$, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = abc$; then use spherical coordinates in uvw-space: $\iiint\limits_G (y^2+z^2)\,dV_{xyz} = abc\iiint\limits_{G_1}(b^2v^2+c^2w^2)\,dV_{uvw} = \int_0^{2\pi}\int_0^{\pi}\int_0^1 abc(b^2\sin^2\phi\sin^2\theta +$ $c^2\cos^2\phi)\rho^4\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\dfrac{abc}{15}(4b^2\sin^2\theta + 2c^2)\,d\theta = \dfrac{4}{15}\pi abc(b^2+c^2)$.

31. $u = \theta = \begin{cases} \cot^{-1}(x/y) & \text{if } y \ne 0 \\ 0 & \text{if } y = 0,\ x > 0 \\ \pi & \text{if } y = 0,\ x < 0 \end{cases}$, $v = r = \sqrt{x^2+y^2}$. Other answers are possible.

32. $u = r = \sqrt{x^2+y^2},\quad v = \dfrac{1}{2}+\dfrac{\theta}{\pi} = \begin{cases} \dfrac{1}{2}+\dfrac{1}{\pi}\tan^{-1}(y/x) & \text{if } x \ne 0 \\ 1 & \text{if } x = 0,\ y > 0 \\ 0 & \text{if } x = 0,\ y < 0 \end{cases}$. Other answers are possible.

33. $u = \dfrac{3}{7}x - \dfrac{2}{7}y,\ v = -\dfrac{1}{7}x + \dfrac{3}{7}y$. Other answers are possible.

34. $u = -x + \dfrac{4}{3}y,\ v = y$. Other answers are possible.

35. Let $u = y-4x, v = y+4x$, then $x = \dfrac{1}{8}(v-u), y = \dfrac{1}{2}(v+u)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{8}$; $\dfrac{1}{8}\iint\limits_S \dfrac{u}{v}\,dA_{uv} = \dfrac{1}{8}\int_2^5\int_0^2 \dfrac{u}{v}\,du\,dv = \dfrac{1}{4}\ln\dfrac{5}{2}$.

36. Let $u = y+x, v = y-x$, then $x = \dfrac{1}{2}(u-v), y = \dfrac{1}{2}(u+v)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$; $-\dfrac{1}{2}\iint\limits_S uv\,dA_{uv} = -\dfrac{1}{2}\int_0^2\int_0^1 uv\,du\,dv = -\dfrac{1}{2}$.

37. Let $u = x-y, v = x+y$, then $x = \dfrac{1}{2}(v+u), y = \dfrac{1}{2}(v-u)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$; the boundary curves of the region S in the uv-plane are $u = 0, v = u$, and $v = \pi/4$; thus $\dfrac{1}{2}\iint\limits_S \dfrac{\sin u}{\cos v}\,dA_{uv} = \dfrac{1}{2}\int_0^{\pi/4}\int_0^v \dfrac{\sin u}{\cos v}\,du\,dv = \dfrac{1}{2}\left[\ln(\sqrt{2}+1) - \dfrac{\pi}{4}\right]$.

38. Let $u = y - x, v = y + x$, then $x = \frac{1}{2}(v - u), y = \frac{1}{2}(u + v)$ so $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$; the boundary curves of the region S in the uv-plane are $v = -u, v = u, v = 1$, and $v = 4$; thus $\frac{1}{2}\iint_S e^{u/v} dA_{uv} = \frac{1}{2}\int_1^4 \int_{-v}^{v} e^{u/v} du\, dv = \frac{15}{4}(e - e^{-1})$.

39. Let $u = \frac{y}{x}$, $v = \frac{x}{y^2}$, then $x = \frac{1}{u^2 v}$, $y = \frac{1}{uv}$ so $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{u^4 v^3}$; $\iint_S \frac{1}{u^4 v^3}\, dA_{uv} = \int_1^4 \int_1^2 \frac{1}{u^4 v^3}\, du\, dv = \frac{35}{256}$.

40. Let $x = 3u, y = 2v$, $\frac{\partial(x,y)}{\partial(u,v)} = 6$; S is the region in the uv-plane enclosed by the circle $u^2 + v^2 = 1$, so $\iint_R (9 - x - y)\, dA = \iint_S 6(9 - 3u - 2v)\, dA_{uv} = 6\int_0^{2\pi}\int_0^1 (9 - 3r\cos\theta - 2r\sin\theta) r\, dr\, d\theta = 54\pi$.

41. $x = u$, $y = \frac{w}{u}$, $z = v + \frac{w}{u}$, $\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{1}{u}$; $\iiint_S \frac{v^2 w}{u}\, dV_{uvw} = \int_2^4 \int_0^1 \int_1^3 \frac{v^2 w}{u}\, du\, dv\, dw = 2\ln 3$.

42. $u = xy$, $v = yz$, $w = xz$, $1 \le u \le 2$, $1 \le v \le 3$, $1 \le w \le 4$, $x = \sqrt{uw/v}$, $y = \sqrt{uv/w}$, $z = \sqrt{vw/u}$, $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{2\sqrt{uvw}} V = \iiint_G dV = \int_1^2 \int_1^3 \int_1^4 \frac{1}{2\sqrt{uvw}}\, dw\, dv\, du = 4(\sqrt{2} - 1)(\sqrt{3} - 1)$.

43. (b) $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \cdot \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} x_u u_x + x_v v_x & x_u u_y + x_v v_y \\ y_u u_x + y_v v_x & y_u u_y + y_v v_y \end{vmatrix} = \begin{vmatrix} x_x & x_y \\ y_x & y_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$

44. $\frac{\partial(u,v)}{\partial(x,y)} = 3xy^4 = 3v$ so $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3v}$; $\frac{1}{3}\iint_S \frac{\sin u}{v}\, dA_{uv} = \frac{1}{3}\int_1^2 \int_\pi^{2\pi} \frac{\sin u}{v}\, du\, dv = -\frac{2}{3}\ln 2$.

45. $\frac{\partial(u,v)}{\partial(x,y)} = 8xy$ so $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{8xy}$; $xy\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = xy \cdot \frac{1}{8xy} = \frac{1}{8}$ so $\frac{1}{8}\iint_S dA_{uv} = \frac{1}{8}\int_9^{16}\int_1^4 du\, dv = \frac{21}{8}$.

46. $\frac{\partial(u,v)}{\partial(x,y)} = -2(x^2+y^2)$, so $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2(x^2+y^2)}$; $(x^4 - y^4)e^{xy}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{x^4 - y^4}{2(x^2+y^2)} e^{xy} = \frac{1}{2}(x^2 - y^2)e^{xy} = \frac{1}{2}ve^u$, so $\frac{1}{2}\iint_S ve^u\, dA_{uv} = \frac{1}{2}\int_3^4 \int_1^3 ve^u\, du\, dv = \frac{7}{4}(e^3 - e)$.

47. Set $u = x + y + 2z, v = x - 2y + z, w = 4x + y + z$, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 18$, and $V = \iiint_R dx\, dy\, dz = \int_{-6}^{6}\int_{-2}^{2}\int_{-3}^{3} \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du\, dv\, dw = 6 \cdot 4 \cdot 12 \cdot \frac{1}{18} = 16$.

48. (a) $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$

(b) $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi.$

49. The main motivation is to change the region of integration to one that has a simple description in either rectangular, polar, cylindrical, or spherical coordinates.

50. First consider the case in which R is defined by $a \le u(x,y) \le b$, $c \le v(x,y) \le d$, for some functions u and v. If we can solve for x and y in terms of u and v, then we can write $\displaystyle\iint_R f(x,y)\,dA_{xy} = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$, where S is the rectangle $a \le u \le b$, $c \le v \le d$. For the more general case in which the boundary curves of R are level curves of more than 2 functions, we can pick 2 of these functions, say $u(x,y)$ and $v(x,y)$, try to solve for x and y in terms of u and v, and rewrite all of the inequalities in terms of u and v. This gives a region S in the uv-plane, with one boundary curve which is a horizontal line segment and one which is a vertical line segment. If we are very lucky, the other boundary curves may also be fairly simple and we may be able to compute the resulting integral over S. See Examples 2 and 3.

Exercise Set 14.8

1. $\displaystyle M = \int_0^1\int_0^{\sqrt{x}}(x+y)\,dy\,dx = \frac{13}{20}$, $\displaystyle M_x = \int_0^1\int_0^{\sqrt{x}}(x+y)y\,dy\,dx = \frac{3}{10}$, $\displaystyle M_y = \int_0^1\int_0^{\sqrt{x}}(x+y)x\,dy\,dx = \frac{19}{42}$, $\displaystyle \overline{x} = \frac{M_y}{M} = \frac{190}{273}$, $\displaystyle \overline{y} = \frac{M_x}{M} = \frac{6}{13}$; the mass is $\dfrac{13}{20}$ and the center of gravity is at $\left(\dfrac{190}{273}, \dfrac{6}{13}\right)$.

2. $\displaystyle M = \int_0^\pi\int_0^{\sin x} y\,dy\,dx = \frac{\pi}{4}$, $\overline{x} = \dfrac{\pi}{2}$ from the symmetry of the density and the region, $\displaystyle M_x = \int_0^\pi\int_0^{\sin x} y^2\,dy\,dx = \frac{4}{9}$, $\displaystyle \overline{y} = \frac{M_x}{M} = \frac{16}{9\pi}$; mass $\dfrac{\pi}{4}$, center of gravity $\left(\dfrac{\pi}{2}, \dfrac{16}{9\pi}\right)$.

3. $\displaystyle M = \int_0^{\pi/2}\int_0^a r^3\sin\theta\cos\theta\,dr\,d\theta = \frac{a^4}{8}$, $\overline{x} = \overline{y}$ from the symmetry of the density and the region,

$\displaystyle M_y = \int_0^{\pi/2}\int_0^a r^4\sin\theta\cos^2\theta\,dr\,d\theta = \frac{a^5}{15}$, $\overline{x} = \dfrac{8a}{15}$; mass $\dfrac{a^4}{8}$, center of gravity $\left(\dfrac{8a}{15}, \dfrac{8a}{15}\right)$.

4. $\displaystyle M = \int_0^\pi\int_0^1 r^3\,dr\,d\theta = \frac{\pi}{4}$, $\overline{x} = 0$ from the symmetry of density and region, $\displaystyle M_x = \int_0^\pi\int_0^1 r^4\sin\theta\,dr\,d\theta = \frac{2}{5}$, $\overline{y} = \dfrac{8}{5\pi}$; mass $\dfrac{\pi}{4}$, center of gravity $\left(0, \dfrac{8}{5\pi}\right)$.

5. $\displaystyle M = \iint_R \delta(x,y)\,dA = \int_0^1\int_0^1 |x+y-1|\,dx\,dy = \int_0^1\left[\int_0^{1-x}(1-x-y)\,dy + \int_{1-x}^1(x+y-1)\,dy\right]dx = \frac{1}{3}$. $\displaystyle \overline{x} = 3\int_0^1\int_0^1 x\delta(x,y)\,dy\,dx = 3\int_0^1\left[\int_0^{1-x}x(1-x-y)\,dy + \int_{1-x}^1 x(x+y-1)\,dy\right]dx = \frac{1}{2}$. By symmetry, $\overline{y} = \dfrac{1}{2}$ as well; center of gravity $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

6. $\displaystyle \overline{x} = \frac{1}{M}\iint_G x\delta(x,y)\,dA$, and the integrand is an odd function of x while the region is symmetric with respect to the y-axis, thus $\overline{x} = 0$; likewise $\overline{y} = 0$.

7. $\displaystyle V = 1, \overline{x} = \int_0^1\int_0^1\int_0^1 x\,dz\,dy\,dx = \frac{1}{2}$, similarly $\overline{y} = \overline{z} = \dfrac{1}{2}$; centroid $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$.

8. $V = \pi r^2 h = 2\pi$, $\overline{x} = \overline{y} = 0$ by symmetry, $\displaystyle \iiint_G z\,dz\,dy\,dx = \int_0^2\int_0^{2\pi}\int_0^1 rz\,dr\,d\theta\,dz = 2\pi$, centroid $= (0,0,1)$.

9. True. This is the definition of "centroid"; see Section 6.7.

10. False. For example, suppose the lamina is the annulus $1 \le r \le 2$ with constant density 1. The centroid is the origin, which is not part of the annulus, so the density is 0 there. But the mass is not 0.

11. False. The coordinates are the first moments about the y- and x-axes, divided by the mass.

12. False. Density in 3-space has units of mass per unit <u>volume</u>.

13. Let $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ in formulas (11) and (12).

14. $\overline{x} = 0$ from the symmetry of the region, $A = \displaystyle\int_0^{2\pi}\int_0^{a(1+\sin\theta)} r\,dr\,d\theta = \frac{3\pi a^2}{2}$, $\overline{y} = \displaystyle\frac{1}{A}\int_0^{2\pi}\int_0^{a(1+\sin\theta)} r^2\sin\theta\,dr\,d\theta = \frac{2}{3\pi a^2}\cdot\frac{5\pi a^3}{4} = \frac{5a}{6}$; centroid $\left(0, \dfrac{5a}{6}\right)$.

15. $\overline{x} = \overline{y}$ from the symmetry of the region, $A = \displaystyle\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \frac{\pi}{8}$, $\overline{x} = \displaystyle\frac{1}{A}\int_0^{\pi/2}\int_0^{\sin 2\theta} r^2\cos\theta\,dr\,d\theta = \frac{8}{\pi}\cdot\frac{16}{105} = \frac{128}{105\pi}$; centroid $\left(\dfrac{128}{105\pi}, \dfrac{128}{105\pi}\right)$.

16. $\overline{x} = 0$ from the symmetry of the region, $A = \dfrac{1}{2}\pi(b^2 - a^2)$, $\overline{y} = \displaystyle\frac{1}{A}\int_0^{\pi}\int_a^{b} r^2\sin\theta\,dr\,d\theta = \frac{1}{A}\frac{2}{3}(b^3 - a^3) = \frac{4(b^3 - a^3)}{3\pi(b^2 - a^2)}$; centroid $\left(0, \dfrac{4(b^3 - a^3)}{3\pi(b^2 - a^2)}\right)$.

17. $\overline{y} = 0$ from the symmetry of the region, $A = \dfrac{1}{2}\pi a^2$, $\overline{x} = \displaystyle\frac{1}{A}\int_{-\pi/2}^{\pi/2}\int_0^{a} r^2\cos\theta\,dr\,d\theta = \frac{1}{A}\frac{2}{3}a^3 = \frac{4a}{3\pi}$; centroid $\left(\dfrac{4a}{3\pi}, 0\right)$.

18. $\overline{x} = 3/2$ and $\overline{y} = 1$ from the symmetry of the region, $\displaystyle\iint_R x\,dA = \overline{x}A = \frac{3}{2}\cdot 6 = 9$, $\displaystyle\iint_R y\,dA = \overline{y}A = 1\cdot 6 = 6$.

19. $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of the region, $V = 1/6$, $\overline{x} = \displaystyle\frac{1}{V}\int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx = 6\cdot\frac{1}{24} = \frac{1}{4}$; centroid $\left(\dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4}\right)$.

20. The solid is described by $-1 \le y \le 1, 0 \le z \le 1 - y^2, 0 \le x \le 1 - z$; $V = \displaystyle\int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} dx\,dz\,dy = \frac{4}{5}$, $\overline{x} = \displaystyle\frac{1}{V}\int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} x\,dx\,dz\,dy = \frac{5}{14}$, $\overline{y} = 0$ by symmetry, $\overline{z} = \displaystyle\frac{1}{V}\int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} z\,dx\,dz\,dy = \frac{2}{7}$; the centroid is $\left(\dfrac{5}{14}, 0, \dfrac{2}{7}\right)$.

21. $\overline{x} = 1/2$ and $\overline{y} = 0$ from the symmetry of the region, $V = \displaystyle\int_0^1\int_{-1}^{1}\int_{y^2}^{1} dz\,dy\,dx = \frac{4}{3}$, $\overline{z} = \displaystyle\frac{1}{V}\iiint_G z\,dV = \frac{3}{4}\cdot\frac{4}{5} = \frac{3}{5}$; centroid $\left(\dfrac{1}{2}, 0, \dfrac{3}{5}\right)$.

22. $\overline{x} = \overline{y}$ from the symmetry of the region, $V = \int_0^2 \int_0^2 \int_0^{xy} dz\,dy\,dx = 4$, $\overline{x} = \dfrac{1}{V}\iiint\limits_G x\,dV = \dfrac{1}{4}\cdot\dfrac{16}{3} = \dfrac{4}{3}$, $\overline{z} = \dfrac{1}{V}\iiint\limits_G z\,dV = \dfrac{1}{4}\cdot\dfrac{32}{9} = \dfrac{8}{9}$; centroid $\left(\dfrac{4}{3}, \dfrac{4}{3}, \dfrac{8}{9}\right)$.

23. $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of the region, $V = \pi a^3/6$, $\overline{x} = \dfrac{1}{V}\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x\,dz\,dy\,dx = \dfrac{1}{V}\int_0^a \int_0^{\sqrt{a^2-x^2}} x\sqrt{a^2-x^2-y^2}\,dy\,dx = \dfrac{1}{V}\int_0^{\pi/2}\int_0^a r^2\sqrt{a^2-r^2}\cos\theta\,dr\,d\theta = \dfrac{6}{\pi a^3}\cdot\dfrac{\pi a^4}{16} = \dfrac{3a}{8}$; this gives us the centroid $\left(\dfrac{3a}{8}, \dfrac{3a}{8}, \dfrac{3a}{8}\right)$.

24. $\overline{x} = \overline{y} = 0$ from the symmetry of the region, $V = 2\pi a^3/3$, $\overline{z} = \dfrac{1}{V}\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z\,dz\,dy\,dx = \dfrac{1}{V}\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \dfrac{1}{2}(a^2-x^2-y^2)\,dy\,dx = \dfrac{1}{V}\int_0^{2\pi}\int_0^a \dfrac{1}{2}(a^2-r^2)r\,dr\,d\theta = \dfrac{3}{2\pi a^3}\cdot\dfrac{\pi a^4}{4} = \dfrac{3a}{8}$; centroid $\left(0, 0, \dfrac{3a}{8}\right)$.

25. $M = \int_0^a \int_0^a \int_0^a (a-x)\,dz\,dy\,dx = \dfrac{a^4}{2}$, $\overline{y} = \overline{z} = \dfrac{a}{2}$ from the symmetry of density and region, $\overline{x} = \dfrac{1}{M}\int_0^a \int_0^a \int_0^a x(a-x)\,dz\,dy\,dx = \dfrac{2}{a^4}\cdot\dfrac{a^5}{6} = \dfrac{a}{3}$; mass $\dfrac{a^4}{2}$, center of gravity $\left(\dfrac{a}{3}, \dfrac{a}{2}, \dfrac{a}{2}\right)$.

26. $M = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h (h-z)\,dz\,dy\,dx = \dfrac{\pi}{2}a^2h^2$, $\overline{x} = \overline{y} = 0$ from the symmetry of density and region, $\overline{z} = \dfrac{1}{M}\iiint\limits_G z(h-z)\,dV = \dfrac{2}{\pi a^2h^2}\cdot\dfrac{\pi a^2h^3}{6} = \dfrac{h}{3}$; mass $\dfrac{\pi a^2h^2}{2}$, center of gravity $\left(0, 0, \dfrac{h}{3}\right)$.

27. $M = \int_{-1}^1 \int_0^1 \int_0^{1-y^2} yz\,dz\,dy\,dx = \dfrac{1}{6}$, $\overline{x} = 0$ by the symmetry of density and region, $\overline{y} = \dfrac{1}{M}\iiint\limits_G y^2z\,dV = 6\cdot\dfrac{8}{105} = \dfrac{16}{35}$, $\overline{z} = \dfrac{1}{M}\iiint\limits_G yz^2\,dV = 6\cdot\dfrac{1}{12} = \dfrac{1}{2}$; mass $\dfrac{1}{6}$, center of gravity $\left(0, \dfrac{16}{35}, \dfrac{1}{2}\right)$.

28. $M = \int_0^3 \int_0^{9-x^2} \int_0^1 xz\,dz\,dy\,dx = \dfrac{81}{8}$, $\overline{x} = \dfrac{1}{M}\iiint\limits_G x^2z\,dV = \dfrac{8}{81}\cdot\dfrac{81}{5} = \dfrac{8}{5}$, $\overline{y} = \dfrac{1}{M}\iiint\limits_G xyz\,dV = \dfrac{8}{81}\cdot\dfrac{243}{8} = 3$, $\overline{z} = \dfrac{1}{M}\iiint\limits_G xz^2dV = \dfrac{8}{81}\cdot\dfrac{27}{4} = \dfrac{2}{3}$; mass $\dfrac{81}{8}$, center of gravity $\left(\dfrac{8}{5}, 3, \dfrac{2}{3}\right)$.

29. **(a)** $M = \int_0^1 \int_0^1 k(x^2+y^2)\,dy\,dx = \dfrac{2k}{3}$, $\overline{x} = \overline{y}$ from the symmetry of density and region, $\overline{x} = \dfrac{1}{M}\iint\limits_R kx(x^2+y^2)dA = \dfrac{3}{2k}\cdot\dfrac{5k}{12} = \dfrac{5}{8}$; center of gravity $\left(\dfrac{5}{8}, \dfrac{5}{8}\right)$.

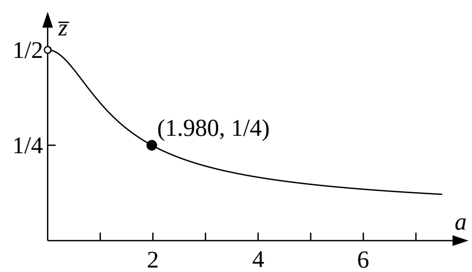
(b) $\overline{y} = 1/2$ from the symmetry of density and region, $M = \int_0^1 \int_0^1 kx\,dy\,dx = \dfrac{k}{2}$, $\overline{x} = \dfrac{1}{M}\iint\limits_R kx^2\,dA = \dfrac{2}{k}\cdot\dfrac{k}{3} = \dfrac{2}{3}$, center of gravity $\left(\dfrac{2}{3}, \dfrac{1}{2}\right)$.

30. **(a)** $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of density and region, $M = \int_0^1\int_0^1\int_0^1 k(x^2+y^2+z^2)\,dz\,dy\,dx = k$, $\overline{x} = \frac{1}{M}\iiint\limits_G kx(x^2+y^2+z^2)\,dV = \frac{1}{k}\cdot\frac{7k}{12} = \frac{7}{12}$; center of gravity $\left(\frac{7}{12},\frac{7}{12},\frac{7}{12}\right)$.

(b) $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of density and region, $M = \int_0^1\int_0^1\int_0^1 k(x+y+z)\,dz\,dy\,dx = \frac{3k}{2}$, $\overline{x} = \frac{1}{M}\iiint\limits_G kx(x+y+z)\,dV = \frac{2}{3k}\cdot\frac{5k}{6} = \frac{5}{9}$; center of gravity $\left(\frac{5}{9},\frac{5}{9},\frac{5}{9}\right)$.

31. $V = \iiint\limits_G dV = \int_0^{\pi}\int_0^{\sin x}\int_0^{1/(1+x^2+y^2)} dz\,dy\,dx \approx 0.666633$, $\overline{x} = \frac{1}{V}\iiint\limits_G x\,dV \approx 1.177406$, $\overline{y} = \frac{1}{V}\iiint\limits_G y\,dV \approx 0.353554$, $\overline{z} = \frac{1}{V}\iiint\limits_G z\,dV \approx 0.231557$.

32. **(b)** Use polar coordinates for x and y to get $V = \iiint\limits_G dV = \int_0^{2\pi}\int_0^a\int_0^{1/(1+r^2)} r\,dz\,dr\,d\theta = \pi\ln(1+a^2)$, $\overline{z} = \frac{1}{V}\iiint\limits_G z\,dV = \frac{a^2}{2(1+a^2)\ln(1+a^2)}$. Thus $\lim\limits_{a\to 0^+}\overline{z} = \frac{1}{2}$; $\lim\limits_{a\to+\infty}\overline{z} = 0$. Also, $\lim\limits_{a\to 0^+}\overline{z} = \frac{1}{2}$; $\lim\limits_{a\to+\infty}\overline{z} = 0$.



(c) Solve $\overline{z} = 1/4$ for a to obtain $a \approx 1.980291$.

33. $M = \int_0^{2\pi}\int_0^3\int_r^3 (3-z)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \frac{1}{2}r(3-r)^2\,dr\,d\theta = \frac{27}{8}\int_0^{2\pi} d\theta = \frac{27\pi}{4}$.

34. $M = \int_0^{2\pi}\int_0^a\int_0^h kzr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{1}{2}kh^2r\,dr\,d\theta = \frac{1}{4}ka^2h^2\int_0^{2\pi} d\theta = \frac{\pi ka^2h^2}{2}$.

35. $M = \int_0^{2\pi}\int_0^{\pi}\int_0^a k\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{1}{4}ka^4\sin\phi\,d\phi\,d\theta = \frac{1}{2}ka^4\int_0^{2\pi} d\theta = \pi ka^4$.

36. $M = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{3}{2}\sin\phi\,d\phi\,d\theta = 3\int_0^{2\pi} d\theta = 6\pi$.

37. $\bar{x} = \bar{y} = 0$ from the symmetry of the region, $V = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r\sqrt{2-r^2} - r^3)\,dr\,d\theta = \frac{\pi}{6}(8\sqrt{2}-7)$, $\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} zr\,dz\,dr\,d\theta = \frac{6}{(8\sqrt{2}-7)\pi}\cdot\frac{7\pi}{12} = \frac{7}{16\sqrt{2}-14}$; centroid $\left(0,0,\frac{7}{16\sqrt{2}-14}\right)$.

38. $\bar{x} = \bar{y} = 0$ from the symmetry of the region, $V = 8\pi/3$, $\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^2\int_r^2 zr\,dz\,dr\,d\theta = \frac{3}{8\pi}\cdot 4\pi = \frac{3}{2}$; centroid $\left(0,0,\frac{3}{2}\right)$.

39. $\bar{y} = 0$ from the symmetry of the region, $V = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\,dz\,dr\,d\theta = 3\pi/2$,

$\bar{x} = \dfrac{2}{V}\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r^2\cos\theta\,dz\,dr\,d\theta\dfrac{4}{3\pi}(\pi) = 4/3$, $\bar{z} = \dfrac{1}{V}\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} zr\,dz\,dr\,d\theta = \dfrac{4}{3\pi}(5\pi/6) = 10/9$; centroid $(4/3, 0, 10/9)$.

40. $M = \int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} zr\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta}\dfrac{1}{2}r(4-r^2)^2\,dr\,d\theta = \dfrac{16}{3}\int_0^{\pi/2}(1-\sin^6\theta)\,d\theta = (16/3)(11\pi/32) = 11\pi/6$.

41. $\bar{x} = \bar{y} = \bar{z}$ from the symmetry of the region, $V = \pi a^3/6$, $\bar{z} = \dfrac{1}{V}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a}\rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{6}{\pi a^3}\cdot\dfrac{\pi a^4}{16} = \dfrac{3a}{8}$; centroid $\left(\dfrac{3a}{8}, \dfrac{3a}{8}, \dfrac{3a}{8}\right)$.

42. $\bar{x} = \bar{y} = 0$ from the symmetry of the region, $V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{64\pi}{3}$,

$\bar{z} = \dfrac{1}{V}\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4}\rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{3}{64\pi}\cdot 48\pi = \dfrac{9}{4}$; centroid $\left(0, 0, \dfrac{9}{4}\right)$.

43. $M = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{1}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\dfrac{1}{4}\sin\phi\,d\phi\,d\theta = \dfrac{1}{8}(2-\sqrt{2})\int_0^{2\pi}d\theta = \dfrac{\pi}{4}(2-\sqrt{2})$.

44. $\bar{x} = \bar{y} = 0$ from the symmetry of density and region, $M = \int_0^{2\pi}\int_0^{1}\int_0^{1-r^2}(r^2+z^2)r\,dz\,dr\,d\theta = \dfrac{\pi}{4}$, $\bar{z} = \dfrac{1}{M}\int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} z(r^2+z^2)r\,dz\,dr\,d\theta = \dfrac{4}{\pi}\cdot\dfrac{11\pi}{120} = \dfrac{11}{30}$; center of gravity $\left(0, 0, \dfrac{11}{30}\right)$.

45. $\bar{x} = \bar{y} = 0$ from the symmetry of density and region, $M = \int_0^{2\pi}\int_0^{1}\int_0^{r} zr\,dz\,dr\,d\theta = \dfrac{\pi}{4}$,

$\bar{z} = \dfrac{1}{M}\int_0^{2\pi}\int_0^{1}\int_0^{r} z^2r\,dz\,dr\,d\theta = \dfrac{4}{\pi}\cdot\dfrac{2\pi}{15} = \dfrac{8}{15}$; center of gravity $\left(0, 0, \dfrac{8}{15}\right)$.

46. $\bar{x} = \bar{y} = 0$ from the symmetry of density and region, $M = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} k\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{\pi k a^4}{2}$, $\bar{z} = \dfrac{1}{M}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} k\rho^4\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \dfrac{2}{\pi k a^4}\cdot\dfrac{\pi k a^5}{5} = \dfrac{2a}{5}$; center of gravity $\left(0, 0, \dfrac{2a}{5}\right)$.

47. $\bar{x} = \bar{z} = 0$ from the symmetry of the region, $V = 54\pi/3 - 16\pi/3 = 38\pi/3$, $\bar{y} = \dfrac{1}{V}\int_0^{\pi}\int_0^{\pi}\int_2^{3}\rho^3\sin^2\phi\sin\theta\,d\rho\,d\phi\,d\theta = \dfrac{1}{V}\int_0^{\pi}\int_0^{\pi}\dfrac{65}{4}\sin^2\phi\sin\theta\,d\phi\,d\theta = \dfrac{1}{V}\int_0^{\pi}\dfrac{65\pi}{8}\sin\theta\,d\theta = \dfrac{3}{38\pi}\cdot\dfrac{65\pi}{4} = \dfrac{195}{152}$; centroid $\left(0, \dfrac{195}{152}, 0\right)$.

48. $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{R}\delta_0 e^{-(\rho/R)^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\dfrac{1}{3}(1-e^{-1})R^3\delta_0\sin\phi\,d\phi\,d\theta = \dfrac{4\pi}{3}(1-e^{-1})\delta_0 R^3$.

49. $I_x = \int_0^{a}\int_0^{b} y^2\delta\,dy\,dx = \dfrac{\delta a b^3}{3}$, $I_y = \int_0^{a}\int_0^{b} x^2\delta\,dy\,dx = \dfrac{\delta a^3 b}{3}$, $I_z = I_x + I_y = \dfrac{\delta ab(a^2+b^2)}{3}$.

50. $I_x = \int_0^{2\pi}\int_0^{a} r^3\sin^2\theta\,\delta\,dr\,d\theta = \dfrac{\delta\pi a^4}{4}$; $I_y = \int_0^{2\pi}\int_0^{a} r^3\cos^2\theta\,\delta\,dr\,d\theta = \dfrac{\delta\pi a^4}{4} = I_x$; $I_z = I_x + I_y = \dfrac{\delta\pi a^4}{2}$.

51. $I_z = \int_0^{2\pi}\int_0^a\int_0^h r^2\delta\, r\, dz\, dr\, d\theta = \delta\int_0^{2\pi}\int_0^a\int_0^h r^3 dz\, dr\, d\theta = \frac{1}{2}\delta\pi a^4 h.$

52. $I_y = \int_0^{2\pi}\int_0^a\int_0^h (r^2\cos^2\theta + z^2)\delta r\, dz\, dr\, d\theta = \delta\int_0^{2\pi}\int_0^a (hr^3\cos^2\theta + \frac{1}{3}h^3 r)\, dr\, d\theta =$
$= \delta\int_0^{2\pi}\left(\frac{1}{4}a^4 h\cos^2\theta + \frac{1}{6}a^2h^3\right) d\theta = \delta\left(\frac{\pi}{4}a^4h + \frac{\pi}{3}a^2h^3\right).$

53. $I_z = \int_0^{2\pi}\int_{a_1}^{a_2}\int_0^h r^2\delta\, r\, dz\, dr\, d\theta = \delta\int_0^{2\pi}\int_{a_1}^{a_2}\int_0^h r^3\, dz\, dr\, d\theta = \frac{1}{2}\delta\pi h(a_2^4 - a_1^4).$

54. $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a (\rho^2\sin^2\phi)\delta\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \delta\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta = \frac{8}{15}\delta\pi a^5.$

55. (a) The solid generated by R_k as it revolves about L is a cylinder of height Δy_k and radius $x_k^* + \frac{1}{2}\Delta x_k$ from which a cylinder of height Δy_k and radius $x_k^* - \frac{1}{2}\Delta x_k$ has been removed, so its volume is $\pi(x_k^* + \frac{1}{2}\Delta x_k)^2\Delta y_k - \pi(x_k^* - \frac{1}{2}\Delta x_k)^2\Delta y_k = 2\pi x_k^*\Delta x_k\Delta y_k = 2\pi x_k^*\Delta A_k$.

(b) From part (a), $V = \iint_R 2\pi x\, dA = 2\pi\iint_R x\, dA$. From equation (13), this equals $2\pi\cdot\overline{x}\cdot$[area of R].

56. (a) $V = \left[\frac{1}{2}\pi a^2\right]\left[2\pi\left(a + \frac{4a}{3\pi}\right)\right] = \frac{1}{3}\pi(3\pi+4)a^3.$

(b) The distance between the centroid and the line is $\frac{\sqrt{2}}{2}\left(a + \frac{4a}{3\pi}\right)$, so $V = \left[\frac{1}{2}\pi a^2\right]\left[2\pi\frac{\sqrt{2}}{2}\left(a + \frac{4a}{3\pi}\right)\right] = \frac{1}{6}\sqrt{2}\pi(3\pi+4)a^3.$

57. $\overline{x} = k$ so $V = \pi ab\cdot 2\pi k = 2\pi^2 abk$.

58. $\overline{y} = 4$ from the symmetry of the region; $A = \int_{-2}^{2}\int_{x^2}^{8-x^2} dy\, dx = \frac{64}{3}$. So $V = \frac{64}{3}\cdot 2\pi\cdot 4 = \frac{512\pi}{3}$.

59. The region generates a cone of volume $\frac{1}{3}\pi ab^2$ when it is revolved about the x-axis, the area of the region is $\frac{1}{2}ab$ so $\frac{1}{3}\pi ab^2 = \frac{1}{2}ab\cdot 2\pi\overline{y}$, $\overline{y} = \frac{b}{3}$. A cone of volume $\frac{1}{3}\pi a^2 b$ is generated when the region is revolved about the y-axis so $\frac{1}{3}\pi a^2 b = \frac{1}{2}ab\cdot 2\pi\overline{x}$, $\overline{x} = \frac{a}{3}$. The centroid is $\left(\frac{a}{3}, \frac{b}{3}\right)$.

60. The centroid of the circle which generates the tube travels a distance $s = \int_0^{4\pi}\sqrt{\sin^2 t + \cos^2 t + \frac{1}{16}}\, dt = \sqrt{17}\pi$, so $V = \pi\left(\frac{1}{2}\right)^2\sqrt{17}\pi = \frac{\sqrt{17}\pi^2}{4}$.

61. It is the point P in the plane of the lamina such that the lamina will balance on any knife-edge passing through P. (If P is in the lamina, then the lamina will also balance on a point of support at P.)

Chapter 14 Review Exercises

3. **(a)** $\displaystyle\iint_R dA$ **(b)** $\displaystyle\iiint_G dV$ **(c)** $\displaystyle\iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$

4. **(a)** $x = a\sin\phi\cos\theta, y = a\sin\phi\sin\theta, z = a\cos\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$.

(b) $x = a\cos\theta, y = a\sin\theta, z = z, 0 \le \theta \le 2\pi, 0 \le z \le h$.

5. $\displaystyle\int_0^1 \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$

6. $\displaystyle\int_0^2 \int_x^{2x} f(x,y)\,dy\,dx + \int_2^3 \int_x^{6-x} f(x,y)\,dy\,dx$

7. **(a)** The transformation sends $(1,0)$ to (a,c) and $(0,1)$ to (b,d). There are two possibilities: either $(a,c) = (2,1)$ and $(b,d) = (1,2)$ or $(a,c) = (1,2)$ and $(b,d) = (2,1)$. So either $a = 2$, $b = 1$, $c = 1$, $d = 2$ or $a = 1$, $b = 2$, $c = 2$, $d = 1$.

(b) For either transformation in part (a), $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = 3$, so the area is $\displaystyle\iint_R dA = \int_0^1\int_0^1 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^1\int_0^1 3\,du\,dv = 3$. The diagonals of R cut it into 4 congruent right triangles. One of these has vertices $(0,0)$, $\left(\dfrac{3}{2},\dfrac{3}{2}\right)$, and $(2,1)$, so its bases have lengths $\dfrac{3}{2}\sqrt{2}$ and $\dfrac{1}{2}\sqrt{2}$ and its area is $\dfrac{1}{2}\cdot\dfrac{3}{2}\sqrt{2}\cdot\dfrac{1}{2}\sqrt{2} = \dfrac{3}{4}$; hence R has area $4\cdot\dfrac{3}{4} = 3$.

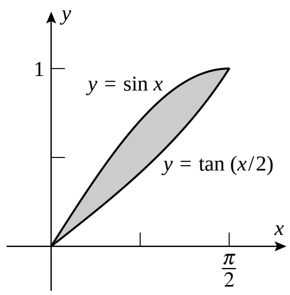
8. If $0 < x$, $y < \pi$ then $0 < \sin\sqrt{xy} \le 1$, with equality only on the hyperbola $xy = \pi^2/4$, so $\displaystyle 0 = \int_0^\pi\int_0^\pi 0\,dy\,dx < \int_0^\pi\int_0^\pi \sin\sqrt{xy}\,dy\,dx < \int_0^\pi\int_0^\pi 1\,dy\,dx = \pi^2$.

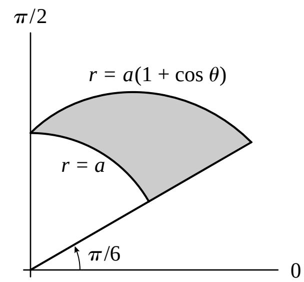
9. $\displaystyle\int_{1/2}^1 2x\cos(\pi x^2)\,dx = \frac{1}{\pi}\sin(\pi x^2)\Big]_{1/2}^1 = -\frac{1}{\sqrt{2}\,\pi}$.

10. $\displaystyle\int_0^2 \frac{x^2}{2}e^{y^3}\Big]_{x=-y}^{2y} dy = \frac{3}{2}\int_0^2 y^2 e^{y^3}\,dy = \frac{1}{2}e^{y^3}\Big]_0^2 = \frac{1}{2}\left(e^8 - 1\right)$.

11. $\displaystyle\int_0^1\int_{2y}^2 e^x e^y\,dx\,dy$

12. $\displaystyle\int_0^\pi\int_0^x \frac{\sin x}{x}\,dy\,dx$

13. 

14. 

15. $2\int_0^8\int_0^{y^{1/3}} x^2\sin y^2\,dx\,dy = \frac{2}{3}\int_0^8 y\sin y^2\,dy = -\frac{1}{3}\cos y^2\Big]_0^8 = \frac{1}{3}(1-\cos 64)\approx 0.20271.$

16. $\int_0^{\pi/2}\int_0^2 (4-r^2)r\,dr\,d\theta = 2\pi.$

17. $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2xy}{x^2+y^2}$, and $r = 2a\sin\theta$ is the circle $x^2+(y-a)^2=a^2$, so $\int_0^a\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}}\dfrac{2xy}{x^2+y^2}dy\,dx =$ $\int_0^a x\left[\ln\left(a+\sqrt{a^2-x^2}\right)-\ln\left(a-\sqrt{a^2-x^2}\right)\right]\,dx = a^2.$

18. $\int_{\pi/4}^{\pi/2}\int_0^2 4r^2(\cos\theta\sin\theta)\,r\,dr\,d\theta = -4\cos 2\theta\Big]_{\pi/4}^{\pi/2} = 4.$

19. $\int_0^2\int_{(y/2)^{1/3}}^{2-y/2} dx\,dy = \int_0^2\left(2-\frac{y}{2}-\left(\frac{y}{2}\right)^{1/3}\right)dy = \left(2y-\frac{y^2}{4}-\frac{3}{2}\left(\frac{y}{2}\right)^{4/3}\right)\Big]_0^2 = \frac{3}{2}.$

20. $A = 6\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = 3\int_0^{\pi/6}\cos^2 3\theta = \frac{\pi}{4}.$

21. $\int_0^{2\pi}\int_0^2\int_{r^4}^{16} r^2\cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^2 r^3(16-r^4)\,dr = 32\pi.$

22. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1\frac{1}{1+\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \left(1-\frac{\pi}{4}\right)\frac{\pi}{2}\int_0^{\pi/2}\sin\phi\,d\phi = \left(1-\frac{\pi}{4}\right)\frac{\pi}{2}(-\cos\phi)\Big]_0^{\pi/2} = \left(1-\frac{\pi}{4}\right)\frac{\pi}{2}.$

23. (a) $\int_0^{2\pi}\int_0^{\pi/3}\int_0^a(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^a\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta.$

(b) $\int_0^{2\pi}\int_0^{\sqrt{3}a/2}\int_{r/\sqrt{3}}^{\sqrt{a^2-r^2}} r^2\,dz\,r dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}a/2}\int_{r/\sqrt{3}}^{\sqrt{a^2-r^2}} r^3\,dz\,dr\,d\theta.$

(c) $\int_{-\sqrt{3}a/2}^{\sqrt{3}a/2}\int_{-\sqrt{(3a^2/4)-x^2}}^{\sqrt{(3a^2/4)-x^2}}\int_{\sqrt{x^2+y^2}/\sqrt{3}}^{\sqrt{a^2-x^2-y^2}}(x^2+y^2)\,dz\,dy\,dx.$

24. (a) $\int_0^4\int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}}\int_{x^2+y^2}^{4x} dz\,dy\,dx$ **(b)** $\int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta}\int_{r^2}^{4r\cos\theta} r\,dz\,dr\,d\theta$

25. $V = \int_0^{2\pi}\int_0^{a/\sqrt{3}}\int_{\sqrt{3}r}^a r\,dz\,dr\,d\theta = 2\pi\int_0^{a/\sqrt{3}} r(a-\sqrt{3}r)\,dr = \frac{\pi a^3}{9}.$

26. The intersection of the two surfaces projects onto the yz-plane as $2y^2+z^2=1$, so
$V = 4\int_0^{1/\sqrt{2}}\int_0^{\sqrt{1-2y^2}}\int_{y^2+z^2}^{1-y^2} dx\,dz\,dy = 4\int_0^{1/\sqrt{2}}\int_0^{\sqrt{1-2y^2}}(1-2y^2-z^2)\,dz\,dy = 4\int_0^{1/\sqrt{2}}\frac{2}{3}(1-2y^2)^{3/2}\,dy = \frac{\sqrt{2}\pi}{4}$.

27. The triangular region R is described by $0\le x\le 1$, $-x\le y\le x$. Hence $S = \iint\limits_R \sqrt{z_x^2+z_y^2+1}\,dA = \int_0^1\int_{-x}^{x}\sqrt{(4x)^2+3^2+1}\,dy\,dx = \int_0^1\int_{-x}^{x}\sqrt{16x^2+10}\,dy\,dx = \int_0^1 2x\sqrt{16x^2+10}\,dx = \frac{1}{24}(16x^2+10)^{3/2}\Big]_0^1 = \frac{1}{12}(13\sqrt{26}-5\sqrt{10})\approx 4.20632$.

28. $\|\mathbf{r}_u\times\mathbf{r}_v\| = \sqrt{2u^2+2v^2+4}$, $S = \iint\limits_{u^2+v^2\le 4}\sqrt{2u^2+2v^2+4}\,dA = \int_0^{2\pi}\int_0^2\sqrt{2r^2+4}\,r\,dr\,d\theta = \frac{8\pi}{3}(3\sqrt{3}-1)$.

29. $(\mathbf{r}_u\times\mathbf{r}_v)\Big|_{\substack{u=1\\v=2}} = \langle -2,-4,1\rangle$, tangent plane $2x+4y-z=5$.

30. $u=-3, v=0$, $(\mathbf{r}_u\times\mathbf{r}_v)\Big|_{\substack{u=-3\\v=0}} = \langle -18,0,-3\rangle$, tangent plane $6x+z=-9$.

32. $x = \frac{1}{10}u+\frac{3}{10}v$ and $y = -\frac{3}{10}u+\frac{1}{10}v$, hence $|J(u,v)| = \left|\left(\frac{1}{10}\right)^2+\left(\frac{3}{10}\right)^2\right| = \frac{1}{10}$, and $\iint\limits_R \frac{x-3y}{(3x+y)^2}\,dA = \frac{1}{10}\int_1^3\int_0^4\frac{u}{v^2}\,du\,dv = \frac{1}{10}\int_1^3\frac{1}{v^2}\,dv\int_0^4 u\,du = \frac{1}{10}\frac{2}{3}8 = \frac{8}{15}$.

33. **(a)** Add u and w to get $x=\ln(u+w)-\ln 2$; subtract w from u to get $y=\frac{1}{2}u-\frac{1}{2}w$, substitute these values into $v=y+2z$ to get $z=-\frac{1}{4}u+\frac{1}{2}v+\frac{1}{4}w$. Hence $x_u=\frac{1}{u+w}$, $x_v=0$, $x_w=\frac{1}{u+w}$; $y_u=\frac{1}{2}$, $y_v=0$, $y_z=-\frac{1}{2}$; $z_u=-\frac{1}{4}$, $z_v=\frac{1}{2}$, $z_w=\frac{1}{4}$, and thus $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{2(u+w)}$.

(b) $V = \iiint\limits_G dV = \int_1^3\int_1^2\int_0^4\frac{1}{2(u+w)}\,dw\,dv\,du = \frac{1}{2}(7\ln 7-5\ln 5-3\ln 3) = \frac{1}{2}\ln\frac{823543}{84375}\approx 1.139172308$.

34. $V=\frac{4}{3}\pi a^3, \bar{d} = \frac{3}{4\pi a^3}\iiint\limits_{\rho\le a}\rho\,dV = \frac{3}{4\pi a^3}\int_0^{\pi}\int_0^{2\pi}\int_0^a\rho^3\sin\phi\,d\rho\,d\theta\,d\phi = \frac{3}{4\pi a^3}\cdot 2\cdot 2\pi\cdot\frac{a^4}{4} = \frac{3}{4}a$.

35. $A = \int_{-4}^4\int_{y^2/4}^{2+y^2/8}dx\,dy = \int_{-4}^4\left(2-\frac{y^2}{8}\right)dy = \frac{32}{3}; \bar{y}=0$ by symmetry;
$\int_{-4}^4\int_{y^2/4}^{2+y^2/8}x\,dx\,dy = \int_{-4}^4\left(2+\frac{1}{4}y^2-\frac{3}{128}y^4\right)dy = \frac{256}{15}$, $\bar{x} = \frac{3}{32}\frac{256}{15} = \frac{8}{5}$; centroid $\left(\frac{8}{5},0\right)$.

36. $A=\pi ab/2, \bar{x}=0$ by symmetry, $\int_{-a}^a\int_0^{b\sqrt{1-x^2/a^2}}y\,dy\,dx = \frac{1}{2}\int_{-a}^a b^2\left(1-\frac{x^2}{a^2}\right)dx = \frac{2ab^2}{3}$, centroid $\left(0,\frac{4b}{3\pi}\right)$.

37. $V=\frac{1}{3}\pi a^2h, \bar{x}=\bar{y}=0$ by symmetry, $\int_0^{2\pi}\int_0^a\int_0^{h-rh/a}rz\,dz\,dr\,d\theta = \pi\int_0^a rh^2\left(1-\frac{r}{a}\right)^2 dr = \frac{\pi a^2h^2}{12}$, centroid $\left(0,0,\frac{h}{4}\right)$.

38. $V = \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} dz\,dy\,dx = \int_{-2}^{2}\int_{x^2}^{4}(4-y)\,dy\,dx = \int_{-2}^{2}\left(8-4x^2+\frac{1}{2}x^4\right)dx = \frac{256}{15}$, $\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} y\,dz\,dy\,dx = \int_{-2}^{2}\int_{x^2}^{4}(4y-y^2)\,dy\,dx = \int_{-2}^{2}\left(\frac{1}{3}x^6-2x^4+\frac{32}{3}\right)dx = \frac{1024}{35}$, $\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} z\,dz\,dy\,dx = \int_{-2}^{2}\int_{x^2}^{4}\frac{1}{2}(4-y)^2\,dy\,dx = \int_{-2}^{2}\left(-\frac{x^6}{6}+2x^4-8x^2+\frac{32}{3}\right)dx = \frac{2048}{105}$, $\bar{x}=0$ by symmetry, centroid $\left(0,\frac{12}{7},\frac{8}{7}\right)$.

Chapter 14 Making Connections

1. **(a)** $I^2 = \left[\int_0^{+\infty} e^{-x^2}\,dx\right]\left[\int_0^{+\infty} e^{-y^2}\,dy\right] = \int_0^{+\infty}\left[\int_0^{+\infty} e^{-x^2}dx\right]e^{-y^2}dy = \int_0^{+\infty}\int_0^{+\infty} e^{-x^2}e^{-y^2}\,dx\,dy =$
$= \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy.$

(b) $I^2 = \int_0^{\pi/2}\int_0^{+\infty} e^{-r^2} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}.$

(c) Since $I>0$, $I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

2. The two quarter-circles with center at the origin and of radius A and $\sqrt{2}A$ lie inside and outside of the square with corners $(0,0),(A,0),(A,A),(0,A)$, so the following inequalities hold:

$$\int_0^{\pi/2}\int_0^{A}\frac{1}{(1+r^2)^2}r\,dr\,d\theta \le \int_0^{A}\int_0^{A}\frac{1}{(1+x^2+y^2)^2}\,dx\,dy \le \int_0^{\pi/2}\int_0^{\sqrt{2}A}\frac{1}{(1+r^2)^2}r\,dr\,d\theta.$$

The integral on the left can be evaluated as $\frac{\pi A^2}{4(1+A^2)}$ and the integral on the right equals $\frac{2\pi A^2}{4(1+2A^2)}$. Since both of these quantities tend to $\frac{\pi}{4}$ as $A\to+\infty$, it follows by sandwiching that $\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \frac{\pi}{4}$.

3. **(a)** 1.173108605 **(b)** $\int_0^{\pi}\int_0^{1} re^{-r^4}\,dr\,d\theta = \pi\int_0^1 re^{-r^4}\,dr \approx 1.173108605.$

4. **(a)** At any point outside the closed sphere $\{x^2+y^2+z^2\le 1\}$ the integrand is negative, so to maximize the integral it suffices to include all points inside the sphere; hence the maximum value is taken on the region $G=\{x^2+y^2+z^2\le 1\}$.

(b) 1.675516

(c) $\int_0^{2\pi}\int_0^{\pi}\int_0^{1}(1-\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{8\pi}{15}.$

5. **(a)** Let S_1 be the set of points (x,y,z) which satisfy the equation $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$, and let S_2 be the set of points (x,y,z) where $x=a(\sin\phi\cos\theta)^3, y=a(\sin\phi\sin\theta)^3, z=a\cos^3\phi$, $0\le\phi\le\pi, 0\le\theta<2\pi$. If (x,y,z) is a point of S_2 then $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}[(\sin\phi\cos\theta)^3+(\sin\phi\sin\theta)^3+\cos^3\phi]=a^{2/3}$, so (x,y,z) belongs to S_1. If (x,y,z) is a point of S_1 then $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$. Let $x_1=x^{1/3}, y_1=y^{1/3}, z_1=z^{1/3}, a_1=a^{1/3}$. Then $x_1^2+y_1^2+z_1^2=a_1^2$, so in spherical coordinates $x_1=a_1\sin\phi\cos\theta, y_1=a_1\sin\phi\sin\theta, z_1=a_1\cos\phi$, with $\theta=\tan^{-1}\left(\frac{y_1}{x_1}\right)=\tan^{-1}\left(\frac{y}{x}\right)^{1/3}, \phi=\cos^{-1}\frac{z_1}{a_1}=\cos^{-1}\left(\frac{z}{a}\right)^{1/3}$. Then $x=x_1^3=a_1^3(\sin\phi\cos\theta)^3=a(\sin\phi\cos\theta)^3$, similarly $y=a(\sin\phi\sin\theta)^3, z=a\cos\phi$ so (x,y,z) belongs to S_2. Thus $S_1=S_2$.

(b) Let $a=1$ and $\mathbf{r}=(\cos\theta\sin\phi)^3\,\mathbf{i}+(\sin\theta\sin\phi)^3\,\mathbf{j}+\cos^3\phi\,\mathbf{k}$, then $S = 8\int_0^{\pi/2}\int_0^{\pi/2}\|\mathbf{r}_\theta\times\mathbf{r}_\phi\|\,d\phi\,d\theta =$

$$72\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^4\phi\cos\phi\sqrt{\cos^2\phi+\sin^2\phi\sin^2\theta\cos^2\theta}\,d\theta\,d\phi \approx 4.4506.$$

6. $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin^3\phi\cos^3\theta & 3\rho\sin^2\phi\cos\phi\cos^3\theta & -3\rho\sin^3\phi\cos^2\theta\sin\theta \\ \sin^3\phi\sin^3\theta & 3\rho\sin^2\phi\cos\phi\sin^3\theta & 3\rho\sin^3\phi\sin^2\theta\cos\theta \\ \cos^3\phi & -3\rho\cos^2\phi\sin\phi & 0 \end{vmatrix} = 9\rho^2\cos^2\theta\sin^2\theta\cos^2\phi\sin^5\phi$, so

$$V = 9\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\cos^2\theta\sin^2\theta\cos^2\phi\sin^5\phi\,d\rho\,d\phi\,d\theta = \frac{4}{35}\pi a^3.$$