Functions

Let X and Y be two sets.

A function f maps **each** element of X to **exactly one** element of Y

Let $S = \underbrace{\{x_1, x_2, x_3, x_4, x_5\}}_{Students}$, $G = \underbrace{\{A, B, C, F\}}_{Grades}$

- Crazy Professor's map: $\Big\{(x_1, F), (x_2, F), (x_3, F), (x_4, F), (x_5, F)\Big\}$
- Another crazy map: $\Big\{(x_1, A), (x_2, A), (x_3, A), (x_4, A), (x_5, A)\Big\}$
- Reasonable map: $\Big\{(x_1, C), (x_2, A), (x_3, B), (x_4, B), (x_5, A)\Big\}$

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$f : X \mapsto Y$ is a subset of $X \times Y$, such that for every $x \in X$, f contains exactly one ordered pair with first component x

Function Graphical Representation

- $f : X \mapsto Y$ can be represented graphically (bipartite graphs)
- '*Draw*' sets X and Y. For $f(x) = y$ draw arrow from $x \in X$ to $y \in Y$

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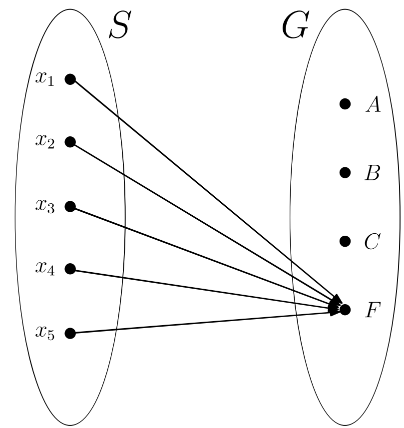

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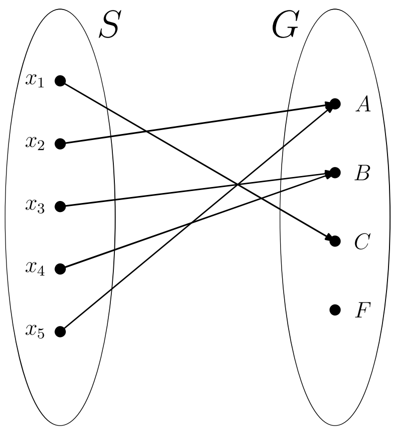

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Functions: Basic Terms

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Let $f : X \mapsto Y$ and let $f(x) = y$

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- Range of f:
  set of images of all elements of X

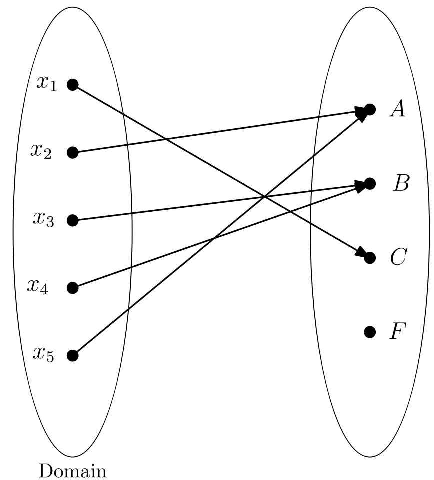

Figure: $f : X \mapsto Y$

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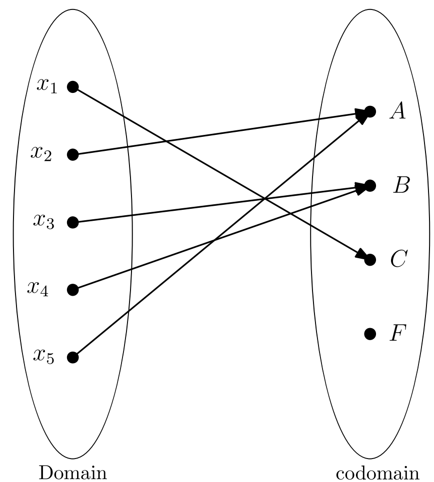

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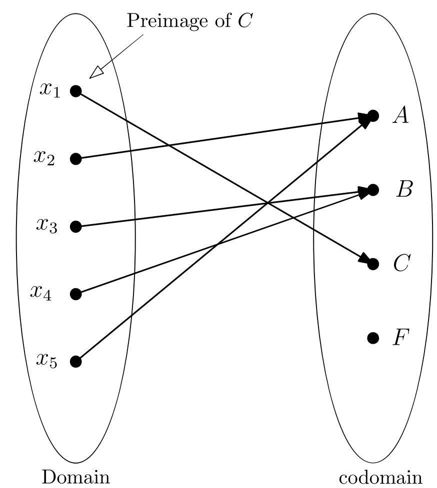

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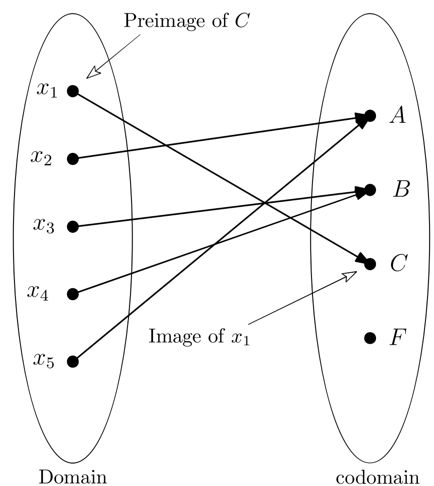

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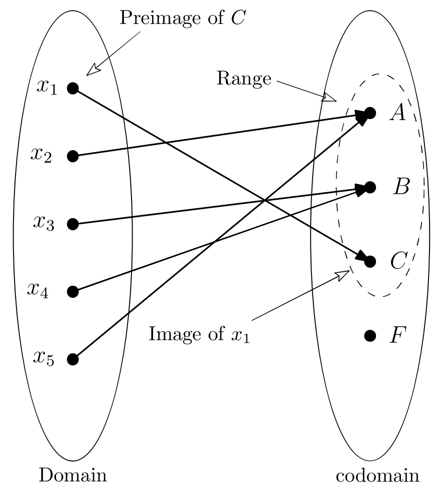

Figure: $f : X \mapsto Y$

Function: Basic Terms

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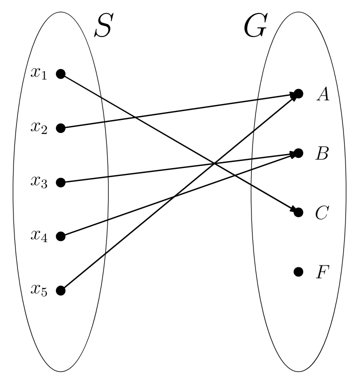

$\boxed{\textbf{ICP 5-5}}$ Range of $f = \{A, B, C\}$? a) True b) False

Function: Basic Terms

- Let $S = \underbrace{\{x_1, x_2, x_3, x_4, x_5\}}_{Students}$, $\quad G = \underbrace{\{A, B, C, F\}}_{Grades}$

- $f : S \mapsto G := \Big\{(x_1, F), (x_2, F), (x_3, F), (x_4, F), (x_5, F)\Big\}$

Function: Basic Terms

- Let $S = \underbrace{\{x_1, x_2, x_3, x_4, x_5\}}_{Students}$, $\quad G = \underbrace{\{A, B, C, F\}}_{Grades}$
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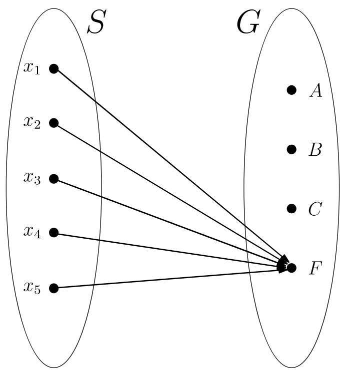

ICP 5-6 Range of $f = \{F\}$?

a) True b) False

Function: Basic Terms

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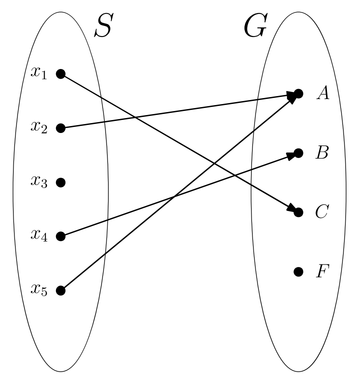

$\boxed{\textbf{ICP 5-7}}$ Range of $f = \{A, B, C\}$? a) True b) False

Function: Basic Terms

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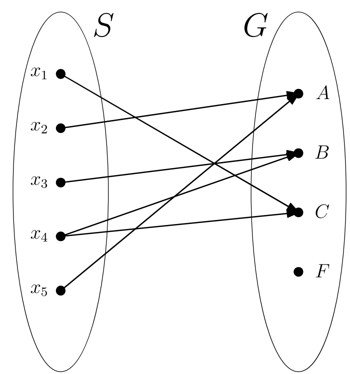

$\boxed{\textbf{ICP 5-8}}$ Range of $f = \{A, B, C\}$?

a) True b) False

Functions: Mapping Rule

For large domains difficult to describe functions with graphs or sets

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Functions can be specified by a mapping rule (formula)

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Functions can be specified by a mapping rule (formula)

Let $S = \underbrace{\{x_1, x_2, x_3, x_4, x_5\}}_{Students}, \qquad G = \underbrace{\{A, B, C, F\}}_{Grades}$

$f : S \mapsto G$ is defined as follows:

$$f(x_i) = \begin{cases} A & if \ score(x_i) \geq 90\% \\ B & if \ 80\% \leq \ score(x_i) < 90\% \\ C & if \ 60\% \leq \ score(x_i) < 80\% \\ F & if \ score(x_i) < 60\% \end{cases}$$

Functions: Mapping Rule

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be given as

$$f(x) = x^2$$

- Domain of f
- Codomain of f
- Image of -3 and 5
- Preimage of $25, 4, -3, 30$
- Range of f

Functions: C++/JAVA

function $\textsc{Sign}(x)$ $\triangleright$ returns int
  if $x < 0$ **then**
    return -1
  else if $x = 0$ **then**
    return 0
  else
    return 1

Functions: C++/JAVA

function SIGN(x) ▷ returns int
  if $x < 0$ **then**
    return -1
  else if $x = 0$ **then**
    return 0
  else
    return 1

- Domain of funct()
- Codomain of funct()
- Image of $-3, 2, 5, 0$
- Preimage of $0, 4, 1, -1$
- Range of funct()
- Functionality of funct()

Functions: Summary

- $f : X \mapsto Y$ maps **each** element of X to **exactly one** element of Y

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- Functions can be represented by
  - Listing set of all (pre-image, image) ordered pairs
  - Bipartite Graph
  - Mapping Rule or Algebraic Expression
  - Programming Code

Functions

Imdad ullah Khan

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Types of functions: One-to-One

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

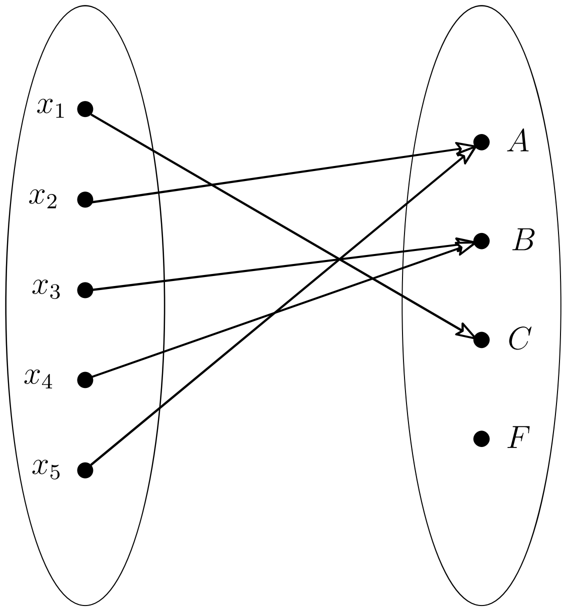
$$\forall\, x_1, x_2 \in X \;\; \big(\, f(x_1) = f(x_2) \rightarrow x_1 = x_2 \,\big)$$

Each element of X is mapped to a unique element of Y

Think of the contrapositive

$$\forall\, x_1, x_2 \in X \;\; \big(x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)\big)$$

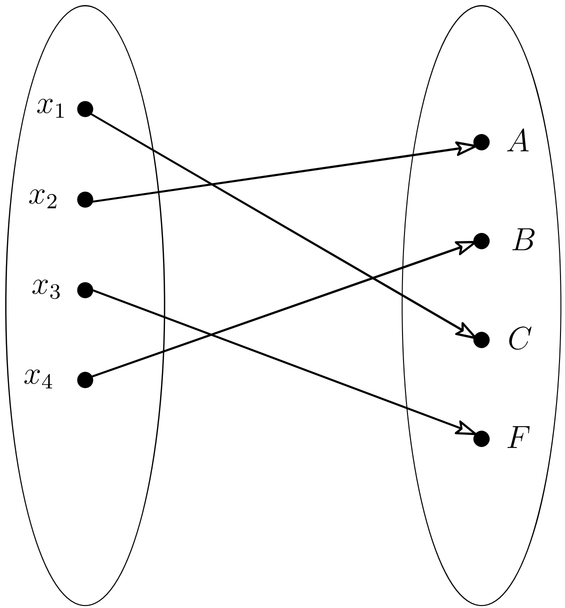
Types of functions: One-to-One



Not one-to-one

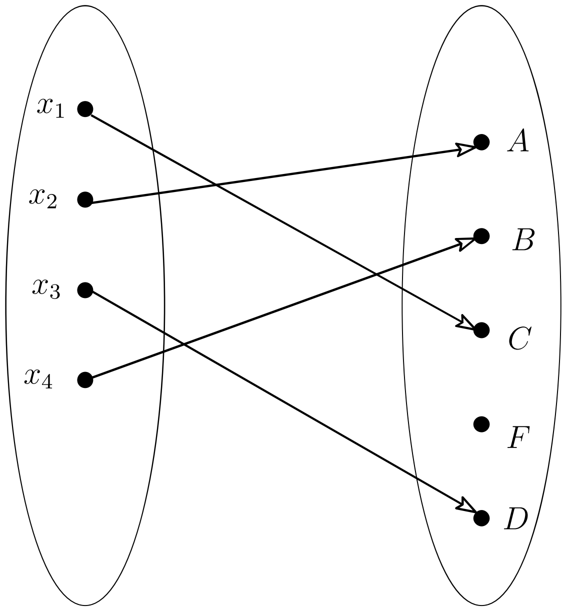
$f(x_2) = f(x_5)$

Types of functions: One-to-One



one-to-one

Types of functions: One-to-One



one-to-one

Types of functions: One-to-One

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) = 2x - 3$$

Is f one-to-one?

$$f(x_1) = f(x_2) \rightarrow 2(x_1) - 3 = 2(x_2) - 3 \rightarrow x_1 = x_2$$

Hence, f is one-to-one

Types of functions: One-to-One

Which of the following functions is one-to-one?

$\boxed{\textbf{ICP 5-9}}$ $f : \mathbb{Z} \mapsto \mathbb{R},\ f(x) = x^2$ a) True b) False

$\boxed{\textbf{ICP 5-10}}$ $f : \mathbb{R} \mapsto \mathbb{R},\ f(x) = x^3$ a) True b) False

$\boxed{\textbf{ICP 5-11}}$ $f : \mathbb{Z} \mapsto \mathbb{Z},\ f(x) = 2x$ a) True b) False

$\boxed{\textbf{ICP 5-12}}$ $f : \mathbb{Z} \mapsto \mathbb{N},\ f(x) = |x|$ a) True b) False

$\boxed{\textbf{ICP 5-13}}$ $f :$ people $\mapsto$ people,
$f(x) =$ father of x a) True b) False

Types of functions: Onto

A function $f : X \mapsto Y$ is **onto** (or **surjective**) iff

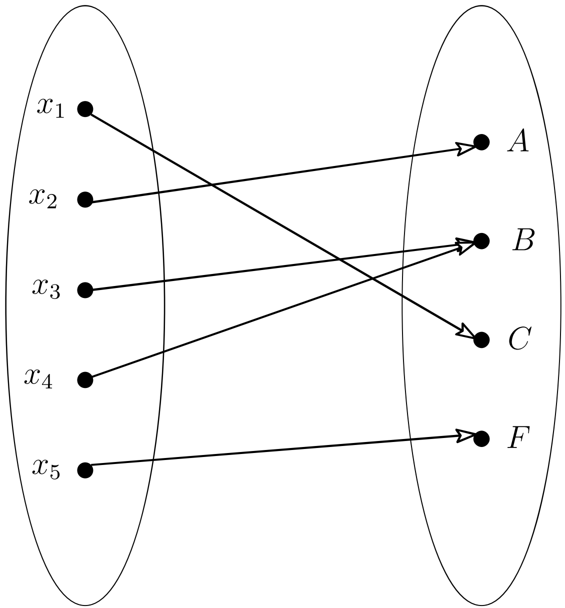
for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$

Each element of Y is assigned to some element of X

$$\forall\, y \in Y\; \exists x \in X \;\; f(x) = y$$

Range = codomain

Types of functions: Onto



onto

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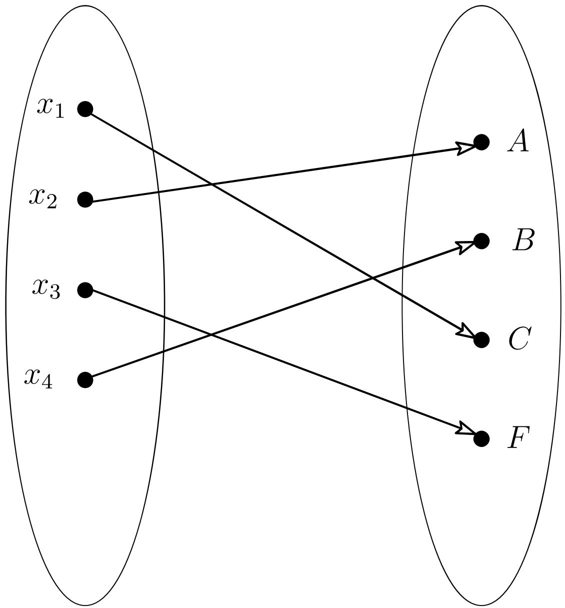

onto

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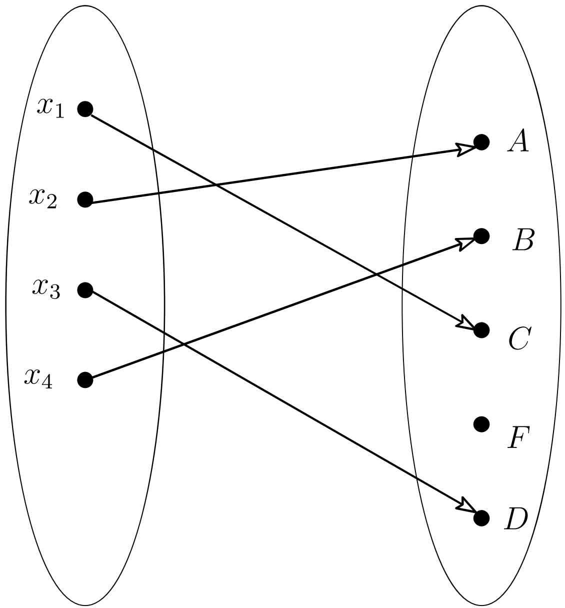


not onto

Types of functions: Onto

Which of the following functions is onto?

ICP 5-14	$f : \mathbb{Z} \mapsto \mathbb{R},\ f(x) = x^2$	a) True	b) False
ICP 5-15	$f : \mathbb{R} \mapsto \mathbb{R},\ f(x) = x^3$	a) True	b) False
ICP 5-16	$f : \mathbb{Z} \mapsto \mathbb{Z},\ f(x) = 2x$	a) True	b) False
ICP 5-17	$f : \mathbb{Z} \mapsto \mathbb{N},\ f(x) = \|x\|$	a) True	b) False
ICP 5-18	f : people $\mapsto$ people, $f(x) =$ father of x	a) True	b) False

Types of functions: Onto

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) = 2x - 3$$

What is the range of f? Is f onto?

We characterize $range(f)$ (a set) from the definition of f and check if it equals codomain

$$\begin{aligned} y \in range(f) &\leftrightarrow \quad y = 2x - 3 \qquad\qquad x \in \mathbb{Z} \\ &\leftrightarrow \quad y = 2(x-2) + 1 \\ &\leftrightarrow \quad y \text{ is odd} \end{aligned}$$

$range(f) \neq codomain(f)$

Hence, f is not onto

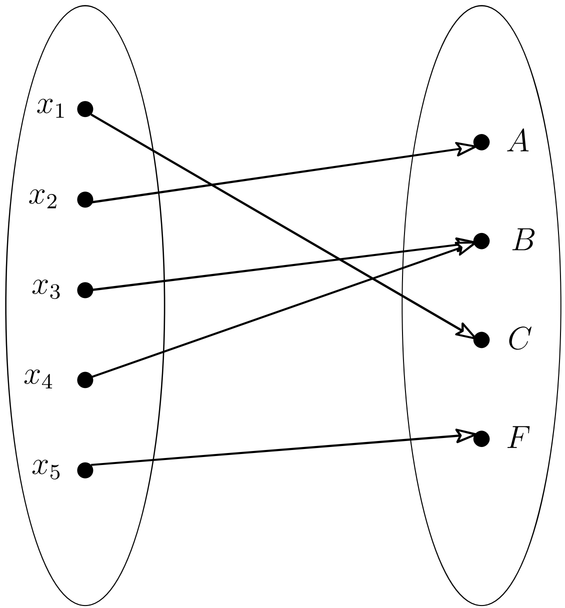
Types of functions: Bijection

A function $f : X \mapsto Y$ is **one-to-one correspondence** (or **bijective**) iff it is **both one-to-one** and **onto**

- Each element of X is mapped to a unique element of Y
- Each element of Y is assigned to some element of X

- if X and Y are finite sets, then $|X| = |Y|$
- $|$**domain**$| = |$**codomain**$|$

Types of functions: Bijection

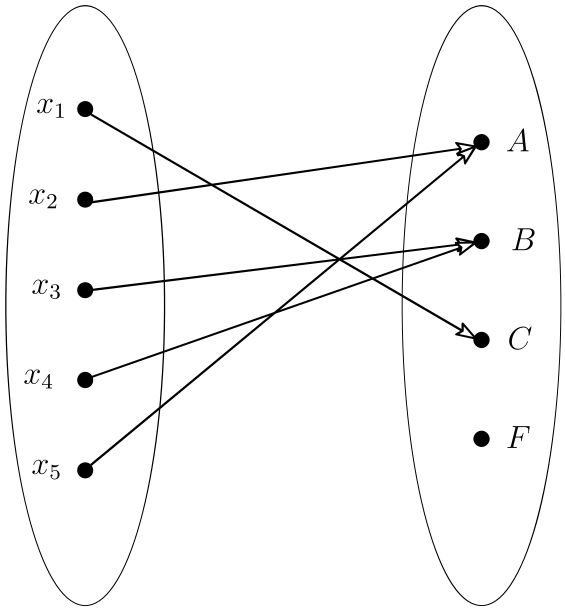


Not bijection

onto

not one-to-one

Types of functions: Bijection



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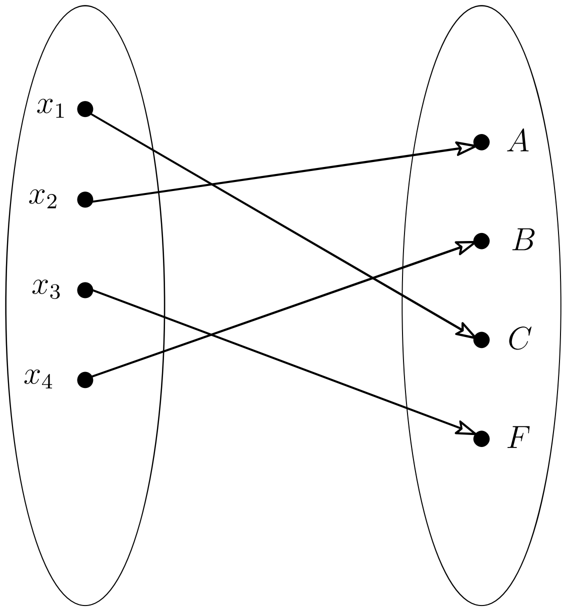

bijection

onto

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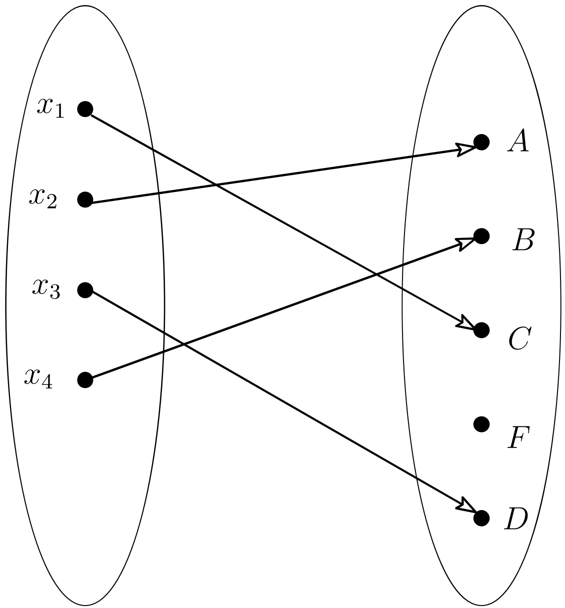
Types of functions: Bijection



Not bijection

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Types of functions: Bijection

Which of the following functions is a bijection?

ICP 5-19	$f : \mathbb{Z} \mapsto \mathbb{R},\ f(x) = x^2$	a) True	b) False
ICP 5-20	$f : \mathbb{R} \mapsto \mathbb{R},\ f(x) = x^3$	a) True	b) False
ICP 5-21	$f : \mathbb{Z} \mapsto \mathbb{Z},\ f(x) = 2x$	a) True	b) False
ICP 5-22	$f : \mathbb{Z} \mapsto \mathbb{N},\ f(x) = \lvert x \rvert$	a) True	b) False
ICP 5-23	f : people $\mapsto$ people, $f(x) =$ father of x	a) True	b) False

Types of functions: Bijection

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be

$$f(x) = 2x - 3$$

Is f a bijection?

- For any $y \in \mathbb{R}$, since ${}^{y+3}/_{2} \in \mathbb{R}$ and $f({}^{y+3}/_{2}) = y$

  $\triangleright$ so f is onto

- $f(x_1) = f(x_2) \rightarrow 2(x_1) - 3 = 2(x_2) - 3 \rightarrow x_1 = x_2$

  $\triangleright$ so f is one-to-one

- Hence, f is a bijection

Types of functions: Cardinalities

If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$

What can we conclude about $|X| \oslash |Y|$,

1. when f is one-to-one?
2. when f is onto?

Types of functions: Summary

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

$$\forall\, x_1, x_2 \in X\ (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$$

A function $f : X \mapsto Y$ is **onto** (or **surjective**) iff

for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$

A function $f : X \mapsto Y$ is **one-to-one correspondence** (or **bijective**) iff

it is **both one-to-one** and **onto**

If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$

Functions

- Ordered tuples and Cartesian Product
- Function and Representations
- Types of Functions
- Composition and Inverse of Function
- Numeric Functions

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Types of functions: Review

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

$$\forall\, x_1, x_2 \in X \; (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$$

A function $f : X \mapsto Y$ is **onto** (or **surjective**) iff

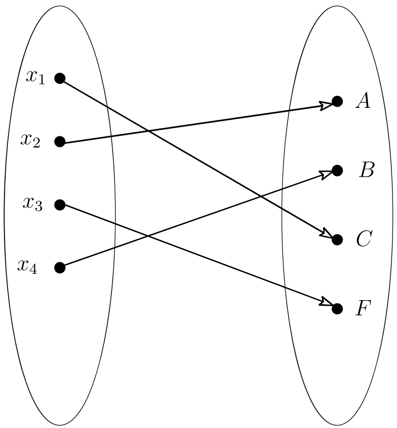
for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$

A function $f : X \mapsto Y$ is **one-to-one correspondence** (or **bijective**) iff

it is **both one-to-one** and **onto**

Inverse of a function

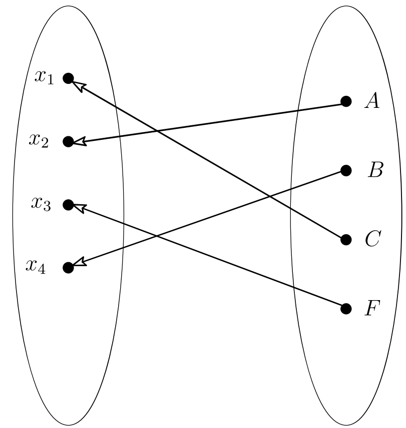
If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection



bijection

onto

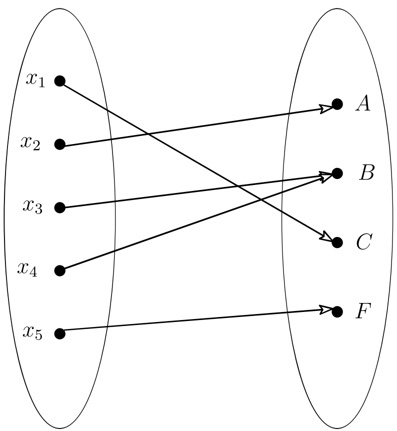
one-to-one



Inverse of a function

If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

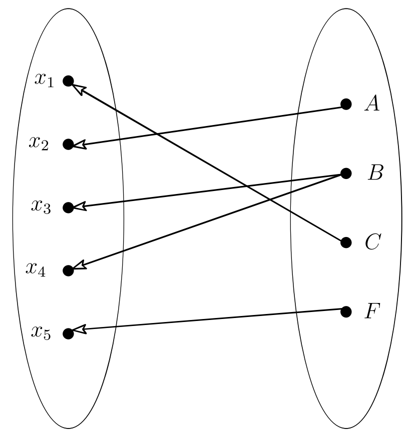
ICP 5-24 Can we say this when f is not one-to-one? a) Yes b) No



Not bijection

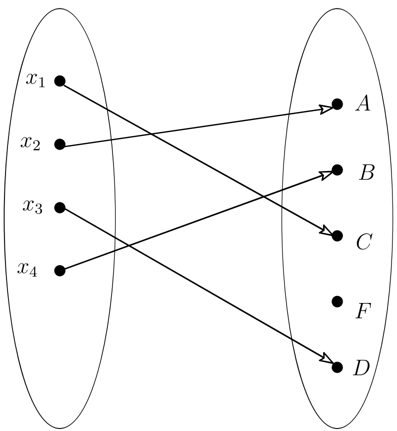
onto

not one-to-one



If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

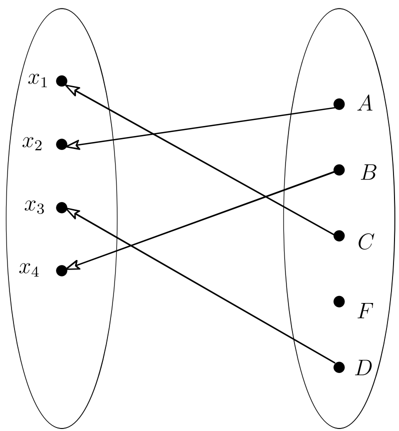
ICP 5-25 Can we say this when f is not onto? a) Yes b) No



Not bijection

not onto

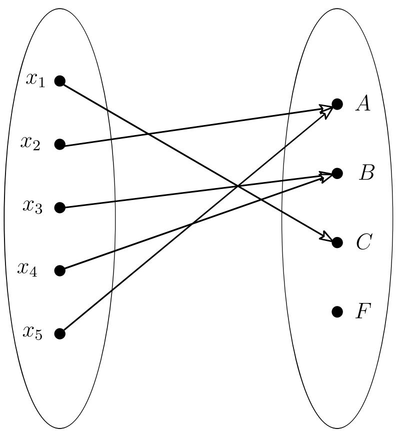
one-to-one



Inverse of a function

If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

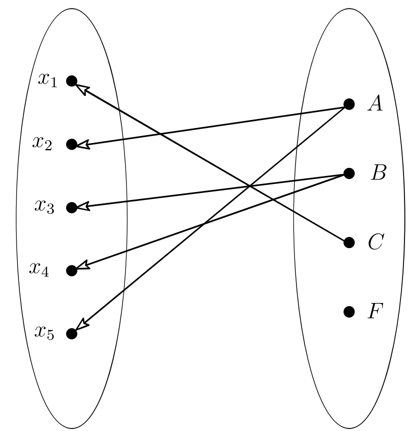
ICP 5-26 Can we say this when f is neither? a) Yes b) No



Not bijection

not onto

not one-to-one



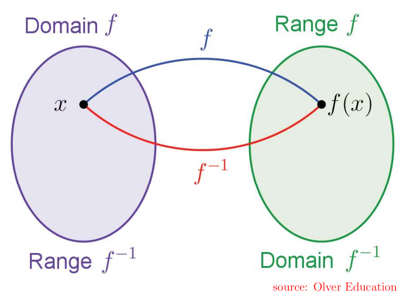
Inverse of function

If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

If $f : X \mapsto Y$ is a bijection, $f^{-1} : Y \mapsto X$ is the inverse function such that

$$f^{-1}(b) = a, \text{ when } f(a) = b$$

If an inverse exists, then $\quad f(a) = b \leftrightarrow f^{-1}(b) = a$



Inverse of function

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be

$$f(x) = 2x - 3$$

Is f a bijection? What is f^{-1}?

For any $y \in \mathbb{R}$, since $y+3/2 \in \mathbb{R}$ and $f(y+3/2) = y$

▷ so f is onto

$f(x_1) = f(x_2) \rightarrow 2(x_1) - 3 = 2(x_2) - 3 \rightarrow x_1 = x_2$

▷ so f is one-to-one

Hence, f is a bijection

Suppose $f^{-1}(y) = x$

Let $y = 2x - 3$ and solve for x, we get

$$f^{-1}(y) = y+3/2$$

Inverse of function

$\boxed{\textbf{ICP 5-27}}$

Let $f : \mathbb{R}^- \mapsto \mathbb{R}^+$ be

$$f(x) = x^2$$

Is f a bijection? What is f^{-1}?

$$\mathbb{R}^- = \{x \in \mathbb{R} \mid x \leq 0\} \quad \text{and} \quad \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$$

$$f^{-1}(y) = -\sqrt{y}$$

Composing functions

The output of one function can be used as input to another function

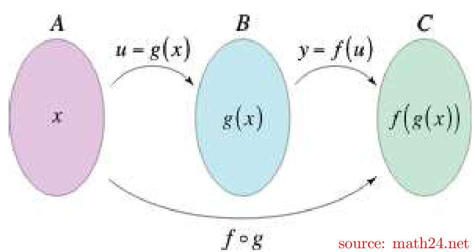
▷ Restriction on range and domain

Let $g : A \mapsto B$ and let $f : B \mapsto C$

The composition of the functions f and g is a function, $(f \circ g) : A \mapsto C$ is

$$(f \circ g)(a) = f(g(a))$$

- The output of g is input to f
- Range of g (actual outputs) must be a subset of domain of f



Composition

- Let $f : \mathbb{R} \mapsto \mathbb{R}$ be $f(x) := x^3$
- Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) := x^2$

- $(f \circ g)(x) : \mathbb{Z} \mapsto \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = (x^2)^3 = x^6$$

What will be $(g \circ f)$?

<u>Range of inner function must be a subset of domain of outer function</u>

Composition

- Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be $f(x) := x^3$
- Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) := x^2$

- $(f \circ g)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(f \circ g)(x) \;=\; f(g(x)) \;=\; f(x^2) \;=\; (x^2)^3 \;=\; x^6$$

- $(g \circ f)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(g \circ f)(x) \;=\; g(f(x)) \;=\; g(x^3) \;=\; (x^3)^2 \;=\; x^6$$

Composition

- Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be $f(x) := x^2$
- Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) := 2x$
- **ICP 5-28** $(f \circ g)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$$

- **ICP 5-29** $(g \circ f)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = 2(x^2) = 2x^2$$

$(f \circ g)(\cdot)$ is not necessarily the same as $(g \circ f)(\cdot)$

Sometimes, the other side of composition may not even be possible

Composing function with itself

A function can be composed with itself

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) = x + 1$$

$$(f \circ f)(x) \;=\; f(f(x)) \;=\; f(x+1) \;=\; (x+1)+1 \;=\; x+2$$

Composing function with itself repeatedly

A function can be composed with itself multiple times

$f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) = x + 1$$

$$(f \circ (f \circ f))(x) = f(f(f(x))) = f(f(x+1)) = f(x+2) = x+3$$

Let $father : People \mapsto People$ be defined as

$$father(x) = \text{father of } x$$

$$father(father(father(x))) = \text{great grand father of } x$$

Composing function with it's inverse

When a function is invertible (when it's a bijection), then it can be composed with it's inverse

$$\text{If } f = g^{-1}\text{ , then } (f \circ g)(x) = f(g(x)) = g^{-1}(g(x)) = x$$

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be $f(x) = x + 1$

Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) = x - 1$

$\triangleright$ $f = g^{-1}$

$$f \circ g(x) = f(g(x)) = f(x-1) = (x-1)+1 = x$$

Function inverse and composition: Summary

- If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection too
- If $f : X \mapsto Y$ is a bijection, $f^{-1} : Y \mapsto X$ is the inverse function such that $f^{-1}(b) = a$, when $f(a) = b$
- Let $g : A \mapsto B$ and let $f : B \mapsto C$. The composition of the functions f and g is a function, $(f \circ g) : A \mapsto C$ is $(f \circ g)(a) = f(g(a))$
- Function can be composed with itself (multiple times) and with it's inverse

Functions

- Ordered tuples and Cartesian Product
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- Numeric Functions

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Numeric Functions

A numeric function is typically of the form $f : \mathbb{R} \mapsto \mathbb{R}$

In discrete mathematics, generally $f : \mathbb{Z} \mapsto \mathbb{Z}$

The most important property of numeric functions (for us)

Both the domain and codomain are totally ordered sets

▷ every pair is comparable, $<, >, \leq, \geq$ are well defined

We can plot $f : \mathbb{R} \mapsto \mathbb{R}$ using the line plot

Monotonic Functions

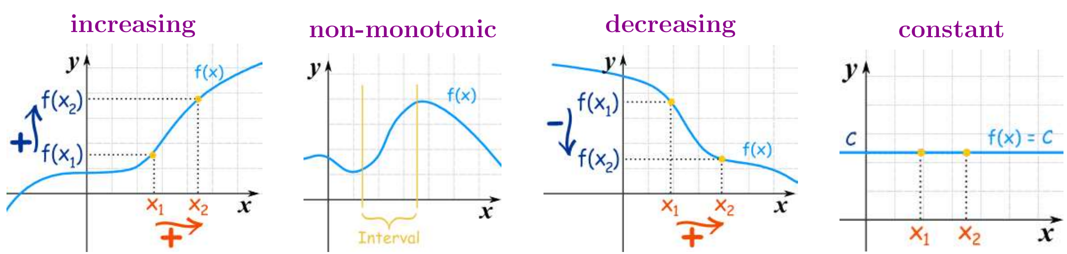
A function $f : \mathbb{R} \mapsto \mathbb{R}$ is increasing if $f(x) \leq f(y)$ whenever $x < y$

$$\forall x \, \forall y \ \ (x < y \ \rightarrow \ f(x) \leq f(y))$$

f is called strictly increasing if $f(x) < f(y)$ when $x < y$

▷ Decreasing and strictly decreasing functions are defined similarly

Collectively, such functions are called *monotonic functions*



source: mathsisfun.com

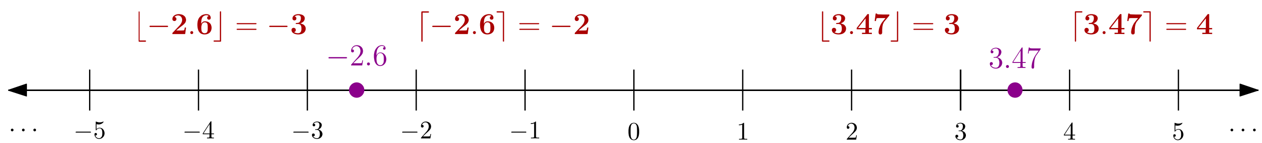
Note: strictly increasing/decreasing functions are necessarily one-to-one

Important Numeric Functions: Ceiling and Floors

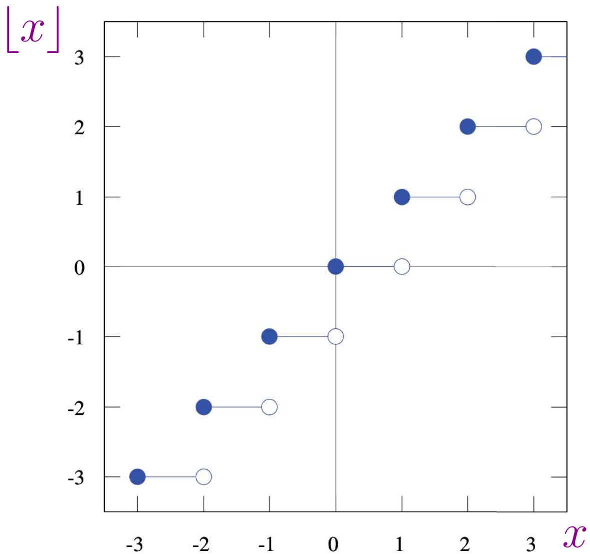
The floor and ceiling functions map real numbers to integers ($\mathbb{R} \mapsto \mathbb{Z}$)

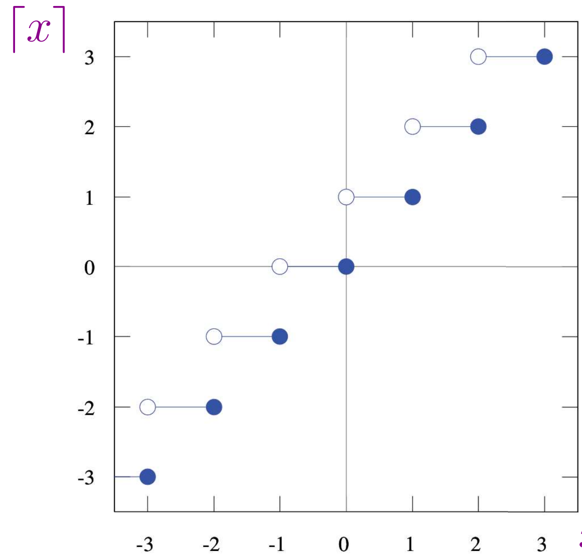
Given $x \in \mathbb{R}$,

- **floor**(x), $\lfloor x \rfloor$ is the largest integer $\leq x$
- **ceiling**(x), $\lceil x \rceil$ is the smallest integer $\geq x$



Important Numeric Functions: Ceiling and Floors





source: Wikipedia

Important Numeric Functions: Ceiling and Floors

ICP 5-30

- $\lfloor 1.7 \rfloor, \quad \lceil 1.7 \rceil$
- $\lfloor 3.2 \rfloor, \quad \lceil 3.2 \rceil$
- $\lfloor 3.5 \rfloor, \quad \lceil 3.5 \rceil$
- $\lfloor 3.9 \rfloor, \quad \lceil 3.9 \rceil$
- $\lfloor 11 \rfloor, \quad \lceil 11 \rceil$

ICP 5-31

- $\lfloor -1.7 \rfloor, \quad \lceil -1.7 \rceil$
- $\lfloor -3.2 \rfloor, \quad \lceil -3.2 \rceil$
- $\lfloor -3.5 \rfloor, \quad \lceil -3.5 \rceil$
- $\lfloor -3.9 \rfloor, \quad \lceil -3.9 \rceil$
- $\lfloor -11 \rfloor, \quad \lceil -11 \rceil$