Linear Algebra

Weeks	Contents/Topics	Remarks	Exercises	CLO's	Tools
Week 1	Introduction, System of Linear equations, Elementary row operation		**1.1** (1-20)	1	Q1, A1, M1, F
Week 2	**Solving system of Linear equations**: Gaussian Elimination and Gauss Jordan methods **Matrix Operations** Elementary Matrices, Methods for finding Inverse, Invertible Matrices,	**Assignment 1**	**1.2** (1-26) **1.5** (1-6, 11-18) **1.6** (1-20)		
Week 3	Diagonal, triangular, and symmetric matrices, Matrix Transformations		**1.7** (1-10, 19-28) **1.8** (1-24, 27-41) (CLO 2)		
Week 4	Application no 1: Network Analysis Determinants and their properties, Minors, Cofactors, Inverse using cofactors, Cramer's Rule	**Quiz 1**	**1.10** (1-4) (CLO 3) **2.1** (1-32) **2.2** (1-23) **2.3**(1-29,31,32)		
Week 5	General Vector Space Subspaces		**4.1** (1,2,9,11, 12) Example: 1-5,7 **4.2** (1-5, 19) Example: 1-6,13		
Week 6	**1st Mid Term Exam**				

Linear Equations 1.1

Linear:
- ↳ no power variables
- ↳ no trig, exoential, logs
- ↳ no variable roots/products

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

a. $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ T b. $x_1 + 3x_2 + x_1x_3 = 2$ F
c. $x_1 = -7x_2 + 3x_3$ T d. $x_1^{-2} + x_2 + 8x_3 = 5$ F
e. $x_1^{3/5} - 2x_2 + x_3 = 4$ F f. $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$ T

2. In each part, determine whether the equation is linear in x and y.

a. $2^{1/3}x + \sqrt{3}y = 1$ T b. $2x^{1/3} + 3\sqrt{y} = 1$ F
c. $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ T d. $\frac{\pi}{7}\cos x - 4y = 0$ F
e. $xy = 1$ F f. $y + 7 = x$ T

3. Using the notation of Formula (7), write down a general linear system of

a. two equations in two unknowns.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{12}x_1 + a_{22}x_2 = b_2$$

find a system of linear equations

5. a. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$2x = 0$
$3x - 4y = 0$
$y = 1$

–8, find the augmented matrix fo

7. a. $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$

$$\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$$

Value of K

19. a. $\begin{bmatrix} 1 & k & -4 \\ 4 & 8 & 2 \end{bmatrix}$

$R_2 \sim 4R_1 - R_2$

$$\begin{bmatrix} 1 & K & -4 \\ 0 & 4K-8 & -18 \end{bmatrix}$$

$x + Ky = -4$
$(4K-8)y = -18$
if $K = 2$ $0 = -18$
$K \neq 2$

9. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

a. $(3, 1, 1)$

?

dont know

$x = 1,\ y = 2$ one solution
$0 = -6,$ no solution
infinitely

Use parametric equations

15. a. $\begin{aligned} 2x - 3y &= 1 \\ 6x - 9y &= 3 \end{aligned}$

$$\begin{bmatrix} 2 & -3 & 1 \\ 6 & -9 & 3 \end{bmatrix} R_2 \sim 3R_1 - R_2$$

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$0 = 0$

$x = \dfrac{1 + 3y}{2}$

let $y = t$

$x = \frac{1}{2} + \frac{3}{2}t$

1.2

$a x_1 + b y_1 = c_1$ → graph → substitution → elimination → matrice ↳ cramers ↳ adjoint method

$a x_2 + b y_2 = c_2$ → forward → backward

gaus elimination → echalon

$$\begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

gaus jordan → reduced echalon

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- change in rows : no change
- change in columns : change in variables
- gaus eq, less eq more variables → infinite solutions

A $\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ B $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ Augmented matrice AB $\begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & 3 \end{bmatrix}$

RULES

1. rows add/sub
2. rows constants mul/div
3. rows do not add/sub constants
4. rows are not multiplied with eachother

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ unique solution

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ infinite solution

$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ no solution

A^{-1} form echalon

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

(identity matrice over the last three columns)

- convert A → reduced echalon
- Perform same operation on identity matrice
- identity matrice left is answer

STANDARD MATRIX

$e_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

first solve simultaneously and find variable then substitute and add values

$$
\left.\begin{array}{llllll}
a_{11}x_1 & + a_{12}x_2 & + \cdots + & a_{1n}x_n & = 0 \\
a_{21}x_1 & + a_{22}x_2 & + \cdots + & a_{2n}x_n & = 0 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1}x_1 & + a_{m2}x_2 & + \cdots + & a_{mn}x_n & = 0
\end{array}\right\} \text{all zero's hence homogenous}
$$

if homogenous invertible then trivial

if not homogenous invertible then nontrivial

infinitey / unique = consistent

no solution = inconsistent

$$
\left[\begin{array}{ccc|c}
1 & 2 & 6 & b_1 \\
0 & 1 & 5 & b_2+b_1 \\
0 & 0 & 1 & \frac{b_3}{h}+b_2+b_1
\end{array}\right]
$$

consistent

$$
\left[\begin{array}{ccc|c}
1 & 2 & 6 & b_1 \\
0 & 1 & 5 & b_2+b_1 \\
0 & 0 & 0 & b_3+b_2+b_1
\end{array}\right]
$$

if non zero here then inconsistent

consistent if $b_3+b_2+b_1=0$

Theorem 1.2.1

Free Variable Theorem for Homogeneous Systems

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n-r$ free variables.

Theorem 1.5.2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

EXAMPLE 2 | Using Elementary Matrices

Consider the matrix

$$
A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} \quad R_3 \sim 3R_1 + R_3
$$

and consider the elementary matrix

$$
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad R_3 \sim 3R_1 + R_3 \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}
$$

may be sign changes

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$
EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}
$$

which is precisely the matrix that results when we add 3 times the first row of A to the third row.

TRIANGLES

$$
\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 6 & 2 \end{bmatrix}
$$

diagonal lower

$$
\begin{bmatrix} 5 & 8 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}
$$

upper

Inverting co-efficient method

EXAMPLE 1 | Solution of a Linear System Using A^{-1}

Consider the system of linear equations

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 5 \\
2x_1 + 5x_2 + 3x_3 &= 3 \\
x_1 + 8x_3 &= 17
\end{aligned}
$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}
$$

In Example 4 of the preceding section, we showed that A is invertible and

$$
A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}
$$

By Theorem 1.6.2, the solution of the system is

$$
\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}
$$

or $x_1 = 1, x_2 = -1, x_3 = 2$.

INVERTIBLE

$$
\begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 5 \\ 2 & 1 & 6 \end{bmatrix}
$$

invertible since no zeros in its diagonal

not invertible since zeros in its diagonal

1.8

find Domain and co-domain

rows = co domain

columns = domain

3×2: $3 \downarrow$ row, $2 \downarrow$ column

		D	CD		D	CD
1.	a. A has size 3×2.	R^2	R^3	b. A has size 2×3.	R^3	R^2
	c. A has size 3×3.	R^3	R^3	d. A has size 1×6.	R^6	R
2.	a. A has size 4×5.	R^5	R^4	b. A has size 5×4.	R^4	R^5
	c. A has size 4×4.	R^4	R^4	d. A has size 3×1.	R^1	R^3

3. **a.** $w_1 = 4x_1 + 5x_2$, $w_2 = x_1 - 8x_2$ — D R^2, CD R^2; 2×2

b. $w_1 = 5x_1 - 7x_2$, $w_2 = 6x_1 + x_2$, $w_3 = 2x_1 + 3x_2$ — D R^2, CD R^3; 3×2

5. **a.** $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ — D R, CD R^2; 2×3, 3×1, 2×1

b. $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

7. **a.** $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$ — D = R^2, CD = R^2

b. $T(x_1, x_2, x_3) = (4x_1 + x_2, x_1 + x_2)$ — D = R^3, CD = R^2

Matrix Multiplication

x-axis	y-axis	y=x
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

find standard matrix

11. **a.** $w_1 = 2x_1 - 3x_2 + x_3$
$w_2 = 3x_1 + 5x_2 - x_3$

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix}$$

b. $w_1 = 7x_1 + 2x_2 - 8x_3$
$w_2 = \quad - x_2 + 5x_3$
$w_3 = 4x_1 + 7x_2 - x_3$

$$\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$$

In Exercises 27–28, the images of the standard basis vectors for R^3 are given for a linear transformation $T: R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

27. $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

Application of Network Analysis

1. The accompanying figure shows a network in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

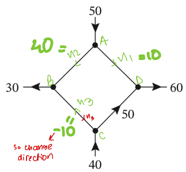


FIGURE Ex-1

Use echalon to solve

	flow in	= flow out
A	50	$= x_1 + x_2$
B	$x_2 + x_3$	$= 30$
C	40	$= x_3 + 50$
D	$x_1 + 50$	$= 60$

rearranging

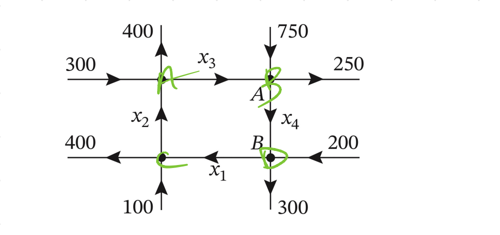
$$x_1 + x_2 = 50$$
$$x_2 + x_3 = 30$$
$$x_3 = -10$$
$$x_1 = 10$$

$$10 + x_2 = 50$$
$$x_2 = 40$$

$x_1 = 10$, $x_2 = 40$, $x_3 = -10$

Q3

ANSWERS



Flow in   Flow out

A	$300 + x_2$	=	$400 + x_3$	$x_2 - x_3 = 100$
B	$x_3 + 750$	=	$250 + x_4$	$x_3 - x_4 = -500$
C	$x_1 + 100$	=	$x_2 + 400$	$x_1 - x_2 = 300$
D	$x_4 + 200$	=	$x_1 + 300$	$x_4 - x_1 = 100$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 0 \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ -1 & 0 & 0 & 1 & 100 \end{array}\right] R_1 + R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & -1 & 0 & 1 & 400 \end{array}\right] R_2 + R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & -1 & 1 & 500 \end{array}\right] R_3 + R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 300$

$x_2 - x_3 = 100$

$x_3 - x_4 = -500$

let $x_4 = t$

$\rightarrow x_3 = -500 + t$

$x_2 = 100 - 500 + t$

$\rightarrow x_2 = -400 + t$

$x_1 = 300 - 400 + t$

$x_1 = -100 + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -100 \\ -400 \\ -500 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Determinants and Cofactors 2.1

dertermiant.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = ad - bc$$

inverse of A:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

arrow method

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}\right]$$

$$\begin{bmatrix} 1 \times 5 \times 9 \\ 2 \times 6 \times 7 \\ 3 \times -4 \times -8 \end{bmatrix} - \begin{bmatrix} 3 \times 5 \times 7 \\ 1 \times 6 \times -8 \\ 2 \times -4 \times 9 \end{bmatrix}$$

Cofactor expansion

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$$

$1[(5)(9)-(6)(-8)]-$
$2[(4)(9)-(7)(6)]+$
$3[(-4)(-8)-(5)(7)]$

Minor:

$$A_{11} = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix} \quad \begin{bmatrix} 5 & 6 \\ 4 & 8 \end{bmatrix} = \det(A) = 16$$

row column

$$M_{32} = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix} \quad \begin{bmatrix} 3 & -4 \\ 2 & 6 \end{bmatrix} = \det(M) : 26$$

determiant of triangular matrices:

lower triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 8 & 6 \end{bmatrix}$$

$\det = 1 \times 4 \times 6$
$= 24$

upper triangular

$$\begin{bmatrix} 5 & 8 & 4 \\ 0 & 4 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

$\det = 5 \times 4 \times 2$
$= 40$

Cofactor:

↳ $C_{11} = (-1)^{1+1} A_{11}$ $= 16$ → from above solution

$C_{11} = 16$

↳ $C_{32} = (-1)^{3+2} M_{32}$ $= 26$ from above solution

$C_{32} = -26$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

ANSWERS

minor and cofactor

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

(M_{23}, C_{23})

$$\begin{bmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{bmatrix}$$

$4(2-14)+1(8-14(4))+6\overset{0}{(4-4)}$

$M_{23} = -96$

$C_{23} = (-1)^{2+3} \quad -96 \Rightarrow M_{23}$

$= 96$

determiant, inverse if investible

5. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$

$d = 12 + 10$

$= 22$ hence invertible

$$A^{-1} = \frac{1}{22}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/11 & -5/22 \\ 1/11 & 3/22 \end{bmatrix}$$

det = 0, find λ

15. $A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$

$(\lambda-2)(\lambda+4)+5=0$

$\lambda^2+4\lambda-2\lambda-8+5=0$

$\lambda^2+2\lambda-3=0$

$\lambda=1,\ \lambda=-3$

can be either

2.2

EXAMPLE 2 | Proportional Rows or Columns

Each of the following matrices has two proportional rows or columns; thus, each has a determinant of zero.

$$\begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

Theorem 2.2.2

Let A be a square matrix. Then $\det(A) = \det(A^T)$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

inverse of A (A^{-1})

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

(identity matrice over the last three columns)

↳ convert A → reduced echalo

↳ perform same operation on identity matrice

↳ identity matrice left is answer

Determiant using Row Reduction

- -1 : interchanging rows
- k : taking k common
- 1 : addion/subtraction

$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{bmatrix}$ det $= -1$ → interchanging

$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{bmatrix}$ det $= (-1)3$ → taking 3 common in first row

$R_1 \sim R_1/3$

$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{bmatrix}$ det $= (-1)(3)(1)$ → subtracting

$R_3 \sim 2R_1 - R_3$

$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{bmatrix}$ det $= (-1)(3)(1)(1)$ → subtracting

$R_3 \sim 10R_2 - R_3$

$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ det $= (-1)(3)(1)(1)(-55)$ → -55 common

$R_3 \sim R_3/-55$

det $= 165$

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d=? by reduction

9. $\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

$3\begin{bmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$ $2R_1+R_2$

$3\begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix}$ row switch ✓

$3(-1)\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{bmatrix}$ $3R_2-R_3$

$3(-1)\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{bmatrix}$

$(-1)3(-11)$

33

evaluate determinant

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$

15. $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $(-1)(-1)(-6)$

$= -6$

2.3

Square matrice (n×n)

↳ det(AB) = det(A) × det(B)

↳ det(A) ≠ 0, then A is invertible

Invertible matrice inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

EXAMPLE 6 | Adjoint of a 3 × 3 Matrix

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

As noted in Example 5, the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

EXAMPLE 7 | Using the Adjoint to Find an Inverse Matrix

Use Formula (6) to find the inverse of the matrix A in Example 6.

Solution We showed in Example 5 that $\det(A) = 64$. Thus,

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

Cramers rule

$4x + 5y = 2$

$11x + y + 2z = 3$

$x + 5y + 2z = 1$

$$\begin{array}{ccc} x & y & z \end{array} \quad a$$

$$\begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$D|A| : \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix} \quad \det = 4(2-10) - 5(22-2) = 4(-8) - 5(20) = -132$$

replace x with a

$$D|A_x| : \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix} \quad \det = 2(2-10) - 5(6-2) = 2(-8) - 5(4) = -36$$

replace y with a

$$D|A_y| : \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \quad \det = 4(6-2) - 2(22-2) = 4(4) - 2(20) = -24$$

replace z with a

$$D|A_z| : \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix} \quad \det = 4(1-15) - 5(11-3) + 2(55-1) = 4(-14) - 5(8) + 2(54) = 12$$

$$x = \frac{D_y}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

$$x = \frac{-36}{-132} = \frac{3}{11} \qquad y = \frac{-24}{-132} = \frac{2}{11} \qquad z = \frac{12}{-132} = -\frac{1}{11}$$

ANSWERS

k for which matrice is invertible

invertible.

15. $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

$(k-3)(k-2) - 4 = 0$

$k^2 - 2k - 3k + 6 - 4 = 0$

$k^2 - 5k + 2 = 0$

if $k \neq : \frac{5 \pm \sqrt{17}}{2}$

then invertible

Cramers rule

26. $x - 4y + z = 6$
$4x - y + 2z = -1$
$2x + 2y - 3z = -20$

$$\begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix} \quad \begin{bmatrix} 6 \\ -1 \\ -20 \end{bmatrix}$$

$$A = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} \quad (3-4) + 4(-12-4) + (8+2) = -55$$

$$A_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} \quad 6(3-4) + 4(3+40) + (-2-20) = 144$$

$$A_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} \quad 1(3+40) - 6(-12-4) \; 1(-80+2) = 61$$

$$A_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = \quad 1(20+2) + 4(-80+2) + 6(8+2) = -230$$

$x = \frac{144}{-55}$ $\quad y = \frac{61}{-55}$ $\quad z = \frac{-230}{-55} = \frac{46}{11}$

REAL VECTOR SPACE 4.1

vector space

1. if you and v are objects in V, then $U+V$ is in V
2. $U+V=V+U$
3. $U+(V+W)=(U+V)+W$
4. $0+U=U+0=0$ 0 is a zero vector
5. $U+(-V)=0$, $(-V)+U=0$
6. KU is scaler then KU is in vector space
7. $K(U+V)=KU+KV$
8. $(K+m)U=KU+mU$
9. $K(mU)=Km(U)$
10. $1U=U$

V is a set of ordered pair of Real number

Q1. $U+V=\underbrace{(U_1+V_1, U_2+V_2)}$ R^2 form: ordered pair

$KU=(0, KU_2)$

a) compute $U+V$ & KU

$U=(-1,2)$ $V=(3,4)$, $K=3$

$U+V=(-1,2)+(3,4)$

$=(2,6)\to R^2$

zero vector space

$0+0=0$, $K0=0$

Is R^n a vector space

↳ $V_1+U_1=V$ closure under addition

↳ $KU=V$ closure under scalar multiplication

↳ $KU=(U_1,0)$ then not a vector space

c) compute $U+V$ & KU

$U=(0,4)$, $V=(1,-3)$, $K=2$

$U+V=(0+1+1, 4-3+1)\to$ formula?

$(2,2)$

d) show that $(0,0)\neq 0$ → take (e_1,e_2)

e) show that $(-1,-1)=0$

$1+?^{\to 0}=1$

$1+?^{\to 1}=1$ multiplication inverse

$(e_1,e_2)+(U_1,U_2)=(U_1,U_2)$

$e_1+U_1+1, e_2+U_2+1)=(U_1,U_2)$

$e_1=-1$ $e_2=-1$

$(e_1,e_2)=0$

$0=(-1,-1)$

f) show that $U+(-U)=0$

$-U=(U_1'+U_2')$

$(U_1,U_2)+(U_1'+U_2')=(-1,-1)$

$(U_1+U_1'+1, U_2+U_2'+1)=(-1,-1)$

$U_1'=-2-U_1$, $U_2'=-2-U_2$

$U=(0,4)$

$-U=(-2-0,-2-4)$

$-U=(-2,-6)$

SUBSPACE

Subspace

u and v vectors are in W → u+v is in W

k scalar and u vector in W → ku is in W

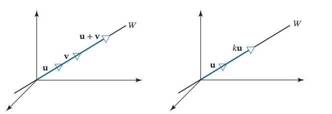
zero subspace

$W = \{0\}$

$0+0=0$, $k0=0$

Planes through origin are subspaces of R^3

lines through origin are subspaces of R^2, R^3



(a) W is closed under addition.

(b) W is closed under scalar multiplication.

Subspaces of R^2	Subspaces of R^3
• {0} • Lines through the origin • R^2	• {0} • Lines through the origin • Planes through the origin • R^3

upper triangular matrices
lower triangular matrices
diagonal matrices

$\Big\}$ subspaces of M_{nm}

not subspace

W of invertible nxn matrice

SPANNING 4.3

Q1) find wether (2,2,2) is linear combination of $v_1 = (0,-2,2)$ and $v_2 = (1,3,-1)$

$c_1 v_1 + c_2 v_2 = v$

$c_1(0,-2,2) + c_2(1,3,-1) = (2,2,2)$

$c_2 = 2$

$-2c_1 + 3c_2 = 2$

$2c_1 - c_2 = 2$

- if c_1, c_2, c_3 value comes then linear combination
- value can be 0

solve simultaneously
OR
Gausian elimination

ii) $v_1 = (2,-1,3)$ $v_2 = (4,1,2)$ $v_3(8,-1,8)$

$$\left[\begin{array}{ccc|c} 2 & 4 & 8 & a \\ -1 & 1 & -1 & b \\ 3 & 2 & 8 & c \end{array}\right]$$

since square matrice

$\det = 2(8+2) - 4(-8+3) + 8(-2-3)$

$= 20 + 20 - 40$

$= 0$

no span

Q2) Determine wether vector span R^3 or not

i) $v_1 = (2,2,2)$ $v_2 = (0,0,3)$ $v_3 = (0,1,1)$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = (a,b,c)$

$c_1(2,2,2) + c_2(0,0,3) + c_3(0,1,1) = (a,b,c)$

$2c_1 + \quad = a$

$2c_1 + c_3 = b$

$2c_1 + 3c_2 + c_3 = c$

square matrice

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & a \\ 2 & 0 & 1 & b \\ 2 & 3 & 1 & c \end{array}\right]$$

$\det = 2(-3) = -6$

hence spanning

Linear Combination

if inconsistent, no linear combination

e.g

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b+c \\ 0 & 1 & \frac{1}{2} & a-b \\ 0 & 0 & 0 & c \end{array}\right]$$

↳ since square matrice no need to make echalon
$\det(A) = 0$

then no span

$\det = ad - bc$

Steps

↳ if square matrice
- $\det = 0$ = no span

↳ make echalon
- no solution = no span

span=0 no span
span≠0 spanning

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

unique solution

consistent
span

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinite solution

consistent
span

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

no solution

not consistent
no span

Q3) suppose that

$v_1=(2,1,0,3)$ $v_2=(3,-1,5,2)$ $v_3=(-1,0,2,1)$

find wether $v=(2,3,-7,3)$ span $\{v_1,v_2,v_3\}$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right] \quad R_1 \sim R_1/2$$

not square matrice

so use echalon

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 1 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 \sim R_1-R_2 \\ \\ R_4 \sim 3R_1-R_4 \end{array}$$

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 1 \\ 0 & 5/2 & 1/2 & -2 \\ 0 & 5 & 2 & -7 \\ 0 & -5/2 & -5/2 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \sim 2/5\, R_2 \\ R_3 \sim R_3 + 2R_4 \end{array}$$

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 1 \\ 0 & 1 & \frac{1}{5} & -4/5 \\ 0 & 0 & -3 & -7 \\ 0 & & & \end{bmatrix}$$

Q4) Determine wether the following polynomial span P_2

$P_1 = 1-x+2x^2$ $P_3 = 5-x+4x^2$

$P_2 = 3+x$ $P_4 = -2-2x+2x^2$

$$\begin{bmatrix} 1 & 5 & 3 & -2 & a \\ -1 & -1 & 1 & -2 & b \\ 2 & 4 & 0 & 2 & c \end{bmatrix}$$

Q5) Determine wether matrices span M_{22}

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} k_1+k_2 & k_2+k_3 \\ k_1+k_4 & k_3+k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

square $\rightarrow$ $\det = 0$

LINEAR INDEPENDENCE 4.4

$u = (1, 2, -1)$
$v = (6, 4, -2)$
$w = (9, 2, 7)$

$w = k_1 u, k_2 v$ where u & v are linear combination
find $w = k_1 u, k_2 v$
$w = (4, -1, 8)$

$(9, 2, 7) = k_1(1, 2, -1), k_2(6, 4, 2)$

$9 = k_1 + 6k_2$, $2 = 2k_1 + 4k_2$, $7 = -k_1 + 2k_2$

$w = (4, -1, 8)$

Solve simultaneously / echalon and equate to 0
Trivial : $k_1, k_2, k_3 = 0$ → linearly independent
Non trivial: $k_1, k_2, k_3 = s, t$ → linearly dependent

if square matrice
$\det = 0$ = non trivial → dependent

Q1) find wether $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ linear combonation of $A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$

$$k_1\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + k_2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_3\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

$4k_1 + k_2 = 6$
$-k_2 + 2k_3 = -8$
$-2k_1 + 2k_2 + k_3 = -1$
$-2k_1 + 3k_2 + 4k_3 = -8$

- $\boxed{k_2 = 6 - 4k_1}$
- $6 - 4k_1 + 2k_3 = -8$
  $2k_3 = -14 + 4k_1$
  $\boxed{k_3 = -7 + 2k_1}$
- $-2k_1 + 12 - 8k_1 - 7 + 2k_1 = -1$

- $k_2 = 6$
- $k_3 = -7$
- $k_1 = 0$

- $0 + 18 - 28 = -8$
  $0 = 2$

$1, 2, -3$

Theorem 4.4.3

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

→ no of eqs
↳ no of unknowns = c_1, c_2, c_3 or vectors

linearly independent = don't lie in plane
linearly dependent = lie in plane

Polynomial = $P(n) = a_0 + a_1 n + a_2 n^2 \ldots$

$P^2(n) = a_0 + a_1 n + a_2 n^2 \ldots$

Q2) find wether $P(n) = 1+n$ linear combonation of $P_1 = 2 + n + n^2$, $P_2 = 1 - n^2$, $P_3 = 1 + 2n$

$(1,1,0)$

$k_1 P_1 + k_2 P_2 + k_3 P_3 = P$

$\underbrace{(2,1,1) \quad (1,0,-1) \quad (1,2,0)}_{\text{converting to vectors}}$

Q4) determine wether the 3 vectors lie on the plane in R^3

$V_1 = (2,-2,0)$
$V_2 = (6,1,4)$
$V_3 = (2,0,-4)$

if all linearly dependent

Q) show that ⇓ linearly dependendent set in R^4

$V_1 = (1,2,3,4)$
$V_2 = (0,1,0,-1)$
$V_3 = (1,3,3,3)$

Q) for what value of λ following vector for $\overbrace{\text{linearly dependent}}^{|A|=0}$ set in R^3

$V_1 = (\lambda, -\frac{1}{2}, -\frac{1}{2})$
$V_2 = (-\frac{1}{2}, \lambda, -\frac{1}{2})$
$V_3 = (-\frac{1}{2}, -\frac{1}{2}, \lambda)$

NOT COMING IN PAPER (I THINK?)

EXAMPLE 8 | Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

This function is not identically zero on the interval $(-\infty, \infty)$ since, for example,

$$W\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = -1$$

Thus, the functions are linearly independent.

EXAMPLE 9 | Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = 1, \mathbf{f}_2 = e^x$, and $\mathbf{f}_3 = e^{2x}$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 2e^{3x}$$

This function is obviously not identically zero on $(-\infty, \infty)$, so $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ form a linearly independent set.

COORDINATION AND BASIS 4.5

consistent

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

unique solution

linear combination

column space

span

consistent

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinite solution

linear combination

column space

span

not consistent

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

no solution

no linear combination

no column space

no span

COORDINATE VECTOR
↳ Solve simultaneously/echalon
↳ find values

$$\begin{bmatrix} \downarrow & \downarrow & \downarrow \end{bmatrix}$$

for SQUARE MATRICE

$\det(A) = 0$	$\det(A) \neq 0$
no span	span
non trivial	trivial
dependent	independent

Basis

↳ linearly independent
↳ span

after echalon

$k_1, k_2, k_3 = s, t$	$k_1, k_2, k_3 = 0$
non trivial	Trivial
dependent	independent
no basis	form basis

Shortcut

linearly independent · spanning · coordinate vector

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 0 & b_1 & 5 \\ 2 & 9 & 3 & 0 & b_2 & -1 \\ 1 & 0 & 4 & 0 & b_3 & 9 \end{array}\right]$$

Show that vector

$v_1=(1,2,1)$
$v_2=(2,9,0)$
$v_3=(3,3,4)$ } formula basis for R^3

find the coordinate vector of $v=(5,-1,9)$ relative the basis $S=\{v_1,v_2,v_3\}$

for linearly independent

$c_1v_1+c_2v_2+c_3v_3=0$

$c_1(1,2,1)+c_2(2,9,0)+c_3(3,3,4)=0$

$c_1+2c_2+3c_3=0$
$2c_1+9c_2+3c_3=0$
$c_1+4c_3=0$

$$\left[\begin{array}{ccc|c}1&2&3&0\\2&9&3&0\\1&0&4&0\end{array}\right]\begin{array}{l}\\R_2\sim R_2-2R_1\\R_3\sim R_3-R_1\end{array}$$

$$\left[\begin{array}{ccc|c}1&2&3&0\\0&5&-3&0\\0&-2&1&0\end{array}\right]\quad R_2\sim \tfrac{1}{5}R_2$$

$$\left[\begin{array}{ccc|c}1&2&3&0\\0&1&-3/5&0\\0&-2&1&0\end{array}\right]\quad R_3\sim R_3+2R_2$$

$$\left[\begin{array}{ccc|c}1&2&3&0\\0&1&-3/5&0\\0&0&1&0\end{array}\right]\to\begin{bmatrix}1&2&3\\0&1&-3/5\\0&0&-1/5\end{bmatrix}$$

$-3/5\,c_3=0$
$c_2=0$
$c_3=0$
$c_1=0$

linearly independent

for spanning

$c_1v_1+c_2v_2+c_3v_3=b$

$c_1(1,2,1)+c_2(2,9,0)+c_3(3,3,4)=(b_1,b_2,b_3)$

$c_1+2c_2+3c_3=b_1$
$2c_1+9c_2+3c_3=b_2$
$c_1+4c_3=b_3$

$$\begin{bmatrix}1&2&3&b_1\\2&9&3&b_2\\1&0&4&b_3\end{bmatrix}$$

det $=-1$

spanning

for coordinate

$c_1v_1+c_2v_2+c_3v_3=(5,-1,9)$

$c_1(1,2,1)+c_2(2,9,0)+c_3(3,3,4)=(5,-1,9)$

Shortcut

$$\begin{array}{cccccc}&&&\text{linearly independent}&\text{spanning}&\text{coordinate vector}\\\end{array}$$
$$\begin{bmatrix}1&2&3&0&b_1&5\\2&9&3&0&b_2&-1\\1&0&4&0&b_3&9\end{bmatrix}$$

Q

$$A = \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & 4 & 5 & 8 & -4 \end{array}\right] \begin{array}{l} \\ R_2 \sim 3R_1 + R_2 \\ R_3 \sim 2R_1 - R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & -8 & -1 & -2 & 2 \end{array}\right]$$

switch rows

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ 0 & -8 & -1 & -2 & 2 \\ 0 & 0 & 5 & 10 & -10 \end{array}\right] \begin{array}{l} \\ R_2 \sim R_2/-8 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ 0 & 1 & \frac{1}{8} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 5 & 10 & -10 \end{array}\right] \begin{array}{l} \\ \\ R_3 \sim R_3/5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ 0 & 1 & \frac{1}{8} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & 2 & -2 \end{array}\right]$$

$$x_1 - 2x_2 + 2x_3 + 3x_4 = -1$$

$$x_2 + \frac{1}{8}x_3 + \frac{1}{4}x_4 = -\frac{1}{4}$$

$$x_3 + 2x_4 = -2$$

let $x_4 = t$

$x_3 = -2 - 2t$

$$x_2 = \cancel{-\frac{1}{4}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}t} - \cancel{\frac{1}{4}t}$$

$x_2 = 0$

$x_1 = -1 + 4 + 4t - 3t$

$x_1 = 3 + t$

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

rank = 3

nulity = 4 - 3

= 1

Dimentions = 2

			(a)	(b)	(c)	(d)	(e)	(f)	(g)
	Size of A:	$m \times n$	3×3	3×3	3×3	5×9	5×9	4×4	6×2
	rank(A)	$= r$	3	2	1	2	2	0	2
	rank($A \mid \mathbf{b}$)	$= s$	3	3	1	2	3	0	2
(i)	dimension of the row space of A	$= r$	3	2	1	2	2	0	2
	dimension of the column space of A	$= r$	3	2	1	2	2	0	2
	dimension of the null space of A	$= n - r$	0	1	2	7	7	4	0
	dimension of the null space of A^T	$= m - r$	0	1	2	3	3	4	4
(ii)	is the system $A\mathbf{x} = \mathbf{b}$ consistent?	Is $r = s$?	Yes	No	Yes	Yes	No	Yes	Yes
(iii)	number of parameters in the general solution of $A\mathbf{x} = \mathbf{b}$	$= n - r$ if consistent	0	-	2	7	-	4	0

ANSWERS

$$\begin{matrix} 2 & 3 \\ 1 & 0 \end{matrix}$$

det = 0 − 3 = −3 span

hence independent trivial

Find the coordinate vector of **w** relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.

a. $\mathbf{u}_1 = (2, -4)$, $\mathbf{u}_2 = (3, 8)$; $\mathbf{w} = (1, 1)$

$$\left[\begin{matrix} 2 & 3 \\ -4 & 8 \end{matrix} \quad \begin{matrix} 1 \\ 1 \end{matrix}\right] \quad /2$$

$$\left[\begin{matrix} 1 & 3/2 \\ -4 & 8 \end{matrix} \quad \begin{matrix} 1/2 \\ 1 \end{matrix}\right] \quad 4R_1 + R_2$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 14 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/14 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/14 \end{bmatrix}$$

↓ coordinate vector

Show that the following polynomials form a basis for P_3.

$$1 + x, \quad 1 - x, \quad 1 - x^2, \quad 1 - x^3$$

$$\begin{array}{c} a \\ u_1 \\ u_2 \\ u_3 \end{array} \left[\begin{array}{cccc|c|c} 1 & 1 & 1 & 1 & 0 & a \\ 1 & -1 & 0 & 0 & 0 & b \\ 0 & 0 & -1 & 0 & 0 & c \\ 0 & 0 & 0 & -1 & 0 & a \end{array}\right]$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↳ span

↳ all 0 so independent

↳ so forms basis

DIMENTIONS/RANK & NULITY 4.6/4.9

Q) $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$

max rank

matrice of $m \times n$

↳ $m \geq n$ rank = n nullity = 0

↳ $m < n$ rank = m nullity = n-m

$n \overset{+}{a}\ r = c$

Rank + nulity = column

- no of non zero rows after reduced echalon
- nulity: no of zero rows
- column: no of unknowns

Rank: no of leading variables

nullity: no of parameters

augmented

column: no of unknowns

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 \\ 3 & 6 & -3 & 3 & 0 \\ 4 & 8 & -4 & 4 & 0 \end{bmatrix} \quad \begin{matrix} \\ R_2 \sim 2R_1 - R_2 \\ R_3 \sim 3R_1 - R_3 \\ R_4 \sim 4R_1 - R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$u_1 + 2u_2 - u_3 + u_4 = 0$

$u_2 = r$, $u_3 = s$, $u_4 = t$

$u_1 = -2r + s - t$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = r \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + s \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + t \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

rank = 1

nulity = column - rank

= 4 - 1

= 3

$$\left.\begin{matrix} v_1 = (-2, 1, 0, 0) \\ v_2 = (1, 0, 1, 0) \\ v_3 = (-1, 0, 0, 1) \end{matrix}\right\} \text{Basis}$$

Dimention = 3

Trivial Solution: $u_1, u_2, u_3 = 0$

Non Trivial solution: $\underbrace{\text{free variables}}_{\text{nulity}}$: s, t, r

BASIS / DIMENTIONS

↳ make echalon

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{1^{st}\text{ vector}} + s \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{2^{nd}\text{ Vector}}$$

Basis → are vectors

← Dimentions = 2

Trivial: $k_1, k_2, k_3 = 0$ → linearly independent → basis

Non trivial: $k_1, k_2, k_3 = s, t$ → linearly dependent → no basis

4.6 ANSWERS

find basis

1.
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ 2R_1+R_2 \\ R_1+R_3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_2-R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis vector

1. linearly Independent
2. span

$\downarrow$

, ,

$c_1v_1 + c_2v_2 + c_3v_3 = b$

$v = (c_1, c_2, c_3) \rightarrow$ Cordinate Vector

$c_1 [\ \ ] + c_2 [\ \ ] + c_3 [\ \ ] = [\ \ ]$

4.7

Change of Basis

Q) $B = \{u_1, u_2\}$ $B' = \{u_1', u_2'\}$

$u_1 = (1,0)$ $u_1' = (1,1)$
$u_2 = (0,1)$ $u_2' = (2,1)$

Q) find transition matrice from B to B'

$$\left[\ \underset{\text{end}}{\text{new Basis}}\ \middle|\ \underset{\text{start}}{\text{old basis}}\ \right] \rightarrow \left[\ I \,/\, \text{transition matrice}\ \right]$$

$$\begin{array}{c} \begin{matrix} u_1' & u_2' & u_1 & u_2 \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

row operation (reduced echalon)

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$P_{B \rightarrow B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$c_1 =$ ____ & $c_2 =$ ____

Q2) find transition matrix from B' to B

$$\begin{array}{c} \begin{matrix} u_1 & u_2 & u_1' & u_2' \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

row echalon reduced

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$P_{B' \rightarrow B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

3 by 3 matrix

$$\begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 & u_1' & u_2' & u_3' \end{matrix} \\ \left[\begin{array}{ccc|ccc} -3 & -3 & 1 & -6 & -2 & -2 \\ 0 & 2 & 6 & -6 & -6 & -3 \\ 3 & -1 & -1 & 0 & 4 & 7 \end{array}\right] \end{array}$$

$$\begin{bmatrix} -3 & -3 & 1 & -6 & -2 & -2 \\ 0 & 2 & 6 & -6 & -6 & -3 \\ 0 & 2 & -2 & 6 & 6 & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 \sim R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} -3 & -3 & 1 & -6 & -2 & -2 \\ 0 & 2 & 6 & -6 & -6 & -3 \\ 0 & 0 & -8 & 12 & 12 & 12 \end{bmatrix} \begin{matrix} \\ R_2 \sim \frac{1}{6} R_2 \\ R_3 \sim \frac{1}{8} R_3 \end{matrix}$$

$$\begin{bmatrix} -3 & -3 & 1 & -6 & -2 & -2 \\ 0 & \frac{1}{3} & 1 & -1 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{12}{8} & -\frac{12}{8} & -\frac{12}{8} \end{bmatrix}$$

not complete

ANSWERS

1. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

a. Find the transition matrix from B' to B.

b. Find the transition matrix from B to B'.

c. Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (11) to compute $[\mathbf{w}]_{B'}$.

d. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

4.

5.

Repeat the directions of Exercise 1 with the same vector **w** but

$$\begin{array}{cccc} u & u & u' & u' \\ \end{array}$$
$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right] \; /2$$

$$\left[\begin{array}{cccc} 1 & 2 & \frac{1}{2} & -\frac{1}{2} \\ 2 & -1 & 3 & -1 \end{array}\right] \; 2R_1 - R_2$$

$$\left[\begin{array}{cccc} 1 & 2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 5 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{2}{5} & 0 \end{array}\right] \; 2R_2 - R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3/10 & 1/2 \\ 0 & 1 & 2/5 & 0 \end{array}\right]$$

transition matrice

4.8 ROW SPACE / COLUMN SPACE / NULL SPACE (BASIS)

1) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

$Ax = 0$

augmented matrice

$\left[\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right]$ $R_2 \sim R_2 - 5R_1$, $R_3 \sim R_3 - 7R_1$

$\left[\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 1 & -19 & 0 \end{array}\right]$ $R_3 \sim R_3 - R_2$

$\left[\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 \sim R_1 + R_2$

make echalon reduced

$\left[\begin{array}{cccc} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - 16x_3 = 0$

$x_2 - 19x_3 = 0$

let $x_3 = t$

$x_2 = 19t$

$x_1 = 16t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \underbrace{\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}}_{\text{basis for null space}}$

Basis for row space (non zero rows) echalon form

$V_1 = (1, 0, -16)$

$V_2 = (0, 1, 19)$

Basis for column space (which contains 1 in column) echalon form

$V_1 = (1, 5, 7)$

$V_2 = (-1, -4, -6)$

column space
↳ echalon reduced
↳ consistent ✓
↳ not consistent ✗

row vectors

$r_1 = 1, -1, 3$

$r_2 = 5, -4, -4$

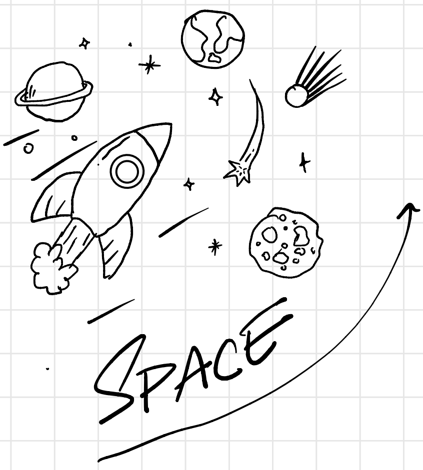
$r_3 = 7, -6, 2$

Column vectors

$c_1 = 1, 5, 7$

$c_2 = -1, -4, -6$

$c_3 = 3, -4, 2$



ANSWERS

is b in column space of A

tors of A.

3. a. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 3 & 2 \end{bmatrix} \quad 2R_1 - R_3$$

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

inconsistent
no column space

In Exercises **11–12**, *a matrix in row echelon form is given. By inspection, find a basis for the row space and for the column space of that matrix.*

11. a. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

row space	column space
(1, 0, 2)	(1, 0, 0)
(0, 0, 1)	(2, 1, 0)

EIGEN values and EIGEN vectors

→ WILL DEFINITELY COME IN PAPER

5.1

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

↳ $\lambda I - A$
↳ $\det = 0$
↳ eigen values insert
↳ equate to 0
↳ eigen vectors

made homogenous system

$(\lambda I - A)x = 0$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det|\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix}$$

$= \lambda\left((\lambda-2)(\lambda-3)\right) + 2(\lambda-2)$

$= (\lambda-2)\left(\lambda(\lambda-3)+2\right) = 0$

$\lambda = 2, \quad \lambda^2 - 3\lambda + 2 = 0$

$\lambda = 2, \lambda = 1$

EIGAN VALUES

for $\lambda = 1$

augmented

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \sim R_2 + R_1 \\ R_3 \sim R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \sim -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

DIAGNASATION

- if 3x3 matrice: 3 Eigan vector
- if 2x2 matrice: 2 Eigan vector

for $\lambda = 2$

augmented

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} R_1 \sim R_1/2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \sim R_1 + R_2 \\ R_3 \sim R_1 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$u_1 + u_3 = 0$

$u_2 = u \qquad u_3 = s$

$u_1 = -s$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = s\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Eigan Vector 2}} + u\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{Eigan Vector 3}}$$

$u_1 + 2u_3 = 0$

$u_2 + u_3 = 0$

let $u_3 = t$

$u_2 = -t$

$u_1 = -2t$

as u^3: 0 0 0 0

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = t\underbrace{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}}_{\text{Eigan vector 1}}$$

if 1 free variable (s,u,t) : eigan vector

if 2 free variable : eigan vectors

Q) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix}$

$\det |\lambda I - A| = 0$ → keep det 0

$\lambda-4((\lambda-1)^2 - 0) - 1(2(\lambda-1)$ take common $\lambda-1$

$(\lambda-1)[(\lambda-4)(\lambda-1) - 2]$

$\lambda-1[\lambda^2 - 4\lambda - 4\lambda + 4 - 2]$

$(\lambda-1)[\lambda^2 - 8\lambda + 2] = 0$

$\underbrace{\lambda = 1, \lambda = 0.43, \lambda = 4.56}$

EIGEN VALUES

Q) $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ → flip signs

sub lamda from diagonal

$\lambda I - A = \begin{bmatrix} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{bmatrix}$

$\det \lambda I - A = \lambda-6((\lambda+2)(\lambda+3)) - 8(\lambda+2)$

$\lambda+2((\lambda-6)(\lambda+3) - 8) = 0$

$\lambda+2(\lambda^2 + 3\lambda - 6\lambda - 18 - 8) = 0$

$\lambda = -2, \lambda^2 - 3\lambda - 26 = 0$

$\lambda = -2, \lambda = -3.8, \lambda = 6.8$

Theorem 5.1.4

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

ishma notes

DIAGONISATION 5.2

Diagonalization

- $\lambda I - A$
- det = 0
- eigen values insert
- equate to 0
- echalon
- eigen vectors
- Diagnisable → solve further if true
  - if 3x3 matrice : 3 Eigan vector
  - if 2x2 matrice : 2 Eigan vector
- P: [E.vector1 E.vector2 ...]
- $D = P^{-1}AP$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- if A is nxn then diagonizable

Similar Matrices

- determiants are same
- ranks are same (after echalon)
- B: $P^{-1}AP$

inverse A^{-1}

2x2 matrice

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

det A = (ad - bc)

3x3 or more

$$\begin{array}{cc} A & I \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$$

- make this equal to I
- this will be A^{-1}

Power of Matrix

- steps for diagonisation
- get P, D
- $A^n = P D^n P^{-1}$

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues.
Eigenspace dimension	If λ is an eigenvalue of A (and hence of $P^{-1}AP$) then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

- , vector, matrice

vectors can be row wise 1
vectors can be column wise 1
→ Transpose 2
→ NULL Vector 2

6

Q1) find E.value, E.vector and wether matrice is diaganisable or not

$$A=\begin{bmatrix}6 & 3 & -8\\ 0 & -2 & 0\\ 1 & 0 & -3\end{bmatrix} \longrightarrow D.\begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}$$

$$\begin{bmatrix}\lambda-6 & -3 & 8\\ 0 & \lambda+2 & 0\\ -1 & 0 & \lambda+3\end{bmatrix}$$

$\det = (\lambda-6)(\lambda+2)(\lambda+3)+8(\lambda+2)$

$(\lambda+2)((\lambda-6)(\lambda+3)+8)=0$

$\lambda=-2$, $\quad \lambda^2+3\lambda-6\lambda-18+8$

$\lambda^2-3\lambda-10=0$

same so wont count

$\lambda=5$, $\lambda=-2$

for $\lambda=5$

$$\left[\begin{array}{ccc|c}-1 & -3 & 8 & 0\\ 0 & 7 & 0 & 0\\ -1 & 0 & 8 & 0\end{array}\right] \begin{array}{l}R_1\sim -R_1\\ \text{after}\\ R_3\sim R_1+R_3\end{array}$$

$$\left[\begin{array}{ccc|c}1 & 3 & -8 & 0\\ 0 & 7 & 0 & 0\\ 0 & 3 & 0 & 0\end{array}\right] R_2\sim R_2/7$$

$$\left[\begin{array}{ccc|c}1 & 3 & -8 & 0\\ 0 & 1 & 0 & 0\\ 0 & 3 & 0 & 0\end{array}\right] R_3\sim 3R_2-R_3$$

$$\left[\begin{array}{ccc|c}1 & 3 & -8 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$u_1+3u_2-8u_3=0$

$u_2=0$

let $u_3=u$

$u_1=8u$

$$\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=u\begin{bmatrix}8\\ 0\\ 1\end{bmatrix}$$

E. Vector 1

for $\lambda=-2$

$$\left[\begin{array}{ccc|c}-8 & -3 & 8 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\end{array}\right] R_1\sim R_1-9R_3$$

$$\left[\begin{array}{ccc|c}1 & -3 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\end{array}\right] R_3\sim R_1+R_3$$

$$\left[\begin{array}{ccc|c}1 & -3 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & -3 & 0 & 0\end{array}\right] R_3\sim R_3-R_1$$

$$\left[\begin{array}{ccc|c}1 & -3 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

$u_1-3u_2-u_3=0$

$u_3=0$

let $u_2=t$

$u_1=3t$

$$\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=t\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}$$

E. Vector 2

not diagnaizable

as 3x3 matrice needs 3 E. vectors

BOOK QUESTIONS

Q. are A and B similar

2) $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$

$d = 16 + 2$
$= 18$

$B: 16 - 2$
$= 14$

not similar

3) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow R$

$d = 1(1)$
$= 1$

$d = 1(1-1)$
$d = 0$

not similar

Q. find matrix P that diagonises A then check by $D = P^{-1}AP$

5) $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{bmatrix}$

$\det = (\lambda - 1)(\lambda + 1) - 0 = 0$

$\lambda = 1 \quad \lambda = -1$

$\lambda = 1$

$\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix}$

$\begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix} - R_1/6$

$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$

$u_1 - \frac{1}{3} u_2 = 0$

let $u_2 = u$

$u_1 = \frac{1}{3} u$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

v_1

$\lambda = -1$

$\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix} - R_1/2$

$\begin{bmatrix} 1 & 0 \\ -6 & 0 \end{bmatrix} \quad 6R_1 + R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$u_1 = 0$

$u_2 = t$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

v_2

diagonizable

$$D = \overset{P^{-1}}{\begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}} \overset{P}{\begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$P = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$

$P^{-1} = \dfrac{1}{\frac{1}{3}} \begin{bmatrix} 1 & 0 \\ -1 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix}$

matrix A^{10}.

17. $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

$$(\lambda I - A) = \begin{bmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{bmatrix}$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2 \quad , \quad \lambda = -3$$

$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} /2 \qquad \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 \\ -2 & 3 \end{bmatrix} 2R_1 + R_2 \qquad \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} /{-3}$$

$$\begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$u_1 = \frac{3}{2}t$ $\qquad$ $u_1 = -t$

$\qquad\qquad\qquad\quad$ $u_2 = t$

$u_2 = t$

$$P = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{\frac{3}{2}+1} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$= \frac{2}{5} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

$$D^{10} = \begin{bmatrix} 2^{10} & 0 \\ 0 & -3^{10} \end{bmatrix}$$

$$\begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & -3^{10} \end{bmatrix} \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

multiply

DYNAMICAL SYSTEM & MAKOV CHAIN 5.5

Steady state vector

↳ $(I-P)q=0$

↳ Echalon

↳ equate probabilities to 1

↳ find free variable and multiply

Stochastic Matrix

↳ sum of each column = 1

Q3) fill in the missing entries of stochastic Matrix

sum of each column = 1

$$\begin{bmatrix} 7/10 & \frac{1}{10} & 1/5 \\ 2/10 & 3/10 & 5/10 \\ 1/10 & 3/5 & 3/10 \end{bmatrix}$$

$\downarrow$ sum=1 $\quad \downarrow$ sum=1 $\quad \downarrow$ sum=1

$7/10 + n + \frac{1}{10} = 1$

$n + \frac{3}{10} + 3/5 = 1$

$\frac{1}{5} + n + 3/10 = 1$

Q) find steady state vector

$$P=\begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix}$$ clovm wise sum = 1

$(I-P)q=0$

I − P q = 0

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0.8&0.1\\0.2&0.9\end{bmatrix}\begin{bmatrix}q_1\\q_2\end{bmatrix}=0$$

$$\begin{bmatrix}0.2&-0.1\\-0.2&0.1\end{bmatrix}\begin{bmatrix}q_1\\q_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0.2&-0.1&0\\-0.2&0.1&0\end{bmatrix} R_1\sim\tfrac{10}{2}R_1$$

$$\begin{bmatrix}1&-\tfrac{1}{2}&0\\-\tfrac{2}{10}&\tfrac{1}{10}&0\end{bmatrix} R_2\sim\tfrac{2}{10}R_1+R_2$$

$$\begin{bmatrix}1&-\tfrac{1}{2}&0\\0&0&0\end{bmatrix}$$

$u_1-\frac{1}{2}u_2=0$

$u_2=t$

$u_1=\frac{1}{2}t$

$$\begin{bmatrix}u_1\\u_2\end{bmatrix}=t\begin{bmatrix}\tfrac{1}{2}\\1\end{bmatrix}$$ sum of these probabilities = 1

$\frac{1}{2}t+1t=1$

$\frac{3}{2}t=1$

$t=\frac{2}{3}$

$$\begin{bmatrix}u_1\\u_2\end{bmatrix}=\frac{2}{3}\begin{bmatrix}\tfrac{1}{2}\\1\end{bmatrix}$$

$$\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\tfrac{1}{3}\\\tfrac{2}{3}\end{bmatrix}$$

Q) find steady state vector

$$P=\begin{bmatrix}0.5&0.4&0.6\\0.2&0.2&0.3\\0.3&0.4&0.1\end{bmatrix}$$

$(I-P)q=0$

$$10\times\begin{bmatrix}0.5&-0.4&-0.6&0\\-0.2&0.8&-0.3&0\\-0.3&-0.4&0.9&0\end{bmatrix}$$

$$10\times\begin{bmatrix}5&-4&-6\\-2&8&-3\\-3&-4&9\end{bmatrix} R_1/5$$

$$\begin{bmatrix}1&-\tfrac{4}{5}&-\tfrac{6}{5}\\-2&8&-3\\-3&-4&9\end{bmatrix}\begin{matrix}\\2R_1+R_2\\3R_1+R_3\end{matrix}$$

$$\begin{bmatrix}1&-\tfrac{4}{5}&-\tfrac{6}{5}\\0&\tfrac{32}{5}&-\tfrac{27}{5}\\0&-\tfrac{32}{5}&\tfrac{27}{5}\end{bmatrix} R_2\sim\tfrac{5}{32}R_2$$

$$\begin{bmatrix}1&-\tfrac{4}{5}&-\tfrac{6}{5}\\0&1&-\tfrac{27}{32}\\0&0&0\end{bmatrix} R_3\sim\tfrac{32}{5}R_2+R_3$$

$u_1-\frac{4}{5}u_2-\frac{6}{5}u_3=0$

$u_2-\frac{27}{32}u_3=0$

$u_3=t$

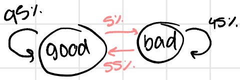
$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=t\begin{bmatrix}\tfrac{15}{8}\\\tfrac{27}{32}\\1\end{bmatrix}$$

$\frac{15}{8}t+\frac{27}{32}t+1t=1$

$\frac{119}{32}t=1$

$t=\frac{32}{119}$

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}\tfrac{60}{119}\\\tfrac{27}{119}\\\tfrac{32}{119}\end{bmatrix}$$

Q13 

transition matrice

b) $x_1 = \begin{array}{cc} & \\ g & b \end{array}$

$$x_1 = \begin{array}{c c} \begin{matrix} g & \quad b \end{matrix} & \\ \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} & \begin{matrix} g \\ b \end{matrix} \end{array} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} n_1 \\ n_2 \end{matrix} = \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.93 \\ 0.07 \end{bmatrix}$$

c) $$x_1 = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 0.77 \\ 0.23 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \begin{bmatrix} 0.77 \\ 0.23 \end{bmatrix} = \begin{bmatrix} 0.858 \\ 0.142 \end{bmatrix}$$

d) $$\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.37 \end{bmatrix}$$

hence 0.63 it will be good tomorrow

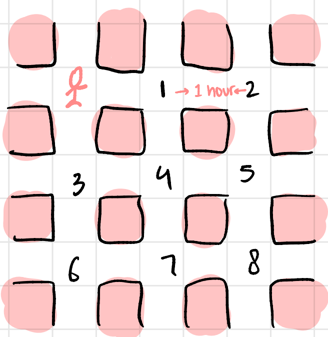
if 5 100 %

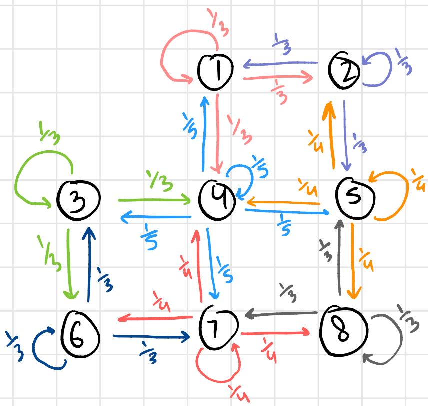
$$
\begin{array}{c|cccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\
\hline
 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 1 \\
 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 2 \\
 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & \frac{1}{3} & 0 & 0 & 3 \\
 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{5} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 4 \\
 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \frac{1}{4} & 0 & 0 & \frac{1}{3} & 5 \\
 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & 0 & 6 \\
 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & 7 \\
 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{3} & 8
\end{array}
\qquad
u_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\qquad
\begin{matrix} 0.0 \\ \boxed{0.25} \\ 0.0 \\ 0.25 \\ 1.25 \\ 0.0 \\ 0.0 \\ 0.25 \end{matrix}
$$

$u_1 = P u_0$ → first hour

$u_2 = P u_1$ → second hour (multiplid second time)

$u_3 = P u_2$ → ...





1 connection $\frac{1}{2}$

2 connection $\frac{1}{3}$

3 connection $\frac{1}{4}$

⋮

INNER PRODUCT 6.1

Real inner product space

$\langle u,v \rangle = \langle v,u \rangle$ Symetry axiom

$\langle u+v,w \rangle = \langle u,w \rangle + \langle v,w \rangle$ Additivity axiom

$\langle ku,v \rangle = k\langle u,v \rangle$ Homogeneity axiom

$\langle v,v \rangle \geq 0$ Positivity axiom

↳ $\langle v,v \rangle = 0$ only if $v = 0$

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 \tag{3}$$

satisfies the four inner product axioms.

Solution

Axiom 1: Interchanging $\mathbf{u}$ and $\mathbf{v}$ in Formula (3) does not change the sum on the right side, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

Axiom 2: If $\mathbf{w} = (w_1, w_2)$, then

$$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle &= 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2 \\ &= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2) \\ &= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2) \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

Axiom 3: $\begin{aligned}\langle k\mathbf{u}, \mathbf{v} \rangle &= 3(ku_1)v_1 + 2(ku_2)v_2 \\ &= k(3u_1v_1 + 2u_2v_2) \\ &= k\langle \mathbf{u}, \mathbf{v} \rangle\end{aligned}$

Axiom 4: Observe that $\langle \mathbf{v}, \mathbf{v} \rangle = 3(v_1v_1) + 2(v_2v_2) = 3v_1^2 + 2v_2^2 \geq 0$ with equality if and only if $v_1 = v_2 = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$.

Real inner product space and k scalar

↳ $\|v\| \geq 0$

↳ $\|kv\| = |k|\,\|v\|$

↳ $d(u,v) = d(v,u)$

↳ $d(u,v) \geq 0$

↳ if $u = v$

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

$\langle u.v \rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ OR $\langle f\ g \rangle = \int_a^b f(u)\, g(u)\, du$

$d(u,v) = \|u-v\| = \sqrt{(u-v).(u-v)}$

↳ distance

$\|u\| = \sqrt{\langle u,u \rangle} = \sqrt{a^2+b^2+c^2}$

↳ norm

BOOK QUESTIONS

Q1) $\langle U,V\rangle = 2U_1V_1 + 3U_2V_2$

$u=(1,1)$ $V=(3,2)$ $W=(0,-1)$, $k=3$

a) $\langle U,V\rangle$

$2(3.1)+3(1.2)$

12

e) $d(U,V)$

$\|U-V\|$

$U-V=(-2,-1)$

$\sqrt{\langle U-V, U-V\rangle}$

$\sqrt{2(-2)(-2)+3(-1)(-1)}$

$8+3$

$\sqrt{11}$

b) $\langle kV, W\rangle$

$KV = 3(3,2)$

$=(9,6)$

$\langle KV, W\rangle = 2(9.0)+3(6.-1)$

$=-18$

c) $\langle U+V, W\rangle$

$U+V=(4,3)$

$\langle U+U, W\rangle = (2.4.0)+(3.3.-1)$

$=-9$

f) $\langle U-kV\rangle$

$KV=(9,6)$

$U-KV=(-8,-5)$

$\sqrt{\langle U-KV\rangle . \langle U-KV\rangle}$

$\sqrt{2(-8)(-8)+3(-5)(-5)}$

$\sqrt{128+75}$

$\sqrt{203}$

d) $\|V\|$

$\sqrt{\langle V.V\rangle}$

$\sqrt{(2.3.3)+(3.2.2)}$

$2.V_1.V_1 + 3.V_2.V_2$

$18+12$

30

$\langle U,U)$

$2(1)(3)+3(1)(2)$

$6+6$

12

$\langle KV, W\rangle$

$KV = 9,6$

$\langle U-KV\rangle$

$U-KV = -8,-5$

$\sqrt{\langle U\rangle . \langle U\rangle}$

$2(-8)^2 + (-5)^2 \times 3$

$\sqrt{203}$

COSINE/ORTHOGONALITY 6.2

cosine angle

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\cdot\|v\|}$$

orthogonality

$$\langle u, v\rangle = 0$$

inner product

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3$$

$$\|u\| = \sqrt{a^2+b^2+c^2}$$

norm

$$\langle v \cdot u\rangle = \square v_1u_1 + \square v_2u_2$$

distance

$$d(u,v) = \|u-v\|$$

norm

$$\|v\| = \sqrt{a^2+b^2+c^2}$$

if norm=1 then its called a unit vector

unit vector

$$\hat{a} = \frac{\vec{v}}{\|v\|}$$

$\vec{a}\cdot\vec{b}$ = scalar — dot product

$\langle\vec{a}\cdot\vec{b}\rangle$ = scalar — inner product

$\vec{a}\times\vec{b}$ = vector

$v = (v_1, v_2)$ $u = (u_1, u_2)$

$v \cdot u = v_1u_1 + v_2u_2$

$\langle v \cdot u\rangle = \square v_1u_1 + \square v_2v_2)$

constant: weighted inner product

1: dot product

e.g $\langle v \cdot u\rangle = 3v_1u_1 + 2v_2u_2$

EXAMPLE 4 | Orthogonal Vectors in P_2

Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q}\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p}\rangle^{1/2} = \left[\int_{-1}^{1} xx\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^2\,dx\right]^{1/2} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q}\rangle^{1/2} = \left[\int_{-1}^{1} x^2x^2\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^4\,dx\right]^{1/2} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q}\rangle = \int_{-1}^{1} xx^2\,dx = \int_{-1}^{1} x^3\,dx = 0$$

Because $\langle \mathbf{p}, \mathbf{q}\rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to the given integral inner product.

EXAMPLE 5 | Theorem of Pythagoras in P_2

In Example 4 we showed that $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to the inner product

$$\langle \mathbf{p}, \mathbf{q}\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

on P_2. It follows from Theorem 6.2.3 that

$$\|\mathbf{p}+\mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$$

Thus, from the computations in Example 4, we have

$$\|\mathbf{p}+\mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

We can check this result by direct integration:

$$\|\mathbf{p}+\mathbf{q}\|^2 = \langle \mathbf{p}+\mathbf{q}, \mathbf{p}+\mathbf{q}\rangle = \int_{-1}^{1} (x+x^2)(x+x^2)\,dx$$

$$= \int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} x^3\,dx + \int_{-1}^{1} x^4\,dx = \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}$$

Book Questions

find the cosine of angle w.r.s.t

Euclidean inner product

1a) $u=(1,-3)$ $v=(2,4)$

$\cos\theta = \frac{(2\times1 + 4(-3))}{(\sqrt{10})(\sqrt{20})}$

$-1/\sqrt{2}$

Q2c) $u=(2,1,7,-1)$ $v=(4,0,0,0)$

$\cos\theta : \frac{8+0+0+0}{\sqrt{55}\ \sqrt{16}}$

$2/\sqrt{55}$

standard inner product

3) $p = -1 + 5u + 2u^2$, $q = 2 + 4u - 9u^2$

$p=(-1,5,2)$ $q=(2,4,-9)$

$\cos\theta : \frac{-2+20-18}{\sqrt{30}\cdot\sqrt{101}}$

0

Euclidean inner product on M_{22}

6) $A=\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ $B=\begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

$\cos\theta = \frac{-6+4-4+6}{\sqrt{30}\cdot\sqrt{30}}$

$= 0$

Q) determine wether orthogonal with respect to Euclidean inner product

7a) $u=(-1,3,2)$ $v=(4,2,-1)$

$\langle u.v \rangle = -4+6-2$

$= 0$ so orthogonal

8b) $u=(-4,6,-10,1)$ $v=(2,1,-2,9)$

$\langle u.v \rangle = -8+6+20+9$

$= 27$ so not orthogonal

Q18) show $u=\begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $v=\begin{bmatrix} 5 \\ -8 \end{bmatrix}$ are orthogonal w.r.s.t inner product or R^2 $A=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

$\begin{bmatrix} 9 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\langle u.v \rangle = 18 - 18 = 0$ so orthogonal

GRAM SCHMIDT / QR DECOMPOSITION 6.3

> WILL COME <

Gram schmidt

$V_1 = U_1$

no constant so dot product

$$V_2 = U_2 - \frac{\langle U_2, V_1 \rangle}{\| V_1 \|^2} V_1$$

$$V_3 = U_3 - \frac{\langle U_3, V_1 \rangle V_1}{\| V_1 \|^2} - \frac{\langle U_3, V_2 \rangle V_2}{\| V_2 \|^2}$$

$$V_4 = U_4 - \frac{\langle U_4, V_1 \rangle V_1}{\| V_1 \|^2} - \frac{\langle U_4, V_2 \rangle V_2}{\| V_2 \|^2} - \frac{\langle U_4, V_3 \rangle V_3}{\| V_3 \|^2}$$

QR Decomposition

- ↳ breaking of matrice
- ↳ U_1, U_2, U_3 → A
- ↳ apply grand schmidt → orthogormal
- ↳ divide by norm → orthonormal → Q
- ↳ $\begin{bmatrix} \langle q_1, U_1 \rangle & \langle q_1, U_2 \rangle & \langle q_1, U_3 \rangle \\ 0 & \langle q_2, U_2 \rangle & \langle q_2, U_3 \rangle \\ 0 & 0 & \langle q_3, U_3 \rangle \end{bmatrix}$ → R
- ↳ $A = QR$

orthogonal

$\langle V_1 . V_2 \rangle = 0$ →

linearly independent

to convert divide by norm

$$\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}$$ →

orthonormal

$\|V_1\| = 1, \ \|V_2\| = 1$

linear combination

↳ check orthogonality

variable dont solve

$$U = \frac{\langle U, V_1 \rangle V_1}{\| V_1 \|^2} + \frac{\langle U, V_2 \rangle V_2}{\| V_2 \|^2} + \frac{\langle U, V_3 \rangle V_3}{\| V_3 \|^2}$$

$U = \_\_ V_1 + \_\_ V_2 + \_\_ V_3$

Coordinate vector

↳ get linear equation

$U_S = (\_\_, \_\_, \_\_)$

BOOK QUESTIONS

Q) orthogonal or orthonormal

1a) $(0,1)$ $(2,0)$

$\langle u \cdot v \rangle = 0 \cdot 0 = 0$ so orthogonal

$\|u\| = 1$ $\quad \|v\| = 2$ not orthonormal

$\qquad\qquad\downarrow$ not 1

Q) if orthonormal convert to orthogonal

5) $A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \end{array}$

$\langle v_1 . v_2 \rangle = 2 - 2 = 0$

$\langle v_2 . v_3 \rangle = 0$

$\langle v_1 . v_3 \rangle = 0$

so orthogonal ✓

$\|v_1\| = \sqrt{2}$

$\|v_2\| = \sqrt{8}$

$\|v_3\| = 5$

$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ orthonormal ✓

Q9) express $u = (-1, 0, 2)$ as linear combination

$v_1 = (2, -2, 1)$ $\quad v_2 = (2, 1, -2)$ $\quad v_3 = (1, 2, 2)$

$\langle v_1 . v_2 \rangle = 0$ $\quad \langle v_2 . v_3 \rangle = 0$ $\quad \langle v_1 . v_3 \rangle = 0$

$n_1 = 3$ $\quad n_2 = 3$ $\quad n_3 = 3$

$\frac{-2+2}{3^2}$ $\quad \frac{-2-4}{3^2}$ $\quad \frac{-1+4}{3^2}$

$0 v_1 \; - \; 2/3\, v_2 \; + \; 1/3\, v_3$

Q13) find coordinate vector of Q9

linear eq found in Q9

$0 v_1 - \frac{2}{3} v_2 + \frac{1}{3} v_3$

$\left(0, -\frac{2}{3}, \frac{1}{3}\right)$

Q27) Gram Schmidt → orthogonal → orthonormal

$u_1 = (1, -3)$ $\quad v_2 = (2, 2)$

$v_1 = (1, -3)$

$v_2 = (2, 2) - \frac{(2-6)}{10} (1, -3)$

$v_2 = (2, 2) + \frac{2}{5} (1, -3)$

$v_2 = \left(\frac{12}{5}, \frac{4}{5}\right)$ → orthogonal

$n_1 = \sqrt{10}$ $\quad n_2 = \sqrt{32/5}$

$\underbrace{v_1 = \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right) \quad v_2 = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)}_{\text{orthonormal basis}}$

Q45) QR Decomposition

$A = \begin{array}{c} \begin{array}{cc} u_1 & v_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \end{array}$ $\quad Q = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \end{array}$

$R = \begin{bmatrix} \overset{q_1 v_1}{\sqrt{5}} & \overset{q_1 v_2}{\sqrt{5}} \\ 0 & \overset{q_2 v_2}{\sqrt{5}} \end{bmatrix}$

Book Questions

44)
$$\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 6 & 1 & -5 \\ 2 & 1 & 1 \\ -2 & -2 & 5 \\ 6 & 8 & -7 \end{bmatrix} \end{array}$$

$$V_1 = (6, 2, -2, 6)$$

$$V_2 = (1,1,-2,8) - \overset{6+2+4+48}{\frac{6\cancel{0}}{8\cancel{0}}}(6,2,-2,6)$$

$$V_2 = (1,1,-2,8) - \left(\tfrac{9}{2}, \tfrac{3}{2}, -\tfrac{3}{2}, \tfrac{9}{2}\right)$$

$$V_2 = \left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right)$$

$$V_3 = (-5,1,5,-7) - \overset{-30+2-10-42}{\frac{-8\cancel{0}}{8\cancel{0}}}(6,2,-2,6) - \overset{\frac{35}{2}-\frac{1}{2}-\frac{5}{2}-\frac{49}{2}}{\frac{-10}{25}}\left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right)$$

$$V_3 = (-5,1,5,-7) + (6,2,-2,6) + \frac{2}{5}\left(-\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right)$$

$$V_3 = \left(-\frac{2}{5}, \frac{14}{5}, \frac{14}{5}, \frac{2}{5}\right)$$

Gram schmidt

$$v_1 = u_1$$

no constant so dot product

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2}$$

$$v_4 = u_4 - \frac{\langle u_4, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_4, v_2 \rangle v_2}{\|v_2\|^2} - \frac{\langle u_4, v_3 \rangle v_3}{\|v_3\|^2}$$

Q1) Using Gram Schmidt Process convert vector into orthogonal

$U_1 = (0, 2, 1, 0)$

$U_2 = (1, -1, 0, 0)$

$U_3 = (1, 2, 0, -1)$

$U_4 = (1\ 0, 0, 1)$

$V_1 = (0, 2, 1, 0)$

$V_2 = (1, -1, 0, 0) - \ -2/5\,(0, 2, 1, 0)$

$V_2 = (1, -1, 0, 0) + (0, 4/5, 2/5, 0)$

$V_2 = (1, -\frac{1}{5}, \frac{2}{5}, 0)$

$V_3 = (1, 2, 0, -1) - \frac{4\,(0, 2, 1, 0)}{5} - \frac{\cancel{3}^{1}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{6}_2}(1, -\frac{1}{5}, \frac{2}{5}, 0)$

$V_3 = (1, 2, 0, -1) - (0, 8/5, 4/5, 0) - (\frac{1}{2}, \frac{-1}{10}, \frac{1}{5}, 0)$

$V_3 = (1/2, 1/2, -1, -1)$

$V_4 = (1, 0, 0, 1) - 0 - 1 \times \frac{5}{6}(1, -1/5, 2/5, 0) - \frac{-1 \times \cancel{2}}{\cancel{2} \times 5}(\frac{1}{2}, \frac{1}{2}, -1, -1)$

$V_4 = (1, 0, 0, 1) - (5/6, -1/6, \frac{1}{3}, 0) + (\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5})$

$V_4 = (4/15, 4/15, -8/15, 4/5)$

using QR decomposition

Q2) $\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$

$\left.\begin{array}{l} \langle v_1 . v_2 \rangle = 2 \\ \langle v_2 . v_3 \rangle = 1 \\ \langle v_1 . v_3 \rangle = 1 \end{array}\right\}$ not 0 so do grandsmidt

orthogonal values

$v_1 = (1,1,1)$ $n = \sqrt{3}$

$v_2 = (-2/3, 1/3, 1/3)$ $n = \sqrt{6}/3$

$v_3 = (0, -1/2, 1/2)$ $n = \sqrt{2}/2$

orthonormal values

$q_1 = \left(\frac{1}{\sqrt{3}}, 1/\sqrt{3}, 1/\sqrt{3}\right)$

$q_2 = \left(-\sqrt{6}/3, \sqrt{6}/6, \sqrt{6}/6\right)$

$q_3 = \left(0, -\sqrt{2}/2, \sqrt{2}/2\right)$

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} 1/\sqrt{3} & -\frac{\sqrt{6}}{3} & 0 \\ 1/\sqrt{3} & \sqrt{6}/6 & -\sqrt{2}/2 \\ 1/\sqrt{3} & \sqrt{6}/6 & \sqrt{2}/2 \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{3} & & \\ 0 & & \\ 0 & 0 & \sqrt{2}/2 \end{bmatrix}}$$

EXAMPLE 10 | *QR*-Decomposition of a 3×3 Matrix

Find a *QR*-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt process with normalization to these column vectors yields the orthonormal vectors (see Example 8)

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, it follows from Formula (16) that R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

from which it follows that a *QR*-decomposition of A is

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

Q) find v_1, v_2, v_3 using Gram Schmidt

$u_1 = (1,1,1)$ $u_2 = (0,1,1)$, $u_3 = (0,0,1)$

↳ $v_1 = (1,1,1)$

$v_2 = (0,1,1) - \frac{2}{3}(1,1,1)$

$v_2 = (0,1,1) - (2/3, 2/3, 2/3)$

↳ $v_2 = (-2/3, 1/3, 1/3)$

$v_3 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}(-2/3, 1/3,$

$v_3 = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$

↳ $v_3 = (0, -\frac{1}{2}, \frac{1}{2})$

DIAGONALIZATION 7.1

if orthogonal

↳ $\det(A) = 1$ or -1

↳ $A \cdot A^T = I$ identity matrice

↳ $A^{-1} = A^T$ (inverse, transpose)

if not orthogonal
no inverse

transpose/inverse/product of a orthogonal matrix is orthogonal

BOOK QUESTIONS

Q) orthogonal? if yes, inverse?

1a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

identity matrice so orthogonal

$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ as $A^T = A^{-1}$

3a) $A = \begin{bmatrix} 0 & 1 & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\det = 1/\sqrt{2}\,(-1)$

$= -1/\sqrt{2} \neq 1$ so not orthogonal

so no inverse

ORTHOGONAL DIAGONISATION 7.2

WILL COME

Orthogonally diagonalizing a symmetric matrix

↳ Eigen value → $\lambda I - A$

↳ Eigen vector → if not diagonisable dont go further

↳ orthogonal → use grand smidth if not orthogonal

↳ orthonormal

↳ $P = [v_1, v_2, v_3]$

↳ $D = P^T A P$ → orthogonally diagonalizable $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

if A is n×n

↳ orthogonally diagonalisable

↳ n eigen vectors

↳ is symmetric

↳ eigen values are real numbers

↳ eigen vectors are orthonormal

Spectral Decomposition

↳ Eigen value → $\lambda I - A$

↳ Eigen vector → if not diagonisable dont go further → λ

↳ orthogonal → use grand smidth if not orthogonal

↳ orthonormal → U

↳ $Ax = \lambda_1 U_1 U_1^T + \lambda_2 U_2 U_2^T \dots$

characteristic equation

$\det(\lambda I - A)$

orthogonal matrix

→ 90° angle so dot product = 0

$$\begin{array}{c} \begin{bmatrix} 2/3 & -1/2 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \\ \begin{matrix} v_1 & \quad v_2 & \quad v_3 \end{matrix} \end{array}$$

$$\left.\begin{array}{l} v_1 \cdot v_2 = 0 \\ v_1 \cdot v_3 = 0 \\ v_2 \cdot v_3 = 0 \end{array}\right\}$$

orthogonal → orthonormal

$$\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}$$

those who are orthogonal dont apply grand smidth

orthonormal matrix

↳ all norm = 1

$$\begin{array}{c} \begin{bmatrix} 2/3 & -1/2 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \\ \begin{matrix} v_1 & \quad v_2 & \quad v_3 \end{matrix} \end{array}$$

$\|v_1\| = 1$

$\|v_2\| = 1$

$\|v_3\| = 1$

Spectral Decomposition

1) Eigen value of Symmectric Matrix
2) Eigen Vector $\rightarrow v_1, v_2, v_3$
3) orthogonal check $\rightarrow v_1, v_2, v_3$ then gramdstich
4) orthonormal vector $\rightarrow q_1, q_2, q_3$
5) Decomposition

not weighted

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T + \dots$$

$$A = \lambda_1 \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \begin{bmatrix} \; & \; & \; \end{bmatrix} + \lambda_2 \begin{bmatrix} \; & \; & \; \end{bmatrix} \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} + \lambda_3 \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \begin{bmatrix} \; & \; & \; \end{bmatrix}$$

Book Questions

Q) find characteristic equation, dimentions

1) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{bmatrix}$

det: $(\lambda-1)(\lambda-4) - 4$

$\lambda^2 - 4\lambda - \lambda + 4$

$\lambda^2 - 5\lambda$ → characteristic equation

$\lambda^2 - 5\lambda = 0$

$\lambda = 5, \lambda = 0$ → dimentions

Q 15) spectral decomposition

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

halfway solved in Q8

$\lambda_1 = 2, \quad \lambda_2 = 4$

$P = \begin{array}{cc} v_1 & v_2 \\ \end{array}$ $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= 2 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2}, & 1/\sqrt{2} \end{bmatrix} + 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= 2 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} + 4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Q) find matrix P that orthogonally diagonalizes A, $P^{-1}AP$?

8) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix} \lambda-3 & -1 \\ -1 & \lambda-3 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-3)(\lambda-3) - 1 = 0$

$\lambda^2 - 3\lambda - 3\lambda + 9 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda = 2, \quad \lambda = 4$

$\lambda = 2$

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim -R_1$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} R_1 + R_2$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$u_1 + u_2 = 0$

$u_2 = t$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 4$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} R_2 + R_1$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$u_1 - u_2 = 0$

$u_2 = t$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{array}{cc} v_1 & v_2 \end{array}$ $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ $\langle v_1 \cdot v_2 \rangle = -1 + 1 = 0$ so orthogonal

no need for grand smidt

$n_1 = \sqrt{2} \quad n_2 = \sqrt{2}$

$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$D = \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P^T} \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P}$

$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

QUADRATIC FORMS 7.3

$x = Py$

$Q = ax_1^2 + bx_2^2 + cx_3^2 + ex_1x_2 + fx_2x_3 + gx_1x_3$

$\div 2$

rows & columns

$$A = \begin{bmatrix} a & e/2 & g/2 \\ e/2 & b & f/2 \\ g/2 & f/2 & c \end{bmatrix}$$

↳ Eigen value $\lambda I - A$

↳ Eigen vector → if not diagonisable dont go further

↳ orthogonal → use grand smidth if not orthogonal

↳ orthonormal

↳ $P: [v_1, v_2, v_3]$

↳ $D = P^T AP$ → orthogonally diagonalizable $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$

↳ $Q = y^T (\underbrace{P^T AP}_{D}) y$

$x^T \quad A \quad x$

BOOK QUESTIONS

Q) Express in matrix notation x^TAx

1b) $4x_1^2 - 9x_2^2 - 6x_1x_2$

$\div 2$

$$A = \begin{bmatrix} 4 & -3 \\ -3 & -9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q) formula for quadratic that dosent use matrices

3) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$2x^2 + 5y^2 - 6xy$

BOOK QUESTION

Q6) find orthogonal change of variables, express Q in terms of new variables

$5u_1^2 + 2u_2^2 + 4u_3^2 + 4u_1u_2$

$4/2$

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Step 1

$$\lambda I - A = \begin{bmatrix} \lambda-5 & -2 & 0 \\ -2 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{bmatrix}$$

$(\lambda-5)[(\lambda-2)(\lambda-4)] + 2(-2(\lambda-4))$

$(\lambda-4)\,[(\lambda-5)(\lambda-2)-4] = 0$

$\lambda = 4, \quad \lambda^2 - 2\lambda - 5\lambda + 10 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$\lambda = 6 \quad \lambda = 1$

Step 2

$\lambda = 4$

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_1 \sim -R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \sim 2R_1 + R_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \sim R_2/6$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$u_1 + 2u_2 = 0$

$u_2 = 0$

$u_3 = t$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 6$

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} R_2 \sim 2R_1 + R_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$u_1 - 2u_2 = 0$

$2u_3 = 0$

$u_2 = t$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} -4 & -2 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} R_1 \sim \frac{R_1}{-4}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} R_2 + 2R_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} R_3 \sim -\frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$u_1 + \frac{1}{2}u_2 = 0$

$u_3 = 0$

$u_2 = t$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

step 3

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$\begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$v_1 \cdot v_2 = 0$

$v_1 \cdot v_3 = 0$ so no need for gram schmidt

$v_2 \cdot v_3 = 0$

step 4 and step 5

$$\begin{array}{ccc} n=1 & n=\sqrt{5} & n=\sqrt{5}/2 \end{array}$$
$$P = \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \end{bmatrix} \quad P^T = \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \quad A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

step 6

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$4y_1^2 + 6y_2^2 + y_3^2$$

$Q = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$

Step 1 Eigen value
Step 2 Eigen vector
Step 3 Convert to orthogonal using Gram Schmidt
Step 4 orthonormal
Step 5 $P \to [\quad]$
Step 6 $D = P^TAP$

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

Step 1

$$\lambda I - A = \begin{bmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda-5 & 4 \\ 2 & 4 & \lambda-5 \end{bmatrix}$$

$(\lambda-2)[(\lambda-5)(\lambda-5)-16]+2(-2(\lambda-5)-8)+2(-8-2(\lambda-5))$

$(\lambda-2)[\lambda^2-5\lambda-5\lambda+25-16]+2(-2\lambda+10-8)+2(-8-2\lambda+10)$

$(\lambda-2)[\lambda^2-10\lambda+9]+2(-2\lambda+2)+2(-2\lambda+2)$

$\lambda^3-10\lambda^2+9\lambda-2\lambda^2+20\lambda-18-4\lambda+4-4\lambda+4$

$\lambda^3-12\lambda^2+21\lambda-10$

$\lambda = 10, \ \lambda = 1, \ \lambda = 1$

if not diagonizable
wont go any further

Step 2

$\lambda = 10$

$$\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} \begin{array}{l} R_1/8 \\ R_2+R_3 \\ \end{array}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 9 & 9 \\ 2 & 4 & 5 \end{bmatrix} \begin{array}{l} \\ \\ 2R_1-R_3 \end{array}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 9 & 9 \\ 0 & -\frac{9}{2} & -\frac{9}{2} \end{bmatrix} \begin{array}{l} \\ R_2/9 \\ \end{array}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 \\ 0 & -\frac{9}{2} & -\frac{9}{2} \end{bmatrix} \begin{array}{l} \\ \\ \frac{2}{9}R_2+R_3 \end{array}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = u$

$x_2 + x_3 = 0$

$x_2 = -u$

$x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 = 0$

$x_1 = -\frac{u}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = u \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{bmatrix} \begin{array}{l} -R_1 \\ R_2+R_3 \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 2 & 4 & -4 \end{bmatrix} \begin{array}{l} \\ \\ 2R_1-R_3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable
so assign variables

let $x_3 = u, x_2 = t$

$x_1 + 2x_2 - 2x_3 = 0$

$x_1 = 2u - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = u \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

step 3

$$\begin{array}{ccc} U_1 & U_2 & U_3 \end{array}$$
$$\begin{bmatrix} -\frac{1}{2} & 2 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$U_1 . U_2 = 0$
$U_2 . U_3 = 0$
$U_1 . U_3 = 0$

no need for grand smidth

step 4 and 5

norm: $\frac{3}{2}$ $\sqrt{5}$ $\sqrt{5}$

$$P = \begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix} \quad P^T = \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

step 6

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

step 7

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$10y_1^2 + y_2^2 + y_3^2$

Q1) find an orthogonal change in variable that eliminate cross product term in this Quadratic form $Q = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$ and express Q in term of the new variable

$$Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & & D_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = D_1 y_1^2 + D_2 y_2^2 + D_3 y_3^2$$

$x = Py$

Q) $3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1$

consistent

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

unique solution

linear combination

column space

span

consistent

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinite solution

linear combination

column space

span

not consistent

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

no solution

no linear combination

no column space

no span

for SQUARE MATRICE

$\det(A)=0$	$\det(A)\neq 0$
no span	span
non trivial	trivial

Basis

↳ linearly independent

↳ span

after echalon

$k_1, k_2, k_3 = s, t$	$k_1, k_2, k_3 = 0$
non trivial	Trivial
dependent	independent
no basis	form basis

BASIS / DIMENTIONS

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

1^{st} vector 2^{nd} Vector

Basis → are vectors

← Dimentions = 2

n a r = c

Rank + nullity = column

- reduce to echalon
- no of non zero rows

no of zero rows

no of unknowns

Diagonalization

- $\lambda I - A$
- $\det = 0$
- eigen values
- eigen vectors
- Diagnisable → solve further if true
  - if 3x3 matrice: 3 Eigan vector
  - if 2x2 matrice: 2 Eigan vector
- P: [E.vector1 E.Vector2 ...]
- $D = P^{-1}AP$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

QR Decomposition

- breaking of matrice
- U_1, U_2, U_3 → A
- orthognormal → grand schmidt
- orthonormal → divide by norm → Q
- $\begin{bmatrix} \langle q_1, U_1\rangle & \langle q_1, U_2\rangle & \langle q_1, U_3\rangle \\ 0 & \langle q_2, U_2\rangle & \langle q_2, U_3\rangle \\ 0 & 0 & \langle q_3, U_3\rangle \end{bmatrix}$ → R
- $A = QR$

Gram schmidt

$V_1 = U_1$

no constant so dot product

$$V_2 = U_2 - \frac{\langle U_2, V_1\rangle}{\|V_1\|^2} V_1$$

$$V_3 = U_3 - \frac{\langle U_3, V_1\rangle V_1}{\|V_1\|^2} - \frac{\langle U_3, V_2\rangle V_2}{\|V_2\|^2}$$

$$V_4 = U_4 - \frac{\langle U_4, V_1\rangle V_1}{\|V_1\|^2} - \frac{\langle U_4, V_2\rangle V_2}{\|V_2\|^2} - \frac{\langle U_4, V_3\rangle V}{\|V_3\|^2}$$

orthogonal matrix

90° angle so dot Product = 0

$$\begin{array}{ccc} & & \\ \end{array}\begin{bmatrix} 2/3 & -1/2 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$V_1 \quad V_2 \quad V_3$

$V_1 \cdot V_2 = 0$

$V_1 \cdot V_3 = 0$

$V_2 \cdot V_3 = 0$

orthogonal → orthonormal

$$\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|}$$

those who are orthogonal dont apply grand smidth

orthonormal matrix

- all norm = 1

$$\begin{bmatrix} 2/3 & -1/2 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$V_1 \quad V_2 \quad V_3$

$\|V_1\| = 1$

$\|V_2\| = 1$

$\|V_3\| = 1$

- Eigen value $\lambda I - A$
- Eigen vector → if not diagonisable dont go further
- orthogonal → use grand smidth if not orthogonal
- orthonormal
- $P: [V_1, V_2, V_3]$

Spectral Decomposition

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T + \ldots$$

Quadratic forms

- $D = P^T A P$ → orthogonally diagonalizable $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ → symmetric matrix
- $Q = y^T (\underbrace{P^T A P}_{D}) y$

$$x^T \quad A \quad x$$

Steady state vector

- $(I - P)q = 0$
- Echalon
- equate Probabilities to 1
- find free variable and multiply

ishma Hafeez notes