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BOOK:

Discrete Mathematics and its applications
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INTRODUCTION

Discrete mathematics is the part of math devoted to the study of discrete objects. Dicrete means consisting of distinct or unconnected elements. It is used whenever objects are counted and when relationship between finite sets is studied. A key reason for the growth in the importance of discrete math is that information is stored & manipulated by computing machines in a discrete fashion.

DISCRETE STRUCTURES

Much of the discrete mathematics is devoted to the study of discrete structures which are used to represent discrete objects. All discrete objects are built up from sets which are a collection of objects.

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Chapter 1:

The Foundation:

Logic, sets and functions

Logics:

The rules of the logic give precise meaning to the mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments.

* Proposition:

A proposition is a statement that is either true or false but not both.

Examples:

1) Washington DC is the capital of USA.
2) $2 + 2 = 3$
3) Read this carefully
4) $x + 1 = 2$

Letters are used to denote proposition just as they are used to denote variables. The conventional letters used are: $p, q, r, s, \ldots$ etc.

A truth value of a proposition is denoted by T if it is a true proposition and by F if it is a false proposition.

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1) $\neg$, Negation, NOT

Let p be a proposition. The statement $\neg p$ is called negation of p if it is the inverse of p.

p	$\neg p$
T	F
F	T

2) $\wedge$, AND, Multiplication

Let p and q be propositions. $p \wedge q$ is true when both are true and is false otherwise. $p \wedge q$ is called the conjunction of p and q.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3) $\vee$, OR, Addition

$p \vee q$ is false when p and q are false and is true otherwise. The $p \vee q$ is called disjunction of p and q.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

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4) XOR, Exclusive OR, ⊕.

The exclusive OR of p and q is denoted by $p \oplus q$. It is true when exactly one of p and q is true and false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

5) Implication, $p \rightarrow q$

The implication $p \rightarrow q$ is false when p is true and q is false and is true otherwise.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Some of the more common ways are:

1) If p then q
2) p implies q
3) If p, q
4)
5)
6)
7)
8)

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6) Biconditional , $p \leftrightarrow q$

The proposition $p \leftrightarrow q$ is true when both p and q have the same truth values and is false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

7) NAND , $\overline{p \wedge q}$, p / q

The inverse of $p \wedge q$ is called NAND and is denoted by $\overline{p \wedge q}$.

p	q	$p \wedge q$	$\overline{p \wedge q}$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

8) NOR , $\overline{p \vee q}$, $p \downarrow q$

It is the inverse of $p \vee q$ and is called NOR.

p	q	$p \vee q$	$\overline{p \vee q}$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

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Homework

$(p \to q) \to r$

p	q	r	$p \to q$	$(p \to q) \to r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

BIT STRING:

Computers represent information using bits. A bit has two possible values namely 0 and 1. A bit can be used to represent a truth value, and since there are two truth values namely true and false, we use '1' bit to represent true and '0' bit to represent false.

A variable is called a boolean variable if its value is either true or false.

A bit string is a sequence of 0 or more bits. The length of the string is ~~that~~ the number of bits in the string.

10110111 → 8 length

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PRACTICE EXERCISES

Q Which of the following are propositions? What are the truth values of those which are proposition?

1) Do not pass Go. NO

2) What time is it? NO

3) There are no red apples in Maine. (F)

4) $4 + x = 5$ NO

5) $x + 1 = 5$, if $x = 1$ (F)

Q Let p and q be the propositions.
p: I bought a lottery ticket this week.
q: I won a million dollar jackpot on friday.

Express each of the following propositions as an english sentence.

1) $\neg p$
I did not buy a lottery ticket this week.

2) $p \vee q$
I bought a lottery ticket this week or I won a million dollar jackpot on Friday.

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3) $p \rightarrow q$

If I buy a lottery ticket this weekend then I will win the million dollar jackpot on Friday.

4) $p \leftrightarrow q$

I will win the million dollar jackpot on Friday if and only if I buy a lottery ticket this weekend.

5) $\neg p \rightarrow \neg q$

If I do not buy a lottery ticket this weekend then I will not win a million dollar jackpot on Friday.

6) $\neg p \wedge \neg q$

I do not buy a lottery ticket this weekend and I will not win a million dollar jackpot on Friday.

7) $\neg p \vee (p \wedge q)$

Either I do not buy a lottery ticket this weekend or I buy a ~~million~~ lottery ticket this weekend and win a million dollar jackpot on Friday.

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Q. Let p, q and r be the propositions.

p: you get an A in final exam.
q: you do every exercise in this book.
r: you get an A in this class

Write the following propositions using p, q, r and logical connectives.

1) You get an A in this class but you do not do every exercise in this book.

$$r \wedge \neg q$$

2) You get an A in the final, you do every exercise in this book and you get an A in this class.

$$p \wedge q \wedge r$$

3) To get an A in this class, it is necessary for you to get an A in the final exam.

$$r \rightarrow p$$

4) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

$$p \wedge \neg q \wedge r$$

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5) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

$$(p \wedge q) \rightarrow r$$

6) You get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

$$r \leftrightarrow (p \vee q)$$

Q. Construct a truth table for each of the following compound propositions.

1) $p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

2) $p \vee \neg p$

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Dated:

3) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

4) $(p \vee q) \rightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

5) $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$

p	q	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$p \leftrightarrow q$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

6) $(p \rightarrow q) \wedge (\neg p \rightarrow q)$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow q$	$(p \rightarrow q) \wedge (\neg p \rightarrow q)$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

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Q. Find the bitwise OR, bitwise AND and bitwise XOR of each of these pairs of bit strings.

1)

```
  0 1 1 0 1 1 0 1 1 0
  1 1 0 0 0 1 1 1 0 1
  -------------------
  1 1 1 0 1 1 1 1 1 1  -> OR
  0 1 0 0 0 1 0 1 0 0  -> AND
  1 0 1 0 1 0 1 0 1 1  -> XOR
```

2)

```
  1 1 1 1 0 0 0 0
  1 0 1 0 1 0 1 0
  ---------------
  1 1 1 1 1 0 1 0  -> OR
  1 0 1 0 0 0 0 0  -> AND
  0 1 0 1 1 0 1 0  -> XOR
```

3)

```
  0 0 0 1 1 1 0 0 0 1
  1 0 0 1 0 0 1 0 0 0
  -------------------
  1 0 0 1 1 1 1 0 0 1  -> OR
  0 0 0 1 0 0 0 0 0 0  -> AND
  1 0 0 0 1 1 1 0 0 1  -> XOR
```

4)

```
  1 1 1 1 1 1 1 1 1 1
  0 0 0 0 0 0 0 0 0 0
  -------------------
  1 1 1 1 1 1 1 1 1 1  -> OR
  0 0 0 0 0 0 0 0 0 0  -> AND
  1 1 1 1 1 1 1 1 1 1  -> XOR
```

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Propositional Equivalances

* **Tautology:** A compound proposition that is always true, no matter what the truth values of the propositions that occur in it.

* **Contradiction:** A compound proposition that is always false, no matter what the truth values of the propositions that occur in it.

* **Contigency:** A statement that is neither a tautology nor a contradiction.

Logical Equivalances

The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.
The notation $p \Leftrightarrow q$ denotes that p and q are logically equivalent.

Example: $p \rightarrow q \Leftrightarrow \neg p \vee q$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Dated:

	NAME	EQUIVALENCE
1.	Identity laws	$p \wedge T \Leftrightarrow p$ $p \vee F \Leftrightarrow p$
2.	Domination laws	$p \vee T \Leftrightarrow T$ $p \wedge F \Leftrightarrow F$
3.	Idempotent laws	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$
4.	Double negation law	$\neg(\neg p) \Leftrightarrow p$
5.	Commutative laws	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$
6.	Associative laws	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
7.	Distributive laws	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
8.	De Morgan's laws	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

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9.	Absorption laws	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$
10.	Negation laws $p \vee \neg p \Leftrightarrow T$ $p \wedge \neg p \Leftrightarrow F$	$p \vee \neg p \Leftrightarrow T$ $p \wedge \neg p \Leftrightarrow F$

Q Show that the following statements are logically equivalent by using laws.

i) $\neg[p \vee (\neg p \vee q)] \Leftrightarrow \neg p \wedge \neg q$

$\neg[p \vee (\neg p \vee q)] = \neg p \wedge \neg(\neg p \wedge q)$ (De Morgan's law)

$= \neg p \wedge [\neg(\neg p) \vee \neg q]$ (De Morgan's law)

$= \neg p \wedge (p \vee \neg q)$ (Double negation)

$= (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ (Distributive law)

$= F \vee (\neg p \wedge \neg q)$ ($\because \neg p \wedge p \Leftrightarrow F$)

$= \neg p \wedge \neg q$ (Identity law)

Hence proved!

Dated:

Q. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$$(p \wedge q) \rightarrow (p \vee q) = \neg(p \wedge q) \vee (p \vee q)$$

$$= (\neg p \vee \neg q) \vee (p \vee q) \quad \text{(De Morgan's)}$$

$$= (\neg p \vee p) \vee (\neg q \vee q) \quad \text{(Commutative law)}$$

$$= T \vee T$$

$$= T \quad \text{(Domination law)}$$

Hence proved!

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Dated:

Q. Show that each of the following is a tautology using truth table.

1) $(p \wedge q) \rightarrow p$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

2) $(p \wedge q) \rightarrow (p \rightarrow q)$

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

3) $[\neg p \wedge (p \vee q)] \rightarrow q$

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Dated:

4) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r) \wedge (q \to r)$	$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

5) $[\cancel{\neg} p \wedge (p \to q)] \to \cancel{\neg} q$

p	q	$\cancel{\neg p}$	$\cancel{\neg q}$	$p \to q$	$\cancel{\neg} p \wedge (p \to q)$	$[\cancel{\neg} p \wedge (p \to q)] \to \cancel{\neg} q$
T	T	F	F	T	$\cancel{F}$ T	T
T	F	F	T	F	F	T
F	T	T	F	T	$\cancel{T}$ F	T
F	F	T	T	T	$\cancel{T}$ F	T

6) $[p \wedge (p \to q)] \to q$

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Dated:

7) $p \rightarrow (p \vee q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

8) $\neg p \rightarrow (p \rightarrow q)$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow (p \rightarrow q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

9) $\neg (p \rightarrow q) \rightarrow \neg q$

p	q	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg(p \rightarrow q) \rightarrow \neg q$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	F	T
F	F	T	T	F	T

Dated:

Q. Show that the following are logically equivalent.

1) $p \leftrightarrow q \iff (p \wedge q) \vee (\neg p \wedge \neg q)$

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	T	F	F	T	F	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	T

2) $\neg(p \leftrightarrow q) \iff p \leftrightarrow \neg q$

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

3) $\neg(p \oplus q) \iff p \leftrightarrow q$

p	q	$p \oplus q$	$\neg(p \oplus q)$	$p \leftrightarrow q$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	~~F~~ T	T

Dated:

DUAL OF COMPOUND PROPOSITIONS

The dual of the compound proposition that contain only the logical operators $\wedge, \vee, \neg$ in the proposition obtained by replacing each $\vee$ by $\wedge$, $\wedge$ by $\vee$, T by F and F by T.
The dual of S is denoted by S^*.

Q. Find the dual of the following propositions.

1) $p \vee \neg q$

$p \wedge \neg q$

2) $(p \wedge \neg q) \vee (q \wedge F)$

$(p \vee \neg q) \wedge (q \vee T)$

3) $p \wedge \neg q \wedge \neg r$

$p \vee \neg q \vee \neg r$

4) $(p \vee F) \wedge (q \vee T)$

$(p \wedge T) \vee (q \wedge F)$

Dated:

PREDICATES AND QUANTIFIERS

Predicates

Statements involving variables such as "$x > 3$", "$x = y + 3$" and "$x + y = z$" are often found in mathematical assertions and in computer programs. These statements are neither true nor false when the value of variables are not specified. The statement '$x > 3$' has two parts:
'x' is the subject of the statement while '> 3' is the predicate which refers to a property that the subject of the statement can have.
We can denote the ~~sys~~ statement "$x > 3$" by $P(x)$ where 'p' denotes the predicate and 'x' is a variable. The statement $P(x)$ is also said to be the value of the propositional function p at x. Once the value has been assigned to the value x, the statement $P(x)$ has a truth value.

Q. Let $Q(x, y)$ denote the statement $x = y + 3$. What are the truth values of the propositions at $Q(1,2)$ and $Q(3,0)$?

For $Q(1,2)$

$\Rightarrow$ $1 = 2 + 3$

$1 = 5$

False

For $Q(3,0)$

$\Rightarrow$ $3 = 0 + 3$

$3 = 3$

True

Dated:

Q. Let $R(x, y, z)$ denote the statement $x+y=z$. What are the truth values of the proposition at $R(1, 2, 3)$ and at $R(0, 0, 1)$?

For $R(1, 2, 3)$

$\Rightarrow 1+2 = 3$

$3 = 3$

True

For $R(0, 0, 1)$

$\Rightarrow 0+0 = 1$

$0 \neq 1$

False

Quantifiers

When all the variables in a propositional function are assigned values, the resulting statement has a truth value. However there is another important way to change propositional function into two propositions which is called quantification.

There are two types of quantifications:

1) Universal Quantification:

Many mathematical statements assert that a property is true for all values of variable. Such a statement is expressed using a universal quantification. The universal quantification of the proposition forms the proposition that is true if and only if $P(x)$ is true for all values of x in the universe of discourse.

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The universe of discourse specify the possible values of the variable x.

Examples:

$x = 1, 2, 3, \ldots\ldots, 100$ universe of discourse

The notation $\forall x\ P(x)$ denoted the universal quantification of $P(x)$. It is expressed as "for all x $P(x)$" or "for every x $P(x)$".

Q. Express the statement "Every student in this class has studied Calculus" as the universal quantification.

Let $P(x)$ denote the statement 'x has studied calculus'.

$$\cancel{\forall x\ P(x)}$$

$$\forall x\ S(x) \rightarrow P(x)$$

Q. Let $P(x)$ be the statement $x + 1 > x$. What is the truth value of quantification $\forall x\ P(x)$ if the universe of discourse is the set of real numbers.

True

Dated: Existential

2) ~~Extensial~~ Quantification.

The existential ~~extensial~~ quantification of $p(x)$ is the proposition where exists an element x in the universe of discourse such that $p(x)$ is true. It is denoted by $\exists x\, P(x)$ and expressed as "there is an x such that $p(x)$" or "there is at least one x such that $P(x)$".

Q. Let $p(x)$ denote the statement $x > 3$. What is the truth value of the quantification $\exists x\, p(x)$ where the universe of discource is the set of real numbers of integers.

When $x = 4$, the truth value of this quantification is true.

Q. Let $Q(x)$ denotes the statement $x = x+1$. What is the truth value of the quantification $\exists x\, Q(x)$ where the universe of discourse is set of real numbers.

False

Dated:

Q. Consider the following statements. The first two are premises and the third is called a conclusion. The entire set is called an argument.

a) All lions are fierce.
b) Some lions do not drink coffee.
c) Some fierce lions do not drink coffee.

Let $P(x) = x$ is lion
$Q(x) = x$ is fierce
$R(x) = x$ drinks coffee.

a) $\forall x \; P(x) \rightarrow Q(x)$
b) $\exists x \; P(x) \wedge \neg R(x)$
c) $\exists x \; Q(x) \wedge \neg R(x)$

Q. Consider the following statements of which first three are premises and fourth is a valid conclusion.

a) All humming birds are richly colored.
b) No large birds live on honey.
c) Birds that do not live on honey are dull in color.
d) Humming birds are small.

Express the statements in the arguments using quantifiers.

Dated:

Let,

$P(x) =$ x is a humming bird

$Q(x) =$ x is large

$R(x) =$ x lives on honey

$S(x) =$ x is richly colored.

a) $\forall x (P(x) \rightarrow S(x))$

b) $\neg \exists x (Q(x) \wedge R(x))$

c) $\forall x (\neg R(x) \rightarrow \neg S(x))$

d) $\forall x (P(x) \rightarrow \neg Q(x))$

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Q. Let $P(x)$ be the statement. x contains the letter 'a'. What are the truth values of the following?

1) P(orange) = True
2) P(lemon) = False
3) P(true) = False
4) P(false) = True

Q. Let $P(x)$ be the statement "x spends more than five hours every week day in class" where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x\ P(x)$

There is a student who spends more than five hours every weekday in class.

b) $\forall x\ P(x)$

Every student spends more than five hours every weekday in class.

c) $\exists x \neg P(x)$

There is a student who does not spend more than five hours every weekday in class.

d) $\forall x \neg P(x)$

No student spends more than five hours every weekday in class.

Dated:

Q. Let $P(x, y)$ be the statement "x has taken class y" where universe of discourse is the set of all students in the class and y is the set of all computer courses in the school. Express each of the following statements in English.

1) $\exists x \, \exists y \; P(x, y)$

There is a student who has taken at least one CS course in school.

2) $\exists x \, \forall y \; P(x, y)$

There is a student who has taken all CS courses in school.

3) $\forall x \, \exists y \; P(x, y)$

Every student has taken at least one CS course in school.

4) $\exists y \, \forall x \; P(x, y)$

There is a CS course which is taken by all students in school.

5) $\forall y \, \exists x \; P(x, y)$

All CS courses are taken by a student in school

6) $\forall x \, \forall y \; P(x, y)$

All CS courses are taken by all students in school

Dated:

Q. Use quantifiers to express each of the following statements.

1) Every computer student needs a course in DS.

$$\forall x \; \exists x \; P(x,y)$$

2) There is a student in this class who owns a personal computer.

$$\exists x \; \exists y \; P(x,y)$$

3) Every student in this class have taken at least one computer course.

$$\forall x \; \exists y P(x,y)$$

4) There is a student in this class who has taken at least one course in computer.

$$\exists x \; \exists y \; P(x,y)$$

5) Every student in this class ~~have~~ has been in every building in campus.

$$\forall x \; \forall y \; P(x,y)$$

6) There is a student in this class who has been in every room of at least one building.

$$\exists x \; \forall y \; \exists z \; P(x,y,z)$$

Dated:

7) Every student in this class have been in at least one room of every building in this campus.

Q. Let $P(x) = x$ is a professor
$Q(x) = x$ is ignorant
$R(x) = x$ is vain

Express each of these statements using quantifiers and logical operators, where the domain consists of all people.

a) No professors are ignorant.

$$\forall x \; P(x) \to \neg Q(x)$$

b) All ignorant people are vain.

$$\forall x \; (Q(x) \to R(x))$$

c) No professors are vain.

$$\forall x \; (P(x) \to \neg R(x))$$

d) Does (c) follow from (a) and (b)?

The conclusion does not follow. There may be vain professors, since the premises do not rule out the possibility that there are vain people besides the ignorant people.

Dated:

Q. Let

$P(x) = x$ is a baby
$Q(x) = x$ is logical
$R(x) = x$ is able to manage a crocodile
$S(x) = x$ is despised

Express each statement using quantifiers and logical operators where domain consists of all people.

a) Babies are illogical.

$$\forall x\, [P(x) \rightarrow \neg Q(x)]$$

b) Nobody is despised who can manage a crocodile.

$$\forall x\, [R(x) \rightarrow \neg S(x)]$$

c) Illogical people are despised.

$$\forall x\, [\neg Q(x) \rightarrow S(x)]$$

d) Babies cannot manage crocodile.

$$\forall x\, [P(x) \rightarrow \neg R(x)]$$

Dated:

Q. Translate each of these statements into logical expressions using predicates, quantifiers and logical connectives.

Let, $P(x) = x$ is perfect
$Q(x) = x$ is your friend

a) No one is perfect

$$\forall x \neg P(x)$$

b) Not everyone is perfect

$$\neg \forall x P(x)$$

c) All your friends are perfect.

$$\forall x (Q(x) \rightarrow P(x))$$

d) At least one of your friends is perfect.

$$\exists x (Q(x) \wedge P(x))$$

e) Everyone is your friend and is perfect.

$$\forall x (Q(x) \wedge P(x))$$

f) Not everybody is your friend or someone is not perfect.

$$[\neg \forall x Q(x)] \vee [\exists x \neg P(x)]$$

Dated:

Chapter 2

Sets

Sets are used to group objects together. Often, the objects in a set have similiar properties. For instance, all the students currently enrolled in your school make up a set. The objects in a set are also called the elements or members of the set. A set is said to contain its elements.

The notation {} is used to represent a set. {a, b, c, d} is a set with four elements.

Set-builder notation:

For instance, the set O of all the +ve integers less than 10 can be written as:

O = {x | x is a +ve integer less than 10}

Equal Sets:

Two sets are said to be equal if and only if they have the same elements. The order of the elements in the set doesn't matter. Therefore if A and B are sets then they are equal if and only if $\forall x \, (x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

$$\{1, 3, 5, 7\} = \{5, 3, 7, 1\}$$

Dated:

Empty Set:

There is a special set that has no elements. This set is called the empty set or null set. It is denoted by {} or ϕ.

Venn diagrams:

Sets can be represented graphically using Venn diagrams. In venn diagram, the universal set U which contains all the objects under consideration is represented by a rectangle. Inside these rectangles, circles or other geometrical figures are used to represent figures.

Subset:

The set A is said to be the subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of B. We see that $A \subseteq B$ ~~is~~ if and only if the quantification $\forall x \{x \in A \to x \in B\}$ is true.

The null set is a subset of every set.

$$\phi \in S$$

We wish to emphasize that a set A is a subset of set B, but $A \neq B$. We write $A \subset B$ and say that A is a proper subset of B.

One way to show that two sets have the same elements is to show that each set is the subset of the other.

$$\left.\begin{matrix} B \subseteq A \\ A \subseteq B \end{matrix}\right\} A = B$$

Dated:

Cardinality of Set:

Let 'S' be a set. If there are exactly 'n' distinct elements in S where n is a +ve integer, the 'S' is a finite set and that 'n' is a cardinality of S. The cardinality of S is denoted by $|S|$.

Example: $\{1, 1, 3, 5, 5\}$

$|S| = 3$

Let S be the set of letters in english alphabet
$|S| = 26$

A set is said to be infinite if it is not finite.
e.g. set of +ve integers

Power Set

Given a set 'S', the power set of 'S' is set of all subsets of set S. The power set of set S is denoted by $P(S)$.

E.g. What is the power set of:

① $\{0, 1, 2\}$

$P(\{0,1,2\}) = 2^n = 2^3 = 8$

$= \{\phi, \{1\}, \{2\}, \{0\}, \{0,1\}, \{0,2\}, \{1,2\}$
$\{0,1,2\}\}$

Dated:

② $P(\phi) = \{\{\phi\}\}$

③ $P(\{\phi\}) = \{\phi, \{\phi\}\}$

Cartesian Product:

Let A and B be the sets. The cartesian product of A & B denoted by $A \times B$, is set of all ordered pairs (a,b) where $a \in A$ & $b \in B$. Hence $A \times B = \{(a,b) / a \in A \wedge b \in B\}$

PRACTICE EXERCISE

Q List the members of sets:

a) {x/x is the real numbers such that $x^2 = 1$}

$\{1, -1\}$

b) {x/x is the positive integer less than 12}

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

c) {x/x is square of integer and x is less than 100}

$\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$

d) {x/x is an integer such that $x^2 = 2$}

ϕ

Dated:

Q Use the set-builder notation to give description of each sets.

1) $\{0, 3, 6, 9, 12\}$

$\{x \mid x$ is an integer divisible by 3 and less than or equal to $12\}$

2) $\{-3, -2, -1, 0, 1, 2, 3\}$

$\{x \mid x$ is an integer between ~~-B~~ -4 and $4\}$

3) $\{m, n, o, p\}$

$\{x \mid x$ is an ~~into~~ alphabet between l and $q\}$

Q Suppose $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, $D = \{4, 6, 8\}$. Determine which of these sets are subsets of each.

$B \subset A$, $C \subset A$, $C \subset D$

Use a Venn diagram to illustrate the relationships.

$B \subset A$	$C \subset A$	$C \subset D$

Dated:

Q. What is |S| of each of the sets:

i) $\{a\} = 1$

ii) $\{\{a\}\} = 1$

iii) $\{a, \{a\}\} = 2$

iv) $\{a, \{a\}, \{\{a\}, \{a\}\}\} = 3$

Q. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

i) ϕ No

ii) $\{\phi, \{a\}\}$ Yes

iii) $\{\phi, \{a\}, \{\phi, a\}\}$ No

iv) $\{\phi, \{a\}, \{b\}, \{a, b\}\}$ Yes

Q. Let $A = \{a, b, c, d\}$, $B = \{y, z\}$. Find:

a) $A \times B$

$A \times B = \{(a,y), (b,y), (c,y), (d,y), (a,z), (b,z), (c,z), (d,z)\}$

b) $B \times A$

$B \times A = \{(y,a), (y,b), (y,c), (y,d), (z,a), (z,b), (z,c), (z,d)\}$

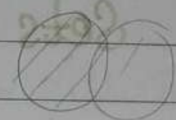
Dated:

Set Operations

1) Union

$(A \cup B)$

$A \cup B = \{x | x \in A \vee x \in B\}$



2) Intersection

$(A \cap B)$

$A \cap B = \{x | x \in A \wedge x \in B\}$

3) Disjoint

Two sets are said to be disjoint if their intersection is empty.

Let $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$

$\Rightarrow \quad A \cap B = \phi$

4) Cardinality of Union of Sets

To find no. of elements in the union of two finite sets A and B, note that $|A| + |B|$ counts each element that is in B but not in A or in A but not in B, exactly once and each element that is in both A and B exactly twice here.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

5) Difference of Sets:

Let A and B be sets. The difference of A and B denoted by $A - B$ is a set containing those elements that are in A but not in B.

Dated:

The difference of A and B is also called complement of B w.r.t A.

$$A-B = \{x \mid x \in A \wedge x \notin B\}$$

6) Complement of Sets

Let A & B be sets. The complement of A denoted by A' or $\bar{A}$ is complement of A w.r.t U.

$$\bar{A} = \{x \mid x \notin A\} \; ; \; \bar{A} = U - A$$

Example:

$A = \{11, 12, \ldots\}$

$U = \{1, 2, 3, \ldots\}$

$A' = \{1, 2, 3, \ldots, 10\}$

SET IDENTITIES

Identity	Name
$A \cap U = A$ $A \cup \phi = A$	Identity laws
$A \cup U = U$ $A \cap \phi = \phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\bar{A})} = A$	Complementation laws

Dated:

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De-Morgan's law
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \phi$	Complement laws

Q Use membership table to prove associative laws.

1) $A \cup (B \cup C) = (A \cup B) \cup C$

A	B	C	$A \cup B$	$B \cup C$	$A \cup (B \cup C)$	$(A \cup B) \cup C$
1	1	1	1	1	1	1
1	1	0	1	1	1	1
1	0	1	1	1	1	1
1	0	0	1	0	1	1
0	1	1	1	1	1	1
0	1	0	1	1	1	1
0	0	1	0	1	1	1
0	0	0	0	0	0	0

Dated:

2) $A \cap (B \cap C) = (A \cap B) \cap C$

A	B	C	$A \cap B$	$B \cap C$	$A \cap (B \cap C)$	$(A \cap B) \cap C$
1	1	1	1	1	1	1
1	1	0	1	0	0	0
1	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	1	0	1	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	0

Q Prove De Morgan's laws with the help of identities.

$\overline{A \cap B}$

Dated:

Q. Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, $C = \{0, 3, 6, 9\}$. What will be $A \cup (B \cup C)$ & $A \cap (B \cap C)$

1) $A \cup (B \cup C)$

$B \cup C = \{0, 1, 2, 3, 4, 6, 9\}$

$A \cup (B \cup C) = \{0, 1, 2, 3, 4, 6, 8, 9\}$

2) $A \cap (B \cap C)$

$B \cap C = \{0, 3\}$

$A \cap (B \cap C) = \{0\}$

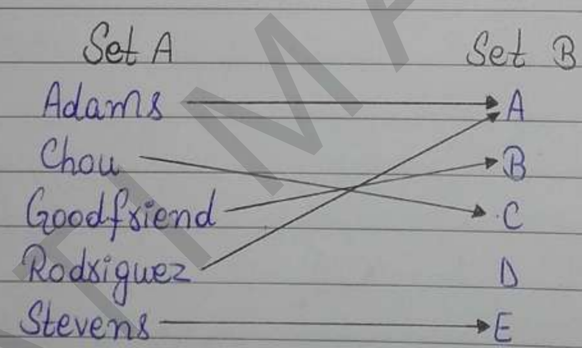
Dated:

FUNCTIONS

The concept of function is extremely important in Discrete math. Functions are used in the definition of such discrete structures as sequences and strings. They are also used to represent how long it takes a computer to solve problems of given size.

Definition # 1:

Let A and B be sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.



Definition # 2

If f is a function from A to B, we say that A is the domain of f and B is the co-domain of f. If $f(a) = b$, we say that b is the image of a and a is a pre-image of b. The range of f is the set of all images of elements of A. Also, if f is a

Dated:

function from A to B, we say that f maps A to B.

Example:

Let f be a function that assigns the last 2 bits of bitstring of length 2 or greater to that string.

Domain of f = bitstring of length 2 or greater

Co-domain or range of f = bitstring of length 2 $\{00, 01, 10, 11\}$

Definition # 3

Let f_1 and f_2 be functions of $A \to R$ then (f_1+f_2) and $(f_1 f_2)$ is also function from $A \to R$ defined by:

$$(f_1+f_2)x = f_1(x) + f_2(x)$$
$$(f_1 f_2)x = f_1(x)\, f_2(x)$$

Example:

$$f_1(x) = x^2$$
$$f_2(x) = x - x^2$$

$$(f_1+f_2)x = x^2 + x - x^2 = x$$
$$(f_1 \cdot f_2)x = x^2(x-x^2) = x^3 - x^4$$

Dated:

Definition # 4

Let f be the function from set A to set B and let S be a subset of A. The image of S is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$ so that

$$f(S) = \{f(s) \mid s \in S\}$$

Example:

Let $A = \{a, b, c, d, e\}$

$B = \{1, 2, 3, 4\}$

with, $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, $f(e) = 1$

The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-One And Onto Functions

Some functions have distinct images at distinct members of their domain. These functions are said to be one-to-one.

Definition # 5

A function f is said to be one-to-one or injective if and only if

$f(x) = f(y) \rightarrow x = y \quad \forall x \,\&\, y$ in domain of f

Dated:

Q1. Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$ is one-to-one.

Yes, it is!

Q2. Determine whether the function $f(x) = x^2$ from the set of all integers to the set of all integers is one-to-one.

No, it is not!

because $f(-1) = 1$ & $f(1) = 1$ but $1 \neq -1$

Q3. Determine whether the function $f(x) = x+1$ is one-to-one.

Yes, it is!

Definition # 6

A function f from A to B is called onto function or surjective if and only if for every every element $b \in B$, there is an element $a \in A$ with, $f(a) = b$. A function f is called a surjection if it is onto i.e. the range & co-domain are equal.

Dated:

Q4. Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is that onto?

Yes, it is onto!

Q5. If $f(x) = x^2$ from set of all integers to set of all integers, is it onto?

No, it is not!

Definition # 7

The function f is one-to-one correspondence or bijection if it is both one-to-one and onto.

Q6. Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$. ~~Is~~ Is f a bijection?

Yes, it is!

Q7. Why is f not a function from $\mathbb{R}$ to $\mathbb{R}$ in the following?

1) $f(x) = 1/x$ not defined for $x = 0$

2) $f(x) = \sqrt{x}$ not defined for -ve values

3) $f(x) = \pm\sqrt{x^2+1}$ not well-defined because two distinct values $\pm$ are assigned to x.

Dated:

Inverse Functions:

Let f be a one-to-one correspondence from set A to set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a of A such that $f(a) = b$. The inverse function of f is given by $f^{-1}(b) = a$. when $f(a) = b$.

Q8. Let f be a function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, $f(c) = 1$. Is f invertible? If yes, what is its inverse?

Yes, it is invertible because it is a bijective func.

$$f^{-1}(1) = c$$
$$f^{-1}(2) = a$$
$$f^{-1}(3) = b$$

Q9. Let f be the function from $\mathbf{Z}$ to $\mathbf{Z}$ such that
$$f(x) = x+1.$$
Is f invertible?

Yes, it is invertible.

Let y be the image of $x+1$

$$f(x) = x+1 \Rightarrow y = x+1$$
$$x = y-1$$
$$f^{-1}(y) = y-1 \qquad \because f^{-1}(y) = x$$

$$f^{-1}(x+1) = x$$

Dated:

Compositions of Functions

Let g be a function from set A to set B.
Let f be a function from set B to set C.

The composition of the function f & g denoted by $(f o g)$ is defined by

$$(f o g)\, a = f(g(a))$$

Q10. Let f & g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g?

$$(f o g)\, x = f(g(x))$$

$$= f(3x+2)$$

$$= 2(3x+2) + 3 \qquad = 6x + 5$$

What is the composition of g and f?

$$(g o f)\, x = g(f(x))$$

$$= g(2x+3)$$

$$= 3(2x+3) + 2 \qquad = 6x + 9 + 2$$

$$= 6x + 11$$

Dated:

Floor And Ceiling Functions

The floor function assigns to the real number x the largest integer that is less than or equal to 'x'. The value of the floor function at 'x' is denoted by $\lfloor x \rfloor$.

Examples: $\lfloor 1/2 \rfloor = 0$, $\lfloor 7 \rfloor = 7$, $\lfloor -1/2 \rfloor = -1$, $\lfloor 3.1 \rfloor = 3$

The ceiling function assigns to the real number x the smallest integer that is greater than or equal to 'x'. The value of the ceiling function at 'x' is denoted by $\lceil x \rceil$.

Examples: $\lceil 1/2 \rceil = 1$, $\lceil 7 \rceil = 7$, $\lceil -1/2 \rceil = 0$, $\lceil 3.1 \rceil = 4$

Q. Find the following values:

1) $\lceil 3/4 \rceil = 1$

2) $\lfloor 7/8 \rfloor = 0$

3) $\lceil -3/4 \rceil = 0$

4) $\lfloor -7/8 \rfloor = -1$

5) $\lceil 3 \rceil = 3$

6) $\lfloor -1 \rfloor = -1$

Dated:

Q. Determine whether each of the following functions from $\{a, b, c, d\}$ to itself is one to one.

1) $f(a) = b$, $f(b) = a$, $f(c) = d$, $f(d) = c$

Yes

2) $f(a) = b$, $f(b) = b$, $f(c) = d$, $f(d) = c$

No

3) $f(a) = d$, $f(b) = a$, $f(c) = b$, $f(d) = d$

Yes

Q. Find fog and gof where

$f(x) = x^2 + 1$

$g(x) = x + 2$

are functions from $\mathbb{R}$ to $\mathbb{R}$.

$$(fog)x = f(g(x))$$

$$= (x+2)^2 + 1$$

$$= x^2 + 4x + 4 + 1$$

$$(fog)x = x^2 + 4x + 5$$

$$(gof)x = g(f(x))$$

$$= (x^2 + 1) + 2$$

$$= x^2 + 3$$

Dated:

SEQUENCES AND SUMMATIONS

Sequences:

A sequence is a discrete structure used to represent an ordered list. A sequence is a function from the subset of a set of integers to a set S.

We use the notation a_n to denote the image of integer n. We call a_n a term of the sequence.

The finite sequences are also called strings.
The length of the string S is the number of terms
The empty string is the string that has no terms, its length is 0.

Consider the sequence $\{a_n\}$ where $a_n = 1/n$

$a_1, a_2, a_3, a_4, \ldots$
$1, 1/2, 1/3, 1/4, \ldots$

Consider the sequence $\{b_n\}$ where $b_n = (-1)^n$

$b_1, b_2, b_3, \ldots$

$-1, 1, -1, 1, \ldots$

Dated:

Summation:

The sum of terms is expressed as

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n = \sum_{j=m}^{n} a_j$$

j = index from m to n.

Geometric Progression:

It is given by:

$$a_n = ar^{n-1}$$

Where,

a = general term

r = common ratio

$$a, ar, ar^2, ar^3, \ldots$$

$$S = \sum_{j=0}^{n} ar^j$$

Dated:

EXERCISE

Q. Find the following terms of the sequence $\{a_n\}$ where

$$a_n = 2(-3)^n + 5^n$$

$a_0 = 2(-3)^0 + 5^0$
$= 2(1) + 1$
$= 3$

$a_1 = 2(-3)^1 + 5^1$
$= 2(-3) + 5$
$= -1$

$a_4 = 2(-3)^4 + 5^4$
$= 2(81) + 625$
$= 787$

$a_5 = 2(-3)^5 + 5^5$
$= 2(-243) + 3125$
$= 2639$

Q. What are the values of following sums:

1) $\sum_{j=1}^{5} j^2$

$= 1 + 4 + 9 + 16 + 25$

$= 55$

2) $\sum_{k=4}^{8} (-1)^k$

$= 1 + (-1) + 1 + (-1) + 1$

$= 1$

3) $\sum_{k=1}^{5} (k+1)$

$= 2 + 3 + 4 + 5 + 6$

$= 20$

4) $\sum_{i=1}^{10} 3$

$= 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3$

$= 30$

Dated:

5) $\sum_{j=0}^{4} (-2)^j$

$= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4$

$= 11$

6) $\sum_{i=1}^{10} (3i)$

$= 3+6+9+12+15+18+21+24+27+30$

$= 165$

7) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

$= (2-1)+(4-2)+(8-4)+(16-8)+(32-16)+(64-32)+(128-64)+(256-128)+(512-256)$

$= 511$

Q. Compute each of the following double sums.

1) $\sum_{i=1}^{4} \sum_{j=1}^{3} (ij)$

$= \sum_{i=1}^{4} (1+2+3)i$

$= \sum_{i=1}^{4} 6i$

$= 6(1+2+3+4)$

$= 6(10)$

$= 60$

2) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i-j)$

$= \sum_{i=1}^{3} (i-1)+(i-2)$

$= \sum_{i=1}^{3} (2i-3)$

$= [2(1)-3]+[2(2)-3]+[2(3)-3]$

$= (2-3)+(4-3)+(6-3)$

$= 3$

Dated:

3) $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i + 2j)$

$= \sum_{i=0}^{3} [3i + 2(0)] + [3i + 2(1)] + [3i + 2(2)]$

$= \sum_{i=0}^{3} (3i + 0 + 3i + 2 + 3i + 4)$

$= \sum_{i=0}^{3} (9i + 6)$

$= [9(0) + 6] + [9(1) + 6] + [9(2) + 6] + [9(3) + 6]$

$= 6 + (9+6) + (18+6) + (27+6)$

$= 78$

4) $\sum_{i=0}^{3} \sum_{j=0}^{2} j$

$= \sum_{i=0}^{3} (0 + 1 + 2)$

$= \sum_{i=0}^{3} (3)$

$= 3 + 3 + 3 + 3$

$= 12$

5) $\sum_{i=0}^{3} \sum_{j=0}^{2} i$

$= \sum_{j=0}^{2} (0+1 + 2 + 3)$

$= \sum_{j=0}^{2} (6)$

$= 6 + 6 + 6$

$= 18$

Dated:

6) $\sum_{i=0}^{2} \sum_{j=0}^{2} i^2 j^3$

$= \sum_{i=0}^{2} i^2 \left[(0)^3 + (1)^3 + (2)^3\right]$

$= \sum_{i=0}^{2} i^2 (0+1+8)$

$= \sum_{i=0}^{2} 9i^2$

$= 9(0)^2 + 9(1)^2 + 9(2)^2$

$= 0 + 9 + 36$

$= 45$

Dated:

Chapter 3

Algorithms

An algorithm is a definite procedure for solving problems using a finite number of steps

1) Describe an algorithm for finding the largest element in a finite sequence of integers.

1) Set the temperory max. equal to the first integer in the sequence.

2) Compare the next integer in the sequence with the temp. max. and if it is larger than the temp. max., set the temp. max. equal to this integer.

3) Repeat the previous step if the sequence has more integers.

4) Stop when there are no integers left in the sequence.

```
procedure max (a1, a2, ....., an : integers)
  max := a1
  for i := 2 to n
     if max < ai then max := ai
  {max is the largest element}
```

Dated:

Chapter 3

The Linear Search:

procedure linear search (x: integer, $a_1, a_2, \ldots, a_n$: distinct integers)

```
i := 1
while (i ≤ n and x ≠ a_i)
i := i + 1
if i ≤ n then location := i
else location := 0
```

{location is the subscript of the term that equals x, or is 0 if x is not found}

The Binary Search

procedure: binary search (x: integer, $a_1, a_2, \ldots, a_n$: increasing integers)

```
i := 1 {i is left endpoint of search interval}
j := n {j is right endpoint of search interval}
while i < j
begin
    m := ⌊(i+j)/2⌋
    if x > a_m then i := m+1
    else j := m
end
if x = a_i then location := i
else location := 0
```

{location is the subscript of the term equal to x or 0 if x is not found}

Dated:

Q. Apply linear search on the following sequence and search element 12.

53, 90, 102, 12, 33, 45, 7, 87, 28, 62

```
i := 1
while (i ≤ 10 and x ≠ ai)
{
    i = i+1;
}
if i ≤ 10 then location := i
else
    location := 0
```

i	a_i	location	
1	53	1	$12 \neq 53$
2	90	2	$12 \neq 90$
3	102	3	$12 \neq 102$
4	12	4	$12 = 12$

location $= 4$

Dated:

Q. Apply binary search and find the element 19 in the list.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	5	6	7	8	10	12	13	15	16	18	19	20	22

mid	low	high
$\frac{1+16}{2} = 8$	1	16

mid value < search value $10 < 19$

low = mid + 1 = 8 + 1
= 9

mid	low	high
$\frac{9+16}{2} = 12$	9	16

$16 < 19$ low = 12 + 1 = 13

mid	low	high
$\frac{13+16}{2} = 14$	13	16

mid value = search value

location = 14

Dated:

Complexity of Algorithms

It means to find out the efficiency of the algorithm. There are two complexities:

1) Time complexity:
An analysis of the time required to solve a problem of a particular size involves the time complexity of the algorithm.

2) Space complexity:
An analysis of the computer memory required involves the space complexity of the algorithm.

Describe the time complexity of the linear search algorithm.

$2n+2$ comparisons are used when x is not present in the list. This is the worst case analysis. By the worst case performance of an algorithm, we mean the largest no. of operations needed to solve the given problem. When x is not present the big-o-notation is O.

Dated:

Describe the time complexity of binary search algorithm.

$2\log n + 2$ comparisons are used to perform a binary search. Its big-O-notation is $O(\log n)$. Binary search is fast.

Describe time complexity of the algorithm of finding the max. element of the set.

Since two comparisons are used for each of the second through the n^{th} elements and one more comparison is used to exit the loop. When $i = n+1$, exactly $2(n-1) + 1 = 2n - 1$ comparisons are used whenever this algorithm is applied.
The algorithm for finding the maximum of the set of elements has the time complexity. Big-O-notation "$O(n)$" is used to express the time complexity.

Describe the average case performance of linear search algorithm.

There are n types of possible inputs when x is known to be in the list. If x is the first term of list, three comparisons are needed: One to determine whether the end of the list is reached, one to compare x and first term and one outside the loop.

Dated:

If x is the second term of the list, two more comparisons are needed. So that a total of five comparisons are used. If x is ith term of the list, two comparisons will be used at each of the i steps of the loop, so that a total of $2i+1$ comparisons are needed. Hence, the average number of comparisons used equals

$$\frac{2(1+2+3+\cdots+n)+1}{n}$$

Since

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\Rightarrow \frac{2[n(n+1)/2]}{n}+1 = n+2$$

Commonly Used Terminology for the Complexity of Algorithm

$O(1)$	~~linear~~ constant complexity
$O(\log n)$	logarithmic complexity
$O(n)$	linear complexity
$O(n \log n)$	$n \log n$ complexity
$O(n^b)$	polynomial complexity
$O(b^n)$ where $b>1$	exponential complexity
$O(n!)$	factorial complexity

Dated:

Integers And Division

Q. Show that 101 is prime.

The only primes not exceeding $\sqrt{101}$ are 2, 3, 5 and 7. Since 101 is not divisible by either of them, hence it is prime.

$$\begin{array}{r|l} 101 & 101 \\ \hline & 1 \end{array}$$

Q. Find the prime factorization of 7007.

$$\begin{array}{r|l} 7 & 7007 \\ 7 & 1001 \\ 11 & 143 \\ 13 & 13 \\ & 1 \end{array}$$

prime factors of $7007 = 7^2 . 11 . 13$

Q. Show that if a is an integer other than 0 then

a) 1 divides a

$a = dq + r$

$a = 1a + 0$

$a = a$

$$\begin{array}{r|l} & a \\ 1 & a \\ & -a \\ \hline & 0 \end{array}$$

b) a divides 0

$a = dq + r$

$a = a(0) + 0$

$a = 0$

$$\begin{array}{r|l} & 0 \\ a & 0 \\ & -0 \\ \hline & 0 \end{array}$$

Dated:

Q. Does 17 divides each of the following:

1) 68

$$\begin{array}{r} 4 \\ 17\overline{)68} \\ \underline{68} \\ 0 \end{array}$$

Yes

2) 84

$$\begin{array}{r} 4 \\ 17\overline{)84} \\ \underline{-68} \\ 16 \end{array}$$

No

3) 357

$$\begin{array}{r} 21 \\ 17\overline{)357} \\ \underline{-357} \\ 0 \end{array}$$

Yes

4) 1001

$$\begin{array}{r} 58 \\ 17\overline{)1001} \\ \underline{-986} \\ 15 \end{array}$$

No

Q. Find prime factors of the following:

1) 39

$$\begin{array}{r|l} 3 & 39 \\ \hline 13 & 13 \\ \hline & 1 \end{array}$$

$39 = 3 \times 13$

2) 81

$$\begin{array}{r|l} 3 & 81 \\ 3 & 27 \\ 3 & 9 \\ 3 & 3 \\ & 1 \end{array}$$

$81 = 3 \times 3 \times 3 \times 3$

3) $101 = 101 \times 1$

$$\begin{array}{r|l} 101 & 101 \\ \hline & 1 \end{array}$$

4) $143 = 11 \times 13$

$$\begin{array}{r|l} 11 & 143 \\ 13 & 13 \\ & 1 \end{array}$$

Dated:

5) 289   6) 899

Q. Are the following integers prime?

1) 19   2) 27

3) 93   4) 101

5) 107   6) 113

Dated:

Q. Evaluate these quantities:

1) $-17 \bmod 2$

1

2) $144 \bmod 7$

4

3) $-221 \bmod 23$

9

4) $155 \bmod 19$

3

Q. What are the GCD and LCM of the following pairs:

1) $(1000, 625)$

$1000 = 2^3 \times 5^{-3}$

$625 = 5^4$

2	1000
2	500
2	250
5	125
5	25
5	5
	1

5	625
5	125
5	25
5	5
	1

$G.C.D = 5^{-3}$

$L.C.M = 2^3 \times 5^4$

Dated:

APPLICATIONS OF CONGRUENCES

1) Hashing Function

It locates the memory location so that data is retrieved quickly.

$$\underbrace{h(\overbrace{k}^{\text{key}})}_{\text{hashing function}} = k \bmod m \rightarrow \text{no. of available memory locations}$$

2) Pseudorandom Numbers

Numbers generated by systematic method are not truly random. They are called pseurandom numbers.

The procedure is linear congruential method.

$$x_{n+1} = (ax_n + c) \bmod m$$

Where,

m = modulus
a = multiplier
c = increament
x_0 = seed

Dated:

Example:

$m = 9,\ a = 7,\ c = 4,\ x_0 = 3$

$x_1 = (7.3 + 4) \bmod 9 = 25 \bmod 9 = 7$
$x_2 = (7.7 + 4) \bmod 9 = 53 \bmod 9 = 8$
$x_3 = (7.8 + 4) \bmod 9 = 60 \bmod 9 = 6$
$x_4 = (7.6 + 4) \bmod 9 = 46 \bmod 9 = 1$
$x_5 = (7.1 + 4) \bmod 9 = 11 \bmod 9 = 2$
$x_6 = (7.2 + 4) \bmod 9 = 18 \bmod 9 = 0$
$x_7 = (7.0 + 4) \bmod 9 = 4 \bmod 9 = 4$
$x_8 = (7.4 + 4) \bmod 9 = 32 \bmod 9 = 5$
$x_9 = (7.5 + 4) \bmod 9 = 39 \bmod 9 = 3$

3) Cryptology

It is the study of secret messages.

plain text $\xrightarrow{\text{encryption}}$ cipher text

Ceasar Cypher Encryption method

$$f(p) = (p+3) \bmod 26$$

a	b	c	d	e	f	g	h	i	j	k	l
0	1	2	3	4	5	6	7	8	9	10	11
m	n	o	p	q	r	s	t	u	v	w	x
12	13	14	15	16	17	18	19	20	21	22	23
y	z										
24	25										

Dated:

Q. What is the secret message produced from the message "MEET YOU IN THE PARK" using Ceasar cipher?

MEET	YOU	IN	THE	PARK
12 4 4 19	24 14 20	8 13	19 7 4	15 0 17 10

Replacing each number by $(p+3) \bmod 26$

15 7 7 22   1 17 23   11 16   20 10 7   18 3 20 13

PHHW BRX LQ WKH SDUN.

Q. What letter replaces the letter 'K' when the function $f(p) = (7p+3) \bmod 26$ is used for encryption?

$$\begin{aligned} f(10) &= [7(10)+3] \bmod 26 \\ &= 73 \bmod 26 \\ &= 21 \\ &= V \end{aligned}$$

Q. Encrypt the message "DO NOT PASS GO" by translating letters into numbers, applying the encryption function given.

1) $f(p) = (p+3) \bmod 26$

DO	NOT	PASS	GO
3 14	13 14 19	15 0 18 18	6 14

Dated:

Replacing each number by $(p+3) \bmod 26$

6 17   16 17 22   18 3 21 21   9 17
G R   Q R W   S D V V   J R

2) $f(p) = (p+13) \bmod 26$

16 1   0 1 6   2 13 5 5   19 1
Q B   A B G   C N F F   T B

3) $f(p) = (3p+7) \bmod 26$

16 23   20 23 12   0 7 9 9   25 23
Q X   U X M   A H J J   Z X

Q. Decrypt these messages using Ceaser cipher.

1) E O X M   M H D Q V
4 14 23 12   12 7 3 16 21

Decrypted:-

1 11 20 9   9 4 0 13 18
B L U J   J E A N S

2) W H V W   W R G D B
22 7 21 22   22 17 6 3 1

Decrypted:

19 4 18 19   19 14 3 0 24
T E S T   T O D A Y

Dated:

Q. Which memory locations are assigned by the hashing function $h(K) = K \bmod 101$

1) 104578690

$h(104578690) = 104578690 \bmod 101 = 58$

2) 432222187

$h(432222187) = 60$

3) 372201919

$h(372201919) = 52$

4) 501338753

$h(501338753) = 3$

Dated:

THE EUCLIDEAN ALGORITM

Q1. Find GCD of the following:

1) (320, 296)

Applying division

1

$$\begin{array}{r} 1 \\ 296\overline{)320} \\ \underline{296} \\ 24 \end{array}$$

$\Rightarrow$ $320 = 296(1) + 24$

2

$$\begin{array}{r} 12 \\ 24\overline{)296} \\ \underline{-288} \\ 8 \end{array}$$

$\Rightarrow$ $296 = 24(12) + 8$

3

$$\begin{array}{r} 3 \\ 8\overline{)24} \\ \underline{-24} \\ 0 \end{array}$$

$\Rightarrow$ $24 = 8(3) + 0$

$\Rightarrow$ G.C.D = 8

2) (346, 592)

Applying division

1

$$\begin{array}{r} 1 \\ 346\overline{)592} \\ \underline{-346} \\ 246 \end{array}$$

$\Rightarrow$ $\cancel{346} = 592 = 346(1) + 246$

2

$$\begin{array}{r} 1 \\ 246\overline{)346} \\ \underline{246} \\ 100 \end{array}$$

$\Rightarrow$ $346 = 246(1) + 100$

3

$$\begin{array}{r} 2 \\ 100\overline{)246} \\ \underline{200} \\ 46 \end{array}$$

$\Rightarrow$ $246 = 100(2) + 46$

Dated:

④ $46 \overline{)100}$ quotient 2; -92; remainder 8

$\Rightarrow 100 = 46(2) + 8$

⑤ $8 \overline{)46}$ quotient 5; -40; remainder 6

$\Rightarrow 46 = 8(5) + 6$

⑥ $6 \overline{)8}$ quotient 1; -6; remainder 2

$\Rightarrow 8 = 6(1) + 2$

⑦ $2 \overline{)6}$ quotient 3; -6; remainder 0

$\Rightarrow 6 = 2(3) + 0$

G.C.D = 2

3) (111, 201)

①

$201 = 101(1) + 100$

$101 \overline{)201}$ quotient 1; -101; remainder 100

②

$101 = 100 + 1$

$100 \overline{)101}$ quotient 1; -100; remainder 1

③

$100 = 1(100) + 0$

$1 \overline{)100}$ quotient 100; -100; remainder 0

G.C.D = 1

Dated:

Euclidean Algorithm And Bezout's Theorem

Q. Show that gcd (324, 148) = 4. Find the integers u and v from Bezout's theorem.

① $324 = 148(2) + 28$
② $148 = 28(5) + 8$
③ $28 = 8(3) + 4$
④ $8 = 4(2) + 0$

$\Rightarrow$ G.C.D = 4 Proved! $4 = 324u + 148v$ — ⊕

Consider eq ①, ② & ③ and solve for remainder

1) $\Rightarrow$ $28 = -148(2) + 324$
2) $\Rightarrow$ $8 = 148 - 28(5)$
3) $\Rightarrow$ $4 = 28 - 8(3)$

Eq ③ as linear combination of 28 & 8

$4 = 28 + 8(-3)$

Back substitution

$4 = 28 + [148 - 28(5)](-3)$

$4 = 28 + 148(-3) + 28(15)$

Dated:

$4 = 28(16) + 148(-3)$

Back substitution

$4 = [324 - 148(2)](16) + 148(-3)$

$4 = 324(16) + 148(-32) + 148(-3)$

$4 = 324(16) + 148(-35)$

Comparing with eq (A)

$\Rightarrow \boxed{u = 16} \quad \boxed{v = -35}$

Q. Show that gcd (15, 36) = 3. And find the integers u and v by bezout's theorem.

① $36 = 15(2) + 6$

② $15 = 6(2) + 3$

③ $6 = 3(2) + 0$

$G.C.D = 3$

$3 = 15u + 36v$ — (A)

Consider eq ① & ② and solve for remainder

1) $\Rightarrow 6 = 36 - 15(2)$

2) $\Rightarrow 3 = 15 - 6(2)$

Dated:

Eq ② as a linear combination of 6 & 15

$$3 = 15 - 6(2)$$

Back substitution

$$3 = 15 + [36 - 15(2)](-2)$$

$$= 15 + 36(-2) + 15(4)$$

$$= 15(5) + 36(-2)$$

Comparing with eq Ⓐ

$\boxed{U = 5}$ $\boxed{V = -2}$

Dated:

METHODS OF PROOF

Rules of Inference

Addition $p \rightarrow (p \vee q)$

Simplification $(p \wedge q) \rightarrow p$

Conjunction $((p) \wedge (q)) \rightarrow (p \wedge q)$

Modus ponens $[p \wedge (p \rightarrow q)] \rightarrow q$

Modus tollens $[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$

Hypothetical syllogism $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Disjunctive syllogism $[(p \vee q) \wedge \neg p] \rightarrow q$

Resolution $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$

Contrapositive $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$

Modus Badus 1) $[(p \rightarrow q) \wedge q] \rightarrow p$ invalid

2) $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ invalid

Dated:

Q1. Which rule of inference is used in each of these arguments?

1) Alice is a math major. Therefore either Alice is a math major or a computer major.

Addition rule

2) Jerry is a math major and a computer major. Therefor Jerry is a math major.

Simplification

3) If it is rainy then the pool will be closed. It is rainy. Therefore, the pool is closed.

Modus ponens

4) If it snows today, the university will close. The university is not closed today. Therefore it did not snow today.

Modus tollens

5) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long then I will sun-burn. Therefore, if I go swimming then I will sun-burn.

hypothetical syllogism

Dated:

Q. Show that the premises "Everyone in this class has taken a course in computer" and "Marla is in this class" imply the conclusion that "Maxla has taken a course in computer."

$\Delta(x)$ = x is in this class
$C(x)$ = x has taken a course in computer

Premises : $\forall x \, (\Delta(x) \to C(x))$
$\Delta(\text{Maxla})$

1)	$\forall x (\Delta(x) \to C(x))$	Premise
2)	$\Delta(\text{Maxla}) \to C(\text{Maxla})$	Universal instantiation
3)	$\Delta(\text{Marla})$	Premise
4)	$C(\text{Maxla})$	Modus Ponen from 2 & 3

Q.

Dated:

Q. ~~Proof~~ Prove that the square of an even no. is an even no. using a direct proof, indirect proof and proof by contradiction.

Direct Proof:

If n is even then n^2 is even.

Assume that the hypothesis (n is even) is true. Then,

$n = 2K$ where K is an integer

It follows that:

$$(n)^2 = (2K)^2$$
$$n^2 = 4K^2$$
$$n^2 = 2(2K^2)$$

$\Rightarrow$ n^2 is an even integer because it is twice of an integer

Indirect Proof:

Taking contrapositive:

If n^2 is odd then n is odd

Assuming hypothesis (n^2 is odd) is true

$$n^2 = (2K+1)^2$$
$$\cancel{n^2 = 4K^2 + 4K + 1}$$
$$\sqrt{n^2} = \sqrt{(2K+1)^2}$$
$$n = 2K+1$$

$\Rightarrow$ n is odd because it is one more than twice an integer.

Dated:

Proof By Contradiction:

Let us assume that n^2 is even and n^2 is odd

By definition

$$n = 2K$$
$$n^2 = 4K^2$$
$$n^2 = 2(2K^2)$$

Hence it must be even!

Q. Prove that sum of two odd integers is even.

Direct Proof:

Let n be an odd integer.

By definition

$$= n + n$$
$$= (2K+1) + (2K+1)$$
$$= 4K + 2$$
$$= 2(2K+1)$$

=> Sum is even because it is twice an integer.

Dated:

Q. Prove that sum of two rational numbers is rational.

If a & b are rational then $a+b$ is rational

Suppose a & b are rational such that

$$a = \frac{l}{m} \qquad b = \frac{q}{r} \quad \text{where } m \neq 0,\ r \neq 0$$

Now

$$a+b = \frac{l}{m} + \frac{q}{r} = \frac{lr + qm}{mr}$$

Since $m \neq 0$, $r \neq 0$ therefore $mr \neq 0$

Hence $a+b$ is a rational number. Proved!

Q. Show that $\sqrt[3]{3}$ is irrational.

Proof by Contradiction:

Suppose that $\sqrt[3]{3}$ is rational

then

$$\sqrt[3]{3} = \frac{a}{b} \Rightarrow 3 = \frac{a^3}{b^3}$$

$$3b^3 = a^3$$

Dated:

MATRICES

Q1. Find AB when

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2+0+16 & 4+0+0 \\ 4+1+4 & 8+1+0 \\ 6+1+0 & 12+1+0 \\ 0+2+8 & 0+2+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18 & 4 \\ 9 & 9 \\ 7 & 13 \\ 10 & 2 \end{bmatrix}$$

Q2. $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Does $AB = BA$?

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+1 & 1+1 \\ 4+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 2+1 \\ 1+2 & 1+1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$

$AB \neq BA$

Join = $A \vee B$

Meet = $A \wedge B$

Dated:

Q. Find join and meet of the following matrices.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{Join } A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{Meet } A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Q. Find boolean product of:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

Dated:

$$A \odot B = \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q. Find A^{-1}.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A \quad \text{—} ①$$

$$|A| = \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = (-3-2) \quad = -5$$

$$\text{Adj } A = \begin{bmatrix} 3 & -2 \\ -1 & -1 \end{bmatrix}$$

$$① \Rightarrow \quad A^{-1} = \frac{1}{-5} \begin{bmatrix} 3 & -2 \\ -1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}$$

Dated:

Chapter 4

Mathematical Induction And Reccursion

Well- Ordering property: The validity of the mathematical induction follows from the following axiom about the set of integers.

"Every non-empty set of non-negative integers has a least element.

Q. Use the well-ordering property to prove the division algorithm.

The division algorithm states that if $\frac{a}{d} = q$

then $a = dq + r$

Let S be the set of non-negative integers of the form $a - dq$ where q is an integer. This set is non-empty since $-dq$ can be made as large as desired. By the well-ordering property S has a least element $r = a - d_0$.
The integer r is non-negative. It is also the case that $r < d$.

Dated:

Q. Use mathematical induction to prove that the sum of first n odd +ve integers is n^2.

Basic Step: $P(1)$ states that the sum of first n odd +ve integers is 1^2. This is true since the sum of first odd +ve integer is 1.

Inductive Step: To show that the proposition $P(n) \rightarrow P(n+1)$ is true for every +ve integer n.

Suppose that $P(n)$ is true for a +ve integer 'n'
i.e.

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$$

We must show that $P(n+1)$ is true assuming that $P(n)$ is true. $P(n+1)$ states:

$$1+3+5+7+\cdots+(2n-1)+(2n+1) = (n+1)^2$$

So assuming that $P(n)$ is true, it follows that

$$[1+3+5+7+\cdots+(2n-1)+(2n+1)] = [1+3+5+7+\cdots+(2n-1)]+(2n+1)$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2$$

Since $P(1)$ is true and the implication $P(n) \rightarrow P(n+1)$ is true for all +ve integers n, the principle of MI shows that $P(n)$ is true for all +ve integers n.

Dated:

Q. Use MI to prove that:

$$1 + 2 + 2^2 + \text{-----} + 2^n = 2^{n+1} - 1$$

for all non-negative integers 'n'.

Basic Step:

P(0) is true since

$$2^0 = 2^{0+1} - 1$$
$$1 = 1$$

Inductive Step:

Assume that P(n) is true and hence it must be shown that P(n+1) is also true

P(n+1) states that:

$$1 + 2 + 2^2 + 2^3 + \text{-----} + 2^n + 2^{n+1} = 2^{(n+1+1)} - 1$$
$$= 2^{(n+2)} - 1$$

By P(n) it follows that:

$$[1 + 2 + 2^2 + \text{----} + 2^n] + 2^{n+1} = (2^{(n+1)} - 1) + 2^{n+1}$$
$$= 2^{n+1+1} - 1$$
$$= 2^{n+2} - 1$$

Hence proved!

Dated:

Q. Use MI to prove the following formula ~~for~~ for the sum of a finite number of terms of a G.P.

$$a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r-1}$$

Basic Step:

P(0) is true since

$$ar^0 = \frac{ar^{0+1} - a}{r-1}$$

$$a(1) = \frac{ar - a}{r-1} = \frac{a\cancel{(r-1)}}{\cancel{r-1}}$$

$$a = a$$

Proved!

Inductive Step:

P(n+1) states that

$$a + ar + ar^2 + \cdots + ar^n + ar^{n+1} = \frac{ar^{(n+1+1)} - a}{r-1}$$

$$= \frac{ar^{(n+2)} - a}{r-1}$$

By P(n)

$$[a + ar + ar^2 + \cdots ar^n] + ar^{n+1} = \frac{ar^{(n+1)} - a}{r-1} + ar^{n+1}$$

$$= \frac{ar^{(n+1)} - a + ar^{n+1}(r-1)}{r-1}$$

Dated:

$$= \frac{\cancel{ar^{(n+1)}} - a + ar^{n+2} - \cancel{ar^{(n+1)}}}{r-1}$$

$$= \frac{ar^{(n+2)} - a}{r-1}$$

Proved!

Q. Use MI to prove that $2^n < n!$ for every +ve integer n where $n \geq 4$.

Basic Step:

$P(4)$ is true since

$$2^4 < 4!$$

$$16 < 24$$

Proved!

Inductive Step:

$P(n+1)$ states that:

$$2^{n+1} < (n+1)!$$

Dated:

Recursively-defined Function

Q. Suppose that f is defined reccursively by
$f(0) = 3$
$f(n+1) = 2f(n) + 3$
Find $f(1)$, $f(2)$, $f(3)$ and $f(4)$.

$f(1) = 2f(0) + 3 = 2(3) + 3 = 6 + 3$
$= 9$

$f(2) = 2f(1) + 3 = 2(9) + 3 = 18 + 3$
$= 21$

$f(3) = 2f(2) + 3 = 2(21) + 3 = 42 + 3$
$= 45$

$f(4) = 2f(3) + 3 = 2(45) + 3 = 90 + 3$
$= 93$

Q. Give an inductive definition of the factorial function $\cancel{f(n) = n!}$ $f(5) = 5!$

$f(n+1) = (n+1) f(n)$

Let $n = 4$

$f(5) = 5 f(4)$
$= 5 . 4 f(3)$
$= 5 . 4 . 3 f(2)$
$= 5 . 4 . 3 . 2 f(1)$
$= 5 . 4 . 3 . 2 . 1 . 1 = 5!$

Dated:

Q. The fibonacci numbers, $f_0, f_1, f_2, \ldots$ are defined by the equations $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$

Find the fibonacci numbers f_2, f_3, f_4, f_5, f_6.

$f_2 = 0 + 1 = 1$
$f_3 = 1 + 1 = 2$
$f_4 = 1 + 2 = 3$
$f_5 = 2 + 3 = 5$
$f_6 = 3 + 5 = 8$

Q. Find $f(1)$, $f(2)$, $f(3)$ and $f(4)$ if $f(n)$ is defined reccursively by $f(0) = 1$ and for $n = 0, 1, 2$

a) $f(n+1) = f(n) + 2$

$f(1) = f(0) + 2 = 1 + 2 = 3$
$f(2) = f(1) + 2 = 3 + 2 = 5$
$f(3) = f(2) + 2 = 5 + 2 = 7$
$f(4) = f(3) + 2 = 7 + 2 = 9$

b) $f(n+1) = 3f(n)$

$f(1) = 3f(0) = 3(1) = 3$
$f(2) = 3f(1) = 3(3) = 9$
$f(3) = 3f(2) = 3(9) = 27$
$f(4) = 3f(3) = 3(27) = 81$

Dated:

Q. Give a reccursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 6n \implies a_1 = 6$

$a_{n+1} = 6(n+1) = 6n+6 = a_n + 6$

b) $a_n = 2n+1 \implies a_1 = 3$

$a_{n+1} = 2(n+1)+1 = 2n+2+1 = (2n+1)+2 = a_n+2$

c) $a_n = 10^n \implies a_1 = 10$

$a_{n+1} = 10^{n+1} = 10^n \cdot 10 = 10a_n$

d) $a_n = 5 \implies a_1 = 5$

$a_{n+1} = a_n$

Q. Let f be the function such that $f(n)$ is the sum of the first n +ve integers.

$f(0) = 0$
$f(1) = 0+1 = 1$
$f(2) = 0+1+2 = 3$

Therefore

$f(n) = f(n-1) + n$

Where,

$f(n)$ is the sum of first n +ve integers

Dated:

Reccursive Algorithm

An algorithm is called reccursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

Q. Give a reccursive algorithm for computing a^n where a is a real number and n is a non-negative integer.

procedure: power (a: non-zero real num., n: non-ve integer)

if $n = 0$ then power $(a, n) := 1$

else power $(a, n) := a \cdot$ power $(a, n-1)$

Q. Give a reccursive algorithm for $n!$

procedure factorial (n: positive integer)

if $n = 1$ then
factorial $(n) := 1$

else
factorial $(n) := n \cdot$ factorial $(n-1)$

Dated:

Q. Give a reccursive algorithm for fibonacci numbers.

procedure fibonacci (n: non-negative integer)

if $n=0$ then fibonacci (0) := 0

else if $n=1$ then fibonacci (1) := 1

else fibonacci (n) := fibonacci ($n-1$) + fibonacci ($n-2$)

Q. Give a reccursive algorithm for finding the sum of first 'n' odd +ve integers

procedure sum of odds (n: positive integer)

if $n=1$ then sum of odds (n) := 1

else sum of odds (n) := sum of odd ($n-1$) + $2n-1$

Q. Give a reccursive algorithm for computing the gcd of two non-negative integers a and b with $a < b$.

procedure gcd (a, b: non-negative integers with $a < b$)

if $a=0$ then gcd (a, b) := b

else gcd (a, b) := gcd (b mod a, a)

Dated:

Pigeon - Hole Principle

Q. Among 100 people, how many were born in the same month?

$N = 100$
$K = 12$

$$\left\lceil \frac{N}{K} \right\rceil = \left\lceil \frac{100}{12} \right\rceil = \lceil 8.3 \rceil = 9$$

Q. What is the min. no. of students required in a DS class to be showed that at least 6 will receive the same grade if there are 5 possible grades?

$$\left\lceil \frac{N}{5} \right\rceil = 6 \qquad \left\lceil \frac{26}{5} \right\rceil = 6$$

$\Rightarrow N = 26$

Q. Show that in any set of 6 classes, there must be 2 that meet on the same day assumming that no classes held on weekend.

$$\left\lceil \frac{6}{5} \right\rceil = 2$$

Dated:

Q. Show that there are at least 4 people in California where the population is 25 million with the same 3 initials who were born on the same day of the year but not necessarily the same year.

$$\left\lceil \frac{25\,000\,000}{26 \times 26 \times 26 \times 366} \right\rceil =$$

$$= \lceil 3.88 \rceil = 4$$

Q. A bowl contains 10 red and 10 blue balls. A woman selects balls at random. How many balls must she select to be sure of getting at least 3 ~~blue~~ balls of same color?

$$\left\lceil \frac{N}{2} \right\rceil = 3$$

$$\left\lceil \frac{5}{2} \right\rceil = 3$$

$$\Rightarrow N = 5$$

Q. You are in a room with 10 other people. There are at least 2 people in the room who know the same number of people in the room.

$$\left\lceil \frac{11}{10} \right\rceil = 2$$

Dated:

Tree Traversal

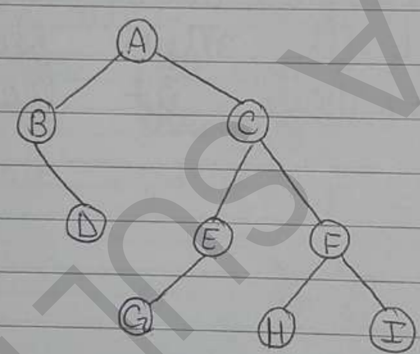
A tree is a connected undirect graph that contains no simple circuits.

1) Pre-order traversal (root -> left -> right)

2) In-order traversal (left -> root -> right)

3) Post-order traversal (left -> right -> root)

(1)



In-order

B - B - A - G - E - C - H - F - I

Pre-order

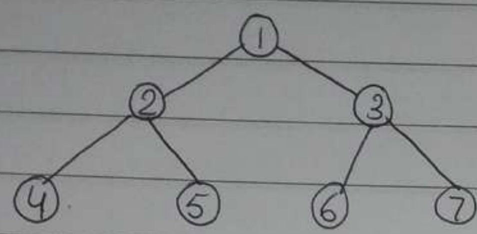
A -> B - D - C - E - G - F - H - I

Post-order

D - B - G - E - H - I - F - C - A

Dated:

2



Pre-order 1-2-4-5-3-6-7
In-order 4-2-5-1-6-3-7
Post-order 4-5-2-6-7-3-1

Dated:

Topics

- ✓ Predicates and Quantifiers
- ✓ Sets
- ✓ Functions
- ✓ Sequences And Summations
- ✓ Algorithms
- Complexity of Algorithms
- ✓ Prime numbers
- ✓ LCM
- ✓ GCD
- ✓ Hashing function
- ✓ Pseudorandom numbers
- ✓ Cryptology
- ✓ Euclidean algorithm
- ✓ Methods of proof
- ✓ Matrices
- Mathematical induction
- ✓ Recursion
- ✓ Pigeon hole principle
- Permutation And Combination
- Binomial theorem
- Probability
- Graphs
- ✓ Dijkstra algorithm
- ✓ Tree Traversal
- ✓ Finite state machine
- ✓ Logic gates

CS-251 Logic Design & Switching Theory

Digital Design → Morris Mano

Digital Fundamentals → Thomas Floyd

Switching theory & logic design → Anand Kumar

Intro to Digital Systems → Thomas Lang

Intro to Digital circuit → Bogart

Number Systems.

igned mag. The MSB of the

The remaining bits represent the mag. of

number same as unsigned binary.

Keep the sign seperate from the magnitude.

leads to difficulties during arithmetic ope.

② Two representations for zero.

MT-273

Course Content:

1) Linear Algebra
2) Ordinary diff eq of 1st order
3) 2nd & higher order diff. eq
4) Partial diff. eq.

Book:

Elementry Linear Algebra → Howard Anton 7th edi

Advanced engg mathematics → Erwin Kreyszing 9th

Diff eq : a modelling prespective → Robert L. Borelli 2nd

Diff eq. by Davis G. Zill

SE by Roger Presman

SE-

Application of systematic, disciplined,
approach to deve operation & maintenance

CS-251 LOGIC DESIGN & SWITCHING THEORY

Reading material

Text Book

Digital Design Morris Mano

Reference books

1- Digital Fundamentals Thomas L. Floyd
2- Introduction to Digital Systems ~~Anand Kumar~~ Thomas Lai
3- Switching theory & logic design → Anand Kumar
4- Intro to digital circuit → Bogart

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→ Anand Kumar
→ Bogart

Number Systems

1- Decimal System
2- Hexadecimal System
3- Binary.
4- Octal.

Conversions:

Decimal to Binary $524)_{10} = \_\_\_\_\_)_2$
Binary to Decimal.

$10111)_2$

$2^4\,2^3\,2^2\,2^1\,2^0$

Hexa   decimal

0
1
2
|
|
9
A
B
C
|
|
F
10 —— 16
|
19
1A
|
|
|
1F
20
|
|
|
30

...lation

...er is the

... the mag.
...unsigned bin...

...ate from the
...ficulties during

...tions for ze...

...on.

Decimal to compliment notation.

1- Represent the no. as unsigned binary
2- Append a '0' bit as MSB.

For -ve no's:

1- Find the compliment notation for the +ve counter part of the number.
2- Take 1's (2's) compliment.

How to take 1's compliment of the binary no:
1- Invert all the bits

How to take 2's compliment of the binary no:

1- Take 1's compli
2- Add a '1' to the LSB.

Shortcut:
1- leave all the trailing zeros unchanged.
2- Leave the 1st '1' unchanged
3- Invert all the remaining bits.

Questions:

1- $24)_{10} = 011000)_{1C}$
2- $24)_{10} = 011000)_{2C}$
3- $-24)_{10} = 100111)_{1C}$
4- $-24)_{10} = 101000)_{2C}$

1. Sign Magnitude representation

$-23)_{10} = \underset{SB}{1}0111)_2$

Convention
0: +ve
1: -ve

1) The LSB of the number is the sign bit.
2) The remaining bits represent the mag. of the number same as unsigned binary

Disadvantage:

1) Keeping the sign separate from the magnitude, leads to difficulties during arithmetic operations.
2) There are two representations for zero

2- Compliment notation:-

- 1's compliment notation.
- 2's compliment notation.

conversions:

For +ve nos:

Decim

1- R
2- App

For

1- Find
+ve
2- Tak

How to
1- ?

How to

1- Take
2- Add

Questions:

1- $24)_{10} =$
2- $24)_{10} =$
3- $-24)_{10} =$
4- $-24)_{10} =$

Compliment notation to decimal conversion

1- Always assign a -ve weight to the MSB.
2- If it is a -ve no. in its 1's compliment notation, add an extra (one) when finding the equivalent.

examples:

1- $\overset{-2^4\,2^3\quad 2^0}{01001})_{1C} = \underline{\quad 9 \quad})_{10}$

2- $01001)_{2C} = \underline{\quad 9 \quad})_{10}$

3- $\overset{-2^4\,2^3\quad 2^0}{11001})_{1C} = \underline{\quad -6 \quad})_{10}$ $\quad -16+8+1+1=-6$

4- $11001)_{2C} = \underline{\quad -7 \quad})_{10}$ $\quad -16+8+\underline{1}=-7$

Representation of zero in compli. notation

1C: $\underline{0000})_{1C} \xrightarrow{1C} 1111 = -0)_{10}$

2C: $\underline{0000})_{2C} \xrightarrow{2C}$

$$\begin{array}{r} 1111 \\ +\ 1 \\ \hline 10000 \\ \hline \end{array}$$

$\xrightarrow{2C} 0000$

Sign extension of signed binary nos.

1- Replicate the MSB of the number.

$11110)_{1C} = 11111110)_{1C}$

$$\begin{matrix} 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 1 & 1 & 1 & 1 & 0 \end{matrix} = -16 + 8 + 4 + 2 = -2+$$

$= -1$

$11111110 = -2.$

$11111101)_{1C}$

$\Updownarrow$

$101)_{1C}$