Exercise # 7.1 Orthogonal Matrices

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Definition: A square matrix A is said to be orthogonal if its transpose is the same as its inverse. i.e if

$$A^{-1} = A^{T}$$

or equivalently, if

$$AA^{T} = A^{T}A = I$$

A matrix transformation $T_A : R^n \longrightarrow R^n$ is said to be an orthogonal transformation or an orthogonal operator if A is an orthogonal matrix.

Example # 01 3x3 Orthogonal Matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$ is orthogonal because:

$$A^{T}.A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example # 02: Rotation and Reflection Matrices are Orthogonal

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$$A^{T}A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ROW VECTORS & COL. VECTORS FORM ORTHONORMAL SETS W.R.T THE EUCLIDEAN INNER PRODUCT.

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Following are equivalent for an $n \times n$ matrix A

a) A is orthogonal.
b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.
c) The column vectors of A form an orthonormal set in R^n with the euclidean inner product.

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a) The transpose of an orthogonal matrix is orthogonal.
b) The ~~inv~~ inverse of an orthogonal matrix is orthogonal.
c) A product of orthogonal matrices is orthogonal.
d) If A is orthogonal, then $\det(A) = \pm 1$

Example # 03 : $\det(A) = \pm 1$ for an Orthogonal Matrix A

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\det(A) = \frac{1}{2} + \frac{1}{2} = \frac{\cancel{2}}{\cancel{2}} = 1$$ orthogonal.

Properties of Orthogonal Transformations

If A is an $n \times n$ matrix, then following are equivalent:
a) A is orthogonal
b) $\|Ax\| = \|x\|$ for all x in R^n
c) $Ax \cdot Ay = x \cdot y$ for all x and y in R^n.

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In each part of exercise 1—4, determine whether the matrix is orthogonal, and if so find its inverse

1- a- $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^T . A = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ orthogonal

$A^{-1} = A^T$

$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b- $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$A^T . A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 + 1/2 & -\cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} \\ -\cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} & \frac{1}{2} + \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ orthogonal

$A^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

2. a $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ orthogonal

$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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b. $$A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5}+\frac{4}{5} & \frac{2}{5}+\frac{2}{5} \\ \frac{2}{5}+\frac{2}{5} & \frac{4}{5}+\frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4/5 \\ \frac{4}{5} & 1 \end{bmatrix}$$ not orthogonal

3. a. $$A = \begin{bmatrix} 0 & 1 & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^{-1} = A^T$ orthogonal.

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

4. b. $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & -1/2 & 0 \\ 0 & 1/\sqrt{3} & 0 & 1 \\ 0 & 1/\sqrt{3} & 1/2 & 0 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & -1/2 & 0 \\ 0 & 1/\sqrt{3} & 0 & 1 \\ 0 & 1/\sqrt{3} & 1/2 & 0 \end{bmatrix}$$

not orthogonal.

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$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ~~not orthogonal matrix.~~

~~So, $A^{-1} = A^{T} =$~~

Q5. In exercise 5–6, show that the matix is orthogonal three ways: first by calculating $A^T.A$, then by using part (b) and (c) of Theorem 7.1.1

$$A = \begin{bmatrix} \frac{4}{5} & 0 & -3/5 \\ \frac{-9}{25} & \frac{4}{5} & -12/25 \\ \frac{12}{25} & \frac{3}{5} & 16/25 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix} \begin{bmatrix} 4/5 & -9/25 & 12/25 \\ 0 & 4/5 & 3/5 \\ -3/5 & -12/25 & 16/25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r_1 = \left[\frac{4}{5}, 0, -\frac{3}{5}\right], \quad r_2 = \left[\frac{-9}{25}, \frac{4}{5}, \frac{-12}{25}\right]$$

$$r_3 = \left[\frac{12}{25}, \frac{3}{5}, \frac{16}{25}\right]$$

$$\langle r_1, r_2 \rangle = \frac{4}{5} \times \frac{-9}{25} + 0 \times \frac{4}{5} + \left(\frac{-3}{5}\right)\left(\frac{-12}{25}\right) = 0$$

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$\langle r_1, r_3 \rangle = 0$ $\quad \langle r_2, r_3 \rangle = 0$

and $\|r_1\| = \|r_2\| = \|r_3\| = 1$

column vectors :

$$c_1 = \begin{bmatrix} 4/5 \\ -9/25 \\ 12/25 \end{bmatrix} \qquad c_2 = \begin{bmatrix} 0 \\ 4/5 \\ 3/5 \end{bmatrix} \qquad c_3 = \begin{bmatrix} -3/5 \\ -12/25 \\ 16/25 \end{bmatrix}$$

$$\langle c_1, c_2 \rangle = \frac{4}{5} \times 0 + \frac{4}{5} \times \frac{-9}{25} + \frac{3}{5} \times \frac{12}{25} = 0$$

$\langle c_2, c_3 \rangle = 0$, $\langle c_1, c_3 \rangle = 0$

and $\|c_1\| = \|c_2\| = \|c_3\| = 1$

Q7. let $T_A : R^3 \to R^3$ be multiplication by the orthogonal matrix in exercise 5. Find $T_A(x)$ for the vector $x = (-2, 3, 5)$ and confirm that $\|T_A(x)\| = \|x\|$ relative to the euclidean inner product.

$$T_A(x) = \begin{bmatrix} \frac{4}{5} & 0 & \frac{-3}{5} \\ \frac{-9}{25} & \frac{4}{5} & \frac{-12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{-23}{5} \\ \frac{18}{25} \\ \frac{101}{25} \end{bmatrix}$$

$$\|T_A(x)\| = \sqrt{\frac{529}{25} + \frac{324}{625} + \frac{10201}{625}} = \sqrt{38}$$

$$\|(x)\| = \sqrt{4 + 9 + 25} = \sqrt{38}$$

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QUADRATIC FORMS

$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2$

$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3$

$a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n^2 +$ (all possible terms $a_kx_ix_j$ in which $i \neq j$)

The terms of the form $a_kx_ix_j$ in which $i \neq j$ are called cross product terms.

ASSOCIATED QUADRATIC FORM

$$x^T Ax = x \cdot Ax = Ax \cdot x$$

$$x^T Ax = [x_1, x_2, \dots x_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & & \\ 0 & & \ddots & \\ 0 & \dots & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \lambda_1 x_1^2 + \lambda_1 x_2^2 + \dots + \lambda_n x_n^2$$

Example #01: Expressing Quadratic Forms in Matrix Notation

In each part, express the quadratic form in the matrix notation $x^T Ax$, where A is symmetric.

(a) $2x^2 + 6xy - 5y^2$

The diagonal entries of A are the coefficients of the squared terms and the off-diagonal entries are half the coefficients of the cross product terms, so

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$$2x^2 + 6xy - 5y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ - & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Q. 1-2: Express the quadratic form in the matrix notation $x^T A x$, where A is a symmetric matrix.

1. a. $3x_1^2 + 7x_2^2$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b. $4x_1^2 - 9x_2^2 - 6x_1x_2$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c. $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2. a. $5x_1^2 + 5x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & \frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

b. $-7x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & -7/2 \\ \frac{-7}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

c. $x_1^2 + x_2^2 - 3x_3^2 - 5x_1x_2 + 9x_1x_3$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{5}{2} & \frac{9}{2} \\ \frac{5}{2} & 1 & 0 \\ \frac{9}{2} & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Q3-4: Find a formula for the quadratic form that doesnot use matrices.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x_1^2 + 5y^2 - 6xy$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -2 & 7/2 & 1 \\ 7/2 & 0 & 6 \\ 1 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2x_1^2 + 3x_3^2 + 7x_1x_2 + 2x_1x_3$$

$$+ 12x_2x_3$$