Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

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◇ all three recurrences iterate $\log_2 n$ times

◇ in each case, size of subproblem in next iteration is half the size in the preceding iteration level

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- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/2) + n,$$
where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

  1. If $a < 2$, then $T(n) = \Theta(n)$.
  2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
  3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$

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Proof

We already proved Case 1 when $a = 1$ in Example 3.
(will not prove it for $1 < a < 2$)
We already proved Case 2 in Example 1.
We will now prove Case 3.

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$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Work at "bottom"

Iterated Work

Total work

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Since $a > 2$, the geometric series is Θ of the largest term.

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1-(a/2)^{\log_2 n}}{1-a/2} = n\Theta((a/2)^{\log_2 n - 1})$$

Total work

- n times the largest term in the geometric series is

$$n\left(\frac{a}{2}\right)^{\log_2 n-1} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{n} = \frac{2}{a}\cdot a^{\log_2 n}$$

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$$\Theta\left(n^{\log_2 a}\right) \qquad \Theta\left(n^{\log_2 a}\right)$$

Example 5 Recap

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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$a = 4$, so the Theorem says that

$$T(n) = \Theta\left(n^{\log_2 a}\right) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

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This matches with the exact answer of $2n^2 - n$.

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The Master Theorem

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/b) + cn^d,$$
where a is a positive integer, $b \geq 1$, c, d are real numbers with c positive and d nonnegative, and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

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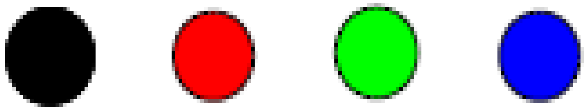

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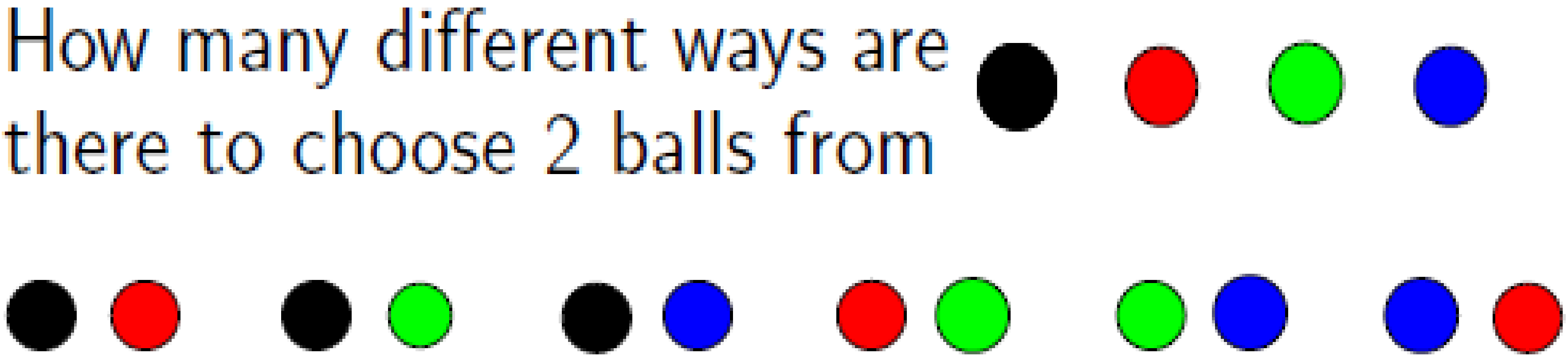

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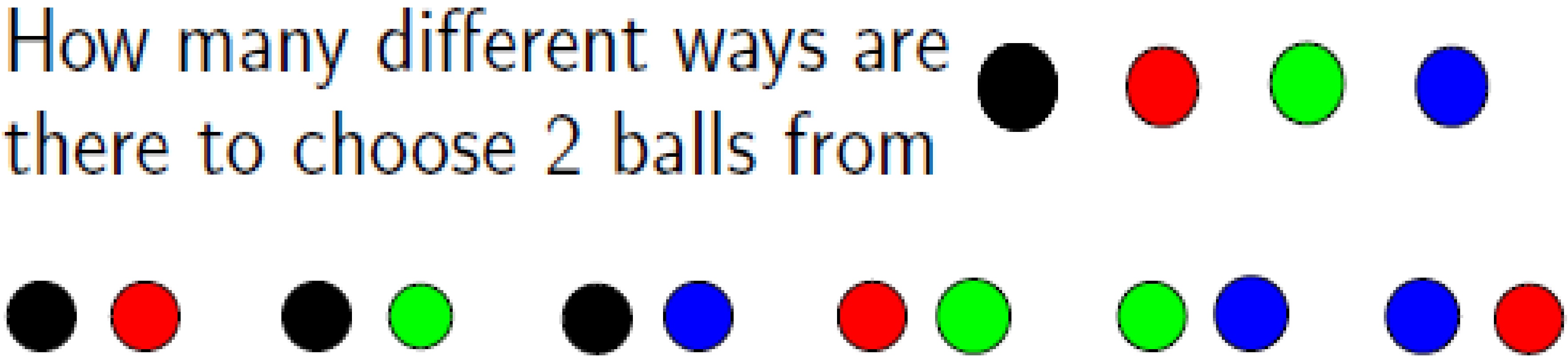

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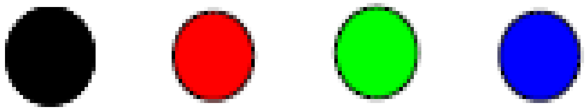

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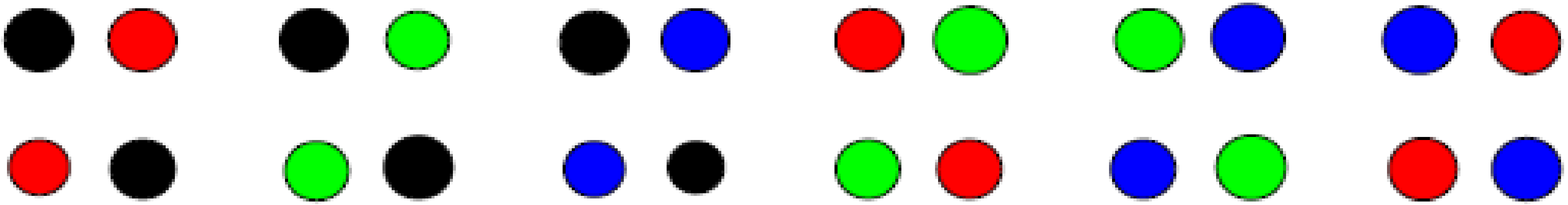

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  - the number of steps in a computer program
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  – simplify the solution by decomposing the problem

Basic Counting Rules

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Example

In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

We may either list all or use the product rule.

$$26 \times 50 = 1300$$

The Product Rule

- **Product Rule**: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is
$$n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$

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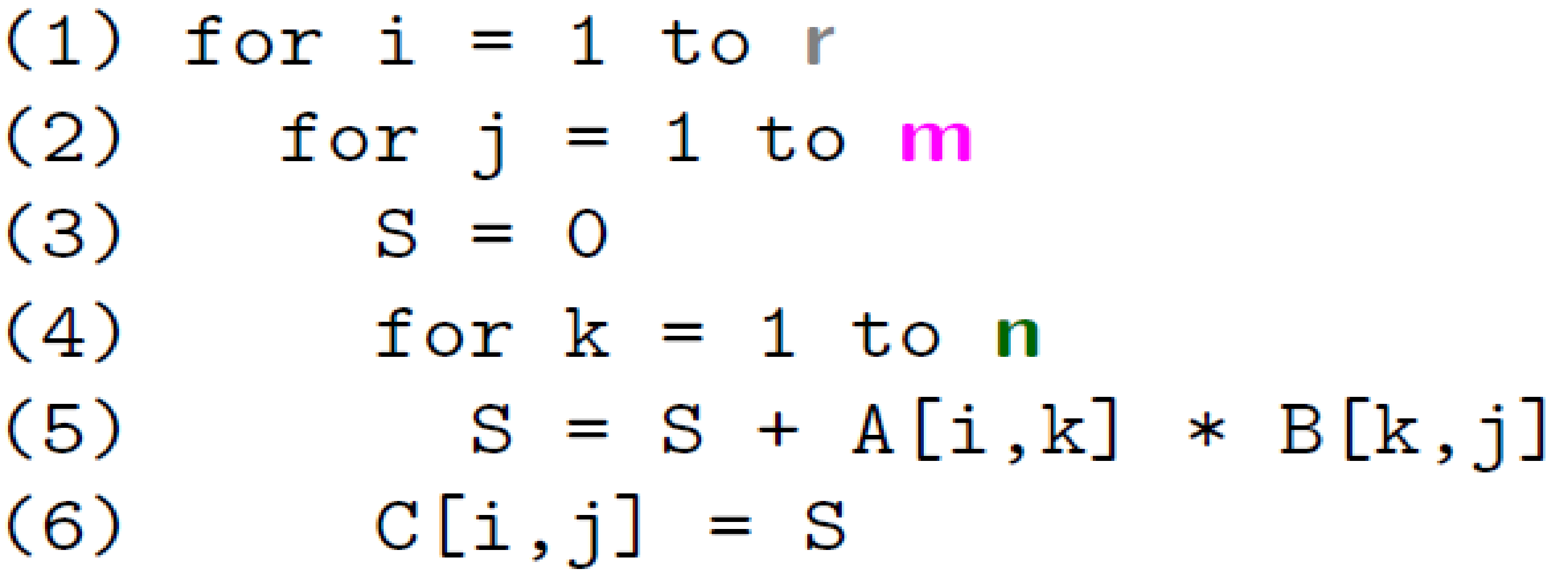
How many one-to-one functions are there from a set with m elements to a set with n elements?

How many onto functions?

The Product Rule

- The following loop is a part of program computing the product of two matrices.



```
for i = 1 to r
  for j = 1 to m
    S = 0
    for k = 1 to n
      S = S + A[i,k] * B[k,j]
    C[i,j] = S
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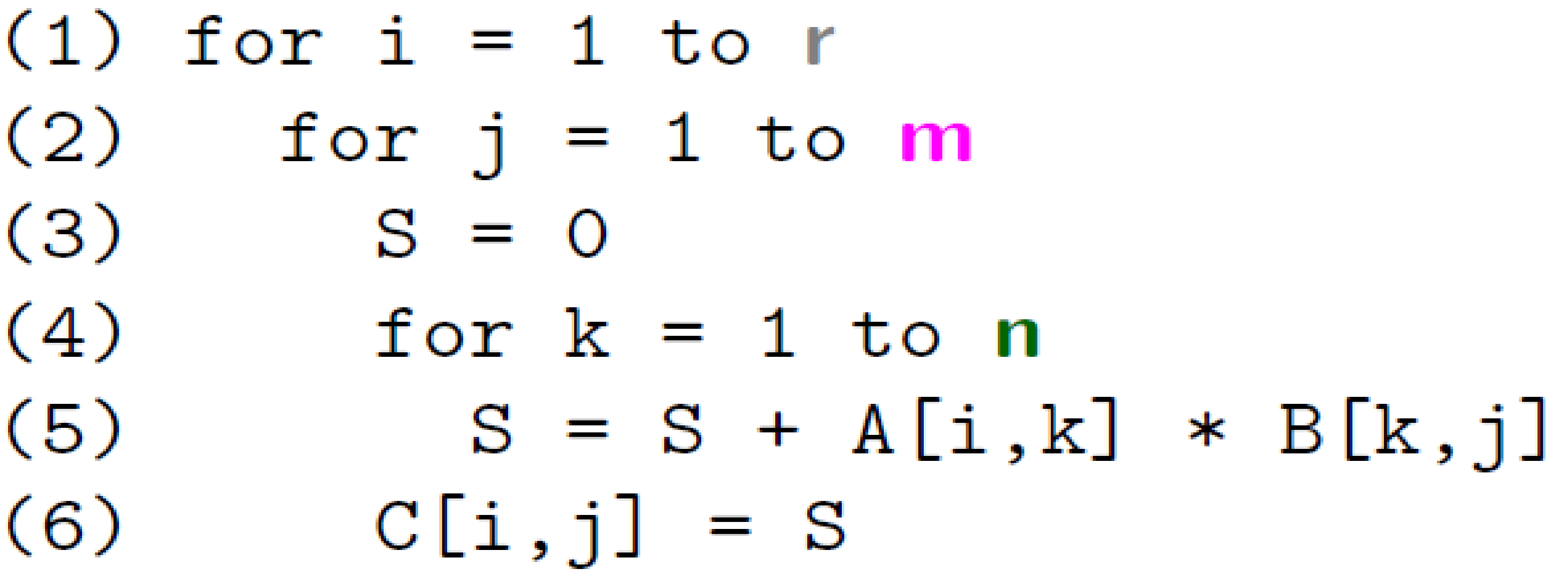

```
(1) for i = 1 to r
(2)   for j = 1 to m
(3)     S = 0
(4)     for k = 1 to n
(5)       S = S + A[i,k] * B[k,j]
(6)     C[i,j] = S
```

How many multiplications (in terms of r, m, n) does this program carry out in total among all iterations of line 5?

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Example

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We may use the sum rule.

$$12 + 5 + 10$$

The Sum Rule

- **Sum Rule**: If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is

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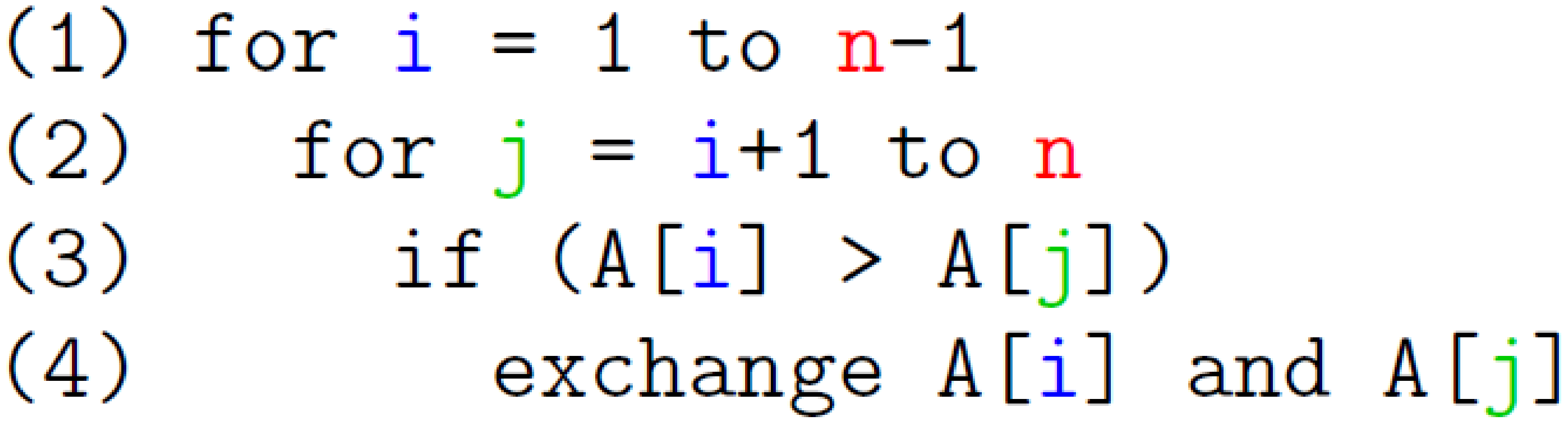

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for i = 1 to n-1
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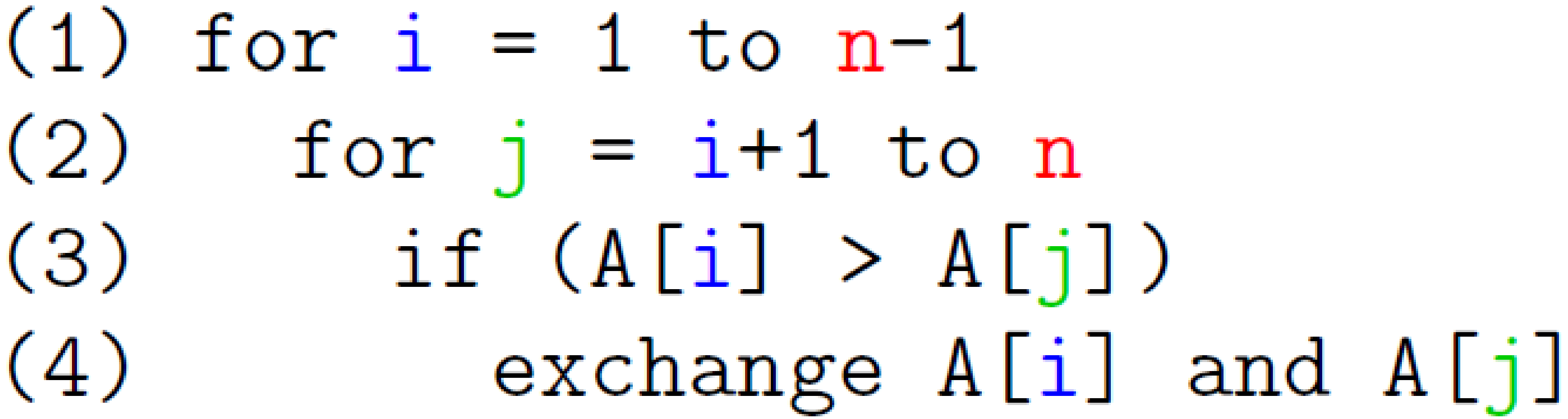

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How many comparisons (in terms of n) does this program carry out in total among all iterations of line 3?

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Each password is 6 to 8 characters long, where each character is a lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

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$$P = P_6 + P_7 + P_8$$

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  What is the number of bit strings of length 4 that do not have two consecutive 1's?

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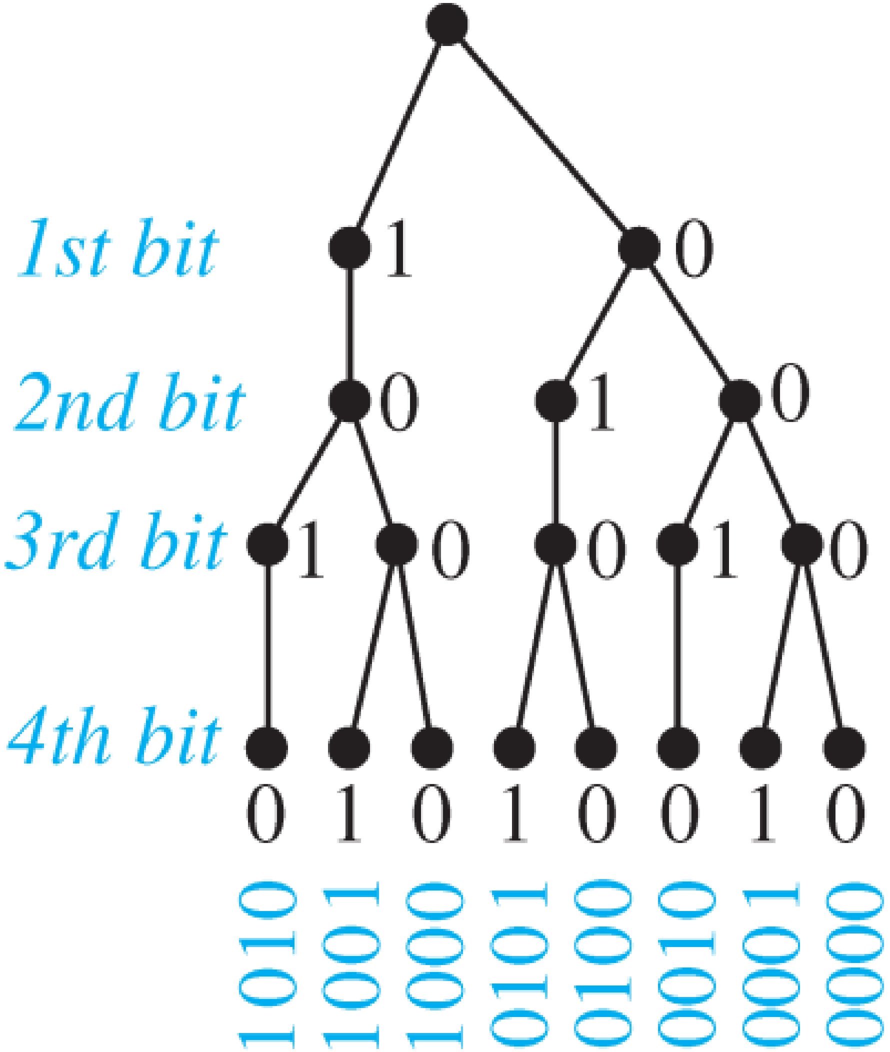

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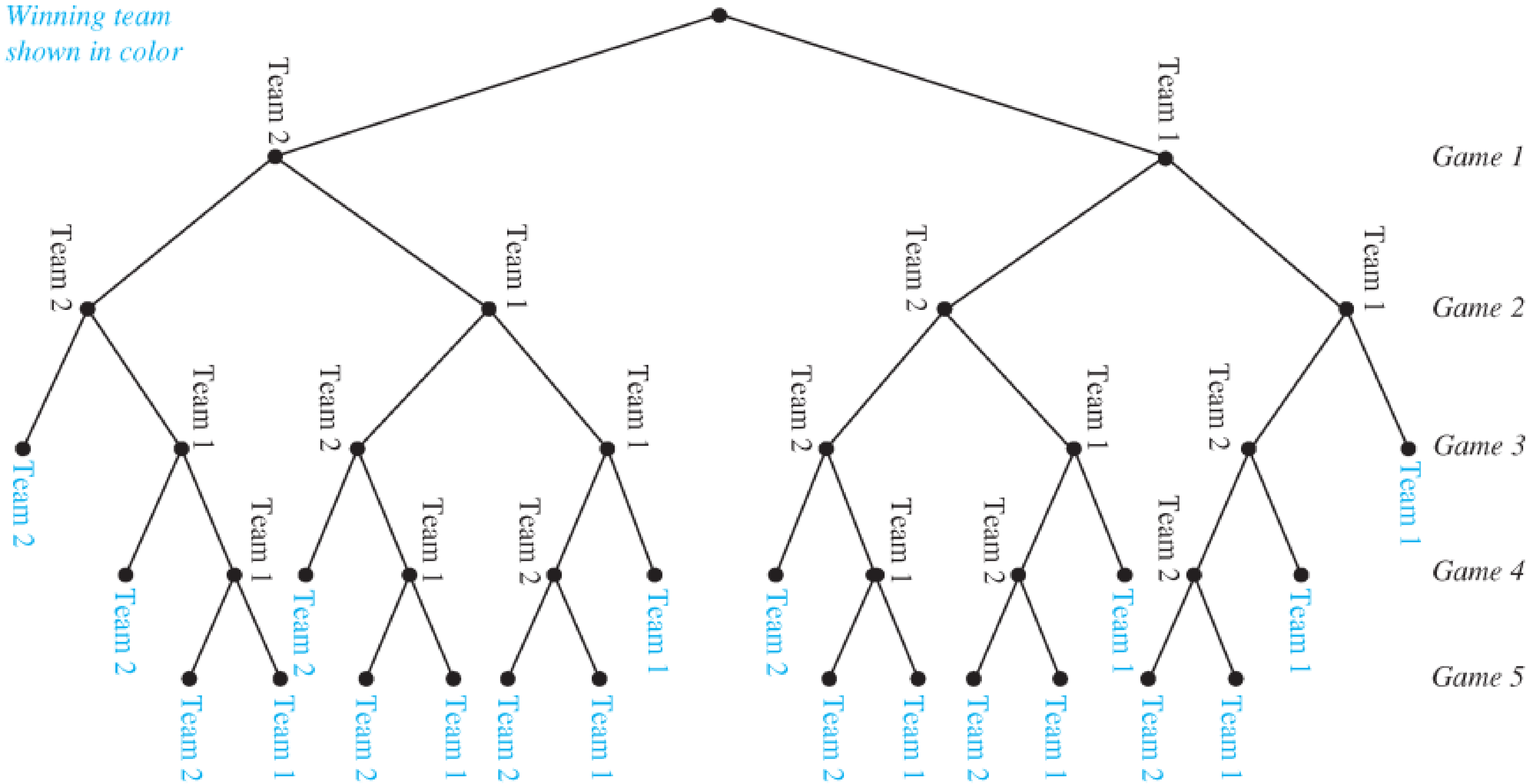

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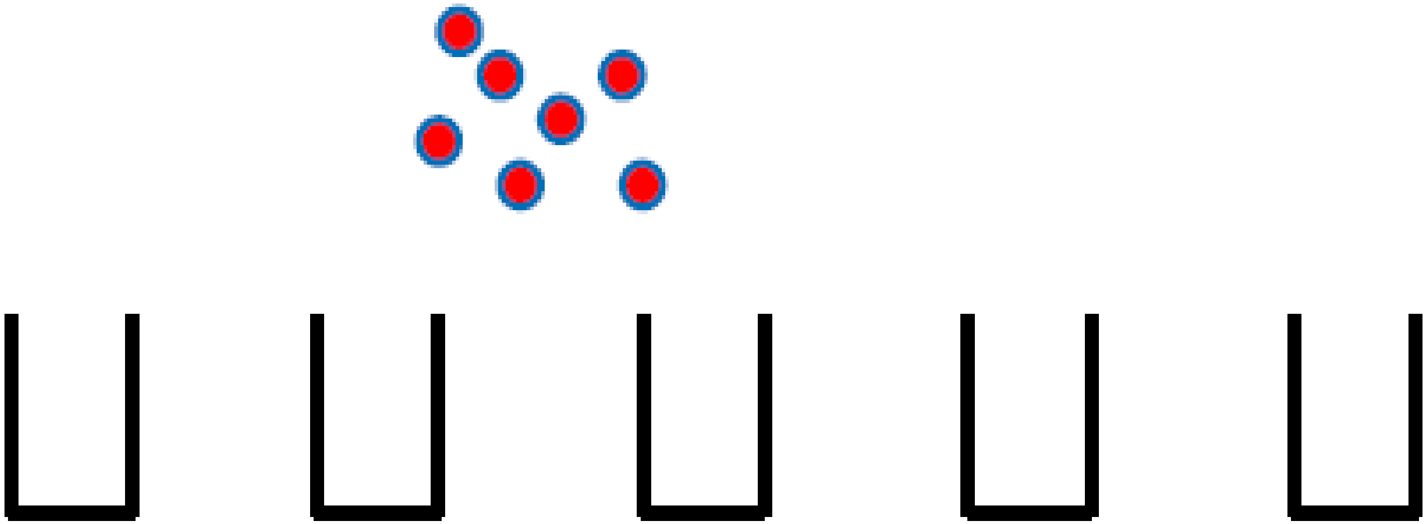

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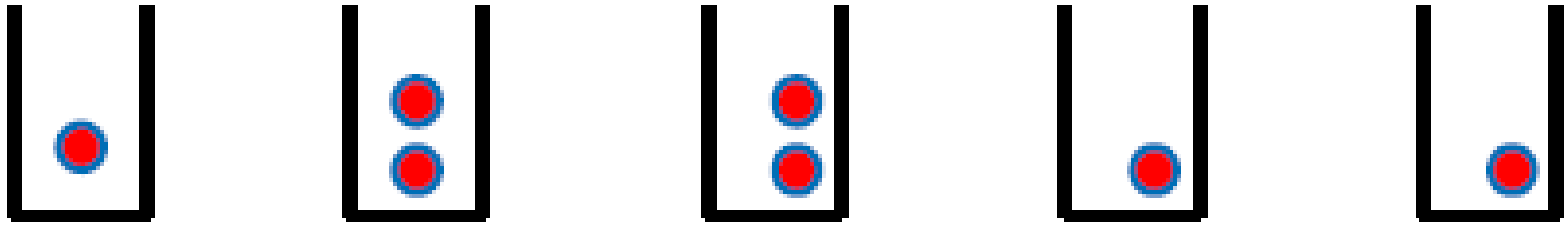

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  Proof by contradiction

  Example

  Assume that there are 367 students. Are there any two people who have the same birthday?

  There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?

Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

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Example

Assume there are 100 students. How many of them were born in the same month?

Bijections and Permutations

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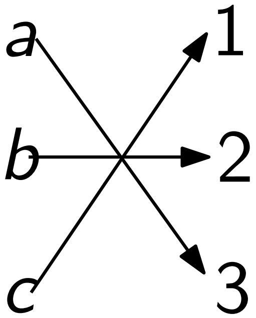

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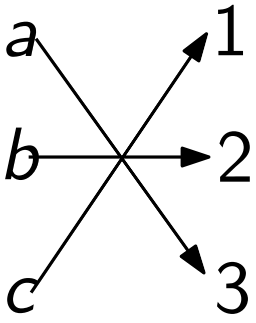

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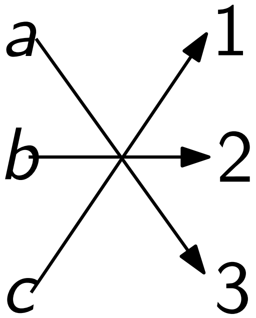

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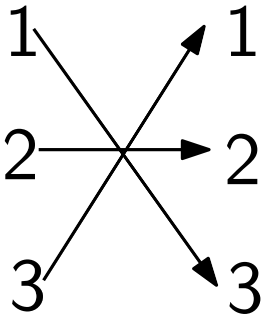

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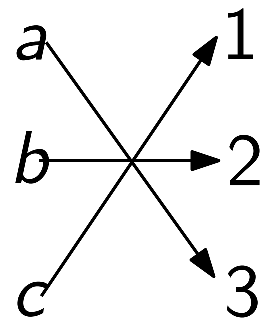
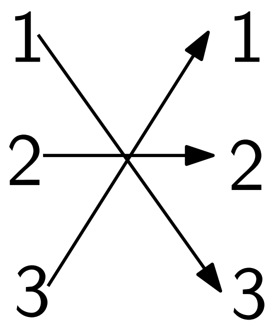
  In a bijection,
  exactly one arrow leaves each item on the left and exactly one arrow arrives at each item on the right.





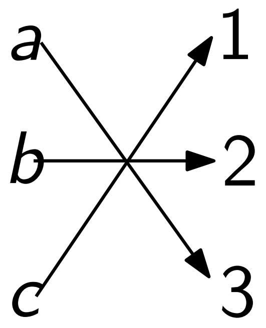
Bijections and Permutations

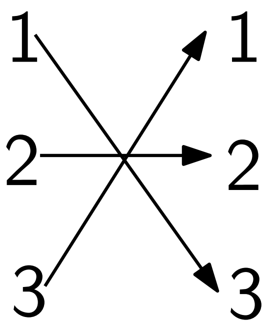
- A function that is both one-to-one and onto is called a *bijection*, or a *one-to-one correspondence*.

  A bijection from a set onto itself is called a *permutation*.

  In a bijection,
  exactly one arrow leaves each item on the left and exactly one arrow arrives at each item on the right.

  Thus,
  the left and right sides must have the same size.





The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by *n* points in the plane.

```
triangleсount = 0
  for i = 1 to n
    for j = i+1 to n
      for k = j+1 to n
        if points i, j, k are not collinear
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```

The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by n points in the plane.

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

Counting Triangles

- 3 points form a triangle if and only if they are non collinear

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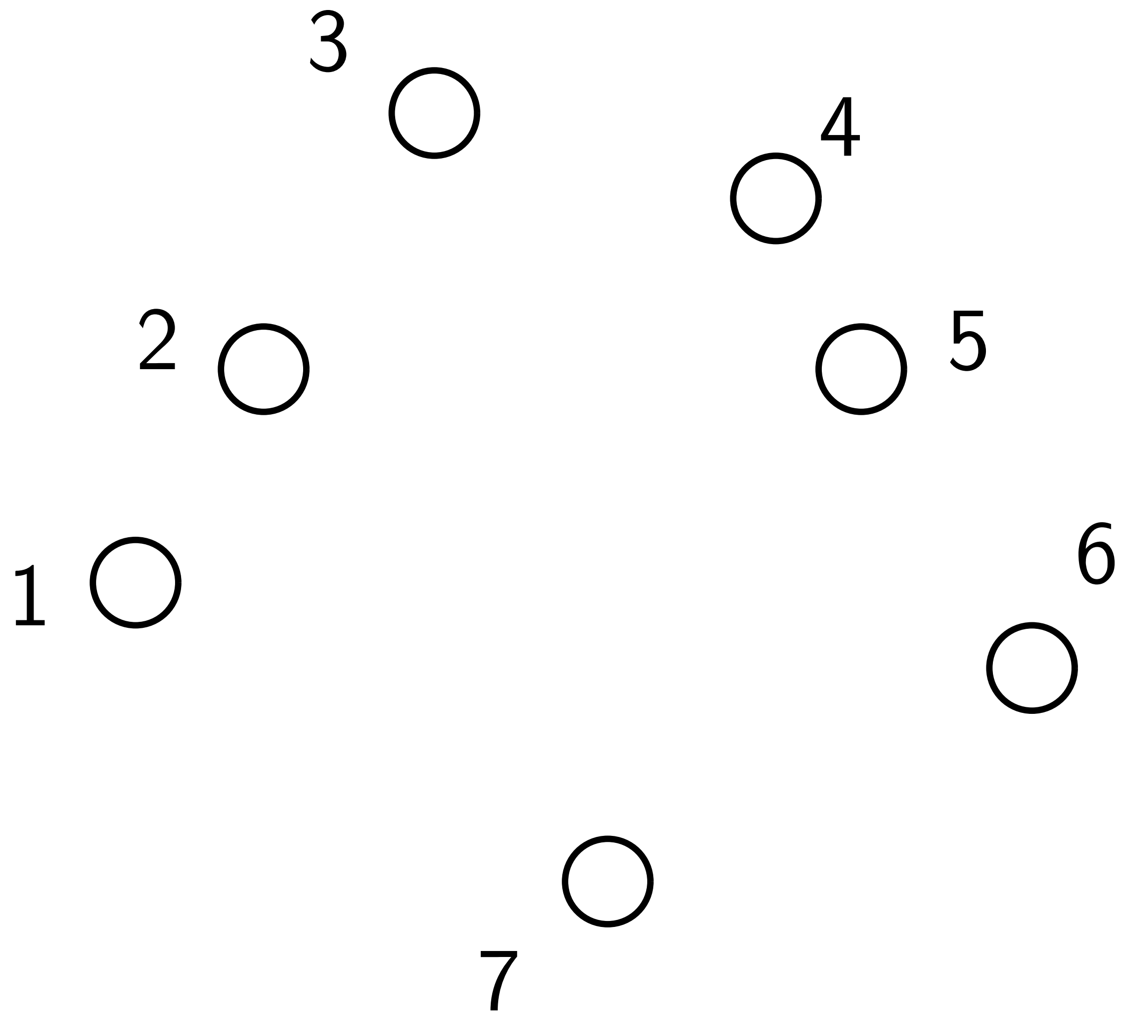

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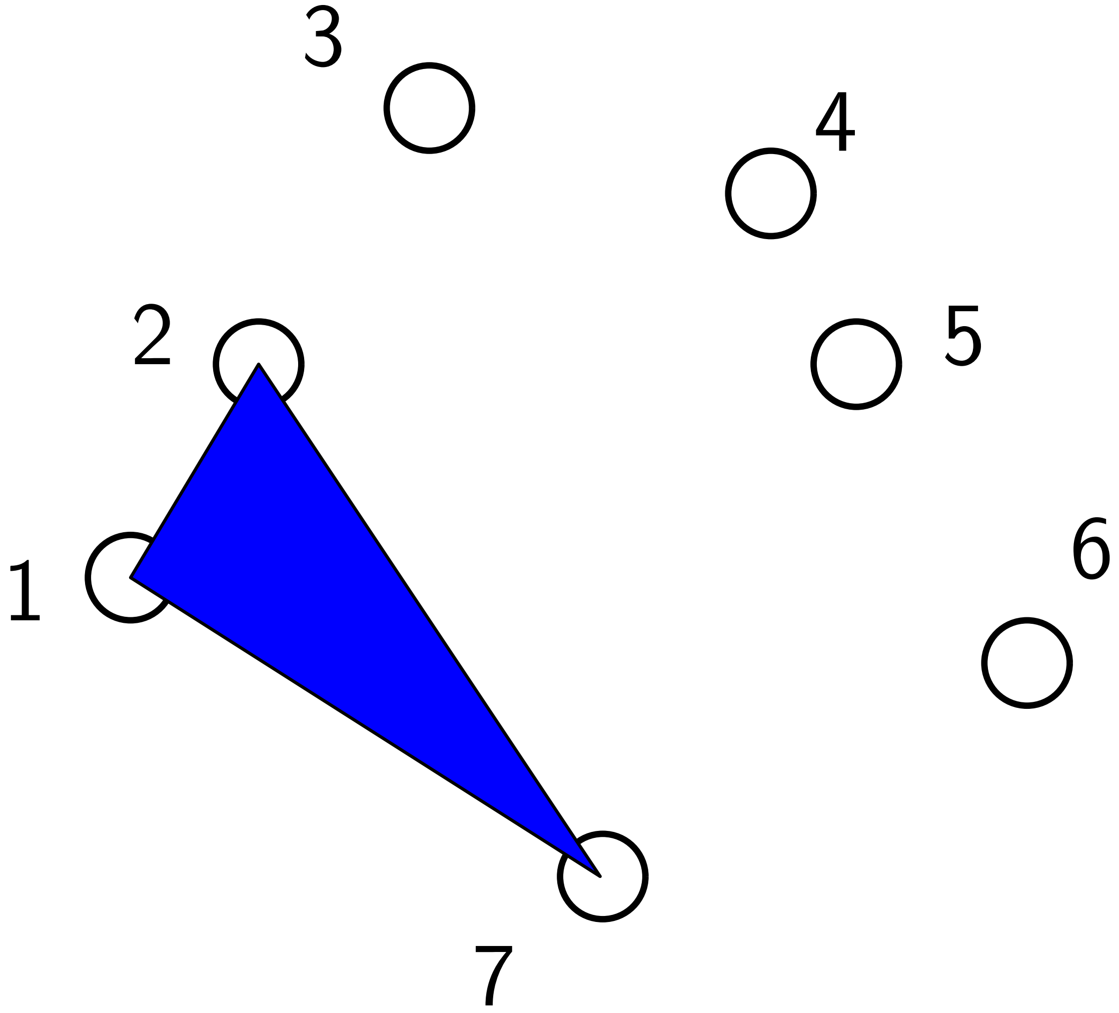

1 – 2 – 7: yes

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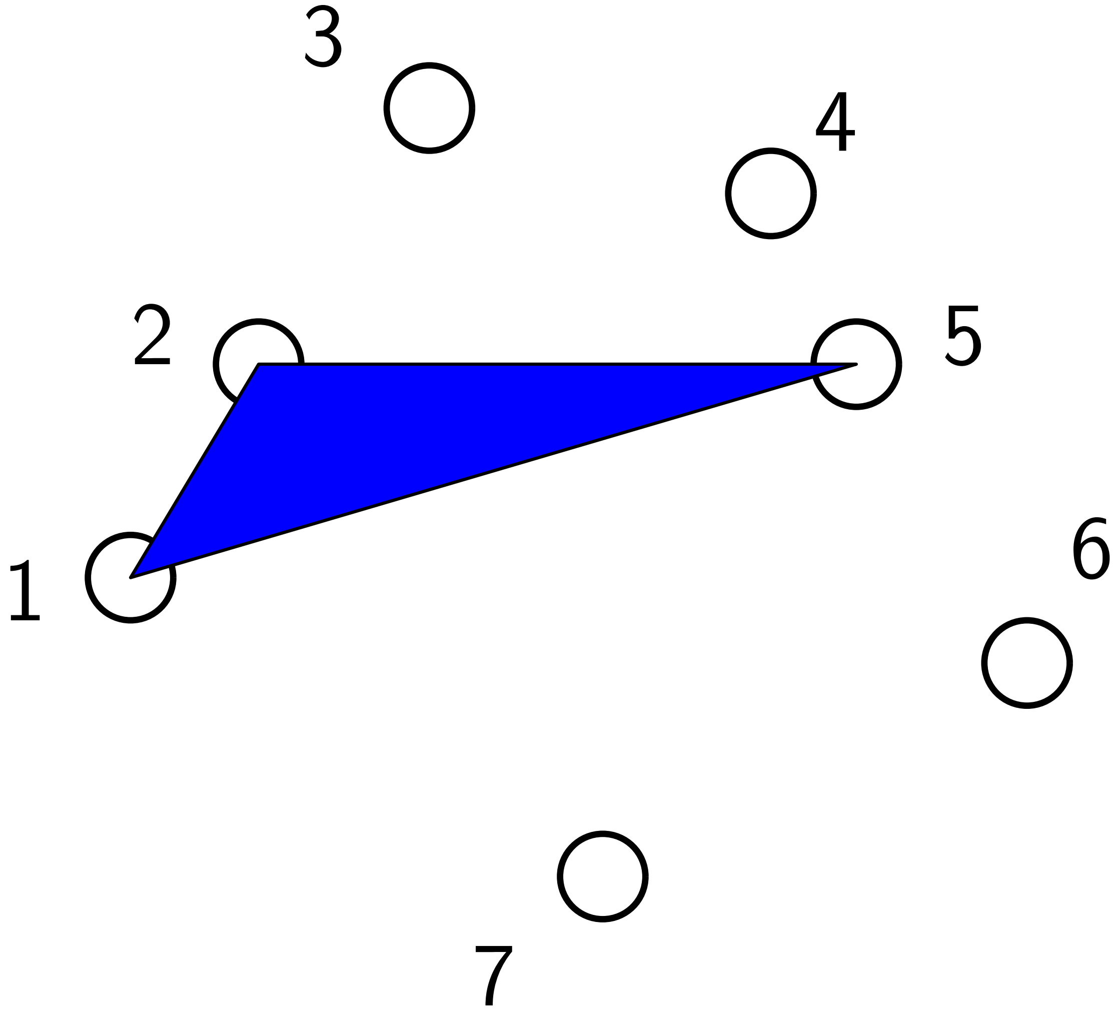

$1 - 2 - 7$: yes

$1 - 2 - 5$: yes

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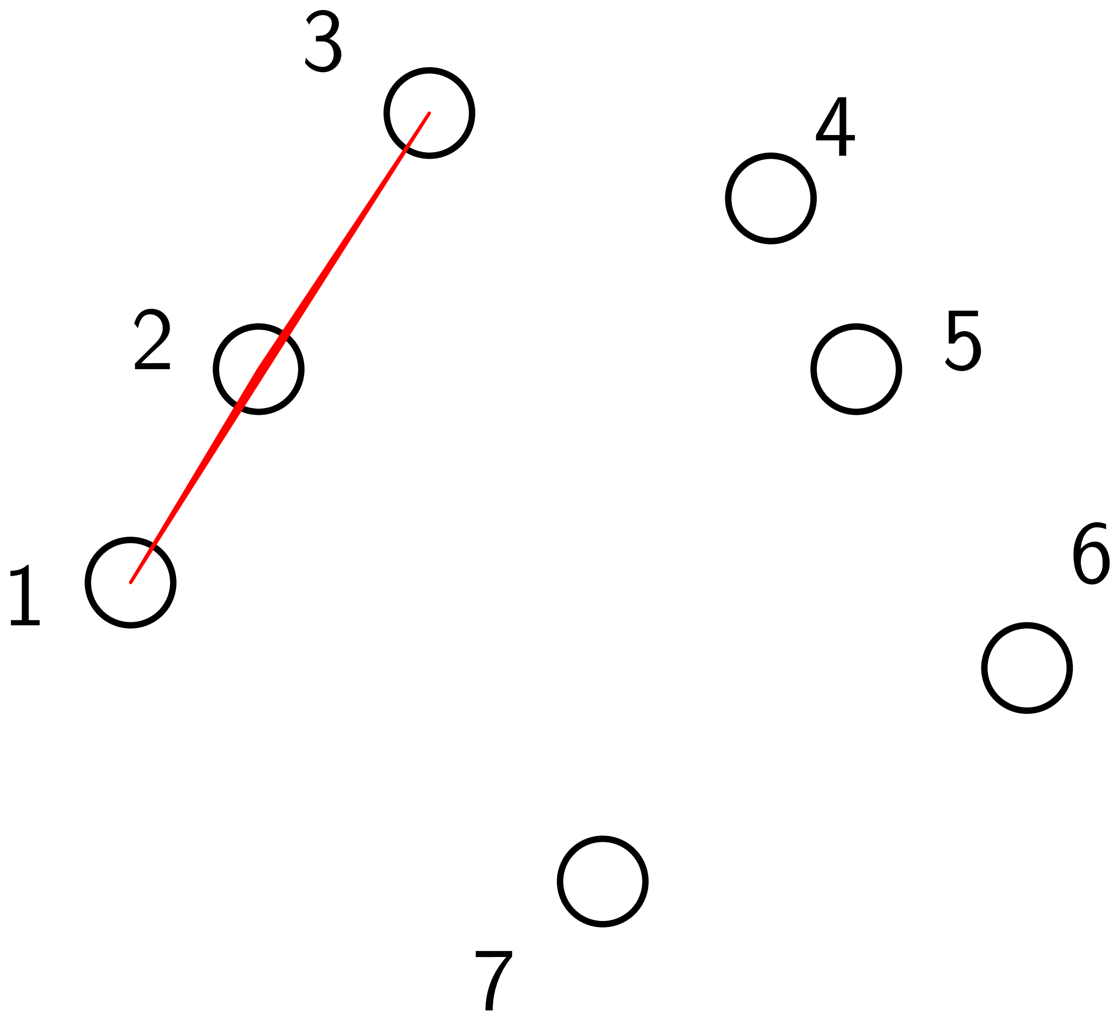

$1 - 2 - 7$: yes

$1 - 2 - 5$: yes

$1 - 2 - 3$: no

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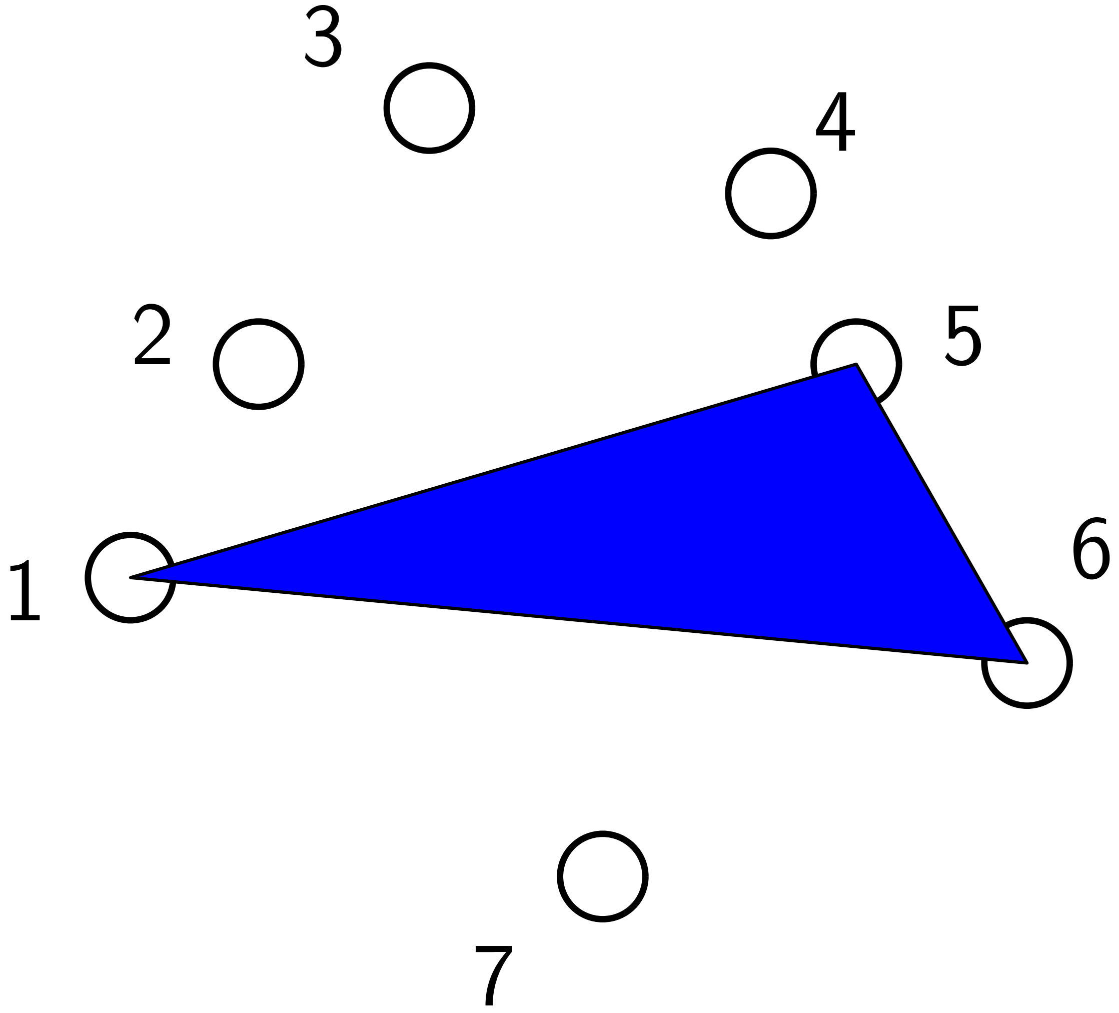

1 – 2 – 7: yes

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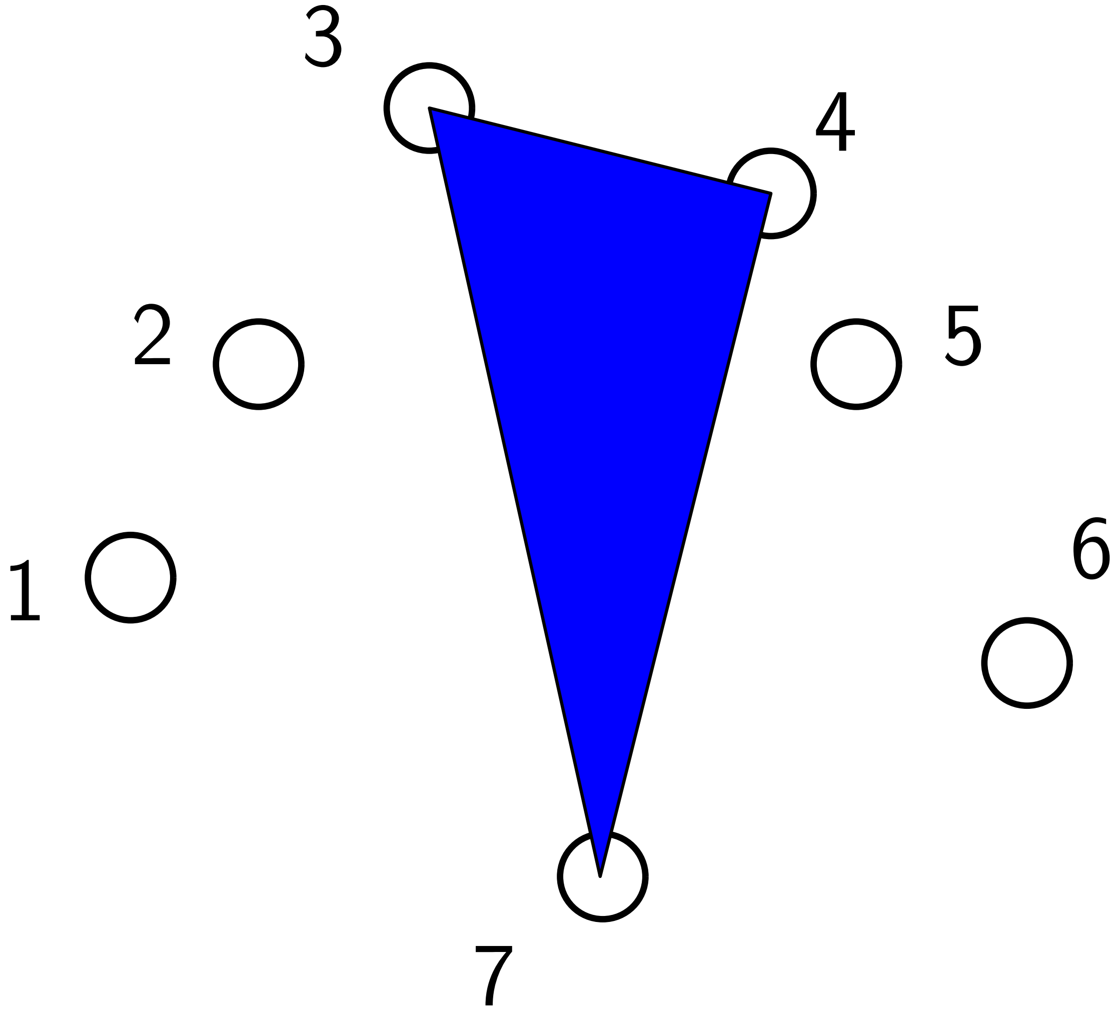

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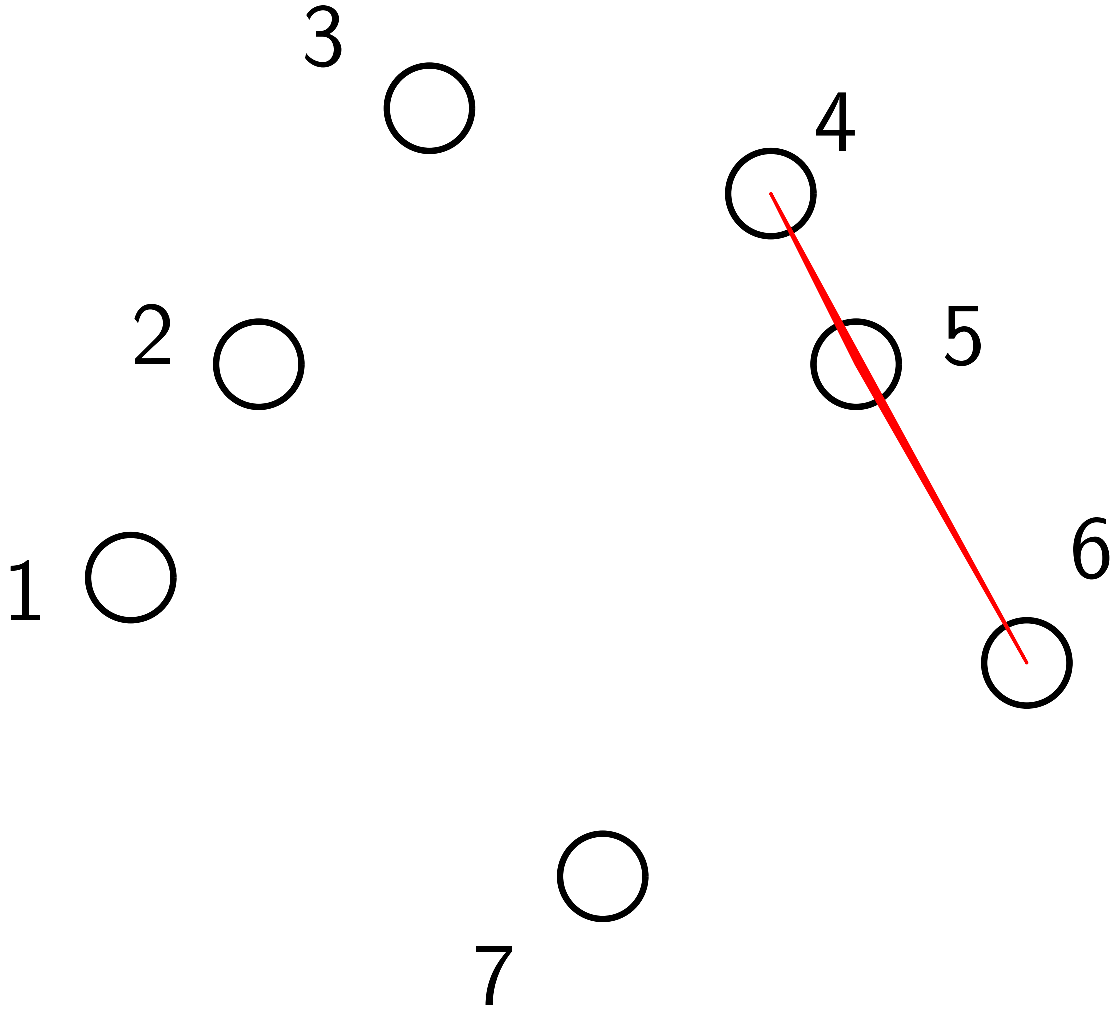

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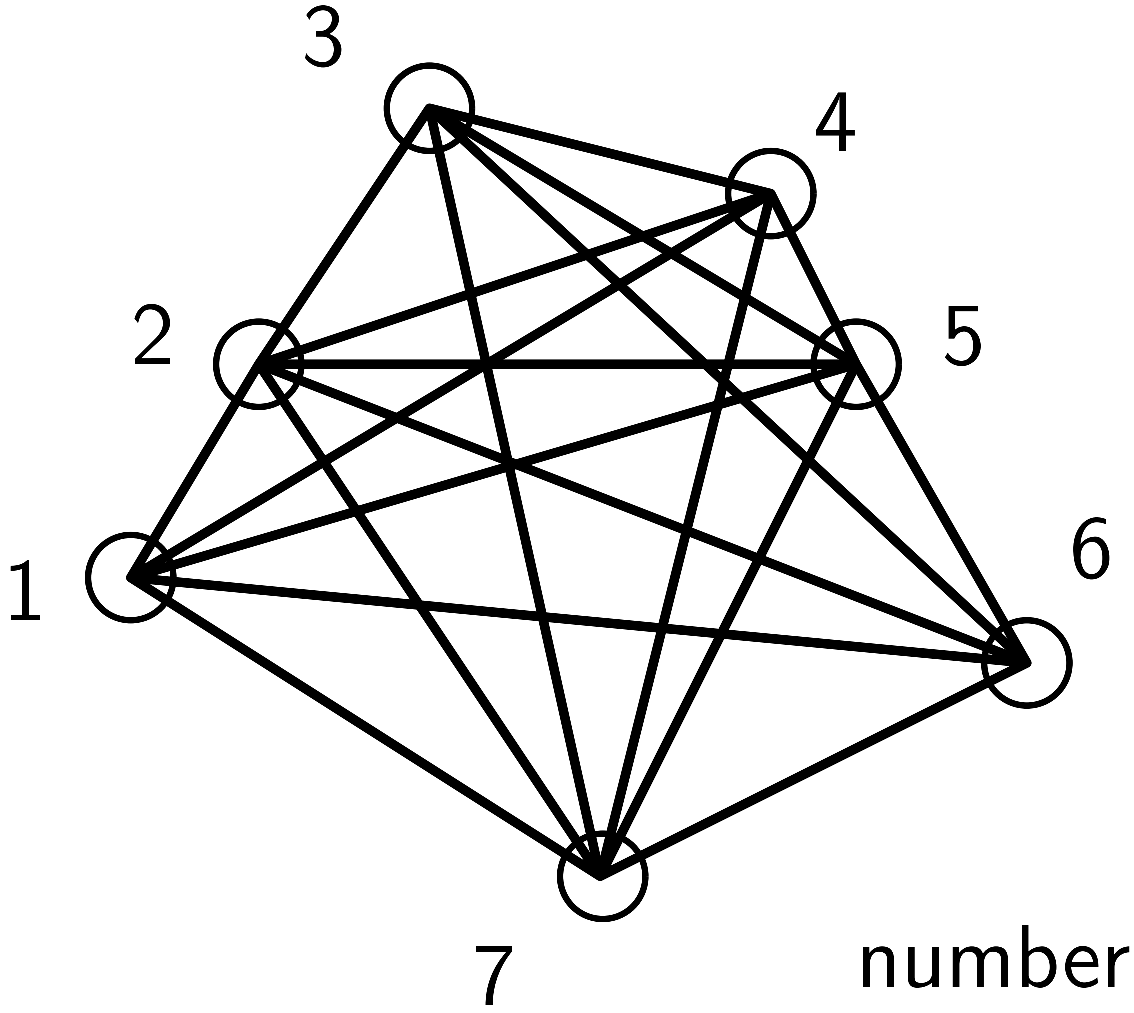

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1 – 5 – 6: yes

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4 – 5 – 6: no

number of triangles: 33

Counting Triangles

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A loop

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are (1,2,3), (1,2,4), (1,3,4), and (2,3,4).

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- Want to compute the number of *increasing triples* (i, j, k) with $1 \leq i < j < k \leq n$.

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Claim: f is a **bijection** (why) so $|X| = |Y|$

f is a bijection because

f is one-to-one

if $(i, j, k) \neq (i', j', k') \Rightarrow f(\,(i, j, k)\,) \neq f(\,(i', j', k'))$

f is onto

if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$ where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

Counting Pairs

- The number of

  increasing pairs (i, j) with $1 \leq i < j \leq n$

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  We actually already saw that $|X| = |Y| = \binom{n}{2}$

The Bijection Principle

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  In practice, in real problems we often only *implicitly* use the bijection and don't *explicitly* describe it

  Currently, we started with the problem of counting the # of increasing triples and changed it to the problem of counting the # of 3-element sets from $\{1, 2, \ldots, n\}$

Inclusion-Exclusion Principle

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$$|A \cup B| = |A| + |B| - |A \cap B|$$

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- **Example**

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  Overcounting!!!

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Overcounting!!!

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Overcounting!!!

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Inclusion-Exclusion Principle

- Two sets

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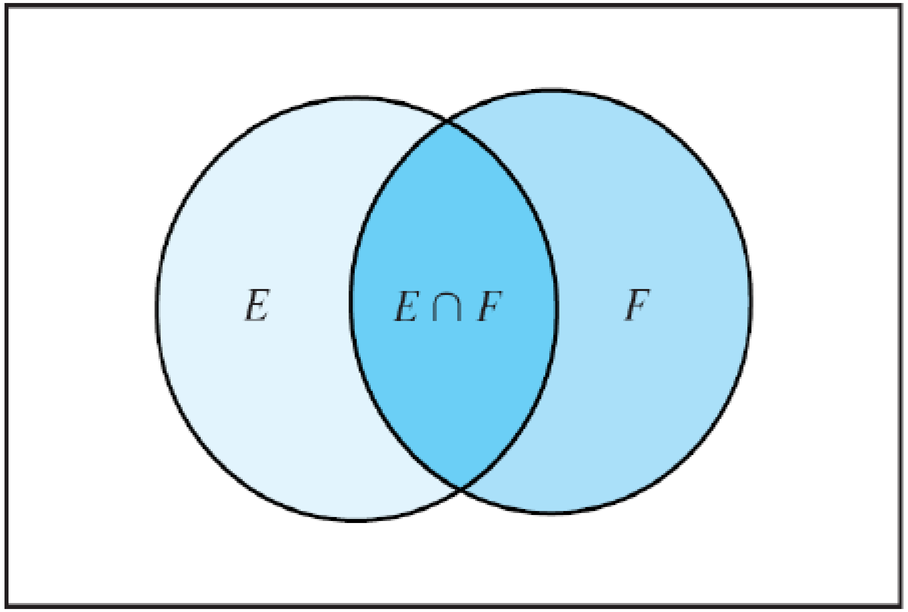

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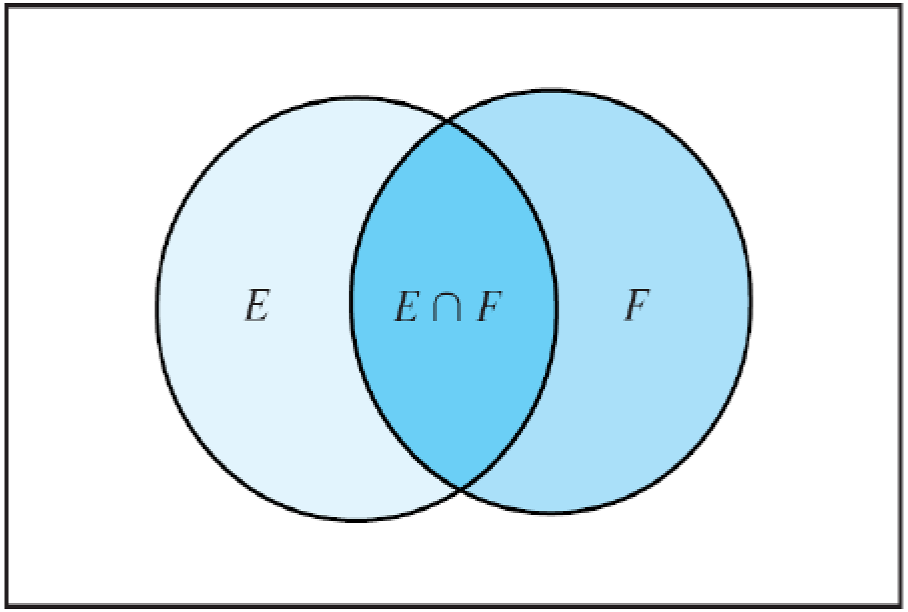
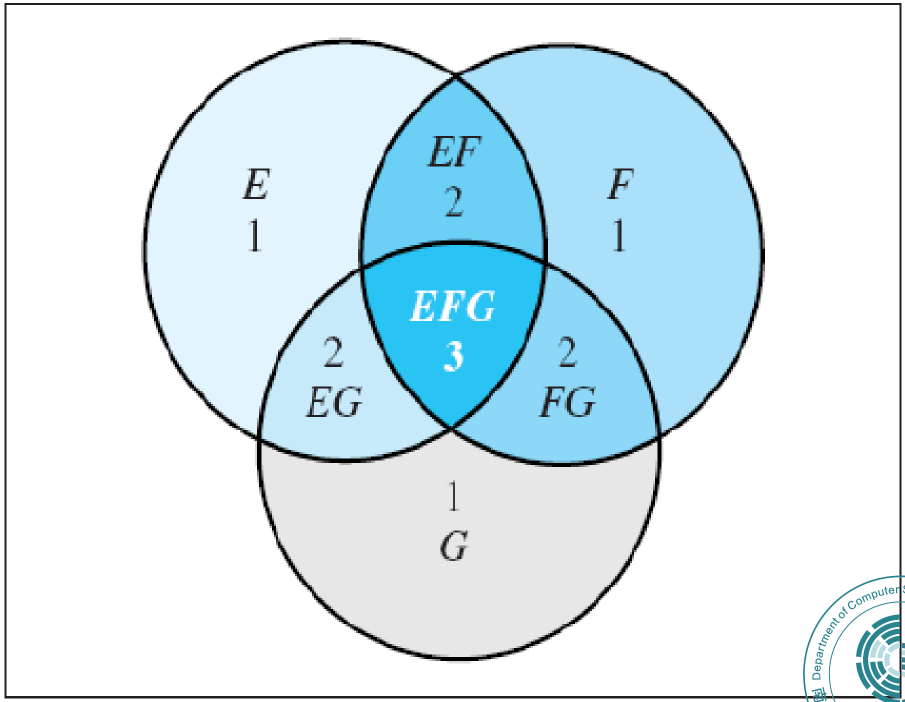


Three sets



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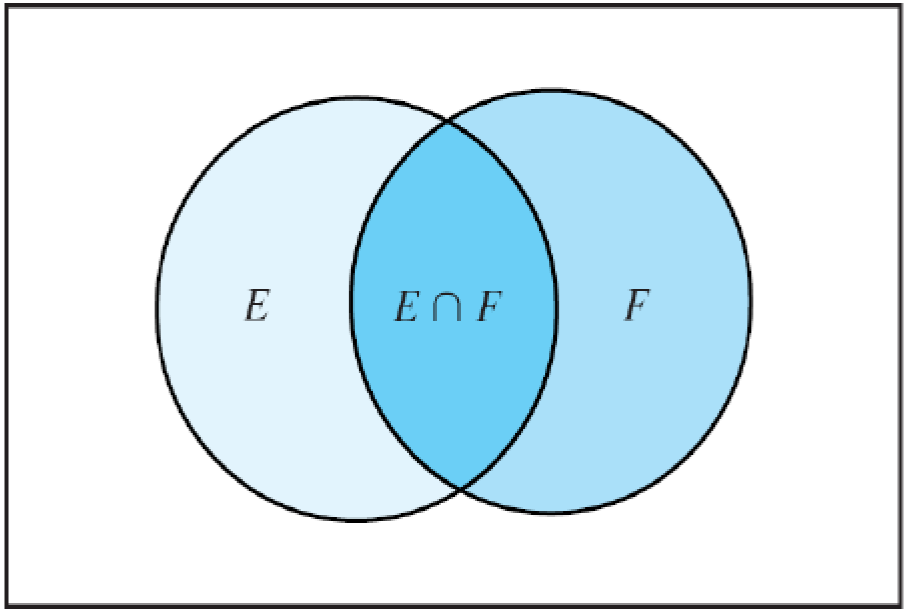
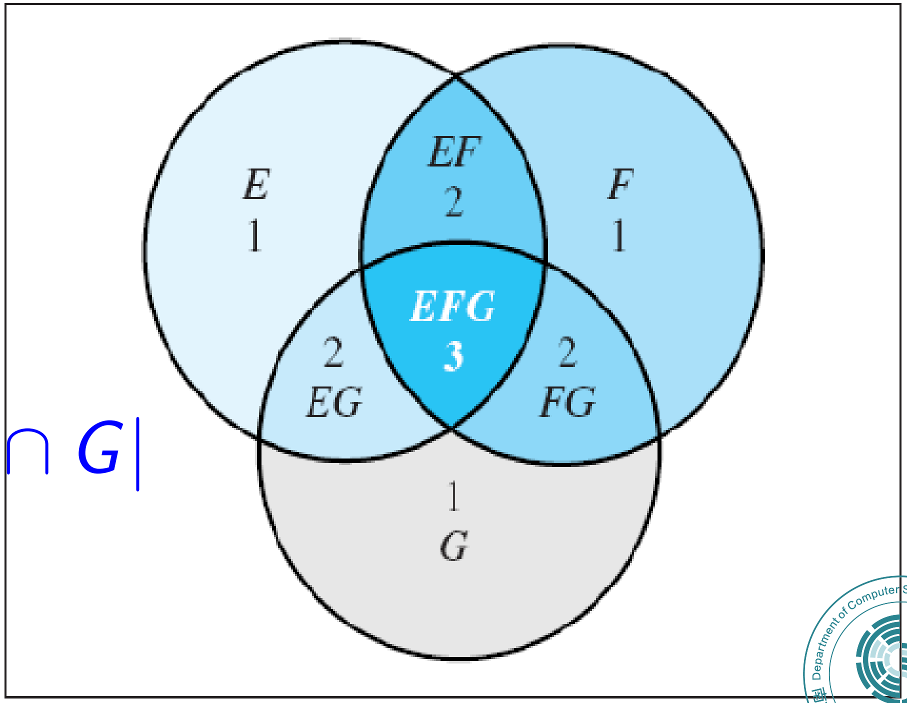


Three sets

$$\begin{aligned} &|E \cup F \cup G| \\ = &|E| + |F| + |G| \\ &-|E \cap F| - |E \cap G| - |F \cap G| \\ &+|E \cap F \cap G| \end{aligned}$$



Inclusion-Exclusion Principle

- $$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

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Base case $(n = 2)$

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Inductive Hypothesis

$$|\cup_{i=1}^{n-1} E_i| = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

Inclusion-Exclusion Principle

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  Set $E = E_1 \cup \cdots \cup E_{n-1}$, and $F = E_n$.

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$$|\cup_{i=1}^{n} E_i| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right|$$

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For the third term, by distributive law,

$$\left|(\cup_{i=1}^{n-1} E_i) \cap E_n\right| = \left|\cup_{i=1}^{n-1} (E_i \cap E_n)\right| = \left|\cup_{i=1}^{n-1} G_i\right|$$

where $G_i = E_i \cap E_n$.

Inclusion-Exclusion Principle

- So far

$$\left|\cup_{i=1}^{n} E_i\right| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\cup_{i=1}^{n-1} G_i\right|$$

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Note that (why?)

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Some discussion:
first summation sums $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do not contain n
$|E_n|$ and second summation together sum $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do contain n

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  This can be used to determine the number of onto functions

  A, B are two sets with $|A| = m$ and $|B| = n$.

  (a) How many onto functions are there from A to B?

  (b) How many functions are there from A to B that map nothing to at least one element of B?

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$= \sum_{k=1}^{n}(-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$

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$= \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)^m$

k-Element Permutations of a Set

- In how many ways can we choose **an ordered triple** of distinct elements from $\{1, 2, \ldots, n\}$?

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  Note that the case of $k = n$ is special;
  An n-element permutation of a set N of size $|N| = n$
  is what we earlier simply called a permutation.

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Ex: When $n = 4$, there are $4 \times 3 \times 2 = 24$
3 -element permutations of $\{1, 2, 3, 4\}$

$$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243$$
$$312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}.$$

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Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a *lexicographic ordering* and is used quite often.

k-Element Permutations of a Set

- **Theorem** If N is a positive integer and k is an integer with $1 \leq k \leq n$, then there are
  $P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$
  k-element permutations with n distinct elements.

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$$(\#\text{ 3-element perms}) = 6 \times (\#\text{ 3-element subsets})$$

$$P(n,3) = 3! \cdot C(n,3)$$

Binomial Coefficient

- **Theorem** For integers n and k with $0 \leq k \leq n$, the number of k-element subsets of an n-element set is
$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

  This is the number of k-combinations of a set with n elements.

Some Properties of Binomial Coefficients

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of k-element subsets of an n-element set.

  $\binom{n}{0} = 1$ only one set of size 0.

  $\binom{n}{n} = 1$ only one set of size n.

  $\binom{n}{k} = \binom{n}{n-k}$ Obvious from equation. Can you think of a simple bijection that explains this?

Some Properties of Binomial Coefficients (cont.)

- $$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

Some Properties of Binomial Coefficients (cont.)

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Use Sum Rule

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S_i = set of all i-subsets of {1,2,. . . ,n}

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Some Properties of Binomial Coefficients (cont.)

- Let $L = L_1 L_2 \ldots L_n$ be a list of size n from $\{0, 1\}$

  If $\mathcal{L} = \text{set of all such lists} \Rightarrow \; |\mathcal{L}| = 2^n$

  There is a *bijection* between $\mathcal{L}$ and P so
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  Define the following function $f : \mathcal{L} \to P$

  If $L \in \mathcal{L}$ then $f(L)$ is the set $S \subseteq \{1, 2, \ldots, n\}$ defined by

$$i \in S \ \Leftrightarrow \ L_i = 1$$

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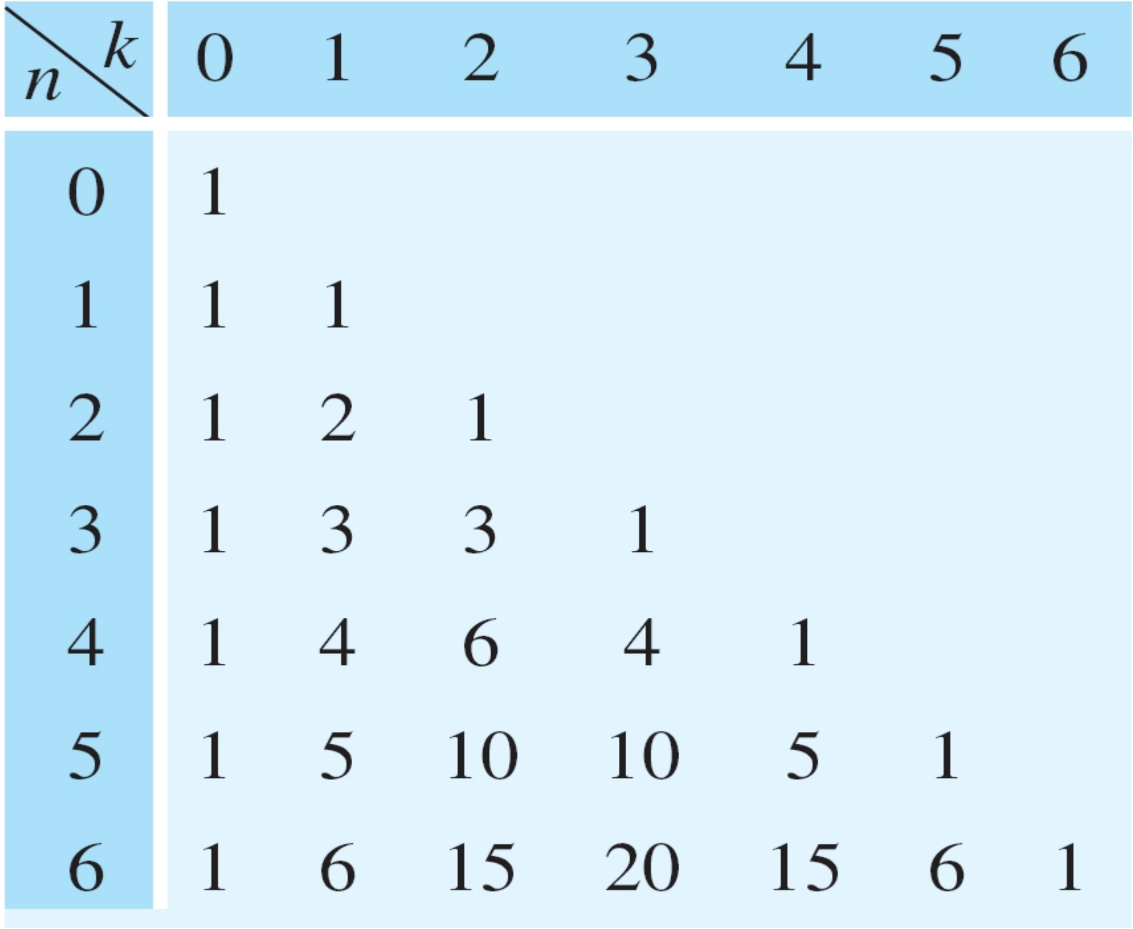
Ex: $n = 5$

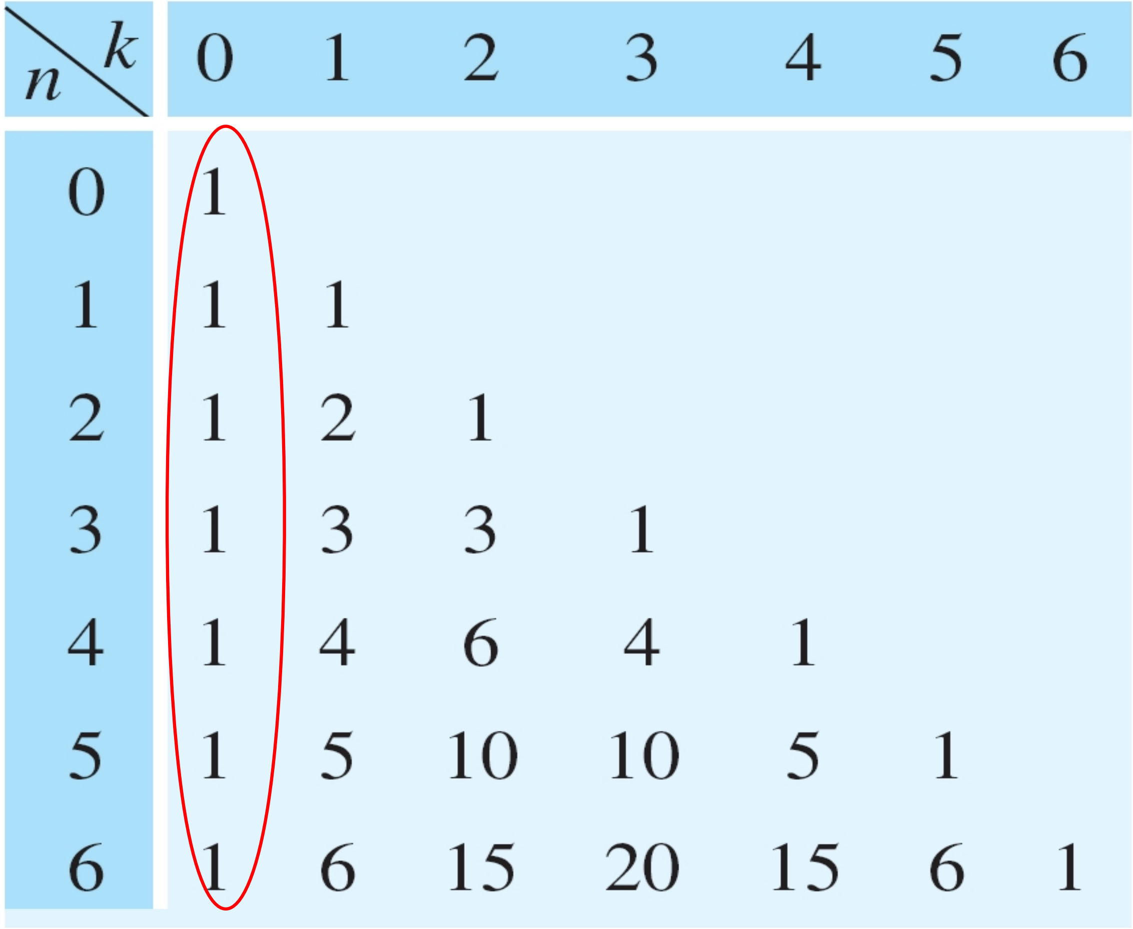
$$f(10101) = \{1, 3, 5\},\; f(11101) = \{1, 2, 3, 5\},\; f(00000) = \emptyset$$

Binomial Coefficients



$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Binomial Coefficients



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0	1						
1	1	1					
2	1	2	1				
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4	1	4	6	4	1		
5	1	5	10	10	5	1	
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Each row begins with a 1 because $\binom{n}{0} = 1$

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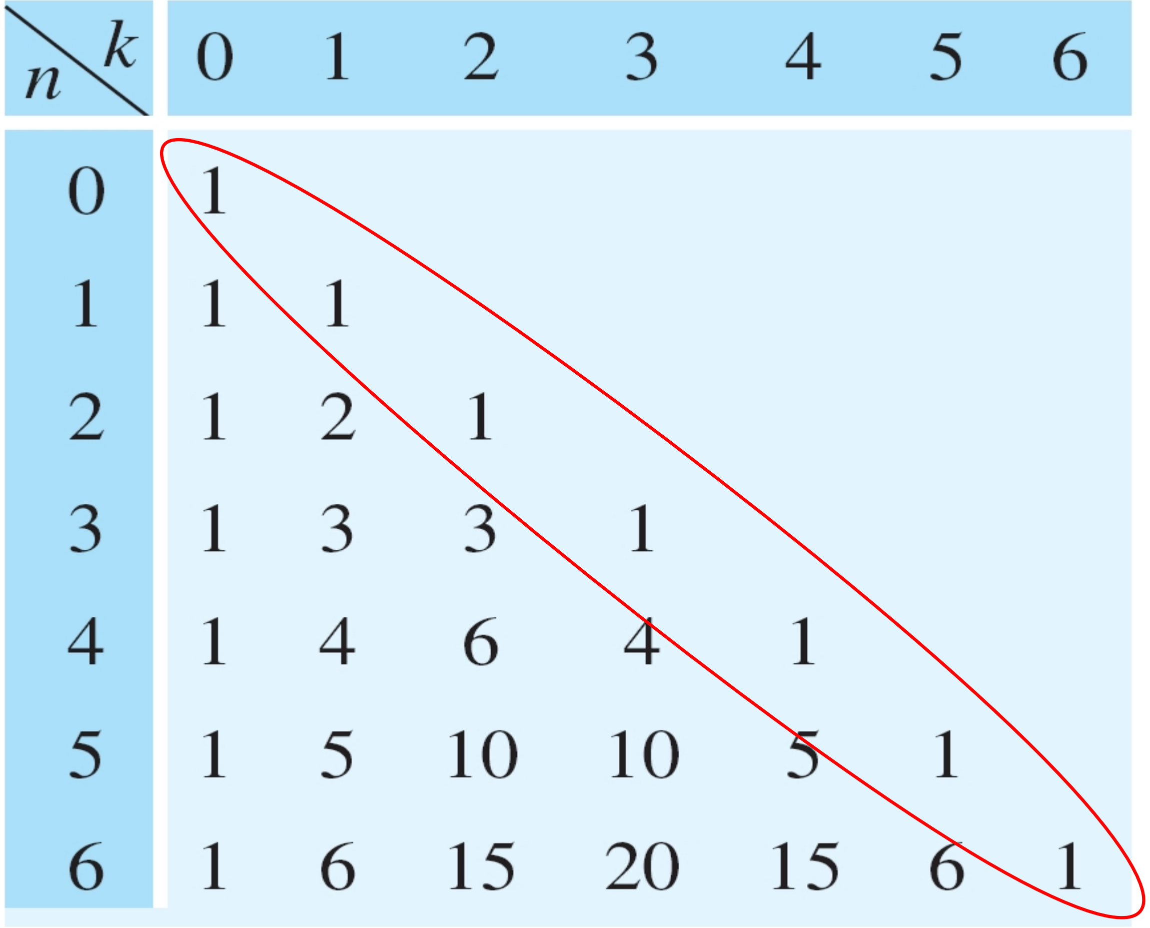

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0	1						
1	1	1					
2	1	2	1				
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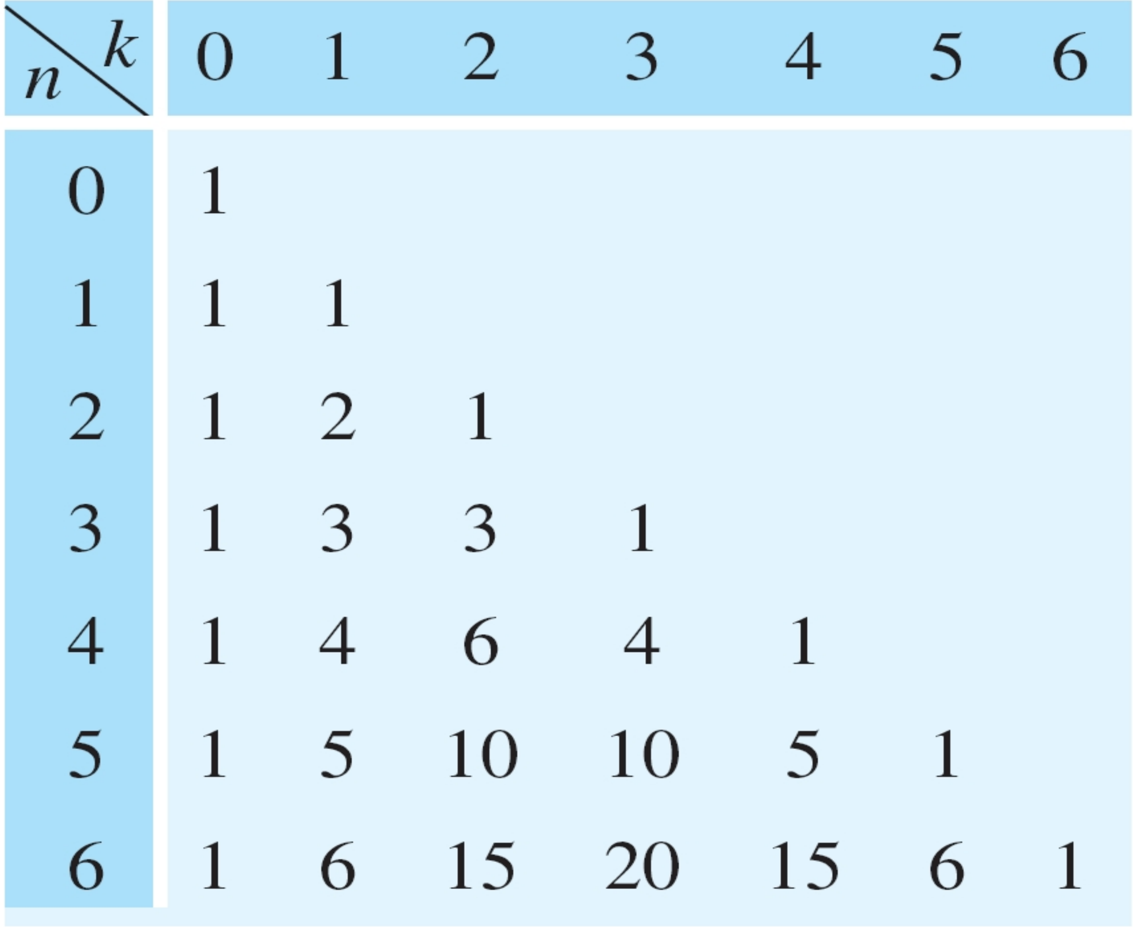

n \ k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
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4	1	4	6	4	1		
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Each row increases at first then decreases.

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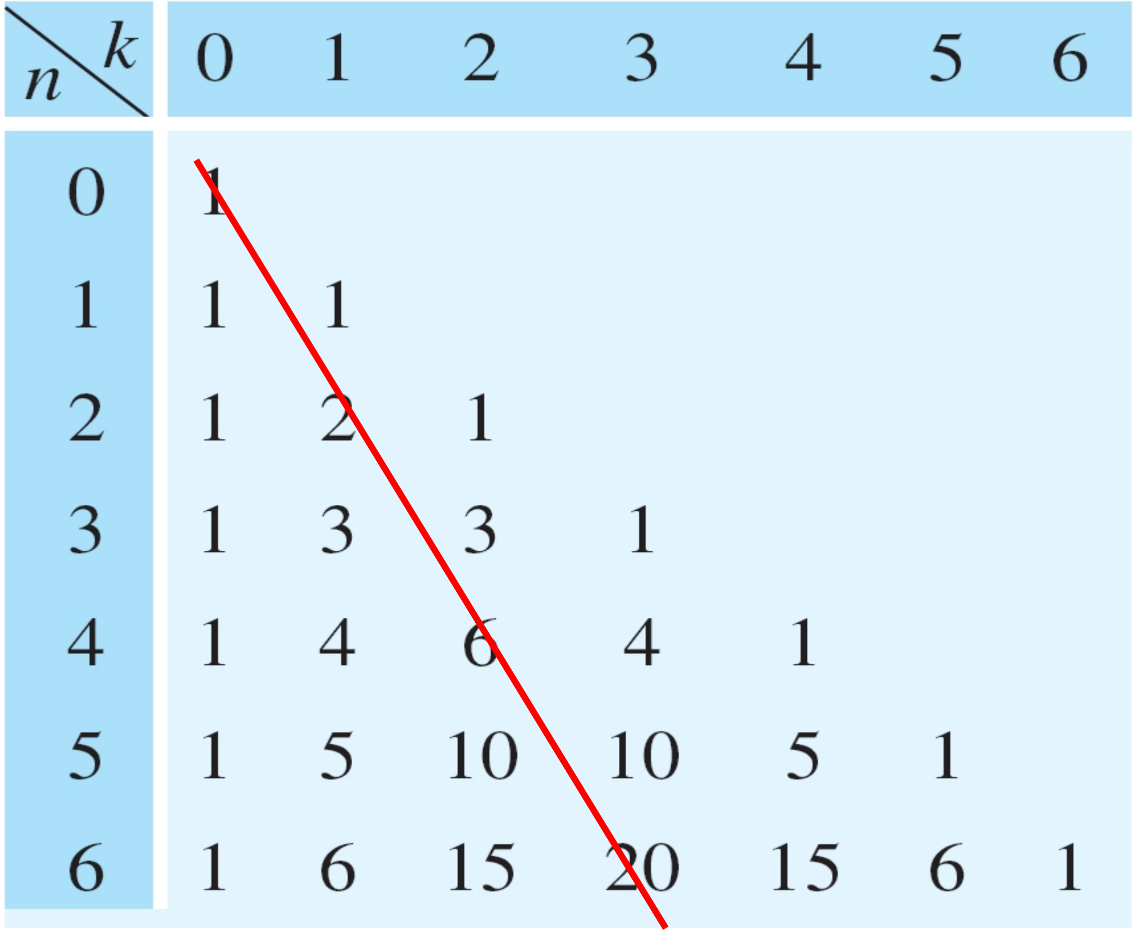

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0	1						
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Each row increases at first then decreases.

Second half of each row is the reverse of the first half.

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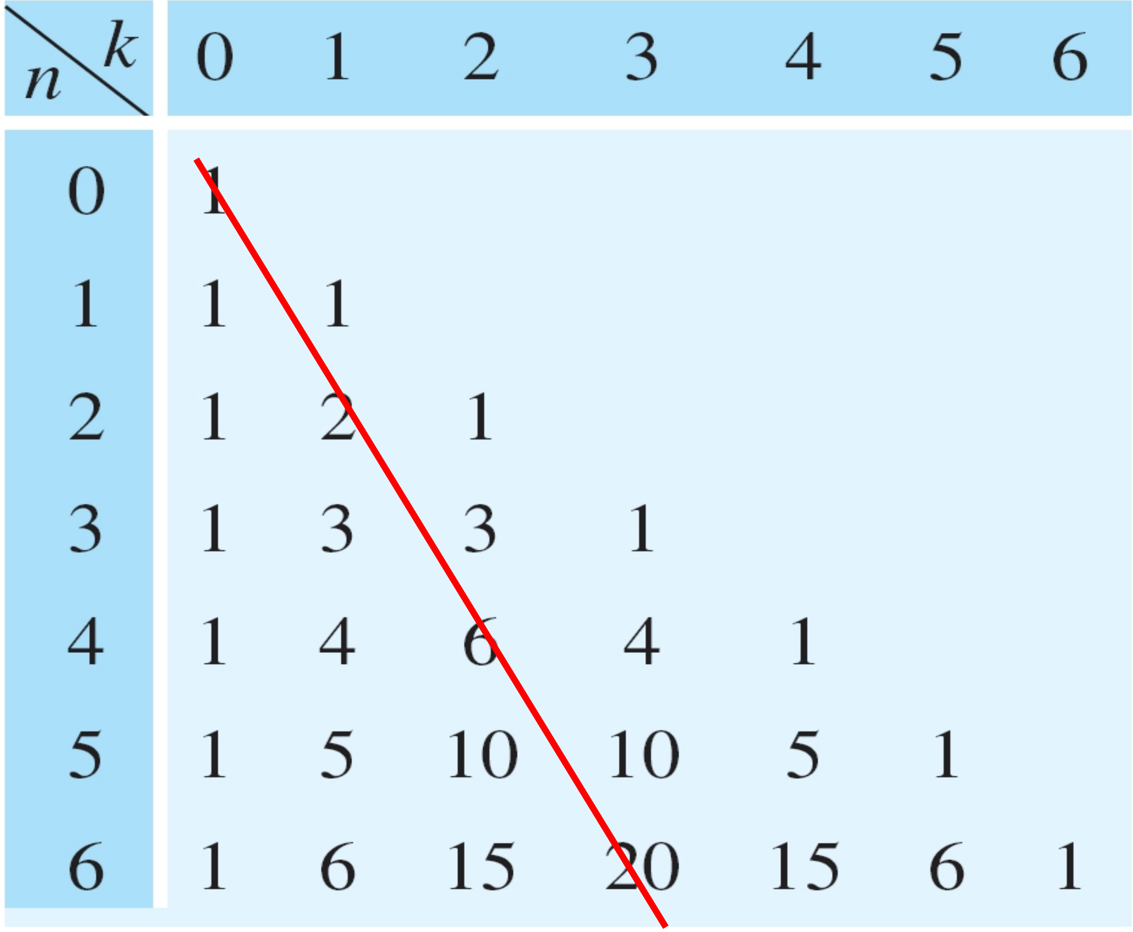

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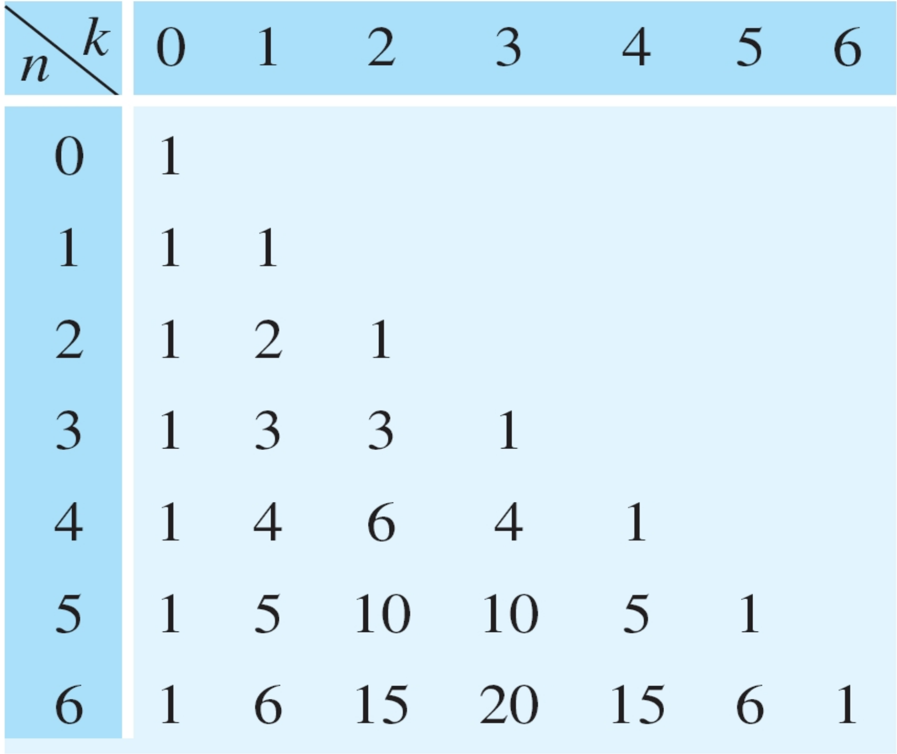
Sum of items on n-th row is 2^n

Pascal's Triangle

Take the table



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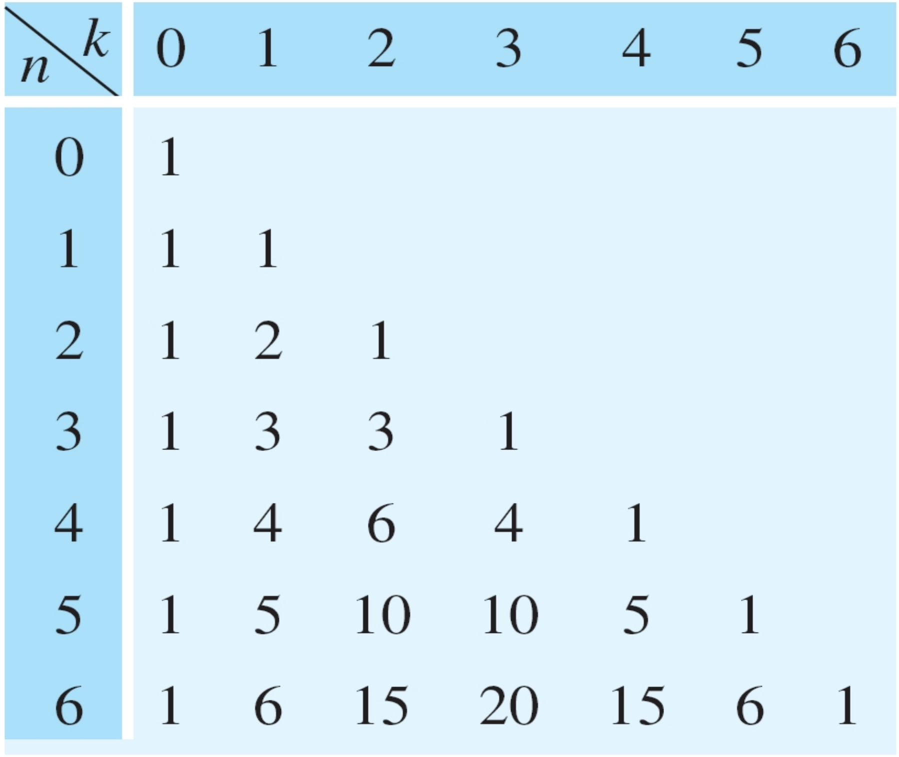

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4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Pascal's Triangle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

What is the next row in the table?

Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Pascal’s Triangle

```
                    1
                 1     1
              1     2     1
           1     3     3     1
        1     4     6     4     1
     1     5    10    10     5     1
  1     6    15    20    15     6     1
1    7    21    35    35    21     7    1
```

Pascal identity

Each (non-1) entry in Pascal’s Triangle is the sum of the two entries directly above it (to left and to right).

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

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  We will use a combinatorial proof.

A Combinatorial Proof

- $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

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$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Therefore, each term (left and right) represents the number of subsets of a particular size chosen from an appropriately sized set.

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- $$\boxed{\binom{n}{k}} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

  Number of k-subsets of an n-element set.

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  Number of $(k-1)$-subsets of an $(n-1)$-element set.

  Number of k-subsets of an $(n-1)$-element set.

Try to use sum principle to explain relationship among these three terms.

Example: $n = 5$, $k = 2$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

A Combinatorial Proof

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Consider $S = \{A, B, C, D, E\}$.

A Combinatorial Proof

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Set S_1 of 2-subsets of S

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\},$$
$$\{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

A Combinatorial Proof

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider $S = \{A, B, C, D, E\}$.

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts.

S_2 the 2-subsets that contain E and

S_3, the set of 2-subsets that do not contain E.

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

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- If n and k are integers satisfying $0 < k < n$, then

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Proof: Apply sum rule.

Let S_1 be set of all k-element subsets.

To apply sum rule, partition S_1 into S_2 and S_3.

Let S_2 be set of k-element subsets that contain x_n.

Let S_3 be set of k-element subsets that don't contain x_n.

Blaise Pascal

Born 1623; Died 1662

French Mathematician

A Founder of Probability Theory

Inventor of one of the first mechanical calculating machines

Pascal Programming Language named for him

Next Lecture

- counting II ...