Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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- **Definition** A *linear homogeneous relation of degree k* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

  where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

  Fact: Assume that the sequences a_n and a'_n both satisfy the recurrence, then $b_n = a_n + a'_n$, $d_n = \alpha a_n$ also satisfy the recurrence, where α is a constant.

  This means: If we find some solutions to a linear homogeneous recurrence, then any linear combination of them will also be a solution.

  So, try to find any solution of the form $a_n = r^n$ that satisfies the recurrence.

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where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant.

◇ Bring $a_n = r^n$ back to the recurrence relation:

$$\text{i.e.,} \quad \begin{aligned} r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}, \\ r^{n-k}&(r^k - c_1 r^{k-1} - \cdots - c_k) = 0 \end{aligned}$$

◇ The solutions to the *characteristic equation* can yield an explicit formula for the sequence.

$$(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$

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  Theorem If this CE has k distinct roots r_i, then the solutions to the recurrence are of the form

  $$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

  for all $n \geq 0$, where the α_i's are constants.

The Case of Degenerate Roots

- **Theorem** If the CE $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0, then
  $$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n,$$
  for all $n \geq 0$ and two constants α_1 and α_2.

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  Exercise.

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  We get $\alpha_1 = 1$ and $\alpha_2 = -1$. Thus,

  $$a_n = 2^n - n2^n$$

The Case of Degenerate Roots in General

- **Theorem** [Theorem 4, p.519] Suppose that there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then

  $$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n,$$

  for all $n \geq 0$ and constants $\alpha_{i,j}$.

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Example

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

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  Fact: Assume that the sequence b_n satisfies the recurrence. Then another sequece a_n satisfies the non-homogeneous recurrence if and only if $h_n = a_n - b_n$ is a sequence that satisfies the associated homogeneous recurrence.

  Idea: We already know how to find h_n. For many common $F(n)$, a solution b_n to the non-homogeneous recurrence is similar to $F(n)$. We then need find $a_n = b_n + h_n$ to the non-homogeneous recurrence that satisfies both recurrence and initial conditions.

- **Theorem** If $a_n = p(n)$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = p(n) + h(n),$$

where $a_n = h(n)$ is any solution to the associated homogeneous recurrence relation

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  Let $p(n) = cn + d$, then
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  We get $c = -1$ and $d = -3/2$. Thus,

  $$p(n) = -n - 3/2$$

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  Let $p(n) = c$, then
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  We get $H(n) = \alpha 2^n - 1$. With the initial condition $H(1) = 1$, we have $\alpha = 1$. Thus, $H(n) = 2^n - 1$.

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  Let $p(k) = c_1 k \cdot 2^k + c_2$, then
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  $c_1 = c_2 = 1$.

  We get $M(k) = \alpha 2^k + k \cdot 2^k + 1$. With the initial condition $M(0) = 0$, we have $\alpha = -1$. Thus, $M(k) = k \cdot 2^k - 2^k + 1$ and
  $T(n) = n \log n - n + 1$.

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Definition The *generating function* for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Generating Functions

- "*Generating functions* are a bridge between *discrete mathematics* , on one hand and *continuous analysis* (particularly complex variable theory) on the other. It is possible to study them solely as tools for solving discrete problems." – Herbert S. Wilf

Generating Functions

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on

Q-Ary Non-Overlapping Codes: A Generating Function Approach

Geyang Wang and Qi Wang, *Member, IEEE*

***Abstract*—Non-overlapping codes are a set of codewords in $\bigcup_{n\geq 2}\mathbb{Z}_q^n$, where $\mathbb{Z}_q=\{0,1,\ldots,q-1\}$, such that the prefix of each codeword is not a suffix of any codeword in the set, including itself; and for variable-length codes, a codeword does not contain any other codeword as a subword. In this paper, we investigate a generic method to generalize binary codes to q-ary ones for $q>2$, and analyze this generalization on the two constructions given by Levenshtein (also by Gilbert; Chee, Kiah, Purkayastha, and Wang) and Bilotta, respectively. The generalization on the former construction gives large non-expandable fixed-length non-overlapping codes whose size can be explicitly determined; the generalization on the latter construction is the first attempt to generate q-ary variable-length non-overlapping codes. More importantly, this generic method allows us to utilize the generating function approach to analyze the cardinality of the underlying q-ary non-overlapping codes. The generating function approach not only enables us to derive new results, e.g., recurrence relations on their cardinalities, new combinatorial interpretations for the constructions, and the superior limit of their cardinalities for some special cases, but also greatly simplifies the arguments for these results. Furthermore, we give an exact formula for the number of fixed-length words that do not contain the codewords in a variable-length non-overlapping code as subwords. This thereby solves an open problem by Bilotta and induces a recursive upper bound on the maximum size of variable-length non-overlapping codes.**

***Index Terms*—Non-overlapping code, variable-length code, generating function.**

(1) No non-empty prefix of each codeword is a suffix of any one, including itself;
(2) For all distinct $\mathbf{u},\mathbf{v}\in S$, $\mathbf{u}$ does not contain $\mathbf{v}$ as a subword.

We say that S is a fixed-length non-overlapping code if $S\subseteq\mathbb{Z}_q^n$, otherwise it is called a variable-length non-overlapping code. In this paper, we consider both fixed-length and variable-length cases. Fixed-length non-overlapping codes have been intensively studied in the literature. Let $C(n,q)$ be the maximum size of a q-ary non-overlapping codes of length n. The main research problems are to construct non-overlapping codes as large as possible in size and to bound $C(n,q)$. The first construction was proposed by Levenshtein in 1964 [2], [3] (Construction 1, see also [4]–[6]). Following the work by de Lind van Wijngaarden and Willink [7] in 2000, Bajic and Stojanovic [8] independently rediscovered binary fixed-length non-overlapping codes (under the name *cross-bifix-free codes*) in 2004. In 2012, Bilotta *et al.* [9] provided a binary construction based on Dyck paths, by which the code size is smaller than Levenshtein's. However, it reveals an interesting connection between non-overlapping codes and other combinatorial objects. In 2013, Chee *et al.* [6] rediscovered Levenshtein's construction (Construction 1), and verified that it is optimal for q =

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$$G(x) = C(m, 0) + \cdots + C(m, m)x^m = (1 + x)^m$$

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$$\begin{aligned} G(x) &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \text{ for } |x| < 1 \\ -xG(x) &= \quad -x - x^2 - x^3 - \cdots - x^n - \cdots \end{aligned}$$

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$$G(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots \text{ for } |x| < 1$$

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G(x) &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \text{ for } |x| < 1 \\
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\end{aligned}
$$

$$G(x) - xG(x) = 1$$

◇ $G(x) = 1/(1-x)^2$ for $|x| < 1$

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Examples

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  $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

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Alternatively, apply the extended binomial theorem:

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k\frac{(n+k-1)\cdots(n+1)n}{k!} = (-1)^k\binom{n+k-1}{k}$$

Useful Generating Functions

$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$

$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$

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$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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- **Problem 1** Find the number of solutions of

$$x_1 + x_2 + x_3 = 17,$$

where x_1, x_2, x_3 are nonnegative integers with $2 \le x_1 \le 5$, $3 \le x_2 \le 6$, $4 \le x_3 \le 7$.

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where x_1, x_2, x_3 are nonnegative integers with $2 \le x_1 \le 5$, $3 \le x_2 \le 6$, $4 \le x_3 \le 7$.

Using *generating functions*, the number is the coefficient of x^{17} in the expansion of

$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$

- **Problem 2** In how many ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

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The coefficient of x^8 in the expansion

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$$C(n+k-1, k) = C(19, 17) = C(19, 2)$$

r-Combinations from a Set

- **Definition** A *k-combination* with repetition allowed, or a *multiset of size k*, chosen from a set of n elements, is an unordered selection of elements with repetition allowed.

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  Read more on pp. 537-548.

- **Problem 4** Use generating functions to find the number of k-combinations of a set with n elements, $C(n, k)$.

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Each of the n elements in the set contributes the term $(1 + x)$ to the generating function $f(x) = \sum_{k=0}^{n} a^k x^k$. Hence, $f(x) = (1 + x)^n$.

Then by the binomial theorem, we have $a_k = \binom{n}{k}$.

Cartesian Product

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_1, b_n), \ldots, (a_m, b_n)\}$$

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Cartesian product defines a set of all ordered arrangements of elements in the two sets.

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  Example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$

  ◇ Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B?

  ◇ Is $Q = \{(1, a), (2, b)\}$ a relation from A to B?

  ◇ Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A?

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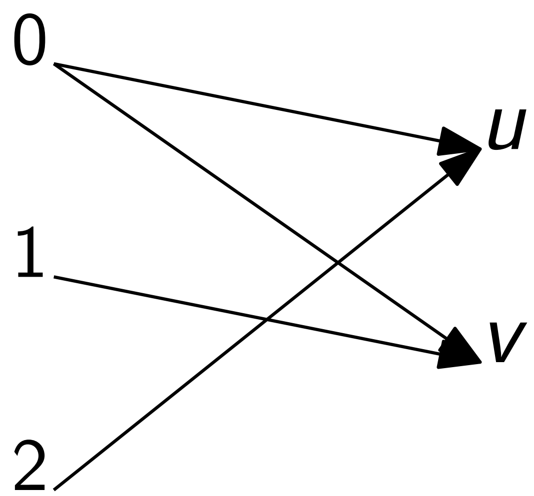

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R	u	v
0	×	×
1	×	
2		×

Relations and Functions

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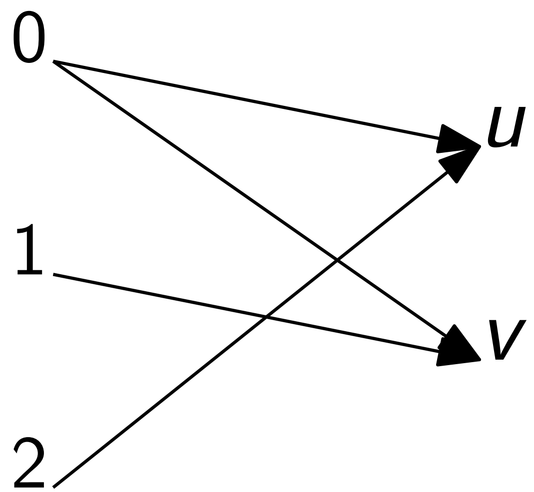

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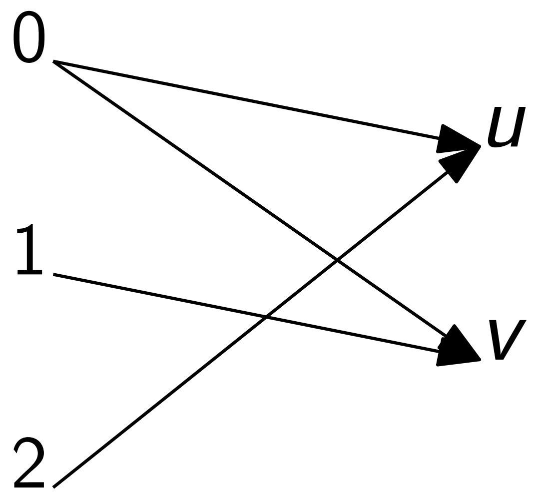

What is the difference between a relation and a function from A to B?

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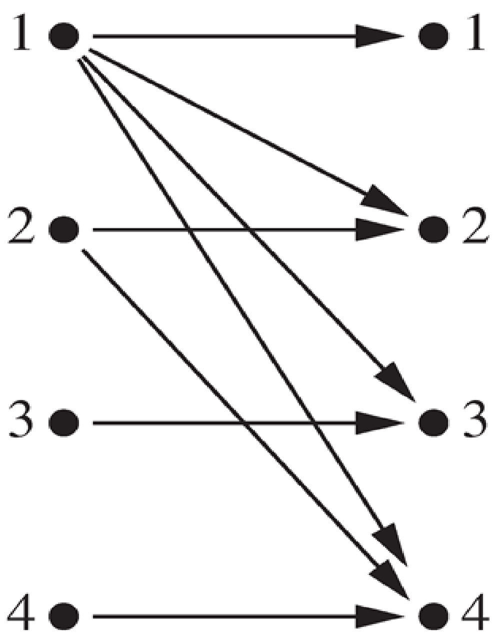

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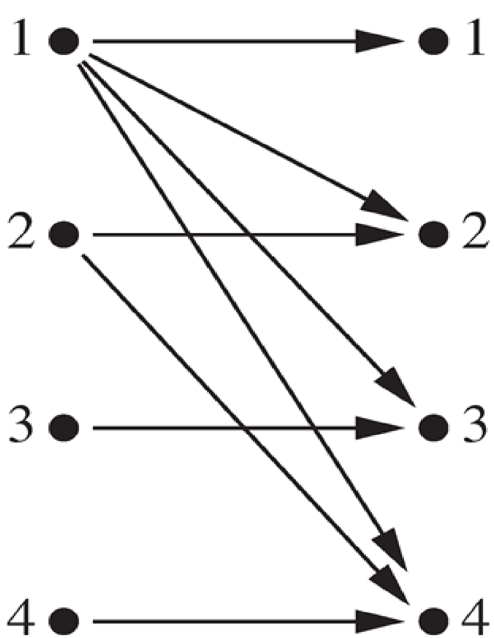

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

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  The number of subsets of a set with k elements is 2^k

Properties of Relations

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  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a | b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} reflexive?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

  Yes. $(1, 1), (2, 2), (3, 3), (4, 4) \in R_{div}$

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A relation R is reflexive if and only if MR has 1 in every position on its main diagonal.

Reflexive Relation

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  No. $(1,1) \notin R$

Properties of Relations

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$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

  Yes. $(1, 1), (2, 2), (3, 3), (4, 4) \notin R_{\neq}$

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Properties of Relations

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  Yes. If $(a, b) \in R_{\neq}$ then $(b, a) \in R_{\neq}$.

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A relation R is symmetric if and only if MR is symmetric.

Properties of Relations

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A relation R is antisymmetric if and only if $m_{ij} = 1$ implies $m_{ji} = 0$ for $i \neq j$.

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Properties of Relations

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Transitive Relation

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- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

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  Combining Relations: Since relations are sets, we can *combine* relations via set operations.

  Set operations: union, intersection, difference, etc.

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- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and
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What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

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  What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

  We may also combine relations by matrix operations.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

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  $S \circ R = \{(1, b), (3, a), (3, b)\}$

Implementation of Composite

- **Example**: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

  $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from A to B

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- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

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Next Lecture

- relation II...