CS215: Discrete Math (H)
2025 Fall Semester Written Assignment # 5
Due: Dec. 22nd, 2025, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b) | a \text{ and } b \text{ have no letters in common}\}$

(2) $R_2 = \{(a, b) | a \text{ and } b \text{ are not the same length}\}$

(3) $R_3 = \{(a, b) | a \text{ is longer than } b\}$

Solution:

(1) Irreflexive, symmetric

(2) Irreflexive, symmetric

(3) Irreflexive, antisymmetric, transitive

$\square$

Q.2 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Solution:

(a) $2^{n(n+1)/2}$

(b) $2^n 3^{n(n-1)/2}$

(c) $2^{n(n-1)}$

(d) $2^{n(n-1)/2}$

(e) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

(f) $3^{n(n-1)/2}$

(g) 2^n

$\square$

Q.3 Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

Solution: R^2 might not be irreflexive. For example, $R = \{(1,2),(2,1)\}$.

$\square$

Q.4 Suppose that R_1 and R_2 are both *reflexive* relations on a set A.

(1) Show that $R_1 \oplus R_2$ is *irreflexive*.

(2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.

(3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Solution:

(1) Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \oplus R_2$ for all $a \in A$. Thus, $R_1 \oplus R_2$ is irreflexive.

(2) Yes. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \cap R_2$

(3) Yes. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \cup R_2$

$\square$

Q.5 Suppose that R is a *symmetric* relation on a set A. Is $\overline{R}$ also symmetric? Explain your answer.

Solution: Under this hypothesis, $\overline{R}$ must also be symmetric. If $(a, b) \in \overline{R}$, then $(a, b) \notin R$, whence (b, a) cannot be in R since R is symmetric. In other words, (b, a) is also contained in $\overline{R}$. Thus, $\overline{R}$ is symmetric.

$\square$

Q.6 Let R_1 and R_2 be *symmetric* relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also be symmetric? Explain your answer.

Solution: Yes. Yes. For both R_1 and R_2, the corresponding 0-1 matrices are both symmetric. Thus. the two matrices representing $R_1 \cap R_2$ and $R_1 \cup R_2$ are also symmetric.

$\square$

Q.7 Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$.

(a) Show that R is an equivalence relation.

(b) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation R?

(c) Give an interpretation of the equivalence classes for the equivalence relation R.

Solution:

(a) For reflexivity, $((a, b), (a, b)) \in R$ because $a \cdot b = b \cdot a$. If $((a, b), (c, d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c, d), (a, b)) \in R$; this tells us that R is symmetric. Finally, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a, b), (e, f)) \in R$; this tells us that R is transitive.

(b) The equivalence classes of $(1, 2)$ is the set of all pairs (a, b) such that the fraction a/b equals $1/2$.

(c) The equivalence classes are the positive rational numbers.

$\square$

Q.8 For the relation R on the set $X = \{(a, b, c) : \ a, b, c \in \mathbb{R}\}$ with $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ if and only if $(a_1, b_1, c_1) = k(a_2, b_2, c_2)$ for some $k \in \mathbb{R} \setminus \{0\}$.

(1) Prove that this is an *equivalence* relation.

(2) Write at least three elements of the equivalence classes $[(1, 1, 1)]$ and $[(1, 0, 3)]$.

(3) Do all the equivalence classes in this relation have the same cardinality?

Solution:

(1) Reflexive: Consider $(a, b, c) \in X$. Note that $(a, b, c) = 1(a, b, c)$. Thus, the relation R is reflexive.

Symmetric: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$. By definition of the relation

$$\begin{aligned} (a_1, b_1, c_1) &= k(a_2, b_2, c_2) \\ \frac{1}{k}(a_1, b_1, c_1) &= (a_2, b_2, c_2). \end{aligned}$$

Since $1/k \in \mathbb{R}$, $(a_2, b_2, c_2)R(a_1, b_1, c_1)$. Thus, the relation is symmetric.

Transitive: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ and $(a_2, b_2, c_2)R(a_3, b_3, c_3)$. By definition of the relation, we have

$$\begin{aligned} (a_1, b_1, c_1) &= j(a_2, b_2, c_2) \\ (a_2, b_2, c_2) &= k(a_3, b_3, c_3) \\ (a_1, b_1, c_1) &= kj(a_3, b_3, c_3) \end{aligned}$$

Since $jk \in \mathbb{R}$, we have $(a_1, b_1, c_1)R(a_3, b_3, c_3)$ and the relation is transitive. To sum up, the relation is an equivalence relation.

(2) We have

$$[(1, 1, 1)] = \{(1, 1, 1), (-1, -1, -1), (2, 2, 2), \ldots\}.$$

$$[(1, 0, 3)] = \{(1, 0, 3), (-1, 0, -3), (2, 0, 6), \ldots\}.$$

(3) No. Note that $[(0,0,0)] = \{(0,0,0)\}$. All the others are infinite.

$\square$

Q.9 Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both *equivalence relations*, then T is also an equivalence relation.

Solution:
We need to show that T is reflexive, symmetric, and transitive.

Reflexive: For any x, we have $(x, x) \in R$ and $(x, x) \in S$, then $(x, x) \in T$.

Symmetric: Suppose that $(x, y) \in T$. This means $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both symmetric, we have $(y, x) \in R$ and $(y, x) \in S$. Then $(y, x) \in T$.

Transitive: Suppose that $(x, y) \in T$ and $(y, z) \in T$. Then $(x, y) \in R$ and $(y, x) \in R$ imply that $(x, z) \in R$. Similarly, we have $(x, z) \in S$. This will imply that $(x, z) \in T$.

$\square$

Q.10 How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?
Solution: 25. There are two possibilities to form exactly three different equivalence classes with 5 elements. One is 3, 1, 1 elements for each equivalence class, and the other is 2, 2, 1 elements for each equivalence class. By counting techniques, there are $\binom{5}{3} + \binom{5}{1} \cdot \binom{4}{2}/2 = 25$.

$\square$

Q.11 Which of these are posets?

(a) $(\mathbf{R}, =)$

(b) $(\mathbf{R}, <)$

(c) $(\mathbf{R}, \leq)$

(d) $(\mathbf{R}, \neq)$

Solution:

(a) Yes. (It is the smallest partial order: reflexivity ensures that very partial order contains at least all pairs (a, b).)

(b) No. It is not reflexive.

(c) Yes.

(d) No. The relations is not reflexive, not antisymmetric, not transitive.

$\square$

Q.12 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Solution:

(a) No. $\propto$ is not symmetric. Let $f(n) = n$ and $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$.

(b) No. $\propto$ is not antisymmetric. Let $f(n) = n$ and $g(n) = 2n$. Then $f = O(g)$ and $g = O(f)$, but $f \neq g$.

(c) No. It is not partial ordering, then not a total ordering.

$\square$

Q.13 For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3 + 5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Solution:

(a) Yes, because the product of positive integers greater than or equal to 2 is less than their sum.

(b) No, because $33 \preceq 26$ and $26 \preceq 33$, but $26 \neq 33$.

(c) No, because $33 \preceq 35$ and $35 \preceq 13$, but we do not have $33 \preceq 13$.

$\square$

Q.14 Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

(a) Prove that $\preceq$ is a partial ordering.

(b) Prove or disprove: $\preceq$ is a total ordering.

Solution:

(a) **Reflexive** For all $x \in \mathbb{R}$, $f(x) \leq f(x)$, so $f \preceq f$.

Antisymmetric Let $f \preceq g$ and $g \preceq f$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq f(x)$ and thus $f(x) = g(x)$. Since this holds for all x, we have $f = g$.

Transitive Let $f \preceq g \preceq h$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq h(x)$, giving $f(x) \leq h(x)$. So, $f \preceq h$.

(b) It is not a total ordering. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\leq -1 = g(1)$ and $g(-1) = 1 \not\leq -1 = f(-1)$. So it is not the case that for all x, $f(x) \leq g(x)$, and it is not the case that for all x, $g(x) \leq f(x)$. That is, these two functions are incomparable.

$\square$

Q.15 We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0, 1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0, 1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.

Solution:

(a) There are many choices here. One is to let $R = \{A_0, A_1, A_2, \ldots\}$ where $A_i = \{j \in \mathbb{N} | j < i\}$. Then R has no maximal element, because for any $A_i \in R$, we have $A_i \not\supseteq A_{i+1} \in R$.

(b) For this we will do the same thing as above in reverse. Let $S = \{B_0, B_1, B_2, \ldots\}$ where $B_i = \{j \in \mathbb{N} | j \geq i\}$. Then S has no minimal element, because for any $B_i \in S$, we have $B_i \not\supseteq B_{i+1}$.

(c) Here we can combine the previous two results. Let $T = \{C_{ij} | i \in \mathbb{N},\ j \in \mathbb{N}\}$ where each $x \in \mathbb{N}$ is in C_{ij} if and only if $x = 2k$ and $k < i$, or $x = 2k+1$ and $K \geq j$. Now T has no minimal or maximal elements, because for any $C_{ij} \in T$, $C_{i,j+1} \not\supseteq C_{ij} \not\supseteq C_{i+1,j}$.

$\square$

Q.16 Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

(1) Find the maximal elements.

(2) Find the minimal elements.

(3) Is there a greatest element?

(4) Is there a least element?

(5) Find all upper bounds of $\{3, 5\}$.

(6) Find the least upper bound of $\{3, 5\}$, if it exists.

(7) Find all lower bounds of $\{15, 45\}$.

(8) Find the greatest lower bound of $\{15, 45\}$, if it exists.

Solution:

(1) By drawing the Hasse diagram, our maximal elements are 24 and 45.

(2) The minimal elements are 3 and 5.

(3) There is no greatest element because this element would have to be a number that all other elements divide. Since our maximal elements are 24 and 45, and they do not divide each other, we do not have a greatest element.

(4) There is no least element because this element would be a number that can divide all other elements. Since our minimal elements are 3 and 5, and they do not divide each other, we do not have a least element.

(5) 15 and 45.

(6) 15.

(7) 3, 5, and 15.

(8) 15.

$\square$

Q.17 Define the relation $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ according to

$$(a, b) \preceq (c, d) \Leftrightarrow (a, b) = (c, d) \text{ or } a^2 + b^2 < c^2 + d^2.$$

Show that $(\mathbb{Z}\times\mathbb{Z}, \preceq)$ is a poset; Construct the Hasse diagram for the subposet $(B, \preceq)$, where $B = \{0, 1, 2\} \times \{0, 1, 2\}$.
Solution: We now prove that $\preceq$ on the set $\mathbb{Z} \times \mathbb{Z}$ is a partial ordering. Obviously, $(a, b) \preceq (a, b)$, and we have $\preceq$ is reflexive; Suppose that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$, then the only possibility is that $(a, b) = (c, d)$. Then $\preceq$ is antisymmetric; Suppose that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$, then we have four possible cases: $(a, b) = (c, d)$ and $c^2 + d^2 < e^2 + f^2$; $(a, b) = (c, d)$ and $(c, d) = (e, f)$; $a^2 + b^2 < c^2 + d^2$ and $(c, d) = (e, f)$; $a^2 + b^2 < c^2 + d^2$ and $c^2 + d^2 < e^2 + f^2$. For each of the four cases above, we have $(a, b) \preceq (e, f)$ and thereby the relation $\preceq$ is transitive.

$\square$

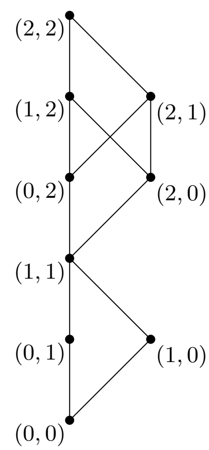

Figure 1: Q.17