CS215: Discrete Math (H)
2025 Fall Semester Written Assignment #1
Due: Oct. 13th, 2025, please submit at the beginning of class

Q.1 Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(1) You get 100 marks on the final, but you do not get an A in this course.

(2) You will get an A in this course if you get 100 marks on the final.

(3) If you do not get 100 marks on the final, then you will not get an A in this course.

(4) Getting 100 marks on the final is sufficient for getting an A in this course.

(5) You get an A in this course, but you do not get 100 marks on the final.

Solution:

(1) $p \wedge \neg q$

(2) $p \rightarrow q$

(3) $\neg p \rightarrow \neg q$

(4) $p \rightarrow q$

(5) $q \wedge \neg p$

$\square$

Q.2 Construct a truth table for each of these compound propositions.

(1) $(p \oplus q) \rightarrow (p \wedge q)$

(2) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

(3) $(p \oplus q) \to (p \oplus \neg q)$

Solution:

p	q	$(p \oplus q) \to (p \wedge q)$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$	$(p \oplus q) \to (p \oplus \neg q)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	T	T	T

$\square$

Q.3 Use truth tables to decide whether or not the following two propositions are equivalent.

(1) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

(2) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(3) $(p \to q) \to r$ and $p \to (q \to r)$

(4) $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$

(5) $(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$ and $q \vee (\neg p \wedge \neg r)$

Solution:

(1) The combined truth table is:

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
F	F	T	T	T	T	F	T
F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F
T	T	T	F	F	F	T	T

By comparing the third and last columns, we have that they are equivalent.

(2) The truth table for $p \to (q \vee r)$ is :

p	q	r	$q \vee r$	$p \to (q \vee r)$
F	F	F	F	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

The truth table for $(p \to q) \vee (p \to r)$ is

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	T	F	T
T	T	T	T	T	T

Since the final columns are the same in both truth tables, we know that these two propositions are equivalent.

(3) The combined truth table is:

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
F	F	F	T	F	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

Since the fifth and last columns are not the same in both truth tables, we know that these two propositions are not equivalent.

(4) The truth table for $(p \vee q) \to r$ is :

p	q	r	$p \vee q$	$(p \vee q) \to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

The truth table for $(p \to r) \wedge (q \to r)$ is

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	T	T	T	T

Since the final columns are the same in both truth tables, we know that these two propositions are equivalent.

(5) The two truth tables are:

p	q	r	$p \to \neg q$	$(p \vee \neg q)$	$r \to (p \vee \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T
F	T	T	T	F	F	F
T	F	F	T	T	T	T
T	F	T	T	T	T	T
T	T	F	F	T	T	F
T	T	T	F	T	T	F

p	q	r	$\neg p \wedge \neg r$	$q \vee (\neg p \wedge \neg r)$
F	F	F	T	T
F	F	T	F	F
F	T	F	T	T
F	T	T	F	T
T	F	F	F	F
T	F	T	F	F
T	T	F	F	T
T	T	T	F	T

Since the final columns are not the same in both truth tables, we know that these two propositions are not equivalent.

$\square$

Q.4 Use logical equivalences to prove the following statements.

(1) $\neg(p \to q) \to \neg q$ is a tautology.

(2) $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are equivalent.

(3) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Solution:

(1) We have

$$\begin{aligned}
&\neg(p \to q) \to \neg q \\
&\equiv \neg\neg(p \to q) \vee \neg q \quad \text{Useful} \\
&\equiv (p \to q) \vee \neg q \quad \text{Double negation} \\
&\equiv (\neg p \vee q) \vee \neg q \quad \text{Useful} \\
&\equiv \neg p \vee (q \vee \neg q) \quad \text{Associative} \\
&\equiv T \quad \text{Domination}
\end{aligned}$$

Therefore, it is a tautology.

(2) We have

$$
\begin{aligned}
&\neg p \rightarrow (q \rightarrow r) \\
\equiv\ & p \vee (q \rightarrow r) \quad \text{Useful and double negation} \\
\equiv\ & p \vee (\neg q \vee r) \quad \text{Useful} \\
\equiv\ & (p \vee \neg q) \vee r \quad \text{Associative} \\
\equiv\ & (\neg q \vee p) \vee r \quad \text{Commutative} \\
\equiv\ & \neg q \vee (p \vee r) \quad \text{Associative} \\
\equiv\ & q \rightarrow (p \vee r) \quad \text{Useful}
\end{aligned}
$$

(3) We have

$$
\begin{aligned}
&(p \rightarrow q) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q)) \\
\equiv\ & \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
\equiv\ & \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan} \\
\equiv\ & \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
\equiv\ & \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
\equiv\ & \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Complement} \\
\equiv\ & \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
\equiv\ & \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
\equiv\ & \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
\equiv\ & (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
\equiv\ & T \vee \neg r \quad \text{Complement} \\
\equiv\ & T \quad \text{Identity.}
\end{aligned}
$$

Thus, it is a tautology.

$\square$

Q.5 Let $F(x, y)$ be the statement "x can fool y", where the domain consists of all people in the world. Use quantifiers to express each of these statement.

(1) Everybody can fool somebody.

(2) There is no one who can fool everybody.

(3) Everyone can be fooled by somebody.

(4) Nancy can fool exactly two people.

(5) There is exactly one person whom everybody can fool.

(6) There is someone who can fool exactly one person besides himself or herself.

Solution:

(1) $\forall x \exists y F(x, y)$

(2) $\neg \exists x \forall y F(x, y)$

(3) $\forall y \exists x F(x, y)$

(4) $\exists y_1 \exists y_2 (F(\text{Nancy}, y_1) \wedge F(\text{Nancy}, y_2) \wedge y_1 \neq y_2 \wedge \forall y (F(\text{Nancy}, y) \rightarrow (y = y_1 \vee y = y_2)))$

(5) $\exists y (\forall x F(x, y) \wedge \forall z (\forall x F(x, z) \rightarrow z = y))$

(6) $\exists x \exists y (x \neq y \wedge F(x, y) \wedge \forall z ((F(x, z) \wedge z \neq x) \rightarrow z = y))$

□

Q.6 Given the following predicate on the set P of all people who ever lived: $\text{Parent}(x, y)$ means x is the parent of y.

(1) Rewrite the following in the language of mathematical logic (you may use the equality/inequality operators):

"All people have two parents."

(2) Based on the definition of Parent, we will *recursively* define the concept of *ancestor*: "An ancestor of a person is one of the person's parents or the ancestor of (at least) one of the person's parents."

Rewrite this definition using the language of mathematical logic. Specifically, you need to provide a necessary and sufficient condition for the predicate $\text{Ancestor}(x, y)$ to be true. Note that you can *recursively* use the predicate $\text{Ancestor}(x, y)$ in the condition itself.

Solution:

(1) $\forall x \in P \; \exists y, z \in P(\text{Parent}(y, x) \wedge \text{Parent}(z, x) \wedge y \neq z)$

(2) $\forall x, y \in P(\text{Ancestor}(x, y) \leftrightarrow (\text{Parent}(x, y) \vee (\exists z \in P(\text{Parent}(z, y) \wedge \text{Ancestor}(x, z)))))$

□

Q.7 For the predicate $P(x, y)$ with two variables x, y, answer the following two questions.

(1) Give an example of a predicate $P(x, y)$ such that $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$ have *different* truth values.

(2) If $\forall y \exists x P(x, y)$ is true, does it necessarily follow that $\exists x \forall y P(x, y)$ is true?

Solution:

(1) Let the domain be the set of natural numbers $\mathbb{N}$, and the predicate $P(x, y)$ denote $x > y$. Then, $\exists x \forall y P(x, y)$ is false, but $\forall y \exists x P(x, y)$ is true.

(2) No. From (1), we know that if $\forall y \exists x P(x, y)$ is true, $\exists x \forall y P(x, y)$ is not necessarily true but false.

□

Q.8 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(1) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

(2) $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$

(3) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$

(4) $\forall x \exists y (P(x, y) \wedge \exists z R(x, y, z))$

Solution:

(1)

$$\begin{aligned}\neg(\exists x\exists yP(x,y)\wedge\forall x\forall yQ(x,y)) &\equiv \neg\exists x\exists yP(x,y)\vee\neg\forall x\forall yQ(x,y)\\ &\equiv \forall x\neg\exists yP(x,y)\vee\exists x\neg\forall yQ(x,y)\\ &\equiv \forall x\forall y\neg P(x,y)\vee\exists x\exists y\neg Q(x,y)\end{aligned}$$

(2)

$$\begin{aligned}\neg\exists x\exists y(Q(x,y)\leftrightarrow Q(y,x)) &\equiv \forall x\neg\exists y(Q(x,y)\leftrightarrow Q(y,x))\\ &\equiv \forall x\forall y\neg(Q(x,y)\leftrightarrow Q(y,x))\\ &\equiv \forall x\forall y(\neg Q(x,y)\leftrightarrow Q(y,x))\end{aligned}$$

(3)

$$\begin{aligned}\neg(\forall x\exists yP(x,y)\vee\forall x\exists yQ(x,y)) &\equiv \exists x\neg\exists yP(x,y)\wedge\exists x\neg\exists yQ(x,y)\\ &\equiv \exists x\forall y\neg P(x,y)\wedge\exists x\forall y\neg Q(x,y).\end{aligned}$$

(4)

$$\begin{aligned}\neg\forall x\exists y(P(x,y)\wedge\exists zR(x,y,z)) &\equiv \exists x\neg\exists y(P(x,y)\wedge\exists zR(x,y,z))\\ &\equiv \exists x\forall y(\neg P(x,y)\vee\forall z\neg R(x,y,z))\end{aligned}$$

□

Q.9

(a) Let P be a proposition in atomic propositions p and q. If $P=\neg(p\leftrightarrow(q\vee\neg p))$, prove that P is equivalent to $\neg p\vee\neg q$.

(b) If P is of any length, using any of the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, prove that P is logically equivalent to a proposition of the from

$$A\Box B,$$

where $\Box$ is one of $\wedge$, $\vee$, $\leftrightarrow$, and A and B are chosen from $\{p,\neg p,q,\neg q\}$.

Solution:

(a) This can be proved by truth table.

Alternatively, we prove it using logical equivalences as follows.

$$\begin{aligned}
P &= \neg(p \leftrightarrow (q \vee \neg p)) \\
&\equiv \neg((p \to (q \vee \neg p)) \wedge ((q \vee \neg p) \to p)) \quad \text{Definition} \\
&\equiv \neg((\neg p \vee (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Useful} \\
&\equiv \neg((\neg p \vee q) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Idempotent}
\end{aligned}$$

For simplicity, let $r = \neg p \vee q$, then we have

$$\begin{aligned}
P &\equiv \neg(r \wedge (\neg r \vee p)) \\
&\equiv \neg r \vee \neg(\neg r \vee p) \quad \text{De Morgan} \\
&\equiv \neg r \vee (r \wedge \neg p) \quad \text{De Morgan and double negation} \\
&\equiv (\neg r \vee r) \wedge (\neg r \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg r \vee \neg p) \quad \text{Negation} \\
&\equiv \neg r \vee \neg p \quad \text{Identity} \\
&\equiv (p \wedge \neg q) \vee \neg p \quad \text{De Morgan} \\
&\equiv (p \vee \neg p) \wedge (\neg q \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg q \vee \neg p) \quad \text{Negation} \\
&\equiv \neg p \vee \neg q \quad \text{Identity.}
\end{aligned}$$

(b) For the proposition P, since the two involved atomic propositions p and q can have at most 4 combinations of truth tables, P has at most 2^4 different forms in terms of truth tables up to logical equivalence. It then suffices to prove that a proposition of the form $A \Box B$ has also 2^4 different forms in terms of truth tables up to logical equivalence.

If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$, and $\Box \in \{\wedge, \vee\}$, then $A \Box B$ has $2 \times 2 \times 2 = 8$ different possible forms. If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$ and $\Box = \leftrightarrow$, then there are two extra different possibilities: $p \leftrightarrow q$ and $p \leftrightarrow \neg q$. Together with $p \vee p \equiv p$, $p \vee \neg p \equiv T$, $q \vee q \equiv q$, $p \wedge \neg p \equiv F$ and similarly $\neg p$, $\neg q$, we will have the $2^4 = 16$ different forms by $A \Box B$. This proves the statement.

$\Box$

Q.10 For the following argument, explain which rules of inference are used for each step.

"Each of five roommates, A, B, C, D, and E, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

Solution:

Let $r(x)$ be "x is one of the five roommates listed", let $d(x)$ be "x has taken a course in discrete mathematics", and let $a(x)$ be "x can take a course in algorithms". We are given premises $\forall x(r(x) \rightarrow d(x)$, $\forall x(d(x) \rightarrow a(x)$, and we want to conclude $\forall x(r(x) \wedge a(x))$.

Step	**Reason**
1. $\forall x(r(x) \rightarrow d(x))$	Hypothesis
2. $r(y) \rightarrow d(y)$	Universal Instantiation using 1.
3. $\forall x(d(x) \rightarrow a(x))$	Hypothesis
4. $d(y) \rightarrow a(y)$	Universal instantiation using 3.
5. $r(y) \rightarrow a(y)$	Hypothetical syllogism using 2. and 4.
6. $\forall x(r(x) \rightarrow a(x))$	Universal generalization using 5.

$\square$

Q.11 Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$.

Solution: We prove by four cases.

Case 1: $x \geq 0$ and $y \geq 0$. Then $|x| + |y| = x + y = |x + y|$.

Case 2: $x < 0$ and $y < 0$. Then $|x| + |y| = -x + (-y) = -(x + y) = |x + y|$.

Case 3: $x \geq 0$ and $y < 0$. Then $|x| + |y| = x + (-y)$. If $x \geq -y$, then $|x + y| = x + y$. But because $y < 0$, $-y > y$, so $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$, then $|x + y| = -(x + y)$. But because $x < 0$, $x \geq -x$, so $|x| + |y| = x + (-y) \geq -x + (-y) = |x + y|$.

Case 4: $x < 0$ and $y \geq 0$. Similar to Case 3.

$\square$

Q.12 Prove or disprove the following.

(1) For two irrational numbers a and b, a^b is also irrational.

(2) For an irrational number a, $\sqrt{a}$ is also irrational.

(3) There is a rational number x and an irrational number y such that x^y is irrational.

Solution:

(1) The statement is false. Consider $\sqrt{2}^{\sqrt{2}}$, if it is rational, then we have a counterexample that a^b is rational for a, b both irrational; if it is not, then we consider $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$, which equals 2, again a counterexample. Alternatively, consider $\sqrt{2}^{2\log_2 3}$, which equals 3, a counterexample, since both $\sqrt{2}$ and $\log_2 3$ are irrational.

(2) The statement is true. Suppose to the contrary that $\sqrt{a}$ is rational, i.e., $\sqrt{a} = \frac{c}{d}$ for two integers c and d. Then we have $a = \frac{c^2}{d^2}$, which is again rational, contradiction. Thus, $\sqrt{a}$ must be irrational.

(3) Let $x = 2$ and $y = \sqrt{2}$. If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$. Then $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2}\cdot(\sqrt{2})/4} = \sqrt{2}$.

$\square$

Q.13 Prove that $\sqrt[3]{2}$ is irrational.
Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

$\square$

Q.14 Prove that between every two rational numbers there is an irrational number.

Solution: By finding a common denominator, we can assume the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$. In particular, $(a+1)/b \leq c/b$. Thus, $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well, which is wrong.

$\square$

Q.15 Let the coefficients of the polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$ be integers. Show that any *real* root of the equation $f(x) = 0$ is either integral or irrational. Note that in your proof, you may direct use the following result without a proof. "**Fact.** If a prime p is a factor of some power of an integer, then it is a factor of that integer."

Solution: Let r be a real root of the polynomial, such that

$$a_0 + a_1r + a_2r^2 + \cdots + a_{n-1}r^{n-1} + r^n = 0.$$

There are three cases: either r is an integer, or r is irrational, or $r = s/t$ for integers s and t which have no common factors and such that $t > 1$. We want to eliminate the last case, so assume to the contrary that it holds for some r.

Substituting s/t for r and multiplying both sides of the above equation by t^m yields:

$$\begin{aligned} a_0t^n + a_1st^{n-1} + a_2s^2t^{n-2} + \cdots + a_{n-1}s^{n-1}t + s^n &= 0, \\ a_0t^n + a_1st^{n-1} + a_2s^2t^{n-2} + \cdots + a_{n-1}s^{n-1}t &= -s^n. \end{aligned}$$

Now since $t > 1$, it must have a prime factor p. The prime p therefore divides each term of the left-hand side of the equation above, so p also divides the right-hand side, $-s^n$. This means that p divides s^n. Then by Fact, p is also a factor of s. Thus, we know that p is a common factor of s and t, contradicting the fact that s and t have no common factors.