CS215: Discrete Math (H)
2025 Fall Semester Written Assignment # 2
Due: Oct. 27th, 2025, please submit at the beginning of class

Q.1 Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(1) $(A - B = A) \to (B \subseteq A)$

(2) $(A - B = \emptyset) \to (A \cap B = B \cap A)$

(3) $(A \subseteq B) \to (|A \cup B| \geq 2|A|)$

Q.2 Let's formulate the "Barber's paradox" in the language of predicate logic. In English, the paradox may be stated as:

"The barber of the village Seville shaves those residents of Seville who do not shave themselves."

Assume that S is the set of all residents of Seville, which includes the barber. We have the following predicates over elements of the set S:

- $Shaves(x, y)$: true if x shaves y, false otherwise.
- $Barber(x)$: true if x is the barber of Seville (you may assume that Seville has just one barber), false otherwise.

Rewrite the statement of the paradox using only these two predicates, along with the notation of mathematical logic. Please also state the reason why the paradox occurs in the logical statement.

Q.3 Prove or disprove the following.

(1) For any three sets A, B, C, $C - (A \cap B) = (C - A) \cap (C - B)$.

(2) For any two sets A, B, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$, where $\mathcal{P}(A)$ denotes the power set of the set A.

(3) For any two sets A, B, $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$, where $\mathcal{P}(A)$ denotes the power set of the set A.

(4) For a function $f : X \to Y$, $f(A \cap B) = f(A) \cap f(B)$, for any two sets $A, B \subseteq X$.

Q.4 Give an example of two uncountable sets A and B such that the intersection $A \cap B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.5 The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. Give an example of two uncountable sets A and B such that the intersection $A \oplus B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.6 Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Q.7 Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Q.8 Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the *composition* function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must f be onto? Explain your answer.

(e) If $f \circ g$ is onto, must g be onto? Explain your answer.

Q.9 Derive the formula for $\sum_{k=1}^{n} k^2$.

Q.10 Derive the formula for $\sum_{k=1}^{n} k^3$.

Q.11 Show that a subset of a countable set is also countable.

Q.12 Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Q.13 If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

Q.14 By the Schröder-Bernstein theorem, prove that $(0, 1)$ and $[0, 1]$ have the same cardinality.

Q.15 If $f_1(x)$ and $f_1(x)$ are functions from the set of positive integers to the set of positive real numbers and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, is $(f_1 - f_2)(x)$ also $\Theta(g(x))$? Either prove that it is or give a counter example.

Q.16 Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.

Q.17 Prove that $n \log n = \Theta(\log n!)$ for all positive integers n.

Q.18

(a) Show that this algorithm determines the number of 1 bits in the bit string S:

Algorithm 1 bit count (S: bit string)

```
count := 0
while S ≠ 0 do
    count := count + 1
    S := S ∧ (S − 1)
end while
return count {count is the number of 1's in S}
```

Here $S - 1$ is the bit string obtained by changing the rightmost 1 bit of S to a 0 and all the 0 bits to the right of this to 1's. [Recall that $S \wedge (S - 1)$ is the bitwise AND of S and $S - 1$.]

(b) How many bitwise *AND* operations are needed to find the number of 1 bits in a string S using the algorithm in part a)?

Q.19

(1) Show that $(\sqrt{2})^{\log n} = O(\sqrt{n})$, where the base of the logarithm is 2.

(2) Arrange the functions

$$n^n,\ (\log n)^2,\ n^{1.0001},\ (1.0001)^n,\ 2^{\sqrt{\log_2 n}},\ n(\log n)^{1001}$$

in a list such that each function is big-O of the next function.

Q.20 Give an example of two increasing functions $f(n)$ and $g(n)$ from the set of positive integers to the set of positive integers such that neither $f(n)$ is $O(g(n))$ nor $g(n)$ is $O(f(n))$.

Q.21 Aliens from another world come to the Earth and tell us that the *3SAT* problem is *solvable* in $O(n^8)$ time. Which of the following statements follow as a consequence?
A. All NP-Complete problems are solvable in polynomial time.
B. All NP-Complete problems are solvable in $O(n^8)$ time.
C. All problems in NP, even those that are not NP-Complete, are solvable in polynomial time.
D. No NP-Complete problem can be solved *faster* than in $O(n^8)$ in the worst case.
E. P = NP.

Q.22 Compare the following pairs of functions in terms of order of growth. In each of the following, determine if $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$. There is **no need** to explain your answers.

	$f(n)$	$g(n)$	
(1)	$(\log_2 n)^a$	n^b	here $a, b > 0$
(2)	$2^{n \log_2 n}$	$10n!$	
(3)	$\sqrt{n}$	$(\log_2 n)^5$	
(4)	$\frac{n^2}{\log_2 n}$	$(n \log_2 n)^4$	
(5)	$\log_2 n$	$\log_2(66n)$	
(6)	$1000(\log_2 n)^{0.9999}$	$(\log_2 n)^{1.001}$	
(7)	n^2	$n(\log_2 n)^{15}$	