Department of
Computer Science and Engineering
计算机科学与工程系

# CS215 DISCRETE MATH

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# Application of Number Theory

- G. H. Hardy (1877 - 1947)

  In his 1940 autobiography *A Mathematician's Apology*, Hardy wrote

  "The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present, **almost as 'useless' as the theory of numbers**."

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  "The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present, **almost as 'useless' as the theory of numbers**."

If he could see the world now, Hardy would be spinning in his grave.

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- At one point, the largest employer of mathematicians in the United States, and probably the world, was the National Security Agency (NSA). The NSA is the largest spy agency in the US (bigger than CIA, Central Intelligence Agency), and has the responsibility for code design and breaking.

# Division

- If $a$ and $b$ are integers with $a \neq 0$, we say that *a divides b* if there is an integer $k$ such that $b = ak$, or equivalently $b/a$ is an integer. In this case, we say that $a$ is a *factor* or *divisor* of $b$, and $b$ is a *multiple* of $a$. (We use the notations $a|b$, $a \nmid b$)

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**Example**

◇ $4 \mid 24$

◇ $3 \nmid 7$

# Divisibility

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- **Question:** Let $n$ and $d$ be two positive integers. How many positive integers not exceeding $n$ are divisible by $d$?

**Answer:** Count the number of integers such that $0 < kd \leq n$. Therefore, there are $\lfloor n/d \rfloor$ such positive integers.

- **Properties**

  Let $a, b, c$ be integers. Then the following hold:

  (i) if $a|b$ and $a|c$, then $a|(b+c)$

  (ii) if $a|b$ then $a|bc$ for all integers $c$

  (iii) if $a|b$ and $b|c$, then $a|c$

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  **Proof**.

# Divisibility

- **Corollary** If $a, b, c$ are integers, where $a \neq 0$, such that $a | b$ and $a | c$, then $a | (mb + nc)$ whenever $m$ and $n$ are integers.

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**Proof**. By part (ii) and part (i) of Properties.

# The Division Algorithm

- If $a$ is an integer and $d$ a positive integer, then there are **unique** integers $q$ and $r$, with $0 \leq r < d$, such that $a = dq + r$. In this case, $d$ is called the *divisor*, $a$ is called the *dividend*, $q$ is called the *quotient*, and $r$ is called the *remainder*.

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In this case, we use the notations $q = a \; div \; d$ and $r = a \bmod d$.

# Congruence Relation

- If $a$ and $b$ are integers and $m$ is a positive integer, then $a$ is *congruent to $b$ modulo $m$ if $m$ divides $a - b$*, denoted by $a \equiv b \pmod{m}$. This is called *congruence* and $m$ is its *modulus*.

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**Example**

- ◇ $15 \equiv 3 \pmod{6}$
- ◇ $-1 \equiv 11 \pmod{6}$

# More on Congruences

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**Proof**.

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# $\pmod{m}$ and $\bmod m$ Notations

- $a \equiv b \pmod{m}$ and $a \bmod m = b$ are different.
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# Congruences of Sums and Products

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# Algebraic Manipulation of Congruences

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$14 \equiv 8 \pmod{6}$ but $7 \not\equiv 4 \pmod{6}$

# Computing the mod Function

- **Corollary** Let $m$ be a positive integer and let $a$ and $b$ be integers. Then

$$(a+b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$
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**Example**

- ◇ $7 +_{11} 9 = ?$
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# Group

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  **Example**:
  $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{M}_{n \times n}, +)$ ?
  $(\mathbb{Z}^*, \times)$, $(\mathbb{Q}^*, \times)$, $(\mathbb{R}^*, \times)$, $(\mathbb{M}^*_{n \times n}, \cdot)$ ?

# Permutation Group

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  For example, $s_3 =< 1, 2, 3 >$
  $P_3 = \{< 1, 2, 3 >, < 1, 3, 2 >, < 2, 1, 3 >, < 2, 3, 1 >, < 3, 1, 2 >, < 3, 2, 1 >\}$

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- Define a binary operation $\circ$ on the elements of $P_n$: for $\rho, \pi \in P_n$, $\pi \circ \rho$ denotes a *re-permutation* of the elements of $\rho$ according to the elements of $\pi$.

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  $$< 1, 2, 3 > \circ\rho = \rho \circ < 1, 2, 3 >= \rho$$

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$(P_n, \circ)$ is not abelian.

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- If the group operation is referred to as addition (*multiplication*), then the group also allows for *subtraction* (*division*).

  $$a - b = a + (-b)$$
  $$a/b = a \cdot b^{-1}$$

# Ring

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  **Identity element** for multiplication: $a1 = 1a = a$
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# Field

- A *field*, denoted by $(F, +, \times)$, is an *integral domain* whose elements satisfy the following additional property.

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  $(\mathbb{Z}, +, \times)$, $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$ ?
  $(\mathbb{Z}_p, +, \times)$ ?

# Representations of Integers

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- We may use *decimal* (*base* 10) or *binary* or *octal* or *hexadecimal* or other notations to represent integers.

- Let $b > 1$ be an integer. Then if $n$ is a positive integer, it can be expressed uniquely in the form $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$, where $k$ is nonnegative, $a_i$'s are nonnegative integers less than $b$. The representation of $n$ is called *the base-$b$ expansion of $n$* and is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$.

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**Example**

◇ $(101011111)_2 = 2^8 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 351$

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**Example**

◇ $(101011111)_2 = (\underline{101}\,\overline{011}\,\underline{111}) = (537)_8$

◇ $(7016)_8 = (\underline{111}\,\overline{000}\,\underline{001}\,\overline{110})_2$

$= (\underline{1110}\,\overline{0000}\,\underline{1110})_2 = (E0E)_{16}$

# Base-$b$ Expansions

$$
\begin{aligned}
n &= a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\
&= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \cdots + a_2 b + a_1) + {\color{red} a_0} \\
&= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \cdots + a_2) + {\color{red} a_1}) + {\color{blue} a_0} \\
&= \cdots
\end{aligned}
$$

# Base-$b$ Expansions

$$\begin{aligned} n &= a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\ &= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \cdots + a_2 b + a_1) + a_0 \\ &= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \cdots + a_2) + a_1) + a_0 \\ &= \cdots \end{aligned}$$

To construct the base-$b$ expansion of an integer $n$,

- Divide $n$ by $b$ to obtain $n = bq_0 + a_0$, with $0 \leq a_0 < b$
- The remainder $a_0$ is the rightmost digit in the base-$b$ expansion of $n$. Then divide $q_0$ by $b$ to get $q_0 = bq_1 + a_1$ with $0 \leq a_1 < b$
- $a_1$ is the second digit from the right. Continue by successively dividing the quotients by $b$ until **the quotient is 0**

# Algorithm: Constructing Base-$b$ Expansions

**procedure** *base b expansion*($n$, $b$: positive integers with $b > 1$)
$q := n$
$k := 0$
**while** ($q \neq 0$)
    $a_k := q$ **mod** $b$
    $q := q$ **div** $b$
    $k := k + 1$
**return**$(a_{k-1}, \ldots, a_1, a_0)\{(a_{k-1} \ldots a_1 a_0)_b$ is base $b$ expansion of $n\}$

# Example

- $(12345)_{10} = (30071)_8$

# Example

- $(12345)_{10} = (30071)_8$

$$12345 = 8 \cdot 1543 + 1$$

$$1543 = 8 \cdot 192 + 7$$

$$192 = 8 \cdot 24 + 0$$

$$24 = 8 \cdot 3 + 0$$

$$3 = 8 \cdot 0 + 3$$

# Binary Addition of Integers

$a = (a_{n-1} a_{n-2} \ldots a_1 a_0), \; b = (b_{n-1} b_{n-2} \ldots b_1 b_0)$

**procedure** *add*($a$, $b$: positive integers)
{the binary expansions of $a$ and $b$ are $(a_{n-1}, a_{n-2}, \ldots, a_0)_2$ and $(b_{n-1}, b_{n-2}, \ldots, b_0)_2$, respectively}
$c := 0$
**for** $j := 0$ to $n - 1$
  $d := \lfloor (a_j + b_j + c)/2 \rfloor$
  $s_j := a_j + b_j + c - 2d$
  $c := d$
$s_n := c$
**return**$(s_0, s_1, \ldots, s_n)${the binary expansion of the sum is $(s_n, s_{n-1}, \ldots, s_0)_2$}

# Binary Addition of Integers

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$O(n)$ bit additions

# Algorithm: Binary Multiplication of Integers

$a = (a_{n-1} a_{n-2} \ldots a_1 a_0)_2, \; b = (b_{n-1} b_{n-2} \ldots b_1 b_0)_2$

$$\begin{aligned} ab &= a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1} 2^{n-1}) \\ &= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1} 2^{n-1}) \end{aligned}$$

**procedure** *multiply*($a$, $b$: positive integers)
{the binary expansions of a and b are $(a_{n-1}, a_{n-2}, \ldots, a_0)_2$ and $(b_{n-1}, b_{n-2}, \ldots, b_0)_2$, respectively}
**for** $j := 0$ to $n - 1$
    **if** $b_j = 1$ **then** $c_j = a$ shifted $j$ places
    **else** $c_j := 0$
{$c_0, c_1, \ldots, c_{n-1}$ are the partial products}
$p := 0$
**for** $j := 0$ to $n - 1$
    $p := p + c_j$
**return** $p$ {p is the value of $ab$}

# Algorithm: Binary Multiplication of Integers

$a = (a_{n-1}a_{n-2}\ldots a_1a_0)_2,\ b = (b_{n-1}b_{n-2}\ldots b_1b_0)_2$

$$\begin{aligned} ab &= a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1}2^{n-1}) \\ &= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1}2^{n-1}) \end{aligned}$$

**procedure** *multiply*($a$, $b$: positive integers)
{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}
**for** $j := 0$ to $n-1$
  **if** $b_j = 1$ **then** $c_j = a$ shifted $j$ places
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**return** $p$ {p is the value of $ab$}

$O(n^2)$ shifts and $O(n^2)$ bit additions

# Algorithm: Computing div and mod

**procedure** *division algorithm* (a: integer, $d$: positive integer)
$q := 0$
$r := |a|$
**while** $r \geq d$
    $r := r - d$
    $q := q + 1$
**if** $a < 0$ and $r > 0$ **then**
    $r := d - r$
    $q := -(q+1)$
**return** $(q, r)$ {$q = a$ **div** $d$ *is the quotient*, $r = a$ **mod** $d$ *is the remainder* }

# Algorithm: Computing div and mod

**procedure** *division algorithm* (a: integer, $d$: positive integer)
$q := 0$
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**return** $(q, r)$ {$q = a$ **div** $d$ *is the quotient*, $r = a$ **mod** $d$ *is the remainder* }

$O(q \log a)$ bit operations. But there exist more efficient algorithms with complextiy $O(n^2)$, where $n = \max(\log a, \log d)$

# Algorithm: Computing div and mod (cont)

- **procedure** $division2\ (a, d \in \mathbb{N},\ d \geq 1)$
  **if** $a < d$
      **return** $(q, r) = (0, a)$
  $(q, r) = division2(\lfloor a/2 \rfloor, d)$
  $q = 2q,\ r = 2r$
  **if** $a$ is odd
      $r = r + 1$
  **if** $r \geq d$
      $r = r - d$
      $q = q + 1$
  **return** $(q, r)$

# Algorithm: Computing div and mod (cont)

- **procedure** *division*2 $(a, d \in \mathbb{N},\ d \geq 1)$
  **if** $a < d$
      **return** $(q, r) = (0, a)$
  $(q, r) = division2(\lfloor a/2 \rfloor, d)$
  $q = 2q,\ r = 2r$
  **if** $a$ is odd
      $r = r + 1$
  **if** $r \geq d$
      $r = r - d$
      $q = q + 1$
  **return** $(q, r)$

  $O(\log q \log a)$ bit operations.

# Algorithm: Binary Modular Exponentiation

$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}} \cdots b^{a_1\cdot 2}\cdot b^{a_0}$$

Successively finds $b \bmod m$, $b^2 \bmod m$, $b^4 \bmod m$, $\ldots$, $b^{2^{k-1}} \bmod m$, and multiplies together the terms $b^{2^j}$ where $a_j = 1$.

**procedure** *modular exponentiation*($b$: integer, $n = (a_{k-1}a_{k-2}\ldots a_1a_0)_2$ , $m$: positive integers)
$x := 1$
$power := b$ **mod** $m$
**for** $i := 0$ to $k-1$
  **if** $a_i = 1$ **then** $x := (x \cdot power)$ **mod** $m$
  $power := (power \cdot power)$ **mod** $m$
**return** $x$ {$x$ equals $b^n$ **mod** $m$ }

# Algorithm: Binary Modular Exponentiation

$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}}\cdots b^{a_1\cdot 2}\cdot b^{a_0}$$

Successively finds $b \bmod m$, $b^2 \bmod m$, $b^4 \bmod m$, $\ldots$, $b^{2^{k-1}} \bmod m$, and multiplies together the terms $b^{2^j}$ where $a_j = 1$.

**procedure** *modular exponentiation*($b$: integer, $n = (a_{k-1}a_{k-2}\ldots a_1a_0)_2$ , $m$: positive integers)
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**for** $i := 0$ to $k - 1$
    **if** $a_i = 1$ **then** $x := (x \cdot power)$ **mod** $m$
    $power := (power \cdot power)$ **mod** $m$
**return** $x$ {$x$ equals $b^n$ **mod** $m$ }

$O((\log m)^2 \log n)$ bit operations

# Next Lecture

- number theory, cryptography ...