Department of
Computer Science and Engineering
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CS215 DISCRETE MATH

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Encoding the Inputs of Problems

- Complexity of a problem is measure w.r.t the size of input.

- In order to formally discuss how hard a problem is, we need to be much more formal than before about the input size of a problem.

Input Size Example: Composite

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  Any integer $n > 0$ can be represented in the binary number system as a string $a_0 a_1 \cdots a_k$ of length $\lceil \log_2(n+1) \rceil$.

  Thus, a natural measure of input size is $\lceil \log_2(n+1) \rceil$ (or just $\log_2 n$)

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  Using fixed length encoding, we write a_i as a binary string of length $m = \lceil \log_2 \max(|a_i| + 1) \rceil$.

  This coding gives an input size nm.

Complexity in terms of Input Size

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  But, note that the input size of this problem is $size(n) = \log_2 n$, so the number of comparisons performed is actually $\Theta(n) = \Theta(2^{size(n)})$, which is exponential.

Input Size Example: Integer Multiplication

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  The minimum inpute size is

  $$s = \lceil \log_2(a+1) \rceil + \lceil \log_2(b+1) \rceil.$$

  A natural choice is to use $t = \log_2 \max(a, b)$ since $\frac{s}{2} \leq t \leq s$.

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  If L is the problem, and x is the input, we will often write $x \in L$ to denote a yes answer and $x \notin L$ to denote a no answer.

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  Examples:

  Knapsack vs. Decision Knapsack (DKnapsack)

Knapsack vs. DKnapsack

- We have a knapsack of capacity W (a positive integer) and n objects with weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$, where v_i and w_i are positive integers.

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  Optimization problem: (Knapsack)

  Find the largest value $\sum_{i \in T} v_i$ of any subset T that fits in the knapsack, i.e., $\sum_{i \in T} w_i \leq W$.

  Decision problem: (DKnapsack)

  Given k, is there a subset of the objects that fits in the knapsack and has total value at least k?

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  First solve the optimization problem, then check the decision problem. If it does, answer yes, otherwise no.

  Thus, if we prove that a given decision problem is hard to solve efficiently, then it is obvious that the optimization problem must be (at least as) hard.

Complexity Classes

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  - ◇ the classification of certain "decision problems" into several classes:
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A. Use polynomial-time algorithms.

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  Example:

  The standard multiplication algorithm has time $O(m_1 m_2)$, where m_1, m_2 denote the number of digits in the two integers, respectively.

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  In terms of the input size, the complexity is $\Theta(2^N N^2)$.

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  In reality, an $O(n^{20})$ algorithm is not really practical.

Polynomial-Time Solvable Problems

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  Definition (The Class P) The class P consists of all decision problems that are solvable in polynomial time. That is, there exists an algorithm that will decide in polynomial time if any given input is a yes-input or a no-input.

The Class P

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  A. You need to prove that there is no polynomial-time algorithm for this problem. (much much harder)

Certificates and Verifying Certificates

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  Verifying a certificate: Given a presumed yes-input and its corresponding certificate, by making use of the given certificate, we verify that the input is actually a yes-input.

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NP – "nondeterministic polynomial-time"

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  Just being able to verify a certificate in polynomial time does not necessarily mean we can tell whether an input is a yes-input or a no-input in polynomial time.

  However, we are still no closer to solving it and do not know the answer. The search for a solution, though, has provided us with deep insights into what distinguishes an "easy" problem from a "hard" one.

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- If **Q** can be reduced to **Q'**,
  then **Q** is "no harder to solve" than **Q'**.

Polynomial-Time Reductions

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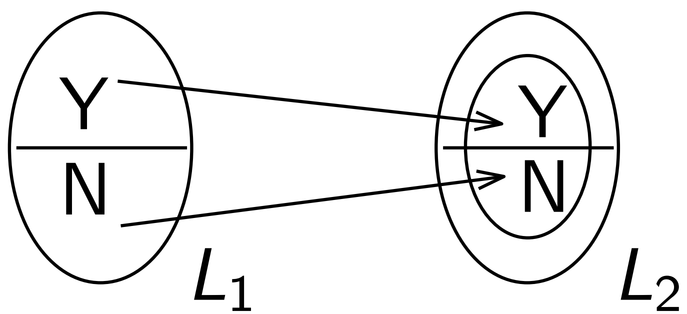
- A *polynomial-time reduction* from L_1 to L_2 is a transformation f with the following two properties:

  (1) f transforms an input x for L_1 into an input $f(x)$ for L_2 s.t.

  a yes-input of L_1 maps to a yes-input of L_2, and a no-input of L_1 maps to a no-input of L_2

  (2) f is computable in *polynomial time* in size(x)



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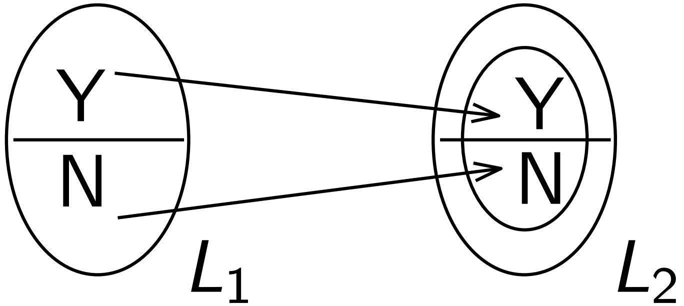

If such an f exists, we say that L_1 is *polynomial-time reducible* to L_2, and write $L_1 \leq_P L_2$.

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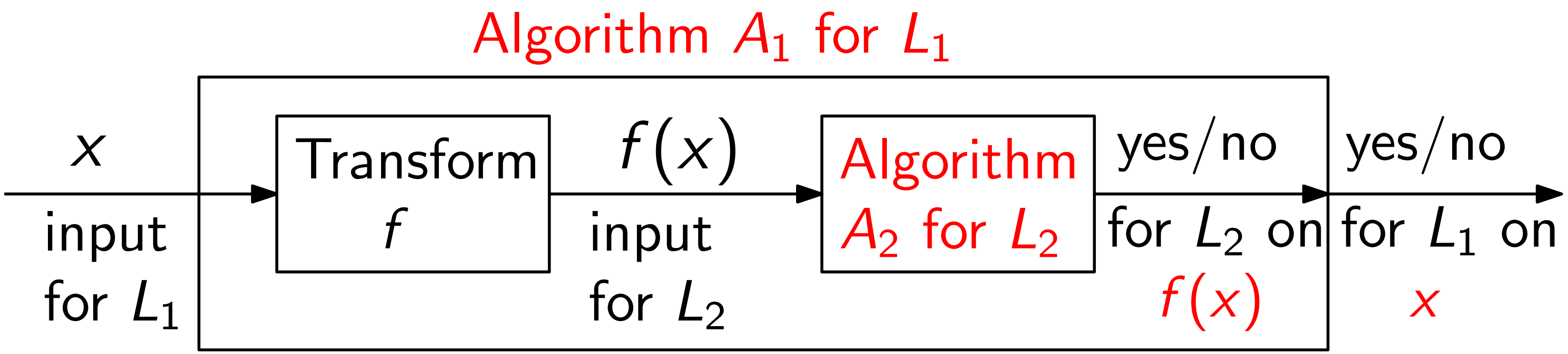

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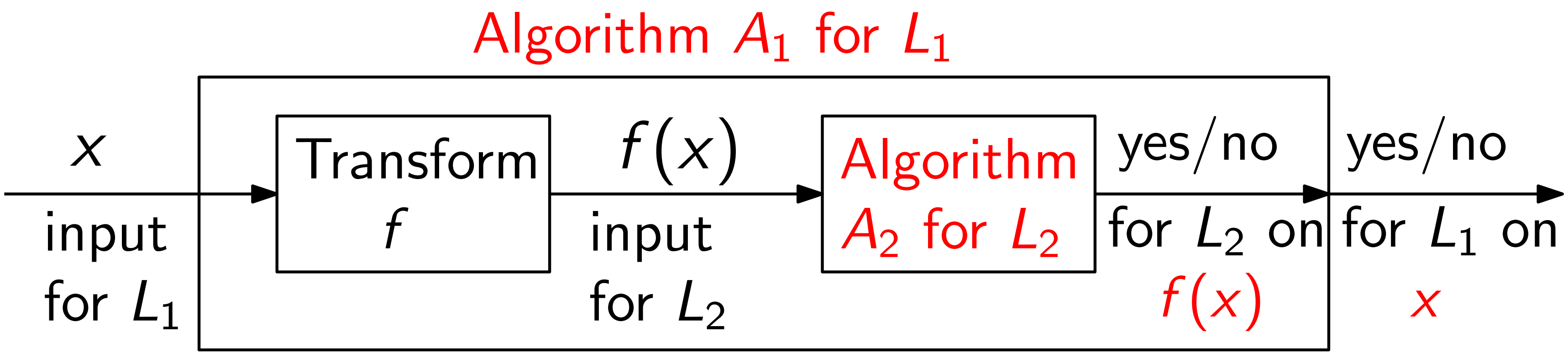

- If A_2 is polynomial-time algorithm, so is A_1

Polynomial-Time Reduction $f : L_1 \to L_2$

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  Proof. $L_2 \in P$ means we have a polynomial-time algorithm A_2 for L_2. Since $L_1 \leq_P L_2$, we have a polynomial-time transformation f mapping input x for L_1 to an input for L_2.

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  Combining these, we get the following polynomial-time algorithm for solving L_1:
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  Note: The converse (if $L_1 \le_P L_2$ and $L_1 \in P$, then $L_2 \in P$) is not true.

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  (1) $L \in NP$
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  Intuitively, *NPC* consists of all the hardest problems in NP.

- **Theorem** Let L be any problem in NPC.

  (1) If there is a polynomial-time algorithm for L, then there is a polynomial-time algorithm for every $L' \in NP$

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NP-Completeness and Its Properties

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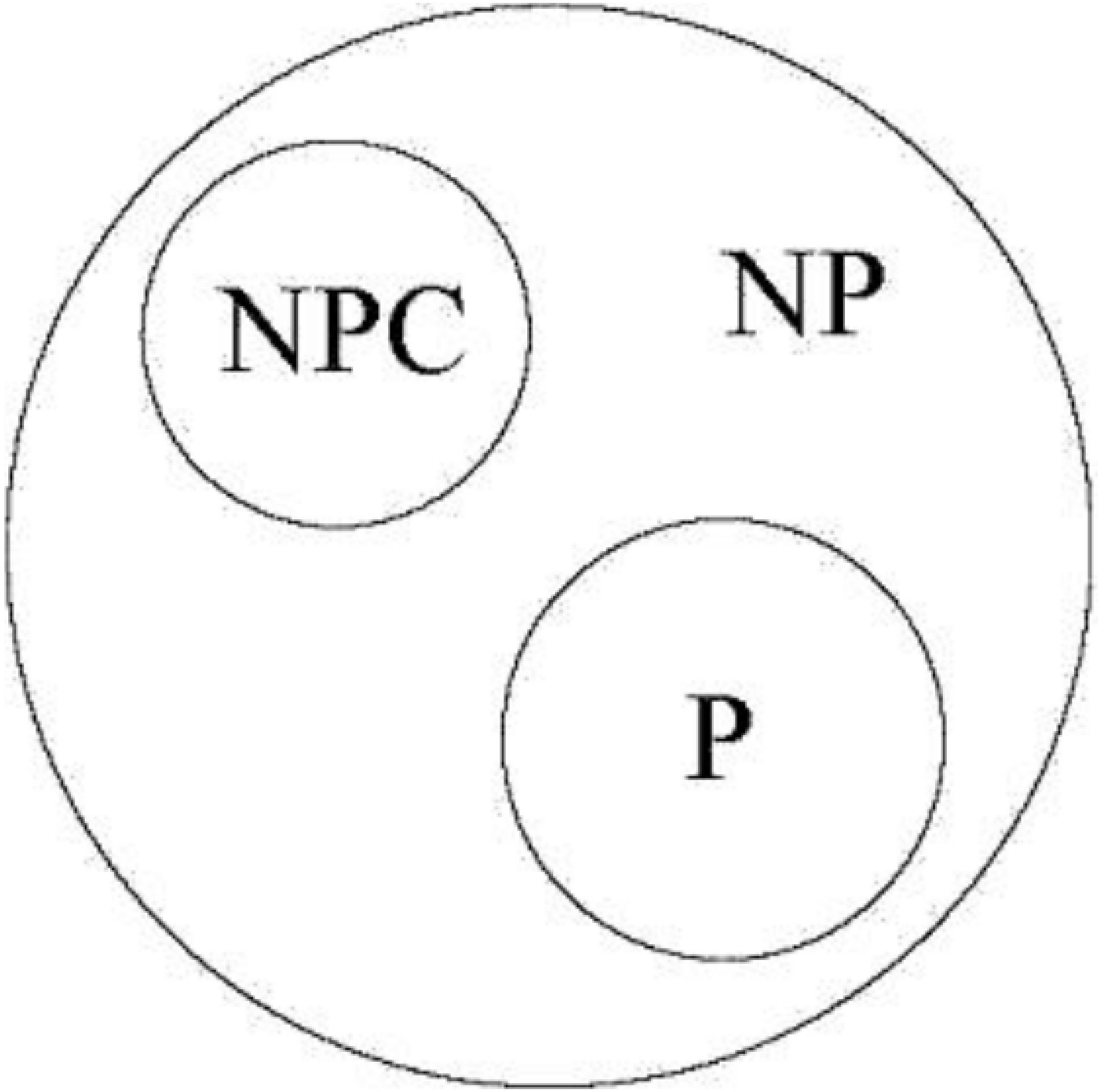
- Either *all* NP-Complete problems are polynomial time solvable, or *all* NP-Complete problems are not polynomial time solvable.

  This is the major reason why we are interested in NP-Completeness.

The Classes P, NP, and NPC



Application of Number Theory

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If he could see the world now, Hardy would be spinning in his grave.

Number Theory

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- At one point, the largest employer of mathematicians in the United States, and probably the world, was the National Security Agency (NSA). The NSA is the largest spy agency in the US (bigger than CIA, Central Intelligence Agency), and has the responsibility for code design and breaking.

Division

- If a and b are integers with $a \neq 0$, we say that a *divides* b if there is an integer k such that $b = ak$, or equivalently b/a is an integer. In this case, we say that a is a *factor* or *divisor* of b, and b is a *multiple* of a. (We use the notations $a|b$, $a \nmid b$)

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Example

- ◇ $4 \mid 24$
- ◇ $3 \nmid 7$

Divisibility

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- **Question:** Let n and d be two positive integers. How many positive integers not exceeding n are divisible by d?

  Answer: Count the number of integers such that $0 < kd \leq n$. Therefore, there are $\lfloor n/d \rfloor$ such positive integers.

- **Properties**

  Let a, b, c be integers. Then the following hold:

  (i) if $a|b$ and $a|c$, then $a|(b+c)$
  (ii) if $a|b$ then $a|bc$ for all integers c
  iii) if $a|b$ and $b|c$, then $a|c$

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Proof.

Divisibility

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  Proof. By part (ii) and part (i) of Properties.

The Division Algorithm

- If a is an integer and d a positive integer, then there are **unique** integers q and r, with $0 \leq r < d$, such that $a = dq + r$. In this case, d is called the *divisor*, a is called the *dividend*, q is called the *quotient*, and r is called the *remainder*.

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  In this case, we use the notations $q = a \ div \ d$ and $r = a \bmod d$.

Congruence Relation

- If a and b are integers and m is a positive integer, then a is *congruent to b modulo m if m divides $a - b$*, denoted by $a \equiv b \pmod{m}$. This is called *congruence* and m is its *modulus*.

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Example

- ◇ $15 \equiv 3 \pmod{6}$
- ◇ $-1 \equiv 11 \pmod{6}$

More on Congruences

- Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

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Proof.

"only if" part

"if" part

$\pmod{m}$ and $\bmod m$ Notations

- $a \equiv b \pmod{m}$ and $a \bmod m = b$ are different.
  - ◇ $a \equiv b \pmod{m}$ is a relation on the set of integers
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  Proof.

Congruences of Sums and Products

- Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

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Algebraic Manipulation of Congruences

- If $a \equiv b \bmod m$, then
  - $c \cdot a \equiv c \cdot b \pmod{m}$?
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$14 \equiv 8 \pmod{6}$ but $7 \not\equiv 4 \pmod{6}$

Computing the mod Function

- **Corollary** Let m be a positive integer and let a and b be integers. Then

$$(a+b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$

$$ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$$

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Next Lecture

- number theory ...