Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

Dr. Qi Wang

Department of Computer Science and Engineering
Office: Room413, CoE South Tower
Email: wangqi@sustech.edu.cn

Cartesian Product

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_1, b_n), \ldots, (a_m, b_n)\}$$

Cartesian Product

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_1, b_n), \ldots, (a_m, b_n)\}$$

Cartesian product defines a set of all ordered arrangements of elements in the two sets.

Binary Relation

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

Binary Relation

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Binary Relation

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

  We use the notation $a \ R \ b$ to denote $(a, b) \in R$, and $a \not R b$ to denote $(a, b) \notin R$.

Binary Relation

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

  We use the notation $a\ R\ b$ to denote $(a, b) \in R$, and $a \not R b$ to denote $(a, b) \notin R$.

  Example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$

  ◇ Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B?

  ◇ Is $Q = \{(1, a), (2, b)\}$ a relation from A to B?

  ◇ Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A?

Representing Binary Relations

- We can graphically represent a binary relation R as:

  if $a\ R\ b$, then we draw an arrow from a to b: $a \rightarrow b$

Representing Binary Relations

- We can graphically represent a binary relation R as:

  if $a \; R \; b$, then we draw an arrow from a to b: $a \rightarrow b$

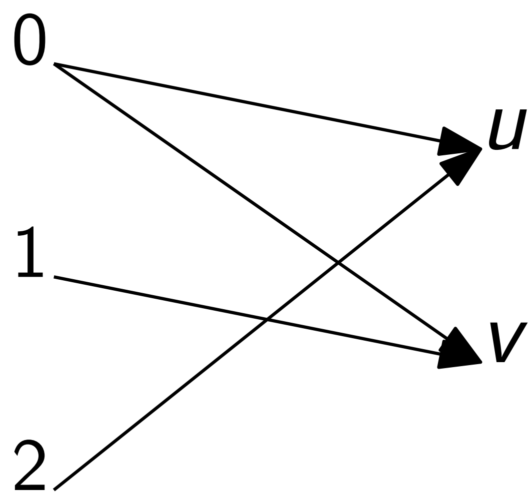
Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. ($R \subseteq A \times B$)

Representing Binary Relations

- We can graphically represent a binary relation R as:

  if $a\ R\ b$, then we draw an arrow from a to b: $a \to b$

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. ($R \subseteq A \times B$)



Representing Binary Relations

- We can also represent a binary relation R by a table showing the ordered pairs of R.

Representing Binary Relations

- We can also represent a binary relation R by a table showing the ordered pairs of R.

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, u), (2, v)\}$. ($R \subseteq A \times B$)

Representing Binary Relations

- We can also represent a binary relation R by a table showing the ordered pairs of R.

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, u), (2, v)\}$. ($R \subseteq A \times B$)

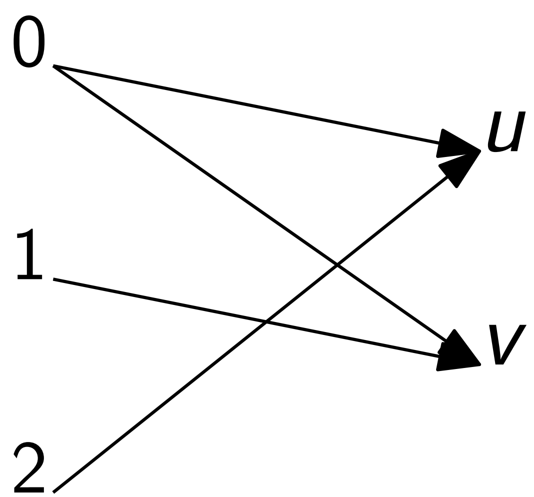
R	u	v
0	×	×
1	×	
2		×

Relations and Functions

- Relations represent one to many relationships between elements in A and B.

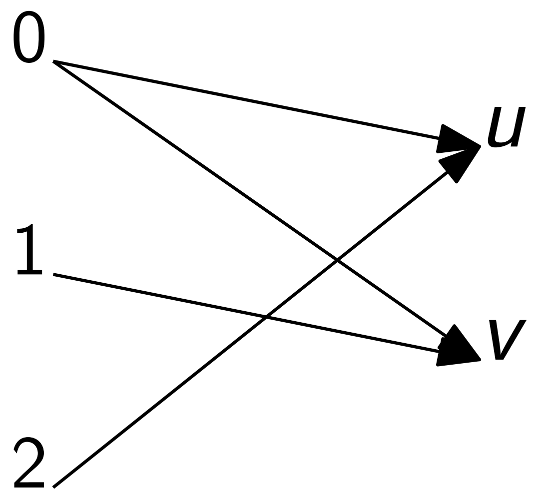
Relations and Functions

- Relations represent one to many relationships between elements in A and B.



Relations and Functions

- Relations represent one to many relationships between elements in A and B.



What is the difference between a relation and a function from A to B?

Relation on the Set

- **Definition**: A relation on the set A is a relation from A to itself.

Relation on the Set

- **Definition**: A relation on the set A is a relation from A to itself.

  Example: Let $A = \{1, 2, 3, 4\}$ and $R_{div} = \{(a, b) : a|b\}$.What does R_{div} consist of?

- **Definition**: A relation on the set A is a relation from A to itself.

  Example: Let $A = \{1, 2, 3, 4\}$ and $R_{div} = \{(a, b) : a|b\}$.What does R_{div} consist of?

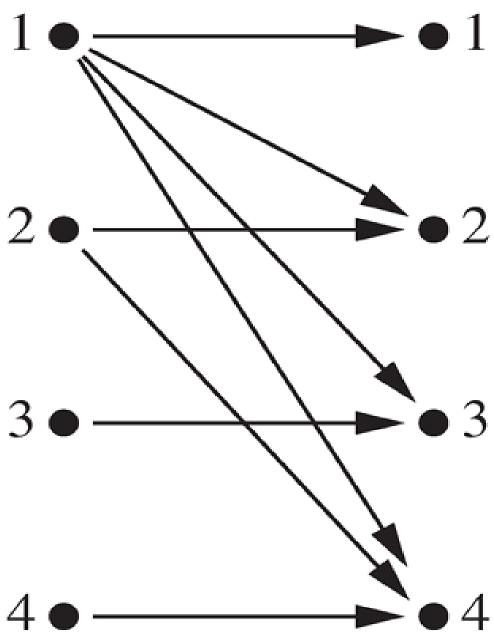
  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

Relation on the Set

- **Definition**: A relation on the set A is a relation from A to itself.

  Example: Let $A = \{1, 2, 3, 4\}$ and $R_{div} = \{(a, b) : a|b\}$.What does R_{div} consist of?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

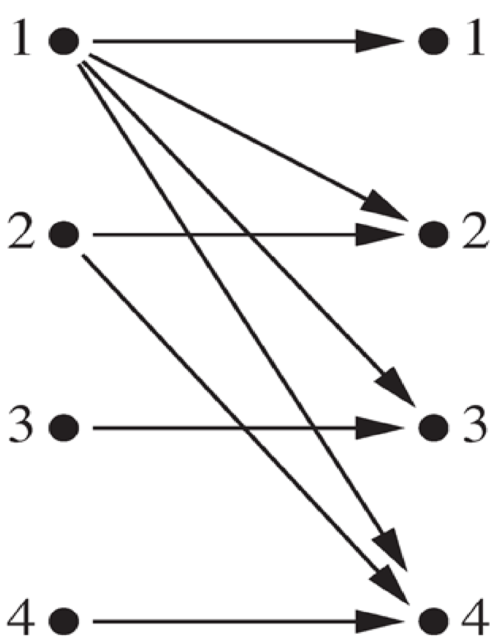


Relation on the Set

- **Definition**: A relation on the set A is a relation from A to itself.

  Example: Let $A = \{1, 2, 3, 4\}$ and $R_{div} = \{(a, b) : a|b\}$.What does R_{div} consist of?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Number of Binary Relations

- **Theorem** The number of binary relations on a set A, where $|A| = n$ is 2^{n^2}.

Number of Binary Relations

- **Theorem** The number of binary relations on a set A, where $|A| = n$ is 2^{n^2}.

  Proof

  If $|A| = n$, then the cardinality of the Cartesian product $|A \times A| = n^2$.

Number of Binary Relations

- **Theorem** The number of binary relations on a set A, where $|A| = n$ is 2^{n^2}.

  Proof

  If $|A| = n$, then the cardinality of the Cartesian product $|A \times A| = n^2$.

  R is a binary relation on A if $R \subseteq A \times A$ (R is subset)

Number of Binary Relations

- **Theorem** The number of binary relations on a set A, where $|A| = n$ is 2^{n^2}.

  Proof

  If $|A| = n$, then the cardinality of the Cartesian product $|A \times A| = n^2$.

  R is a binary relation on A if $R \subseteq A \times A$ (R is subset)

  The number of subsets of a set with k elements is 2^k

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a | b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} reflexive?

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} reflexive?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} reflexive?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

  Yes. $(1, 1), (2, 2), (3, 3), (4, 4) \in R_{div}$

Reflexive Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

Reflexive Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Reflexive Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

A relation R is reflexive if and only if MR has 1 in every position on its main diagonal.

Reflexive Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Reflexive Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

  Is R reflexive?

Reflexive Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

  Is R reflexive?

  No. $(1,1) \notin R$

Properties of Relations

- **Irreflexive Relation**: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

- **Irreflexive Relation**: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

Properties of Relations

- **Irreflexive Relation**: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

  Is $R_{\neq}$ irreflexive?

- **Irreflexive Relation**: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

  Is $R_{\neq}$ irreflexive?

  $$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

- **Irreflexive Relation**: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

  Is $R_{\neq}$ irreflexive?

  $$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

  Yes. $(1,1),(2,2),(3,3),(4,4) \notin R_{\neq}$

Irreflexive Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Irreflexive Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

$$\text{MR} = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Irreflexive Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is irreflexive if and only if MR has 0 in every position on its main diagonal.

Properties of Relations

- **Symmetric Relation**: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

- **Symmetric Relation**: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

Properties of Relations

- **Symmetric Relation**: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} symmetric?

Properties of Relations

- **Symmetric Relation**: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} symmetric?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

- **Symmetric Relation**: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} symmetric?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

  No. $(1, 2) \in R_{div}$ but $(2, 1) \notin R$

Symmetric Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Symmetric Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

  $$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

  Is $R_{\neq}$ symmetric?

Symmetric Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

  $$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

  Is $R_{\neq}$ symmetric?

  Yes. If $(a, b) \in R_{\neq}$ then $(b, a) \in R_{\neq}$.

Symmetric Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}.$$

Symmetric Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Symmetric Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is symmetric if and only if MR is symmetric.

Properties of Relations

- **Antisymmetric Relation**: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

- **Antisymmetric Relation**: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

  Example: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

- **Antisymmetric Relation**: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

Example: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Is R antisymmetric?

- **Antisymmetric Relation**: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

Example: Assume that $R = \{(1, 2), (2, 2), (3, 3)\}$ on $A = \{1, 2, 3, 4\}$.

Is R antisymmetric?

Yes.

Antisymmetric Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Antisymmetric Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

$$\mathrm{MR} = \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

Antisymmetric Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

$$MR = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

A relation R is antisymmetric if and only if $m_{ij} = 1$ implies $m_{ji} = 0$ for $i \neq j$.

Antisymmetric Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

Antisymmetric Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} antisymmetric?

Antisymmetric Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} antisymmetric?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

Antisymmetric Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} antisymmetric?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

  Yes. If $a|b$ and $b|a$, then $a = b$.

Antisymmetric Relation

- **Example**: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} antisymmetric?

  $R_{div} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$.

  Yes. If $a|b$ and $b|a$, then $a = b$.

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Properties of Relations

- **Transitive Relation**: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

- **Transitive Relation**: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

Properties of Relations

- **Transitive Relation**: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} transitive?

- **Transitive Relation**: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} transitive?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

- **Transitive Relation**: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

  Example: Assume that $R_{div} = \{(a, b) : a|b\}$ on $A = \{1, 2, 3, 4\}$.

  Is R_{div} transitive?

  $R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

  Yes. If $a|b$ and $b|c$, then $a|c$.

Transitive Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Transitive Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4), \\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Is $R_{\neq}$ transitive?

Transitive Relation

- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

  $$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

  Is $R_{\neq}$ transitive?

  No. $(1,2),(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$.

Transitive Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Transitive Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

  Is R transitive?

Transitive Relation

- **Example**: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

  Is R transitive?

  Yes.

Combining Relations

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Combining Relations

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

  Combining Relations: Since relations are sets, we can *combine* relations via set operations.

Combining Relations

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

  Combining Relations: Since relations are sets, we can *combine* relations via set operations.

  Set operations: union, intersection, difference, etc.

Combining Relations

- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and $R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$, $R_2 = \{(1, v), (3, u), (3, v)\}$

Combining Relations

- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and $R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$, $R_2 = \{(1, v), (3, u), (3, v)\}$

What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

Combining Relations

- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and
  $R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$,
  $R_2 = \{(1, v), (3, u), (3, v)\}$

  What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

  We may also combine relations by matrix operations.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

  Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

  Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$

  $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$

  $S = \{(0, b), (1, a), (2, b)\}$

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

  Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$

  $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$

  $S = \{(0, b), (1, a), (2, b)\}$

  $S \circ R = \{(1, b), (3, a), (3, b)\}$

Implementation of Composite

- **Example**: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

  $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from A to B

  $S = \{(1, a), (3, b), (3, a)\}$ is a relation from B to C

Implementation of Composite

- **Example**: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

  $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from A to B

  $S = \{(1, a), (3, b), (3, a)\}$ is a relation from B to C

  $S \circ R = \{(1, b), (1, a), (2, a)\}$

Implementation of Composite

- **Example**: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

  $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from A to B

  $S = \{(1, a), (3, b), (3, a)\}$ is a relation from B to C

  $S \circ R = \{(1, b), (1, a), (2, a)\}$

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$$

Implementation of Composite

- **Example**: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

  $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from A to B

  $S = \{(1, a), (3, b), (3, a)\}$ is a relation from B to C

  $S \circ R = \{(1, b), (1, a), (2, a)\}$

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

Implementation of Composite

- **Example**: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$
  $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from A to B
  $S = \{(1, a), (3, b), (3, a)\}$ is a relation from B to C

  $S \circ R = \{(1, b), (1, a), (2, a)\}$

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

Composite of Relations

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Composite of Relations

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by
$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$
Example: Let $A = \{1, 2, 3, 4\}$, and
$R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$

Composite of Relations

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Example: Let $A = \{1, 2, 3, 4\}$, and $R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$

$R^1 = R$

Composite of Relations

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by
$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Example: Let $A = \{1, 2, 3, 4\}$, and $R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$

$R^1 = R$

$R^2 = R \circ R = \{(1, 3), (1, 4), (2, 3), (3, 3)\}$

Composite of Relations

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Example: Let $A = \{1, 2, 3, 4\}$, and $R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$

$R^1 = R$

$R^2 = R \circ R = \{(1, 3), (1, 4), (2, 3), (3, 3)\}$

$R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$

Composite of Relations

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Example: Let $A = \{1, 2, 3, 4\}$, and $R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$

$$R^1 = R$$

$$R^2 = R \circ R = \{(1, 3), (1, 4), (2, 3), (3, 3)\}$$

$$R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$$

$$R^4 = R^3 \circ R = \{(1, 3), (2, 3), (3, 3)\}$$

Composite of Relations

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Example: Let $A = \{1, 2, 3, 4\}$, and $R = \{(1,2), (2,3), (2,4), (3,3)\}$

$R^1 = R$

$R^2 = R \circ R = \{(1,3), (1,4), (2,3), (3,3)\}$

$R^3 = R^2 \circ R = \{(1,3), (2,3), (3,3)\}$

$R^4 = R^3 \circ R = \{(1,3), (2,3), (3,3)\}$

$R^k =?$ for $k > 3$

Transitive Relation and R^n

- **Theorem** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Transitive Relation and R^n

- **Theorem** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof.

Transitive Relation and R^n

- **Theorem** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof.

"if" part: In particular, $R^2 \subseteq R$.

Transitive Relation and R^n

- **Theorem** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof.

"if" part: In particular, $R^2 \subseteq R$.

If $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, we have $(a, c) \in R^2 \subseteq R$.

Transitive Relation and R^n

- **Theorem** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof.

"if" part: In particular, $R^2 \subseteq R$.

If $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, we have $(a, c) \in R^2 \subseteq R$.

"only if" part: by induction.

Number of Reflexive Relations

- **Theorem** The number of reflexive relations on a set A with $|A| = n$ is $2^{n(n-1)}$.

Number of Reflexive Relations

- **Theorem** The number of reflexive relations on a set A with $|A| = n$ is $2^{n(n-1)}$.

  Proof. A reflexive relation R on A must contain all pairs (a, a) for every $a \in A$.

Number of Reflexive Relations

- **Theorem** The number of reflexive relations on a set A with $|A| = n$ is $2^{n(n-1)}$.

  Proof. A reflexive relation R on A must contain all pairs (a, a) for every $a \in A$.

  All other pairs in R are of the form (a, b) with $a \neq b$ s.t. $a, b \in A$.

Number of Reflexive Relations

- **Theorem** The number of reflexive relations on a set A with $|A| = n$ is $2^{n(n-1)}$.

  Proof. A reflexive relation R on A must contain all pairs (a, a) for every $a \in A$.

  All other pairs in R are of the form (a, b) with $a \neq b$ s.t. $a, b \in A$.

  How many of these pairs are there?

Number of Reflexive Relations

- **Theorem** The number of reflexive relations on a set A with $|A| = n$ is $2^{n(n-1)}$.

  Proof. A reflexive relation R on A must contain all pairs (a, a) for every $a \in A$.

  All other pairs in R are of the form (a, b) with $a \neq b$ s.t. $a, b \in A$.

  How many of these pairs are there?

  How many subsets on $n(n-1)$ elements are there?

Summary on Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Summary on Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

Summary on Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

Summary on Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

Summary on Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

  Transitive Relation: A relation R on a set A is called *reflexive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Representing Relations

- Some ways to represent *n*-ary relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$

Representing Relations

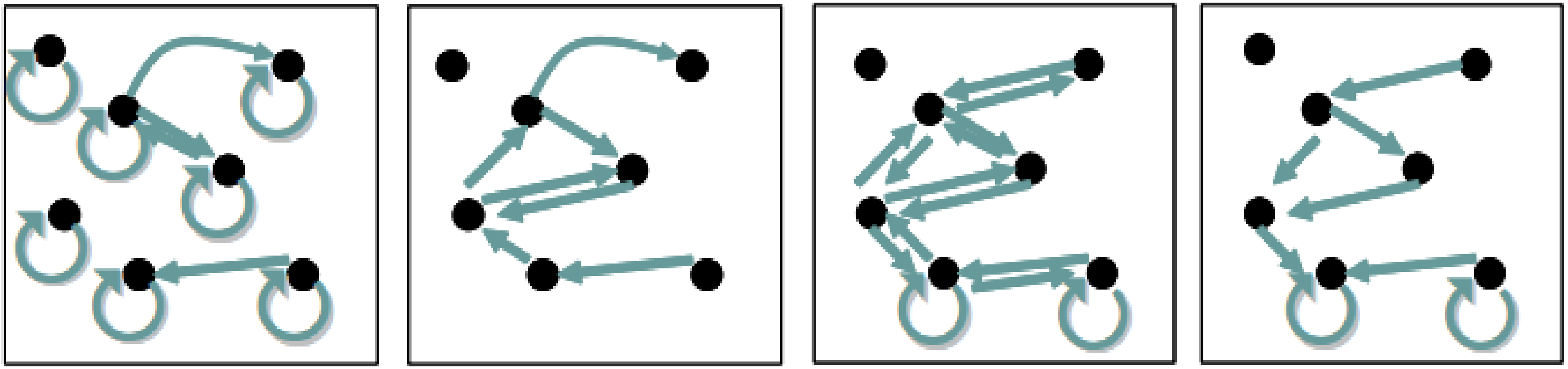
- Some ways to represent *n*-ary relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$

- Some special ways to represent binary relations:
  - with a *zero-one matrix*
  - with a *directed graph*

Representing Relations

- Some ways to represent *n-ary* relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$

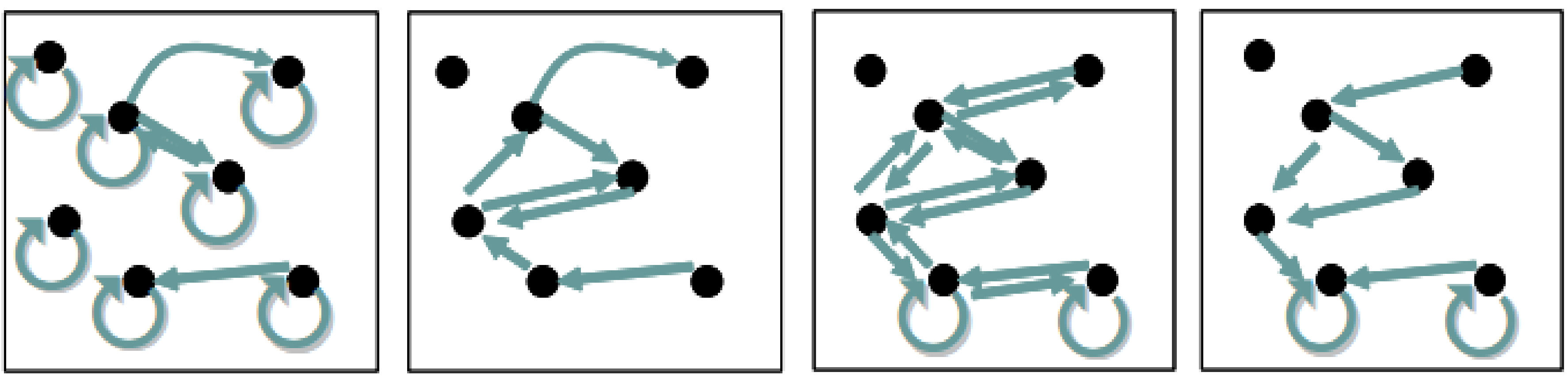
- Some special ways to represent binary relations:
  - with a *zero-one matrix*
  - with a *directed graph*



Representing Relations

- Some ways to represent *n-ary* relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$

- Some special ways to represent binary relations:
  - with a *zero-one matrix*
  - with a *directed graph*



reflexive   irreflexive   symmetric   antisymmetric

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why?

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ are not in R.

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ are not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ are not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ are not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ are not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

  The minimal set $S \supseteq R$ is called *the reflexive closure of R*.

Reflexive Closure

- The set *S* is called *the reflexive closure of R* if it:

Reflexive Closure

- The set S is called *the reflexive closure of* R if it:
  - ◇ contains R
  - ◇ is reflexive
  - ◇ is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Closures on Relations

- Relations can have different properties:
  - reflexive
  - symmetric
  - transitive

Closures on Relations

- Relations can have different properties:
  - reflexive
  - symmetric
  - transitive

We define:

- reflexive closures
- symmetric closures
- transitive closures

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q $(S \subseteq Q)$ with property P that contains R $(R \subseteq Q)$.

  S is the **minimal set** containing R satisfying the property P.

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (symmetric closure)

  $R = \{(1,2),(1,3),(2,2)\}$ on $A = \{1,2,3\}$

  What is the symmetric closure S of R?

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q $(S \subseteq Q)$ with property P that contains R $(R \subseteq Q)$.

  S is the **minimal set** containing R satisfying the property P.

  Example: (symmetric closure)

  $R = \{(1,2),(1,3),(2,2)\}$ on $A = \{1,2,3\}$

  What is the symmetric closure S of R?

  $R = \{(1,2),(1,3),(2,2)\} \cup \{(2,1),(3,1)\}$

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (transitive closure)

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Is R transitive?

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (transitive closure)

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Is R transitive? No

  How to make it transitive?

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (transitive closure)

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Is R transitive? No

  How to make it transitive?

  $S = \{(1,2),(2,2),(2,3)\} \cup \{(1,3)\}$

Transitive Closure

- We can represent the relation on the graph.

Transitive Closure

- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$

Transitive Closure

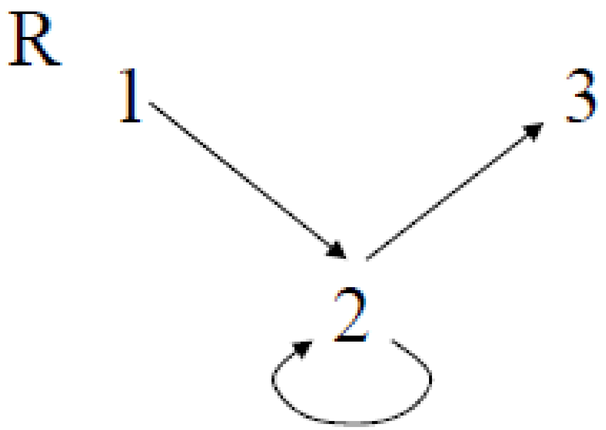
- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$



Transitive Closure

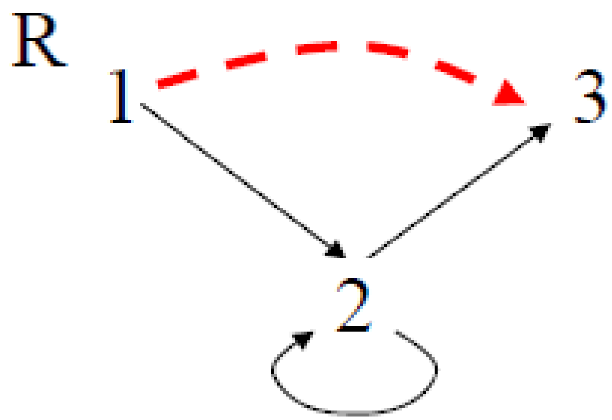
- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$



Transitive Closure

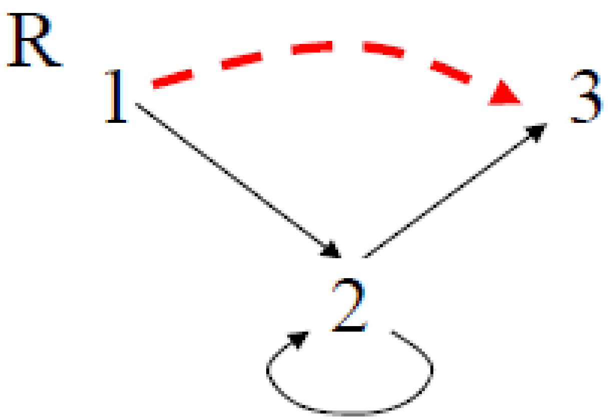
- We can represent the relation on the graph.

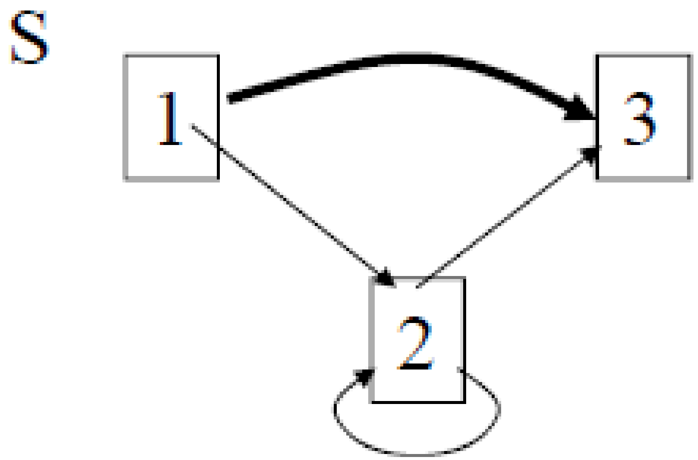
  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$





Paths in Directed Graphs

- **Definition** A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

Paths in Directed Graphs

- **Definition** A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

  Theorem Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Path Length

- **Theorem** Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

  Proof (by induction)

Path Length

- **Theorem** Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

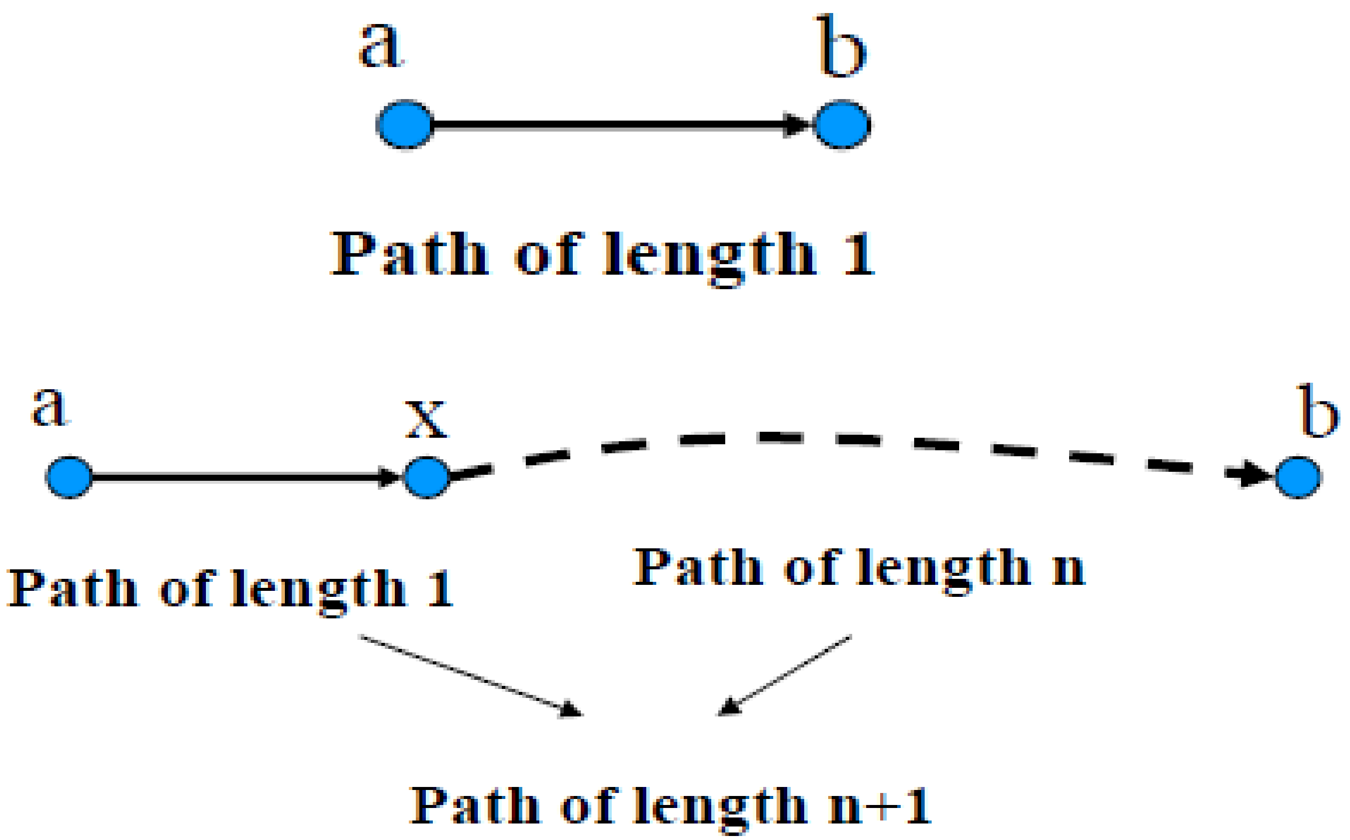
  Proof (by induction)

Path of length 1

Path Length

- **Theorem** Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Proof (by induction)



Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

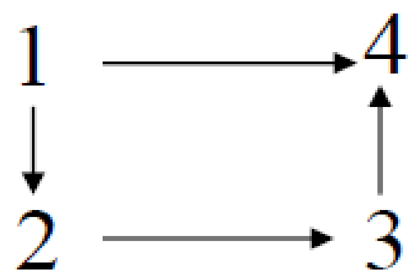
Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:

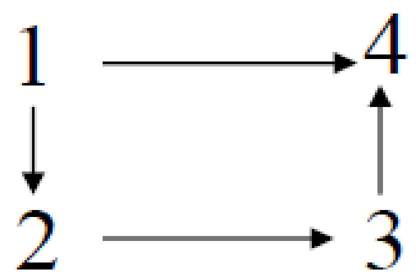
$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$



- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:



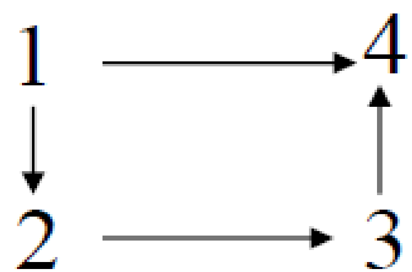
$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R^2 = \{(1, 3), (2, 4)\}$$

Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:



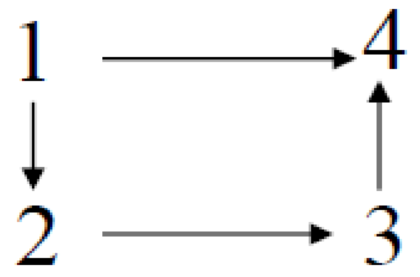
$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R^2 = \{(1, 3), (2, 4)\}$$
$$R^3 = \{(1, 4)\}$$

Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:

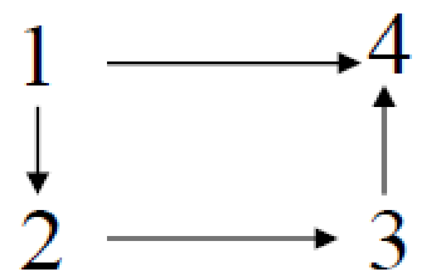


$$\begin{aligned} A &= \{1, 2, 3, 4\} \\ R &= \{(1, 2), (1, 4), (2, 3), (3, 4)\} \\ R^2 &= \{(1, 3), (2, 4)\} \\ R^3 &= \{(1, 4)\} \\ R^4 &= \emptyset \end{aligned}$$

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:



$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R^2 = \{(1, 3), (2, 4)\}$$
$$R^3 = \{(1, 4)\}$$
$$R^4 = \emptyset$$

$$R^* = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

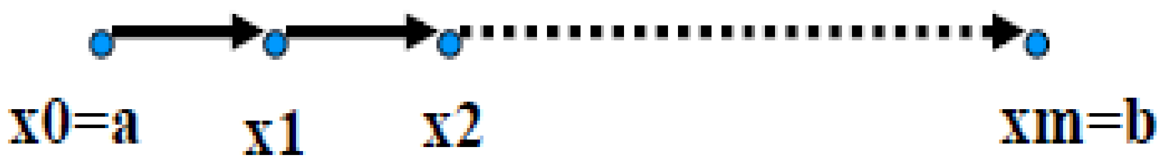
  Proof (by intuition)

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

  Proof (by intuition)

  ◇ There are at most n different elements we can visit on a path if the path does not have loops

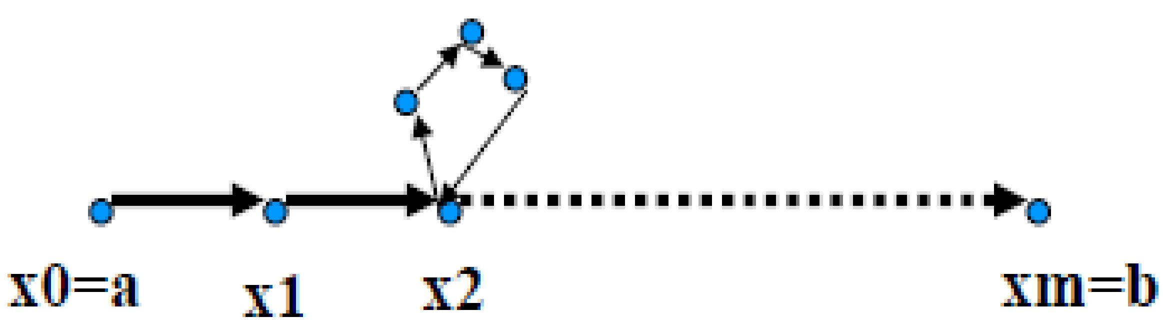


- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

  Proof (by intuition)

  ◇ There are at most n different elements we can visit on a path if the path does not have loops

  ◇ Loops may increase the length but the same node is visited more than once

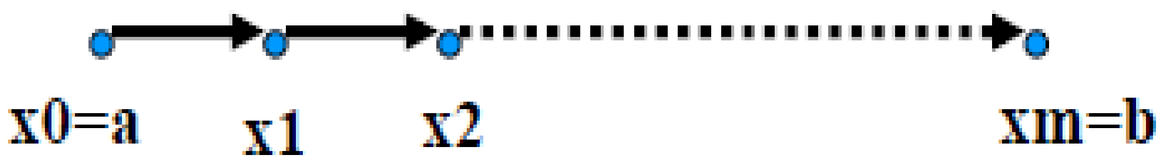


Connectivity

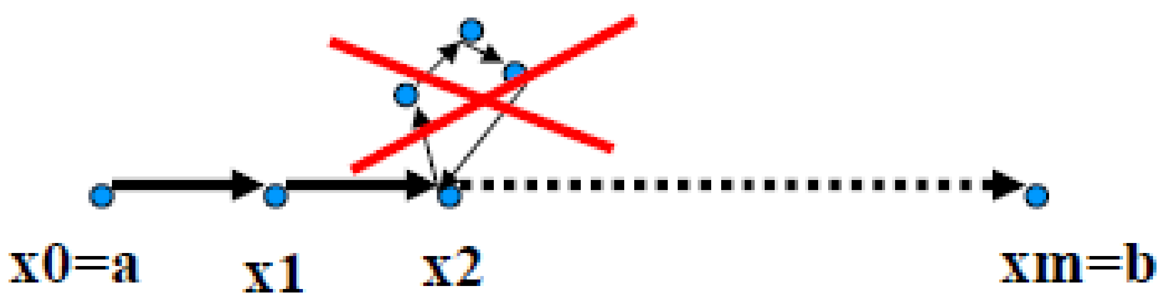
- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

  Proof (by intuition)

  ◇ There are at most n different elements we can visit on a path if the path does not have loops



  ◇ Loops may increase the length but the same node is visited more than once



Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Recall Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

  Proof

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

2. Suppose that S is a transitive relation containing R.

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

2. Suppose that S is a transitive relation containing R.

Then S^n is also transitive and $S^n \subseteq S$. Why?

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

2. Suppose that S is a transitive relation containing R.

Then S^n is also transitive and $S^n \subseteq S$. Why?

We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

Find Transitive Closure

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Find Transitive Closure

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \mathbf{M}_R^{[n]}$$

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \mathbf{M}_R^{[n]}$$

Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{M}_{R^*} =?$

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
for $i := 2$ to n
  $A := A \odot \mathbf{M}_R$
  $B := B \vee A$
return B
// B is the zero-one matrix for R^*

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
for $i := 2$ to n
  $A := A \odot \mathbf{M}_R$
  $B := B \vee A$
return B
// B is the zero-one matrix for R^*

This algorithm takes $\Theta(n^4)$ time.

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
for $i := 2$ to n
  $A := A \odot \mathbf{M}_R$
  $B := B \vee A$
return B
// B is the zero-one matrix for R^*

This algorithm takes $\Theta(n^4)$ time. Why?

Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$W := \mathbf{M}_R$;
for $k := 1$ to n
  for $i := 1$ to n
    for $j := 1$ to n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W
// W is the zero-one matrix for R^*

Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$W := \mathbf{M}_R$;
for $k := 1$ to n
  for $i := 1$ to n
    for $j := 1$ to n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W
// W is the zero-one matrix for R^*

$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.

$$W_{ij}^{[k]} = W_{ij}^{[k-1]} \vee \left(W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]} \right)$$

Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$W := \mathbf{M}_R$;
for $k := 1$ to n
  for $i := 1$ to n
    for $j := 1$ to n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W
// W is the zero-one matrix for R^*

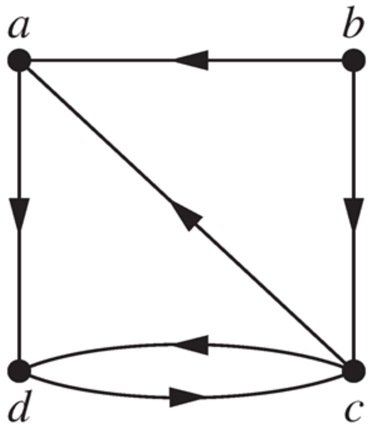
$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.

$$W_{ij}^{[k]} = W_{ij}^{[k-1]} \vee \left(W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]}\right)$$

This algorithm takes $\Theta(n^3)$ time.

Example

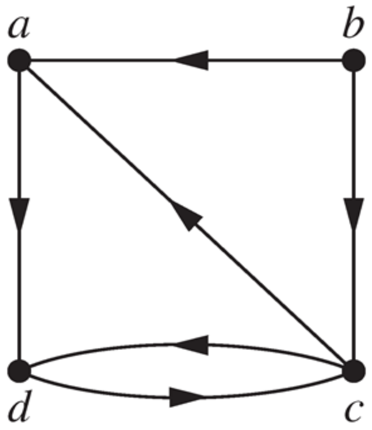
Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.

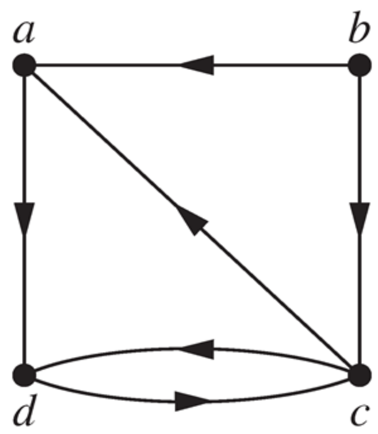


Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



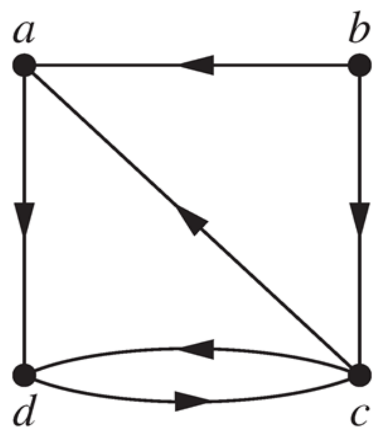
Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

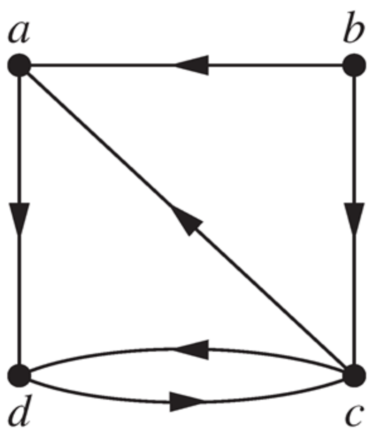
$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ \color{red}{1} & 0 & 1 & \color{red}{1} \end{bmatrix}$$

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



Let $v_1 = a,\ v_2 = b,\ v_3 = c,\ v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

n-ary Relations

- **Definition** An n-ary relation R on sets $A_1, \ldots, A_n$, written as $R : A_1, \ldots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

n-ary Relations

- **Definition** An n-ary relation R on sets $A_1, \ldots, A_n$, written as $R : A_1, \ldots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

  – The sets A_i's are called the *domains* of R.

n-ary Relations

- **Definition** An n-ary relation R on sets $A_1, \ldots, A_n$, written as $R : A_1, \ldots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

  – The sets A_i's are called the *domains* of R.

  – The *degree* of R is n.

n-ary Relations

- **Definition** An n-ary relation R on sets $A_1, \ldots, A_n$, written as $R : A_1, \ldots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

  – The sets A_i's are called the *domains* of R.

  – The *degree* of R is n.

  – R is *functional* in domain A_i if it contains at most one n-tuple $(\cdots, a_i, \cdots)$ for any value a_i within domain A_i.

Relational Databases

- A *relational database* is essentially an n-ary relation R.

Relational Databases

- A *relational database* is essentially an n-ary relation R.
- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.

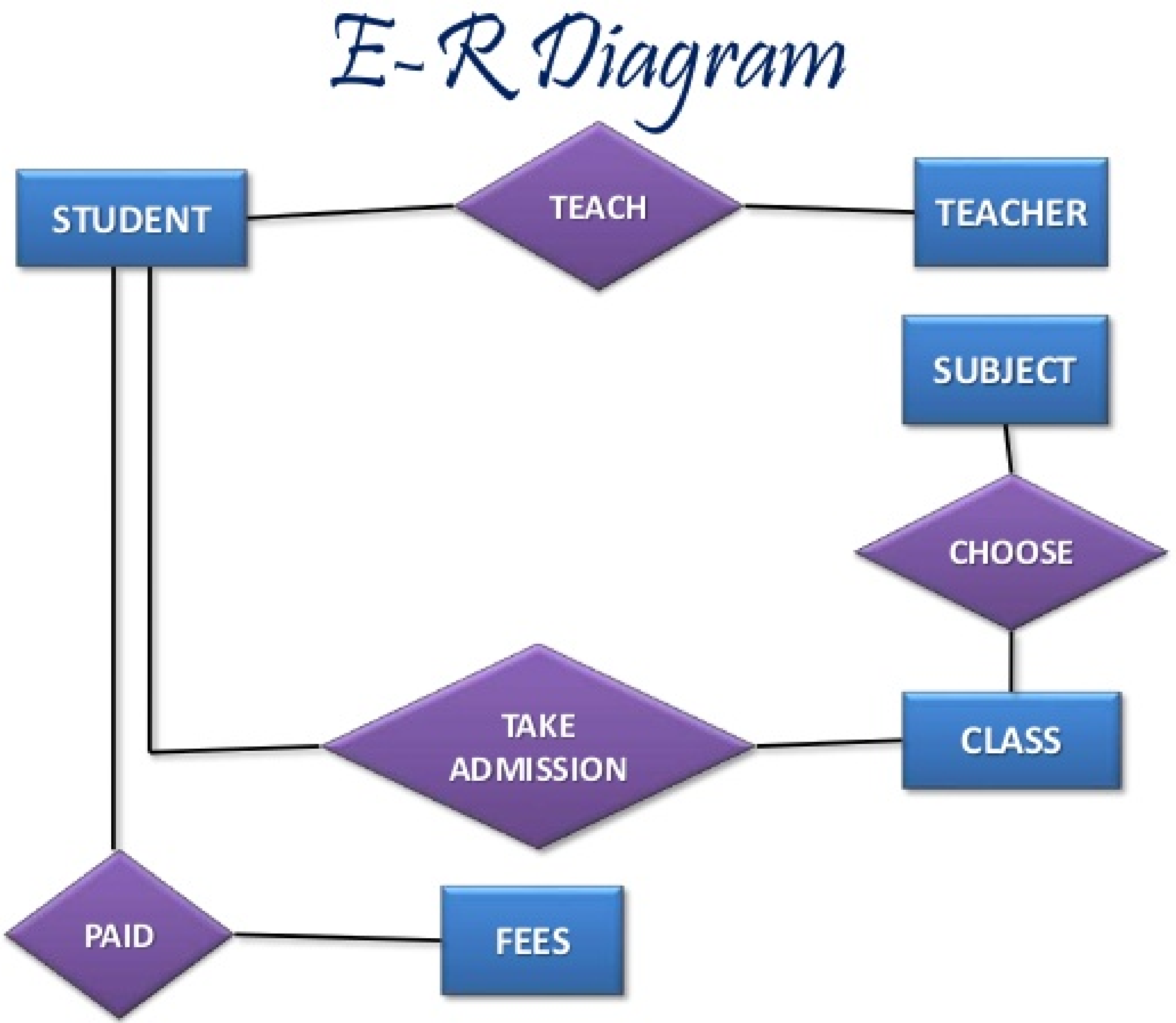
Relational Databases

- A *relational database* is essentially an n-ary relation R.

- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.

- A *composite key* for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

Relational Databases



Selection Operators

- Let A be any *n-ary domain* $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any *condition* (predicate) on elements (n-tuples) of A.

Selection Operators

- Let A be any *n*-ary domain $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any *condition* (predicate) on elements (n-tuples) of A.

- The *selection operator* s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.

Selection Operators

- Let A be any *n-ary domain* $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any *condition* (predicate) on elements (n-tuples) of A.

- The *selection operator* s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.

  – $\forall R \subseteq A$,

$$
\begin{aligned}
s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\
&= \{a \in R \mid s_C(a) = T\}.
\end{aligned}
$$

Selection Operator Example

- Suppose that we have a domain

$$A = StudentName \times Standing \times SocSecNos$$

Selection Operator Example

- Suppose that we have a domain

$$A = \mathit{StudentName} \times \mathit{Standing} \times \mathit{SocSecNos}$$

- Suppose that we have a domain

$$\mathit{UpperLevel}(\mathit{name},\ \mathit{standing},\ \mathit{ssn}) :\equiv [(\mathit{standing} = \mathit{junior}) \vee (\mathit{standing} = \mathit{senior})]$$

Selection Operator Example

- Suppose that we have a domain

$$A = \mathit{StudentName} \times \mathit{Standing} \times \mathit{SocSecNos}$$

- Suppose that we have a domain

$$\mathit{UpperLevel}(\mathit{name}, \ \mathit{standing}, \ \mathit{ssn}) \\ :\equiv \quad [(\mathit{standing} = \mathit{junior}) \vee (\mathit{standing} = \mathit{senior})]$$

- Then, $s_{\mathit{UpperLevel}}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.

  i.e., where $1 \leq i_k \leq n$ for all $1 \leq k \leq m$.

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any *n*-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.

  i.e., where $1 \leq i_k \leq n$ for all $1 \leq k \leq m$.

- Then the *projection operator* on n-tuples

  $$P_{\{i_k\}} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

  is defined by

  $$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

Projection Example

- Suppose that we have a tenary domain

  $Cars = Model \times Year \times Color \ (n = 3)$

Projection Example

- Suppose that we have a tenary domain

  $$Cars = Model \times Year \times Color \ (n = 3)$$

- Consider the index sequence $\{i_k\} = \{1, 3\}$ $(m = 2)$

Projection Example

- Suppose that we have a tenary domain

  $$Cars = Model \times Year \times Color \ (n = 3)$$

- Consider the index sequence $\{i_k\} = \{1, 3\}$ $(m = 2)$

- Then the projection $P_{\{i_k\}}$ simply maps each tuple $(a_1, a_2, a_3) = (model, year, color)$ to its image:

  $$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

Projection Example

- Suppose that we have a tenary domain

  $$Cars = Model \times Year \times Color \ (n = 3)$$

- Consider the index sequence $\{i_k\} = \{1, 3\}$ $(m = 2)$
- Then the projection $P_{\{i_k\}}$ simply maps each tuple $(a_1, a_2, a_3) = (model, year, color)$ to its image:

  $$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of $model/color$ combinations available.

Join Operator

- Puts two relations together to form a sort of *combined relation*.

Join Operator

- Puts two relations together to form a sort of *combined relation*.

- If the tuple (A, B) appears in R_1, and the tuple (B, C) appears in R_2, then the tuple (A, B, C) appears in the *join* $J(R_1, R_2)$.

Join Operator

- Puts two relations together to form a sort of *combined relation*.
- If the tuple (A, B) appears in R_1, and the tuple (B, C) appears in R_2, then the tuple (A, B, C) appears in the *join* $J(R_1, R_2)$.
- A, B, C can also be sequences of elements rather that single elements.

Join Example

- Suppose that R_1 is a teaching assignment table, relating *Professors* to *Courses*.

Join Example

- Suppose that R_1 is a teaching assignment table, relating *Professors* to *Courses*.

- Suppose that R_2 is a room assignment table relating *Courses* to *Rooms* and *Times*.

Join Example

- Suppose that R_1 is a teaching assignment table, relating *Professors* to *Courses*.
- Suppose that R_2 is a room assignment table relating *Courses* to *Rooms* and *Times*.
- Then $J(R_1, R_2)$ is like your class schedule, listing $(professor, course, room, time)$.

Next Lecture

- relation II ...