Department of Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

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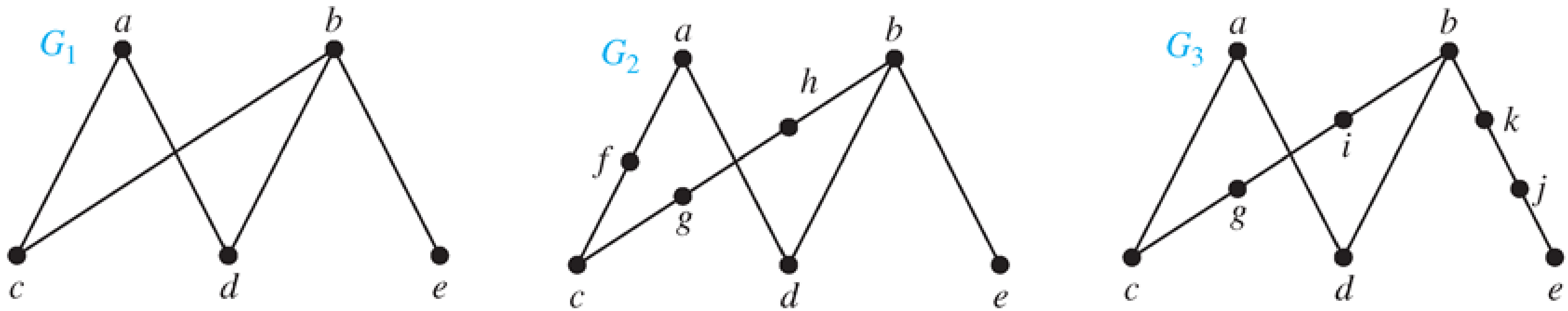

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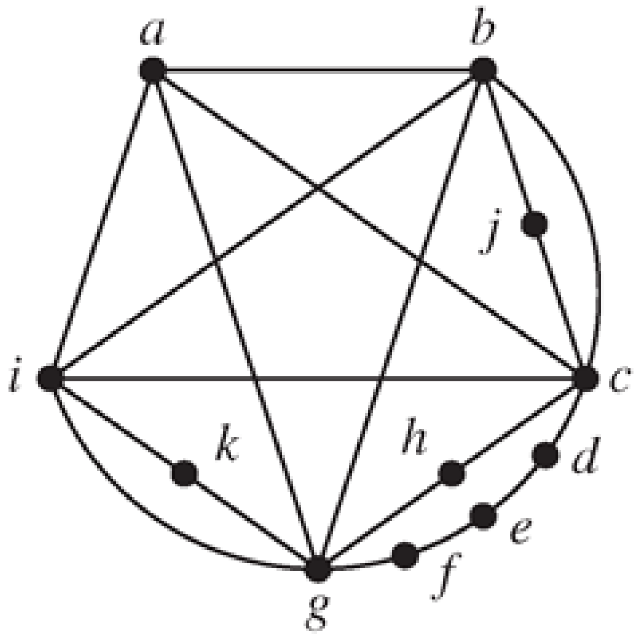
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples



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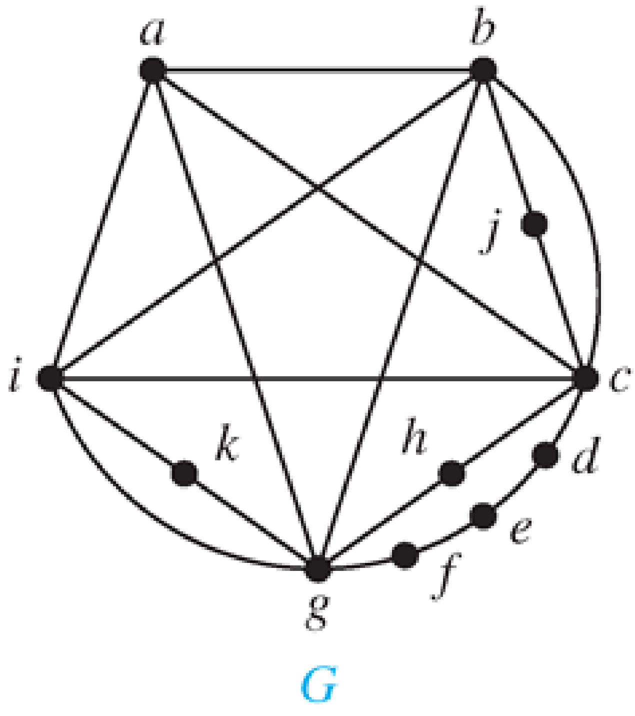

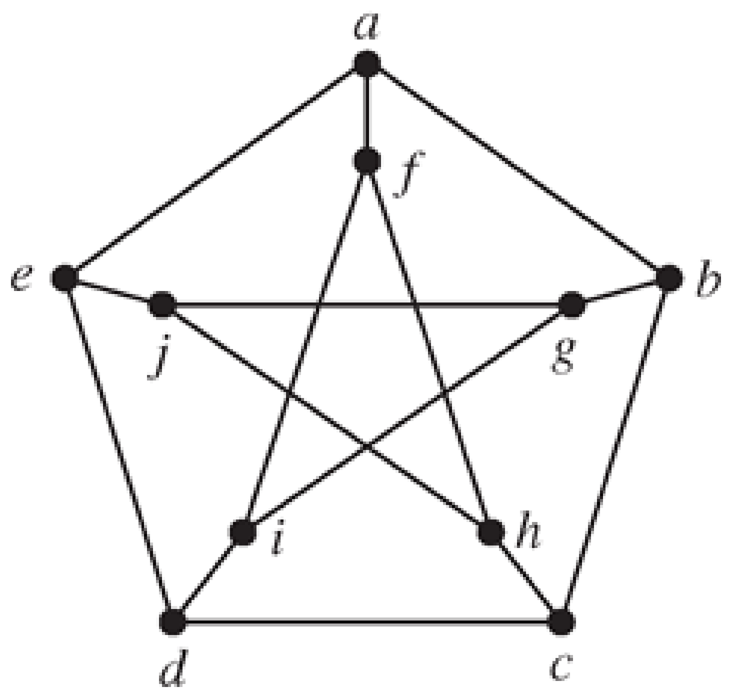

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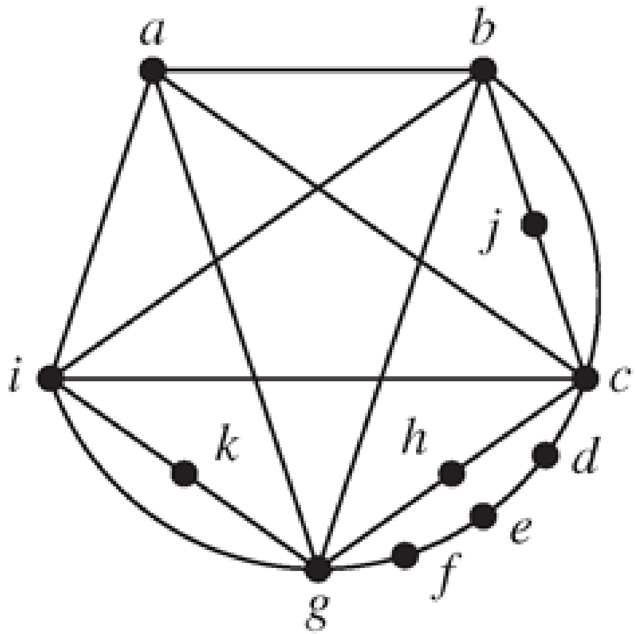

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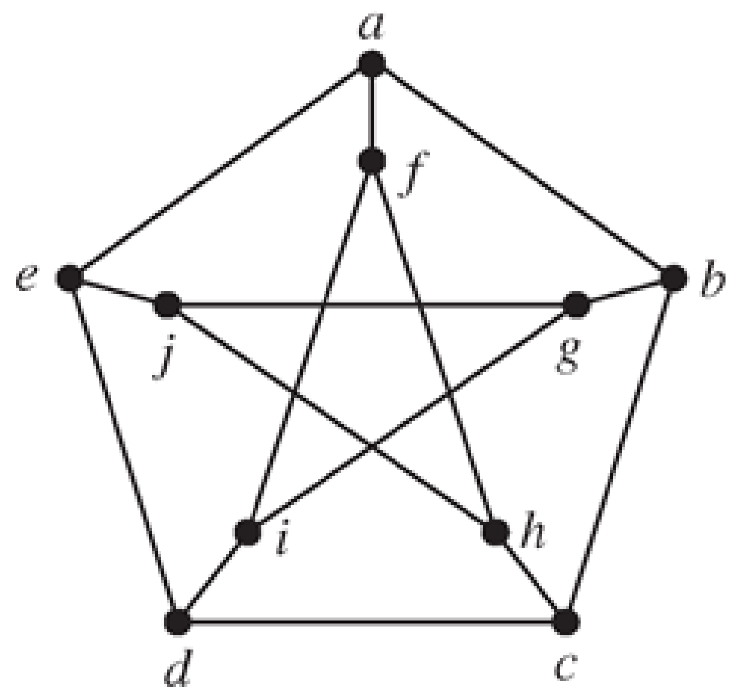
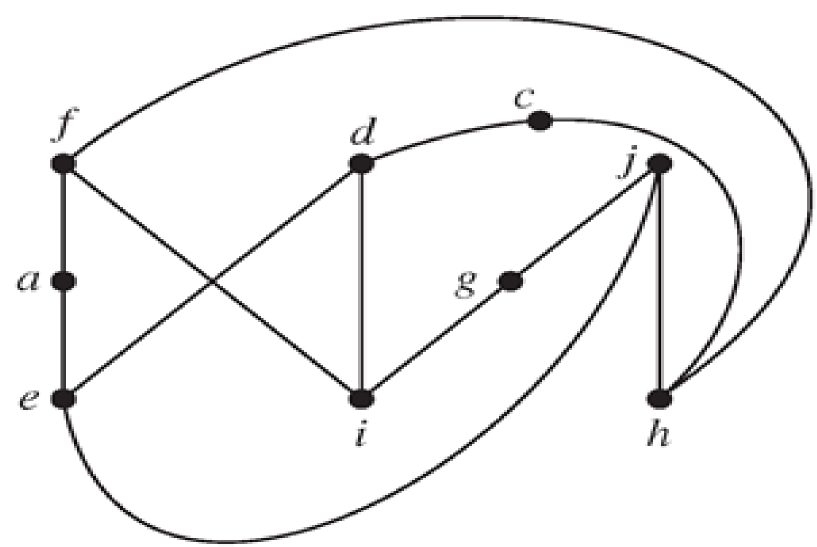
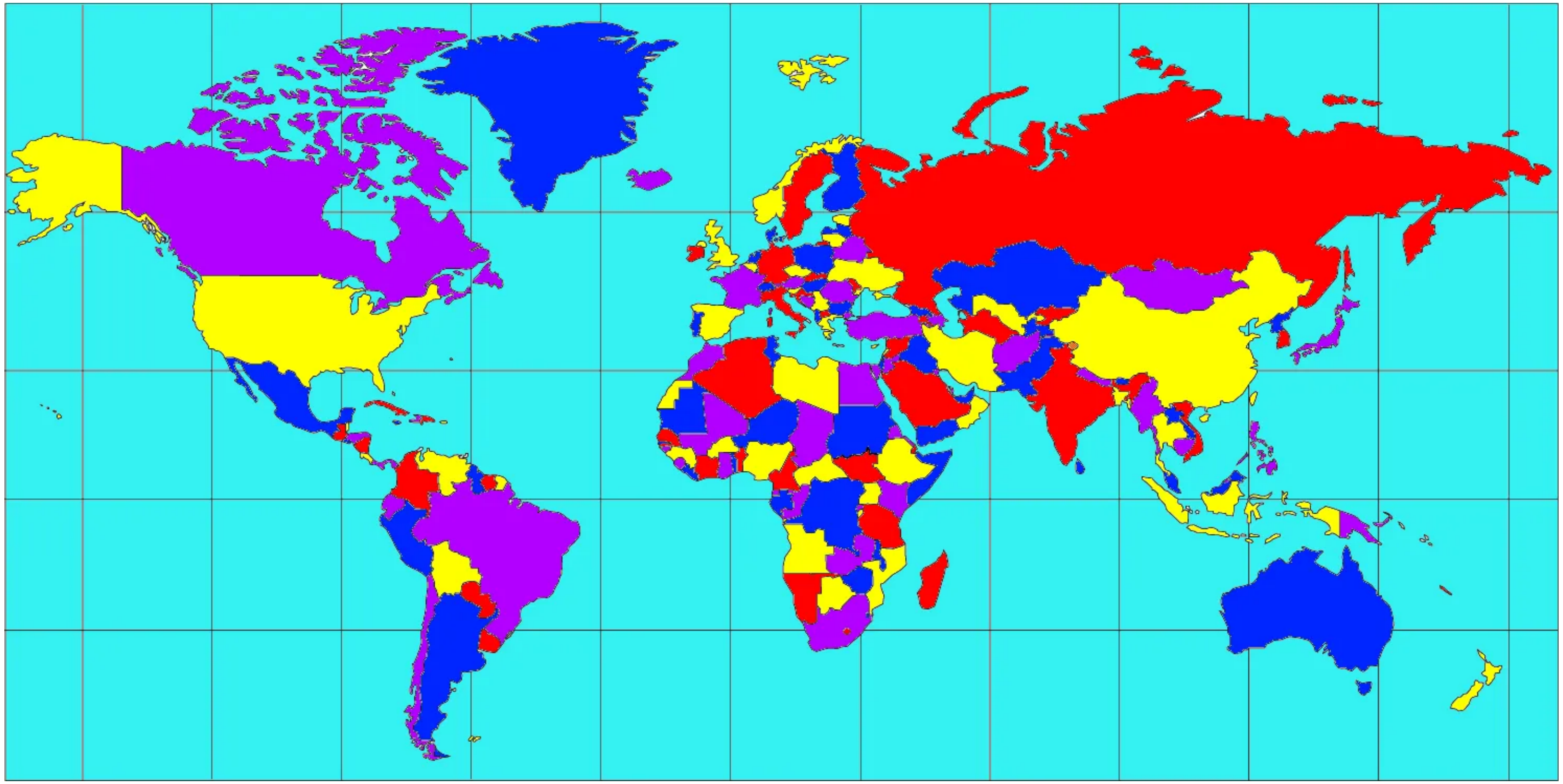




Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



Graph Coloring

- **Four-color theorem**

  ◇ first proposed by Francis Guthrie in 1852

  ◇ his brother Frederick Guthrie told Augustus De Morgan

  ◇ De Morgan wrote to William Hamilton

  ◇ Alfred Kempe proved it incorrectly in 1879

  ◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*

  ◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)

  ◇ Kempe's incorrect proof serves as a basis

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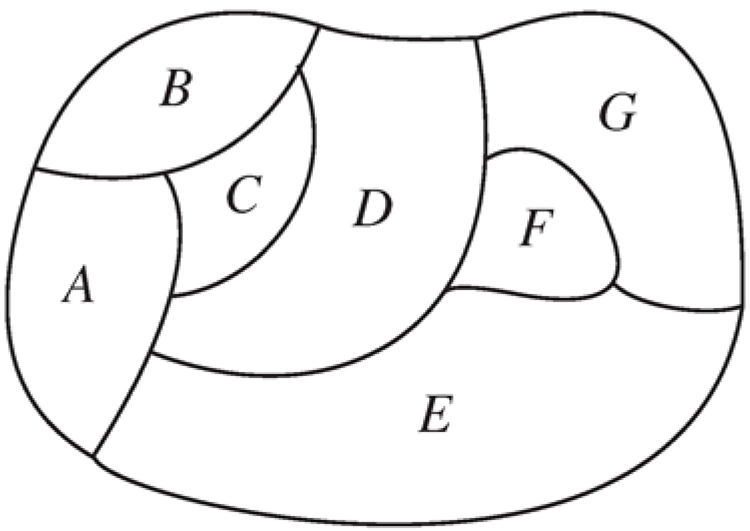

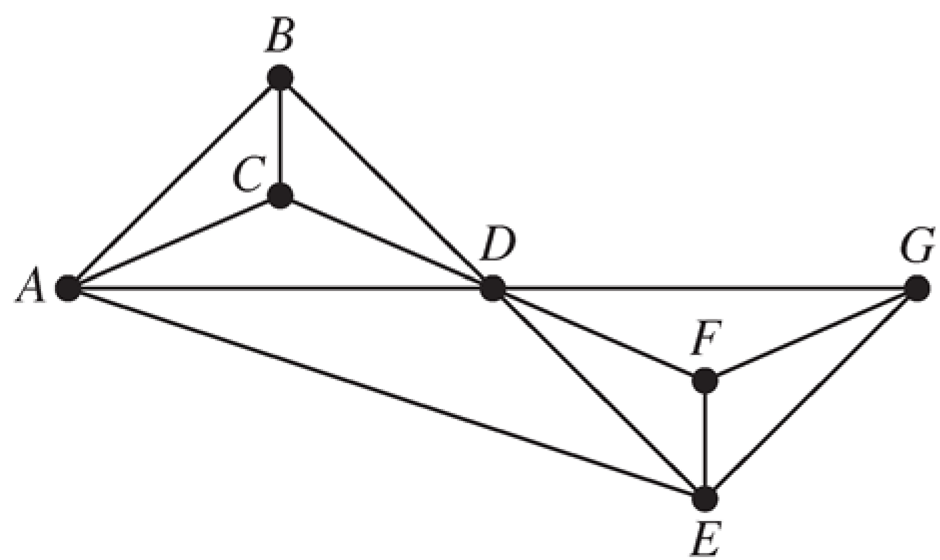

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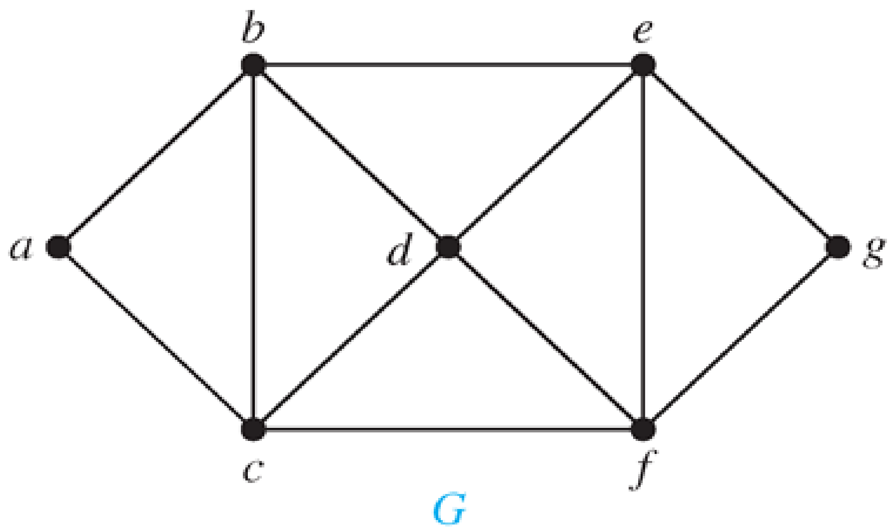

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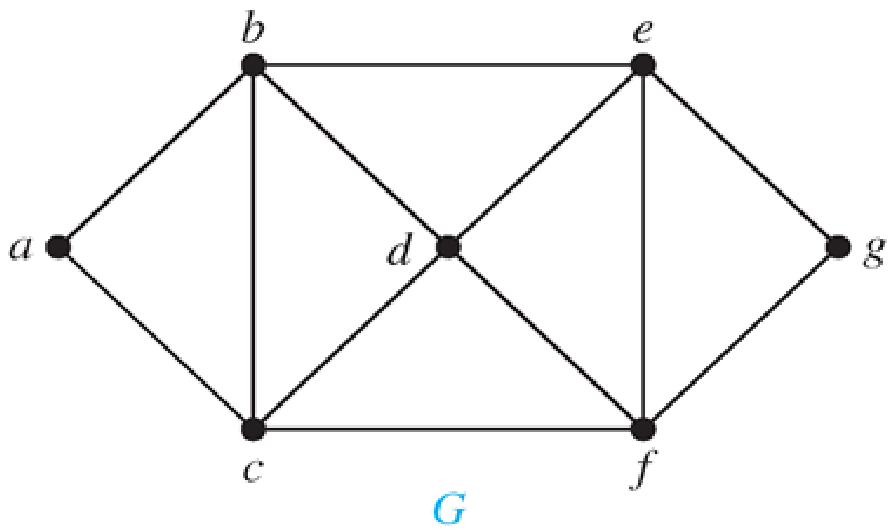

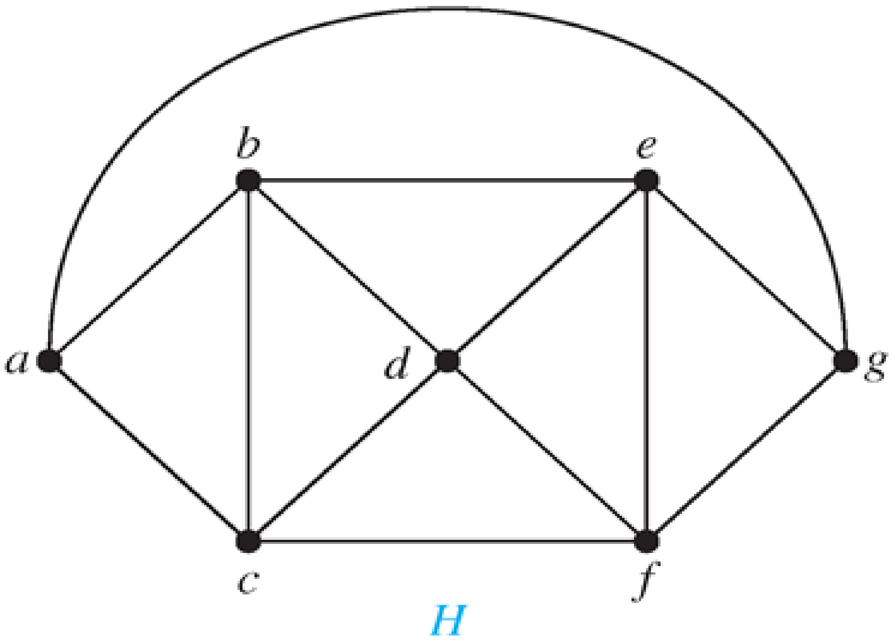

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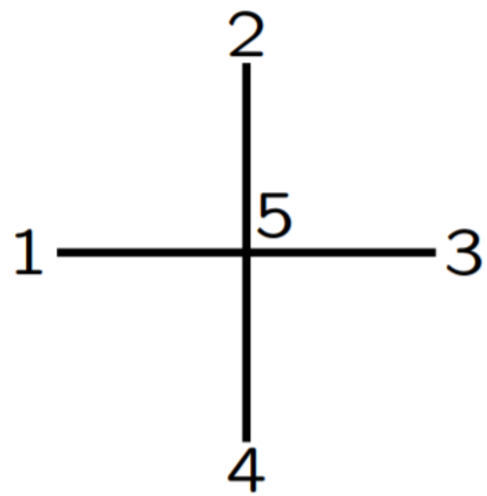
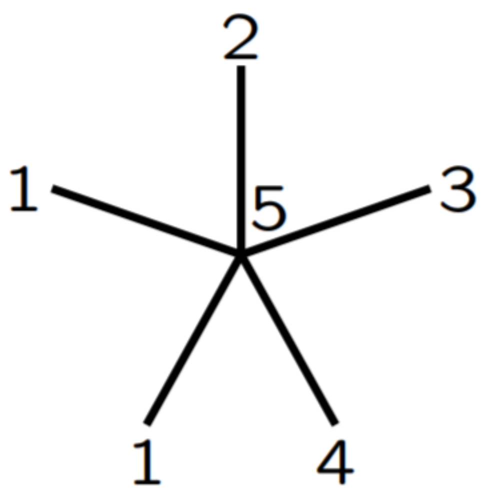
  If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we can pick an available color for it.





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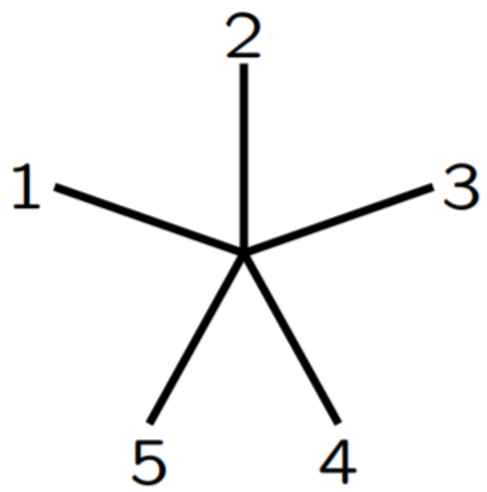

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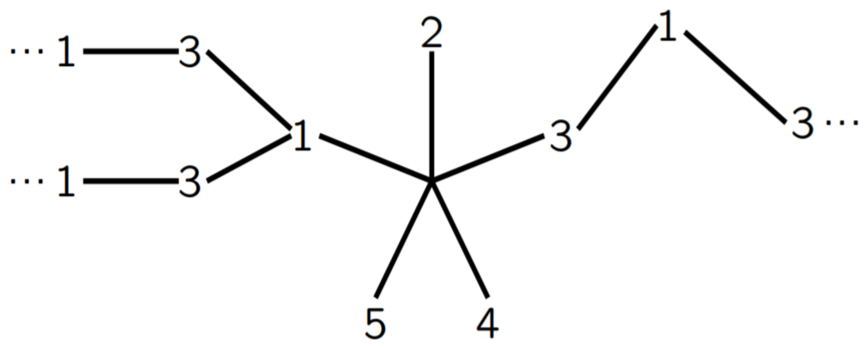

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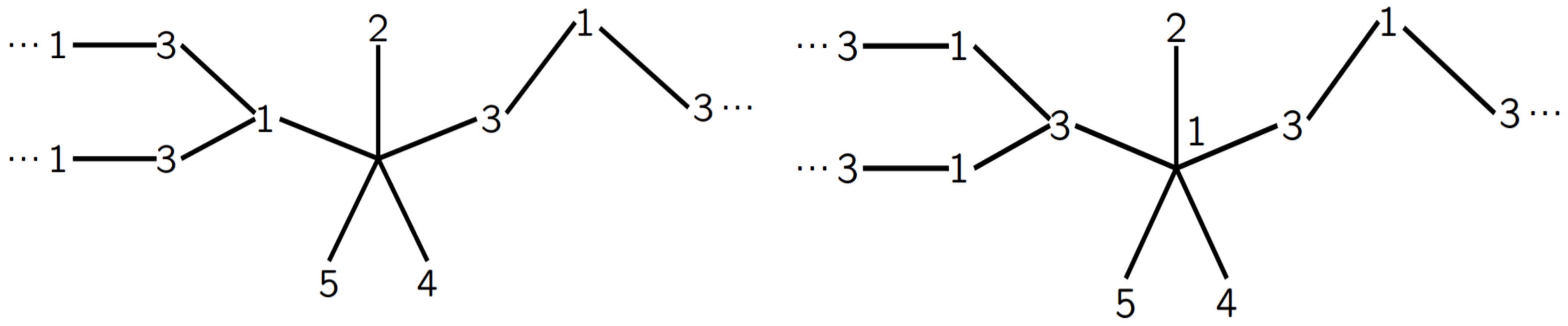

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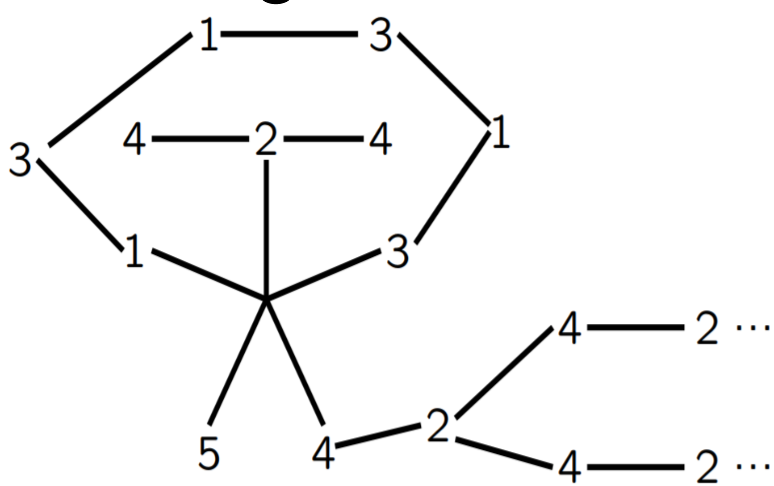

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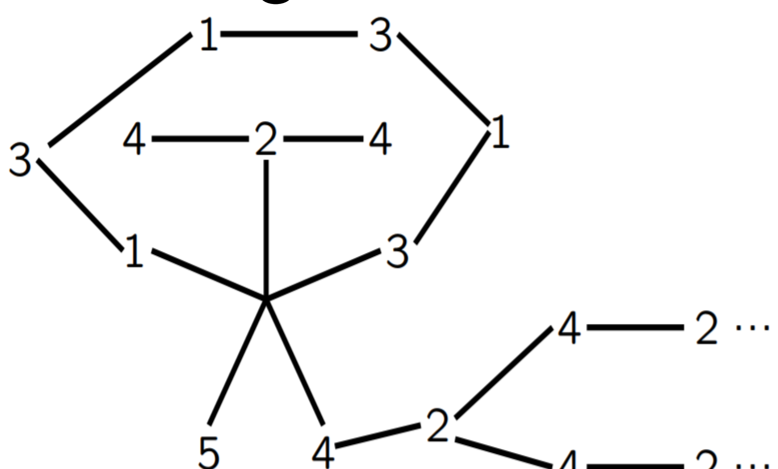

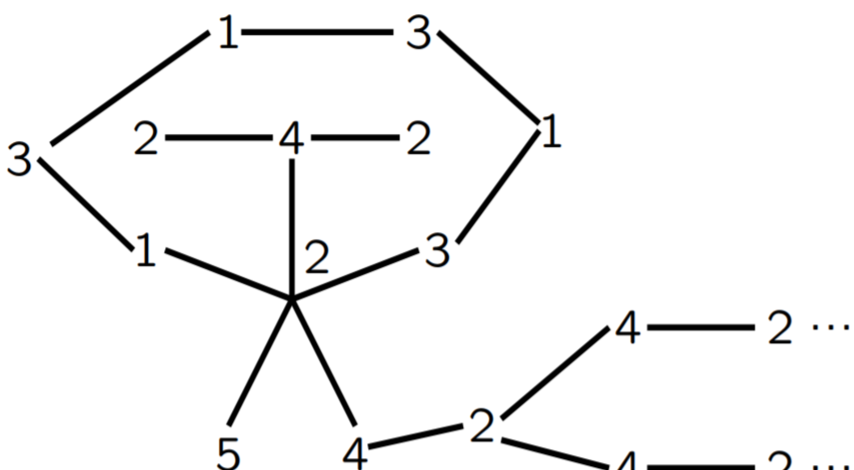

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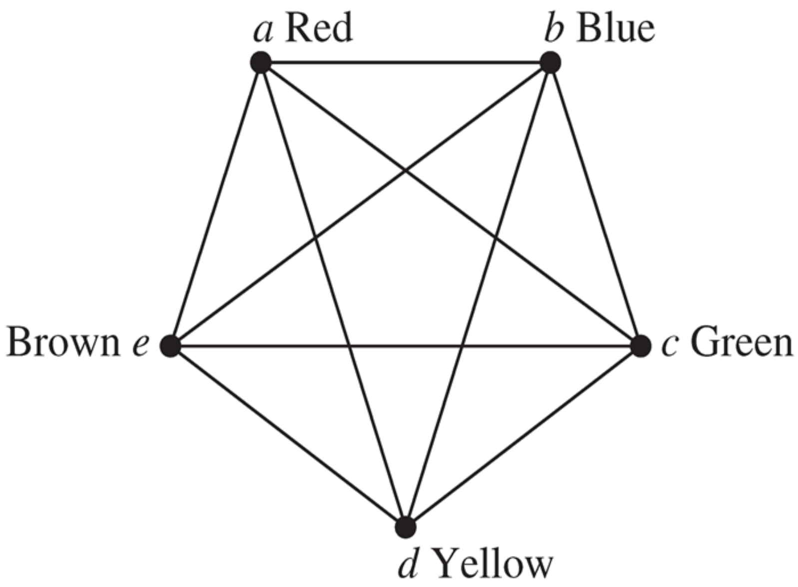

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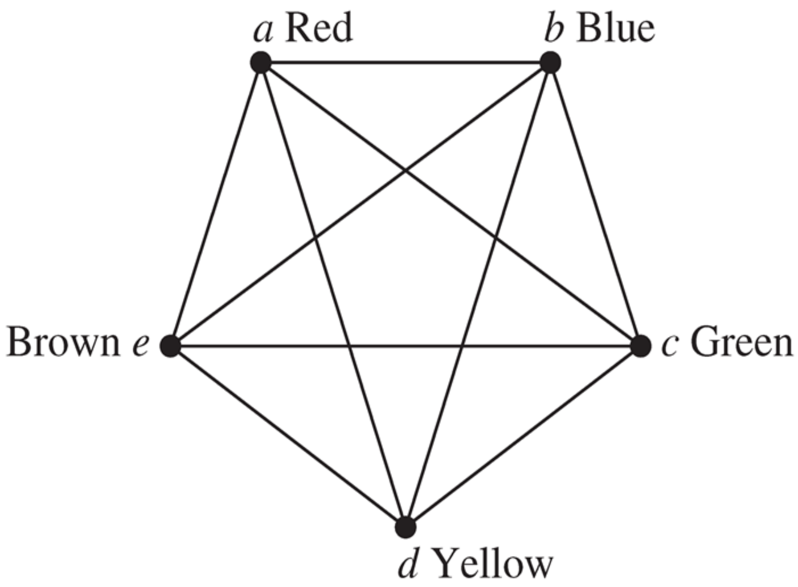

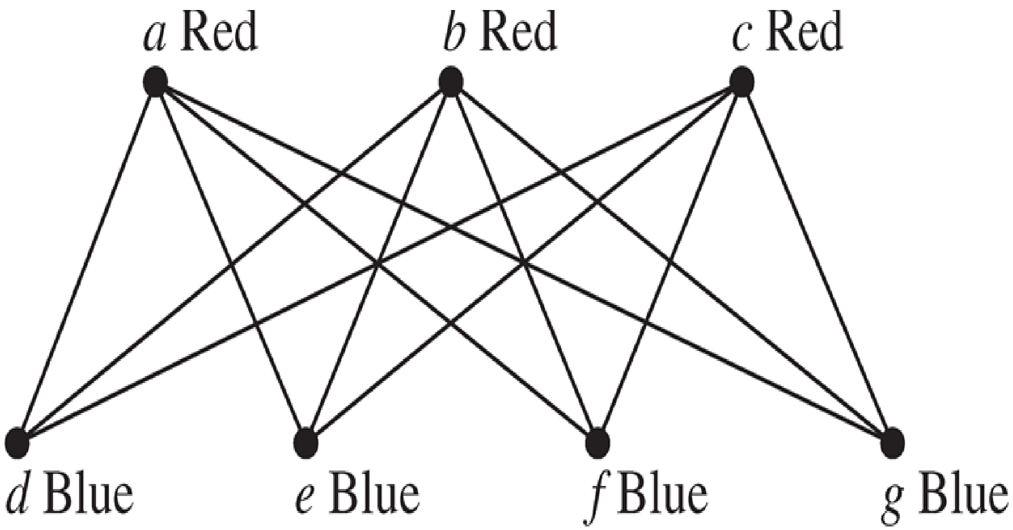

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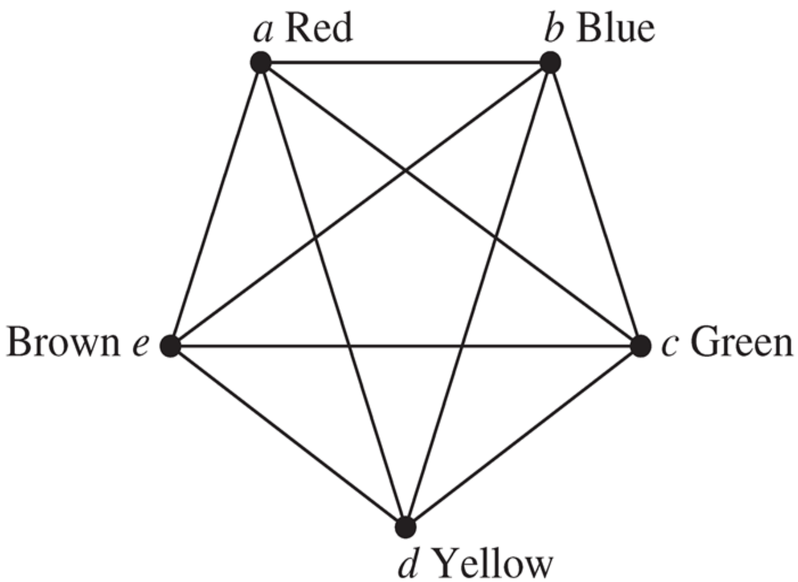

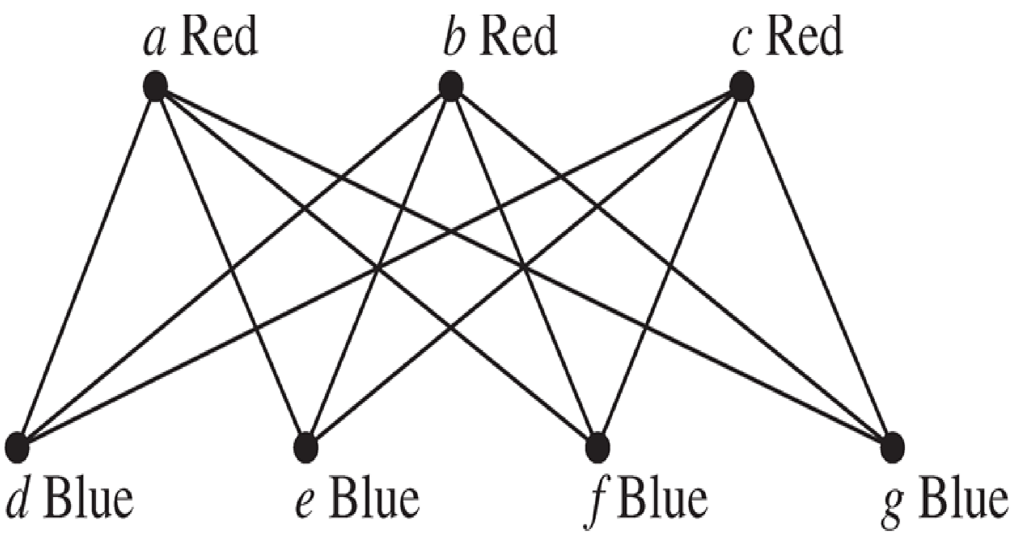

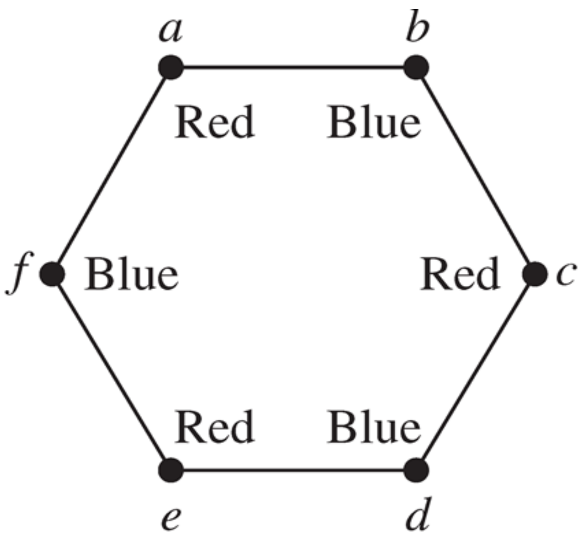

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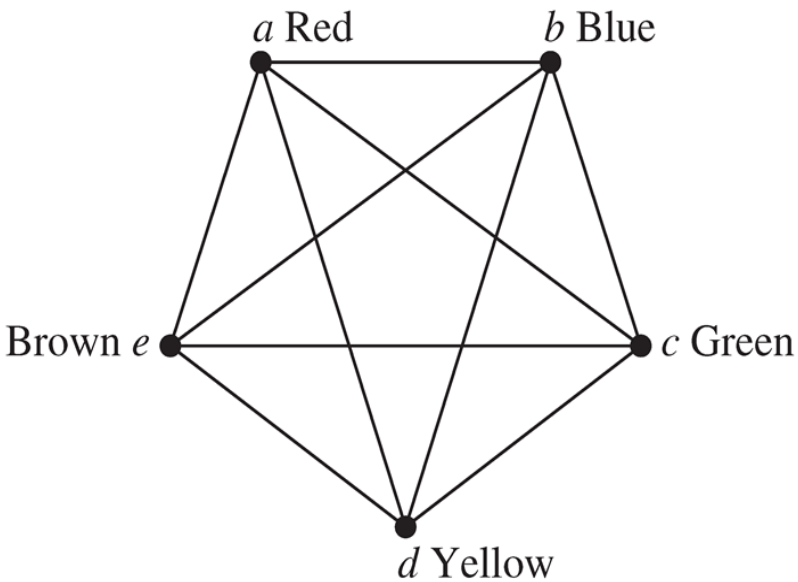
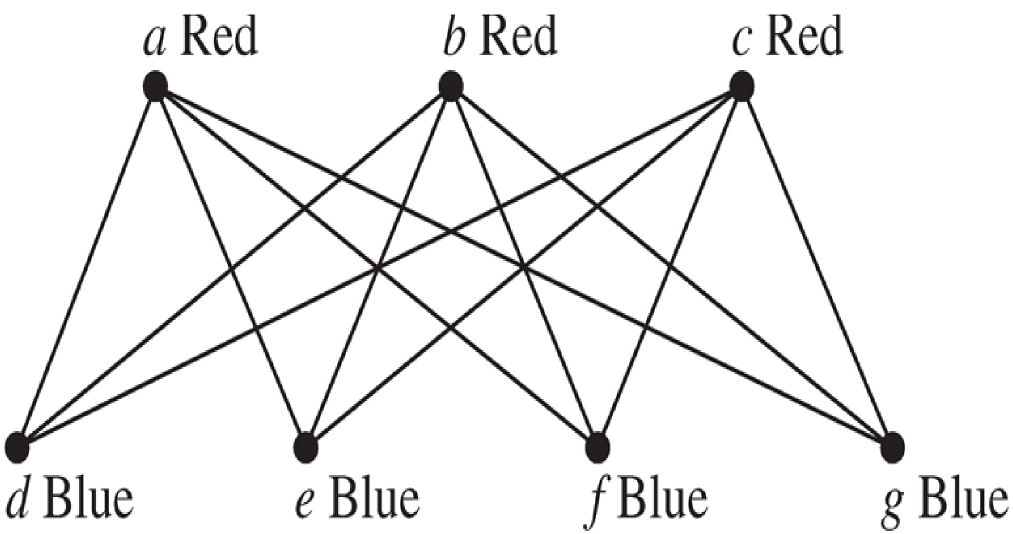
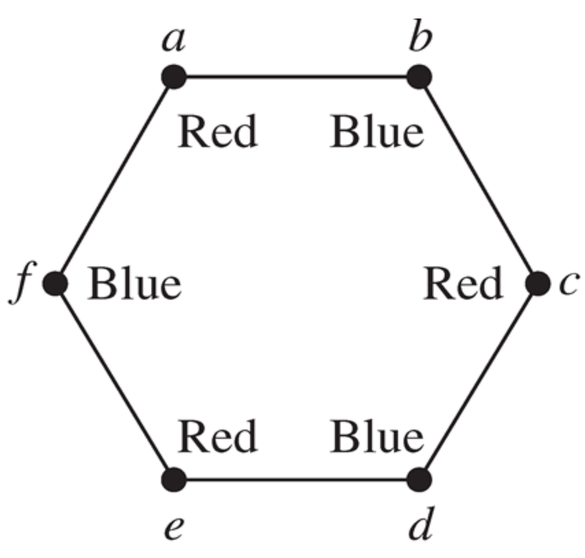
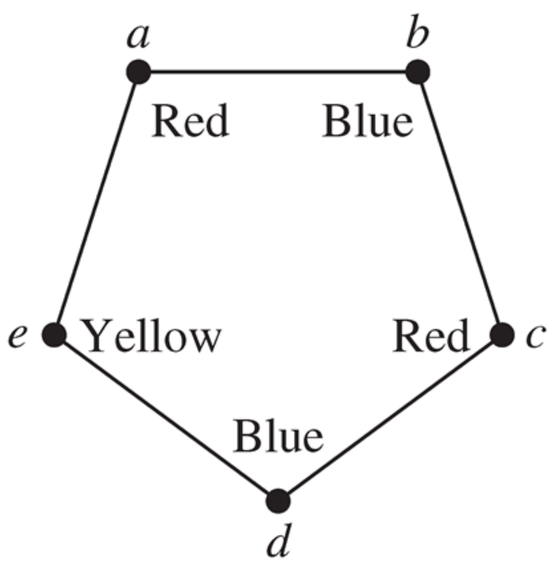




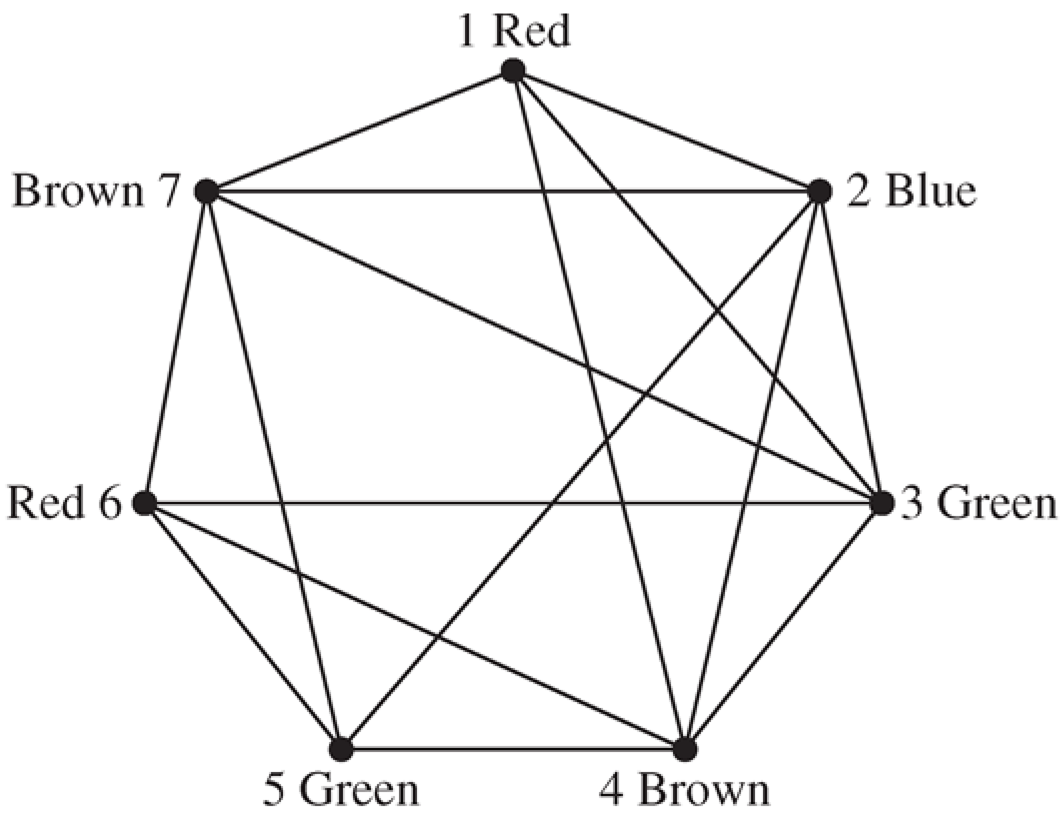




Applications of Graph Coloring

- Scheduling Final Exams

  Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

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  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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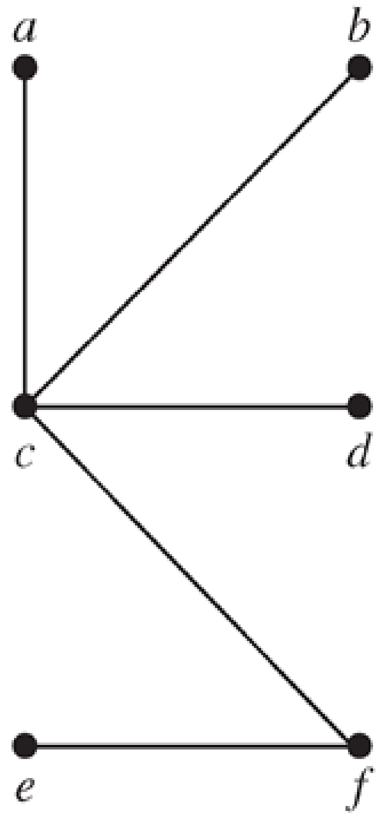

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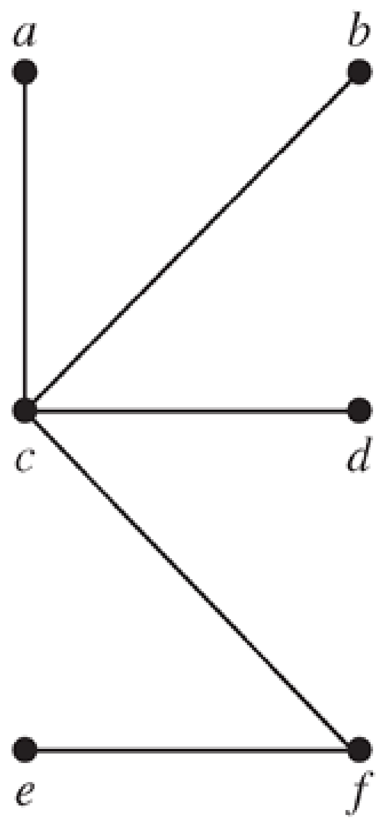

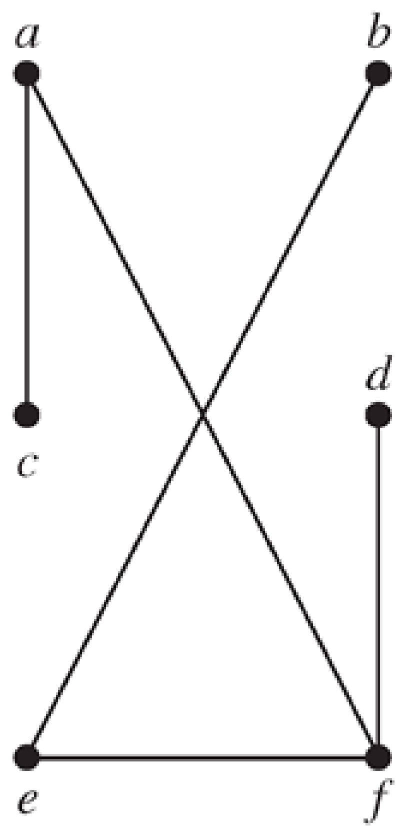

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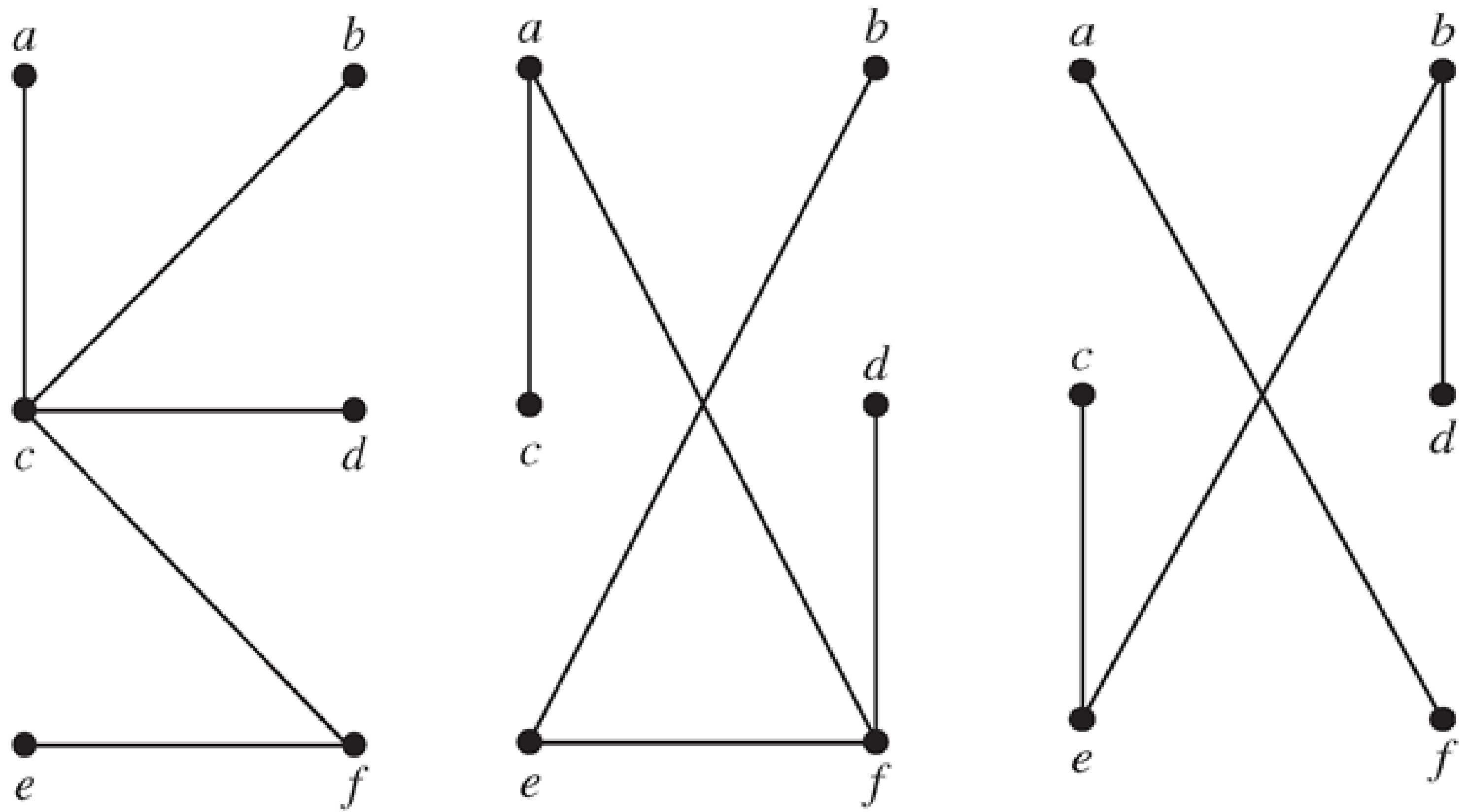

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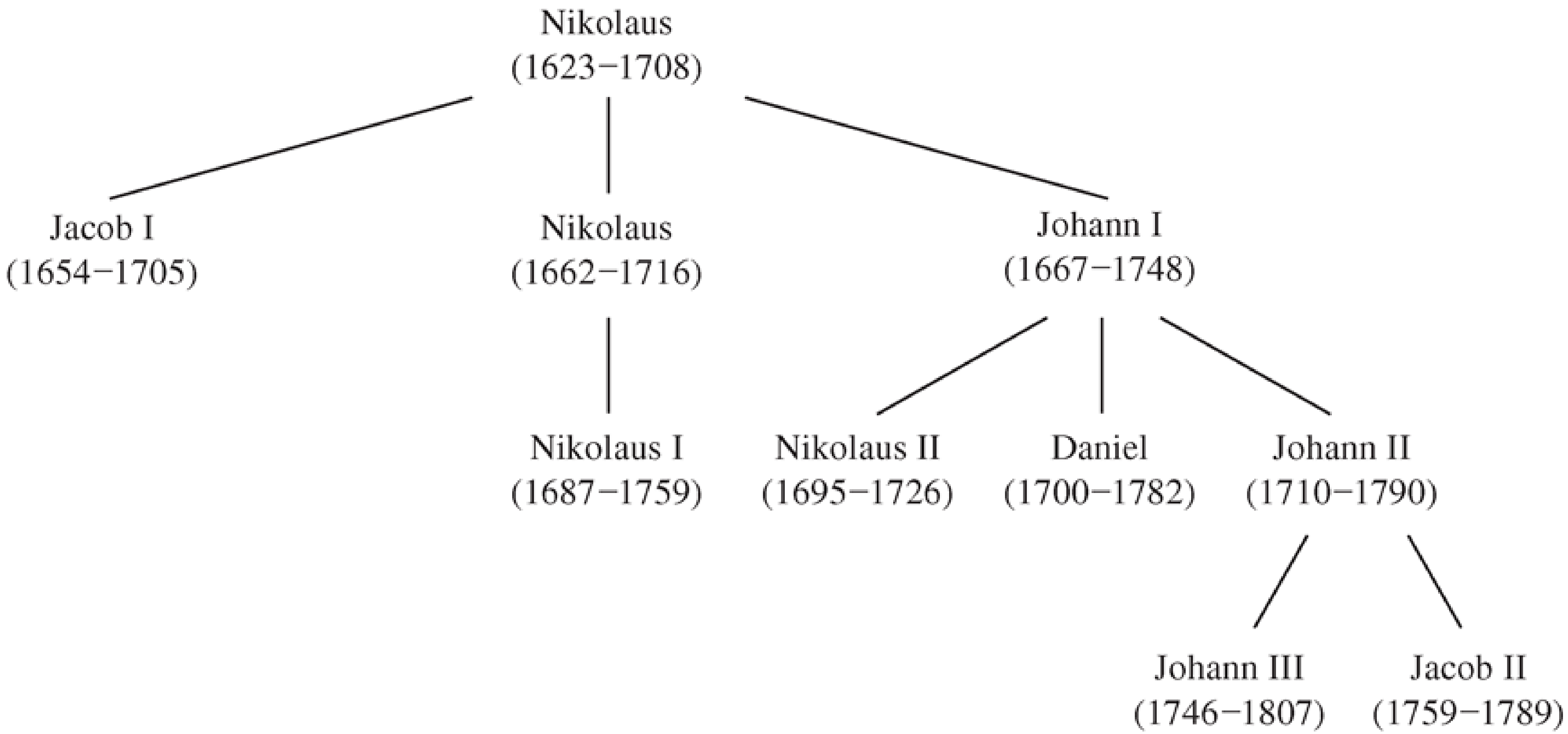

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Two properties of tree: connected, no circuit

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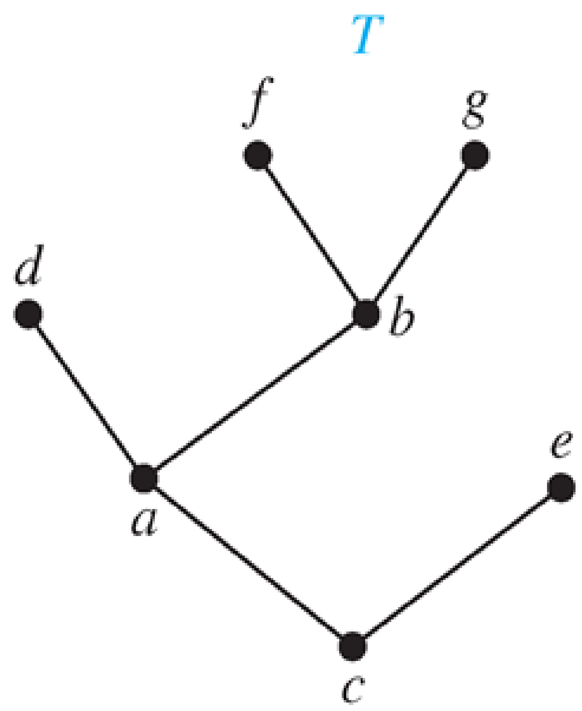

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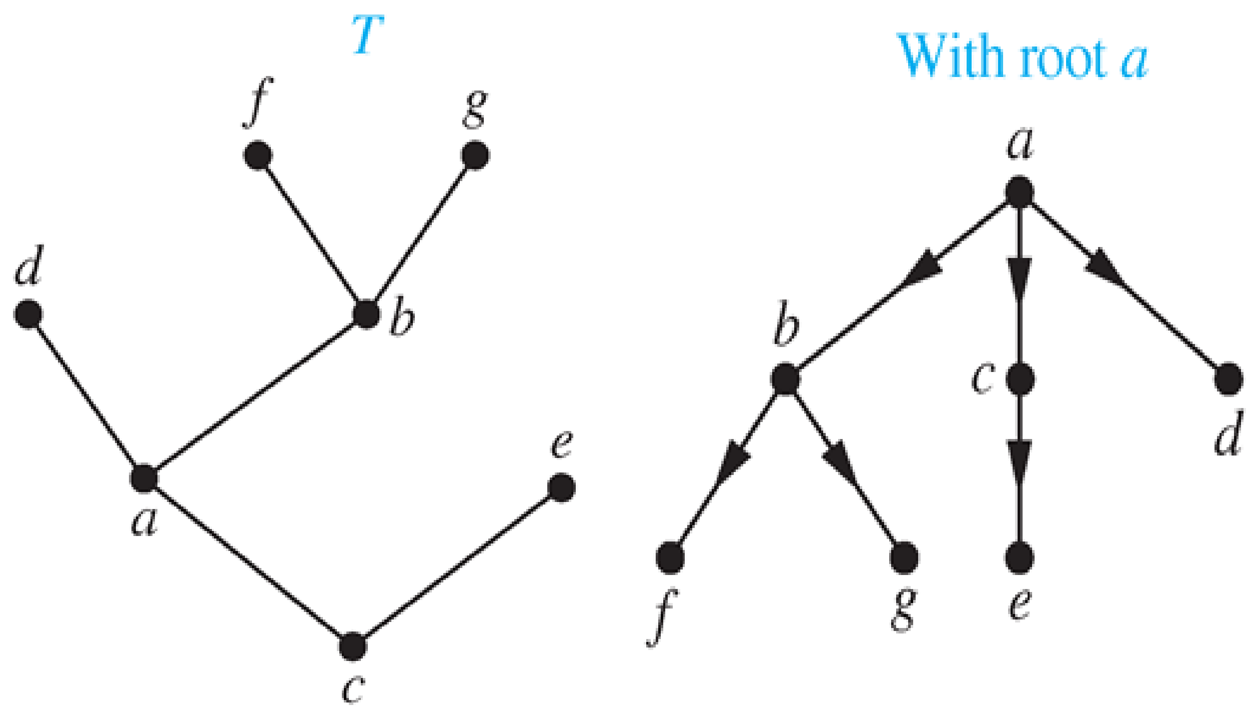

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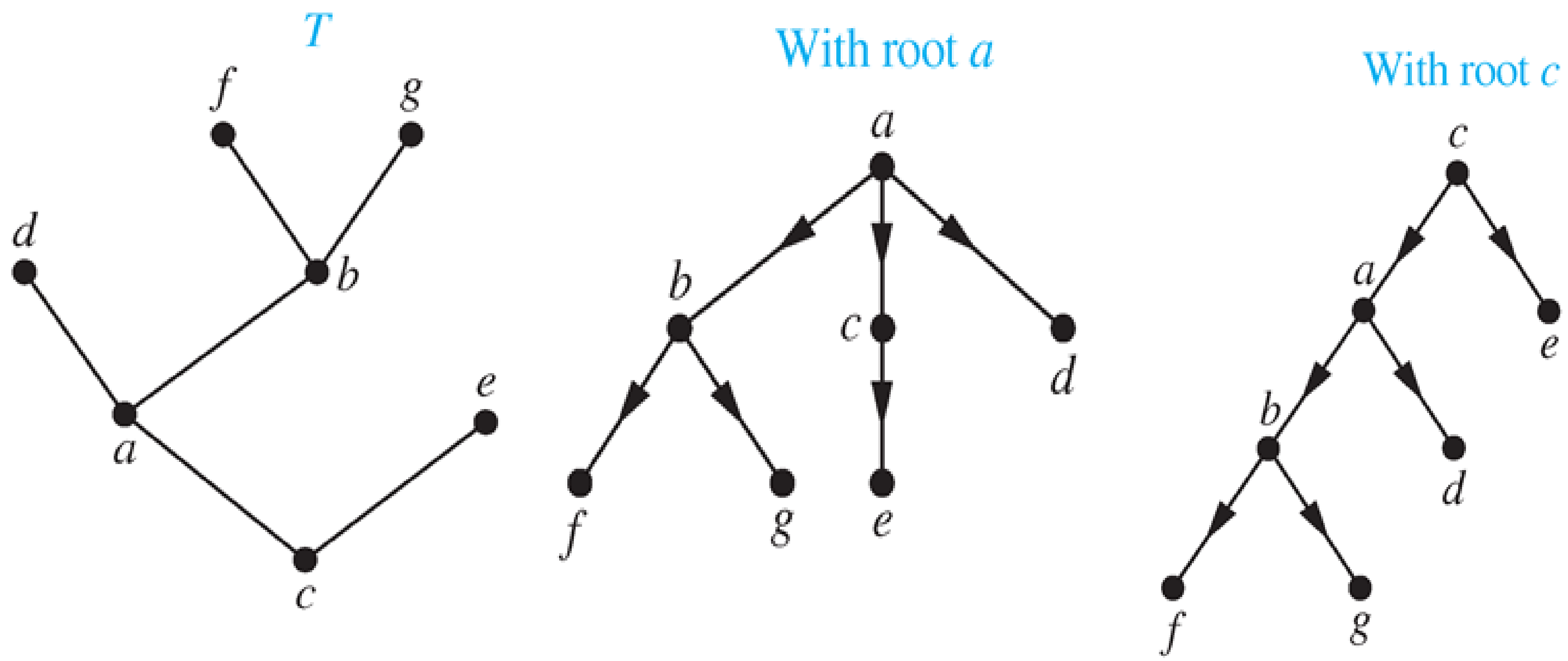

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  subtree with a as its root: consists of a and its descendants and all edges incident to these descendants

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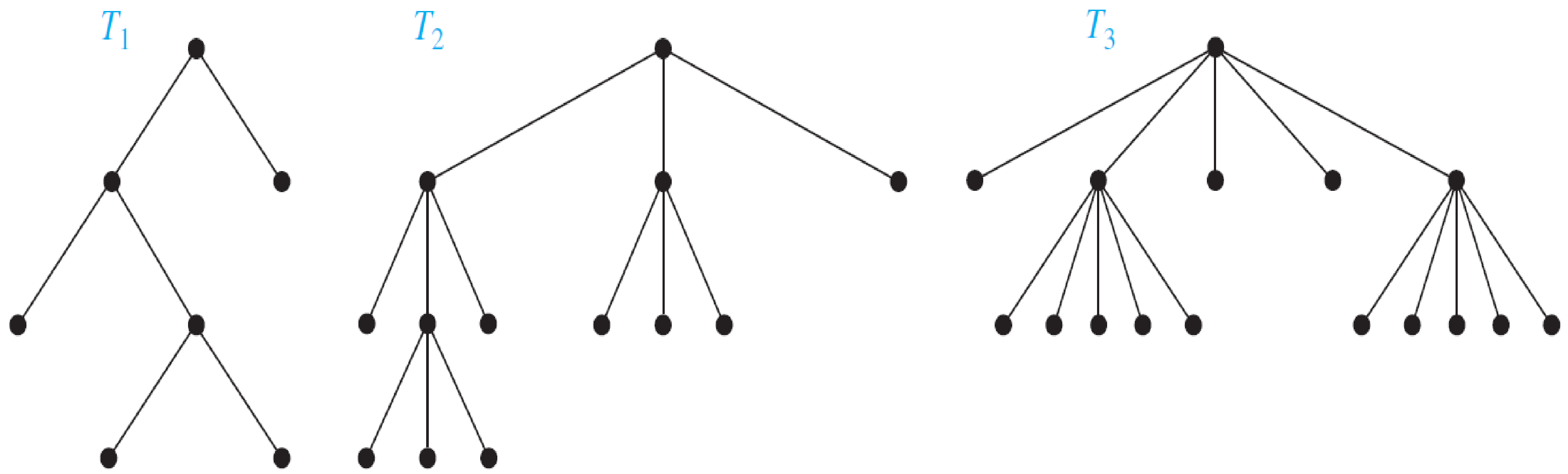
  left subtree, *right subtree*

Full m-Ary Trees

- **Definition** A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. In particular, an m-ary tree with $m = 2$ is called a *binary tree*.



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Level and Height

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  The *height* of a rooted tree is the maximum of the levels of the vertices.

  Definition A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h-1$. (differ no greater than 1)

The Number of Leaves

- **Theorem** There are at most m^h leaves in an m-ary tree of height h.

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Binary Trees

- **Definition** A *binary tree* is an ordered rooted tree where each internal tree has two children, the first is called the *left child* and the second is the *right child*. The tree rooted at the left child of a vertex is called the *left subtree* of this vertex, and the tree rooted at the right child of a vertex is called the *right subtree* of this vertex.

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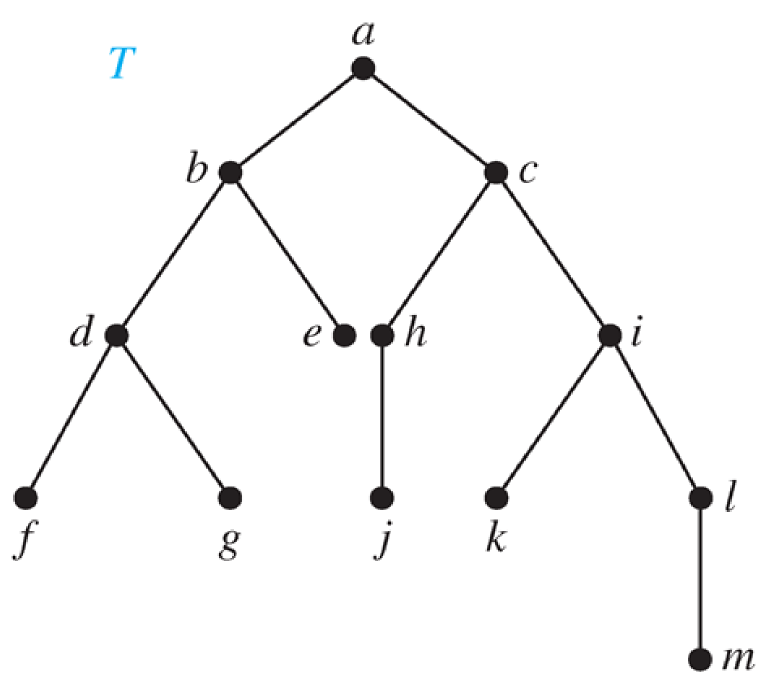

Tree Traversal

- The procedures for systematically visiting every vertex of an ordered tree are called *traversals*.

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  The three most commonly used traversals are *preorder traversal*, *inorder traversal*, *postorder traversal*.

Preorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *preorder traversal* begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.

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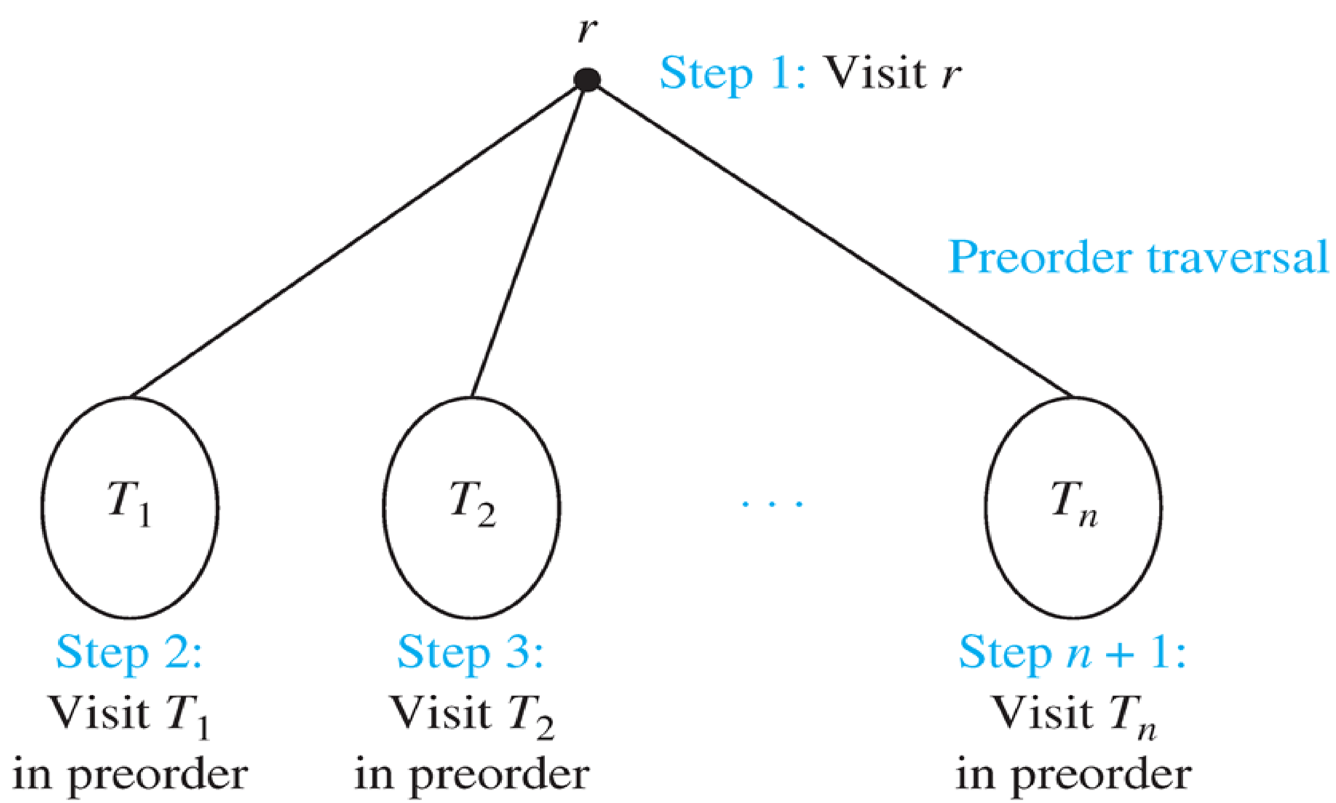
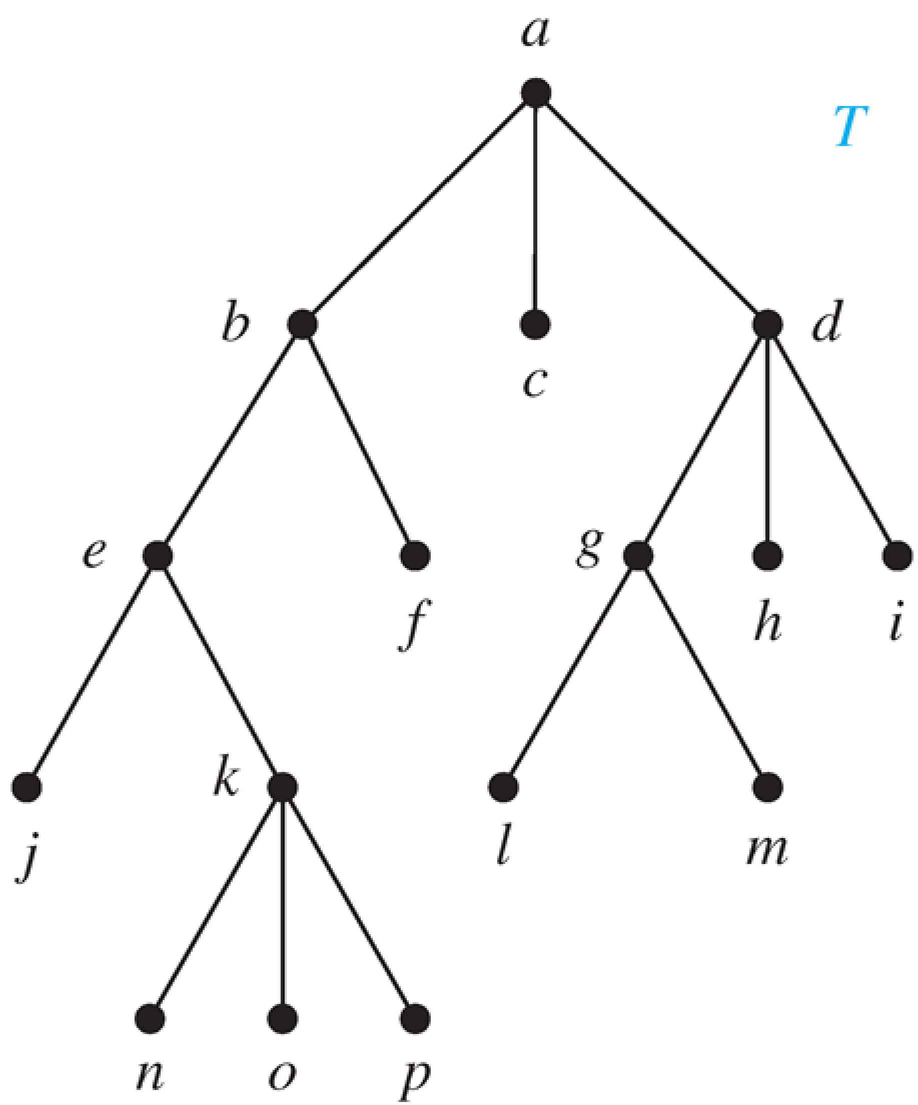


Preorder Traversal

- **Example**



Preorder Traversal

procedure *preorder* (T: ordered rooted tree)
$r :=$ root of T
list r
for each child c of r from left to right
    $T(c) :=$ subtree with c as root
    $preorder(T(c))$

Inorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.

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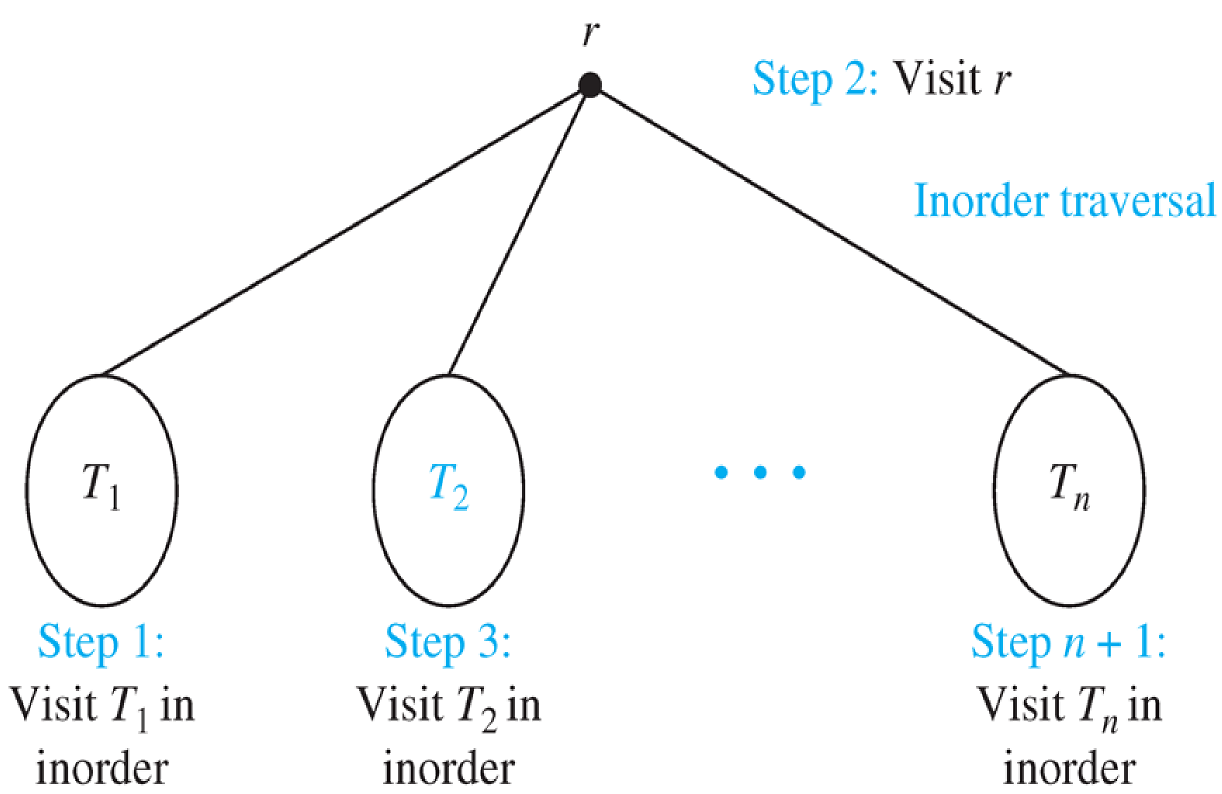
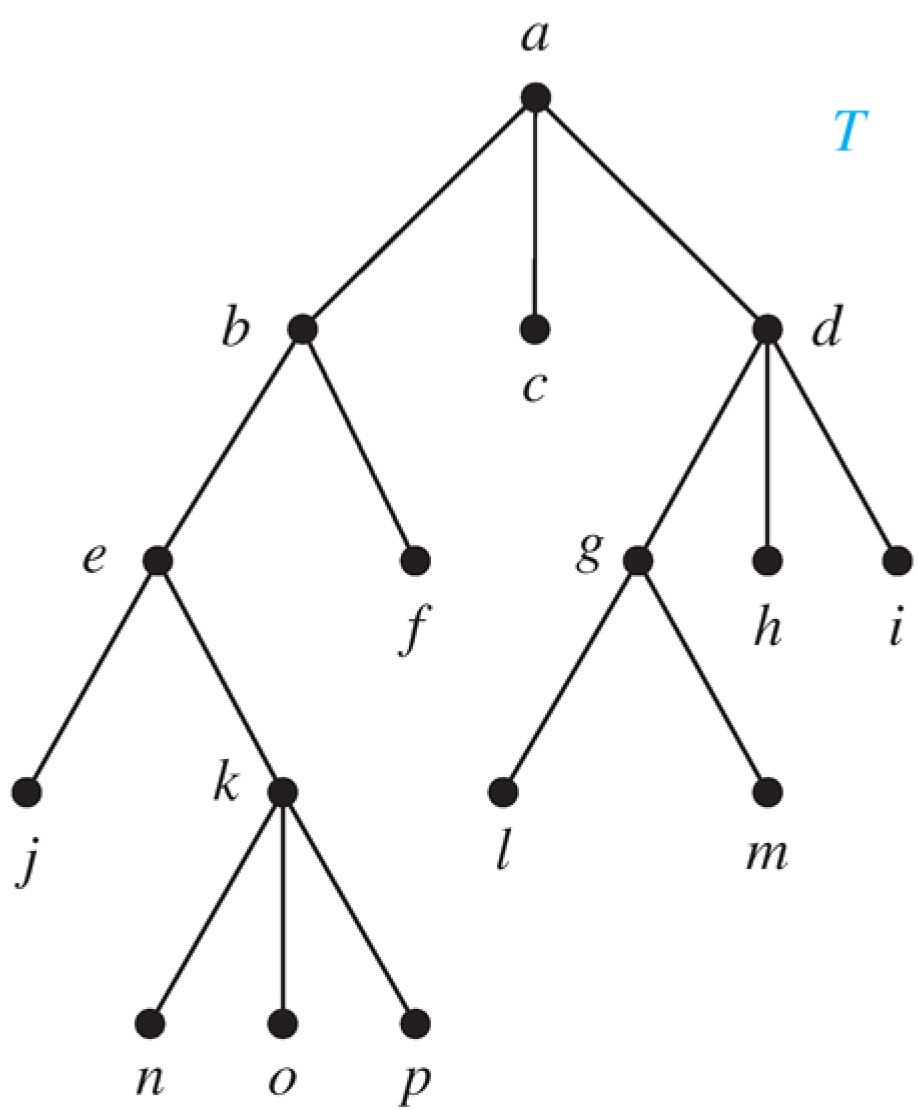


Inorder Traversal

- **Example**



Inorder Traversal

```
procedure inorder (T: ordered rooted tree)
r := root of T
if r is a leaf then list r
else
    l := first child of r from left to right
    T(l) := subtree with l as its root
    inorder(T(l))
    list(r)
    for each child c of r from left to right
        T(c) := subtree with c as root
        inorder(T(c))
```

Postorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.

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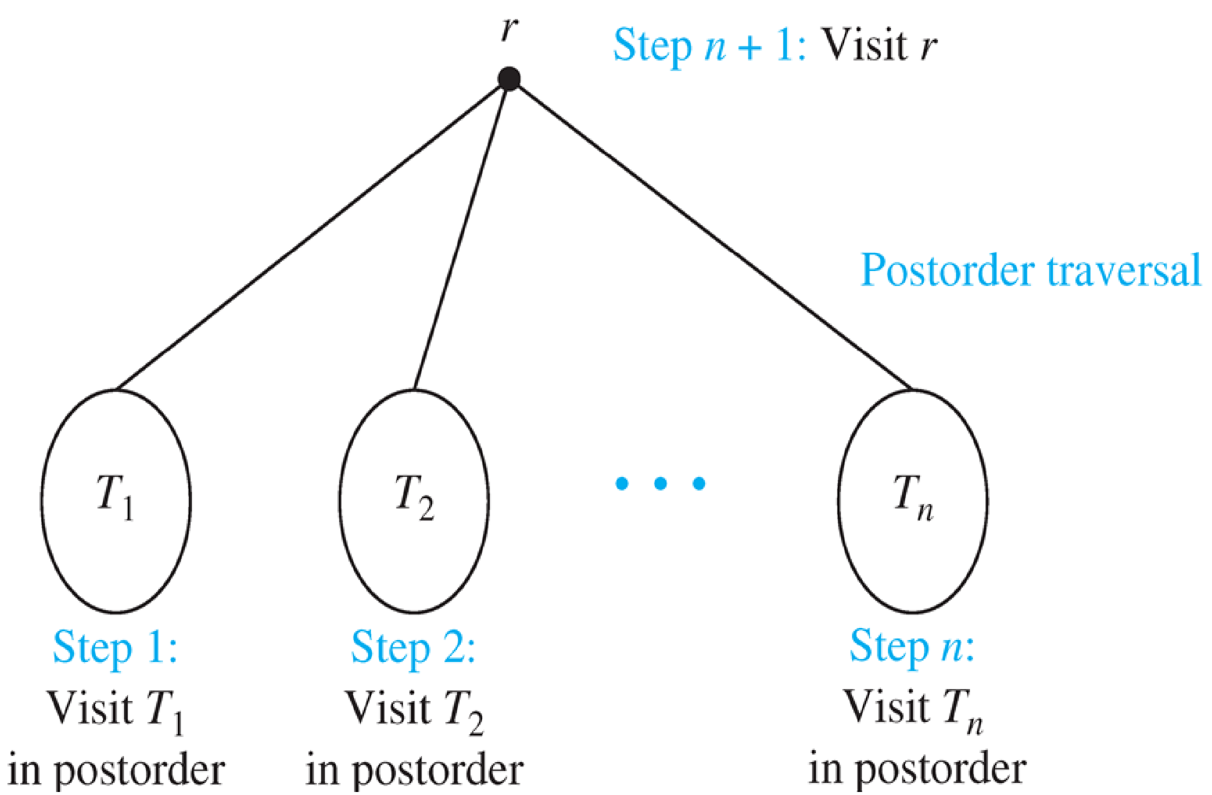

Postorder Traversal

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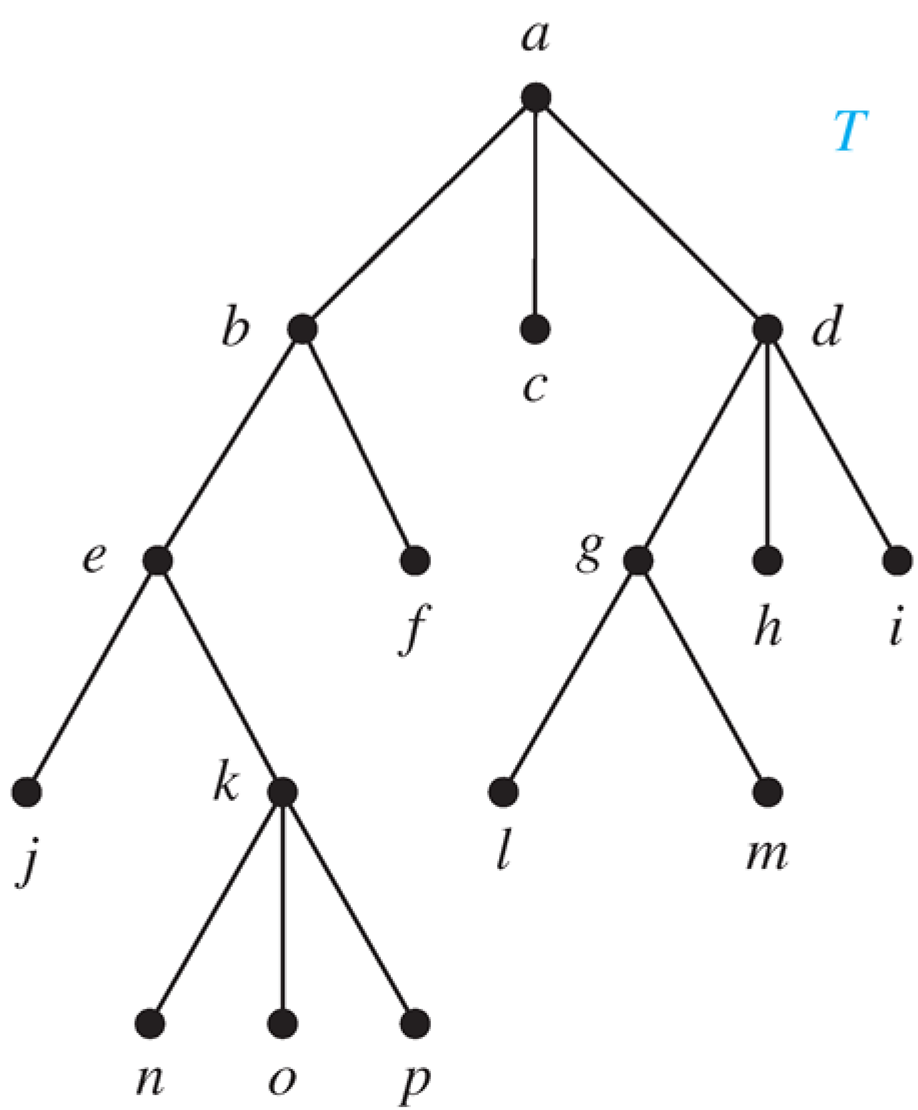
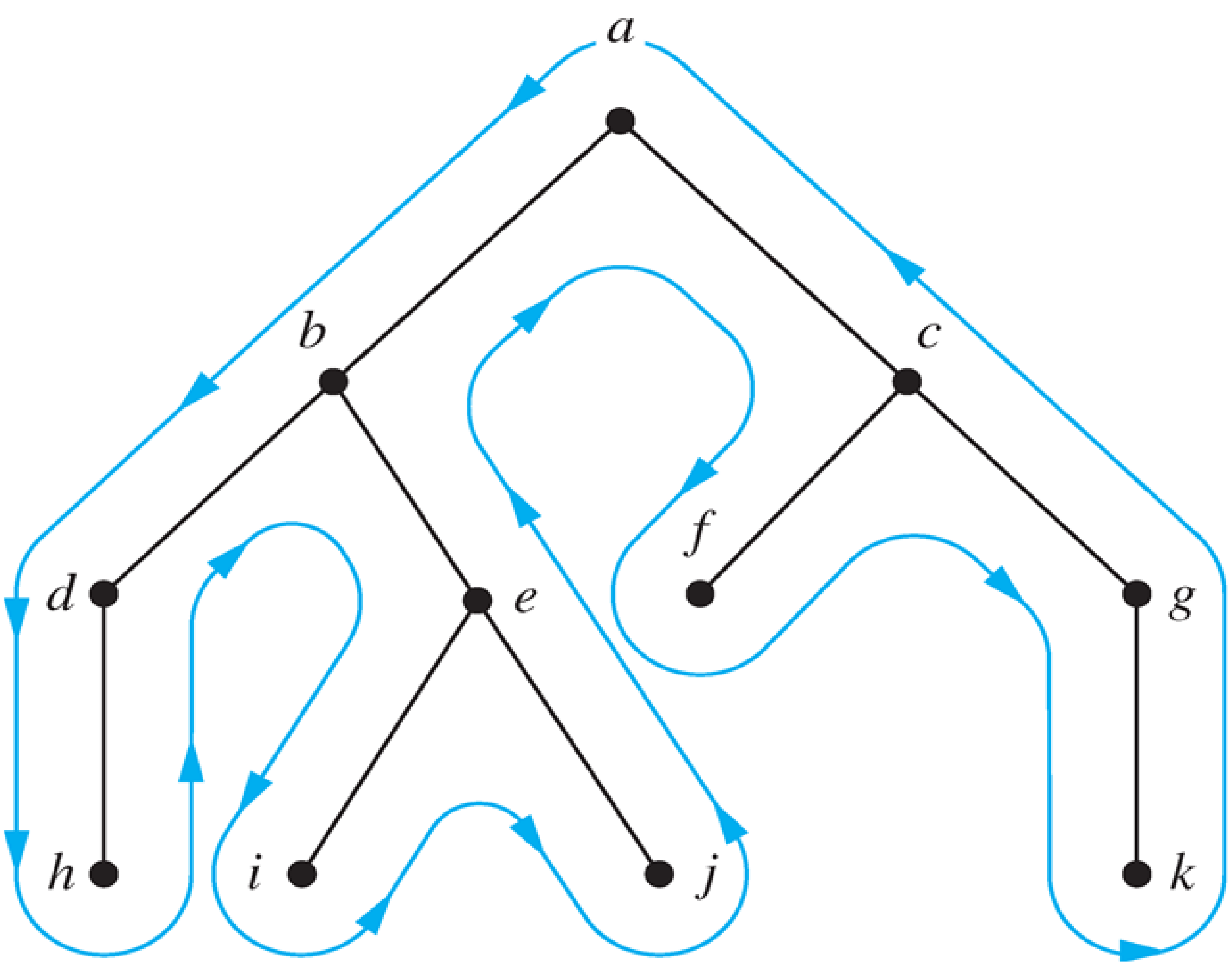


Postorder Traversal

procedure *postordered* (T: ordered rooted tree)
r := root of T
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    postorder($T(c)$)
list r

Preorder, Inorder, Postorder Traversal



Expression Trees

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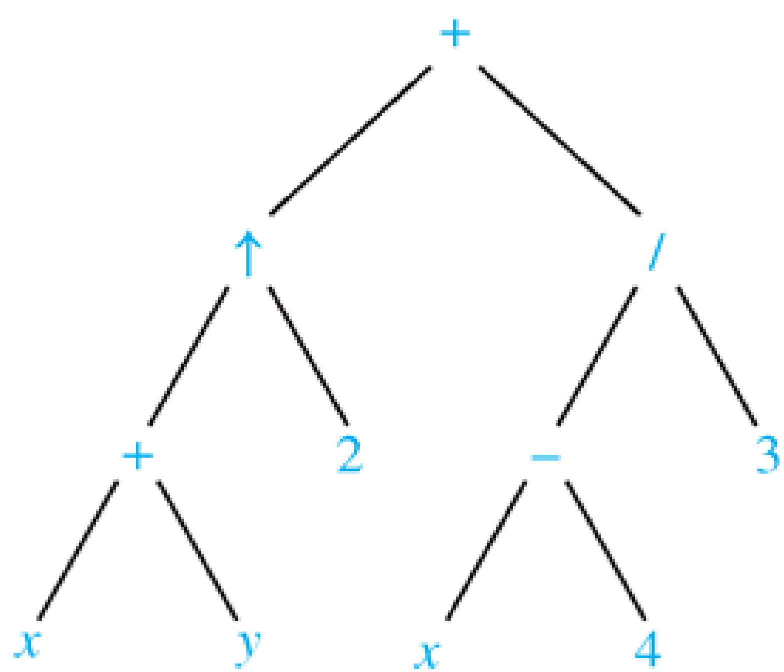

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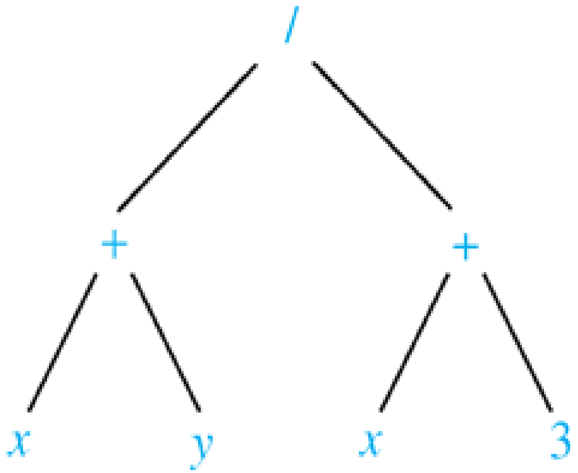

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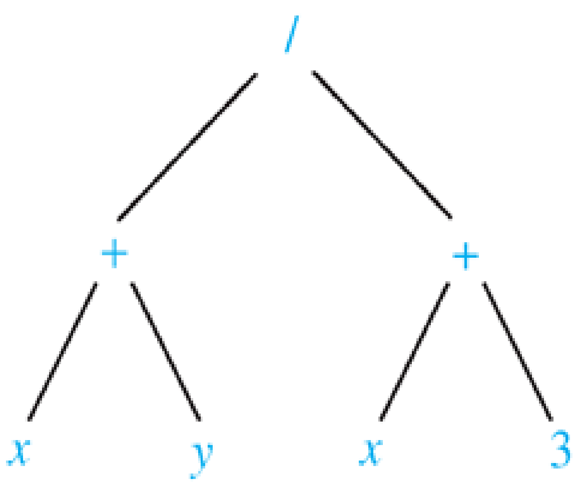

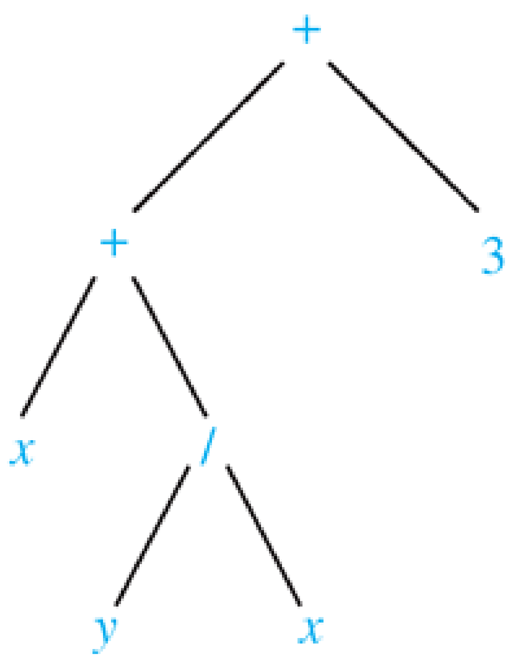

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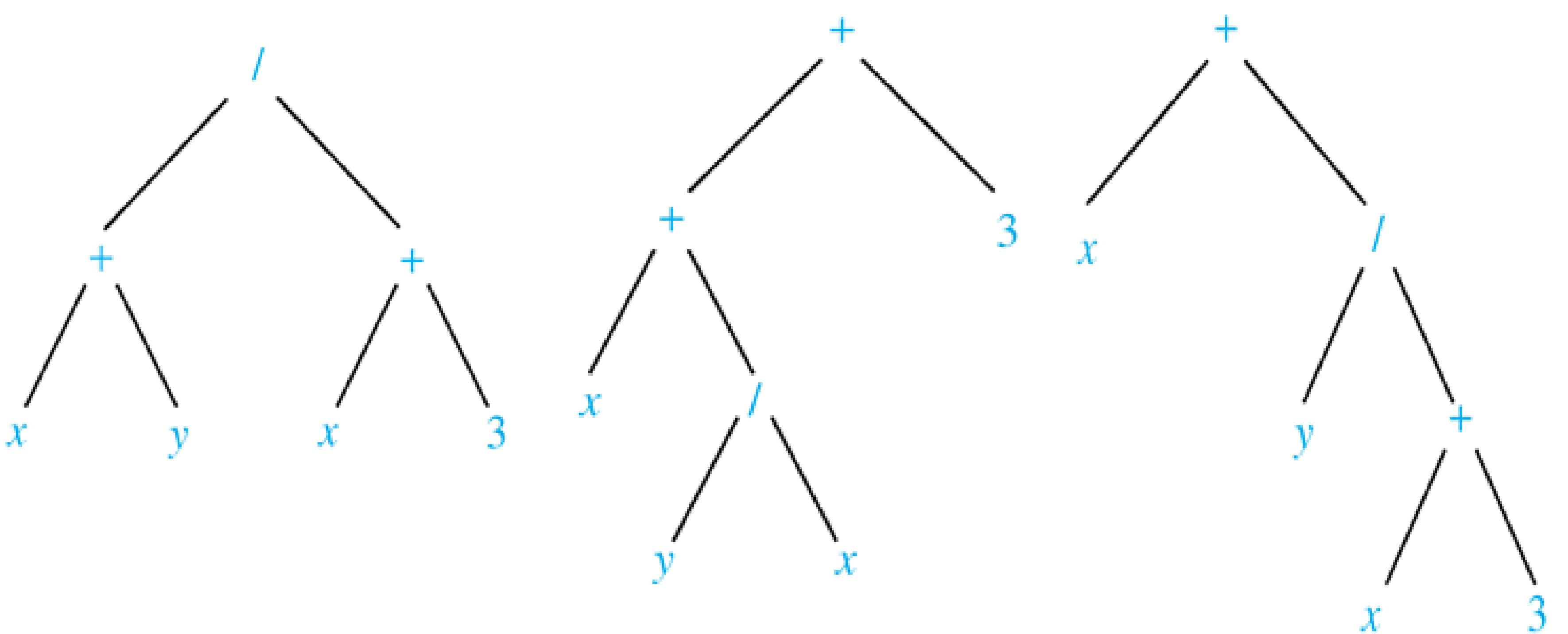

Prefix Notation

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  Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

Prefix Notation

- **Example**

  $+ \; - \; * \, 2 \; 3 \; 5 \; / \; \uparrow \; 2 \; 3 \; 4$

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$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8/4 = 2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

$$+ \quad \underbrace{- \quad 6 \quad 5}_{6 - 5 = 1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1 + 2 = 3}$$

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Postfix Notation

- **Example**

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$$\underbrace{7 \quad 6 \quad -}_{7 - 6 = 1} \quad 4 \quad \uparrow \quad 9 \quad 3 \quad / \quad +$$

$$\underbrace{1 \quad 4 \quad \uparrow}_{1^4 = 1} \quad 9 \quad 3 \quad / \quad +$$

$$1 \quad \underbrace{9 \quad 3 \quad /}_{9 / 3 = 3} \quad +$$

$$\underbrace{1 \quad 3 \quad +}_{1 + 3 = 4}$$

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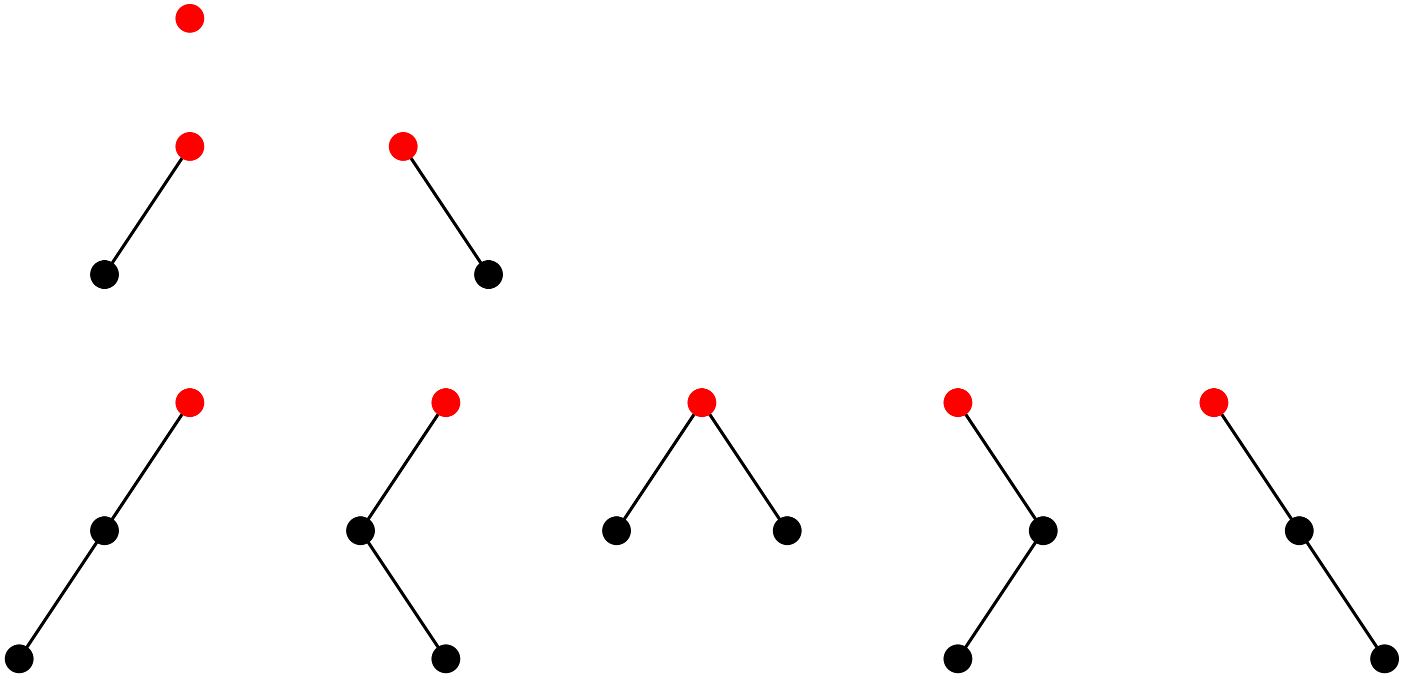

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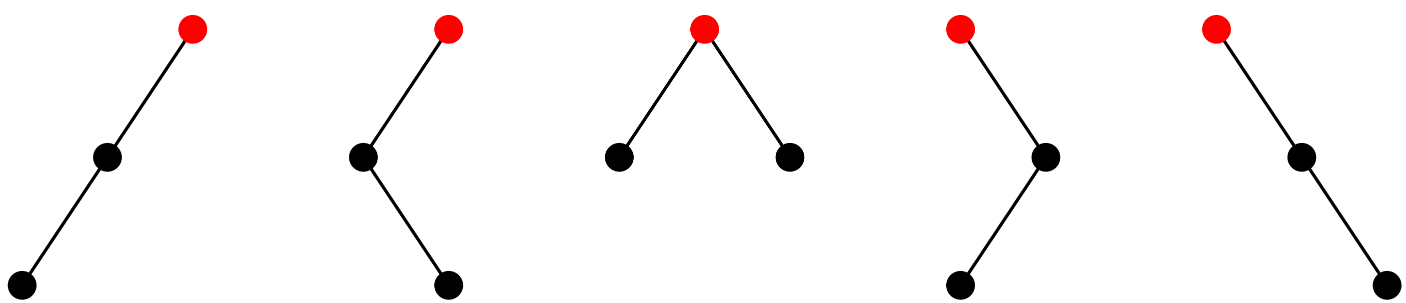

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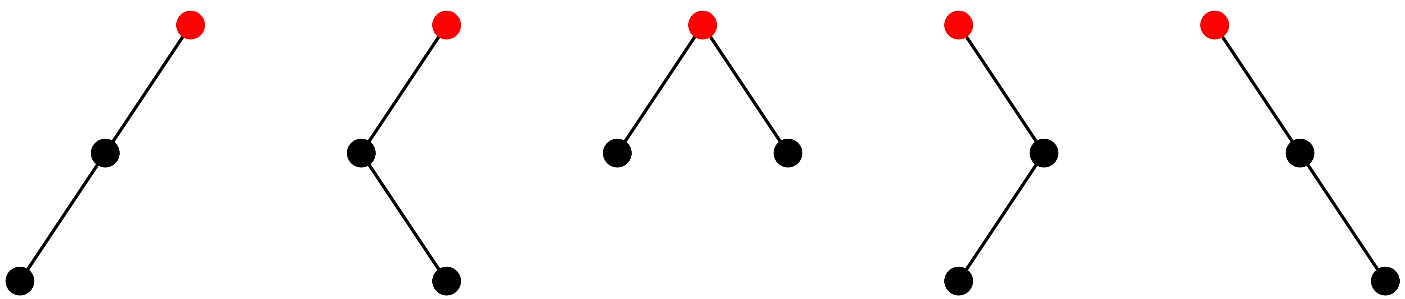

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How to find a formula for C_n?

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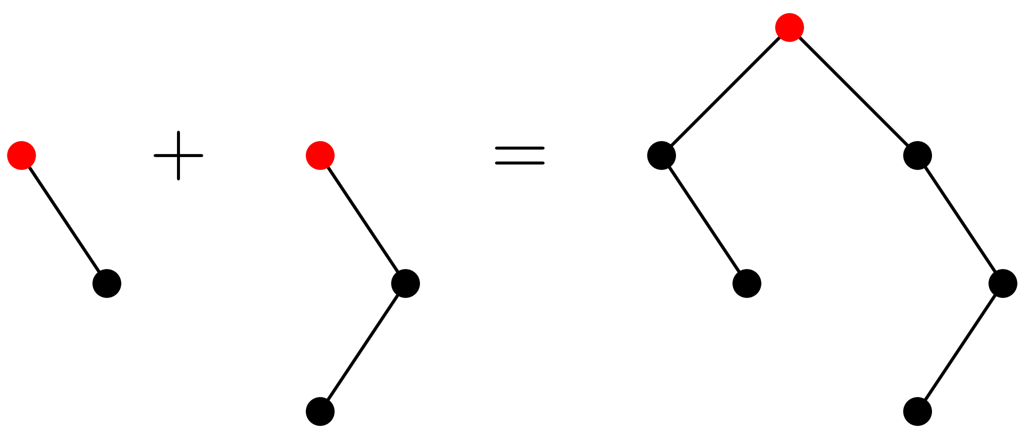

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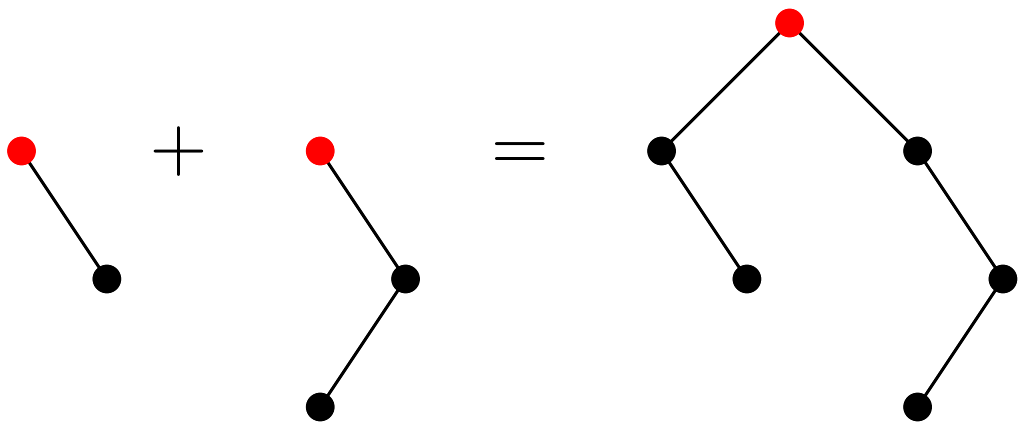

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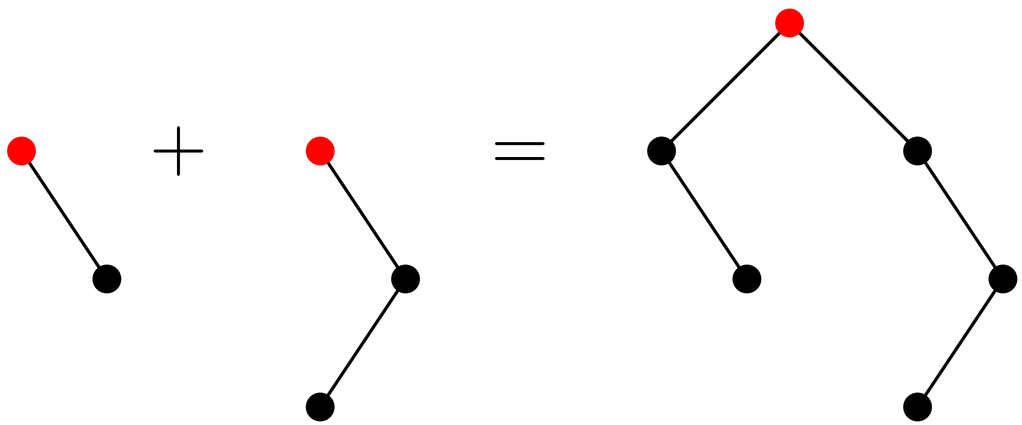

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For example, $C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 1 * 2 + 1 * 1 + 2 * 1 = 5.$

Catalan Numbers: Using Generating Functions

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  When $x \to 0$, $\frac{1+\sqrt{1-4x}}{2x} \to \infty$ and $\frac{1-\sqrt{1-4x}}{2x} \to 1$.

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  C_n – the coefficient of x^n in the expansion of f.

Catalan Numbers

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$$\sqrt{1-4x} = (1+(-4x))^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n.$$

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Then we have $C_n = \frac{1}{n+1}\binom{2n}{n}$.

This is called the *n*-th *Catalan number*.

Catalan Numbers: Related Problems

- **Theorem** The number of sequences $a_1, \ldots, a_{2n}$ of $2n$ terms that can be formed using exactly n $+1$'s and exactly n -1's whose partial sums are always nonnegative, i.e., $a_1 + a_2 + \cdots + a_k \geq 0$ for any $1 \leq k \leq 2n$, equals the n-th Catalan number C_n.

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R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015. Includes 214 combinatorial interpretations of C_n, and 68 additional problems!

Spanning Trees

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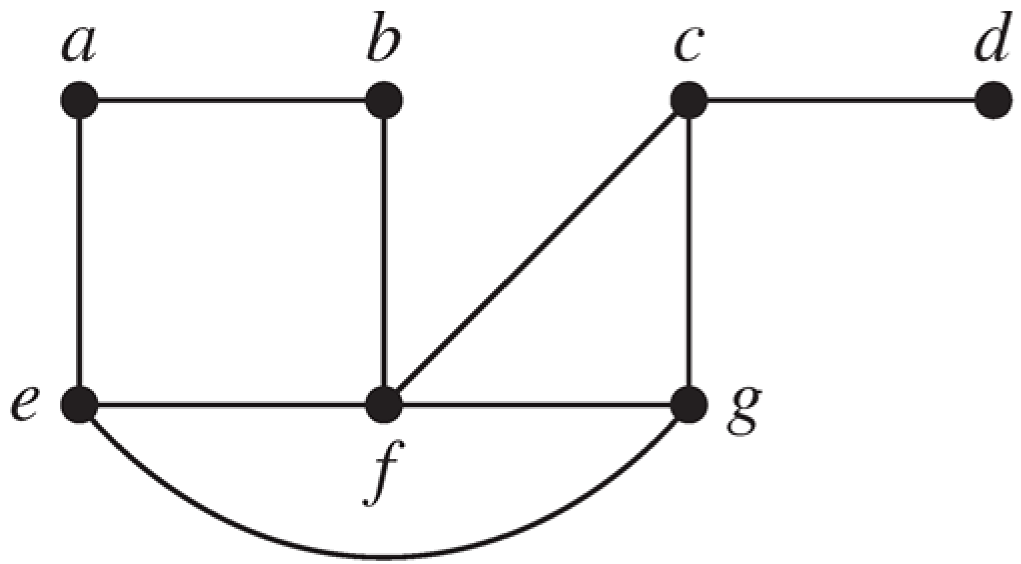

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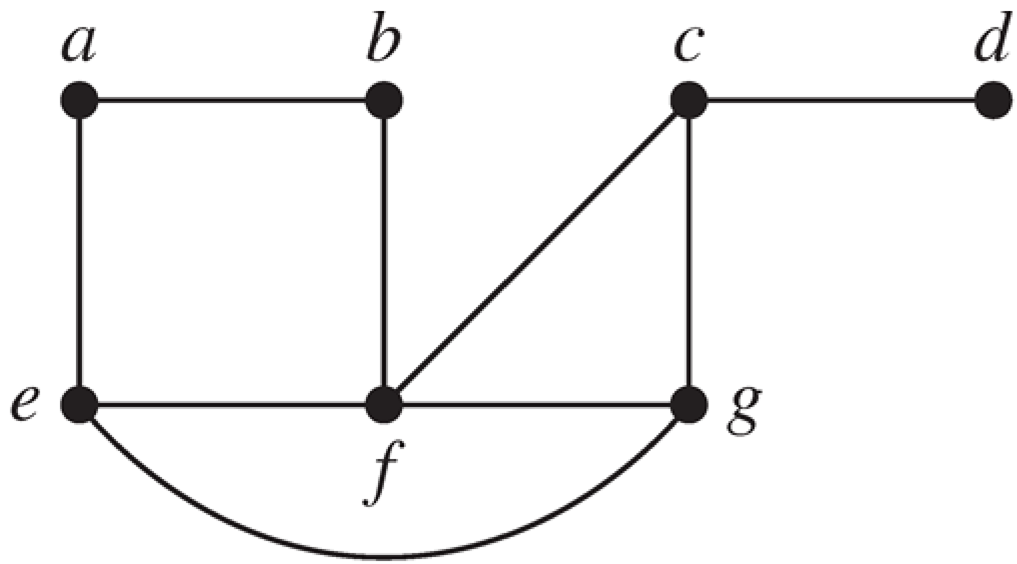

remove edges to avoid circuits

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  Continue adding to this path as long as possible.

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  ◇ Otherwise, move back to some vertex to repeat this procedure (*backtracking*)

Depth-First Search

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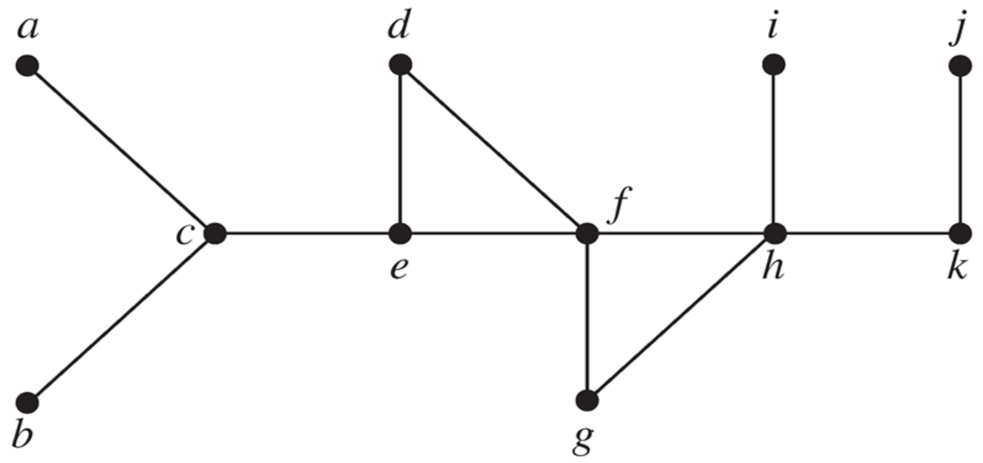

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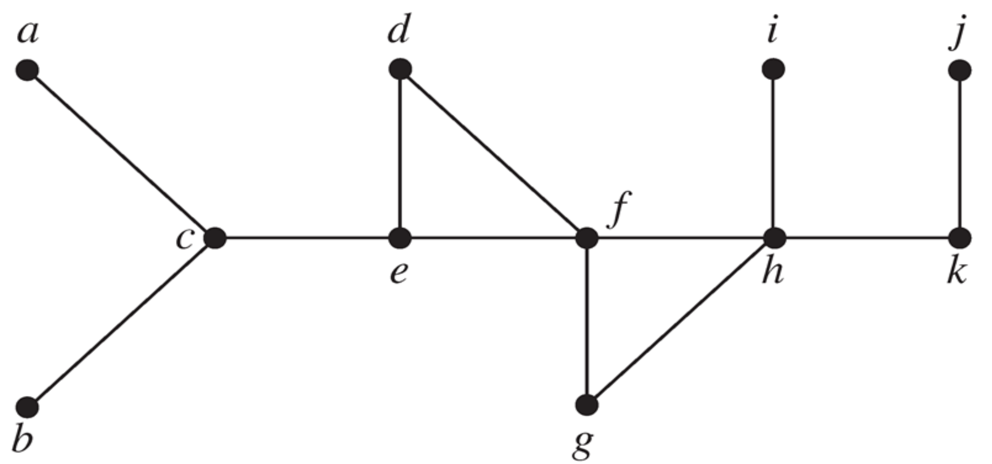

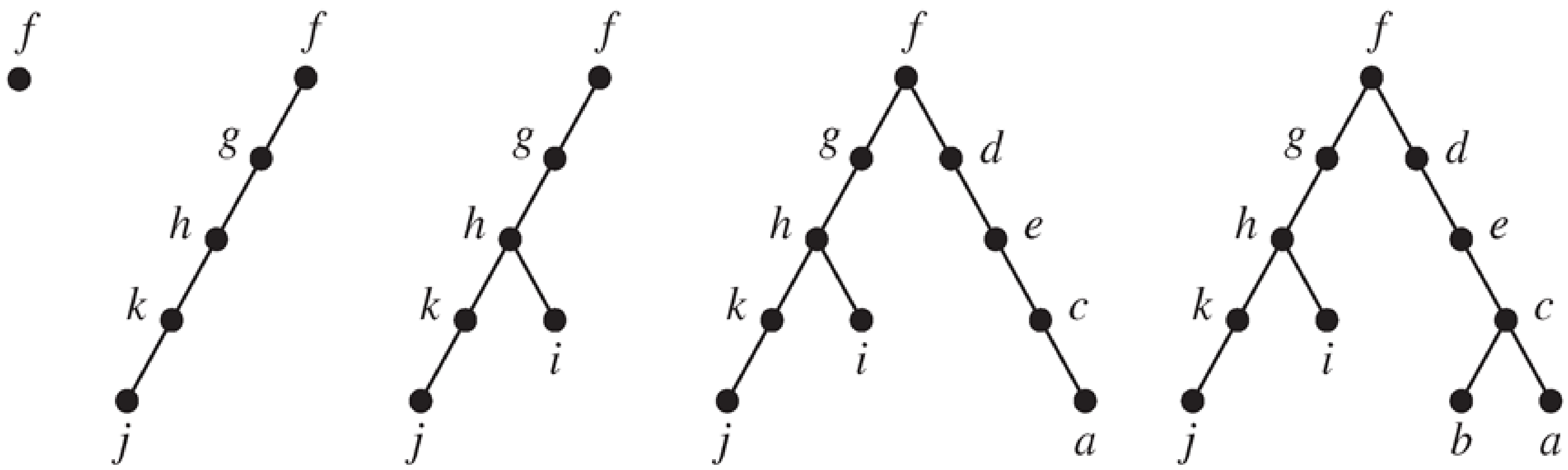

Depth-First Search Algorithm

procedure *DFS*(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of the vertex v_1
visit(v_1)

procedure *visit*(v: vertex of G)
for each vertex w adjacent to v and not yet in T
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time complexity: $O(e)$

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  - ◇ First arbitrarily choose a vertex of the graph as the root.
  - ◇ Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges
  - ◇ For each vertex added at the previous level, add edge incident to this vertex, as long as it does not produce a simple circuit.
  - ◇ Continue in this manner until all vertices have been added.

Breadth-First Search

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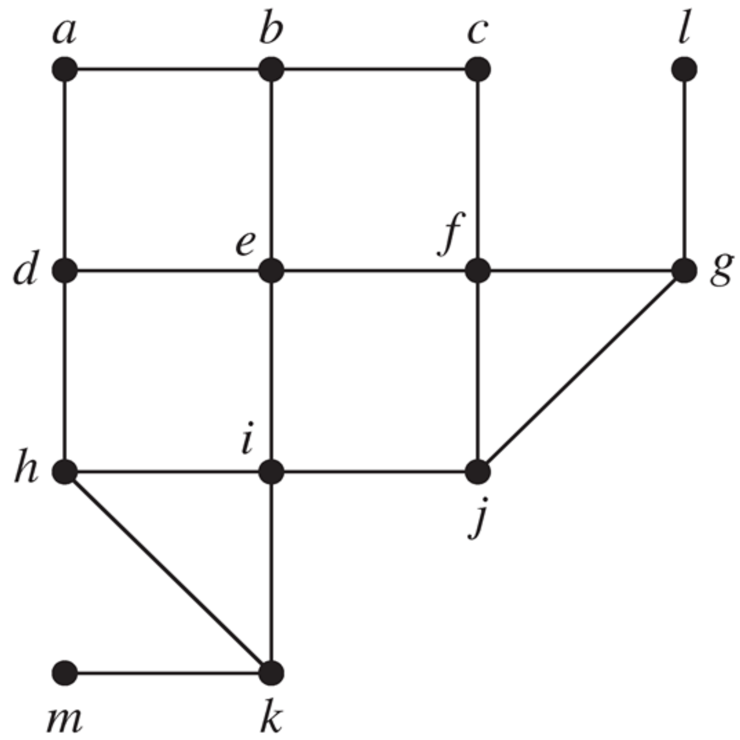

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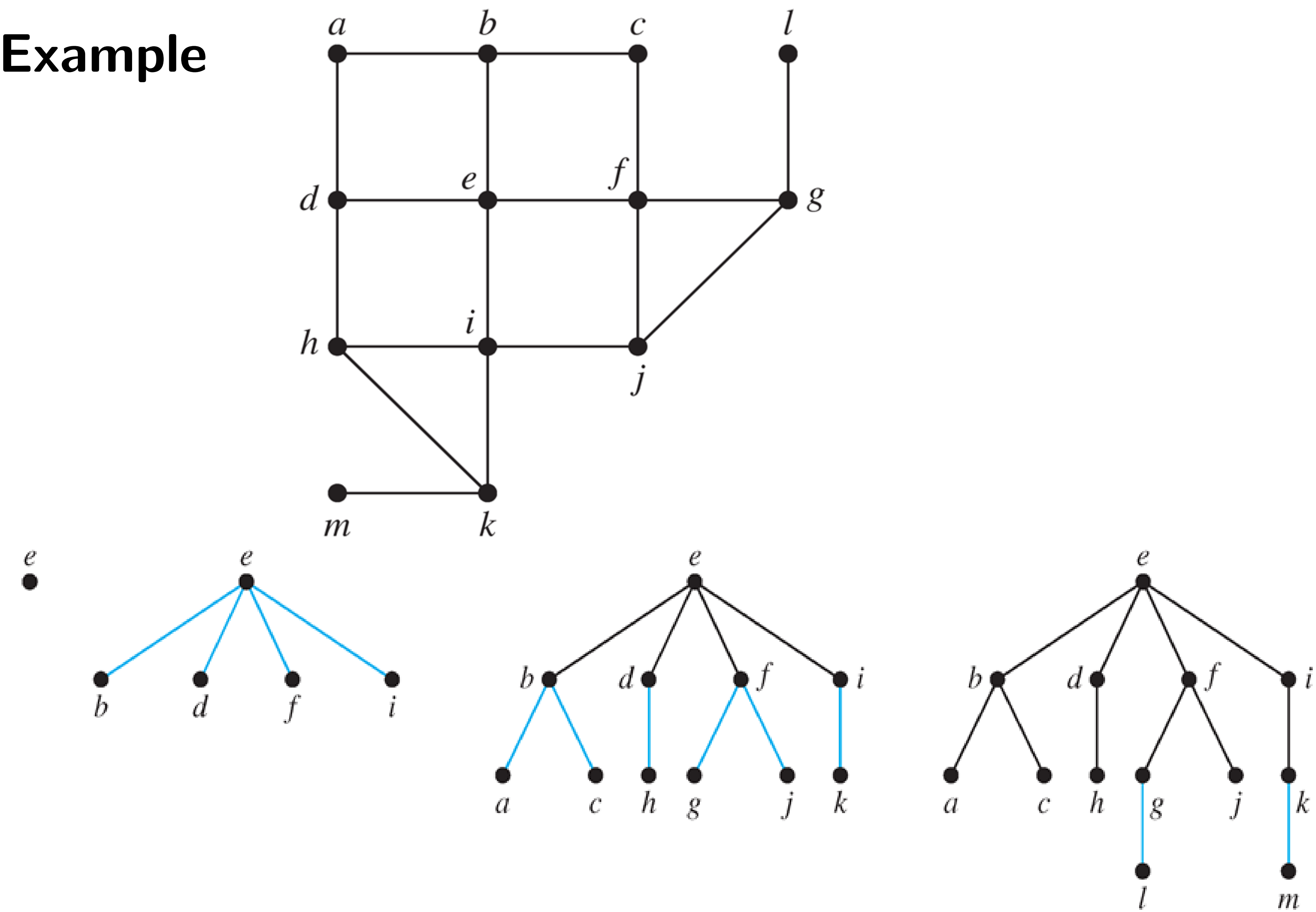

Breadth-First Search

procedure *BFS*(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of the vertex v_1
L := empty list *visit*(v_1)
put v_1 in the list L of unprocessed vertices
while L is not empty
    remove the first vertex, v, from L
    for each neighbor w of v
        if w is not in L and not in T **then**
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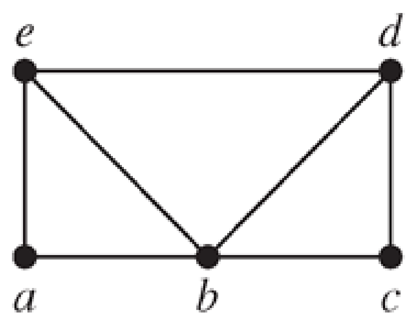
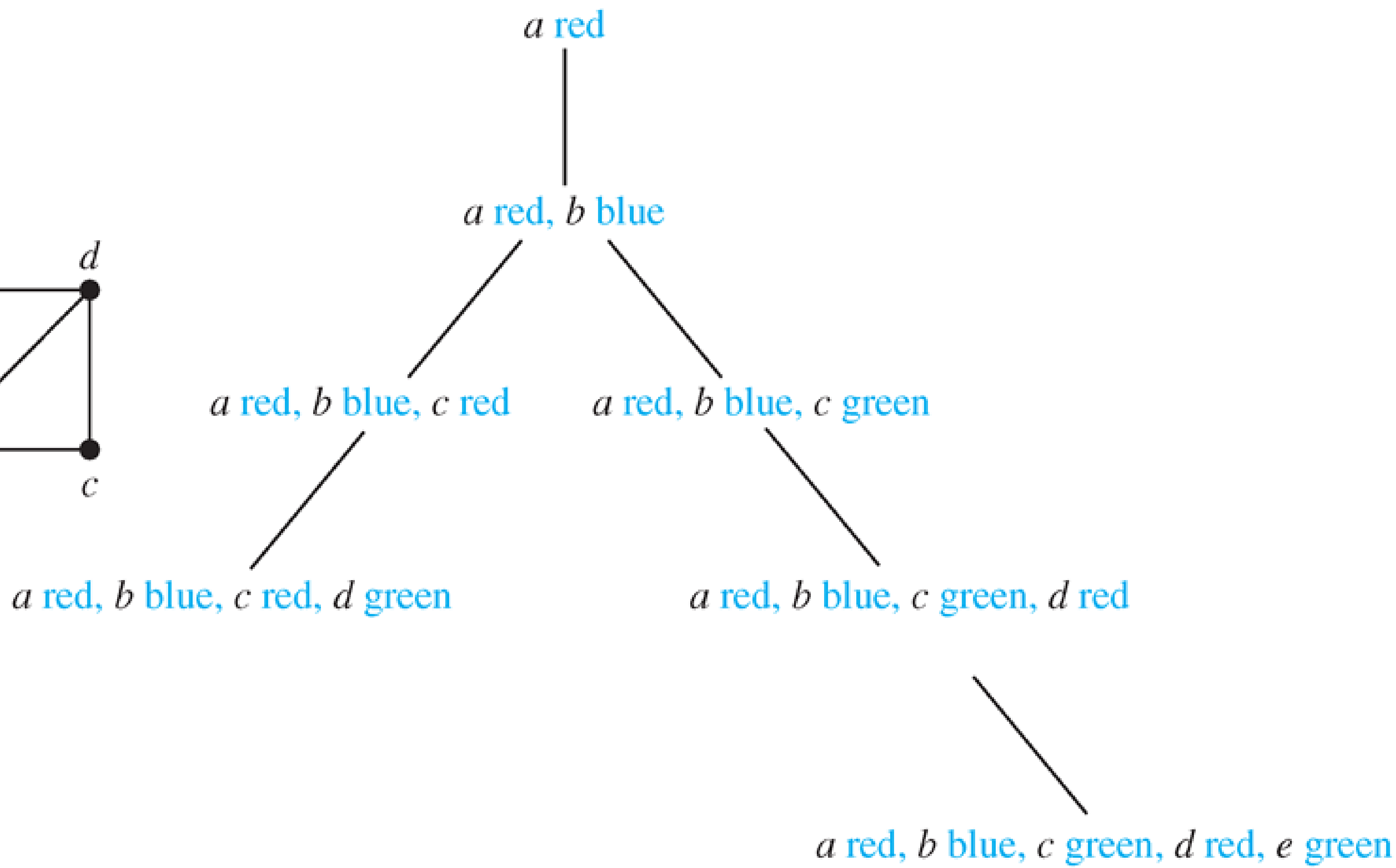
  graph coloring, sums of subsets, ...


e
d
a
b
c


a red
a red, b blue
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a red, b blue, c red, d green
a red, b blue, c green, d red
a red, b blue, c green, d red, e green

Applications of DFS, BFS

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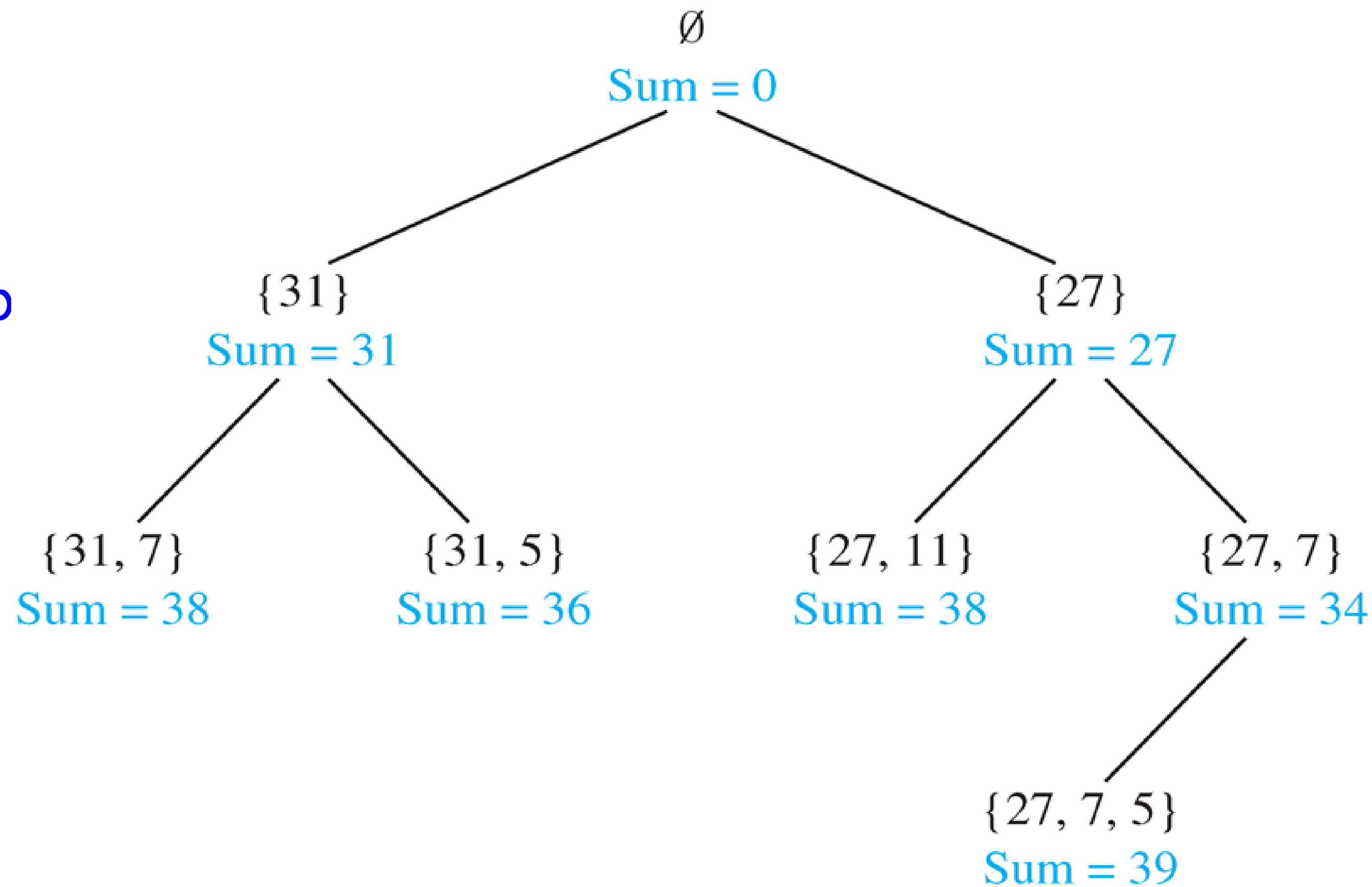
  find

  grap



find a subset of $\{31, 27, 15, 11, 7, 5\}$ with the sum 39

Minimum Spanning Trees

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  two greedy algorithms:
  Prim's Algorithm, Kruscal's Algorithm

Prim's Algorithm

ALGORITHM 1 Prim's Algorithm.

procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
for $i := 1$ **to** $n - 2$
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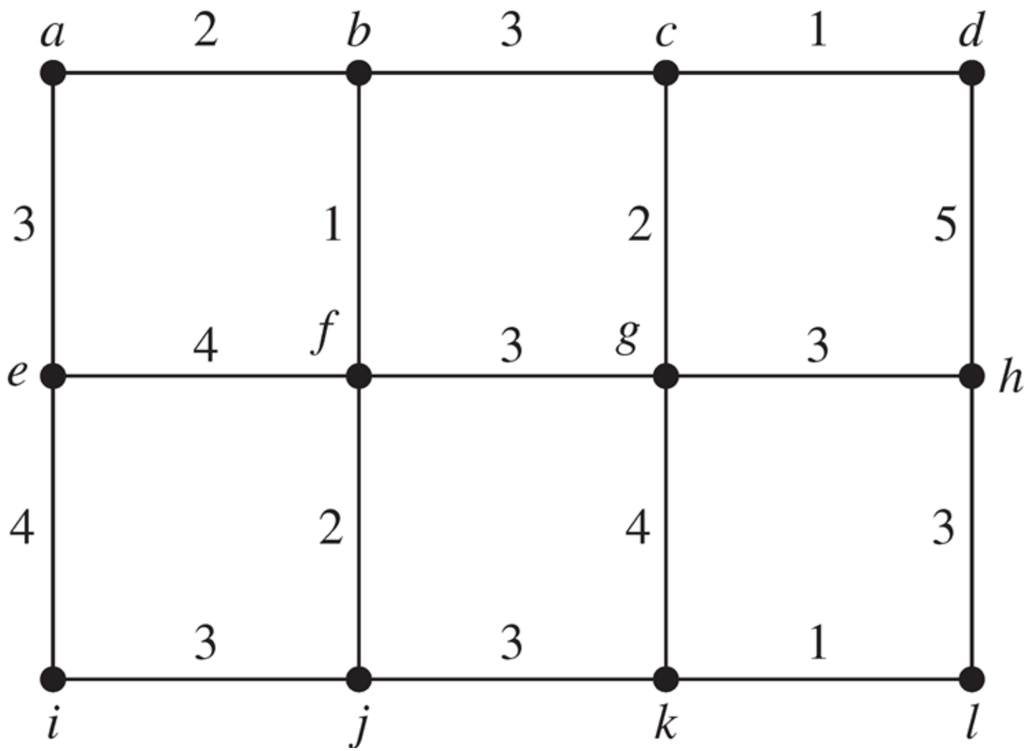
We can maintain a *heap* of all the edges with at least one endpoint in T, and in each iteration, we do Extract-Mins until we see an edge that has one endpoint in T and one endpoint not in T.

time complexity: $e \log v$

- **Example**



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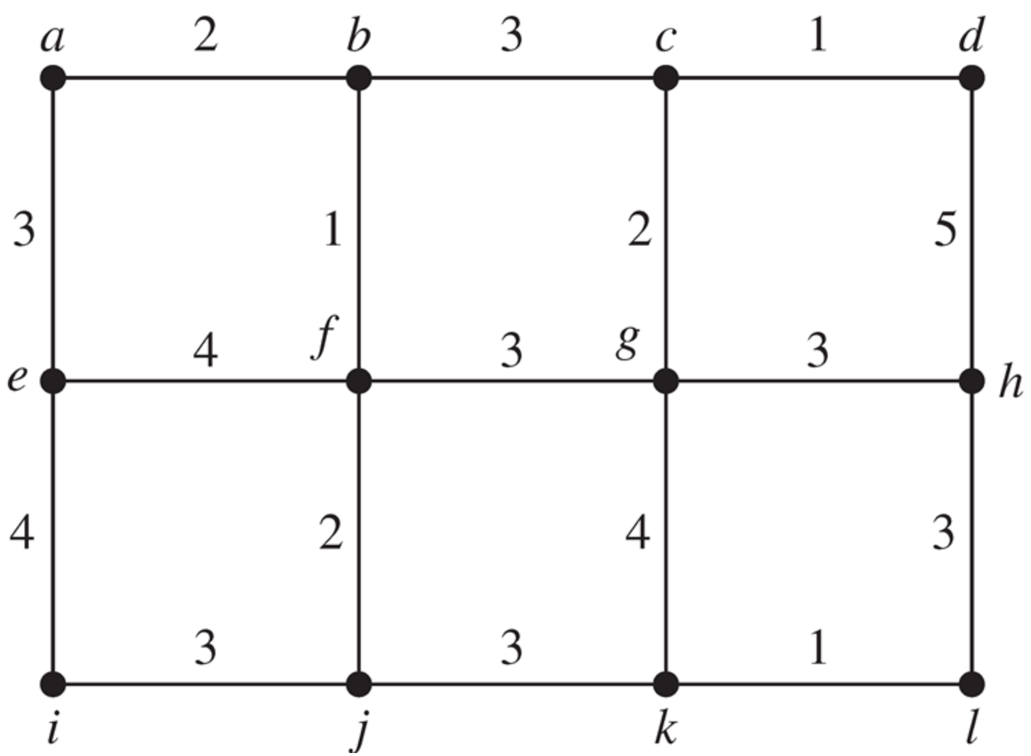

Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1
	Total:	24

Prim's Algorithm: Correctness

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  Since Prim's algorithm has chosen to add e, we have $w(e) \leq w(e')$. So if we add e to M and remove e' from M, we will have a new tree M' whose total weight $\leq$ that of M, and $T \cup \{e\} \subset M'$.

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see *CLRS* / *Algorithm Design*, J. Kleinberg, E. Tardos

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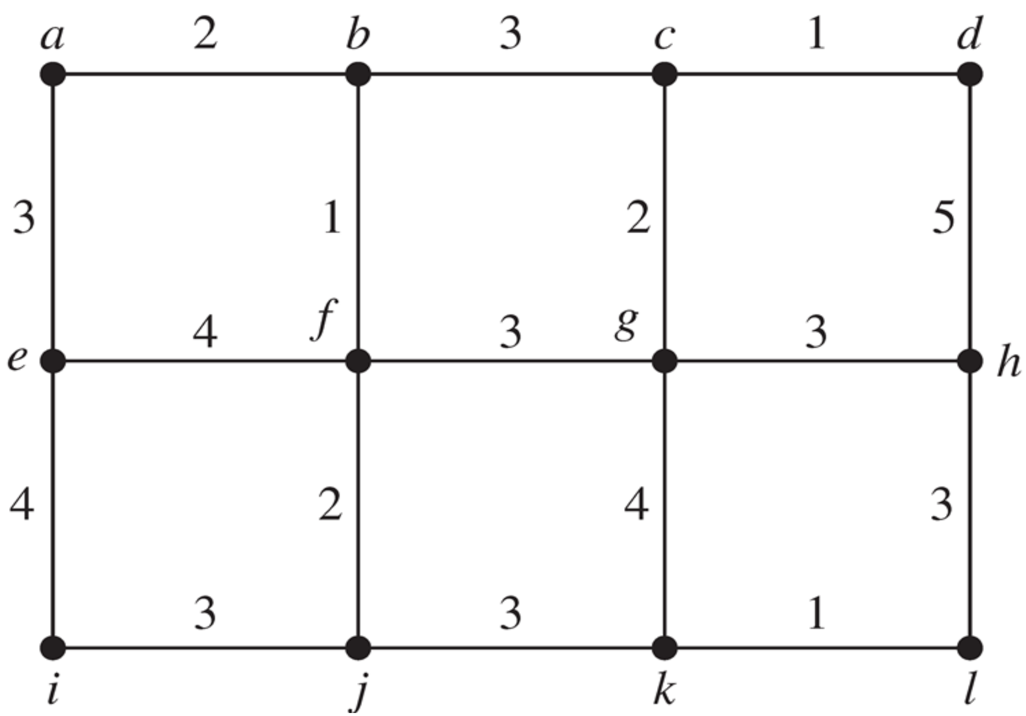

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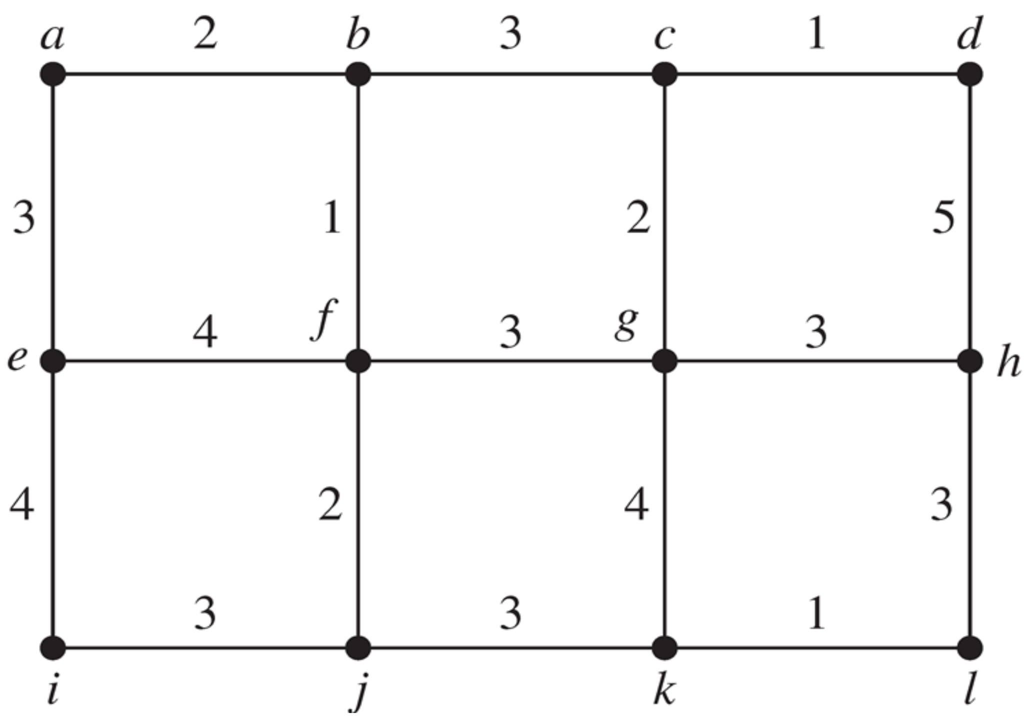

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  Theorem Let $(S, \bar{S})$ be an arbitrary cut, and let e be an edge across the cut (one endpoint in S, the other in $\bar{S}$) that has the smallest weight of all edges cross the cut. Then there must be an MST T containing e.

  Theorem Let $(S, \bar{S})$ be an arbitrary cut, and let E' be the set of edges across the cut of minimum weight ($w(e) = w(e')$ for any two edges $e, e' \in E'$ and $w(e) < w(e')$ for any $e \in E'$ and $e' \notin E'$). Let T be an arbitrary MST. Then T must contain some edge in E'.

Next Lecture

- reduction, review ...