



CS215 DISCRETE MATH

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Linear Congruences

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Systems of linear congruences have been studied since ancient times.

今有物不知其数 三三数之剩二 五五数之剩三 七七数之剩二 问物几何

About 1500 years ago, the Chinese mathematician Sun-Tsu asked: “There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?”

Modular Inverse

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One method of solving linear congruences makes use of an inverse \bar{a} if it exists. From $ax \equiv b \pmod{m}$, it follows that $\bar{a}ax \equiv \bar{a}b \pmod{m}$ and then $x \equiv \bar{a}b \pmod{m}$.

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When does an inverse of a modulo m exist?

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Proof. Since $\gcd(a, m) = 1$, there are integers s and t such that $sa + tm = 1$. Hence $sa + tm \equiv 1 \pmod{m}$. Since $tm \equiv 0 \pmod{m}$, it follows that $sa \equiv 1 \pmod{m}$. This means that s is an inverse of a modulo m .

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How to prove the uniqueness of the inverse?

How to find inverses?

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Example. Find an inverse of 101 modulo 4620.

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Example. Find an inverse of 101 modulo 4620.

$$4620 = 45 \cdot 101 + 75$$

$$101 = 1 \cdot 75 + 26$$

$$75 = 2 \cdot 26 + 23$$

$$26 = 1 \cdot 23 + 3$$

$$23 = 7 \cdot 3 + 2$$

$$3 = 1 \cdot 2 + 1$$

$$2 = 2 \cdot 1$$

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$$75 = 2 \cdot 26 + 23$$

$$26 = 1 \cdot 23 + 3$$

$$23 = 7 \cdot 3 + 2$$

$$3 = 1 \cdot 2 + 1$$

$$2 = 2 \cdot 1$$

$$1 = 3 - 1 \cdot 2$$

$$1 = 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$$

$$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$$

$$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = 26 \cdot 26 - 9 \cdot 75$$

$$1 = 26 \cdot (101 - 1 \cdot 75) - 9 \cdot 75$$

$$= 26 \cdot 101 - 35 \cdot 75$$

$$1 = 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$$

$$= -35 \cdot 4620 + 1601 \cdot 101$$

Using Inverses to Solve Congruences

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Solution: We found that -2 is an inverse of 3 modulo 7 . Multiply both sides of the congruence by -2 , we have $x \equiv -8 \equiv 6 \pmod{7}$.

Number of Solutions to Congruences *

- **Theorem*** Let $d = \gcd(a, m)$ and $m' = m/d$. The congruence $ax \equiv b \pmod{m}$ has solutions if and only if $d|b$. If $d|b$, then there are exactly d solutions. If x_0 is a solution, then the other solutions are given by $x_0 + m', x_0 + 2m', \dots, x_0 + (d - 1)m'$.

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Proof.

- 1) “only if”: If x_0 is a solution, then $ax_0 - b = km$. Thus, $ax_0 - km = b$. Since d divides $ax_0 - km$, we must have $d|b$.
- 2) “if”: Suppose that $d|b$. Let $b = kd$. There exist integers s, t such that $d = as + mt$. Multiply both sides by k . Then $b = ask + mtk$. Let $x_0 = sk$. Then $ax_0 \equiv b \pmod{m}$.
- 3) “ $\# = d$ ”: $ax_0 \equiv b \pmod{m}$ $ax_1 \equiv b \pmod{m}$ imply that $m|a(x_1 - x_0)$ and $m'|a'(x_1 - x_0)$. This implies further that $x_1 = x_0 + km'$, where $k = 0, 1, \dots, d - 1$.

The Chinese Remainder Theorem

- About 1500 years ago, the Chinese mathematician Sun-Tsu asked:
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The Chinese Remainder Theorem

- **Theorem** (*The Chinese Remainder Theorem*) Let m_1, m_2, \dots, m_n be pairwise relatively prime positive integers greater than 1 and a_1, a_2, \dots, a_n arbitrary integers. Then the system

$$x \equiv a_1 \pmod{m_1}$$

$$x \equiv a_2 \pmod{m_2}$$

...

$$x \equiv a_n \pmod{m_n}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \dots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k , an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_n M_n y_n.$$

It is checked that x is a solution to the n congruences.

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How to prove the **uniqueness** of the solution modulo m ?

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Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$,
 $M_3 = m/7 = 15$.

$$35 \cdot 2 \equiv 1 \pmod{3}$$

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七子团圆正月半，除百零五便得知。
-- 程大位《算法统要》(1593年)

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Fermat's Little Theorem

- **Theorem (Fermat's little theorem)** : Let p be a prime, and let x be an integer such that $x \not\equiv 0 \pmod{p}$. Then

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$$\{1, 2, \dots, p-1\} = \{x, 2x, \dots, x(p-1) \pmod{p}\}$$

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Theorem * There is a primitive root modulo n if and only if $n = 2, 4, p^e$ or $2p^e$, where p is an odd prime.

Q : proof? The number of primitive roots? *

Number Theory and Cryptography

- Division, Primes
- Congruence
- Greatest Common Divisor (GCD)
- Euler's Theorem / Fermat's Little Theorem

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$$a = \color{blue}{d}q + r$$

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$$a = d q + r \quad q = a \text{ div } d \quad r = a \text{ mod } d$$

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■ Greatest Common Divisor (GCD)

Find the GCD of 286 and 503.

$\gcd(503, 286)$	$503 = 1 \cdot 286 + 217$	$1 = 10 - 1 \cdot 9$
$= \gcd(286, 217)$	$286 = 1 \cdot 217 + 69$	$1 = 7 \cdot 10 - 1 \cdot 69$
$= \gcd(217, 69)$	$217 = 3 \cdot 69 + 10$	$1 = 7 \cdot 217 - 22 \cdot 69$
$= \gcd(69, 10)$	$69 = 6 \cdot 10 + 9$	$1 = 29 \cdot 217 - 22 \cdot 286$
$= \gcd(10, 9)$	$10 = 1 \cdot 9 + 1$	$1 = 29 \cdot 503 - 51 \cdot 286$
$= 1$	$9 = 9 \cdot 1$	

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find the modular inverse

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Number Theory Summary

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$$x^{\phi(n)} \equiv 1 \pmod{n} \text{ if } \gcd(x, n) = 1$$

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Modular Arithmetic in CS

- Modular arithmetic and congruencies are used in CS:
 - ◊ Pseudorandom number generators
 - ◊ Hash functions
 - ◊ Cryptography

Pseudorandom Number Generators

■ *Linear congruential method*

We choose four numbers:

- ◊ the modulus m
- ◊ multiplier a
- ◊ increment c
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We generate a sequence of numbers $x_1, x_2, \dots, x_n, \dots$ with $0 \leq x_i < m$ by using the congruence

$$x_{n+1} = (ax_n + c) \pmod{m}$$

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Pseudorandom Number Generators

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Example:

- Assume : $m=9, a=7, c=4, x_0 = 3$
- $x_1 = 7*3+4 \pmod{9} = 25 \pmod{9} = 7$
- $x_2 = 53 \pmod{9} = 8$
- $x_3 = 60 \pmod{9} = 6$
- $x_4 = 46 \pmod{9} = 1$
- $x_5 = 11 \pmod{9} = 2$
- $x_6 = 18 \pmod{9} = 0$
-

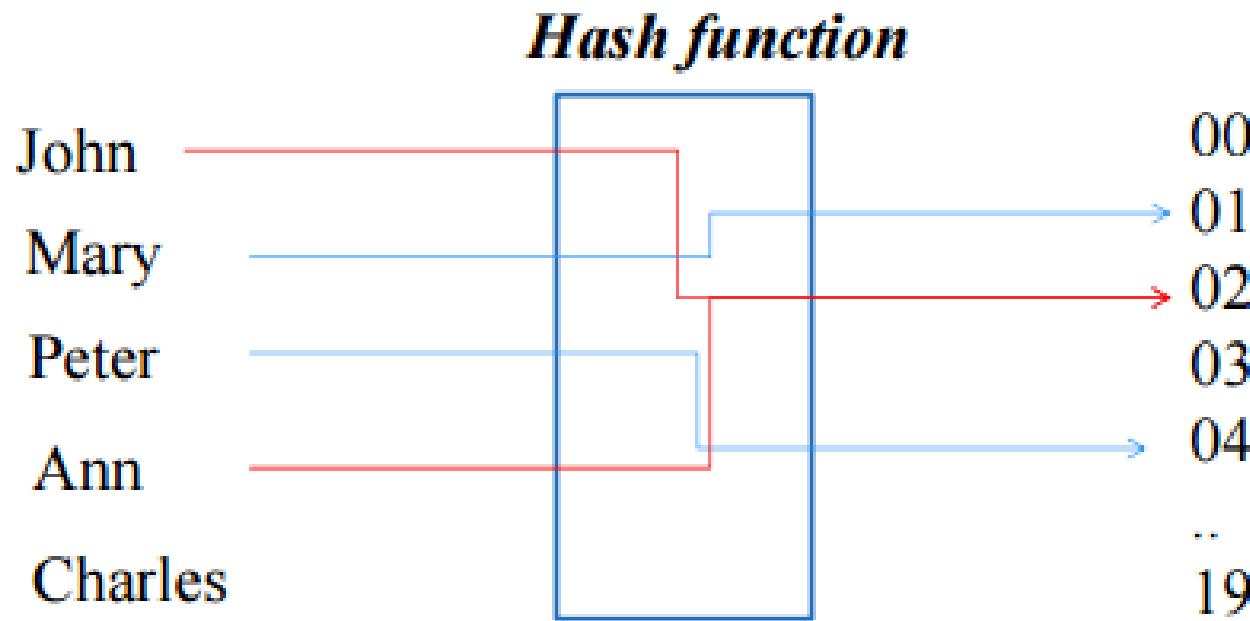
Hash Functions

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Example:



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- **Problem:** Given a large collection of records, how can we store and find a record quickly?

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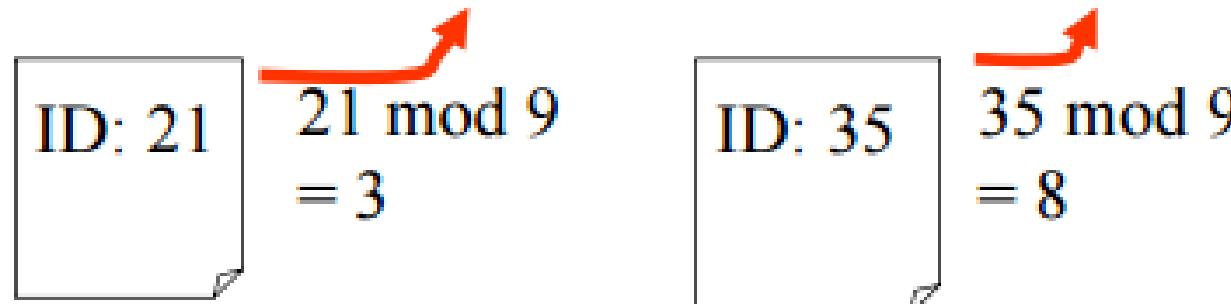
Solution: Use a hash function, calculate the location of the record based on the record's ID.

Example: A common hash function is

- $h(k) = k \bmod n$,

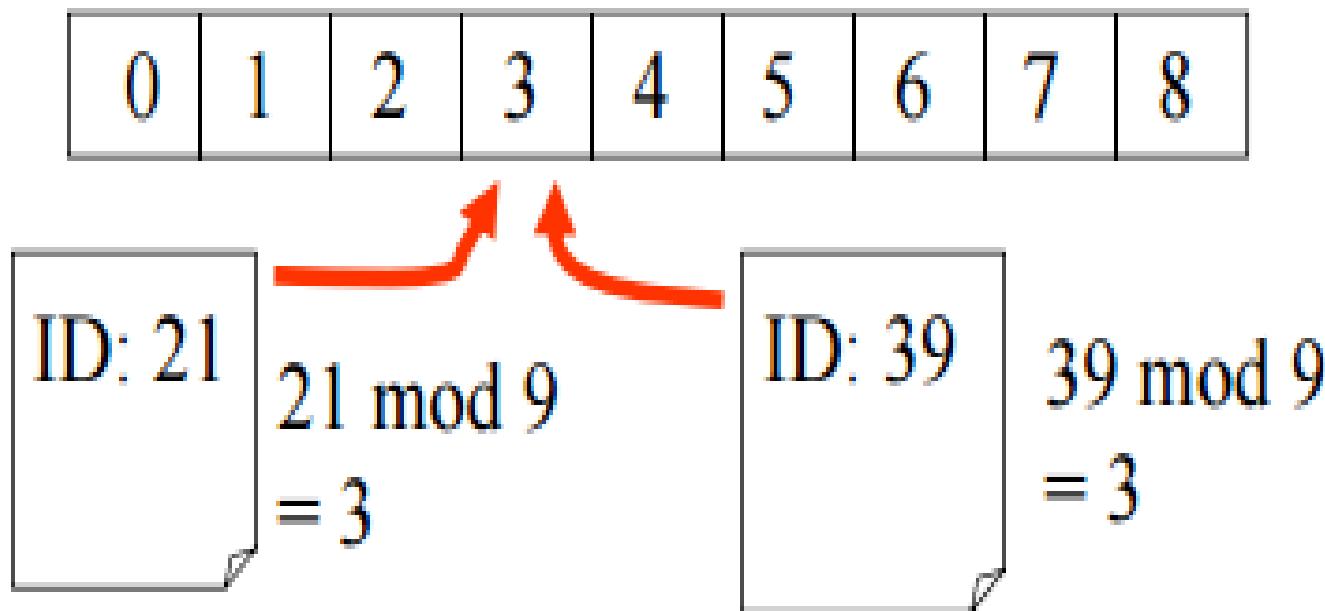
where n is the number of available storage locations.

0	1	2	3	4	5	6	7	8
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Hash Functions

- Two records mapped to the same location



Hash Functions

- Solution 1: move to the next available location

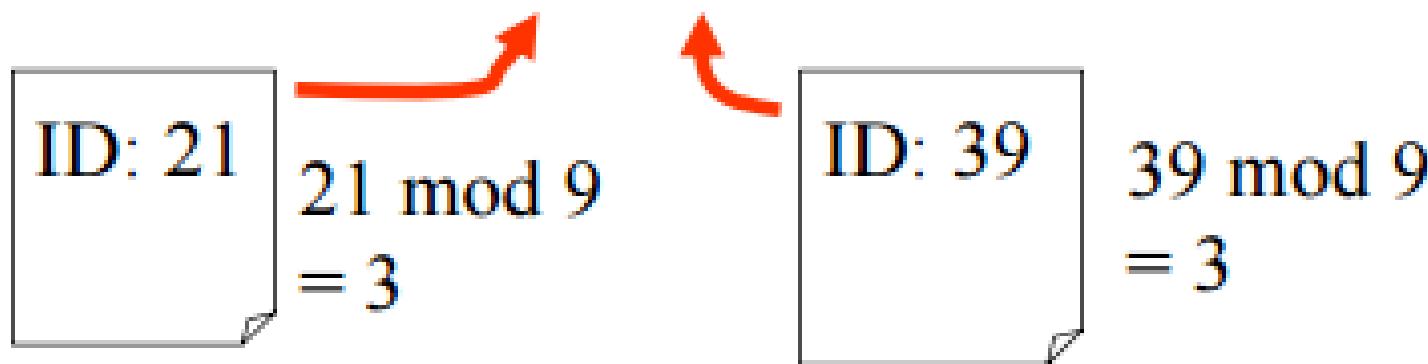
try

$$h_0(k) = k \bmod n$$

$$h_1(k) = (k+1) \bmod n$$

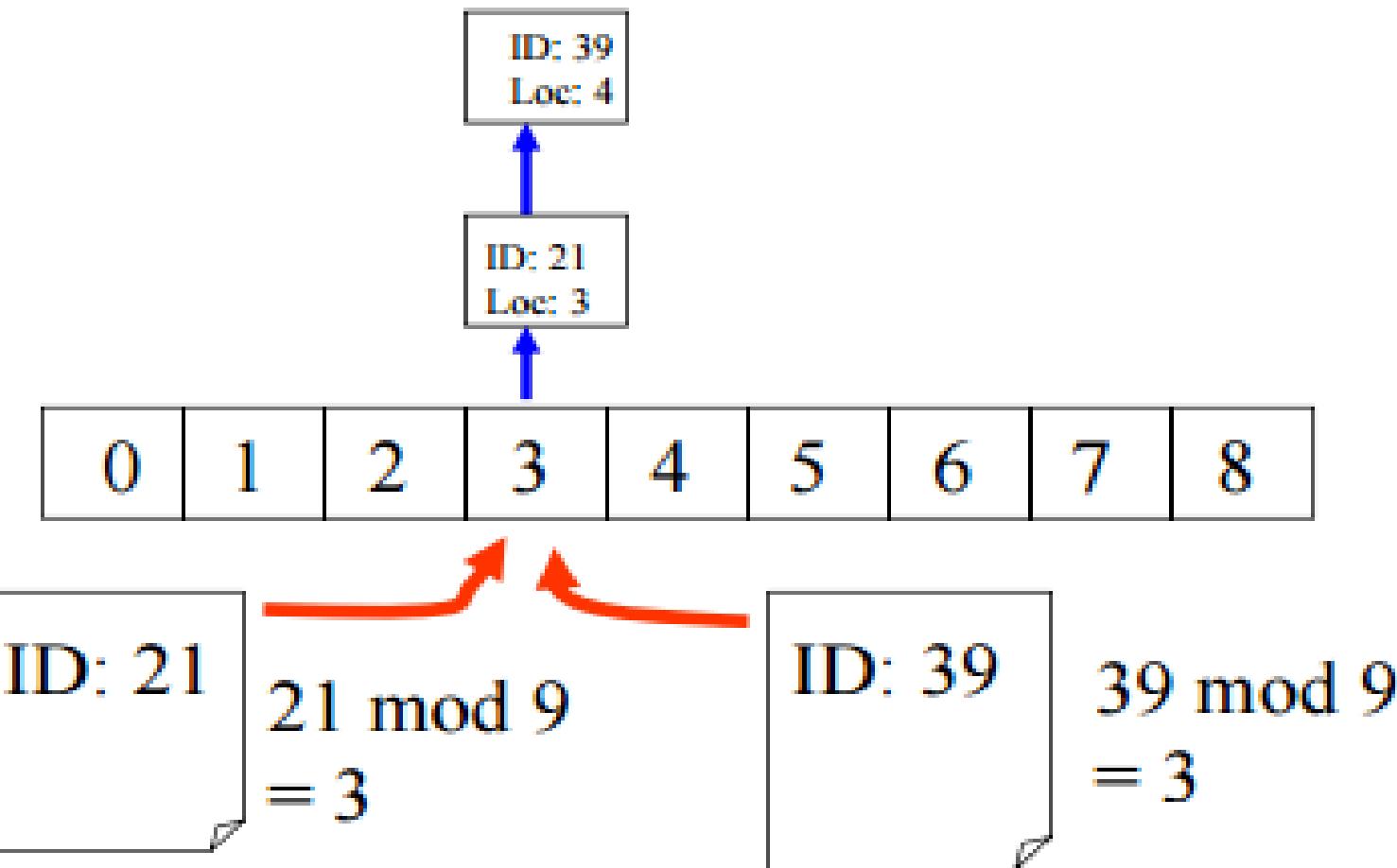
...

$$h_m(k) = (k+m) \bmod n$$



Hash Functions

- **Solution 2:** remember the exact location in a secondary structure that is searched sequentially



Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Cryptocurrency, e.g., bitcoin

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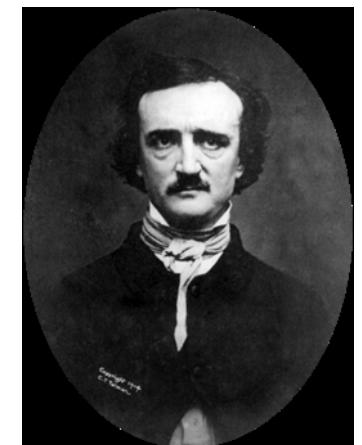
(secret) (writing)

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The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).



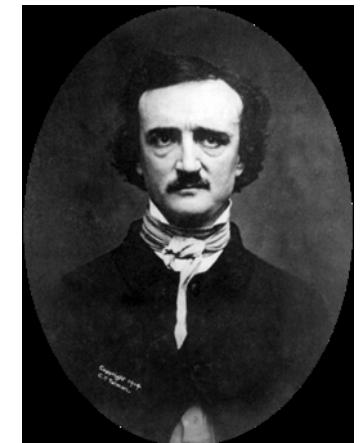
Cryptography

- History of almost 4000 years (from 1900 B.C.)

Cryptography = kryptos + graphos
(secret) (writing)

The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).

“Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve.” – 1941



Cryptography

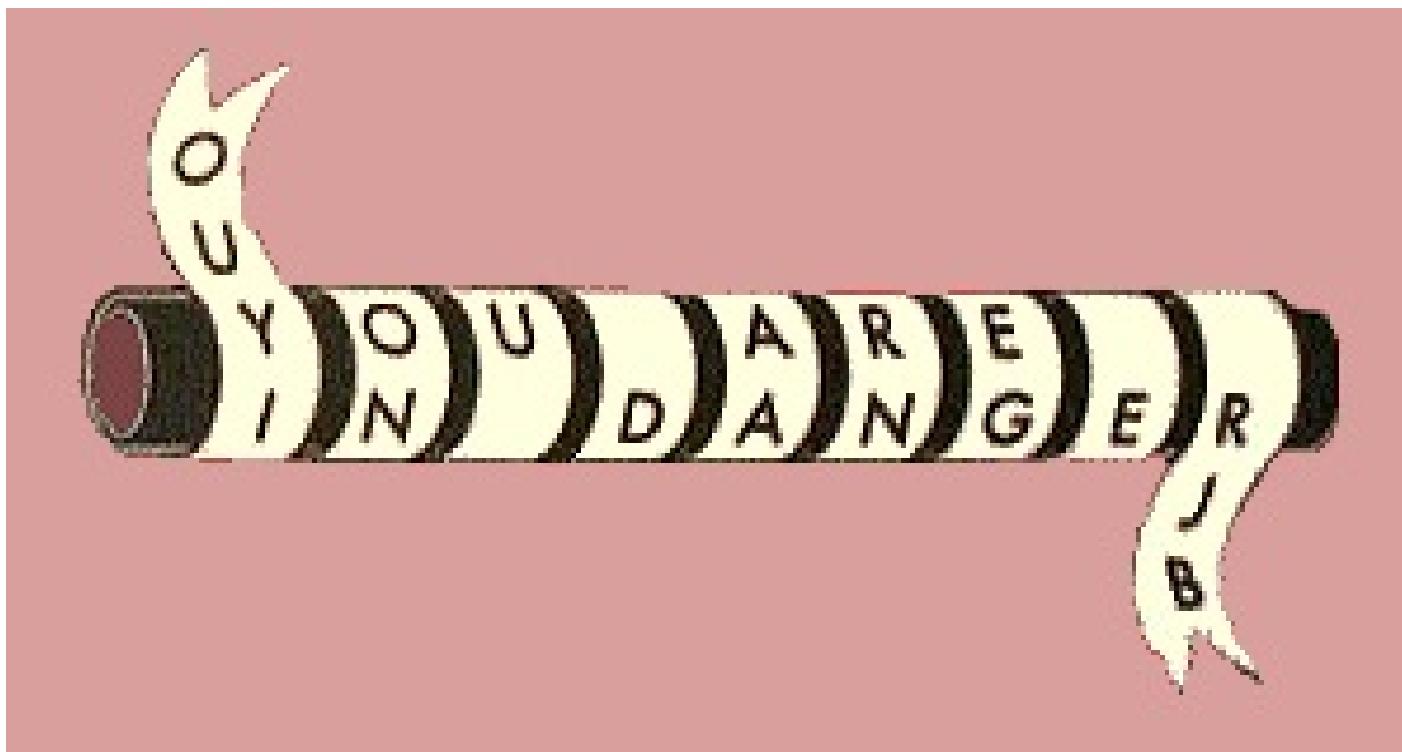
- One-sentence definition:

“Cryptography is the practice and study of techniques for secure communication in the presence of third parties called *adversaries*.” – Ronald L. Rivest



Some Examples

- In 405 BC, the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



Some Examples

- The Greeks also invented a cipher which changed **letters** to **numbers**. A form of this code was still being used during *World War I*.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Some Examples

- Caesar Cipher (after the name of JULIUS CAESAR)



VENI, VIDI, VICI

YHQL YLGL YLFL

Some Examples

- Morse Code: created by Samuel Morse in 1838

Morse Alphabet	
A	• —
B	— • • •
C	— • — •
D	— • •
E	•
F	• • — •
G	— — •
H	• • • •
I	• •
J	• — — —
K	— • —
L	• — • •
M	— —
N	— •
O	— — —
P	• — — — •
Q	— — • —
R	• — •
S	• • •
T	—
U	• • —
V	• • • —
W	• — —
X	— • • —
Y	— • — —
Z	— — • •
Full stop (.)	• — • — • —
Break signal or fresh line	— • • • —
Apostrophe (')	• — — — — •
Hyphen (-)	— • • • • —
Exclamation (!)	— — • • — —
Interrogation (?)	• • — — • •
Underline (_____)	• • — — • —
Parenthesis ()	— • — — • —
Inverted commas (" ")	• — • • — •

Some Examples

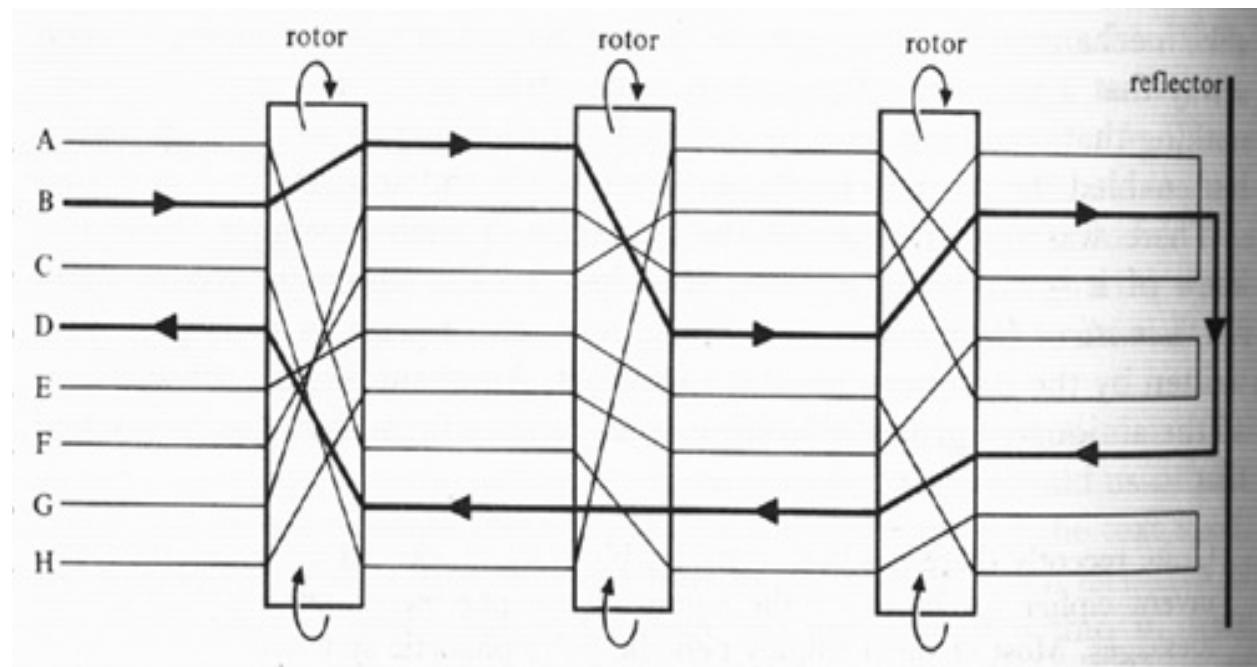
- Crytograms from the Chinese gold bars



<http://www.iacr.org/mis.../china/china.html>

Some Examples

- Enigma, Germany coding machine in *World War II*.



Some Examples

- Sigaba, used by U.S. during *World War II*.



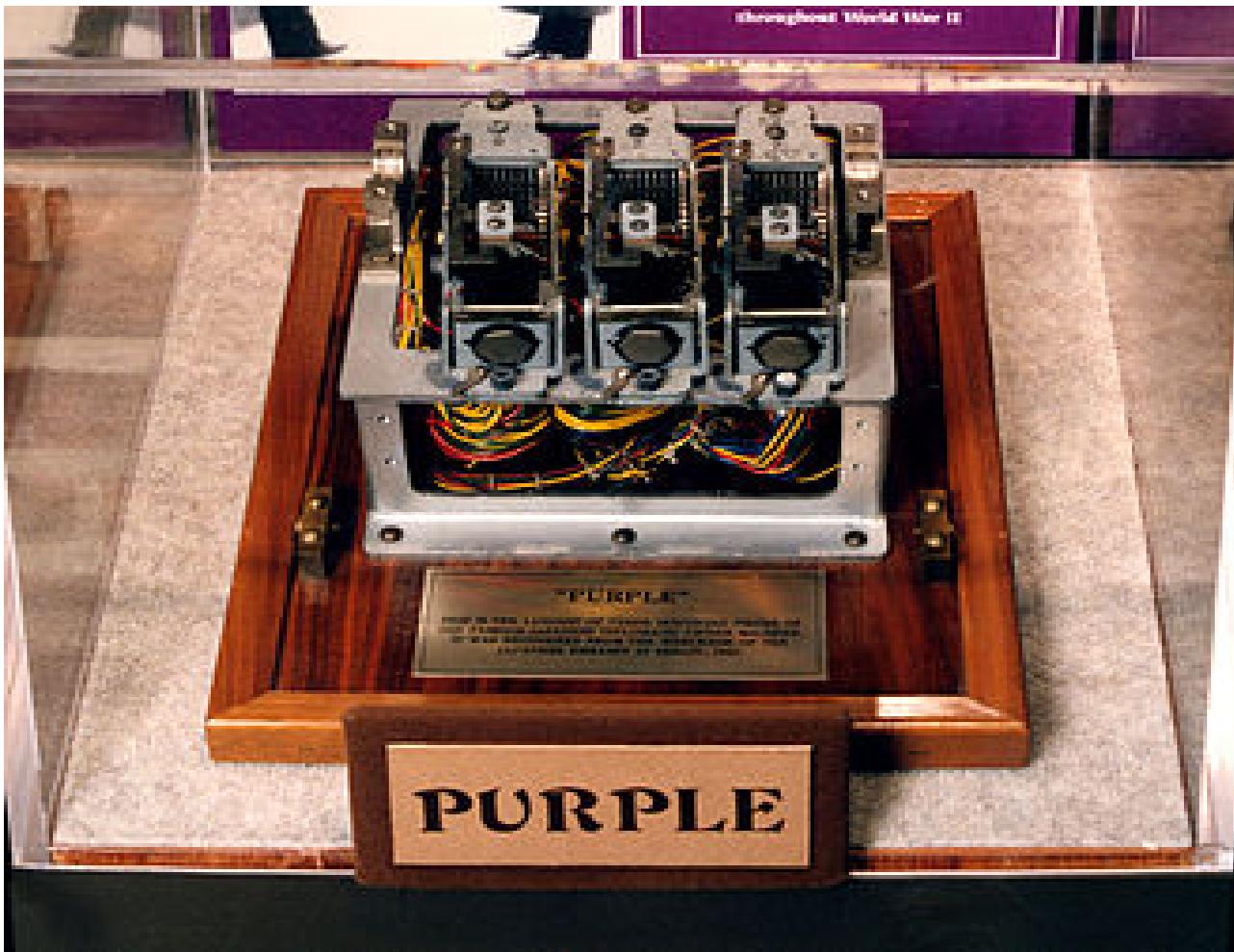
Some Examples

- Japanese “Enigma” Rotor Cipher Machine



Some Examples

- Japanese Purple Machine (97-shiki obun inji-ki)



People Working in Breaking Codes



Alan Turing
(1912-1954)



Claude E. Shannon
(1916-2001)

Cryptography History

- History (until 1970's)

“*Symmetric*” cryptography

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They need agree in advance on the *secret key k*.

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Q: How can they do this?

Cryptography History

■ History (until 1970's)

“*Symmetric*” cryptography



They need agree in advance on the *secret key k*.

Q: How can they do this?

Q: What if Bob could send Alice a “special key” useful only for **encryption** but no help for **decryption**?

Caesar Cipher

- Key: $k = 0, 1, \dots, 25$

Encryption: encode i as $(i + k) \bmod 26$

Decryption: decode j as $(j - k) \bmod 26$

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plaintext: SEND REINFORCEMENT

Key: 2

ciphertext: UGPF TGKPHQTEGOGPV

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ciphertext: UGPF TGKPHQTEGOGPV

Problem: only 26 possibilities for keys!

Kerchoff's Principle (1883): System should be secure even if algorithms are known, as long as key is secret.

Substitution Cipher

- Key: table mapping each letter to another letter

A	B	C		Z
V	R	E		D

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- Key: table mapping each letter to another letter

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Encryption & Decryption: letter by letter according to table

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However, substitution cipher is still **insecure!**

Key observation: can recover plaintext using *statistics* on *letter frequencies*.

Substitution Cipher

■ Table 1: Relative frequencies of the letters of the English language

Letter	Relative Frequency (%)	Letter	Relative Frequency (%)
a	8.167	n	6.749
b	1.492	o	7.507
c	2.782	p	1.929
d	4.253	q	0.095
e	12.702	r	5.987
f	2.228	s	6.327
g	2.015	t	9.056
h	6.094	u	2.758
i	6.966	v	0.978
j	0.153	w	2.360
k	0.772	x	0.150
l	4.025	y	1.974
m	2.406	z	0.074

Substitution Cipher

Table 2: Number of Diagraphs Expected in 2,000 Letters of English Text

th	-	50	at	-	25	st	-	20
er	-	40	en	-	25	io	-	18
on	-	39	es	-	25	le	-	18
an	-	38	of	-	25	is	-	17
re	-	36	or	-	25	ou	-	17
he	-	33	nt	-	24	ar	-	16
in	-	31	ea	-	22	as	-	16
ed	-	30	ti	-	22	de	-	16
ne	-	30	to	-	22	rt	-	16
ha	-	26	it	-	20	ve	-	16

Table 3: The 15 Most Common Trigraphs in the English Language

1	-	the	6	-	tio	11	-	edt
2	-	and	7	-	for	12	-	tis
3	-	tha	8	-	nde	13	-	oft
4	-	ent	9	-	has	14	-	sth
5	-	ion	10	-	nce	15	-	men

Substitution Cipher

- LIVITCSWPIYVEWHEVSRIQMXLEYVEOIEWHRXEXIPFE
MVEWHKVSTYLXZIXLIKIIXPPIJVSZEYPERRGERIMWQL
MGLMXQERIWGPSRIHMXQEREKI

Substitution Cipher

- LIVITCSWPIYVEWHEVSRIQMXXLEYVEOIEWHRXEXIPFEMVEWHKVSTYXLXZIXLIKIIXPPIJVSZEYPERRGERIMWQLMGLMXQERIWGPSRIHMXQEREKI

I – *most common letter*

LI – *most common pair*

XLI – *most common triple*

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I – *most common letter*

I = e

LI – *most common pair*

L = h

XLI – *most common triple*

X = t

Substitution Cipher

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MVEWHKVSTYlxzixlikiiXPIJVsZEYPERRGERIMWQL
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V = r

E = a

Y = g

Substitution Cipher

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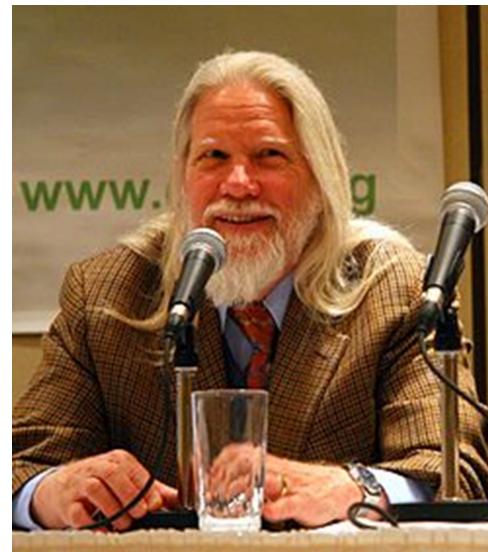
Y = g

HereUpOnLeGrandAroseWithAGraveAndStatelyAirAndBroug
MeTheBeetleFromAGlassCaselnWhichItWasEnclosedIt-
WasABe

Cryptography History

- History (from 1976)
 - ◊ W. Diffie, M. Hellman, “New direction in cryptography”, *IEEE Transactions on Information Theory*, vol. 22, pp. 644-654, 1976.

“We stand today on the brink of a revolution in cryptography.”



Bailey W. Diffie

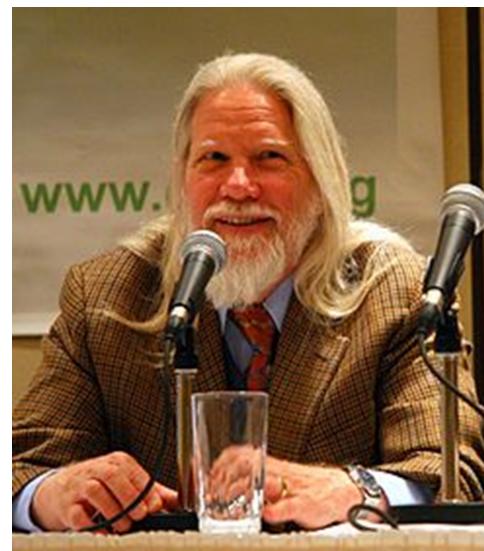
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2015 Turing Award

Bailey W. Diffie

Martin E. Hellman

2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to modern cryptography . Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography," ^[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today. ^[40]
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Public Key Cryptography

- Alice wants to send a message to Bob



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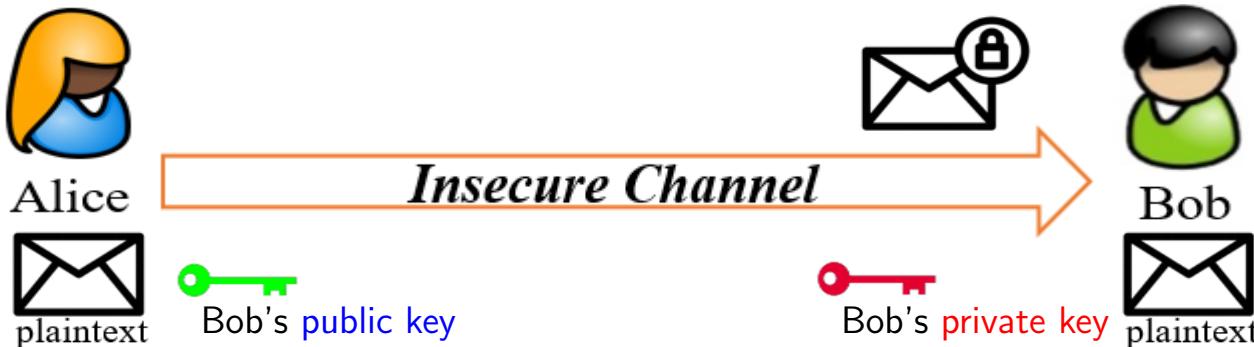
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Ronald L. Rivest



Adi Shamir



Leonard M. Adleman

R. Rivest, A. Shamir, L. Adleman, “A method for obtaining digital signatures and public-key cryptosystems”,
Communications of the ACM, vol. 21-2, pages 120-126, 1978.

RSA Public Key Cryptosystem

■ Rivest-Shamir-Adleman

2002 **Turing Award**

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Pick two **large** primes, p and q . Let $n = pq$, then $\phi(n) = (p - 1)(q - 1)$. Encryption and decryption keys e and d are selected such that

- $\gcd(e, \phi(n)) = 1$
- $ed \equiv 1 \pmod{\phi(n)}$

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$C = M^e \pmod{n}$ (RSA **encryption**)

$M = C^d \pmod{n}$ (RSA **decryption**)

RSA Public Key Cryptosystem

- $C = M^e \text{ mod } n$ (RSA encryption)
- $M = C^d \text{ mod } n$ (RSA decryption)

Theorem (Correctness) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \leq x < n$,

$$x^{ed} \equiv x \pmod{n}.$$

RSA Public Key Cryptosystem

- $C = M^e \bmod n$ (RSA encryption)
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Q : How to prove this?

RSA Public Key Cryptosystem: Example

Parameters:	p	q	n	$\phi(n)$	e	d
	5	11	55	40	7	23

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	5	11	55	40	7	23

Public key: (7, 55)

Private key: 23

Encryption: $M = 28, C = M^7 \text{ mod } 55 = 52$

Decryption: $M = C^{23} \text{ mod } 55 = 28$

RSA Public Key Cryptosystem: Parameters

Parameters: p q n $\phi(n)$ e d

Public key: (e, n)

Private key: d

$p, q, \phi(n)$ must be kept **secret!**

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Comment: It is believed that determining $\phi(n)$ is equivalent to factoring n . Meanwhile, determining d given e and n , appears to be at least as time-consuming as the integer factoring problem.

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CS 208 – Algorithm Design and Analysis

The Security of the RSA

In practice, RSA keys are typically 1024 to 2048 bits long.

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Q : Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \text{lcm}(p - 1, q - 1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Applications of RSA

- SSL/TLS protocol

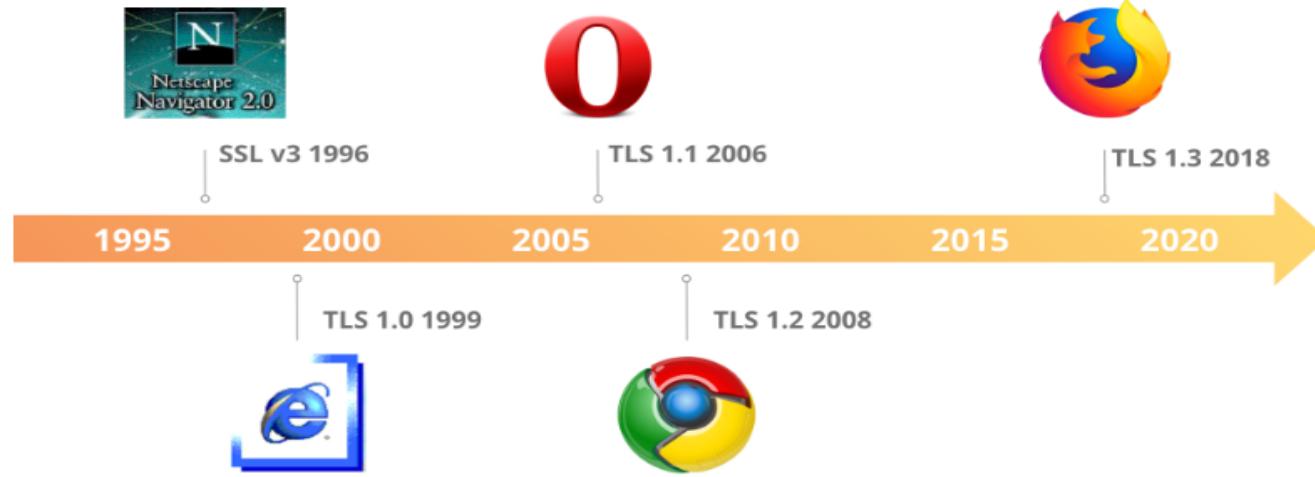
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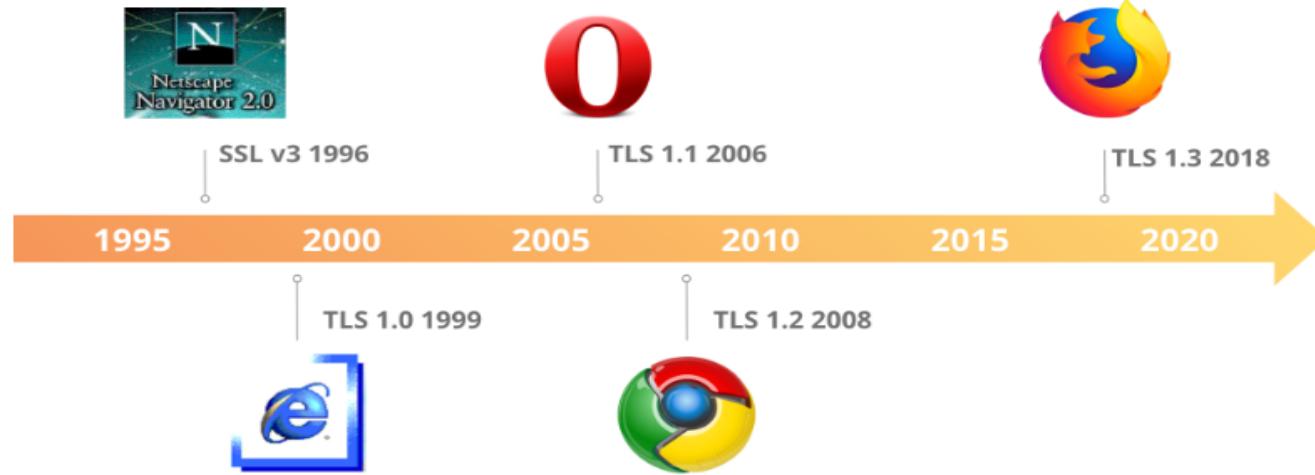
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Applications of RSA

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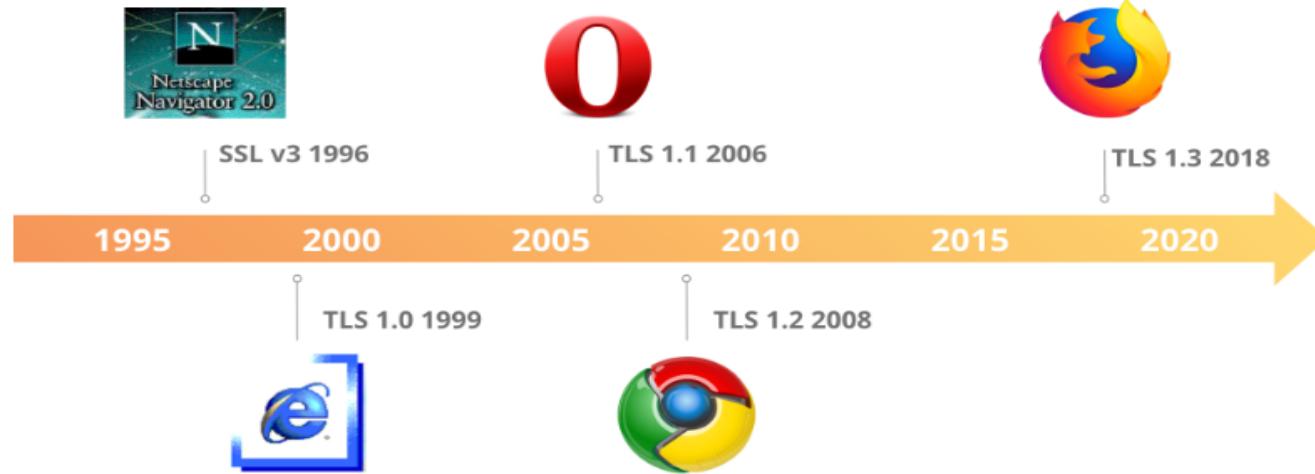


Key exchange/agreement and authentication

Algorithm	SSL 2.0	SSL 3.0	TLS 1.0	TLS 1.1	TLS 1.2	TLS 1.3
RSA	Yes	Yes	Yes	Yes	Yes	No
DH-RSA	No	Yes	Yes	Yes	Yes	No
DHE-RSA (forward secrecy)	No	Yes	Yes	Yes	Yes	Yes
ECDH-RSA	No	No	Yes	Yes	Yes	No
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Applications of RSA

SSL/TLS protocol



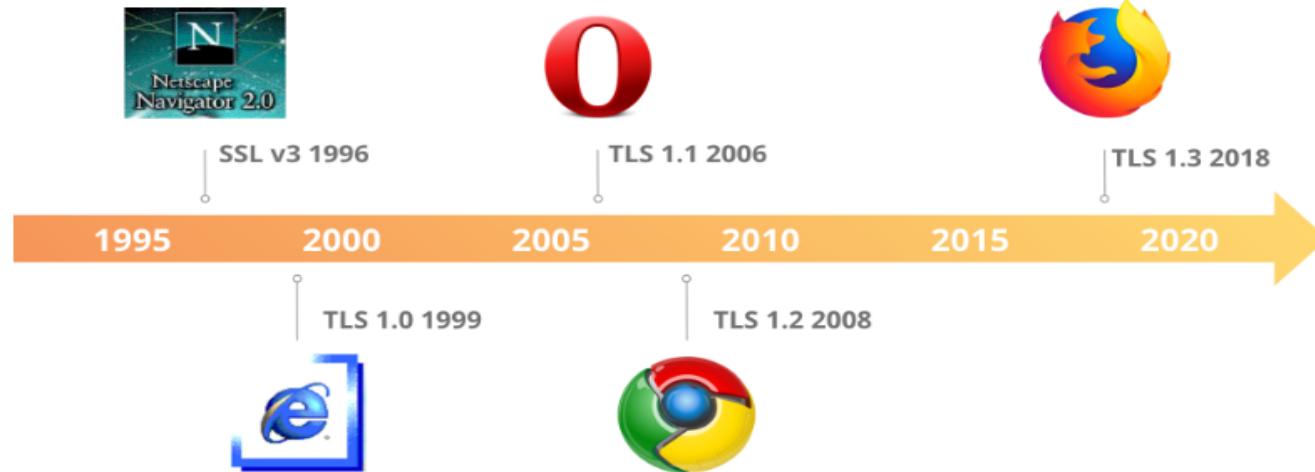
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CS 305 – Computer Networks

Applications of RSA

SSL/TLS protocol



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CS 305 – Computer Networks

CS 403 – Cryptography and Network Security

Using RSA for Digital Signature

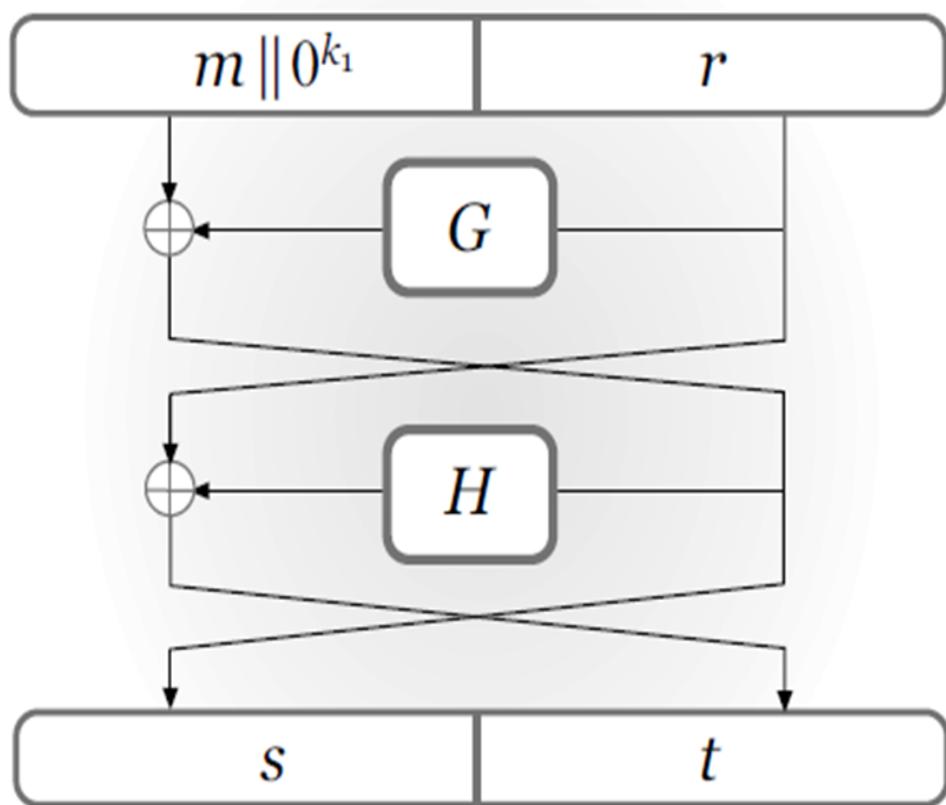
$$S = M^d \bmod n \text{ (RSA signature)}$$

$$M = S^e \bmod n \text{ (RSA verification)}$$

Why?

RSA-OAEP Standard

- RSA-OAEP (Optimal Asymmetric Encryption Padding) is *IND-CCA2 secure*.
- PKCS#1 V2, RFC2437 Standard



Next Lecture

- cryptography ...

