


Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Euler Paths and Circuits

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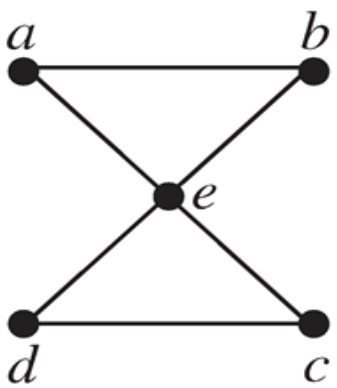
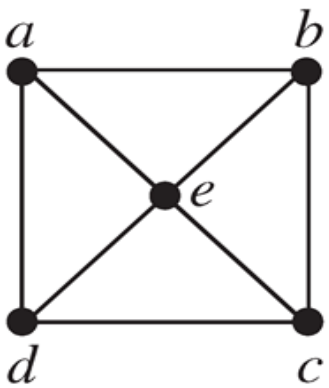
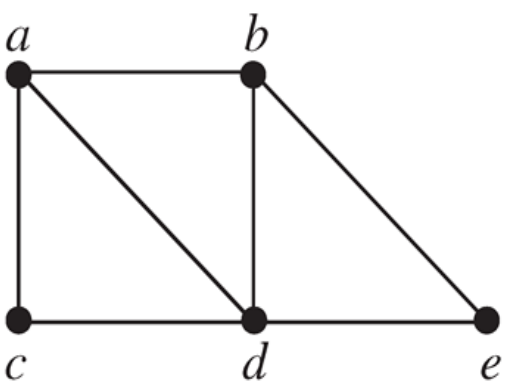


G_1



G_2



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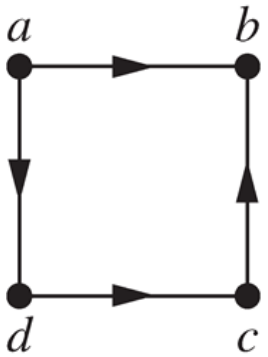

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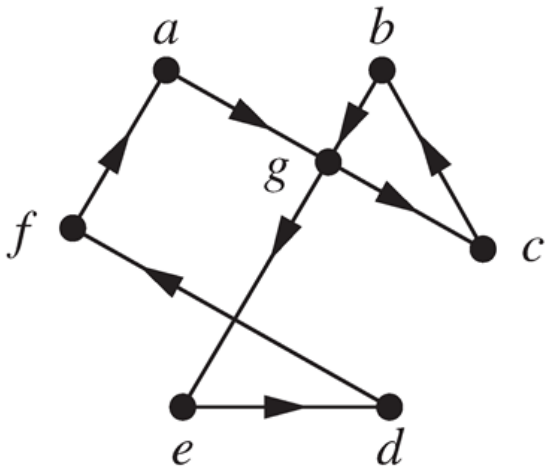

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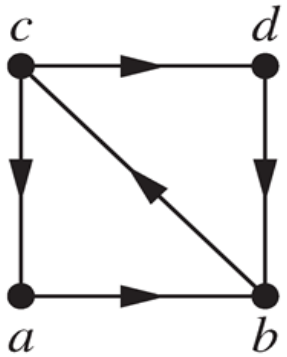

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- ◇ The initial vertex and the final vertex of an Euler path have odd degree.

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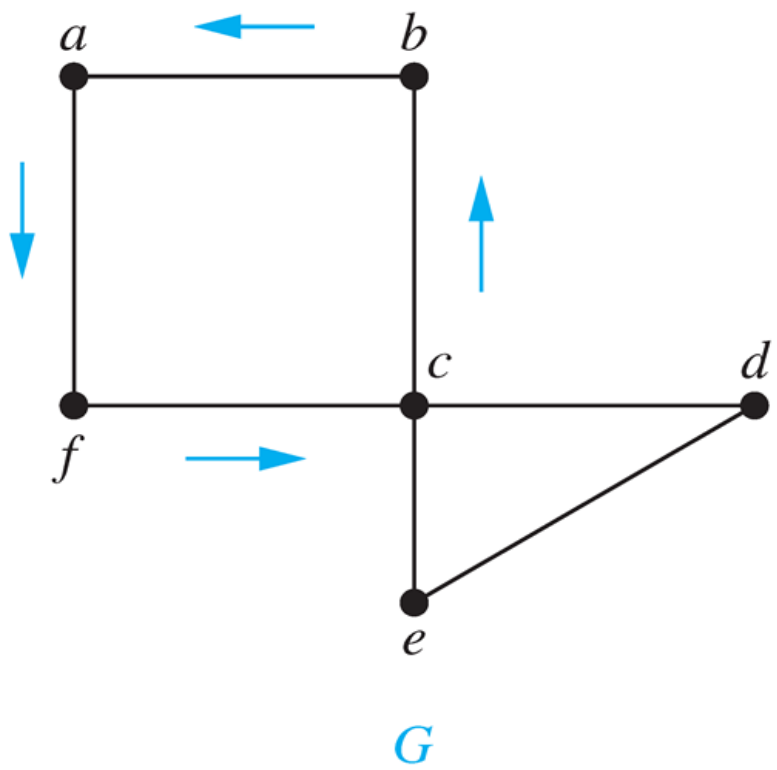

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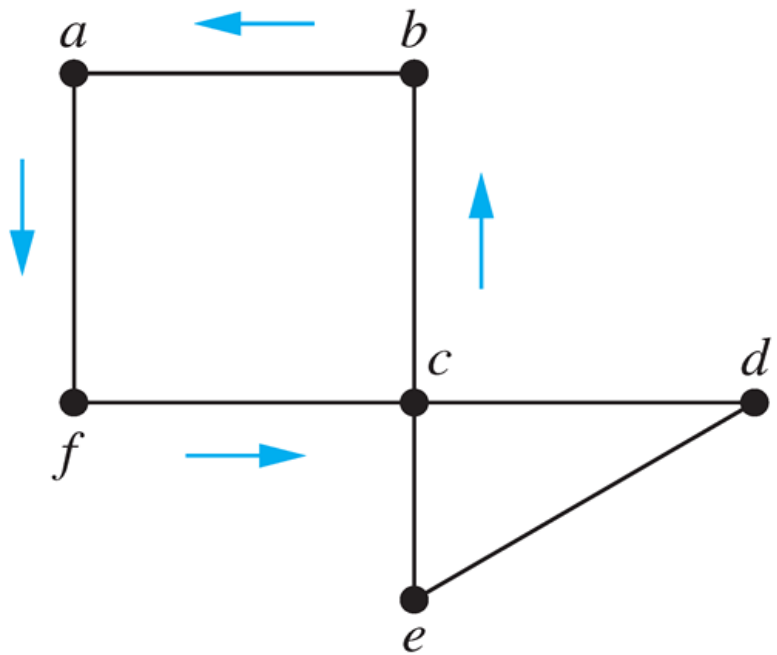

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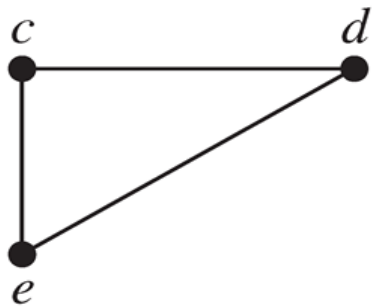

H

Algorithm for Constructing an Euler Circuit

procedure *Euler*(*G*: connected multigraph with all vertices of even degree)
circuit := a circuit in *G* beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex.
H := *G* with the edges of this circuit removed
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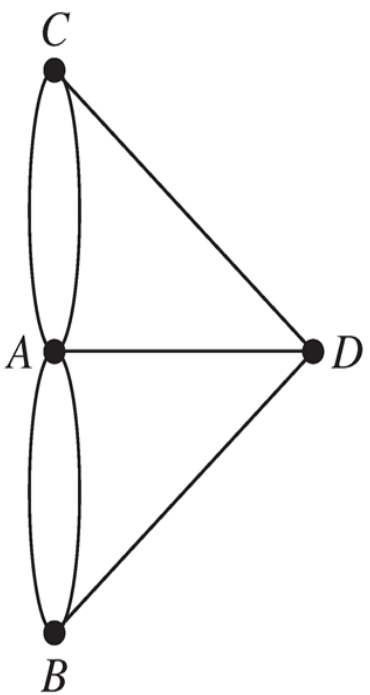

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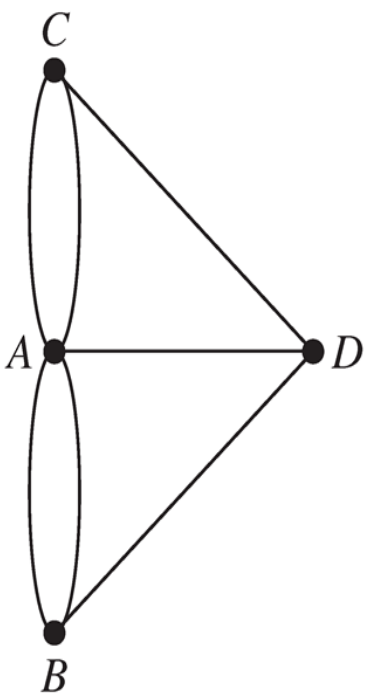

No Euler circuit

Euler Circuits and Paths

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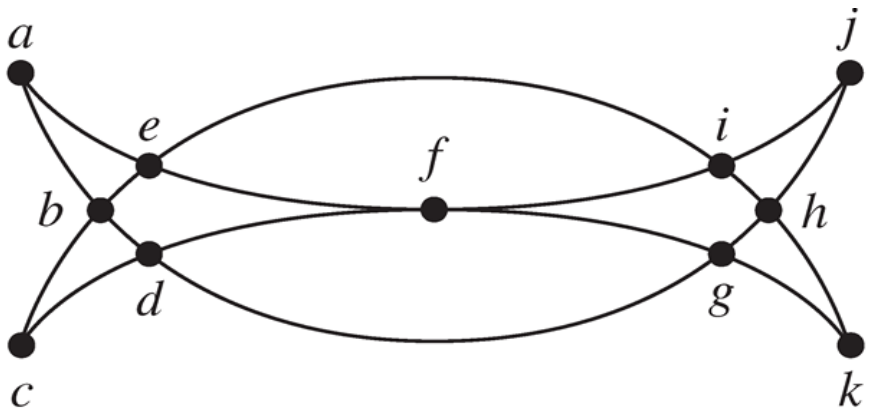

G

FIGURE 6 **Mohammed's Scimitars.**

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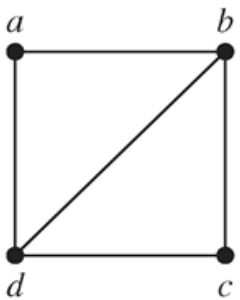

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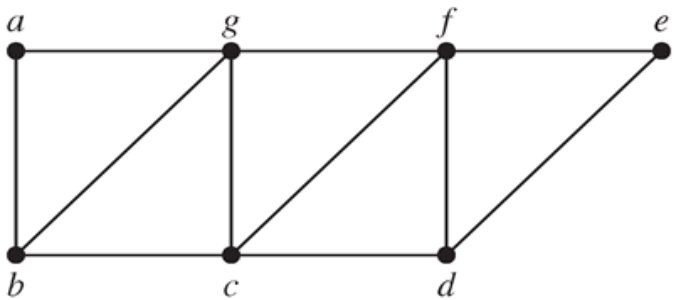

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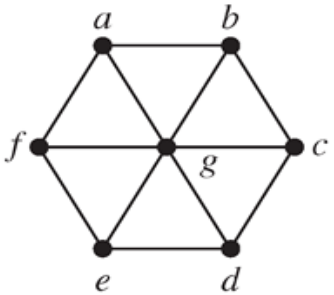

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Applications of Euler Paths and Circuits

- Finding a path or circuit that traverses each
  - ◇ street in a neightborhood
  - ◇ road in a transportation network
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Chinese Postman Problem Meigu Guan [60']

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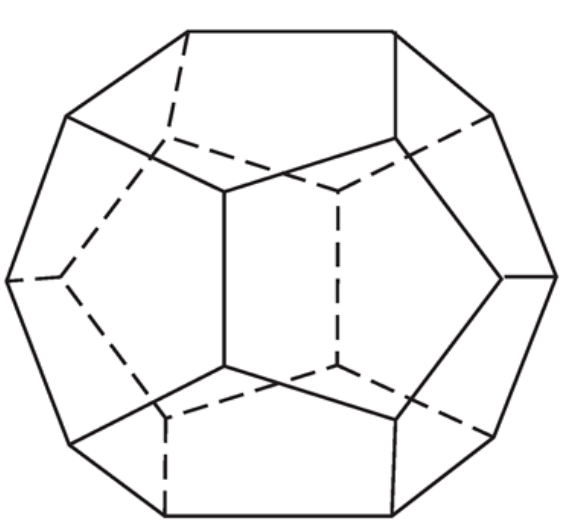
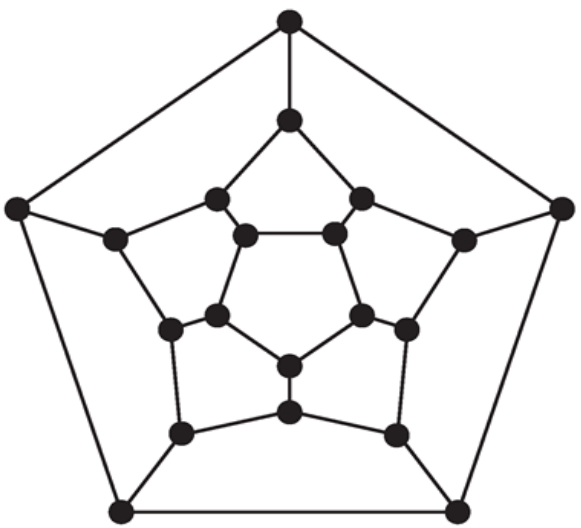


(a)



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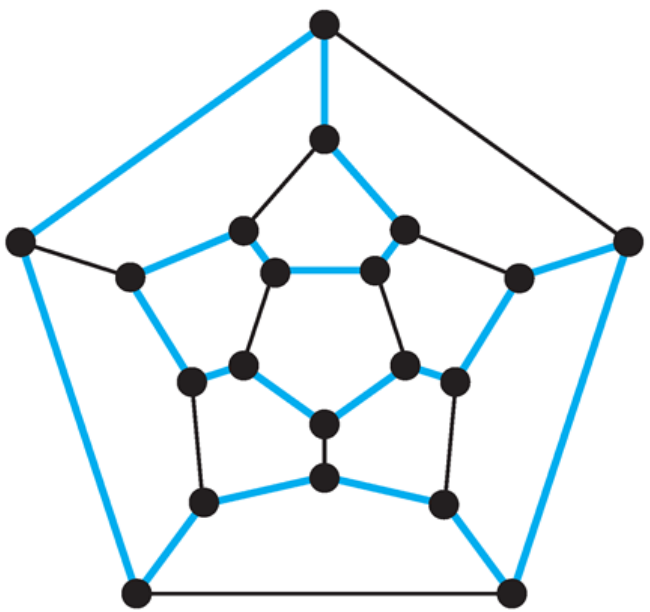

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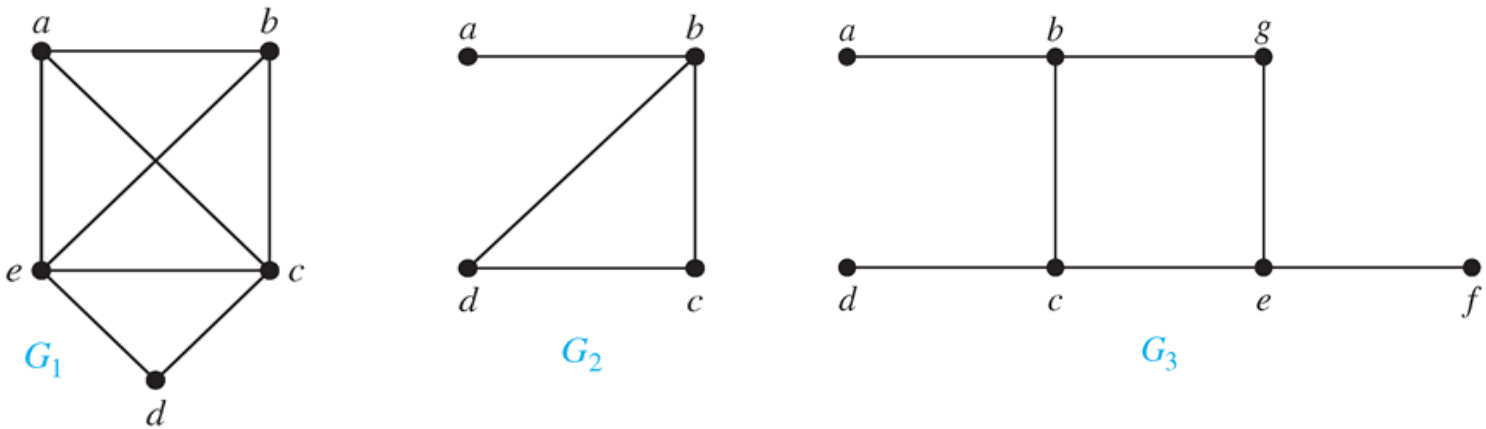

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  Hamilton path problem $\in$ NPC

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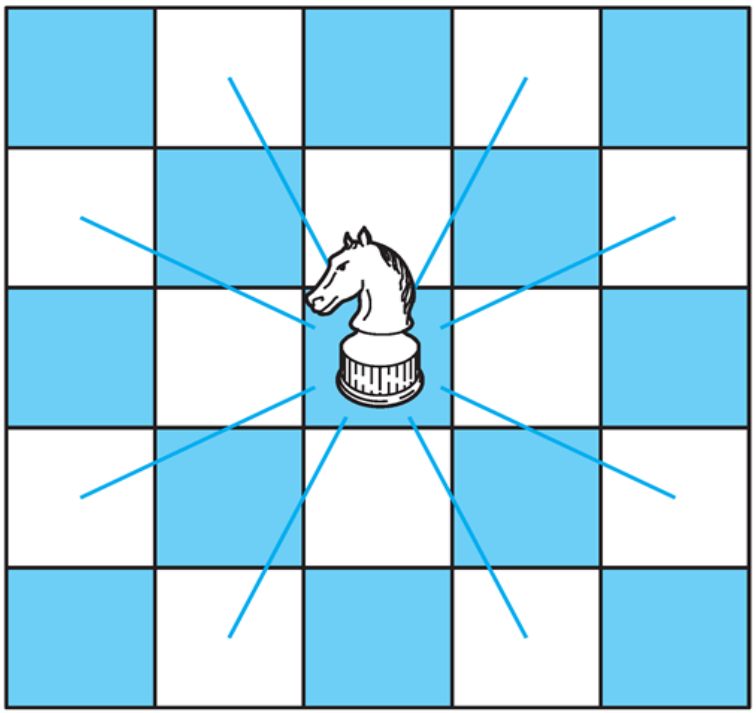

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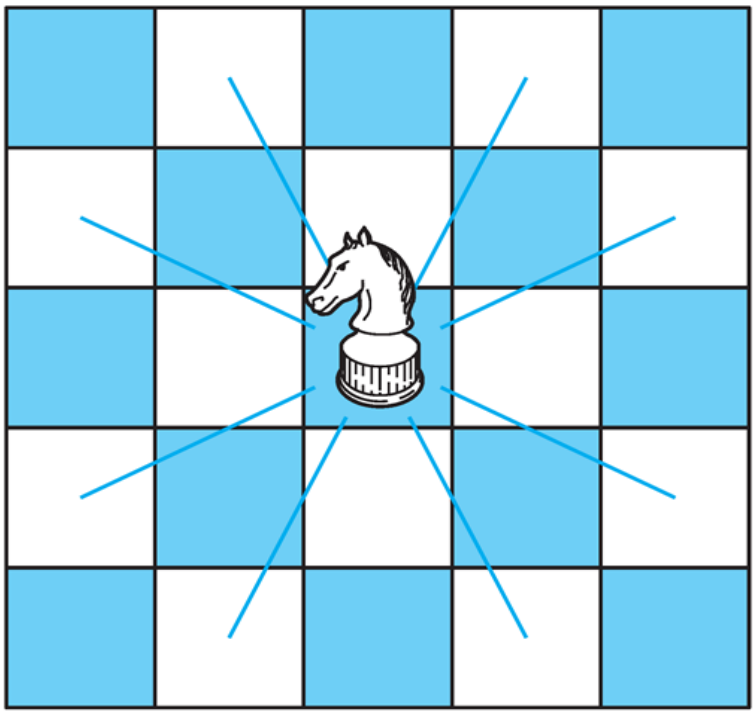

What about in 6×6 chessboard?

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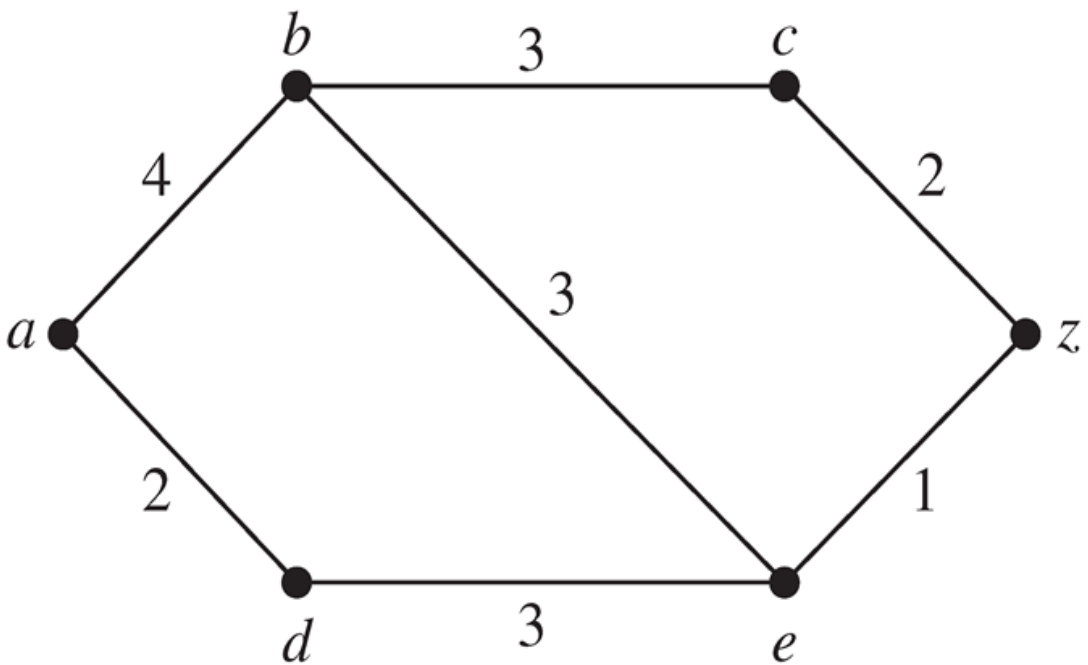

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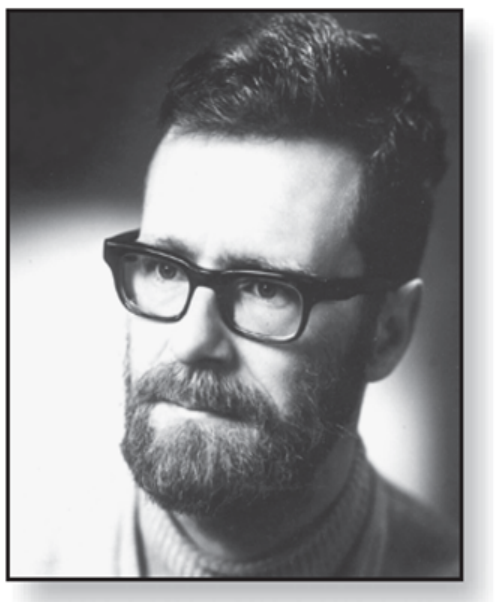

Edsger Wybe Dijkstra

Dijkstra's Algorithm

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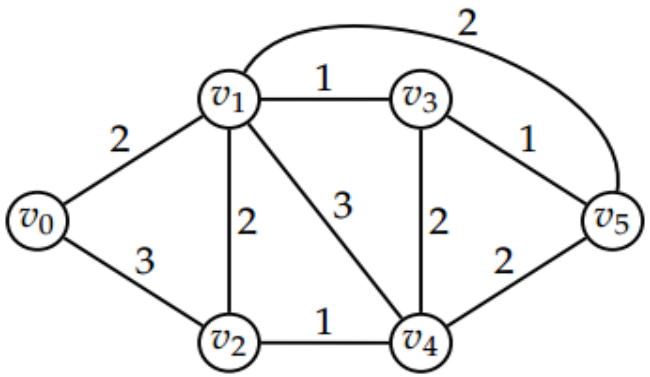
Example



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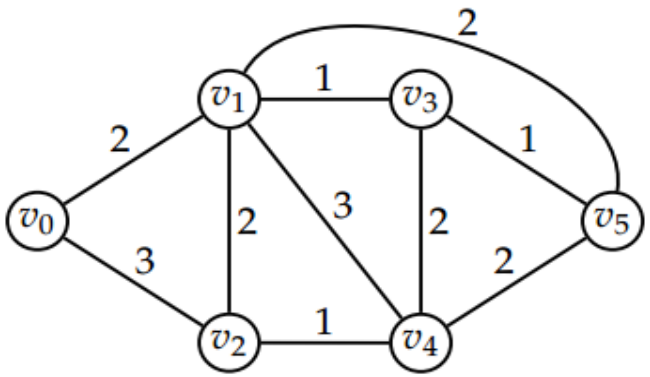

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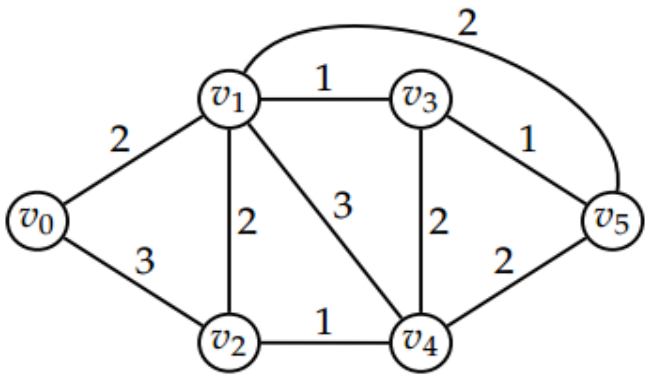

v_0	v_1	v_2	v_3	v_4	v_5
0	∞	∞	∞	∞	∞

$d(v_0) = 0$, all other $d(v) = \infty$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

  let $v \notin S$ be the vertex with the least value $d(v)$,

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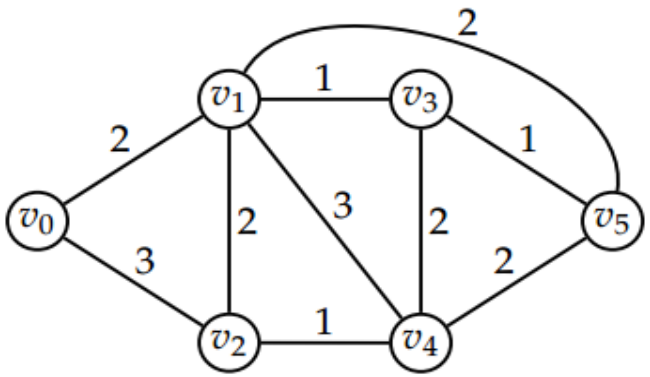
Example



v_0	v_1	v_2	v_3	v_4	v_5
0	∞	∞	∞	∞	∞

$i = 0$

$d(v_1) = \min\{\infty, 2\} = 2$, $d(v_2) = \min\{\infty, 3\} = 3$

Dijkstra's Algorithm

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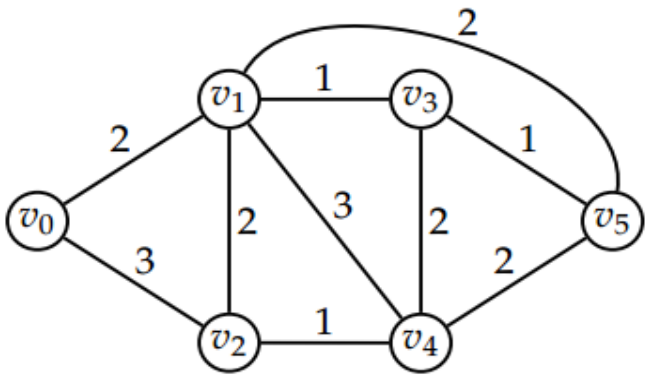

v_0	v_1	v_2	v_3	v_4	v_5
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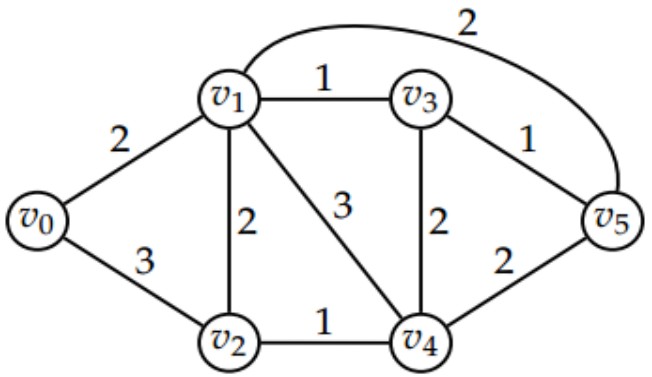

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	∞	∞	∞

$i = 1$

$d(v_2) = \min\{3, d(v_1) + \alpha(v_1 v_2)\} = \min\{3, 4\} = 3$,

$d(v_3) = 2 + 1 = 3$, $d(v_4) = 2 + 3 = 5$,

$d(v_5) = 2 + 2 = 4$

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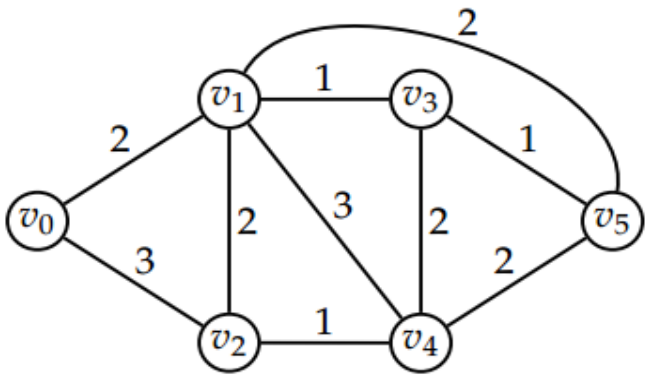

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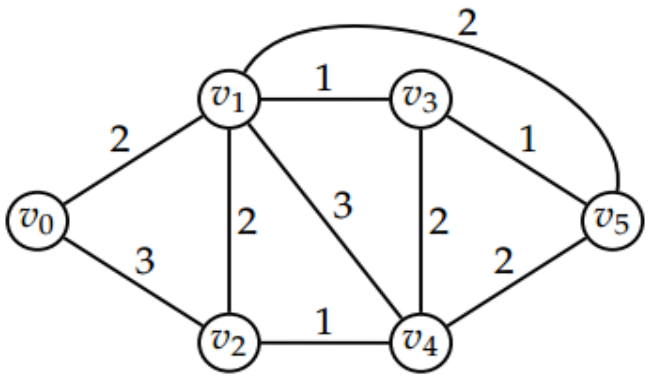

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0	2	3	3	5	4

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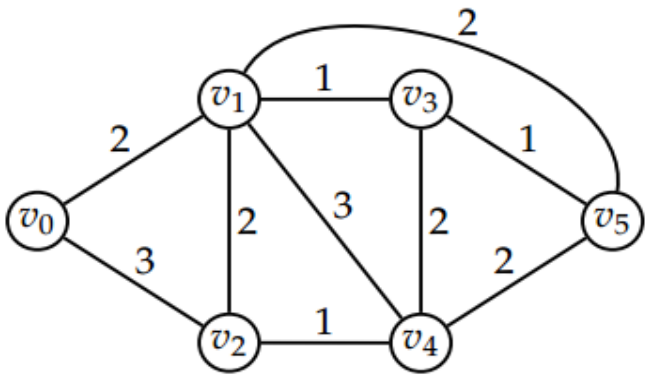

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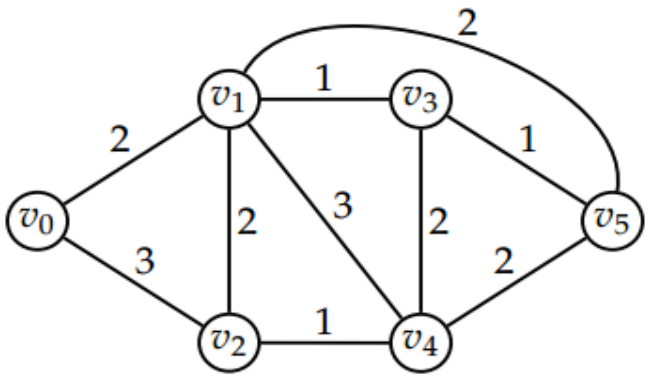

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 3$

$d(v_4) = \min\{4, 3 + 2\} = 4$,

$d(v_5) = \min\{4, 3 + 1\} = 4$

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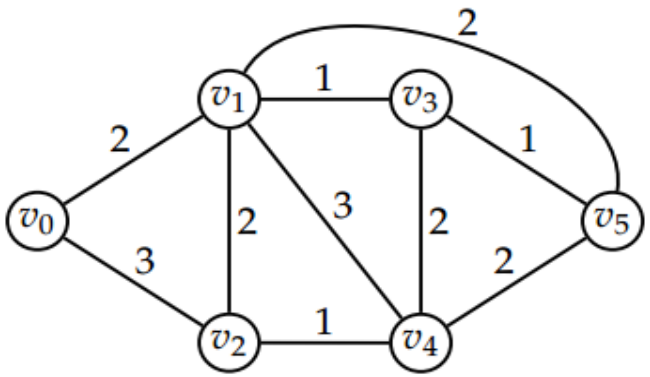

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

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$d(v_5) = \min\{4, 4 + 2\} = 4$

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- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

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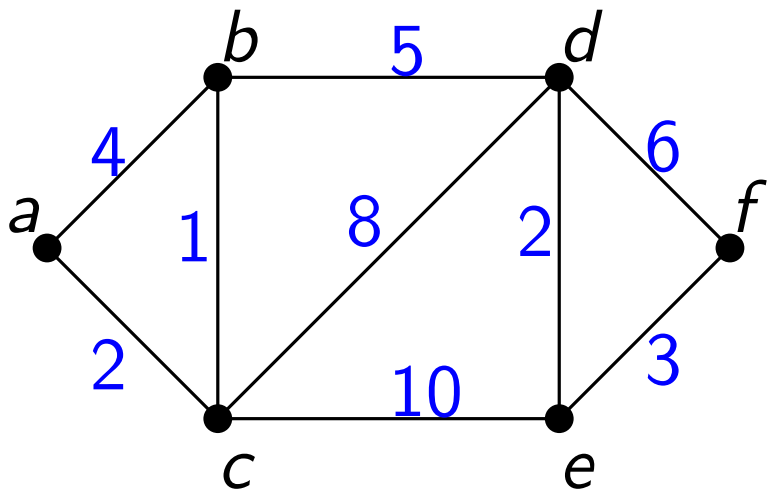
  read the Textbook p.712 – p.714

Another Example



The Single-Source Shortest Paths (SSSP) Problem

- Dijkstra's algorithm

  $O(v^2)$ using an *array* [Dijkstra 1956]

  $O(e + v \log v)$ using a *Fibonacci heap min-priority queue* [Fredman & Tarjan 1984]

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- New result

The Single-Source Shortest Paths (SSSP) Problem

Negative-Weight Single-Source Shortest Paths in Near-linear Time

Aaron Bernstein* Danupon Nanongkai† Christian Wulff-Nilsen‡

Abstract

We present a randomized algorithm that computes single-source shortest paths (SSSP) in $O(m \log^8(n) \log W)$ time when edge weights are integral and can be negative.[1] This essentially resolves the classic negative-weight SSSP problem. The previous bounds are $\tilde{O}((m+n^{1.5}) \log W)$ [BLNPSSSW FOCS'20] and $m^{4/3+o(1)} \log W$ [AMV FOCS'20]. Near-linear time algorithms were known previously only for the special case of planar directed graphs [Fakcharoenphol and Rao FOCS'01].

In contrast to all recent developments that rely on sophisticated continuous optimization methods and dynamic algorithms, our algorithm is simple: it requires only a simple graph decomposition and elementary combinatorial tools. In fact, ours is the first combinatorial algorithm for negative-weight SSSP to break through the classic $\tilde{O}(m\sqrt{n} \log W)$ bound from over three decades ago [Gabow and Tarjan SICOMP'89].

The Single-Source Shortest Paths (SSSP) Problem

Breaking the Sorting Barrier for Directed Single-Source Shortest Paths

Ran Duan * Jiayi Mao * Xiao Mao † Xinkai Shu ‡ Longhui Yin *

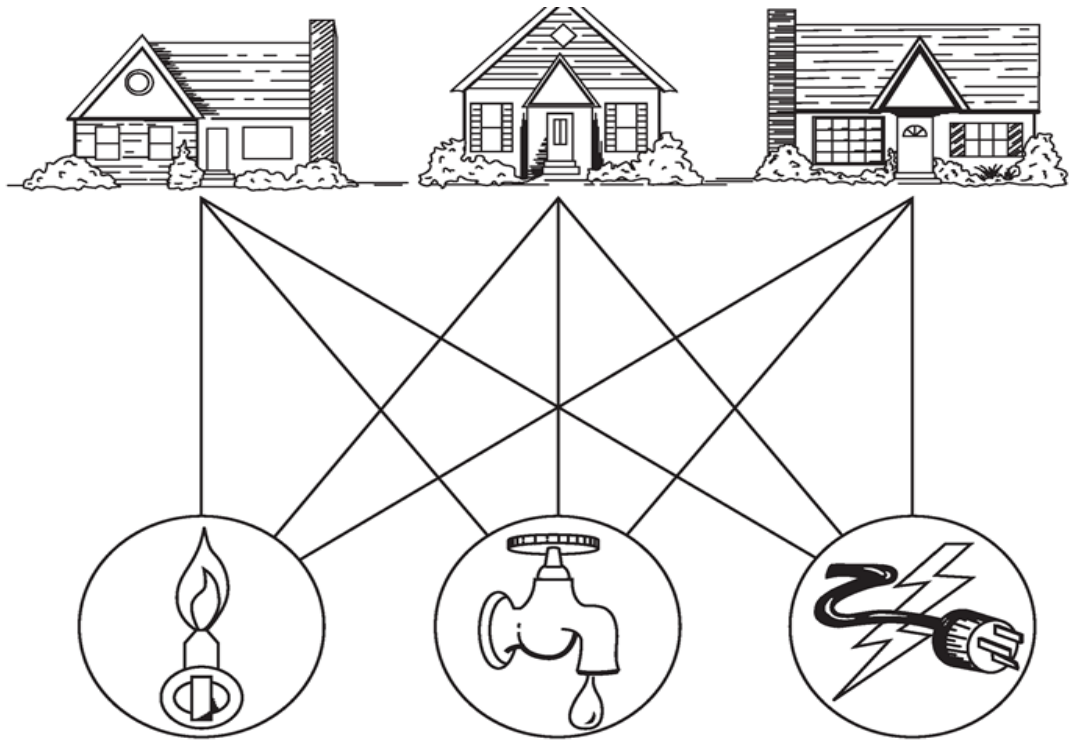
July 31, 2025

Abstract

We give a deterministic $O(m \log^{2/3} n)$-time algorithm for single-source shortest paths (SSSP) on directed graphs with real non-negative edge weights in the comparison-addition model. This is the first result to break the $O(m + n \log n)$ time bound of Dijkstra's algorithm on sparse graphs, showing that Dijkstra's algorithm is not optimal for SSSP.

Planar Graphs

- Join three houses to each of three seperate utilities.



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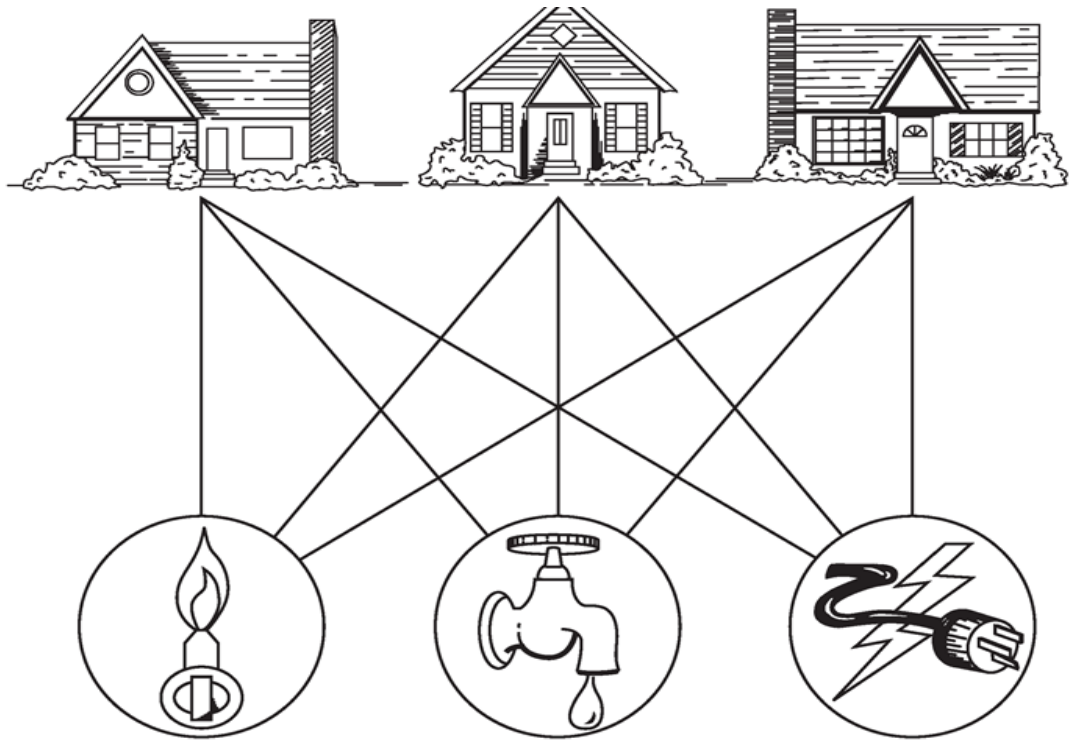

Can this graph be drawn in the plane s.t. no two of its edges cross?

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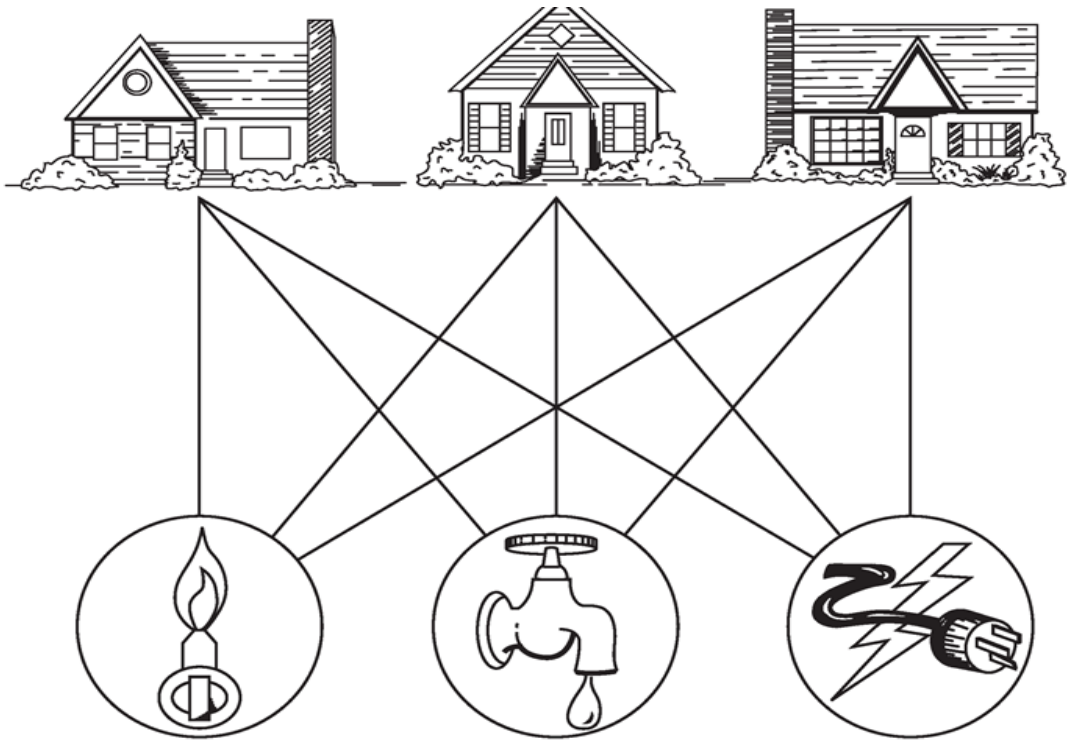

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- **Definition** A graph is called *planar* if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of the graph.

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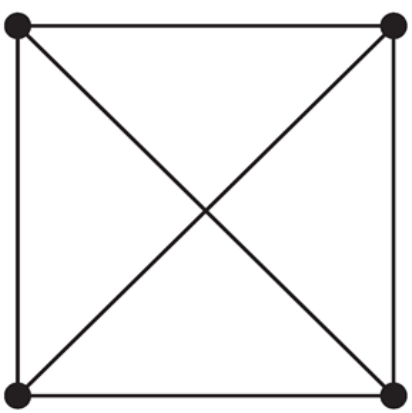

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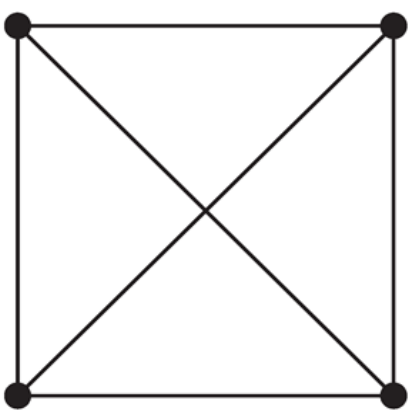

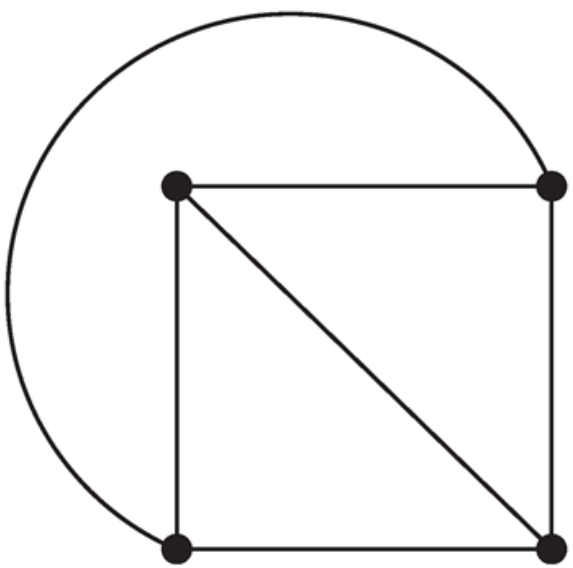

Planar Graphs

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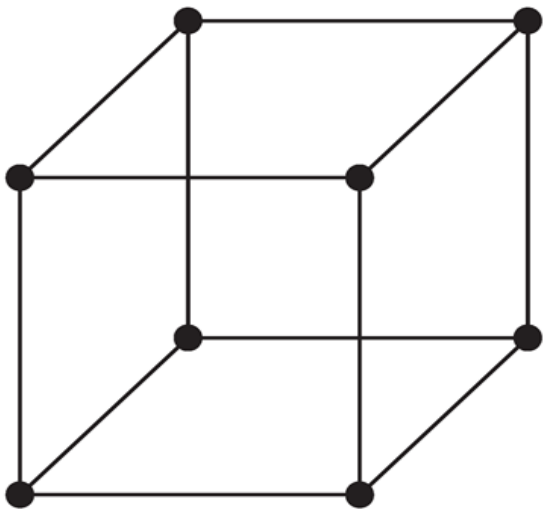

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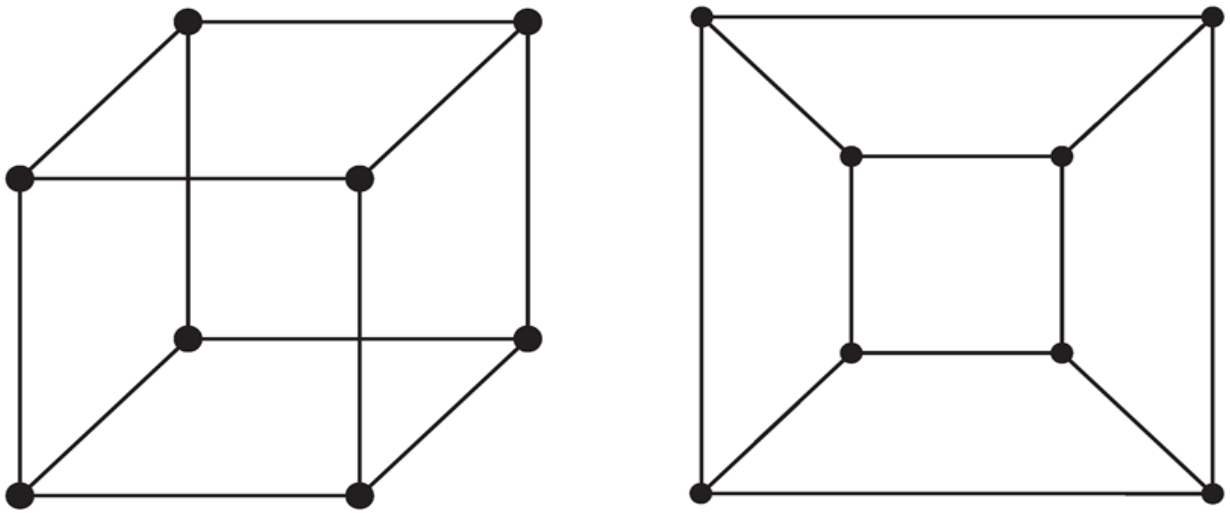

Planar Graphs

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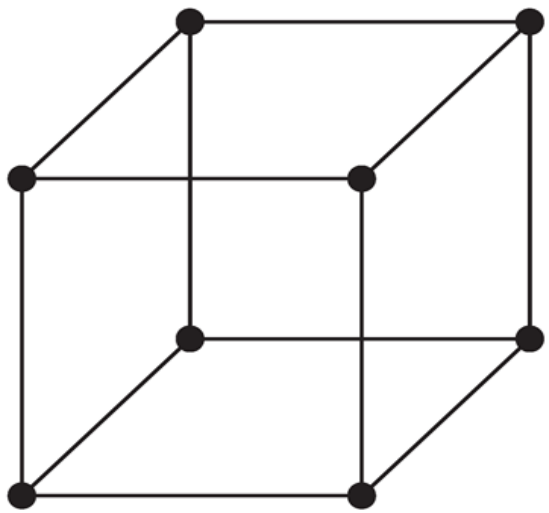
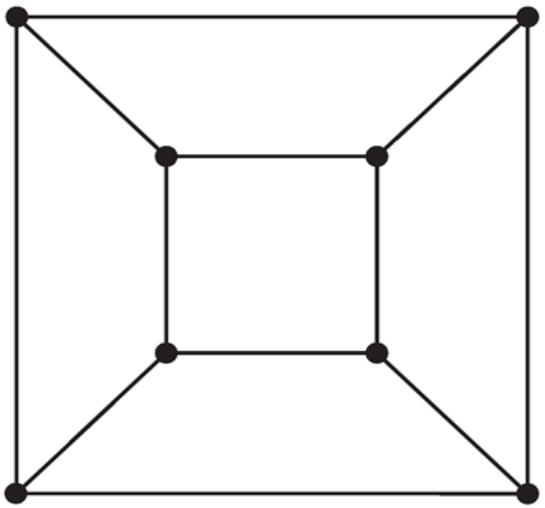
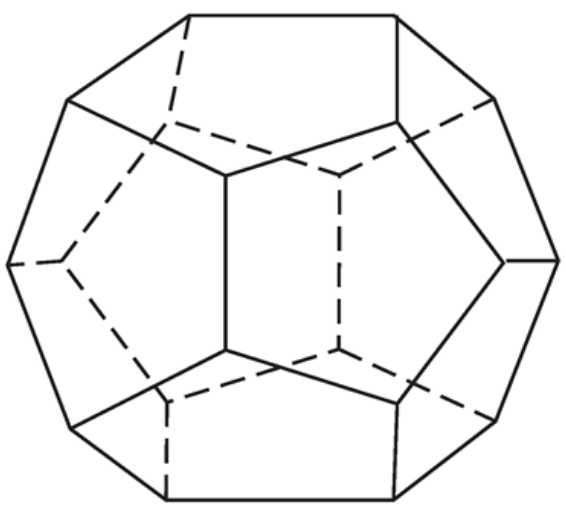


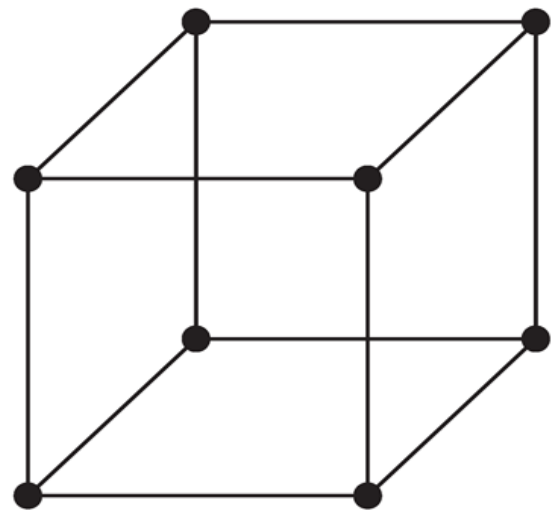


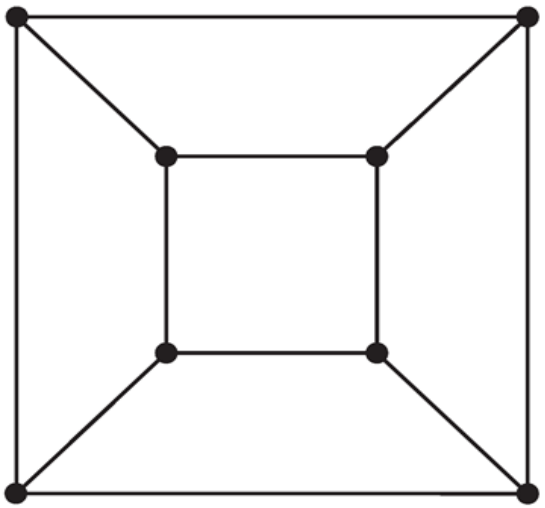


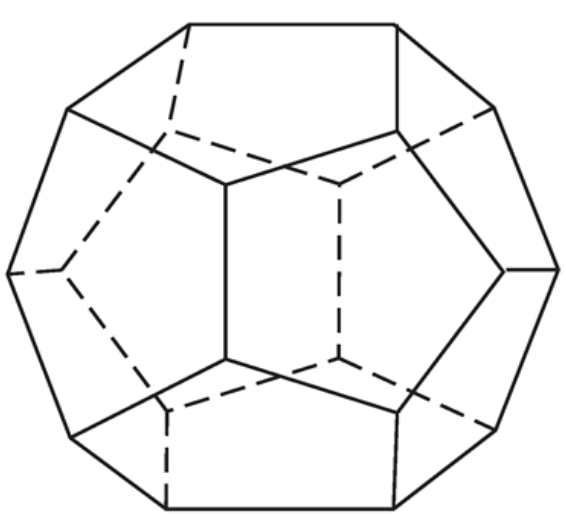
(a)

Planar Graphs

- **Example**







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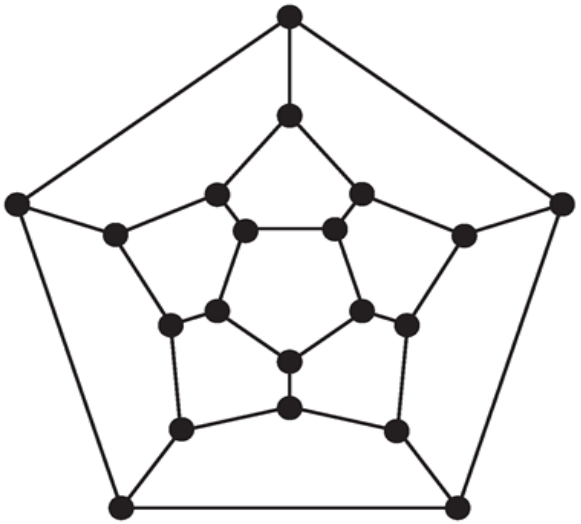

(b)

Planar Graphs

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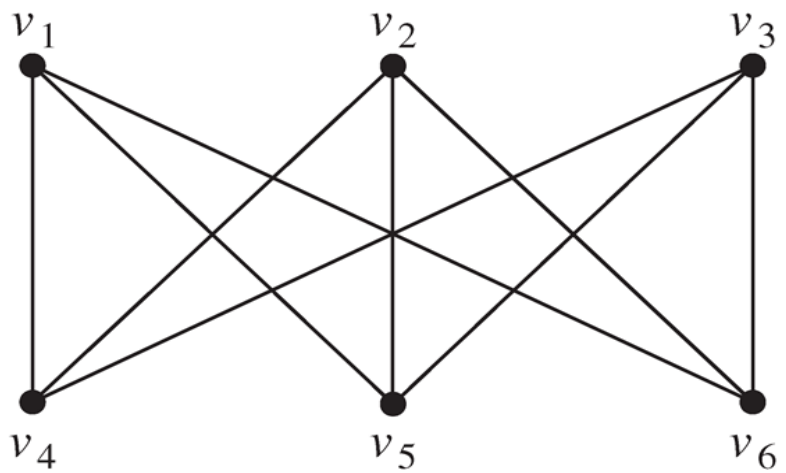

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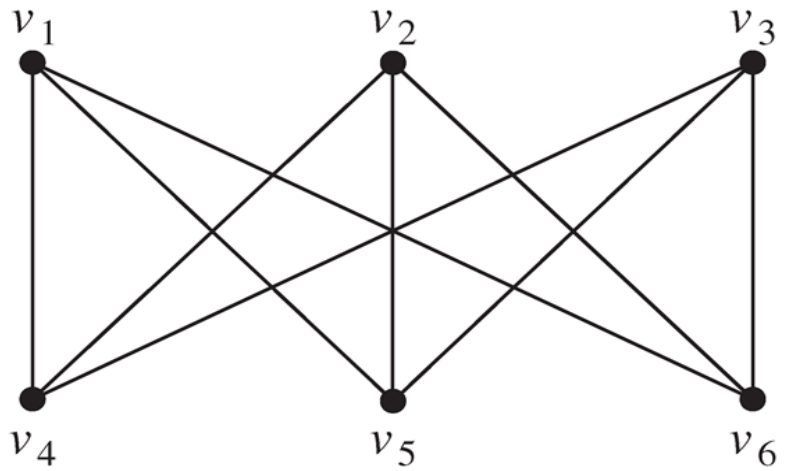

Applications

◇ IC design
◇ design of road networks

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (by induction)

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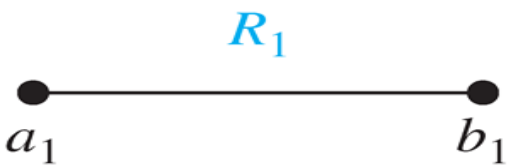

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- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

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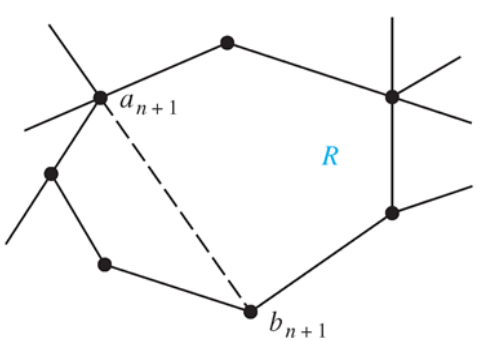

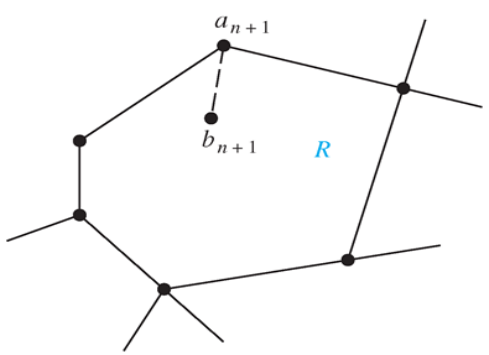

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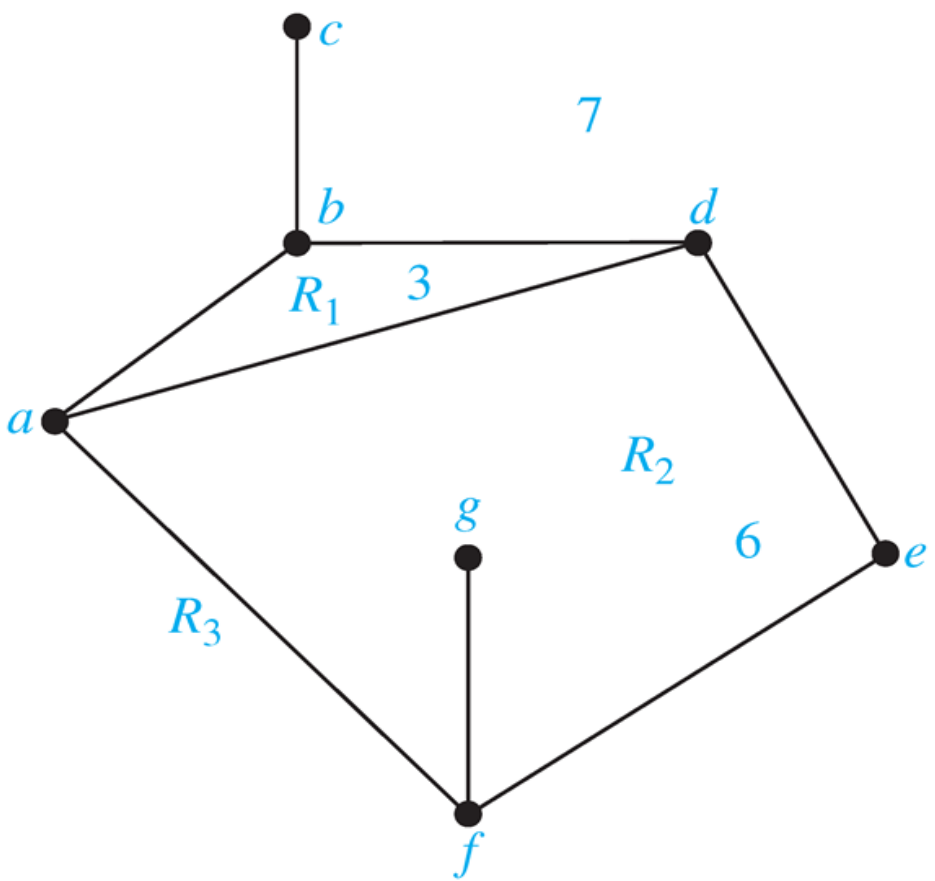

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  Proof similar to that of Corollary 1.

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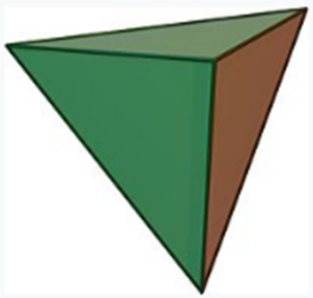
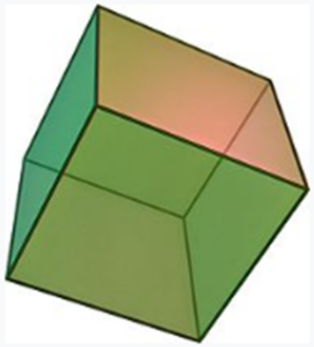
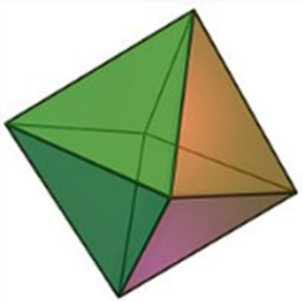
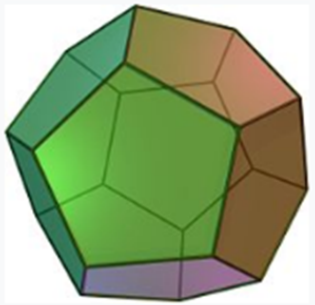
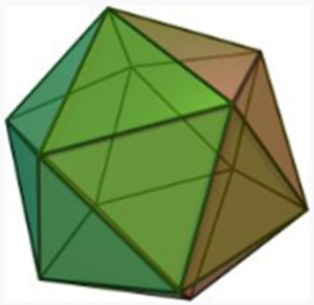
Using Corollary 3

Corollary 2 is used in the proof of Five Color Theorem.

Only 5 Platonic Solids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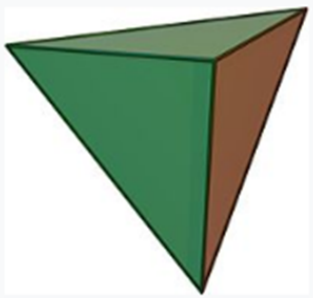	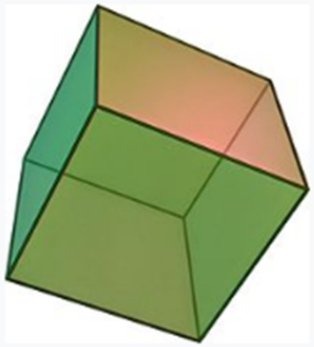	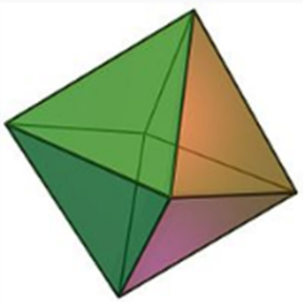	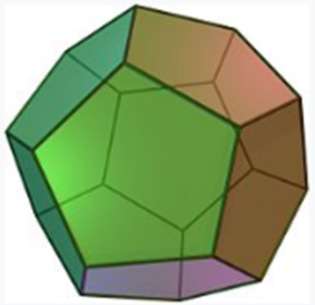	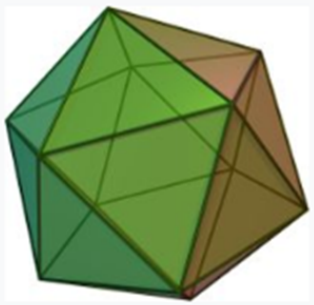
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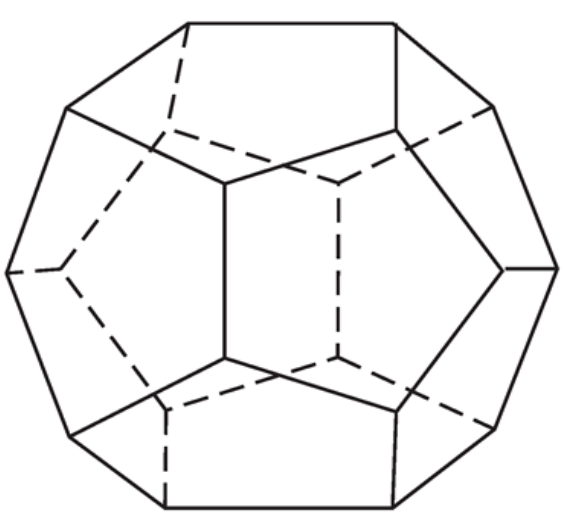

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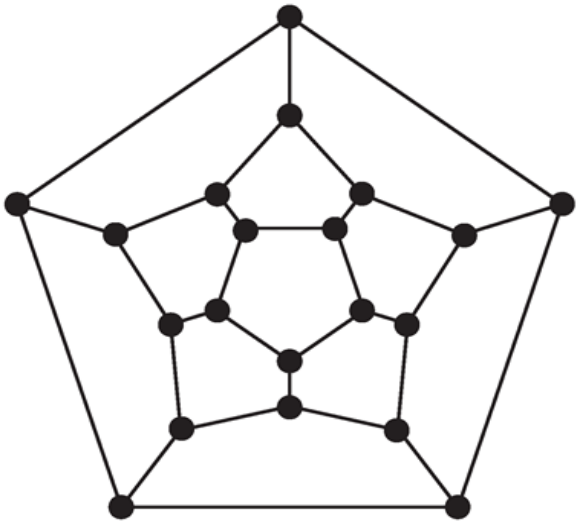

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Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

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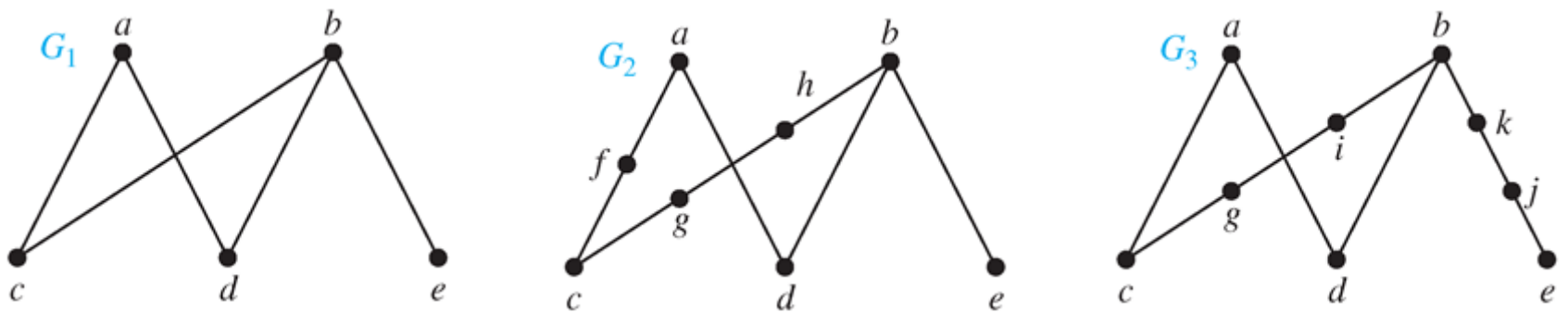

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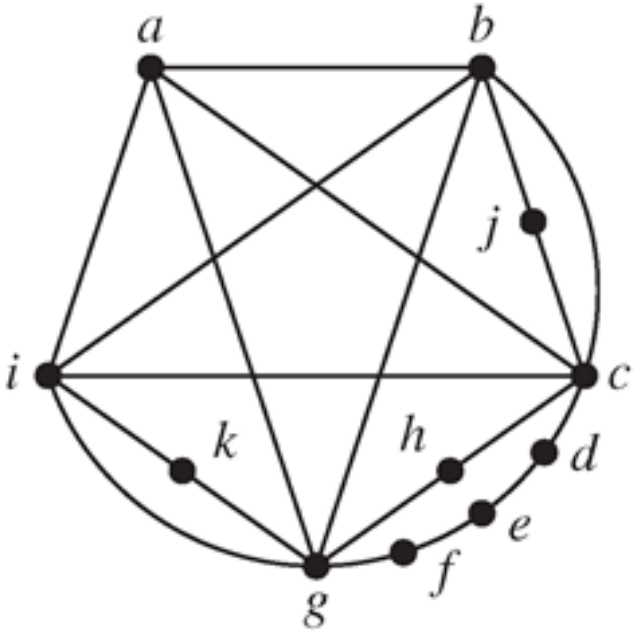
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples



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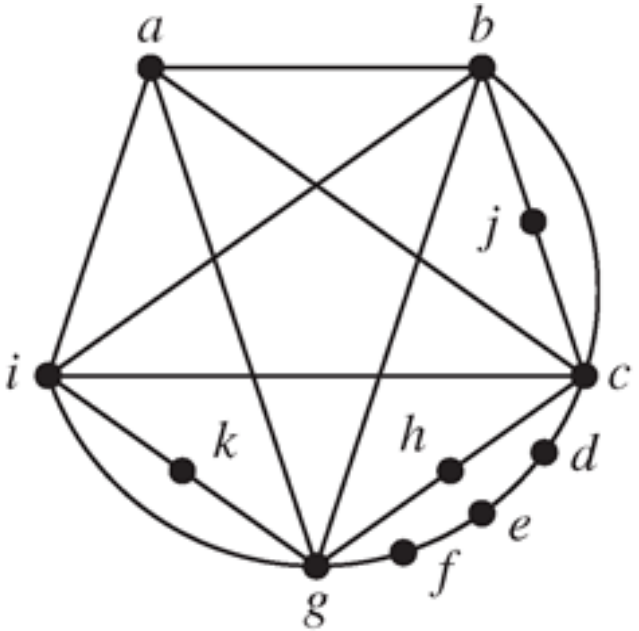

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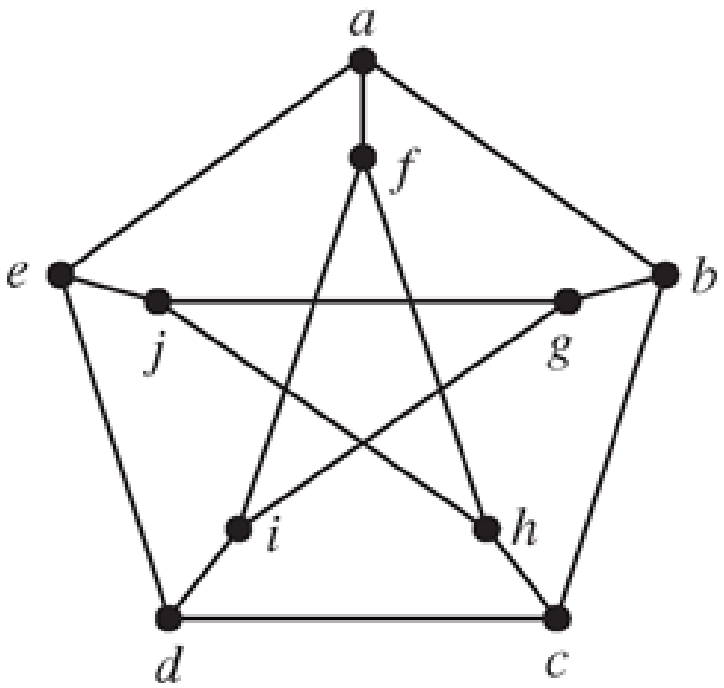

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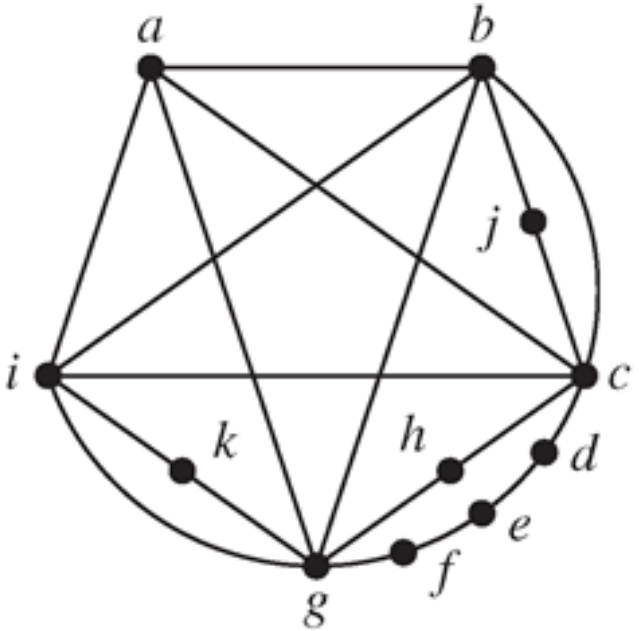

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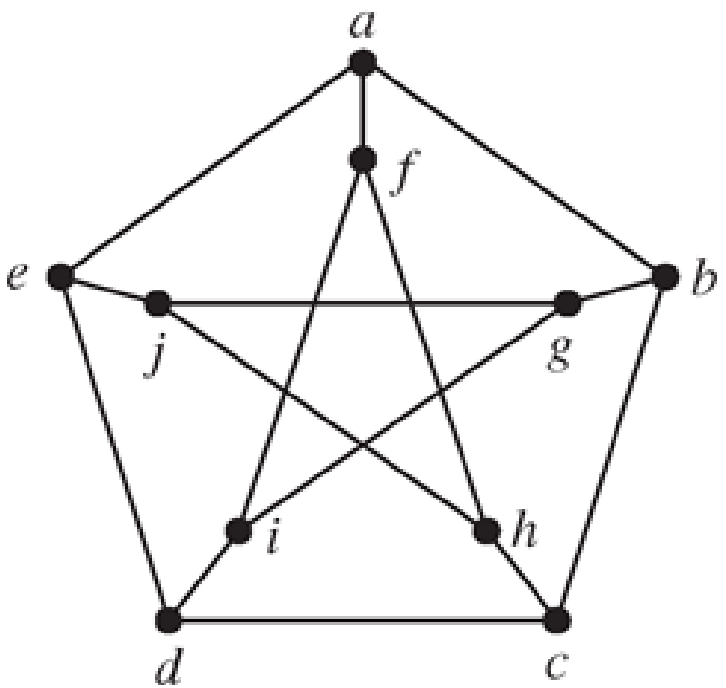
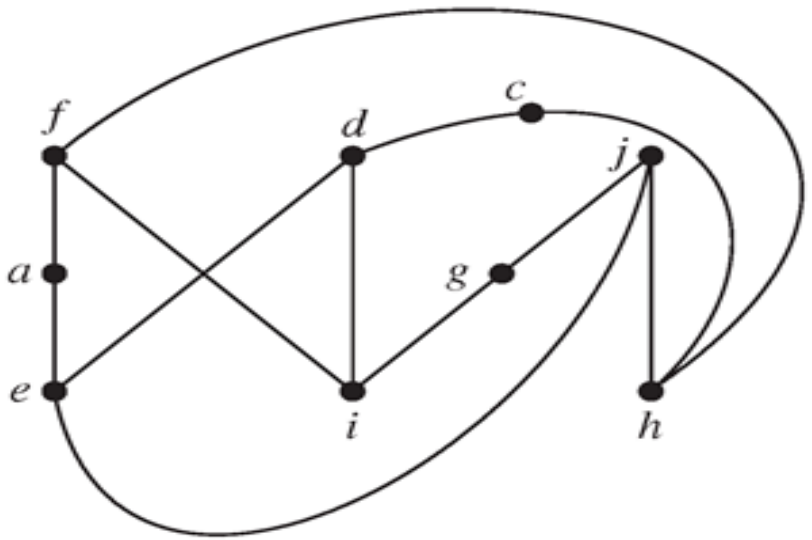
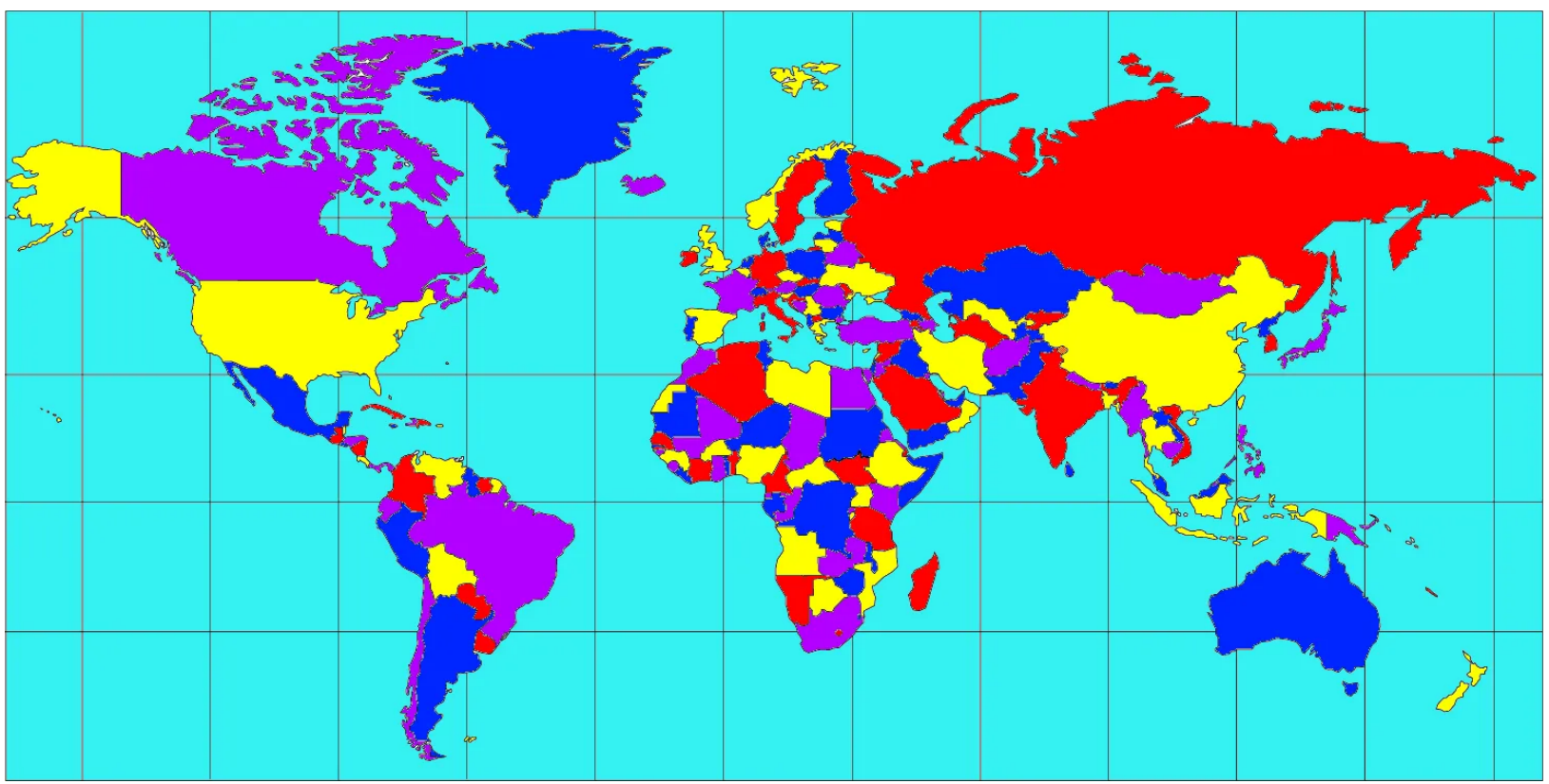




Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



Graph Coloring

- **Four-color theorem**
  - ◇ first proposed by Francis Guthrie in 1852
  - ◇ his brother Frederick Guthrie told Augustus De Morgan
  - ◇ De Morgan wrote to William Hamilton
  - ◇ Alfred Kempe proved it incorrectly in 1879
  - ◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*
  - ◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)
  - ◇ Kempe's incorrect proof serves as a basis

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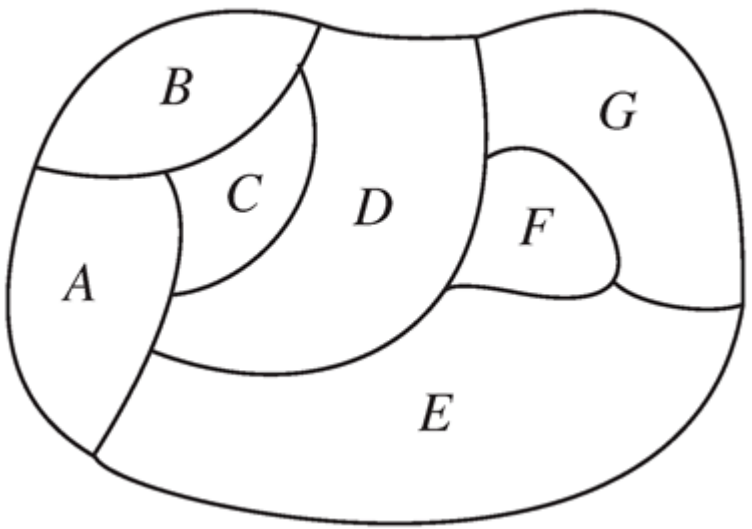

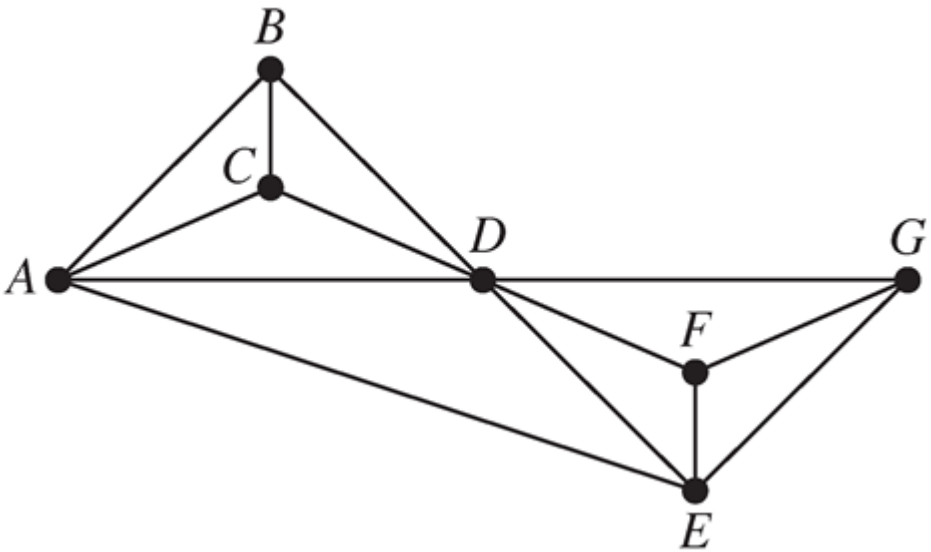

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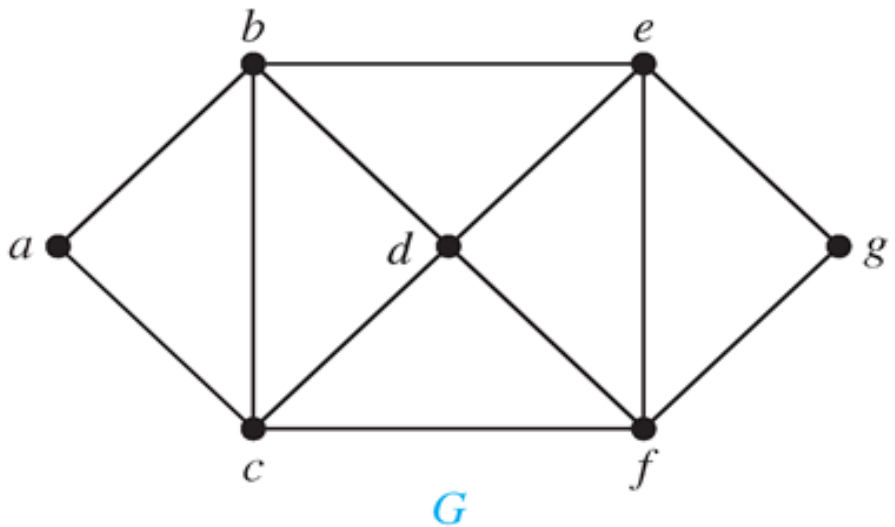

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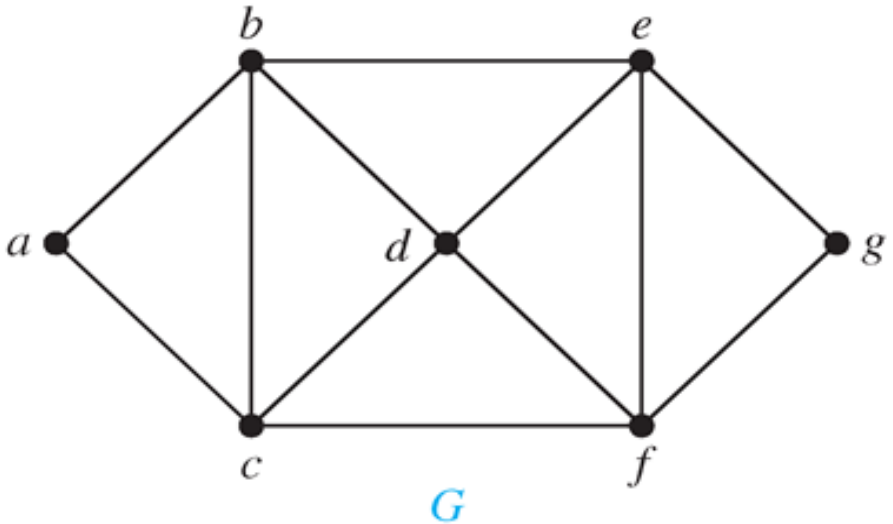

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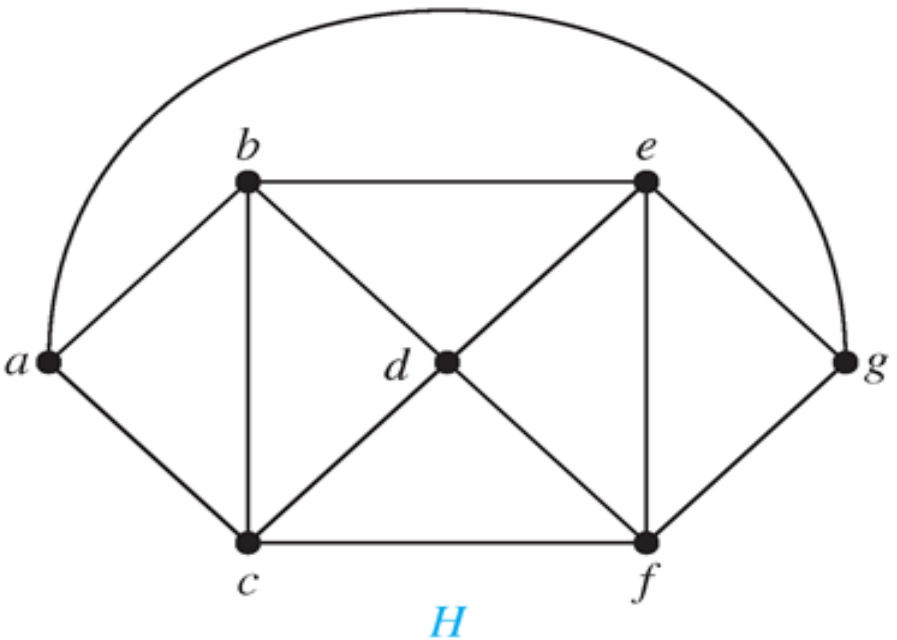

H

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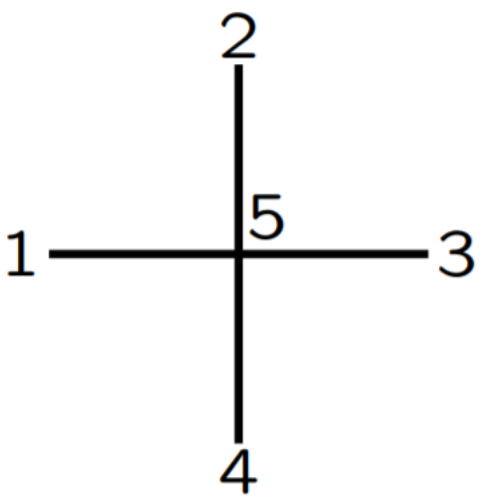
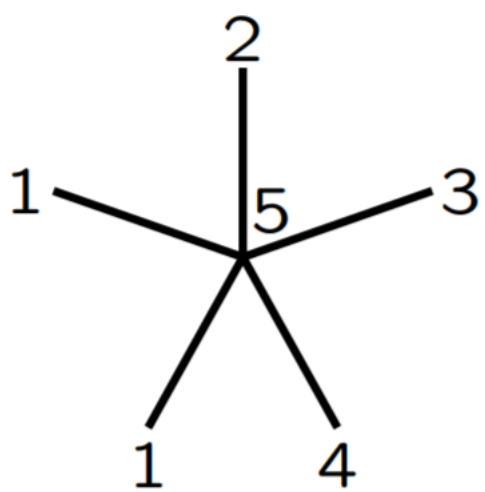
- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we can pick an available color for it.





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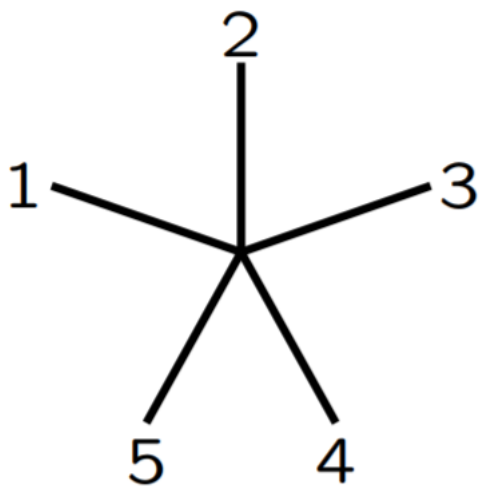
  If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the "special" vertex (degree 5) 1 to 5 (in order).



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  We make a subgraph out of all the vertices colored 1 or 3. If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.

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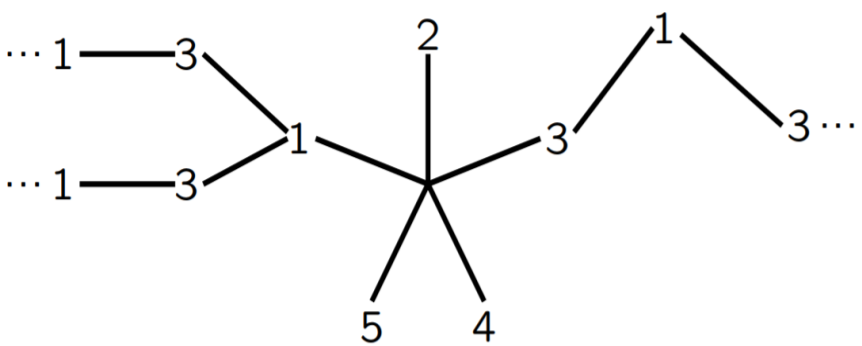

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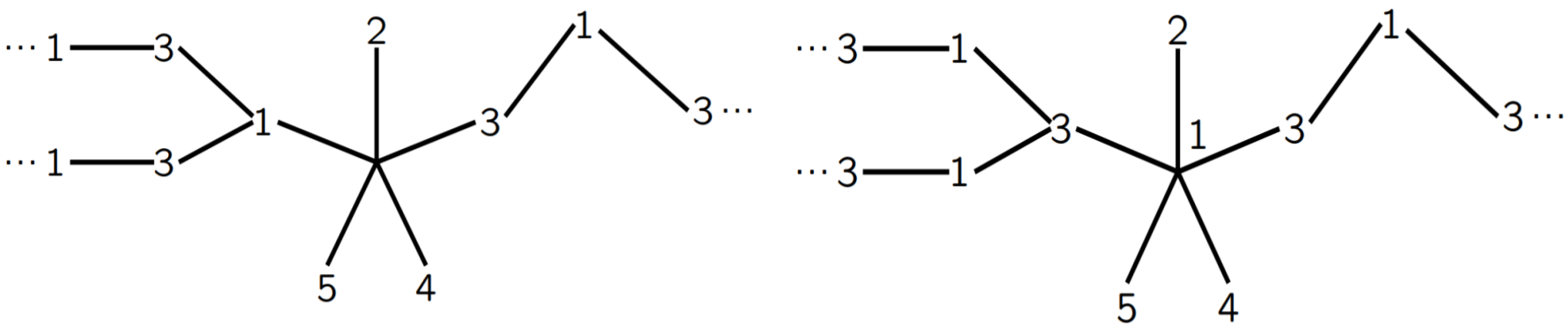

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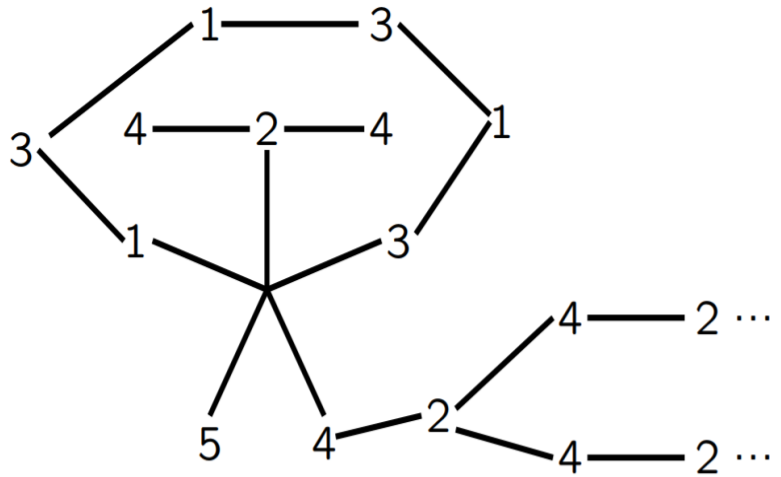

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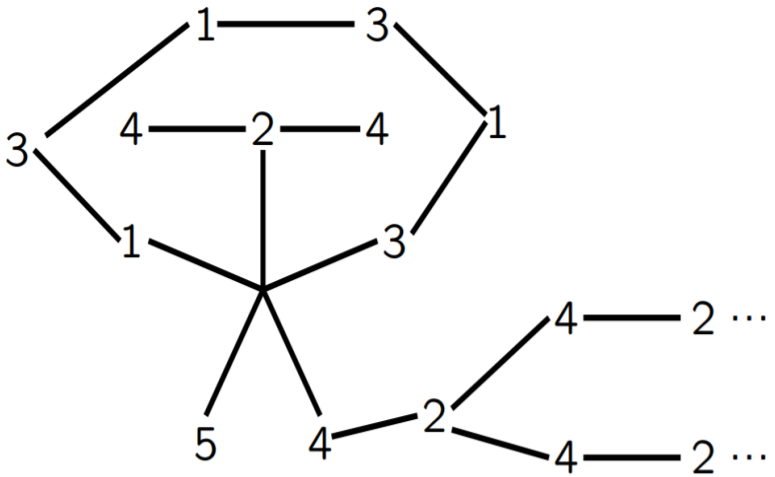

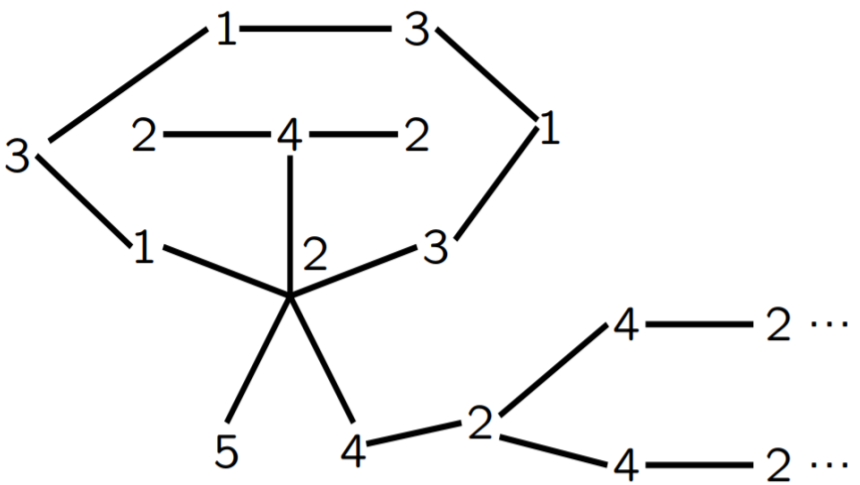

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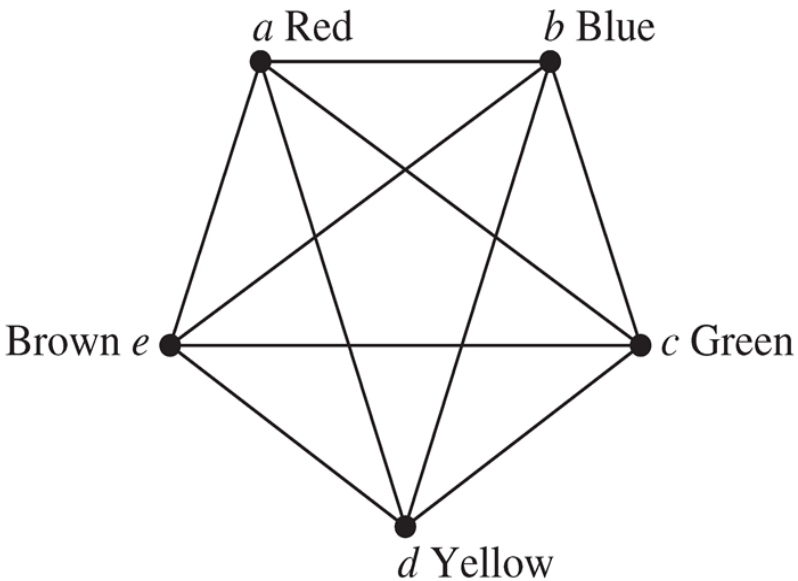

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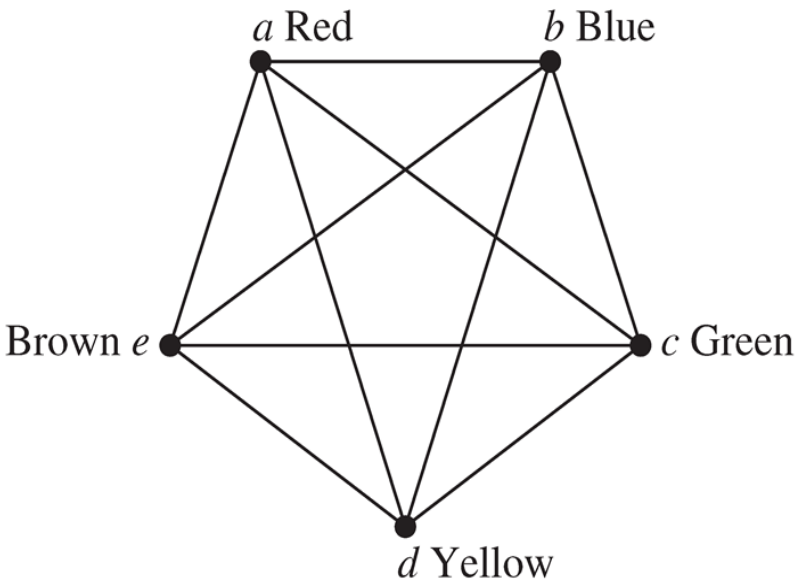

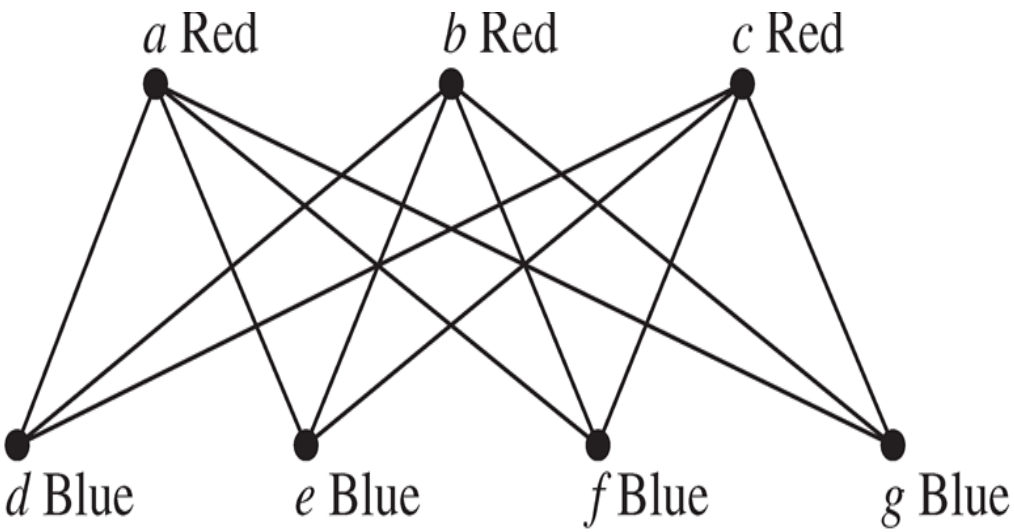

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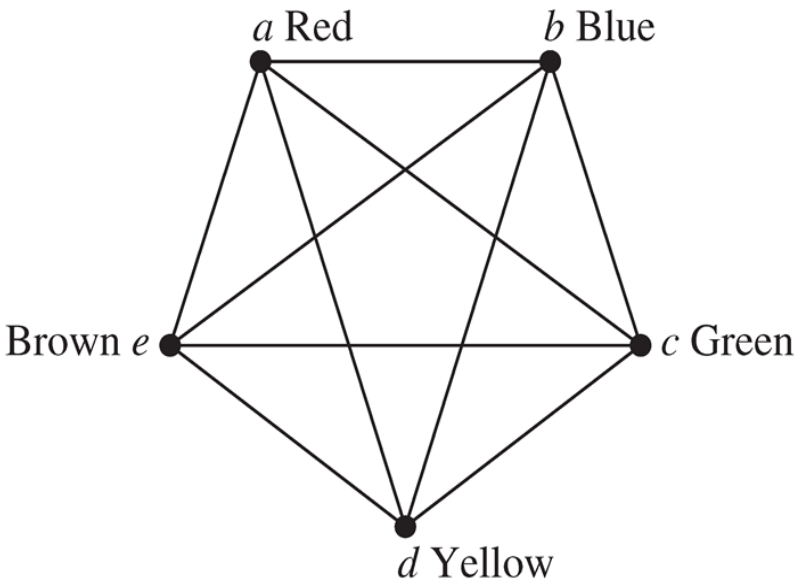

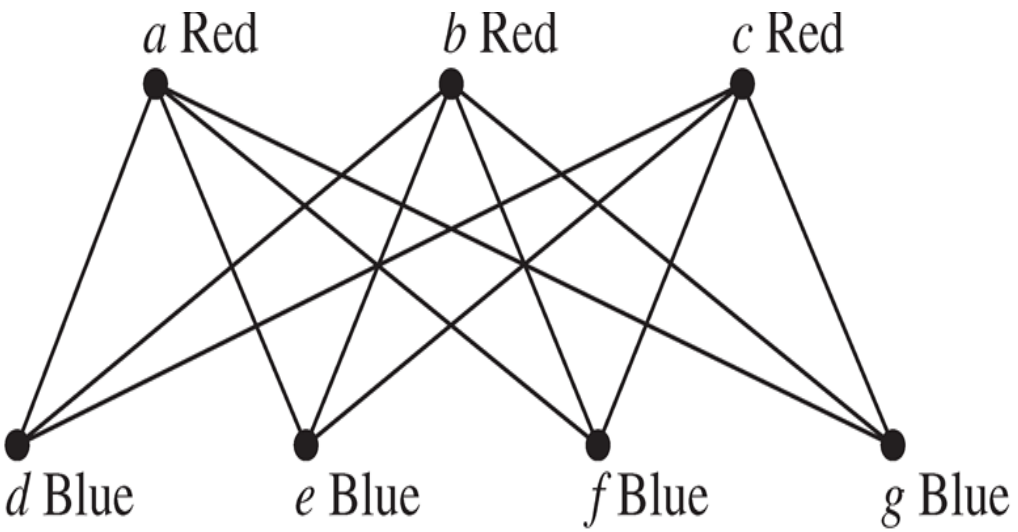

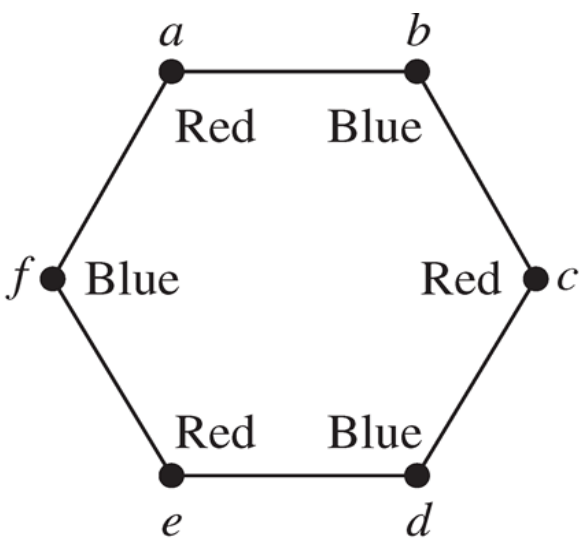

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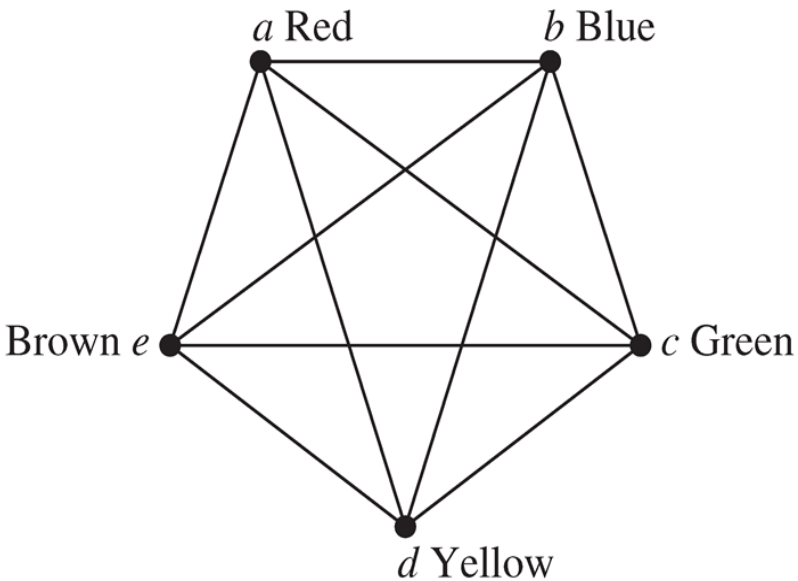
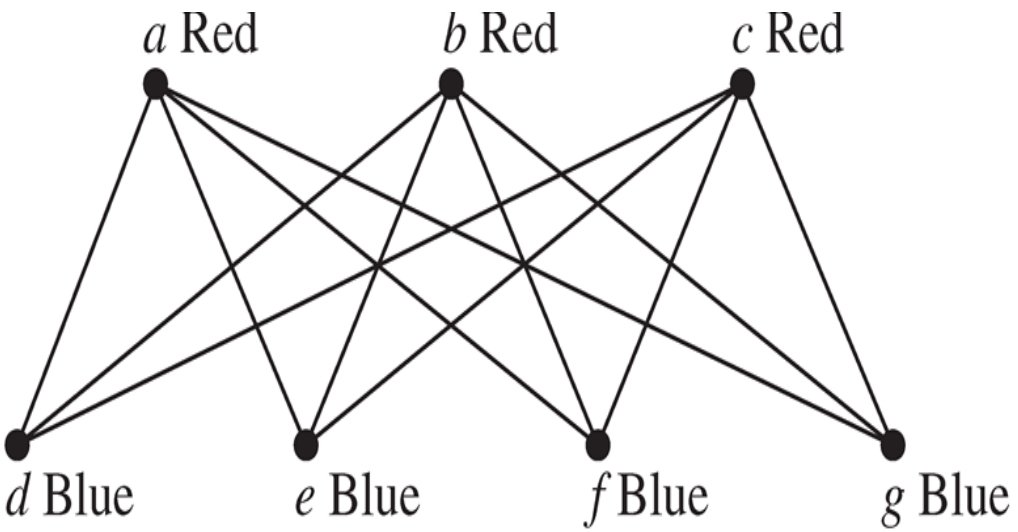
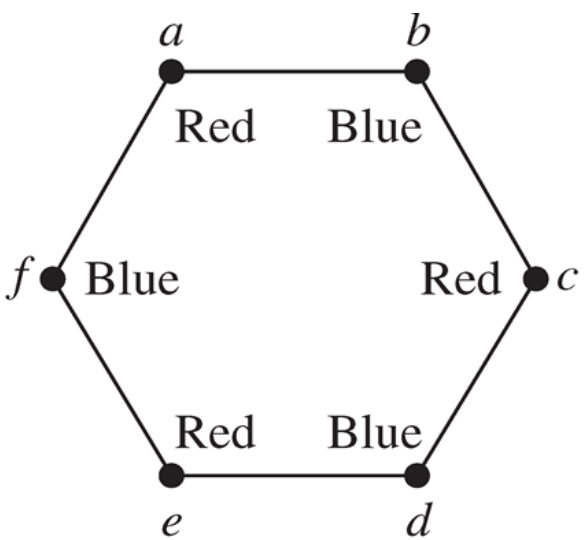
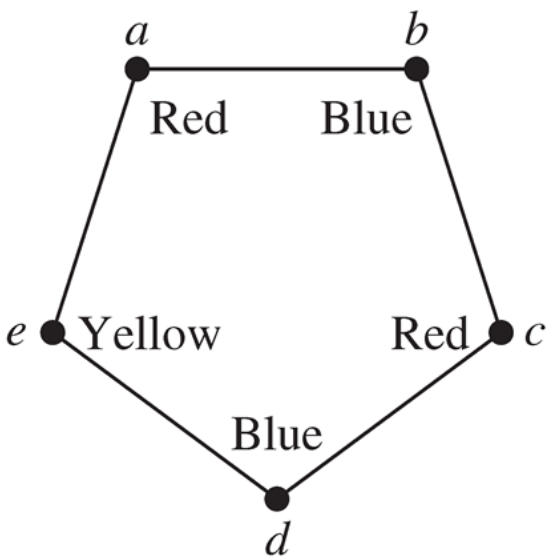




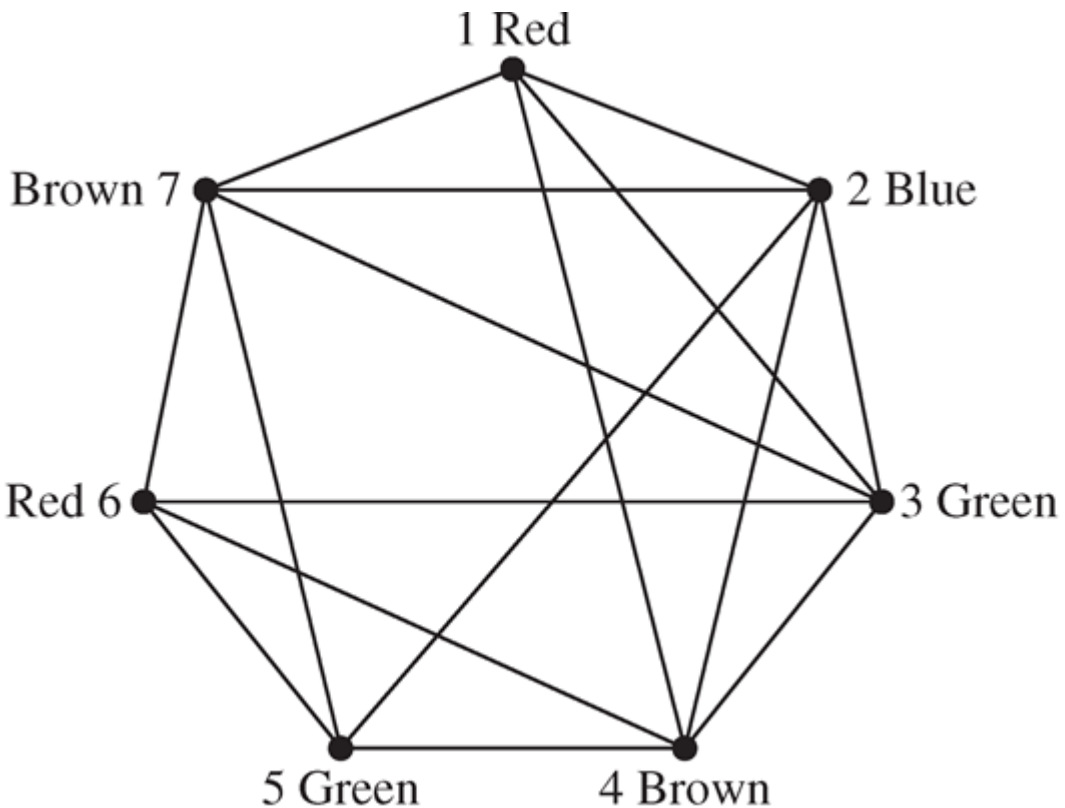




Applications of Graph Coloring

- Scheduling Final Exams

  Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Coloring

- Channel Assignments

  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Graph Coloring $\in$ NPC

Next Lecture

- tree ...