


CS215 DISCRETE MATH

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Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

  Transitive Relation: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

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Recall Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

Connectivity

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Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

2. Suppose that S is a transitive relation containing R.

Then S^n is also transitive and $S^n \subseteq S$. Why?

We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

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Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{M}_{R^*} = ?$

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n - 1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
for $i := 2$ to n
  $A := A \odot \mathbf{M}_R$
  $B := B \vee A$
return B
// B is the zero-one matrix for R^*

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Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
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$W := \mathbf{M}_R$;
for $k := 1$ to n
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$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.

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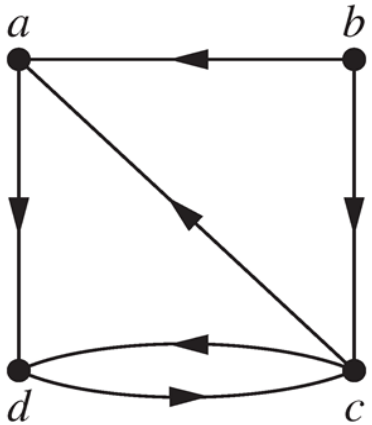
Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



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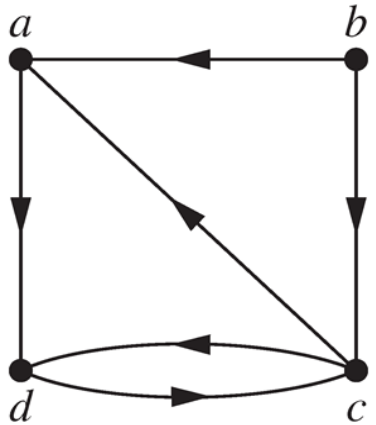

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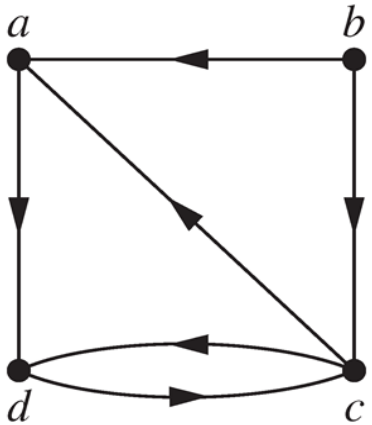

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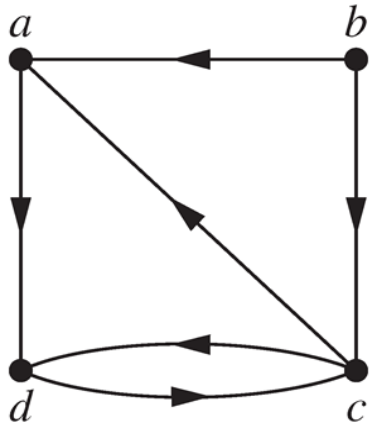

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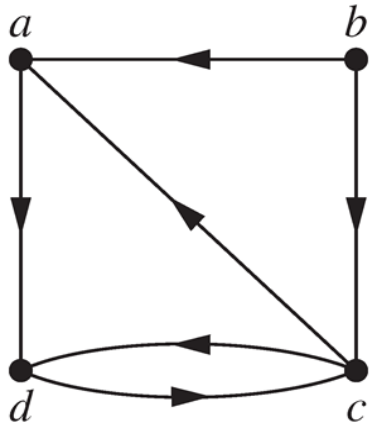

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n-ary Relations

- **Definition** An n-ary relation R on sets $A_1, \ldots, A_n$, written as $R : A_1, \ldots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

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  – R is *functional* in domain A_i if it contains at most one n-tuple $(\cdots, a_i, \cdots)$ for any value a_i within domain A_i.

Relational Databases

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- A *composite key* for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

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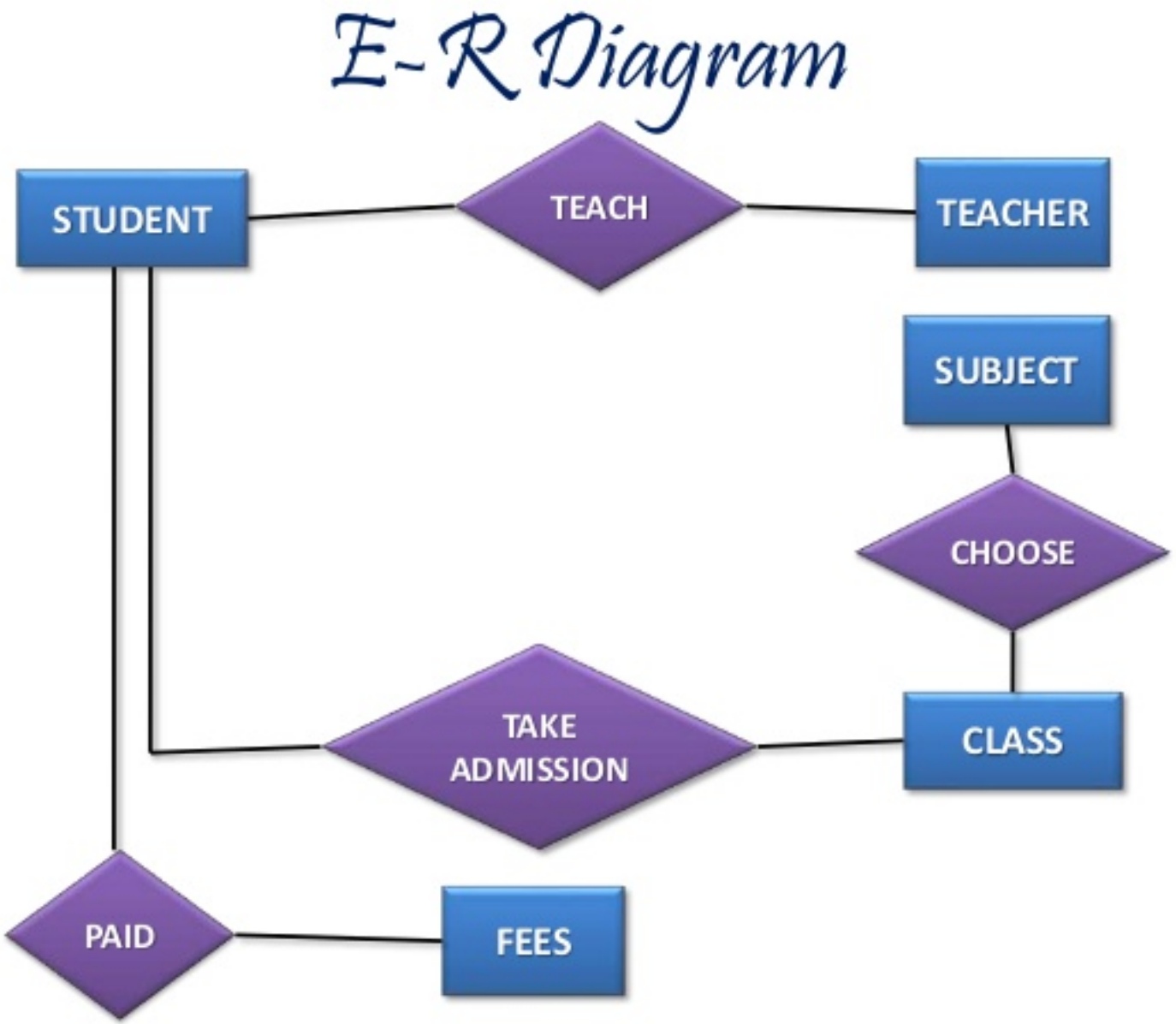

Selection Operators

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  – $\forall R \subseteq A$,

$$\begin{aligned} s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\ &= \{a \in R \mid s_C(a) = T\}. \end{aligned}$$

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- Then, $s_{\mathit{UpperLevel}}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.

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- Then the *projection operator* on n-tuples

  $$P_{\{i_k\}} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

  is defined by

  $$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

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- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of $model/color$ combinations available.

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- A, B, C can also be sequences of elements rather that single elements.

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- Suppose that R_2 is a room assignment table relating *Courses* to *Rooms* and *Times*.
- Then $J(R_1, R_2)$ is like your class schedule, listing (*professor*, *course*, *room*, *time*).

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  R has the following pairs:

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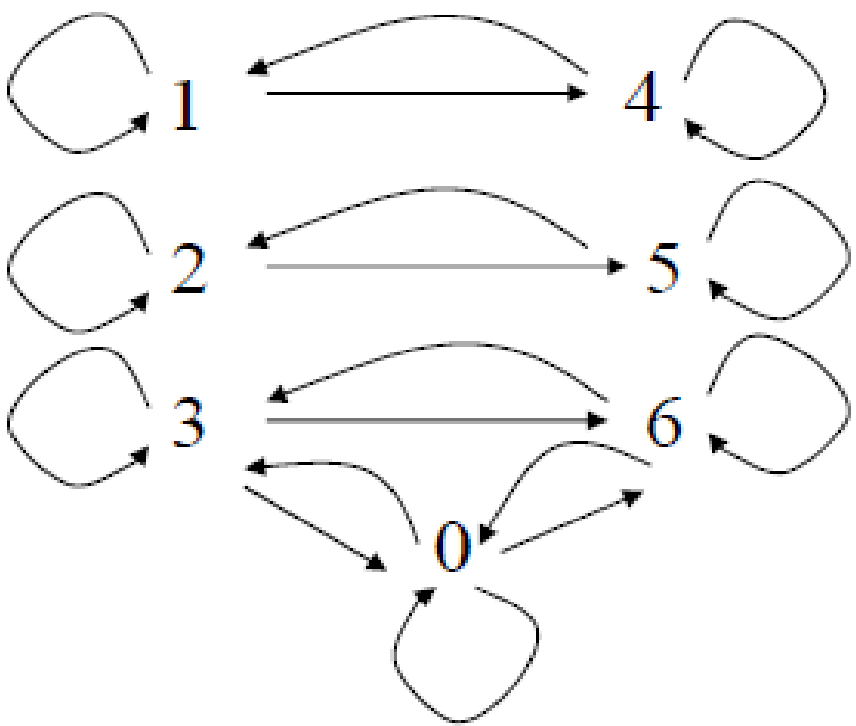

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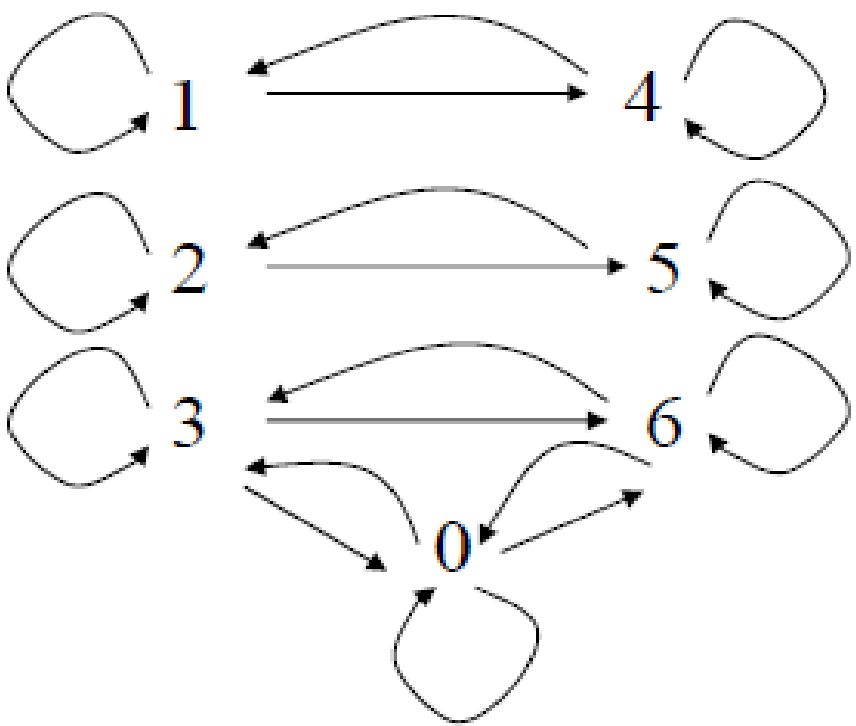

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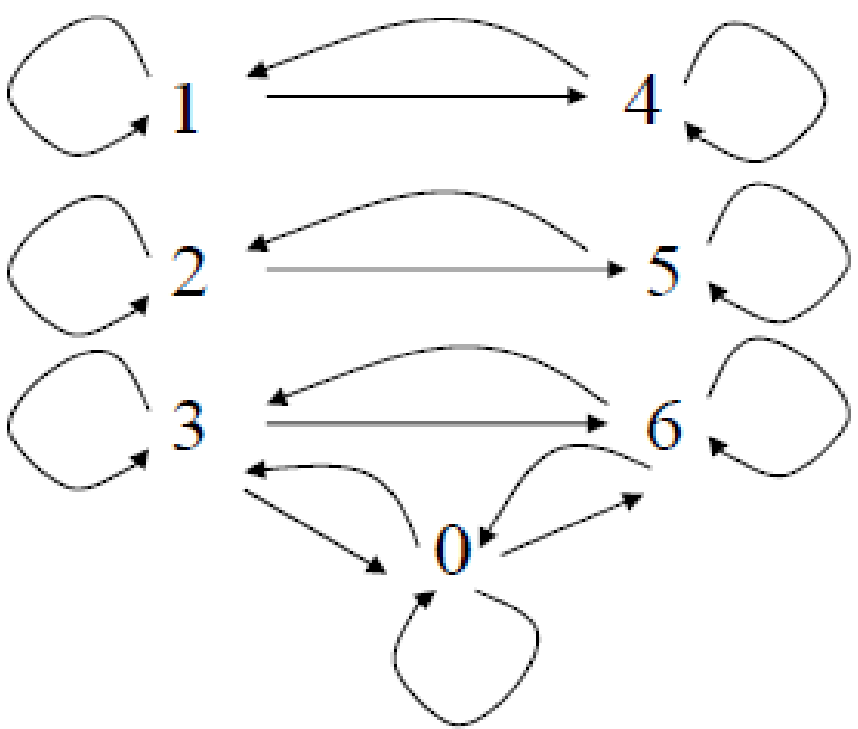
Is R reflexive? Yes

Is R symmetric?



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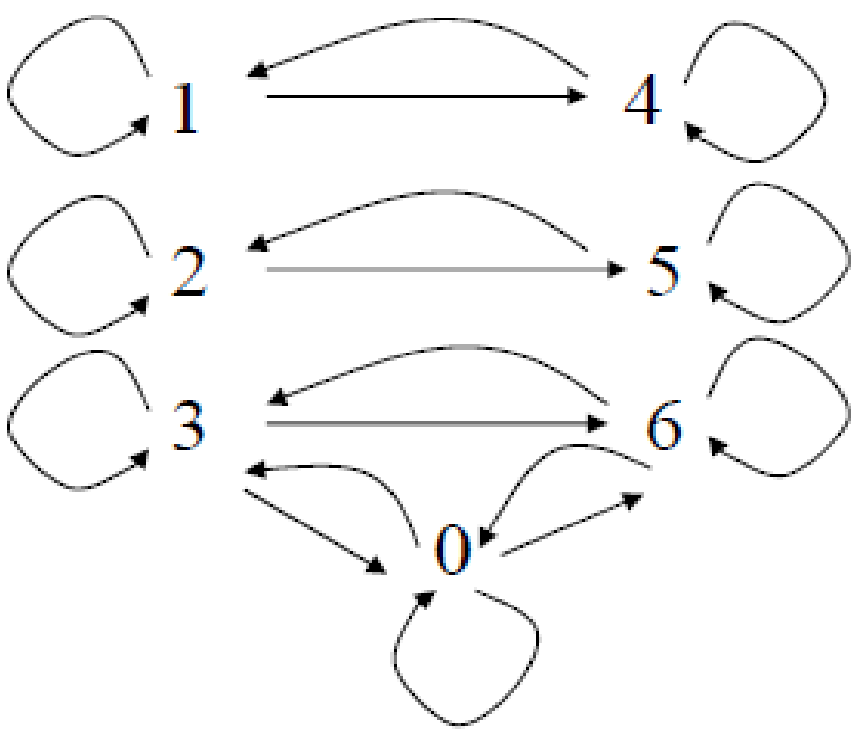

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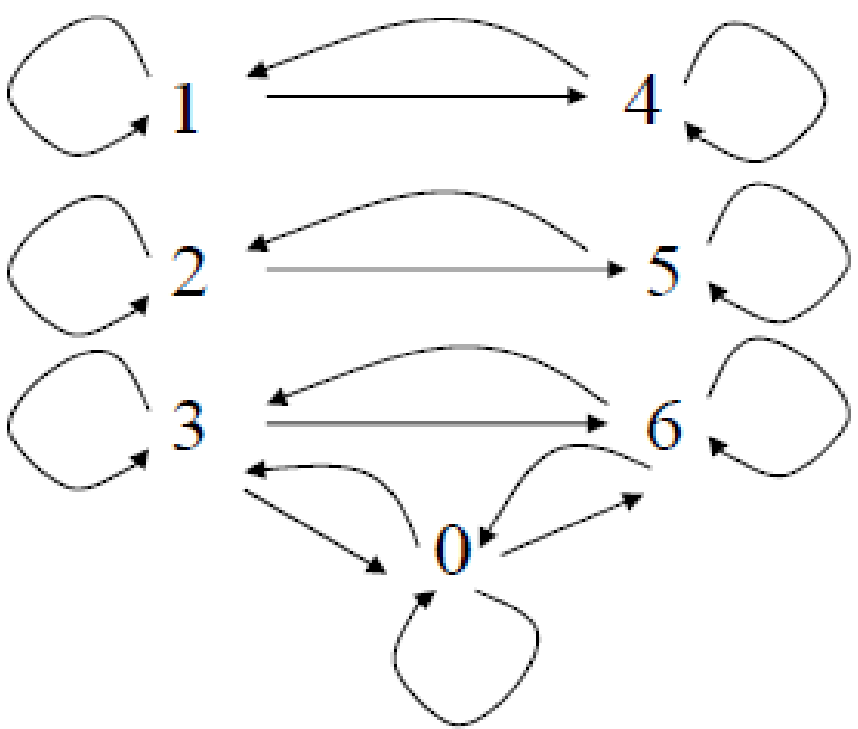
Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive?



Equivalence Relation

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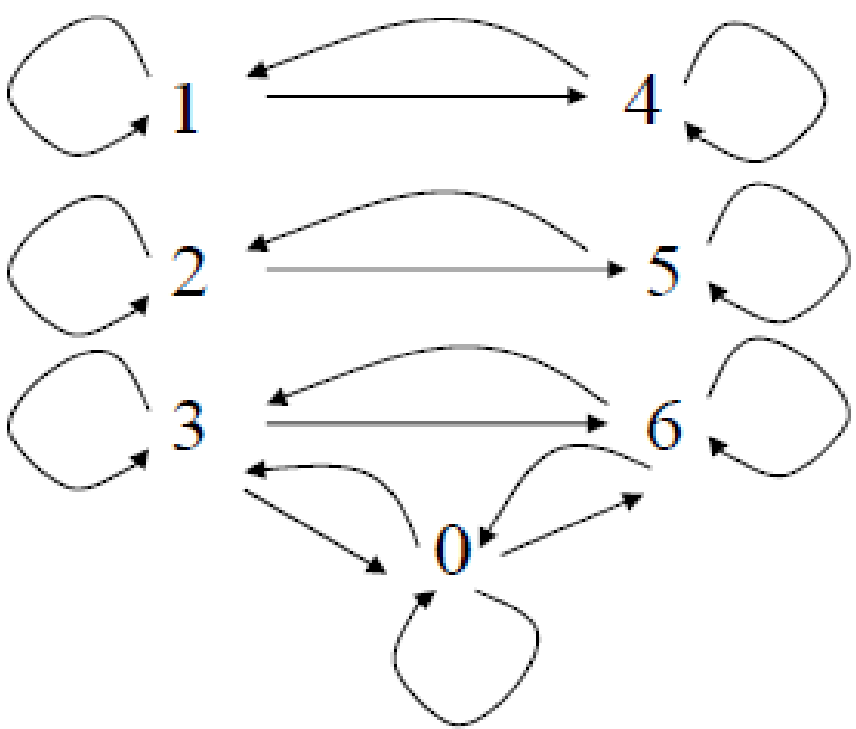

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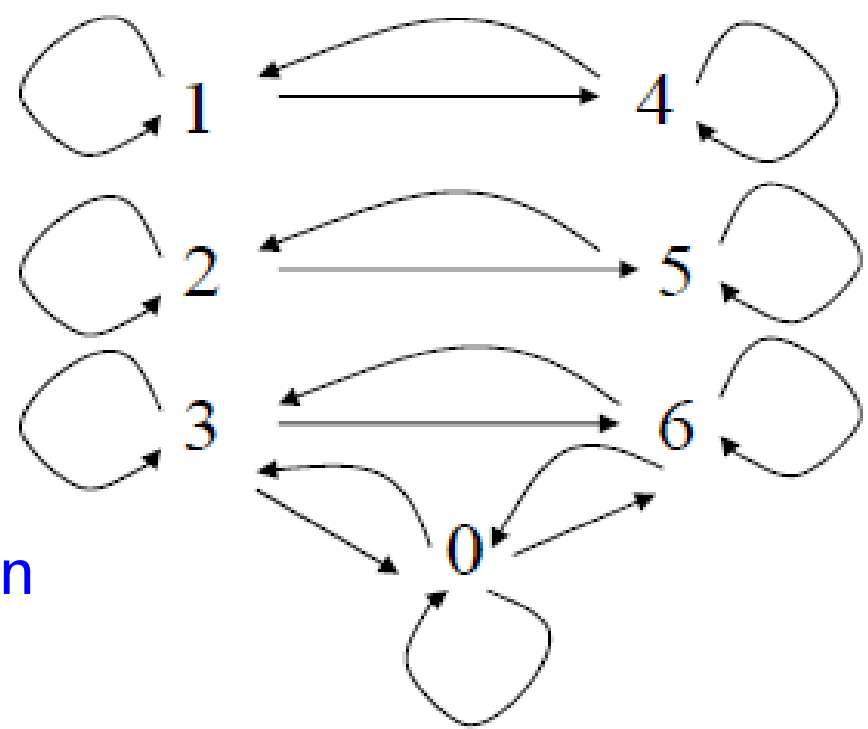

R is an equivalence relation

Examples of Equivalence Relations

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  "Strings a and b have the same length."

  "Integers a and b have the same absolute value."

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  $[a]$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

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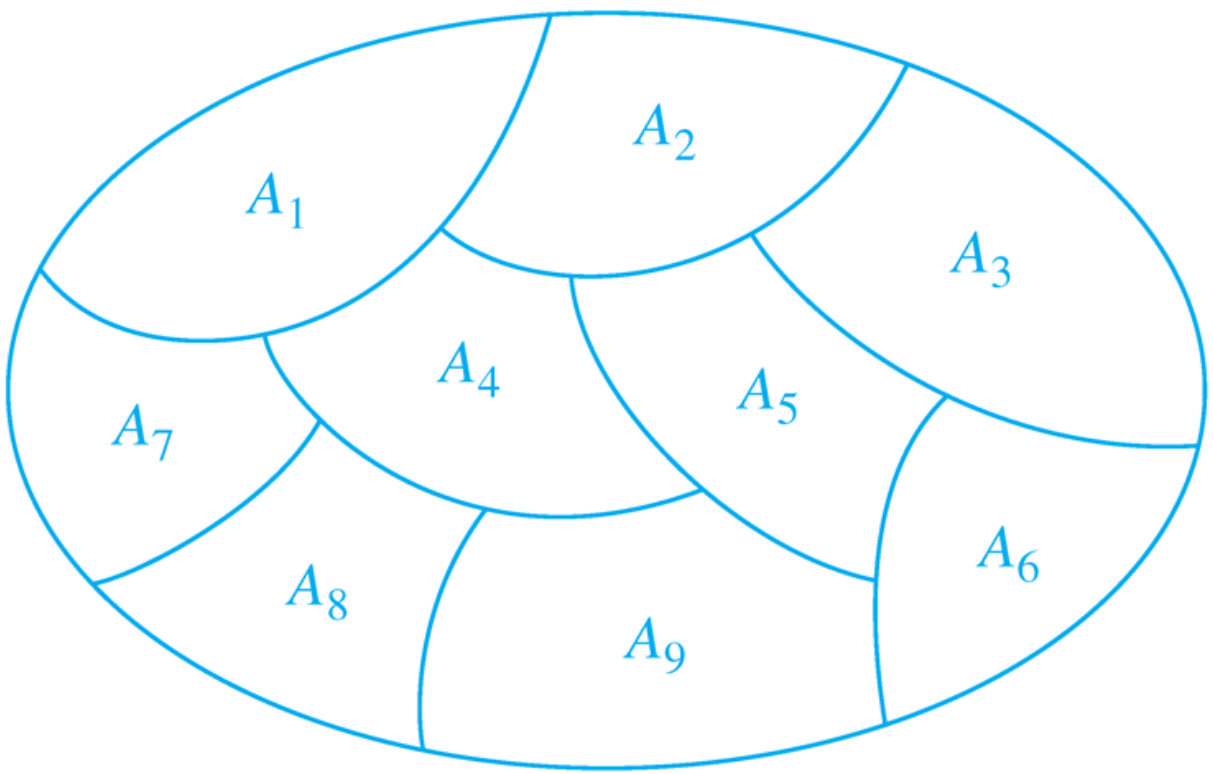

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Theorem Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be a partition of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

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  $2, 4$ are comparable, $3, 5$ are incomparable.

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  - ◇ *discreet* $\prec$ *discrete*
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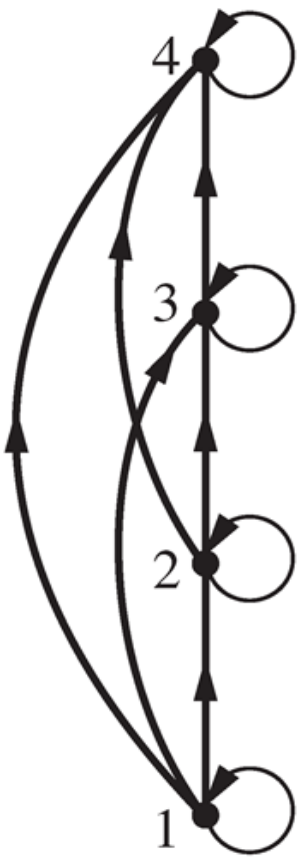
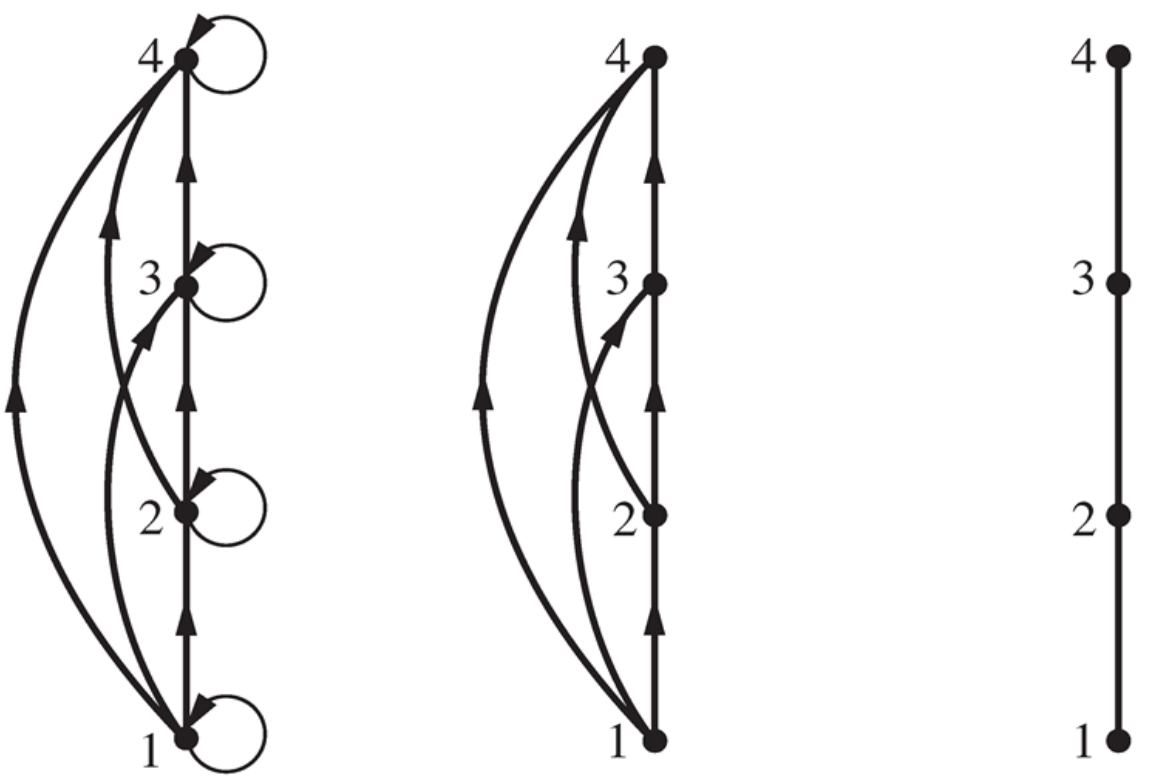


Hasse Diagram

- (a) A partial ordering. The loops are due to the reflexive property

  (b) The edges that must be present due to the transitive property are deleted

  (c) The Hasse diagram for the partial ordering (a)



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  - ◇ Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

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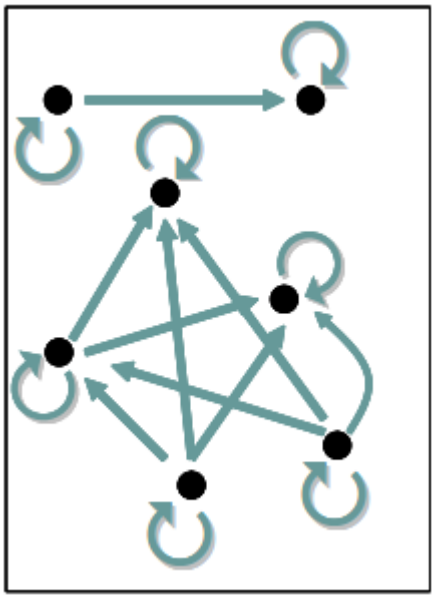

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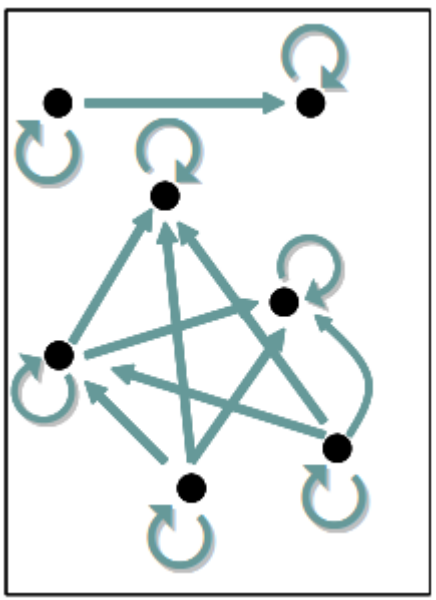

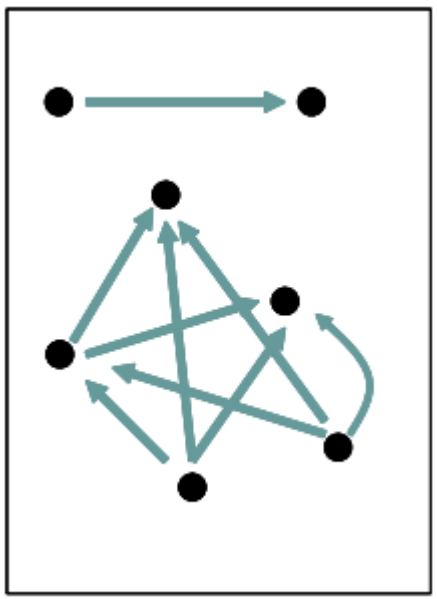

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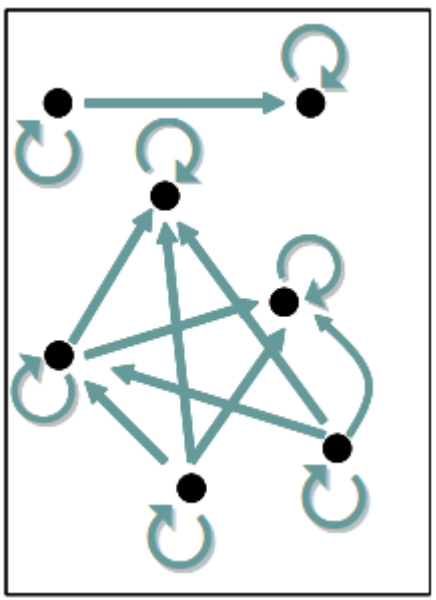

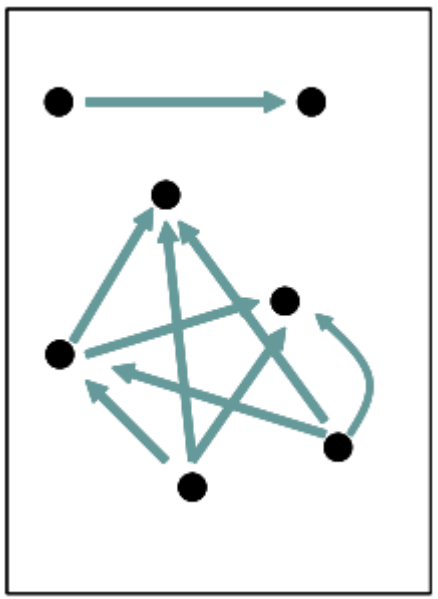

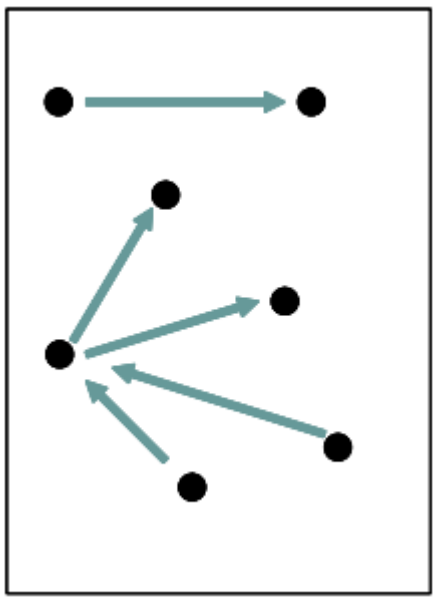

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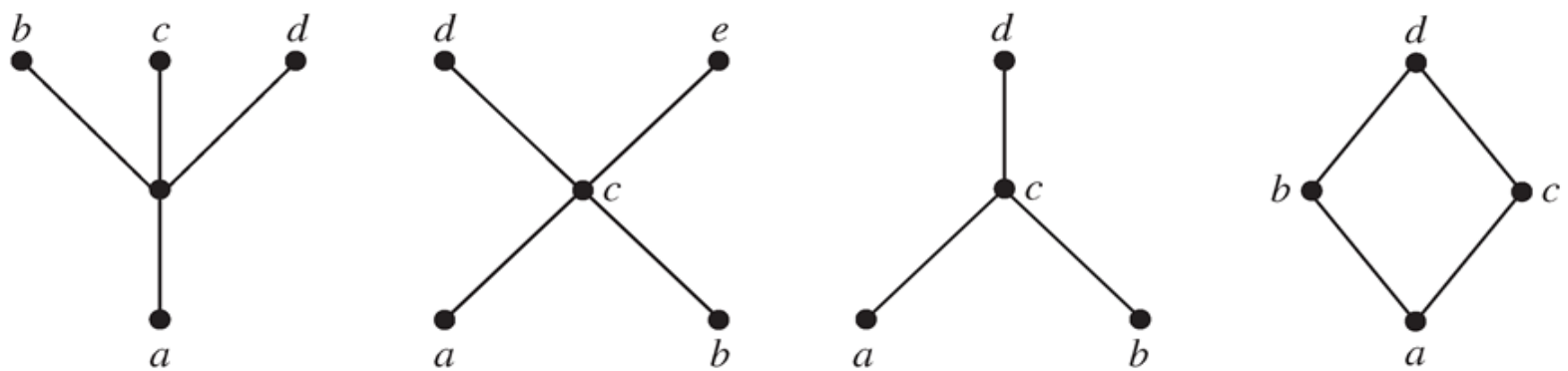

(a) (b) (c) (d)

Maximal and Minimal Elements

- **Definition** Let A be a subset of a poset $(S, \preccurlyeq)$.
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Example Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

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p.620, Theorem 1

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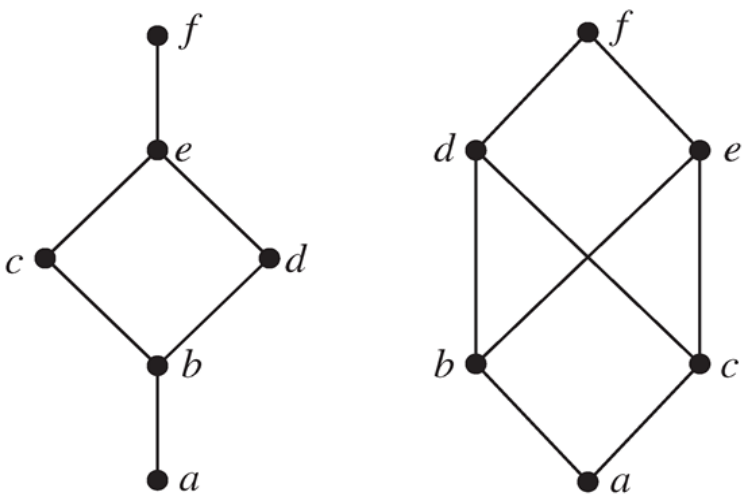

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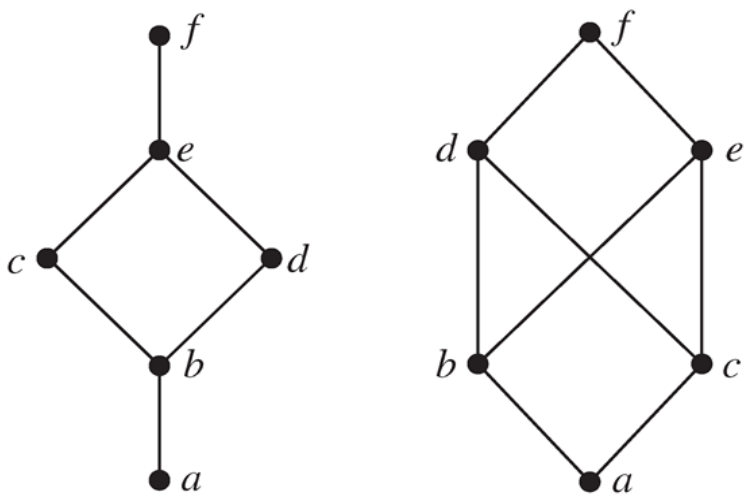

Example Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Topological Sorting

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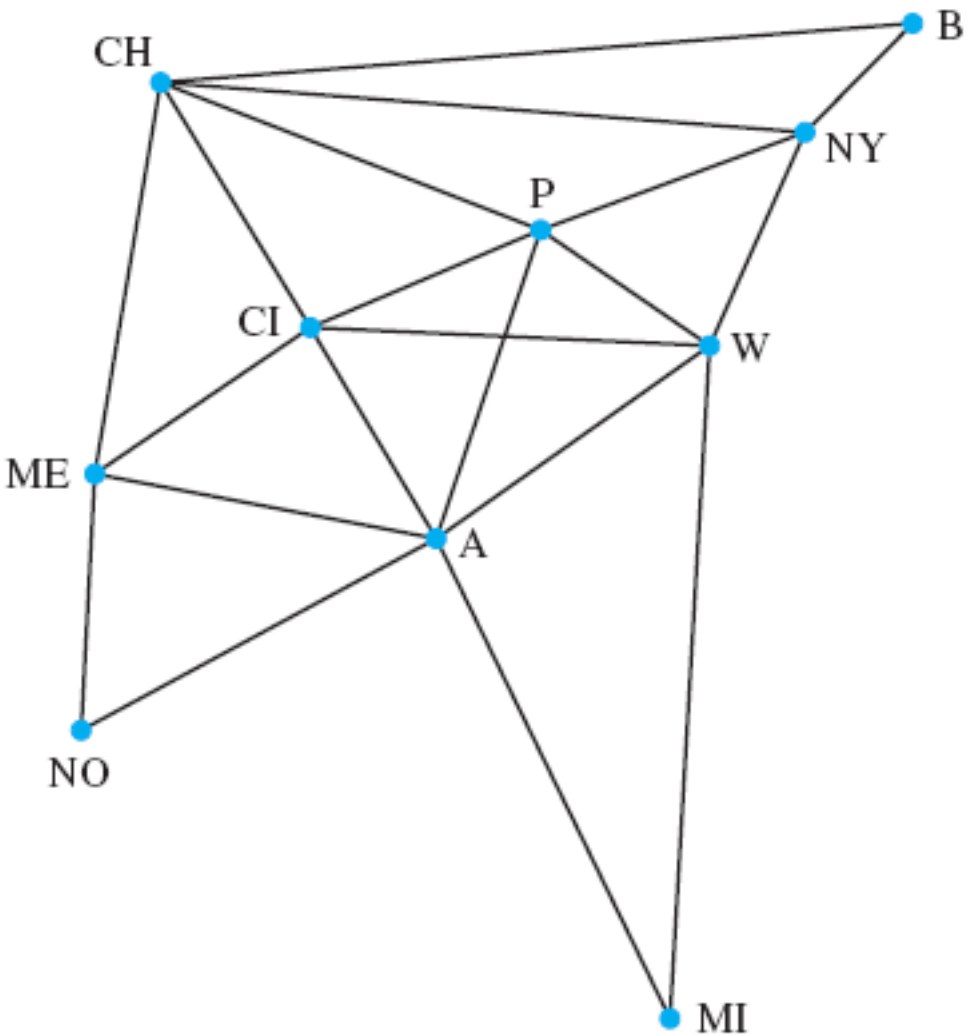
  Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever $a \; R \; b$. $\preccurlyeq$ is said *compatible with* R.

Topological Sorting for Finite Posets

procedure topological_sort (S: finite poset)

$k := 1$;
while $S \neq \emptyset$
  $a_k :=$ a minimal element of S
  $S := S \setminus \{a_k\}$
  $k := k + 1$
end while
// $\{a_1, a_2, \ldots, a_n\}$ is a compatible total ordering of S

Example



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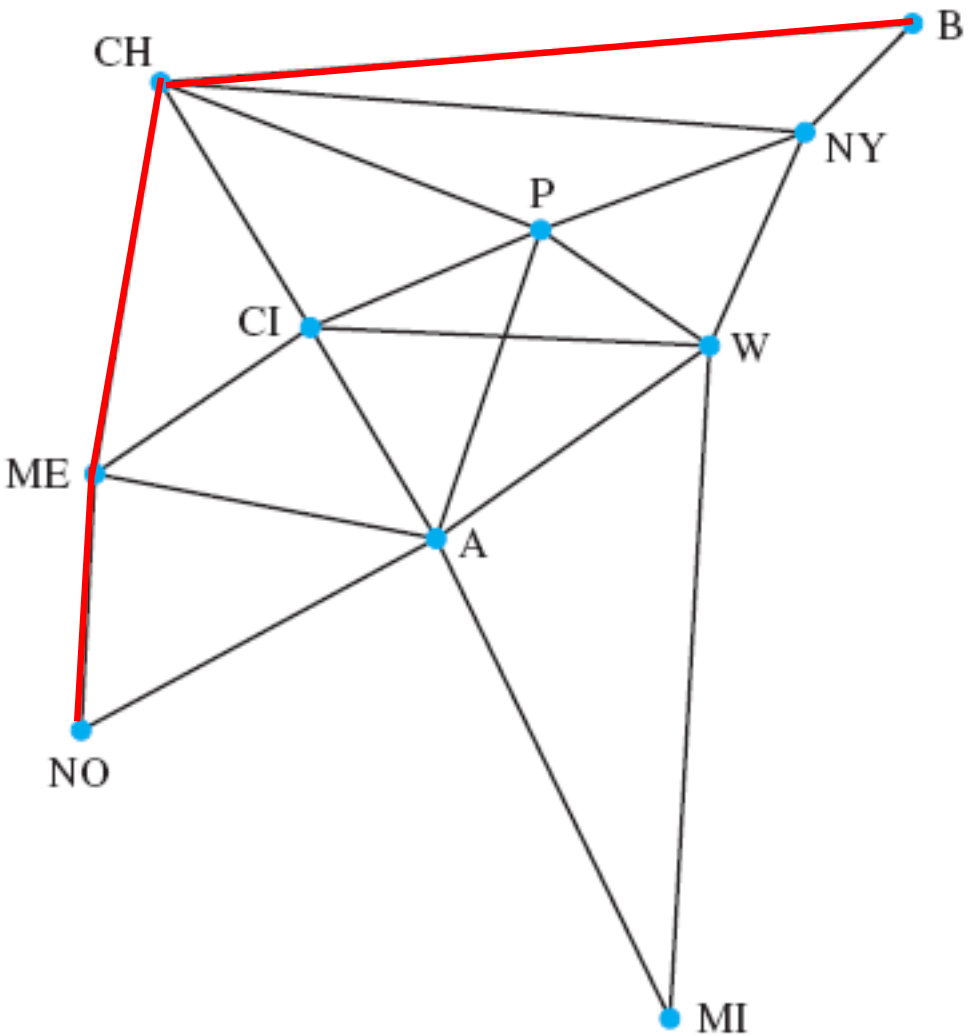

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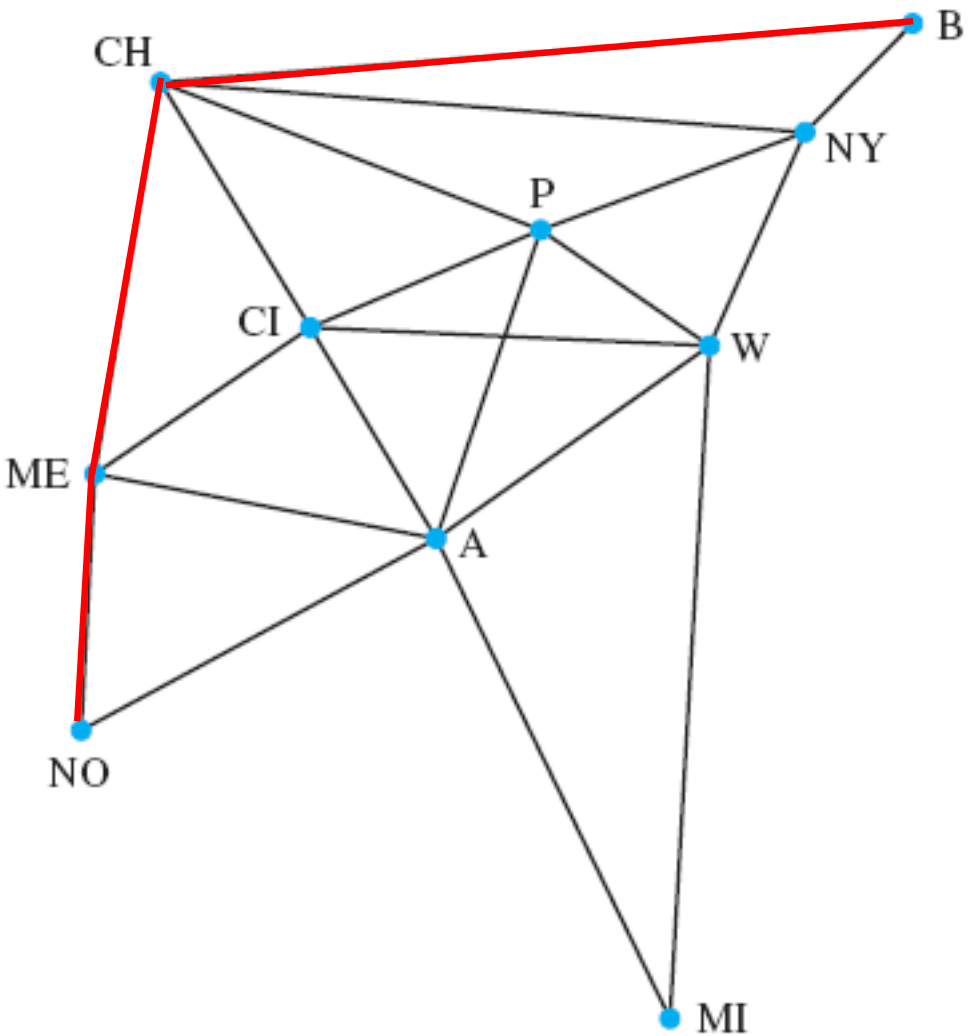

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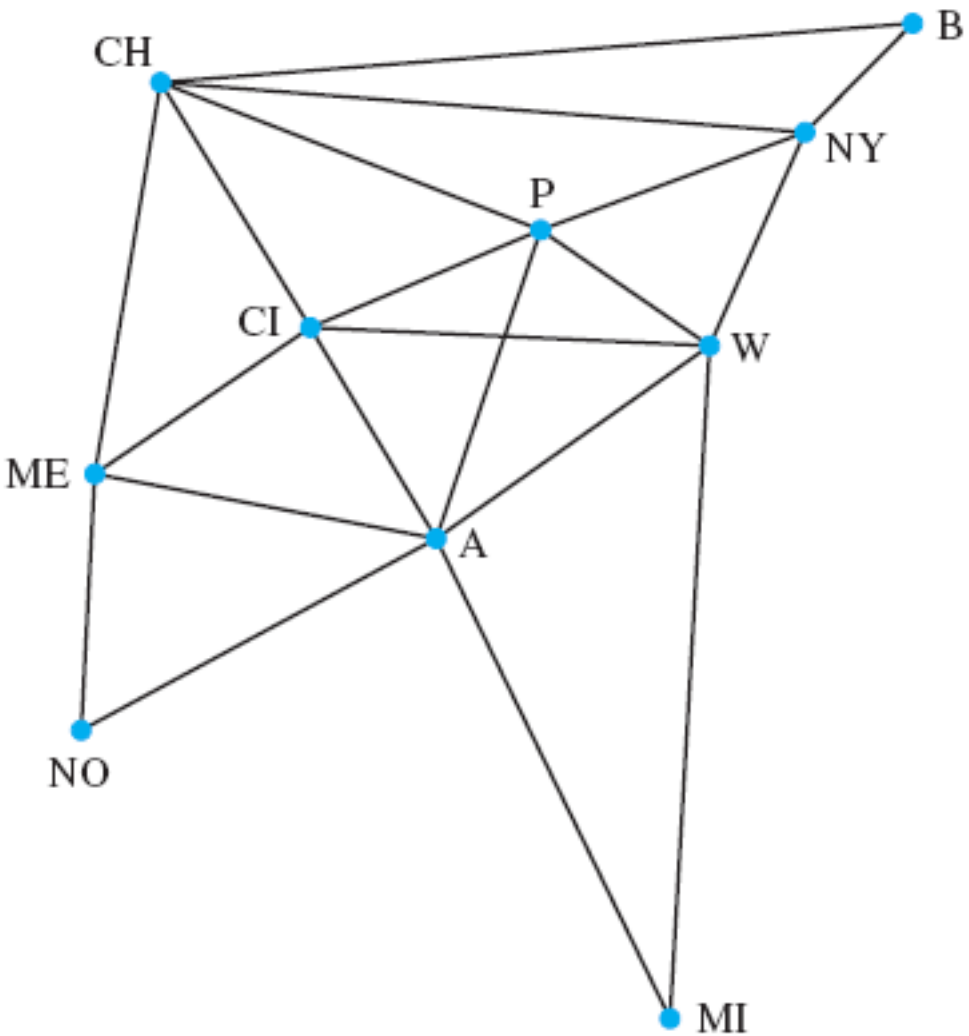

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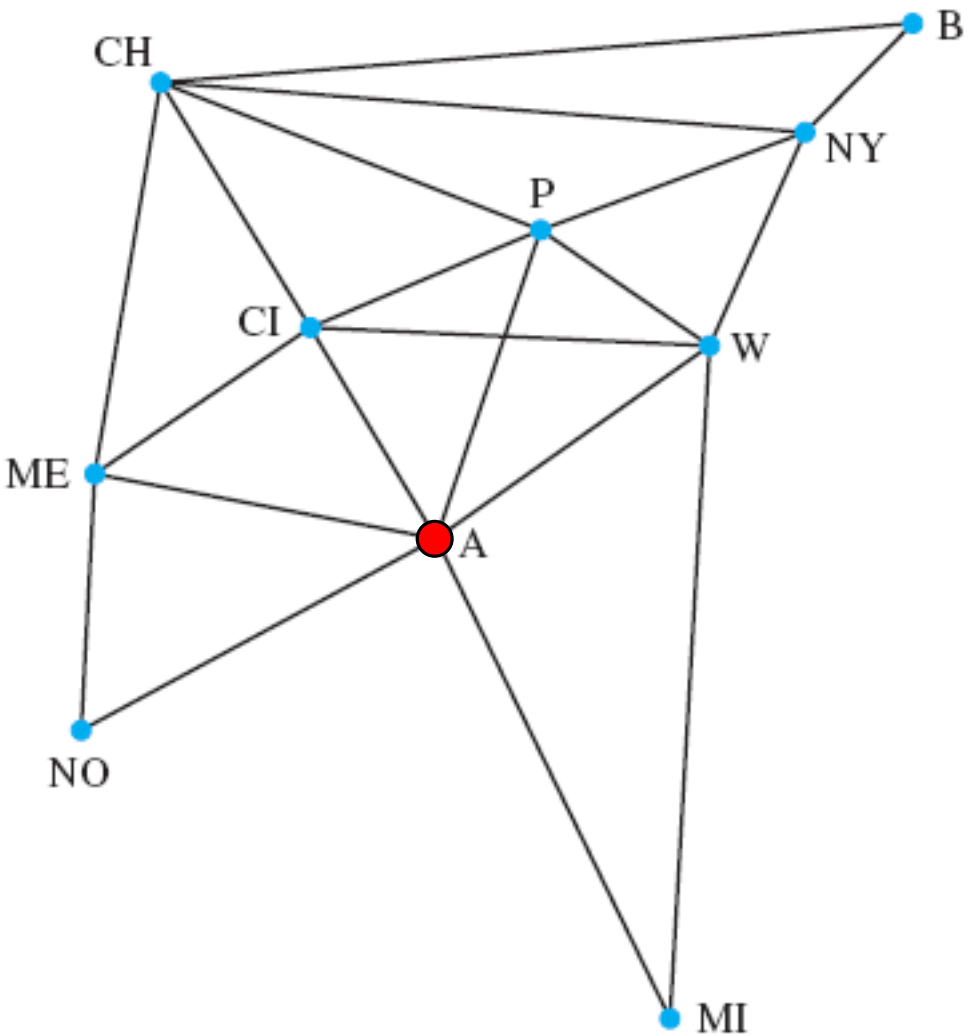

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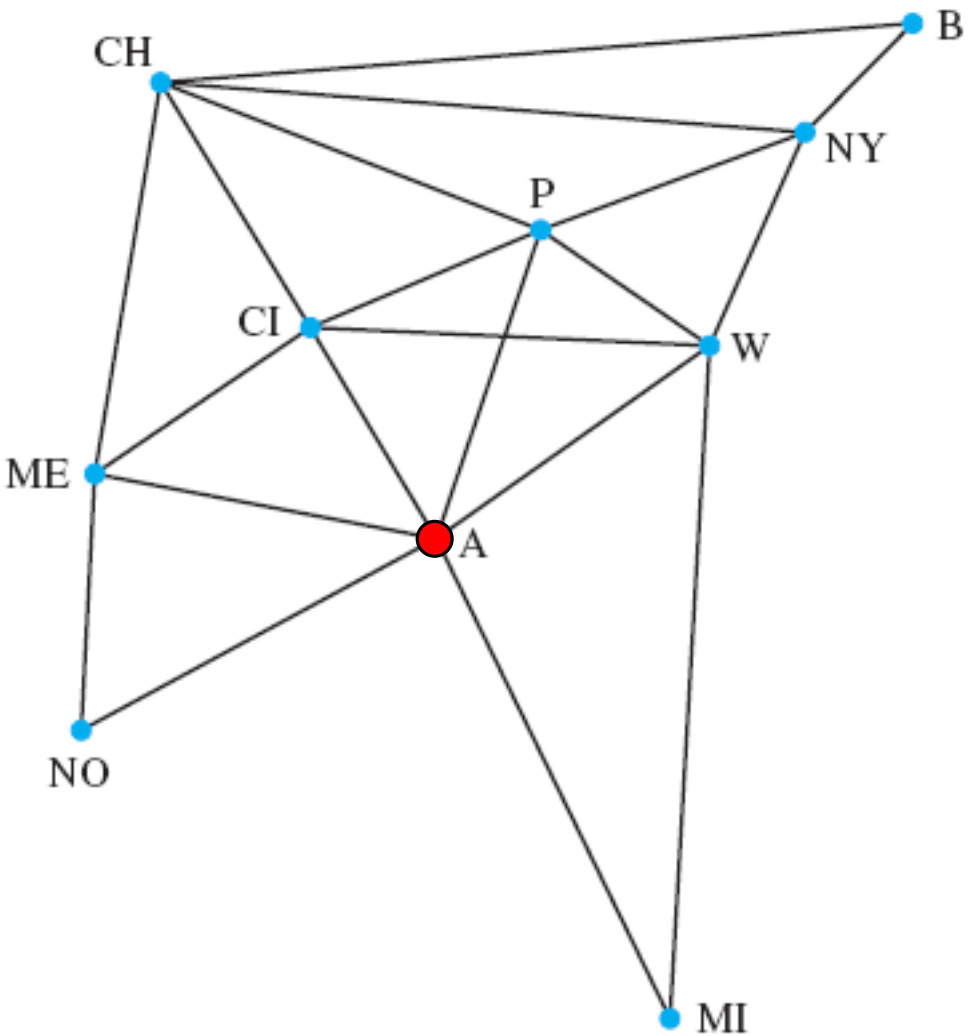

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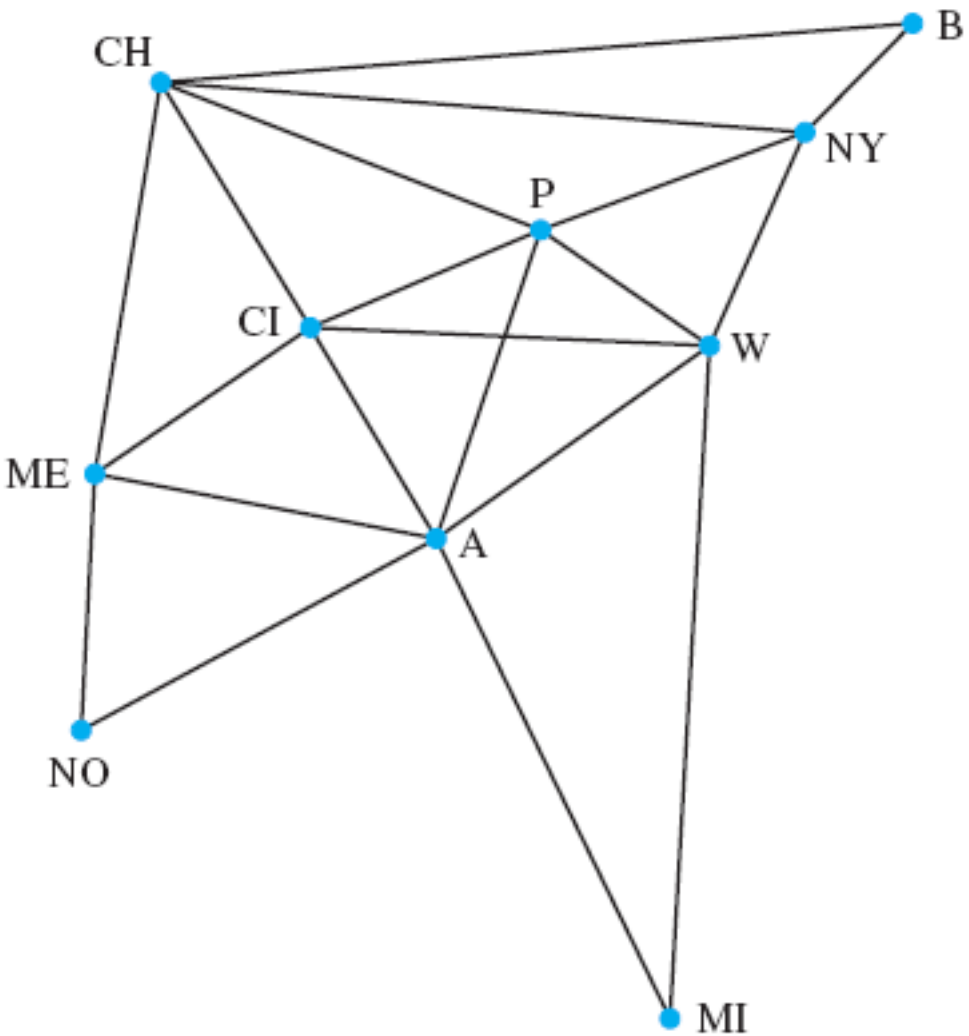

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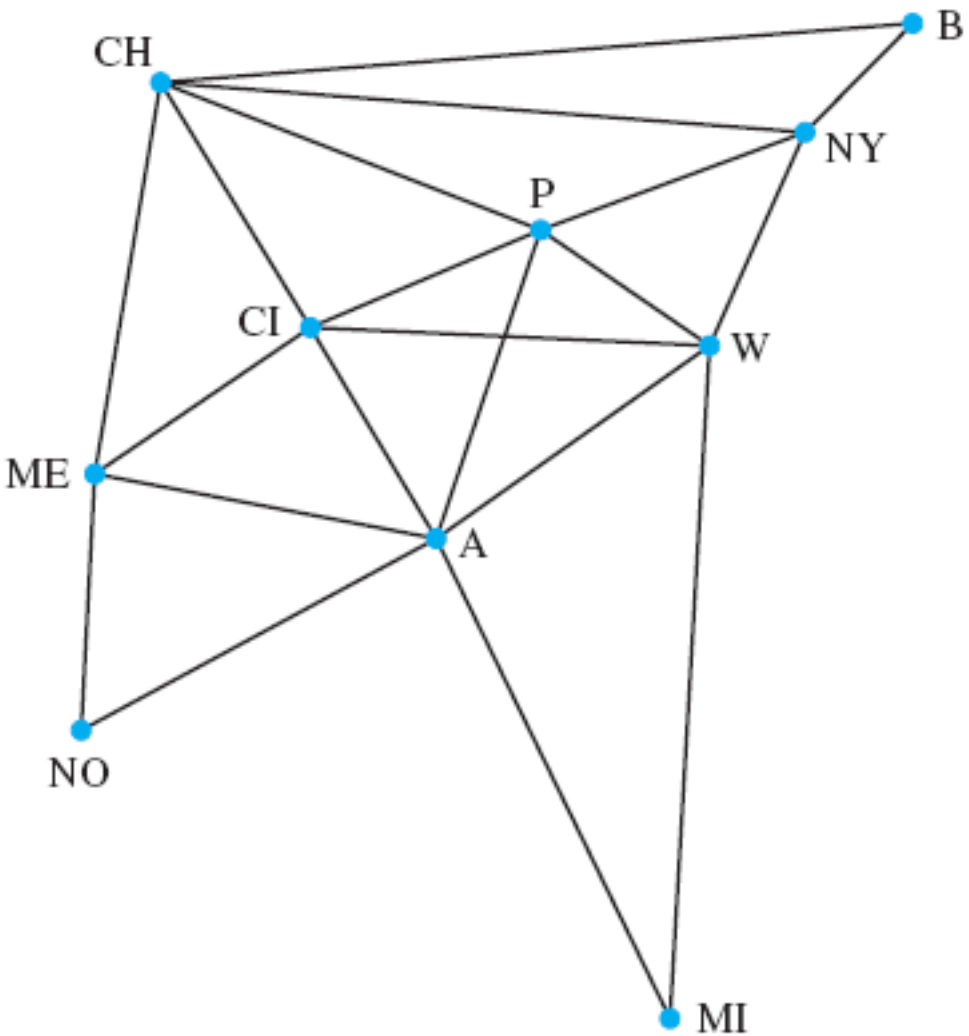

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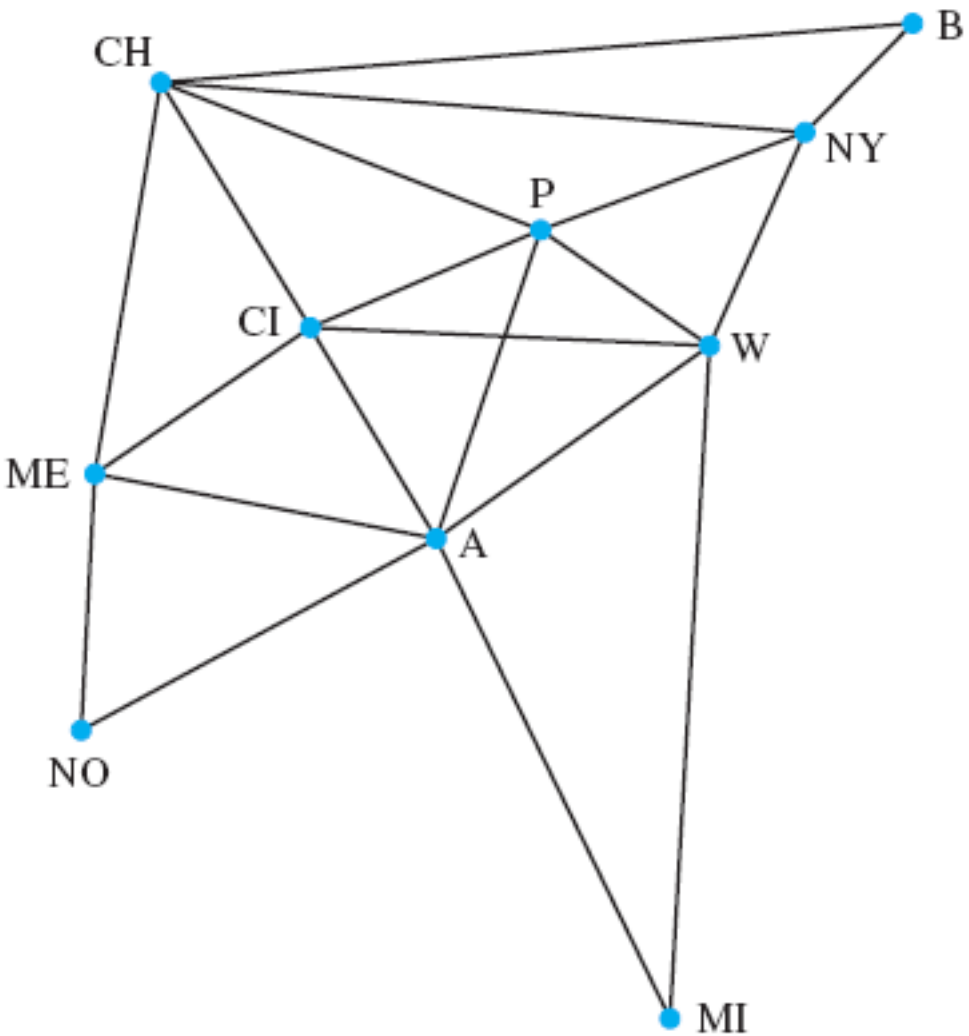

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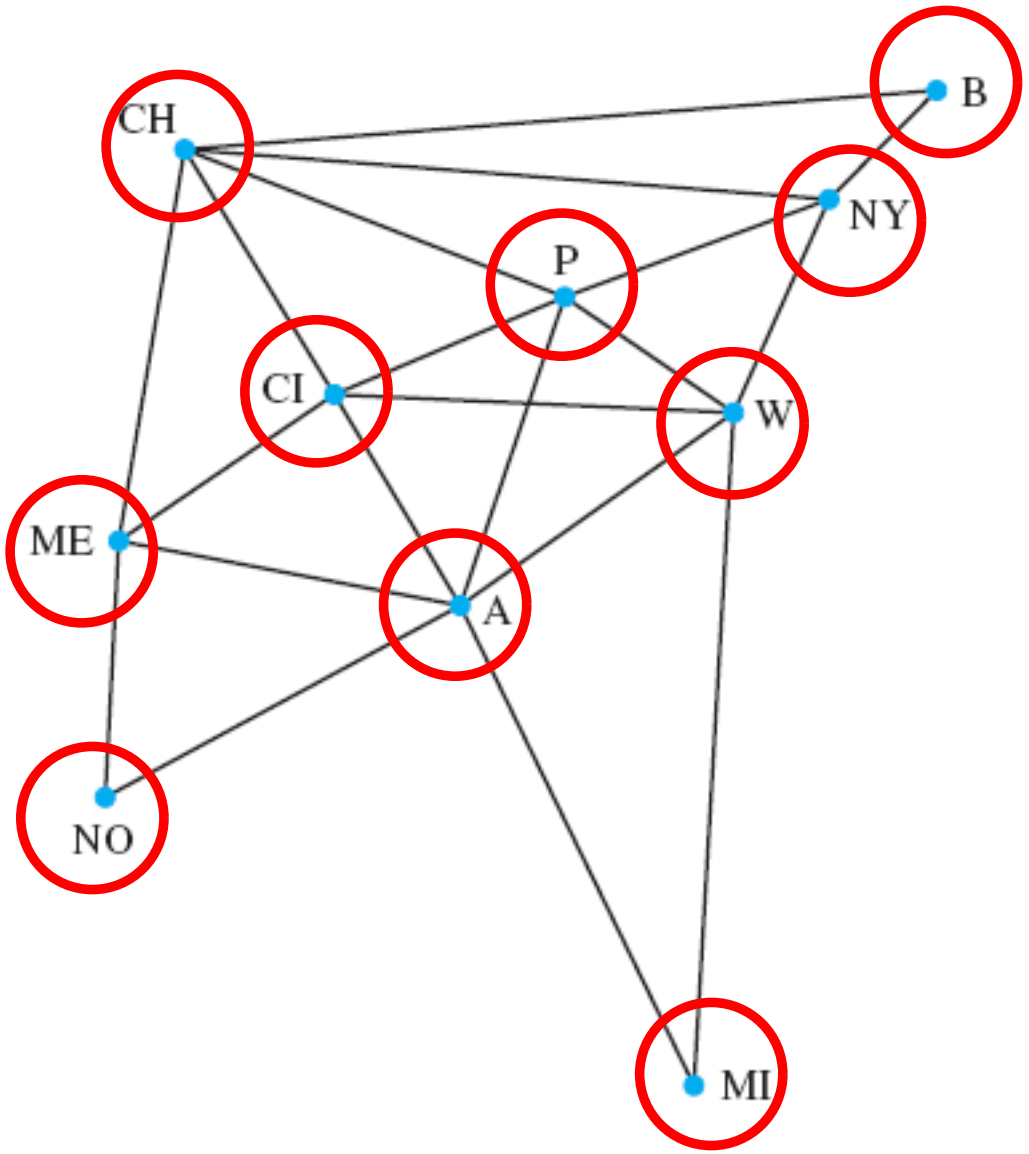

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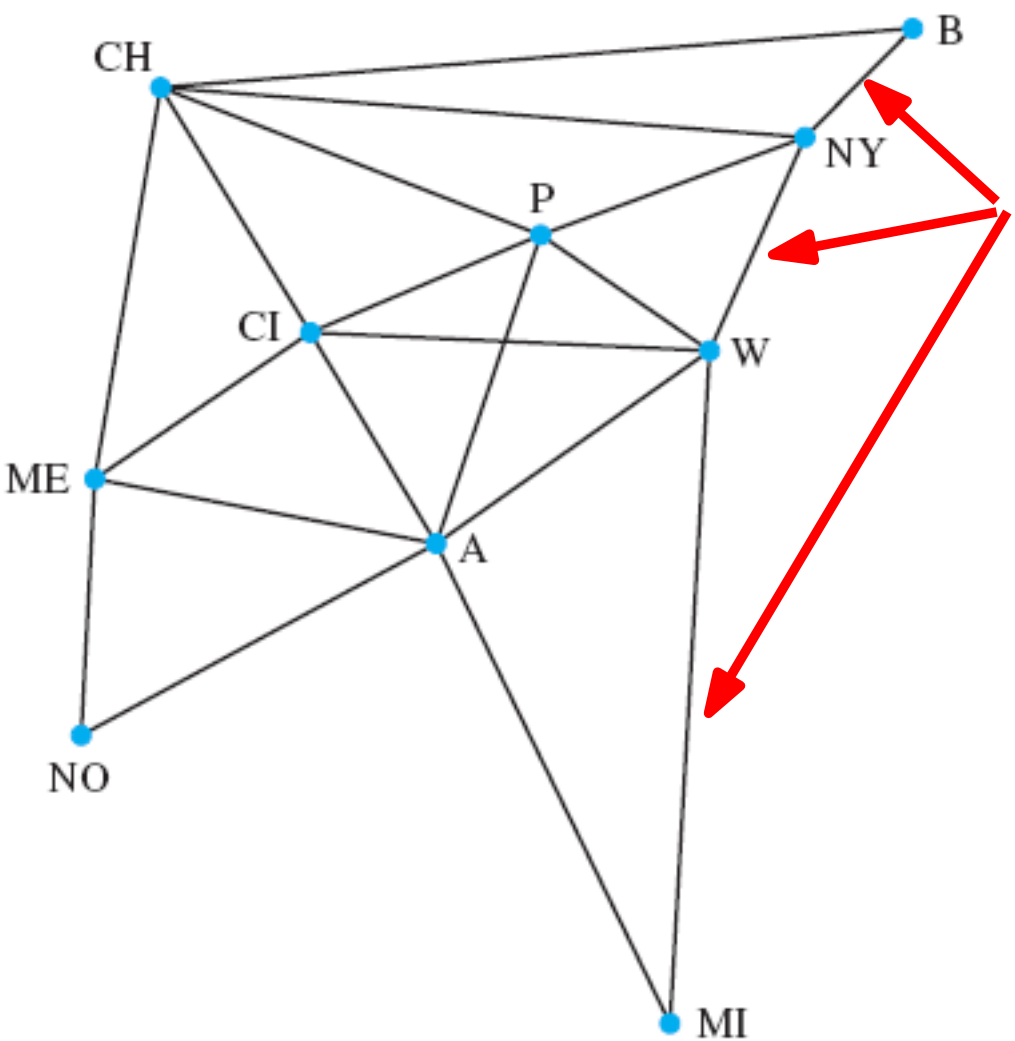

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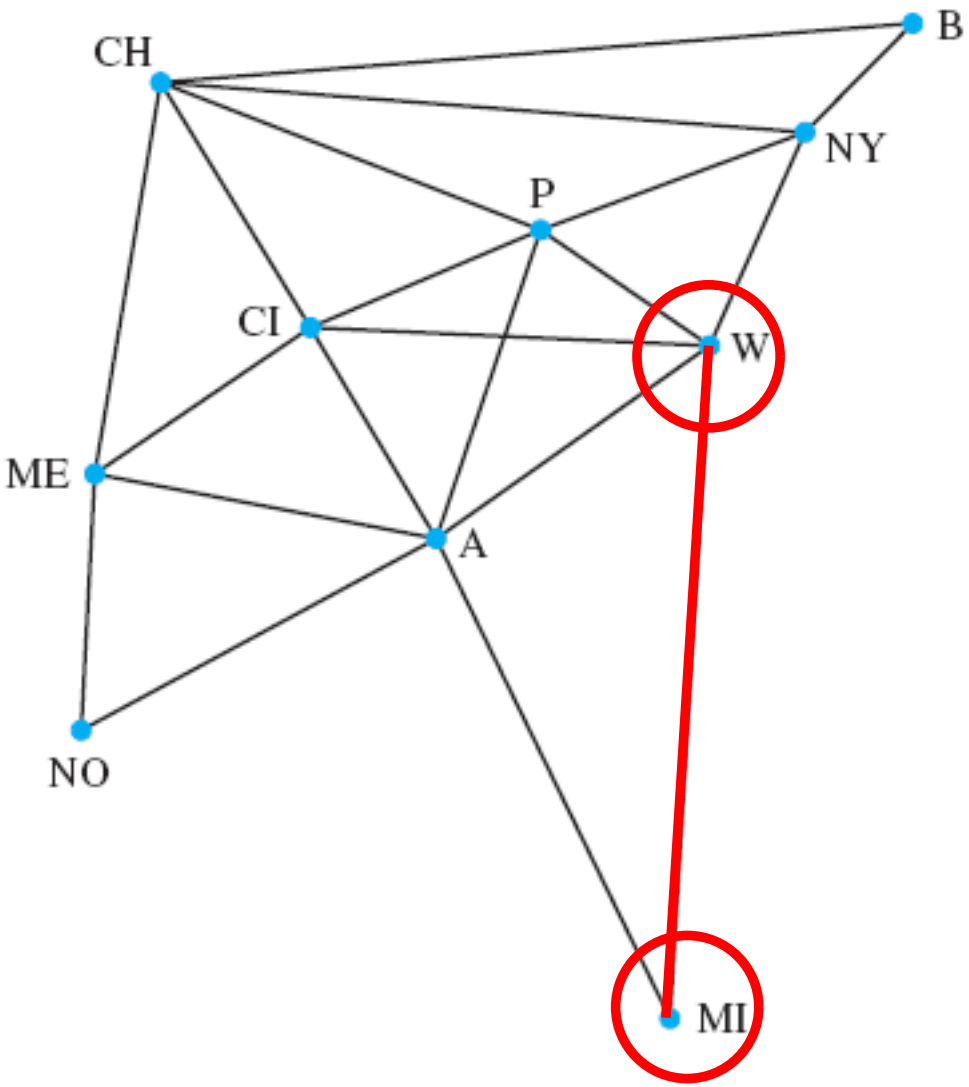

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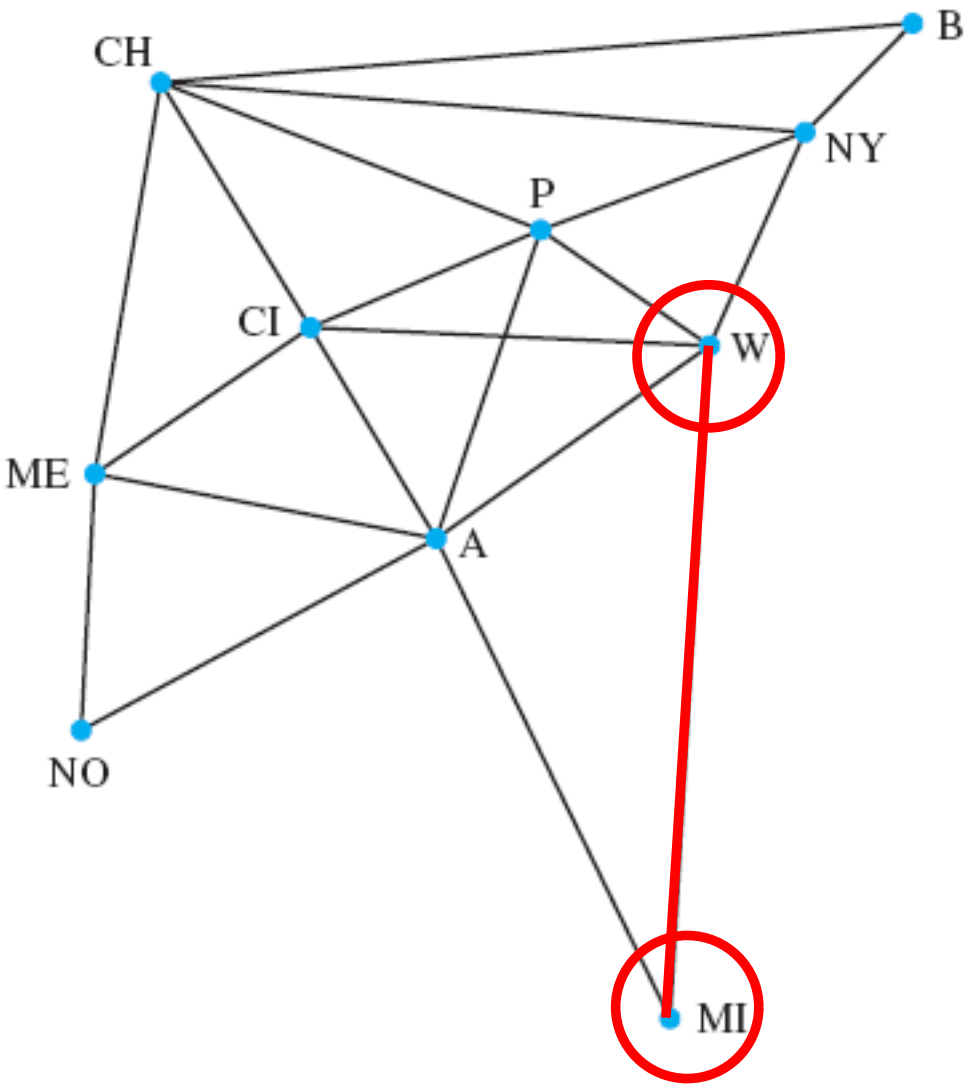

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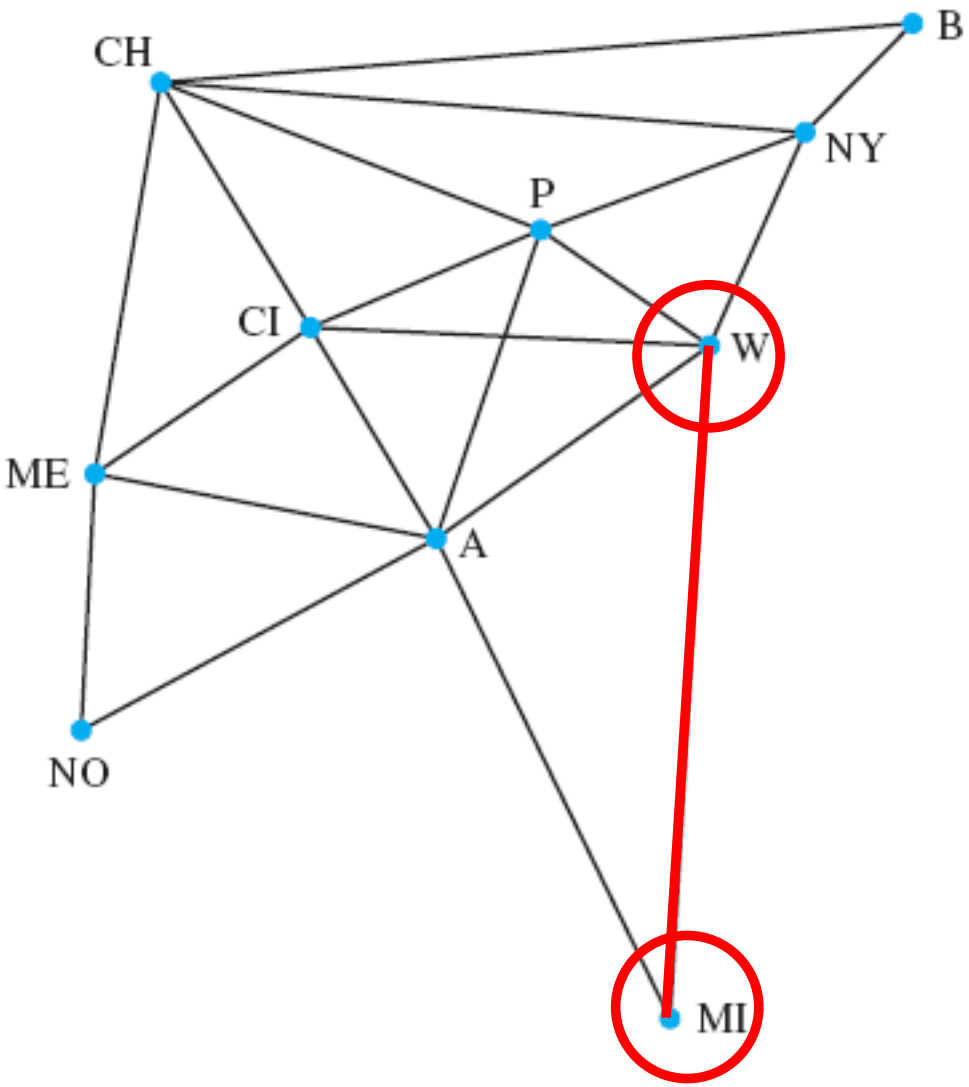

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When a vertex is an endpoint of an edge, we say that the edge and the vertex are incident to each other

Definition of a Graph

- **Definition**. A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to be *incident to* (or *connect* its endpoints.

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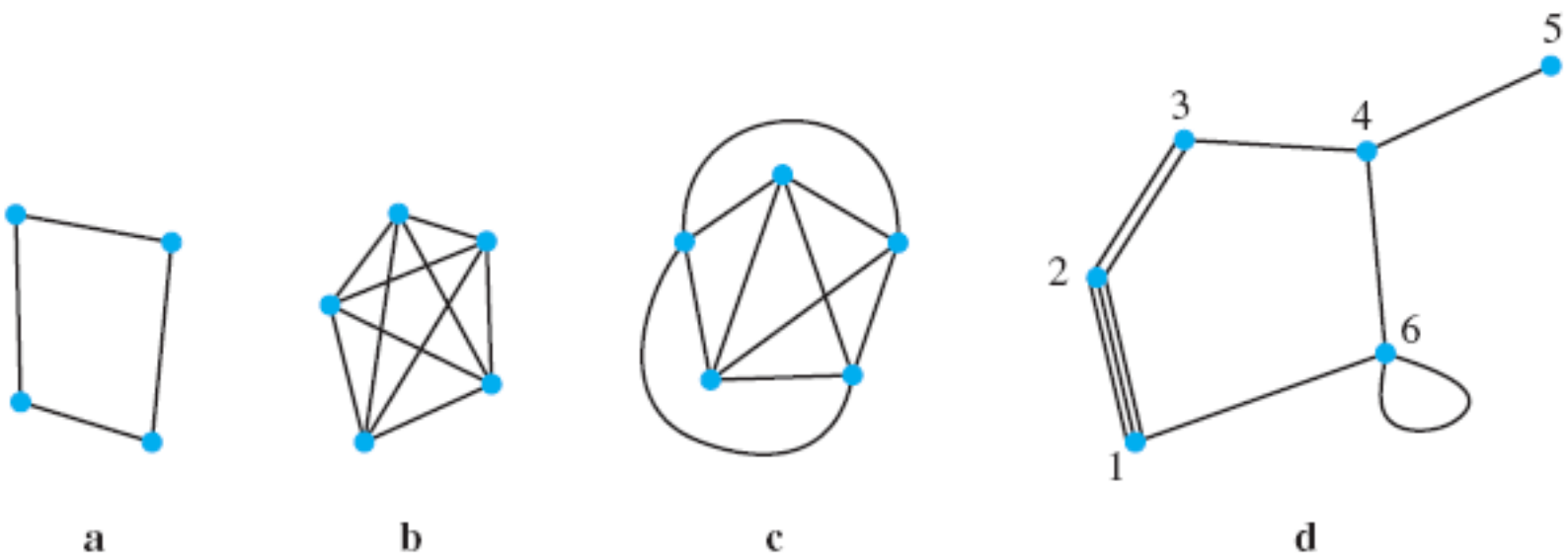

More Definitions

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  A graph with n vertices that has an edge between each pair of vertices

Graphs

- Graphs and graph theory can be used to model:
  - ◇ Computer networks
  - ◇ Social networks
  - ◇ Communication networks
  - ◇ Information networks
  - ◇ Software design
  - ◇ Transportation networks
  - ◇ Biological networks

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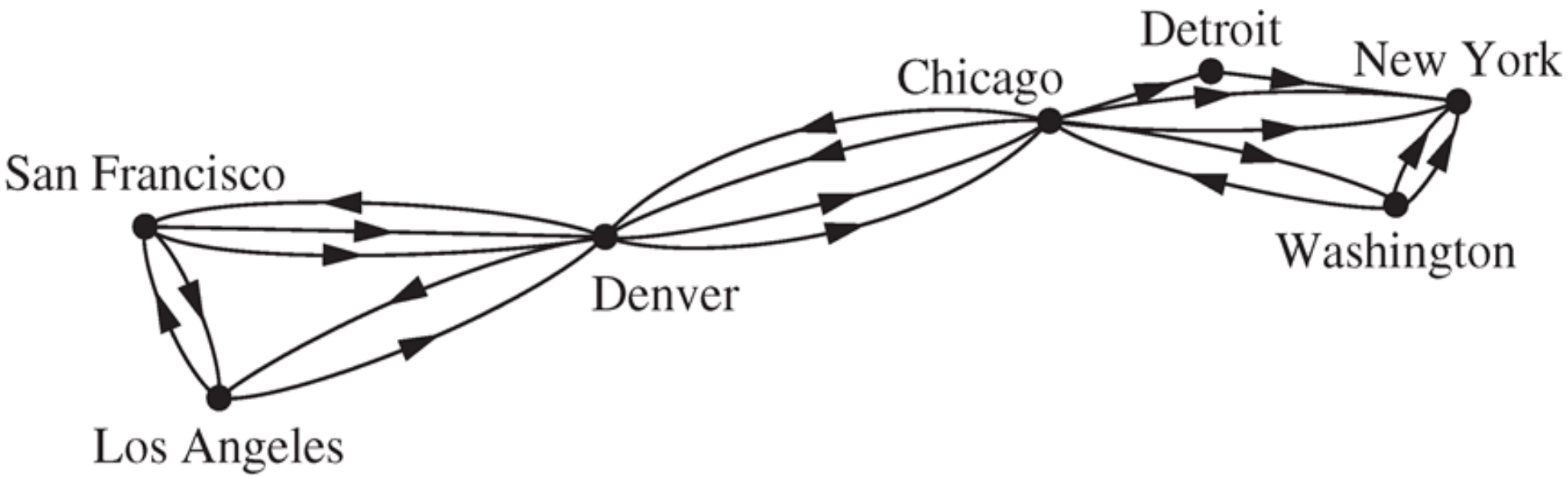

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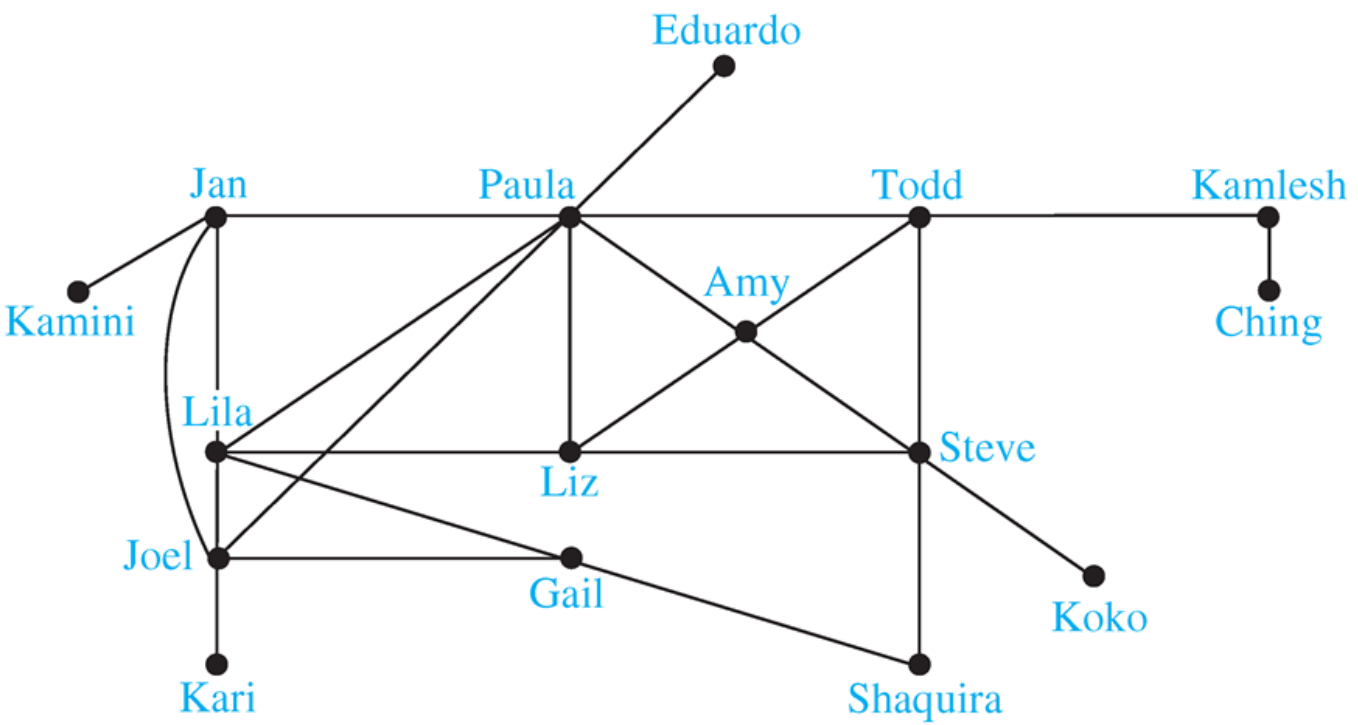

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  Example

  - the Hollywood graph
  - the Erdős number

The Erdős Number

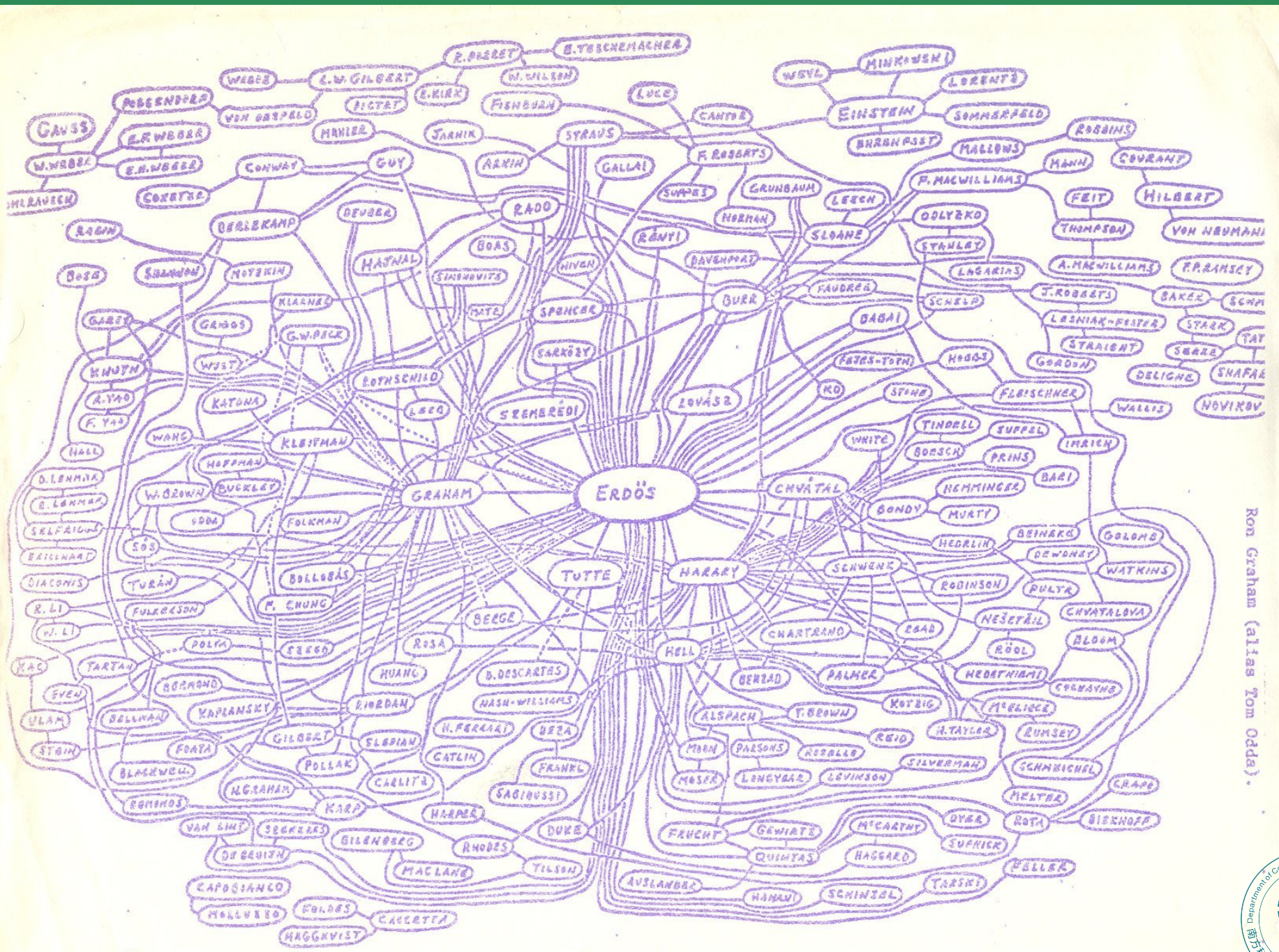

Figure 1

To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

The Erdős Number

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Statistics on Mathematical Collaboration, 1903-2016

	#Laureates	#Erdős	%Erdős	Min	Max	Average	Median
Fields Medal	56	56	100.0%	2	6	3.36	3
Nobel Economics	76	47	61.84%	2	8	4.11	4
Nobel Chemistry	172	42	24.42%	3	10	5.48	5
Nobel Medicine	210	58	27.62%	3	12	5.50	5
Nobel Physics	200	159	79.50%	2	12	5.63	5

Undirected Graphs

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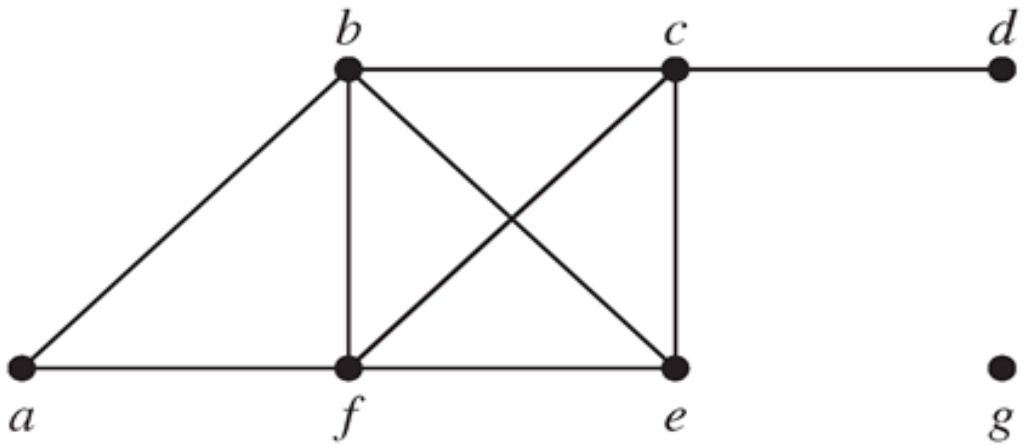
Undirected Graphs

- **Example**: What are the degrees and neightborhoods of the vertices in the graph G?



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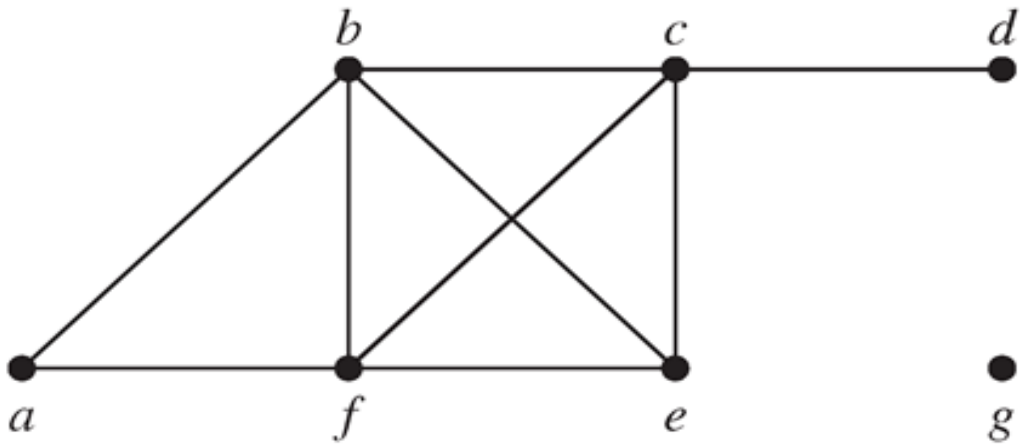

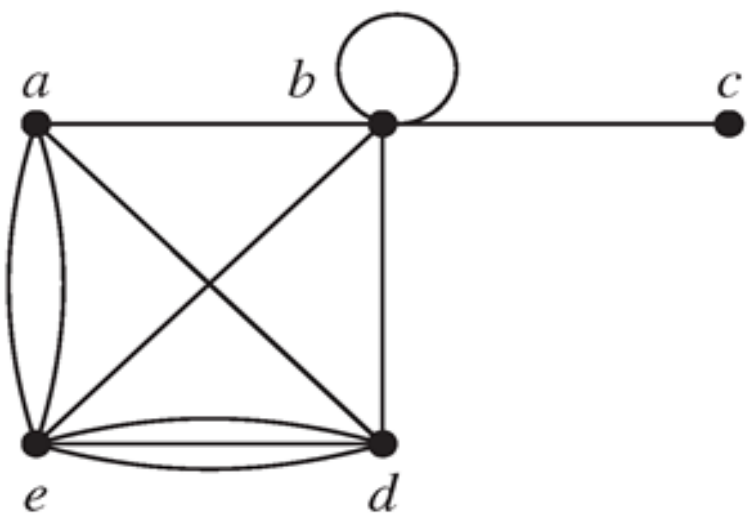

Undirected Graphs

- **Theorem 1** (Handshaking Theorem) If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof

Undirected Graphs

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Directed Graphs

- **Definition** An *directed graph* $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of directed edges. Each edge is an ordered pair of vertices. The directed edge (u, v) is said to start at u and end at v.

  Definition Let (u, v) be an edge in G. Then u is the *initial vertex* of the edge and is *adjacent to* v and v is the *terminal vertex* of this edge and is *adjacent from* u. The initial and terminal vertices of a loop are the same.

Directed Graphs

- **Definition** The *in-degree* of a vertex v, denoted by $\deg^-(v)$, is the number of edges which terminate at v. The *out-degree* of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

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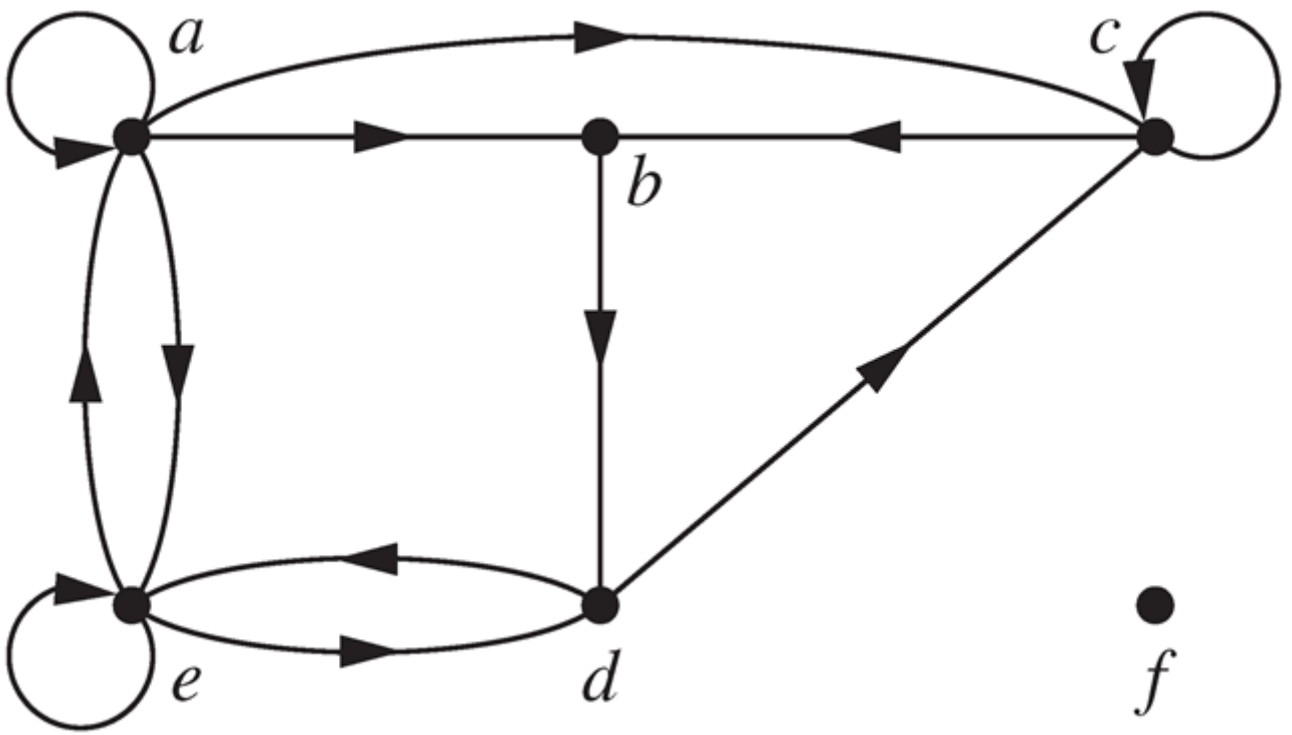

Directed Graphs

- **Theorem 3** Let $G = (V, E)$ be a graph with directed edges. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

Proof

Complete Graphs

- A *complete graph* on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

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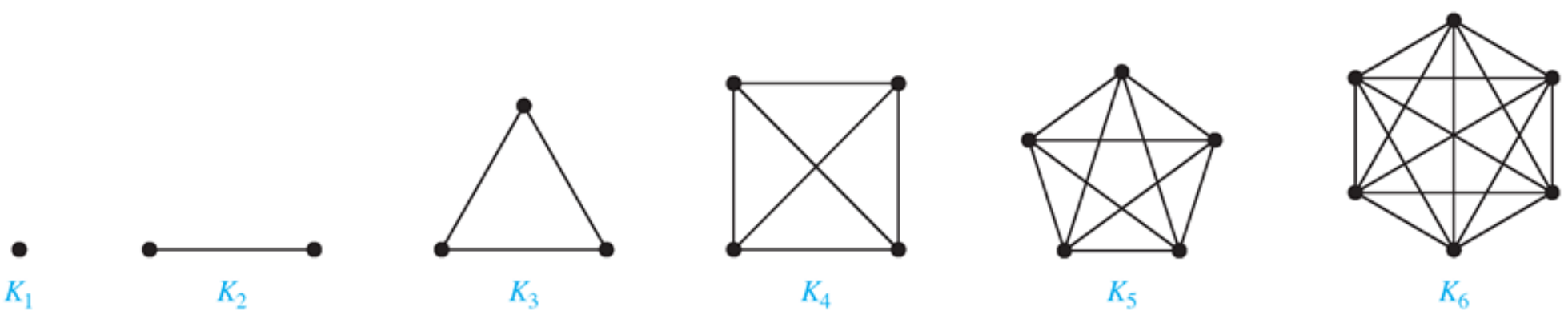

Cycles

- A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

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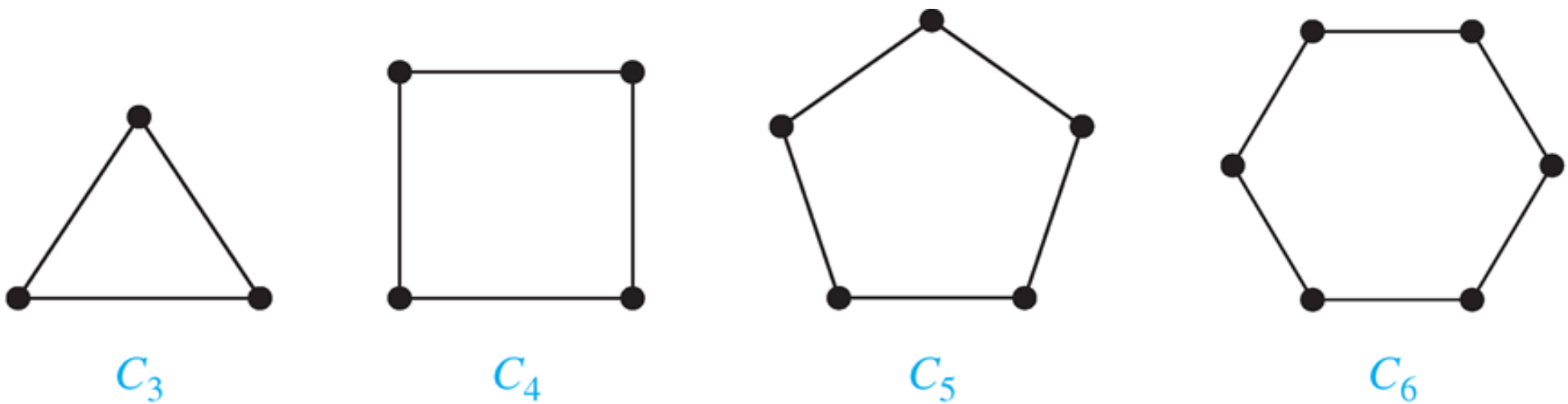

Wheels

- A *wheel* W_n is obtained by adding an additional vertex to a cycle C_n.

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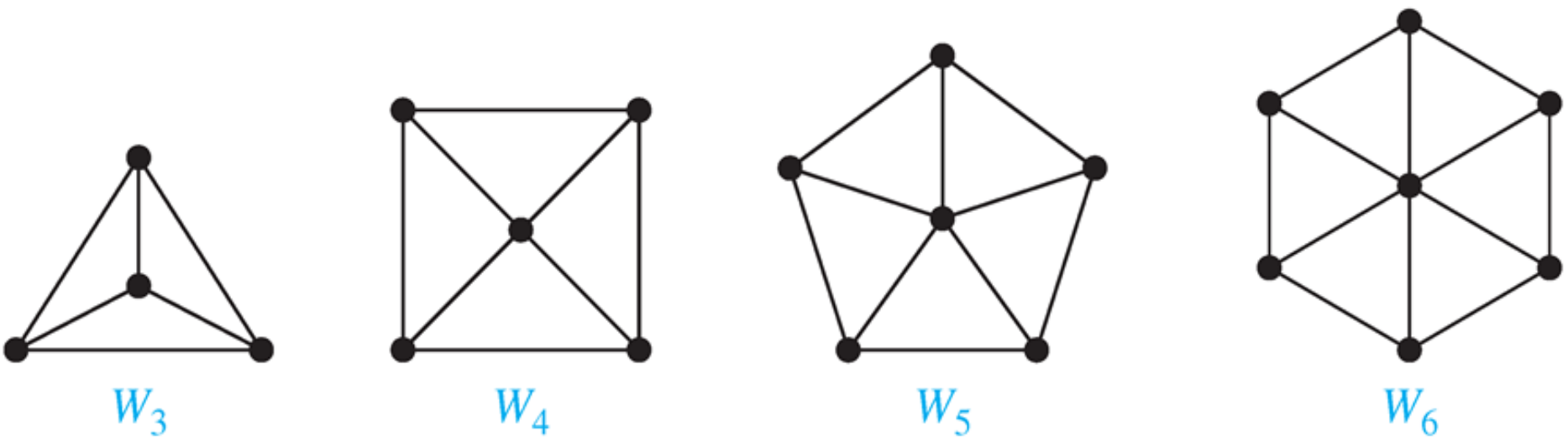

N-dimensional Hypercube

- An *n-dimensional hypercube*, or *n-cube*, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.

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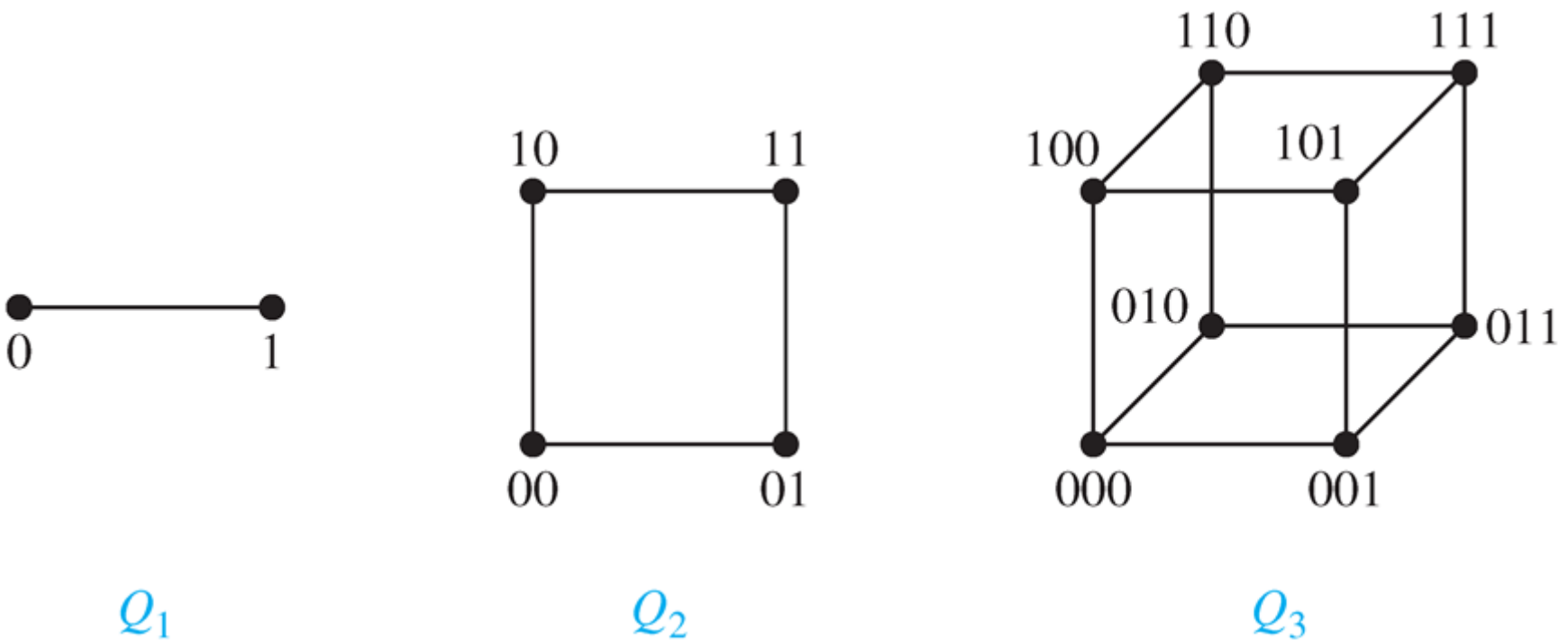

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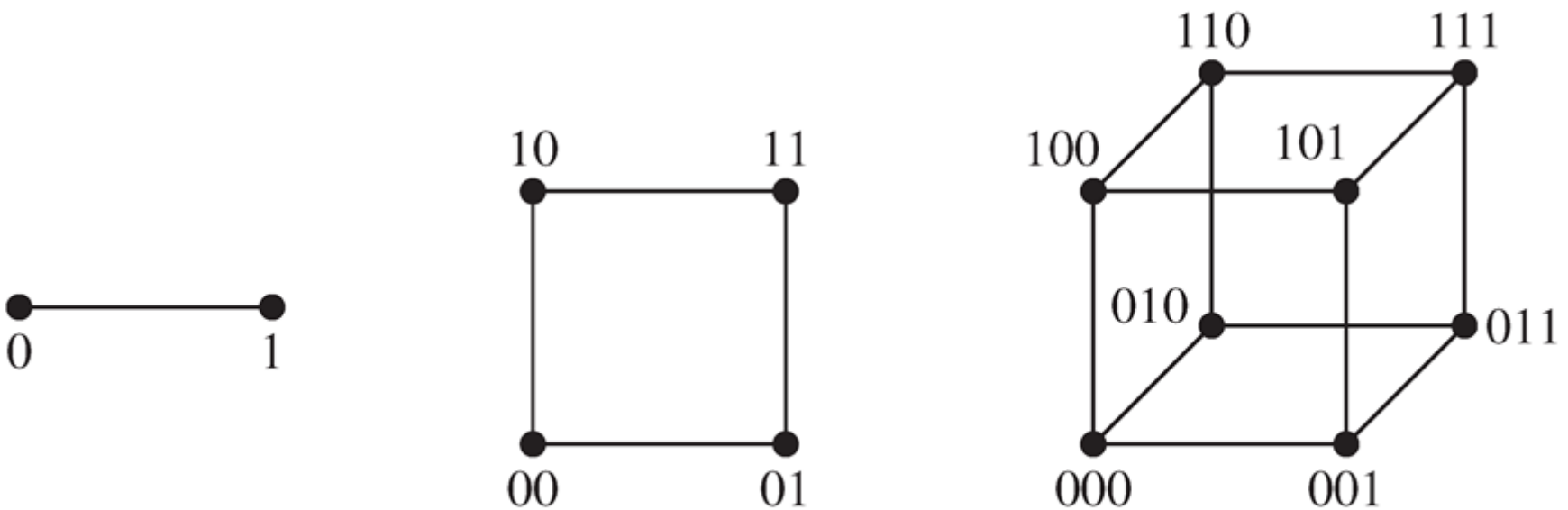

How many vertices? How many edges?

Bipartite Graphs

- **Definition** A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.

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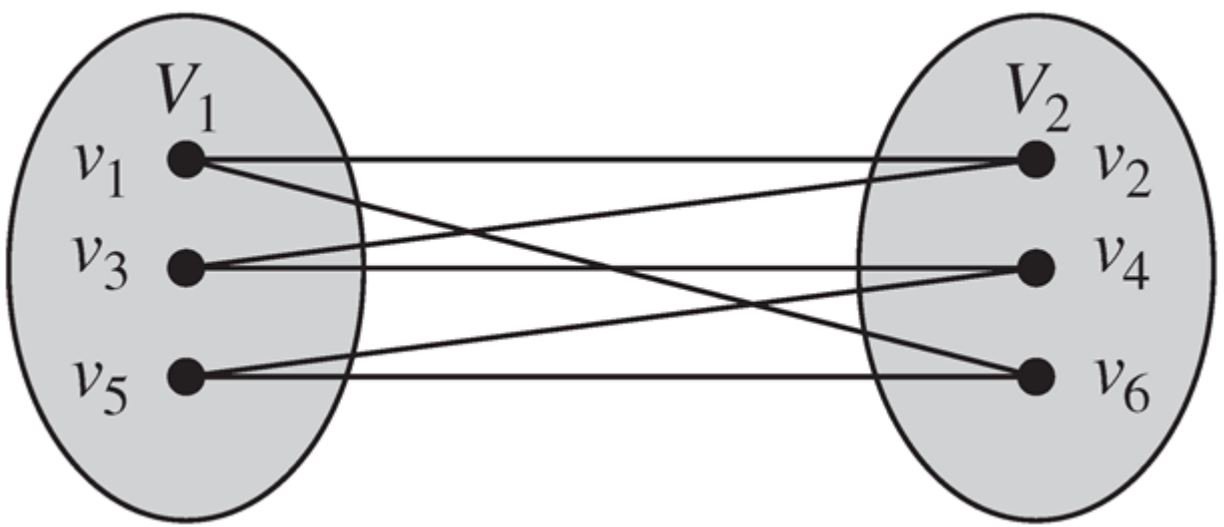
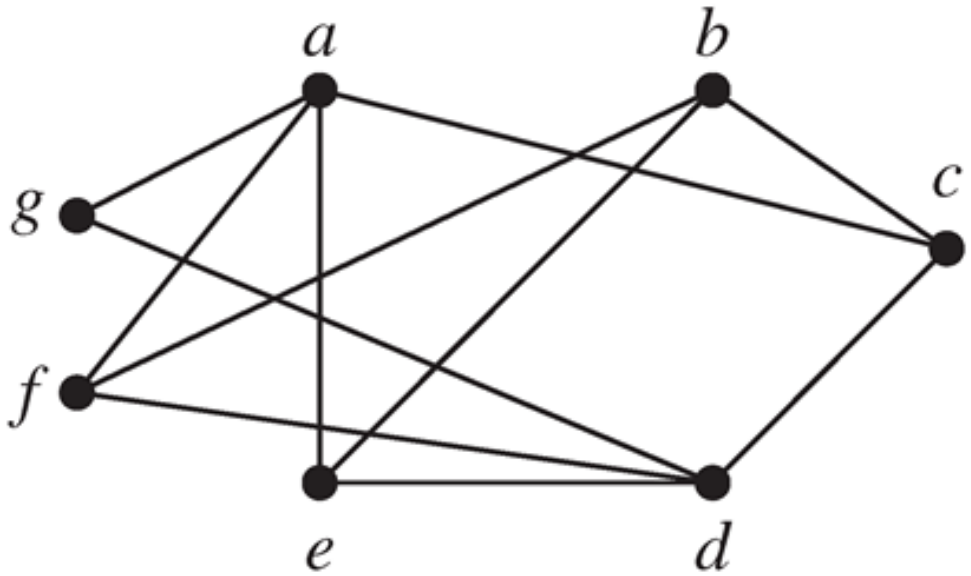
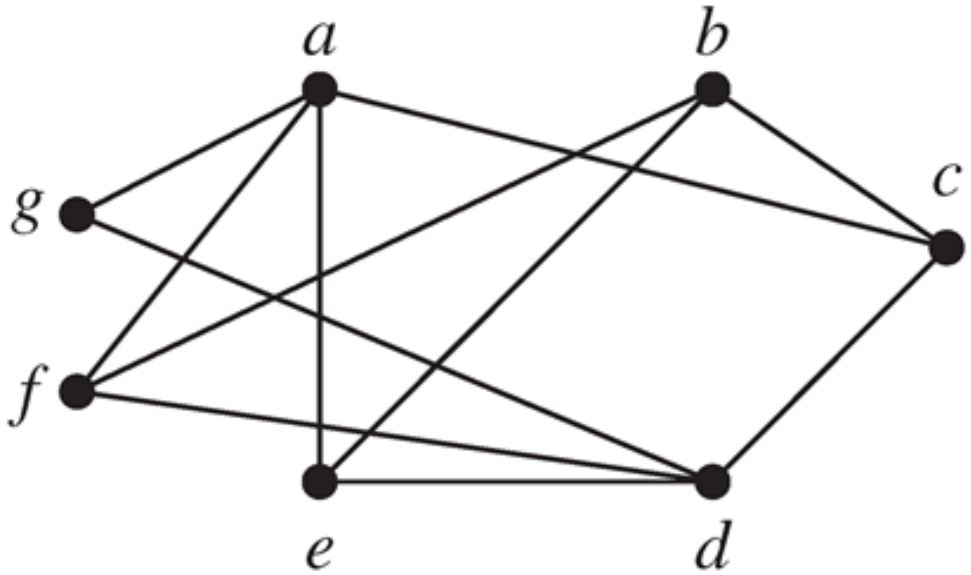
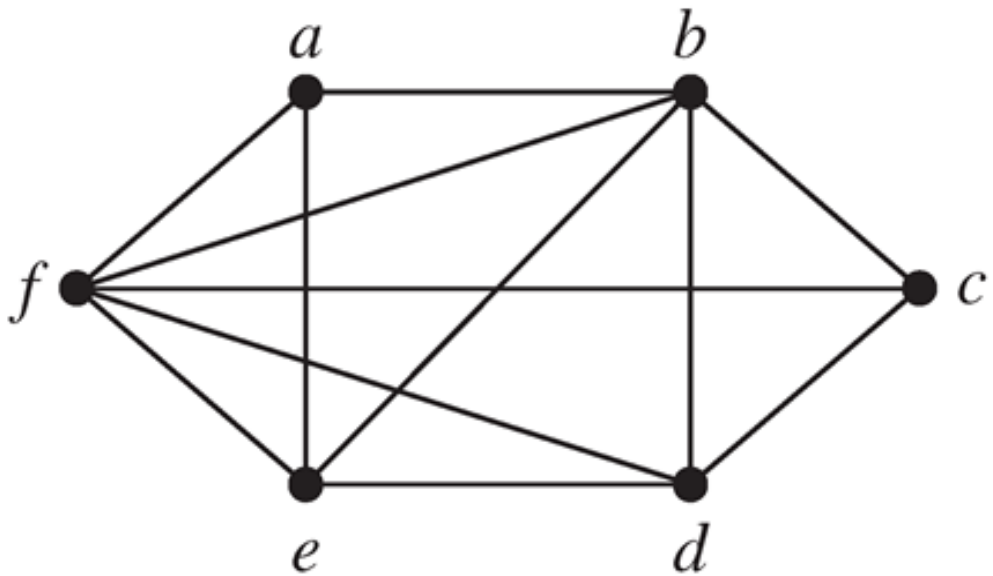


Bipartite Graphs



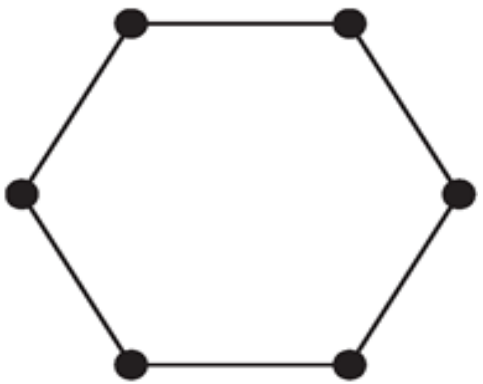
Bipartite Graphs





Bipartite Graphs

- **Example** Show that C_6 is bipartite.



C_6

Bipartite Graphs

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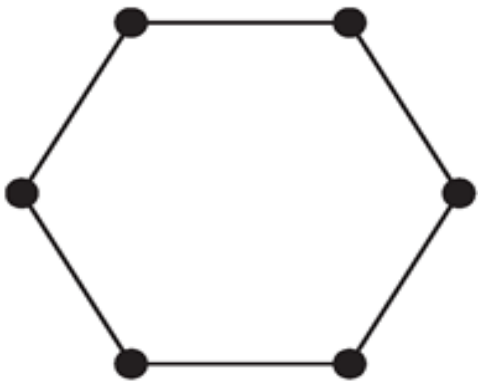
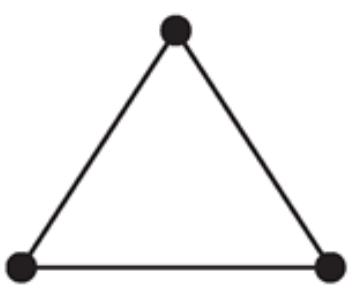


C_6

Example Show that C_3 is not bipartite.



C_3

Complete Bipartite Graphs

- **Definition** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

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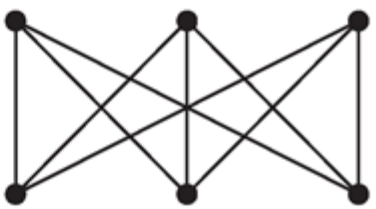
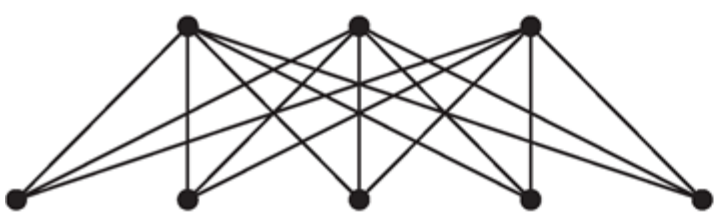
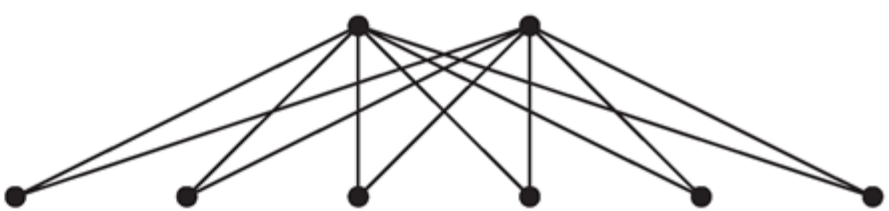


$K_{3,3}$



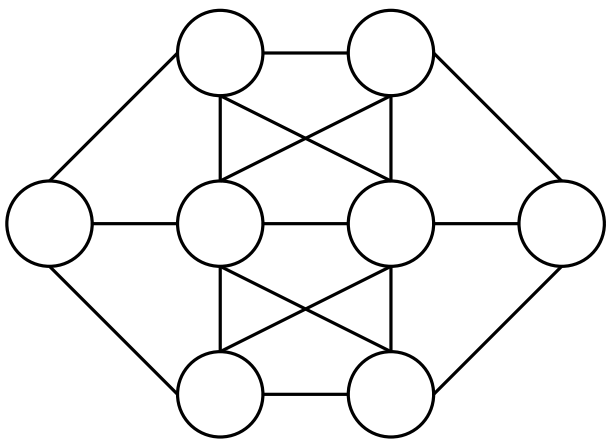
$K_{3,5}$



$K_{2,6}$

Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that no letter is adjacent to a letter that is next to it in the alphabet.



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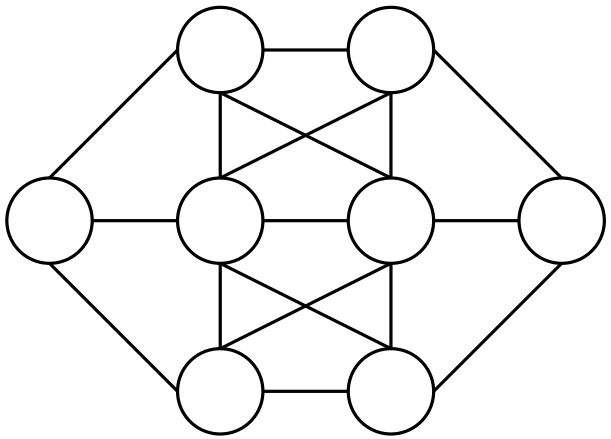

- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

Next Lecture

- graph ...