Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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NP-complete Problems

- Class **NP** vs Class **P**
  - **P**: decision problems solvable in polynomial time
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- Approximation Algorithm
  Natural idea: settle for *non-optimal* solutions for these "hard" problems, if we can find such close-to-the-optimal solutions reasonably fast.

Satisfiability Problem

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- **Definition** A *Boolean formula* is a logical formula consisting of
  - Boolean variables (0 = false, 1 = true),
  - logical operations
    - $\neg x$: Negation
    - $x \vee y$: Disjunction
    - $x \wedge y$: Conjunction

  With the truth table defined by:

x	y	$\neg x$	$x \vee y$	$x \wedge y$
0	0	1	0	0
0	1		1	0
1	0	0	1	0
1	1		1	1

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Example. $f(x, y, z) = (x \wedge (y \vee \neg z)) \vee (\neg y \wedge z \wedge \neg x)$

x	y	z	$(x \wedge (y \vee \neg z))$	$(\neg y \wedge z \wedge \neg x)$	$f(x, y, z)$
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0	1	0	0	0	0
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The assignment, $x = 1, y = 1, z = 0$ makes $f(x,y,z)$ true, and hence it is satisfiable.

Satisfiable

- **Example.** $f(x,y) = (x \vee y) \wedge (\neg x \vee y) \wedge (x \vee \neg y) \wedge (\neg x \vee \neg y)$

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Definition For a fixed k, Boolean formulas in the following form are called k-conjunctive normal form (k-CNF):

$$f_1 \wedge f_2 \wedge \cdots \wedge f_n$$

where each f_i is of the form $f_i = y_{i,1} \vee y_{i,2} \vee \cdots \vee y_{i,k}$, and each $y_{i,j}$ is a variable or the negation of a variable.

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  Theorem 2SAT $\in$ Class **P**
  Proof. We will show how to solve 2SAT efficiently using path searches in graphs.

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  Proof.
  Use some basic search algorithms in graph theory (DFS/BFS).

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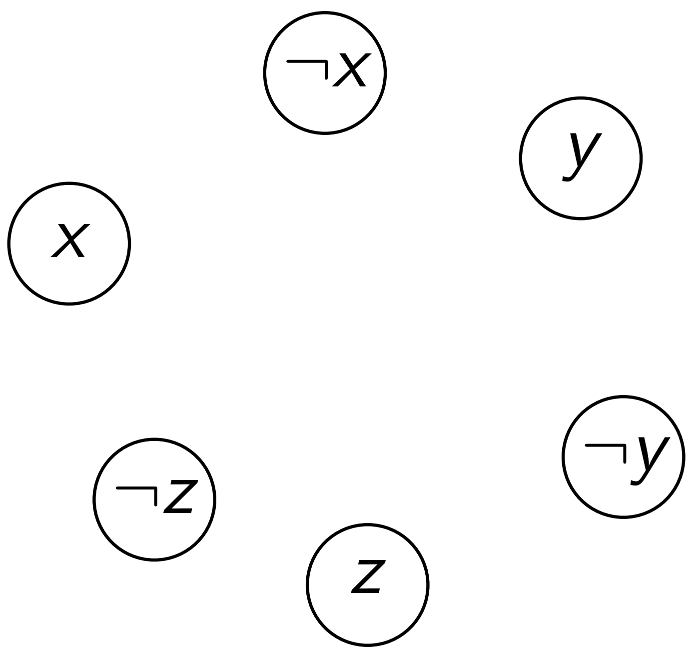

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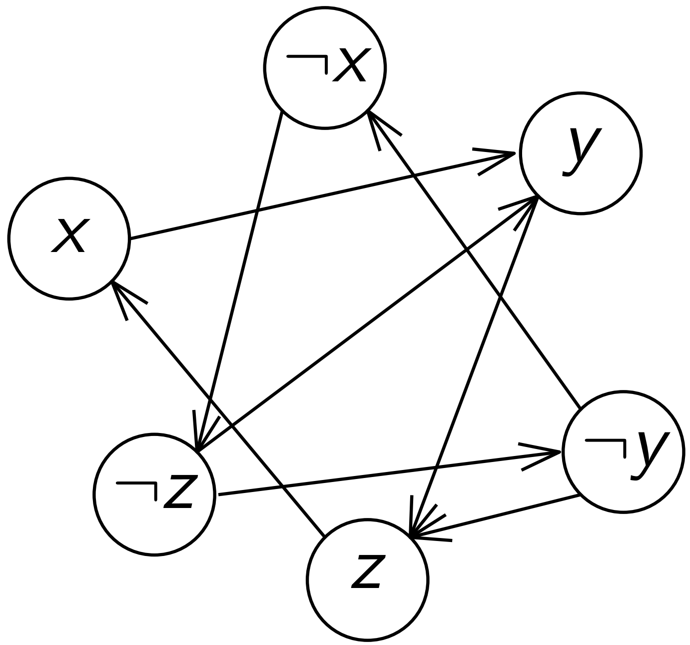

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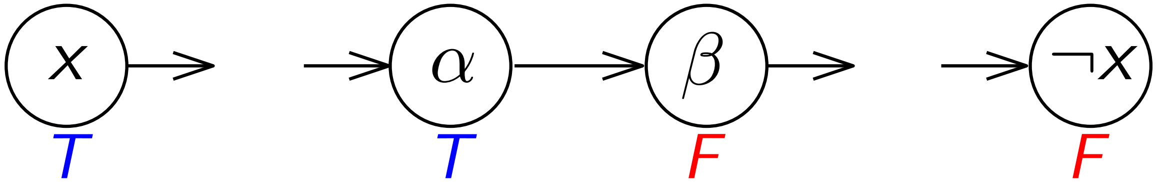

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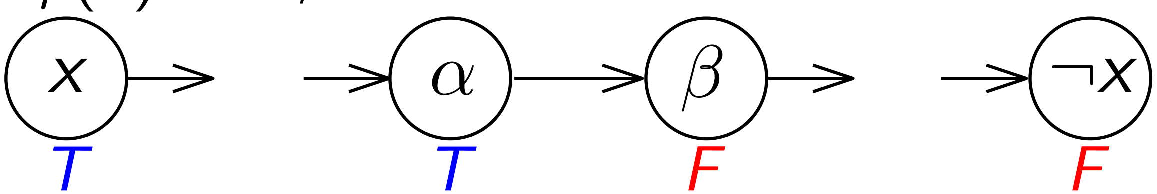

$(\neg\alpha \vee \beta)$ is false!

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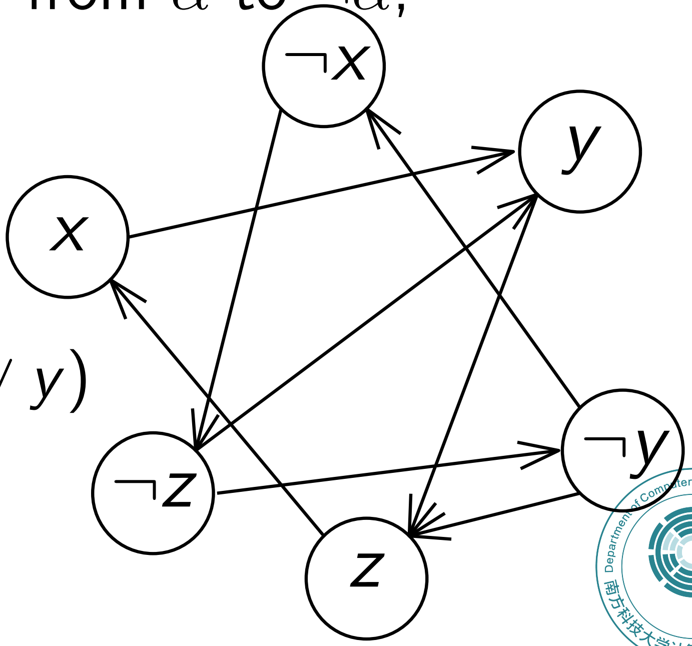

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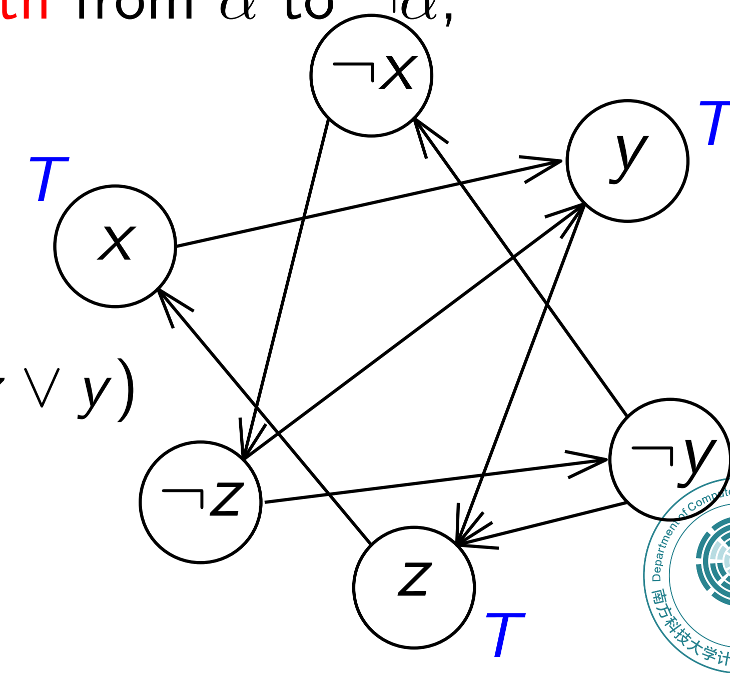

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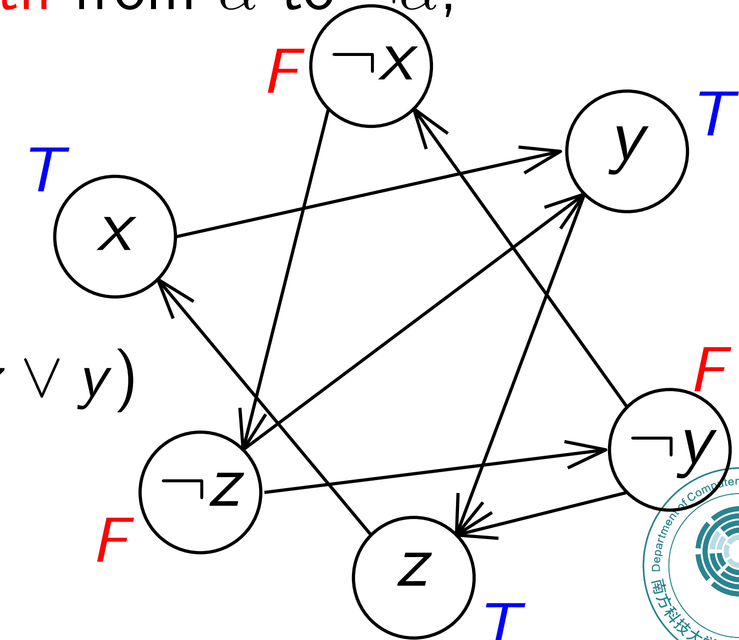

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  Theorem A 2-CNF formula f is satisfiable if and only if
  there are no paths from x to $\neg x$ or from $\neg x$ to x for any literal x.

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- An efficient algorithm for **2SAT** is the following.
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  - Output NO if any of these tests succeeds.
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  a yes-input of L_1 maps to a yes-input of L_2, and a no-input of L_1 maps to a no-input of L_2

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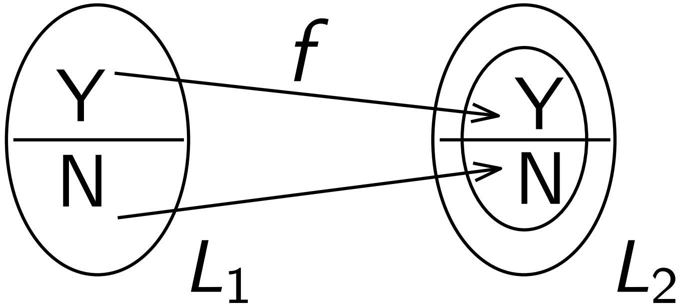

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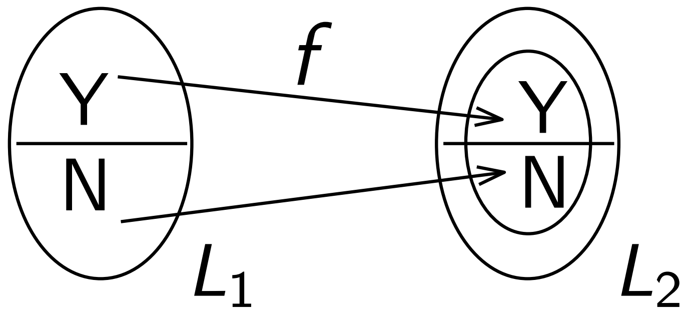

If such an f exists, we say that L_1 is *polynomial-time reducible* to L_2, and write $L_1 \leq_P L_2$.

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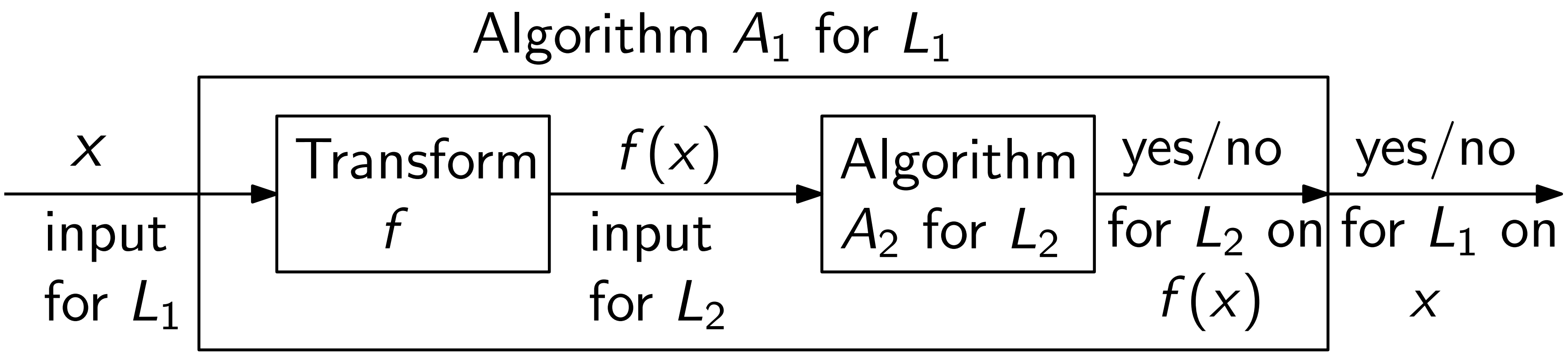

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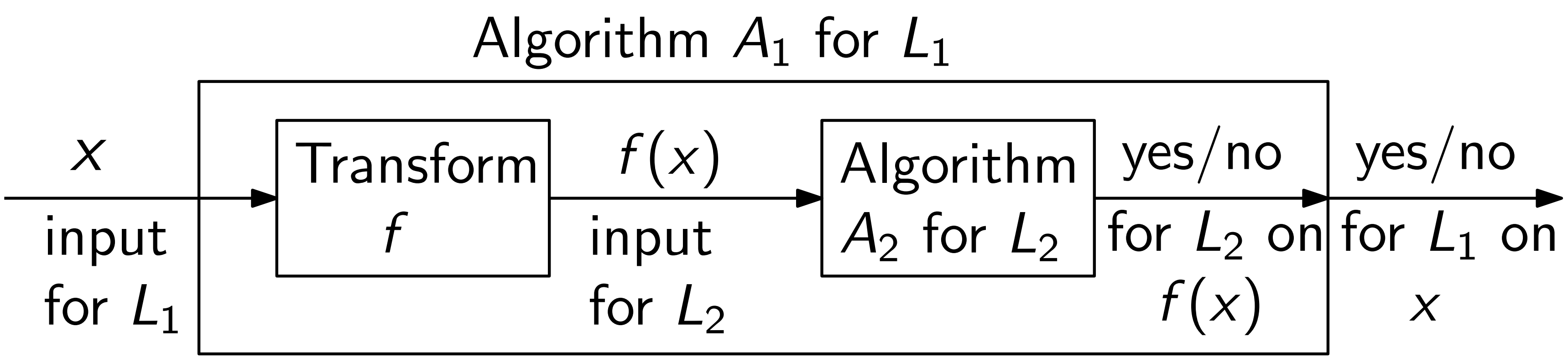

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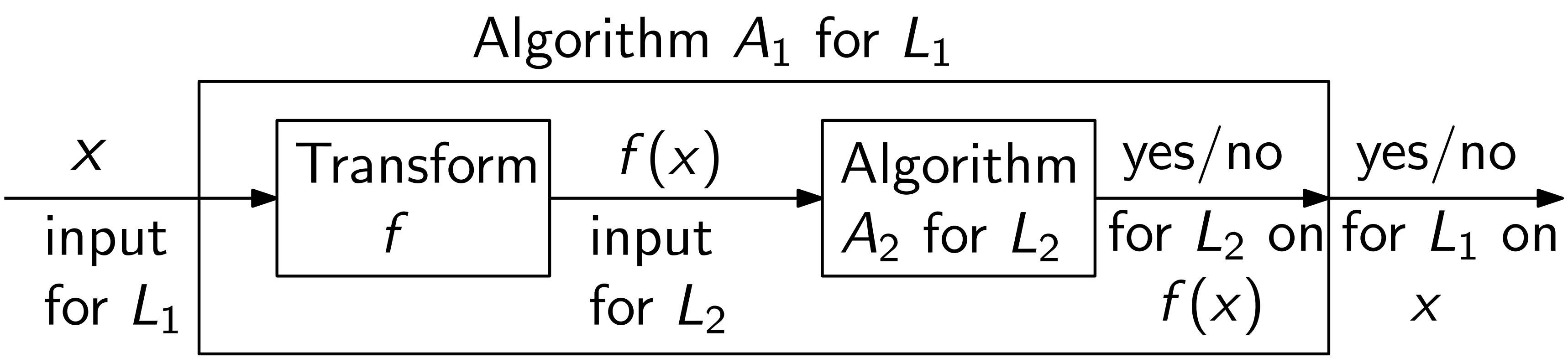

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Theorem If $L_1 \leq_P L_2$ and $L_2 \in P$, then $L_1 \in P$

Lemma If $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then $L_1 \leq_P L_3$.

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  However, due to the transitivity property of $\leq_P$, we can do the following to prove a decision problem $L \in NPC$:
  – prove $L \in NP$ (usually easy)
  – for some $L_0 \in NPC$, prove $L_0 \leq_P L$

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– By definition, it requires to show every $L' \in NP$, $L' \leq_P L$.
– But there are infinitely many problems in NP, so how can we argue there exists a reduction from every L' to L?

However, due to the transitivity property of $\leq_P$, we can do the following to prove a decision problem $L \in NPC$:
– prove $L \in NP$ (usually easy)
– for some $L_0 \in NPC$, prove $L_0 \leq_P L$

Proof. Let L' be any problem in NP. Since $L_0 \in NPC$, by definition we have $L' \leq_P L_0$. Since $L_0 \leq_P L$, then by transitivity, we have $L' \leq_P L$.

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  We will prove:
  3SAT $\leq_P$ DCLIQUE
  DCLIQUE $\leq_P$ DVC

CLIQUE

- **Definition** A *clique* in an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices s.t. each pair $u, v \in V'$ is connected by an edge $(u, v) \in E$. In other words, a clique is a complete subgraph of G.

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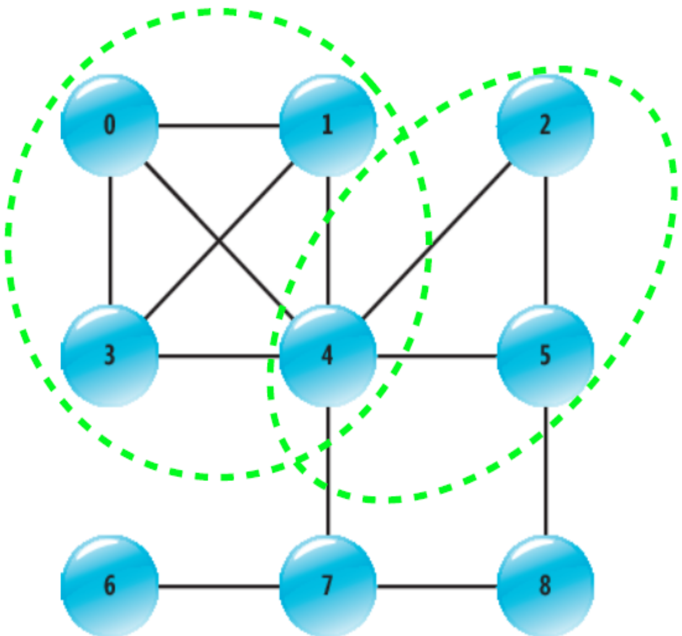

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  We will define a polynomial transformation f from 3SAT to DCLIQUE $f : \phi \mapsto (G, k)$ that builds a graph G and integer k s.t. ϕ is a Yes-input to 3SAT if and only if (G, k) is a Yes-input to DCLIQUE.

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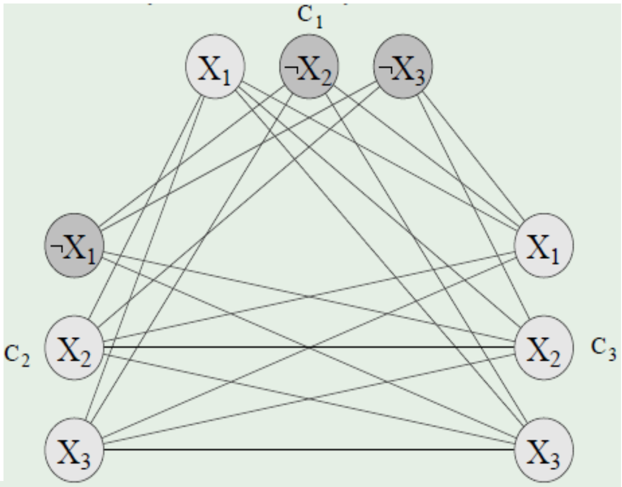

$$\phi = C_1 \wedge C_2 \wedge C_3$$
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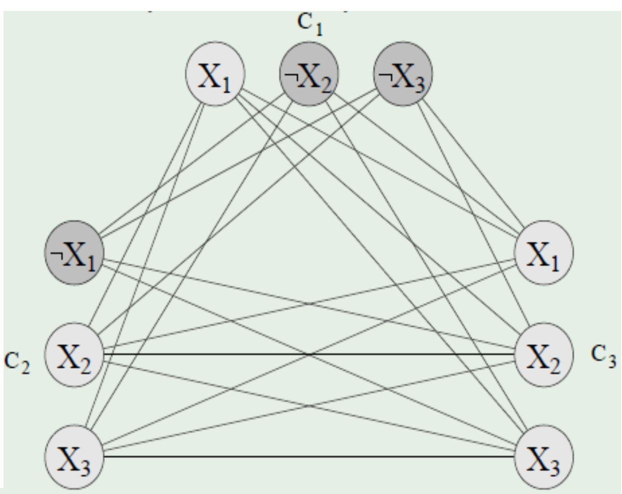

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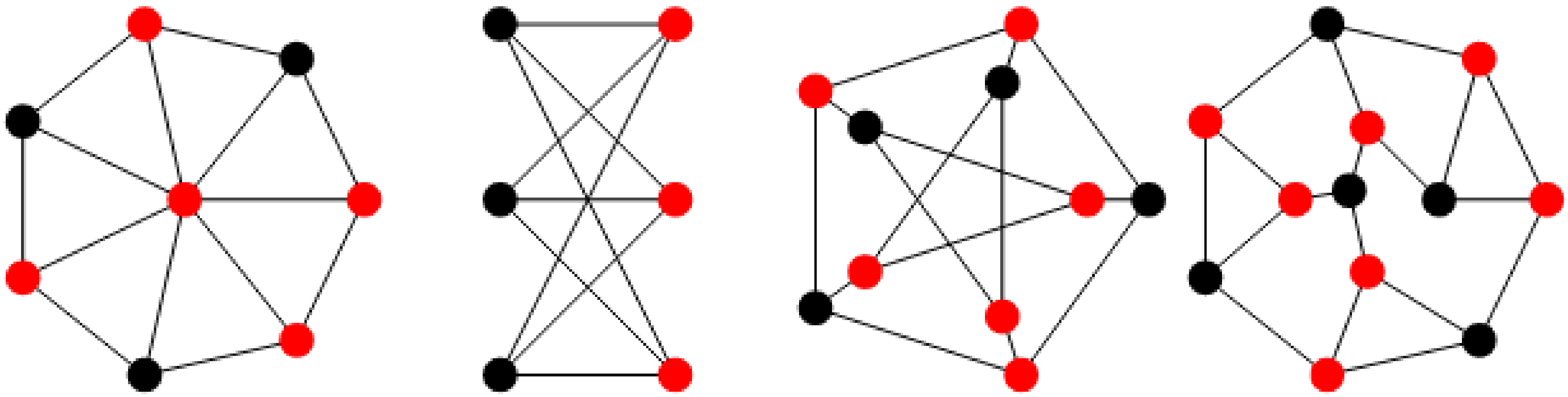

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Definition The *complement* of a graph $G = (V, E)$ is defined by $\overline{G} = (V, \overline{E})$ where

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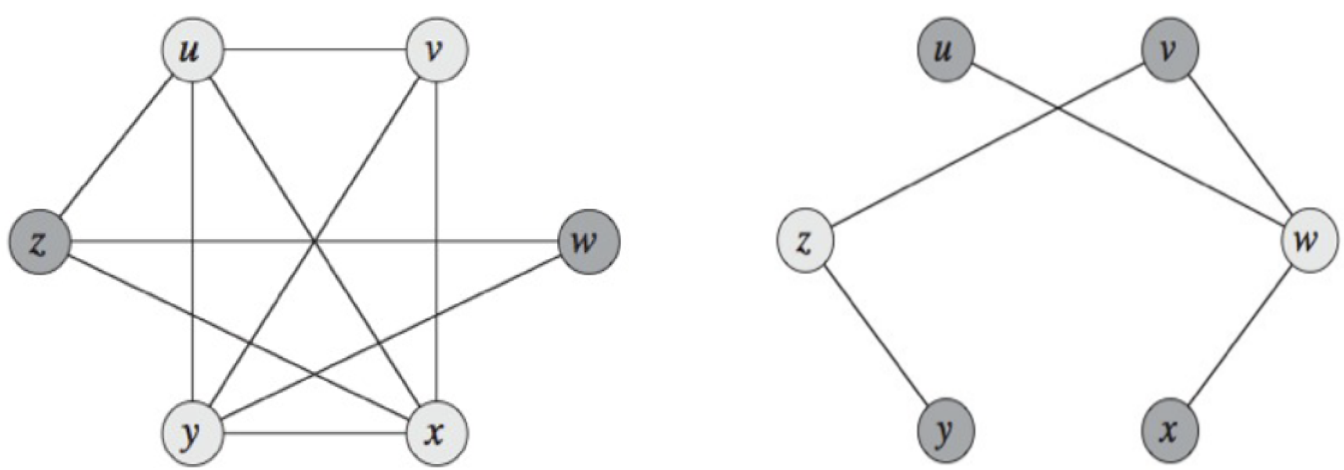

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  The reduction takes polynomial time

  A *clique* of size k in G $\Rightarrow$ a *vertex cover* of size $|V| - k$ in $\overline{G}$
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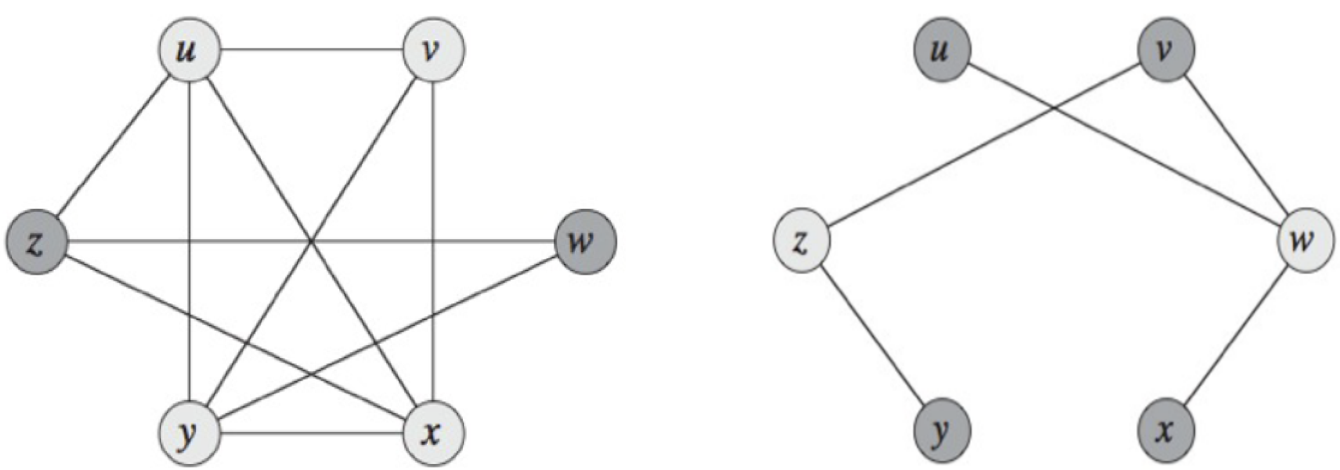

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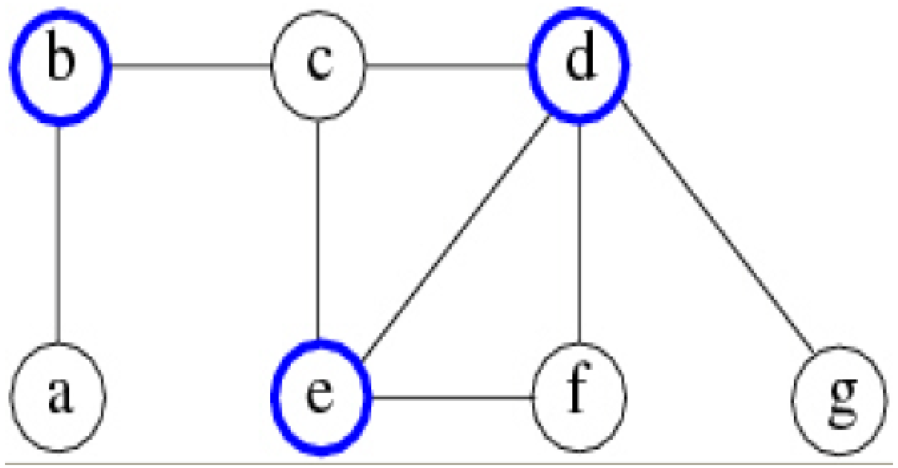

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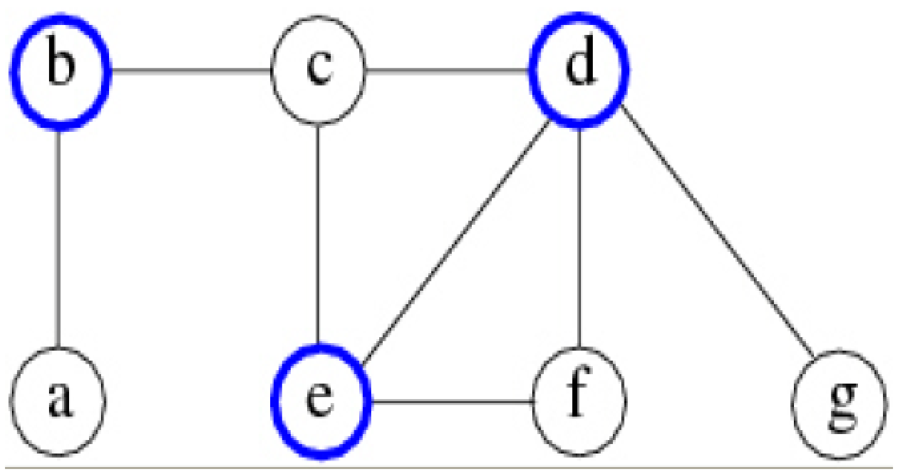

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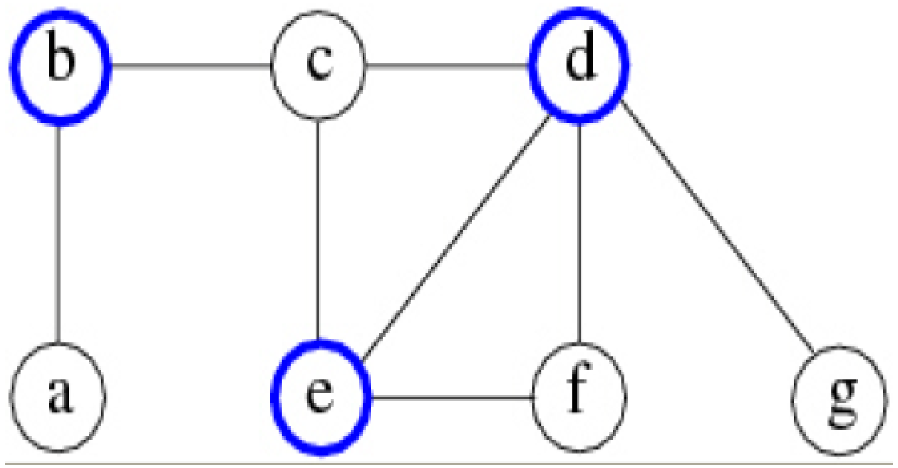
  It is very unlikely to give an exact polynomial time algorithm (Why?)



An Approximation Algorithm for VC

Approx-Vertex-Cover(G=(V, E))

```
C = empty-set;
E'= E;
while E' is not empty do do
    let (u, v) be any edge in E'          (*);
    add u and v to C;
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But, how good is C?

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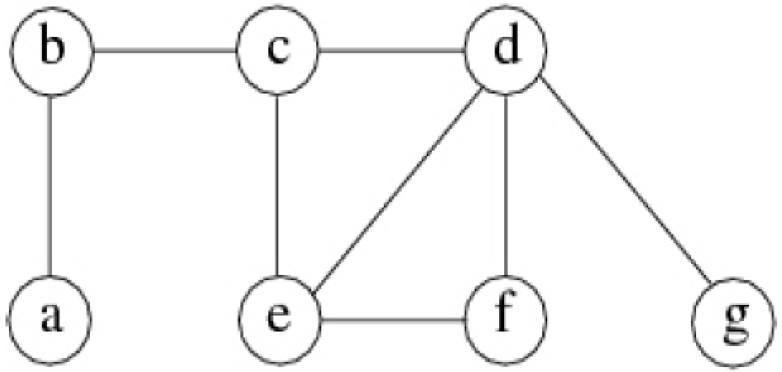

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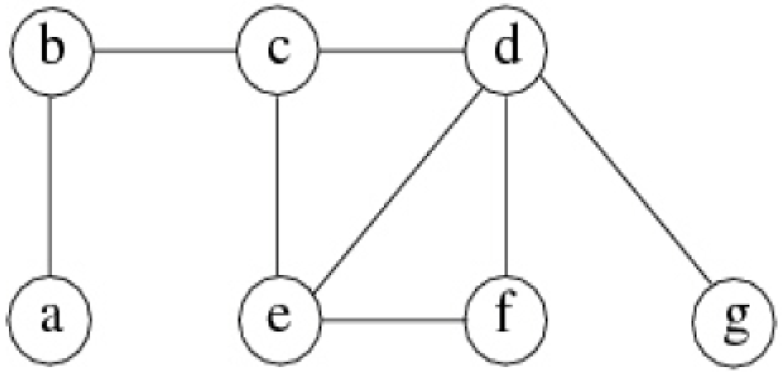

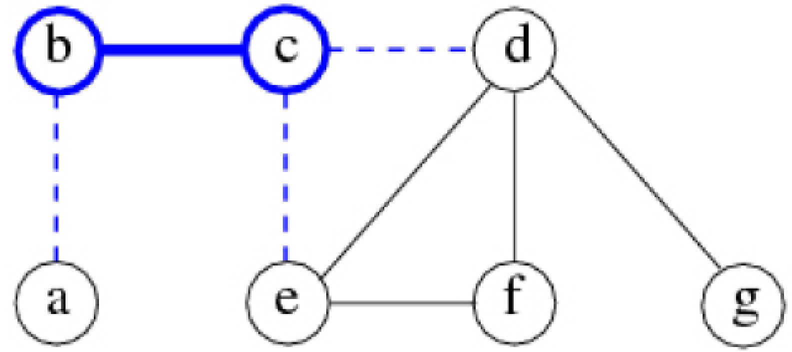

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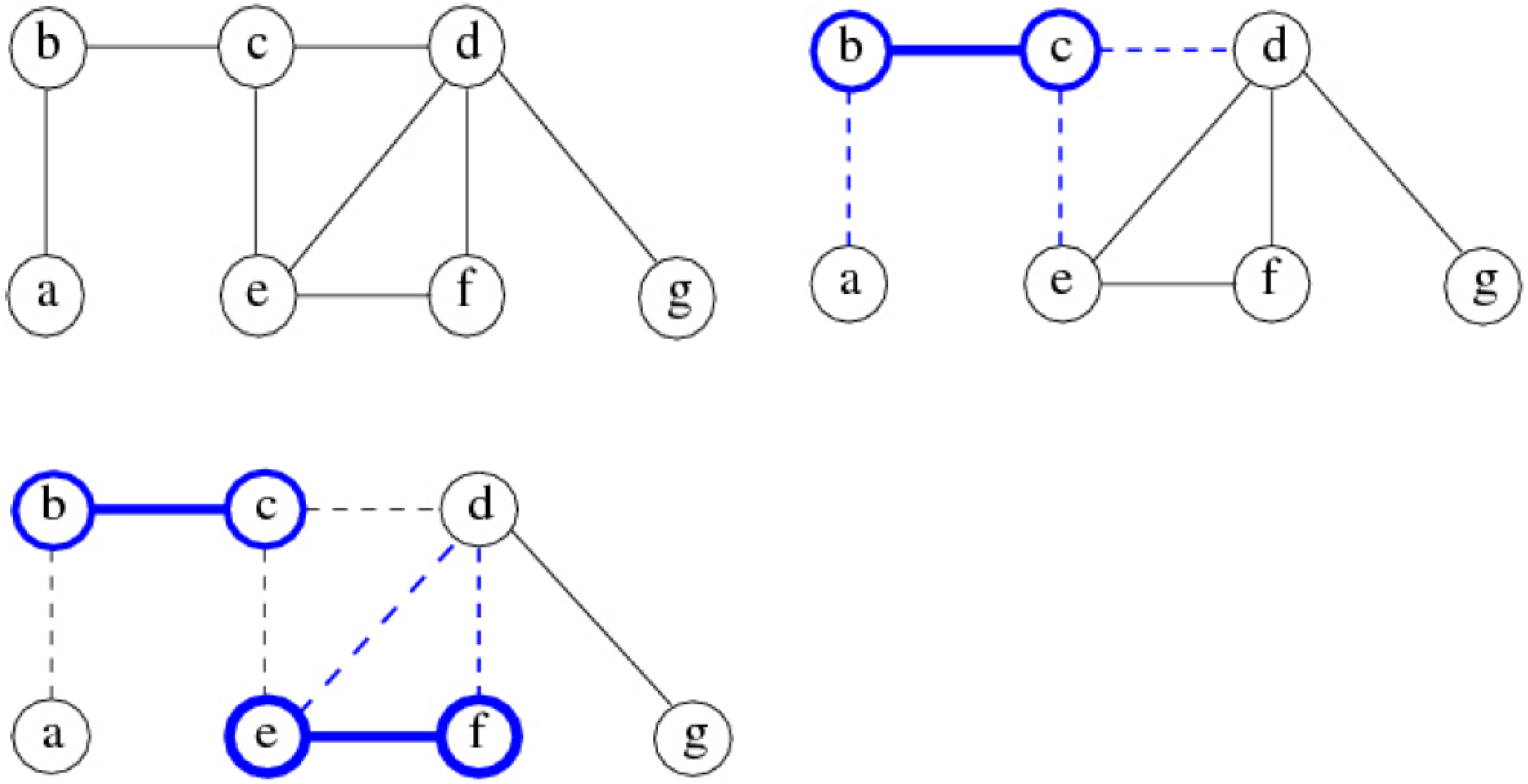

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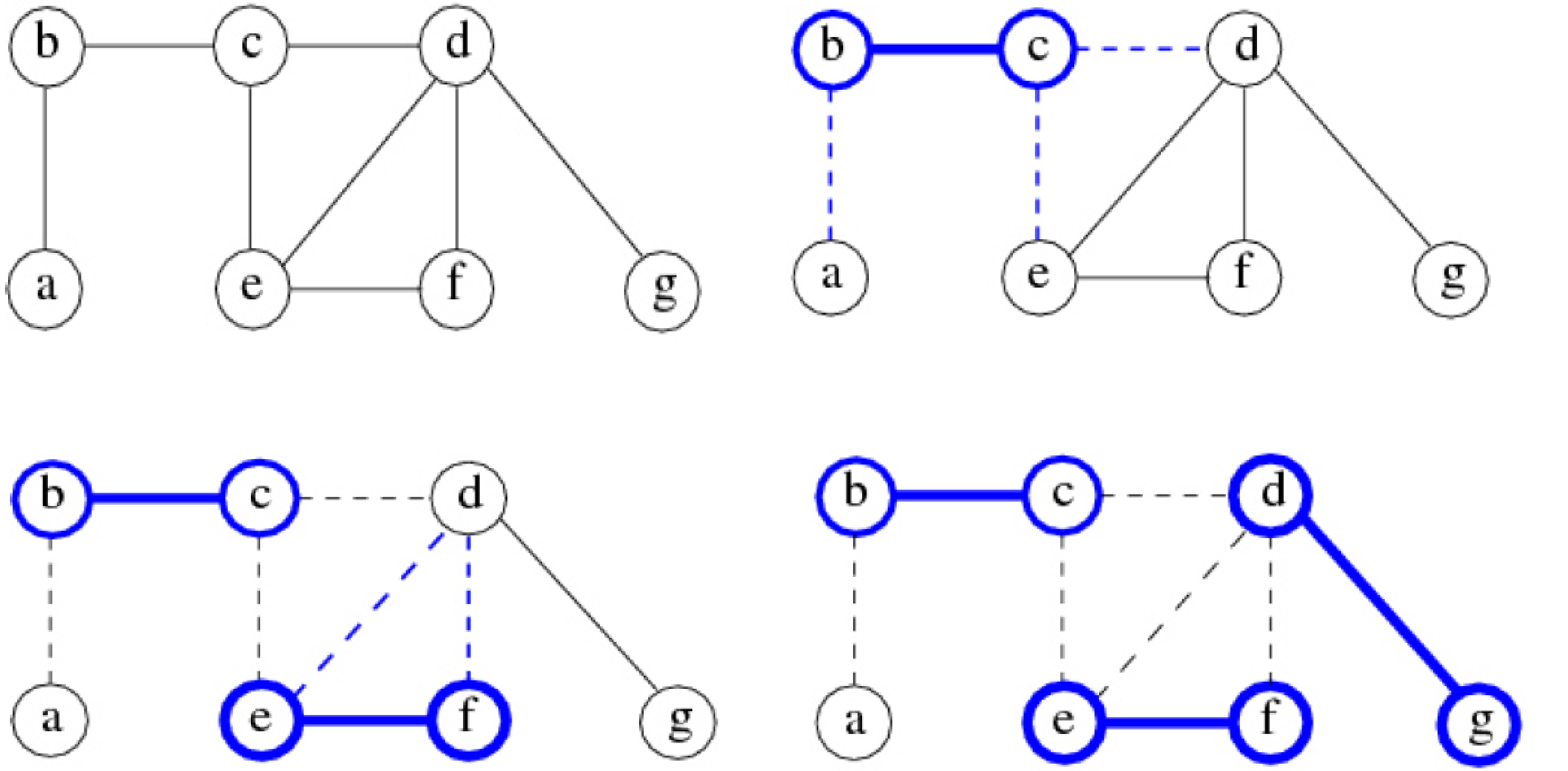

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The optimal vertex cover C^* must cover every edge in M, so $|C^*| \geq |M|$. But notice that the algorithm returns a vertex set of size $2|M|$. Therefore, we have
$$|C| = 2|M| \leq 2|C^*|.$$

Field

- A *field* is a set $\mathbb{F}$ equipped with two operations, *addition* $(+)$ and *multiplication* $(\cdot)$, and two special elements $0, 1$, s.t.:
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  - The properties can be verified

Every $a \in \mathbb{F}_p^*$ has a *multiplicative inverse*: since $a \in \mathbb{F}_p^*$ and p is a prime, we have $\gcd(a, p) = 1$, and by extended Euclidean algorithm, there exist x, y s.t. $ax + py = 1$, and then $x = a^{-1} \bmod p$.

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- Any finite field $\mathbb{F}$ is a *finite dimensional vector space* over $\mathbb{F}_p$, with $n = \dim_{\mathbb{F}_p}(\mathbb{F})$, $|\mathbb{F}| = p^n$, i.e., the cardinality of $\mathbb{F}$ must be a prime power.

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- An *irreducible polynomial* $f(x)$ of degree m is chosen:
  $f(x)$ cannot be factered as a product of binary polynomials each of degree less than m
  - *Addition*: usual
  - *Multiplication*: modulo $f(x)$

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$$\begin{array}{llll}
\mathbf{0} & z^2 & z^3 & z^3+z^2 \\
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z & z^2+z & z^3+z & z^3+z^2+z \\
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$z + 1$	$z^2 + z + 1$	$z^3 + z + 1$	$z^3 + z^2 + z + 1$

  - *Addition*: $(z^3 + z^2 + 1) + (z^2 + z + 1) = z^3 + z$
  - *Subtraction*: $(z^3 + z^2 + 1) - (z^2 + z + 1) = z^3 + z$
  - *Multiplication*: $(z^3 + z^2 + 1) \cdot (z^2 + z + 1) = z^5 + z + 1 = z^2 + 1$
  - *Inversion*: $(z^3 + z^2 + 1)^{-1} = z^2$

    since $(z^3 + z^2 + 1) \cdot z^2 = z^5 + z^4 + z^2 = 1 \bmod z^4 + z + 1$

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  The finite field $\mathbb{F}_{2^4}$ can be viewed as a *vector space* over $\mathbb{F}_2$.
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  If $\psi : z \mapsto c$ is an *ismorphism* between K_1 and K_2, then $f_1(c) \equiv 0 \pmod{f_2}$ for some $c \in K_2$. The choices for c are $z^2 + z$, $z^2 + z + 1$, $z^3 + z^2$, and $z^3 + z^2 + 1$.

Extension Fields and Subfields

- Let p be a prime and $m \geq 2$. Let $\mathbb{F}_p[z]$ denote the set of all polynomials in the variable z with coefficients from $\mathbb{F}_p$. Let $f(z)$ be an *irreducible polynomial of degree m in $\mathbb{F}_p[z]$*.

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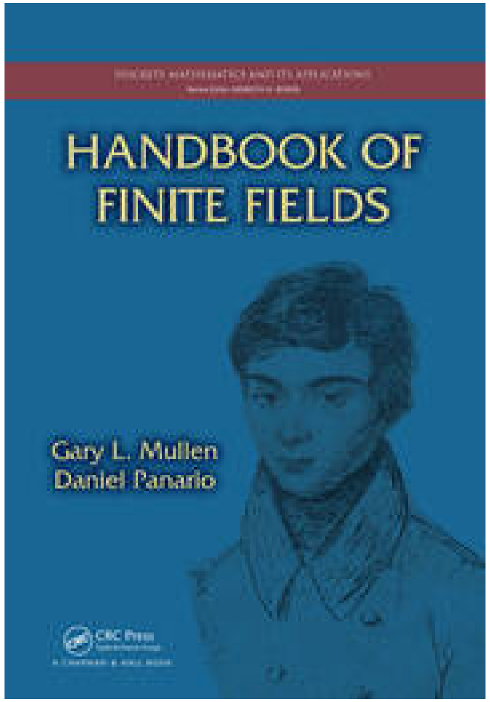
- A finite field $\mathbb{F}_{p^m}$ has precisely one subfield of order p^ℓ for each positive divisor ℓ of m.
  The elements of this subfield are the elements $a \in \mathbb{F}_{p^m}$ satisfying $a^{p^\ell} = a$; Conversely, every subfield of $\mathbb{F}_{p^m}$ has order p^ℓ for some positive divisor ℓ of m.

Applications of Finite Fields



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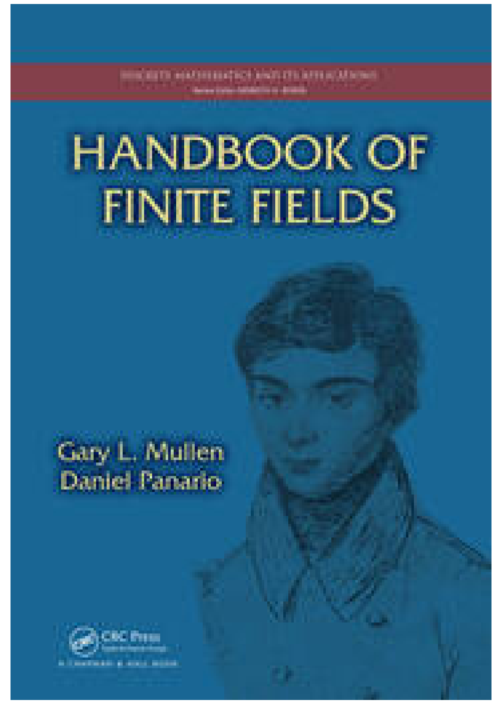

coding theory, cryptography, combinatorics, data storage systems, simulation, communications, signal design, ...

Review

01. Propositional Logic
02. Predicate Logic
03. Mathematical Proofs
04. Sets
05. Functions
06. Complexity of Algorithms
07. Number Theory
    Groups, Rings and Fields
08. Cryptography
09. Mathematical Induction
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- Quantified statements

  universal, existential, equivalence

Methods of Proving Theorems

- Basic methods to prove theorems:
  - *direct proof*
    - $p \to q$ is proved by showing that if p is true then q follows
  - *proof by contrapositive*
    - show the contrapositive $\neg q \to \neg p$
  - *proof by contradiction*
    - show that $(p \land \neg q)$ contradicts the assumptions
  - *proof by cases*
    - give proofs for all possible cases
  - *proof of equivalence*
    - $p \leftrightarrow q$ is replaced with $(p \to q) \land (q \to p)$

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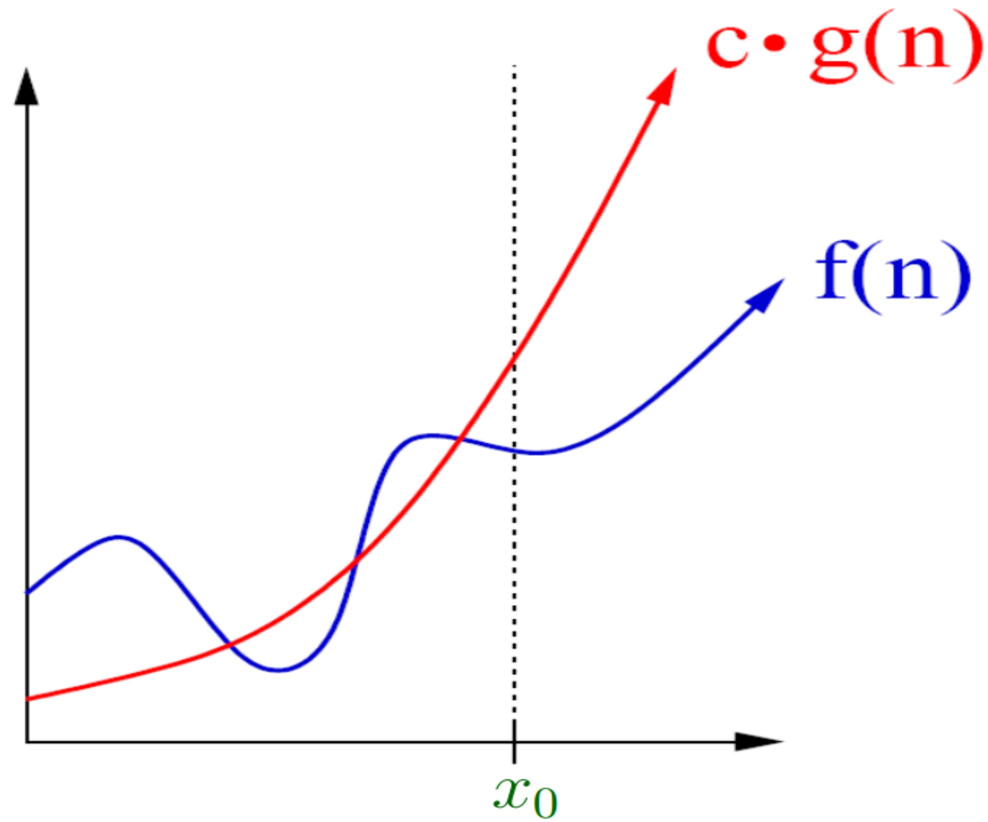
- counting the number of such functions?

Big-O Notation

- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(n) = O(g(n))$ (reads: $f(n)$ is O of $g(n)$), if there exist some positive constants C and k such that $|f(n)| \leq C|g(n)|$, whenever $n > k$.



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$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 2 \pmod{7} \end{aligned}$$

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Cryptography

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  RSA cryptosystem

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3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Recurrence

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  bottom up or top down

Recurrence

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  prove by induction, complexity, ...

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- **Definition** An *r-combination* with repetition allowed, or a *multiset of size r*, chosen from a set of n elements, is an unordered selection of elements with repetition allowed.

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Graphs & Trees

- Basic concepts

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  connected graph, simple graph, isomophism, chromatic number, planar graph, Euler circuit, Hamilton circuit, shortest path, bipartite graph, complete graph, special graphs (K_n, $K_{m,n}$, C_n, W_n, Q_n), m-ary tree, tree traversal, spanning tree ...

Good Luck!