6.6

1. $\vec{r}'(t) = (a\cos t, a\sin t, 2bt)$

$\vec{r}''(t) = (-a\sin t, a\cos t, 2b)$

2. 设 $\vec{r}(t) = (x_1(t), \cdots, x_n(t))$ 为单位向量

则 $x_1(t)^2 + \cdots + x_n(t)^2 = 1 \quad \cdots ①$

① 两边对 t 求导：$2x_1(t)x_1'(t) + \cdots + 2x_n(t)x_n'(t) = 0$

即 $2\vec{r}(t)\cdot\vec{r}'(t) = 0$，$\therefore \vec{r}(t) \perp \vec{r}'(t)$

几何意义：单位球面上一条曲线的切线与该处半径垂直

3. (1)(2)(3)(5) 显然：只需把 $\vec{r}(t)$ 的分量设出来

(4) $$(\vec{r}_1\times\vec{r}_2)'(t_0) = \lim_{t\to t_0}\frac{\vec{r}_1(t)\times\vec{r}_2(t) - \vec{r}_1(t_0)\times\vec{r}_2(t_0)}{t-t_0}$$

$$= \lim_{t\to t_0}\left(\vec{r}_1(t)\times\frac{\vec{r}_2(t)-\vec{r}_2(t_0)}{t-t_0}\right) + \lim_{t\to t_0}\left(\frac{\vec{r}_1(t)-\vec{r}_1(t_0)}{t-t_0}\times\vec{r}_2(t_0)\right)$$

$$= \left(\lim_{t\to t_0}\vec{r}_1(t)\right)\times\left(\lim_{t\to t_0}\frac{\vec{r}_2(t)-\vec{r}_2(t_0)}{t-t_0}\right) + \left(\lim_{t\to t_0}\frac{\vec{r}_1(t)-\vec{r}_1(t_0)}{t-t_0}\right)\times\vec{r}_2(t_0)$$

$$= \vec{r}_1(t_0)\times\vec{r}_2'(t_0) + \vec{r}_1'(t_0)\times\vec{r}_2(t_0)$$

4. $\vec{r}(t) = (a\cos t, a\sin t, bt)$

$\vec{r}'(t) = (-a\sin t, a\cos t, b)$

$\vec{k} = (0,0,1)$ // z 轴，$\dfrac{\vec{r}'(t)\cdot\vec{k}}{|\vec{r}'(t)|\,|\vec{k}|} = \dfrac{b}{\sqrt{a^2+b^2}}$ 为定值

5. $\vec{r}'(t) = \left(\frac{1}{(1+t)^2}, -\frac{1}{t^2}, 2t\right)$，$t>0$ 时 $\vec{r}'(t)\neq\vec{0}$，$\therefore \vec{r}(t)$ 光滑

t^2 在 $t>0$ 时严格单调，$\therefore \vec{r}(t)$ 不自交，$\therefore$ 简单

6. (1) $\vec{r}(t)=(a\sin^2 t,\ b\sin t\cos t,\ c\cos^2 t)$

$\vec{r}\,'(t)=(a\sin 2t,\ b\cos 2t,\ -c\sin 2t)$ $\quad \vec{r}(\frac{\pi}{4})=(\frac{a}{2},\frac{b}{2},\frac{c}{2})$

$\vec{r}\,'(\frac{\pi}{4})=(a,0,-c)$

切线：$\vec{R}=\vec{r}(\frac{\pi}{4})+t\vec{r}\,'(\frac{\pi}{4})$，即 $\begin{cases} x=\frac{a}{2}+at \\ y=\frac{b}{2} \\ z=\frac{c}{2}-ct \end{cases}$

法平面：$(\vec{R}-\vec{r}(\frac{\pi}{4}))\cdot\vec{r}\,'(\frac{\pi}{4})=0$，即 $a(x-\frac{a}{2})-c(z-\frac{c}{2})=0$

(2) $\vec{r}(t)=(t-\cos t,\ 3+\sin^2 t,\ 1+\cos 3t)$ $\quad \vec{r}(\frac{\pi}{2})=(\frac{\pi}{2},4,1)$

$\vec{r}\,'(t)=(1+\sin t,\ \sin 2t,\ -3\sin 3t)$ $\quad \vec{r}\,'(\frac{\pi}{2})=(2,0,3)$

切线：$\vec{R}=\vec{r}(\frac{\pi}{2})+t\vec{r}\,'(\frac{\pi}{4})$，即 $\begin{cases} x=\frac{\pi}{4}+2t \\ y=4 \\ z=1+3t \end{cases}$ ($\vec{R}$为(x,y,z)即自变量)

法平面：$(\vec{R}-\vec{r}(\frac{\pi}{2}))\cdot\vec{r}\,'(\frac{\pi}{2})=0$，即 $2(x-\frac{\pi}{2})+3(z-1)=0$

7. (1) $\vec{r}=(u\cos v,\ u\sin v,\ av)$，$\vec{r}_u=(\cos v,\ \sin v,\ 0)$，$\vec{r}_v=(-u\sin v,\ u\cos v,\ a)$

切平面：$\vec{r}(u_0,v_0)+t\vec{r}_u(u_0,v_0)+s\vec{r}_v(u_0,v_0)$ $\quad (t,s\in\mathbb{R})$

法向量：$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & a \end{vmatrix}=a\sin v\,\vec{i}-a\cos v\,\vec{j}+u\vec{k}$

$=(a\sin v,\ -a\cos v,\ u)$

法线：$\vec{r}(u_0,v_0)+t\vec{n}(u_0,v_0)$ $\quad (t\in\mathbb{R})$

(2) $\vec{r}=(a\sin\theta\cos\varphi,\ b\sin\theta\sin\varphi,\ c\cos\theta)$

$\vec{r}_\theta=(a\cos\theta\cos\varphi,\ b\cos\theta\sin\varphi,\ -c\sin\theta)$

$\vec{r}_\varphi=(-a\sin\theta\sin\varphi,\ b\sin\theta\cos\varphi,\ 0)$

$\vec{n}=\vec{r}_\theta\times\vec{r}_\varphi=(bc\sin^2\theta\cos\varphi,\ ac\sin^2\theta\sin\varphi,\ ab\sin\theta\cos\theta)$

(1)另解：先求出 $\vec{n}$，切平面方程为 $(\vec{R}-\vec{r}(u_0,v_0))\cdot\vec{n}(u_0,v_0)=0$

8. (1) $z=f(x,y)=\sqrt{x^2+y^2}-xy$

$f'_x(x,y)=\frac{x}{\sqrt{x^2+y^2}}-y$, $f'_x(3,4)=-\frac{17}{5}$

$f'_y(x,y)=\frac{y}{\sqrt{x^2+y^2}}-x$, $f'_y(3,4)=-\frac{11}{5}$

$\vec{n}=(-f'_x,-f'_y,1)$, $\therefore \vec{n}(3,4)=(\frac{17}{5},\frac{11}{5},1)$

法线：$\vec{R}=(3,4,-7)+t(\frac{17}{5},\frac{11}{5},1)$，即 $\begin{cases}x=3+\frac{17}{5}t\\ y=4+\frac{11}{5}t\\ z=-7+t\end{cases}$ $(t\in\mathbb{R})$

切平面：$(\vec{R}-(3,4,-7))\cdot(\frac{17}{5},\frac{11}{5},1)=0$ 即 $\frac{17}{5}(x-3)+\frac{11}{5}(y-4)+(z+7)=0$

(2) $z=f(x,y)=\arctan\frac{y}{x}$

$f'_x(x,y)=-\frac{y}{x^2+y^2}$, $f'_x(1,1)=-\frac{1}{2}$

$f'_y(x,y)=\frac{x}{x^2+y^2}$, $f'_x(1,1)=\frac{1}{2}$

$\vec{n}=(-f'_x,-f'_y,1)$, $\therefore \vec{n}(1,1)=(\frac{1}{2},-\frac{1}{2},1)$

(3) $F(x,y,z)=e^z-z+xy-3=0$

$F_x=y$, $F_y=x$, $F_z=e^z-1$

$\therefore \vec{n}=(F_x,F_y,F_z)=(y,x,e^z-1)$, $\vec{n}(2,1,0)=(1,2,0)$

(4) $F(x,y,z)=4+r-x-y-z=0$ $(r=\sqrt{x^2+y^2+z^2})$

$F_x=\frac{x}{r}-1$, $F_y=\frac{y}{r}-1$, $F_z=\frac{z}{r}-1$

$\vec{n}=(\frac{x}{r}-1,\frac{y}{r}-1,\frac{z}{r}-1)$, $\vec{n}(2,3,6)=(-\frac{5}{7},-\frac{4}{7},-\frac{1}{7})$

9. $x-y+2z=0$ 可写成 $(x,y,z)\cdot(1,-1,2)=0$ 即 $(x,y,z)\perp(1,-1,2)$

∴ 其法向量为 $\vec{N}=(1,-1,2)$

椭球面为 $F(x,y,z)=x^2+2y^2+z^2-1=0$

法向量 $\vec{n}=(2x,4y,2z)$ $(\vec{n}=(F_x,F_y,F_z))$

需要 $\vec{n}(x,y,z)\parallel\vec{N}$ (平面平行⇔法向相同)

设 $\vec{n}(x,y,z)=k\vec{N}$ 则 $\begin{cases}x=k/2\\ y=-k/4\\ z=k\end{cases}$

代入 $x^2+2y^2+z^2=1 \Rightarrow \frac{k^2}{4}+2\frac{k^2}{16}+k^2=1 \Rightarrow k=\sqrt{\frac{8}{11}}$

∴ $x=\sqrt{\frac{2}{11}}$, $y=-\sqrt{\frac{1}{22}}$, $z=\sqrt{\frac{8}{11}}$, 切平面为 $(\vec{R}-(x,y,z))\cdot\vec{n}(x,y,z)=0$

10. $F(x,y,z)=z-xy$, $\vec{n}=(F_x,F_y,F_z)=(-y,-x,1)$

$\{x+3y+z=0\}$ 的法向为 $(1,3,1)$

∴ 令 $(-y,-x,1)\parallel(1,3,1)$, 得 $x=-3$, $y=-1$

∴ 该点为 $(-3,-1,3)$, 法向为 $(1,3,1)$

∴ 法线为 $\begin{cases}x=-3+t\\ y=-1+3t\\ z=3+t\end{cases}$ $(t\in\mathbb{R})$

11. $L:\frac{x-6}{2}=\frac{y-3}{1}=\frac{z-\frac{1}{2}}{-1}$, 故 L 过 $P(6,3,\frac{1}{2})$ 且方向向量为 $\vec{v}=(2,1,-1)$

椭球面为 $F(x,y,z)=x^2+2y^2+3z^2-21=0$

法向 $\vec{n}=(2x,4y,6z)$

若 P 在切平面内, 则 $(6,3,\frac{1}{2})-(x,y,z)\perp(2x,4y,6z)$

即 $(x-6,y-3,z-\frac{1}{2})\cdot(2x,4y,6z)=0$, 即 $2(x^2+2y^2+3z^2)=12x+12y+3z$

∴ $2\times21=12x+12y+3z$, $4x+4y+z=14$ …①

若 L 在切平面内, 则 $\vec{v}\perp\vec{n}$, 故 $4x+4y-6z=0$ …②

由①②得 $z=2$, $x+y=3$

又 $x^2+2y^2+3z^2=21$, ∴ $x^2+2y^2=9$, 即 $(x,y,z)=(3,0,2)$ 或 $(1,2,2)$

12. $z=x^2+y^2$ 法向 $\vec{n}=(-z_x,-z_y,1)=(-2x,-2y,1)$

故 $\vec{n}(1,-2)=(-2,4,1)$ 为 π 的法向且 $(1,-2,5)\in\pi$

$\begin{cases} x+y+b=0 \\ x+ay-z-3=0 \end{cases} \Rightarrow (-b,0,-b-3)\in L$ 且 $(0,-b,-ab-3)\in L$

$\therefore (-b,0,-b-3)$、$(0,-b,-ab-3)\in\pi$

$\therefore \begin{cases} (-b,0,-b-3)-(1,-2,5)\perp(-2,4,1) \\ (0,-b,-ab-3)-(1,-2,5)\perp(-2,4,1) \end{cases}$

即 $a=-1$, $b=-2$

13. 只需证交点处法向垂直

交点处 $ax=by$

$x^2+y^2+z^2-ax=0$ 法向为 $(2x-a,2y,2z)$

$x^2+y^2+z^2-by=0$ 法向为 $(2x,2y-b,2z)$

$(2x-a,2y,2z)\cdot(2x,2y-b,2z)=4(x^2+y^2+z^2)-2ax-2by=0 \quad (ax=by)$

14. 只需证 $(2,-3,1)$ 处两曲面的法向平行

$x+2y-\ln z+4=0$ 法向为 $(1,2,-\frac{1}{z})=(1,2,-1)$

$x^2-xy-8x+z+5=0$ 法向为 $(2x-y-8,-x,1)=(-1,-2,1)$

15. $z=f(x,y)=xe^{\frac{x}{y}}$, 法向: $\overset{\vec{n}}{\parallel}\ (-f_x,-f_y,1)=(-(1+\frac{x}{y})e^{\frac{x}{y}},\frac{x^2}{y^2}e^{\frac{x}{y}},1)$

而 $\vec{n}(x,y)\cdot(x,y,z)=-(1+\frac{x}{y})e^{\frac{x}{y}}\cdot x+\frac{x^2}{y^2}e^{\frac{x}{y}}\cdot y+1\cdot xe^{\frac{x}{y}}=0$

$\therefore (x,y,z)\perp\vec{n}(x,y)$, 即 $(0,0,0)-(x,y,z)\perp\vec{n}$

$\therefore (0,0,0)$ 在切平面内

16. (1) $f(x,y)=x^3y+xy^3+x^2y^2-3=0$

$\therefore$ 法向 $\vec{n}=(f'_x, f'_y)=(3x^2y+y^3+2xy^2, x^3+3xy^2+2x^2y)$

$\therefore \vec{n}(1,1)=(6,6)\parallel(1,1)$，$\therefore$ 法线为 $x=y$

切线为过 $(1,1)$ 的垂直于 $x=y$ 的线，故为 $y=2-x$

(2) $f(x,y)=x+2y-\cos xy$

$\vec{n}=(f'_x, f'_y)=(1+y\sin xy, 2+x\sin xy)$

$\therefore \vec{n}(1,0)=(1,2)$，$\therefore$ 法线为 $y=2x-2$

切线为过 $(1,0)$ 的垂直于 $y=2x-2$ 的线，故为 $y=\frac{1}{2}-\frac{x}{2}$

17. (1) $y^2+z^2=25$ 的法向 $\vec{n_1}=(0,2y,2z)$

$x^2+y^2=10$ 的法向 $\vec{n_2}=(2x,2y,0)$

$\therefore$ 交线的切方向 $\vec{t_0}=\vec{n_1}\times\vec{n_2}=4(-yz, xz, -xy)$

$\vec{t_0}(1,3,4)=4(-12,4,-3)$

$\therefore$ 切线为 $\begin{cases} x=1-12t \\ y=3+4t \\ z=4-3t \end{cases}$ $(t\in\mathbb{R})$

法平面为 $((x,y,z)-(1,3,4))\cdot(-12,4,-3)=0$

(2) $2x^2+3y^2+z^2=47$ 的法向 $\vec{n_1}=(4x,6y,2z)$

$x^2+2y^2-z=0$ 的法向 $\vec{n_2}=(2x,4y,-1)$

$\therefore$ 交线的切向 $\vec{t_0}=\vec{n_1}\times\vec{n_2}=(-6y-8yz, 4xz+4x, 16xy-12xy)$

$\therefore \vec{t_0}(-2,1,6)=(-54,-56,-8)=-2(27,28,4)$