7.6书后习题答案

1.

(1)

设$x=a\sin\theta\cos\varphi \quad y=b\sin\theta\sin\varphi \quad z=c\cos\theta \qquad 0\leqslant\varphi\leqslant 2\pi \quad ,0\leqslant\theta\leqslant\pi$

$\dfrac{\partial(x,y)}{\partial(\theta,\varphi)}=\begin{vmatrix} a\cos\theta\cos\varphi & b\cos\theta\sin\varphi \\ -a\sin\theta\sin\varphi & b\sin\theta\cos\varphi \end{vmatrix}=ab\sin\theta\cos\theta$

$\Rightarrow\oiint(x^2+y^2)\,dx\,dy=\int_0^{2\pi}\int_0^{\pi}(a\sin\theta\cos\varphi+b^2\sin^2\theta\sin\varphi+c\cos\theta)\cdot ab\sin\theta\cos\theta\,d\theta d\varphi$

$=2\pi abc\int_0^{\pi}\sin\theta\cos^2\theta\,d\theta=\frac{4}{3}\pi abc$

(2)

设$x=R\sin\theta \quad z=R\cos\theta \quad y=t \quad ,0\leqslant t\leqslant h \qquad 0\leqslant\theta\leqslant\pi$

$\dfrac{\partial(x,y)}{\partial(\theta,t)}=\begin{vmatrix} R\cos\theta & 0 \\ 0 & 1 \end{vmatrix}=R\cos\theta$

$\Rightarrow\iint\limits_S xyz\,\mathrm{dx}\,\mathrm{dy}=\int_0^{\pi}\int_0^{h}R^2\cos\theta\sin\theta\cdot t\cdot R\cos\theta\,dt\,d\theta$

$=\int_0^{\pi}R^3\cos^2\theta\sin^2\theta\,d\theta\cdot\int_0^{h}t\,\mathrm{dt}=\frac{1}{3}h^2R^3$

(3)

设$x=R\sin\theta\cos\varphi \quad y=R\sin\theta\sin\varphi \quad z=R\cos\theta \quad ,0\leqslant\theta\leqslant\pi \quad \frac{\pi}{2}\leqslant\varphi\leqslant\frac{3\pi}{2}$

$\dfrac{\partial(y,z)}{\partial(\theta,\varphi)}=\begin{vmatrix} R\cos\theta\sin\varphi & -R\sin\theta \\ R\sin\theta\cos\varphi & 0 \end{vmatrix}=R^2\sin^2\theta\cos\varphi$

$\Rightarrow\iint\limits_S xy^2z^2\,\mathrm{dy}\,\mathrm{dz}=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{\pi}R\sin\theta\cos\varphi\cdot R^2\sin^2\theta\sin^2\varphi\cdot R^2\sin^2\theta\cos\varphi\,d\theta d\varphi$

$=R^7\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{\pi}\sin^5\theta\cos^2\theta\sin^2\varphi\cos^2\varphi\,d\theta d\varphi=R^7\int_0^{\pi}\sin^5\theta\cos^2\theta\,d\theta\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin^2\varphi\cos^2\varphi\,d\varphi$

$=R^7\cdot\frac{16}{105}\cdot\frac{\pi}{8}=\frac{2\pi}{105}R^7$

(4)

设$x=\sin\theta\cos\varphi \quad y=\sin\theta\sin\varphi \quad z=\cos\theta \quad ,0\leqslant\theta\leqslant\frac{\pi}{2} \quad 0\leqslant\varphi\leqslant 2\pi$

$\dfrac{\partial(x,z)}{\partial(\theta,\varphi)}=\sin^2\theta\sin\varphi$

$\Rightarrow\iint\limits_S yz\,\mathrm{dz}\,\mathrm{dx}=\iint\limits_S\sin\theta\sin\varphi\cdot\cos\theta\cdot\sin^2\theta\sin\varphi\,d\theta d\varphi$

$=\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta\cdot\int_0^{2\pi}\sin^2\varphi\,d\varphi=\frac{1}{4}\cdot\pi=\frac{\pi}{4}$

(5)

由对称性可知：$\iint\limits_S x^2\,\mathrm{dy}\,\mathrm{dz}+y^2\mathrm{dz}\,\mathrm{dx}+z^2\mathrm{dx}\,\mathrm{dy}=3\iint\limits_S z^2\mathrm{dx}\,\mathrm{dy}$

$\Rightarrow$原式$=3\iint\limits_S(1-x-y)^2\mathrm{dx}\,\mathrm{dy}=3\int_0^1\int_0^{1-x}(1-x-y)^2\mathrm{dy}=3\int_0^1(1-x)^3\mathrm{dx}=\frac{1}{4}$

(6)

由对称性可知$\iint\limits_S(y-z)\,\mathrm{dy}\,\mathrm{dz}=\iint\limits_S(x-z)\,\mathrm{dx}\,\mathrm{dz}$

$\Rightarrow$原式$=\iint\limits_S(x-y)\,\mathrm{dx}\,\mathrm{dy}$

设$x=z\cos\theta\quad y=z\sin\theta\qquad z=z\qquad,0\leqslant\theta\leqslant2\pi\quad0\leqslant z\leqslant1$

$\frac{\partial(x,y)}{\partial(z,\theta)}=z$

$\Rightarrow$原式$=\int_0^{2\pi}\int_0^1 z(\cos\theta-\sin\theta)\cdot z\,\mathrm{dz}\,d\theta=0$

(7)

设$x=a\sin\theta\cos\varphi\quad y=a\sin\theta\sin\varphi\quad z=a\cos\theta\quad,0\leqslant\theta\leqslant\frac{\pi}{2}\quad0\leqslant\varphi\leqslant\pi$

$\frac{\partial(y,z)}{\partial(\theta,\varphi)}=a^2\sin^2\theta\cos\varphi$

$\frac{\partial(x,z)}{\partial(\theta,\varphi)}=-a^2\sin^2\theta\sin\varphi$

$\frac{\partial(x,y)}{\partial(\theta,\varphi)}=a^2\sin\theta\cos\theta$

$\Rightarrow$原式$=a^5\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta\cos^2\varphi+\sin^5\theta\cos^2\varphi\sin^2\varphi+\sin^3\theta\cos^2\theta\cos^2\varphi\sin^2\varphi d\theta d\varphi$

$=a^5\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta+\sin^5\theta\cos^2\varphi\sin^2\varphi d\theta d\varphi=a^5\cdot\left(\frac{4}{15}\pi+\frac{2}{15}\pi\right)=\frac{2}{5}\pi a^5$

(8)

$\oiint\limits_S f(x)\,\mathrm{dy}\,\mathrm{dz}+g(z)\,\mathrm{dz}\,\mathrm{dx}+h(z)\,\mathrm{dx}\,\mathrm{dy}=\iiint\limits_V\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz}$

$=bc(f(a)-f(0))+ac(g(b)-g(0))+ab(h(c)-h(0))$

2.

设通量为M

$M=\iint\limits_S\vec{v}\cdot\vec{n}\,\mathrm{ds}=\iiint\limits_V(\nabla.\vec{v})\,\mathrm{dv}=\iiint\limits_V x^2+1\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz}=\frac{1}{3}abc(3+a^2)$

3.

(1)

原式$=\iiint\limits_V(3x^2+3y^2+3z^2)\,\mathrm{dx\,dy\,dz}$

设$x=r\sin\theta\cos\varphi\quad,y=r\sin\theta\sin\varphi\quad,z=r\cos\theta\quad;0\leqslant r\leqslant R\quad,0\leqslant\varphi\leqslant2\pi\quad,0\leqslant\theta\leqslant\pi$

$\Rightarrow\dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=\begin{vmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta\\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0\end{vmatrix}=r^2\sin\theta$

$\Rightarrow$原式$=\int_0^R\int_0^{2\pi}\int_0^{\pi}3r^2\cdot r^2\sin\theta d\theta d\varphi\,\mathrm{dr}=\dfrac{12}{5}\pi R^3$

(2)

原式$=\iiint\limits_V(y+z+x)\,\mathrm{dx\,dy\,dz}=\int_0^z\int_0^{1-z}\int_0^{1-y-z}(x+y+z)\,\mathrm{dx\,dy\,dz}$

$\quad=\dfrac{1}{2}\int_0^1\int_0^{1-z}1-(y+z)^2\,\mathrm{dy\,dz}=\dfrac{1}{6}\int_0^1 2-3z+z^3\,\mathrm{dz}=\dfrac{1}{8}$

(3)

原式$=\iiint\limits_V(1+1+1)\,\mathrm{dx\,dy\,dz}=3\iiint\limits_V\mathrm{dx\,dy\,dz}=4\pi\mathrm{abc}$

(4)

原式$=2\iiint\limits_V x+y+z\,\mathrm{dx\,dy\,dz}$

设$x=a+r\sin\theta\cos\varphi\quad,y=b+r\sin\theta\sin\varphi\quad,z=c+r\cos\theta$

其中　$0\leqslant r\leqslant R\quad,0\leqslant\varphi\leqslant2\pi\quad,0\leqslant\theta\leqslant\pi$

$\Rightarrow\dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=r^2\sin\theta$

$\Rightarrow$原式$=2\iiint\limits_V[a+b+c+r(\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta)]r^2\sin\theta\,\mathrm{drd}\theta d\varphi$

$\quad=2(a+b+c)\cdot\dfrac{4}{3}\pi R^3+2\int_0^R r^3\int_0^{2\pi}\int_0^{\pi}\sin^2\theta(\cos\varphi+\sin\varphi)+\sin\theta\cos\theta d\theta d\varphi$

$\quad=\dfrac{8}{3}\pi(a+b+c)R^3$

(5)

原式$=3\iiint\limits_V\mathrm{dx\,dy\,dz}=3\int_0^1\int_0^{1-z}\int_0^{1-y-z}\mathrm{dx\,dy\,dz}=3\int_0^1\int_0^{1-z}(1-z-y)\,\mathrm{dy\,dz}$

$\quad=\dfrac{3}{2}\int_0^1(1-z)^2\,\mathrm{dz}=\dfrac{1}{2}$

(6)

原式 $=\iiint\limits_V y^2+z^2+x^2 \mathrm{dx\,dy\,dz}$

设 $x=r\sin\theta\cos\varphi \quad ,y=r\sin\theta\sin\varphi \quad ,z=\frac{1}{2}+r\cos\theta;0\leqslant r\leqslant\frac{1}{2} \quad ,0\leqslant\varphi\leqslant 2\pi \quad ,0\leqslant\theta\leqslant\pi$

$\Rightarrow\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=r^2\sin\theta$

$\Rightarrow$ 原式 $=\iiint\limits_V \left(r^2+r\cos\theta+\frac{1}{4}\right)r^2\sin\theta\,\mathrm{dr}\,d\theta d\varphi$

$=\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{1}{2}}r^4\sin\theta+r^3\cos\theta\sin\theta+\frac{1}{4}r^2\sin\theta\,\mathrm{dr}\,d\theta d\varphi$

$=\frac{\pi}{15}$

(7)

设 S': $x^2+y^2\leqslant a^2$

原式 $=\iiint\limits_V 0\mathrm{dx\,dy\,dz}-\iint\limits_{S'}(y^2+z^2)\,\mathrm{dy\,dz}+(z^2+x^2)\,\mathrm{dz\,dx}+(x^2+y^2)\,\mathrm{dx\,dy}$

$=-\iint\limits_{S'}(x^2+y^2)\,\mathrm{dx\,dy}$

设 $x=r\cos\theta \quad ,y=r\sin\theta \quad ;0\leqslant r\leqslant a \quad ,0\leqslant\theta\leqslant 2\pi$

原式 $=-\int_0^{2\pi}\int_0^a r^3\mathrm{dr}\,d\theta=\frac{\pi}{2}a^4$

(8)

设 S': $z=1,x^2+y^2\leqslant 1$

原式 $=\iiint\limits_V 3\,\mathrm{dx\,dy\,dz}-\iint\limits_{S'}(1-y)\,\mathrm{dx\,dy}$

对 $\iiint\limits_V 3\,\mathrm{dx\,dy\,dz}$ 设 $z=z \quad ,x=r\cos\theta \quad ,y=r\sin\theta;0\leqslant\theta\leqslant 2\pi\,,0\leqslant r\leqslant\sqrt{2}\,,0\leqslant z\leqslant 1$

$\Rightarrow\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=r$

$\Rightarrow\iiint\limits_V 3\,\mathrm{dx\,dy\,dz}=\iiint\limits_V 3r\,\mathrm{dr}\,d\theta\,\mathrm{dz}=3\int_0^1\mathrm{dz}\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}r\,\mathrm{dr}=\frac{3}{2}\pi$

对 $\iint\limits_{S'}(1-y)\,\mathrm{dx\,dy}$ 设 $x=r\cos\theta \quad ,y=r\sin\theta \quad ;0\leqslant\theta\leqslant 2\pi\,,0\leqslant r\leqslant 1$

$\Rightarrow\frac{\partial(x,y)}{\partial(r,\theta)}=r$

$\Rightarrow\iint\limits_{S'}(1-y)\,\mathrm{dx\,dy}=\int_0^{2\pi}\int_0^1(1-r\sin\theta)\,r\,\mathrm{dr}\,d\theta=\pi$

$\Rightarrow$ 原式 $=\frac{3}{2}\pi-\pi=\frac{1}{2}\pi$

4.

(1)

通量：$\Omega=\oiint\limits_S \vec{F}\cdot\vec{n}\,\mathrm{dS}=\oiint\limits_S -km\dfrac{\vec{r}}{r^3}\cdot\vec{n}\,\mathrm{dS}$

在曲面内部取一以M为中心直径为δ的小球，设球面为S_1,剩余曲面为S_2

$$\Rightarrow\oiint\limits_S \vec{F}\cdot\vec{n}\,\mathrm{dS}=\oiint\limits_{S_1}\vec{F}\cdot\vec{n}\,\mathrm{dS}+\oiint\limits_{S_2}\vec{F}\cdot\vec{n}\,\mathrm{dS}$$

$$\oiint\limits_{S_1}\vec{F}\cdot\vec{n}\,\mathrm{dS}=-\frac{km}{\delta^3}\iint\limits_S x\,\mathrm{dy}\,\mathrm{dz}+\mathrm{ydz}\,\mathrm{dx}+z\,\mathrm{dx}\,\mathrm{dy}=-\frac{km}{\delta^3}\iiint\limits_V 3\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz}$$

$$=-\frac{km}{\delta^3}\cdot 4\pi\delta^3=-4\pi km$$

由已知得$\oiint\limits_{S_2}\vec{F}\cdot\vec{n}\,\mathrm{dS}=0$

$$\Rightarrow\oiint\limits_S \vec{F}\cdot\vec{n}\,\mathrm{dS}=\oiint\limits_{S_1}\vec{F}\cdot\vec{n}\,\mathrm{dS}+\oiint\limits_{S_2}\vec{F}\cdot\vec{n}\,\mathrm{dS}=-4\pi km+0=-4\pi km$$

(2)

对于任意一个不包围质点M的曲面设为S，

一定可以等于两个包围质点M的曲面S_1与S_2相减。

即$S=S_1-S_2$

$$\Rightarrow\oiint\limits_S -km\frac{\vec{r}}{r^3}\cdot\vec{n}\,\mathrm{dS}=\oiint\limits_{S_1}-km\frac{\vec{r}}{r^3}\cdot\vec{n}\,\mathrm{dS}-\oiint\limits_{S_2}-km\frac{\vec{r}}{r^3}\cdot\vec{n}\,\mathrm{dS}=0$$

(3)

对于质点处的以M为球心的一半球$R=\delta$

$$\oiint\limits_S \vec{F}\cdot\vec{n}\,\mathrm{dS}=-\frac{km}{\delta^3}\iint\limits_S x\,\mathrm{dy}\,\mathrm{dz}+\mathrm{ydz}\,\mathrm{dx}+z\,\mathrm{dx}\,\mathrm{dy}=-\frac{km}{\delta^3}\iiint\limits_V 3\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz}$$

$$=-\frac{km}{\delta^3}\cdot 2\pi\delta^3=-2\pi km$$

当光滑封闭曲面S含有质点M时，挖下一个小圆盘$R=\delta$，得到曲面S^-

并且在曲面外部补充一个半球面S_1构成一封闭曲面Γ

$$\Rightarrow\oiint\limits_\Gamma \vec{F}\cdot\vec{n}\,\mathrm{dS}=\oiint\limits_{S^-}\vec{F}\cdot\vec{n}\,\mathrm{dS}+\oiint\limits_{S_1}\vec{F}\cdot\vec{n}\,\mathrm{dS}=\oiint\limits_{S^-}\vec{F}\cdot\vec{n}\,\mathrm{dS}-2\pi km=-4\pi km$$

$$\lim_{\delta\to 0}\oiint\limits_{S^-}\vec{F}\cdot\vec{n}\,\mathrm{dS}=\oiint\limits_S\vec{F}\cdot\vec{n}\,\mathrm{dS}=-2\pi km$$

5.

$$\frac{1}{3}\oiint\limits_S \mathrm{x}\mathrm{dy}\,\mathrm{dz}+y\,\mathrm{dx}\,\mathrm{dz}+z\,\mathrm{dx}\,\mathrm{dy}=\frac{1}{3}\iiint\limits_V 3\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz}=V$$

6.

$$\cos(\widehat{\vec{c}\cdot\vec{n}})=\frac{\vec{c}\cdot\vec{n}}{|\vec{c}|}$$

$$\Rightarrow\oiint\limits_S\cos(\widehat{\vec{c}\cdot\vec{n}})\ \mathrm{dS}=\frac{1}{|\vec{c}|}\cdot\oiint\limits_S\vec{c}\cdot\vec{n}\,\mathrm{dS}=\frac{1}{|\vec{c}|}\iiint\limits_V\nabla\cdot\vec{c}\,\mathrm{dV}=0$$

7.

高斯公式：

$$\oiint\limits_S P(x,y)\,\mathrm{dy}\,\mathrm{dz}+Q(x,y)\,\mathrm{dz}\,\mathrm{dx}=\iiint\limits_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz}$$

由已知得

$$\iiint\limits_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz}=\iint\limits_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{dx}\,\mathrm{dy}$$

$$\oiint\limits_S P(x,y)\,\mathrm{dy}\,\mathrm{dz}=\oint_L P(x,y)\,\mathrm{dy}$$

$$\oiint\limits_S Q(x,y)\,\mathrm{dx}\,\mathrm{dz}=\oint_L Q(x,y)\,\mathrm{dx}$$

$$\Rightarrow\oint_L P(x,y)\,\mathrm{dy}+\oint_L Q(x,y)\,\mathrm{dx}=\iint\limits_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{dx}\,\mathrm{dy}$$

8.解

(1) $\int_L y\,\mathrm{dx}+z\,\mathrm{dy}+x\,\mathrm{dz}=\iint\limits_S\begin{vmatrix}\mathrm{dy}\,\mathrm{dz}&\mathrm{dz}\,\mathrm{dx}&\mathrm{dx}\,\mathrm{dy}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y&z&x\end{vmatrix}=\iint\limits_S-\mathrm{dy}\,\mathrm{dz}-\mathrm{dz}\,\mathrm{dx}-\mathrm{dx}\,\mathrm{dy}=-\frac{3}{2}$

(2)

$$\int_L(y-z)\,\mathrm{dx}+(z-x)\,\mathrm{dy}+(x-y)\,\mathrm{dz}=\iint\limits_S\begin{vmatrix}\mathrm{dy}\,\mathrm{dz}&\mathrm{dz}\,\mathrm{dx}&\mathrm{dx}\,\mathrm{dy}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y-z&z-x&x-y\end{vmatrix}$$

$$=\iint\limits_S-2\mathrm{dy}\,\mathrm{dz}-2\mathrm{dz}\,\mathrm{dx}-2\mathrm{dx}\,\mathrm{dy}=-2\pi ah-2\pi a^2$$

(3)

$$\int_L (y^2-z^2)\,\mathrm{dx}+(z^2-x^2)\,\mathrm{dy}+(x^2-y^2)\,\mathrm{dz}=\iint_S \begin{vmatrix} \mathrm{dy\,dz} & \mathrm{dz\,dx} & \mathrm{dx\,dy} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2-z^2 & z^2-x^2 & x^2-y^2 \end{vmatrix}$$

$$=\iint_S -2(y+z)\,\mathrm{dy\,dz}-2(x+z)\,\mathrm{dz\,dx}-2(x+y)\,\mathrm{dx\,dy}$$

由对称性可知

上式$=-6\iint_S (x+y)\,\mathrm{dx\,dy}=-12\iint_S x\,\mathrm{dx\,dy}$

设S_1: $0\leqslant x\leqslant a \quad ,0\leqslant y\leqslant a \quad ,z=0 \quad ;S_2$: $0\leqslant x\leqslant \frac{a}{2} \quad ,0\leqslant y\leqslant \frac{a}{2} \quad ,z=0$

S_3: $\frac{a}{2}\leqslant x\leqslant a \quad ,\frac{a}{2}\leqslant y\leqslant a \quad ,z=0,$ 且$x+y\geqslant \frac{3}{2}a$

$\Rightarrow S=S_1-S_2-S_3$

$\iint_{S_1} x\,\mathrm{dx\,dy}=\frac{1}{2}a^3 \quad ;\iint_{S_2} x\,\mathrm{dx\,dy}=\int_0^{\frac{a}{2}}\int_0^{\frac{a}{2}-y} x\,\mathrm{dx\,dy}=\frac{a^3}{48} \quad ;$

$\iint_{S_3} x\,\mathrm{dx\,dy}=\int_{\frac{a}{2}}^{a}\int_{\frac{3}{2}a-y}^{a} x\,\mathrm{dx\,dy}=\frac{5}{48}a^3$

$\Rightarrow$原式$=-12\left(\frac{1}{2}a^3-\frac{a^3}{48}-\frac{5}{48}a^3\right)=-\frac{9}{2}a^3$

(4)

$\int_L y^2\mathrm{dx}+\mathrm{xy\,dy}+\mathrm{xz\,dz}=\oiint_S -z\,\mathrm{dz\,dx}-\mathrm{y dx\,dy}=-2\iint_{S'} y\,\mathrm{dx\,dy}$

其中S': $x^2+(y-1)^2\leqslant 1, z=0$

设$x=r\cos\theta \quad y=r\sin\theta \quad$;其中$r\leqslant 2\cos\theta \qquad -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$

$\Rightarrow \iint_{S'} y\,\mathrm{dx\,dy}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\sin\theta\,\mathrm{dr}\,d\theta=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta\sin\theta\,d\theta=0$

$\Rightarrow$原式$=-2\cdot 0=0$

(5)

$$\vec{n}=\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$$

$$\Rightarrow \int_L (y^2-y)\,\mathrm{dx}+(z^2-z)\,\mathrm{dy}+(x^2-x)\,\mathrm{dz}=\oiint_S \begin{vmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2-y & z^2-z & x^2-x \end{vmatrix} dS$$

$$=\oiint_S [3-2(x+y+z)]\cdot\frac{\sqrt{3}}{3}dS=\sqrt{3}\iint_S dS=\sqrt{3}\pi a^2$$

(6)

$$\int_L (y^2-z^2)\,\mathrm{dx}+(2z^2-x^2)\,\mathrm{dy}+(3x^2-y^2)\,\mathrm{dz}$$

$$=\oiint_S (-2y-4z)\,\mathrm{dy}\,\mathrm{dz}-(2z+6x)\,\mathrm{dz}\,\mathrm{dx}-(2x+2y)\,\mathrm{dx}\,\mathrm{dy}$$

$$=-2\oiint_S (y+2z)\,\mathrm{dy}\,\mathrm{dz}+(z+3x)\,\mathrm{dz}\,\mathrm{dx}+(x+y)\,\mathrm{dx}\,\mathrm{dy}$$

$$\oiint_S (x+y)\,\mathrm{dx}\,\mathrm{dy}=\int_0^1\int_{x-1}^{1-x}x+y\,\mathrm{dy}\,\mathrm{dx}+\int_{-1}^0\int_{-x-1}^{1+x}x+y\,\mathrm{dy}\,\mathrm{dx}=0$$

当$x+y=1$时，$z=1$ ；当$x+y=-1$时，$z=3$

当$x-y=1$时，$2x+z=3$ ；当$x-y=-1$时，$2x+z=1$

$$\Rightarrow\oiint_S (z+3x)\,\mathrm{dz}\,\mathrm{dx}=\int_0^1\int_1^{3-2x}z+3x\,\mathrm{dz}\,\mathrm{dx}+\int_{-1}^0\int_{1-2x}^3 z+3x\,\mathrm{dz}\,\mathrm{dx}=4$$

$$\oiint_S (y+2z)\,\mathrm{dy}\,\mathrm{dz}=\int_0^1\int_1^{3-2y}y+2z\,\mathrm{dz}\,\mathrm{dy}+\int_{-1}^0\int_{1-2y}^3 y+2z\,\mathrm{dz}\,\mathrm{dy}=8$$

$\Rightarrow$原式$=-2\cdot(4+8+0)=-24$

9.

(1)

$$\int_L x^2y^3\mathrm{dx}+\mathrm{dy}+z\,\mathrm{dz}=\oiint_S -3x^2y^2\,\mathrm{dx}\,\mathrm{dy}=\iint_S -3x^2y^2\,\mathrm{dx}\,\mathrm{dy}$$

设$x=r\cos\theta$ ，$y=r\sin\theta$ ；$0\leqslant r\leqslant R$ ，$0\leqslant\theta\leq\pi$

$$\frac{\partial(x,y)}{\partial(r,\theta)}=r;$$

$\Rightarrow$原式$=-3\int_0^{2\pi}\int_0^R r^5\cos^2\theta\sin^2\theta\,\mathrm{dr}\,d\theta=-\frac{\pi}{8}R^6$

(2)

$$\int_L x^2y^3\mathrm{dx}+\mathrm{dy}+z\,\mathrm{dz}=\oiint_S -3x^2y^2\,\mathrm{dx}\,\mathrm{dy}=\iint_{S'} -3x^2y^2\,\mathrm{dx}\,\mathrm{dy}$$

S': $x^2+y^2\leq R^2$

$\Rightarrow$两者结果相同

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$$\Gamma=\oint_L\vec{c}\,dl=\oiint_S(\nabla\times\vec{c})\,\mathrm{dS}=\oiint_S 0\,dS=0$$

11.

$$M=\oiint_S\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+z^2 & z^2+x^2 & x^2+y^2\end{vmatrix}\mathrm{dS}=2\iint_S(y-z)\cos\alpha+(z-x)\cos\beta+(x-y)\cos\gamma\,\mathrm{dS}$$

$\because$圆

$$\Rightarrow\cos\alpha=\frac{x}{r}\quad;\cos\beta=\frac{y}{r}\quad;\cos\gamma=\frac{z}{r}$$

$$\Rightarrow M=2\iint_S(y-z)\cdot\frac{x}{r}+(z-x)\cdot\frac{y}{r}+(x-y)\cdot\frac{z}{r}\,\mathrm{dS}=0$$

（完结）

撒花！！！