$\S 10.1$

1\. (a) $\int_{-\pi}^{\pi} 1\cdot \sin nx\,dx \overset{奇}{=} 0$ $\qquad \int_{-\pi}^{\pi} 1\cdot\cos nx\,dx = \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} = 0$

(b) $\int_{-\pi}^{\pi} \sin nx\sin mx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}[\cos(m-n)x - \cos(n+m)x]\,dx = 0 \quad (m\neq n)$

$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}[\cos(m-n)x + \cos(m+n)x]\,dx = 0 \quad (m\neq n)$

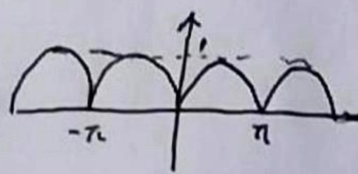
(c) $\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}[\sin(n+m)x + \sin(n-m)x]\,dx = 0$

2\. $\int_{-\pi}^{\pi}\sin^2 nx\,dx = \int_{-\pi}^{\pi}\frac{1-\cos 2nx}{2}\,dx = \pi$

$\int_{-\pi}^{\pi}\cos^2 nx\,dx = \int_{-\pi}^{\pi}\frac{1+\cos 2nx}{2}\,dx = \pi$

3\. (1) $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \qquad b_n = 0$

$= \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\cos mx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos mx\,dx$



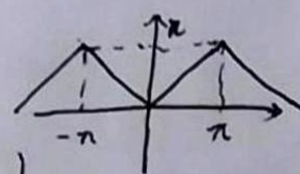
$= \frac{1}{\pi}\int_0^{\pi}[\sin(1-m)x + \sin(1+m)x]\,dx$

$= \frac{2[1-(-1)^{m+1}]}{\pi(1-m^2)}$

$|\sin x| \sim \frac{2}{\pi} + \sum_{m=1}^{+\infty}\frac{2[1-(-1)^{m+1}]}{\pi(1-m^2)}\cos mx$ 在 $\mathbb{R}$ 上绝对一致收敛

(2) $f(x)$ 偶. $\quad b_n = 0$

$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}x\frac{\sin nx}{n}\Big|_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx$



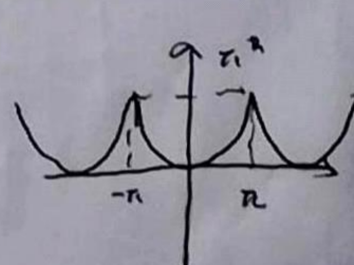
$= \frac{2}{n\pi}\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{2[(-1)^n-1]}{n^2\pi} \quad (n\geq 1)$

$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = 2$

$|x| \sim 1 + \sum_{n=0}^{+\infty}\frac{2[(-1)^n-1]}{n^2\pi}\cos nx$ 在 $\mathbb{R}$ 上绝对一致收敛

(3) $f(x)$ 偶 $\quad b_n = 0.$

$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{2\sin nx}{\pi n}x^2\Big|_0^{\pi} - \int_0^{\pi}\frac{2\sin nx}{n\pi}\cdot 2x\,dx$



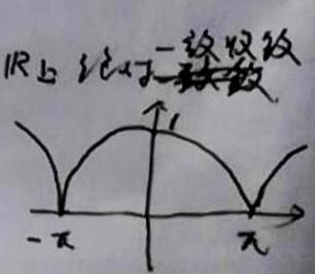
$= \frac{2\cos nx}{n^2\pi}\cdot 2x\Big|_0^{\pi} - \int_0^{\pi}\frac{2\cos nx}{n^2\pi}\cdot 2\,dx$

$= \frac{4\cdot(-1)^n}{n^2} \quad (n\geq 1)$

$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}$ $\qquad x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{+\infty}\frac{4(-1)^n}{n^2}\cos nx$ 在 $\mathbb{R}$ 上 绝对一致收敛

(4) $f(x)$ 偶. $\quad b_n = 0$

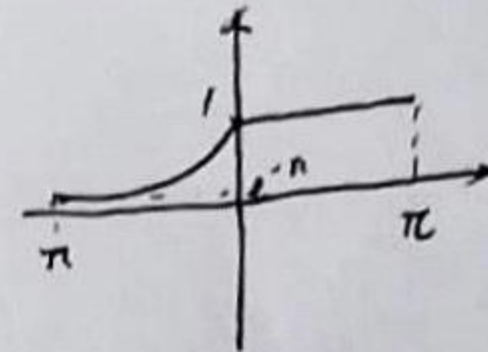
$a_n = \frac{2}{\pi}\int_0^{\pi}\cos\frac{x}{2}\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(\cos(\frac{1}{2}-n)x + \cos(\frac{1}{2}+n)x)\,dx$



$= \frac{(-1)^{n+1}}{\pi(n^2-\frac{1}{4})} \quad n\geq 0.$

$$\cos\frac{x}{2} \sim \frac{2}{\pi} + \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{\pi(n^2-\frac{1}{4})} \cos nx \quad \alpha$$

在R上 绝对一致收敛



(5)

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} \sin nx\,dx + \int_0^{\pi} e^x \sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n-1}{n} + \frac{ne^{\pi}[1-(-1)^n]}{n^2+1}\right)$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} \cos nx\,dx + \int_0^{\pi} e^x \cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(0 + \frac{ae^{\pi}[(-1)^n-1]}{n^2+1}\right) = \frac{e^{\pi}[(-1)^n-1]}{\pi(n^2+1)} \quad (n \geq 1)$$

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0} dx + \int_0^{\pi} e^x dx\right) = \frac{1}{\pi}(\pi + e^{\pi} - 1)$$

$$\Rightarrow f(x) \sim \frac{(\pi+e^{\pi}-1)}{2\pi} + \sum_{n=1}^{+\infty}\left[\frac{e^{\pi}[(-1)^n-1]}{(n^2+1)\pi}\cos nx + \left(\frac{(-1)^n-1}{n\pi} + \frac{ne^{\pi}[1-(-1)^n]}{\pi(n^2+1)}\right)\sin nx\right]$$

$$\longrightarrow \begin{cases} f(x) & x \neq (2n+1)\pi \,. \\ \dfrac{1+e^{-\pi}}{2} & x = (2n+1)\pi \quad n \in \mathbb{Z} \end{cases}$$

~~$x = n\pi,\ n = \pm 1, \pm 2$~~

4. $x=(2k-1)\pi$，$k\in\mathbb{Z}$ 时 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx=0$

$x\neq(2k-1)\pi$，$k\in\mathbb{Z}$ 时，$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}=\sum_{n=1}^{\infty}\frac{2\cos(n+1)\pi\sin nx}{2n}=-\sum_{n=1}^{\infty}\frac{\sin[(x-\pi)n]+\sin[(x+\pi)n]}{2n}$

$$\left|\sum_{k=1}^{n}\sin k(x-\pi)\right|=\frac{\left|\cos\frac{x-\pi}{2}-\cos[(n+\frac{1}{2})(x-\pi)]\right|}{\left|2\sin\frac{x-\pi}{2}\right|}\leq\frac{1}{\left|\sin\frac{x-\pi}{2}\right|}$$

$$\left|\sum_{k=1}^{n}\sin k(x+\pi)\right|=\frac{\left|\cos\frac{x+\pi}{2}-\cos[(n+\frac{1}{2})(x+\pi)]\right|}{\left|2\sin\frac{x+\pi}{2}\right|}\leq\frac{1}{\left|\sin\frac{x+\pi}{2}\right|}$$

$\frac{1}{n}\downarrow 0$ 故由 Dirichlet 判别法：$\sum_{n=1}^{\infty}\frac{\sin[(x-\pi)n]}{2n}$ $\sum_{n=1}^{\infty}\frac{\sin[(x+\pi)n]}{2n}$ 收敛

故 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$ 收敛。

5. 令 $g(t)=f(\frac{T}{2\pi}t)$ （即 $x=\frac{T}{2\pi}t$）

$$g(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)\qquad\begin{cases}a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\cos nt\,dt\\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\sin nt\,dt\end{cases}$$

$t=\frac{2\pi}{T}x$ 代入

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi}{T}x+b_n\sin\frac{2n\pi}{T}x)$$

$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos\frac{2n\pi}{T}x\,dx\qquad b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\sin\frac{2n\pi}{T}x\,dx$$

6. $T=b-a$. $\quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi}{b-a}x+b_n\sin\frac{2n\pi}{b-a}x)$

$$a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi}{b-a}x\,dx\qquad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}x\,dx$$

7. (1) 偶开拓：$b_n=0$. $\quad a_n=\frac{2}{\pi}\int_0^{\pi}(1-\sin\frac{x}{2})\cos nx\,dx=\frac{1}{(\frac{1}{4}-n^2)\pi}$ $(n\geq1)$

$$a_0=\frac{2}{\pi}\int_0^{\pi}(1-\sin\frac{x}{2})dx=\frac{2(\pi-2)}{\pi}$$

$$\frac{\pi-2}{\pi}+\sum_{n=1}^{\infty}\frac{\cos nx}{(\frac{1}{4}-n^2)\pi}\to f(x).\quad\forall x\in[0,\pi]$$

(2) 奇开拓 $a_n=0$.

$$\cancel{b_n=\frac{2}{T}\int_0^T\frac{x}{3}\sin nx\,dx=\frac{2}{T}\left(-\frac{\cos nx}{n}\cdot\frac{x}{3}\right)\Big|_0^T}$$

$$b_n=\frac{2}{T}\int_0^T\frac{x}{3}\sin(\frac{n\pi}{T}x)dx=\frac{-2x}{3n\pi}\cos\frac{n\pi}{T}x\Big|_0^T+\frac{2}{3n\pi}\int_0^T(\cos\frac{n\pi}{T}x)\,dx$$

$$=\frac{2T}{3n\pi}\cdot(-1)^{n+1}$$

$$\sum_{n=1}^{\infty}\frac{2T}{3n\pi}(-1)^{n+1}\sin nx\to f(x).\quad\forall x\in[0,T]$$

(3) $a_n=\frac{1}{l}\int_{-l}^{l}e^{ax}\cos\frac{n\pi}{l}x\,dx=\frac{(-1)^n(e^{al}-e^{-al})\cdot a}{a^2+\frac{n^2\pi^2}{l^2}}$ $(n\geq1)$

$$b_n=\frac{1}{l}\int_{-l}^{l}e^{ax}\sin\frac{n\pi}{l}x\,dx=-\frac{(-1)^n(e^{al}-e^{-al})\cdot\frac{n\pi}{l}}{a^2+\frac{n^2\pi^2}{l^2}}$$

$$a_0=\frac{1}{l}\int_{-l}^{l}e^{ax}dx=\frac{e^{al}-e^{-al}}{al}$$

$$\frac{e^{al}-e^{-al}}{2al}+\sum_{n=1}^{+\infty}\left[\frac{(-1)^n(e^{al}-e^{-al})a}{a^2+\frac{n^2\pi^2}{l^2}}\cos\frac{n\pi}{l}x+\frac{(-1)^{n+1}(e^{al}-e^{-al})\frac{n\pi}{l}}{a^2+\frac{n^2\pi^2}{l^2}}\sin\frac{n\pi}{l}x\right]$$

$$\longrightarrow\begin{cases} f(x). & x\in(-l,l)\\ \frac{e^{al}+e^{-al}}{2} & x\in\{-l,l\}\end{cases}$$

(4) 偶开拓： $b_n=0$

$$a_0=\frac{2}{2}\left(\int_0^1 dx+\int_1^2 -dx\right)=0$$

$$a_n=\int_0^1\cos\left(\frac{n\pi}{2}x\right)dx-\int_1^2\cos\left(\frac{n\pi}{2}x\right)dx=\frac{4\sin\frac{n\pi}{2}}{n\pi}$$

$$\sum_{n=1}^{+\infty}\frac{4\sin\frac{n\pi}{2}}{n\pi}\longrightarrow\begin{cases} f(x) & x\in[0,1)\cup(1,2]\\ 0 & x=1\end{cases}$$

(5) 奇开拓 ： $a_n=0$

$$b_n=\frac{2}{3}\left(\int_0^1 x\sin\frac{n\pi}{3}x\,dx+\int_1^2\sin\frac{n\pi}{3}x\,dx+\int_2^3(3-x)\sin\frac{n\pi}{3}x\,dx\right)$$

$$=\frac{6(\cos\frac{n\pi}{3}-\cos\frac{n\pi}{3})}{n\pi}+\frac{6\sin\frac{n\pi}{3}+6\sin\frac{2n\pi}{3}}{n^2\pi^2}$$

$$\sum_{n=1}^{+\infty}b_n\sin\frac{n\pi}{3}x\longrightarrow f(x)\qquad\forall x\in[0,3]$$

8\. (1) 正弦： $a_n=0.$ $b_n=\frac{2}{\pi}\int_0^\pi x\sin x\sin nx\,dx=\frac{1}{\pi}\int_0^\pi x[\cos(n-1)x-\cos(n+1)x]dx$

$$=\frac{1}{\pi}\left[\frac{(-1)^{n-1}-1}{(n-1)^2}+\frac{(-1)^n\pi}{n+1}\right]$$

$$f(x)\sim\sum_{n=1}^{+\infty}\frac{1}{\pi}\left[\frac{(-1)^{n-1}-1}{(n-1)^2}+\frac{(-1)^n\pi}{n+1}\right]\sin nx$$

余弦. $b_n=0.$ $a_n=\frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx=\frac{1}{\pi}\int_0^\pi x[\sin(1-n)x+\sin(1+n)x]dx$

$$=\frac{(-1)^n\cdot 2}{1-n^2}$$

$$f(x)\sim 1+\sum_{n=1}^{+\infty}\frac{(-1)^n\cdot 2}{1-n^2}\cos nx$$

(2) 正弦： $a_n=0.$ $b_n=\frac{2}{\pi}\int_0^\pi 2x^2\sin nx\,dx$

$$=\frac{4}{\pi}\left[\frac{(-1)^{n+1}\cdot\pi^2}{n}+\frac{2\cdot(-1)^n-2}{n^3}\right]$$

$$f(x)\sim\sum_{n=1}^{+\infty}\frac{4}{\pi}\left[\frac{(-1)^{n+1}\pi^2}{n}+\frac{2\cdot(-1)^n-2}{n^3}\right]\sin nx$$

余弦 . $b_n=0.$ $a_0=\frac{2}{\pi}\int_0^\pi 2x^2dx=\frac{4\pi^2}{3}$

$$a_n=\frac{2}{\pi}\int_0^\pi 2x^2\cos nx\,dx=\frac{8\cdot(-1)^n}{n^2}$$

$$f(x) \sim \frac{2\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{8(-1)^n}{n^2} \cos nx$$

(3) 正弦：$a_n = 0$.

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx = \frac{2}{l}\int_0^{\frac{l}{2}} A \sin\frac{n\pi x}{l} dx = \frac{2A}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(x) \sim \sum_{n=1}^{+\infty} \frac{2A}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \sin\frac{n\pi}{l}x$$

余弦：$b_n = 0$.

$$a_0 = \frac{2}{l}\int_0^l f(x) dx = \frac{2}{l}\cdot\frac{l}{2}\cdot A = A$$

$$a_n = \frac{2}{l}\int_0^{\frac{l}{2}} A\cos\frac{n\pi x}{l} dx = \frac{2A}{n\pi}\sin\frac{n\pi}{2}$$

$$f(x) \sim \frac{A}{2} + \sum_{n=1}^{+\infty}\left(\frac{2A}{n\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi}{l}x$$

(4) 正弦：$a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^{2h}\left(1-\frac{x}{2h}\right)\sin nx\, dx = \frac{2}{\pi}\left[\frac{1-\cos 2nh}{n} + \frac{\cos 2nh}{n} - \frac{\sin 2nh}{n^2}\cdot\frac{1}{2h}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n} - \frac{\sin(2nh)}{2hn^2}\right]$$

$$f(x) \sim \sum_{n=1}^{+\infty}\frac{2}{\pi}\left(\frac{1}{n} - \frac{\sin(2nh)}{2hn^2}\right)\sin nx$$

余弦：$b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{2h}\left(1-\frac{x}{2h}\right)dx = \frac{2h}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{2h}\left(1-\frac{x}{2h}\right)\cos nx\, dx = \frac{2}{\pi}\left[\frac{\sin 2nh}{n} - \frac{\sin 2nh}{n} - \frac{\cos 2nh - 1}{2n^2h}\right]$$

$$= \frac{-\cos 2nh + 1}{n^2h\pi}$$

$$f(x) \sim \frac{h}{\pi} + \sum_{n=1}^{+\infty}\frac{1-\cos 2nh}{n^2h\pi}\cos nx$$

9. (1) $a_0 = \int_{-\pi}^{\pi} f(x)dx = \int_0^{\pi} f(x)dx + \int_{-\pi}^{0} f(t-\pi)\,d(t-\pi) = \int_0^{\pi} f(x)dx + \int_0^{\pi} -f(t)dt = 0$

$a_{2k} = \int_{-\pi}^{\pi} f(x)\cos 2kx\,dx = \int_0^{\pi} f(x)\cos 2kx\,dx + \int_{\pi}^{0} f(t-\pi)\cos 2k(t-\pi)\,d(t-\pi)$

$= \int_0^{\pi} |f(x)\cos 2kx - f(x)\cos 2kx|\,dx = 0$

$b_{2k} = \int_{-\pi}^{\pi} f(x)\sin 2kx\,dx = \int_0^{\pi} f(x)\sin 2kx\,dx + \int_{-\pi}^{0} f(x)\sin 2kx\,dx$

$= \int_0^{\pi} f(x)\sin 2kx - f(x)\sin 2k(x-\pi)\,dx = 0$

(2)

$a_{2k-1} = \int_{-\pi}^{\pi} f(x)\cos(2k-1)x\,dx = \int_0^{\pi} f(x)\cos(2k-1)x + f(x-\pi)\cos[(2k-1)(x-\pi)]\,dx$

$= 0$

同理 $b_{2n-1} = 0$

10. $\bar{a}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+h)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi+h}^{\pi+h} f(x)\cos n(x-h)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos nx\cos nh + \sin nx\sin nh)\,dx$

$= \cos nh\, a_n + b_n \sin nh$

类似可证 $\bar{b}_n = b_n\cos nh - a_n \sin nh$

11. $f(x)$ 为偶函数 故 $b_n = 0$.

$a_0 = \frac{2}{\pi}\int_0^{\pi} (1-x^2)dx = 2 - \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} (1-x^2)\cos nx\,dx = \frac{4(-1)^{n+1}}{n^2}$

$f(x) \sim 1 - \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4(-1)^{n-1}}{n^2}\cos nx$

(1) $1 = 1 - \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4(-1)^{n-1}}{n^2} \Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}$

(2) $\frac{(2-\frac{2\pi^2}{3})^2}{2} + \sum_{n=1}^{+\infty} \frac{16}{n^4} = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = 2(1 - \frac{2\pi^2}{3} + \frac{\pi^4}{5})$

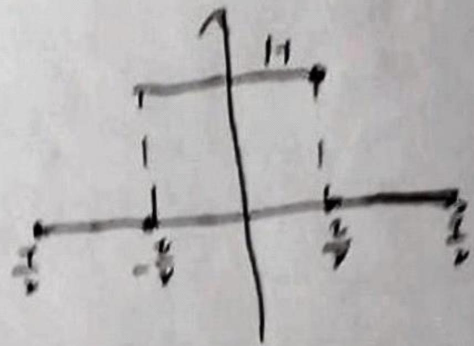
$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{\frac{2\pi^4}{5} - \frac{2\pi^4}{9}}{16} = \frac{\pi^4}{90}$

12. $f(x)$奇. 故 $a_n = 0$. $b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{2[1-(-1)^n]}{n\pi}$

$f(x) \sim \sum_{n=1}^{+\infty} \frac{2[1-(-1)^n]}{n\pi}\sin nx = \sum_{n=1}^{+\infty} \frac{2\sin(2n-1)x}{(2n-1)\pi}$

$\Rightarrow 1 = \sum_{n=1}^{+\infty} \frac{2(-1)^{n-1}}{(2n-1)\pi} \Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{2}$

3. $b_n = 0$.



$$a_0 = \frac{2}{\pi}\int_0^{\frac{\tau}{2}} H\,dx = \frac{\tau H}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\frac{\tau}{2}} H\cos\frac{2n\pi}{T}x\,dx$$

$$= \frac{HT}{n\pi^2}\sin\frac{n\pi\tau}{T}$$

$$f(x) = \frac{\tau H}{2\pi} + \sum_{n=1}^{\infty}\frac{HT}{n\pi^2}\sin\frac{n\pi\tau}{T}\cos\frac{2n\pi}{T}x$$

14、15、16. 注意到 $f_n \subset L[-\pi,\pi]$ 且 F_n 关于加法、数乘封闭.

~~故 $f_n \subset L[-\pi,\pi]$，且 F_n 为子空间~~.

故证完 19、20、21 之后，自然地 14、15、16 成立.

17. (1) 偶延拓：$b_n = 0$， $a_0 = \frac{2}{\pi}\int_0^a dx = \frac{2a}{\pi}$

$$a_n = \frac{2}{\pi}\int_0^a \cos nx\, dx = \frac{2\sin na}{n\pi}$$

$$\Rightarrow \frac{a}{\pi} + \sum_{n=1}^{+\infty} \frac{2\sin na}{n\pi}\cos nx \longrightarrow \begin{cases} f(x) & x\in(-\pi,\pi)\setminus\{\pm a\} \\ \frac{1}{2} & x\in\{\pm a\}\end{cases}$$

$$\frac{\left(\frac{2a}{\pi}\right)^2}{2} + \sum_{n=1}^{+\infty}\left(\frac{2\sin na}{n\pi}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{2a}{\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2} = \frac{(a-a^2)\pi}{2}$$

(2) $$\sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{\cos^2 na}{n^2} = \frac{\pi^2}{6} - \frac{(a-a^2)\pi}{2}$$

18. $$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\pi}\int_{-\pi}^{x} f(x+t)f(t)\,dt\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos nx\,dx\right)dt$$

$$\overset{10.}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[a_n\cos nt + b_n\sin nt\right]dt.$$

$$= a_n^2 + b_n^2 \quad (n\geq 1) \qquad A_0 = a_0^2$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[b_n\cos nt - a_n\sin nt\right]dt$$

$$= b_n a_n - a_n b_n = 0.$$

$F(x) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} A_n\cos nx$. 故 $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(t)\,dt = F(0) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} A_n = \frac{a_0^2}{2} + \sum_{n=1}^{+\infty} a_n^2 + b_n^2$

19. ⓪ $f, g \in L^2[-\pi,\pi]$ $\alpha, \beta \in \mathbb{R}$.

$$\int_{-\pi}^{\pi}(\alpha f+\beta g)^2 dx = \alpha^2\int_{-\pi}^{\pi}f^2dx+\beta^2\int_{-\pi}^{\pi}g^2dx+2\alpha\beta\int_{-\pi}^{\pi}fg\,dx$$

C-S不等式

$$\le \alpha^2\int_{-\pi}^{\pi}f^2dx+\beta^2\int_{-\pi}^{\pi}g^2dx+2\alpha\beta\left(\int_{-\pi}^{\pi}f^2dx\right)^{\frac12}\left(\int_{-\pi}^{\pi}g^2dx\right)^{\frac12} < +\infty$$

$\Rightarrow$ $\alpha f+\beta g \in L^2[-\pi,\pi]$

① $\forall f, g \in L^2[-\pi,\pi]$. $f(x)+g(x) = g(x)+f(x)$ $\forall x \in [-\pi,\pi]$

$\Rightarrow$ $f+g = g+f$

② $\forall f, g, h \in L^2[-\pi,\pi]$. $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$.

$\Rightarrow$ $(f+g)+h = f+(g+h)$

③ $f\equiv 0$ (a.e.) 则 $f+g=g$ $\forall g \in L^2[-\pi,\pi]$

④ $\forall f$. $\exists -f$ s.t. $f+(-f)\equiv 0$.

⑤ $\forall \alpha \in \mathbb{R}$. $f, g \in L^2[-\pi,\pi]$ $\alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x)$

$\Rightarrow$ $\alpha(f+g) = \alpha f+\alpha g$

⑥ $\forall \alpha, \beta \in \mathbb{R}$. $f \in L^2[-\pi,\pi]$. $(\alpha+\beta)f(x) = \alpha f(x)+\beta f(x)$

$\Rightarrow$ $(\alpha+\beta)f = \alpha f+\beta f$

⑦ $\forall \alpha, \beta \in \mathbb{R}$. $f \in L^2[-\pi,\pi]$. $(\alpha\beta)f(x) = \alpha(\beta f(x))$

$\Rightarrow$ $(\alpha\beta)f = \alpha(\beta f)$

⑧ $\forall f \in L^2[-\pi,\pi]$. $1\cdot f(x) = f(x)$.

$\Rightarrow$ $1\cdot f = f$.

20. 正定性：~~$\langle f(x), f(x)\rangle = \int_{-\pi}^{\pi} f(x)g(x)$~~

$\langle f(x), f(x)\rangle = \int_{-\pi}^{\pi} f^2(x)dx \ge 0$. 取等号 $\Leftrightarrow$ $f(x)=0$ a.e.

对称性：$\langle f(x), g(x)\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx = \int_{-\pi}^{\pi} g(x)f(x)dx = \langle g(x), f(x)\rangle$

线性性：$\langle f(x), \alpha g(x)+\beta h(x)\rangle = \int_{-\pi}^{\pi} f(x)(\alpha g(x)+\beta h(x))dx$

$= \alpha\int_{-\pi}^{\pi} f(x)g(x)dx + \beta\int_{-\pi}^{\pi} f(x)h(x)dx$

$= \alpha\langle f(x), g(x)\rangle + \beta\langle f(x), h(x)\rangle$

21. $\|f\| = \langle f, f\rangle^{\frac12}$ 正定性：$\|f\|\ge 0$. $\|f\|=0 \Leftrightarrow \langle f,f\rangle = 0 \Leftrightarrow f=0$ a.e.

三角不等式. $\|f+g\|^2 = \langle f+g, f+g\rangle = \|f\|^2+\|g\|^2+2\langle f, g\rangle \overset{\text{C-S不等式}}{\le} (\|f\|+\|g\|)^2$

齐性：$\|cf\| = \langle cf, cf\rangle^{\frac12} = |c|\langle f, f\rangle^{\frac12} = |c|\,\|f\|$