7.7书后习题答案

1.

(1)

$$\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)\Bigg|_{(1,1,-2)}=(yz,xz,xy)\Bigg|_{(1,1,-2)}=(-2,-2,1)$$

$$\hat{\vec{l}}=\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)$$

$$\Rightarrow\frac{\partial u}{\partial\hat{\vec{l}}}=(-2,-2,1)\cdot\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)=0$$

(2)

$$\operatorname{grad}u=\nabla u=(3,2z,2z+2y)$$

$$\operatorname{grad}u\,|_{(1,-2,2)}=(3,4,0)$$

$\Rightarrow$方向导数最大值5　最小值为$-5$

2.

(1)

$$\operatorname{div}\vec{v}=\nabla\cdot\vec{v}=6x+3y^2+z^2+xy-6xz$$

$$\nabla\cdot\vec{v}\,|_M=6+16-14=8$$

(2)

$$\operatorname{div}\vec{v}\,|_M=\nabla\cdot\vec{v}\,|_M=2x\,\mathrm{siny}+2y\,\mathrm{sinxz}-\sin z\cdot xy\cdot\cos(\mathrm{cosz})$$

3.

(1)

$$\operatorname{div}[(\vec{r}\cdot\vec{w})\vec{w}]=\nabla\cdot[(\vec{r}\cdot\vec{w})\vec{w}]=(\vec{r}\cdot\vec{w})(\nabla\cdot\vec{w})+\vec{w}\cdot\nabla(\vec{r}\cdot\vec{w})=\vec{w}\cdot\nabla(\vec{r}\cdot\vec{w})=\vec{w}\cdot\vec{w}$$

(2)

$$\operatorname{div}[(\vec{r}\cdot\vec{w})\vec{r}]=\nabla\cdot[(\vec{r}\cdot\vec{w})\vec{r}]=(\vec{r}\cdot\vec{w})(\nabla\cdot\vec{r})+\vec{r}\cdot\nabla(\vec{r}\cdot\vec{w})=3(\vec{r}\cdot\vec{w})+\vec{r}\cdot\vec{w}=4\vec{r}\cdot\vec{w}$$

(3)

$$\operatorname{div}\left(\frac{\vec{r}}{r}\right)=\frac{1}{r}\nabla\cdot\vec{r}+\left(\nabla\frac{1}{r}\right)\cdot\vec{r}=\frac{3}{r}-\frac{\hat{\vec{r}}}{r^2}\cdot\vec{r}=\frac{2}{r}$$

(4)

$$\operatorname{div}(\vec{w}\times\vec{r})=\nabla\cdot(\vec{w}\times\vec{r})=\vec{r}\cdot(\nabla\times\vec{w})-\vec{w}\cdot(\nabla\times\vec{r})=0-0=0$$

(5)

$$\text{div}\,(r^2\vec{w}) = \nabla(r^2\vec{w}) = \nabla r^2\cdot\vec{w} + r^2(\nabla\cdot\vec{w}) = 2\vec{r}\cdot\vec{w}$$

(6)

$$\text{div}\,[f(r)\vec{r}] = \nabla f(r)\cdot\vec{r} + f(r)\,(\nabla\cdot\vec{r}) = rf'(r) + 3f(r)$$

4.

(1)

$$\nabla\cdot\nabla\varphi = \nabla^2\varphi = 6xy^4z^2 + 12x^3y^2z^2 + 2x^3y^4$$

(2)

$$\nabla u = ((x+1)yze^{(x+y+z)},\, (y+1)xze^{(x+y+z)},\, (z+1)xye^{(x+y+z)})$$

$$\nabla\cdot\nabla u = (x+2)yze^{(x+y+z)} + (y+2)xze^{(x+y+z)} + (z+2)yxe^{(x+y+z)}$$

5.

(1)

$$\text{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\vec{i} - 2x\vec{j} - 2y\vec{k}$$

(2)

$$\text{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^y + y & e^y + z & y + 2ze^y \end{vmatrix} = 2ze^y\vec{i} - (1 + xe^y)\vec{k}$$

6

(1)

$$\begin{aligned}\text{rot}\,(\vec{w}\times\vec{r}) = \nabla\times(\vec{w}\times\vec{r}) &= \vec{w}\,(\nabla\cdot\vec{r}) + (\vec{r}\cdot\nabla)\,\vec{w} - (\nabla\cdot\vec{w})\,\vec{r} - \vec{w}\cdot\nabla\vec{r} \\ &= 3\vec{w} + 0 - 0 - \vec{w} = 2\vec{w}\end{aligned}$$

(2)

$$\text{rot}[f(r)\vec{r}] = \nabla f(r)\times\vec{r} + f(r)\,(\nabla\times\vec{r}) = 0 + 0 = 0$$

(3)

$$\text{rot}[f(r)\vec{w}] = \nabla f(r)\times\vec{w} + f(r)\,(\nabla\times\vec{w}) = f'(r)\cdot\hat{\vec{r}}\times\vec{w} = \frac{1}{r}f'(r)\,(\vec{r}\times\vec{w})$$

(4)

用张量做

设用$\dfrac{\partial}{\partial x_p}$代表$\nabla$ ，$w_j$代表$\vec{w}$ ，$r_i$代表$\vec{r}$

$\Rightarrow \nabla\times[\vec{r}\times f(r)\vec{w}]$可以用$\varepsilon_{pkq}\dfrac{\partial}{\partial x_p}(\varepsilon_{kij}\,r_i f(r)\,w_j)$表示

$$\Rightarrow \varepsilon_{pkq}\frac{\partial}{\partial x_p}(\varepsilon_{kij}\,r_i f(r)\,w_j)=(\delta_{qi}\delta_{pj}-\delta_{qj}\delta_{pi})\frac{\partial}{\partial x_p}(r_i\,f(r)\,w_j)$$

$$=\frac{\partial}{\partial x_j}(r_q\,f(r)\,w_j)-\frac{\partial}{\partial x_p}(r_p\,f(r)\,w_q)$$

$$=f(r)\frac{\partial}{\partial x_j}(r_q\,w_j)+r_q w_j\frac{\partial}{\partial x_j}(f(r))-f(r)\frac{\partial}{\partial x_p}(r_p\,w_q)-r_p\,w_q\frac{\partial}{\partial x_p}(f(r))$$

$$=f(r)\left[r_q\frac{\partial w_j}{\partial x_j}+w_j\frac{\partial r_q}{x_j}\right]+r_q w_j f'(r)\frac{r_j}{r}-f(r)\left[r_p\frac{\partial w_q}{\partial x_p}+w_q\frac{\partial r_p}{\partial x_p}\right]-r_p\,w_q f'(r)\frac{r_p}{r}$$

$$\Rightarrow \nabla\times[\vec{r}\times f(r)\vec{w}]$$

$$=f(r)[\vec{r}(\nabla\cdot\vec{w})+(\vec{w}\cdot\nabla)\vec{r}]+f'(r)(\vec{w}\cdot\vec{r})\frac{\vec{r}}{r}-f(r)[(\vec{r}\cdot\nabla)\vec{w}+\vec{w}(\nabla\cdot\vec{r})]-\vec{w}f'(r)r$$

$$=f(r)[\vec{0}+\vec{w}]+f'(r)(\vec{w}\cdot\vec{r})\frac{\vec{r}}{r}-f(r)[\vec{0}+3\vec{w}]-\vec{w}f'(r)r$$

$$=-2f(r)\vec{w}+f'(r)(\vec{w}\cdot\vec{r})\frac{\vec{r}}{r}-\vec{w}f'(r)r$$

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(1) $\vec{F}=-y\vec{i}+x\vec{j}$

$l_1$: $w=\int_{l_1}\vec{F}\cdot d\vec{l}=\int_{(0,0)}^{(1,0)}\vec{F}\cdot d\vec{l}+\int_{(1,0)}^{(1,1)}\vec{F}\cdot d\vec{l}=0+\int_0^1(y\vec{i}+\vec{j})\vec{j}\,dy=1$

$l_2$: $w=\int_{l_2}\vec{F}\cdot d\vec{l}=\int_{l_2}(-y\vec{i}+x\vec{j})(dx\,\vec{i}+2x\,dx\,\vec{j})=\int_0^1 x^2\,\mathrm{dx}=\dfrac{1}{3}$

$l_3$: $w=\int_{l_3}\vec{F}\cdot d\vec{l}=\int_0^1(-x\vec{i}+x\vec{j})(dx\,\vec{i}+dx\vec{j})=0$

$l_4$: $w=\int_{l_4}\vec{F}\cdot d\vec{l}=\int_{(0,0)}^{(0,1)}\vec{F}\cdot d\vec{l}+\int_{(0,1)}^{(1,1)}\vec{F}\cdot d\vec{l}$

$$=\int_{(0,0)}^{(1,0)}(-y\vec{i}+x\vec{j})\cdot\vec{j}\,\mathrm{dy}+\int_{(1,0)}^{(1,1)}(-y\vec{i}+x\vec{j})\cdot\vec{i}\,\mathrm{dx}=0+\int_0^1-1\,dx=-1$$

沿不同路径积分积分值不等

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ -y&x&0\end{vmatrix}=2\vec{k}\neq 0$$

$\Rightarrow$场有旋，无源，非有势场$\Rightarrow$沿不同路径积分积分值不等

(2) $\vec{F}=2xy\,\vec{i}+x^2\,\vec{j}$

$l_1$: $w=\int_{l_1}\vec{F}\cdot d\vec{l}=\int_{(0,0)}^{(1,0)}\vec{F}\cdot d\vec{l}+\int_{(1,0)}^{(1,1)}\vec{F}\cdot d\vec{l}=0+\int_0^1(y\,\vec{i}+\vec{j})\,\vec{j}\,dy=1$

$l_2$: $w=\int_{l_2}\vec{F}\cdot d\vec{l}=\int_0^1(2x^3\,\vec{i}+x^2\,\vec{j})\,(dx\,\vec{i}+2x\,dx\,\vec{j})=\int_0^1 4x^3\,\mathrm{dx}=1$

$l_3$: $w=\int_{l_3}\vec{F}\cdot d\vec{l}=\int_0^1(2x^2\,\vec{i}+x^2\,\vec{j})\,(dx\,\vec{i}+dx\vec{j})=\int_0^1 3x^2\,\mathrm{dx}=1$

$l_4$: $w=\int_{l_4}\vec{F}\cdot d\vec{l}=\int_{(0,0)}^{(0,1)}\vec{F}\cdot d\vec{l}+\int_{(0,1)}^{(1,1)}\vec{F}\cdot d\vec{l}$

$$=\int_{(0,0)}^{(1,0)}(2xy\vec{i}+x^2\,\vec{j})\cdot\vec{j}\,\mathrm{dy}+\int_{(1,0)}^{(1,1)}(2xy\vec{i}+x^2\,\vec{j})\cdot\vec{i}\,\mathrm{dx}=0+\int_0^1 2x\,dx=1$$

沿不同路径积分积分值相等

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \mathrm{2xy} & x^2 & 0\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

8.

(1)

$\vec{F}=(2x+y)\vec{i}+(x+4y+2z)\,\vec{j}+(2y-6z)\vec{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+y & x+4y+2z & 2y-6z\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L'$: $y=0, x+z=a$, 从$(a,0,0)\rightarrow(0,0,a)$

$\Rightarrow$原式$=\int_0^a-(2x+y)+(2y-6z)\,dz=\int_0^a-4z-2a\,\mathrm{dz}=-4a^2$

(2)

$\vec{F}=(x^2-yz)\vec{i}+(y^2-xz)\,\vec{j}+(z^2-xy)\vec{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-yz & y^2-xz & z^2-\mathrm{xy}\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L'$: $x=a, y=0$, 从$(a,0,0)\rightarrow(a,0,h)$

$\Rightarrow$原式$=\int_0^h z^2\,dz=\frac{1}{3}h^3$

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(1)

$$\nabla\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x\,\mathrm{cosy}-y^2\mathrm{sinx} & 2y\,\mathrm{cosx}-x^2\mathrm{siny} & 0\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场

设势函数为$\varphi$

$$\Rightarrow\begin{cases}\dfrac{\partial\varphi}{\partial x}=2x\,\mathrm{cosy}-y^2\mathrm{sinx}\\ \dfrac{\partial\varphi}{\partial y}=2y\,\mathrm{cosx}-x^2\mathrm{siny}\\ \dfrac{\partial\varphi}{\partial z}=0\end{cases}$$

$\Rightarrow\varphi=x^2\sin y+y^2\cos x+c\quad ,c$为常数

(2)

$$\nabla\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ yz(2x+y+z) & xz(2y+z+x) & xy(2z+x+y)\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场

设势函数为$\varphi$

$$\Rightarrow\begin{cases}\dfrac{\partial\varphi}{\partial x}=yz(2x+y+z)\\ \dfrac{\partial\varphi}{\partial y}=xz(2y+z+x)\\ \dfrac{\partial\varphi}{\partial z}=xy(2z+x+y)\end{cases}$$

$\Rightarrow\varphi=xyz(x+y+z)+c\quad ,c$为常数

(3)

$$\nabla\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x(x^2+y^2+z^2) & y(x^2+y^2+z^2) & z(x^2+y^2+z^2)\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场

设势函数为$\varphi$

$$\Rightarrow\begin{cases}\dfrac{\partial\varphi}{\partial x}=x(x^2+y^2+z^2)\\ \dfrac{\partial\varphi}{\partial y}=y(x^2+y^2+z^2)\\ \dfrac{\partial\varphi}{\partial z}=z(x^2+y^2+z^2)\end{cases}$$

$\Rightarrow\varphi=\dfrac{1}{4}(x^4+y^4+z^4)+\dfrac{1}{2}(x^2y^2+y^2z^2+x^2z^2)+c\quad ,c$为常数

10.

$\vec{F}=(x^2+5ay+3yz)\vec{i}+(5x+3axz-2)\vec{j}+[(a+2)xy-4z]\vec{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2+5ay+3yz & 5x+3axz-2 & (a+2)xy-4z\end{vmatrix}$$

$$=(5+3az-5a-3z)\vec{k}-[(a+2)y-3y]\vec{j}+[(a+2)x-3ax]\vec{i}$$

$\because\vec{F}$为有势场

$$\therefore\begin{cases}5+3az-5a-3z=0\\(a+2)y-3y=0\\(a+2)x-3ax=0\end{cases}$$

$\Rightarrow a=1$

$\Rightarrow\vec{F}=(x^2+5y+3yz)\vec{i}+(5x+3xz-2)\vec{j}+(3xy-4z)\vec{k}$

设势函数为$\varphi$

$$\Rightarrow\begin{cases}\dfrac{\partial\varphi}{\partial x}=x^2+5y+3yz\\ \dfrac{\partial\varphi}{\partial y}=5x+3xz-2\\ \dfrac{\partial\varphi}{\partial z}=3xy-4z\end{cases}$$

$\Rightarrow\varphi=\dfrac{1}{3}x^3+3xyz+5yz-2y-2z^2+c\quad,c$为常数

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(1)

$$\begin{cases}\dfrac{\partial u}{\partial x}=3x^2+6\mathrm{xy}^2\\ \dfrac{\partial u}{\partial y}=6x^2y-4y^3\end{cases}$$

$\Rightarrow u=x^3+3x^2y^2-y^4+c\quad,c$为常数

(2)

$$\begin{cases}\dfrac{\partial u}{\partial x}=x^2-2yz\\ \dfrac{\partial u}{\partial y}=y^2-2xz\\ \dfrac{\partial u}{\partial z}=z^2-2xy\end{cases}$$

$\Rightarrow u=\dfrac{1}{3}(x^3+y^3+z^3)-2xyz+c\quad,c$为常数

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(1)

$\vec{F}=(x-y)\vec{i}+(y-x)\vec{j}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x-y & y-x & 0\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L'\colon x=y$, 从$(0,0)\to(1,1)$

$\Rightarrow$原式$=0$

(2)

$$\vec{F}=\left(\frac{1}{y}\sin\frac{x}{y}-\frac{y}{x^2}\cos\frac{y}{x}+1\right)\vec{i}+\left(\frac{1}{x}\cos\frac{x}{y}-\frac{x}{y^2}\sin\frac{x}{y}+\frac{1}{y^2}\right)\vec{j}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{1}{y}\sin\frac{x}{y}-\frac{y}{x^2}\cos\frac{y}{x}+1 & \frac{1}{x}\cos\frac{x}{y}-\frac{x}{y^2}\sin\frac{x}{y}+\frac{1}{y^2} & 0\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L'\colon x=y$, 从$(1,1)\to(2,2)$

$\Rightarrow$原式$=\int_1^2\frac{1}{x}\sin1-\frac{1}{x}\cos1+1+\frac{1}{x}\cos1-\frac{1}{x}\sin1+\frac{1}{x^2}\,dx=\frac{3}{2}$

(3)

$$\vec{F}=\frac{x}{\sqrt{x^2+y^2}}\vec{i}+\frac{y}{\sqrt{x^2+y^2}}\vec{j}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L_1\colon y=0\quad,(1,0)\to(6,0)\qquad;L_2\colon x=6\quad,(6,0)\to(6,3)$

$\Rightarrow$原式$=\int_1^6 1\,dx+\int_0^3\frac{y}{\sqrt{36+y^2}}\,\mathrm{dy}=6+\frac{1}{2}(\sqrt{45}-\sqrt{36})=\frac{3}{2}\sqrt{5}+3$

(4)

$$\vec{F}=x\vec{i}+y\vec{j}+z^3\vec{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z^3\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L_1\colon y=0,z=2\quad,(0,0,2)\to(2,0,2)\quad;L_2\colon x=z=2\quad,(2,0,2)\to(2,3,2)$

$\qquad L_3\colon x=2,y=3\quad,(2,3,2)\to(2,3,-4)$

$\Rightarrow$原式$=\int_0^2 x\,\mathrm{dx}+\int_0^3 y^2\,dy-\int_2^{-4}z^3\,\mathrm{dz}=2+9-\frac{1}{4}(4^4-2^4)=2+9-60=-49$

(5)

$$\vec{F}=\left(1-\frac{1}{y}+\frac{y}{z}\right)\vec{i}+\left(\frac{x}{z}+\frac{x}{y^2}\right)\vec{j}-\frac{xy}{z^2}\vec{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 1-\frac{1}{y}+\frac{y}{z} & \frac{x}{z}+\frac{x}{y^2} & \frac{xy}{z^2}\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L'\colon x=y=z$, 从$(1,1,1)\to(2,2,2)$

$\Rightarrow$原式$=\int_1^2 1-\frac{1}{x}+1+1+\frac{1}{x}-1\,\mathrm{dx}=\int_1^2 2dx=2$

(6)

$$\vec{F}=\frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i}+\frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j}+\frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}}\end{vmatrix}=0$$

$\Rightarrow$场无旋，有源，是有势场$\Rightarrow$沿不同路径积分积分值相等

$\Rightarrow$取路径: $L_1\colon y=y_1, z=z_1\quad,(x_1,y_1,z_1)\to(x_2,y_1,z_1)\quad;$

$L_2\colon x=x_2, z=z_1\quad,(x_2,y_1,z_1)\to(x_2,y_2,z_1)$

$L_3\colon x=x_2, y=y_2\quad,(x_2,y_2,z_1)\to(x_2,y_2,z_2)$

$$\Rightarrow\text{原式}=\int_{x_1}^{x_2}\frac{x}{\sqrt{x^2+y_1^2+z_1^2}}\,\mathrm{dx}+\int_{y_1}^{y_2}\frac{y}{\sqrt{x_2^2+y^2+z_1^2}}\,dy-\int_{z_1}^{z_2}\frac{z}{\sqrt{x_2^2+y_2^2+z^2}}\,\mathrm{dz}$$

$$=\sqrt{x_2^2+y_1^2+z_1^2}-\sqrt{x_1^2+y_1^2+z_1^2}+\sqrt{x_2^2+y_2^2+z_1^2}-\sqrt{x_2^2+y_1^2+z_1^2}+\sqrt{x_2^2+y_2^2+z_2^2}-\sqrt{x_2^2+y_2^2+z_1^2}$$

$$=\sqrt{x_2^2+y_2^2+z_2^2}-\sqrt{x_1^2+y_1^2+z_1^2}$$

由题目可知：$\sqrt{x_2^2+y_2^2+z_2^2}=\sqrt{x_1^2+y_1^2+z_1^2}$

$\Rightarrow$原式$=0$

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(1)

即证：$\vec{h}=xf(x^2+y^2)\,\vec{i}+yf(x^2+y^2)\,\vec{j}$有对应的势函数，设势函数为$\varphi$

$$\begin{cases}\frac{\partial\varphi}{\partial x}=xf(x^2+y^2) & 1\\ \frac{\partial\varphi}{\partial y}=yf(x^2+y^2) & 2\end{cases}$$

设$F(u)$为$f(u)$的原函数

$\Rightarrow \varphi = \dfrac{1}{2} F(x^2+y^2) + C \quad C$为常数

$\because \varphi$存在

$\Rightarrow \int_L x f(x^2+y^2)\, dx + y f(x^2+y^2)\, \mathrm{dy} = 0$

(2)

$\int_L f\left(\sqrt{x^2+y^2+z^2}\right)(x\,\mathrm{dx} + \mathrm{ydy} + z\,\mathrm{dz}) = \dfrac{1}{2}\int_L f\left(\sqrt{x^2+y^2+z^2}\right) d(x^2+y^2+z^2)$

令$\sqrt{x^2+y^2+z^2} = u$,

原式$= \int_L f(u)\, u\, \mathrm{du} = 0$

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$\Gamma = \oint_L \vec{B} \cdot d\vec{l} = \oiint\limits_S (\nabla \times \vec{B}) \cdot \vec{n}\mathrm{dS}$

$\vec{B} = \dfrac{-2I_y}{x^2+y^2} dx + \dfrac{2I_x}{x^2+y^2} dy$

$\nabla \times \vec{B} = 0$

$\Rightarrow \Gamma = 0$

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即证：$\vec{h} = (f'(x) + 6f(x) + e^{-2x})\, y\, \vec{i} + f'(x)\, \vec{j}$有对应的势函数，设势函数为$\varphi$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = (f'(x) + 6f(x) + e^{-2x})\, y & 1 \\ \dfrac{\partial \varphi}{\partial y} = f'(x) & 2 \end{cases}$$

设$F(x)$为$f(x)$的原函数

$1 \Rightarrow \varphi = f(x)\, y + 6yF(x) - \dfrac{1}{2} e^{-2x} y + g(y)$

$\Rightarrow \dfrac{\partial \varphi}{\partial y} = f(x) + 6F(x) - \frac{1}{2} e^{-2x} + g'(y) = f'(x)$

$\Rightarrow g'(y)$为常数，设$g'(y) = C_1$

$\Rightarrow f(x) + 6F(x) - \frac{1}{2} e^{-2x} + C_1 = f'(x)$

$\Rightarrow F(x) = C_2 e^{3x} + C_3 e^{-2x} + \dfrac{1}{6} C_1 + \dfrac{1}{10} x e^{-2x}$ ,其中$C_1, C_2, C_3$为常数。

$\Rightarrow f(x) = F'(x) = 3C_2 e^{3x} - 2C_3 e^{-2x} + \left(\dfrac{1}{10} - \dfrac{1}{5} x\right) e^{-2x}$

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(1)

因为线积分与路径无关，所以存在势函数$\varphi$, 设$\mathrm{A}(x)$为$\alpha'(x)$的原函数

$$\begin{cases} \dfrac{\partial\varphi}{\partial x}=P=[2x\alpha'(x)+\beta'(x)]y^2-2y\beta(x)\tan 2x & 1 \\ \dfrac{\partial\varphi}{\partial y}=Q=[\alpha'(x)+4x\alpha'(x)]y+\beta(x) & 2 \end{cases}$$

由2知$\Rightarrow \varphi=\dfrac{1}{2}[\alpha'(x)+4x\alpha'(x)]y^2+\beta(x)y+f(x)$

由1知$\Rightarrow \varphi=y^2[2x\alpha(x)-2\mathrm{A}(x)+\beta(x)]-2y\int\beta(x)\tan 2x\,\mathrm{dx}+g(y)$

$$\Rightarrow\begin{cases} 2x\alpha(x)-2\mathrm{A}(x)+\beta(x)=\dfrac{1}{2}[\alpha'(x)+4x\alpha'(x)] \\ \beta(x)=2\int\beta(x)\tan 2x\,\mathrm{dx}+C & C\text{为常数} \end{cases}$$

$$\Rightarrow\begin{cases} \beta(x)=2\cos 2x+C_1 & C_1\text{为常数} \\ \alpha(x)=(1+x)\sin 2x+C_1 & C_1\text{为常数} \end{cases}$$

$\because\alpha(0)=0, \alpha'(0)=2, \beta(0)=2$

$$\Rightarrow\begin{cases} \beta(x)=2\cos 2x \\ \alpha(x)=(1+x)\sin 2x \end{cases}$$

(2)

$\alpha'(x)=2(1+x)\cos 2x+\sin 2x$

$\Rightarrow P=y^2(2x(1+x)\cos 2x+2\cos 2x+\sin x\cdot 2x)-4y\sin 2x$

$\quad =y^2[(2x^2+2x+2)\cos 2x+2x\sin x]-4y\sin 2x$

$\quad Q=[2(1+x)\cos 2x+(4x+5)\sin 2x]y+2\cos 2x$

取积分路径$L$: 沿$y$轴从$(0,0)$到$(0,2)$

$\Rightarrow$原式$=\int_0^2 2y+2\,\mathrm{dy}=4+4=8$

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$\because\int_L 2xy\,\mathrm{dx}+Q(x,y)\,dy$与路径无关

左式取路径$L$: 沿$x=0$.从$(0,0)$到$(0,1)$, 沿$y=1$, 从$(0,1)$到$(t,1)$

$\Rightarrow\int_{(0,0)}^{(t,1)}2xy\,\mathrm{dx}+Q(x,y)\,dy=\int_0^1 Q(0,y)\,\mathrm{dy}+\int_0^t 2x\,\mathrm{dx}=\int_0^1 Q(0,y)\,\mathrm{dy}+t^2$

右式取路径$L$: 沿$x=0$.从$(0,0)$到$(0,t)$, 沿$y=t$, 从$(0,t)$到$(1,t)$

$\Rightarrow\int_{(0,0)}^{(1,t)}2xy\,\mathrm{dx}+Q(x,y)\,dy=\int_0^t Q(0,y)\,\mathrm{dy}+\int_0^1 2tx\,\mathrm{dx}=\int_0^t Q(0,y)\,\mathrm{dy}+t$

$\Rightarrow\int_0^t Q(0,y)\,\mathrm{dy}+t=\int_0^1 Q(0,y)\,\mathrm{dy}+t^2$

$\because\int_L 2xy\,\mathrm{dx}+Q(x,y)\,dy$与路径无关

$\Rightarrow\dfrac{\partial Q}{\partial x}=2x$

$\Rightarrow Q=x^2+f(y)$

$\Rightarrow\int_0^t f(y)\,\mathrm{dy}+t=\int_0^1 f(y)\,\mathrm{dy}+t^2$

$\Rightarrow\int_1^t f(y)\,\mathrm{dy}=t^2-t$

$\Rightarrow f(y)=2y-1$

$\Rightarrow Q=x^2+2y-1$

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(1)

原式$=d\left(\dfrac{1}{2}x^2y^2+2xy-2y\sin x+y\cos x\right)=0$

$\Rightarrow\dfrac{1}{2}x^2y^2+2xy-2y\sin x+y\cos x=C\quad$，$C$为常数

(2)

$2xy\,\mathrm{dx}+(y^2-x^2)\,dy=0$

设$y\neq 0$

$\Rightarrow\dfrac{2x}{y}\mathrm{dx}+\left(1-\dfrac{x^2}{y^2}\right)\mathrm{dy}=0$

$\Rightarrow d\left(y+\dfrac{x^2}{y}\right)=0$

$\Rightarrow y+\dfrac{x^2}{y}=C$

$\Rightarrow x^2+y^2=Cy\,(y\neq 0)$，$C$为常数

若$y=0$则$x=0$

满足$x^2+y^2=Cy$，$C$为常数

$\Rightarrow$综上$x^2+y^2=Cy$，$C$为常数

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$\because$全微分

$\Rightarrow\dfrac{\partial}{\partial x}[f'(x)+e^x\cos y+2x]=\dfrac{\partial}{\partial y}[e^x\sin y+x^2y+f(x,y)]$

$\Rightarrow f''(x)+e^x\cos y+2-e^x\cos y-x^2-f(x)=0$

$\Rightarrow f''(x)-f(x)=x^2-2$

$\because f(0)=0, f'(0)=2$

$\Rightarrow f(x)=e^x-e^{-x}-x^2$

$\Rightarrow f'(x)=e^x+e^{-x}-2x$

$\Rightarrow$全微分: $[e^x\mathrm{siny}+e^x y-e^{-x}y]dx+[e^x+e^{-x}+e^x\mathrm{cosy}]\mathrm{dy}=0$

$\Rightarrow d[e^x\mathrm{siny}+e^x y+e^{-x}y]=0$

$\Rightarrow e^x\mathrm{siny}+e^x y+e^{-x}y=C \quad C$为常数

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$$\begin{cases}\frac{\partial u}{\partial x}=2xy(x^4+y^2)^\lambda\\ \frac{\partial u}{\partial y}=-x^2(x^4+y^2)^\lambda\end{cases}$$

因为$\vec{v}$是梯度

$\Rightarrow\nabla\times\vec{v}=0$

$$\Rightarrow-\frac{\partial}{\partial x}[x^2(x^4+y^2)^\lambda]-\frac{\partial}{\partial y}[2xy(x^4+y^2)^\lambda]=0$$

$\Rightarrow 4x(x^4+y^2)^\lambda+\lambda\cdot 4x(x^4+y^2)^\lambda=0$

$\Rightarrow\lambda=-1$

$$\Rightarrow\begin{cases}\frac{\partial u}{\partial x}=\frac{2xy}{x^4+y^2}\\ \frac{\partial u}{\partial y}=-\frac{x^2}{x^4+y^2}\end{cases}$$

$$\Rightarrow u=\arctan\left(\frac{x^2}{y}\right)+C \quad ,C\text{为常数。}$$

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(1)

$$\nabla[\vec{w}\cdot f(x)\vec{r}]=\vec{w}\cdot\nabla(f(r)\vec{r})+f(r)\vec{r}\cdot\nabla\vec{w}=f(r)\vec{w}\cdot\nabla\vec{r}+(\vec{w}\cdot\vec{r})\nabla f(r)$$

$$=f(r)\vec{w}+(\vec{w}\cdot\vec{r})f'(r)\frac{\vec{r}}{r}$$

用张量做

设用$\dfrac{\partial}{\partial x_i}$代表$\nabla$ ，$w_j$代表$\vec{w}$ ，$r_j$代表$\vec{r}$

$$\frac{\partial}{\partial x_i}(w_j f(r) r_j) = w_j \frac{\partial}{\partial x_i}(f(r) r_j) + f(r) r_j \frac{\partial w_j}{\partial x_i} = w_j f(r) \frac{\partial r_j}{\partial x_i} + w_j r_j \frac{\partial f(r)}{\partial x_i}$$

$$= w_j r_j \frac{\partial f(r)}{\partial x_i} + f(r) w_j \frac{\partial r_j}{\partial x_j} = w_j r_j \frac{\partial f(r)}{\partial x_i} + f(r) w_i$$

(2)

$$\nabla \cdot [\vec{w} \times f(r) \vec{r}] = f(r) \nabla \cdot [\vec{w} \times \vec{r}] + (\vec{w} \times \vec{r}) \cdot \nabla (f(r))$$

$$= f(r) \nabla \cdot [\vec{w} \times \vec{r}] + (\vec{w} \times \vec{r}) \cdot f'(r) \frac{\vec{r}}{r} = f(r)\ \vec{w} \cdot (\nabla \times \vec{r}) + (\vec{w} \times \vec{r}) \cdot f'(r) \frac{\vec{r}}{r}$$

$$\because (\vec{w} \times \vec{r}) \cdot \vec{r} = 0 \quad , \nabla \times \vec{r} = \vec{0}$$

$$\therefore \nabla \cdot [\vec{w} \times f(r) \vec{r}] = 0$$

(3)

用张量做

设用$\dfrac{\partial}{\partial x_p}$代表$\nabla$ ，$w_i$代表$\vec{w}$ ，$r_j$代表$\vec{r}$

$\Rightarrow \nabla \times [\vec{w} \times f(r) \vec{r}]$可以用$\varepsilon_{pkq} \dfrac{\partial}{\partial x_p}(\varepsilon_{kij}\, w_i f(r) r_j)$表示

$$\Rightarrow \varepsilon_{pkq} \frac{\partial}{\partial x_p}(\varepsilon_{kij}\, w_i f(r) r_j) = (\delta_{qi}\delta_{pj} - \delta_{qj}\delta_{pi}) \frac{\partial}{\partial x_p}(w_i\, f(r) r_j)$$

$$= \frac{\partial}{\partial x_j}(w_q\, f(r) r_j) - \frac{\partial}{\partial x_p}(w_p\, f(r) r_q)$$

$$= f(r) \frac{\partial}{\partial x_j}(w_q r_j) + w_q r_j \frac{\partial}{\partial x_j}(f(r)) - f(r) \frac{\partial}{\partial x_p}(w_p r_q) - w_p r_q \frac{\partial}{\partial x_p}(f(r))$$

$$= f(r) \left[ w_q \frac{\partial r_j}{\partial x_j} + r_j \frac{\partial w_q}{x_j} \right] + w_q r_j f'(r) \frac{r_j}{r} - f(r) \left[ w_p \frac{\partial r_q}{\partial x_p} + r_q \frac{\partial w_p}{\partial x_p} \right] - w_p r_q f'(r) \frac{r_p}{r}$$

$$\Rightarrow \nabla \times [\vec{w} \times f(r) \vec{r}]$$

$$= f(r) [\vec{w}(\nabla \cdot \vec{r}) + (\vec{r} \cdot \nabla) \vec{w}] + \vec{w} f'(r) r - f(r) [(\vec{w} \cdot \nabla) \vec{r} + \vec{r}(\nabla \cdot \vec{w})] - (\vec{w} \cdot \vec{r}) f'(r) \frac{\vec{r}}{r}$$

$$= 3\, \vec{w} f(r) + 0 + \vec{w}\, r f'(r) - \vec{w} f(r) + 0 - (\vec{w} \cdot \vec{r}) f'(r) \frac{\vec{r}}{r}$$

$$= 2 \vec{w} f(r) + \vec{w}\, r f'(r) - (\vec{w} \cdot \vec{r}) f'(r) \frac{\vec{r}}{r}$$

（完结）

撒花！！！