1. (1) $A(\lambda)=\frac{1}{\pi}\int_0^T kx\cos\lambda x\,dx=\frac{k}{\pi}\cdot\frac{1}{\lambda}\int_0^T x\,d\sin\lambda x=\frac{k}{\pi\lambda^2}(\lambda T\sin\lambda T+\cos\lambda T-1)$

$B(\lambda)=\frac{1}{\pi}\int_0^T kx\sin\lambda x\,dx=\frac{k}{\pi\lambda^2}(\sin\lambda T-\lambda T\cos\lambda T)$

在$x=T$处，傅里叶积分收敛于 $\frac{f(T^+)+f(T^-)}{2}=\frac{kT}{2}\neq f(T)$

故 $\int_0^{+\infty}[A(\lambda)\cos\lambda x+B(\lambda)\sin\lambda x]\,d\lambda=\begin{cases} f(x), & x\neq T\\ \frac{kT}{2}, & x=T\end{cases}$

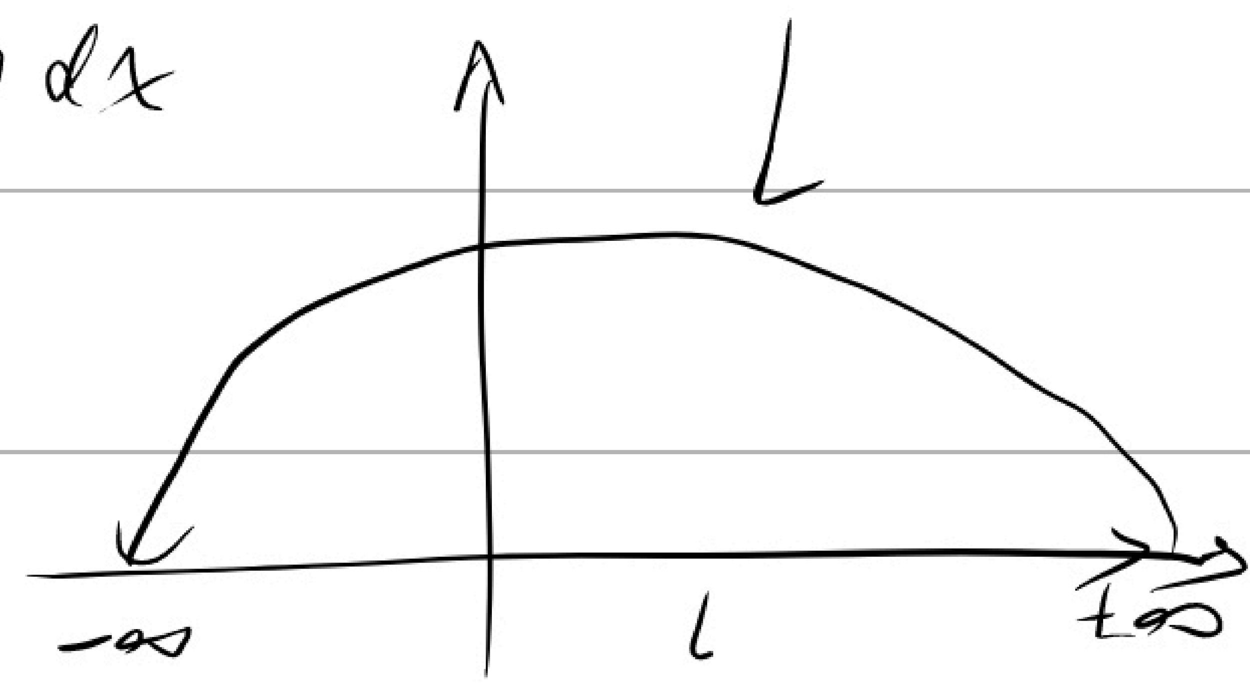
(2) $A(\lambda)=0$, $B(\lambda)=\frac{1}{\pi}\int_{-1}^{1}\operatorname{sgn}x\cdot\sin\lambda x\,dx=\frac{2(1-\cos\lambda)}{\pi\lambda}$

$f(x)=\frac{2}{\pi}\int_0^{+\infty}\frac{1-\cos\lambda}{\lambda}\sin\lambda x\,d\lambda$

(3) $A(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}\cos\lambda x\,dx$

下面用到复变函数知识：$\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}(\cos\lambda x+i\sin\lambda x)\,dx$

$=\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}e^{i\lambda x}\,dx$



$\frac{1}{a^2+x^2}e^{i\lambda x}$ 在上半平面有一阶极点 $x=ai$.

沿着大圆弧L有 $\int_L\frac{1}{a^2+x^2}e^{i\lambda x}\,dx=0$（若尔当引理）

闭路 $\oint\frac{1}{a^2+x^2}e^{i\lambda x}\,dx=2\pi i\,\mathrm{Res}\left[\frac{1}{a^2+x^2}e^{i\lambda x},ai\right]=2\pi i\,\frac{e^{-\lambda a}}{2ai}=\frac{\pi}{ae^{\lambda a}}$

故 $\int_l\frac{1}{a^2+x^2}e^{i\lambda x}\,dx=\oint\frac{1}{a^2+x^2}e^{i\lambda x}\,dx=\frac{\pi}{ae^{\lambda a}}$

$\Rightarrow \int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}e^{i\lambda x}\,dx=\mathrm{Re}\left[\frac{\pi}{ae^{\lambda a}}\right]=\frac{\pi}{ae^{\lambda a}}$

$B(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}\sin\lambda x\,dx=0$

故 $f(x)=\frac{1}{\pi}\int_0^{+\infty}\frac{\pi}{ae^{\lambda a}}\cos\lambda x\,d\lambda=\frac{1}{a}\int_0^{+\infty}e^{-a\lambda}\cos\lambda x\,d\lambda$

(4) $A(\lambda)=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos x)\cos\lambda x\,dx=\frac{2}{\pi}\left(\frac{1}{\lambda}\sin\frac{\lambda\pi}{2}-\frac{1}{2(\lambda+1)}\cos\frac{\lambda\pi}{2}+\frac{1}{2(\lambda-1)}\cos\frac{\lambda\pi}{2}\right)$

$=\frac{2}{\pi}\left(\frac{1}{\lambda}\sin\frac{\lambda\pi}{2}-\frac{1}{1-\lambda^2}\cos\frac{\lambda\pi}{2}\right)$

$B(\lambda)=0$.

$f(x)=\frac{2}{\pi}\int_0^{+\infty}\left(\frac{1}{\lambda}\sin\frac{\lambda\pi}{2}-\frac{1}{1-\lambda^2}\cos\frac{\lambda\pi}{2}\right)\cos\lambda x\,d\lambda$

(5) $A(\lambda)=\frac{1}{\pi}\int_{-\infty}^{0}e^{x}\cos\lambda x\,dx+\frac{1}{\pi}\int_0^{+\infty}e^{-x}\cos\lambda x\,dx=\frac{2}{\pi(1+\lambda^2)}$

$B(\lambda)=0$

$f(x)=\frac{2}{\pi}\int_0^{+\infty}\frac{1}{1+\lambda^2}\cos\lambda x\,dx$

2、$\nu\in(0,+\infty)$时，$S_\nu(x)$为常义积分，$(0,\nu)$上无瑕点，$S_\nu(x)$绝敛

$\nu\to+\infty$时，$\frac{2}{\pi}\int_0^{+\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\begin{cases}1, & |x|<1\\ \frac{1}{2}, & x=\pm1\\ 0, & |x|>1\end{cases}$ （P326页）

吉布斯现象：部分傅里叶积分在函数间断点附近形成一个尖峰，当ν很大时，逼近的误差$\max\limits_{[-1,1]}|f(x)-S_\nu(x)|$趋于一个常数，大约等于0.09

3.(1) $F(\lambda)=-2i\int_0^{+\infty}xe^{-ax}\sin\lambda x\,dx=\frac{2i}{a}\int_0^{+\infty}x\sin\lambda x\,de^{-ax}$

$=-\frac{2i}{a}\int_0^{+\infty}e^{-ax}\sin\lambda x\,dx-\frac{2i\lambda}{a}\int_0^{+\infty}e^{-ax}x\cos\lambda x\,dx$

$=-\frac{2i}{a}\cdot\frac{\lambda}{a^2+\lambda^2}-\frac{2\lambda i}{a^2}\int_0^{+\infty}e^{-ax}\cos\lambda x\,dx+\frac{2\lambda^2 i}{a^2}\int_0^{+\infty}e^{-ax}x\sin\lambda x\,dx$

$\Rightarrow\ (1+\frac{\lambda^2}{a^2})F(\lambda)=\frac{2\lambda i}{a(a^2+\lambda^2)}+\frac{2\lambda i}{a^2}\cdot\frac{a}{\lambda^2+a^2}$

$\Rightarrow\ F(\lambda)=-\frac{4\lambda a i}{(\lambda^2+a^2)^2}$

(2) $F(\lambda)=2\int_0^{+\infty}e^{-ax}\cos bx\cos\lambda x\,dx=\int_0^{+\infty}e^{-ax}[\cos(b+\lambda)x+\cos(b-\lambda)x]\,dx$

$$=\frac{a}{a^2+(b+\lambda)^2}+\frac{a}{a^2+(b-\lambda)^2}$$

(3) $F(\lambda)=2\int_0^{\frac{\pi}{2}}\cos x\cos\lambda x\,dx=\int_0^{\frac{\pi}{2}}[\cos(\lambda+1)x\,dx+\cos(1-\lambda)x\,dx$

$$=\frac{1}{1+\lambda}\sin\left(\frac{\pi}{2}+\frac{\pi\lambda}{2}\right)+\frac{1}{1-\lambda}\sin\left(\frac{\pi}{2}-\frac{\pi\lambda}{2}\right)$$

$$=\frac{2}{1-\lambda^2}\cos\frac{\pi\lambda}{2}$$

(4) $F(\lambda)=2\int_0^1(1-x)\cos\lambda x\,dx=2\int_0^1\cos\lambda x\,dx-2\int_0^1 x\cos\lambda x\,dx$

$$=\frac{2}{\lambda}\sin\lambda-2\left(\frac{\sin\lambda}{\lambda}+\frac{\cos\lambda-1}{\lambda^2}\right)=\frac{2(1-\cos\lambda)}{\lambda^2}$$

4. (1) $A(\lambda)=\frac{2}{\pi}\int_0^{+\infty}e^{-x}\cos\lambda x\,dx=\frac{2}{\pi(1+\lambda^2)}$

$$f(x)=\frac{2}{\pi}\int_0^{+\infty}\frac{1}{1+\lambda^2}\cos\lambda x\,d\lambda$$

(2) $B(\lambda)=\frac{2}{\pi}\int_0^{+\infty}e^{-x}\sin\lambda x\,dx=\frac{2\lambda}{\pi(1+\lambda^2)}$

$$f(x)=\frac{2}{\pi}\int_0^{+\infty}\frac{\lambda}{1+\lambda^2}\sin\lambda x\,dx$$

5. 当$x<0$时补充定义 $f(x)=f(-x)=\frac{1}{\sqrt{-x}}$

$F(\lambda)=2\int_0^{+\infty}\frac{1}{\sqrt{x}}\cos\lambda x\,dx=\frac{2}{\sqrt{\lambda}}\int_0^{+\infty}\frac{\cos u}{\sqrt{u}}du=\sqrt{\frac{2\pi}{\lambda}}$ （P_{283} 菲涅耳积分）

6. 本函数微分方法：简单证明一阶导形.

$$\mathcal{F}[f'(x)](\lambda)=\int_{-\infty}^{+\infty}f'(x)e^{-i\lambda x}dx=\int_{-\infty}^{+\infty}e^{-i\lambda x}df(x)=$$

$f(x)e^{-i\lambda x}\Big|_{-\infty}^{+\infty} + i\lambda\int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}dx = i\lambda\int_{-\infty}^{+\infty} f(x)e^{-i\lambda x} = i\lambda\, \mathcal{F}[f(x)](\lambda)$

像函数微分法：$F'(\lambda) = \int_{-\infty}^{+\infty} f(x)\dfrac{de^{-i\lambda x}}{d\lambda}dx = \int_{-\infty}^{+\infty} -ixf(x)e^{i\lambda x}dx$

$= \mathcal{F}[-ixf(x)](\lambda)$

7. $\mathcal{F}[f(ax)](\lambda) = \int_{-\infty}^{+\infty} f(ax)e^{-i\lambda x}dx = \frac{1}{a}\int_{-\infty}^{+\infty} f(t)e^{-\frac{i\lambda t}{a}}dt = \frac{1}{a}\mathcal{F}[f(x)](\frac{\lambda}{a})$，$a>0$

$a<0$ 时 $\mathcal{F}[f(ax)](\lambda) = \frac{1}{a}\int_{+\infty}^{-\infty} f(t)e^{-\frac{i\lambda t}{a}}dt = -\frac{1}{a}\mathcal{F}[f(x)](\frac{\lambda}{a})$，$a<0$.

综上 $\forall a\neq 0$ $\mathcal{F}[f(ax)](\lambda) = \frac{1}{|a|}\mathcal{F}[f(x)](\frac{\lambda}{a})$

8. ①交换律 $f(x)*g(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt$，令 $\tau = x-t$，则 $dt = -d\tau$

$f*g(x) = -\int_{+\infty}^{-\infty} f(\tau)\cdot g(x-\tau)d\tau = \int_{-\infty}^{+\infty} g(x-\tau)f(\tau)d\tau = g*f(x)$

②分配律 $f_1(x)*[f_2(x)+f_3(x)] = \int_{-\infty}^{+\infty} f_1(x-t)[f_2(t)+f_3(t)]dt$

$= \int_{-\infty}^{+\infty} f_1(x-t)f_2(t)dt + \int_{-\infty}^{+\infty} f_1(x-t)f_3(t)dt = f_1*f_2(x) + f_1*f_3(x)$

由于卷积满足交换律，故 $[f_2(x)+f_3(x)]*f_1(x) = f_2*f_1(x) + f_3*f_1(x)$

③结合律 $[f_1(x)*f_2(x)]*f_3(x) = \int_{-\infty}^{+\infty}[f_1(\tau)*f_2(\tau)]f_3(x-\tau)d\tau$

$= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f_1(t)f_2(\tau-t)dt\right]f_3(x-\tau)d\tau$

$= \int_{-\infty}^{+\infty} f_2(\tau-t)f_3(x-\tau)d\tau\int_{-\infty}^{+\infty} f_1(t)dt$，令 $u = \tau - t$，则

上式 $= \int_{-\infty}^{+\infty} f_2(u)f_3(x-u-t)du\int_{-\infty}^{+\infty} f_1(t)dt$

$= \int_{-\infty}^{+\infty}[f_2*f_3(x-t)]f_1(t)dt = f_1(x)*[f_2(x)*f_3(x)]$. 证毕

9. $f*g(x)=\int_{-\infty}^{+\infty} f(x-t)g(t)\,dt$

$\int_{-\infty}^{+\infty} f*g(x)\,dx = \int_{-\infty}^{+\infty} f(x-t)\,dx \cdot \int_{-\infty}^{+\infty} g(t)\,dt$，令 $u=x-t$.

则上式 $= \int_{-\infty}^{+\infty} f(u)\,du \cdot \int_{-\infty}^{+\infty} g(t)\,dt$.

$\because f(x), g(x)$ 在 $(-\infty,+\infty)$ 上绝对可积，故 $\int_{-\infty}^{+\infty} f(u)\,du \cdot \int_{-\infty}^{+\infty} g(t)\,dt$ 有界

即 $f*g(x)$ 也在整个实数轴上绝对可积.

10. $\hat{g}(\lambda) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t)f(x+t)e^{-i\lambda x}\,dt\,dx$

$= \int_{-\infty}^{+\infty} f(x+t)e^{-i\lambda x}\,dx \int_{-\infty}^{+\infty} f(t)\,dt$

$= \int_{-\infty}^{+\infty} f(x+t)e^{-i\lambda(x+t)}\,dx \int_{-\infty}^{+\infty} f(t)e^{-i\lambda t}\,dt$

$= F^2(\lambda)$

$g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^2(\lambda)e^{i\lambda x}\,d\lambda = \int_{-\infty}^{+\infty} f(t)f(x+t)\,dt$

令 $x=0$ 得 $\int_{-\infty}^{+\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^2(\lambda)\,d\lambda$

11. 记 $\hat{u}(t,\lambda)=\int_{-\infty}^{+\infty} u(t,x)e^{-i\lambda x}\,dx$，设 $\hat{\varphi}(\lambda)=\mathcal{F}[e^{-x^2}](\lambda)$

对方程进行 Fourier 变换得 $\begin{cases} \dfrac{d\hat{u}}{dt} = -\frac{1}{4}\lambda^2\hat{u} \\ \hat{u}\big|_{t=0} = \hat{\varphi}(\lambda) \end{cases}$

$t\tau^2 + \frac{x-\tau^2}{t}$

$\Rightarrow \hat{u}(t,\lambda) = \hat{\varphi}(\lambda)e^{-\frac{1}{4}\lambda^2 t}$

$u(t,x) = F^{-1}[\hat{\varphi}(\lambda)e^{-\frac{1}{4}\lambda^2 t}] = F^{-1}[\hat{\varphi}(\lambda)] * F^{-1}[e^{-\frac{1}{4}\lambda^2 t}]$

$= e^{-x^2} * \frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{t}}$

$= \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty} e^{-\tau^2}e^{-\frac{(x-\tau)^2}{t}}\,d\tau$

(2,(1)) $\frac{\partial u}{\partial t} = \frac{1}{2}[-af'(x-at)+af'(x+at)]+\frac{1}{2a}[ag(x+at)+ag(x-at)]$

$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}a^2[f''(x-at) + f''(x+at)]+\frac{a}{2}[g'(x+at)-g'(x-at)]$

$\frac{\partial u}{\partial x} = \frac{1}{2}[f'(x-at)+f'(x+at)]+\frac{1}{2a}[g(x+at)-g(x-at)]$

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}[f''(x-at)+f''(x+at)]+\frac{1}{2a}[g'(x+at-g'(x-at)]$

$\Rightarrow \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$

初始条件: $u|_{t=0} = \frac{1}{2}[f(x)+f(x)]+\frac{1}{2a}\int_x^x g(s)\,ds = f(x)$

$u_t|_{t=0} = \frac{1}{2}[-af'(x)+af'(x)]+\frac{1}{2}[g(x)+g(x)] = g(x)$

(2) 参考《数学物理方程》—季孝达 P139

例 4.1.2 无限长弦的自由振动

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, & t>0, -\infty<x<+\infty\ , \\ u|_{t=0} = \varphi(x)\ , & \frac{\partial u}{\partial t}|_{t=0} = \psi(x)\ . \end{cases}$$

解 设 $\widehat{u}(t,\lambda) = F[u(t,x)]$, 对方程和初始条件作 Fourier 变换, 得像函数满足的初值问题

$$\begin{cases} \frac{\mathrm{d}^2\widehat{u}}{\mathrm{d}t^2} = -a^2\lambda^2\widehat{u}\ , \\ \widehat{u}|_{t=0} = \widehat{\varphi}(\lambda)\ , \quad \frac{\mathrm{d}\widehat{u}}{\mathrm{d}t}|_{t=0} = \widehat{\psi}(\lambda)\ , \end{cases}$$

其中, $\widehat{\varphi}(\lambda) = F[\varphi(x)]$, $\widehat{\psi}(\lambda) = F[\psi(x)]$.

该方程的通解为

$$\widehat{u}(t,\lambda) = A(\lambda)\mathrm{e}^{\mathrm{i}\lambda at} + B(\lambda)\mathrm{e}^{-\mathrm{i}\lambda at}\ .$$

代入初始条件, 确定

$$A(\lambda) = \frac{1}{2}\left[\widehat{\varphi}(\lambda) + \frac{1}{\mathrm{i}\lambda a}\widehat{\psi}(\lambda)\right]\ ,$$

$$B(\lambda) = \frac{1}{2}\left[\widehat{\varphi}(\lambda) - \frac{1}{\mathrm{i}\lambda a}\widehat{\psi}(\lambda)\right]\ .$$

代入通解, 得到像函数

$$\widehat{u}(t,\lambda) = \frac{1}{2}\left[\widehat{\varphi}(\lambda)\mathrm{e}^{\mathrm{i}\lambda at} + \widehat{\varphi}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda at}\right] + \frac{1}{2a\mathrm{i}\lambda}\left[\widehat{\psi}(\lambda)\mathrm{e}^{\mathrm{i}\lambda at} - \widehat{\psi}(\lambda)e^{-\mathrm{i}\lambda at}\right]\ .$$

对像函数作反变换, 由 Fourier 变换的积分公式 (4.1.8)

$$F\Big[\int_{-\infty}^{x} f(\xi)\mathrm{d}\xi\Big] = \frac{1}{\mathrm{i}\lambda}\tilde{f}(\lambda)$$

和位移公式 (4.1.6)

$$F\big[f(x+\xi)\big] = \widehat{f}(\lambda)\mathrm{e}^{\mathrm{i}\lambda\xi} \ ,$$

得

$$u(t,x) = \frac{1}{2}\big[\varphi(x+at)+\varphi(x-at)\big] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi \ ,$$

13. 解 $\int_{-\infty}^{+\infty}|f(x)|^2dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty}e^{\frac{-x^2}{2\sigma^2}}dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}dt = \frac{1}{\sqrt{\pi}}\cdot\sqrt{\pi} = 1$

$\hat{f}(\lambda) = (8\pi\sigma^2)^{\frac{1}{4}}e^{-\sigma^2\lambda^2}$

$\int_{-\infty}^{+\infty}|\hat{f}(\lambda)|^2d\lambda = \sqrt{8\pi\sigma^2}\int_{-\infty}^{+\infty}e^{-2\sigma^2\lambda^2}d\lambda = 2\sqrt{\pi}\int_{-\infty}^{+\infty}e^{-t^2}dt = 2\pi$

$\Delta_0 f = \int_{-\infty}^{+\infty}x^2|f(x)|^2dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty}x^2e^{-\frac{x^2}{2\sigma^2}}dx = \frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}t^2e^{-t^2}dt$

$= \frac{-\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}te^{-t^2} = -\frac{\sigma^2}{\sqrt{\pi}}te^{-t^2}\Big|_{-\infty}^{+\infty} + \frac{\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}dt = \sigma^2$

同理 $\Delta_0\hat{f} = \int_{-\infty}^{+\infty}\lambda^2|\hat{f}(\lambda)|^2d\lambda = \sqrt{8\pi\sigma^2}\int_{-\infty}^{+\infty}\lambda^2e^{-2\sigma^2\lambda^2}d\lambda = \frac{1}{4\sigma^2}$

于是有 $\Delta_0 f\cdot\Delta_0\hat{f} = \frac{1}{4}$. 即高斯函数使得海森堡不等式取到等号