6.3习题答案

1.

(1)

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}=\left(e^t+\frac{2t}{1+(1+t^2)^2}\right)\cdot y\cdot x^{y-1}=y\cdot x^{y-1}\left(e^{x^y}+\frac{2x^y}{1+(1+x^{2y})^2}\right)$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=\left(e^t+\frac{2t}{1+(1+t^2)^2}\right)\cdot x^y\cdot\ln^x=x^y\cdot\ln^x\left(e^{x^y}+\frac{2x^y}{1+(1+x^{2y})^2}\right)$

(2)

$u=e^{r^2z}=e^{r^{(s+2)}}$

$\Rightarrow\frac{\partial u}{\partial r}=e^{r^{(s+2)}}\cdot(s+2)\cdot r^{(s+1)}\quad;\frac{\partial u}{\partial s}=e^{r^{(s+2)}}\cdot r^{(s+2)}\cdot\ln^r$

(3)

$\frac{\mathrm{du}}{\mathrm{dx}}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}=\frac{a\,e^{ax}(y-z)}{a^2+1}+\frac{e^{ax}}{a^2+1}\cdot a\,\mathrm{cosx}+\frac{e^{ax}}{a^2+1}\cdot\mathrm{sinx}=e^{ax}\mathrm{sinx}$

(4)

设$f=\rho^2+\varphi^2+\theta^2$

$\Rightarrow\frac{\partial u}{\partial\varphi}=\frac{\partial f}{\partial\rho}\cdot\frac{\partial\rho}{\partial\varphi}+\frac{\partial f}{\partial\varphi}=2\tan\varphi\theta\cdot\sec^2\varphi\theta\cdot\theta+2\varphi\qquad;\frac{\partial u}{\partial\theta}=\frac{\partial f}{\partial\rho}\cdot\frac{\partial\rho}{\partial\theta}+\frac{\partial f}{\partial\theta}=2\tan\varphi\theta\cdot\sec^2\varphi\theta\cdot\varphi+2\vartheta$

2.

(1)

$\frac{\mathrm{du}}{\mathrm{dt}}=f_1'\cdot\frac{\mathrm{dx}}{\mathrm{dt}}+f_2'\cdot\frac{\mathrm{dy}}{\mathrm{dt}}=3t^2f_1'+4\mathrm{tf}_2'$

(2)

$\frac{\mathrm{du}}{\mathrm{dt}}=f_1'\cdot\frac{\mathrm{dx}}{\mathrm{dt}}+f_2'\cdot\frac{\mathrm{dy}}{\mathrm{dt}}+f_3'\cdot\frac{\mathrm{dz}}{\mathrm{dt}}=f_1'\cdot\mathrm{cost}-f_2'\cdot\mathrm{sint}+f_3'\cdot e^t$

(3)

$\frac{\partial u}{\partial x}=f_1'\cdot 2x+f_2'\cdot y\cdot e^{xy};$

$\frac{\partial^2u}{\partial x\partial y}=\frac{\partial^2u}{\partial y\partial x}=\frac{\partial}{\partial y}(2xf_1'+y\cdot e^{xy}f_2')=-4\mathrm{xy}\,f_{11}''+(2x^2-2y^2)\,e^{xy}f_{12}''+x\mathrm{ye}^{2xy}f_{22}''+(1+xy)\,e^{xy}f_1'$

(4)

$\frac{\partial u}{\partial x}=f_1'+2xf_2'$

$\frac{\partial^2u}{\partial x^2}=f_{11}''+4x^2f_{22}''+4xf_{12}''+2\,f_2'$

$\frac{\partial^2u}{\partial x\partial y}=\frac{\partial^2u}{\partial y\partial x}=\frac{\partial}{\partial y}(f_1'+2xf_2')=f_{11}''+(2x+2y)\,f_{12}''+4xyf_{22}''$

(5)

$\frac{\partial u}{\partial x}=\frac{1}{y}f_1'\qquad;\frac{\partial u}{\partial y}=-\frac{x}{y^2}f_1'+\frac{1}{z}f_2'\qquad;\frac{\partial u}{\partial z}=-\frac{y}{z^2}f_2'$

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{1}{y}f_1'\right)=-\frac{1}{y^2}f_1'-\frac{x}{y^3}f_{11}''+\frac{1}{\mathrm{yz}}f_{12}''$

$\frac{\partial^2 u}{\partial y\partial z}=\frac{\partial}{\partial y}\left(-\frac{y}{z^2}f_2'\right)=-\frac{1}{z^2}f_2'+\frac{x}{z^2y}f_{12}''-\frac{y}{z^3}f_{22}''$

$\frac{\partial^3 u}{\partial x\partial y\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\partial z}\right)=-\frac{1}{z^2}\cdot\frac{1}{y}f_{12}''+\frac{1}{z^2y}f_{12}''+\frac{x}{z^2y}\cdot\frac{1}{y}f_{112}'''-\frac{y}{z^3}\cdot\frac{1}{y}f_{122}'''=\frac{x}{z^2y^2}f_{112}'''-\frac{1}{z^3}f_{122}'''$

(6)

$\frac{\partial u}{\partial x}=f_1'+yf_2'+yzf_3'$ ；$\frac{\partial u}{\partial y}=xf_2'+xzf_3'$ ；$\frac{\partial u}{\partial z}=xyf_3'$

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=f_{11}''+2yf_{12}''+2yzf_{13}''+2y^2zf_{23}''+y^2f_{22}''+y^2z^2f_{33}''$

同理可得：$\frac{\partial^2 u}{\partial y^2}=x^2f_{22}''+2x^2zf_{23}''+x^2z^2f_{33}''$ ；$\frac{\partial^2 u}{\partial z^2}=x^2y^2f_{33}''$

$\frac{\partial^2 u}{\partial x\partial y}=f_2'+zf_3'+xf_{12}''+xyf_{22}''+2xyzf_{23}''+xzf_{13}''+xyz^2f_{33}''$

3.解

$\frac{\partial u}{\partial x}=f'\cdot(y\varphi_1'+\varphi_2')=yf'\varphi_1'+f'\varphi_2'$

$\frac{\partial u}{\partial y}=f'\cdot(x\varphi_1'+\varphi_2')=xf'\varphi_1'+f'\varphi_2'$

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=f'\varphi_1'+xyf''(\varphi_1')^2+xf''\varphi_2'\varphi_1'+xyf'\varphi_{11}''+xf'\varphi_{21}''+f''y\varphi_1'\varphi_2'+f''(\varphi_2')^2+f'\varphi_{22}''$

4.解

$\frac{\partial z}{\partial x}=f'y$ ；$\frac{\partial z}{\partial y}=f'x$

$\Rightarrow x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=xyf'-xyf'=0$

5.解

$\frac{\partial z}{\partial x}=f'\cdot\frac{1}{x}$ ；$\frac{\partial z}{\partial y}=-\frac{1}{y^2}f'$

$\Rightarrow x\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=f'-f'=0$

6.解

$\frac{\partial u}{\partial x}=\varphi+x\cdot\varphi'+y\psi'$

$\frac{\partial u}{\partial y}=x\cdot\varphi'+\psi+y.\psi'$

$\frac{\partial^2 u}{\partial x^2}=2\varphi'+x\varphi''+y\psi''$

$\frac{\partial^2 u}{\partial y^2}=2\psi'+x\varphi''+y\psi''$

$\frac{\partial^2 u}{\partial x\partial y}=\varphi'+x\varphi''+\psi'+y\psi''$

$\Rightarrow\frac{\mathrm{d}^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$

7.

$\frac{\partial u}{\partial r}=-e^{a\theta}\sin(a\ln^{r})\cdot\frac{a}{r}$

$\frac{\partial u}{\partial \theta}=a\cdot e^{a\theta}\cos(a\ln^{r})$

$\frac{\partial^2 u}{\partial r^2}=\frac{a}{r^2}\cdot e^{a\theta}\sin(a\ln^{r})-\frac{a^2}{r^2}e^{a\theta}\cos(a\ln^{r})$

$\frac{\partial^2 u}{\partial \theta^2}=a^2e^{a\theta}\cos(a\ln^{r})$

$\Rightarrow\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0$

8.

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}=\frac{\partial u}{\partial \xi}+3\frac{\partial u}{\partial \eta};$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta};$

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial \xi^2}+6\frac{\partial^2 u}{\partial \xi\partial \eta}+9\frac{\partial^2 u}{\partial \eta^2};$

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}-2\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{\partial^2 u}{\partial \eta^2};$

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial \xi^2}+2\frac{\partial^2 u}{\partial \xi\partial \eta}-3\frac{\partial^2 u}{\partial \eta^2};$

$\Rightarrow\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=16\frac{\partial^2 u}{\partial \xi\partial \eta}+8\frac{\partial u}{\partial \xi}=0;$

$\Rightarrow\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{1}{2}\frac{\partial u}{\partial \xi}=0$

9.和第8题比较类似

$\frac{\partial u}{\partial x}=(1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta};$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta};$

$\frac{\partial^2 u}{\partial x^2}=\sin x\frac{\partial u}{\partial \xi}-\sin x\frac{\partial u}{\partial \eta}+(1-\cos x)^2\frac{\partial^2 u}{\partial \xi^2}+2(1-\cos^2 x)\frac{\partial^2 u}{\partial \xi\partial \eta}+(1+\cos x)^2\frac{\partial^2 u}{\partial \eta^2}$

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}-2\frac{\partial^2 u}{\partial \xi\partial \eta}$

$\frac{\partial^2 u}{\partial x\partial y}=(1-\cos x)\frac{\partial^2 u}{\partial \xi^2}+2\cos x\frac{\partial^2 u}{\partial \xi\partial \eta}-(1+\cos x)\frac{\partial^2 u}{\partial \eta^2}$

$\Rightarrow\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}\cos x-\frac{\partial^2 u}{\partial y^2}\sin^2 x-\frac{\partial u}{\partial y}\sin x=4\frac{\partial^2 u}{\partial \xi\partial \eta}=0$

$\Rightarrow\frac{\partial^2 u}{\partial \xi\partial \eta}=0$

10.

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\quad;\frac{\partial z}{\partial y}=-2\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}$

$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}+2\frac{\partial^2 z}{\partial v\partial u}\qquad;\frac{\partial^2 z}{\partial y^2}=4\frac{\partial^2 z}{\partial u^2}-4a\frac{\partial^2 z}{\partial u\partial v}+a^2\frac{\partial^2 z}{\partial v^2}\qquad;\frac{\partial^2 z}{\partial x\partial y}=-2\frac{\partial^2 z}{\partial u^2}+(a-2)\frac{\partial^2 z}{\partial x\partial v}+a\frac{\partial^2 z}{\partial v^2}$

$\Rightarrow6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=(5a+10)\frac{\partial^2 z}{\partial u^2}+(6+a-a^2)\frac{\partial^2 z}{\partial v^2}=0$

由题意知：

$$\begin{cases} 5a+10\neq 0 \\ 6+a-a^2=0 \end{cases}$$

$\Rightarrow a=3$

11.证明：

必要性：

$\because f(tx,ty,tz)=t^k f(x,y,z)\,(t>0)$

两边对t求导

$xf'_{tx}(tx,ty,tz)+yf'_{ty}(tx,ty,tz)+zf'_{tz}(tx,ty,tz)=kt^{k-1}f(x,y,z)$

$\Rightarrow txf'_{tx}(tx,ty,tz)+tyf'_{ty}(tx,ty,tz)+tzf'_{tz}(tx,ty,tz)=kt^k f(x,y,z)=kf(tx.ty.tz)$

$\Rightarrow xf'_x(x,y,z)+yf'_y(x,y,z)+zf'_z(x,y,z)=kf(x,y,z)$

下证充分性:

令$\varphi(t)=\frac{f(tx,ty,tz)}{t^k}$

则$\varphi'(t)=\frac{(xf'_{tx}+yf'_{ty}+zf'_{tz})t^k-kt^{k-1}f(tx,ty,tz)}{t^{2k}}=\frac{(txf'_{tx}+tyf'_{ty}+zt\,f'_{tz})t^k-kt^k f(tx,ty,tz)}{t^{2k+1}}=0$

$\Rightarrow \varphi(t)=\varphi(1)$

$\Rightarrow f(x,y,z)=\frac{f(tx,ty,tz)}{t^k}$

12.解

$\because \frac{\partial z}{\partial y}=x^2+2y$

$\Rightarrow z=x^2y+y^2+f(x)$

$\because z(x,x^2)=1=2x^4+f(x)$

$\Rightarrow f(x)=1-2x^4$

$\Rightarrow z=x^2y+y^2+1-2x^4$

13.解

$\because \frac{\partial u}{\partial x}=x$

$\Rightarrow u=\frac{1}{2}x^2+\varphi(y)$

$\because u=\frac{1}{2}x^2+\varphi(x^2)=1$

$\Rightarrow \varphi(y)=1-\frac{1}{2}y$

$\Rightarrow u=\frac{1}{2}x^2-\frac{1}{2}y+1$

$\Rightarrow \frac{\partial u}{\partial y}=\frac{1}{2}$

14.解

$\because u(x,2x)=x$

两边同时对x求导

$u'_x(x,2x)+2u'_y(x,2x)=1$

$\Rightarrow u'_y(x,2x)=\frac{1}{2}(1-x^2)$

$\Rightarrow u''_{xy}(x,2x)+2u''_{yy}(x,2x)=-x \qquad ;u''_{xx}(x,2x)+2u''_{xy}(x,2x)=2x$

$\because u''_{xx}=u''_{yy}$

$\Rightarrow u''_{xx}=u''_{yy}=-\frac{4}{3}x \quad ;u''_{\mathrm{xy}}=\frac{5}{3}x$

15.

(1)

$du=f'\cdot(dx+dy)$

(2)

$du=f'_1d\xi+f'_2d\eta=f'_1(ydx+xdy)+f'_2\left(\frac{1}{y}dx-\frac{x}{y^2}dy\right)$

$\Rightarrow du=\left(yf'_1+\frac{1}{y}f'_2\right)dx+\left(xf'_1-\frac{x}{y^2}f'_2\right)dy$

(3)

$du=f'_1dx+f'_2dy+f'_3dz=f'_1dt+2tf'_2dt+3t^2f'_3dt=(f'_1+2tf'_2+3t^2f'_3)dt$

(4)

$d\xi=2xdx+2ydy \quad ;d\eta=2xdx-2ydy \quad ;d\zeta=2ydx+2xdy$

$du=f'_1d\xi+f'_2d\eta+f'_3d\zeta=(2xf'_1+2xf'_2+2yf'_3)dx+(2yf'_1-2yf'_2+2xf'_3)dy$

(完)