6.3习题答案

1.

(1)

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}=\left(e^t+\frac{2t}{1+(1+t^2)^2}\right)\cdot y\cdot x^{y-1}=y\cdot x^{y-1}\left(e^{x^y}+\frac{2x^y}{1+(1+x^{2y})^2}\right)$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=\left(e^t+\frac{2t}{1+(1+t^2)^2}\right)\cdot x^y\cdot\ln^x=x^y\cdot\ln^x\left(e^{x^y}+\frac{2x^y}{1+(1+x^{2y})^2}\right)$

(2)

$u=e^{r^2z}=e^{r^{(s+2)}}$

$\Rightarrow\frac{\partial u}{\partial r}=e^{r^{(s+2)}}\cdot(s+2)\cdot r^{(s+1)}\quad;\frac{\partial u}{\partial s}=e^{r^{(s+2)}}\cdot r^{(s+2)}\cdot\ln^r$

(3)

$\frac{\mathrm{du}}{\mathrm{dx}}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}=\frac{a\,e^{ax}(y-z)}{a^2+1}+\frac{e^{ax}}{a^2+1}\cdot a\,\mathrm{cosx}+\frac{e^{ax}}{a^2+1}\cdot\mathrm{sinx}=e^{ax}\mathrm{sinx}$

(4)

设$f=\rho^2+\varphi^2+\theta^2$

$\Rightarrow\frac{\partial u}{\partial\varphi}=\frac{\partial f}{\partial\rho}\cdot\frac{\partial\rho}{\partial\varphi}+\frac{\partial f}{\partial\varphi}=2\tan\varphi\theta\cdot\sec^2\varphi\theta\cdot\theta+2\varphi\qquad;\frac{\partial u}{\partial\theta}=\frac{\partial f}{\partial\rho}\cdot\frac{\partial\rho}{\partial\theta}+\frac{\partial f}{\partial\theta}=2\tan\varphi\theta\cdot\sec^2\varphi\theta\cdot\varphi+2\vartheta$

2.

(1)

$\frac{\mathrm{du}}{\mathrm{dt}}=f_1'\cdot\frac{\mathrm{dx}}{\mathrm{dt}}+f_2'\cdot\frac{\mathrm{dy}}{\mathrm{dt}}=3t^2f_1'+4\mathrm{tf}_2'$

(2)

$\frac{\mathrm{du}}{\mathrm{dt}}=f_1'\cdot\frac{\mathrm{dx}}{\mathrm{dt}}+f_2'\cdot\frac{\mathrm{dy}}{\mathrm{dt}}+f_3'\cdot\frac{\mathrm{dz}}{\mathrm{dt}}=f_1'\cdot\mathrm{cost}-f_2'\cdot\mathrm{sint}+f_3'\cdot e^t$

(3)

$\frac{\partial u}{\partial x}=f_1'\cdot 2x+f_2'\cdot y\cdot e^{xy};$

$\frac{\partial^2u}{\partial x\partial y}=\frac{\partial^2u}{\partial y\partial x}=\frac{\partial}{\partial y}(2x\,f_1'+y\cdot e^{xy}\,f_2')=-4\mathrm{xy}\;f_{11}''+(2x^2-2y^2)\,e^{xy}\,f_{12}''+x\,\mathrm{y}\mathrm{e}^{2xy}f_{22}''+(1+xy)\,e^{xy}\,f_2'$

(4)

$\frac{\partial u}{\partial x}=f_1'+2x\,f_2'$

$\frac{\partial^2u}{\partial x^2}=f_{11}''+4x^2f_{22}''+4x\,f_{12}''+2\,f_2'$

$\frac{\partial^2u}{\partial x\partial y}=\frac{\partial^2u}{\partial y\partial x}=\frac{\partial}{\partial y}(f_1'+2x\,f_2')=f_{11}''+(2x+2y)\,f_{12}''+4xy\,f_{22}''$

(5)

$\frac{\partial u}{\partial x}=\frac{1}{y}f_1'\qquad;\frac{\partial u}{\partial y}=-\frac{x}{y^2}\,f_1'+\frac{1}{z}f_2'\qquad;\frac{\partial u}{\partial z}=-\frac{y}{z^2}f_2'$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y} f_1'\right) = -\frac{1}{y^2} f_1' - \frac{x}{y^3} f_{11}'' + \frac{1}{\mathrm{yz}} f_{12}''$

$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(-\frac{y}{z^2} f_2'\right) = -\frac{1}{z^2} f_2' + \frac{x}{z^2 y} f_{12}'' - \frac{y}{z^3} f_{22}''$

$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = -\frac{1}{z^2} \cdot \frac{1}{y} f_{12}'' + \frac{1}{z^2 y} f_{12}'' + \frac{x}{z^2 y} \cdot \frac{1}{y} f_{112}''' - \frac{y}{z^3} \cdot \frac{1}{y} f_{122}''' = \frac{x}{z^2 y^2} f_{112}''' - \frac{1}{z^3} f_{122}'''$

(6)

$\frac{\partial u}{\partial x} = f_1' + y f_2' + y z f_3' \qquad ; \frac{\partial u}{\partial y} = x f_2' + x z f_3' \qquad ; \frac{\partial u}{\partial z} = x y f_3'$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = f_{11}'' + 2 y f_{12}'' + 2 y z f_{13}'' + 2 y^2 z f_{23}'' + y^2 f_{22}'' + y^2 z^2 f_{33}''$

同理可得：$\frac{\partial^2 u}{\partial y^2} = x^2 f_{22}'' + 2 x^2 z f_{23}'' + x^2 z^2 f_{33}'' \qquad ; \frac{\partial^2 u}{\partial z^2} = x^2 y^2 f_{33}''$

$\frac{\partial^2 u}{\partial x \partial y} = f_2' + z f_3' + x f_{12}'' + x y f_{22}'' + 2 x y z f_{23}'' + x z f_{13}'' + x y z^2 f_{33}''$

3.解

$\frac{\partial u}{\partial x} = f' \cdot (y \varphi_1' + \varphi_2') = y f' \varphi_1' + f' \varphi_2'$

$\frac{\partial u}{\partial y} = f' \cdot (x \varphi_1' + \varphi_2') = x f' \varphi_1' + f' \varphi_2'$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = f' \varphi_1' + x y f'' (\varphi_1')^2 + x f'' \varphi_2' \varphi_1' + x y f' \varphi_{11}'' + x f' \varphi_{21}'' + f'' y \varphi_1' \varphi_2' + f'' (\varphi_2')^2 + f' \varphi_{22}''$

4.解

$\frac{\partial z}{\partial x} = f' y \qquad ; \frac{\partial z}{\partial y} = f' x$

$\Rightarrow x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = x y f' - x y f' = 0$

5.解

$\frac{\partial z}{\partial x} = f' \cdot \frac{1}{x} \qquad ; \frac{\partial z}{\partial y} = -\frac{1}{y^2} f'$

$\Rightarrow x \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = f' - f' = 0$

6.解

$\frac{\partial u}{\partial x} = \varphi + x \cdot \varphi' + y \psi'$

$\frac{\partial u}{\partial y} = x \cdot \varphi' + \psi + y . \psi'$

$\frac{\partial^2 u}{\partial x^2} = 2 \varphi' + x \varphi'' + y \psi''$

$\frac{\partial^2 u}{\partial y^2} = 2 \psi' + x \varphi'' + y \psi''$

$\frac{\partial^2 u}{\partial x \partial y} = \varphi' + x \varphi'' + \psi' + y \psi''$

$\Rightarrow \frac{\mathrm{d}^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

7.

$\frac{\partial u}{\partial r} = -e^{a\theta}\sin(a\ln^r)\cdot\frac{a}{r}$

$\frac{\partial u}{\partial \theta} = a\cdot e^{a\theta}\cos(a\ln^r)$

$\frac{\partial^2 u}{\partial r^2} = \frac{a}{r^2}\cdot e^{a\theta}\sin(a\ln^r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\ln^r)$

$\frac{\partial^2 u}{\partial \theta^2} = a^2 e^{a\theta}\cos(a\ln^r)$

$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$

8.

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + 3\frac{\partial u}{\partial \eta};$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta};$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 6\frac{\partial^2 u}{\partial \xi\partial \eta} + 9\frac{\partial^2 u}{\partial \eta^2};$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi\partial \eta} + \frac{\partial^2 u}{\partial \eta^2};$

$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\partial \eta} - 3\frac{\partial^2 u}{\partial \eta^2};$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 16\frac{\partial^2 u}{\partial \xi\partial \eta} + 8\frac{\partial u}{\partial \xi} = 0;$

$\Rightarrow \frac{\partial^2 u}{\partial \xi\partial \eta} + \frac{1}{2}\frac{\partial u}{\partial \xi} = 0$

9.和第8题比较类似

$\frac{\partial u}{\partial x} = (1-\cos x)\frac{\partial u}{\partial \xi} + (1+\cos x)\frac{\partial u}{\partial \eta};$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta};$

$\frac{\partial^2 u}{\partial x^2} = \sin x\frac{\partial u}{\partial \xi} - \sin x\frac{\partial u}{\partial \eta} + (1-\cos x)^2\frac{\partial^2 u}{\partial \xi^2} + 2(1-\cos^2 x)\frac{\partial^2 u}{\partial \xi\partial \eta} + (1+\cos x)^2\frac{\partial^2 u}{\partial \eta^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \xi\partial \eta}$

$\frac{\partial^2 u}{\partial x\partial y} = (1-\cos x)\frac{\partial^2 u}{\partial \xi^2} + 2\cos x\frac{\partial^2 u}{\partial \xi\partial \eta} - (1+\cos x)\frac{\partial^2 u}{\partial \eta^2}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y}\cos x - \frac{\partial^2 u}{\partial y^2}\sin^2 x - \frac{\partial u}{\partial y}\sin x = 4\frac{\partial^2 u}{\partial \xi\partial \eta} = 0$

$\Rightarrow \frac{\partial^2 u}{\partial \xi\partial \eta} = 0$

10.

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\quad;\frac{\partial z}{\partial y} = -2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial v\partial u}\qquad;\frac{\partial^2 z}{\partial y^2} = 4\frac{\partial^2 z}{\partial u^2} - 4a\frac{\partial^2 z}{\partial u\partial v} + a^2\frac{\partial^2 z}{\partial v^2}\qquad;\frac{\partial^2 z}{\partial x\partial y} = -2\frac{\partial^2 z}{\partial u^2} + (a-2)\frac{\partial^2 z}{\partial x\partial v} + a\frac{\partial^2 z}{\partial v^2}$

$\Rightarrow 6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y} - \frac{\partial^2 z}{\partial y^2} = (5a+10)\frac{\partial^2 z}{\partial u\partial v} + (6+a-a^2)\frac{\partial^2 z}{\partial v^2} = 0$

由题意知：

$\begin{cases} 5a+10\neq 0 \\ 6+a-a^2=0 \end{cases}$

$\Rightarrow a=3$

11.证明：

必要性：

$\because f(tx,ty,tz)=t^k f(x,y,z)\,(t>0)$

两边对t求导

$xf'_{tx}(tx,ty,tz)+yf'_{ty}(tx,ty,tz)+zf'_{tz}(tx,ty,tz)=kt^{k-1}f(x,y,z)$

$\Rightarrow txf'_{tx}(tx,ty,tz)+tyf'_{ty}(tx,ty,tz)+tzf'_{tz}(tx,ty,tz)=kt^k f(x,y,z)=kf(tx.ty.tz)$

$\Rightarrow xf'_x(x,y,z)+yf'_y(x,y,z)+zf'_z(x,y,z)=kf(x,y,z)$

下证充分性:

令$\varphi(t)=\frac{f(tx,ty,tz)}{t^k}$

则$\varphi'(t)=\frac{(xf'_{tx}+yf'_{ty}+zf'_{tz})t^k-kt^{k-1}f(tx,ty,tz)}{t^{2k}}=\frac{(txf'_{tx}+tyf'_{ty}+zt\,f'_{tz})t^k-kt^kf(tx,ty,tz)}{t^{2k+1}}=0$

$\Rightarrow\varphi(t)=\varphi(1)$

$\Rightarrow f(x,y,z)=\frac{f(tx,ty,tz)}{t^k}$

12.解

$\because\frac{\partial z}{\partial y}=x^2+2y$

$\Rightarrow z=x^2y+y^2+f(x)$

$\because z(x,x^2)=1=2x^4+f(x)$

$\Rightarrow f(x)=1-2x^4$

$\Rightarrow z=x^2y+y^2+1-2x^4$

13.解

$\because\frac{\partial u}{\partial x}=x$

$\Rightarrow u=\frac{1}{2}x^2+\varphi(y)$

$\because u=\frac{1}{2}x^2+\varphi(x^2)=1$

$\Rightarrow\varphi(y)=1-\frac{1}{2}y$

$\Rightarrow u=\frac{1}{2}x^2-\frac{1}{2}y+1$

$\Rightarrow\frac{\partial u}{\partial y}=\frac{1}{2}$

14.解

$\because u(x,2x)=x$

两边同时对x求导

$u'_x(x,2x)+2u'_y(x,2x)=1$

$\Rightarrow u'_y(x,2x)=\frac{1}{2}(1-x^2)$

$\Rightarrow u''_{xy}(x,2x)+2u''_{yy}(x,2x)=-x \qquad ;u''_{xx}(x,2x)+2u''_{xy}(x,2x)=2x$

$\because u''_{xx}=u''_{yy}$

$\Rightarrow u''_{xx}=u''_{yy}=-\frac{4}{3}x \quad ;u''_{\mathrm{xy}}=\frac{5}{3}x$

15.

(1)

$du=f'\cdot(dx+dy)$

(2)

$du=f'_1d\xi+f'_2d\eta=f'_1(ydx+xdy)+f'_2\left(\frac{1}{y}dx-\frac{x}{y^2}dy\right)$

$\Rightarrow du=\left(yf'_1+\frac{1}{y}f'_2\right)dx+\left(xf'_1-\frac{x}{y^2}f'_2\right)dy$

(3)

$du=f'_1dx+f'_2dy+f'_3dz=f'_1dt+2tf'_2dt+3t^2f'_3dt=(f'_1+2tf'_2+3t^2f'_3)dt$

(4)

$d\xi=2xdx+2ydy \quad ;d\eta=2xdx-2ydy \quad ;d\zeta=2ydx+2xdy$

$du=f'_1d\xi+f'_2d\eta+f'_3d\zeta=(2xf'_1+2xf'_2+2yf'_3)dx+(2yf'_1-2yf'_2+2xf'_3)dy$

(完)