多变量6.5节答案

1.(1) $f(5+k,6+k)=(5+h)^3+(6+k)^2-6(5+h)(6+k)-39(5+h)+18(6+k)+4$

$f(5,6)=-102$

$\Delta f=f(5+k,6+k)-f(5,6)=h^3+15h^2+k^2-6hk$

(2) $f(1+h,-1+k)=(1+h)^2(-1+k)+(1+h)(-1+k)^2-2(1+h)(-1+k)$

$f(1,-1)=2$

$\Delta f=h-3k-h^2-2hk+k^2+h^2k+hk^2$

2.$f'_x(1,-2)=f'_y(1,-2)=0, f''_{xx}(1,-2)=4, f''_{yy}(1,-2)=-2, f''_{xy}(1,-2)=-1$

$f(1,-2)=5$

$f(x,y)=2(x-1)^2-(y+2)^2-(x-1)(y+2)+5$

3.(1) $f'_x(0,0)=f''_{xx}(0,0)=f''_{xx}(0,0)=0, f'_y(0,0)=1, f''_{yy}(0,0)=-1, f'''_{yyy}(0,0)=2$

$(0,0)$处$\dfrac{\partial^2 f}{\partial x\partial y}=1, \dfrac{\partial^3 f}{\partial x^2\partial y}=1, \dfrac{\partial^3 f}{\partial x\partial y^2}=-1, f(0,0)=0$

$\Rightarrow f(x,y)=y+\dfrac{1}{2}(2xy-y^2)+\dfrac{1}{6}(3x^2y-3xy^2+y^3)+R_3$

(2)一元函数泰勒展开$\sqrt{1-t}=1-\dfrac{1}{2}t-\dfrac{1}{8}t^2+R_2(t)$

$\Rightarrow f(x,y)=1-\dfrac{1}{2}(x^2+y^2)-\dfrac{1}{8}(x^2+y^2)^2+R_4$

(3) $\dfrac{1}{1-x-y+xy}=\dfrac{1}{(1-x)(1-y)}=\dfrac{1}{x-y}\left(\dfrac{1}{1-x}-\dfrac{1}{1-y}\right)=\dfrac{\sum_{n=0}^{\infty}(x^n-y^n)}{x-y}$

第n阶为$\dfrac{x^{n+1}-y^{n+1}}{x-y}$

$f(x,y)=1+(x+y)+\cdots+\dfrac{x^{n+1}-y^{n+1}}{x-y}+R_n$

(4) $f(x,y)=e^{x+y}=1+x+y+\cdots+\dfrac{1}{n!}(x+y)^n+R_n$

(5) $f(x,y)=\sin(x^2+y^2)=\sum\limits_{k=0}^{n}\dfrac{(-1)^k(x^2+y^2)^{2k+1}}{(2k+1)!}+R_{4k+2}$,展开到了$4k+2$阶

(6) $\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$处$f'_x=\dfrac{1}{2}, f'_y=\dfrac{1}{2}, f''_{xx}=-\dfrac{1}{2}, f''_{yy}=-\dfrac{1}{2}, f''_{xy}=\dfrac{1}{2}, f\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)=\dfrac{1}{2}$

$\Rightarrow f(x,y)=\dfrac{1}{2}+\dfrac{1}{2}\left(x-\dfrac{\pi}{4}\right)+\dfrac{1}{2}\left(y-\dfrac{\pi}{4}\right)+\dfrac{1}{2}\left[-\dfrac{1}{2}\left(x-\dfrac{\pi}{4}\right)^2+\left(x-\dfrac{\pi}{4}\right)\left(y-\dfrac{\pi}{4}\right)-\dfrac{1}{2}\left(y-\dfrac{\pi}{4}\right)^2\right]+R_2$

(7) $f'_x=yx^{y-1}, f''_{xx}=y(y-1)x^{y-2}, f''_{xy}=x^{y-1}+yx^{y-1}\ln x$

$f'_y=x^y\ln x, f''_{yy}=x^y\ln^2 x$

在$(1,1)$处$f'_x=1, f''_{xx}=0, f''_{xy}=1, f'_y=0, f''_{yy}=0, f(1,1)=1$

$\Rightarrow f(x,y)=1+(x-1)+(x-1)(y-1)+R_2$

4.$3z_x'z^2-2z-2xz_x=0\Rightarrow z_x'=\dfrac{2z}{3z^2-2x}$; $z_x'(1,1)=2$

$3z_y'z^2-2xz_y'+1=0\Rightarrow z_y'=\dfrac{1}{2x-3z^2}$, $z_y'(1,1)=-1$

$z_{xx}''=2\dfrac{z_x'(3z^2-2x)-z(6zz_x'-2)}{(3z-2x)^2}$, $z_{xx}''(1,1)=-16$

$z_{yy}''=\dfrac{6zz_y^{'}}{(3z^2-2x)^2}$, $z_{yy}''(1,1)=-6$

$z_{xy}''=\dfrac{6zz_x'-2}{(3z^2-2x)^2}$, $z_{xy}''(1,1)=10$

$\Rightarrow z(x,y)=1+2(x-1)-(y-1)-8(x-1)^2+5(y-1)^2-6(x-1)(y-1)$

5.(1) $f(x,y)=\dfrac{\cos x}{\cos y}$, 则$f(x,y)$在原点附近无穷可微，且

$f(0,0)=1, f_x'(0,0)=0, f_y'(0,0)=0, f_{xx}''(0,0)=-1, f_{xy}''(0,0)=0, f_{yy}''(0,0)=1$

$\Rightarrow\dfrac{\cos x}{\cos y}=1-\dfrac{1}{2}x^2+\dfrac{1}{2}y^2+o(x^2+y^2)$

当$|x|,|y|$充分小时，$\dfrac{\cos x}{\cos y}\approx 1-\dfrac{1}{2}(x^2-y^2)$

(2) 变量代换$\begin{cases} a=x+y \\ b=x-y \end{cases}$

一元函数泰勒展开$\arctan(1+x)\approx\dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2$

$\Rightarrow \arctan\left(\dfrac{1+a}{1-b}\right)=\arctan[(1+a)(1+b+b^2+\cdots+b^n)]$

$=\arctan(1+a+b+ab+b^2+R_2)$

$\approx\dfrac{\pi}{4}+\dfrac{1}{2}(a+b+\mathrm{ab}+b^2)-\dfrac{1}{4}(a+b)^2$

$=\dfrac{\pi}{4}+\dfrac{1}{2}(2x+x^2-y^2+x^2+y^2-2xy)-x^2$

$=\dfrac{\pi}{4}+x-xy$

6.(1) 驻点方程$\begin{cases} f_x'=4-2x=0 \\ f_y'=-4-2y=0 \end{cases}\Rightarrow$ 解得驻点$(2,-2)$

在驻点处二阶偏导$A=f_{xx}''=-2, C=f_{yy}''=-2, B=f_{xy}^{''}=0, \mathrm{AC}-B^2>0$

$\Rightarrow(2,-2)$处有极大值$f(-2,-2)=8$

(2) 驻点方程$\begin{cases} y-\dfrac{50}{x^2}=0 \\ x-\dfrac{20}{y^2}=0 \end{cases}\Rightarrow$ 驻点$(5,2)$

$A=f_{xx}''(5,2)=\dfrac{4}{5}>0, C=f_{yy}''(5,2)=5, B=f_{xy}''(5,2)=1, \mathrm{AC}-B^2=3>0$

在$(5,2)$处有极小值$f(5,2)=30$

(3)驻点方程$\left\{\begin{array}{c} 2e^{2x}(x+2y+y^2)+e^{2x}=0 \\ e^{2x}(x+2y)=0 \end{array}\right.\Rightarrow$驻点$\left(\frac{1}{2},1\right)$

$A=f''_{xx}\left(\frac{1}{2},-1\right)=2e, C=f''_{yy}\left(\frac{1}{2},-1\right)=2e, B=f''_{xy}\left(\frac{1}{2},-1\right)=e, \mathrm{AC}-B^2>0$

$\left(\frac{1}{2},1\right)$处有极小值$f\left(\frac{1}{2},1\right)=-\frac{1}{2}e$

(4)驻点方程$\left\{\begin{array}{l} \sqrt{1+y}+\dfrac{y}{2\sqrt{1+x}}=0 \\ \dfrac{x}{2\sqrt{1+y}}+\sqrt{1+x}=0 \end{array}\right.\Rightarrow$驻点$\left(-\frac{2}{3},-\frac{2}{3}\right)$

$A=f''_{xx}\left(-\frac{2}{3},-\frac{2}{3}\right)=\frac{\sqrt{3}}{2}, C=f''_{yy}\left(-\frac{2}{3},-\frac{2}{3}\right)=\frac{\sqrt{3}}{2}, B=f''_{xy}\left(-\frac{2}{3},-\frac{2}{3}\right)=\sqrt{3}$

$\mathrm{AC}-B^2<0$, 故不存在极值

7.(1)设$F(x,y)=(x^2+y^2)-a^2(x^2-y^2)$则$F_x=4x(x^2+y^2)-2ax, F_y=4y(x^2+y^2)+2a^2y.$

$\frac{dy}{dx}=-\frac{4x(x^2+y^2)-2a^2x}{4y(x^2+y^2)+2a^2y}=0$

解得$x=0$或$y^2=\frac{a^2}{2}-x^2$

$x=0$代入原方程得$y=0$, 此时$F_y=0$, 故$x=0$舍弃

$y^2=\frac{a^2}{2}-x^2$代入原方程解得驻点$\left(\pm\sqrt{\frac{3}{8}}a,\sqrt{\frac{1}{8}}a\right)$, $\left(\pm\sqrt{\frac{3}{8}}a,-\sqrt{\frac{1}{8}}a\right)$

$y''=-\frac{1}{[2y(x^2+y^2)+a^2y]^2}\{[2y(x^2+y^2)+a^2y](6x^2+2y^2+4xyy'-a^2)-[2x(x^2+y^2-a^2x)](4xy+2x^2y'+6y^2y'+a^2y')\}$

在驻点处$x^2+y^2=\frac{a^2}{2}, y'=0\Rightarrow y''=-\frac{2x^2}{a^2y}$, 可以看出$y''$与$y$异号

在$\left(\pm\sqrt{\frac{3}{8}}a,\sqrt{\frac{1}{8}}a\right)$处$y''<0$, 有极大值，在$\left(\pm\sqrt{\frac{3}{8}}a,-\sqrt{\frac{1}{8}}a\right)$处$y''>0$，有极小值

(2)两边对x求偏导　$4x+2zz_x'+8z+8xz_x'-z_x'=0$

对y求偏导　$4y+2zz_y'+8xz_y'-z_y'=0$

令$z_x'=z_y'=0$得$y=0, x=-2z$, 代入原方程得$8z^2+z^2-16z^2-z+8=0$

$\Rightarrow z=-\frac{8}{7}, z=1$, 故$x=\frac{16}{7}$或$-2$，驻点为$\left(\frac{16}{7},0\right)$或$(-2,0)$

对上面两个偏导数求偏导得　$4+2(z_x')^2+2zz''_{xx}+8z_x'+8z_x'+8xz''_{xx}-z''_{xx}=0$

$4+2(z_y')^2+2zz''_{yy}+8xz''_{yy}-z''_{yy}=0$

$2z_x'z_y'+2zz''_{xy}+8z_y'+8xz''_{yy}-z''_{xy}=0$

$\left(\dfrac{16}{7},0\right)$处$A=z''_{xx}=-\dfrac{4}{15},B=z''_{xy}=0,C=z''_{yy}=-\dfrac{4}{15},\mathrm{AC}-B^2>0$

$\Rightarrow z(x,y)$在$\left(\dfrac{16}{7},0\right)$取得极大值$-\dfrac{8}{7}$

$(-2,0)$处$A=z''_{xx}=\dfrac{4}{15},B=z''_{xy}=0,C=z''_{yy}=\dfrac{4}{15},\mathrm{AC}-B^2>0$

$\Rightarrow z(x,y)$在$(-2,0)$处取得极小值1

(3)配方得$(x-1)^2+(y+1)^2+(z-2)^2=4^2$,这是一个球面，球心在$(1,-1,2)$，半径是4，由几何意义，$z$在$(1,-1,6)$取得极大值6，在$(1,-1,-2)$有极小值$-2$

8.(1)u物理意义为点到原点距离的平方

故z有极小值$\left|\dfrac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right|^2=\dfrac{a^2b^2}{a^2+b^2}$

$$\begin{cases}\dfrac{x}{a}+\dfrac{y}{b}=1\\ y=\dfrac{b}{a}x\end{cases}\Rightarrow \text{极值点}\left(\frac{ab^2}{a^2+b^2},\frac{a^2b}{a^2+b^2}\right)$$

(2)辅助函数$F(x,y,z,\lambda)=x+y+z+\lambda\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}-1\right)$

$$\begin{cases}F'_x=1-\dfrac{\lambda}{x^2}=0\\ F'_y=1-\dfrac{\lambda}{y^2}=0\\ F'_z=1-\dfrac{\lambda}{z^2}=0\end{cases}\Rightarrow \text{联立}\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1\text{得驻点}(3,3,3)$$

由题意u有极小值$u(3,3,3)=9$

(3)辅助函数$F(x,y,z,\lambda)=\sin x\sin y\sin z+\lambda\left(x+y+z-\dfrac{\pi}{2}\right)$

$$\begin{cases}F'_x=\sin y\sin z\cos x+\lambda=0\\ F'_y=\sin x\sin z\cos y+\lambda=0\\ F'_y=\sin x\sin y\cos z+\lambda=0\end{cases}$$

联立$x+y+z=\dfrac{\pi}{2}$得$x=y=z=\dfrac{\pi}{6}$

由题意有极大值$u\left(\dfrac{\pi}{6},\dfrac{\pi}{6},\dfrac{\pi}{6}\right)=\dfrac{1}{8}$

(4)$z=x-y,u=-xy(x+y)$辅助函数$F(x,y)=-xy(x+y)+\lambda(2x^2+2y^2+2xy-1)$

$F'_x=-2xy-y^2+4\lambda x+2\lambda y=0,F'_y=-x^2-2xy+4\lambda y+2\lambda x=0\Rightarrow x^3=y^3$

联立$2x^2+2y^2+2xy-1=0$得$x=y=\pm\frac{\sqrt{6}}{6}$

驻点为$\left(\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},-\frac{\sqrt{6}}{3}\right),\left(-\frac{\sqrt{6}}{6},-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3}\right)$

在$\left(\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},-\frac{\sqrt{6}}{3}\right)$有极小值$-\frac{\sqrt{6}}{18}$, $\left(-\frac{\sqrt{6}}{6},-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3}\right)$处有极大值$\frac{\sqrt{6}}{18}$

9.(1)解方程组$\left\{\begin{array}{l} z_x'=2x=0 \\ z_y'=-2y=0 \end{array}\right.\Rightarrow$驻点$(0,0)$

$A=z_{xx}''=2, B=z_{xy}''=0, C=z_{yy}''=-2, \mathrm{AC}-B^2<0$, $(0,0)$不是极值点

在边界$x^2+y^2=4$上，$z=2x^2-4\in[-4,4]$

在$(0,\pm2)$处z有最小值，$(\pm2,0)$处z有最大值4

(2) $\left\{\begin{array}{l} z_x'=2x-y=0 \\ z_y'=-x+2y=0 \end{array}\right.\Rightarrow$驻点$(0,0)$

$A=z_{xx}''=0, C=z_{yy}''=2, B=0, \mathrm{AC}-B^2>0$, 在$(0,0)$处$z$有极小值0

边界$x+y=1$上$z=1-3x(1-x)=3x^2-3x+1=3\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\geqslant\frac{1}{4}$

$x-y=1$上$z=1+x(x-1)=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\geqslant\frac{3}{4}$

$x+y=-1$上$z=1+3x(x+1)=3\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\geqslant\frac{1}{4}$

$y-x=1$上$z=1+x(x+1)=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\geqslant\frac{3}{4}$

边界点$(1,0)$ $(0,1)$ $(-1,0)$ $(0,-1)$函数值都等于$1\Rightarrow z_{\max}=1$

$z_{\min}=z(0,0)=0$

(3) $\left\{\begin{array}{l} z_x'=\cos x-\cos(x+y)=0 \\ z_y'=\cos y-\cos(x+y)=0 \end{array}\right.\Rightarrow\cos x=\cos y$

所以$x=y$或$x+y=2\pi$

区域内部$x=y,\Rightarrow\cos x-\cos 2x=0$解得$x=y=\frac{2\pi}{3}$且$z\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)=\frac{3\sqrt{3}}{2}$

边界上函数值均为0，故z在$\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)$取得最大值$\frac{3\sqrt{3}}{2}$, 边界上取得最小值0

(4) $\left\{\begin{array}{l} z_x'=8xy-3x^2y-2xy^2=0 \\ z_y'=4x^2-x^3-2x^2y=0 \end{array}\right.\Rightarrow$内部驻点$(2,1)$

$z(2,1)=4$

边界上$x=0$时$z=0$, $y=0$时$z=0$

$x+y=6$时$z=-2x^2(6-x)=2(x^3-6x^2)$, 由$z'=6x^2-24x=0\Rightarrow x=4$

$x=4$时$z=-64, x=0$时$z=0, x=0$时$z=0$

z的最值只能在上述值中取得$z_{\min}=z(4,2)=-64, z_{\max}=z(2,1)=4$

10.设点的坐标为(x,y,z)

$d^2=(x-1)^2+(y-1)^2+(z-1)^2+(x-2)^2+(y-3)^2+(z-4)^2$

$=2(x^2+y^2+z^2)-6x-8y-10z+32$

$F(x,y,z)=2(x^2+y^2+z^2)-6x-8y-10z+32+\lambda(3x-2z)$

$$\begin{cases} F_x'=4x-6+3\lambda=0 \\ F_y'=4y-8=0 \\ F_z'=4z-10-2\lambda=0 \end{cases} \Rightarrow 联立3x-2z=0得驻点\left(\frac{21}{13},2,\frac{63}{26}\right)$$

由题意最小值一定存在故此驻点就是最小值点

11.由方程组知$x^2+y^2=2, y\in[-\sqrt{2},\sqrt{2}]\Rightarrow z=(2+y^2)\in[2,4]$

最小值2对应点$(\pm\sqrt{2},0,0)$最大值4对应点$(0,\pm\sqrt{2},4)$

12.沿着过原点曲线$y=ax^2, f(x,y)=(3a-2a^2-1)x^4=-(2a-1)(a-1)x^4$

$\frac{1}{2}<a<1$时，$(0,0)$为极小值，$a<\frac{1}{2}, a>1$时$(0,0)$为极大值

故$(0,0)$不是$f(x,y)$的极值点

沿着直线$y=kx, f(x,y)=u(x)=3kx^3-x^4-2k^2x^2$

$u'=6kx^2-4x^3-4k^2x=0\Rightarrow(0,0)$为驻点

$u''(0,0)=-4k^2$

$k\neq0, u''(0,0)<0, (0,0)$为$u$的极大值点

$k=0$时，$u''(0,0)=0$，但$u'''(0,0)=-24<0, (0,0)$为极大值点

沿直线$x=0, f(x,y)=-2y^2\leqslant0, (0,0)$也为极大值点

故沿过$(0,0)$的每一条直线$(0,0)$都是极大值点

13.帐篷体积$V=\pi R^2H+\frac{1}{3}\pi R^2h=V_0$,面积$S=2\pi RH+\frac{1}{2}\cdot2\pi R\sqrt{R^2+h^2}$

引入辅助函数$F(R,H,h,\lambda)=2\pi RH+\pi R\sqrt{R^2+h^2}+\lambda\left(\pi R^2H+\frac{1}{3}\pi R^2h-V_0\right)$

$$\begin{cases} F_R'=2\pi H+\pi\sqrt{h^2+R^2}+\dfrac{\pi R^2}{\sqrt{h^2+R^2}}+\lambda\left(2\pi RH+\dfrac{2}{3}\pi Rh\right)=0 \quad (1) \\ F_H'=2\pi R+\lambda\pi R^2=0 \quad (2) \\ F_h'=\dfrac{\pi \mathrm{Rh}}{\sqrt{h^2+R^2}}+\dfrac{1}{3}\pi R^2\lambda=0 \quad (3) \\ F_\lambda'=\pi R^2H+\dfrac{1}{3}\pi R^2h-V_0=0 \quad (4) \end{cases}$$

由(2)得　$\lambda R=-2$　(5)

由(3)得 $\frac{h}{\sqrt{h^2+R^2}}+\frac{1}{3}R\lambda=0$

(5)代入上式得$\dfrac{h}{\sqrt{h^2+R^2}}=\dfrac{2}{3}\Rightarrow h=\dfrac{2}{\sqrt{5}}R$

(5)代入(1)得$-2H+\dfrac{1}{6}h+\dfrac{2R^2}{3h}=0,$

将$h=\dfrac{2}{\sqrt{5}}R$代入上式得$R=\sqrt{5}H\Rightarrow h=2H$

故$R=\sqrt{5}H$，$h=2H$时所用篷布最省

14.设六面体连接一顶点的三条棱长为x,y,z，则$4(x+y+z)=12a$

三棱长相互垂直时体积最大，即$V=xyz$在$x+y+z=3a$下的条件极值问题

$V=xy(3a-x-y)=3axy-x^2y-xy^2$

$$\begin{cases} V_x'=3ay-2xy-y^2=0 \\ V_y'=3ax-x^2-2xy=0 \end{cases}\Rightarrow x=y=a,\text{此时}z=a$$

由题意$x=y=z=a$时有最大体积a^3

或者$xyz\leqslant\left(\dfrac{x+y+z}{3}\right)^2=a^3$

15.椭圆上一点(x_0,y_0)的切线方程为$\dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1$

设椭圆上一点$P\,(a\cos\theta,b\sin\theta)$,则切线方程为$\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$

与两轴围成面积$S=\dfrac{1}{2}\cdot\dfrac{a}{\cos\theta}\cdot\dfrac{b}{\sin\theta}=\dfrac{ab}{\sin2\theta}$

$\theta=\dfrac{\pi}{4}$时于点$\left(\dfrac{a}{\sqrt{2}},\dfrac{b}{\sqrt{2}}\right)$取得最大值$ab$

16.$u=\displaystyle\sum_{i=1}^{n}\left[(x-x_i)^2+(y-y_i)^2\right]$

另$u_x'=u_y'=0$得驻点$\left(\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}x_i,\dfrac{1}{n}\sum_{i=1}^{n}y_i\right)$

驻点处$A=u_{xx}''=2n,B=u_{xy}''=0,C=u_{yy}''=2n,\mathrm{AC}-B^2>0$，该点有极小值

$\Rightarrow(x_0,y_0)=\left(\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}x_i,\dfrac{1}{n}\sum_{i=1}^{n}y_i\right)$

17.设内接长方体第一卦限内顶点为(x,y,z)

长方体体积$V=8xyz,\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

另$F=8xyz+\lambda\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)$

$$\begin{cases} F'_x = 8yz + \dfrac{2\lambda x}{a^2} = 0 \\ F'_y = 8xz + \dfrac{2\lambda y}{b^2} = 0 \\ F'_z = 8xy + \dfrac{2\lambda y}{c^2} = 0 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \end{cases} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$$

即$x = \dfrac{\sqrt{3}}{3}a, y = \dfrac{\sqrt{3}}{3}b, z = \dfrac{\sqrt{3}}{3}c$

由题意最大长方体体积$V = 8xyz = \dfrac{8\sqrt{3}}{9}abc$

18.设(x, y, z)为椭球面上任意一点，到平面距离

$$d = \frac{|x + y + 2z - 9|}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}|x + y + 2z - 9|$$

$$F = 6d^2 + \lambda(x^2 + 4y^2 + 4z^2 - 4) = (x + y + 2z - 9)^2 + \lambda(x^2 + 4y^2 + 4z^2 - 4)$$

$$\begin{cases} F'_x = 2(x + y + 2z - 9) + 2\lambda x = 0 \\ F'_y = 2(x + y + 2z - 9) + 8\lambda y = 0 \\ F'_z = 4(x + y + 2z - 9) + 8\lambda z = 0 \end{cases} \Rightarrow \begin{cases} x = 4y \\ z = 2y \end{cases}$$

联立$x^2 + 4y^2 + 4z^2 = 4$得驻点$\left(\dfrac{4}{3}, \dfrac{1}{3}, \dfrac{2}{3}\right), \left(-\dfrac{4}{3}, -\dfrac{1}{3}, -\dfrac{2}{3}\right)$

由题意最近点为$\left(\dfrac{4}{3}, \dfrac{1}{3}, \dfrac{2}{3}\right)$，最远点为$\left(-\dfrac{4}{3}, -\dfrac{1}{3}, -\dfrac{2}{3}\right)$