1. (1) $I=\int_0^1 \frac{x^b-x^a}{\ln x}dx=\int_0^1\int_a^b x^t\,dt\,dx=\int_a^b\int_0^1 x^t\,dx\,dt=\int_a^b \frac{1}{t+1}dt=\ln\frac{b+1}{a+1}$

(2) 取 $b=3$、$a=1$，$I=\ln 2$

2. $\iint_D f(x)f(y-x)\,dxdy=\iint\limits_{\substack{0\le x\le 2\\ 0\le y-x\le 2}} \sin x\cdot\sin(y-x)\,dxdy=\int_0^2\left(\int_x^{x+2}\sin(y-x)\,dy\right)\sin x\,dx$

$=\int_0^2(1-\cos 2)\sin x\,dx=(1-\cos 2)^2$

3. 设 $G(t)=\int_0^1\left(e^{\frac{1}{2}f(x)}+te^{-\frac{1}{2}f(x)}\right)^2dx=\int_0^1 e^{f(x)}dx+2t+t^2\int_0^1 e^{-f(x)}dx$

而 $\left(e^{\frac{1}{2}f(x)}+te^{-\frac{1}{2}f(x)}\right)^2\geqslant 0$，故其积分 $\geqslant 0$

$\therefore G(t)\geqslant 0$，故 $\Delta=2^2-4\int_0^1 e^{f(x)}dx\int_0^1 e^{-f(x)}dx\le 0$

4. 由中值定理，$\exists\xi\in(0,1)$ st $g(\xi)=\int_0^1 g(x)\,dx$

$\therefore \int_0^1 f(x)g(x)\,dx-\int_0^1 f(x)\,dx\int_0^1 g(x)\,dx=\int_0^1 f(x)(g(x)-g(\xi))\,dx=\int_0^\xi f(x)(g(x)-g(\xi))\,dx+\int_\xi^1 f(x)(g(x)-g(\xi))\,dx$

$\geqslant f(\xi)\int_0^\xi(g(x)-g(\xi))\,dx+f(\xi)\int_\xi^1(g(x)-g(\xi))\,dx$

$=0$

5. $f(t)=t^4+2\int_0^t\int_0^{2\pi} r^2f(r)\,r\,d\theta\,dr=t^4+4\pi\int_0^t r^3f(r)\,dr\ \cdots$ ①

$\therefore f'(t)=4t^3+4\pi t^3f(t)$，即 $\frac{f'(t)}{1+\pi f(t)}=4t^3$，$\frac{1}{\pi}(\ln(1+\pi f(t)))'=4t^3=(t^4)'$

$\therefore \frac{1}{\pi}\ln(1+\pi f(t))=t^4+C$，即 $1+\pi f(t)=Ce^{\pi t^4}$

① 中取 $t=0$ 得 $f(0)=0$，故 $1+\pi\cdot 0=Ce^{\pi\cdot 0^4}$ 即 $C=1$

$\therefore f(t)=\frac{e^{\pi t^4}-1}{\pi}$

6. $F(t)=\iint\limits_{x^2+y^2\le t^2} f(x^2+y^2)\,dxdy=\int_0^{2\pi}d\theta\int_0^t f(r^2)\,r\,dr=\pi\int_0^t f(r^2)(r^2)'\,dr=\pi\int_0^{t^2}f(u)\,du$

$\therefore F'(t)=\pi f(t^2)$，$F'(0)=\pi$

7. $F(t)=\int_{\Omega_t} z^2\,dV+\int_{\Omega_t} f(x^2+y^2)\,dV=\frac{1}{3}\cdot\pi t^2+\iint\limits_{x^2+y^2\le t^2} f(x^2+y^2)\,dxdy$

$=\frac{\pi}{3}t^2+2\pi\int_0^t f(r^2)\,r\,dr=\frac{\pi}{3}t^2+\pi\int_0^{t^2} f(u)\,du$

$\therefore \frac{F(t)}{t^2}=\frac{\pi}{3}+\pi\frac{1}{t^2}\int_0^{t^2} f(u)\,du$

$\lim\limits_{t\to0}\frac{F(t)}{t^2}=\frac{\pi}{3}+\pi f(0)=\frac{\pi}{3}$

($\lim\limits_{x\to0}\frac{1}{x}\int_0^x f(t)\,dt=f(0)$ ，用洛必达)

8. 由条件，$\frac{\partial}{\partial y}\left(y^2f(x)+2ye^x+2yg(x)\right)=\frac{\partial}{\partial x}\left(2yg(x)+2f(x)\right)$

即 $2yf(x)+2e^x+2g(x)=2yg'(x)+2f'(x)$

即 $y\left(f(x)-g'(x)\right)=f'(x)-g(x)-e^x$

x、y 任意，故 $\begin{cases} f(x)-g'(x)=0 \\ f'(x)-g(x)-e^x=0 \end{cases}$

$\therefore f(x)=g'(x)=(f'(x)-e^x)'=f''(x)-e^x$

解得 $f(x)=Ae^x+Be^{-x}+\frac{1}{2}xe^x$ ，故 $g(x)=f'(x)-e^x=(A-\frac{1}{2})e^x-Be^{-x}+\frac{1}{2}xe^x$

$f(0)=g(0)=0 \Rightarrow A=\frac{1}{4}$ ，$B=-\frac{1}{4}$

$\therefore f(x)=(\frac{1}{4}+\frac{1}{2}x)e^x-\frac{1}{4}e^{-x}$ ，$g(x)=(-\frac{1}{4}+\frac{1}{2}x)e^x+\frac{1}{4}e^{-x}$

取曲线 $\begin{cases} x=t \\ y=t \end{cases}$ $(0\le t\le1)$ ，$\sqrt{x'(t)^2+y'(t)^2}=\sqrt{2}$

则积分 $=\int_0^1\left(t^2f(t)+2te^t+2tg(t)+2tg(t)+2f(t)\right)\sqrt{2}\,dt$

$=\sqrt{2}\int_0^1 \cdots\cdots$ 代入即可

9.