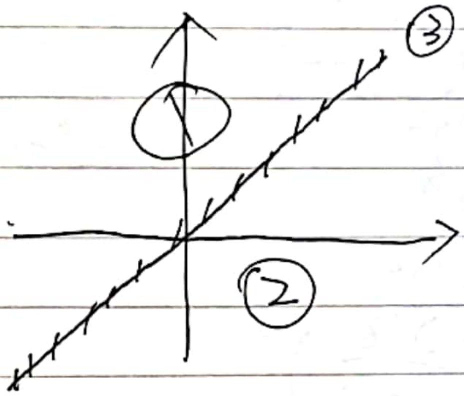
①②

1. 设 $f(x,y)=\begin{cases} \dfrac{e^{x-y}-1}{x-y}, & x>y \\ 1, & x=y \\ \dfrac{\sin x-\sin y}{x-y}, & x<y \end{cases}$

讨论 f 的连续性

可将 E^2 分为两个区域与一个边界，①，②为区域，③为直线 $y=x$



1°. 则对 $\forall (x_0,y_0)\in$ ①，$\exists\, r>0$ 使得 $B((x_0,y_0),r)\subset$ ①

易知 $f(x,y)$ 在 (x_0,y_0) 处连续

同理对 $\forall\,(x_1,y_1)\in$ ②，f 在 (x_1,y_1) 处连续.

2° 下考虑 $(t,t)\in$ ③

$$\lim_{\substack{x\to t\\ y\to t}}\frac{e^{x-y}-1}{x-y}=\lim_{s\to 0}\frac{e^s-1}{s}=1$$

$$\left(\because\ |x-y|\le |x-t|+|y-t| \xrightarrow[y\to t]{x\to t} 0\right)$$

$$\lim_{\substack{x\to t\\ y\to t}}\frac{\sin x-\sin y}{x-y}=\lim_{\substack{x\to t\\ y\to t}}\frac{2\cos\frac{x+y}{2}\sin\frac{x-y}{2}}{x-y}=\cos t$$

故 f 在 (t,t) 处连续 $\Leftrightarrow \cos t=1 \Leftrightarrow t=2k\pi,\ k\in\mathbb{Z}$.

综上 f 的连续点为 $\{(x,y)\mid x>y\}\cup\{(x,y)\mid x<y\}$

$\cup\,\{(t,t)\mid t=2k\pi,\ k\in\mathbb{Z}\}$

2. 设 $z=\ln(e^x+e^y)$. 验证

(1) $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$

(2) $\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=0$

Pr: (1) $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{e^x}{e^x+e^y}+\frac{e^y}{e^x+e^y}=1$

(2) $\frac{\partial^2 z}{\partial x^2}=\frac{e^x(e^x+e^y)-e^{2x}}{(e^x+e^y)^2}=\frac{e^{x+y}}{(e^x+e^y)^2}$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{-e^y e^x}{(e^x+e^y)^2}$

故 $\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2$

3. $u=f(x,y,z)$, $y=\varphi(x,r)$, $z=h(x,y,r)$. 当把 u 看成 x,r 的复合函数时. 求 $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x}=f_1+f_2\frac{\partial y}{\partial x}+f_3\frac{\partial z}{\partial x}$$
$$=f_1+f_2\varphi_1+f_3[h_1+h_2\varphi_1]$$
$$=f_1+f_2\varphi_1+f_3h_1+f_3h_2\varphi_1$$

其中 $f_i=\frac{\partial f}{\partial x_i}\cancel{(x,y,z)}\ \cancel{xe_1+ye_2+ze_3}$

3. $u=f(x,y,z)$，$y=\varphi(x,r)$，$z=h(x,y,r)$. 把 u 看成 x, r 的复合函数时. 求 $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = f_1 + f_2\frac{\partial y}{\partial x} + f_3\frac{\partial z}{\partial x}$$

$$= f_1 + f_2\varphi_1 + f_3[h_1 + h_2\varphi_1]$$

$$= f_1 + f_2\varphi_1 + f_3h_1 + f_3h_2\varphi_1$$

4. 若 $u=\varphi(x+\psi(y))$. 试证:

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}$$

$$LHS = \varphi'\big|_{x+\psi(y)} \cdot \varphi''\big|_{x+\psi(y)} \cdot \psi'\big|_{y}$$

$$RHS = \varphi'\big|_{x+\psi(y)} \cdot \psi'\big|_{y} \cdot \varphi''\big|_{x+\psi(y)}$$

$$LHS = RHS$$

⑤ ⑩

5. $u=f(x,y,z)$ 当 $x=r\cos\theta,\ y=r\sin\theta,\ z=z$

试证 $\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}-r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta$$

$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial^2 u}{\partial x^2}\cos^2\theta+\frac{\partial^2 u}{\partial y\partial x}\cos\theta\sin\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\theta+\frac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta$$

$$\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta+\frac{\partial^2 u}{\partial y\partial x}-r^2\sin\theta\cos\theta-\frac{\partial u}{\partial x}r\cos\theta$$
$$+\frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta+\frac{\partial^2 u}{\partial x\partial y}-r^2\sin\theta\cos\theta-\frac{\partial u}{\partial x}r\sin\theta$$

故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}$

Date.

$$\frac{dy}{dx}=f_x+f_t\left(\frac{\partial t}{\partial x}+\frac{\partial t}{\partial y}\frac{dy}{dx}\right)$$

对$F(x,y,t)=0$两边关于x求导

$$F_x+F_y\frac{dy}{dx}+F_t\left(\frac{\partial t}{\partial x}+\frac{\partial t}{\partial y}\frac{dy}{dx}\right)=0$$

$$\frac{dy}{dx}=f_x+f_t\frac{-F_t-F_y\frac{dy}{dx}}{F_t}$$

$$\frac{dy}{dx}=\frac{f_xF_t-F_xf_t}{F_t+f_tF_y}.$$

7. 设 $y=y(x)$ 是由 $F(x,y)=0$ 所确定的函数. 证明:

$$\frac{d^2y}{dx^2}=-\frac{\frac{\partial^2F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2-2\frac{\partial^2F}{\partial x\partial y}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}+\frac{\partial^2F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}$$

Pr: $\because$ $F_1+F_2\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

$$\frac{d^2y}{dx^2}=\frac{\left[-\frac{\partial^2F}{\partial x^2}-\frac{\partial^2F}{\partial y\partial x}\frac{dy}{dx}\right]\frac{\partial F}{\partial y}+\frac{\partial F}{\partial x}\frac{\partial^2F}{\partial y^2}\frac{dy}{dx}+\frac{\partial F}{\partial x}\frac{\partial^2F}{\partial x\partial y}}{\left(\frac{\partial F}{\partial y}\right)^2}$$

代入 $\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$ 得

$$\frac{d^2y}{dx^2}=\frac{-\frac{\partial^2F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2+2\frac{\partial^2F}{\partial x\partial y}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}-\frac{\partial^2F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}$$

8. 求下列函数的极值或条件极值

(1) $f(x,y)=xy\sqrt{1-x^2-y^2}$

<1> 若 x,y 有 0 或 $x^2+y^2=1$ 则 $f=0$

<2> Else. $\frac{\partial f}{\partial x}=y\sqrt{1-x^2-y^2}-x^2y(1-x^2-y^2)^{-\frac{1}{2}}$

$$=\frac{y(1-2x^2-y^2)}{\sqrt{1-x^2-y^2}}=0 \Leftrightarrow 2x^2+y^2=1$$

同理 $\frac{\partial f}{\partial y}=0 \Leftrightarrow 2y^2+x^2=1$

$\Rightarrow x=y=\pm\frac{1}{\sqrt{3}}$

则~~知~~验证 $f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ 为极大值 $\frac{\sqrt{3}}{9}$. $f(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})$ 取极小值 $-\frac{\sqrt{3}}{9}$

(2) $f(x,y)=\sin x+\sin y+\cos(x+y)$ $(0\le x\le\frac{\pi}{4}, 0\le y\le\frac{\pi}{4})$

$$\frac{\partial f}{\partial x}=\cos x-\sin(x+y)=0$$

$$\frac{\partial f}{\partial y}=\cos y-\sin(x+y)=0$$

$\Rightarrow \cos x=\cos y \Rightarrow x=y$

$\cos x=\sin 2x=\cos(\frac{\pi}{2}-2x)$

$\Rightarrow x=\frac{\pi}{6}$

$f(\frac{\pi}{6},\frac{\pi}{6})=\frac{3}{2}$ 取极大值.

(3) $u=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$，要求 $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{a^2}$

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2\le 3\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=\frac{3}{a^2}$$

$$\left|\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right|\le\frac{\sqrt{3}}{|a|}$$

且当 $x=y=z=\sqrt{3}|a|$ 时 取极大值

$x=y=z=-\sqrt{3}|a|$ 时 取极小值.

9. 求 $z=x^2+y^2-12x+16y$ 在有界闭域 $x^2+y^2\le 25$ 上的最大,最小值.

$z=(x-6)^2+(y+8)^2-100$

最大值

$x^2+y^2-12x+16y$

$\le x^2+y^2+12|x|+16|y|$

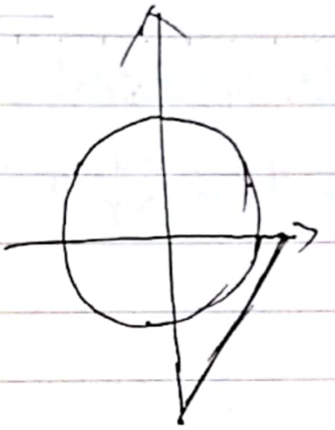
$\le 25+\sqrt{(12^2+16^2)(x^2+y^2)}$

$\le 25+\sqrt{400\times 25}=25+\cancel{100}\ 100=\cancel{2}\ 125$

取等当且仅当 $\frac{|x|}{12}=\frac{|y|}{16}$ $x^2+y^2=25$ $\Leftrightarrow \begin{cases}x=-3\\ y=4\end{cases}$

最大值为 ~~2~~ 125

最小值



图象法 $x=3$ $y=-4$

$z=-75$

极小值为 -75

10. 求平面 $Ax+By+Cz=0$ 与柱面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a,b>0)$ 相交所成的椭圆的面积.

问题等价于求 $u^2=x^2+y^2+z^2$ 在 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 且 $Ax+By+Cz=0$ 下的极值

令 $F=x^2+y^2+z^2+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\mu(Ax+By+Cz)$

$$F_x = 2x+\frac{2\lambda}{a^2}x+\mu A=0$$
$$F_y = 2y+\frac{2\lambda}{b^2}y+\mu B=0$$
$$F_z = 2z+\mu C=0$$
$$F_\lambda = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$
$$F_\mu = Ax+By+Cz=0$$

$$2(x^2+y^2+z^2)+2\lambda=0$$
$$\lambda=-u^2$$
$$\left(2-\frac{2u^2}{a^2}\right)x+\mu A=0$$

若 $u^2=a^2$ 则矛盾

故 $u^2\neq a^2$ $\quad x=\frac{\mu A}{\frac{2u^2}{a^2}-2}=\frac{\mu A a^2}{2u^2-2a^2}$

同理得 $y=\frac{\mu B b^2}{2u^2-2b^2}$ $\quad z=-\frac{\mu C}{2}$

$$\frac{\mu A^2a^2}{2u^2-2a^2}+\frac{\mu B^2b^2}{2u^2-2b^2}+-\frac{\mu C^2}{2}=0$$

$$\mu A^2a^2(u^2-b^2)+\mu B^2b^2(u^2-a^2)-\mu C^2(u^2-a^2)(u^2-b^2)=0$$

$$u_1^2u_2^2=\frac{\mu A^2a^2b^2+\mu B^2b^2a^2+\mu C^2a^2b^2}{\mu C^2}$$

$$=\frac{a^2b^2(A^2+B^2+C^2)}{C^2}$$

$$\pi u_1u_2=\pi\frac{ab\sqrt{A^2+B^2+C^2}}{C}$$

若
故

同理

14. 设曲面 $S: \sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}\ (a>0)$

(1) 证明: S上任意点处的切平面与各坐标轴截距之和等于a.

法向量 $\vec{n}: \left(\frac{1}{2}x^{-\frac{1}{2}}, \frac{1}{2}y^{-\frac{1}{2}}, \frac{1}{2}z^{-\frac{1}{2}}\right)$

设 $(x_0, y_0, z_0) \in S$

则 $\vec{l} = (x-x_0, y-y_0, z-z_0)$

使得 $\vec{n}\cdot\vec{l}=0$

$\Leftrightarrow (x-x_0)\frac{1}{2}x_0^{-\frac{1}{2}}+(y-y_0)\frac{1}{2}y_0^{-\frac{1}{2}}+(z-z_0)\frac{1}{2}z_0^{-\frac{1}{2}}=0$

取 $y=z=0$

$(\hat{x}-x_0)x_0^{-\frac{1}{2}} = y_0^{\frac{1}{2}}+z_0^{\frac{1}{2}}$

$\hat{x} = x_0+(x_0y_0)^{\frac{1}{2}}+(x_0z_0)^{\frac{1}{2}}$

故 $\hat{x}+\hat{y}+\hat{z} = x_0+y_0+z_0+2(x_0y_0)^{\frac{1}{2}}+2(x_0z_0)^{\frac{1}{2}}+2(y_0z_0)^{\frac{1}{2}}$

$= (\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})^2 = a.$

(2) 即 ~~$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$~~ $\hat{x}+\hat{y}+\hat{z}=a$

$f(\hat{x},\hat{y},\hat{z}) = \hat{x}\hat{y}\hat{z} \le \left(\frac{x+y+z}{3}\right)^3 = \left(\frac{a}{3}\right)^3$

且当 $\hat{x}=\hat{y}=\hat{z}=\frac{a}{3}$ 时取最大值.