Testing on x1 family Resonant Orbit in Barred Galaxy

Problem: Runge-kutta 4th order does not conserved Jacobi Energy.

In rotating potential, neither the hamiltonian nor angular momentum conserved, but the only way is the conservation of jacobi energy,

$$E_J = \frac{1}{2}|\dot{x}|^2 + \Phi_{eff}(x)$$

$$\Phi_{eff}(x) \equiv \phi(x) - \frac{1}{2}|\Omega_b \times x|^2$$

Since, we have considered the logarithmic potential for galaxy as well as bar potential. Therefore, our final form of potential is in the form,

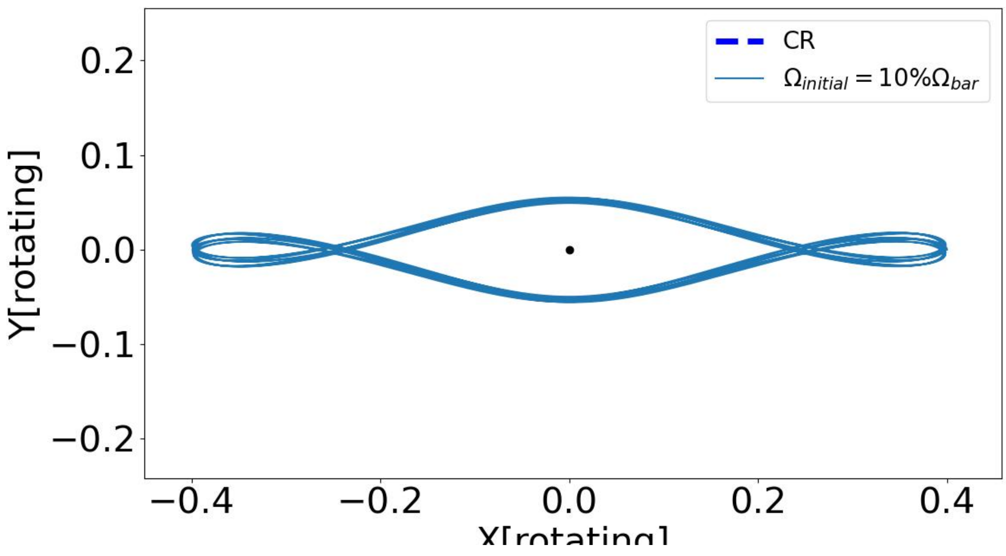
$$\phi(x) = \frac{1}{2}v_c^2 \ln(r_c^2 + x^2 + (\frac{y^2}{q^2})) + \varepsilon\, r^{1/2}(16 - r)\cos(2 \times \arctan(y/x))$$

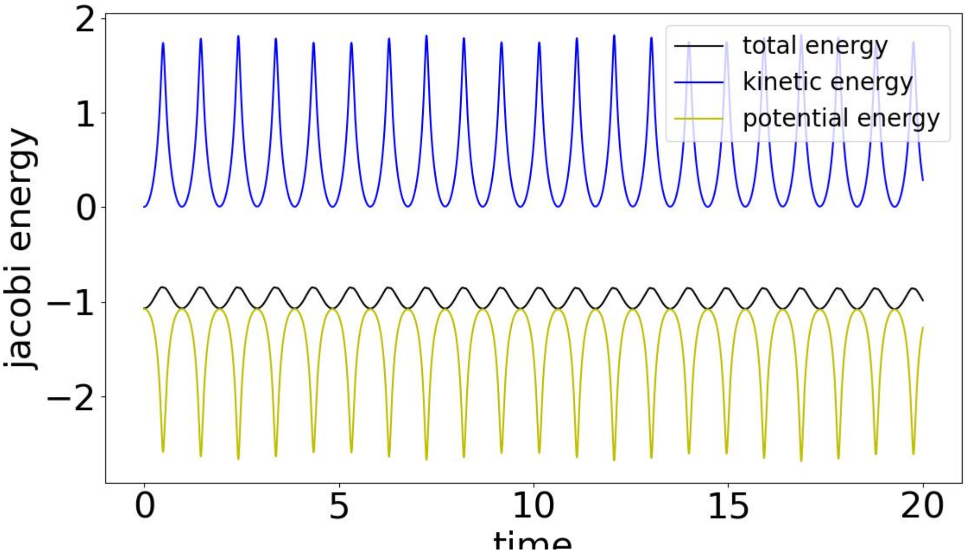
Here $r = \sqrt{x^2 + y^2}$; and $\varepsilon = 0.00001$ for weak bar [1]. Also, vc=omega=1.0, rc=0.03, q=0.8 are taken from literature [2]. We applied the RK4 method to compute x1 family orbits and its corresponding jacobi energy as shown in below figure. Here, we can see the jacobi energy is not conserved, due to the numerical limitation. We have tested the same orbit while decreasing the step size and increase the RK orders upto 5 and 6. But still, we could not able to conserved the quantity. The new method called "**Relaxation Runge Kutta**" as given by,

$$u(t_n + \gamma_n \Delta t) \approx u_\gamma^{n+1} = u^n + \gamma_n \Delta t \sum_{j=1}^{s} b_j f(t_n + c_j \Delta t, y_j)$$

The only difference between RK and RRK is the factor γ_n that multiplies the step size. While the choice of γ_n determines how far to step in RK direction. From this point of view γ_n is similar to the relaxation parameter used in some iterative algebraic solvers. The condition of γ_n is given by [3],

$$\gamma_n = \begin{cases} 1 \quad \cdots\cdots\cdots\cdots \quad \left\|\sum_{j=1}^{s} b_j f_j\right\|^2 = 0 \\ \dfrac{2\sum_{i,j=1}^{s} b_i a_{ij} < f_i, f_j >}{\sum_{i,j=1}^{s} b_i b_j < f_i, f_j >} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \left\|\sum_{j=1}^{s} b_j f_j\right\|^2 \neq 0 \end{cases}$$





Reference:
[1] Orbits in Weak and Strong Bars (1980) by Contopoulos & Papayannopoulos
[2] Galactic Dynamics 2nd edition by Binney and Tremaine
[3] Relaxation Runge-Kutta Methods: Conservation and stability for Inner-Product Norms (2019) by David I. Ketcheson