Fundamentals of Orbits in the Spherical Mass Distribution:

In general, spherical mass distribution is stated as the gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of distribution were only concentrated at the center. Orbits are the trajectories that bodies travel on, under the influence of gravity in a gravitational field. These trajectories can be computed using Newton's second law as,

$$m\ddot{x} = mg(x)$$

Where g(x) is the gravitational field, m is the mass of the body, and $\ddot{x}$ is the acceleration vector. The orbital trajectory is fully determined by its initial phase-space coordinate w0 = (x0, v0), that can numerically solved by converting the second order differential equation to the first order and using a standard Euler Method or for more accuracy Runge Kutta 4th order on each time step size. Also, the mass of the body so-called test-particle is so small that it does not affect the evolution of gravitational field cause by the central source.

In my initial attempts, I used the cartesian coordinate to evolve trajectories. However for spherical mass distribution, potential functions are normally in spherical coordinates as,

Keplerian Potential (*point mass distribution*): $\varphi(r) = -\frac{GM}{r}$

Spherical Harmonic Potential (*constant density mass distribution*): $\varphi(r) = \frac{1}{2}\omega^2 r^2$; ω^2 is azimuthal frequency

Isochrone Potential: $\varphi(r) = -\frac{GM}{b+\sqrt{r^2+b^2}}$; b is scale length

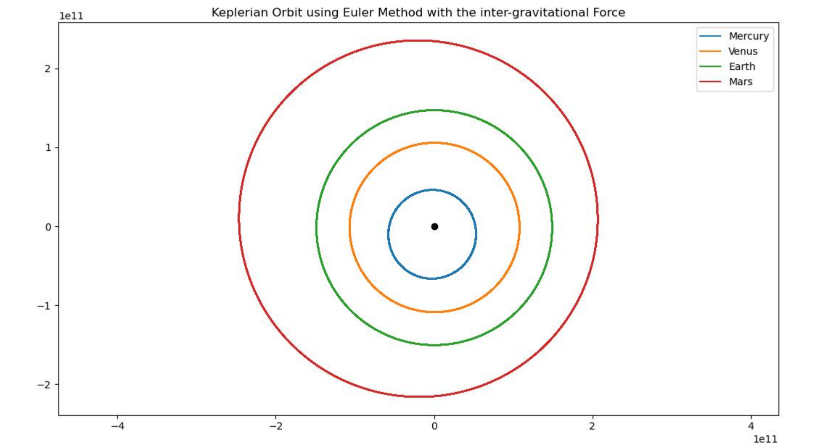
That is converted into Forces as a gradient of potential,

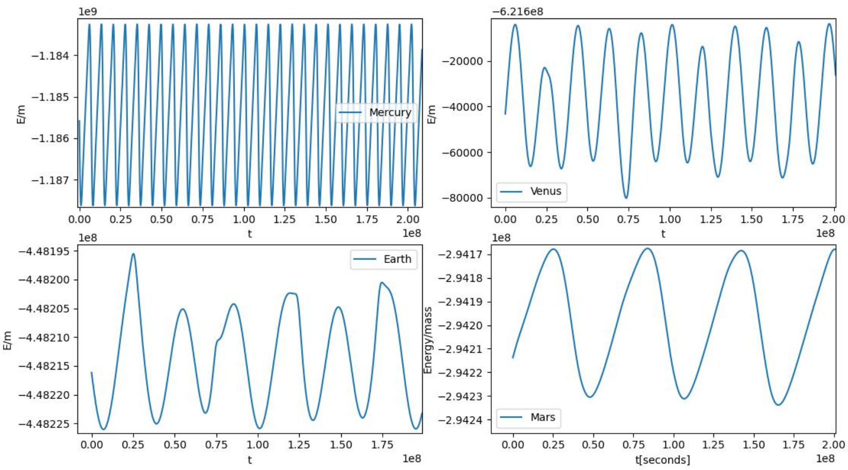
$F(x) = \frac{\partial\varphi(r)}{\partial r} . \frac{\partial r}{\partial x}$; $F(y) = \frac{\partial\varphi(r)}{\partial r} . \frac{\partial r}{\partial y}$

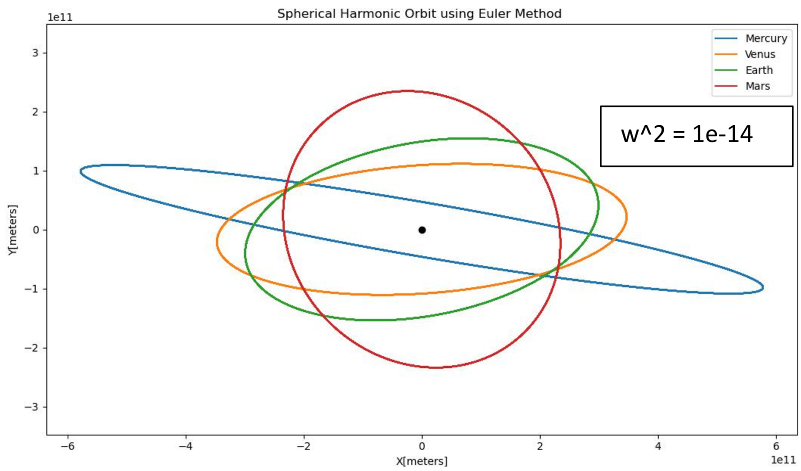
The radial energy and angular momentum are in forms,

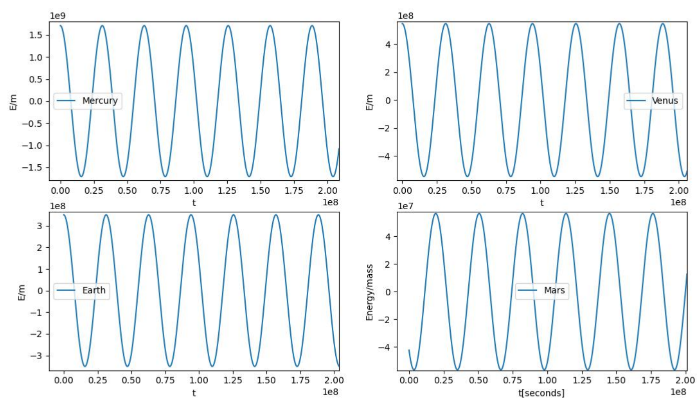
$$E = \frac{\dot{r}^2}{2} + \varphi(r)$$

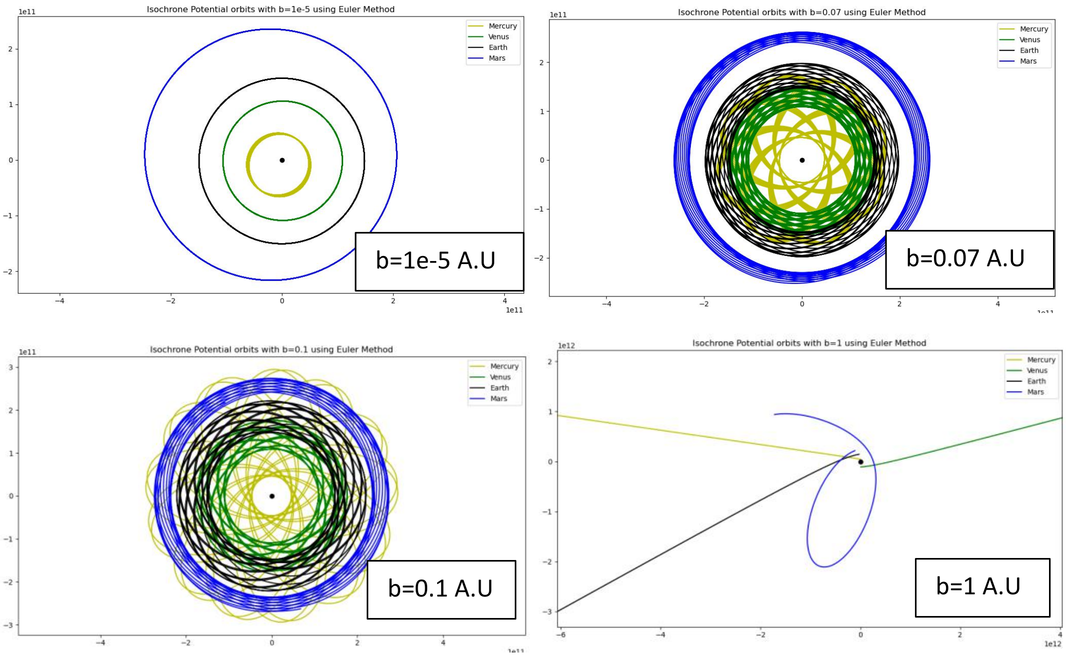
$$L = r \times \dot{r} = x.v_y - y.v_x$$


Keplerian Orbit using Euler Method with the inter-gravitational Force
1e11
Mercury
Venus
Earth
Mars
1e11


1e9
E/m
Mercury
t
1e8
−6.216e8
E/m
Venus
t
1e8
1e8
E/m
Earth
t
1e8
1e8
Energy/mass
Mars
t[seconds]
1e8


Spherical Harmonic Orbit using Euler Method
1e11
Mercury
Venus
Earth
Mars
w^2 = 1e-14
Y[meters]
X[meters]
1e11


1e9
E/m
Mercury
t
1e8
1e8
E/m
Venus
t
1e8
1e8
E/m
Earth
t
1e8
1e7
Energy/mass
Mars
t[seconds]
1e8



Keplerian orbits are also evaluated by using the orbital elements such that;

$\mathrm{M(t)} = \mathrm{Mo} + \Delta t\sqrt{\frac{GM}{a^3}}$; M(t) is the Mean anomaly with 2nd term is mean motion of Keplers

$E_{j+1} = E_j - \frac{E_j - e\sin E_j - M(t)}{1 - e\cos E_j}$; Relation of Eccentric Anomaly and Mean Anomaly

$\nu(t) = 2\arctan\left(\frac{\sqrt{1+e}}{\sqrt{1-e}}\tan\left(\frac{E_{j+1}}{2}\right)\right)$; True anomaly

$r_c(t) = a(1 - e\cos E_{j+1})$; Distance to central body

$o(t) = \begin{pmatrix} o_x(t) \\ o_y(t) \end{pmatrix} = r_c(t)\begin{pmatrix} \cos\nu(t) \\ \sin\nu(t) \end{pmatrix}$; Positions in Orbital Frame

$\dot{o}(t) = \begin{pmatrix} \dot{o}_x(t) \\ \dot{o}_y(t) \end{pmatrix} = \frac{\sqrt{GMa}}{r_c(t)}\begin{pmatrix} -\sin E_{j+1} \\ \sqrt{1-e^2}\cos E_{j+1} \end{pmatrix}$; Velocities in Orbital Frame with z-axis perpendicular to O.P.