Axisymmetic electro-elastic field in a heterogeneous annular plate with a uniform load

Abstract

This paper presents electro-elastic field in a heterogeneous circular plate subjected to a load uniformly distributed on the lower boundary. Direct displacement method is employed to derive the coupling electro-elastic field so as to get the explicit expression of stress and displacement. Finally, numerical calculations are performed to validate the present analysis and to investigate the influence of the material heterogeneity and the multi-field coupling.

Keywords: Heterogeneous circular plate, Piezoelectric materials, Direct displacement method

1. Basic equations

The basic equations for the axisymmetric problem of a transversely isotropic FGM body, referred to a cylindrical coordinate system (r, θ, z) with the z axis being normal to the plane of isotropy, are read as follow

balance equation

$$
\begin{aligned}
&\sigma_{r,r} + \tau_{rz,z} + r^{-1}(\sigma_r - \sigma_\theta) = 0, \\
&\tau_{rz,r} + r^{-1}\tau_{rz} + \sigma_{z,z} = 0, \\
&D_{r,r} + r^{-1}D_r + D_{z,z} = 0,
\end{aligned}
\tag{1}
$$

where u and w denote the radial and axial displacement components, respectively.

Constitutive equation

$$
\begin{aligned}
&\sigma_r = c_{11}u_{,r} + c_{12}r^{-1}u + c_{13}w_{,z} + e_{31}\phi_{,z}, \\
&\sigma_\theta = c_{12}u_{,r} + c_{11}r^{-1}u + c_{13}w_{,z} + e_{31}\phi_{,z}, \\
&\sigma_z = c_{13}(u_{,r} + r^{-1}u) + c_{33}w_{,z} + e_{33}\phi_{,z}, \\
&\tau_{rz} = c_{44}(u_{,z} + w_{,r}) + e_{15}\phi_{,r}, \\
&D_r = e_{15}(u_{,z} + w_{,r}) - \xi_{11}\phi_{,r}, \\
&D_z = e_{31}(u_{,r} + r^{-1}u) + e_{33}w_{,z} - \xi_{33}\phi_{,z},
\end{aligned}
\tag{2}
$$

where σ_r, σ_θ, σ_z and τ_{ij} are the stress components.

Boundary conditions of upper and lower surfaces

$$
\tau_{rz}(r, \pm h/2) = 0,\ D_z(z, \pm h/2) = 0,\ \sigma_z(r, h/2) = 0,\ \sigma_z(r, -h/2) = -q,
\tag{3}
$$

2. Direct displacement method

By employing the direct displacement method, we assume generalized displacements

$$
\begin{aligned}
u(r,z) &= r^{-1}u_{-1}(z) + ru_1(z) + r^3u_3(z) + A_1(z) r \ln\frac{r}{r_1},\\
w(r,z) &= w_0(z) + r^2 w_2(z) + r^4 w_4(z) + B_0(z)\ln\frac{r}{r_1} + B_2(z) r^2 \ln\frac{r}{r_1},\\
\phi(r,z) &= \phi_0(z) + r^2\phi_2(z) + C_0(z)\ln\frac{r}{r_1},
\end{aligned}
\tag{4}
$$

where $u_i(z)$, $w_i(z)$, $\phi_i(z)$, $A_i(z)$, $B_i(z)$ and $C_i(z)$ are referred to the displacement functions. The justification of Eq. (4) is discussed in Appendix A.

Substituting Eq. (4) into Eq. (2) leads to the expressions for the stress components

$$
\begin{aligned}
\sigma_r =&(c_{11}+c_{12})u_1 + c_{11}A_1 + c_{13}w_{0,z} + e_{31}\phi_{0,z}\\
&+ (c_{12}-c_{11})u_{-1}r^{-2} + \big[(3c_{11}+c_{12})u_3 + c_{13}w_{2,z} + e_{31}\phi_{2,z}\big]r^2 + c_{13}w_{4,z}r^4\\
&+ \big[(c_{11}+c_{12})A_1 + c_{13}B_{0,z} + e_{31}C_{0,z}\big]\ln\frac{r}{r_1} + c_{13}B_{2,z}r^2\ln\frac{r}{r_1}\\
\sigma_\theta =&(c_{11}+c_{12})u_1 + c_{12}A_1 + c_{13}w_{0,z} + e_{31}\phi_{0,z}\\
&+ (c_{11}-c_{12})u_{-1}r^{-2} + \big[(c_{11}+3c_{12})u_3 + c_{13}w_{2,z} + e_{31}\phi_{2,z}\big]r^2 + c_{13}w_{4,z}r^4\\
&+ \big[(c_{11}+c_{12})A_1 + c_{13}B_{0,z} + e_{31}C_{0,z}\big]\ln\frac{r}{r_1} + c_{13}B_{2,z}r^2\ln\frac{r}{r_1}\\
\sigma_z =&2c_{13}u_1 + c_{13}A_1 + c_{33}w_{0,z} + e_{33}\phi_{0,z}\\
&+ (4c_{13}u_3 + c_{33}w_{2,z} + e_{33}\phi_{2,z})r^2 + c_{33}w_{4,z}r^4\\
&+ (2c_{13}A_1 + c_{33}B_{0,z} + e_{33}C_{0,z})\ln\frac{r}{r_1} + c_{33}B_{2,z}r^2\ln\frac{r}{r_1}\\
\tau_{rz} =&\big[c_{44}(u_{-1,z}+B_0) + e_{15}C_0\big]r^{-1} + \big[c_{44}(u_{1,z}+2w_2+B_2) + 2e_{15}\phi_2\big]r\\
&+ c_{44}(u_{3,z}+4w_4)r^3 + c_{44}(A_{1,z}+2B_2)r\ln\frac{r}{r_1}\\
D_r =&[e_{15}(u_{-1,z}+B_0) - \xi_{11}C_0]r^{-1} + [e_{15}(u_{1,z}+2w_2+B_2) - 2\xi_{11}\phi_2]r\\
&+ e_{15}(u_{3,z}+4w_4)r^3 + e_{15}(A_{1,z}+2B_2)r\ln\frac{r}{r_1}\\
D_z =&2e_{31}u_1 + e_{31}A_1 + e_{33}w_{0,z} - \xi_{33}\phi_{0,z}\\
&+ (4e_{31}u_3 + e_{33}w_{2,z} - \xi_{33}\phi_{2,z})r^2 + e_{33}w_{4,z}r^4\\
&+ (2e_{31}A_1 + e_{33}B_{0,z} - \xi_{33}C_{0,z})\ln\frac{r}{r_1} + e_{33}B_{2,z}r^2\ln\frac{r}{r_1}
\end{aligned}
\tag{5}
$$

Substituting stress Eq. (5) into balance Eq. (1) and comparing the coefficients of r^m and $r^m\ln(r/r_1)$, we can obtain the following differential equations system

$$
\big[c_{44}(u_{-1,z}+B_0) + e_{15}C_0\big]_{,z} + 2c_{11}A_1 + c_{13}B_{0,z} + e_{31}C_{0,z} = 0 \tag{6a}
$$

$$\left[c_{44}(u_{1,z}+2w_2+B_2)+2e_{15}\phi_2\right]_{,z}+8c_{11}u_3+2c_{13}w_{2,z}+c_{13}B_{2,z}+2e_{31}\phi_{2,z}=0 \tag{6b}$$

$$\left[c_{44}u_{3,z}+4c_{44}w_4\right]_{,z}+4c_{13}w_{4,z}=0 \tag{6c}$$

$$\left[c_{44}A_{1,z}+2c_{44}B_2\right]_{,z}+2c_{13}B_{2,z}=0 \tag{6d}$$

$$\begin{aligned}&\left[2c_{13}u_1+c_{13}A_1+c_{33}w_{0,z}+e_{33}\phi_{0,z}\right]_{,z}\\&+c_{44}(2u_{1,z}+A_{1,z}+4w_2+4B_2)+4e_{15}\phi_2=0\end{aligned} \tag{6e}$$

$$\left[4c_{13}u_3+c_{33}w_{2,z}+e_{33}\phi_{2,z}\right]_{,z}+4c_{44}(u_{3,z}+4w_4)=0 \tag{6f}$$

$$(c_{33}w_{4,z})_{,z}=0 \tag{6g}$$

$$\left[2c_{13}A_1+c_{33}B_{0,z}+e_{33}C_{0,z}\right]_{,z}+2c_{44}A_{1,z}+4c_{44}B_2=0 \tag{6h}$$

$$(c_{33}B_{2,z})_{,z}=0 \tag{6i}$$

$$\begin{aligned}&\left[2e_{31}u_1+e_{31}A_1+e_{33}w_{0,z}-\xi_{33}\phi_{0,z}\right]_{,z}\\&+e_{15}(2u_{1,z}+A_{1,z}+4w_2+4B_2)-4\xi_{11}\phi_2=0\end{aligned} \tag{6j}$$

$$\left[4e_{31}u_3+e_{33}w_{2,z}-\xi_{33}\phi_{2,z}\right]_{,z}+4e_{15}(u_{3,z}+4w_4)=0 \tag{6k}$$

$$(e_{33}w_{4,z})_{,z}=0 \tag{6l}$$

$$\left[2e_{31}A_1+e_{33}B_{0,z}-\xi_{33}C_{0,z}\right]_{,z}+2e_{15}A_{1,z}+4e_{15}B_2=0 \tag{6m}$$

$$(e_{33}B_{2,z})_{,z}=0 \tag{6n}$$

Similarly, substitution of stress Eq. (5) into boundary Eq. (3) and comparison between the coefficients of r^m and $r^m\ln(r/r_1)$ gives following boundary conditions

$$\left[c_{44}(u_{-1,z}+B_0)+e_{15}C_0\right]\Big|_{z=\pm h/2}=0 \tag{7a}$$

$$\left[c_{44}(u_{1,z}+2w_2+B_2)+2e_{15}\phi_2\right]\Big|_{z=\pm h/2}=0 \tag{7b}$$

$$\left[c_{44}u_{3,z}+4c_{44}w_4\right]\Big|_{z=\pm h/2}=0 \tag{7c}$$

$$\left[c_{44}A_{1,z}+2c_{44}B_2\right]\Big|_{z=\pm h/2}=0 \tag{7d}$$

$$\left[2e_{31}u_1+e_{31}A_1+e_{33}w_{0,z}-\xi_{33}\phi_{0,z}\right]\Big|_{z=\pm h/2}=0 \tag{7e}$$

$$\left[4e_{31}u_3+e_{33}w_{2,z}-\xi_{33}\phi_{2,z}\right]\Big|_{z=\pm h/2}=0 \tag{7f}$$

$$\left[e_{33}w_{4,z}\right]\Big|_{z=\pm h/2}=0 \tag{7g}$$

$$\left[2e_{31}A_1+e_{33}B_{0,z}-\xi_{33}C_{0,z}\right]\Big|_{z=\pm h/2}=0 \tag{7h}$$

$$\left[e_{33}B_{2,z}\right]\Big|_{z=\pm h/2}=0 \tag{7i}$$

$$[4c_{13}u_3 + c_{33}w_{2,z} + e_{33}\phi_{2,z}]\big|_{z=\pm h/2} = 0 \tag{7j}$$

$$[c_{33}w_{4,z}]\big|_{z=\pm h/2} = 0 \tag{7k}$$

$$[2c_{13}A_1 + c_{33}B_{0,z} + e_{33}C_{0,z}]\big|_{z=\pm h/2} = 0 \tag{7l}$$

$$[c_{33}B_{2,z}]\big|_{z=\pm h/2} = 0 \tag{7m}$$

$$\begin{aligned} &\big[2c_{13}u_1 + c_{13}A_1 + c_{33}w_{0,z} + e_{33}\phi_{0,z}\big]\Big|_{z=\ h/2} = -q \\ &\big[2c_{13}u_1 + c_{13}A_1 + c_{33}w_{0,z} + e_{33}\phi_{0,z}\big]\Big|_{z=-h/2} = 0 \end{aligned} \tag{7n}$$

Then, we will derive the explicit expressions of the generalized displacement functions according to the combination of Eq. (6) and Eq. (7).

From Eqs. (6g), (6i), (6l) (6n) and Eqs. (7g), (7i), (7k), (7m), we can obtain that both $B_{2,z}$ and $w_{4,z}$ are zeros, so we can write them as

$$\begin{aligned} &B_2 = \gamma_{b2} \\ &w_4 = \gamma_{w4}, \end{aligned} \tag{8}$$

where γ_{b2} and γ_{w4} are integral constants. Then, the combination of Eqs. (6c) and (7c) gives $c_{44}u_{3,z}+4c_{44}\gamma_{w4} = 0$, while the combination of Eqs. (6d) and (7d) gives $c_{44}A_{1,z}+2c_{44}\gamma_{b2} = 0$, so we can describe u_3 and A_1 as

$$\begin{aligned} &u_3(z) = -4\gamma_{w4}z + \gamma_{u3} \\ &A_1(z) = -2\gamma_{b2}z + \gamma_{a1} \end{aligned} \tag{9}$$

where where γ_{u3} and γ_{a1} are integral constants.

Substituting Eq. (9) into Eqs. (6f) and (6k), and combining Eqs. (7f) and (7j), we obtain

$$\begin{aligned} &4c_{13}u_3 + c_{33}w_{2,z} + e_{33}\phi_{2,z} = 0 \\ &4e_{31}u_3 + e_{33}w_{2,z} - \xi_{33}\phi_{2,z} = 0 \end{aligned} \tag{10}$$

Solving above equation to obtain $w_{2,z}$ and $\phi_{2,z}$, and integrating them, we can obtain

$$\begin{aligned} &w_2(z) = 16\gamma_{w4}\hat{f}_1^1(z) - 4\gamma_{u3}\hat{f}_1^0(z) + \gamma_{w2} \\ &\phi_2(z) = 16\gamma_{w4}\hat{f}_2^1(z) - 4\gamma_{u3}\hat{f}_2^0(z) + \gamma_{\phi 2}, \end{aligned} \tag{11}$$

where γ_{w2} and $\gamma_{\phi 2}$ are integral constants, and f_1^n and f_2^n are defined as

$$\begin{aligned} &\hat{f}_1^n(z) = \int_{-h/2}^{z} f_1^n(\eta)\mathrm{d}\eta, \quad \text{where} \quad f_1^n(z) = \frac{c_{13}\xi_{33} + e_{33}e_{31}}{e_{33}^2 + \xi_{33}c_{33}}z^n, \quad n = 0, 1 \\ &\hat{f}_2^n(z) = \int_{-h/2}^{z} f_2^n(\eta)\mathrm{d}\eta, \quad \text{where} \quad f_2^n(z) = \frac{e_{33}c_{13} - c_{33}e_{31}}{e_{33}^2 + \xi_{33}c_{33}}z^n, \quad n = 0, 1 \end{aligned} \tag{12}$$

Similarly, the combination of Eqs. (6h), (6m), (7h) and (7l) gives

$$\begin{aligned}&2c_{13}A_1 + c_{33}B_{0,z} + e_{33}C_{0,z} = 0\\&2e_{31}A_1 + e_{33}B_{0,z} - \xi_{33}C_{0,z} = 0\end{aligned} \tag{13}$$

and its solution can be written as

$$\begin{aligned}&B_0(z) = 4\gamma_{b2}\hat{f}_1^1(z) - 2\gamma_{a1}\hat{f}_1^0(z) + \gamma_{b0}\\&C_0(z) = 4\gamma_{b2}\hat{f}_2^1(z) - 2\gamma_{a1}\hat{f}_2^0(z) + \gamma_{c0},\end{aligned} \tag{14}$$

where γ_{b0} and γ_{c0} are integral constants.

Substituting Eqs. (9) and (14) into Eqs. (6a), and integrating it will result in

$$\left[c_{44}(u_{-1,z} + B_0) + e_{15}C_0\right] = 4\gamma_{b2}K_1(z) - 2\gamma_{a1}K_0(z) \tag{15}$$

where K_i is define as

$$K_n(z) = \int_{-h/2}^{z} \left(c_{11} - c_{13}\frac{c_{13}\xi_{33} + e_{33}e_{31}}{e_{33}^2 + \xi_{33}c_{33}} - e_{31}\frac{e_{33}c_{13} - c_{33}e_{31}}{e_{33}^2 + \xi_{33}c_{33}}\right)\eta^n \mathrm{d}\eta \quad (n = 0, 1, 2). \tag{16}$$

According the boundary condition in Eq. (7a), we have

$$2\gamma_{b2}K_1(\frac{h}{2}) - \gamma_{a1}K_0(\frac{h}{2}) = 0. \tag{17}$$

Defining function form as following

$$\hat{f}_3^n(z) = \int_{-h/2}^{z} f_3^n(\eta)d\eta, \quad \text{where} \quad f_3^n(z) = c_{44}^{-1}K_n(z) - c_{44}\hat{f}_1^n(z) - e_{15}\hat{f}_2^n(z), \quad n = 0, 1 \tag{18}$$

we can solve Eq. (15) to obtain u_{-1} as

$$u_{-1}(z) = 4\gamma_{b2}\hat{f}_3^1(z) - 2\gamma_{a1}\hat{f}_3^0(z) - \gamma_{b0}\hat{f}_4(z) - \gamma_{c0}\hat{f}_5(z) + \gamma_{u-1}, \tag{19}$$

where γ_{u-1} is integral constant, and

$$\begin{aligned}&\hat{f}_4(z) = \int_{-h/2}^{z} f_4(\eta)\mathrm{d}\eta, \quad \text{where} \quad f_4(\eta) = c_{44}\\&\hat{f}_5(z) = \int_{-h/2}^{z} f_5(\eta)\mathrm{d}\eta, \quad \text{where} \quad f_5(\eta) = e_{15}\end{aligned} \tag{20}$$

Similarly, Substitution of Eqs. (8), (9) and (11) into Eqs. (6b) leads to

$$c_{44}(u_{1,z} + 2w_2 + B_2) + 2e_{15}\phi_2 = 32\gamma_{w4}K_1(z) - 8\gamma_{u3}K_0(z). \tag{21}$$

According the boundary condition in Eq. (7a), we can obtain

$$4\gamma_{w4}K_1(\frac{h}{2}) - \gamma_{u3}K_0(\frac{h}{2}) = 0. \tag{22}$$

Solving Eq. (21) can obtain u_1 as

$$u_1(z) = 32\gamma_{w4}\hat{f}_3^1(z) - 8\gamma_{u3}\hat{f}_3^0(z) - 2\gamma_{w2}\hat{f}_4(z) - 2\gamma_{\phi 2}\hat{f}_5(z) - \gamma_{b2}\hat{f}_4(z) + \gamma_{u1}, \tag{23}$$

where γ_{u1} is integral constant.

Substituting Eqs. (23), (9) (8) and (11) into Eqs. (6e) and (6j), and integrating them results in

$$\begin{aligned} &c_{13}(2u_1 + A_1) + c_{33}w_{0,z} + e_{33}\phi_{0,z} = -64\gamma_{w4}\hat{K}_1(z) + 16\gamma_{u3}\hat{K}_0(z) \\ &e_{31}(2u_1 + A_1) + e_{33}w_{0,z} - \xi_{33}\phi_{0,z} = -64\gamma_{w4}\hat{H}_1(z) + 16\gamma_{u3}\hat{H}_0(z) + 4\gamma_{\phi 2}\hat{f}_6(z) \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\hat{K}_n(z) = \int_{-h/2}^{z} K_n(z)\mathrm{d}\eta \qquad n = 0, 1 \\ &\hat{H}_n = \int_{-h/2}^{z} H_n(\eta)\mathrm{d}\eta \ \text{ where } \ H_n(z) = \frac{e_{15}}{c_{44}}K_n(z) - \left(\frac{e_{15}^2}{c_{44}} + \xi_{11}\right)\hat{f}_2^n(z) \quad n = 0, 1 \\ &\hat{f}_6(z) = \int_{-h/2}^{z} f_6(\eta)\mathrm{d}\eta \ \text{ where } \ f_6(z) = \frac{e_{15}^2}{c_{44}} + \xi_{11} \end{aligned} \tag{25}$$

The boundary conditions in Eqs. (7e) and (7n) require that

$$\begin{aligned} &-64\gamma_{w4}\hat{K}_1(\frac{h}{2}) + 16\gamma_{u3}\hat{K}_0(\frac{h}{2}) = q \\ &-16\gamma_{w4}\hat{H}_1(\frac{h}{2}) + 4\gamma_{u3}\hat{H}_0(\frac{h}{2}) + \gamma_{\phi 2}\hat{f}_6(\frac{h}{2}) = 0 \end{aligned} \tag{26}$$

Substituting Eqs. (23) and (9) into Eqs. (24) arrives at

$$\begin{aligned} c_{33}w_{0,z} + e_{33}\phi_{0,z} = &-64\gamma_{w4}R_1 + 16\gamma_{u3}R_0 \\ &+ 4\gamma_{\phi 2}c_{13}\hat{f}_4(z) + 4\gamma_{w2}c_{13}\hat{f}_5(z) + 2\gamma_{b2}c_{13}\hat{f}_4(z) + 2\gamma_{b2}c_{13}z \\ &- \gamma_{a1}c_{13} - 2\gamma_{u1}c_{13} \\ e_{33}w_{0,z} - \xi_{33}\phi_{0,z} = &-64\gamma_{w4}S_1 + 16\gamma_{u3}S_0 \\ &+ 4\gamma_{\phi 2}e_{31}\hat{f}_4(z) + 4\gamma_{w2}e_{31}\hat{f}_5(z) + 2\gamma_{b2}e_{31}\hat{f}_4(z) + 2\gamma_{b2}e_{31}z \\ &- \gamma_{a1}e_{31} - 2\gamma_{u1}e_{31} + 4\gamma_{\phi 2}\hat{f}_6(z) \end{aligned} \tag{27}$$

where

$$\begin{aligned} &R_n(z) = \hat{K}_n(z) + c_{13}\hat{f}_3^n(z) \qquad n = 0, 1 \\ &S_n(z) = \hat{H}_n^*(z) + e_{31}\hat{f}_3^n(z) \qquad n = 0, 1 \end{aligned} \tag{28}$$

Soling Eq. (27), we can obtain w_0 and ϕ_0 as

$$\begin{aligned} w_0 =& -64\gamma_{w4}\hat{f}_7^1 + 16\gamma_{u3}\hat{f}_7^0 + 4\gamma_{\phi 2}(\hat{f}_9^4 + f_7) + 4\gamma_{w2}\hat{f}_9^4 + 2\gamma_{b2}(\hat{f}_9^4 + \hat{f}_{11}^1) \\ & - \gamma_{a1}\hat{f}_{11}^0 - 2\gamma_{u1}\hat{f}_{11}^0 + \gamma_{w0} \\ \phi_0 =& -64\gamma_{w4}\hat{f}_8^1 + 16\gamma_{u3}\hat{f}_8^0 + 4\gamma_{\phi 2}(\hat{f}_{10}^4 + f_8) + 4\gamma_{w2}\hat{f}_{10}^5 + 2\gamma_{b2}(\hat{f}_{10}^4 + \hat{f}_{12}^1) \\ & - \gamma_{a1}\hat{f}_{12}^0 - 2\gamma_{u1}\hat{f}_{12}^0 + \gamma_{\phi 0} \end{aligned} \tag{29}$$

where

$$\begin{aligned} &\hat{f}_7^n(z) = \int_{-h/2}^{z} f_7^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_7^n(z) = \frac{e_{33}R_n(z) + \xi_{33}S_n(z)}{e_{33}^2 + c_{33}\xi_{33}} \qquad n = 0,1 \\ &\hat{f}_8^n(z) = \int_{-h/2}^{z} f_8^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_8^n(z) = \frac{e_{33}S_n(z) - c_{33}R_n(z)}{e_{33}^2 + c_{33}\xi_{33}} \qquad n = 0,1 \\ &\hat{f}_9^n(z) = \int_{-h/2}^{z} f_9^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_9^n(z) = \frac{e_{33}c_{13} + \xi_{33}e_{31}}{e_{33}^2 + c_{33}\xi_{33}}\hat{f}_n(z) \qquad n = 4,5 \\ &\hat{f}_{10}^n(z) = \int_{-h/2}^{z} f_{10}^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_{10}^n(z) = \frac{e_{33}e_{31} - c_{33}c_{13}}{e_{33}^2 + c_{33}\xi_{33}}\hat{f}_n(z) \qquad n = 4,5 \\ &\hat{f}_{11}^n(z) = \int_{-h/2}^{z} f_9^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_{11}^n(z) = \frac{e_{33}c_{13} + \xi_{33}e_{31}}{e_{33}^2 + c_{33}\xi_{33}}z^n \qquad n = 0,1 \\ &\hat{f}_{12}^n(z) = \int_{-h/2}^{z} f_9^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_{12}^n(z) = \frac{e_{33}e_{31} - c_{33}c_{13}}{e_{33}^2 + c_{33}\xi_{33}}z^n \qquad n = 0,1 \\ &\hat{f}_{13}(z) = \int_{-h/2}^{z} f_{13}^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_{13}(z) = \frac{\xi_{33}}{e_{33}^2 + c_{33}\xi_{33}}\hat{f}_6(z) \\ &\hat{f}_{14}(z) = \int_{-h/2}^{z} f_{14}^n(\eta)\mathrm{d}\eta \quad \text{where} \quad f_{14}(z) = \frac{e_{33}}{e_{33}^2 + c_{33}\xi_{33}}\hat{f}_6(z) \end{aligned} \tag{30}$$

3. The elastic field in the plate

Introducing related functions into Eqs.(4) gives rise to the expressions for the displacements as

$$
\begin{aligned}
u =& \gamma_{a1}(-2r^{-1}\hat{f}_3^0 + r\ln\frac{r}{r_1}) + \gamma_{b0}(r^{-1}\hat{f}_4) + \gamma_{b2}(4r^{-1}\hat{f}_3^1 - r\hat{f}_4 - 2r\ln\frac{r}{r_1}z) \\
&+ \gamma_{u-1}(r^{-1}) + \gamma_{u1}(r) + \gamma_{u3}(-8r\hat{f}_3^0 + r^3) + \gamma_{w2}(-2r\hat{f}_4) + \gamma_{w4}(32r\hat{f}_3^1 - 4r^3 z) \\
&+ \gamma_{\phi 2}(-2r\hat{f}_5) \\
w =& \gamma_{a1}(-\hat{f}_{11}^0 - 2\ln\frac{r}{r_1}\hat{f}_1^0) + \gamma_{b0}\ln\frac{r}{r_1} + \gamma_{b2}(2\hat{f}_9^4 + 2\hat{f}_{11}^1 + 4\ln\frac{r}{r_1}\hat{f}_1^1 + r^2\ln\frac{r}{r_1}) \\
&+ \gamma_{u1}(-2\hat{f}_{11}^0) + \gamma_{u3}(16\hat{f}_7^0 - 4r^2\hat{f}_1^0) \\
&+ \gamma_{w0} + \gamma_{w2}(4\hat{f}_9^5 + r^2) + \gamma_{w4}(-64\hat{f}_7^1 + 16r^2\hat{f}_1^1 + r^4) \\
&+ \gamma_{\phi 2}(4\hat{f}_9^4 + 4\hat{f}_{13}) \\
\phi =& \gamma_{a1}(-\hat{f}_{12}^0 - 2\ln\frac{r}{r_1}\hat{f}_2^0) + \gamma_{b2}(2\hat{f}_{10}^4 + 2\hat{f}_{12}^1 + 4\ln\frac{r}{r_1}\hat{f}_2^1) + \gamma_{c0}\ln\frac{r}{r_1} \\
&+ \gamma_{u1}(-2\hat{f}_{12}^0) + \gamma_{u3}(16\hat{f}_8^0 - 4r^2\hat{f}_2^0) \\
&+ \gamma_{w2}(4\hat{f}_{10}^5 + r^2) + \gamma_{w4}(-64\hat{f}_8^1 + 16r^2\hat{f}_2^1) \\
&+ \gamma_{\phi 0} + \gamma_{\phi 2}(4\hat{f}_{10}^4 + 4\hat{f}_{14})
\end{aligned}
\tag{31}
$$

The stress can be written as

$$
\begin{aligned}
\sigma_r =&\gamma_{a1}(2c_{11}r^{-2}\hat{f}_3^0 + c_{11}\ln\frac{r}{r_1} + c_{11} - 2c_{12}r^{-2}\hat{f}_3^0 + c_{12}\ln\frac{r}{r_1} - c_{13}f_{11}^0 - 2c_{13}\ln\frac{r}{r_1}f_1^0 \\
&- e_{31}f_{12}^0 - 2e_{31}\ln\frac{r}{r_1}f_2^0) \\
&+ \gamma_{b0}(-c_{11}r^{-2}\hat{f}_4 + c_{12}r^{-2}\hat{f}_4) \\
&+ \gamma_{b2}(-4c_{11}r^{-2}\hat{f}_3^1 - c_{11}\hat{f}_4 - 2c_{11}\ln\frac{r}{r_1}z - 2c_{11}z + 4c_{12}r^{-2}\hat{f}_3^1 - c_{12}\hat{f}_4 - 2c_{12}\ln\frac{r}{r_1}z \\
&+ 2c_{13}f_9^4 + 2c_{13}f_{11}^1 + 4c_{13}\ln\frac{r}{r_1}f_1^1 + 2e_{31}f_{10}^4 + 2e_{31}f_{12}^1 + 4e_{31}\ln\frac{r}{r_1}f_2^1) \\
&+ \gamma_{u-1}(-c_{11}r^{-2} + c_{12}r^{-2}) \\
&+ \gamma_{u1}(c_{11} + c_{12} - 2c_{13}f_{11}^0 - 2e_{31}f_{12}^0) \\
&+ \gamma_{u3}(-8c_{11}\hat{f}_3^0 + 3c_{11}r^2 - 8c_{12}\hat{f}_3^0 + c_{12}r^2 + 16c_{13}f_7^0 - 4c_{13}r^2f_1^0 + 16e_{31}f_8^0 - 4e_{31}r^2f_2^0) \\
&+ \gamma_{w2}(-2c_{11}\hat{f}_4 - 2c_{12}\hat{f}_4 + 4e_{31}f_{10}^5) \\
&+ \gamma_{w4}(32c_{11}\hat{f}_3^1 - 12c_{11}r^2z + 32c_{12}\hat{f}_3^1 - 4c_{12}r^2z - 64c_{13}f_7^1 + 16c_{13}r^2f_1^1 - 64e_{31}f_8^1 + 16e_{31}r^2f_2^1) \\
&+ \gamma_{\phi 2}(-2c_{11}\hat{f}_5 - 2c_{12}\hat{f}_5 + 4c_{13}f_9^4 + 4c_{13}f_{13} + 4e_{31}f_{10}^4 + 4e_{31}f_{14}) \\
\sigma_\theta =&\gamma_{a1}(2c_{12}r^{-2}\hat{f}_3^0 + c_{12}\ln\frac{r}{r_1} + c_{12} - 2c_{11}r^{-2}\hat{f}_3^0 + c_{11}\ln\frac{r}{r_1} - c_{13}f_{11}^0 - 2c_{13}\ln\frac{r}{r_1}f_1^0 \\
&- e_{31}f_{12}^0 - 2e_{31}\ln\frac{r}{r_1}f_2^0) \\
&+ \gamma_{b0}(-c_{12}r^{-2}\hat{f}_4 + c_{11}r^{-2}\hat{f}_4) \\
&+ \gamma_{b2}(-4c_{12}r^{-2}\hat{f}_3^1 - c_{12}\hat{f}_4 - 2c_{12}\ln\frac{r}{r_1}z - 2c_{12}z + 4c_{11}r^{-2}\hat{f}_3^1 - c_{11}\hat{f}_4 - 2c_{11}\ln\frac{r}{r_1}z \\
&+ 2c_{13}f_9^4 + 2c_{13}f_{11}^1 + 4c_{13}\ln\frac{r}{r_1}f_1^1 + 2e_{31}f_{10}^4 + 2e_{31}f_{12}^1 + 4e_{31}\ln\frac{r}{r_1}f_2^1) \\
&+ \gamma_{u-1}(-c_{12}r^{-2} + c_{12}r^{-2}) \\
&+ \gamma_{u1}(c_{11} + c_{12} - 2c_{13}f_{11}^0 - 2e_{31}f_{12}^0) \\
&+ \gamma_{u3}(-8c_{12}\hat{f}_3^0 + 3c_{12}r^2 - 8c_{11}\hat{f}_3^0 + c_{11}r^2 + 16c_{13}f_7^0 - 4c_{13}r^2f_1^0 + 16e_{31}f_8^0 - 4e_{31}r^2f_2^0) \\
&+ \gamma_{w2}(-2c_{12}\hat{f}_4 - 2c_{11}\hat{f}_4 + 4e_{31}f_{10}^5) \\
&+ \gamma_{w4}(32c_{12}\hat{f}_3^1 - 12c_{12}r^2z + 32c_{11}\hat{f}_3^1 - 4c_{11}r^2z - 64c_{13}f_7^1 + 16c_{13}r^2f_1^1 - 64e_{31}f_8^1 + 16e_{31}r^2f_2^1) \\
&+ \gamma_{\phi 2}(-2c_{12}\hat{f}_5 - 2c_{11}\hat{f}_5 + 4c_{13}f_9^4 + 4c_{13}f_{13} + 4e_{31}f_{10}^4 + 4e_{31}f_{14}) \\
\sigma_z =&\gamma_{a1}(2c_{13}\ln\frac{r}{r_1} + c_{13} + c_{13}\ln\frac{r}{r_1} - c_{33}f_{11}^0 - 2c_{33}\ln\frac{r}{r_1}f_1^0 - e_{33}f_{12}^0 - 2e_{33}\ln\frac{r}{r_1}f_2^0) \\
&+ \gamma_{b2}(-2c_{13}\hat{f}_4 - 4c_{13}\ln\frac{r}{r_1}z - 2c_{13}z \\
&+ 2c_{33}f_9^4 + 2c_{33}f_{11}^1 + 4c_{33}\ln\frac{r}{r_1}f_1^1 + 2e_{33}f_{10}^4 + 2e_{33}f_{12}^1 + 4e_{33}\ln\frac{r}{r_1}f_2^1) \\
&+ \gamma_{u1}(2c_{13} - 2c_{33}f_{11}^0 - 2e_{33}f_{12}^0) \\
&+ \gamma_{u3}(-16c_{13}\hat{f}_3^0 + 4c_{13}r^2 + 16c_{33}f_7^0 - 4c_{33}r^2f_1^0 + 16e_{33}f_8^0 - 4e_{33}r^2f_2^0) \\
&+ \gamma_{w2}(-4c_{13}\hat{f}_4 + 4e_{33}f_{10}^5) \\
&+ \gamma_{w4}(64c_{13}\hat{f}_3^1 - 16c_{13}r^2z - 64c_{33}f_7^1 + 16c_{33}r^2f_1^1 - 64e_{33}f_8^1 + 16e_{33}r^2f_2^1) \\
&+ \gamma_{\phi 2}(-4c_{13}\hat{f}_5 + 4c_{33}f_9^4 + 4c_{33}f_{13} + 4e_{33}f_{10}^4 + 4e_{33}f_{14})
\end{aligned}
$$

$$(32)$$

$$
\begin{aligned}
\tau_{rz} =&\gamma_{a1}(-2c_{44}r^{-1}f_3^0-2c_{44}\hat{f}_1^0-2e_{15}\hat{f}_2^0)\\
&+\gamma_{b0}(c_{44}r^{-1}f_4+c_{44})\\
&+\gamma_{b2}(4c_{44}r^{-1}f_3^1-c_{44}rf_4-2c_{44}r\ln\frac{r}{r_1}+4c_{44}\hat{f}_1^1+2c_{44}r\ln\frac{r}{r_1}+c_{44}r^2+4e_{15}\hat{f}_2^1)\\
&+\gamma_{u3}(-8c_{44}rf_3^0-8c_{44}r\hat{f}_1^0-8e_{15}r\hat{f}_2^0)\\
&+\gamma_{w2}(-2c_{44}rf_4+2c_{44}r+2e_{15}r)\\
&+\gamma_{w4}(32c_{44}rf_3^1-4c_{44}r^3+32c_{44}r\hat{f}_1^1+4c_{44}r^3+32e_{15}r\hat{f}_2^1)\\
&+\gamma_{\phi2}(-2c_{44}rf_5)\\
D_r =&\gamma_{a1}(-2e_{15}r^{-1}f_3^0-2e_{15}\hat{f}_1^0+2\xi_{11}\hat{f}_2^0)\\
&+\gamma_{b0}(e_{15}r^{-1}f_4-\xi_{11})\\
&+\gamma_{b2}(e_{15}r^{-1}f_3^1-e_{15}rf_4-2e_{15}r\ln\frac{r}{r_1}+4e_{15}\hat{f}_1^1+2e_{15}r\ln\frac{r}{r_1}+e_{15}r^2-4\xi_{11}\hat{f}_2^1)\\
&+\gamma_{u3}(-8e_{15}rf_3^0-8e_{15}r\hat{f}_1^0+8\xi_{11}r\hat{f}_2^0)\\
&+\gamma_{w2}(-2e_{15}rf_4+2e_{15}r-2\xi_{11}r)\\
&+\gamma_{w4}(32e_{15}rf_3^1-4e_{15}r^3+32e_{15}r\hat{f}_1^1+4e_{15}r^3-32\xi_{11}r\hat{f}_2^1)\\
&+\gamma_{\phi2}(-2e_{15}rf_5)\\
D_z =&\gamma_{a1}(2e_{31}\ln\frac{r}{r_1}+e_{31}+e_{31}\ln\frac{r}{r_1}-e_{33}f_{11}^0-2e_{33}\ln\frac{r}{r_1}f_1^0+\xi_{33}f_{12}^0+2\xi_{33}\ln\frac{r}{r_1}f_2^0)\\
&+\gamma_{b2}(-2e_{31}\hat{f}_4-4e_{31}\ln\frac{r}{r_1}z-2e_{31}z\\
&+2e_{33}f_9^4+2e_{33}f_{11}^1+4e_{33}\ln\frac{r}{r_1}f_1^1-2\xi_{33}f_{10}^4-2\xi_{33}f_{12}^1-4\xi_{33}\ln\frac{r}{r_1}f_2^1)\\
&+\gamma_{u1}(2e_{31}-2e_{33}f_{11}^0+2\xi_{33}f_{12}^0)\\
&+\gamma_{u3}(-16e_{31}\hat{f}_3^0+4e_{31}r^2+16e_{33}f_7^0-4e_{33}r^2f_1^0-16\xi_{33}f_8^0+4\xi_{33}r^2f_2^0)\\
&+\gamma_{w2}(-4e_{31}\hat{f}_4-4\xi_{33}f_{10}^5)\\
&+\gamma_{w4}(64e_{31}\hat{f}_3^1-16e_{31}r^2z-64e_{33}f_7^1+16e_{33}r^2f_1^1+64\xi_{33}f_8^1-16\xi_{33}r^2f_2^1)\\
&+\gamma_{\phi2}(-4e_{31}\hat{f}_5+4e_{33}f_9^4+4e_{33}f_{13}-4\xi_{33}f_{10}^4-4\xi_{33}f_{14})
\end{aligned}
\tag{33}
$$

It is easy to obtain the expressions for the radial resultant force $N(r)$ and the bending moment $M(r)$ by integrating Eq.(32) as

$$
\begin{aligned}
N(r)=\int_{-h/2}^{h/2}\sigma_r\mathrm{d}z =&\gamma_{a1}L_{a1}^0+\gamma_{b0}L_{b0}^0++\gamma_{b2}L_{b2}^0+\gamma_{u-1}L_{u-1}^0+\gamma_{u1}L_{u1}^0+\gamma_{u3}L_{u3}^0\\
&+\gamma_{w0}L_{u3}^0+\gamma_{w4}L_{w4}^0+\gamma_{\phi2}L_{\phi2}^0\\
M(r)=\int_{-h/2}^{h/2}\sigma_r z\mathrm{d}z =&\gamma_{a1}L_{a1}^1+\gamma_{b0}L_{b0}^1++\gamma_{b2}L_{b2}^1+\gamma_{u-1}L_{u-1}^1+\gamma_{u1}L_{u1}^1+\gamma_{u3}L_{u3}^1\\
&+\gamma_{w0}L_{u3}^1+\gamma_{w4}L_{w4}^1+\gamma_{\phi2}L_{\phi2}^1
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
L_{a1}^{n} =& \int_{-h/2}^{h/2} z^{n}(2c_{11}r^{-2}\hat{f}_{3}^{0} + c_{11}\ln\frac{r}{r_1} + c_{11} - 2c_{12}r^{-2}\hat{f}_{3}^{0} + c_{12}\ln\frac{r}{r_1} - c_{13}f_{11}^{0} - 2c_{13}\ln\frac{r}{r_1}f_{1}^{0} \\
&- e_{31}f_{12}^{0} - 2e_{31}\ln\frac{r}{r_1}f_{2}^{0})z^{n}\mathrm{d}z \\
L_{b0}^{n} =& \int_{-h/2}^{h/2} (-c_{11}r^{-2}\hat{f}_{4} + c_{12}r^{-2}\hat{f}_{4})z^{n}\mathrm{d}z \\
L_{b2}^{n} =& \int_{-h/2}^{h/2} (-4c_{11}r^{-2}\hat{f}_{3}^{1} - c_{11}\hat{f}_{4} - 2c_{11}\ln\frac{r}{r_1}z - 2c_{11}z + 4c_{12}r^{-2}\hat{f}_{3}^{1} - c_{12}\hat{f}_{4} - 2c_{12}\ln\frac{r}{r_1}z \\
&+ 2c_{13}f_{9}^{4} + 2c_{13}f_{11}^{1} + 4c_{13}\ln\frac{r}{r_1}f_{1}^{1} + 2e_{31}f_{10}^{4} + 2e_{31}f_{12}^{1} + 4e_{31}\ln\frac{r}{r_1}f_{2}^{1})z^{n}\mathrm{d}z \\
L_{u-1}^{n} =& \int_{-h/2}^{h/2} (-c_{11}r^{-2} + c_{12}r^{-2})z^{n}\mathrm{d}z \\
L_{u1}^{n} =& \int_{-h/2}^{h/2} (c_{11} + c_{12} - 2c_{13}f_{11}^{0} - 2e_{31}f_{12}^{0})z^{n}\mathrm{d}z \\
L_{u3}^{n} =& \int_{-h/2}^{h/2} (-8c_{11}\hat{f}_{3}^{0} + 3c_{11}r^{2} - 8c_{12}\hat{f}_{3}^{0} + c_{12}r^{2} + 16c_{13}f_{7}^{0} - 4c_{13}r^{2}f_{1}^{0} + 16e_{31}f_{8}^{0} - 4e_{31}r^{2}f_{2}^{0})z^{n}\mathrm{d}z \\
L_{w2}^{n} =& \int_{-h/2}^{h/2} (-2c_{11}\hat{f}_{4} - 2c_{12}\hat{f}_{4} + 4e_{31}f_{10}^{5})z^{n}\mathrm{d}z \\
L_{w4}^{n} =& \int_{-h/2}^{h/2} (32c_{11}\hat{f}_{3}^{1} - 12c_{11}r^{2}z + 32c_{12}\hat{f}_{3}^{1} - 4c_{12}r^{2}z - 64c_{13}f_{7}^{1} \\
&+ 16c_{13}r^{2}f_{1}^{1} - 64e_{31}f_{8}^{1} + 16e_{31}r^{2}f_{2}^{1})z^{n}\mathrm{d}z \\
L_{\phi2}^{n} =& \int_{-h/2}^{h/2} (-2c_{11}\hat{f}_{5} - 2c_{12}\hat{f}_{5} + 4c_{13}f_{9}^{4} + 4c_{13}f_{13} + 4e_{31}f_{10}^{4} + 4e_{31}f_{14})z^{n}\mathrm{d}z
\end{aligned}
\tag{35}
$$

According to Eq. (33), the resultant shear force $Q(r)$ can be obtained as

$$
\begin{aligned}
Q(r) =& \int_{-h/2}^{h/2} \tau_{rz}\mathrm{d}z \\
=&\gamma_{a1}\Gamma_{a1} + \gamma_{b0}\Gamma_{b0} + +\gamma_{b2}\Gamma_{b2} + \gamma_{u3}\Gamma_{u3} + \gamma_{w2}\Gamma_{w2} + \gamma_{w4}\Gamma_{w4} + \gamma_{\phi2}\Gamma_{\phi2}
\end{aligned}
\tag{36}
$$

where

$$
\begin{aligned}
\Gamma_{a1} &= \int_{-h/2}^{h/2} z^n(-2c_{44}r^{-1}f_3^0 - 2c_{44}\hat{f}_1^0 - 2e_{15}\hat{f}_2^0)\mathrm{d}z \\
\Gamma_{b0} &= \int_{-h/2}^{h/2} (c_{44}r^{-1}f_4 + c_{44})\mathrm{d}z \\
\Gamma_{b2} &= \int_{-h/2}^{h/2} (4c_{44}r^{-1}f_3^1 - c_{44}rf_4 - 2c_{44}r\ln\frac{r}{r_1} + 4c_{44}\hat{f}_1^1 + 2c_{44}r\ln\frac{r}{r_1} + c_{44}r^2 + 4e_{15}\hat{f}_2^1)\mathrm{d}z \\
\Gamma_{u3} &= \int_{-h/2}^{h/2} (-8c_{44}rf_3^0 - 8c_{44}r\hat{f}_1^0 - 8e_{15}r\hat{f}_2^0)\mathrm{d}z \\
\Gamma_{w2} &= \int_{-h/2}^{h/2} (-2c_{44}rf_4 + 2c_{44}r + 2e_{15}r)\mathrm{d}z \\
\Gamma_{w4} &= \int_{-h/2}^{h/2} (32c_{44}rf_3^1 - 4c_{44}r^3 + 32c_{44}r\hat{f}_1^1 + 4c_{44}r^3 + 32e_{15}r\hat{f}_2^1)\mathrm{d}z \\
\Gamma_{\phi 2} &= \int_{-h/2}^{h/2} (-2c_{44}rf_5)\mathrm{d}z
\end{aligned}
\tag{37}
$$

4. Integral constants

There are total 12 integral constants to be determined. They are γ_{a1}, γ_{b0}, γ_{b2}, γ_{c0}, γ_{u-1}, γ_{u1}, γ_{u3}, γ_{w0}, γ_{w2}, γ_{w4}, $\gamma_{\phi 0}$ and $\gamma_{\phi 2}$.

4.1. Simply supported plate

For a FGM plate with simply supported edges at $r = r_i$(i = 0, 1), the cylindrical boundary conditions are

$$
w(r_i, 0) = 0,\ \phi(r_i, 0) = 0,\ N(r_i) = 0,\ M(r_i) = 0\ (i = 0, 1) \tag{38}
$$

Introducing Eqs. (31) and (34) into Eq. (38) results in eight equations, corresponding to boundary conditions at $r = r_0$and $r = r_1$. Furthermore, combining another four conditions in Eqs. (17), (22) and (26), there constitute twelve simultaneous linear algebraic equations, which can fully determine the explicit expression forms of displacements.

4.2. Clamped plate

In this circumstance, the boundary conditions at $r = r_i$ (i=0, 1) are

$$
u(r_i, 0) = 0\ w(r_i, 0) = 0,\ w_{,r}(r_i, 0) = 0,\ \phi(r_i, 0) = 0,\ (i = 0, 1) \tag{39}
$$

Substituting Eqs. (31) and (34) into the conditions above, and linking with Eqs. (17), (22) and (26) form 12 linear algebraic equations eventually. Therefore, all the integral constants in the displacement functions are determined.

4.3. Simply supported-clamped plate

The boundary conditions are

$$N(r_0) = 0,\ M(r_0) = 0,\ w_{,r}(r_0, 0) = 0,\ \phi(r_0, 0) = 0, \tag{40}$$

$$u(r_1, 0) = 0\ w(r_1, 0) = 0,\ w_{,r}(r_1, 0) = 0,\ \phi(r_1, 0) = 0, \tag{41}$$

The integral constants in the displacement functions can be determined when combining Eqs. (40) and (41) with Eqs. (17), (22) and (26).

4.4. Clamped-simply supported plate

In this case, the plate is internally clamped and externally simply supported, the boundary conditions $r = r_i$(i=0,1) are then presented as

$$u(r_0, 0) = 0\ w(r_0, 0) = 0,\ w_{,r}(r_0, 0) = 0,\ \phi(r_0, 0) = 0, \tag{42}$$

$$N(r_1) = 0,\ M(r_1) = 0,\ w_{,r}(r_1, 0) = 0,\ \phi(r_1, 0) = 0, \tag{43}$$

The integral constants in the displacement functions can be determined in the same way.

4.5. Free-clamped plate

In this instance, the boundary conditions at the cylindrical surfaces read

$$N(r_0) = 0,\ M(r_0) = 0,\ Q(r_0) = 0,\ \phi(r_0, 0) = 0, \tag{44}$$

$$u(r_1, 0) = 0\ w(r_1, 0) = 0,\ w_{,r}(r_1, 0) = 0,\ \phi(r_0, 0) = 0, \tag{45}$$

Hence all the integral constants are fixed by solving Eqs. (44), (45) (17), (22) and (26) simultaneously. On the basis of analysis above, the displacements and stresses of the plates with diverse edge conditions can be expressed in an explicit fashion.

Appendix A. Discussion on the form of Eq.(4)

This section is devoted to justification of the form of the displacements in Eq. (4). According to the form of displacements in a homogeneous annular plate presented in [45], we assume

$$\begin{aligned}
u(r, z) &= \sum_{m=-\infty}^{\infty} r^m \left[u_m(z) + A_m(z) \ln(r/r_1)\right] \\
w(r, z) &= \sum_{m=-\infty}^{\infty} r^m [w_m(z) + B_m(z) \ln(r/r_1)] \\
\phi(r, z) &= \sum_{m=-\infty}^{\infty} r^m [\phi_m(z) + C_m(z) \ln(r/r_1)]
\end{aligned} \tag{A.1}$$

where $u_m(z), w_m(z), \phi_m(z), A_m(z), B_m(z)$ and $C_m(z)$ are referred to as the displacement functions.

Introducing Eq. (A.1) into constitutive equation in Eq. (2) obtain the stress and electric displacement as

$$\begin{aligned}\sigma_r = \sum_{m=-\infty}^{\infty} r^m\{&[(m+1)c_{11}+c_{12}]u_{m+1}+c_{11}A_{m+1}+c_{13}w_{m,z}+e_{31}\phi_{m,z}\\&+[(m+1)c_{11}A_{m+1}+c_{12}A_{m+1}+c_{13}B_{m,z}+e_{31}C_{m,z}]\ln(r/r_1)\}\end{aligned} \tag{A.2a}$$

$$\begin{aligned}\sigma_\theta = \sum_{m=-\infty}^{\infty} r^m\{&[(m+1)c_{12}+c_{11}]u_{m+1}+c_{12}A_{m+1}+c_{13}w_{m,z}+e_{31}\phi_{m,z}\\&+[(m+1)c_{12}A_{m+1}+c_{11}A_{m+1}+c_{13}B_{m,z}+e_{31}C_{m,z}]\ln(r/r_1)\}\end{aligned} \tag{A.2b}$$

$$\begin{aligned}\sigma_z = \sum_{m=-\infty}^{\infty} r^m\{&[(m+2)c_{13}u_{m+1}+c_{13}A_{m+1}+c_{33}w_{m,z}+e_{33}\phi_{m,z}\\&+[(m+2)c_{13}A_{m+1}+c_{33}B_{m,z}+e_{33}C_{m,z}]\ln(r/r_1)\}\end{aligned} \tag{A.2c}$$

$$\begin{aligned}\tau_{rz} = \sum_{m=-\infty}^{\infty} r^m\{&c_{44}u_{m,z}+(m+1)c_{44}w_{m+1}+c_{44}B_{m+1}+(m+1)e_{15}\phi_{m+1}\\&+e_{15}C_{m+1}+[c_{44}A_{m,z}+(m+1)c_{44}B_{m+1}+(m+1)e_{15}C_{m+1}]\ln(r/r_1)\}\end{aligned} \tag{A.2d}$$

$$\begin{aligned}D_r = \sum_{m=-\infty}^{\infty} r^m\{&e_{15}u_{m,z}+(m+1)e_{15}w_{m+1}+e_{15}B_{m+1}-(m+1)\xi_{11}\phi_{m+1}\\&-\xi_{11}C_{m+1}+[e_{15}A_{m,z}+(m+1)e_{15}B_{m+1}-(m+1)\xi_{11}C_{m+1}]\ln(r/r_1)\}\end{aligned} \tag{A.2e}$$

$$\begin{aligned}D_z = \sum_{m=-\infty}^{\infty} r^m\{&(m+2)e_{31}u_{m+1}+e_{31}A_{m+1}+e_{33}w_{m,z}-\xi_{33}\phi_{m,z}\\&+[(m+2)e_{31}A_{m+1}+e_{33}B_{m,z}-\xi_{33}C_{m,z}]\ln(r/r_1)\}\end{aligned} \tag{A.2f}$$

Substituting Eq. (A.2) into equilibrium equation Eq. (1) and comparing the coefficients of r^m and $r^m\ln(r/r_1)$ at the two ends, we obtain

$$\begin{aligned}&[c_{44}A_{m+1,z}+(m+2)c_{44}B_{m+2}+(m+2)e_{15}C_{m+2}]_{,z}\\&+(m+2)(m+4)c_{11}A_{m+3}+(m+2)c_{13}B_{m+2,z}+(m+2)e_{31}C_{m+2,z}=0\end{aligned} \tag{A.3a}$$

$$\begin{aligned}&[c_{44}u_{m+1,z}+(m+2)c_{44}w_{m+2}+c_{44}B_{m+2}+(m+2)e_{15}\phi_{m+2}+e_{15}C_{m+2}]_{,z}\\&+(m+2)(m+4)c_{11}u_{m+3}+2(m+3)c_{11}A_{m+3}+(m+2)c_{13}w_{m+2,z}\\&+c_{13}B_{m+2,z}+(m+2)e_{31}\phi_{m+2,z}+e_{31}C_{m+2,z}=0\end{aligned} \tag{A.3b}$$

$$\begin{aligned}&[(m+2)c_{13}A_{m+1}+c_{33}B_{m,z}+e_{33}C_{m,z}]_{,z}\\&+(m+2)c_{44}A_{m+1,z}+(m+2)^2c_{44}B_{m+2}+(m+2)^2e_{15}C_{m+2}=0\end{aligned} \tag{A.3c}$$

$$
\begin{aligned}
&[(m+2)c_{13}u_{m+1}+c_{13}A_{m+1}+c_{33}w_{m,z}+e_{33}\phi_{m,z}]_{,z}\\
&+(m+2)c_{44}u_{m+1,z}+c_{44}A_{m+1,z}+(m+2)^2c_{44}w_{m+2}+2(m+2)c_{44}B_{m+2}\\
&+(m+2)^2e_{15}\phi_{m+2}+2(m+2)e_{15}C_{m+2}=0
\end{aligned}
\tag{A.3d}
$$

$$
\begin{aligned}
&[(m+2)e_{31}A_{m+1}+e_{33}B_{m,z}-\xi_{33}C_{m,z}]_{,z}\\
&+(m+2)e_{15}A_{m+1,z}+(m+2)^2e_{15}B_{m+2}-(m+2)^2\xi_{11}C_{m+2}=0
\end{aligned}
\tag{A.3e}
$$

$$
\begin{aligned}
&[(m+2)e_{31}u_{m+1}+e_{31}A_{m+1}+e_{33}w_{m,z}-\xi_{33}\phi_{m,z}]_{,z}\\
&+(m+2)e_{15}u_{m+1,z}+e_{15}A_{m+1,z}+(m+2)^2e_{15}w_{m+2}+2(m+2)e_{15}B_{m+2}\\
&-(m+2)^2\xi_{11}\phi_{m+2}-2(m+2)\xi_{11}C_{m+2}=0
\end{aligned}
\tag{A.3f}
$$

Substituting Eqs. (A.2c), (A.2d) and (A.2e) into Eq. (3) and comparing the coefficients of r^m and $r^m\ln(r/r_1)$, we can derive

$$
[c_{44}A_{m+1,z}+(m+2)c_{44}B_{m+2}+(m+2)e_{15}C_{m+2}]\big|_{z=\pm h/2}=0
\tag{A.4a}
$$

$$
\begin{aligned}
&[c_{44}u_{m+1,z}+(m+2)c_{44}w_{m+2}+c_{44}B_{m+2}+(m+2)e_{15}\phi_{m+2}\\
&+e_{15}C_{m+2}]\big|_{z=\pm h/2}=0
\end{aligned}
\tag{A.4b}
$$

$$
[(m+2)e_{31}A_{m+1}+e_{33}B_{m,z}-\xi_{33}C_{m,z}]\big|_{z=\pm h/2}=0
\tag{A.4c}
$$

$$
[(m+2)e_{31}u_{m+1}+e_{31}A_{m+1}+e_{33}w_{m,z}-\xi_{33}\phi_{m,z}]\big|_{z=\pm h/2}=0
\tag{A.4d}
$$

$$
[(m+2)c_{13}A_{m+1}+c_{33}B_{m,z}+e_{33}C_{m,z}]\big|_{z=\pm h/2}=0
\tag{A.4e}
$$

$$
\begin{aligned}
&[(m+2)c_{13}u_{m+1}+c_{13}A_{m+1}+c_{33}w_{m,z}+e_{33}\phi_{m,z}]\big|_{z=h/2}=-q\delta_{m0}\\
&[(m+2)c_{13}u_{m+1}+c_{13}A_{m+1}+c_{33}w_{m,z}+e_{33}\phi_{m,z}]\big|_{z=-h/2}=0
\end{aligned}
\tag{A.4f}
$$

where δ_{m0} is the Kronecker delta.

When $m\neq-2,\ m\neq-4$, if we let

$$
A_{m+1}=0,\ u_{m+1}=0,\ B_m=0,\quad w_m=0, C_m=0,\ \phi_m=0,
\tag{A.5}
$$

and substitute Eq. (A.5) into Eq. (A.3), we can obtain

$$
\begin{aligned}
&A_{m+3}=0,\ u_{m+3}=0,\ B_{m+2}=0,\\
&w_{m+2}=0,\ C_{m+2}=0,\ \phi_{m+2}=0.
\end{aligned}
\tag{A.6}
$$

According to Eqs. (A.5) and . (A.6), we assume all the items with m being odd number, i.e. $m=2n-1,\ n\in Z$, will vanish, thus we have

$$
\begin{aligned}
&A_{2n+2}=0,\ u_{2n+2}=0,\ B_{2n+2}=0,\\
&w_{2n+1}=0,\ \phi_{2n+1}=0,\ C_{2n+1}=0.\qquad n\in Z
\end{aligned}
\tag{A.7}
$$

In this case, the boundary conditions in Eq. (A.4) are automatically satisfied. For the case of m is even number, we also assume

$$\begin{aligned} &A_{2n+3}=0,\ u_{2n+3}=0,\ B_{2n+2}=0,\\ &w_{2n+2}=0,\ C_{2n+2}=0,\ \phi_{2n+2}=0. \qquad n\in Z \ \text{ and } \ |n|>2 \end{aligned} \tag{A.8}$$

without loss of generality. In this case, the boundary conditions are also satisfied.

Therefor, we should discuss the cases of $m=-4,-2,0,2,$ and 4 step by step to determine which items in Eq. (A.1) should be retained.

Case $m=-4$

For $m=-4$, using $A_{-3}=0,\ u_{-3}=0,\ B_{-4},\ w_{-4}=0, \phi_{-4}=0,\ C_{-4}=0,$, according Eq. (A.8) with $n=-3$, we can obtain from Eqs. (A.3) that

$$\left[c_{44}B_{-2}+c_{44}C_{-2}\right]_{,z}+c_{13}B_{-2,z}+e_{31}C_{-2,z}=0 \tag{A.9a}$$

$$\begin{aligned} &\left[-2c_{44}w_{-2}+c_{44}B_{-2}-2e_{15}\phi_{-2}+e_{15}C_{-2}\right]_{,z}\\ &-2c_{11}A_{-1}-2c_{13}w_{-2,z}+c_{13}B_{-2,Z}-2e_{31}\phi_{-2,z}+e_{31}C_{-2,z}=0 \end{aligned} \tag{A.9b}$$

$$c_{44}B_{-2}+e_{15}C_{-2}=0 \tag{A.9c}$$

$$c_{44}w_{-2}-c_{44}B_{-2}+e_{15}\phi_{-2}-e_{15}C_{-2}=0 \tag{A.9d}$$

$$e_{15}B_{-2}-\xi_{11}C_{-2}=0 \tag{A.9e}$$

$$e_{15}w_{-2}-e_{15}B_{-2}-\xi_{11}\phi_{-2}+\xi_{11}C_{-2}=0 \tag{A.9f}$$

Solving Eq. (A.9) can give

$$A_{-1}=0,\ B_{-2}=0,\ C_{-2}=0,\ w_{-2}=0,\ \phi_{-2}=0. \tag{A.10}$$

Case $m=-2$

When $m=-2$, except for Eq. (A.3b) a in Eq. (A.3), all the other items are satisfied automatically. In this case, Eq. (A.3b) can be rewritten as

$$\left[c_{44}u_{-1,z}+c_{44}B_0+e_{15}c_0\right]_{,z}+2c_{11}A_1+c_{13}B_{0,z}+e_{31}C_{0,z}=0, \tag{A.11}$$

and the corresponding boundary condition in Eq. (A.4b) can be rewritten as

$$\left[c_{44}u_{-1,z}+c_{44}B_0+e_{15}C_0\right]\big|_{z=\pm h/2}=0. \tag{A.12}$$

Case $m=4$

In the case of m=4, by making use of $A_7=0,\ u_7=0,\ B_6,\ w_6=0, \phi_6=0,\ C_6=0.$ according Eq. (A.8) with $n=3$, we can obtain from Eqs. (A.3) that

$$\left[c_{44}A_{5,z}\right]_{,z}=0 \tag{A.13a}$$

$$[c_{44}u_{5,z}]_{,z} = 0 \tag{A.13b}$$

$$[6c_{13}A_5 + c_{33}B_{4,z} + e_{33}C_{4,z}]_{,z} + 6c_{44}A_{5,z} = 0 \tag{A.13c}$$

$$[6c_{13}u_5 + c_{13}A_5 + c_{33}w_{4,z} + e_{33}\phi_{4,z}]_{,z} + 6c_{44}u_{5,z} + c_{44}A_{5,z} = 0 \tag{A.13d}$$

$$[6e_{31}A_5 + e_{33}B_{4,z} - \xi_{33}\phi_{4,z}]_{,z} + 6e_{15}A_{5,z} = 0 \tag{A.13e}$$

$$[6e_{31}u_5 + e_{31}A_5 + e_{33}w_{4,z} - \xi_{33}\phi_{4,z}]_{,z} + 6e_{15}u_{5,z} + e_{15}A_{5,z} = 0. \tag{A.13f}$$

According to the boundary in Eq. (A.4), we can obtain

$$[c_{44}A_{5,z}]\big|_{z=\pm h/2} = 0 \tag{A.14a}$$

$$[c_{44}u_{5,z}]\big|_{z=\pm h/2} = 0 \tag{A.14b}$$

$$[6c_{31}A_5 + c_{33}B_{4,z} + e_{33}c_{4,z}]\big|_{z=\pm h/2} = 0 \tag{A.14c}$$

$$[6c_{31}u_5 + c_{31}A_5 + c_{33}w_{4,z} + e_{33}\phi_{m,z}]\big|_{z=\pm h/2} = 0 \tag{A.14d}$$

$$[6e_{31}A_5 + e_{33}B_{4,z} - \xi_{33}c_{4,z}]\big|_{z=\pm h/2} = 0 \tag{A.14e}$$

$$[6e_{31}u_5 + e_{31}A_5 + e_{33}w_{4,z} - \xi_{33}\phi_{4,z}]\big|_{z=\pm h/2} = 0. \tag{A.14f}$$

Integrating Eqs. (A.13a) and (A.13b) and with the aid of Eqs. (A.14a) and (A.14b), respectively, we obtain

$$u_{5,z} = 0,\ A_{5,z} = 0, \tag{A.15}$$

which immediately gives

$$u_5 = a_1,\ A_5 = a_2. \tag{A.16}$$

Then, integrating Eqs.(A.13c), (A.13d), (A.13e) and (A.13f) with the aid of Eqs. (A.14c), (A.14d), (A.14e) and (A.14f), respectively, we can obtain

$$c_{33}B_{4,z} + e_{33}c_{4,z} = -6c_{31}a_2 \tag{A.17a}$$

$$c_{33}w_{4,z} + e_{33}\phi_{m,z} = -6c_{31}a_1 - c_{31}a_2 \tag{A.17b}$$

$$e_{33}B_{4,z} - \xi_{33}c_{4,z} = -6e_{31}a_2 \tag{A.17c}$$

$$e_{33}w_{4,z} - \xi_{33}\phi_{4,z} = -6e_{31}a_1 - e_{31}a_2 \tag{A.17d}$$

Solving Eq. (A.17) gives

$$B_{4,z} = -6a_2\frac{e_{31}e_{33} + \xi_{33}c_{31}}{e_{33}^2 + \xi_{33}c_{33}} \tag{A.18a}$$

$$C_{4,z} = -6a_2\frac{e_{33}c_{31} - c_{33}e_{31}}{e_{33}^2 + \xi_{33}c_{33}} \tag{A.18b}$$

$$w_{4,z} = (-6a_1 + a_2)\frac{e_{31}e_{33} + \xi_{33}c_{31}}{e_{33}^2 + \xi_{33}c_{33}} \tag{A.18c}$$

$$\phi_{4,z} = (-6a_1 + a_2)\frac{e_{33}c_{31} - c_{33}e_{31}}{e_{33}^2 + \xi_{33}c_{33}}. \tag{A.18d}$$

Case $m = 2$

When $m = 2$, Eqs. (A.3) can be rewritten as

$$[c_{44}A_{3,z} + 4c_{44}B_4 + 4e_{15}C_4]_{,z} + 24c_{11}A_5 + 4c_{13}B_{4,z} + 4e_{31}C_{4,z} = 0 \tag{A.19a}$$

$$\begin{aligned}&\left[c_{44}u_{3,z} + 4c_{44}w_4 + c_{44}B_4 + 4e_{15}\phi_4 + e_{15}C_4\right]_{,z}\\ &+ 24c_{11}u_5 + 10c_{11}A_5 + 4c_{13}w_{4,z} + c_{13}B_{4,z} + 4e_{31}\phi_{4,z} + e_{31}C_{4,z} = 0\end{aligned} \tag{A.19b}$$

$$[4c_{13}A_3 + c_{33}B_{2,z} + e_{33}C_{2,z}]_{,z} + 4c_{44}A_{3,z} + 16c_{44}B_4 + 16e_{15}C_4 = 0 \tag{A.19c}$$

$$\begin{aligned}&[4c_{13}u_3 + c_{13}A_3 + c_{33}w_{2,z} + e_{33}\phi_{2,z}]_{,z}\\ &+ 4c_{44}u_{3,z} + c_{44}A_{3,z} + 16c_{44}w_4 + 8c_{44}B_4 + 16e_{15}\phi_4 + 8c_{44}C_4 = 0\end{aligned} \tag{A.19d}$$

$$[4e_{31}A_3 + e_{33}B_{2,z} - \xi_{33}C_{2,z}]_{,z} + 4e_{15}A_{3,z} + 16e_{15}B_4 - 16\xi_{11}C_4 = 0 \tag{A.19e}$$

$$\begin{aligned}&\left[4c_{13}u_3 + c_{13}A_3 + e_{33}w_{2,z} - \xi_{33}\phi_{2,z}\right]_{,z}\\ &+ 4e_{15}u_{3,z} + e_{15}A_{3,z} + 16e_{15}w_4 + 8e_{15}B_4 - 16\xi_{11}\phi_4 + 8\xi_{11}C_4 = 0\end{aligned} \tag{A.19f}$$

Correspondingly, the boundary conditions in Eq. (A.4) can be rewritten as

$$[c_{44}A_{3,z} + 4c_{44}B_4 + 4e_{15}C_4]\big|_{z=\pm h/2} = 0 \tag{A.20a}$$

$$[c_{44}u_{3,z} + 4c_{44}w_4 + c_{44}B_4 + 4e_{15}\phi_4 + e_{15}C_4]\big|_{z=\pm h/2} = 0 \tag{A.20b}$$

$$[4e_{31}A_3 + e_{33}B_{2,z} - \xi_{33}C_{2,z}]\big|_{z=\pm h/2} = 0 \tag{A.20c}$$

$$[4e_{31}u_3 + e_{31}A_3 + e_{33}w_{2,z} - \xi_{33}\phi_{2,z}]\big|_{z=\pm h/2} = 0 \tag{A.20d}$$

$$[4c_{13}A_3 + c_{33}B_{2,z} + e_{33}C_{2,z}]\big|_{z=\pm h/2} = 0 \tag{A.20e}$$

$$[4c_{13}u_3 + c_{13}A_3 + c_{33}w_{2,z} + e_{33}\phi_{2,z}]\big|_{z=\pm h/2} = 0 \tag{A.20f}$$

Substituting $A_5 = a_2$ and Eqs. (A.18a) and (A.18b) into Eq. (A.19a), and with the aid of boundary condition in Eq. (A.20a), we can obtain

$$\begin{aligned}&[4c_{44}B_4 + c_{44}A_{3,z}) + 4e_{15}C_4\big|_{z=h/2}\\ &= -24a_2\int_{-h/2}^{h/2}\left[c_{11} - \frac{e_{31}e_{33}e_{13} + c_{31}^2c_{33} + e_{33}c_{31}e_{31} - e_{31}^2c_{33}}{e_{33}^2 + \xi_{33}c_{33}}\right]d\eta\\ &= 0,\end{aligned} \tag{A.21}$$

which results in

$$A_5 = a_2 = 0. \tag{A.22}$$

Thus, both $B_{4,z}$ and $C_{4,z}$ in Eqs. (A.18a) and (A.18b) will be zeros, and we can let $B_4 = a_3$ and $C_4 = a_4$. Then form Eq. (A.22) we can obtain

$$A_{3,z} = -4a_4\frac{e_{15}}{c_{44}} - 4a_3 \tag{A.23}$$

Similarly, substituting $u_5 = a_1$ and Eqs. (A.18c) and (A.18d) into Eq. (A.19b), and with the aid of boundary condition in Eq. (A.20b), we can obtain

$$\begin{aligned}&\left[c_{44}u_{3,z} + 4c_{44}w_4 + c_{44}B_4 + 4e_{15}\phi_4 + e_{15}C_4\right]\big|_{z=h/2} \\ &= -24a_1\int_{-h/2}^{h/2}\left[c_{11} - c_{13}\frac{e_{31}e_{33}+\xi_{33}c_{31}}{e_{33}^2+\xi_{33}c_{33}} - e_{31}\frac{e_{33}c_{31}-c_{33}e_{31}}{e_{33}^2+\xi_{33}c_{33}}\right]d\eta \quad = 0,\end{aligned} \tag{A.24}$$

which indicates

$$u_5 = a_1 = 0. \tag{A.25}$$

Then both $w_{4,z}$ and $\phi_{4,z}$ in Eqs. (A.18a) and (A.18b) will be zeros, and we can write them as $w_4 = a_5$ and $\phi_4 = a_6$, respectively.

Combining Eq. (A.19d) and Eq. (A.20f), we can derive

$$\left[4c_{13}u_3 + c_{13}A_3 + c_{33}w_{2,z} + e_{33}\phi_{2,z}\right]\Big|_{z=h/2} = 4a_4\int_{-h/2}^{h/2} e_{15}d\eta = 0, \tag{A.26}$$

which lead to

$$C_4 = a_4 = 0. \tag{A.27}$$

Then, integrating Eq. (A.23) will give

$$A_3 = -4a_3z + a_7, \tag{A.28}$$

where a_7 is a constant. Similarly, the combination of Eq. (A.19f) and Eq. (A.20d) gives

$$\left[4c_{13}u_3 + c_{13}A_3 + e_{33}w_{2,z} - \xi_{33}\phi_{2,z}\right]\Big|_{z=h/2} = 16a_6\int_{-h/2}^{h/2}\left(\frac{e_{15}^2}{c_{44}} + c_{31}\right)d\eta = 0, \tag{A.29}$$

and leads to

$$\phi_4 = a_6 = 0 \tag{A.30}$$

Then, noticing $A_{3,z} = -a_3$ and $B_4 = a_3$, from Eq. (A.19c) and Eq. (A.19e), we can obtain simultaneous equations about $B_{2,z}$ and $C_{2,z}$ as

$$c_{33}B_{2,z} + e_{33}C_{2,z} = -4c_{13}A_3 \tag{A.31a}$$

$$e_{33}B_{2,z} - \xi_{33}C_{2,z} = -4e_{31}A_3. \tag{A.31b}$$

And then solution for $B_{2,Z}$ and $C_{2,Z}$ can be expressed as

$$B_{2,z} = 16a_3 z\frac{e_{31}e_{33} + \xi_{33}c_{31}}{e_{33}^2 + \xi_{33}c_{33}} - 4a_7\frac{e_{31}e_{33} + \xi_{33}c_{31}}{e_{33}^2 + \xi_{33}c_{33}} \tag{A.32a}$$

$$C_{2,z} = 16a_3 z\frac{c_{13}e_{33} - c_{33}e_{31}}{e_{33}^2 + \xi_{33}c_{33}} - 4a_7\frac{e_{31}e_{33} + \xi_{33}c_{31}}{e_{33}^2 + \xi_{33}c_{33}} \tag{A.32b}$$

Case $m = 0$

In the end, for the case of $m = 0$, Eqs. (A.3) will be

$$\left[c_{44}A_{1,z} + 2c_{44}B_2 + 2e_{15}C_2\right]_{,z} + 8c_{11}A_3 + 2c_{13}B_{2,z} + 2e_{31}C_{2,z} = 0 \tag{A.33a}$$

$$\begin{aligned}&\left[c_{44}u_{1,z} + 2c_{44}w_2 + c_{44}B_2 + 2e_{15}\phi_2 + e_{15}C_2\right]_{,z}\\ &+ 8c_{11}u_3 + 6c_{11}A_3 + 2c_{13}w_{2,z} + c_{13}B_{2,z} + 2e_{31}\phi_{2,z} + e_{31}C_{2,z} = 0\end{aligned} \tag{A.33b}$$

$$\left[2c_{13}A_1 + c_{33}B_{0,z} + e_{33}C_{0,z}\right]_{,z} + c_{44}A_{1,z} + 4c_{44}B_2 + 4e_{15}C_2 = 0 \tag{A.33c}$$

$$\begin{aligned}&\left[2c_{13}u_1 + c_{13}A_1 + c_{33}w_{0,z} + e_{33}\phi_{0,z}\right]_{,z}\\ &+ 2c_{44}u_{1,z} + c_{44}A_{1,z} + 4c_{44}w_2 + 4c_{44}B_2 + 4e_{15}\phi_2 + 4e_{15}C_2 = 0\end{aligned} \tag{A.33d}$$

$$\left[2e_{31}A_1 + e_{33}B_{1,z} - \xi_{33}C_{0,z}\right]_{,z} + e_{15}A_{1,z} + 4e_{15}B_2 - 4\xi_{11}C_2 = 0 \tag{A.33e}$$

$$\begin{aligned}&\left[2e_{31}u_1 + e_{31}A_1 + e_{33}w_{0,z} - \xi_{33}\phi_{0,z}\right]_{,z}\\ &+ 2e_{15}u_{1,z} + e_{15}A_{1,z} + 4e_{15}w_2 + 4e_{15}B_2 - 4\xi_{11}\phi_2 - 4\xi_{11}C_2 = 0\end{aligned} \tag{A.33f}$$

Correspondingly, the boundary condition in Eqs. (A.4) becomes

$$\left[c_{44}A_{1,z} + 2c_{44}B_2 + 2e_{15}C_2\right]\Big|_{z=\pm h/2} = 0 \tag{A.34a}$$

$$\left[c_{44}u_{1,z} + 2c_{44}w_2 + c_{44}B_2 + 2e_{15}\phi_2 + e_{15}C_2\right]\Big|_{z=\pm h/2} = 0 \tag{A.34b}$$

$$\left[2e_{31}A_1 + e_{33}B_{1,z} - \xi_{33}C_{0,z}\right]\Big|_{z=\pm h/2} = 0 \tag{A.34c}$$

$$\left[2e_{31}u_1 + e_{31}A_1 + e_{33}w_{0,z} - \xi_{33}\phi_{0,z}\right]\Big|_{z=\pm h/2} = 0 \tag{A.34d}$$

$$\left[2c_{13}A_1 + c_{33}B_{0,z} + e_{33}C_{0,z}\right]\Big|_{z=\pm h/2} = 0 \tag{A.34e}$$

$$\begin{aligned}\left[2c_{13}u_1 + c_{13}A_1 + c_{33}w_{0,z} + e_{33}\phi_{0,z}\right]\Big|_{z=\ h/2} &= -p\\ \left[2c_{13}u_1 + c_{13}A_1 + c_{33}w_{0,z} + e_{33}\phi_{0,z}\right]\Big|_{z=-h/2} &= \ 0\end{aligned} \tag{A.34f}$$

Substituting Eqs. (A.28) and (A.32) into Eq. (A.34a), and with the aid of Eq. (A.34a), we can obtain

$$2c_{44}B_2 + c_{44}A_{1,z} + 2e_{15}C_2\Big|_{z=h/2} = -32a_3K_1\left(\frac{h}{2}\right) + 8a_7K_0\left(\frac{h}{2}\right) \ = 0. \tag{A.35}$$

Similarly, the combination of Eq. (A.33c) and Eq. (A.34e) gives

$$\begin{aligned}&2c_{13}A_1 + c_{33}B_{0,z} + e_{33}C_{0,z}\\&= 4a_3\left[\frac{h}{2}K_1\left(\frac{h}{2}\right) - K_2\left(\frac{h}{2}\right)\right] - a_7\left[\frac{h}{2}K_0\left(\frac{h}{2}\right) - K_1\left(\frac{h}{2}\right)\right] = 0\end{aligned} \tag{A.36}$$

For the linear equations system composed of Eqs. and , its trivial is not zero, i.e., $4K_1^2\left(\frac{h}{2}\right) - 4K_0\left(\frac{h}{2}\right)K_2\left(\frac{h}{2}\right) \neq 0$, so its solution is

$$\begin{aligned}&B_4 = a_3 = 0\\&A_3 = a_7 = 0.\end{aligned} \tag{A.37}$$

Furthermore, from Eq. (A.32) we can obtain $B_{2,z} = 0$ and $C_{2,z} = 0$. Then, letting $B_2 = a_8$ and $C_2 = a_9$, from Eq. (Appendix A) we can obtain

$$A_{1,z} = -\frac{2e_{15}a_9}{c_{44}} - 2a_8. \tag{A.38}$$

Integrating Eq. (A.33c) with the boundary condition in Eq. (A.41c) gives

$$\left[2e_{31}A_1 + e_{33}B_{1,z} - \xi_{33}C_{0,z}\right]\Big|_{z=h/2} = -4a_9\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{e_{15}^2}{c_{44}} + \xi_{11}\right)d\eta = 0, \tag{A.39}$$

which leads to

$$C_2 = a_9 = 0. \tag{A.40}$$

In summary, we prove that the displacement components for the electroelasticity annular plate under uniform load on its upper surface take the following forms

$$u(r,z) = r^{-1}u_{-1}(z) + ru_1(z) + r^3u_3(z) + A_1(z)r\ln\frac{r}{r_1}, \tag{A.41a}$$

$$w(r,z) = w_0(z) + r^2w_2(z) + r^4w_4(z) + B_0(z)\ln\frac{r}{r_1} + B_2(z)r^2\ln\frac{r}{r_1}, \tag{A.41b}$$

$$\phi(r,z) = \phi_0(z) + r^2\phi_2(z) + C_0(z)\ln\frac{r}{r_1}, \tag{A.41c}$$

which is the same as Eq. (4)