Name ____________________________

# Final Exam
## Intertemporal Choice
## Fall, 2015
## Answers

You are expected to answer all parts of all questions. If you cannot solve part of a question, *do not give up.* The exam is written so that you should be able to answer later parts even if you are stumped by earlier parts.

Write all answers on the exam itself; if you run out of room, use the back of the previous page.

# Part I. Long Question.

**Theory and Evidence on the Effects of Credit Expansion.**
Aydin (2015) reports the results of an experiment in which consumers with credit cards received an exogenous increase in their credit limit. This question asks you to think about what results you would expect from such an experiment if all the consumers behaved according to a modified version of the `TractableBufferStock` model.

The modification permits the model to analyze a change in credit availability. In order to explicitly incorporate a borrowing constraint that can be relaxed, the model must be changed in one respect: We will assume that unemployed consumers, rather than receiving income of zero, instead receive an 'unemployment insurance' (UI) benefit whose value in the first period ('period 1') of unemployment (the 'replacement rate') is $\mu$ times the value of their income in the last period of employment ('period 0'); the benefit thereafter grows by the same factor $\Gamma$ as the aggregate wage.[1] To simplify the analysis, suppose that the government finances the UI program by some windfall source of revenue (say, an oil discovery) and thus does no increase in taxes on employed consumers is required to finance the new program.

1. Explain the 'growth impatience condition' and show that, if it holds for the unemployed consumer, the PDV of UI benefits that an employed consumer knows he will receive if he becomes unemployed next period, relative to his current income, is:

$$\underline{h} = \left(\frac{\mu}{1-\Gamma/\mathsf{R}}\right)/\mathsf{R} \tag{1}$$

   where $\Gamma$ is the growth factor for aggregate wages, and $\mathsf{R}$ is the interest factor.

   *Answer:*

   The 'growth impatience condition' is explained in detail in `TractableBufferStock`, and the generalized version that applies in the presence of permanent shocks to income is articulated in Carroll (2020 (Forthcoming)).

   Relative to his last 'employed' income, a consumer who is unemployed receives income of $\mu$ in the first period of unemployment, $\mu\Gamma$ in the second period, and so on; the PDV as of date 1 is therefore:

$$\mu + \mu(\Gamma/\mathsf{R}) + \mu(\Gamma/\mathsf{R})^2 + \dots = \left(\frac{\mu}{1-\Gamma/\mathsf{R}}\right).$$

   The question asked for the value of this, discounted back to the period prior to unemployment, so we divide by $\mathsf{R}$ to obtain (1).

2. Explain why, in a model with no exogenously imposed credit constraints, the creation of the UI program (an increase from $\mu = 0$ to $0 < \mu < 1$) implies that the consumer will be willing to borrow, but will never choose to allow his end-of-period

[1] This follows the approach elaborated in Carroll, Slacalek, and Sommer (2019).

cash-on-hand to fall to $-\underline{h}$ or below. Explain why the creation of the UI system can be interpreted as relaxing a 'natural borrowing constraint.'

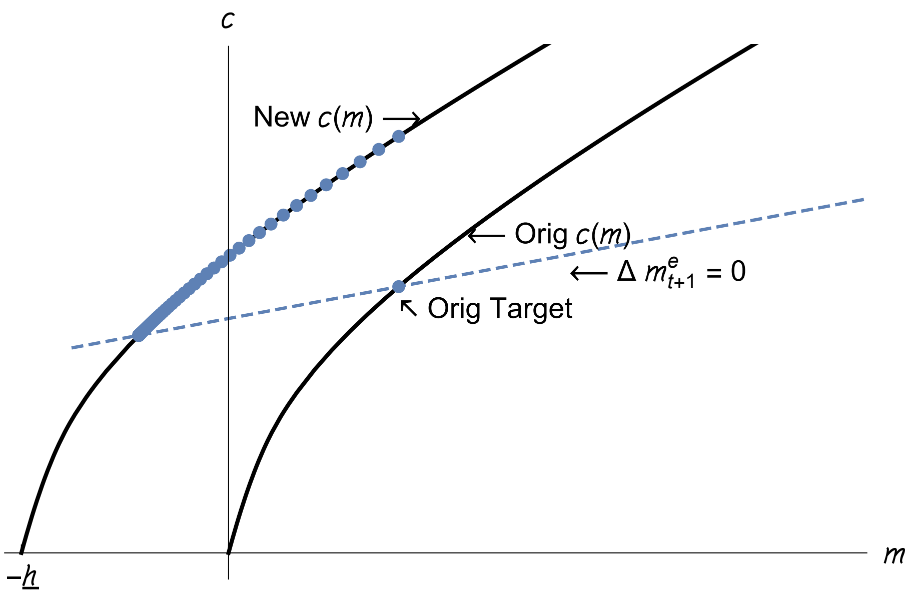
*Answer:*

> `TractableBufferStock` assumes that the consumer cannot default. In the presence of a UI system, the maximum amount that the unemployed consumer could repay would be his entire UI benefit every period; the PDV of those payments is $\underline{h}$. If in period $t$ an employed consumer were to borrow the amount $\underline{h}/\mathsf{R}$, the debt would grow to a value of $\underline{h}$ by period $t+1$, and debt repayment would exhaust all income forever, leaving the consumer to consume zero forever. Since this would generate negative infinite utility, the consumer will never go into debt by an amount as large as $\underline{h}$. This borrowing constraint is 'natural' because it arises from a combination of standard considerations: Precautionary motives prevent a consumer from wanting consumption to reach zero, and the budget constraint must be satisfied. The creation of the UI system 'relaxes' this constraint because it permits a consumer whose maximum borrowing, prior to the UI system, was zero, to borrow any amount less than $\underline{h}$ and still be able to repay.

3. Assume that before the creation of the UI system at date 1, the employed consumer was at the target level of market resources, $m_0 = \check{m}_0$. (For convenience, henceforth please assume that the growth factor for aggregate wages is $\Gamma = 1$.)

   a) Show how the consumption function diagram changes for the employed consumer, and show a series of dots that trace out the immediate and eventual effects on consumption and market resources.

   *Answer:*



b) Show that, if the MPC at the target level of wealth, $\check{\kappa}_0$, is much greater than the interest rate (as is true for plausible calibrations of the model for impatient consumers), the infinite-horizon 'propensity to consume' out of the 'windfall' represented by the introduction of the UI program is (very) approximately $-\mathsf{r}\underline{h}$. That is, show that, if $\check{c}_0$ was the target level of consumption before the expansion and $\check{c}_\infty$ is the new target toward which consumption tends, then $\check{c}_\infty - \check{c}_0 \approx -\mathsf{r}\underline{h}$. Explain why eventually consumption is *lower* as a result of the relaxation of the constraint.

*Answer:*

If $\Gamma = 1$ then the formula for the $\Delta m^e = 0$ locus in TractableBufferStock reduces to:

$$c = (1 - \mathsf{R}^{-1})m + \mathsf{R}^{-1} \tag{2}$$

so

$$\begin{aligned} c_\infty - c_0 &= \overbrace{(m_\infty - m_0)}^{\Delta \check{m}}(1 - \mathsf{R}^{-1}) \\ &= (\Delta \check{m})(\mathsf{R} - 1)/\mathsf{R} \\ &= (\Delta \check{m})(\mathsf{r}/\mathsf{R}) \\ &\approx \mathsf{r}(\Delta \check{m}) \end{aligned}$$

where the last line follows because with a small interest rate, $\mathsf{r}/\mathsf{R} \approx \mathsf{r}$.

As $m^e \uparrow \infty$, the MPC $\kappa \equiv \mathrm{c}'(m^e)$ approaches its value in the perfect foresight case, approximately $\underline{\kappa} \approx \mathsf{r} - \rho^{-1}(\mathsf{r} - \vartheta)$; the return impatience condition guarantees that $\underline{\kappa} > \mathsf{r}$, and concavity of the consumption function implies that at any $m < \infty$ the actual MPC $\kappa$ is greater than $\underline{\kappa}$, so for any $m < \infty$ we know that $\kappa(m) > \mathsf{r}$.

Call the MPC at the initial target level of wealth $\check{\kappa}_0 > \underline{\kappa} > \mathsf{r}$. Then the effect on $c$ of an increase in wealth of amount $\underline{h}$ would be approximately $\underline{h}\check{\kappa}_0$.

Under the assumption that the consumption function is approximately linear around the target level of wealth, we can calculate the amount by which $m^e$ would need to decline in order to reduce consumption so that it again matches the sustainable amount, as follows. For each unit reduction in $m^e$, actual consumption falls by (approximately) $\check{\kappa}_0$ while sustainable consumption falls by $\mathsf{r}$. We need to eliminate the increase in the *excess* of actual over sustainable consumption. That is, we need to find the offsetting $\Delta m^e$ such that

$$\begin{aligned} (\check{\kappa}_0 - \mathsf{r})(\Delta m^e) &\approx -\check{\kappa}_0 \underline{h} \\ \Delta \check{m} &\approx -\left(\frac{\check{\kappa}_0}{\check{\kappa}_0 - \mathsf{r}}\right)\underline{h} \end{aligned}$$

but if $\breve{\kappa}_0 >> \mathsf{r}$ (the 'much greater' assumption in the premise of the question), this can be even more crudely approximated (assuming $\breve{\kappa}_0/(\breve{\kappa}_0 - \mathsf{r}) \approx 1$) as

$$\Delta\check{m} \approx -\underline{h}.$$

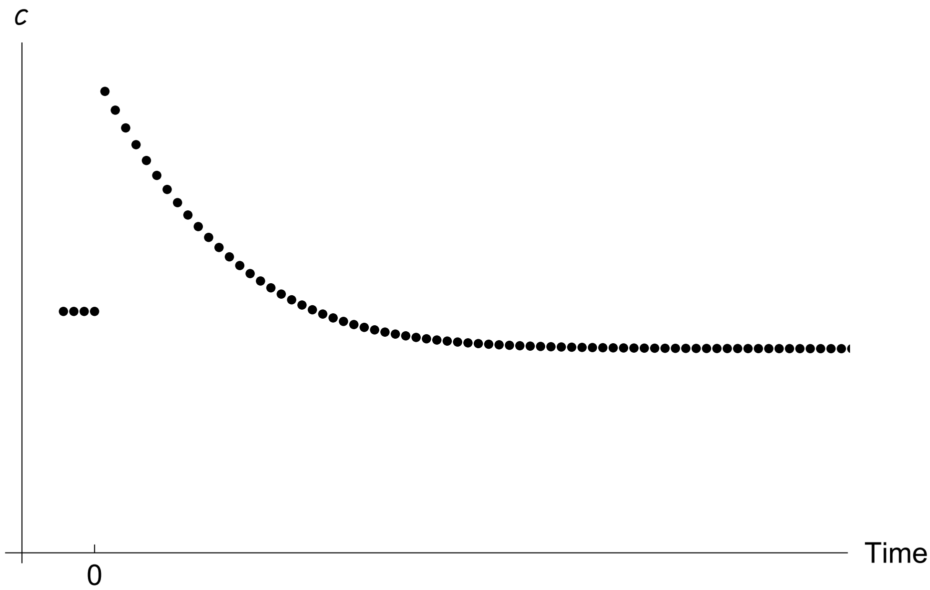
(**?** argues for a calibration of the model in which the annual MPC for the median consumer is about 0.40, so even for an interest rate of 0.03 this is $0.40/0.37 \approx 1.1$.)

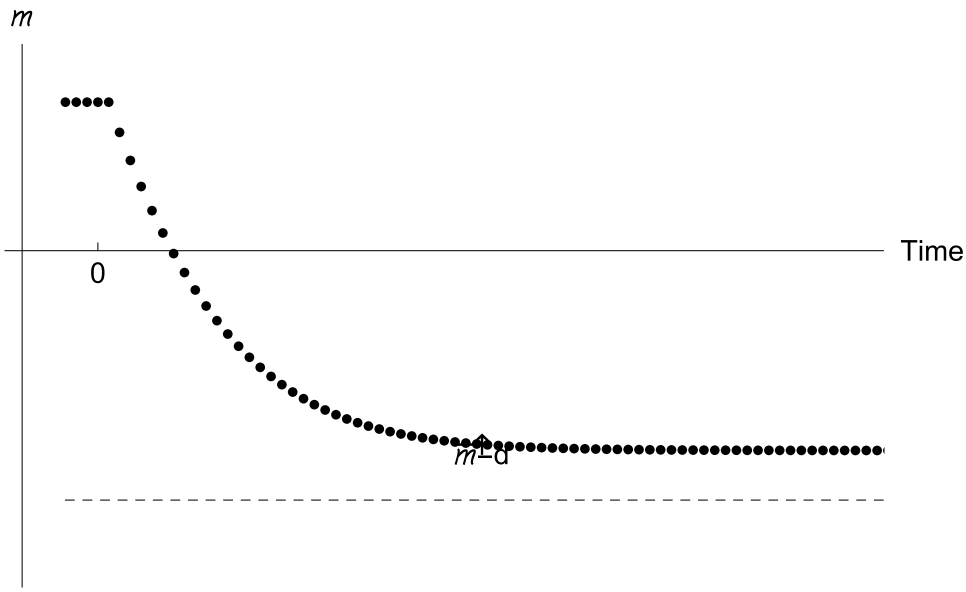
If the target level of wealth falls by $\underline{h}$, then according to (3), the amount by which consumption must fall is approximately $\mathsf{r}\underline{h}$.

c) Draw diagrams showing the time series paths for $c$ and $m$, and indicate the eventual new 'steady-state' levels of $c$ and $m$. How does concavity of the consumption function modify the results from what they would be if the consumption function were linear?

*Answer:*

The figures show the paths; the steady-state level of $c$ ends up being below the initial value and the steady-state value of $m$ ends up being lower. Concavity of the $c(m^e)$ function reduces both the decline in $m^e$ and the decline in $\check{c}$ compared to the reductions that would occur if the consumption function were linear. This is because the precautionary motive intensifies as the level of assets declines, which makes people hold a bit more assets than they would in the absence of the concavity induced by precaution.






d) Consider an increase in credit availability that occurs at date 0. Out of such an increase, define the 'marginal propensity to *have consumed*' (MPTHC) by the date $n > 0$ as the amount by which market wealth $m$ has declined divided by the amount by which credit was originally increased. That is, for a consumer whose wealth at date 0 was at its target value, $m_0 = \check{m}$, the marginal propensity to *have consumed* by date $n$ would be $(m_0 - m_n)/\underline{h}$. Carroll (2001) shows that, in a model with a more realistic income process (transitory and permanent shocks calibrated to the PSID), a relaxation of an exogenous liquidity constraint results in a change in the target level of $m$ that is about the same size as the change in the availability of credit. In the simplified model considered here, the results in Carroll (2001) would correspond to the proposition that $\lim_{n \uparrow \infty} m_0 - m_n \approx \underline{h}$. Under this assumption, draw a graph that shows the evolution of the MPTHC from date 0 toward date $\infty$.

*Answer:*

Under the assumption that the result in Carroll (2001) holds in this model, the graph should be upward sloping, concave, and asymptoting to 1 as $n$ approaches infinity.

4. Give some intuition for why the response to a relaxation of an 'artificial' borrowing constraint (such as a limitation imposed by lenders that requires the debt-to-income ratio to be smaller than a certain amount) is likely to be similar to the relaxation of the 'natural' borrowing constraint described above. (You can take this to be true; your answer will be judged by the quality of your intuitive explanation for why).

*Answer:*

In the absence of uncertainty, the consumer's impatience will cause him to drive market resources to the lowest level allowed by either the natural or the artificial borrowing constraint (whichever is tighter). When uncertainty is added to the problem, in the model with only a natural and no artificial borrowing constraint, the consumer will aim to have a target amount of 'buffering capacity' and so will behave in such a way as to make market resources tend toward some amount $\check{m} > -\underline{h}$. But in the presence of uncertain future income, the knowledge that a constraint might bind in the future induces a consumer who is not constrained today to accumulate a buffer stock of assets for precisely the same (cognitive) reason: To reduce the (anticipated) pain that would occur in some future period. While constraints plus uncertainty are numerically and analytically more difficult to handle, the emotional logic of the model is the same in both cases: Worry about the future induces precautionary behavior today. Thus, one would expect, qualitatively at least, for the relaxation of the constraint to have qualitatively similar effects whether the constraint was 'natural' or 'artificial.'

5. Aydin (2015) defines the *marginal propensity to consume out of liquidity* as 'the dollar response of debt to a \$1 change in borrowing capacity.' Given your answer to the question above about the MPTHC, critique this definition, particularly with respect to what it has to say about the time horizon.

   *Answer:*

   The chief point I am looking for here is that the MPCL is not well defined without a timeframe. The concept is inherently one that requires a specification of the interval over which the amount in question has been consumed; similarly, a vehicle's speed cannot be measured in kilometers; it must be measured in kilometers per unit of time, like kilometers per hour. To be fair, Aydin does discuss the 'short-term' and 'long-term' MPCL's in the body of the paper, but these terms are not very well defined.

6. Given his definition of the MPCL, Aydin (2015) makes the following statements about what his data show. For each statement, discuss the claimed result by deciding whether it is either (a) consistent with the results above from the tractable model; (b) inconsistent with the results above; or (c) ambigious in whether it is consistent or inconsistent with the model.

   a) 'MPCL is significantly larger than zero for three quarters of the population, most of whom are not immediately constrained.'

      *Answer:*

      Consistent (or at least, not inconsistent). If the tractable model applied to all consumers (that is, all consumers were impatient), in fact, that model would imply that at all horizons and for all consumers, the MPTHC is greater than zero; but for persons with a large amount of wealth or over a short time horizon, it could be *close* to zero. Thus, whether it will be 'significantly' larger than zero will depend on sample size, time horizon, the number of people who are sufficiently impatient to have a high MPC at their target wealth, etc.

b) 'In the short-run, MPCL exhibits a clear heterogeneity in cash-on-hand, exclusively in line with a concave consumption function.'

*Answer:*

Consistent. The consumption function is concave in the tractable model.

c) 'In the long-run, MPCL is mean-reverting: an increase in credit capacity does not permanently shift leverage, but individuals accumulate debt only to deaccumulate it after 6-18 months.'[2]

*Answer:*

It seems that what Aydin means by "mean-reverting" is that eventually the MPCL goes to zero. That is, while consumers run up debt initially, they will eventually repay all of that debt. If that is what is meant, the result is inconsistent; the MPTHC is monotonically increasing in the tractable model.

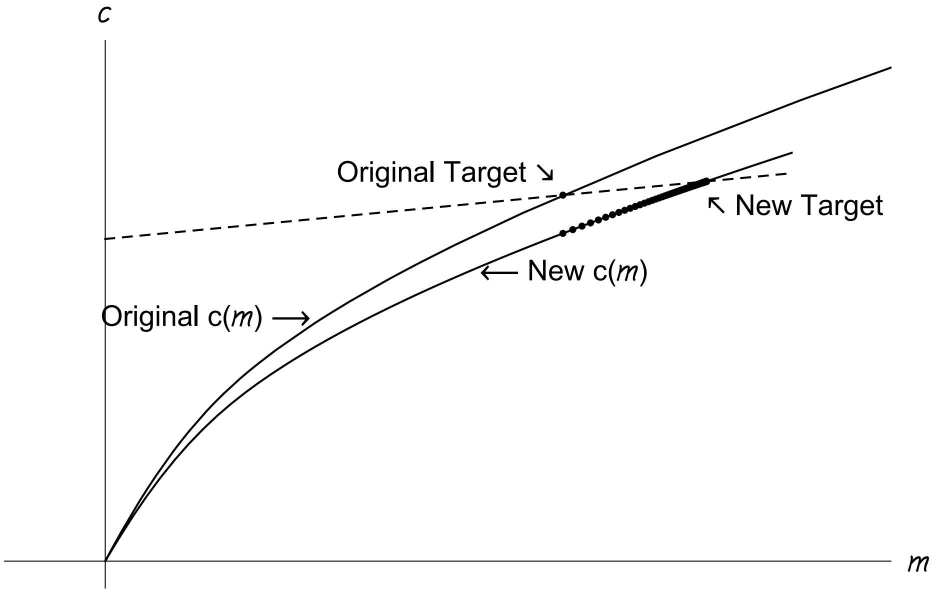
The final part of this question invites you to contemplate consequences of the paper's findings for the analysis of the Great Recession. Assume for the purposes of the discussion that the paper's results suggest that there are two types of consumers: Higly impatient ones, with a low target buffer stock of assets; and mildly impatient ones whose buffer stock is very large. Mian and Sufi in a series of papers, and especially Mian, Rao, and Sufi (2013), have suggested that the credit boom that led up to 2007 largely reflected an increase in credit to the 'impatient' consumers who had small buffer stocks of assets, and that the post-2007 contraction in credit was concentrated on the same set of consumers. Recent work by Adelino, Schoar, and Severino (2015) challenges this proposition, arguing instead that much of the credit boom was directed toward consumers with substantial assets and income.

7. Draw the consumption function diagrams for highly impatient and for mildly impatient consumers, and use those diagrams to explain what the model says about why it might matter, for the consumption response, whether the credit boom was directed toward patient consumers (with initially large amounts of assets) or toward impatient consumers (with initially small amounts of assets).

   *Answer:*

[2] It must be noted that Aydin's evidence for this proposition seems weak. The figure cited as proof does not appear to me to exhibit any robust evidence of a tendency to decumulate. But, your task is to evaluate whether the proposition as stated is consistent with the model.

Below is a figure (reproduced from Carroll, Slacalek, and Sommer (2019)) that shows how the phase diagram changes if the degree of impatience suddenly *decreases*; the two consumption functions are therefore the functions associated with a less-impatient and a more-impatient consumer whose budget resources are otherwise identical. (Obviously, the consumer who consumes more at any given level of $m$ is the more impatient one.) This is the opposite of the experiment asked in the question, so the figure is the inverse of the one you were asked to draw: 'Old' and 'new' are reversed.)




The figure makes it clear that a given leftward shift of the consumption function (that is, a given expansion of credit availability) will have a larger effect on spending for the impatient than for the patient consumers (because, for both sets of consumers, the function shifts left by the same amount; but for any given initial $m$, the MPC $\kappa$ is larger for the impatient than for the patient consumers, so a given horizontal shift implies a greater change in consumption). The point is only reinforced by the assumption that each group of consumers was initially at its target level of wealth, since the concavity of the consumption function means that that, for either function, the effect of a leftward shift is greater when the amount of $m$ is smaller. This explains the importance of the distribution of the increase in credit availability; it is more plausible to attribute the consumption boom from 2001-2007 solely to credit expansion if that credit expansion went to consumers who were both impatient and with low levels of initial wealth. If, instead, most of the credit went to consumers who already had substantial amounts of wealth, it is harder to believe that the whole consumption boom was attributable to the expansion of credit.

8. Some work has argued that another factor was involved in the consumption boom and subsequent bust: Uncertainty. Specifically, in the years leading up to 2007, macroeconomists were writing about the 'great moderation' in macroeconomic performance, the unemployment rate was stable at near-record low levels for many years, and direct measures of consumer uncertainty registered at low levels. After 2007, uncertainty by various measures spiked to near record high levels. Suppose that the degree of uncertainty faced by everyone is the same. Does the model imply that one group or the other (the 'highly impatient' or the 'mildly impatient') might react more to the increase in uncertainty? Sketch some implications for the research agenda.

   *Answer:*

   > The key point here is that the expansion and subsequent contraction of credit availability is not the only possible mechanism that might be used to explain the movements of consumption during the boom and then bust. A decline in uncertainty ('the great moderation') would boost everyone's consumption, while an increase in uncertainty would reduce everyone's consumption. However, it is likely that the effect of a given change in the degree of uncertainty will be larger for the 'patient' than for the 'impatient' consumers. This can be most easily seen by thinking about the extreme case where the impatient consumers are so impatient that they maintain only a tiny buffer stock, or none at all. In that case, their ability to respond to a change in uncertainty is very small, since they are already spending almost all of their income in most periods. In contrast, patient consumers who care a lot about the future could plausibly have a big consumption response to a change in uncertainty, particularly if it was of unknown duration. The point you were expected to make here is that one way to distinguish the Mian, Rao, and Sufi (2013) story from, say, a story in which fluctuations in uncertainty played a bigger role, is to look at how the patterns of consumption changed *differently* for different groups. If one could show that most of the change in consumption was by impatient low-wealth types, that supports the MRS story. If instead most of the change in consumption was accounted for by the behavior of consumers with substantial amounts of net worth, that would suggest that some other factor than a direct credit channel (such as a change in uncertainty) was important.

# Part II. Discussion Question.

**Open Source Software and Knowledge Capital.** "Open Source" software is created largely by computer programmers who are not directly paid for their contributions. Some of the most central components of the infrastructure of the internet are open-source (for example, the Apache web server software that powers perhaps half of the world's webservers; the FireFox browser; the LaTeX typographical system; and much more).

Explaining the production of valuable services by volunteer labor is a challenge for neoclassical economics (to say the least!). This question asks you to make a first stab at the job, using the Lucas (1988) growth model with an aggregate production function

$$Y = AK^{\alpha}(\ell h L)^{1-\alpha} \tag{3}$$

Recall that in that model, "human capital" $h$ accumulates according to

$$\dot{h}/h \;\; = \;\; \phi(1-\ell) \tag{4}$$

where $\ell$ is the proportion of time that people spend working.

1. Write a paragraph or two about reasons for and against a reinterpretation of Lucas's model in which what he calls "human capital" is thought of as "knowledge capital" produced by people in their free-time contributions to open source knowledge projects. Comment, in particular, about what you need to assume about the parameter that captures the degree of externalities in aggregate $h$ in order for this interpretation to have any force.

   *Answer:*

   > This is covered in the handout. The essential distinction is between human capital, which is embodied in an individual mind and hence cannot be accumulated forever without bound, and knowledge capital, which goes into things like textbooks and can be accumulated without bound. This carries over into whether we can interpret $\ell$ as a proportion of the population or a proportion of an individual's time.

2. Suppose a government concludes that the value of open source software in boosting aggregate productivity is very large. Under the appropriate assumptions (that is, the assumptions that make the Lucas model sketched above appropriate for addressing the question), suppose the government wants to evaluate a subsidy that will increase the leisure time of computer programmers (in the belief that in their leisure time they will create more open-source software).

   - Assume that only computer programmers contribute to knowledge capital in the relevant sense (everybody else just contributes raw labor). Also, programmers are "born that way" (they can't help becoming programmers, and nobody else can become one) so that their proportion in the population remains constant at $\Pi$. Show that under these assumptions the effective aggregate production function becomes

$$y = \hat{A}k^{\alpha}(\ell h)^{1-\alpha} \tag{5}$$

where $h$ is the knowldege capital per capita of the computer programmers and $\ell$ is the proportion of their hours they spend working.

*Answer:*

First note that if only a fraction $1-\Pi$ of the population are workers then the production function becomes

$$\begin{aligned} Y &= AK^{\alpha}(\ell h\{(1-\Pi)L\})^{1-\alpha} && (6)\\ Y &= AK^{\alpha}(\ell h)^{1-\alpha}(1-\Pi)^{1-\alpha}LL^{-\alpha} && (7)\\ y &= Ak^{\alpha}(\ell h)^{1-\alpha}(1-\Pi)^{1-\alpha} && (8)\\ y &= \hat{A}k^{\alpha}(\ell h)^{1-\alpha} && (9) \end{aligned}$$

where $y$ and $k$ are per capita terms, and $\hat{A} = A(1-\Pi)^{1-\alpha}$

- Explain why the optimal policy for the government is likely to involve subsidizing the programmers *not* to work. Discuss how would the appropriate subsidy depends (qualitatively) on the different parameters in the model (including the Cobb-Douglas parameter for capital's share in output).

  *Answer:*

  Ultimately the programmers' contribution to knowledge creation is more important than their raw labor. As discussed further below, more raw labor has a level effect, but more time spent in knowledge creating activities has a growth effect. To the extent that this growth happens in the future and hence depends on preferences towards the future, the trade-off will depend on $\beta$; a higher value will imply a higher subsidy, as the consumers care more about future growth. As capital's share of income declines and capital becomes less important to output, human capital becomes more important, and so as $\alpha$ falls the subsidy should rise. A larger $\Pi$ will mean that a single dollar of subsidy will go farther, since it increases $\hat{A}$ and hence the marginal product of human capital. Finally, a higher $\phi$ means that the externalities from human capital accumulation are larger, and so a higher subsidy is warranted.

- Now suppose that labor market imperfections mean that the company that a programmer works for is able to temporarily capture a substantial portion of the "knowledge" benefits of the programmer's "free time" programming activities, but the amount that the company captures depends on the number of hours that the programmer spends working at the company (the $\ell$). Will the company agree with the government about the optimal size of the subsidy? Explain why or why not, and discuss what determines whether the company is likely to benefit (on net) from the externalities mentioned above.

*Answer:*

No, the company has two incentives for $\ell$ to be higher: the value of the raw labor and the greater ability to capture benefits from the programmer's "free time." The social planner only has the former incentive, and since she takes the externality into account she has incentives to reduce $\ell$. Hence the optimal subsidy is likely larger than what the company would prefer. The of the disagreement between the company's and the government's evaluation depends on the size of the externality and how much of the rents the company can capture before its patent/monopoly rights expire.

- Assuming that parameter values are such that the subsidy to leisure of programmers has a positive net value, would the model suggest that such a subsidy would have a positive effect on the level of output, on the growth rate of output, or both?

  *Answer:*

  Both. The subsidy would lower $\ell$ the moment it went into effect, which would directly lower the level of output; however, since we have assumed that that parameter values are such that the subsidy has a positive net value, the benefits of higher knowledge capital would offset this and output would increase (although this might take time to materialize, since $h$ moves continuously). Since knowledge capital is accumulating at a faster rate after the subsidy, growth is higher.

- How might such a subsidy compare, in its effects, to a more traditional policy like a permanent investment tax credit?

  *Answer:*

  In a standard growth model, a permanent ITC will increase investment flows and hence increase capital to a new steady state level. In a model like Romer's with positive externalities from capital investment, there is also a permanent increase in the steady state growth rate. The subsidy here is like the ITC of the Romer model, in the sense that there is a permanent growth effect. However, since this subsidy has direct labor market effects, there is also a level effect that depends on the parameters.

- Explain why the government's optimization problem becomes much harder if some people can pretend to be computer programmers even though their only real talent is in English Literature and they hate programming and in their free time they read Shakespeare instead of coding the latest update to Firefox.

  *Answer:*

  The government faces the problem that they would be paying Lit people to withdraw their raw labor from the market, reducing output, but not gaining any of the knowledge capital in return. In this case they need to deal additionally with the adverse selection problem, meaning that they need to either design optimal screening or sorting devices, or pay verification costs to companies who monitor people claiming to be programmers. In either case, as in the imperfect capital markets a la Romer, the contract design problem becomes much more difficult.

# Part III. An Entrepreneurial Country.

The handout EntrepreneurPF combines results from PerfForesightCRRA and qModel to produce a model of the behavior of an entrepreneur who owns a business whose investments incur costs of adjustment. You should be able to skim the handout to get the gist of how the model works (since it is basically a mashup of two frameworks that you are already familiar with). Use the model in the handout to answer the following questions, under the following assumptions:

- We are now reinterpreting the model to describe a small open economy run by a benevolent social planner, rather than a firm run by an entrepreneur
- Leading up to period $t$, the economy was in equilibrium with a stable and positive amount of monetary resources $\bar{m}$ and a constant (equilibrium) capital stock $\bar{k}$.
- For all three of the experiments below, show the dynamics of consumption, investment, national net worth, and the aggregate capital stock following the shock at period $t$, and explain why the results look the way they do.

1. Suppose the country experiences a war that suddenly destroys 10 percent of its physical physical capital stock, so that $k_t = 0.9\bar{k}$.

   *Answer:*

   > The key insight about the perfect foresight entrepreneur's model is that Fisherian separation holds. So, any question that is about investment can be answered identically to the answer in a $\varrho$ model, and any question about consumption reflects the answer that would apply in PerfForesightCRRA.
   >
   > So, for investment and capital, the answer is identical to the experiment in which 10 percent of the entrepreneur's capital stock is destroyed in the handout.
   >
   > For consumption the answer is that there is an instantaneous jump in $c$ to the level consistent with the new PDV of profits, and consumption is projected never to change thereafter.

2. Suppose that year $t$ was 2008 and the country had accumulated a substantial amount of monetary assets so that $a_{t-1} >> 0$, but those assets happened to be deposited in Lehmann Brothers right at the time it collapsed. The value of those assets therefore falls to zero.

   *Answer:*

   > In this case, there is no impairment in the productivity of the physical assets, so this is equivalent to a wealth shock in PerfForesightCRRA: An immediate and permanent change in $c$.

3. Suppose now that what happens at date $t$ is that another country starts producing the same goods that this country has produced before, and the consequence is that this country's production function for the firm suddenly, permanently, and unexpectedly drops; specifically, leading up to period $t$ the firm was in steady state, but in periods $t+1$ and beyond the production function will be $\mathrm{f}_{\geq}(k) = \Psi \mathrm{f}_{<}(k)$ for some $\Psi < 1$ where $\mathrm{f}_{<}$ and $\mathrm{f}_{\geq}$ indicate the production functions before and after the increase in productivity.

   *Answer:*

   > In this case, the decline in productivity has effects on investment in the capital stock identical to those that transpire in `qModel` in response to a productivity shock.
   >
   > This event reduces the PDV of profits from the entrepreneurial venture. The consequence is identical to the consequence of any decline in net worth (the sum of human and nonhuman wealth) in `PerfForesightCRRA`: An immediate downward jump in consumption followed by no further changes.

Consider a firm characterized by the following:

| | | |
|---|---|---|
| $k_t$ | - | Firm's capital stock at the beginning of period $t$ |
| $\mathrm{f}(k)$ | - | The firm's total output depends only on $k$ |
| $i_t$ | - | Investment in period $t$ |
| $\mathrm{j}(i, k)$ | - | AdJustment costs associated with investment $i$ given capital $k$ |
| $\xi_t = i_t + \mathrm{j}_t$ | - | eXpenditures (purchases plus adjustment costs) on investment |
| $\beta = 1/\mathsf{R}$ | - | Discount factor for future profits (inverse of interest factor) |

Suppose that the firm's goal is to pick the sequence $\mathrm{i}_t$ that solves:

$$\mathrm{e}(k_t) = \max_{\{\mathrm{i}\}_t^\infty} \sum_{n=0}^{\infty} \beta^n \left( f_{t+n} - i_{t+n} - \mathrm{j}_{t+n} \right) \tag{10}$$

subject to the transition equation for capital,

$$k_{t+1} = (k_t + i_t)\daleth \tag{11}$$

where $\daleth = (1 - \delta)$ is the amount of capital left after one period of depreciation at rate $\delta$.[3] $\mathrm{e}_t$ is the value of the profit-maximizing firm: If capital markets are efficient this is the equity value that the firm would command if somebody wanted to buy it.

The firm's Bellman equation can be written:
*Answer:*

$$\begin{aligned}
\mathrm{e}_t(k_t) &= \max_{\{\mathrm{i}\}_t^\infty} \sum_{n=0}^{\infty} \beta^n \left( f_{t+n} - i_{t+n} - \mathrm{j}_{t+n} \right) \\
&= \max_{\{i_t\}} \; f_t - i_t - \mathrm{j}(i_t, k_t) + \beta \left[ \max_{\{i\}_{t+1}^\infty} \sum_{n=0}^{\infty} \beta^n \left( f_{t+1+n} - i_{t+1+n} - \mathrm{j}_{t+1+n} \right) \right] \\
&= \max_{\{i_t\}} \; f_t - i_t - \mathrm{j}(i_t, k_t) + \beta \mathrm{e}_{t+1}\left( (k_t + i_t)\daleth \right)
\end{aligned}$$

Define $\mathrm{j}_t^i$ as the derivative of adjustment costs with respect to the level of investment.

The first order condition for optimal investment implies:
*Answer:*

$$\begin{aligned}
0 &= -1 - j_t^i + \daleth\beta \mathrm{e}_{t+1}^k(k_{t+1}) \\
1 + j_t^i &= \daleth\beta \mathrm{e}_{t+1}^k(k_{t+1})
\end{aligned} \tag{12}$$

In words: The marginal cost of an additional unit of investment (the LHS) should be equal to the discounted marginal value of the resulting extra capital (the RHS).

[3] There are some small differences between the formulation of the model here and in `qModel`. Here, investment costs are paid at the time of investment and the depreciation factor applies to $(k_t + i_t)$ rather than just $k_t$. These changes simplify the computational solution without changing any key results.

*Answer:*

The Envelope theorem says

$$
\begin{aligned}
\mathrm{e}^k_t(k_t) &= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \beta \mathrm{e}^k_{t+1}(k_{t+1}) \overbrace{\left(\frac{\partial k_{t+1}}{\partial k_t}\right)}^{\daleth} \\
&= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \underbrace{\beta \daleth \mathrm{e}^k_{t+1}(k_{t+1})}_{=(1+\mathrm{j}^i_t) \text{ from } (12)}
\end{aligned}
$$

So the corresponding $t+1$ equation can be substituted into (12) to obtain

$$
(1 + \mathrm{j}^i_t) = \left(\mathrm{f}^k(k_{t+1}) + (1 + \mathrm{j}^i_{t+1} - \mathrm{j}^k_{t+1})\right) \daleth \beta \tag{13}
$$

which is the Euler equation for investment.

Now suppose that a steady state exists in which the capital stock is at its optimal level and is not adjusting, so costs of adjustment are zero: $\mathrm{j}_t = \mathrm{j}_{t+1} = j^i_t = j^i_{t+1} = j^k_t = j^k_{t+1} = 0$.

*Answer:*

If $j^i_t = j^i_{t+1} = j^k_{t+1}$ then (13) reduces to

$$
\begin{aligned}
1 &= \overbrace{\beta}^{=\mathsf{R}^{-1}} \daleth \left[\mathrm{f}^k(\check{k}) + 1\right] \\
\mathsf{R} &= \daleth(\mathrm{f}^k(\check{k}))
\end{aligned} \tag{14}
$$

so that the capital stock is equal to the value that causes its marginal product to match the interest factor, after compensating for depreciation.

Another way to analyze this problem is in terms of the marginal value of capital, $\lambda_t \equiv \mathrm{e}^k_t(k_t)$.

*Answer:*

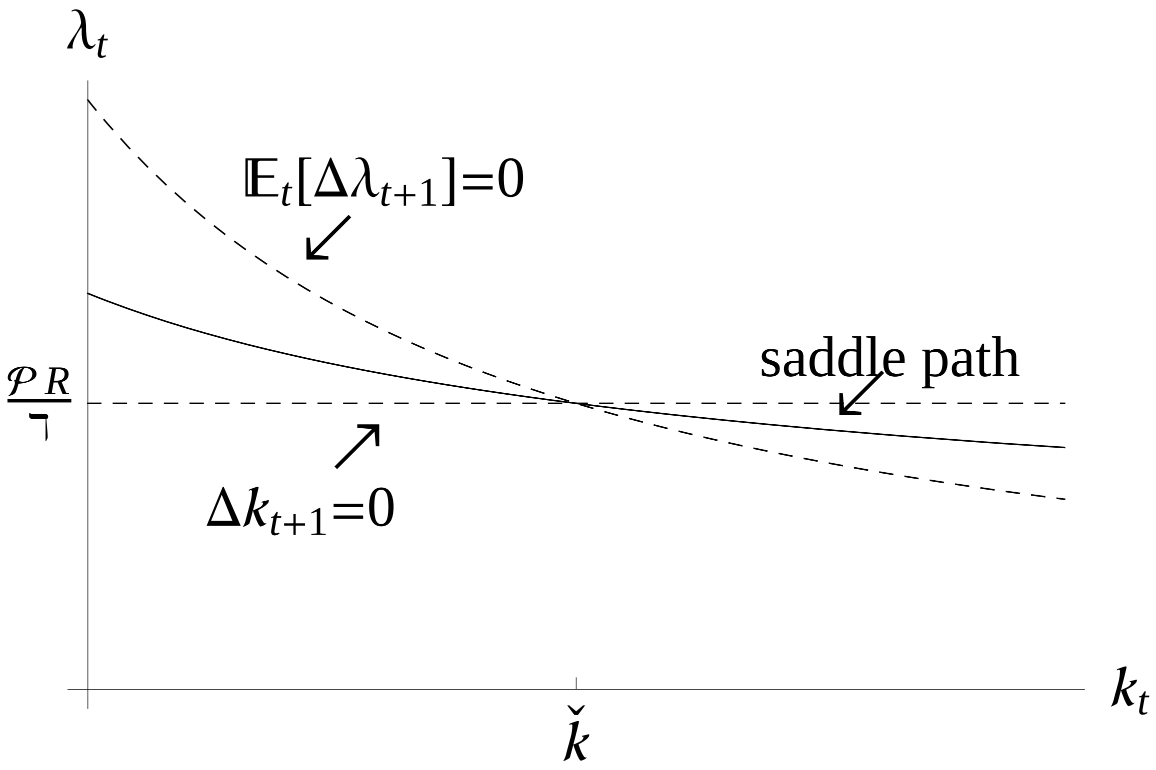
Rewrite (13) as

$$
\begin{aligned}
\lambda_t &= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \beta\daleth(\lambda_t + \lambda_{t+1} - \lambda_t) \\
&= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \beta\daleth(\lambda_t + \Delta\lambda_{t+1}) \\
(1 - \beta\daleth)\lambda_t &= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \Delta\lambda_{t+1} \\
\lambda_t &= \frac{\mathrm{f}^k(k_t) - \mathrm{j}^k_t + \Delta\lambda_{t+1}}{(1 - \beta\daleth)}
\end{aligned} \tag{15}
$$

and the phase diagram is constructed using the $\Delta\lambda_{t+1} = 0$ locus. In the vicinity of the steady state, we can assume $\mathrm{j}^k_t \approx 0$ in which case the $\Delta\lambda_{t+1} = 0$ locus becomes

$$
\lambda_t = \frac{\mathrm{f}^k(k_t)}{(1 - \beta\daleth)} \tag{16}
$$

**Figure 1** Phase Diagram




which implies (since $\mathrm{f}^k(k_t)$ is downward sloping in $k_t$) that the $\Delta\lambda_t = 0$ locus (that is, the $\lambda_t(k_t)$ function that corresponds to $\Delta\lambda_t = 0$) is downward sloping.

The phase diagram is depicted below.

The steady state of the model will be the point at which $k_{t+1} = k_t = \check{k}$, implying from (11) a steady-state investment rate of

$$\begin{aligned} \check{k} &= (\check{k} + \check{\imath})\daleth \\ \check{\imath} &= (1 - \daleth)\check{k}/\daleth = (\delta/\daleth)\check{k} \end{aligned} \tag{17}$$

and solving (14) for $\mathrm{f}^k(\check{k})$

$$\left(\frac{(1-\beta\daleth)}{\beta\daleth}\right) = \mathrm{f}^k(\check{k}) \tag{18}$$

which can be substituted into (16) to obtain the steady-state value of $\lambda$:

$$\check{\lambda} = \left(\frac{\mathsf{R}}{\daleth}\right). \tag{19}$$

We now wish to modify the problem in two ways. First, we have been assuming that the firm has only physical capital, and no financial assets. Second, we have been assuming that the manager running the firm only cares about the PDV of profits; suppose instead we want to assume that the firm is a small business run by an entrepreneur who must live off the dividends of the firm, and thus they are maximizing the discounted sum of utility from dividends $\mathrm{u}(c_t)$ rather than just the level of discounted profits. (Note that we designate dividends by $c_t$; dividends were not explicitly chosen in the $\varrho$-model version of the problem, because the Modigliani-Miller theorem says that the firm's value is unaffected by its dividend policy).

We call the maximizer running this firm the 'entrepreneur.' The entrepreneur's level of monetary assets $m_t$ evolves according to

$$m_{t+1} = f_{t+1} + (m_t - i_t - \mathrm{j}_t - c_t)\,\mathsf{R}. \tag{20}$$

That is, next period the firm's money is next period's profits plus the return factor on the money at the beginning of this period, minus this period's investment and associated adjustment costs, minus dividends paid out (which, having been paid out, are no longer part of the firm's money).

The entrepreneur's Bellman equation can now be written

$$\mathrm{v}_t(k_t, m_t) = \max_{\{i_t, c_t\}} \quad \mathrm{u}(c_t) + \beta \mathrm{v}_{t+1}(k_{t+1}, m_{t+1})$$

*Answer:*

Value is simply the discounted sum of utility from future dividends:

$$\begin{aligned}
\mathrm{v}_t(k_t, m_t) &= \max_{\{i,c\}_t^\infty} \sum_{n=0}^{\infty} \beta^n \mathrm{u}(c_{t+n}) \\
&= \max_{\{i,c\}_t^\infty} \left( \mathrm{u}(c_t) + \beta \sum_{n=0}^{\infty} \beta^n \mathrm{u}(c_{t+1+n}) \right) \\
&= \max_{\{i_t,c_t\}} \quad \mathrm{u}(c_t) + \beta \mathrm{v}_{t+1}(k_{t+1}, m_{t+1}).
\end{aligned}$$

Assume that f and j do not depend directly on $m_t$. That is, their partial derivatives with respect to $m_t$ are zero.

Then we will have

*Answer:*

FOC wrt $c_t$:

$$\mathrm{u}'(c_t) = \mathsf{R}\beta v_{t+1}^m \tag{21}$$

Envelope wrt $m_t$:

$$v_t^m = \mathsf{R}\beta v_{t+1}^m \tag{22}$$

and combining the FOC with the Envelope theorem we get the usual

$$\begin{aligned}
v_t^m &= \mathsf{R}\beta v_{t+1}^m \\
&= \mathrm{u}'(c_t) \\
&= \mathsf{R}\beta \mathrm{u}'(c_{t+1}) \\
&= \mathrm{u}'(c_{t+1})
\end{aligned}$$

where the last line follows because we have assumed $\mathsf{R}\beta = 1$.

Now note that the value function can be rewritten as

$$\mathrm{v}_t(k_t, m_t) = \max_{\{i_t, m_{t+1}\}} \mathrm{u}((f_{t+1} - m_{t+1})/\mathsf{R} + m_t - i_t - \mathrm{j}_t) + \beta \mathrm{v}_{t+1}(k_{t+1}, m_{t+1})$$

*Answer:*

This holds because maximizing with respect to $m_{t+1}$ (subject to the accumulation equation) is equivalent to maximizing with respect to the components of $m_{t+1}$.

For the version in (23) the FOC with respect to $i_t$ is

$$\mathrm{u}'(c_t)((1 + \mathrm{j}^i_t) - f^k_{t+1}\daleth/\mathsf{R}) = \daleth\beta v^k_{t+1} \tag{23}$$

*Answer:*

This holds because the derivative of the RHS of (23) with respect to $i_t$ is

$$\mathrm{u}'(c_t)\left(\left(\frac{\partial f_{t+1}}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial i_t}\right)/\mathsf{R} - \frac{\partial i_t}{\partial i_t} - \frac{\partial \mathrm{j}_t}{\partial i_t}\right) + \beta\left(\frac{\partial k_{t+1}}{\partial i_t}\right)\mathrm{v}^k_{t+1}(k_{t+1}, m_{t+1}) \tag{24}$$

(remember that $m_{t+1}$ is a control variable and thus its derivative with respect to investment is zero) so the FOC translates to

$$\mathrm{u}'(c_t)(f^k_{t+1}\daleth/\mathsf{R} - 1 - \mathrm{j}^i_t) + \beta\daleth v^k_{t+1} = 0 \tag{25}$$

which reduces to (23).

Now we can use the envelope theorem with respect to $k_t$ to show that

$$v^k_t = \mathrm{u}'(c_t)(f^k_{t+1}\daleth/\mathsf{R} - \mathrm{j}^k_t) + \beta\daleth v^k_{t+1} \tag{26}$$

*Answer:*

This can be seen by directly taking the derivative of the RHS of (23) with respect to $k_t$:

$$\mathrm{u}'(c_t)\left(\left(\frac{\partial f_{t+1}}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial k_t}\right)/\mathsf{R} - \frac{\partial \mathrm{j}_t}{\partial k_t}\right) + \beta\left(\frac{\partial k_{t+1}}{\partial k_t}\right)v^k_{t+1} \tag{27}$$

and noting that the Envelope theorem tells us the derivatives with respect to the controls $m_{t+1}$ and $i_t$ are zero while $\partial k_{t+1}/\partial k_t = \daleth$.

Now we can combine (23) and (26) to derive the Euler equation for investment

$$(1 + \mathrm{j}^i_t) = \daleth\beta\left[\mathrm{f}^k(k_{t+1}) + (1 + \mathrm{j}^i_{t+1} - \mathrm{j}^k_{t+1})\right]. \tag{28}$$

*Answer:*

To see this, start with the Envelope theorem,

$$\begin{aligned} v_t^k &= \mathrm{u}'(c_t)(f_{t+1}^k ⅂/\mathsf{R} - \mathrm{j}_t^k) + \overbrace{⅂\beta v_{t+1}^k}^{=\mathrm{u}'(c_t)((1+\mathrm{j}_t^i)-f_{t+1}^k⅂/\mathsf{R}) \text{ from } (23)} \\ &= \mathrm{u}'(c_t)(f_{t+1}^k ⅂/\mathsf{R} - \mathrm{j}_t^k) + \mathrm{u}'(c_t)((1+\mathrm{j}_t^i) - f_{t+1}^k ⅂/\mathsf{R}) \\ &= \mathrm{u}'(c_t)\left(1 + \mathrm{j}_t^i - \mathrm{j}_t^k\right) \end{aligned} \tag{29}$$

which means that we can rewrite (23) substituting the rolled-forward version of (29)

$$\begin{aligned} \mathrm{u}'(c_t)((1+\mathrm{j}_t^i) - f_{t+1}^k ⅂/\mathsf{R}) &= ⅂\beta v_{t+1}^k \\ &= ⅂\beta \mathrm{u}'(c_{t+1})\left(1 + \mathrm{j}_{t+1}^i - \mathrm{j}_{t+1}^k\right) \\ (1+\mathrm{j}_t^i) &= ⅂\beta\left[\mathrm{f}^k(k_{t+1}) + (1 + \mathrm{j}_{t+1}^i - \mathrm{j}_{t+1}^k)\right] \end{aligned}$$

where the last line follows because with $\mathsf{R}\beta = 1$ we know that $c_{t+1} = c_t$ implying $\mathrm{u}'(c_{t+1}) = \mathrm{u}'(c_t)$.

*Answer:*

Since behavior (for either a firm manager or a consumer) is determined by Euler equations, and the Euler equations for both consumption and investment are identical in this model to the Euler equations for the standard models, there is no observable consequence for investment of the fact that the firm is being run by a utility maximizer, and there is no observable consequence for consumption of the fact that the consumer owns a business enterprise with costly capital adjustment.

Now consider a firm of this kind that happens to have arrived in period $t$ with positive monetary assets $m_t > 0$ and with capital equal to the steady-state target value $k_t = \check{k}$. Suppose that a thief steals all the firm's monetary assets.

*Answer:*

The consequences for the firm are depicted below in figure 2.

Dividends follow a random walk. Thus, there is a one-time downward adjustment to the level of dividends to reflect the stolen money. Thereafter dividends are constant, as are monetary assets (which are constant at zero forever).

The theft of the money has no effect on investment or the capital stock, because the firm's investment decisions are made on the basis of whether they are profitable and the theft of the money has no effect on the profitability of investments.

Now consider another kind of shock: The firm's main building gets hit by a meteor, destroying some of the firm's capital stock.

*Answer:*

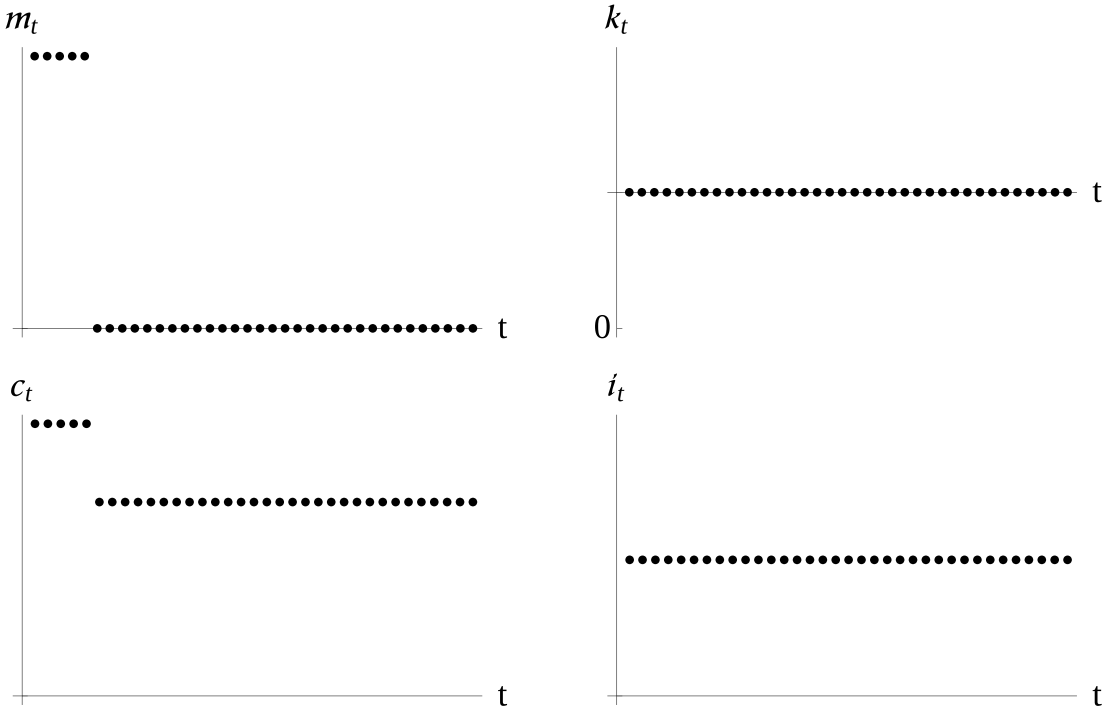



**Figure 2** Negative shock to $m_t$

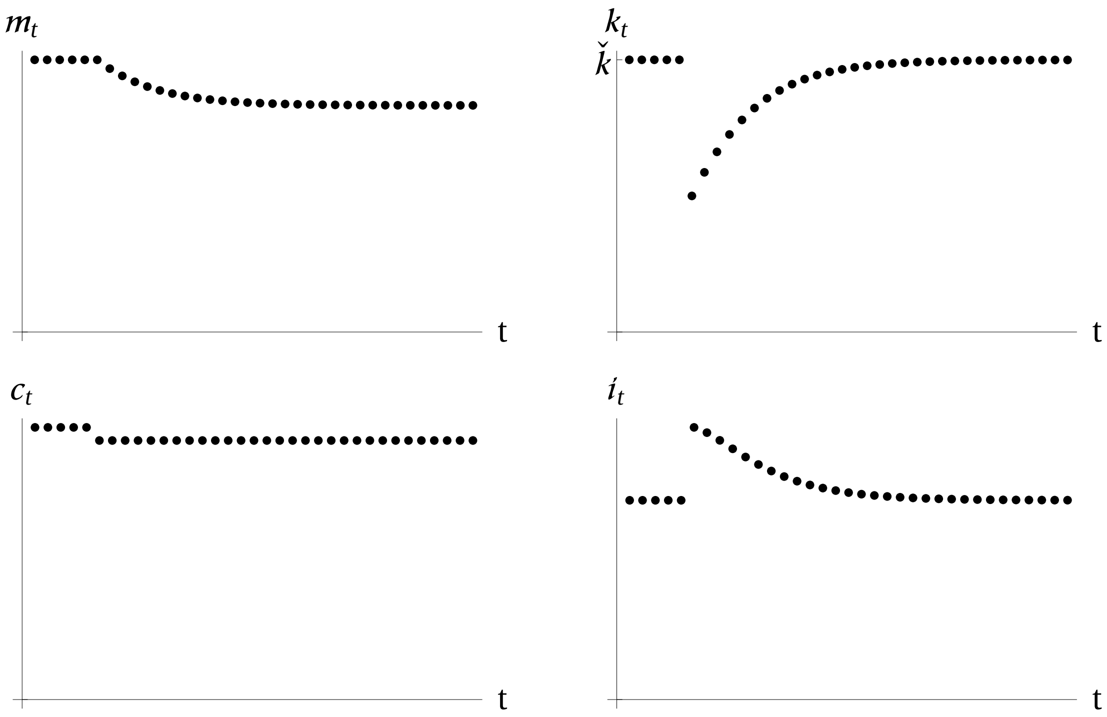



**Figure 3** Negative shock to $k_t$

The results are depicted below in 3.

Again, because dividends follow a random walk, what the firm's managers do is to assess the effect of the meteor shock on the firm's total value and they adjust the level of dividends downward immediately to the sustainable new level of dividends. Thereafter there is no change in the level of dividends.

Investment is more complicated. The firm's capital stock is obviously reduced below its steady-state value by the meteor, so there must be a period of high investment expenditures to bring capital back toward its steady state. However, the firm started out with monetary assets of zero. Therefore the high initial investment expenditures will be paid for by borrowing, driving the firm's monetary assets to a permanent negative value (the firm goes into debt to pay for its rebuilding). Gradually over time the capital stock is rebuilt back to its target level, and investment expenditures return to zero (or the level consistent with replacing depreciated capital).

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