**Horizon-Invariance of Portfolio Shares (Samuelson (1969))**

Personal finance advisors often recommend that the proportion of wealth invested in risky assets should decline as a person ages. This question examines whether that advice is explained by the benchmark CRRA Merton-Samuelson portfolio choice model.

Consider a consumer with CRRA utility function $\mathrm{u}(c) = -c^{1-\rho}$ with risk aversion parameter $\rho > 1$. At the end of period $t$ this consumer has assets $a_t$ and is trying to decide how much to invest in a risky asset that earns stochastic return $\log \mathbf{R}_{t+1} = \mathbf{r}_{t+1} \sim \mathcal{N}(\mathbf{r} - \sigma^2/2, \sigma^2_{\mathbf{r}})$ compared to a safe asset that earns a log return $\mathsf{r} = 0$ (so that $\mathsf{R} = 1$).

For a consumer ending period $t$ with assets $a_t$, suppose that expected value is a function of the form

$$\mathfrak{v}_t(a_t) \quad = \quad -a_t^{1-\rho}\, \mathbb{E}_t \left[ e^{\mathbf{r}_{t+1}\varsigma(1-\rho)} \right] \tag{1}$$

where $\mathbf{r}$ is the equity premium (because $\mathsf{r} = 0$) and the optimal portfolio share in the risky asset is

$$\varsigma \quad = \quad \frac{\mathbf{r}}{\rho \sigma^2_{\mathbf{r}}}. \tag{2}$$

(This will be true, for example, if period $t+1$ is the last period of the consumer's life).

1. Show that if

   $$\mathrm{v}_t(m_t) \quad = \quad \mathrm{u}(c_t) + \beta \mathfrak{v}_t(m_t - c_t) \tag{3}$$

   then the value function as of the time when the consumption decision is made is

   $$\mathrm{v}_t(m_t) \quad = \quad -m_t^{1-\rho}\zeta \tag{4}$$

   for some constant $\zeta$. (Hint: Solve for the consumption function using the FOC, defining $\alpha = \mathbb{E}_t \left[ e^{\mathbf{r}_{t+1}\varsigma(1-\rho)} \right]$, to show that optimal consumption is a constant fraction $\gamma$ of $m_t$.)

   It is a standard result in portfolio theory that if the value function in period $t+1$ has the generic form $\mathrm{v}_{t+1}(m_{t+1}) = -m_{t+1}^{1-\rho}\zeta$ and the consumer faces a portfolio investment choice like the one outlined above, and the consumer will receive no income in period $t+1$ except the income on his capital (that is, no labor or pension or transfer income), then the optimal portfolio share to invest in the risky asset at the end of period $t$ is given by (2). (This is the Merton-Samuelson model).

2. Given this information, does the Merton-Samuelson model support or contradict the advice provided by financial advisors? Explain your conclusions.

3. One key feature of reality that is omitted from the Merton-Samuelson model is the fact that people derive income from sources other than their portfolio of risky and riskless financial assets. Considering human wealth as a riskless asset (and ignoring the possibility of liquidity constraints), does the pattern of human wealth over the lifetime suggest any pattern of portfolio shares in risky versus riskless financial assets?

4. Another possible feature of reality omitted from the usual economic analysis is that the coefficient of relative risk aversion might vary by age. Suppose that intrinsic risk aversion increases with age. Would this make the model more consistent or less consistent with the advice given by financial advisors?

# References

SAMUELSON, PAUL A. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51, 239–46.