Horizon-Invariance of Portfolio Shares (Samuelson (1969))

Personal finance advisors often recommend that the proportion of wealth invested in risky assets should decline as a person ages. This question examines whether that advice is explained by the benchmark CRRA Merton-Samuelson portfolio choice model.

Consider a consumer with CRRA utility function $\mathrm{u}(c) = -c^{1-\rho}$ with risk aversion parameter $\rho > 1$. At the end of period t this consumer has assets a_t and is trying to decide how much to invest in a risky asset that earns stochastic return $\log \mathbf{R}_{t+1} = \mathbf{r}_{t+1} \sim \mathcal{N}(\mathbf{r} - \sigma^2/2, \sigma^2_{\mathbf{r}})$ compared to a safe asset that earns a log return $\mathsf{r} = 0$ (so that $\mathsf{R} = 1$).

For a consumer ending period t with assets a_t, suppose that expected value is a function of the form

$$\mathfrak{v}_t(a_t) \quad = \quad -a_t^{1-\rho}\,\mathbb{E}_t\left[e^{\mathbf{r}_{t+1}\varsigma(1-\rho)}\right] \tag{1}$$

where $\mathbf{r}$ is the equity premium (because $\mathsf{r} = 0$) and the optimal portfolio share in the risky asset is

$$\varsigma \quad = \quad \frac{\mathbf{r}}{\rho\sigma^2_{\mathbf{r}}}. \tag{2}$$

(This will be true, for example, if period $t+1$ is the last period of the consumer's life).

1. Show that if

   $$\mathrm{v}_t(m_t) \quad = \quad \mathrm{u}(c_t) + \beta\mathfrak{v}_t(m_t - c_t) \tag{3}$$

   then the value function as of the time when the consumption decision is made is

   $$\mathrm{v}_t(m_t) \quad = \quad -m_t^{1-\rho}\zeta \tag{4}$$

   for some constant ζ. (Hint: Solve for the consumption function using the FOC, defining $\alpha = \mathbb{E}_t\left[e^{\mathbf{r}_{t+1}\varsigma(1-\rho)}\right]$, to show that optimal consumption is a constant fraction γ of m_t.)

   Answer:

   Defining $\alpha = \mathbb{E}_t\left[e^{\mathbf{r}_{t+1}\varsigma(1-\rho)}\right]$, decision-period value is

   $$\mathrm{v}_t(m_t) \quad = \quad \max_{c_t} \; -c_t^{1-\rho} - \beta a_t^{1-\rho}\alpha \tag{5}$$

   with FOC

   $$\begin{aligned} c_t^{-\rho} &= (m_t - c_t)^{-\rho}\alpha\beta && (6)\\ c_t &= (m_t - c_t)(\alpha\beta)^{-1/\rho} && (7)\\ (1 + (\alpha\beta)^{-1/\rho})c_t &= m_t(\alpha\beta)^{-1/\rho} && (8)\\ c_t &= \underbrace{\left(\frac{(\alpha\beta)^{-1/\rho}}{(1 + (\alpha\beta)^{-1/\rho})}\right)}_{\equiv\gamma} m_t && (9) \end{aligned}$$

   so

   $$\mathrm{v}_t(m_t) \quad = \quad -(\gamma m_t)^{1-\rho} - \beta((1-\gamma)m_t)^{1-\rho}\alpha \tag{10}$$

$$= \quad -m_t^{1-\rho}\underbrace{\left(\gamma^{1-\rho} + (1-\gamma)^{1-\rho}\alpha\beta\right)}_{\equiv\zeta} \tag{11}$$

It is a standard result in portfolio theory that if the value function in period $t+1$ has the generic form $\mathrm{v}_{t+1}(m_{t+1}) = -m_{t+1}^{1-\rho}\zeta$ and the consumer faces a portfolio investment choice like the one outlined above, and the consumer will receive no income in period $t+1$ except the income on his capital (that is, no labor or pension or transfer income), then the optimal portfolio share to invest in the risky asset at the end of period t is given by (2). (This is the Merton-Samuelson model).

2. Given this information, does the Merton-Samuelson model support or contradict the advice provided by financial advisors? Explain your conclusions.

   Answer:

   > What this result means is that the share of your total wealth allocated to risky assets is always constant and equal to (2), at every age no matter how far you are from T. So this contradicts the advice from financial advisors, unless there is some other kind of wealth not encompassed in the model as specified above.

3. One key feature of reality that is omitted from the Merton-Samuelson model is the fact that people derive income from sources other than their portfolio of risky and riskless financial assets. Considering human wealth as a riskless asset (and ignoring the possibility of liquidity constraints), does the pattern of human wealth over the lifetime suggest any pattern of portfolio shares in risky versus riskless financial assets?

   Answer:

   > The model as described does not include human wealth. However, if human wealth were perfectly certain and there were no liquidity constraints, it would be equivalent to current market resources, and would therefore operate like an unobserved part of m_t. Since the proportion of total wealth accounted for by human wealth declines as people get older, if we imagine a component h_t that varies with age and is incorporated in a measure of total wealth, the constant portfolio share in risky assets (including human wealth) implies that as the perfectly certain human wealth declines, the share of observed wealth invested in the risky asset may decline.

4. Another possible feature of reality omitted from the usual economic analysis is that the coefficient of relative risk aversion might vary by age. Suppose that intrinsic risk aversion increases with age. Would this make the model more consistent or less consistent with the advice given by financial advisors?

   Answer:

This is another potential explanation for the advice of the personal finance gurus. The equation for the portfolio share invested in risky assets declines directly as ρ increases (cf. (2)), so if ρ increases with age it would be natural to expect ς to decline.

References

Samuelson, Paul A. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51, 239–46.