A Two-Asset Savings Model with an Income-Contribution Scheme

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Abstract

This paper contains the highlights from the REMARK file in Code>Python folder.

Keywords Lifecycle, Portfolio Choice, Social Security, Open Source

GitHub: http://github.com/econ-ark/REMARK/REMARKS/CGMPortfolio
(In GitHub repo, see /Code for tools for solving and simulating the model)

CLICK HERE for an interactive Jupyter Notebook that uses the Econ-ARK/HARK toolkit to produce our figures (warning: it may take several minutes to launch). Information about citing the toolkit can be found at Acknowleding Econ-ARK.

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All numerical results herein were produced using the Econ-ARK/HARK toolkit; for further reference options see Acknowleding Econ-ARK. Thanks to Chris Carroll and Sylvain Catherine for comments and guidance.

1 Introduction

This paper develops a two-asset consumption-savings model and serves as the documentation of an open-source implementation of methods to solve and simulate it in the HARK toolkit (Carroll, Kaufman, Kazil, Palmer, and White, 2018). The model represents an agent who can save using two different assets—one risky and the other risk-free—to insure against fluctuations in his income, but faces frictions to transferring funds between assets. The flexibility of its implementation and its inclusion in the HARK toolkit will allow users to adapt the model to realistic life-cycle calibrations, and also to embedded it in heterogeneous-agents macroeconomic models.

2 The model

I now discuss the main components of the model informally, and leave its full recursive mathematical representation for Section 3.

2.0.1 Time, mortality, and utility

Time advances in discrete steps that I will index with t. The model can be used in both infinite and finite-horizon versions.

Agents face an exogenous risk of death δ_t each period, which becomes certain at the maximum age of the finite-horizon version. There are no intentional bequests; agents will consume all of their resources if they reach the last period, but they can leave accidental bequests upon premature death.

In each period, agents derive utility from consumption only. Their utility function follows a constant relative risk aversion specification. Formally, for a level of consumption C, the agent derives instant utility

$$u(C) = \frac{C^{1-\rho}}{1-\rho}. \tag{1}$$

2.0.2 Income process

Agents supply labor inelastically. Their labor earnings $Y_{i,t}$ are the product of a permanent component $P_{i,t}$ and a transitory stochastic component $\theta_{i,t}$ as in Carroll (1997), where i indexes different agents. Formally,

$$\begin{aligned} \ln Y_{i,t} &= \ln P_{i,t} + \ln \theta_{i,t} \\ \ln P_{i,t} &= \ln P_{i,t-1} + \ln \Gamma_{i,t} + \ln \psi_{i,t} \end{aligned}$$

where $\Gamma_{i,t}$ is a deterministic growth factor that can capture life-cycle patterns in earnings, and $\ln \psi_{i,t} \sim \mathcal{N}(-\sigma^2_{\psi,t}/2, \sigma_{\psi,t})$ is a multiplicative shock to permanent income[1].

[1]The mean of the shock is set so that $E[\psi_{i,t}] = 1$.

The transitory component $\theta_{i,t}$ is a mixture that models unemployment and other temporal fluctuations in income as

$$\ln \theta_{i,t} = \begin{cases} \ln \mathcal{U}, & \text{With probability } \mho \\ \ln \tilde{\theta}_{i,t} \sim \mathcal{N}(-\sigma_{\theta,t}^2/2, \sigma_{\theta,t}), & \text{With probability } 1 - \mho, \end{cases}$$

with $\mho$ representing the probability of unemployment and $\mathcal{U}$ the replacement factor of unemployment benefits.

This specification of the income process is parsimonious and flexible enough to accommodate life-cycle patterns in income growth and volatility, transitory unemployment and exogenous retirement. Introduced by Carroll (1997), this income specification is common in studies of life-cycle wealth accumulation and portfolio choice; see e.g., Cagetti (2003); Cocco, Gomes, and Maenhout (2005); Fagereng, Gottlieb, and Guiso (2017). The specification has also been used in studies of income volatility, which have yielded calibrations of its stochastic shocks' distributions (see e.g., Carroll, 1992; Carroll and Samwick, 1997; Sabelhaus and Song, 2010)

2.0.3 Financial assets and frictions

Agents smooth their consumption by saving and have two assets available for this purpose. The first is a risk-free liquid account with constant per-period return factor R. The second has a stochastic return factor $\tilde{R}$ that is log-normally distributed and independent across time. Various interpretations such as stocks, a retirement fund, or entrepreneurial capital could be given to the risky asset. Importantly, consumption must be paid for using funds from the risk-free account. The levels of risk-free and risky assets owned by the agent will both be state variables, denoted with $M_{i,t}$ and $N_{i,t}$ respectively.

Portfolio rebalancing takes place by moving funds between the risk-free and risky accounts. These flows are one of the agents' control variables and are denoted as $D_{i,t}$, with $D_{i,t} > 0$ representing a movement of funds from the risk-free to the risky account. Withdrawals from the risky account are subject to a constant-rate tax τ which can represent, for instance, capital-gains realization taxes or early retirement-fund withdrawal penalties. In sum, denoting post-rebalancing asset levels with $\tilde{\cdot}$,

$$\begin{aligned} \tilde{M}_{i,t} &= M_{i,t} - D_{i,t}(1 - 1_{[D_{i,t} \leq 0]}\tau) \\ \tilde{N}_{i,t} &= N_{i,t} + D_{i,t}. \end{aligned}$$

At any given period, an agent might not be able to rebalance his portfolio. This ability is governed by an exogenous stochastic shock that is realized at the start of the period

$$\text{Adj}_t \sim \text{Bernoulli}(p_t),$$

with $\text{Adj}_t = 1$ meaning that the agent can rebalance and $\cancel{\text{Adj}}_t = 1$ ($\text{Adj}_t = 0$) forcing him to set $D_{i,t} = 0$. This friction is a parsimonious way to capture the fact that portfolio rebalancing is costly and households do it sporadically. Recent studies have advocated for (Giglio, Maggiori, Stroebel, and Utkus, 2021) and used (Luetticke, 2021) this kind of rebalancing friction.

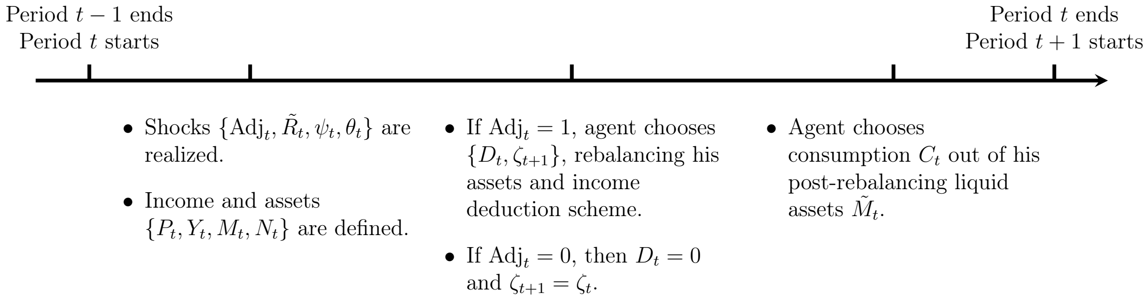

Figure 1 Summary of the Model's Timing

To partially evade the possibility of being unable to rebalance their accounts, agents can use an income deduction scheme. By default, labor income ($Y_{i,t}$) is deposited to the risk-free liquid account at the start of every period. However, agents can pre-commit to have a fraction $\zeta_t \in [0,1]$ of their income diverted to their risky account instead. This fraction can be tweaked by the agent whenever $\mathrm{Adj}_t = 1$; otherwise it stays at its previous value, $\zeta_{t+1} = \zeta_t$.

2.0.4 Timing

Figure 1 summarizes the timing of stochastic shocks and optimizing decisions that occur within a period of the life cycle model.

3 Recursive representation of the model

Individual subscripts i are dropped for simplicity. The value function for an agent who is not allowed to rebalance his portfolio at time t is

$$\begin{aligned}
V_t^{\cancel{\mathrm{Adj}}}(M_t, N_t, P_t, \zeta_t) = \max_{C_t} u(C_t) + p_{t+1}\beta\delta_{t+1}E_t\left[V_{t+1}^{\mathrm{Adj}}\left(M_{t+1}, N_{t+1}, P_{t+1}\right)\right] + \\
(1-p_{t+1})\beta\delta_{t+1}E_t\left[V_{t+1}^{\cancel{\mathrm{Adj}}}\left(M_{t+1}, N_{t+1}, P_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\text{Subject to:}\\
0 \leq &C_t \leq M_t\\
A_t =& M_t - C_t\\
M_{t+1} =& RA_t + (1-\zeta_{t+1})Y_{t+1}\\
N_{t+1} =& \tilde{R}_{t+1}N_t + \zeta_{t+1}Y_{t+1}\\
P_{t+1} =& \Gamma_{t+1}\psi_{t+1}P_t\\
Y_{t+1} =& \theta_{t+1}P_{t+1}\\
\zeta_{t+1} =& \zeta_t
\end{aligned}$$

and that of agent who is allowed to rebalance is

$$V_t^{\text{Adj}}(M_t, N_t, P_t) = \max_{C_t, D_t, \zeta_{t+1}} u(C_t) + p_{t+1}\beta\delta_{t+1}E_t\left[V_{t+1}^{\text{Adj}}(M_{t+1}, N_{t+1}, P_{t+1})\right] +$$
$$(1 - p_{t+1})\beta\delta_{t+1}E_t\left[V_{t+1}^{\cancel{\text{Adj}}}(M_{t+1}, N_{t+1}, P_{t+1}, \zeta_{t+1})\right]$$

Subject to:

$$-N_t \leq D_t \leq M_t, \quad \zeta_{t+1} \in [0, 1], \quad 0 \leq C_t \leq \tilde{M}_t$$

$$\begin{aligned}
\tilde{M}_t &= M_t - D_t\left(1 - 1_{[D_t \leq 0]}\tau\right) \\
\tilde{N}_t &= N_t + D_t \\
A_t &= \tilde{M}_t - C_t \\
M_{t+1} &= RA_t + (1 - \zeta_{t+1})Y_{t+1} \\
N_{t+1} &= \tilde{R}_{t+1}\tilde{N}_t + \zeta_{t+1}Y_{t+1} \\
P_{t+1} &= \Gamma_{t+1}\psi_{t+1}P_t \\
Y_{t+1} &= \theta_{t+1}P_{t+1}
\end{aligned}$$

The problem can be normalized by permanent income, following Carroll (2020). Using lower case variables to denote their upper-case counterparts normalized by permanent income ($x_t \equiv X_t/P_t$) and defining $\tilde{\Gamma}_t = \Gamma_t\psi_t$, we can write normalized problems

$$v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t) = \max_{c_t} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v_{t+1}^{\text{Adj}}(m_{t+1}, n_{t+1})\right] +$$
$$(1 - p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v_{t+1}^{\cancel{\text{Adj}}}(m_{t+1}, n_{t+1}, \zeta_{t+1})\right]$$

Subject to:

$$\begin{aligned}
0 &\leq c_t \leq m_t \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{R}{\tilde{\Gamma}_{t+1}}a_t + (1 - \zeta_{t+1})\theta_{t+1} \\
n_{t+1} &= \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}}n_t + \zeta_{t+1}\theta_{t+1} \\
\zeta_{t+1} &= \zeta_t
\end{aligned} \tag{2}$$

and

$$v_t^{\text{Adj}}(m_t, n_t) = \max_{c_t, d_t, \zeta_{t+1}} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v_{t+1}^{\text{Adj}}(m_{t+1}, n_{t+1})\right] + (1-p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v_{t+1}^{\cancel{\text{Adj}}}(m_{t+1}, n_{t+1}, \zeta_{t+1})\right]$$

$$\text{Subject to:}$$

$$-n_t \le d_t \le m_t, \quad \zeta_{t+1} \in [0,1], \quad 0 \le c_t \le \tilde{m}_t$$

$$\begin{aligned}
\tilde{m}_t &= m_t - d_t\left(1 - 1_{[d_t \le 0]}\tau\right) \\
\tilde{n}_t &= n_t + d_t \\
a_t &= \tilde{m}_t - c_t \\
m_{t+1} &= \frac{R}{\tilde{\Gamma}_{t+1}} a_t + (1-\zeta_{t+1})\theta_{t+1} \\
n_{t+1} &= \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t + \zeta_{t+1}\theta_{t+1}
\end{aligned} \tag{3}$$

It can be shown that

$$V_t^{\text{Adj}}(M_t, N_t, P_t) = P_t^{1-\rho}v_t^{\text{Adj}}(m_t, n_t), \quad V_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = P_t^{1-\rho}v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t),$$

and that the policy functions of both problems are related through

$$C_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = P_t c_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t), \quad C_t^{\text{Adj}}(M_t, N_t, P_t) = P_t c_t^{\text{Adj}}(m_t, n_t),$$
$$D_t(M_t, N_t, P_t) = P_t d_t(m_t, n_t), \quad \zeta_{t+1}(M_t, N_t) = \zeta_{t+1}(m_t, n_t).$$

Therefore, the model's implementation solves the problem in normalized form, and rescales the relevant states and choices using permanent income when simulating.

3.1 Partition into stages

An additional insight that facilitates solving the model is that the three decisions that an agent might take in a period (rebalancing his assets, choosing his income deduction fraction and consuming) can be seen as happening sequentially. This is convenient because:

- The sub-problems are easier to solve than the multi-choice full problem.
- Since the non-adjusting agent only chooses his consumption and we must solve his problem for every combination of $(m_t, n_t, \zeta_t = \zeta_{t+1})$, we can re-utilize his solution by expressing the adjusting agent's problem as

  $$v_t^{\text{Adj}}(m_t, n_t) = \max_{\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}} v_t^{\cancel{\text{Adj}}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}).$$

  This insight is similar to the "nested" reformulation suggested by Druedahl (2020).

To start re-expressing the problem, I take the order of the ageent's decisions to be: rebalance assets, define income contribution share and finally consume. I denote the stages at which these decisions are taken with Reb, Sha, and Cns respectively. I will use $v^{\texttt{Reb}}(\cdot)$, $v^{\texttt{Sha}}(\cdot)$ and $v^{\texttt{Cns}}(\cdot)$ to represent *stage value functions.*

I now present each stage in detail, working backwards in time.

3.1.1 Consumption stage, Cns

An agent who takes his assets and income contribution share as given is one who can not adjust them and can only choose his consumption. This corresponds to the problem of the non-adjusting agent (Equation 2). The important facts to realize at this stage are that

$$v_t^{\text{Adj}}(m_t, n_t) = v_t^{\texttt{Reb}}(m_t, n_t), \quad v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t) = v_t^{\texttt{Cns}}(m_t, n_t, \zeta_t),$$

with the first fact being true because we have assumed that asset rebalancing is the first decision that an adjusting agent takes, and he assumes that his subsequent decisions will be optimal. Therefore, the consumption stage problem is

$$\begin{aligned} v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}) = \max_{c_t} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Reb}}\left(m_{t+1}, n_{t+1}\right)\right] + \\ (1-p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Cns}}\left(m_{t+1}, n_{t+1}, \zeta_{t+1}\right)\right] \\ \text{Subject to:} \qquad\qquad\qquad\qquad \\ 0 \leq c_t \leq \tilde{m}_t \\ a_t = \tilde{m}_t - c_t \\ m_{t+1} = \frac{R}{\tilde{\Gamma}_{t+1}} a_t + (1-\zeta_{t+1})\theta_{t+1} \\ n_{t+1} = \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t + \zeta_{t+1}\theta_{t+1} \end{aligned} \tag{4}$$

3.1.2 Income contribution share stage, Sha

An agent with the option to set his income contribution share will do so taking his asset allocation as given and assuming that he will optimally pick his consumption in the next stage. His problem is

$$v_t^{\texttt{Sha}}(\tilde{m}_t, \tilde{n}_t) = \max_{\zeta_{t+1}\in[0,1]} v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}) \tag{5}$$

3.1.3 Rebalancing stage, `Reb`

The first decision that an agent takes, if allowed, is how to reallocate his assets. At this stage, his value function is

$$
\begin{aligned}
v_t^{\texttt{Reb}}(m_t, n_t) = &\max_{d_t} v_t^{\texttt{Sha}}(\tilde{m}_t, \tilde{n}_t) \\
&\text{Subject to:} \\
-n_t \leq &d_t \leq m_t \\
\tilde{m}_t = &m_t - d_t \left(1 - 1_{[d_t \leq 0]}\tau\right) \\
\tilde{n}_t = &n_t + d_t
\end{aligned}
\tag{6}
$$

The solution to this stage problem will be the policy function $d_t(\cdot,\cdot)$ that gives the optimal flow from risk-free to risky assets, which can be negative. However, it is convenient to define a normalized policy function $\underline{d}_t$ as

$$
\underline{d}_t(m,n) = \begin{cases} d_t(m,n)/m, & \text{if } d_t(m,n) \geq 0 \\ d_t(m,n)/n, & \text{if } d_t(m,n) < 0 \end{cases}
$$

so that $-1 \leq \underline{d}(m,n) \leq 1$ for all (m,n).

4 Examples of Solutions and Simulations

This section examines various instances of the model under different specifications for lives' lengths and financial frictions. The main purpose of these exercises is to illustrate the model's capabilities and the implications of financial frictions for asset allocations.

4.1 Infinite-horizon

For a first exercise, consider the infinite-horizon version of the model. In this version, parameters are constant and the optimization problem that the agent solves every period is—up to the values of state variables—the same. I set the parameters to the values reported in Table 1.

The infinite-horizon solution is obtained by value-function iteration backwards in time. I define trivial starting value functions $v_T^{\text{Adj}}(\cdot)$ and $v_T^{\cancel{\text{Adj}}}(\cdot)$ and use Equations 3 and 2 to find $v_{T-1}^{\text{Adj}}(\cdot)$ and $v_{T-1}^{\cancel{\text{Adj}}}(\cdot)$. I repeat this process until the value functions obtained in successive iterations are similar enough.

To illustrate the effect of the different financial frictions in the model, I present solutions for three different parametrizations.

- **Base**: a version without financial frictions. The probability of being able to rebalance is $p = 1$ and the risky withdrawal tax rate is $\tau = 0$.

- **Tax**: a version with a risky withdrawal tax of 10% ($\tau = 0.1$). The probability of being able to rebalance is $p = 1$.

Table 1 Infinite Horizon Example Calibration

Name in HARK	Mathematical Symbol	Value
`CRRA`	ρ	5.0
`Rfree`	R	1.03
`DiscFac`	β	0.9
`LivPrb`	δ	0.98
`PermGroFac`	Γ	1.01
`PermShkStd`	σ_ψ	0.1
`TranShkStd`	σ_θ	0.1
`UnempPrb`	$\mho$	0.05
`IncUnemp`	$\mathcal{U}$	0.3
`RiskyAvg`	$E[\tilde{R}]$	1.08
`RiskyStd`	$\sqrt{V[\tilde{R}]}$	0.18
`AdjustPrb`	p	Varying
`tau`	τ	Varying

- **Calvo**: a version with a stochastic inability to rebalance that occurs with a 75% chance ($p = 0.25$). The risky withdrawal is $\tau = 0$.

The first stage of an agent's period (if he is allowed to rebalance) is to solve the asset-rebalancing problem (Equation 6). The solution to this problem is the rebalancing fraction function $\underline{d}(m, n)$, which I present in Figure 2 for the different parametrizations and various combinations of (m, n). The figure shows how individuals with low risk-free resources withdraw funds from their risky accounts in order to finance their consumption. The version of the model with the withdrawal tax has regions where $\underline{d} = 0$ as a result of the tax's asymmetry—these are regions where the marginal utility of the risk-free asset lies between the pre-tax and post-tax marginal utility of risky assets. Finally, as illustrated by the figure, the stochastic inability to rebalance pushes agents to keep less funds in the risky asset, withdrawing them at higher rates when they get the chance.

The second stage in a period, which an agent also participates in only if he draws $\text{Adj}_t = 1$, consists of choosing the income contribution share ζ. Figure 3 presents the policy function $\zeta(\tilde{m}, \tilde{n})$ for various $(\tilde{m}, \tilde{n})$ combinations. The income deduction scheme becomes irrelevant in the "Base" and "Tax" versions of the model, since agents are always able to rebalance their assets ($p = 1$) and thus pre-committing funds has o benefits. The figure shows that when $p < 1$ agents will make use of the system, especially if the agent has low risky asset balances.

The final stage in the agent's problem, or the only one if $\text{Adj}_t = 0$, is to choose how much of his risk-free resources to consume. Figure 4 presents the consumption functions for different combinations of post-rebalancing assets $(\tilde{m}, \tilde{n})$ and income contribution fractions ζ. Consumption is increasing in both assets and financial frictions reduce the level of consumption at any given state. A first characteristic to note is that the

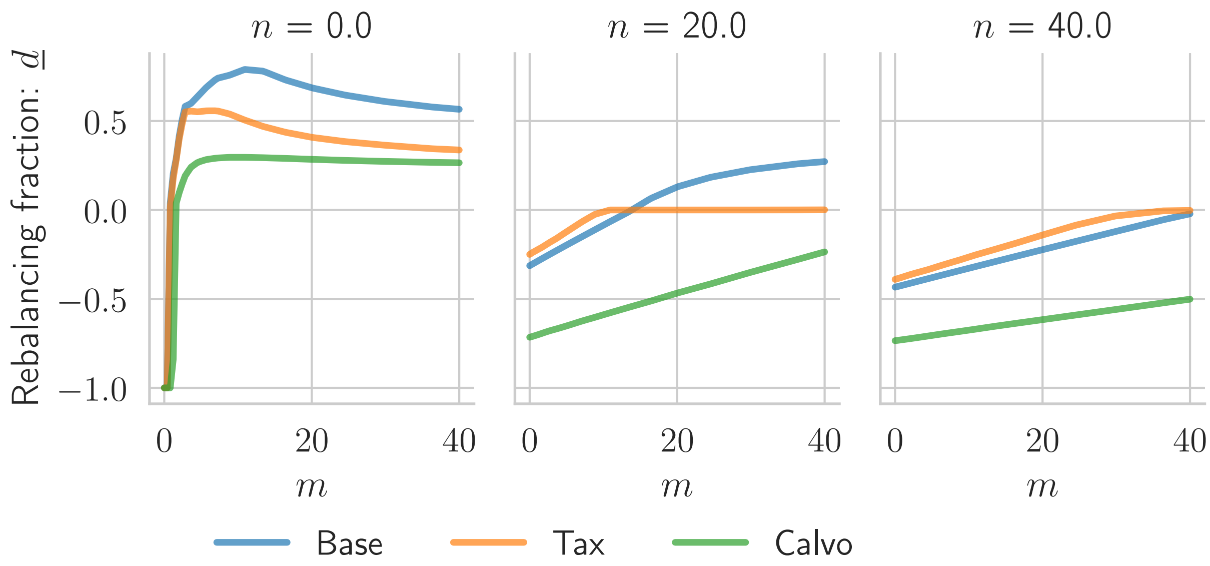

Figure 2 Optimal rebalancing fraction $\underline{d}$ in the infinite horizon model.

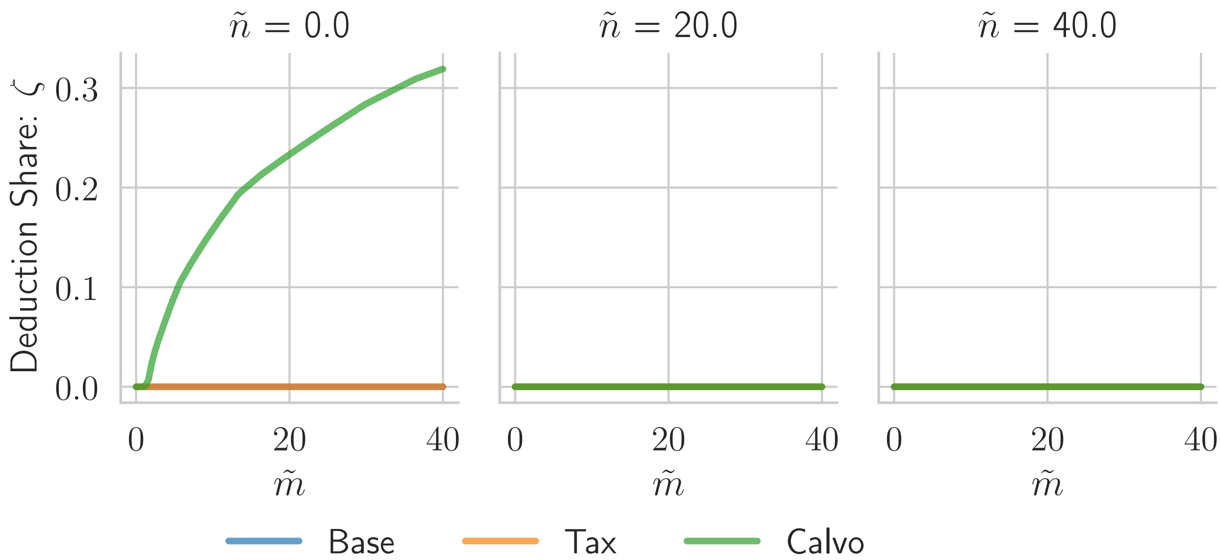

Figure 3 Optimal income contribution share ζ in the infinite horizon model.

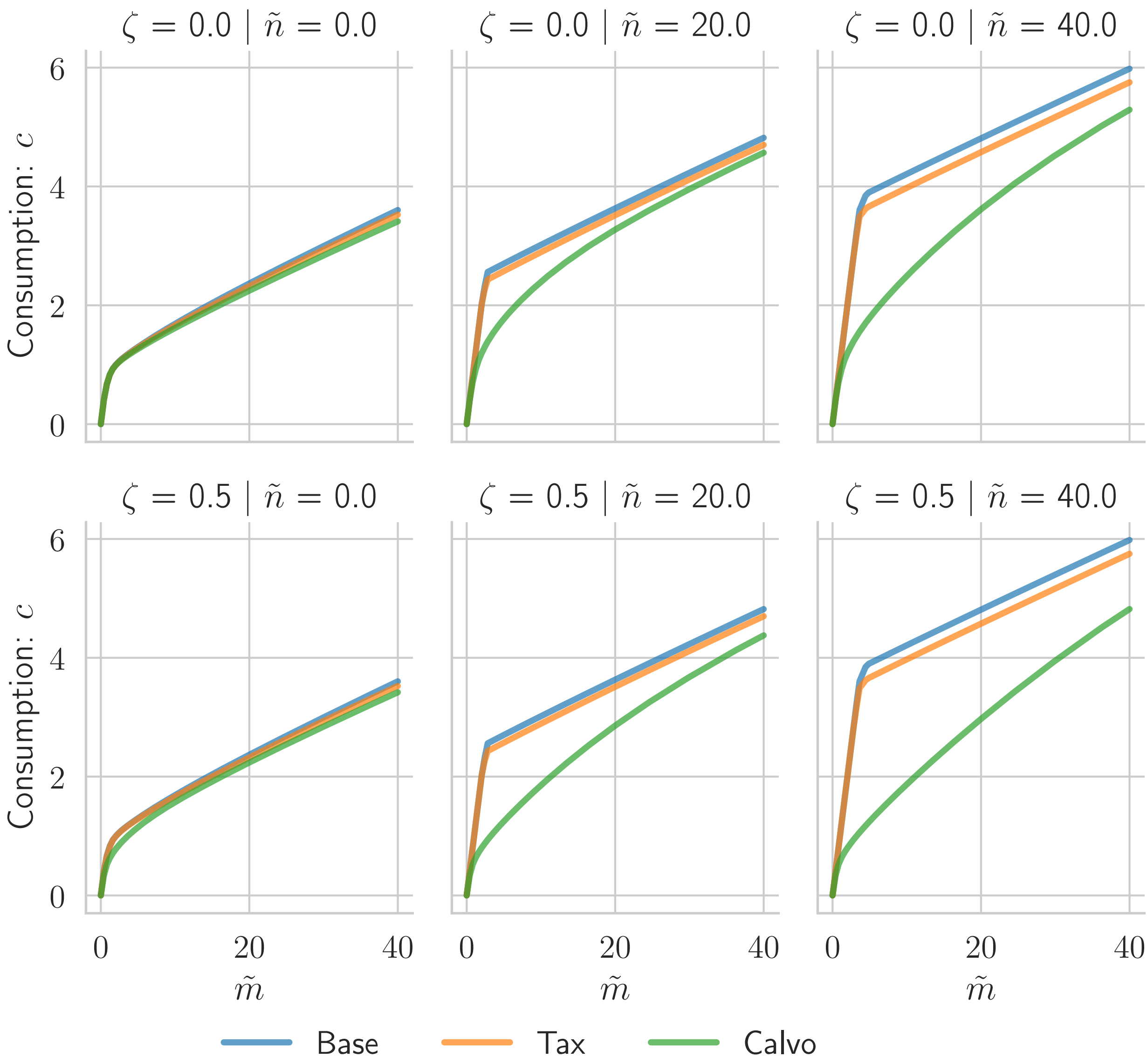

Figure 4 Optimal consumption c in the infinite horizon model.

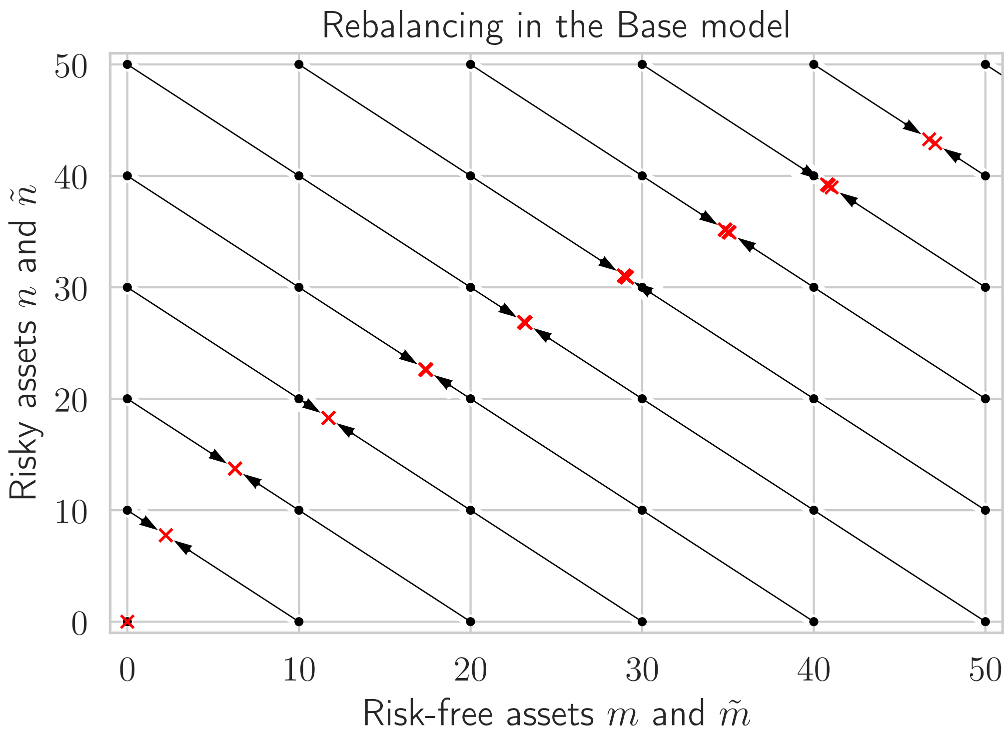

Figure 5 Asset rebalancing in the Base model

contribution share has no effect on the consumption functions of the "Base" model and little effect in those of the "Tax" model. Since these agents know they will be able to rebalance their assets, the only relevance of where their income is initially deposited comes from potentially paying the rebalancing tax. A second notable aspect is the interaction of financial frictions and saving choices. The figure shows that each of the frictions produces leftward shifts in the points at which the consumer switches from consuming all of his liquid resources ($c = \tilde{m}$) to saving ($c < \tilde{m}$).

Figures 5, 6 and 7 give a visual representation of the agents' reallocation decisions and the effect that financial frictions have on them. For various starting asset allocations (m, n) represented with black dots, each of the plots displays the optimal post-rebalancing allocation $(\tilde{m}, \tilde{n})$ with a red cross, joining initial and final positions with arrows. Figure 5 shows that in the frictionless "Base" model every initial point in a $m + n = x$ line maps to the same post-rebalancing portfolio, and that these portfolios lean towards risk-free assets as total wealth increases (they start above the 45-degree line but move below it as $m + n$ increases). Figure 6 introduces the rebalancing tax, which creates a region in which the agent does not rebalance his portfolio. This zone of inaction is a different representation of the flat regions of the $\underline{d}(\cdot)$ function in Figure 2. The tax also rotates the south-east pointing arrows, as a unit of n now transforms into less than a unit of m. Finally, Figure 7 removes the tax and introduces the stochastic inability

Figure 6 Asset rebalancing in the "Calvo" model.

to rebalance, which closes the inaction zone but makes the risky asset less desirable, resulting in an optimal portfolio trajectory that is closer to the horizontal axis.

4.2 Life-Cycle finite horizon

The finite-horizon version of the model represents agents that live for up to a maximum finite number of periods and might face different parametrizations of their problem (taxes, income patterns, etc) in each period. This version is well-suited for representing microeconomic life-cycle problems of savings and portfolio decisions, and for using its agents in overlapping-generations macroeconomic models. This section presents life-cycle simulations of the model under different parametrizations for financial frictions, showing the effect that they produce on wealth accumulation and portfolio choice.

I maintain the preferences (ρ, β) and financial-asset properties (R, $E[\tilde{R}]$, $V[\tilde{R}]$) from Table 1. Survival probabilities δ, and income's growth factor and volatilities (Γ, σ_ψ, σ_θ) are now time-varying and calibrated to represent individuals who enter the model at age 25, retire exogenously at 65, and live to a maximum age of 90. Survival probabilities come from the 2004 SSA life-tables for males. Income growth factors and volatilities come from the calibration for high-school graduates in Cagetti (2003). The parametrizations of the model differ in their assumptions about financial frictions. I present the following four configurations:

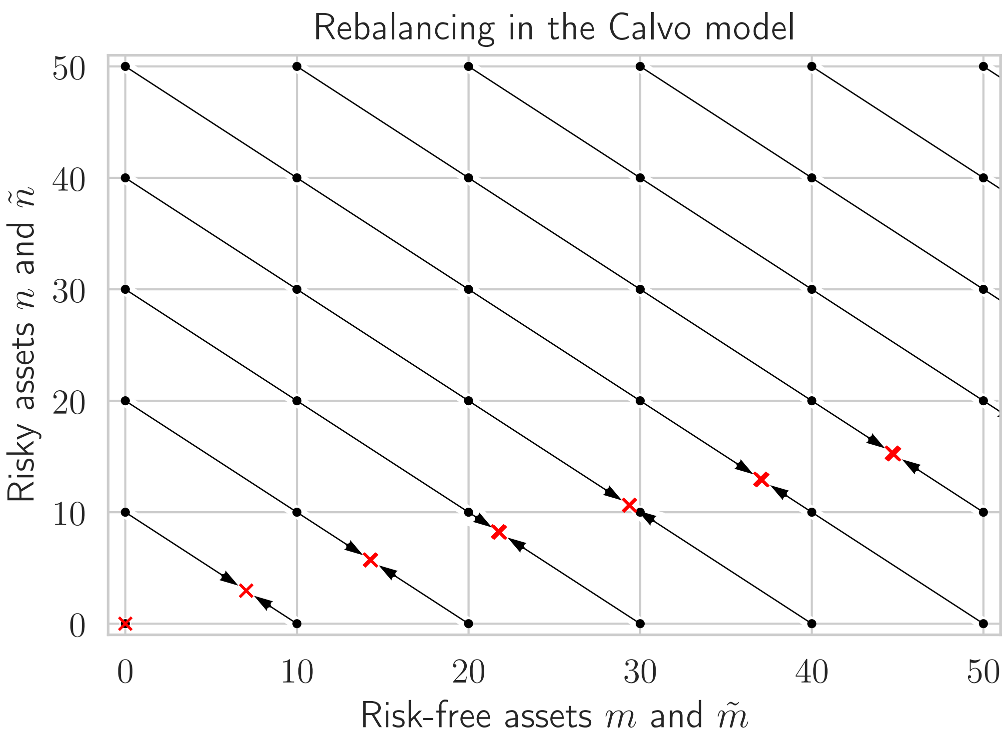

Figure 7 Asset rebalancing in the model with a withdrawal tax.

- **Base**: the probability of being able to rebalance is $p = 1$ and the risky withdrawal tax rate is $\tau = 0$, both constant throughout the agents' lives.
- **Tax**: the risky withdrawal tax is constant at 10% and the agents can always rebalance their portfolios.
- **Calvo**: there is no risky withdrawal tax, but there is only a 25% chance that agents can rebalance their portfolios every year.
- **Retirement**: there is no risky withdrawal tax, but the agents' ability to rebalance their portfolio is time-varying; they can rebalance their assets and pick their income-deduction scheme for certain when they enter the model at age 25, but then they have no chance of doing so again ($p = 0$) until they retire. After retirement at age 65, they can always rebalance their portfolio ($p = 1$).

I simulate populations of 50 agents for 5,000 periods with each of the parametrizations. Each time an agent dies, he is replaced by a new 25-year-old. I calculate age profiles of various variables of interest by finding their averages across all the simulated observations of every given age. Figure 8 presents the age profiles of permanent income, asset balances, the risky share of savings, and the income deduction share for all parametrizations.

The age profiles of the "Base" and "Tax" parametrizations are similar. The risky share of savings, which I define as $\tilde{N}/(A + \tilde{M})$, follows the same pattern as in Cocco, Gomes,

Figure 8 Age profiles of variables of interest in life-cycle calibrations

and Maenhout (2005), starting high and declining with age. The main difference between both calibrations is that the tax makes agents' risky share lower when they are young (because of an unwillingness to pay the tax if they are forced to draw from their scarce wealth) and higher after retirement as the tax incentivizes them to withdraw their funds more slowly. Since in both calibrations the agent can rebalance his portfolio each period, there is no reason to use the income deduction scheme.

Matters change substantially with the stochastic inability to rebalance. As Figure 8 shows, agents under the "Calvo" calibration accumulate less risky asset balances. In part, this is due to the mechanical effect of a reduced chance of being able to shift funds into the risky asset. However, risky assets also become less desirable at young ages since they are worse at insuring agents against their income fluctuations—they might be barred from accessing their funds precisely at a period when they fall unemployed. The uncertain ability to extract funds from the risky asset also reduces its appeal as a vehicle for retirement savings, which is evidenced in a low income deduction share. As a result of investing less in the risky asset, agents in this calibration accumulate less total wealth and can afford only a lower level of consumption.

The last calibration, "Retirement", shows what happens when the rebalancing friction negates the risky asset's use as a buffer for income fluctuations at younger ages but maintains its attractiveness as a vehicle for retirement savings. In this calibration, agents use the income-deduction scheme to progressively build up their stock of risky assets. The average contribution share is high enough that these agents accumulate the most risky assets by retirement out of all calibrations. When they retire, as they can finally access their risky funds, they afford a higher level of consumption than the other types of agent.

These exercises show how the different financial frictions that the model can accommodate generate different consumption, savings, and portfolio patterns.

5 Conclusion

6 Puzzles and Questions

7 Robustness Analyses

Appendices

A First order conditions and value-function derivatives

The computational solution of the model uses the stage-problems' first-order conditions extensively. This appendix writes the first-order conditions and the value-function derivatives that appear in them explicitly.

A.0.1 Consumption stage, `Cns`

The first order condition for an interior solution ($c < \tilde{m}$) of the consumption stage problem (Equation 4) is

$$\begin{aligned} u'(c_t) = p_{t+1}\beta R\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{-\rho}\frac{\partial v_{t+1}^{\texttt{Reb}}}{\partial\tilde{m}_{t+1}}\right] + \\ (1-p_{t+1})\beta R\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{-\rho}\frac{\partial v_{t+1}^{\texttt{Cns}}}{\partial\tilde{m}_{t+1}}\right] \end{aligned}$$

The derivatives of the stage value function are

$$\frac{\partial v_t^{\texttt{Cns}}(m_t, n_t, \zeta_{t+1})}{\partial\tilde{m}_t} = u'(c_t) \tag{7}$$

$$\begin{aligned} \frac{\partial v_t^{\texttt{Cns}}(m_t, n_t, \zeta_{t+1})}{\partial\tilde{n}_t} = p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{R}_{t+1}\tilde{\Gamma}_{t+1}^{-\rho}\frac{\partial v_{t+1}^{\texttt{Reb}}}{\partial\tilde{m}_{t+1}}\right] + \\ (1-p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{R}_{t+1}\tilde{\Gamma}_{t+1}^{-\rho}\frac{\partial v_{t+1}^{\texttt{Cns}}}{\partial\tilde{m}_{t+1}}\right] \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{\partial v_{t+1}^{\texttt{Cns}}(m_t, n_t, \zeta_{t+1})}{\partial\zeta_{t+1}} = p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}\theta_{t+1}\left(\frac{\partial v_{t+1}^{\texttt{Reb}}}{\partial\tilde{n}_{t+1}} - \frac{\partial v_{t+1}^{\texttt{Reb}}}{\partial\tilde{m}_{t+1}}\right)\right] + \\ (1-p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}\theta_{t+1}\left(\frac{\partial v_{t+1}^{\texttt{Cns}}}{\partial\tilde{n}_{t+1}} - \frac{\partial v_{t+1}^{\texttt{Cns}}}{\partial\tilde{m}_{t+1}}\right) + \tilde{\Gamma}_{t+1}^{1-\rho}\frac{\partial v_{t+1}^{\texttt{Cns}}}{\partial\zeta_{t+2}}\right] \end{aligned} \tag{9}$$

A.0.2 Income contribution share stage, `Sha`

The first order condition for an interior solution ($\zeta \in (0,1)$) of the income contribution share stage (Equation 5) is

$$0 = \frac{\partial v_t^{\texttt{Cns}}}{\partial\zeta_{t+1}}, \tag{10}$$

and the derivatives of the stage value function are

$$\frac{\partial v_t^{\texttt{Sha}}(\tilde{m}_t,\tilde{n}_t)}{\partial \tilde{m}_t} = \frac{\partial v_t^{\texttt{Cns}}}{\partial \tilde{m}_t}, \qquad \frac{\partial v_t^{\texttt{Sha}}(\tilde{m}_t,\tilde{n}_t)}{\partial \tilde{n}_t} = \frac{\partial v_t^{\texttt{Cns}}}{\partial \tilde{n}_t}$$

A.0.3 Rebalancing stage, `Reb`

The first order condition for a solution of the type $d \in [(-n,0) \cup (0,m)]$the rebalancing stage problem (Equation 6) is

$$\left(1 - 1_{[d_t \leq 0]}\tau\right) \frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{m}} = \frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{n}}, \tag{11}$$

and a necessary condition for a solution of the type $d = 0$ is

$$(1-\tau)\,\frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{m}} \leq \frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{n}} \leq \frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{m}} \tag{12}$$

The derivatives of the stage value function are

$$\frac{\partial v_t^{\texttt{Reb}}(m_t,n_t)}{\partial m_t} = \max\left\{\frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{m}}, \frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{n}}\right\} \tag{13}$$

$$\frac{\partial v_t^{\texttt{Reb}}(m_t,n_t)}{\partial n_t} = \max\left\{(1-\tau)\frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{m}}, \frac{\partial v_t^{\texttt{Sha}}}{\partial \tilde{n}}\right\} \tag{14}$$

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