A Two-Asset Savings Model with an Income-Contribution Scheme REMARK

May 8, 2021

Mateo Velásquez-Giraldo

Abstract

This paper contains the highlights from the REMARK file in Code>Python folder.

Keywords Lifecycle, Portfolio Choice, Social Security, Replication

GitHub: http://github.com/econ-ark/REMARK/REMARKS/CGMPortfolio
(In GitHub repo, see `/Code` *for tools for solving and simulating the model)*

CLICK HERE for an interactive Jupyter Notebook that uses the Econ-ARK/HARK toolkit to produce our figures (warning: it may take several minutes to launch). Information about citing the toolkit can be found at Acknowleding Econ-ARK.

[1]Contact: mvelasq2@jhu.edu, matthew.zahn@jhu.edu, Department of Economics, 590 Wyman Hall, Johns Hopkins University, Baltimore, MD 21218, https://t.co/uaflostQyF?amp=1, http://matthewvzahn.com.

All numerical results herein were produced using the Econ-ARK/HARK toolkit; for further reference options see Acknowleding Econ-ARK. Thanks to Chris Carroll and Sylvain Catherine for comments and guidance.

1 Introduction

2 The base model

2.1 Life-Cycle Model

The life-cycle model resembles that in **?**, with modifications to the portfolio allocation problem and the income process. I now discuss the main components of the model, which I summarize in its formal recursive representation in Appendix **??**.

2.1.1 Lifespan, mortality, and utility

Agents enter the model at age 24 and can live up to a maximum age of 100. Each period, they face an exogenous risk of death that becomes certain at the maximum age. There are no intentional bequests: agents will consume all of their resources if they make it to the maximum age, but can leave accidental bequests upon premature death.

In each period, agents derive utility from consumption only. Their utility function follows a constant relative risk aversion specification. Formally, for a level of consumption C, the agent derives instant utility

$$u(C) = \frac{C^{1-\rho}}{1-\rho}. \tag{1}$$

2.1.2 Income process

Agents supply labor inelastically and retire exogenously at age 65. Pre-retirement labor earnings $Y_{i,t}$ are the product of a permanent component $P_{i,t}$ and a transitory stochastic component $\theta_{i,t}$ as in **?**. Formally,

$$\begin{aligned} \ln Y_{i,t} &= \ln P_{i,t} + \ln \theta_{i,t} \\ \ln P_{i,t} &= \ln P_{i,t-1} + \ln \Gamma_{i,t} + \ln \psi_{i,t} \end{aligned}$$

where $\Gamma_{i,t}$ is a deterministic growth factor that captures life-cycle patterns in earnings, and $\ln \psi_{i,t} \sim \mathcal{N}(-\sigma^2_{\psi,t}/2, \sigma_{\psi,t})$ is a multiplicative shock to permanent income[1].

The transitory component $\theta_{i,t}$ is a mixture that models unemployment and other temporal fluctuations in income as

$$\ln \theta_{i,t} = \begin{cases} \ln \mathcal{U}, & \text{With probability } \mho \\ \ln \tilde{\theta}_{i,t} \sim \mathcal{N}(-\sigma^2_{\theta,t}/2, \sigma_{\theta,t}), & \text{With probability } 1-\mho, \end{cases}$$

with $\mho$ representing the probability of unemployment and $\mathcal{U}$ the replacement factor of unemployment benefits.

The sequences of growth factors $\{\Gamma_{i,t}\}_{t=25}^{100}$ vary by educational attainment, and their calibration comes from **?**. Figure 1 displays the income paths that individuals with different education levels would experience in the absence of shocks[2]. The decline at

[1] The mean of the shock is set so that $E[\psi_{i,t}] = 1$.

[2] This is $P_{i,24} \times \prod_{j=25}^{t} \Gamma_{i,t}$.

Figure 1 Deterministic Component of Income by Education Level

Figure 2 Age Profiles of Income Volatility

age 65 corresponds to retirement and after this point income becomes deterministic, following $Y_{i,t} = P_{i,t} = P_{i,t-1}\Gamma_{i,t}$.

An important advance in life-cycle modeling has been the recognition that the volatility of income varies with age. **?** estimate age profiles of both transitory and permanent shock volatilities, which I incorporate as the values of $\{\sigma_{\theta,t}, \sigma_{\psi,t}\}_{t=24}^{64}$. Figure 2 presents these volatilities, which are large and have a clear downward trend in age. Income fluctuations are the main motive for saving at early ages in the model and therefore these volatility patterns will have implications for portfolio choice.

2.1.3 Financial assets and frictions

Agents smooth their consumption by saving and have two assets available for this purpose. The first is a risk-free liquid account with constant per-period return factor R. The second represents stocks and has a stochastic return factor $\tilde{R}$ that agents view as log-normally distributed and independent across time. Importantly, consumption must be paid for using funds from the risk-free liquid account. The levels of risk-free and risky assets owned by the agent will both be state variables, denoted with $M_{i,t}$ and $N_{i,t}$ respectively.

Portfolio rebalancing takes place by moving funds between the risk-free and risky accounts. These flows are one of the agents' control variables and are denoted as $D_{i,t}$, with $D_{i,t} > 0$ representing a movement of funds from the risk-free to the risky account. Withdrawals from the risky account are taxed at a constant rate τ, representing capital taxes or early retirement fund withdrawal penalties. In sum, denoting post-rebalancing levels with $\tilde{\cdot}$,

$$\begin{aligned}\tilde{M}_{i,t} &= M_{i,t} - D_{i,t}(1 - 1_{[D_{i,t} \leq 0]}\tau)\\ \tilde{N}_{i,t} &= N_{i,t} + D_{i,t}.\end{aligned}$$

At any given period, an agent might not be able to rebalance his portfolio. This ability is governed by an exogenous stochastic shock that is realized at the start of the period

$$\text{Adj}_t \sim \text{Bernoulli}(p_t),$$

with $\text{Adj}_t = 1$ meaning that the agent can rebalance and $\cancel{\text{Adj}}_t = 1$ ($\text{Adj}_t = 0$) forcing him to set $D_{i,t} = 0$. This friction is a parsimonious way to capture the fact that portfolio rebalancing is costly and households do it sporadically **?**.

To partially evade the possibility of being unable to rebalance their accounts, agents can use an income deduction scheme. By default, labor income and retirement benefits

Figure 3 Portfolio Rebalancing in the HRS

"Frictionless" corresponds to a version of the model in which agents can always adjust their portfolio ($\{p_t\}_{t=24}^{100} = 1$) and are not subject to taxes ($\tau = 0$). "Tax" is a version in which $\tau = 0.1$. "Calvo" is a version in which agents have a 25% chance of being able to rebalance in every period ($\{p_t\}_{t=24}^{100} = 0.25$). "Tax + Calvo" is a version with the two previous frictions added simultaneously.

Figure 4 Frictions and Their Interaction With Beliefs.

($Y_{i,t}$) are deposited to the risk-free liquid account at the start of the period. However, agents can pre-commit to have a fraction $\zeta_t \in [0, 1]$ of their income diverted to their risky account instead. This fraction can be tweaked by the agent whenever $\mathrm{Adj}_t = 1$; otherwise it stays at its previous value, $\zeta_{t+1} = \zeta_t$.

These frictions to stockholding represent a departure from the usual modeling approach that is followed in the literature. The most usual friction to stockholding in the literature is a fixed monetary cost that agents must pay every period in which they hold stocks. This framework's convenience comes from only having to track total wealth as a state variable—as opposed to tracking both $M_{i,t}$ and $N_{i,t}$—as agents risky-asset positions are liquidated every period before they decide whether to enter the stock market again.

While convenient, this framework has problematic implications. First, it assumes that agents liquidate their portfolios and decide whether to re-enter the stock market every period, with the friction placed on this decision. This implication contradicts the fact that people rarely rebalance their portfolios as shown by Figure 3. One can imagine that if a person starts a period with 90% of their wealth in stocks, the path of least resistance for them would be to preserve their portfolio instead of fully liquidating it. A second problem that this friction introduces in heterogeneous agent models is selection; an entry barrier limits stockholders to be those whose eagerness overcomes the cost and therefore it raises the share of wealth invested in stocks conditional on participation above the already-too-high prediction of the frictionless model.

Figure 4 shows the effects of the frictions in my model on the share of wealth in stocks conditional on participation, and their interaction with beliefs about the equity premium. A major challenge in the life-cycle portfolio literature is explaining the low share of wealth that young people place in stocks (compare Figures **??** and 4). Figure 4 shows that pessimism about the equity premium combined with either the tax or stochastic rebalancing friction can substantially reduce the conditional share of young agents, flattening its age profile and bringing it closer to the data (see Figure **??**). The interaction between pessimistic beliefs and frictions to rebalancing will be my strategy for improving the fit of the life cycle model.

2.1.4 Timing

Figure 5 summarizes the timing of stochastic shocks and optimizing decisions that occur within a period of the life cycle model.

Figure 5 Summary of Timing in the Life Cycle Model

3 Recursive representation of the model

Individual subscripts are dropped for simplicity. The value function for an agent who is not allowed to rebalance his portfolio at time t is

$$
\begin{aligned}
V_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = \max_{C_t} u(C_t) + p_{t+1}\beta\delta_{t+1}E_t\left[V_{t+1}^{\text{Adj}}\left(M_{t+1}, N_{t+1}, P_{t+1}\right)\right] + \\
(1 - p_{t+1})\beta\delta_{t+1}E_t\left[V_{t+1}^{\cancel{\text{Adj}}}\left(M_{t+1}, N_{t+1}, P_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}
$$

$$
\text{Subject to:}
$$

$$
\begin{aligned}
0 \leq & C_t \leq M_t \\
A_t = & M_t - C_t \\
M_{t+1} = & RA_t + (1 - \zeta_{t+1})Y_{t+1} \\
N_{t+1} = & \tilde{R}_{t+1}N_t + \zeta_{t+1}Y_{t+1} \\
P_{t+1} = & \Gamma_{t+1}\psi_{t+1}P_t \\
Y_{t+1} = & \theta_{t+1}P_{t+1} \\
\zeta_{t+1} = & \zeta_t
\end{aligned}
$$

and that of agent who is allowed to rebalance is

$$
\begin{aligned}
V_t^{\text{Adj}}(M_t, N_t, P_t) = \max_{C_t, D_t, \zeta_{t+1}} u(C_t) + p_{t+1}\beta\delta_{t+1}E_t\left[V_{t+1}^{\text{Adj}}\left(M_{t+1}, N_{t+1}, P_{t+1}\right)\right] + \\
(1 - p_{t+1})\beta\delta_{t+1}E_t\left[V_{t+1}^{\cancel{\text{Adj}}}\left(M_{t+1}, N_{t+1}, P_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}
$$

$$
\text{Subject to:}
$$

$$
-N_t \leq D_t \leq M_t, \quad \zeta_{t+1} \in [0, 1], \quad 0 \leq C_t \leq \tilde{M}_t
$$

$$
\begin{aligned}
\tilde{M}_t = & M_t - D_t\left(1 - 1_{[D_t \leq 0]}\tau\right) \\
\tilde{N}_t = & N_t + D_t \\
A_t = & \tilde{M}_t - C_t \\
M_{t+1} = & RA_t + (1 - \zeta_{t+1})Y_{t+1} \\
N_{t+1} = & \tilde{R}_{t+1}\tilde{N}_t + \zeta_{t+1}Y_{t+1} \\
P_{t+1} = & \Gamma_{t+1}\psi_{t+1}P_t \\
Y_{t+1} = & \theta_{t+1}P_{t+1}
\end{aligned}
$$

The problem can be normalized by permanent income, following **?**. Using lower case variables to denote their upper-case counterparts normalized by permanent income ($x_t =$

X_t/P_t) and defining $\tilde{\Gamma}_t = \Gamma_t \psi_t$, we can write normalized problems

$$\begin{aligned}
v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t) = \max_{c_t} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\text{Adj}}\left(m_{t+1}, n_{t+1}\right)\right] + \\
(1-p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\cancel{\text{Adj}}}\left(m_{t+1}, n_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\text{Subject to:}\\
0 \leq &c_t \leq m_t\\
a_t =&m_t - c_t\\
m_{t+1} =&\frac{R}{\tilde{\Gamma}_{t+1}} a_t + (1-\zeta_{t+1})\theta_{t+1}\\
n_{t+1} =&\frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} n_t + \zeta_{t+1}\theta_{t+1}\\
\zeta_{t+1} =&\zeta_t
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
v_t^{\text{Adj}}(m_t, n_t) = \max_{c_t, d_t, \zeta_{t+1}} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\text{Adj}}\left(m_{t+1}, n_{t+1}\right)\right] + \\
(1-p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\cancel{\text{Adj}}}\left(m_{t+1}, n_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\text{Subject to:}\\
-n_t \leq d_t \leq m_t, \quad \zeta_{t+1} \in&[0,1], \quad 0 \leq c_t \leq \tilde{m}_t\\
\\
\tilde{m}_t =&m_t - d_t\left(1 - 1_{[d_t \leq 0]}\tau\right)\\
\tilde{n}_t =&n_t + d_t\\
a_t =&\tilde{m}_t - c_t\\
m_{t+1} =&\frac{R}{\tilde{\Gamma}_{t+1}} a_t + (1-\zeta_{t+1})\theta_{t+1}\\
n_{t+1} =&\frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t + \zeta_{t+1}\theta_{t+1}
\end{aligned}$$

It can be shown that

$$V_t^{\text{Adj}}(M_t, N_t, P_t) = P_t^{1-\rho} v_t^{\text{Adj}}(m_t, n_t), \quad V_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = P_t^{1-\rho} v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t),$$

and that the policy functions of both problems are related through

$$\begin{aligned}
C_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = P_t c_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t), \quad C_t^{\text{Adj}}(M_t, N_t, P_t) = P_t c_t^{\text{Adj}}(m_t, n_t),\\
D_t(M_t, N_t, P_t) = P_t d_t(m_t, n_t), \quad \zeta_{t+1}(M_t, N_t) = \zeta_{t+1}(m_t, n_t).
\end{aligned}$$

Therefore, I solve the normalized problem and re-scale its solutions to obtain the original problem's solutions.

3.1 Partition into stages

An additional insight to facilitate solving the model is that the three decisions that an agent might take in a period (rebalancing his assets, choosing his income deduction fraction and consuming) can be seen as happening sequentially. This is convenient because:

- The sub-problems are easier to solve than the multi-choice full problem.
- Since the non-adjusting agent only chooses his consumption and we must solve his problem for every combination of $(m_t, n_t, \zeta_t = \zeta_{t+1})$, we can re-utilize his solution by expressing the adjusting agent's problem as

$$v_t^{\text{Adj}}(m_t, n_t) = \max_{\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}} v_t^{\cancel{\text{Adj}}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}).$$

To start re-expressing the problem, take the order of the decisions to be: rebalance assets, then define income contribution share, then consume, and denote the stages at which these decisions are taken with `Reb`, `Sha`, and `Cns`. I will use $v^{\texttt{Reb}}(\cdot)$, $v^{\texttt{Sha}}(\cdot)$ and $v^{\texttt{Cns}}(\cdot)$ to represent *stage value functions.*

I now present each stage in detail, working backwards in time.

3.1.1 Consumption stage, `Cns`

An agent who takes his assets and income contribution share as given is one who can not adjust them and can only choose his consumption. This corresponds to the problem of the non-adjusting agent (Equation 2). The important facts to realize at this stage are that

$$v_t^{\text{Adj}}(m_t, n_t) = v_t^{\texttt{Reb}}(m_t, n_t), \quad v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t) = v_t^{\texttt{Cns}}(m_t, n_t, \zeta_t),$$

with the first fact being true because we have assumed that asset rebalancing is the first decision that an adjusting agent takes, and he assumes that his subsequent decisions will be optimal. Therefore, the consumption stage problem is

$$\begin{aligned}
v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}) = \max_{c_t} u(c_t) + p_{t+1}\beta\delta_{t+1} E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Reb}}\left(m_{t+1}, n_{t+1}\right)\right] + \\
(1 - p_{t+1})\beta\delta_{t+1} E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Cns}}\left(m_{t+1}, n_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}$$

Subject to:

$$\begin{aligned}
0 \leq & c_t \leq \tilde{m}_t \\
a_t = & \tilde{m}_t - c_t \\
m_{t+1} = & \frac{R}{\tilde{\Gamma}_{t+1}} a_t + (1 - \zeta_{t+1})\theta_{t+1} \\
n_{t+1} = & \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t + \zeta_{t+1}\theta_{t+1}
\end{aligned}$$

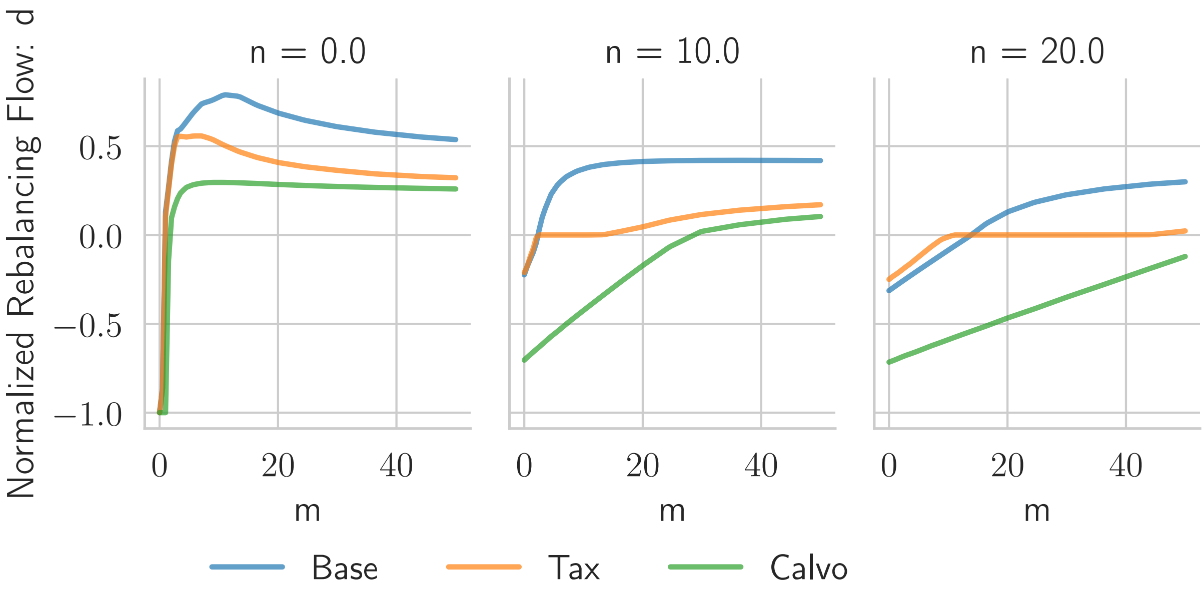

Figure 6 Convergence of the Consumption Rules

3.1.2 Income contribution share stage, `Sha`

An agent with the option to set his income contribution share will do so taking his asset allocation as given and assuming that he will optimally pick his consumption in the next stage. His problem is

$$v_t^{\texttt{Sha}}(\tilde{m}_t, \tilde{n}_t) = \max_{\zeta_{t+1}\in[0,1]} v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1})$$

3.1.3 Rebalancing stage, `Reb`

The first decision that an agent takes, if allowed, is how to reallocate his assets.

$$v_t^{\texttt{Reb}}(m_t, n_t) = \max_{d_t} v_t^{\texttt{Sha}}(\tilde{m}_t, \tilde{n}_t)$$

Subject to:

$$\begin{aligned} -n_t \leq & d_t \leq m_t \\ \tilde{m}_t =& m_t - d_t\left(1 - 1_{[d_t\leq 0]}\tau\right) \\ \tilde{n}_t =& n_t + d_t \end{aligned}$$

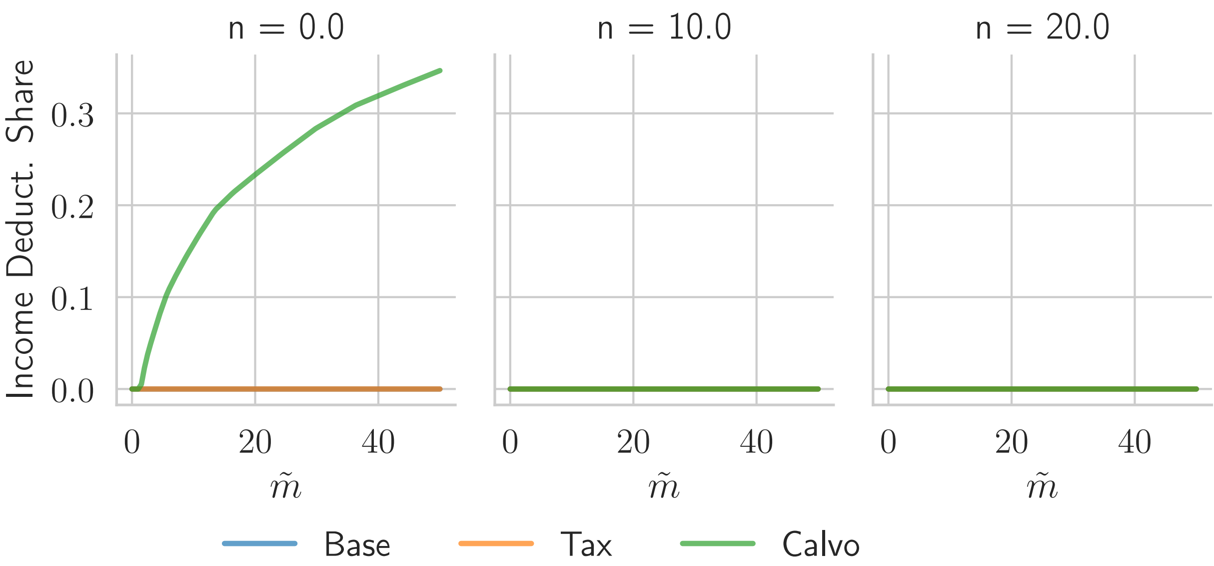

Figure 7 Convergence of the Consumption Rules

4 Solutions

4.1 Infinite-horizon

5 Conclusion

6 Puzzles and Questions

7 Robustness Analyses

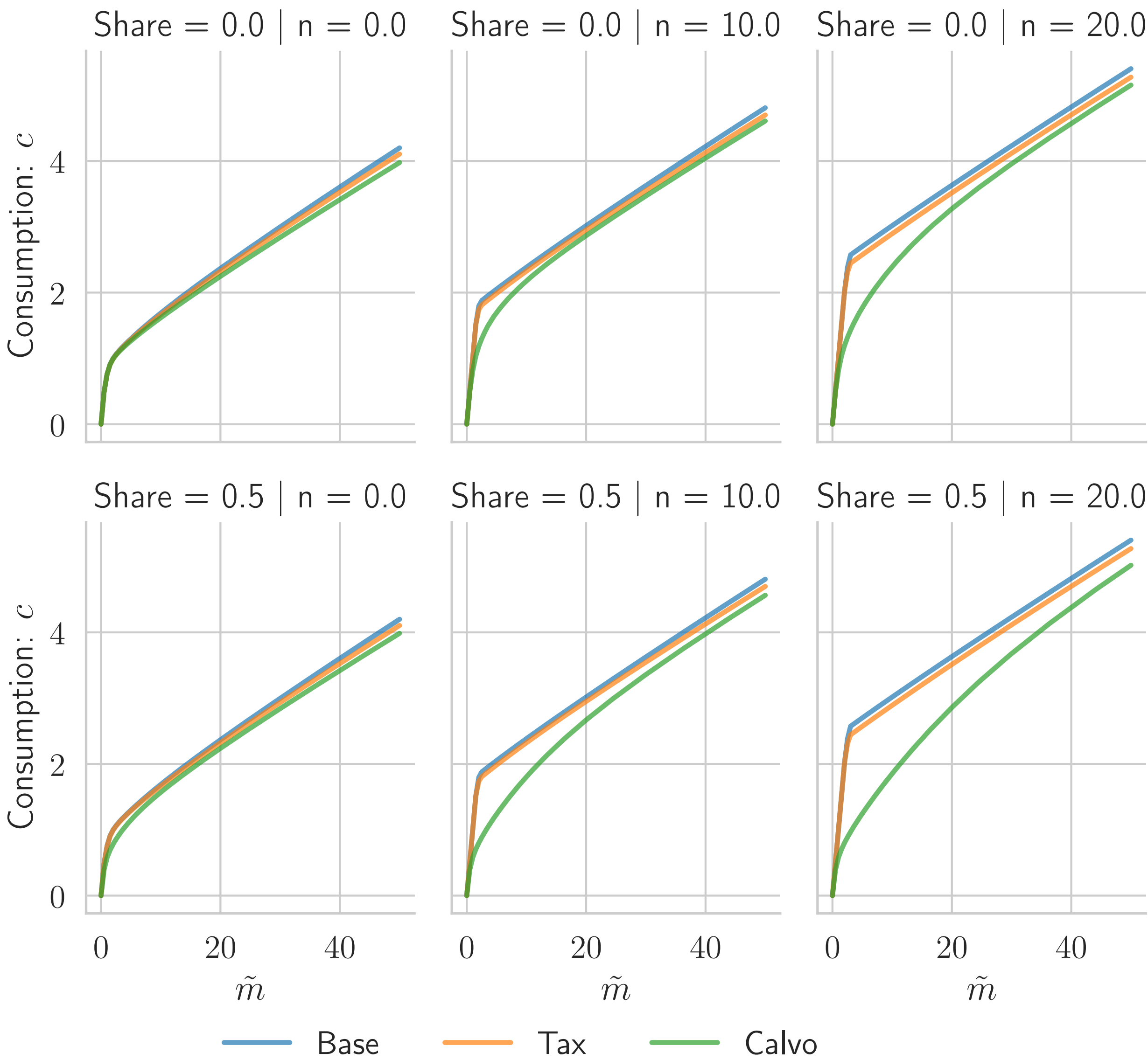

Figure 8 Convergence of the Consumption Rules