A Two-Asset Savings Model with an Income-Contribution Scheme REMARK

May 16, 2021

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Abstract

This paper contains the highlights from the REMARK file in Code>Python folder.

Keywords Lifecycle, Portfolio Choice, Social Security, Replication

GitHub: `http://github.com/econ-ark/REMARK/REMARKS/CGMPortfolio`
(In GitHub repo, see `/Code` *for tools for solving and simulating the model)*

CLICK HERE for an interactive Jupyter Notebook that uses the Econ-ARK/HARK toolkit to produce our figures (warning: it may take several minutes to launch). Information about citing the toolkit can be found at Acknowleding Econ-ARK.

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All numerical results herein were produced using the Econ-ARK/HARK toolkit; for further reference options see Acknowleding Econ-ARK. Thanks to Chris Carroll and Sylvain Catherine for comments and guidance.

1 Introduction

This paper develops a two-asset consumption-savings model and serves as the documentation of an open-source implementation of its solution and simulation methods in the HARK toolkit (?). The model represents an agent who can save using two different assets—one risky and the other risk-free—to insure against fluctuations in his income, but faces frictions to transferring funds between assets. The flexibility of its implementation and its inclusion in the HARK toolkit allows users to adapt the model to realistic life-cycle calibrations, and also to embedded it in heterogeneous-agents macroeconomic models.

Macro applications The last decade of research in macroeconomics has experienced and increased interest in how households allocate their savings to assets with different risk exposures and liquidity. Heterogeneous asset allocations have been shown to be determinant to the effects of fiscal and monetary policy (**??**).

Micro applications

2 The model

I now discuss the main components of the model informally, and leave its full recursive mathematical representation for Section 3.

2.0.1 Time, mortality, and utility

Time advances in discrete steps that I will index with t. The model can be used in both infinite and finite-horizon versions.

Agents face an exogenous risk of death δ_t each period, which becomes certain at the maximum age of the finite-horizon version. There are no intentional bequests; agents will consume all of their resources if they reach the last period, but they can leave accidental bequests upon premature death.

In each period, agents derive utility from consumption only. Their utility function follows a constant relative risk aversion specification. Formally, for a level of consumption C, the agent derives instant utility

$$u(C) = \frac{C^{1-\rho}}{1-\rho}. \tag{1}$$

2.0.2 Income process

Agents supply labor inelastically. Their labor earnings $Y_{i,t}$ are the product of a permanent component $P_{i,t}$ and a transitory stochastic component $\theta_{i,t}$ as in **?**, where i indexes different agents. Formally,

$$\begin{aligned}
\ln Y_{i,t} &= \ln P_{i,t} + \ln \theta_{i,t} \\
\ln P_{i,t} &= \ln P_{i,t-1} + \ln \Gamma_{i,t} + \ln \psi_{i,t}
\end{aligned}$$

where $\Gamma_{i,t}$ is a deterministic growth factor that can capture life-cycle patterns in earnings, and $\ln\psi_{i,t} \sim \mathcal{N}(-\sigma^2_{\psi,t}/2, \sigma_{\psi,t})$ is a multiplicative shock to permanent income[1].

The transitory component $\theta_{i,t}$ is a mixture that models unemployment and other temporal fluctuations in income as

$$\ln\theta_{i,t} = \begin{cases} \ln\mathcal{U}, & \text{With probability } \mho \\ \ln\tilde{\theta}_{i,t} \sim \mathcal{N}(-\sigma^2_{\theta,t}/2, \sigma_{\theta,t}), & \text{With probability } 1-\mho, \end{cases}$$

with $\mho$ representing the probability of unemployment and $\mathcal{U}$ the replacement factor of unemployment benefits.

This specification of the income process is parsimonious and flexible enough to accommodate life-cycle patterns in income growth and volatility, transitory unemployment and exogenous retirement. Introduced by **?**, this income specification is common in studies of life-cycle wealth accumulation and portfolio choice; see e.g., **???**. The specification has also been used in studies of income volatility, which have yielded calibrations of its stochastic shocks' distributions (see e.g., **???**)

2.0.3 Financial assets and frictions

Agents smooth their consumption by saving and have two assets available for this purpose. The first is a risk-free liquid account with constant per-period return factor R. The second has a stochastic return factor $\tilde{R}$ that is log-normally distributed and independent across time. Various interpretations such as stocks, a retirement fund, or entrepreneurial capital could be given to the risky asset. Importantly, consumption must be paid for using funds from the risk-free account. The levels of risk-free and risky assets owned by the agent will both be state variables, denoted with $M_{i,t}$ and $N_{i,t}$ respectively.

Portfolio rebalancing takes place by moving funds between the risk-free and risky accounts. These flows are one of the agents' control variables and are denoted as $D_{i,t}$, with $D_{i,t} > 0$ representing a movement of funds from the risk-free to the risky account. Withdrawals from the risky account are subject to a constant-rate tax τ which can represent, for instance, capital-gains realization taxes or early retirement-fund withdrawal penalties. In sum, denoting post-rebalancing asset levels with $\tilde{\cdot}$,

$$\begin{aligned} \tilde{M}_{i,t} &= M_{i,t} - D_{i,t}(1 - 1_{[D_{i,t}\leq 0]}\tau) \\ \tilde{N}_{i,t} &= N_{i,t} + D_{i,t}. \end{aligned}$$

At any given period, an agent might not be able to rebalance his portfolio. This ability is governed by an exogenous stochastic shock that is realized at the start of the period

$$\text{Adj}_t \sim \text{Bernoulli}(p_t),$$

with $\text{Adj}_t = 1$ meaning that the agent can rebalance and $\cancel{\text{Adj}}_t = 1$ ($\text{Adj}_t = 0$) forcing him to set $D_{i,t} = 0$. This friction is a parsimonious way to capture the fact that portfolio rebalancing is costly and households do it sporadically. Recent studies have advocated for (**?**) and used (**?**) this kind of rebalancing friction.

[1]The mean of the shock is set so that $E[\psi_{i,t}] = 1$.

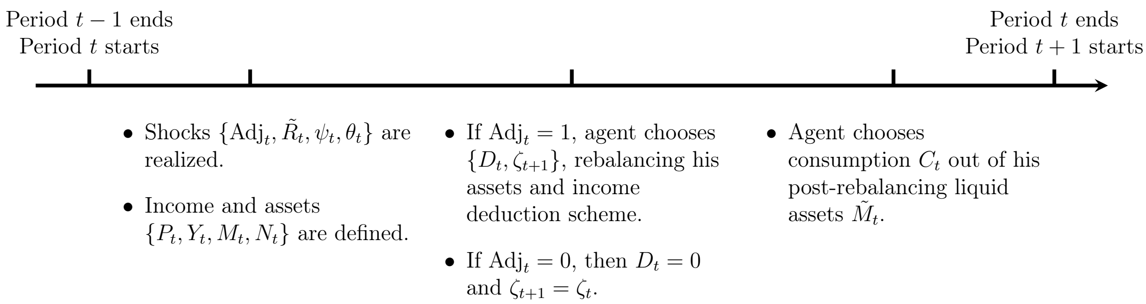

Figure 1 Summary of the Model's Timing

To partially evade the possibility of being unable to rebalance their accounts, agents can use an income deduction scheme. By default, labor income ($Y_{i,t}$) is deposited to the risk-free liquid account at the start of every period. However, agents can pre-commit to have a fraction $\zeta_t \in [0, 1]$ of their income diverted to their risky account instead. This fraction can be tweaked by the agent whenever $\text{Adj}_t = 1$; otherwise it stays at its previous value, $\zeta_{t+1} = \zeta_t$.

2.0.4 Timing

Figure 1 summarizes the timing of stochastic shocks and optimizing decisions that occur within a period of the life cycle model.

3 Recursive representation of the model

Individual subscripts are dropped for simplicity. The value function for an agent who is not allowed to rebalance his portfolio at time t is

$$
\begin{aligned}
V_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = \max_{C_t} u(C_t) + p_{t+1}\beta\delta_{t+1}E_t\left[V_{t+1}^{\text{Adj}}\left(M_{t+1}, N_{t+1}, P_{t+1}\right)\right] + \\
(1 - p_{t+1})\beta\delta_{t+1}E_t\left[V_{t+1}^{\cancel{\text{Adj}}}\left(M_{t+1}, N_{t+1}, P_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\text{Subject to:}\\
0 \leq& C_t \leq M_t\\
A_t =& M_t - C_t\\
M_{t+1} =& RA_t + (1 - \zeta_{t+1})Y_{t+1}\\
N_{t+1} =& \tilde{R}_{t+1}N_t + \zeta_{t+1}Y_{t+1}\\
P_{t+1} =& \Gamma_{t+1}\psi_{t+1}P_t\\
Y_{t+1} =& \theta_{t+1}P_{t+1}\\
\zeta_{t+1} =& \zeta_t
\end{aligned}
$$

and that of agent who is allowed to rebalance is

$$
\begin{aligned}
V_t^{\text{Adj}}(M_t, N_t, P_t) = \max_{C_t, D_t, \zeta_{t+1}} u(C_t) + p_{t+1}\beta\delta_{t+1}E_t\left[V_{t+1}^{\text{Adj}}\left(M_{t+1}, N_{t+1}, P_{t+1}\right)\right] + \\
(1-p_{t+1})\beta\delta_{t+1}E_t\left[V_{t+1}^{\cancel{\text{Adj}}}\left(M_{t+1}, N_{t+1}, P_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}
$$

Subject to:

$$
-N_t \leq D_t \leq M_t, \quad \zeta_{t+1} \in [0,1], \quad 0 \leq C_t \leq \tilde{M}_t
$$

$$
\begin{aligned}
\tilde{M}_t =& M_t - D_t\left(1 - 1_{[D_t \leq 0]}\tau\right) \\
\tilde{N}_t =& N_t + D_t \\
A_t =& \tilde{M}_t - C_t \\
M_{t+1} =& RA_t + (1-\zeta_{t+1})Y_{t+1} \\
N_{t+1} =& \tilde{R}_{t+1}\tilde{N}_t + \zeta_{t+1}Y_{t+1} \\
P_{t+1} =& \Gamma_{t+1}\psi_{t+1}P_t \\
Y_{t+1} =& \theta_{t+1}P_{t+1}
\end{aligned}
$$

The problem can be normalized by permanent income, following **?**. Using lower case variables to denote their upper-case counterparts normalized by permanent income ($x_t \equiv X_t/P_t$) and defining $\tilde{\Gamma}_t = \Gamma_t\psi_t$, we can write normalized problems

$$
\begin{aligned}
v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t) = \max_{c_t} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v_{t+1}^{\text{Adj}}\left(m_{t+1}, n_{t+1}\right)\right] + \\
(1-p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v_{t+1}^{\cancel{\text{Adj}}}\left(m_{t+1}, n_{t+1}, \zeta_{t+1}\right)\right]
\end{aligned}
$$

Subject to:

$$
\begin{aligned}
0 \leq& c_t \leq m_t \\
a_t =& m_t - c_t \\
m_{t+1} =& \frac{R}{\tilde{\Gamma}_{t+1}}a_t + (1-\zeta_{t+1})\theta_{t+1} \\
n_{t+1} =& \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}}n_t + \zeta_{t+1}\theta_{t+1} \\
\zeta_{t+1} =& \zeta_t
\end{aligned} \tag{2}
$$

and

$$v_t^{\text{Adj}}(m_t, n_t) = \max_{c_t, d_t, \zeta_{t+1}} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\text{Adj}}\left(m_{t+1}, n_{t+1}\right)\right] +$$
$$(1 - p_{t+1})\beta\delta_{t+1}E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\cancel{\text{Adj}}}\left(m_{t+1}, n_{t+1}, \zeta_{t+1}\right)\right]$$

Subject to:

$$-n_t \leq d_t \leq m_t, \quad \zeta_{t+1} \in [0, 1], \quad 0 \leq c_t \leq \tilde{m}_t$$

$$\begin{aligned}
\tilde{m}_t &= m_t - d_t\left(1 - 1_{[d_t \leq 0]}\tau\right) \\
\tilde{n}_t &= n_t + d_t \\
a_t &= \tilde{m}_t - c_t \\
m_{t+1} &= \frac{R}{\tilde{\Gamma}_{t+1}} a_t + (1 - \zeta_{t+1})\theta_{t+1} \\
n_{t+1} &= \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t + \zeta_{t+1}\theta_{t+1}
\end{aligned}$$

It can be shown that

$$V_t^{\text{Adj}}(M_t, N_t, P_t) = P_t^{1-\rho} v_t^{\text{Adj}}(m_t, n_t), \quad V_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = P_t^{1-\rho} v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t),$$

and that the policy functions of both problems are related through

$$C_t^{\cancel{\text{Adj}}}(M_t, N_t, P_t, \zeta_t) = P_t c_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t), \quad C_t^{\text{Adj}}(M_t, N_t, P_t) = P_t c_t^{\text{Adj}}(m_t, n_t),$$
$$D_t(M_t, N_t, P_t) = P_t d_t(m_t, n_t), \quad \zeta_{t+1}(M_t, N_t) = \zeta_{t+1}(m_t, n_t).$$

Therefore, the model's implementation solves the problem in normalized form, and re-scales the relevant states and choices using permanent income when simulating.

3.1 Partition into stages

An additional insight that facilitates solving the model is that the three decisions that an agent might take in a period (rebalancing his assets, choosing his income deduction fraction and consuming) can be seen as happening sequentially. This is convenient because:

- The sub-problems are easier to solve than the multi-choice full problem.
- Since the non-adjusting agent only chooses his consumption and we must solve his problem for every combination of $(m_t, n_t, \zeta_t = \zeta_{t+1})$, we can re-utilize his solution by expressing the adjusting agent's problem as

  $$v_t^{\text{Adj}}(m_t, n_t) = \max_{\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}} v_t^{\cancel{\text{Adj}}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}).$$

  This insight is similar to the "nested" reformulation suggested by **?**.

To start re-expressing the problem, I take the order of the ageent's decisions to be: rebalance assets, define income contribution share and finally consume. I denote the stages at which these decisions are taken with `Reb`, `Sha`, and `Cns` respectively. I will use $v^{\texttt{Reb}}(\cdot)$, $v^{\texttt{Sha}}(\cdot)$ and $v^{\texttt{Cns}}(\cdot)$ to represent *stage value functions*.

I now present each stage in detail, working backwards in time.

3.1.1 Consumption stage, `Cns`

An agent who takes his assets and income contribution share as given is one who can not adjust them and can only choose his consumption. This corresponds to the problem of the non-adjusting agent (Equation 2). The important facts to realize at this stage are that

$$v_t^{\text{Adj}}(m_t, n_t) = v_t^{\texttt{Reb}}(m_t, n_t), \quad v_t^{\cancel{\text{Adj}}}(m_t, n_t, \zeta_t) = v_t^{\texttt{Cns}}(m_t, n_t, \zeta_t),$$

with the first fact being true because we have assumed that asset rebalancing is the first decision that an adjusting agent takes, and he assumes that his subsequent decisions will be optimal. Therefore, the consumption stage problem is

$$\begin{aligned} v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1}) = \max_{c_t} u(c_t) + p_{t+1}\beta\delta_{t+1}E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Reb}}\left(m_{t+1}, n_{t+1}\right)\right] + \\ (1 - p_{t+1})\beta\delta_{t+1}E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Cns}}\left(m_{t+1}, n_{t+1}, \zeta_{t+1}\right)\right] \end{aligned}$$

$$\begin{aligned} &\text{Subject to:} \\ 0 \leq &c_t \leq \tilde{m}_t \\ a_t =& \tilde{m}_t - c_t \\ m_{t+1} =& \frac{R}{\tilde{\Gamma}_{t+1}} a_t + (1 - \zeta_{t+1})\theta_{t+1} \\ n_{t+1} =& \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t + \zeta_{t+1}\theta_{t+1} \end{aligned}$$

3.1.2 Income contribution share stage, `Sha`

An agent with the option to set his income contribution share will do so taking his asset allocation as given and assuming that he will optimally pick his consumption in the next stage. His problem is

$$v_t^{\texttt{Sha}}(\tilde{m}_t, \tilde{n}_t) = \max_{\zeta_{t+1} \in [0,1]} v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t, \zeta_{t+1})$$

Table 1 Infinite Horizon Example Calibration

Name in HARK	Mathematical Symbol	Value
`CRRA`	ρ	5.0
`Rfree`	R	1.03
`DiscFac`	β	0.9
`LivPrb`	δ	0.98
`PermGroFac`	Γ	1.01
`PermShkStd`	σ_ψ	0.1
`TranShkStd`	σ_θ	0.1
`UnempPrb`	$\mho$	0.05
`IncUnemp`	$\mathcal{U}$	0.3
`RiskyAvg`	$E[\tilde{R}]$	1.08
`RiskyStd`	$\sqrt{V[\tilde{R}]}$	0.18
`AdjustPrb`	p	Varying
`tau`	τ	Varying

3.1.3 Rebalancing stage, `Reb`

The first decision that an agent takes, if allowed, is how to reallocate his assets. At this stage, his value function is

$$
\begin{aligned}
v_t^{\texttt{Reb}}(m_t, n_t) &= \max_{d_t} v_t^{\texttt{Sha}}(\tilde{m}_t, \tilde{n}_t) \\
\text{Subject to:} & \\
-n_t \leq & d_t \leq m_t \\
\tilde{m}_t =& m_t - d_t \left(1 - 1_{[d_t \leq 0]}\tau\right) \\
\tilde{n}_t =& n_t + d_t
\end{aligned}
$$

4 Examples of Solutions and Simulations

4.1 Infinite-horizon

4.2 Life-Cycle finite horizon

5 Conclusion

6 Puzzles and Questions

7 Robustness Analyses

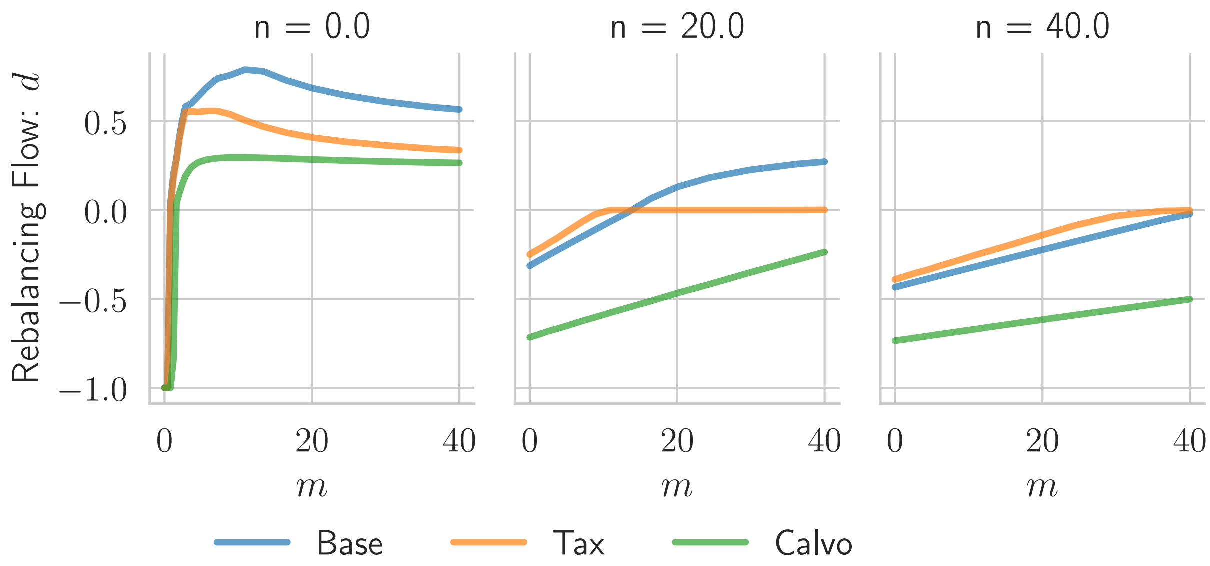

Figure 2 Convergence of the Consumption Rules

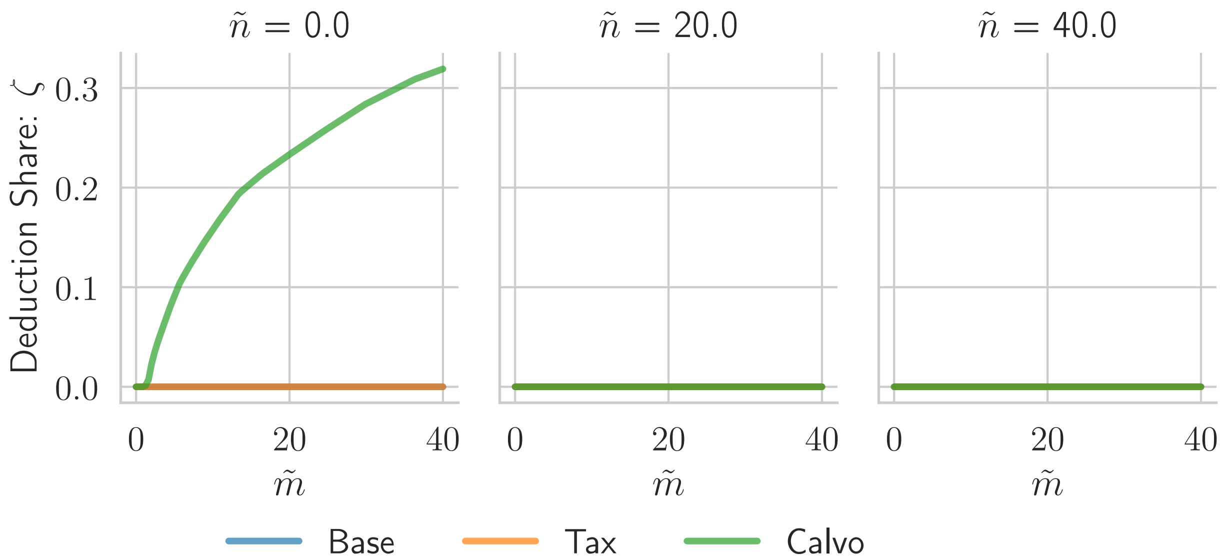

Figure 3 Convergence of the Consumption Rules

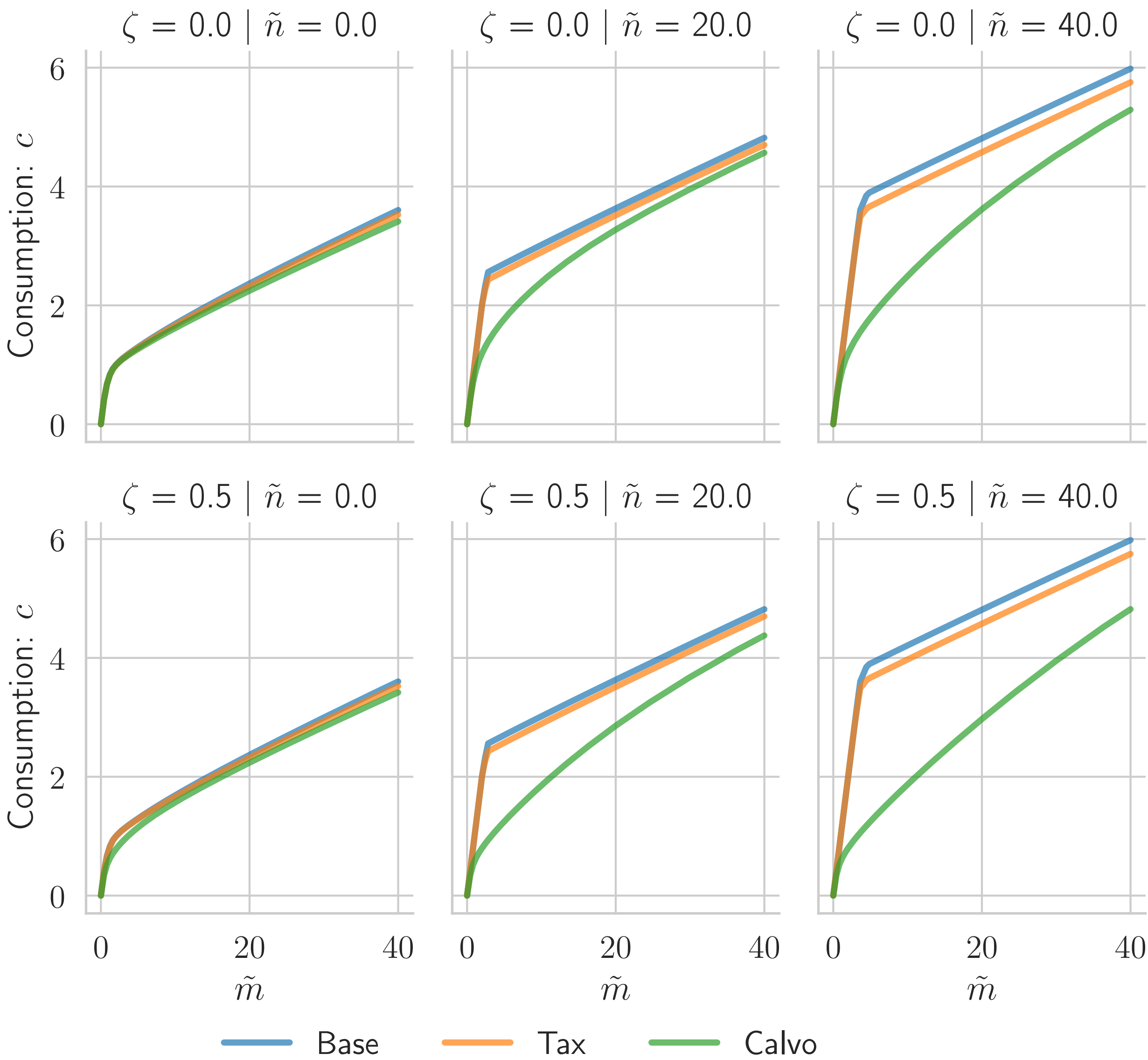

Figure 4 Convergence of the Consumption Rules

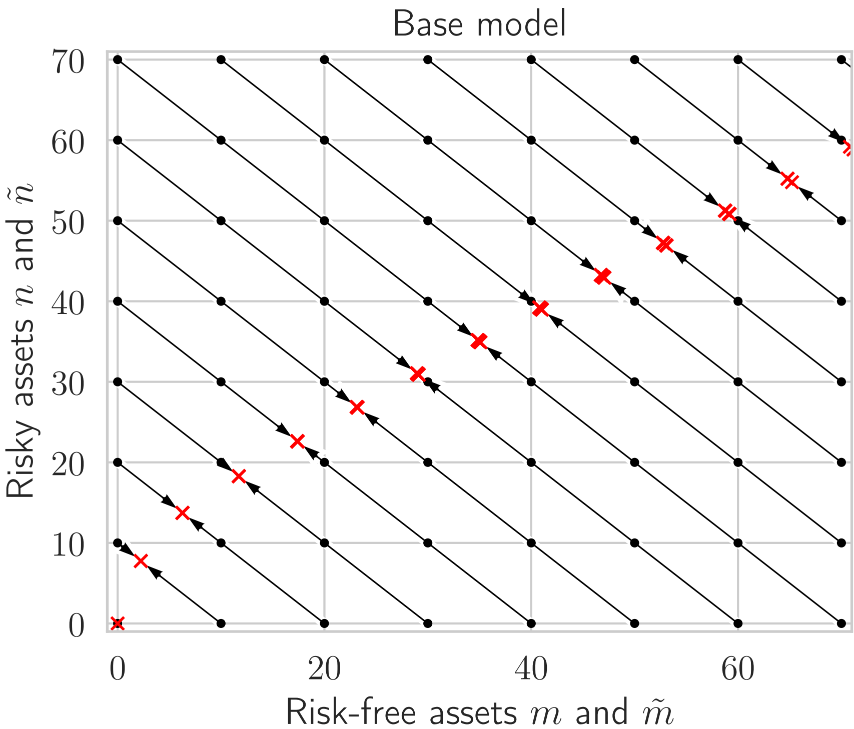

Figure 5 Convergence of the Consumption Rules

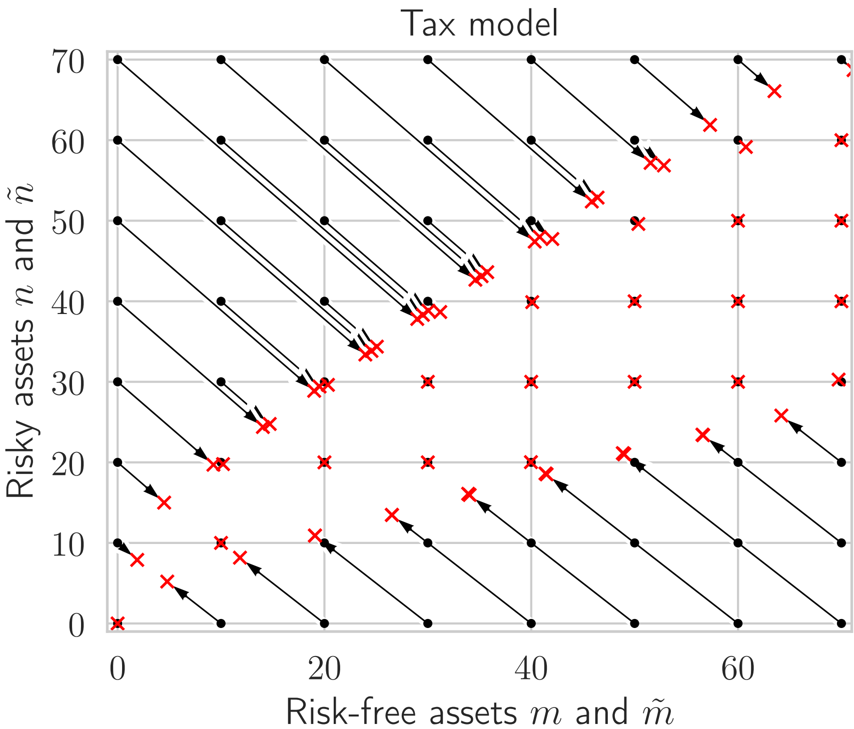

Figure 6 Convergence of the Consumption Rules

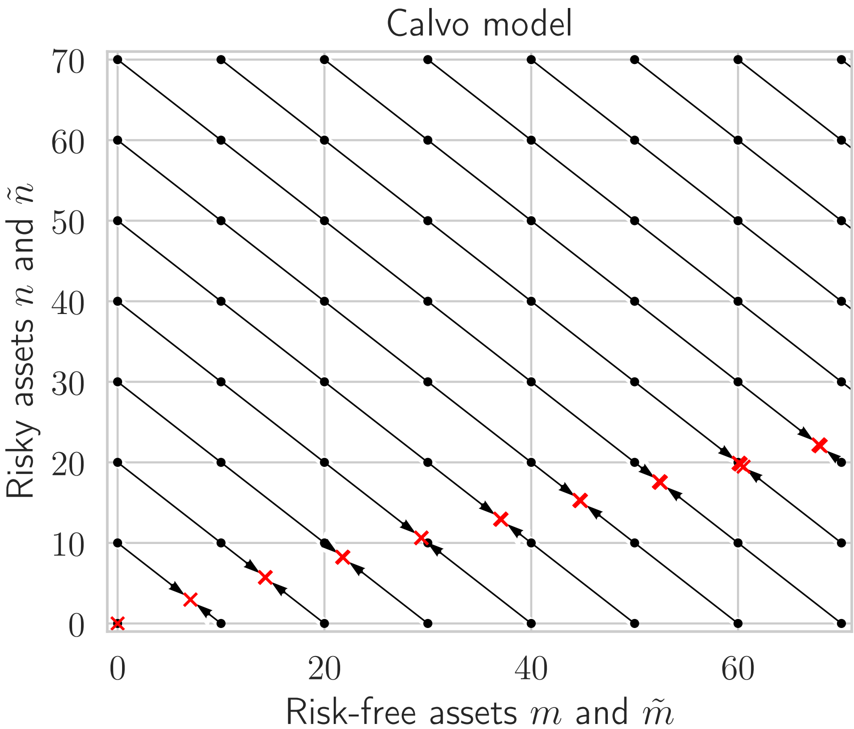

Figure 7 Convergence of the Consumption Rules

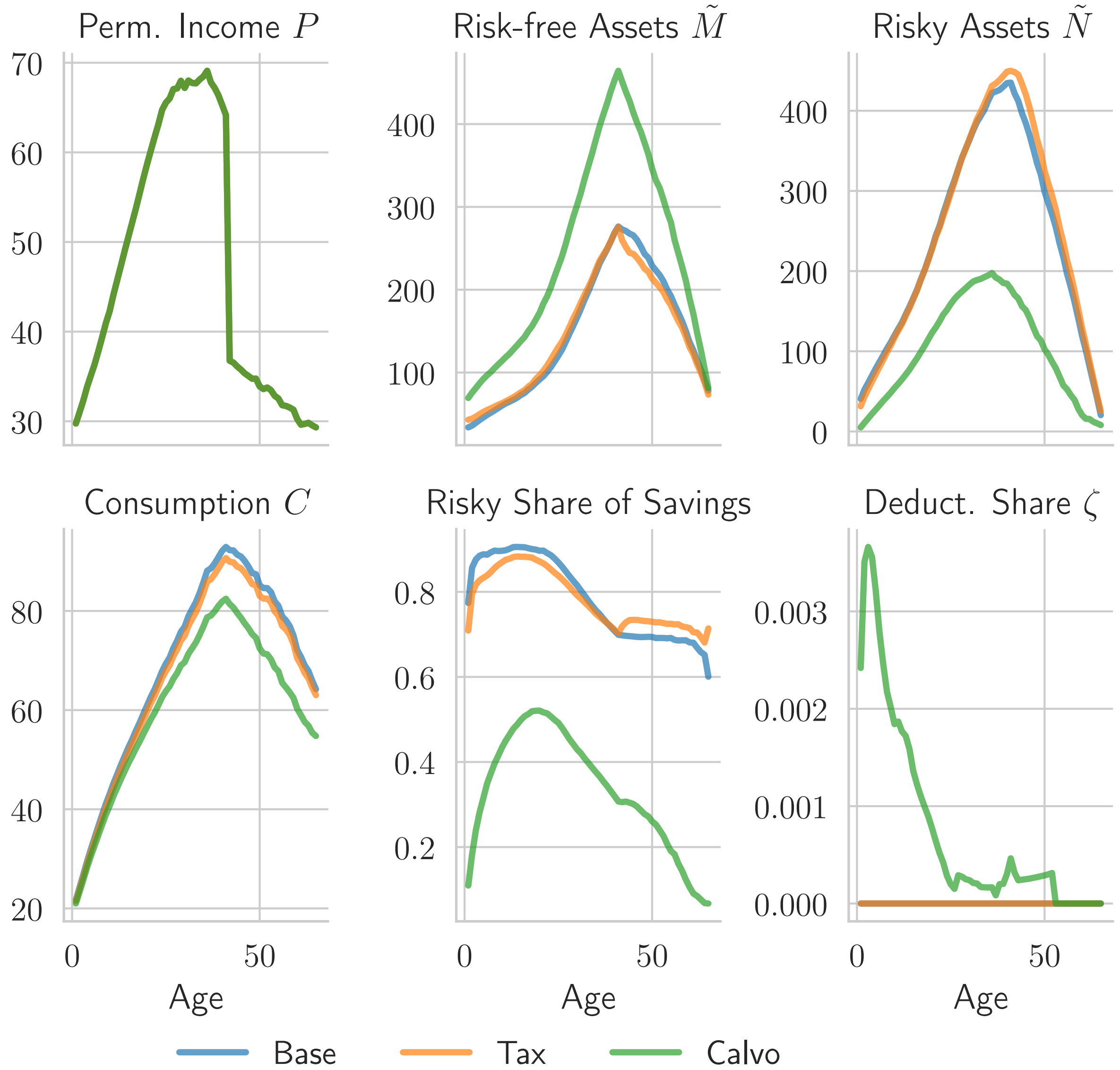

Figure 8 Convergence of the Consumption Rules

References