Exercise sheet week 2

This exercise sheet will **not** be graded.

1 Killing-Yano tensors

An anti-symmetric rank-2 tensor $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is called a **Killing-Yano tensor** if it satisfies

$$\nabla_\alpha \omega_{\beta\gamma} + \nabla_\beta \omega_{\alpha\gamma} = 0.$$

a. Show that for any Killing-Yano tensor $\omega_{\alpha\beta}$, the symmetric tensor $\omega_{\alpha\gamma}\omega_\beta{}^\gamma$ is a Killing tensor. (In some sense Killing-Yano tensors can be thought of as the "square root" of a Killing tensor.))
b. Show that for any Killing-Yano tensor $\omega_{\mu\nu}$ and geodesic $x^\mu(s)$, the (co-)vector $V_\mu = \omega_{\mu\alpha}\frac{dx^\alpha}{ds}$ is parallel transported only the geodesic, i.e.

$$\frac{dx^\alpha}{ds}\nabla_\alpha V^\mu = 0.$$

Note: Kerr also has a Killing-Yano tensor.

2 Vortical solutions revisited

In class we introduced "vortical" geodesic solutions as solutions for which $z = \cos\theta$ oscillates between $0 \leq z_1 \leq z_2 \leq 1$. We argued that these solutions exist only when

$$\mathcal{E}^2 - \mu > 0 \tag{1}$$
$$\mathcal{L}^2 \leq a^2(\mathcal{E}^2 - \mu) \tag{2}$$
$$-(|\mathcal{L}| - |a|\sqrt{\mathcal{E}^2 - \mu}|)^2 \leq \mathcal{Q} \leq 0 \tag{3}$$

Let's examine these solutions a little further.
a. Define

$$\begin{aligned}\hat{Q} &= \frac{Q}{a^2(E^2 + \mu)}\\ \hat{L}^2 &= \frac{L^2}{a^2(E^2 + \mu)}\\ u &= \cos^2\theta\\ \hat{P}_\theta &= -\frac{P_\theta}{a^2(E^2 + \mu)}\end{aligned}$$

Show that we this notation we get

$$\hat{P}_\theta(u) = u^2 + \left(\hat{Q} + \hat{L}^2 - 1\right)u - \hat{Q}.$$

b. Find the zeroes of $\hat{P}_\theta(u)$.
c. Under what conditions are both roots u_1 and u_2 real lie in the range $0 < u_1 < u_2 < 1$? Can you recover the mentioned conditions for vortical solutions.
We now turn the radial part of vortical solution.
d. Show that for vortical solutions

$$P_r \geq 2M(Q + (L - aE)^2 + \mu r^2).$$

e. Show that for vortical null geodesics ($\mu = 0$) that $P_r > 0$. What does this mean for their radial solutions? Sketch a vortical null trajectory in the maximally extended Penrose diagram for Kerr.
f. Show that for vortical timelike geodesics ($\mu > 0$) we always have that $P_r > 0$ outside r_-.