# Chapter 2

# Geodesics in Kerr Spacetime

## 2.1 The geodesic equation

**Literature:** Harmark, Sec. 1.3.3 and 1.4.6; Carroll Sec. 3.3 and 3.4

In the first GR course you saw the geodesic equation

$$\frac{D}{d\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\nabla_\alpha\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\ \alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0. \tag{2.1}$$

Solving this equation can be made easier by first identifying constants of motion, the existence of which is interwined with the notion of spacetime symmetries.

## 2.2 Symmetries and Killing vectors

**Literature:** Carroll Sec. 3.8 and Appendix B, Wald Appendix C

A symmetry of a spacetime $M$ is a automorphism $\phi : M \to M$ that leaves the spacetime "invariant". Infinitesimal automorphisms are given by vector fields in the following way. Suppose we have a (smooth) vector field $V^\mu$, we can construct a family of automorphisms $\phi^t_{V^\mu}$ indexed by a variable $t \in \mathbb{R}$ by mapping each event $p \in M$ to a new event $p'$ by following the integral curves of $V^\mu$ for a time $t$. (I.e. we solve the differential equation $\frac{\mathrm{d}x^\mu}{\mathrm{d}dt} = V^\mu$ with initial condition $x(0) = p$, and set $p' = x(t)$.)

We can try to ask ourselves the question how does a $T_{\mu_1\ldots\mu_n}$ change along the integral lines of $V^\mu$. Naively one may try to write the down the derivative

$$\lim_{t\to 0}\frac{T_{\mu_1\ldots\mu_n}(p') - T_{\mu_1\ldots\mu_n}(p)}{t}.$$

However, such an expression does not make any mathematical sense. The tensors $T_{\mu_1\ldots\mu_n}(p')$ and $T_{\mu_1\ldots\mu_n}(p)$ belong to (the tensor product of) the (co)tangent space at different points in $M$. Consequently, we cannot add (or subtract) them. To get another object that lives at $p$, we can consider the induced by $\phi^t_{V^\mu}$ on the tensor bundles (the push forward), or more specifically its inverse the pull back $(\phi^t_{V^\nu})^*$, which in terms of components is given by

$$\left(\left(\phi^t_{V^\nu}\right)^* T\right)_{\mu_1\ldots\mu_n}(p) = \frac{\partial p'^{\alpha_1}}{\partial p^{\mu_1}} \cdots \frac{\partial p'^{\alpha_n}}{\partial p^{\mu_n}} T_{\alpha_1\ldots\alpha_n}(p').$$

This allows us to define a derivative that is defined covariantly and expresses how much a tensor field changes in the direction of a vector field $V^\mu$.

**Definition 2.** Given a vector field $V^\mu$ and a tensor field $T_{\mu_1\ldots\mu_n}$ we can define the *Lie*[1] *derivative* of $T_{\mu_1\ldots\mu_n}$ w.r.t. $V^\mu$ at an event $p$ as follows,

$$\mathcal{L}_{V^\nu} T_{\mu_1\ldots\mu_n}(p) = \lim_{t\to 0} \frac{\left((\phi^t_{V^\nu})^* T_{\mu_1\ldots\mu_n}\right)(p) - T_{\mu_1\ldots\mu_n}(p)}{t}$$

Note that this notion of derivative does not require the existence of a metric. This makes it suitable to explore the symmetries of the metric itself. In particular, if we calculate the Lie derivative of a metric tensor $g_{\mu\nu}$ we find

$$\begin{aligned}
\mathcal{L}_{K^\lambda} g_{\mu\nu} &= \lim_{t\to 0} \frac{\left((\phi^t_{V^\nu})^* g_{\mu\nu}\right)(p) - g_{\mu\nu}(p)}{t} \\
&= \lim_{t\to 0} \frac{\frac{\partial p'^\alpha}{\partial p^\mu}\frac{\partial p'^\beta}{\partial p^\nu} g_{\alpha\beta}(p') - g_{\mu\nu}(p)}{t} \\
&= \lim_{t\to 0} \frac{\left(\delta^\alpha_\mu + t\partial_\mu V^\alpha\right)\left(\delta^\beta_\nu + t\partial_\nu V^\beta\right)\left(g_{\alpha\beta}(p) + tV^\gamma \partial_\gamma g_{\alpha\beta}(p)\right) - g_{\mu\nu}(p)}{t} \\
&= g_{\alpha\nu}\partial_\mu V^\alpha + g_{\mu\beta}\partial_\nu V^\beta + V^\gamma \partial_\gamma g_{\mu\nu} \\
&= \partial_\mu V_\nu - V^\alpha \partial_\mu g_{\alpha\nu} + \partial_\nu V_\mu - V^\beta \partial_\nu g_{\mu\beta} + V^\gamma \partial_\gamma g_{\mu\nu} \\
&= \partial_\mu V_\nu + \partial_\nu V_\mu - V^\alpha\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) \\
&= \partial_\mu V_\nu + \partial_\nu V_\mu - 2V_\alpha \Gamma^\alpha_{\mu\nu} \\
&= \nabla_\mu V_\nu + \nabla_\nu V_\mu = 2\nabla_{(\mu} V_{\nu)}
\end{aligned}$$

We are now ready to introduce the notion of a symmetry of spacetime $(M, g_{\mu\nu})$.

**Definition 3.** Let $K^\mu$ be a vector field on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$. $K^\mu$ is called a *Killing*[2] vector (field) if equivalently:

[1] Pronounced "Lee" after 19th century Norwegian mathematician Sophus Lie.
[2] After the 19th century German mathematician, Wilhelm Killing.

1. $\mathcal{L}_{K^\lambda} g_{\mu\nu} = 0$
2. $\nabla_{(\mu} K_{\nu)} \equiv \frac{1}{2}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = 0$

Killing vectors encode the symmetries of a spacetime geometry. Sometimes coordinates make it easy to find Killing vectors, as described by the following lemma.

**Lemma 1.** *If the components of a metric $g_{\mu\nu}$ in some particular coordinates do not depend on the coordinate $k$, then $\left(\frac{\partial}{\partial k}\right)^\mu$ is a Killing vector field.*

*Proof.* Left as an exercise to the reader. □

This leads to the main result that will help us solve the geodesic equation in Kerr.

**Lemma 2.** *Let $x^\mu(\lambda)$ be a geodesic on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$, and let $K^\mu$ be a Killing vector field, then the quantity $\mathcal{K} = K_\alpha \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$ is conserved along the geodesic $x^\mu(\lambda)$.*

*Proof.*

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}\lambda} &= \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\nabla_\alpha \mathcal{K} && (2.2)\\
&= \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\nabla_\alpha\left(K_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right) && (2.3)\\
&= \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\nabla_\alpha K_\beta + K_\beta \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\nabla_\alpha \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} && (2.4)\\
&= \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\nabla_{(\alpha} K_{\beta)} + 0 && (2.5)\\
&= 0. && (2.6)
\end{aligned}$$

□

The notion of a Killing vector field has a generalization to tensors for hire rank that satisfy similar properties.

**Definition 4.** Let $K_{\mu\nu}$ be a symmetric tensor field on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$. $K_{\mu\nu}$ is called a *Killing tensor* (field) if

$$\nabla_{(\lambda} K_{\mu\nu)} = 0.$$

Unlike Killing vectors, Killing tensors do not have an interpretation in terms of spacetime symmetries. Nonetheless, they still lead to constants of motion.

**Lemma 3.** *Let $x^\mu(\lambda)$ be a geodesic on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$, and let $K^{\mu\nu}$ be a Killing tensor field, then the quantity $\mathcal{K} = K^{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}$ is conserved along the geodesic $x^\mu(\lambda)$.*

*Proof.* Left as an exercise to the reader. $\square$