Chapter 2

Geodesics in Kerr Spacetime

2.1 The geodesic equation

Literature: Harmark, Sec. 1.3.3 and 1.4.6; Carroll Sec. 3.3 and 3.4

In the first GR course you saw the geodesic equation

$$\frac{D}{ds}\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\nabla_\alpha \frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0. \tag{2.1}$$

Solving this equation can be made easier by first identifying constants of motion, the existence of which is intertwined with the notion of spacetime symmetries.

2.2 Symmetries and Killing vectors

Literature: Carroll Sec. 3.8 and Appendix B, Wald Appendix C

A symmetry of a spacetime M is a automorphism $\phi : M \to M$ that leaves the spacetime "invariant". Infinitesimal automorphisms are given by vector fields in the following way. Suppose we have a (smooth) vector field V^μ, we can construct a family of automorphisms $\phi^t_{V^\mu}$ indexed by a variable $t \in \mathbb{R}$ by mapping each event $p \in M$ to a new event p' by following the integral curves of V^μ for a time t. (I.e. we solve the differential equation $\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = V^\mu$ with initial condition $x(0) = p$, and set $p' = x(t)$.)

We can try to ask ourselves the question how does a $T_{\mu_1\ldots\mu_n}$ change along the integral lines of V^μ. Naively one may try to write the down the derivative

$$\lim_{t\to 0}\frac{T_{\mu_1\ldots\mu_n}(p') - T_{\mu_1\ldots\mu_n}(p)}{t}.$$

However, such an expression does not make any mathematical sense. The tensors $T_{\mu_1\ldots\mu_n}(p')$ and $T_{\mu_1\ldots\mu_n}(p)$ belong to (the tensor product of) the (co)tangent space at different points in M. Consequently, we cannot add (or subtract) them. To get another object that lives at p, we can consider the map induced by $\phi^t_{V^\mu}$ on the tensor bundles (the push forward), or more specifically its inverse (the pull back) $(\phi^t_{V^\nu})^*$, which in terms of components is given by

$$\left((\phi^t_{V^\nu})^* T\right)_{\mu_1\ldots\mu_n}(p) = \frac{\partial p'^{\alpha_1}}{\partial p^{\mu_1}} \cdots \frac{\partial p'^{\alpha_n}}{\partial p^{\mu_n}} T_{\alpha_1\ldots\alpha_n}(p').$$

This allows us to define a derivative that is defined covariantly and expresses how much a tensor field changes in the direction of a vector field V^μ.

Definition 2. Given a vector field V^μ and a tensor field $T_{\mu_1\ldots\mu_n}$ we can define the **Lie**[1] **derivative** of $T_{\mu_1\ldots\mu_n}$ w.r.t. V^μ at an event p as follows,

$$\mathcal{L}_{V^\nu} T_{\mu_1\ldots\mu_n}(p) = \lim_{t\to 0} \frac{\left((\phi^t_{V^\nu})^* T_{\mu_1\ldots\mu_n}\right)(p) - T_{\mu_1\ldots\mu_n}(p)}{t}$$

Note that this notion of derivative does not require the existence of a metric. This makes it suitable to explore the symmetries of the metric itself. In particular, if we calculate the Lie derivative of a metric tensor $g_{\mu\nu}$ we find

$$\begin{aligned}
\mathcal{L}_{K^\lambda} g_{\mu\nu} &= \lim_{t\to 0} \frac{\left((\phi^t_{V^\nu})^* g_{\mu\nu}\right)(p) - g_{\mu\nu}(p)}{t} \\
&= \lim_{t\to 0} \frac{\frac{\partial p'^\alpha}{\partial p^\mu}\frac{\partial p'^\beta}{\partial p^\nu} g_{\alpha\beta}(p') - g_{\mu\nu}(p)}{t} \\
&= \lim_{t\to 0} \frac{\left(\delta^\alpha_\mu + t\partial_\mu V^\alpha\right)\left(\delta^\beta_\nu + t\partial_\nu V^\beta\right)\left(g_{\alpha\beta}(p) + tV^\gamma\partial_\gamma g_{\alpha\beta}(p)\right) - g_{\mu\nu}(p)}{t} \\
&= g_{\alpha\nu}\partial_\mu V^\alpha + g_{\mu\beta}\partial_\nu V^\beta + V^\gamma \partial_\gamma g_{\mu\nu} \\
&= \partial_\mu V_\nu - V^\alpha \partial_\mu g_{\alpha\nu} + \partial_\nu V_\mu - V^\beta \partial_\nu g_{\mu\beta} + V^\gamma \partial_\gamma g_{\mu\nu} \\
&= \partial_\mu V_\nu + \partial_\nu V_\mu - V^\alpha\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) \\
&= \partial_\mu V_\nu + \partial_\nu V_\mu - 2V_\alpha \Gamma^\alpha_{\mu\nu} \\
&= \nabla_\mu V_\nu + \nabla_\nu V_\mu = 2\nabla_{(\mu} V_{\nu)}.
\end{aligned}$$

We are now ready to introduce the notion of a symmetry of spacetime $(M, g_{\mu\nu})$.

Definition 3. Let K^μ be a vector field on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$. K^μ is called a **Killing**[2] vector (field) if equivalently:

[1] Pronounced "Lee" after 19th century Norwegian mathematician Sophus Lie.

[2] After the 19th century German mathematician, Wilhelm Killing.

1. $\mathcal{L}_{K^\lambda} g_{\mu\nu} = 0$

2. $\nabla_{(\mu} K_{\nu)} \equiv \frac{1}{2}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = 0$

Killing vectors encode the symmetries of a spacetime geometry. Sometimes coordinates make it easy to find Killing vectors, as described by the following lemma.

Lemma 1. *If the components of a metric $g_{\mu\nu}$ in some particular coordinates do not depend on the coordinate k, then $\left(\frac{\partial}{\partial k}\right)^\mu$ is a Killing vector field.*

Proof. Left as an exercise to the reader. □

This leads to the main result that will help us solve the geodesic equation in Kerr.

Lemma 2. *Let $x^\mu(s)$ be a geodesic on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$, and let K^μ be a Killing vector field, then the quantity $\mathcal{K} = K_\alpha \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}$ is conserved along the geodesic $x^\mu(s)$.*

Proof.

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}s} &= \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \nabla_\alpha \mathcal{K} && (2.2)\\
&= \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \nabla_\alpha \left(K_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \right) && (2.3)\\
&= \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \nabla_\alpha K_\beta + K_\beta \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \nabla_\alpha \frac{\mathrm{d}x^\beta}{\mathrm{d}s} && (2.4)\\
&= \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \nabla_{(\alpha} K_{\beta)} + 0 && (2.5)\\
&= 0. && (2.6)
\end{aligned}$$

□

The notion of a Killing vector field has a generalization to tensors for higher rank that satisfy similar properties.

Definition 4. Let $K_{\mu\nu}$ be a symmetric tensor field on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$. $K_{\mu\nu}$ is called a **Killing tensor** (field) if

$$\nabla_{(\lambda} K_{\mu\nu)} = 0.$$

There are a number of trivial examples of Killing tensors.

Example 1. For any (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$, the metric tensor itself $g_{\mu\nu}$ is a Killing tensor field.

Example 2. Suppose V^μ and W^μ are Killing vector fields, then $K_{\mu\nu} = V_{(\mu}W_{\nu)}$ is a Killing tensor.

Unlike Killing vectors, Killing tensors do not have an interpretation in terms of spacetime symmetries. Nonetheless, they still lead to constants of motion.

Lemma 3. *Let $x^\mu(s)$ be a geodesic on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$, and let $K^{\mu\nu}$ be a Killing tensor field, then the quantity $\mathcal{K} = K_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}$ is conserved along the geodesic $x^\mu(s)$.*

Proof. Left as an exercise to the reader. □

Example 3. When applied to the metric $g_{\mu\nu}$ this lemma reproduces the familiar result that the norm of the tangent vector to a geodesic $g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}$ is preserved along a geodesic.

2.3 Constants of Motion of Kerr geodesics

From the explicit expression for the Kerr metric in Boyer-Lindquist coordinates (1.16), we can immediately infer that the Kerr metric has two Killing vector fields $\left(\frac{\partial}{\partial t}\right)^\mu$ and $\left(\frac{\partial}{\partial \phi}\right)^\mu$, related to the time translation and axial symmetries of the spacetime, respectively. These are all the independent Killing vector fields that the Kerr metric has. (More precisely, all Killing vectors of Kerr can be written as a linear combination of $\left(\frac{\partial}{\partial t}\right)^\mu$ and $\left(\frac{\partial}{\partial \phi}\right)^\mu$).

These Killing symmetries lead to two constants of motion

$$\mathcal{E} = -\left(\frac{\partial}{\partial t}\right)_\alpha \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = -g_{t\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}, \tag{2.7}$$

$$\mathcal{L} = \left(\frac{\partial}{\partial \phi}\right)_\alpha \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = g_{\phi\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}. \tag{2.8}$$

When the affine parameter s is chosen such that $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ equals the four momentum p^μ of the particle following the geodesic, $\mathcal{E}$ is equal to the energy of the particle and $\mathcal{L}$ is equal to the component of the orbital angular momentum along the symmetry axis of the Kerr geometry. By analogy, we will refer to $\mathcal{E}$ and $\mathcal{L}$ as the **energy** and **angular momentum** regardless of the chosen affine parameter.

As for any spacetime the metric $g_{\mu\nu}$ is a Killing tensor leading to our third constant of motion, the invariant mass squared

$$\mu = -g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}. \tag{2.9}$$

It turns out that Kerr spacetime has an additional "hidden" symmetry in the form of a Killing tensor. This Killing tensor is given by

$$K_{\mu\nu} = \Sigma\left(\ell_\mu n_\nu + \ell_\nu n_\mu\right) + r^2 g_{\mu\nu}, \tag{2.10}$$

where ℓ^μ and n^ν are principal null vectors[3] of the Kerr geometry. In Boyer-Lindquist coordinates their components can be written,

$$\ell^\mu = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad \text{and} \tag{2.11}$$

$$n^\mu = \left(\frac{r^2+a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma}\right). \tag{2.12}$$

This gives rise to a fourth constant of motion, the **Carter constant**

$$\mathcal{K} = K_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}. \tag{2.13}$$

This constant of motion has the rough interpretation as the "total angular momentum squared" of the particle.

Any functional combination of constants of motion is also a constant of motion. It is conventional to use this freedom to replace $\mathcal{K}$ with the constant

$$\mathcal{Q} = \mathcal{K} - \left(\mathcal{L} - a\mathcal{E}\right)^2, \tag{2.14}$$

which (somewhat confusingly) is also referred to as "the Carter constant". (We will be following this tradition.)

[3] This means that they are null vectors k^μ, which satisfy $k^\alpha k^\beta k_{[\mu}C_{\nu]\alpha\beta[\lambda}k_{\rho]} = 0$, which expresses that k^μ is an eigenvector of the Weyl curvature tensor $C_{\mu\nu\rho\sigma}$ in some suitable sense.

2.4 Separating the geodesic equations

So with constants of motion $(\mu, \mathcal{E}, \mathcal{L}, \mathcal{Q})$ in hand, we have four equations that a geodesic must satisfy

$$\mu = g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}, \tag{2.15}$$

$$\mathcal{Q} + (\mathcal{L} - a\mathcal{E})^2 = K_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}, \tag{2.16}$$

$$\mathcal{E} = -g_{t\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}, \tag{2.17}$$

$$\mathcal{L} = g_{\phi\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}. \tag{2.18}$$

So, if we fix values of $(\mu, \mathcal{E}, \mathcal{L}, \mathcal{Q})$, we have four equations for four unknowns (the components of $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}$), and we can solve these equations. After some straightforward (but tedious) algebra we find

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = \frac{\left(\mathcal{E}(r^2+a^2) - a\mathcal{L}\right)^2 - \Delta\left(\mathcal{Q} + (\mathcal{L} - a\mathcal{E})^2 + \mu r^2\right)}{\Sigma^2} \tag{2.19}$$

$$\left(\frac{\mathrm{d}\cos\theta}{\mathrm{d}s}\right)^2 = \frac{\mathcal{Q} - \cos^2\theta\Big(a^2(\mu - \mathcal{E}^2)\sin^2\theta + \mathcal{L}^2 + \mathcal{Q}\Big)}{\Sigma^2} \tag{2.20}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\frac{r^2+a^2}{\Delta}\big(\mathcal{E}(r^2+a^2) - a\mathcal{L}\big) - a^2\mathcal{E}\sin^2\theta + a\mathcal{L}}{\Sigma}, \text{ and} \tag{2.21}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{\frac{a}{\Delta}\left(\mathcal{E}(r^2+a^2) - a\mathcal{L}\right) + \frac{\mathcal{L}}{\sin^2\theta} - a\mathcal{E}}{\Sigma}. \tag{2.22}$$

This is already a huge improvement over the general geodesic equations in that only first order derivatives appear. Moreover, t and ϕ do not appear at all on the right hand side of the equations. If we can manage to solve the equations for r and θ the solutions for t and ϕ can be found by direct integration. Finally, the equations for r and θ appear almost entirely decoupled, apart from the Σ in the denominators the right hand side of the radial (r) equation depends only on r and the polar (θ) equation depends only on θ. Having decoupled equations would be nice since we could then solve each equation independently. To fully decouple the equations we introduce a new time parameter, the **Mino-Carter time** λ defined by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}s} = \frac{1}{\Sigma}. \tag{2.23}$$

With this choice our equations become

$$\begin{aligned}
\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 &= \big(\mathcal{E}(r^2+a^2)-a\mathcal{L}\big)^2 - \Delta\left(\mathcal{Q}+(\mathcal{L}-a\mathcal{E})^2+\mu r^2\right) = P_r(r)\\
\left(\frac{\mathrm{d}\cos\theta}{\mathrm{d}\lambda}\right)^2 &= \mathcal{Q}-\cos^2\theta\Big(a^2(\mu-\mathcal{E}^2)\sin^2\theta+\mathcal{L}^2+\mathcal{Q}\Big) \qquad = P_\theta(\cos\theta)\\
\frac{\mathrm{d}t}{\mathrm{d}\lambda} &= \frac{r^2+a^2}{\Delta}\big(\mathcal{E}(r^2+a^2)-a\mathcal{L}\big) - a^2\mathcal{E}\sin^2\theta + a\mathcal{L} \qquad = T_r(r)+T_\theta(\cos\theta)\\
\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} &= \frac{a}{\Delta}\left(\mathcal{E}(r^2+a^2)-a\mathcal{L}\right)+\frac{\mathcal{L}}{\sin^2\theta}-a\mathcal{E} \qquad = \Phi_r(r)+\Phi_\theta(\cos\theta).
\end{aligned}$$

These equations are completely decoupled. Moreover, the right hand side of the t and ϕ equations separate as the sum of something that depends on r with something that depends on θ. Consequently, these can be integrated separately for different radial and polar solutions.

The equations for r and $\cos\theta$ each individually take the form of a particle moving in a 1-dimensional potential. The functions P_r and P_θ are therefore known as the **radial** and **polar potentials**. Note that instead of the equations involving the square of the first derivatives (for r and θ) we can easily return to second order differential equations (which now will also be decoupled) by taking the derivative with respect to λ to yield

$$\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2} = \frac{1}{2}P_r'(r), \quad \text{and} \tag{2.24}$$

$$\frac{\mathrm{d}^2\cos\theta}{\mathrm{d}\lambda^2} = \frac{1}{2}P_\theta'(\cos\theta). \tag{2.25}$$

2.5 The polar equation

$$\left(\frac{\mathrm{d}\cos\theta}{\mathrm{d}\lambda}\right)^2 = \mathcal{Q}-\cos^2\theta\Big(a^2(\mu-\mathcal{E}^2)\sin^2\theta+\mathcal{L}^2+\mathcal{Q}\Big) = P_\theta(\cos\theta) \tag{2.26}$$

Because the left-hand side of the equation is a square, we can only find solutions if the right-hand side is not negative, i.e. when $P_\theta(\cos\theta)\geq 0$. We can therefore learn about the possible solutions by studying the roots of the polynomial $P_\theta(z)$. Since $P_\theta(z)$ is a fourth order polynomial in $z=\cos\theta$, there at most 4 real roots. Moreover, since $P_\theta(z)$ is an even function of z these roots come in pairs $z=\pm z_i$. Consequently,

$$P_\theta(z) = a^2(\mu-\mathcal{E}^2)(z^2-z_1^2)(z^2-z_2^2). \tag{2.27}$$

If we evaluate $P_\theta(z)$ at the poles $z=\pm1$, we find that $P_\theta(\pm1)=-\mathcal{L}^2$. We thus immediately learn geodesics can only reach the poles when $\mathcal{L}=0$.

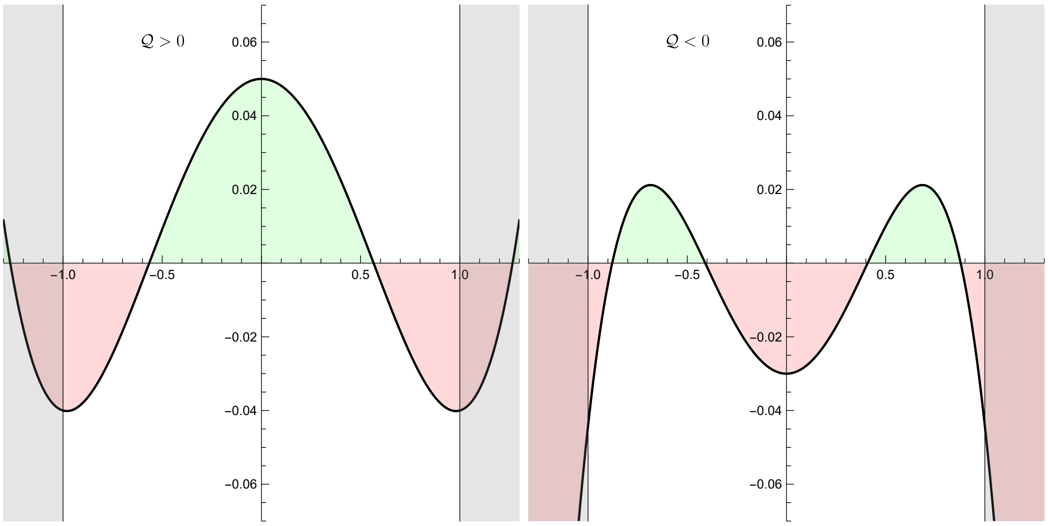

Figure 2.1: The polar potential P_θ when $\mathcal{Q}$ is positive (on the left) or negative (on the right). In the positive $\mathcal{Q}$ case, we find solutions where $\cos\theta$ oscillates around the equator ($\cos\theta = 0$) between $\pm z_1$. In the negative $\mathcal{Q}$ case, we can get vortical solutions where $\cos\theta$ oscillates between $z_1 < z_2$ (or $-z_2 < -z_1$), but never crosses the equator.

When we evaluate $P_\theta(z)$ on the equator $\cos\theta = 0$ we find that $P_\theta(0) = \mathcal{Q}$. Therefore, we only find solutions that visit the equator when $\mathcal{Q} \geq 0$. If $\mathcal{Q} > 0$ and $\mathcal{L} \neq 0$, there must be an odd number of zeroes between 0 and 1, and the same number between -1 and 0. Since there are at most 4 zeroes, there is exactly one such zero z_1, and the only solutions oscillate around the equator between $\pm z_1$. In the Schwarzschild ($a \to 0$) limit, these solutions describe trajectories whose orbital plane is inclined relative to the equator of the coordinate system. As such these solutions are generally known as **inclined** trajectories. In Kerr this orbital plane precesses, leading to another common name **precessing** trajectories.

If $\mathcal{Q} = 0$, then $z = 0$ is a double root of P_θ. This implies that $x = \cos\theta = 0$ is constant is a solution to the differential equation. These solutions stay in the Kerr equatorial plane and are therefore known as **equatorial** trajectories.

If $\mathcal{Q} < 0$ then no solution is possible near the equator. In this case, it is only possible for P_θ to be positive somewhere in the range $-1 < \cos\theta < 1$, if P_θ has exactly two zeroes $0 < z_1 < z_2 \leq 1$. We can easily see that this is only possible if $\mu < \mathcal{E}^2$ by considering the behavior of P_θ at large z. At large z, $P_\theta = a^2(\mu - \mathcal{E}^2)z^4 + \mathcal{O}(z^2)$. So, if $\mu > \mathcal{E}^2$ the polar potential becomes positive at large z. Consequently, there must be at least one zero between $z = 1$ (where P_θ is negative) and $z = \infty$, and there cannot be four roots in

Type	$\mu - \mathcal{E}^2$	$\frac{\mathcal{L}^2}{a^2(\mathcal{E}^2-\mu)}$	$\mathcal{Q}$	$z = \cos\theta$
Equatorial	any	any	0	0
Inclined	any	any	> 0	$-z_1 < z < z_1$
Vortical	< 0	≤ 1	$-1 \leq \frac{\mathcal{Q}}{\left(\lvert\mathcal{L}\rvert-\lvert a\rvert\sqrt{\mathcal{E}^2-\mu}\right)^2} \leq 0$	$z_1 < \lvert z\rvert < z_2$

Table 2.1: Classification of solutions of the polar equation.

the range $-1 < z < 1$. The condition $\mu < \mathcal{E}^2$ is by itself not sufficient the existence of two zeroes $0 < z_1 < z_2 \leq 1$. One can show (but we will not here) that such roots exist if and only if the following conditions are met

$$\mathcal{L}^2 \leq a^2(\mathcal{E}^2 - \mu) \text{ and } -\left(\lvert\mathcal{L}\rvert - \lvert a\rvert\sqrt{\mathcal{E}^2 - \mu}\right)^2 \leq \mathcal{Q} \leq 0. \tag{2.28}$$

Solutions of this type are referred to as **vortical** trajectories.

2.6 The radial equation

The decoupled equation for radial motion is given by

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \left(\mathcal{E}(r^2 + a^2) - a\mathcal{L}\right)^2 - \Delta\left(\mathcal{Q} + (\mathcal{L} - a\mathcal{E})^2 + \mu r^2\right) = P_r(r). \tag{2.29}$$

It has the same, overall structure as the polar equation. In particular, we can only have solutions when the radial potential P_r is non-negative, and the radial potential is a fourth order polynomial in r, which therefore has four zeroes r_i (0,2, or 4 of which may be real-valued) and can be written

$$P_r = (\mathcal{E}^2 - \mu)(r - r_1)(r - r_2)(r - r_3)(r - r_4). \tag{2.30}$$

As with the polar equation it will be useful to examine the value of P_r at specific values of r. We start with $r = 0$, and observe that $P_r(0) = -a^2\mathcal{Q}$. Consequently, solutions to the radial equation can only reach $r = 0$ if $\mathcal{Q} \leq 0$, i.e. only equatorial and vortical trajectories can ever reach $r = 0$. Of these, only the equatorial ($\mathcal{Q} = 0$) solutions can ever reach the curvature singularity at $\theta = \pi/2$. We thus find that of all geodesics in Kerr only a measure zero subset hit the singularity. All others miss it entirely, some of them passing through the ring singularity to the $r < 0$ region (all of them vortical) and some of them scattering back due to centrifugal barrier.

Next we turn our attention to the horizon $r_\pm$, since Δ vanishes here we find

$$P_r = \left(\mathcal{E}(r_\pm^2 + a^2) - a\mathcal{L}\right)^2 \geq 0. \tag{2.31}$$

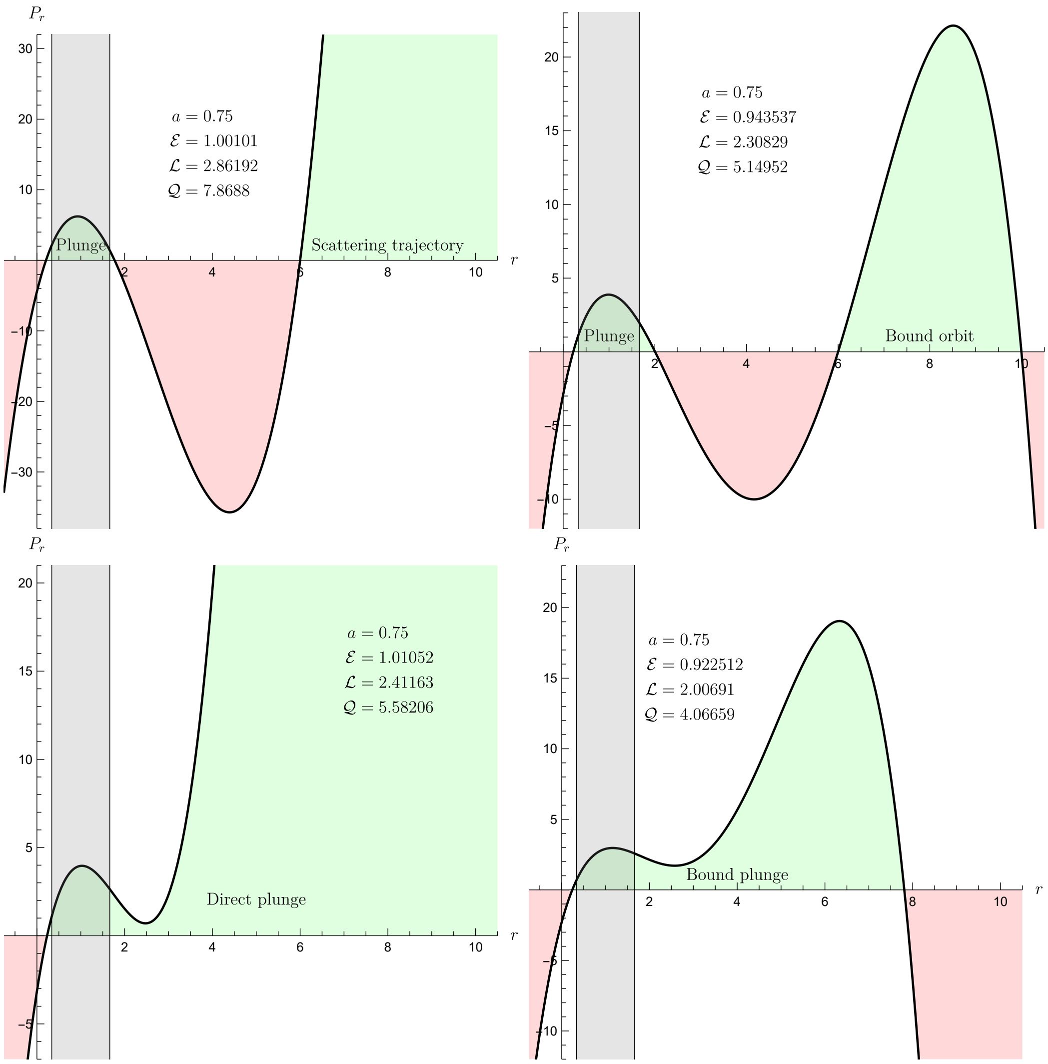

Figure 2.2: The radial potential P_r for a scattering trajectory (top left), bound orbit (top right), direct plunge (bottom left), and bound plunge (bottom left). Note the existence of secondary "deeply bound" plunge solutions in the top plots.

Consequently, for all values of $(\mu, \mathcal{E}, \mathcal{L}, \mathcal{Q})$ that allow the existence of solutions to the polar equation, we have solutions that cross the horizons. Moreover, since r is timelike between the horizons, we cannot have any zeroes between $r_\pm$ for timelike $\mu > 0$ and null $\mu = 0$ solutions. This also implies that if $\mathcal{Q} > 0$ then there must exist at least one (and at most three) zeroes between the inner horizon r_- and $r = 0$.

Last we turn our attention to the behavior of P_r as r approaches $\pm\infty$. Expand P_r we find $P_r = (\mathcal{E}^2 - \mu)r^4 + \mathcal{O}(r^3)$. Consequently, in order for a geodesic to reach infinity $(\mathcal{I}^\pm, i^\pm, i^0)$ it must have $\mathcal{E}^2 > \mu$. For this reason geodesics satisfying this condition are referred to as **unbound**. Note that for null geodesics $\mu = 0$; they are necessarily unbound.

For unbound $\mathcal{E}^2 > \mu$ geodesics the radial potential must have an even number of zeroes between the outer horizon r_+ and $r = \infty$. When there no zeroes in this region the solution corresponds to a trajectory that starts at infinity and dives directly into the black hole. These solutions are known as **direct plunges**.

If there are two zeroes $r_1 > r_2$ in the region outside of the black hole, solutions correspond to trajectories starting from infinity, and scattering of the black hole potential back to infinity. This trajectories are known as **scattering** or **hyperbolic** trajectories.

Geodesics with $\mathcal{E}^2 < \mu$ are only possible for timelike trajectories with $\mu > 0$, and known as **bound** trajectories because they do not possess enough energy to reach infinity. The existence of a polar solution implies that for bound orbits we necessarily have that $\mathcal{Q} \geq 0$, and therefore that there exists at least one zero inside the inner horizon. Since the radial potential changes sign between the outer horizon r_+ and infinity there must be either one or three zeroes outside the outer horizon.

If there exists just one zero $r_+ < r_1 < \infty$ the solution corresponds to a trajectory starting at the past horizon moving outward to r_1 and then diving back into the black hole. These trajectories are known as **(bound) plunges**. Of course, these geodesics do not stop at the horizon. If followed into the black hole, they will also cross the inner horizon before encountering a turning point in the inner region, and proceeding back outward through the inner horizon and exiting the outer horizon into a new parallel universe in the maximally extended Kerr solution, after which the above repeats ad infinitum with the geodesic eventually (according to its own proper time) visiting infinitely many copies of the external universe.

If there are three zeroes $r_+ < r_3 < r_2 < r_1 < \infty$ then the solution oscillates between r_1 and r_2 to form a **bound (eccentric) orbit**. These orbits are often identified by the **eccentricity** ($e = \frac{r_1 - r2}{r_1 + r2}$) and **semi-latus**

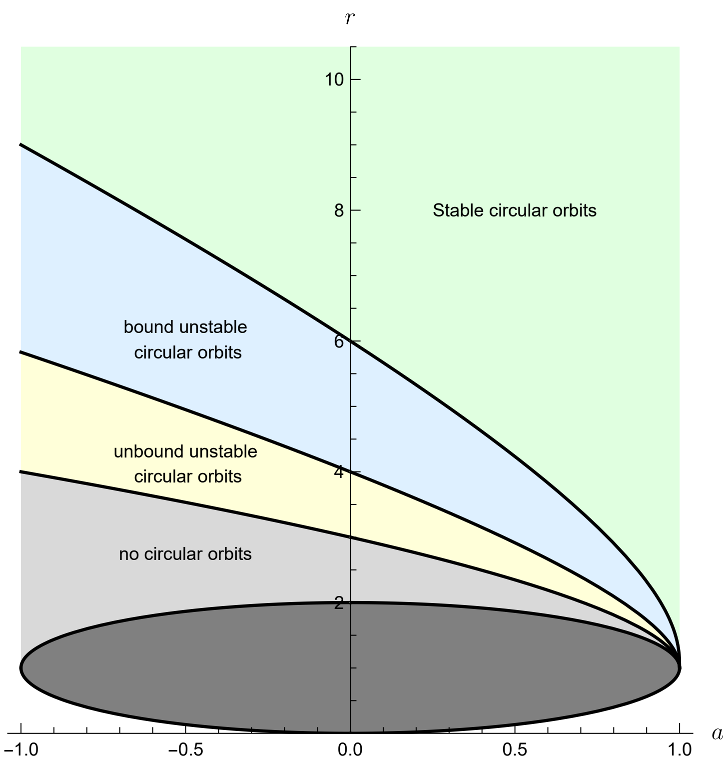

Figure 2.3: The location of the ISCO, IBCO, and light ring for equatorial orbits in Kerr.

rectum ($p = \frac{2r_1r2}{r_1+r2}$).

Note that both in the case of bound orbits and scattering orbits, there exists a secondary solution with the same constants of motion which oscillates between the remaining turning point outside the horizon and the zero inside the inner horizon (if $\mathcal{Q} > 0$). These solutions behave similarly to the bound plunges and are therefore known as **deeply bound plunges**.

2.6.1 Circular orbits

Circular orbits are geodesics for which the radial potential allows a constant solution r_o. This requires not only that $P_r(r_o) = 0$ (which ensures that $\frac{\mathrm{d}r}{\mathrm{d}\lambda} = 0 = \frac{\mathrm{d}r}{\mathrm{d}s}$), but also that $P_r'(r_o) = 0$ (which ensures that $\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2} = 0 = \frac{\mathrm{d}^2r}{\mathrm{d}s^2}$). This therefore requires that r_o is a double zero of the radial potential. A double zero can be either a maximum or minimum of the radial potential. In the former case the circular orbit is **stable**; any small perturbation will produce an eccentric orbit with small eccentricity. In the later case, the circular orbit is **unstable**; any perturbation leads to a wildly different orbit.

Stable circular orbits occur as the limit of eccentric orbits as the zeroes r_1 and r_2 merge. Unstable circular orbits are formed by merging the inner turning point of a scattering or bound orbit with the outer turning point of a deeply bound plunge.

Stable circular orbits can only be found outside a certain radius, the **innermost stable circular orbit** or **ISCO**. Inside this radius one can still find circular orbits, but the become increasingly unstable with larger and larger energies. Initially, these unstable circular orbits have low enough energies, that even after a perturbation the particle will stay bound to the black hole. However, below another radius, the **innermost bound circular orbit** or **IBCO**, the energy becomes large enough that perturbations can send the particle scattering to infinity. Finally, the energy of the circular orbits diverges at the **light ring**, this radius can only be approached if one simultaneously allows the mass μ to go to zero, and the geodesic to become null. Inside the light ring no circular orbits are possible (outside the event horizon).

2.7 Null geodesics and black hole shadows

Coming soon.