Hand-in sheet 2

Guidelines

- This hand-in sheet will be graded. It will count for 20% of the total grade. Please hand-in your answers before Noon Friday 14 June 2024.
- Answers may be handed in physically to or digitally through e-mail to: jose.ezquiaga@nbi.ku.dk.
- You are allowed (and encouraged) to work in groups to discuss the exercises. However, everyone must write and hand-in their own answers (Show your work!).
- You are also allowed to use computer algebra software (e.g. Mathematica) and coding in general (e.g. jupyter notebook of pyhton) to aid with your computations. However, if you do, please include your notebooks with your answer.
- Please make sure that your answers (and any supplemental material such as code) are comprehensible!

Exercise 1: Radiation of angular momentum

We have studied that GWs emit energy. However, they also radiate angular momentum. The energy momentum carried out by GW emission is[1]

$$\frac{dJ^i}{dt} = \frac{c^3}{32\pi G}\int d\Omega r^2 \langle -\epsilon_{ikl}x^k \dot{h}^{TT}_{ab}\partial^l h^{ab}_{TT} + 2\epsilon_{ikl}\dot{h}^{al}_{TT}h^{TT}_{ak}\rangle \,, \tag{0.0.1}$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol ($\epsilon_{ijk} = -\epsilon_{jik} = \epsilon_{jki}$, $\epsilon_{iij} = 0$). This variation of the total angular momentum $\vec{J}$ can be split into the orbital angular momentum $\vec{L}$ and the spin $\vec{S}$:

$$\frac{dJ^i}{dt} = \frac{dL^i}{dt} + \frac{dS^i}{dt} \,, \tag{0.0.2}$$

which map directly to the first and second term of the first equation.

1. Compute the leading order contribution to the orbital angular and spin emitted by a GW in terms of the multipole moments of the source in the far zone. How does it compare to the energy emission?
2. In the near zone, compute the variation of orbital angular of the source in the limit of small velocities $v/c \ll 1$.
3. Apply your results to the inspiral phase of a quasi-circular binary.

[1] You are welcome to try to derive this formula as well, but this goes beyond the scope of this exercise

Exercise 2: Elliptic orbits

Let us consider a binary made of point particles with masses m_1 and m_2, total mass $M_{\rm tot} = m_1 + m_2$ and reduced mass $\mu = m_1 m_2/(m_1 + m_2)$. We want to study the GWs emitted in an elliptic Keplerian orbit. The orbit is described by a semi-major axis a and an orbital eccentricity e. Given the conservation of angular momentum $\vec{L}$ that restrict the motion to a plane, it is useful to introduce polar coordinates in the plane of the orbit (r, ψ). The modulus of the angular momentum L is determined by

$$L = \mu r^2 \dot{\psi} \,, \tag{0.0.3}$$

and the energy is given by

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{G\mu m}{r} \,. \tag{0.0.4}$$

There are many relations that you can obtain about elliptic orbit. The key ones for this problem are the relations between e and a and the modulus of the angular momentum L and the energy E:

$$e^2 = 1 + \frac{2EL^2}{G^2 m^2 \mu^3} \,, \tag{0.0.5}$$

$$a = \frac{Gm\mu}{2|E|} \,. \tag{0.0.6}$$

Similarly important, for a reference frame with (x, y) in the plane of the orbit, the second mass moments are given by

$$M_{11} = \mu r^2 \cos^2\psi \,, \tag{0.0.7}$$

$$M_{22} = \mu r^2 \sin^2\psi \,, \tag{0.0.8}$$

$$M_{12} = M_{21} = \mu r^2 \sin\psi \cos\psi \,. \tag{0.0.9}$$

1. Compute the GWs polarizations h_+ and $h_\times$ for a far away observer at distance r on the direction of the orbital anhular momentum

2. Compute the orbital energy and angular momentum radiated per unit time (for this exercise 1 will be useful). How does it compare to the circular orbit?

3. Compute the evolution of the semi-major axis a and eccentricity e as a function of time.

4. Given what you have learned, should we expect to observe eccentric binaries with current detectors?

Bibliography