Black holes & gravitational waves – Exam 2024

Guidelines

- This is a take-home exam. It will be graded. It will count for 60% of the total grade. Please hand-in your answers before **Noon Thursday 20 June 2024**.
- Answers may be handed in physically to Jose M. Ezquiaga (office Cc11) or digitally through e-mail to:
maarten.meent@nbi.ku.dk & jose.ezquiaga@nbi.ku.dk.
- You must do the exam individually
- You are allowed to use computer algebra software (e.g. Mathematica) and coding in general (e.g. jupyter notebook of pyhton) to aid with your computations. However, if you do, please include your notebooks with your answer.
- Please make sure that your answers (and any supplemental material such as code) are comprehensible!
- The bonus questions are optional. They are included because they complete the "story" of the question.

Question 1. (In)variance of the Weyl scalars

The most general infinitesimal transformation of a null tetrad $(\ell^\mu, n^\mu, m^\mu, \bar{\mu})$ that preserves the orthogonality/normalization relationships:

- $\ell^\alpha \ell_\alpha = n^\alpha n_\alpha = m^\alpha m_\alpha = \bar{m}^\alpha \bar{m}_\alpha = 0$
- $\ell^\alpha n_\alpha = -1$
- $m^\alpha \bar{m}_\alpha = 1$
- all other inner products are zero

is given by

$$
\begin{aligned}
\ell^\mu &\mapsto \ell^\mu + \epsilon \left(-c\ell^\mu + b^* m^\mu + b\bar{m}^\mu\right) + \mathcal{O}(\epsilon^2)\\
n^\mu &\mapsto n^\mu + \epsilon \left(cn^\mu + a^* m^\mu + a\bar{m}^\mu\right) + \mathcal{O}(\epsilon^2)\\
m^\mu &\mapsto m^\mu + \epsilon \left(a\ell^\mu + bn^\mu + idm^\mu\right) + \mathcal{O}(\epsilon^2)\\
\bar{m}^\mu &\mapsto \bar{m}^\mu + \epsilon \left(a^*\ell^\mu + b^* n^\mu - id\bar{m}^\mu\right) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

For $(a, b) \in \mathbb{C}$ and $(c, d) \in \mathbb{R}$. Such transformations are known as *infinitesimal tetrad rotations.*

They can be split in three elementary types of transformations:

Type I (Keeping ℓ^μ fixed. Controlled by a)

$$\begin{aligned}
\ell^\mu &\mapsto \ell^\mu + \mathcal{O}(\epsilon^2) \\
n^\mu &\mapsto n^\mu + \epsilon\left(a^* m^\mu + a\bar{m}^\mu\right) + \mathcal{O}(\epsilon^2) \\
m^\mu &\mapsto m^\mu + \epsilon a \ell^\mu + \mathcal{O}(\epsilon^2) \\
\bar{m}^\mu &\mapsto \bar{m}^\mu + \epsilon a^* \ell^\mu + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Type II (Keeping n^μ fixed. Controlled by b)

$$\begin{aligned}
\ell^\mu &\mapsto \ell^\mu + \epsilon\left(b^* m^\mu + b\bar{m}^\mu\right) + \mathcal{O}(\epsilon^2) \\
n^\mu &\mapsto n^\mu + \mathcal{O}(\epsilon^2) \\
m^\mu &\mapsto m^\mu + \epsilon b n^\mu + \mathcal{O}(\epsilon^2) \\
\bar{m}^\mu &\mapsto \bar{m}^\mu + \epsilon b^* n^\mu + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Type III (Preserving ℓ^μ-n^μ and m^μ- $\bar{m}^\mu$ planes. Controlled by c and d.)

$$\begin{aligned}
\ell^\mu &\mapsto \ell^\mu - \epsilon c \ell^\mu + \mathcal{O}(\epsilon^2) \\
n^\mu &\mapsto n^\mu + \epsilon c n^\mu + \mathcal{O}(\epsilon^2) \\
m^\mu &\mapsto m^\mu + i\epsilon d m^\mu + \mathcal{O}(\epsilon^2) \\
\bar{m}^\mu &\mapsto \bar{m}^\mu - i\epsilon d \bar{m}^\mu + \mathcal{O}(\epsilon^2)
\end{aligned}$$

a. Show that the Weyl curvature tensor satisfies $C_{\mu\alpha\beta\nu}\ell^{(\alpha}n^{\beta)} = C_{\mu\alpha\beta\nu}m^{(\alpha}\bar{m}^{\beta)}$.

b. Show that the Weyl scalars ψ_0, ψ_1, ψ_2, ψ_3, and ψ_4 transform under type I tetrad rotations as:

$$\begin{aligned}
\psi_0 &\mapsto \psi_0 + \mathcal{O}(\epsilon^2) \\
\psi_1 &\mapsto \psi_1 + \epsilon a^* \psi_0 + \mathcal{O}(\epsilon^2) \\
\psi_2 &\mapsto \psi_2 + 2\epsilon a^* \psi_1 + \mathcal{O}(\epsilon^2) \\
\psi_3 &\mapsto \psi_3 + 3\epsilon a^* \psi_2 + \mathcal{O}(\epsilon^2) \\
\psi_4 &\mapsto \psi_4 + 4\epsilon a^* \psi_3 + \mathcal{O}(\epsilon^2)
\end{aligned}$$

c. Show that the Weyl scalars ψ_0, ψ_1, ψ_2, ψ_3, and ψ_4 transform under type II tetrad rotations as:

$$\begin{aligned}
\psi_0 &\mapsto \psi_0 + 4\epsilon b \psi_1 + \mathcal{O}(\epsilon^2) \\
\psi_1 &\mapsto \psi_1 + 3\epsilon b \psi_2 + \mathcal{O}(\epsilon^2) \\
\psi_2 &\mapsto \psi_2 + 2\epsilon b \psi_3 + \mathcal{O}(\epsilon^2) \\
\psi_3 &\mapsto \psi_3 + \epsilon b \psi_4 + \mathcal{O}(\epsilon^2) \\
\psi_4 &\mapsto \psi_4 + \mathcal{O}(\epsilon^2)
\end{aligned}$$

d. Show that the Weyl scalars ψ_0, ψ_1, ψ_2, ψ_3, and ψ_4 transform under type III tetrad rotations as:

$$\begin{aligned}
\psi_0 &\mapsto \psi_0 - 2\epsilon(c - id)\psi_0 + \mathcal{O}(\epsilon^2) \\
\psi_1 &\mapsto \psi_1 - \epsilon(c - id)\psi_1 + \mathcal{O}(\epsilon^2) \\
\psi_2 &\mapsto \psi_2 + \mathcal{O}(\epsilon^2) \\
\psi_3 &\mapsto \psi_3 + \epsilon(c - id)\psi_3 + \mathcal{O}(\epsilon^2) \\
\psi_4 &\mapsto \psi_4 + 2\epsilon(c - id)\psi_4 + \mathcal{O}(\epsilon^2)
\end{aligned}$$

e. In the case of the Kerr metric with the Kinnersley tetrad, which Weyl scalars are invariant under infinitesimal tetrad rotations at linear order in ϵ.

f. The Weyl scalars transform as scalars under spacetime gauge transformations ξ^μ (Note that the Lie derivative of a scalar $\mathcal{L}_\xi \psi$ is simply the directional derivative $\xi^\alpha \partial_\alpha \psi$.) For the Kerr metric with Kinnersley tetrad, the perturbations of which Weyl scalars are gauge invariant?

g. For the Kerr metric with Kinnersley tetrad in Boyer-Lindquist coordinates, the Weyl scalar ψ_2^{Kerr} is equal to $\frac{-M}{(r - ia\cos\theta)^3}$. Suppose we have calculated a perturbation to the Kerr metric and found the perturbed Weyl scalar $\psi_2 = \psi_2^{\text{Kerr}} + \delta\psi_2$. Find the spacetime gauge transformation ξ^μ such that $\psi_2 = \psi_2^{\text{Kerr}}$.

Question 2: Detecting a compact binary coalescence

During the course we have studied the gravitational waves produced during the inspiral phase of a compact binary. We have then learned how these waves propagate in the Universe and what imprint they leave on a ground-base detector. It is now the time to explore what are the capabilities of such detectors to measure the properties of the signal. This is essential if we want to understand how such detections could be used to learn about astrophysics, cosmology and fundamental physics.

A compact binary at cosmological distances emits a GW during the inspiral phase whose polarizations are given (at leading order) by the following expressions in the detector's frame:

$$h_+(t) = \frac{4}{d_L}\left(\frac{G\mathcal{M}_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\left(\frac{1+\cos^2\iota}{2}\right)\cos(\Phi(t))\,, \tag{0.0.1}$$

$$h_\times(t) = \frac{4}{d_L}\left(\frac{G\mathcal{M}_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\cos\iota\sin(\Phi(t))\,, \tag{0.0.2}$$

where d_L is the luminosity distance, ι is the inclination angle and the phase is given by

$$\Phi(t) = -2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-5/8}(t_c - t)^{5/8} + \Phi_0\,. \tag{0.0.3}$$

Recall that $\mathcal{M}_z = (1+z)\mathcal{M}_c$ is the redshifted chirp mass and $f_{\rm gw}(t)$ the frequency evolution measured at the detector.

At the detector one measures a strain which is a projection into the antenna pattern functions,

$$h(t) = F_+h_+(t) + F_\times h_\times(t)\,, \tag{0.0.4}$$

where $F_{+,\times}$ are a function of the sky position and polarization angle $\{\theta,\varphi,\psi\}$.

1. Show that the strain can be written as

$$h(t) = \frac{4}{d_{\rm eff}}\left(\frac{G\mathcal{M}_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\cos\left[\Phi(t)+\chi\right] \tag{0.0.5}$$

   by finding the values of the effective distance $d_{\rm eff}$ and the phase χ. With a single detector can we disentangle the GW distance from the inclination and sky position? For generic sky positions, will we think that the source is closer or further away?

2. Compute the Fourier transform of the strain (using the stationary phase approximation),

$$\tilde{h}(f) = \int dt\, h(t)e^{i2\pi ft}\,, \tag{0.0.6}$$

   and show that it only depends on four parameters: an effective distance d, the redshifted chirp mass $\mathcal{M}_z$, a reference time t_0 and a reference phase ϕ_0. Collectively, they form a 4D parameter space $\vec{\theta} = \{d, \mathcal{M}_z, t_0, \phi_0\}$. What is the scaling of the amplitude and complex phase $\tilde{h}(f) = \tilde{A}(f,\vec{\theta})e^{i\Psi(f,\vec{\theta})}$ with frequency? What is the range of frequencies in which this expression is valid?

3. Using the frequency domain strain that you have just derived, compute the Fisher matrix
$$\Gamma_{ij} = (\partial_i h|\partial_j h)\,, \tag{0.0.7}$$
where $\partial_i = \partial/\partial\theta^i$ indicates a derivative in the parameter space $\vec{\theta} = \{\ln d^{-1}, \ln \mathcal{M}_z, t_0, \phi_0\}$ and $(a|b)$ is the noise-weighted inner product of $a(t)$ and $b(t)$:
$$(a|b) = 4\mathrm{Re}\int_0^\infty \frac{\tilde{a}^*(f)b(f)}{S_n(f)}df\,. \tag{0.0.8}$$
Note that the logarithms are useful to get fractional errors. Express each component of the matrix in terms of the signal-to-noise ratio $\rho^2 = (h|h)$ and introduce the notation for the frequency moments
$$\bar{f}^\alpha \equiv \frac{(f^\alpha h|h)}{(h|h)} \tag{0.0.9}$$
in order to conveniently encapsulate the integral terms with powers of the frequency. Since in the inspiral phase the frequency domain waveform scales as a power-law with the frequency, then $\bar{f}^\alpha$ only depends on the frequency and $S_n(f)$.

4. Assuming that the power spectral density is constant over a frequency range and infinite otherwise,
$$S_n(f) = \begin{cases} S_0 & f_0 \leqslant f \leqslant f_{\max}\,, \\ \infty & \text{otherwise}\,, \end{cases} \tag{0.0.10}$$
with $S_0 = 10^{-48}\mathrm{Hz}^{-1}$, $f_0 = 20\mathrm{Hz}$, and $f_{\max} = 1000\mathrm{Hz}$, allows you to compute numerically the frequency moments $\bar{f}^\alpha$. Then, compute the fractional errors in the distance and redshifted chirp mass, $\Delta d/d$ and $\Delta\mathcal{M}_z/\mathcal{M}_z$, and the absolute errors in the reference phase and time, Δt_0 and $\Delta\phi_0$, as a function of the SNR and chirp mass.[1] Finally, express the numerical values of these measurement uncertainties for a reference binary neutron star in the local universe ($z \ll 1$) with $m_1 = m_2 = 1.4M_\odot$ and $\rho = 10$. Compare your results with the errors obtained in the first detected binary neutron star, GW170817.

5. **Bonus:** How does the error in the chirp mass scales with the number of cycles? what does this mean when we compare expected errors for binary neutron stars and binary black holes?

6. **Bonus:** Assuming that you have observed a multi-messenger binary neutron star in the local Universe ($z \ll 1$) that allows you to obtain a spectroscopic redshift with negligible error, what SNR would you need to obtain a 1% measurement of H_0?

7. **Bonus:** If your GW were to be gravitationally lensed by a galaxy producing multiple images, what should you expect in terms of observed value and the measurement errors of the parameters of each image?

[1] Remember that you are allowed to invert the Fisher matrix with Mathematica for example.