

Lecture Notes Black Holes and Gravitational Waves

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Contents

1	Penrose Diagrams	3
1.1	Conformal transformations	3
1.2	Minkowski space	3
1.3	Schwarzschild	6
1.3.1	Definition of a Black Hole	8
1.3.2	Spherical collapse	9
1.4	The Kerr Black Hole	10
1.4.1	Collapse of a rotating black hole	12
2	Geodesics in Kerr Spacetime	15
2.1	The geodesic equation	15
2.2	Symmetries and Killing vectors	15

Conventions

We will be using the following conventions through out these notes.

- We work in geometric units such that $G = c = 1$.
- We utilize *abstract index notation* to refer to tensors. This means that abstract tensors are always denoted with indices that indicate the type and rank of the tensor. This does not signify that any particular coordinate basis has been chosen. Repeated indices of opposite type will denote contractions.

Chapter 1

Penrose Diagrams

Literature:

- Reall, Chap. 5
- Carroll, Sec. 5.6, 5.7 and Appendix H

In this chapter we will develop the tools to describe the causal structure of a spacetime, allowing us to define what it means for a spacetime to contain a black hole.

1.1 Conformal transformations

Given a metric $g_{\mu\nu}$ on a spacetime M , we can define a new metric $\bar{g}_{\mu\nu}$ as $\bar{g}_{\mu\nu} = \omega(x)^2 g_{\mu\nu}$, where $\omega : M \rightarrow \mathbb{R}$ is a positive smooth function. This new metric $\bar{g}_{\mu\nu}$ in general defines a new spacetime geometry. Nonetheless, this new geometry will agree with the old geometry on the notions of curves being “timelike”, “spacelike”, or “null”, i.e. they will agree on the possible causal relations between different spacetime events; the *causal structure* of the spacetime. The transformation from $g_{\mu\nu}$ to $\bar{g}_{\mu\nu}$ encoded by the function $\omega(x)$ is known as a *conformal transformation*, since it preserves the angles (but not lengths) of the geometry.

The idea is to use these conformal transformations to take a (generally infinite) spacetime and map it to a compact spacetime (which we can draw on a page) with the same causal structure.

1.2 Minkowski space

We start with the Minkowski space. In spherical coordinates the Minkowski metric reads

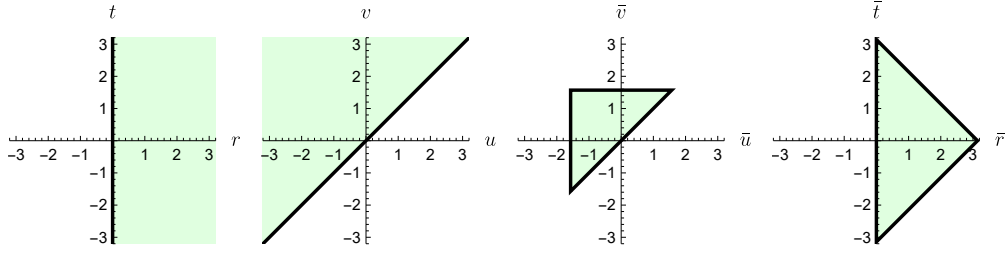


Figure 1.1: Illustration of the ranges of the different coordinates used in this section.

$$\eta_{\mu\nu} = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.1)$$

with the coordinate ranges (which will become important later on) $t \in (-\infty, \infty)$, $r \in [0, \infty)$, $\theta \in [0, \pi]$, and $\phi \in [-\pi, \pi]$. Our first step is to define lightcone coordinates

$$u = t - r, \quad \text{and} \quad v = t + r, \quad (1.2)$$

which gives the metric

$$\eta_{\mu\nu} = -dudv + \frac{1}{4}(v - u)^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.3)$$

with ranges $-\infty < u \leq v < \infty$. We can compactify the range of our coordinates by choosing new coordinates (\bar{u}, \bar{v}) defined by,

$$u = \tan \bar{u}, \quad \text{and} \quad v = \tan \bar{v}, \quad (1.4)$$

which reduces the ranges to $-\pi/2 < \bar{u} \leq \bar{v} < \pi/2$, and gives the metric

$$\eta_{\mu\nu} = (2 \cos \bar{u} \cos \bar{v})^{-2} (-4d\bar{u}d\bar{v} + \sin^2(\bar{v} - \bar{u})(d\theta^2 + \sin^2\theta d\phi^2)). \quad (1.5)$$

To find the conformal compactification of Minkowski space, we finally define the function

$$\omega = 2 \cos \bar{u} \cos \bar{v}, \quad (1.6)$$

and the coordinates

$$\bar{t} = \bar{u} + \bar{v}, \quad \text{and} \quad \bar{r} = \bar{v} - \bar{u}, \quad (1.7)$$

to find the metric

$$\bar{\eta}_{\mu\nu} = -d\bar{t}^2 + d\bar{r}^2 + \sin^2 \bar{r}(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.8)$$

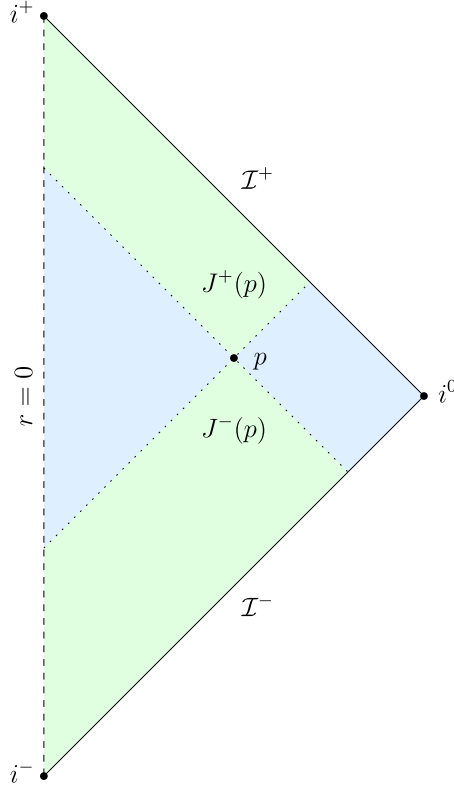


Figure 1.2: The Penrose diagram for Minkowski space.

The astute reader may recognize this as the line element of the Einstein static universe, $\mathbb{R} \times S^3$ (see Harmark Ex. 4.3). However, the crucial difference lies in the range of the coordinates, in particular $0 \leq \bar{r} < \pi$ and $-\pi + \bar{r} < \bar{t} < \pi - \bar{r}$ (see Fig. 1.1). By projecting onto the (\bar{r}, \bar{t}) -plane, we obtain the *Penrose diagram* for Minkowski space, shown in Fig. 1.2. Each interior point of the shown region represents a 2-sphere in the full spacetime. The dashed boundary at $\bar{r} = 0$ represents the symmetry axis at $r = 0$ of the original coordinates, and the outer boundary denotes the points originally “at infinity”.

To understand the structure of “infinity” we can consider the behavior of radial geodesics in the diagram. By construction of the diagram all radial null geodesics travel along 45 degree lines. Any null geodesic will originate at \mathcal{I}^- , travel along an inwards 45 degree straight line until it reaches $r = 0$, and reflect back to an outwards 45 degree line until it reaches \mathcal{I}^+ . We therefore refer to \mathcal{I}^- as *past null infinity*, and \mathcal{I}^+ as *future null infinity*. In a similar vein, all radial spacelike geodesics start and end at i^0 , which we call *spatial*

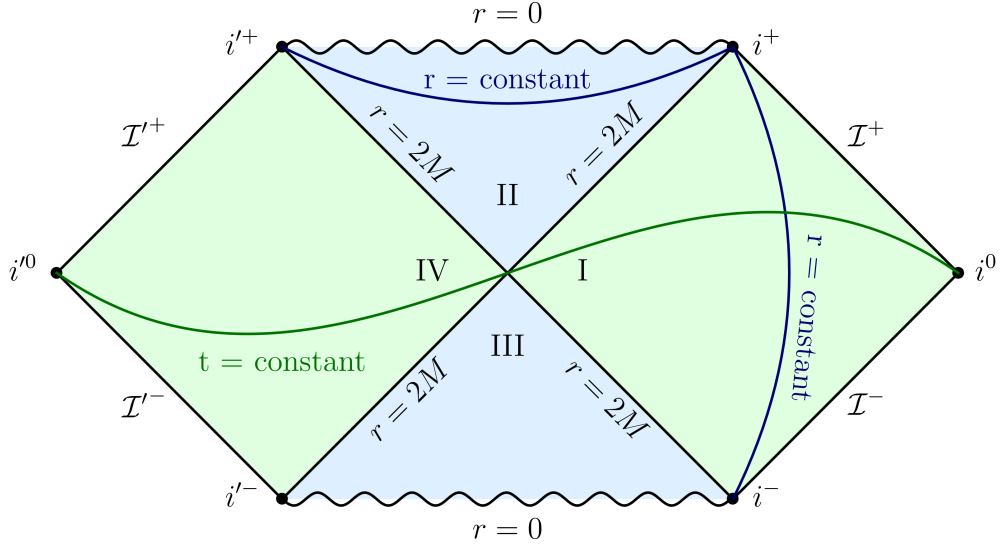


Figure 1.3: The Penrose diagram for the maximally extended Schwarzschild spacetime.

infinity, and timelike geodesics start at i^- and end at i^+ , which are called *past* and *future timelike infinity*, respectively.

A spacetime whose Penrose diagram has a structure near infinity resembling¹ that of Minkowski space is called *asymptotically flat*.

Finally, we use the Penrose diagram of Minkowski space to visualize to more important concepts related to the causal structure of spacetime. Given an event $p \in M$, the set of all events that can be connected to p with a future-directed causal (i.e. timelike or null) curve ending in p , is called the *causal past of p* , and denoted $J^-(p)$. In the Penrose diagram the intersection of $J^-(p)$ with a constant (θ, ϕ) -slice is simply the 45-degree wedge below the point p as shown in Fig. 1.2. Similarly, the set of all events that can be connected to p with a future-directed causal (i.e. timelike or null) curve starting at p , is called the *causal future of p* , and denoted $J^+(p)$.

1.3 Schwarzschild

In the course by Harmark you were introduced to the Schwarzschild metric,

¹For a more precise technical discussion of what “resembling” means in this context, see Chapter 5 of Reall’s lecture notes.

$$g_{\mu\nu} = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.9)$$

This metric has a (coordinate) singularity at $r = 2M$. Consequently, the a priori domain of the coordinates is $-\infty < t < \infty$ and $2M < r < \infty$. By changing the time coordinate to (advanced) Eddington-Finkelstein time

$$v = t + r^* \quad \text{with} \quad r^* = r + 2M \log\left(\frac{r - 2M}{2M}\right), \quad (1.10)$$

you were shown that the validity of the Schwarzschild metric can be extended to $0 < r < \infty$ for infalling observers. The same reasoning can be applied when using the retarded Eddington-Finkelstein time coordinate $u = t - r^*$ to show that there exists an (inequivalent) extension of Schwarzschild to the past of outgoing observers. When using both u and v to form double null coordinates we obtain

$$g_{\mu\nu} = -\left(1 - \frac{2M}{r(u,v)}\right) du dv + r(u,v)^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.11)$$

Even with the maximal coordinate range $-\infty < u, v < \infty$ these coordinates correspond only to the $2M < r < \infty$ Schwarzschild patch. The $r = 2M$ boundary corresponds to the $u \rightarrow \infty$ and $v \rightarrow -\infty$ limits. To find coordinates that cover all of the extended spacetime we introduce rescaled null coordinates

$$u = -4M \log(-\underline{u}) \quad \text{and} \quad v = 4M \log(\underline{v}). \quad (1.12)$$

In these coordinates the original Schwarzschild patch is covered by $-\infty < \underline{u} < 0$ and $0 < \underline{v} < \infty$. With these coordinates the Schwarzschild metric takes its *Kruskal-Szekeres* form,

$$g_{\mu\nu} = -\frac{32M}{r(\underline{u}, \underline{v})} e^{-\frac{r}{2M}} d\underline{u} d\underline{v} + r(\underline{u}, \underline{v})^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.13)$$

where r is defined implicitly through the relation $\underline{u} \underline{v} = (r/(2M) - 1)e^{r/(2M)}$. This form is manifestly smooth at $\underline{u} = 0$ and $\underline{v} = 0$ (i.e. $r = 2M$), and be extended beyond this point.

To produce the Penrose diagram we can again compactify the range of \underline{u} and \underline{v} by introducing

$$\underline{u} = \tan \bar{u}, \quad \text{and} \quad \underline{v} = \tan \bar{v}, \quad (1.14)$$

and an appropriate conformal factor ω such that we obtain a compactified metric of the form

$$\bar{g}_{\mu\nu} = -2d\bar{u}d\bar{v} + R(\bar{u}, \bar{v})^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.15)$$

(the explicit form of ω and R is messy and not particularly illuminating).

The resulting Penrose diagram is shown in Fig. 1.3. The original Schwarzschild patch (marked “I”) is asymptotically flat. The original (advanced) Eddington-Finkelstein coordinates extend the metric to region “II” allowing objects to fall into the black hole and reach the singularity at $r = 0$. The corresponding construction with the retarded time extends the metric to region “III”, allowing the worldline to extend to the singularity in their past.

Finally, we are presented with a second asymptotically flat region (“IV”), complete with its own copies of future and past null infinity (\mathcal{I}^+ and \mathcal{I}^-), future and past timelike infinity (i'^+ and i'^-), and spacelike infinity i'^0 . There are no causal curves that connect regions I and IV. So, for an observer in region I, region IV might as well not exist, and vice versa. There are however spacelike curves that connect regions I and IV. This “worm hole” between “parallel universes” is known as an *Einstein-Rosen bridge*. Because it consists solely of spacelike curves, there is no way to transverse it.

1.3.1 Definition of a Black Hole

We are now ready to give a precise definition of a black hole:

Definition 1. Let $(M, g_{\mu\nu})$ be a spacetime that is asymptotically flat at null infinity. The *black hole region* is $\mathcal{B} = M \setminus J^-(\mathcal{I}^+)$. The boundary of \mathcal{B} , $\partial\mathcal{B}$ is called the *future event horizon*, \mathcal{H}^+ . Similarly, $\mathcal{W} = M \setminus J^+(\mathcal{I}^-)$ is the *white hole region*, and its boundary $\partial\mathcal{W}$, the *past event horizon*, \mathcal{H}^- .

In more plain English a black hole consists of those events in spacetime from which no signal can reach future null infinity, a white hole consists of those events which cannot be reached by any signal starting at past null infinity. A direct consequence of this definition is that an event horizon is always a null-surface.

Concretely, when we look at the Penrose diagram for Schwarzschild in Fig. 1.3, we see that the black hole region \mathcal{B} corresponding to \mathcal{I}^+ consists of the union of regions II and IV, and the future event horizon \mathcal{H}^+ is the boundary $\bar{u} = 0$, where $r = 2M$. The white hole region \mathcal{W} is the union of regions III and IV, and the past event horizon is \mathcal{H}^- is the boundary $\bar{v} = 0$, where $r = 2M$ as well.

Note that since the maximally extended Schwarzschild spacetime has two asymptotically flat regions, each region has its own black (white) hole regions.

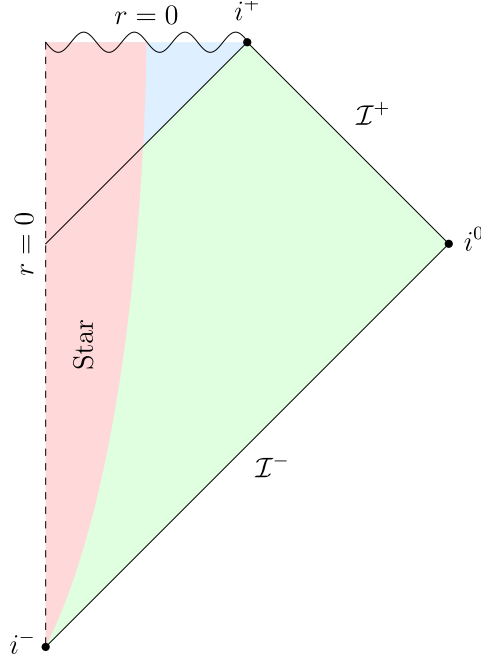


Figure 1.4: The Penrose diagram for a spherically symmetric star collapsing to a black hole.

Relative to \mathcal{I}^+ the black hole consists of the unions of I and II, and relative to \mathcal{I}^- the white hole consists of regions I and III.

1.3.2 Spherical collapse

The maximally extended Schwarzschild solution contains some fantastical features (wormholes, parallel universes, etc.). A good question to ask ourselves is which of these features arise from the strict symmetry constraints (vacuum, spherical symmetry, static), and which features survive in a more realistic scenario. To this end we display the Penrose diagram for a spherically symmetric star collapsing to a black hole in Fig. 1.4. Outside of the star, the spacetime is spherically symmetric and vacuum, and therefore — by Birkhoff's theorem — given by the Schwarzschild solution. The interior of the star, which encapsulates $r = 0$ at early times, is given by a more regular solution.

We see that the only fantastical feature that remain is the formation of an event horizon, black hole region, and singularity. All the more science fiction sounding features like white holes, worm holes, and parallel universes have disappeared due to the inclusion of a matter distribution.

1.4 The Kerr Black Hole

The Kerr solution for a rotating black hole was introduced in Sec. 3.3 of Harmark's notes. In Boyer-Lindquist coordinates it is given by

$$g_{\mu\nu} = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mar}{\Sigma} \sin^2 \theta dt d\phi \quad (1.16)$$

$$+ \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,$$

with

$$\Sigma(r, \theta) = r^2 + a^2 \cos^2 \theta, \quad \text{and} \quad (1.17)$$

$$\Delta(r) = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-). \quad (1.18)$$

This metric has a trivial coordinate singularity at $\theta = 0$ and $\theta = \pi$, coordinate singularities at $r = r_{\pm} = M \pm \sqrt{M^2 - a^2}$ corresponding to horizons, and a true curvature ring singularity at $r = 0$ and $\theta = \pi/2$.

The coordinate singularities at $r = r_{\pm}$ divide the metric in three disjoint patches ($r_+ < r < \infty$, $r_- < r < r_+$, and $-\infty < r < r_-$) that, a priori, are unrelated. The different patches can be related to each other using a similar procedure as in Schwarzschild using advanced and retarded time coordinates. We will not discuss the details here², instead we simply present the resulting Penrose diagram in Fig. 1.5.

From region I we can again traverse into a black hole by crossing $r = r_+$ into region II. In region II the r direction is timelike and we can only proceed to smaller values of r until we cross $r = r_-$ into region V (or V'). In region the r -direction is again spacelike and worldlines can both proceed to smaller r or (as will be the case for almost all geodesics) back outward to larger r until we cross $r = r_-$ again and enter region III'. Here r is again timelike and we can only proceed further outward until we cross $r = r_+$ into region I', a new distinct copy of region I (or its parallel universe region IV'). From here we can continue ad infinitum to find an infinity stack of new asymptotically flat regions.

In region V we could have also proceeded to $r = 0$. Only for $\theta = \pi/2$ is this a singularity. For all other values of θ we are free to cross into the $r < 0$ region. This region is again asymptotically flat but now in the $r \rightarrow -\infty$ limit. However, in this region there is no horizon to shield future null infinity from the singularity at $r = 0$, the singularity is *naked*. To make matters worse, this region also has closed timelike curves, breaking causality.

²For the full details see [1]

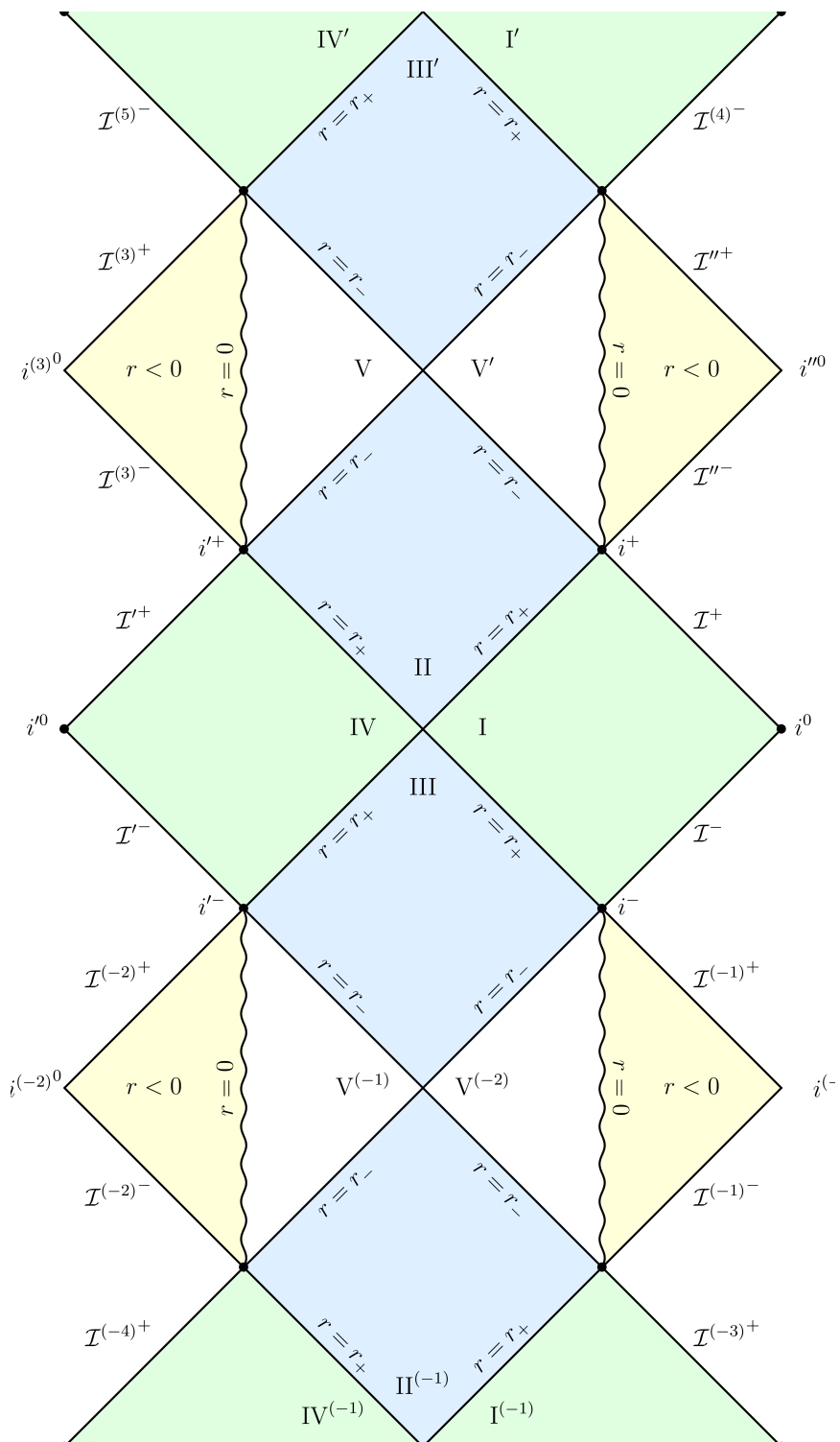


Figure 1.5: The Penrose diagram for a maximally extended Kerr black hole.

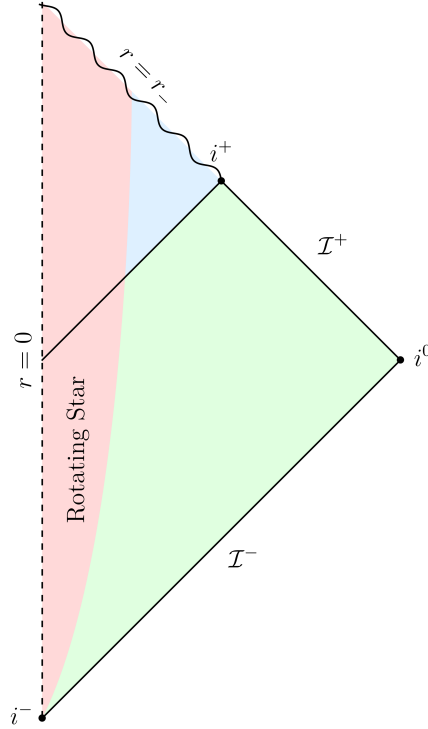


Figure 1.6: A sketch of the Penrose diagram for a rotating star collapsing to a Kerr black hole.

1.4.1 Collapse of a rotating black hole

Maximally extended Kerr is even more fantastical than its non-rotating counter part. It is therefore a good idea to regain our grounding and look at a more realistic collapse scenario. Unfortunately, there is no equivalent of Birkhoff's theorem for rotating geometries. Consequently, we are not guaranteed that the spacetime outside of the star is exactly Kerr. Instead we have to make do with the much weaker result from the no hair theorem that after the collapse the region outside the horizon has to settle down to a member of the Kerr(-Newman) family.

Most of the really fantastical features of maximally extended Kerr manifest themselves after crossing the $r = r_-$ boundary. How much should we really trust the solution after this point. The first thing to note is that this boundary is a *Cauchy horizon*, meaning that if we try to construct the solution for initial data on some timelike slice this is the absolute limit we can evolve or solutions before singularities start to appear somewhere in our

domain of dependence. We should be very wary of the analytically extended version of the solution beyond this point, because it cannot be obtained through a natural evolution. Second, we note that if we take an event on the Cauchy horizon between region II and V' its causal history $J^-(p)$ will actually contain all of region I (for some value of θ). An observer crossing this line is therefore confronted with an infinity amount of history in a finite amount of time. Consequently, even the tiniest perturbation of the spacetime is likely to be blue shifted to something singular. It is therefore conjectured that this Cauchy surface will become singular for generic perturbations away from Kerr.

Based on these conjectures we sketch what the Penrose diagram for a rotating collapsing star likely looks like in Fig. 1.6. The final result looks remarkably like the Schwarzschild case, avoiding the most fantastical behaviour of the maximally extended Kerr.

Chapter 2

Geodesics in Kerr Spacetime

2.1 The geodesic equation

Literature: Harmark, Sec. 1.3.3 and 1.4.6; Carroll Sec. 3.3 and 3.4

In the first GR course you saw the geodesic equation

$$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = \frac{dx^\alpha}{d\lambda} \nabla_\alpha \frac{dx^\mu}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0. \quad (2.1)$$

Solving this equation can be made easier by first identifying constants of motion, the existence of which is intertwined with the notion of spacetime symmetries.

2.2 Symmetries and Killing vectors

Literature: Carroll Sec. 3.8 and Appendix B, Wald Appendix C

A symmetry of a spacetime M is a automorphism $\phi : M \rightarrow M$ that leaves the spacetime “invariant”. Infinitesimal automorphisms are given by vector fields in the following way. Suppose we have a (smooth) vector field V^μ , we can construct a family of automorphisms $\phi_{V^\mu}^t$ indexed by a variable $t \in \mathbb{R}$ by mapping each event $p \in M$ to a new event p' by following the integral curves of V^μ for a time t . (I.e. we solve the differential equation $\frac{dx^\mu}{dt} = V^\mu$ with initial condition $x(0) = p$, and set $p' = x(t)$.)

We can try to ask ourselves the question how does a $T_{\mu_1 \dots \mu_n}$ change along the integral lines of V^μ . Naively one may try to write the down the derivative

$$\lim_{t \rightarrow 0} \frac{T_{\mu_1 \dots \mu_n}(p') - T_{\mu_1 \dots \mu_n}(p)}{t}.$$

However, such an expression does not make any mathematical sense. The tensors $T_{\mu_1 \dots \mu_n}(p')$ and $T_{\mu_1 \dots \mu_n}(p)$ belong to (the tensor product of) the

(co)tangent space at different points in M . Consequently, we cannot add (or subtract) them. To get another object that lives at p , we can consider the induced by $\phi_{V^\mu}^t$ on the tensor bundles (the push forward), or more specifically its inverse the pull back $(\phi_{V^\nu}^t)^*$, which in terms of components is given by

$$((\phi_{V^\nu}^t)^* T)_{\mu_1 \dots \mu_n}(p) = \frac{\partial p'^{\alpha_1}}{\partial p^{\mu_1}} \cdots \frac{\partial p'^{\alpha_n}}{\partial p^{\mu_n}} T_{\alpha_1 \dots \alpha_n}(p').$$

This allows us to define a derivative that is defined covariantly and expresses how much a tensor field changes in the direction of a vector field V^μ .

Definition 2. Given a vector field V^μ and a tensor field $T_{\mu_1 \dots \mu_n}$ we can define the *Lie¹ derivative* of $T_{\mu_1 \dots \mu_n}$ w.r.t. V^μ at an event p as follows,

$$\mathcal{L}_{V^\nu} T_{\mu_1 \dots \mu_n}(p) = \lim_{t \rightarrow 0} \frac{((\phi_{V^\nu}^t)^* T_{\mu_1 \dots \mu_n})(p) - T_{\mu_1 \dots \mu_n}(p)}{t}$$

Note that this notion of derivative does not require the existence of a metric. This makes it suitable to explore the symmetries of the metric itself. In particular, if we calculate the Lie derivative of a metric tensor $g_{\mu\nu}$ we find

$$\begin{aligned} \mathcal{L}_{K^\lambda} g_{\mu\nu} &= \lim_{t \rightarrow 0} \frac{((\phi_{V^\nu}^t)^* g_{\mu\nu})(p) - g_{\mu\nu}(p)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{\partial p'^{\alpha}}{\partial p^{\mu}} \frac{\partial p'^{\beta}}{\partial p^{\nu}} g_{\alpha\beta}(p') - g_{\mu\nu}(p)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(\delta_\mu^\alpha + t \partial_\mu V^\alpha)(\delta_\nu^\beta + t \partial_\nu V^\beta)(g_{\alpha\beta}(p) + t V^\gamma \partial_\gamma g_{\alpha\beta}(p)) - g_{\mu\nu}(p)}{t} \\ &= g_{\alpha\nu} \partial_\mu V^\alpha + g_{\mu\beta} \partial_\nu V^\beta + V^\gamma \partial_\gamma g_{\mu\nu} \\ &= \partial_\mu V_\nu - V^\alpha \partial_\mu g_{\alpha\nu} + \partial_\nu V_\mu - V^\beta \partial_\nu g_{\mu\beta} + V^\gamma \partial_\gamma g_{\mu\nu} \\ &= \partial_\mu V_\nu + \partial_\nu V_\mu - V^\alpha (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) \\ &= \partial_\mu V_\nu + \partial_\nu V_\mu - 2V_\alpha \Gamma_{\mu\nu}^\alpha \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu = 2\nabla_{(\mu} V_{\nu)} \end{aligned}$$

We are now ready to introduce the notion of a symmetry of spacetime $(M, g_{\mu\nu})$.

Definition 3. Let K^μ be a vector field on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$. K^μ is called a *Killing² vector (field)* if equivalently:

¹Pronounced “Lee” after 19th century Norwegian mathematician Sophus Lie.

²After the 19th century German mathematician, Wilhelm Killing.

1. $\mathcal{L}_{K^\lambda} g_{\mu\nu} = 0$
2. $\nabla_{(\mu} K_{\nu)} \equiv \frac{1}{2}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = 0$

Killing vectors encode the symmetries of a spacetime geometry. Sometimes coordinates make it easy to find Killing vectors, as described by the following lemma.

Lemma 1. *If the components of a metric $g_{\mu\nu}$ in some particular coordinates do not depend on the coordinate k , then $\left(\frac{\partial}{\partial k}\right)^\mu$ is a Killing vector field.*

Proof. Left as an exercise to the reader. \square

This leads to the main result that will help us solve the geodesic equation in Kerr.

Lemma 2. *Let $x^\mu(\lambda)$ be a geodesic on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$, and let K^μ be a Killing vector field, then the quantity $\mathcal{K} = K_\alpha \frac{dx^\alpha}{d\lambda}$ is conserved along the geodesic $x^\mu(\lambda)$.*

Proof.

$$\frac{d\mathcal{K}}{d\lambda} = \frac{dx^\alpha}{d\lambda} \nabla_\alpha \mathcal{K} \quad (2.2)$$

$$= \frac{dx^\alpha}{d\lambda} \nabla_\alpha \left(K_\beta \frac{dx^\beta}{d\lambda} \right) \quad (2.3)$$

$$= \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \nabla_\alpha K_\beta + K_\beta \frac{dx^\alpha}{d\lambda} \nabla_\alpha \frac{dx^\beta}{d\lambda} \quad (2.4)$$

$$= \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \nabla_{(\alpha} K_{\beta)} + 0 \quad (2.5)$$

$$= 0. \quad (2.6)$$

\square

The notion of a Killing vector field has a generalization to tensors for hire rank that satisfy similar properties.

Definition 4. Let $K_{\mu\nu}$ be a symmetric tensor field on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$. $K_{\mu\nu}$ is called a *Killing tensor* (field) if

$$\nabla_{(\lambda} K_{\mu\nu)} = 0.$$

Unlike Killing vectors, Killing tensors do not have an interpretation in terms of spacetime symmetries. Nonetheless, they still lead to constants of motion.

Lemma 3. *Let $x^\mu(\lambda)$ be a geodesic on a (pseudo)-Riemannian manifold $(M, g_{\mu\nu})$, and let $K^{\mu\nu}$ be a Killing tensor field, then the quantity $\mathcal{K} = K^{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$ is conserved along the geodesic $x^\mu(\lambda)$.*

Proof. Left as an exercise to the reader. □

Bibliography

- [1] Robert H. Boyer and Richard W. Lindquist. Maximal analytic extension of the Kerr metric. *J. Math. Phys.*, 8:265, 1967.