Chapter 3

Black Hole Perturbation Theory

3.1 Perturbation theory in GR

Globally speaking, when applying perturbation theory to general relativity we want to make the statement that some given spacetime $(M, g_{\mu\nu})$ is equal to some background spacetime $(\bar{M}, \bar{g}_{\mu\nu})$ plus something "small". However, the equation

$$(M, g_{\mu\nu}) = (\bar{M}, \bar{g}_{\mu\nu}) + \text{"something small"}$$

doesn't really make much mathematical sense. To make sense of such a comparison we will need a map $\phi : \bar{M} \to M$ that identifies the points of $\bar{M}$ with those of M. This makes it possible to compare any tensors (such as the metric) that "live" on M with tensors that live in $\bar{M}$ by considering the pull back of the object. E.g. we can consider the difference between the pull-back of $g_{\mu\nu}$ and compare it with $\bar{g}_{\mu\nu}$

$$\delta_\phi g_{\mu\nu} = \phi^* g_{\mu\nu} - \bar{g}_{\mu\nu},$$

and ask ourselves whether this is small. However, we should now wonder how this comparison depends on the choice of ϕ. What would change if we had chosen a different map $\varphi : \bar{M} \to M$ identify the points of $\bar{M}$ and M? To answer this question note that if ϕ and φ are both diffeomorphisms than there exists a unique automorphism $\psi : \bar{M} \to \bar{M}$ such that $\varphi = \phi \circ \psi$, which is generated by some vector field ξ^μ.

To compare $\delta_\phi g_{\mu\nu}$ and $\delta_\varphi g_{\mu\nu}$ we first note that starting from the same image point $p \in m$, the differences $\delta_\varphi g_{\mu\nu}$ and $\delta_\phi g_{\mu\nu}$ "live" at difference points

$\bar{p}, \bar{p}' \in \bar{M}$ satisfying $\bar{p}' = \psi(\bar{p})$. So to compare the two we need to pullback $\delta_\phi g_{\mu\nu}$ to $\bar{p}$ using ψ^*. In the limit that ξ^μ is small we get

$$\begin{aligned}
\delta_\varphi g_{\mu\nu} - \psi^* \delta_\phi g_{\mu\nu} &= \varphi^* g_{\mu\nu} - \bar{g}_{\mu\nu} - \psi^*(\phi^* g_{\mu\nu} - \bar{g}_{\mu\nu}) \\
&= (\phi \circ \psi)^* g_{\mu\nu} - \bar{g}_{\mu\nu} - \psi^*(\phi^* g_{\mu\nu} - \bar{g}_{\mu\nu}) \\
&= \psi^* \phi^* g_{\mu\nu} - \bar{g}_{\mu\nu} - \psi^* \phi^* g_{\mu\nu} + \psi^* \bar{g}_{\mu\nu} \\
&= \psi^* \bar{g}_{\mu\nu} - \bar{g}_{\mu\nu} \\
&= \mathcal{L}_{\xi^\mu} \bar{g}_{\mu\nu} + \mathcal{O}\left(\xi^2\right) \\
&= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \mathcal{O}\left(\xi^2\right)
\end{aligned}$$

So, fundamentally the comparison of $g_{\mu\nu}$ with $\bar{g}_{\mu\nu}$ is ambiguous up to a Lie derivative of $\bar{g}_{\mu\nu}$. In to context of perturbation theory this is known as a **gauge** ambiguity. More generally, when comparing any tensor $T_{\mu_1 \cdots \mu_n}$ with a background tensor $\bar{T}_{\mu_1 \cdots \mu_n}$ that comparison has a gauge ambiguity $\mathcal{L}_{\xi^\mu} \bar{T}_{\mu_1 \cdots \mu_n}$.

Lemma 4. *If a background tensor $\bar{T}_{\mu_1 \cdots \mu_n}$ is zero, then its perturbations are free from gauge ambiguities. A quantity free from gauge ambiguities is called **gauge invariant**.*

Note that this gauge freedom in perturbation theory is logically distinct from the general coordinate freedom of general relativity. However, in practice the two are closely related, since we tend to construct the identification map ϕ by choosing similar coordinates on $\bar{M}$ and M and identifying the points with the same coordinate values. Consequently, the ambiguity ψ in this identification map simply becomes the ambiguity of choosing the coordinates.

Moving forward we will drop the fancy notation involving identification maps, and work with the understanding that a suitably identification map has been chosen and used to pull back all tensors to the background manifold $\bar{M}$. Consequently, all tensors will be tensors on the manifold $\bar{M}$ and all raising and lowering operations are understood to use the background metric $\bar{g}_{\mu\nu}$. Moreover, any perturbations are understood as being ambiguous up to gauge transformations defined by gauge vectors ξ^μ

Using these conventions we are now ready to write that

$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu} + \mathcal{O}(\epsilon^2)$

where $h_{\mu\nu}$ is a symmetric rank-2 tensor living on $\bar{M}$ known as the (first-order) **metric perturbation**, and $\epsilon > 0$ is a small number that we will use to keep track of the orders in our perturbation theory. The metric perturbations are ambiguous up to transformations

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu.$$

We can try to fix this gauge freedom by imposing addition conditions on $h_{\mu\nu}$.

So, suppose now we want to know what the inverse perturbed metric $g^{\mu\nu}$ looks like. Note $g^{\mu\nu}$ is now considered a tensor of $\bar{M}$, but cannot be obtained by simply raising the indices on $g_{\mu\nu}$. We should therefore be using a different symbol. Lets write

$$g^{\mu\nu} = A^{\mu\nu} + \epsilon B^{\mu\nu} + \mathcal{O}(\epsilon^2).$$

The inverse metric will still have to satisfy

$$\delta_\mu{}^\nu = g_{\mu\alpha} g^{\alpha\nu} = (\bar{g}_{\mu\alpha} + \epsilon h_{\mu\alpha})(A^{\alpha\nu} + \epsilon B^{\alpha\nu}) = \bar{g}_{\mu\alpha} A^{\alpha\nu} + \epsilon(h_{\mu\alpha} A^{\alpha\nu} + \bar{g}_{\mu\alpha} B^{\alpha\nu}) + \mathcal{O}(\epsilon^2)$$

This equation will have to be satisfied order-by-order in ϵ. Giving us

$$\begin{aligned} \delta_\mu{}^\nu &= \bar{g}_{\mu\alpha} A^{\alpha\nu}, \quad \text{and} \\ 0 &= h_{\mu\alpha} A^{\alpha\nu} + \bar{g}_{\mu\alpha} B^{\alpha\nu}, \end{aligned}$$

which we can solve to find

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \epsilon h^{\mu\nu} + \mathcal{O}(\epsilon^2)$$

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3.2 Linearized Einstein Equation

Literature: Wald Sec. 7.5

We now want to make our way to the perturbed form of the Einstein equation. For this we first observe that both ∇_μ and $\bar{\nabla}_\mu$ are covariant derivatives, but compatible with different metrics. Consequently, we have that their difference

$$\left(\nabla_\mu - \bar{\nabla}_\mu\right) V^\nu = (\Gamma^\nu_{\mu\alpha} - \bar{\Gamma}^\nu_{\mu\alpha}) V^\alpha = \epsilon C^\nu{}_{\mu\alpha} V^\alpha$$

is represent by a rank 3-tensor $C^\nu{}_{\mu\alpha}$.

Equivalently, we could express this as

$$\nabla_\mu V^\nu = \bar{\nabla}_\mu V^\nu + \epsilon C^\nu{}_{\mu\alpha} V^\alpha.$$

This is similar as the usual expression for the covariant derivative, but with the partial derivative replaced by the background covariant derivative and the Christoffel symbols $\Gamma^\nu_{\mu\alpha}$ replaced by the tensor $C^\nu{}_{\mu\alpha}$.

Compatibility, of ∇_μ with the metric $g_{\mu\nu}$, yields an expression for $C^\nu{}_{\mu\alpha}$ in the same way we obtained our expression for the Christoffel symbols of the Levi-Civita connection (see e.g. Theorem 3.1.1 in Wald)

$$\begin{aligned}\epsilon C^\lambda{}_{\mu\nu} &= \frac{1}{2} g^{\lambda\alpha}\left(\bar\nabla_\mu g_{\nu\alpha} + \bar\nabla_\nu g_{\alpha\mu} - \bar\nabla_\alpha g_{\mu\nu}\right)\\ &= \epsilon\frac{1}{2}\bar g^{\lambda\alpha}\left(\bar\nabla_\mu h_{\nu\alpha} + \bar\nabla_\nu h_{\alpha\mu} - \bar\nabla_\alpha h_{\mu\nu}\right) + \mathcal{O}(\epsilon^2).\end{aligned}$$

The Riemann tensor will now be given

$$R^\lambda{}_{\sigma\mu\nu} = \bar R^\lambda{}_{\sigma\mu\nu} + \epsilon\left(\bar\nabla_\mu C^\lambda{}_{\nu\sigma} - \bar\nabla_\nu C^\lambda{}_{\mu\sigma}\right) + \epsilon^2\left(C^\lambda{}_{\mu\alpha}C^\alpha{}_{\nu\sigma} - C^\lambda{}_{\nu\alpha}C^\alpha{}_{\mu\sigma}\right).$$

The Einstein equation can be written

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

where $T = g^{\alpha\beta}T_{\alpha\beta}$ the trace of the energy-momentum tensor. Lets assume we are expanding around a background metric $\bar g_{\mu\nu}$ that is itself a solutions to the vacuum Einstein equation (e.g. Kerr), such that $\bar R_{\mu\nu} = 0$. We then find

$$\begin{aligned}R_{\mu\nu} &= R^\alpha{}_{\mu\alpha\nu}\\ &= \epsilon\left(\bar\nabla_\alpha C^\alpha{}_{\mu\nu} - \bar\nabla_\nu C^\alpha{}_{\alpha\mu}\right) + \mathcal{O}(\epsilon^2)\\ &= \frac{\epsilon}{2}\bar g^{\alpha\beta}\big(\bar\nabla_\alpha\left(\bar\nabla_\mu h_{\nu\beta} + \bar\nabla_\nu h_{\beta\mu} - \bar\nabla_\beta h_{\mu\nu}\right)\\ &\qquad - \bar\nabla_\nu\left(\bar\nabla_\alpha h_{\mu\beta} + \bar\nabla_\mu h_{\beta\alpha} - \bar\nabla_\beta h_{\alpha\mu}\right)\big) + \mathcal{O}(\epsilon^2)\\ &= \frac{\epsilon}{2}\left(-\bar\Box h_{\mu\nu} - \bar\nabla_\nu\bar\nabla_\mu h + 2\bar\nabla^\alpha\bar\nabla_{(\mu}h_{\nu)\alpha}\right) + \mathcal{O}(\epsilon^2)\\ &= \frac{\epsilon}{2}\left(-\bar\Box h_{\mu\nu} - \bar\nabla_\nu\bar\nabla_\mu h + 2\bar\nabla_{(\mu}\bar\nabla^\alpha h_{\nu)\alpha} + 2\bar R^\alpha{}_{\mu\nu}{}^\beta h_{\alpha\beta}\right) + \mathcal{O}(\epsilon^2),\end{aligned}$$

where $\bar\Box = \bar\nabla^\alpha\bar\nabla_\alpha$, and $h = h_{\alpha\beta}\bar g^{\alpha\beta}$ is the trace of the metric perturbation, and in the last line we used that $\bar\nabla_\mu\bar\nabla_\nu h_{\alpha\beta} = \bar\nabla_\nu\bar\nabla_\mu h_{\alpha\beta} + \bar R_{\mu\nu\beta}{}^\gamma h_{\alpha\gamma} + \bar R_{\mu\nu\alpha}{}^\gamma h_{\gamma\beta}$. Moreover, since the background was vacuum the energy momentum tensor $T_{\mu\nu}$ must also be order ϵ. Consequently, the order ϵ of the Einstein equation becomes,

$$\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\bar\nabla_{(\mu}\bar\nabla_{\nu)}h - \bar\nabla_{(\mu}\bar\nabla^\alpha h_{\nu)\alpha} - \bar R^\alpha{}_{\mu\nu}{}^\beta h_{\alpha\beta} = -8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \quad (3.1)$$

We can further simplify this by using the gauge freedom in the metric perturbation. By choosing the Lorenz[1] gauge condition

$$\bar\nabla^\alpha h_{\mu\alpha} = \frac{1}{2}\bar\nabla_\mu h, \quad (3.2)$$

[1]After the 19th century Danish physicist Ludvig Lorenz, not the 19th century Dutch physicist Hendrik Lorentz.

the linearized Einstein equation becomes

$$\frac{1}{2}\Box h_{\mu\nu} - \bar{R}^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta} = -T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}T. \tag{3.3}$$

In this gauge, the linearized Einstein equation explicitly takes the form of a wave equation. However, we should not be deceived by the apparent simplicity of this equation, it still consists of 10 coupled second order partial differential equations, and solving them is generally hard. Sometimes, the symmetries of the background help. For example, for a Schwarzschild background the equations can be solved through separation of variables, allowing the solution to be found mode-by-mode. For each mode, it then reduces to a set of 10 coupled ordinary differential equations. But even in Kerr, there is no known way the separate the variables.

It is therefore worth to consider a different approach.

3.3 The Penrose Wave Equation

Instead of a wave equation for the metric we will pursue a wave equation for the curvature instead. The Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ in 4 dimensions has 20 degrees of freedom. Ten of these are encoded by the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, which through the Einstein equation are algebraicly determined by the energy-momentum content. The remaining 10 degrees of freedom are captured by the trace-free part of the Riemann tensor, the so-called Weyl tensor

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - g_{\alpha[\mu}R_{\nu]\beta} - g_{\beta[\mu}R_{\nu]\alpha} + \frac{R}{3}g_{\alpha[\mu}g_{\nu]\beta}. \tag{3.4}$$

It is defined in such a way that any contraction of its indices produces zero, while retaining the symmetries of $R_{\mu\nu\alpha\beta}$,

$$C_{\mu\nu\alpha\beta} = C_{[\mu\nu][\alpha\beta]}, \tag{3.5}$$

$$C_{\mu\nu\alpha\beta} = C_{\alpha\beta\mu\nu}. \tag{3.6}$$

$$C_{\mu\nu\alpha\beta} = C_{\mu[\nu\alpha\beta]}. \tag{3.7}$$

Moreover, when $R_{\mu\nu} = 0$, we get that $C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}$.

The Weyl tensor therefore must capture all the propagating curvature degrees of freedom, and it would be great if we could write a wave equation for it. To achieve this we first consider the Bianchi identity

$$\nabla_\gamma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} = 0. \tag{3.8}$$

Taking the divergence of this identity yields

$$\Box R_{\alpha\beta\mu\nu} + \nabla^\gamma\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla^\gamma\nabla_\beta R_{\gamma\alpha\mu\nu} = 0. \tag{3.9}$$

The box operator already gives this the appearance of a wave equation. We can commute the two covariant derivatives using the Ricci identity

$$\begin{aligned}\nabla_\mu\nabla_\nu R_{\alpha\beta\gamma\delta} - \nabla_\nu\nabla_\mu R_{\alpha\beta\gamma\delta} = \\ R_{\mu\nu\alpha}{}^{\lambda}R_{\lambda\beta\gamma\delta} + R_{\mu\nu\beta}{}^{\lambda}R_{\alpha\lambda\gamma\delta} + R_{\mu\nu\gamma}{}^{\lambda}R_{\alpha\beta\lambda\delta} + R_{\mu\nu\delta}{}^{\lambda}R_{\alpha\beta\gamma\lambda}\end{aligned} \tag{3.10}$$

This yields

$$\begin{aligned}\Box R_{\alpha\beta\mu\nu} + 2R^{\gamma}{}_{\alpha\mu}{}^{\lambda}R_{\gamma\beta\nu\lambda} - 2R^{\gamma}{}_{\alpha\nu}{}^{\lambda}R_{\gamma\beta\mu\lambda} + 2R^{\gamma}{}_{\alpha\beta}{}^{\lambda}R_{\gamma\lambda\mu\nu} \\ - R_{\alpha}{}^{\lambda}R_{\lambda\beta\mu\nu} + R_{\beta}{}^{\lambda}R_{\lambda\alpha\mu\nu} - \nabla_\alpha\nabla^\gamma R_{\gamma\beta\mu\nu} + \nabla_\beta\nabla^\gamma R_{\gamma\alpha\mu\nu} = 0.\end{aligned} \tag{3.11}$$

Next by contracting the Bianchi identity itself we find the following identity for the divergence of the Riemann tensor

$$\nabla^\gamma R_{\gamma\nu\alpha\beta} = \nabla_\alpha R_{\beta\nu} - \nabla_\beta R_{\alpha\nu} = 2\nabla_{[\alpha}R_{\beta]\nu}. \tag{3.12}$$

Using this last identity, we can eliminate the divergences of the Riemann curvature in favour of the Ricci tensor to obtain the **Penrose wave equation**

$$\begin{aligned}\Box R_{\alpha\beta\mu\nu} + 2R^{\gamma}{}_{\alpha\mu}{}^{\lambda}R_{\gamma\beta\nu\lambda} - 2R^{\gamma}{}_{\alpha\nu}{}^{\lambda}R_{\gamma\beta\mu\lambda} + 2R^{\gamma}{}_{\alpha\beta}{}^{\lambda}R_{\gamma\lambda\mu\nu} \\ - R_{\alpha}{}^{\lambda}R_{\lambda\beta\mu\nu} + R_{\beta}{}^{\lambda}R_{\lambda\alpha\mu\nu} - 2\nabla_\alpha\nabla_{[\mu}R_{\nu]\beta} + 2\nabla_\beta\nabla_{[\mu}R_{\nu]\alpha} = 0.\end{aligned} \tag{3.13}$$

This equation is build from identities that hold for any pseudo-Riemannian manifold, and therefore this equation is satisfied for the curvature tensor obtained from any metric. If in addition we impose that the metric satisfies the vacuum Einstein equation $R_{\mu\nu} = 0$, all the terms in the second line vanish. So, for vacuum solutions this turns into a wave equation for the Weyl tensor.

$$\Box C_{\alpha\beta\mu\nu} + 2C^{\gamma}{}_{\alpha\mu}{}^{\lambda}C_{\gamma\beta\nu\lambda} - 2C^{\gamma}{}_{\alpha\nu}{}^{\lambda}C_{\gamma\beta\mu\lambda} + 2C^{\gamma}{}_{\alpha\beta}{}^{\lambda}C_{\gamma\lambda\mu\nu} = 0. \tag{3.14}$$

More generally, we could replace all occurrences of the Ricci tensor with the trace reversed energy-momentum tensor using the Einstein equation to find the wave equation for the curvature coupled to matter.

Let us now consider perturbations around a vacuum solution of the Einstein equation with an energy-momentum source that is order ϵ. Writting $C_{\alpha\beta\mu\nu} = \bar{C}_{\alpha\beta\mu\nu} + \epsilon\, \delta C_{\alpha\beta\mu\nu} + \mathcal{O}(\epsilon^2)$, the background Weyl curvature will satisfy Eq. (3.14).

To obtain the linear in ϵ part of the equation (3.13) we make the following observations

- The linear in ϵ part of the second line in Eq. (3.13) is constructed entirely from $\bar{C}_{\alpha\beta\mu\nu}$, $T_{\mu\nu}$, the background metric $\bar{g}_{\mu\nu}$, and the background covariant derivative $\bar{\nabla}_\mu$.

- The linear in ϵ part of the Riemann curvature tensor $\delta R_{\alpha\beta\mu\nu}$, consists of $\delta C_{\alpha\beta\mu\nu}$ plus terms constructed from $T_{\mu\nu}$ and the background metric $\bar{g}_{\mu\nu}$.

- We can write the action of the box operator on a generic 4-tensor $T_{\alpha\beta\mu\nu}$ as $\Box T_{\alpha\beta\mu\nu} = \bar{\Box} T_{\alpha\beta\mu\nu} + \epsilon B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'} T_{\alpha'\beta'\mu'\nu'}$ where $B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'}$ is formed as a linear operator on $h_{\mu\nu}$ constructed entirely from $\bar{g}_{\mu\nu}$ and $\bar{\nabla}_\mu$.

- Combining the last two observations we note that we can write the linear in ϵ part of $\Box R_{\alpha\beta\mu\nu}$ as $\bar{\Box}\delta C_{\alpha\beta\mu\nu} + B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'} \bar{C}_{\alpha'\beta'\mu'\nu'}$ plus terms depending only on

So, if in the linear in ϵ part of the equation (3.13), we take all the terms that depend only on $T_{\mu\nu}$, the background metric $\bar{g}_{\mu\nu}$, and the background covariant derivative $\bar{\nabla}_\mu$, move them to the right hand side and collectively call them $S[T]_{\alpha\beta\mu\nu}$, we get

$$\begin{aligned}
\bar{\Box}\delta C_{\alpha\beta\mu\nu} + B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'} \bar{C}_{\alpha'\beta'\mu'\nu'} + 2\bar{C}^{\gamma}{}_{\beta\nu}{}^{\lambda}\delta C_{\gamma\alpha\mu\lambda} + 2\bar{C}^{\gamma}{}_{\alpha\mu}{}^{\lambda}\delta C_{\gamma\beta\nu\lambda} \\
-2\bar{C}^{\gamma}{}_{\beta\mu}{}^{\lambda}\delta C_{\gamma\alpha\nu\lambda} - 2\bar{C}^{\gamma}{}_{\alpha\nu}{}^{\lambda}\delta C_{\gamma\beta\mu\lambda} + 2\bar{C}^{\gamma}{}_{\mu\nu}{}^{\lambda}\delta C_{\gamma\lambda\alpha\beta} + 2\bar{C}^{\gamma}{}_{\alpha\beta}{}^{\lambda}\delta C_{\gamma\lambda\mu\nu} \\
= S[T]_{\alpha\beta\mu\nu}.
\end{aligned} \tag{3.15}$$

This does not seem like much progress. While it looks like a wave equation for $\delta C_{\alpha\beta\mu\nu}$ it mixes with the equally unknown metric perturbation $h_{\mu\nu}$ in a non-trivial way. To make this useful in someway, we are going to need a miracle.

3.4 The Weyl scalars

The Weyl curvature tensor with its many components seems a highly inefficient way of capturing 10 degrees of freedom. The Newman-Penrose formalism provides a more compact way of describing these degrees of freedom. It starts by choosing a tetrad of null vectors $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ with properties that:

- Each vector is null, i.e. $l^\alpha l_\alpha = n^\alpha n_\alpha = m^\alpha m_\alpha = \bar{m}^\alpha \bar{m}_\alpha = 0$

- $l^\alpha n_\alpha = -1$ and $m^\alpha \bar{m}_\alpha = 1$

- All other legs are orthogonal.
- m^μ and $\bar{m}^\mu$ are complex valued and $\bar{m}^\mu$ is the complex conjugate of m^μ.

These conditions imply that

$$g_{\mu\nu} = -2l_{(\mu}n_{\nu)} + 2m_{(\mu}\bar{m}_{\nu)}.$$

This tetrad can used to encode the 10 degrees of freedom of the Weyl curvature in 5 complex scalar fields

$$\psi_0 = C_{\alpha\beta\gamma\delta}\, \ell^\alpha m^\beta \ell^\gamma m^\delta, \tag{3.16a}$$
$$\psi_1 = C_{\alpha\beta\gamma\delta}\, \ell^\alpha m^\beta \ell^\gamma n^\delta, \tag{3.16b}$$
$$\psi_2 = C_{\alpha\beta\gamma\delta}\, \ell^\alpha m^\beta \bar{m}^\gamma n^\delta, \tag{3.16c}$$
$$\psi_3 = C_{\alpha\beta\gamma\delta}\, \ell^\alpha n^\beta \bar{m}^\gamma n^\delta, \tag{3.16d}$$
$$\psi_4 = C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta, \tag{3.16e}$$

It is always possible to choose l^μ such that $\psi_0 = 0$. Such a null vector is called a **principal null vector** of the spacetime. Generically, a 4-dimensional spacetime is expected to have 4-linearly independent principal null vectors. If two principal null vectors happen to coincide than ψ_1 will vanish in addition to ψ_0, and l^μ is called a double principal null vector.

Kerr spacetime has two double principal null vectors[2]. If we choose both l^μ and n^μ to be a double principal null vector then we get that $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$.

A popular choice for the tetrad in Kerr that achieves this is the **Kinnersley tetrad**

$$\ell^\alpha = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \tag{3.17}$$
$$n^\alpha = \left(\frac{r^2 + a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma}\right), \tag{3.18}$$
$$m^\alpha = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right) \qquad \text{and} \tag{3.19}$$
$$\bar{m}^\alpha = \frac{-1}{\sqrt{2}(r - ia\cos\theta)}\left(ia\sin\theta, 0, -1, \frac{i}{\sin\theta}\right). \tag{3.20}$$

[2]In the Petrov classification of spacetimes this means that Kerr is type D.

When considering perturbed spacetimes, we need to choose a tetrad both in the background spacetime and in the perturbed spacetime. The requirement that the tetrad remains null an orthonormal fixes both tetrads up to a local Lorentz transformation. This remain freedom introduces an additional gauge freedom.

3.5 Teukolsky Equation

A miracle happens when we try to project equation (3.15) on to a tetrad basis adapted to the double principal null vectors of Kerr spacetime. When considering the ψ_0 component, first it happens that commuting the tetrad legs with the background box operator produces terms which exactly cancel the $h_{\mu\nu}$ dependent terms in the B tensor. Second, the dependence on all other components of $\delta C_{\alpha\beta\mu\nu}$ drops out, leaving a decoupled wave equation for ψ_0. A similar thing happens when trying to project on $(r - ia\cos\theta)^4\psi_4$. Introducing, the notation $\Phi_2 = \psi_0$ and $\Phi_{-2} = (r - ia\cos\theta)^4\psi_4$, using the Kinnersley tetrad the decoupled equations are given by[3],

$$\begin{aligned} -\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\Phi_s}{\partial t^2} - \frac{4Mar}{\Delta}\frac{\partial^2\Phi_s}{\partial t\partial\varphi} - \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\Phi_s}{\partial\varphi^2} \\ +\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\Phi_s}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi_s}{\partial\theta}\right) \\ +2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\Phi_s}{\partial\varphi} + 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\Phi_s}{\partial t} \\ -(s^2\cot^2\theta - s)\Phi_s = S_s[T_{\mu\nu}], \end{aligned} \tag{3.21}$$

where $S_s[T_{\mu\nu}]$ is a second order differential operator that produces the projected source term for $T_{\mu\nu}$. This it the **Teukolsky equation**. (It turns out that this same equation with $s = \pm 1$ describes solutions to the Maxwell-equations on a curved background. With $s = 0$ it describes a massless scalar field, and with $s = 1/2$ a massless Dirac field.)

The perturbed Weyl scalars ψ_0 and ψ_4 have some useful properties in Kerr spacetimes.

First of all they are gauge invariant. Since they are scalar quantities whose background values are zero they insensitive to normal gauge transformations. Moreover, infinitesimal rotations of the tetrad only mix the background values of ψ_0 and ψ_1 into the perturbed value of ψ_0 (or ψ_4 and ψ_3 into ψ_4). Since

[3] For the gory details of how to get here see [3].

both background values vanish, infinitesimal rotations of the tetrad cannot change ψ_0 (or ψ_4).

Second, it turns out that ψ_0 (or ψ_4) contain almost all there is to know about metric perturbation in Kerr spacetime, as expressed by the following theorem.

Theorem (Wald [4])**.** *Suppose $h_{\mu\nu}$ and $h'_{\mu\nu}$ are (suitably well-behaved) solutions to the Einstein equation, such that the perturbation to the Weyl scalar ψ_0 (or ψ_4) is the same for both perturbations. Then*

$$h_{\mu\nu} - h'_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)} + c_M \frac{\partial g^{\mathrm{Kerr}}}{\partial M} + c_a \frac{\partial g^{\mathrm{Kerr}}}{\partial a},$$

for some gauge vector ξ_μ and constants c_M and c_a.

So, ψ_0 "knows" everything about the vacuum metric perturbation expect the gauge (since it is itself gauge invariant) and perturbation of the mass and spin of the Kerr background (since $\psi_0 = 0$ on any Kerr background.) Furthermore, we get,

Corollary.

$$\psi_0[h] = \psi_0[h'] \Leftrightarrow \psi_4[h] = \psi_4[h'].$$

3.5.1 Separation of Variables

Note that the left-hand side of the Teukolsky equation does not contain any explicit dependence on t and ϕ. We can thus do a Fourier transform with respect to t and ϕ to get

$$\Phi_s(t, r, \theta, \phi) = \int \mathrm{d}\omega \sum_m \hat{\Phi}_{sm\omega}(r, \theta) e^{i(m\phi - \omega t)}, \tag{3.22}$$

and

$$\begin{aligned} &\Delta^{-s} \frac{\partial}{\partial r}\left(\Delta^{s+1} \frac{\partial \hat{\Phi}_{sm\omega}}{\partial r}\right) \\ &+\Big(\frac{\left(\omega(r^2+a^2) - am - is(r-M)\right)^2 + s^2(r-M)^2}{\Delta} + 2am\omega + 4is\omega r\Big)\hat{\Phi}_{sm\omega} \\ &+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \hat{\Phi}_{sm\omega}}{\partial\theta}\right) - \Big(\frac{(m + s\cos\theta)^2}{\sin^2\theta} + a^2\omega^2 \sin^2\theta + 2sa\omega\cos\theta - s\Big)\hat{\Phi}_{sm\omega} \\ &= \hat{S}_{sm\omega}[T_{\mu\nu}]. \end{aligned} \tag{3.23}$$

The left-hand side of the equation separates into part with only explicit dependence on r and a second part that depends on θ.

The polar operator appearing on the left-hand side,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{(m+s\cos\theta)^2}{\sin^2\theta}-a^2\omega^2\sin^2\theta-2sa\omega\cos\theta+s, \quad (3.24)$$

is a Sturm-Liouville operator. Consequently, it has a real eigenvalues $(-\lambda_{slm\omega}-2ma\omega)$ with $l\geq l_0\max(|s|,|m|)$ such that $0\leq\lambda_{sl_0m\omega}<\lambda_{s(l_0+1)m\omega}<\ldots$, and corresponding real-valued eigenfunctions $S_{slm\omega}(\theta)$ that satisfy

$$\int_{-1}^{1}S_{slm\omega}(\theta)S_{sl'm\omega}(\theta)\,\mathrm{d}\cos\theta=\delta_{ll'},$$

and span the space of continuous functions of the interval $[0,\pi]$.

This functions are known as spin-weighted spheroidal harmonics with spin-weight s and spheroidicity $a\omega$. In the limit $a\omega\to 0$, these reduce to $\lambda_{slm\omega}=l(l+1)-s(s+1)$, and the more familiar spin-weighted spherical harmonics $Y_{slm}(\theta,\phi)$ (when paired with the $e^{im\phi}$ from the Fourier expansion.

We can therefore further expand $\hat{\Phi}_{sm\omega}(r,\theta)$ and source $\hat{S}_{sm\omega}[T_{\mu\nu}]$ in the basis of spin-weighted spheroidal harmonics

$$\hat{\Phi}_{sm\omega}(r,\theta)=\sum_{l\geq|s|}R_{slm\omega}(r)S_{slm\omega}(\theta)e^{i(m\phi-\omega t)}, \quad (3.25)$$

and

$$\hat{S}_{sm\omega}[T_{\mu\nu}](r,\theta)=\sum_{l\geq|s|}\hat{S}_{slm\omega}[T_{\mu\nu}](r)S_{slm\omega}(\theta)e^{i(m\phi-\omega t)}. \quad (3.26)$$

Plugging this into (3.23), we find that the radial coefficients satisfy

$$\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial R_{slm\omega}}{\partial r}\right)-V_{slm\omega}R_{slm\omega}=\hat{S}_{slm\omega}[T_{\mu\nu}], \quad (3.27)$$

with

$$V_{slm\omega}=-\frac{\left(\omega(r^2+a^2)-am-is(r-M)\right)^2+s^2(r-M)^2}{\Delta}-4is\omega r+\lambda_{slm\omega}. \quad (3.28)$$

This is the **radial Teukolsky equation**.

3.5.2 Homogeneous solutions, Asymptotic Behaviour, and Boundary conditions

$$\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial R_{slm\omega}}{\partial r}\right)-V_{slm\omega}R_{slm\omega}=0, \quad (3.29)$$

To study the asymptotic behaviour of homogeneous (i.e.$\hat{S}_{slm\omega} = 0$) solutions to the radial Teukolsky equation near the horizon and near infinity, it is useful to make the follow transformation

$$\tilde{R} = \Delta^{s/2}\sqrt{r^2+a^2}R \tag{3.30}$$

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{r^2+a^2}{\Delta}, \tag{3.31}$$

$$r^* = r + \frac{Mr_+}{\sqrt{M^2-a^2}}\log\frac{r-r_+}{2M} - \frac{Mr_-}{\sqrt{M^2-a^2}}\log\frac{r-r_-}{2M}, \tag{3.32}$$

where r^* is the Kerr tortoise coordinate. This turns the homogeneous radial equation into (suppressing the subscripts $slm\omega$ for brevity).

$$\frac{\mathrm{d}^2\tilde{R}}{\mathrm{d}r^{*2}} + U\tilde{R} = 0, \tag{3.33}$$

with

$$U = \frac{\left(\omega(r^2+a^2) - am - is(r-M)\right)^2 + s^2(r-M)^2 + \Delta(4is\omega r - \lambda)}{(r^2+a^2)^2} - G^2 - \frac{\mathrm{d}G}{\mathrm{d}r^*}, \tag{3.34}$$

and

$$G = \frac{s(r-M)}{r^2+a^2} + \frac{r\Delta}{(r^2+a^2)^2}. \tag{3.35}$$

For large r (and thus $r^* \to \infty$), we get that

$$U = \omega^2 + \frac{2is\omega}{r} + \mathcal{O}(r^{-2}) \tag{3.36}$$

Consequently, we get the asymptotic solutions near infinity

$$\tilde{R} \underset{r\to\infty}{\propto} \frac{1}{r^{\pm s}}\exp\left(\pm i\omega r^*\right) \tag{3.37}$$

our translating back to our original radial field R

$$R \underset{r\to\infty}{\propto} \frac{1}{r^{2s+1}}\exp\left(i\omega r^*\right) \quad \text{or} \quad R \underset{r\to\infty}{\propto} \frac{1}{r}\exp\left(-i\omega r^*\right). \tag{3.38}$$

Near the horizon as $r \to r_+$ (and $r^* \to -\infty$), we get

$$U = \left(k - is\frac{\sqrt{M^2-a^2}}{2Mr_+}\right)^2 + \mathcal{O}\,\Delta, \tag{3.39}$$

where

$$k = \omega - m\Omega_+ \tag{3.40}$$

is the frequency of the mode shifted by the rotation frequency of the outer horizon $\Omega_+ = \frac{a}{2Mr_+}$. Consequently, the asymptotic behaviour of the solutions near the horizon is

$$\tilde{R} \underset{r\to r_+}{\propto} \exp\left(\pm i\left(k - is\frac{\sqrt{M^2-a^2}}{2Mr_+}\right)r^*\right) \tag{3.41}$$

$$= \exp\left(\pm s\frac{\sqrt{M^2-a^2}}{2Mr_+}r^*\right)\exp\left(\pm ikr^*\right) \tag{3.42}$$

$$\approx \Delta^{\pm s/2}\exp\left(\pm ikr^*\right), \tag{3.43}$$

where we used that near the horizon

$$r^* \approx \frac{Mr_+}{\sqrt{M^2-a^2}}\log\frac{r-r_+}{2M} \approx \frac{Mr_+}{\sqrt{M^2-a^2}}\log\Delta. \tag{3.44}$$

For our original field R this translates to

$$R \underset{r\to r_+}{\propto} \exp\left(ikr^*\right) \quad \text{or} \quad R \underset{r\to r_+}{\propto} \Delta^{-s}\exp\left(-ikr^*\right). \tag{3.45}$$

So, we find that a general solution of the homogeneous radial Teukolsky equation $R_{slm\omega}$ has the following asymptotic behaviour (for $\omega \neq 0$)

$$R_{slm\omega} = \begin{cases} \frac{A^{\mathcal{I}+}_{slm\omega}}{r^{2s+1}}\exp\left(i\omega r^*\right) + \frac{A^{\mathcal{I}-}_{slm\omega}}{r}\exp\left(-i\omega r^*\right) & \text{as } r\to\infty \\ A^{\mathcal{H}-}_{slm\omega}\exp\left(ikr^*\right) + A^{\mathcal{H}+}_{slm\omega}\Delta^{-s}\exp\left(-ikr^*\right) & \text{as } r\to r_+ \end{cases} \tag{3.46}$$

The A coefficients have the following interpretations:

- $A^{\mathcal{I}+}_{slm\omega}$ is (related to) the amplitude of GWs in the solution $R_{slm\omega}$ traveling **out** to future null infinity $\mathcal{I}^+$.
- $A^{\mathcal{I}-}_{slm\omega}$ is (related to) the amplitude of GWs in the solution $R_{slm\omega}$ coming **in** from past null infinity $\mathcal{I}^-$.
- $A^{\mathcal{H}+}_{slm\omega}$ is (related to) the amplitude of GWs in the solution $R_{slm\omega}$ falling **down** the future event horizon $\mathcal{H}^+$.
- $A^{\mathcal{H}-}_{slm\omega}$ is (related to) the amplitude of GWs in the solution $R_{slm\omega}$ coming **up** from the past event horizon $\mathcal{H}^-$.

The homogeneous radial Teukolsky equation is a second-order ordinary differential equation. Hence it has a two dimensional space of solutions. We can use our knowledge about the asymptotic behaviour of a general solution at infinity and the horizon to impose boundary conditions to identify the elements of a basis for this solution space. The most popular choice involves putting boundary conditions on past null infinity $\mathcal{I}^-$ and the past horizon $\mathcal{H}^-$. The two basis elements in this basis are:

- The **ingoing** solution $R^{\rm in}_{slm\omega}$ is the homogeneous solution that only has waves coming in from past null infinity $\mathcal{I}^-$, but no waves coming up from the past horizon $\mathcal{H}^-$. It is defined by setting $A^{\mathcal{H}^-}_{slm\omega} = 0$ and normalized by $A^{\mathcal{H}^+}_{slm\omega} = 1$.

- The **upgoing** solution $R^{\rm up}_{slm\omega}$ is the homogeneous solution that has waves coming up from the past event horizon $\mathcal{H}^-$, but no waves coming in from past null infinity $\mathcal{I}^-$. It is defined by setting $A^{\mathcal{I}^-}_{slm\omega} = 0$ and normalized by $A^{\mathcal{H}^+}_{slm\omega} = 1$.

Together these two span the solutions for the homogeneous radial Teukolsky equation for any real non-zero frequency ω.

In addition people sometimes use the "time-reverse" solutions for their basis:

- The **outgoing** solution $R^{\rm out}_{slm\omega}$ defined by setting $A^{\mathcal{H}^+}_{slm\omega} = 0$ and normalized by $A^{\mathcal{H}^-}_{slm\omega} = 1$.

- The **downgoing** solution $R^{\rm down}_{slm\omega}$ defined by setting $A^{\mathcal{I}^+}_{slm\omega} = 0$ and normalized by $A^{\mathcal{H}^-}_{slm\omega} = 1$.

Generically, any pair of $\left(R^{\rm in}_{slm\omega}, R^{\rm out}_{slm\omega}, R^{\rm up}_{slm\omega}, R^{\rm down}_{slm\omega}\right)$ can be used as a basis, and the other two can be written in terms of them, e.g.

$$R^{\rm down}_{slm\omega} \propto R^{\rm in}_{slm\omega} - R^{\rm up}_{slm\omega}, \quad \text{and} \tag{3.47}$$

$$R^{\rm up}_{slm\omega} \propto R^{\rm out}_{slm\omega} - R^{\rm down}_{slm\omega}. \tag{3.48}$$

3.5.3 Particular solutions

$$\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial R_{slm\omega}}{\partial r}\right) - V_{slm\omega}R_{slm\omega} = \hat{S}_{slm\omega}[T_{\mu\nu}], \tag{3.49}$$

To find particular solutions to the inhomogeneous Teukolksy equation we can follow a Green's function approach. If we first solve

$$\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial G_{slm\omega}(r,r_0)}{\partial r}\right) - V_{slm\omega}G_{slm\omega}(r,r_0) = \delta(r-r_0), \tag{3.50}$$

the particular solution for a general source is then given by

$$R_{slm\omega} = \int_{r_+}^{\infty} G(r, r_0) \hat{S}_{slm\omega}[T_{\mu\nu}] \, \mathrm{d}r_0. \tag{3.51}$$

To solve for the Green's function we first need to choose boundary conditions. Usually the physically relevant choice are the "retarded" boundary conditions, where for large r, the Green's function $G(r, r_0)$ satisfies upgoing boundary conditions (i.e. no waves from $\mathcal{I}^-$), and incoming boundary conditions (no waves from $\mathcal{H}^-$) near the horizon. With these choices the Green's function is given by

$$G_{slm\omega}(r, r_0) = \begin{cases} \dfrac{R^{\mathrm{up}}_{slm\omega}(r_0) R^{\mathrm{in}}_{slm\omega}(r)}{W[R^{\mathrm{in}}_{slm\omega}, R^{\mathrm{up}}_{slm\omega}](r_0)} & r < r_0 \\ \dfrac{R^{\mathrm{in}}_{slm\omega}(r_0) R^{\mathrm{up}}_{slm\omega}(r)}{W[R^{\mathrm{in}}_{slm\omega}, R^{\mathrm{up}}_{slm\omega}](r_0)} & r > r_0, \end{cases} \tag{3.52}$$

where $W[R^{\mathrm{in}}_{slm\omega}, R^{\mathrm{up}}_{slm\omega}](r_0)$ is the Wronskian

$$W[R^{\mathrm{in}}_{slm\omega}, R^{\mathrm{up}}_{slm\omega}](r) = R^{\mathrm{in}}_{slm\omega} \frac{\mathrm{d}R^{\mathrm{up}}_{slm\omega}}{\mathrm{d}r} - \frac{\mathrm{d}R^{\mathrm{in}}_{slm\omega}}{\mathrm{d}r} R^{\mathrm{up}}_{slm\omega}. \tag{3.53}$$

3.6 Quasinormal Modes

In the last section, we discussed using $R^{\mathrm{in}}_{slm\omega}$ and $R^{\mathrm{up}}_{slm\omega}$ as a basis for all vacuum, and how they can be used to construct general particular solutions. However, what happens when $R^{\mathrm{in}}_{slm\omega} = R^{\mathrm{up}}_{slm\omega}$? Can there be vacuum solutions that simultaneously satisfy upgoing and ingoing boundary conditions, i.e. where there are no incoming waves from either past null infinity or the past horizon? For generic ω the answer is no, but for special values of ω the answer is yes. These frequencies are known as the **quasinormal modes** (or **QNMs**) of a black hole. These QNMs act like a "finger print" of the black hole.

We start by citing an important result regard a Kerr black hole's QNMs

Theorem (Kerr mode stability)**.** *All quasinormal modes of a Kerr black hole have* $\operatorname{im}\omega < 0$*.*

This theorem guarantees that the time dependence of any Kerr QNM ($\exp(-i\omega t)$) is decaying in nature, and there are no vacuum modes that grow out of control.

This also means that for each l and m there will be a QNM with the largest imaginary frequency. This will be the longest lived mode with those

l and m, and is called the **fundamental** mode. The QNMs with a certain l and m are numbered by natural numbers n in order of descending imaginary part (i.e. $n = 0$ is the fundamental mode.) The modes with $n > 0$ are referred to as **overtones** (despite generally having a lower real part of the frequency.)

Lemma 5. *If ω_{lmn} is a quasinormal mode, then so is $-\omega_{l(-m)n}*$.*

The intuitive picture of quasinormal modes is that they consist of waves travelling along the lightring of the Kerr black hole, decaying because of the unstable nature of this orbit. For small enough wave packets, i.e. large l, we indeed see that $\mathrm{re}\,\omega \approx l\Omega_{LR}$ and $\mathrm{im}\,\omega$ roughly corresponds to the Lyapunov exponent characterizing how unstable it is.

Bibliography

[1] Black Hole Perturbation Toolkit. (bhptoolkit.org).

[2] Robert H. Boyer and Richard W. Lindquist. Maximal analytic extension of the Kerr metric. **J. Math. Phys.**, 8:265, 1967.

[3] Michael P. Ryan. Teukolsky equation and Penrose wave equation. **Phys. Rev. D**, 10:1736–1740, 1974.

[4] Robert M. Wald. On perturbations of a Kerr black hole. **J. Math. Phys.**, 14(10):1453–1461, 1973.