Reminder: Euclidean spaces

- We work in n-dimensional space $x = (x_1, x_2, ..., x_n)$.
- 2D: $v = (x_1, x_2) = (x, y)$
- 3D: $v = (x_1, x_2, x_3) = (x, y, z)$

Scalar product

If we have $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, then:

$$x \cdot y = \langle x, y \rangle = x_1 y_1 + ... + x_n y_n$$

Cross product (3D)

$x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

Scalar Field

A scalar field

$$F : \mathbb{R} \to \mathbb{R}, x \to F(x)$$

assigns a scalar value to every point.

Examples: temperature, chemical concentration, pressure, probability distribution.

Given a scalar field $F^n \to \mathbb{R}$, the level set of the value $c \in \mathbb{R}$ is:

$$\{x \in \mathbb{R}^n : f(x) = c\}$$

Vector Field

A vector field assigns a vector to each point in space:

$$F : \mathbb{R}^n \to \mathbb{R}^n, x \to (F_1(x), ..., F_n(x))$$

Examples:

- $F : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \to (-x_2, x_1)$
- $F : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \to (2x_1, x_2 + x_3, x_1 x_2 x_3)$

How to visualize vector fields? Put one arrow for each point in space.

- constant vector field
- rotational vector field
- source vector field

Gradient, Hessian, Laplacian

If $F : \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the gradient is the vector field

$$\nabla F = (\partial x_1 f, \partial x_2 f, ..., \partial x_n f)$$

$$\nabla = \begin{pmatrix} \partial x_1 \\ \partial x_2 \\ ... \\ \partial x_n \end{pmatrix}$$

$$\text{grad } f = \nabla f = \begin{pmatrix} \partial x_1 f \\ \partial x_2 f \\ ... \\ \partial x_n f \end{pmatrix}$$

Example:

$$f(x_1, x_2) = x_1^3 + x_1 x_2^5$$

$$\nabla f(x_1, x_2) = (3x_1^2 + x_2^5, x_1 \cdot 5x_2^4)$$

- Geometrically, ∇f points in the direction of the steepest increae of f
- The directional derivative of f in direction $v \in \mathbb{R}^n$ is:

$$D_v f = \nabla f \cdot v = \partial x_1 f \cdot v + ... + \partial x_n f \cdot v_n$$

The value $D_v f$ tells us how f changes as we move in direction v.

- Along any level set line, the gradient is orthogonal.

Example:

$f(x_1, x_2) = x_1^2 + x_2^2$

$\nabla f(x_1, x_2) = (2x_1, 2x_2)$

Level set of $c = 1$.

If $F : \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the Hessian of f is a matrix field:

$$\nabla^2 f = \begin{pmatrix} \partial_1 \partial_1 f & \partial_1 \partial_2 f & ... & \partial_1 \partial_n f \\ ... & ... & ... & ... \\ \partial_n \partial_1 f & ... & ... & \partial_n \partial_n f \end{pmatrix}$$

- The Hessian is symmetric if all 2nd directives are continuous
- Physical/geometric interpretation contains information on the curvature of the scalar field

If $F : \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the Laplacian of f is the scalar field:

$$\Delta f = \sum_{i=1}^{n} \partial_i \partial_i f = \partial_i \partial_i f + \partial_2 \partial_2 f + ... + \partial_n \partial_n f$$

- Sum of the diagonal entries of the Hessian
- Physically relevant in modeling diffusion

Poisson problem : $-\Delta u = f$ (u is an unknown scalar field, while f is known, it's a differential equation).

Example:

$$f(x, y, z) = xy^3 e^z$$

$$\nabla f(x) = (y^3 e^z, 3xy^2 e^z, xy^3 e^z)$$

$$\partial x \partial x f = 0$$

$$\partial x \partial y f = 3y^2 e^z$$

$$\partial x \partial z f = y^3 e^z$$

$$\partial y \partial y f = 6xye^z$$

$$\partial y \partial z f = 3xy^2 e^z$$

$$\partial z \partial z f = xy^3 e^z$$

$$\nabla^2 f = \begin{pmatrix} 0 & 3y^2 e^z & y^3 e^z \\ 3y^2 e^z & 6xye^z & 3xy^2 e^z \\ y^3 e^z & 3xy^2 e^z & xy^3 e^z \end{pmatrix}$$

$$\Delta f = 0 + 6xye^z + xy^3 e^z = e^z xy(6 + y^2)$$

Example:

$$F(x) = \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}, x \in \mathbb{R}^n$$

First partial derivatives:

$$\partial_i F(x) = \partial_i \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left(\sum_{i=1}^{n} x_i^2 \right)^{-\frac{1}{2}} \cdot 2x_i = \frac{1}{\|x\|} x_i = \frac{x_i}{\|x\|}$$

Notice the singularity at $x = 0$. It looks like the gradient is not defined but in fact it can be removed by continuous extension of the first derivatives.

Second partial derivatives (case disjunction):

$$\partial_i \partial_i f(x) = \partial_i \left(\frac{x_i}{\|x\|} \right) = \partial_i (x_i \cdot \|x\|^{-1}) = \frac{1}{\|x\|} + x_i \cdot \left(-\frac{1}{\|x\|^2} (\partial_i \ \|x\|) \right)$$

$$= \frac{1}{\|x\|} + x_i \cdot - \frac{1}{\|x\|^2} \cdot \frac{x_i}{\|x\|} = \frac{1}{\|x\|} - x^2 \cdot \frac{1}{\|x\|^3}$$

$$\partial_j \partial_i f(x) = \partial_j \left(\frac{x_i}{\|x\|} \right) = 0 + x_i \cdot - \frac{1}{\|x\|^2} (\partial_j \ \|x\|) = x_i \cdot - \frac{1}{\|x\|^2} \frac{x_j}{\|x\|} = \frac{-x_i x_j}{\|x\|^3}$$

All partial derivatives of order 2, we can build the Hessiam matrix.

We want the Laplacian too:

$$\Delta f(x) = \sum_{i=1}^{n} \partial_i \partial_i f(x) = \sum_{i=i}^{n} \frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} = \frac{n}{\|x\|} - \frac{\sum_{i=1}^{n} x_i^2}{\|x\|^3}$$

$$= \frac{n}{\|x\|} - \frac{\|x\|^2}{\|x\|^3} = \frac{n-1}{\|x\|}$$

Divergence

Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n, x \to \left(F_1(x), ..., F_{n(x)}\right)$

The divergence is the vector field:

$$\text{div } F = \sum_{i=1}^{n} \partial_i F_i = \partial_1 F_1 + \partial_2 F_2 + ... + \partial_n F_n$$

- formally, div $F = \nabla \cdot F$
- the Laplacian is the divergence of the gradient: $\Delta F = \text{div } \nabla F$

Example:

$$\text{div } \left(x_1^2 x_2, x_2^3, e^{x_3}\right) = 2x_1 x_2 + 3x_2^2 + e^{x_3}$$

Rotation or curl of vector fields

If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a 3D vector field, $F = (F_1, F_2, F_3)$, then the curl/rotation is a 3D vector field:

$$\text{curl } F = \text{rot } F = \begin{pmatrix} -\partial_3 F_2 + \partial_2 F_3 \\ -\partial_1 F_3 + \partial_3 F_1 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix}$$

Formally, curl $F = \nabla \times F$.

Only works in 3D. There is a rotation in 2D:

If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a 2D vector fireld, then the rotation/curl of F is a scalar field.

$$\text{curl } F = \text{rot } F = -\partial_2 F_1 + \partial_1 F_2$$

Motivation: we formally extend the vector field with a third coordinate $F_3 = 0$.

$$\tilde{F} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \Rightarrow \text{curl } F = \begin{pmatrix} -\partial_3 F_2 + \partial_2 F_3 \\ -\partial_1 F_3 + \partial_3 F_1 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix}$$

Examples: the divergence measures the presence of sinks and sources ("puits et sources"), while rotation measures the presence of a spin.