Combinatorics

We want to choose `k` elements among `n` elements.

	Repetition not allowed	**Repetition allowed**
Order does not matter	$$\frac{n!}{(n-k)!k!}$$ (`n` choices, then `n-1` choices, etc. and we stop at `n-k`, we also remove `n!` because order does not matter and `n!` is the number of permutations)	$$\binom{n-1+r}{n-1}$$ It's stars and bars method. We want `r` stars + `n` stars for each box. Then, we transform `n-1` of these stars into bars to separate the `r` stars into boxes.
Order matters	$$n!$$ (`n` choices, then `n-1` choices, etc.)	$$n^k$$ Cartesian product.

Probabilities

Solving a probability problem

- list possible outcomes, define the probability space
- sometimes we keep a general Ω and different $\mathcal{F}$ depending on the point of view (colorblind/not colorblind, etc.)

Terminology

- Ω is the **sample space**, containing all possible outcomes ω.
- $\mathcal{F}$ is an **event space** (there are multiple event spaces!). It is a set of the subsets of Ω. The powerset of Ω includes all $\mathcal{F}$. $|\mathcal{F}| = 2^{|\Omega|}$ only if Ω is finite.

$\mathcal{F}$ is also called a sigma-algebra.

Example for a fair die:

- **Sample space:** $\{1, 2, 3, 4, 5, 6\}$
- **Events**: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, ..., \{1, 2, 3, 4, 5, 6\}$ note that this is only for one throw! $\{a, b\}$ is read "getting a or getting b".
- **Event space:** all events

Axioms

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Generalization of the OR between events:

$$P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < ... < i_r < n} \left(A_{i_1} \cap ... \cap A_{i_r}\right)$$

(les i_k doivent donc être différents)

Conditional probabilities:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Law of total probability

Let $\{B_i\}_{i=1}^{\infty}$ be pairwise disjoint events, and let $A \subset \cup_{i=1}^{\infty} B_i$ then:

$$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i) = \sum_{i=1}^{\infty} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

Bayes' Theorem:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

but we can replace $P(B)$ by what we know, low of total probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}$$

Derangement

$$!n = \sum_{i=0}^{n} \frac{(-1)^i}{i!}, n \geq 0$$

Si on veut un élément non dérangé parmis n:

$n \cdot \frac{1}{n} \cdot !(n-1)$

Si on en veut r:

(comment on choisit les r bien rangés $\cdot$ le fait qu'ils soient bien rangés et que les restants soient dérangés)

$$\binom{n}{r} \cdot \frac{(n-r)!}{n!} \cdot !(n-r)$$

Independence

If A and B are independent then $\mathbb{P}(A|B) = \mathbb{P}(A)$.

They are independent iff $\mathbb{P}(A \cup B) = \mathbb{P}(A)\mathbb{P}(B)$.

If two events are disjoint they can not be independent unless their respective probabilities are 0.

Pairwise Independence

If you take any two events, they are independent.

Random variable (notation)

We will use Y to denote a random variable.
We will use y to denote a specific value that Y can take.

$Y : \Omega \to \mathbb{R}$ is a function.

$$D_Y = \{x \in \mathbb{R} : \exists \omega \in \Omega \quad \text{s.t. } X(\omega) = y\}$$

D_Y is called the support of X. If D_Y is countable, then Y is a discrete random variable.

$\mathbb{P}(Y = y) \equiv p_Y(y)$

p_Y is called the probability mass function.

A probability distribution is a table or a graph that provides $p(y) \forall y$.

For everything to hold, $\sum_y p(y) = 1$.

Expected Value

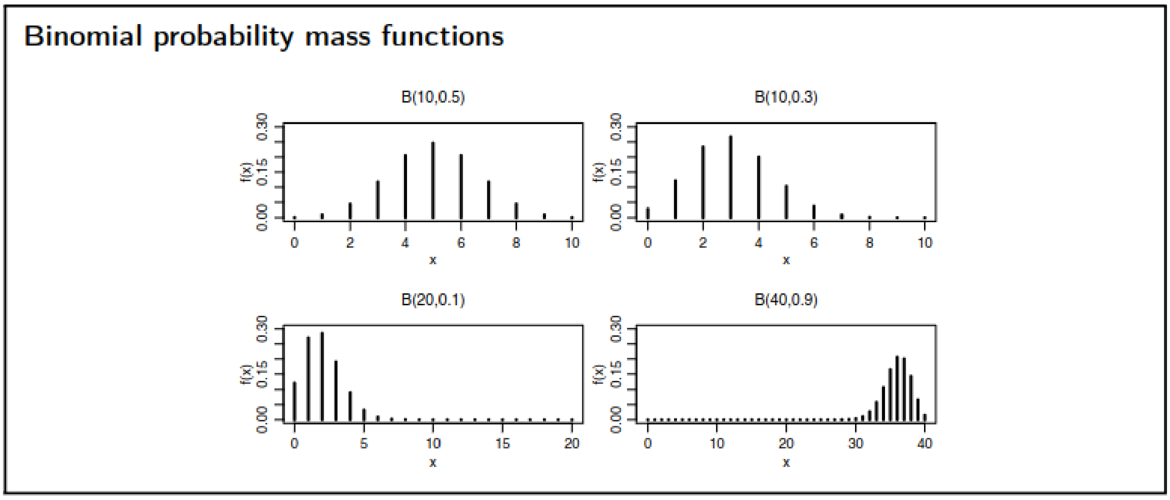
$$E(Y) = \sum_y y p(y)$$

Binomial Random Variable

A binomial random variable X has PMF:

$$f(x) = \binom{n}{x} p^{x(1-p)^{n-x}}, x = 0, 1, ..., n, n \in \mathbb{N}, 0 \leq p \leq 1$$

We write $X \sim B(n, p)$ and call n the denomiator and p probability of success.



Geometric distribution

$$f_{X(x)} = p(1-p)^{x-1}$$

1 success and $x - 1$ failures. the probability to have exactly one success

Memorylessness :

$$P(X > n + m \mid X > m) = P(X > n)$$

Thanks to independence.

Negative binomial distribution

Loi binomiale négative

- **Définition** : La loi binomiale négative généralise la loi géométrique. Elle modélise le nombre d'essais nécessaires pour obtenir un certain nombre de succès r (au lieu d'un seul succès comme dans la loi géométrique), avec une probabilité de succès constante p à chaque essai.
- **Paramètres** : r (nombre de succès désirés) et p (probabilité de succès).
- **Exemple** : Si on lance une pièce de monnaie et qu'on souhaite savoir combien de lancers sont nécessaires pour obtenir 3 faces, on utilise la loi binomiale négative.
- **Formule** :

  $$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

  où k est le nombre total d'essais nécessaires pour obtenir r succès.

Discrete uniform distribution

Poisson random variable