Combinatorics

We want to choose `k` elements among `n` elements.

	Repetition not allowed	**Repetition allowed**
Order does not matter	$$\frac{n!}{(n-k)!k!}$$ (`n` choices, then `n-1` choices, etc. and we stop at `n-k`, we also remove `n!` because order does not matter and `n!` is the number of permutations)	$$\binom{n-1+r}{n-1}$$ It's stars and bars method. We want `r` stars + `n` stars for each box. Then, we transform `n-1` of these stars into bars to separate the `r` stars into boxes.
Order matters	$$n!$$ (`n` choices, then `n-1` choices, etc.)	$$n^k$$ Cartesian product.

Probabilities

Solving a probability problem

- list possible outcomes, define the probability space
- sometimes we keep a general Ω and different $\mathcal{F}$ depending on the point of view (colorblind/not colorblind, etc.)

Terminology

- Ω is the **sample space**, containing all possible outcomes ω.
- $\mathcal{F}$ is an **event space** (there are multiple event spaces!). It is a set of the subsets of Ω. The powerset of Ω includes all $\mathcal{F}$. $|\mathcal{F}| = 2^{|\Omega|}$ only if Ω is finite.

$\mathcal{F}$ is also called a sigma-algebra.

Example for a fair die:

- **Sample space:** $\{1, 2, 3, 4, 5, 6\}$
- **Events**: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, ..., \{1, 2, 3, 4, 5, 6\}$ note that this is only for one throw! $\{a, b\}$ is read "getting a or getting b".
- **Event space:** all events

Axioms

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Generalization of the OR between events:

$$P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < ... < i_r < n} \left(A_{i_1} \cap ... \cap A_{i_r}\right)$$

(les i_k doivent donc être différents)

Conditional probabilities:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Law of total probability

Let $\{B_i\}_{i=1}^{\infty}$ be pairwise disjoint events, and let $A \subset \cup_{i=1}^{\infty} B_i$ then:

$$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i) = \sum_{i=1}^{\infty} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

Bayes' Theorem:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

but we can replace $P(B)$ by what we know, low of total probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}$$