

## Reminder: Euclidean spaces

- We work in n-dimensional space  $x = (x_1, x_2, \dots, x_n)$ .
- 2D:  $v = (x_1, x_2) = (x, y)$
- 3D:  $v = (x_1, x_2, x_3) = (x, y, z)$

Calculer un produit vectoriel :

$$\det \begin{pmatrix} + & - & + \\ + & \text{axe1} & \text{axe2} & \text{axe3} \\ - & \text{vect1x} & \text{vect1y} & \text{vect1z} \\ + & \text{vect2x} & \text{vect2y} & \text{vect2z} \end{pmatrix}$$

## Scalar product

If we have  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ , then:

$$x \cdot y = \langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$$

## Cross product (3D)

$$x = (x_1, x_2, x_3)$$

$$y = (y_1, y_2, y_3)$$

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

## Scalar Field

A scalar field

$$F : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow F(x)$$

assigns a scalar value to every point.

Examples: temperature, chemical concentration, pressure, probability distribution.

Given a scalar field  $F^n \rightarrow \mathbb{R}$ , the level set of the value  $c \in \mathbb{R}$  is:

$$\{x \in \mathbb{R}^n : f(x) = c\}$$

## Vector Field

A vector field assigns a vector to each point in space:

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \rightarrow (F_1(x), \dots, F_n(x))$$

Examples:

- $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x_1, x_2) \rightarrow (-x_2, x_1)$
- $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x_1, x_2) \rightarrow (2x_1, x_2 + x_3, x_1 x_2 x_3)$

How to visualize vector fields? Put one arrow for each point in space.

- constant vector field
- rotational vector field
- source vector field

## Gradient, Hessian, Laplacian

If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a scalar field, then the gradient is the vector field

$$\nabla F = (\partial x_1 f, \partial x_2 f, \dots, \partial x_n f)$$

$$\nabla = \begin{pmatrix} \partial x_1 \\ \partial x_2 \\ \dots \\ \partial x_n \end{pmatrix}$$

$$\text{grad } f = \nabla f = \begin{pmatrix} \partial x_1 f \\ \partial x_2 f \\ \dots \\ \partial x_n f \end{pmatrix}$$

Example:

$$f(x_1, x_2) = x_1^3 + x_1 x_2^5$$

$$\nabla f(x_1, x_2) = (3x_1^2 + x_2^5, x_1 \cdot 5x_2^4)$$

- Geometrically,  $\nabla f$  points in the direction of the steepest increase of  $f$
- The directional derivative of  $f$  in direction  $v \in \mathbb{R}^n$  is:

$$D_v f = \nabla f \cdot v = \partial x_1 f \cdot v + \dots + \partial x_n f \cdot v_n$$

The value  $D_v f$  tells us how  $f$  changes as we move in direction  $v$ .

- Along any level set line, the gradient is orthogonal.

Example:

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$\nabla f(x_1, x_2) = (2x_1, 2x_2)$$

Level set of  $c = 1$ .

If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a scalar field, then the Hessian of  $f$  is a matrix field:

$$\nabla^2 f = \begin{pmatrix} \partial_1 \partial_1 f & \partial_1 \partial_2 f & \dots & \partial_1 \partial_n f \\ \dots & \dots & \dots & \dots \\ \partial_n \partial_1 f & \dots & \dots & \partial_n \partial_n f \end{pmatrix}$$

- The Hessian is symmetric if all 2nd derivatives are continuous
- Physical/geometric interpretation contains information on the curvature of the scalar field

If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a scalar field, then the Laplacian of  $f$  is the scalar field:

$$\Delta f = \sum_{i=1}^n \partial_i \partial_i f = \partial_1 \partial_1 f + \partial_2 \partial_2 f + \dots + \partial_n \partial_n f$$

- Sum of the diagonal entries of the Hessian
- Physically relevant in modeling diffusion

Poisson problem :  $-\Delta u = f$  ( $u$  is an unknown scalar field, while  $f$  is known, it's a differential equation).

Example:

$$f(x, y, z) = xy^3e^z$$

$$\nabla f(x) = (y^3e^z, 3xy^2e^z, xy^3e^z)$$

$$\partial_x \partial_x f = 0$$

$$\partial_x \partial_y f = 3y^2e^z$$

$$\partial_x \partial_z f = y^3e^z$$

$$\partial_y \partial_y f = 6xye^z$$

$$\partial_y \partial_z f = 3xy^2e^z$$

$$\partial_z \partial_z f = xy^3e^z$$

$$\nabla^2 f = \begin{pmatrix} 0 & 3y^2e^z & y^3e^z \\ 3y^2e^z & 6xye^z & 3xy^2e^z \\ y^3e^z & 3xy^2e^z & xy^3e^z \end{pmatrix}$$

$$\Delta f = 0 + 6xye^z + xy^3e^z = e^z xy(6 + y^2)$$

Example:

$$F(x) = \|x\| = \sqrt{\sum_{i=1}^n x_i^2}, x \in \mathbb{R}^n$$

First partial derivatives:

$$\partial_i F(x) = \partial_i \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left( \sum_{i=1}^n x_i^2 \right)^{-\frac{1}{2}} \cdot 2x_i = \frac{1}{\|x\|} x_i = \frac{x_i}{\|x\|}$$

Notice the singularity at  $x = 0$ . It looks like the gradient is not defined but in fact it can be removed by continuous extension of the first derivatives.

Second partial derivatives (case disjunction):

$$\begin{aligned} \partial_i \partial_i f(x) &= \partial_i \left( \frac{x_i}{\|x\|} \right) = \partial_i (x_i \cdot \|x\|^{-1}) = \frac{1}{\|x\|} + x_i \cdot \left( -\frac{1}{\|x\|^2} (\partial_i \|x\|) \right) \\ &= \frac{1}{\|x\|} + x_i \cdot -\frac{1}{\|x\|^2} \cdot \frac{x_i}{\|x\|} = \frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} \end{aligned}$$

$$\partial_j \partial_i f(x) = \partial_j \left( \frac{x_i}{\|x\|} \right) = 0 + x_i \cdot -\frac{1}{\|x\|^2} (\partial_j \|x\|) = x_i \cdot -\frac{1}{\|x\|^2} \frac{x_j}{\|x\|} = \frac{-x_i x_j}{\|x\|^3}$$

All partial derivatives of order 2, we can build the Hessian matrix.

We want the Laplacian too:

$$\Delta f(x) = \sum_{i=1}^n \partial_i \partial_i f(x) = \sum_{i=1}^n \frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} = \frac{n}{\|x\|} - \frac{\sum_{i=1}^n x_i^2}{\|x\|^3}$$

$$= \frac{n}{\|x\|} - \frac{\|x\|^2}{\|x\|^3} = \frac{n-1}{\|x\|}$$

## Divergence

Given a vector field  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \rightarrow (F_1(x), \dots, F_n(x))$

The divergence is the vector field:

$$\operatorname{div} F = \sum_{i=1}^n \partial_i F_i = \partial_1 F_1 + \partial_2 F_2 + \dots + \partial_n F_n$$

- formally,  $\operatorname{div} F = \nabla \cdot F$
- the Laplacian is the divergence of the gradient:  $\Delta F = \operatorname{div} \nabla F$

Example:

$$\operatorname{div} (x_1^2 x_2, x_2^3, e^{x_3}) = 2x_1 x_2 + 3x_2^2 + e^{x_3}$$

## Rotation or curl of vector fields

If  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a 3D vector field,  $F = (F_1, F_2, F_3)$ , then the curl/rotation is a 3D vector field:

$$\operatorname{curl} F = \operatorname{rot} F = \begin{pmatrix} -\partial_3 F_2 + \partial_2 F_3 \\ -\partial_1 F_3 + \partial_3 F_1 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix}$$

Formally,  $\operatorname{curl} F = \nabla \times F$ .

Only works in 3D. There is a rotation in 2D:

If  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a 2D vector field, then the rotation/curl of  $F$  is a scalar field.

$$\operatorname{curl} F = \operatorname{rot} F = -\partial_2 F_1 + \partial_1 F_2$$

Motivation: we formally extend the vector field with a third coordinate  $F_3 = 0$ .

$$\tilde{F} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \Rightarrow \operatorname{curl} F = \begin{pmatrix} -\partial_3 F_2 + \partial_2 F_3 \\ -\partial_1 F_3 + \partial_3 F_1 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix}$$

Examples: the divergence measures the presence of sinks and sources (“puits et sources”), while rotation measures the presence of a spin.

## Curve

$$\int_a^b f(x) dx = \text{integrate } f \text{ over interval } [a, b]$$

Now we generalize this:

$$\int_{\gamma} f dl = \text{integrate } f \text{ over curve } \gamma$$



A curve is a function

$$\gamma : [a, b] \rightarrow \mathbb{R}^n, t \rightarrow \gamma(t)$$

We may also think as  $\gamma(t)$  as a position in time. The image of  $\gamma(t)$  is written  $\Gamma(t)$ .

Some examples:

$\gamma(t) : [0, T] \rightarrow \mathbb{R}^3$  can be the 3D position of a drone flying.

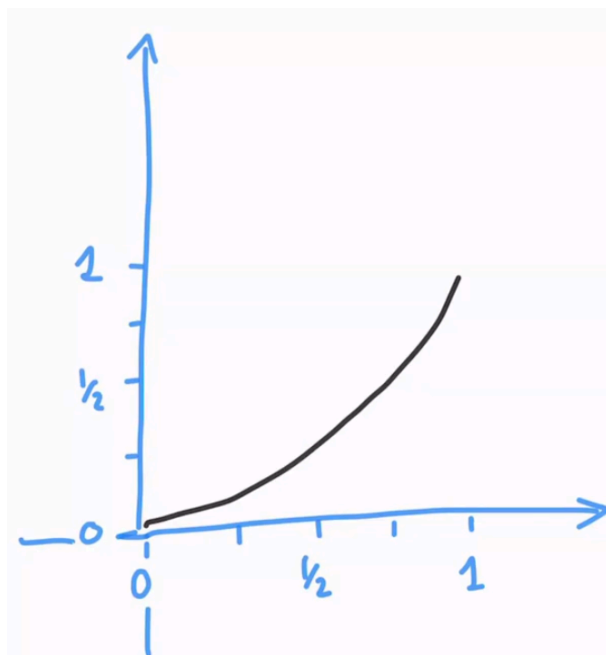
$\gamma(t) : [0, 1] \rightarrow \mathbb{R}^2$  can be the position at  $t\%$  of a car travel on a map.

$\gamma(t) : [0, 2\pi] \rightarrow \mathbb{R}^2, t \rightarrow (\cos(t), \sin(t))$  is the parametrization of the unit circle (as  $t$  progresses, we travel the unit circle).

Two functions can represent the same curve!

$$\gamma_1(t) : [0, 1] \rightarrow \mathbb{R}^2, t \rightarrow (t, t^2)$$

$$\gamma_2(t) : [0, 1] \rightarrow \mathbb{R}^2, t \rightarrow (\sqrt{t}, t)$$



### Notions and definitions

We call a curve

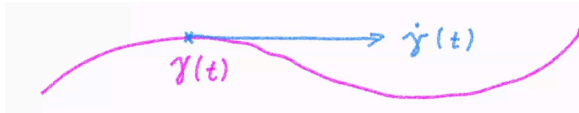
$$\gamma(t) : [a, b] \rightarrow \mathbb{R}^n, t \rightarrow (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$$

- **simple**: if it does not self-intersect (formally,  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  is injective)
- **closed**: if  $\gamma(a) = \gamma(b)$
- **differentiable** if  $\gamma_1(t), \dots, \gamma_n(t)$  are differentiable

- **regular** if the curve is differentiable and the vector  $\forall t (\gamma_1'(t), \dots, \gamma_n'(t)) \neq \vec{0}$  (the derivatives are never 0 all together). It means that curve never comes to a full stop, they always keep moving.

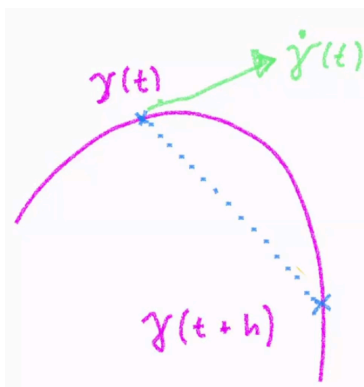
### Tangential vectors and speed

The tangent vector of a curve  $\gamma(t)$  is:  $\dot{\gamma}(t) = (\dot{\gamma}_1(t), \dots, \dot{\gamma}_n(t))$  and the speed is:  $|\dot{\gamma}(t)| = \sqrt{(\dot{\gamma}_1(t))^2 + \dots + (\dot{\gamma}_n(t))^2}$



Similar to the definition of the derivative:

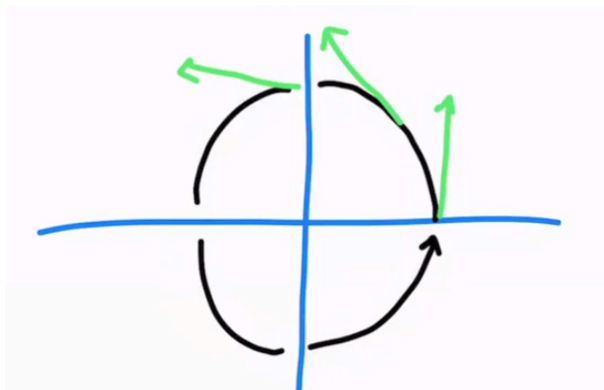
$$\dot{\gamma}(t) = \lim_{h \rightarrow 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$



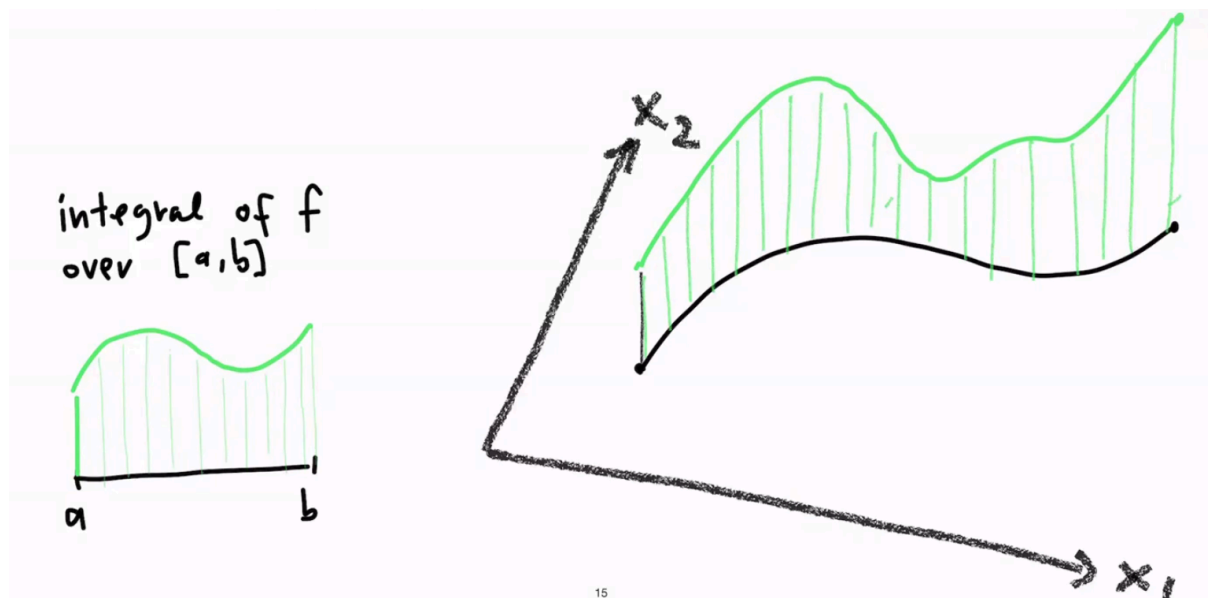
Example:

$$\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2, t \rightarrow (\cos(t), \sin(t))$$

$$\dot{\gamma}(t) = (-\sin(t), \cos(t))$$



## Curve Integrals



Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a curve Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a scalar field

The integral of  $f$  over  $\Gamma$  is:

$$\begin{aligned} \int_{\Gamma} f dl &:= \int_a^b (f \circ \gamma)(t) |\dot{\gamma}(t)| dt \\ &= \int_a^b (f \circ \gamma)(t) \sqrt{(\dot{\gamma}_1(t))^2 + \dots + (\dot{\gamma}_n(t))^2} dt \end{aligned}$$

The curve integral only depends on the curve  $\Gamma$ , not  $\gamma$  (which is what we want, we need the curve, not the parametrization, see eg. where we had 2 functions for one curve).

- where  $\gamma$  is slow,  $\dot{\gamma}$  is small
- where  $\gamma$  is fast,  $\dot{\gamma}$  is large

En fait si la fonction va très lentement, on va “utiliser” une grande partie de notre portion de  $a$  vers  $b$  pour la tracer, mais on réduit dcp le facteur.

If  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  is a simple regular curve, then  $\int_{\Gamma} F dl$  only depends on  $\Gamma$ .

And it should! After all,  $\gamma$  is just a parametrization and  $\Gamma$  is the “physical” object.

### Curve integrals of vector fields

Given a curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  and a vector field  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  we define:

$$\begin{aligned} \int_{\Gamma} \vec{F} dl &:= \int_a^b \vec{F}(\gamma(t)) \cdot \dot{\gamma}(t) dt \\ &= \int_a^b F_1(\gamma(t)) \cdot \dot{y}_1(t) + \dots + F_n(\gamma(t)) \cdot \dot{y}_n(t) dt \end{aligned}$$

Example:

Suppose  $\vec{F} = \nabla f$  is the gradient of scalar field

$$\int_{\Gamma} \vec{F} dl = \int_{\Gamma} \nabla f dl = \int_a^b \nabla f(\gamma(t)) \cdot \dot{\gamma}(t) dt$$

We observe

$$\begin{aligned} (f \circ \gamma)' &= \nabla f(\gamma(t)) \cdot \dot{\gamma}(t) \quad (\text{see Analysis II}) \\ &= \int_a^b (f \circ \gamma)'(t) dt = f(\gamma(b)) - f(\gamma(a)) \end{aligned}$$

One more perspective of curve integrals for vector fields. Suppose  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$ .

$$\int_{\Gamma} F dl = \int_a^b F(\gamma(t)) \cdot \dot{\gamma}(t) dt = \int_a^b F(\gamma(t)) \cdot \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} \|\dot{\gamma}(t)\| dt$$

At any time  $t$ , the vector  $\tau(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$  is the unit tangent vector at time  $t$ .

$$\int_{\Gamma} F dl = \int_a^b F(\gamma(t)) \cdot \tau(t) \cdot \|\dot{\gamma}(t)\| dt = \int_{\gamma} F \cdot \tau dl$$

We can break our integral into pieces of simple and regular differentiable curves.

$$\int_{\Gamma} f dl = \int_a^b f \cdot \gamma \cdot |\dot{\gamma}| dt = \int_b^c f \cdot \gamma \cdot |\dot{\gamma}| dt + \int_c^d f \cdot \gamma \cdot |\dot{\gamma}| dt$$

## Conservative vector fields and their potentials

Let  $F : \Omega \rightarrow \mathbb{R}^n$  be a vector field,  $\Omega \subset \mathbb{R}^n$  open.

Does there exist a potential  $f$  of  $F$  over  $\Omega$ , i.e.  $f \in C^1(\Omega, \mathbb{R})$  such that  $\nabla f = F$ ?

**Theorem:** Let  $\Omega \subset \mathbb{R}^n$  be open and  $\vec{F} \in C^1(\Omega, \mathbb{R}^n)$ ,  $F = (F_1, F_2, \dots, F_n)$ . If  $\vec{F}$  has a potential then  $\partial_i F_j = \partial_j F_i$ ,  $1 \leq i, j \leq n$ .

**Proof:** if  $F$  admits a potential  $f \in C^1(\Omega, \mathbb{R})$ , then already  $f \in C^2(\Omega, \mathbb{R})$ . Given  $1 \leq i, j \leq n$ , we see:

$$\partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j F_i$$

using Schwarz.

**Remark:** This is a necessary condition but not a sufficient one. We use that  $\text{Hess}(f)$  is symmetric.

We call  $F \in C^1(\Omega, \mathbb{R}^n)$  conservative if  $\partial_j F_i = \partial_i F_j$ ,  $1 \leq i, j \leq n$ .

Let  $\Omega \subset \mathbb{R}^n$ . We call this set:

- **convex** if  $\forall (x, y) \in \Omega$  the line segment from  $x$  to  $y$  is within  $\Omega$ .
- **star-shaped** if  $\exists z \in \Omega \forall x \in \Omega$  the line segment from  $z$  to  $x$  is within  $\Omega$ .

Formally:

$$[x, y] := \{tx + (1-t)y : t \in [0, 1]\}$$

line segment from  $x$  to  $y$ . Image of the curve  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ ,  $t \rightarrow tx + (1-t)y$

$\Omega$  convex  $:\Leftrightarrow \forall (x, y) \in \Omega : [x, y] \subset \Omega$

$\Omega$  star-shaped  $:\Leftrightarrow \exists z \forall x \in \Omega : [z, x] \subset \Omega$



Theorem: let  $\Omega \subset \mathbb{R}^n$  be open and star-shaped with respect to  $z \in \Omega$ . If  $F \in C'(\Omega, \mathbb{R}^n)$  is conservative, then  $\vec{F}$  has a potential  $f \in C^2(\Omega, \mathbb{R})$ .

$$\begin{aligned} f(x) &:= \int_0^1 F(z + t(x - z)) \cdot (x - z) dt \\ &= \int_y F dl \quad \text{where } y : [0, 1] \rightarrow \mathbb{R}^n : t \rightarrow z + t(x - z) \end{aligned}$$

It depends on the choice of  $z$ !

Convex and star-shaped domains are important but simple. What if the domain has holes?

$$\Omega = \{x \in \mathbb{R}^3 \mid \|x\| > 1\}, \Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$$

Example:

Can we find a potential for

$$F(x, y) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

Smooth vector field over  $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$

we check that it is curl-free and that

$$\int_y F dl = 0$$

for some simple closed regular curve around  $(0, 0)$ .

- check whether  $\partial_x F_y = \partial_y F_x$
- consider closed simple regular curve  $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2, t \rightarrow (\cos(t), \sin(t))$ ,  $\int_\gamma F dl = 0$

There exists a potential! How to find it?

Suppose  $f \in C'(\Omega, \mathbb{R})$  with  $\nabla f = F$ . For any closed simple regular curve  $\gamma$   $\int_\gamma F dl = f(\gamma(b)) - f(\gamma(a))$ .

We construct a potential. Here we use a piecewise regular curves.

- We pick  $f(1, 0) = C$ .
- For any  $x > 0$ , let  $\gamma_x : [0, 1] \rightarrow \mathbb{R}^2, t \rightarrow (1 + t(x - 1), 0)$ . Obviously the straight line from  $(1, 0)$  to  $(x, 0)$ .

$$f(x, 0) - f(1, 0) = \int_{\gamma_x} F dl = \dots = x - 1 \text{ then } f(x, 0) = x + f(1, 0) - 1$$

- Next, suppose  $(x, y) = (r \cos(\theta), r \sin(\theta))$  for  $r > 0$ , and  $0 \leq \theta < 2\pi$ . Let  $\delta_{x,y}$