Reminder: Euclidean spaces

- We work in n-dimensional space $x = (x_1, x_2, ..., x_n)$.
- 2D: $v = (x_1, x_2) = (x, y)$
- 3D: $v = (x_1, x_2, x_3) = (x, y, z)$

Calculer un produit vectoriel :

$$\det \begin{pmatrix} & + & \text{-} & + \\ + & \text{axe1} & \text{axe2} & \text{axe3} \\ \text{-} & \text{vect1x} & \text{vect1y} & \text{vect1z} \\ + & \text{vect2x} & \text{vect2y} & \text{vect2z} \end{pmatrix}$$

Scalar product

If we have $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, then:

$$x \cdot y = \langle x, y \rangle = x_1 y_1 + ... + x_n y_n$$

Cross product (3D)

$x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

Scalar Field

A scalar field

$$F : \mathbb{R} \to \mathbb{R}, x \to F(x)$$

assigns a scalar value to every point.

Examples: temperature, chemical concentration, pressure, probability distribution.

Given a scalar field $F^n \to \mathbb{R}$, the level set of the value $c \in \mathbb{R}$ is:

$$\{x \in \mathbb{R}^n : f(x) = c\}$$

Vector Field

A vector field assigns a vector to each point in space:

$$F : \mathbb{R}^n \to \mathbb{R}^n, x \to (F_1(x), ..., F_n(x))$$

Examples:

- $F : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \to (-x_2, x_1)$
- $F : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \to (2x_1, x_2 + x_3, x_1 x_2 x_3)$

How to visualize vector fields? Put one arrow for each point in space.

- constant vector field
- rotational vector field
- source vector field

Gradient, Hessian, Laplacian

If $F : \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the gradient is the vector field

$$\nabla F = (\partial x_1 f, \partial x_2 f, ..., \partial x_n f)$$

$$\nabla = \begin{pmatrix} \partial x_1 \\ \partial x_2 \\ \dots \\ \partial x_n \end{pmatrix}$$

$$\text{grad } f = \nabla f = \begin{pmatrix} \partial x_1 f \\ \partial x_2 f \\ \dots \\ \partial x_n f \end{pmatrix}$$

Example:

$$f(x_1, x_2) = x_1^3 + x_1 x_2^5$$

$$\nabla f(x_1, x_2) = (3x_1^2 + x_2^5, x_1 \cdot 5x_2^4)$$

- Geometrically, ∇f points in the direction of the steepest increae of f
- The directional derivative of f in direction $v \in \mathbb{R}^n$ is:

$$D_v f = \nabla f \cdot v = \partial x_1 f \cdot v + ... + \partial x_n f \cdot v_n$$

The value $D_v f$ tells us how f changes as we move in direction v.

- Along any level set line, the gradient is orthogonal.

Example:

$f(x_1, x_2) = x_1^2 + x_2^2$

$\nabla f(x_1, x_2) = (2x_1, 2x_2)$

Level set of $c = 1$.

If $F : \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the Hessian of f is a matrix field:

$$\nabla^2 f = \begin{pmatrix} \partial_1 \partial_1 f & \partial_1 \partial_2 f & \dots & \partial_1 \partial_n f \\ \dots & \dots & \dots & \dots \\ \partial_n \partial_1 f & \dots & \dots & \partial_n \partial_n f \end{pmatrix}$$

- The Hessian is symmetric if all 2nd directives are continuous
- Physical/geometric interpretation contains information on the curvature of the scalar field

If $F : \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the Laplacian of f is the scalar field:

$$\Delta f = \sum_{i=1}^{n} \partial_i \partial_i f = \partial_i \partial_i f + \partial_2 \partial_2 f + ... + \partial_n \partial_n f$$

- Sum of the diagonal entries of the Hessian
- Physically relevant in modeling diffusion

Poisson problem : $-\Delta u = f$ (u is an unknown scalar field, while f is known, it's a differential equation).

Example:

$$f(x, y, z) = xy^3e^z$$

$$\nabla f(x) = (y^3e^z, 3xy^2e^z, xy^3e^z)$$

$$\partial x \partial x f = 0$$

$$\partial x \partial y f = 3y^2e^z$$

$$\partial x \partial z f = y^3e^z$$

$$\partial y \partial y f = 6xye^z$$

$$\partial y \partial z f = 3xy^2e^z$$

$$\partial z \partial z f = xy^3e^z$$

$$\nabla^2 f = \begin{pmatrix} 0 & 3y^2e^z & y^3e^z \\ 3y^2e^z & 6xye^z & 3xy^2e^z \\ y^3e^z & 3xy^2e^z & xy^3e^z \end{pmatrix}$$

$$\Delta f = 0 + 6xye^z + xy^3e^z = e^z xy(6 + y^2)$$

Example:

$$F(x) = \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}, x \in \mathbb{R}^n$$

First partial derivatives:

$$\partial_i F(x) = \partial_i \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left(\sum_{i=1}^{n} x_i^2 \right)^{-\frac{1}{2}} \cdot 2x_i = \frac{1}{\|x\|} x_i = \frac{x_i}{\|x\|}$$

Notice the singularity at $x = 0$. It looks like the gradient is not defined but in fact it can be removed by continuous extension of the first derivatives.

Second partial derivatives (case disjunction):

$$\partial_i \partial_i f(x) = \partial_i \left(\frac{x_i}{\|x\|} \right) = \partial_i (x_i \cdot \|x\|^{-1}) = \frac{1}{\|x\|} + x_i \cdot \left(-\frac{1}{\|x\|^2} (\partial_i \ \|x\|) \right)$$

$$= \frac{1}{\|x\|} + x_i \cdot -\frac{1}{\|x\|^2} \cdot \frac{x_i}{\|x\|} = \frac{1}{\|x\|} - x^2 \cdot \frac{1}{\|x\|^3}$$

$$\partial_j \partial_i f(x) = \partial_j \left(\frac{x_i}{\|x\|} \right) = 0 + x_i \cdot -\frac{1}{\|x\|^2} (\partial_j \ \|x\|) = x_i \cdot -\frac{1}{\|x\|^2} \frac{x_j}{\|x\|} = \frac{-x_i x_j}{\|x\|^3}$$

All partial derivatives of order 2, we can build the Hessiam matrix.

We want the Laplacian too:

$$\Delta f(x) = \sum_{i=1}^{n} \partial_i \partial_i f(x) = \sum_{i=i}^{n} \frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} = \frac{n}{\|x\|} - \frac{\sum_{i=1}^{n} x_i^2}{\|x\|^3}$$

$$= \frac{n}{\|x\|} - \frac{\|x\|^2}{\|x\|^3} = \frac{n-1}{\|x\|}$$

Divergence

Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n, x \to \left(F_1(x), ..., F_{n(x)}\right)$

The divergence is the vector field:

$$\text{div } F = \sum_{i=1}^{n} \partial_i F_i = \partial_1 F_1 + \partial_2 F_2 + ... + \partial_n F_n$$

- formally, div $F = \nabla \cdot F$
- the Laplacian is the divergence of the gradient: $\Delta F = \text{div } \nabla F$

Example:

$$\text{div } \left(x_1^2 x_2, x_2^3, e^{x_3}\right) = 2x_1 x_2 + 3x_2^2 + e^{x_3}$$

Rotation or curl of vector fields

If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a 3D vector field, $F = (F_1, F_2, F_3)$, then the curl/rotation is a 3D vector field:

$$\text{curl } F = \text{rot } F = \begin{pmatrix} -\partial_3 F_2 + \partial_2 F_3 \\ -\partial_1 F_3 + \partial_3 F_1 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix}$$

Formally, curl $F = \nabla \times F$.

Only works in 3D. There is a rotation in 2D:

If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a 2D vector fireld, then the rotation/curl of F is a scalar field.

$$\text{curl } F = \text{rot } F = -\partial_2 F_1 + \partial_1 F_2$$

Motivation: we formally extend the vector field with a third coordinate $F_3 = 0$.

$$\tilde{F} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \Rightarrow \text{curl } F = \begin{pmatrix} -\partial_3 F_2 + \partial_2 F_3 \\ -\partial_1 F_3 + \partial_3 F_1 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix}$$

Examples: the divergence measures the presence of sinks and sources (“puits et sources”), while rotation measures the presence of a spin.

Curve

$$\int_a^b f(x) dx = \text{integrate f over interval } [a, b]$$

Now we generalize this:

$$\int_\gamma f dl = \text{integrate f over curve } \gamma$$

A curve is a function

$$\gamma : [a, b] \to \mathbb{R}^n, t \to \gamma(t)$$

We may also think as $\gamma(t)$ as a position in time. The image of $\gamma(t)$ is written $\Gamma(t)$.

Some examples:

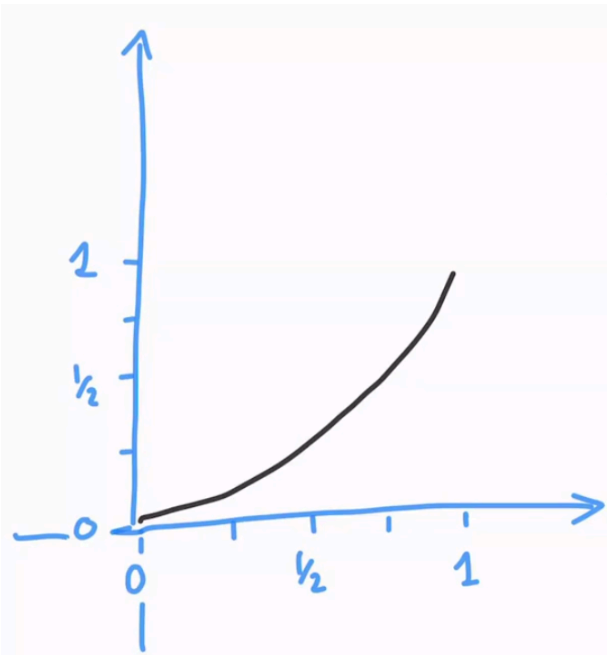
$\gamma(t) : [0, T] \to \mathbb{R}^3$ can be the 3D position of a drone flying.

$\gamma(t) : [0, 1] \to \mathbb{R}^2$ can be the position at $t\%$ of a car travel on a map.

$\gamma(t) : [0, 2\pi] \to \mathbb{R}^2, t \to (\cos(t), \sin(t))$ is the parametrization of the unit circle (as t progresses, we travel the unit circle).

Two functions can represent the same curve!

$\gamma_1(t) : [0, 1] \to \mathbb{R}^2, t \to (t, t^2)$
$\gamma_2(t) : [0, 1] \to \mathbb{R}^2, t \to \left(\sqrt{t}, t\right)$



Notions and definitions
We call a curve

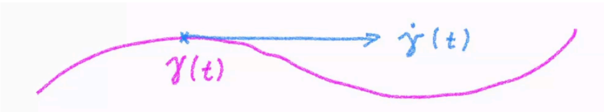
$\gamma(t) : [a, b] \to \mathbb{R}^n, t \to \left(\gamma_1(t), \gamma_2(t), ..., \gamma_{n(t)}\right)$

- **simple**: if it does not self-intersect (formally, $\gamma : [a, b] \to \mathbb{R}^n$ is injective)
- **closed**: if $\gamma(a) = \gamma(b)$
- **differentiable** if $\gamma_1(t), ..., \gamma_{n(t)}$ are differentiable

- **regular** if the curve is differentiable and the vector $\forall t \left(\dot{\gamma_1}(t), ..., \dot{\gamma_{n(t)}}\right) \neq \vec{0}$ (the derivatives are never 0 all together). It means that curve never comes to a full stop, they always keep moving.

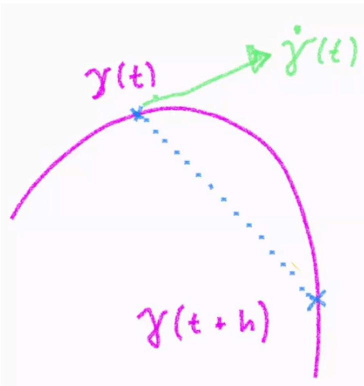
Tangential vectors and speed

The tangent vector of a curve $\gamma(t)$ is: $\dot{\gamma(t)} = \left(\dot{\gamma_1}(t), ..., \dot{\gamma_2}(t)\right)$ and the speed is: $|\dot{\gamma(t)}| = \sqrt{\left(\dot{\gamma_1}(t)^2, ..., \dot{\gamma_2}(t)^2\right)}$



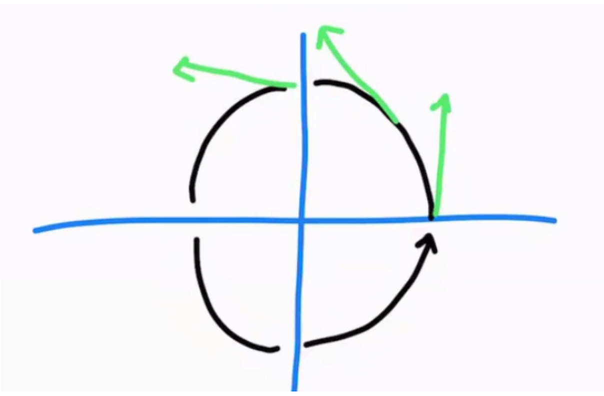
Similar to the definition of the derivative:

$$\dot{\gamma(t)} = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

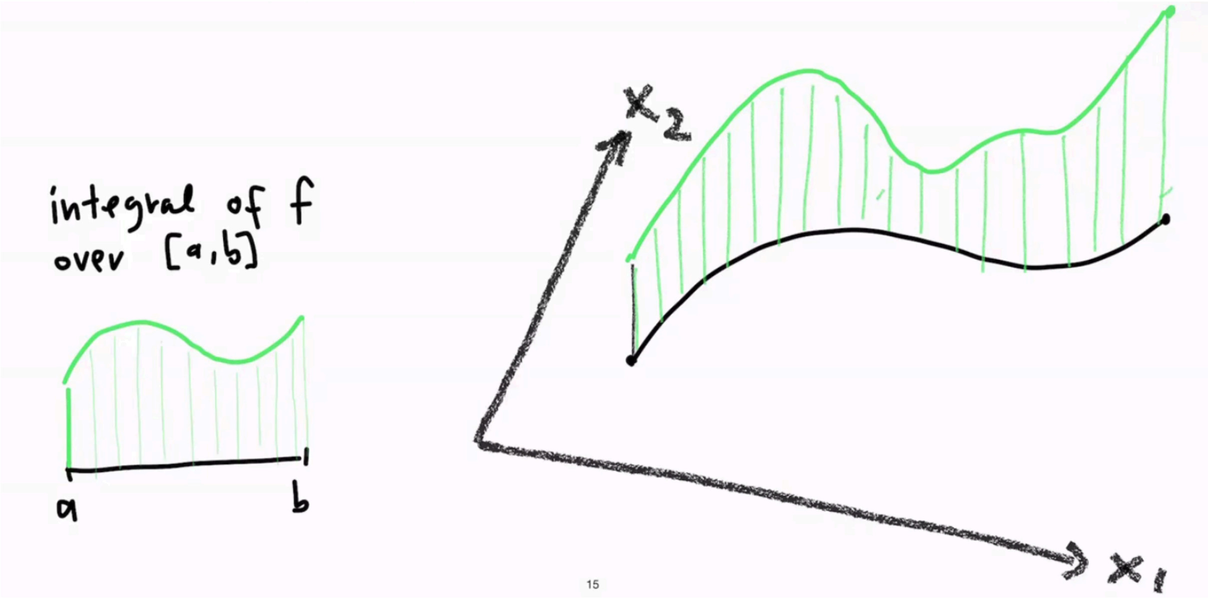


Example:

$\gamma : [0, 2\pi] \to \mathbb{R}^2, t \to (\cos(t), \sin(t))$
$\dot{\gamma(t)} = (-\sin(t), \cos(t))$



Curve Integrals



Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve Let $f : \mathbb{R}^n \to R$ be a scalar field

The integral of f over Γ is:

$$\int_\Gamma f dl := \int_a^b (f \circ \gamma)(t)\ |\dot{\gamma(t)}|\ dt$$

$$= \int_a^b (f \circ \gamma)(t) \sqrt{\left(\dot{\gamma_1(t)}^2 + ... + \dot{\gamma_n}(t)^2\right)} dt$$

The curve integral only depends on the curve Γ, not γ (which is what we want, we need the curve, not the parametrization, see eg. where we had 2 functions for one curve).

- where γ is slow, $\dot{\gamma}$ is small
- where γ is fast, $\dot{\gamma}$ is large

En fait si la fonction va très lentement, on va "utiliser" une grande partie de notre portion de a vers b pour la tracer, mais on réduit dcp le facteur.

If $\gamma : [a, b] \to \mathbb{R}^n$ is a simple regular curve, then $\int_\Gamma F dl$ only depends on Γ.

And it should! After all, γ is just a parametrization and Γ is the "physical" object.

Curve integrals of vector fields

Given a curve $\gamma : [a, b] \to \mathbb{R}^n$ and a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ we define:

$$\int_\Gamma \vec{F} dl := \int_a^b \vec{F}(\gamma(t)) \cdot \dot{y}(t) dt$$

$$= \int_a^b F_1(\gamma(t)) \cdot \dot{y_1}(t) + ... + F_n(\gamma(t)) \cdot \dot{y_n}(t) dt$$

Example:

Suppose $\vec{F} = \nabla f$ is the gradient of scalar field

$$\int_\Gamma \vec{F} dl = \int_\Gamma \nabla f dl = \int_a^b \nabla f(\gamma(t)) \cdot \dot{\gamma}(t) dt$$

We observe

$$(f \circ y)' = \nabla f(\gamma(t)) \cdot \dot{\gamma}(t) \qquad \text{(see Analysis II)}$$

$$= \int_a^b (f \circ \gamma)'(t) dt = f(\gamma(b)) - f(\gamma(a))$$

One more perspective of curve integrals fo vector fields. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ and a curve $\gamma : [a, b] \to \mathbb{R}^n$.

$$\int F dl = \int_a^b F\big(\gamma(t) \cdot \dot{\gamma(t)}\big) = \int_a^b F(\gamma(t)) \cdot \frac{\dot{\gamma(t)}}{\|\dot{\gamma(t)}\|} \, \|\dot{\gamma(t)}\| \, dt$$

At any time t, the vector $\tau(t) = \frac{\dot{\gamma(t)}}{\|\dot{\gamma(t)}\|}$ is the unit tangent vector at time t.

$$\int_\Gamma F dl = \int_a^b F(\gamma(t)) \cdot \tau(t) \cdot \|\dot{\gamma(t)}\| \, dt = \int_\gamma F \cdot \tau dl$$

We can break our integral into pieces of simple and regular differentiable curves.

$$\int_\Gamma f dl = \int_a^b f \cdot y \, |\dot{\gamma}| \, dt = \int_b^c f \cdot y \cdot |\dot{\gamma}| \, dt + \int_c^d f \cdot \gamma \cdot |\dot{\gamma}| \, dt$$

Conservative vector fields and their potentials

Let $F : \Omega \to \mathbb{R}^n$ be a vector field, $\Gamma \subset \mathbb{R}^n$ open.

Does there exist a potential f of F over Ω, i.e. $f \in C'(\Omega, \mathbb{R})$ such that $\nabla f = F$?

Theorem: Let $\Omega \subset \mathbb{R}^n$ be open and $\vec{F} \in C'(\Omega, \mathbb{R}^n)$, $F = (F_1, F_2, ..., F_n)$. If $\vec{F}$ has a potential then $\partial_i F_j = \partial_j F_i$, $1 \leq i, j \leq n$.

Proof: if F admits a potential $f \in C'(\Omega, \mathbb{R})$, then already $f \in C^2(\Omega, \mathbb{R})$. Given $1 \leq i, j \leq n$, we see:

$$\partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j F_i$$

using Schwarz.

Remark: This is a necessary condition but not a sufficient one. We use that $\text{Hess}(f)$ is symmetric.

We call $F \in C'(\Omega, \mathbb{R}^n)$ conservative if $\partial_j F_i = \partial_i F_j$, $i \leq i, j \leq n$.

Let $\Omega \in \mathbb{R}^n$. We call this set:

- **convex** if $\forall (x, y) \in \Omega$ the line segment from x to y is within Ω.
- **star-shaped** if $\exists z \in \Omega \forall x \in \Omega$ the line segment from z to x is within Ω.

Formally:

$$[x, y] := \{tx + (1-t)y : t \in [0, 1]\}$$

line segment from x to y. Image of the curve $\gamma : [0, 1] \to \mathbb{R}^n, t \to tx + (1-t)y$

Ω convex $:\Leftrightarrow \forall (x, y) \in \Omega : [x, y] \in \Omega$
Ω star-shaped $:\Leftrightarrow \exists z \forall x \in \Omega : [z, x] \in \Omega$

Theorem: let $\Omega \subset \mathbb{R}^n$ be open and star-shaped with respect to $z \in \Omega$. If $F \in C'(\Omega, \mathbb{R}^n)$ is conservative, then $\vec{F}$ has a potential $f \in C^2(\Omega, \mathbb{R})$.

$$f(x) := \int_0^1 F(z + t(x - z)) \cdot (x - z) dt$$

$$= \int_y F dl \quad \text{where } y : [0, 1] \to \mathbb{R}^n : t \to z + t(x - z)$$

It depends on the choice of z!

Convex and star-shaped domains are important but simple. What if the domain has holes?

$$\Omega = \left\{x \in \mathbb{R}^3 \mid \|x\| > 1\right\}, \Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$$