Computational Methods

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The purpose of this assignment is to derive reasonable no-arbitrage prices of a couple of simple derivatives written on an asset that presents a "non-standard" dynamics.

In the attached spreadsheet, you will find a time series $\{X_t\}_{t=0}^{N}$ consisting of $N+1$ observations of the daily price of the underlying asset of interest.

1 Assignment

1.1 Problem 1

Question: Looking at both the realized series at your disposal and at the SDE in (1), which kind of market variable (an equity price, a commodity price, an interest rate, ...) can be reasonably represented by X? First of all, you want to verify whether, conditional on your model holding true, the shocks to the price of the underlying are indeed normally distributed as the $dW_t^{\mathbb{Q}} \sim \mathcal{N}(0, dt)$ term would imply.

Answer: The Stochastic Differential Equation which describes the assumed price dynamic of the underlying of interest is given by

$$dX_t = \kappa X_t dt + \sigma \sqrt{X_t}\, dW_t^{\mathbb{Q}} \tag{1}$$

where $\left\{dW_t^{\mathbb{Q}}\right\}_{t\geq 0}$ is a standard Brownian motion under a risk-neutral probability measure and $\Theta = (\kappa, \sigma)$ two non-negative parameters. From initial analysis we can infer:

- it is a particular type of a Square-root diffusion process. Therefore, the process is similar to the Cox-Ingersoll-Ross (CIR) model but without mean reversion, since both the drift and diffusion terms are proportional to X_t
- non-negativity of the underlying price $X_t \geq 0 \quad \forall t$, due to the square root diffusion term $\sigma\sqrt{X_t}$
- exponential growth of the process X_t, due to the drift term κX_t in case of $\kappa > 0$ or no deterministic trend in case of $\kappa = 0$

We have identified two different market variables that could be described by X_t:

- **Implied Volatility of Highly Speculative Assets**

  This model could describe the implied volatility of options on highly speculative assets, such as cryptocurrencies. Unlike the Heston model, which assumes a mean-reverting CIR process for variance, this model allows volatility to grow indefinitely. This is particularly relevant in speculative markets, where volatility tends to "explode" rather than revert to a long-term mean.

- **Interest Rates in Emerging Economies**

  Alternatively, X_t could represent interest rates in emerging markets, particularly in environments characterized by hyperinflation, structural interest rate growth and unstable monetary policies. Unlike developed economies, where interest rates typically exhibit mean-reverting behavior, emerging markets often experience prolonged trends without convergence to an equilibrium level.

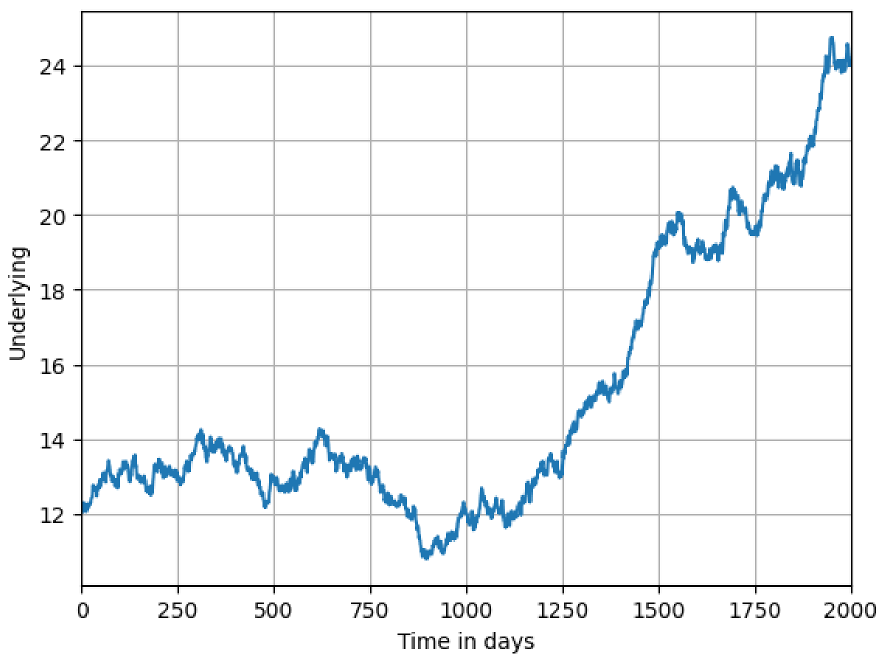

Figure 1: Dynamics of the price of the underlying during the sampled period

By plotting the underlying for all time in Figure 1, we can distinguish three distinct dynamics. From day 0 to day 750 (roughly), we observe a volatile and low growth trend. This is followed by a decay-like dynamic that lasts until day 800 (roughly), and a subsequent sharp growth trend. The dynamics and values assumed by the underlying appear to be more supportive of the emerging markets interest rate hypothesis.

1.2 Problem 2

Question: Conditioning on the first realized value x_0 and on an educated guess for the parameters Θ, use a Euler discretization of the SDE in (1) to recursively compute the time series of the standard normally distributed shocks $\{z_t\}_{t=1}^{N} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. Then, study their unconditional distribution. Do they actually look normally distributed? After this preliminary check, you need to find the best possible estimates for the parameters Θ.

Answer: In order to verify that the shocks in our model are normally distributed as the $dW_t \sim \mathcal{N}(0, dt)$ term would imply, we rearrange the stochastic differential equation 1 and obtain the following:

$$dX_t = \kappa X_t dt + \sigma\sqrt{X_t} dW_t^{\mathbb{Q}}$$
$$dW_t^{\mathbb{Q}} = \frac{dX_t - \kappa X_t dt}{\sigma\sqrt{X_t}}$$
$$Z = \frac{1}{\sqrt{dt}} \frac{dX_t - \kappa X_t dt}{\sigma\sqrt{X_t}} \sim \mathcal{N}(0,1)$$

By applying Euler discretization, we extract dX_t from the data taking $dX_t = X_{t+\Delta t} - X_t$, where we use $\Delta t = \frac{1}{250}$. This allows us to calculate the time series of the shocks$\left\{z_t^{\mathbb{Q}}\right\}_{t=1}^{N}$ conditioned on the value of x_0.

We make the following educated guess on the parameters Θ:

$$\kappa = 0.2$$
$$\sigma = 0.5$$

To assess the normality of the standardized shocks, first of all we compute the distribution parameters, getting the following results:

$$\mu = -0.065$$
$$\sigma = 0.988$$

In addition, we rely on graphical analysis through the histogram with the normal probability density function and the normal Q-Q plot. These graphs provide visual evidence of the distributional properties of z_t and help identify potential deviations from normality.

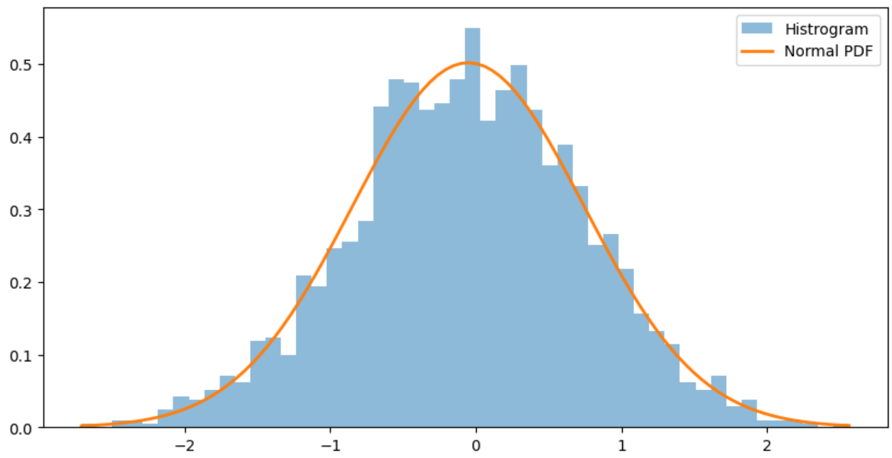

Figure 2: Empirical Distribution of Standardized Shocks vs Gaussian PDF

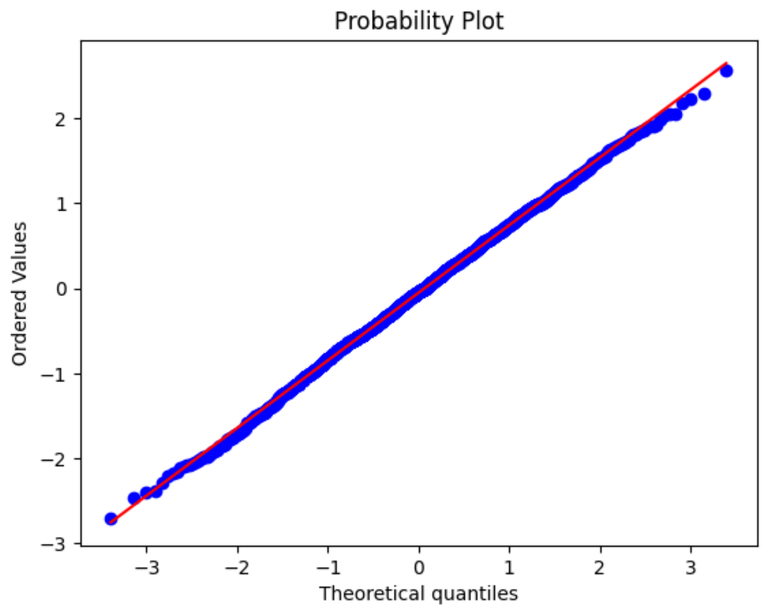

Figure 3: Q-Q Plot of Standardized Shocks against Standard Normal Distribution

The visual analysis confirms that the standardized shocks z_t follow a standard normal distribution, consistent with the assumption that the term $dW_t^{\mathbb{Q}}$ in the model is the differential of a standard Brownian motion. We can therefore proceed with the next steps of the analysis under this assumption.

1.3 Problem 3

Question: Relying on the normality assumption of the shocks $\{z_t\}_{t=1}^N$ from the previous point, recursively derive the expected value and the variance of $\widehat{X}_{t+\Delta t}$ as a function of $\widehat{X}_t$ and Θ. Namely, find $a(\widehat{X}_t;\Theta)$ and $b(\widehat{X}_t;\Theta)$ such that

$$\widehat{X}_{t+\Delta t} \sim N(a(\widehat{X}_t;\Theta), b(\widehat{X}_t;\Theta))$$

Then, find the MLE-based estimate of Θ.

Having found the best possible estimates for the parameters Θ, you can turn to the pricing problems below. Assume that the current spot price of the underlying is the last available observation of the time series at your disposal. Namely, set $X(0) = x_N$.

The first derivative to price is a plain vanilla European call option on X with maturity $T = 0.6$ and strike price $K = X(0)$. Assuming that the constant risk-free rate for this economy is equal to $r = 2\%$, you need to compute

$$\pi_0^E = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(X_T - K)^+] \tag{2}$$

Answer: In order to obtain the MLE-based estimate of Θ we discretize the underlying stochastic process X_t:

$$\begin{aligned} X_{t+\Delta t} &= X_t + dX_t \\ &= X_t + \kappa X_t \Delta t + \sigma\sqrt{X_t}\, dW_t^{\mathbb{Q}} \\ &= X_t + \kappa X_t \Delta t + \sigma\sqrt{X_t}\,\sqrt{\Delta t}\,\mathcal{N}(0,1) \end{aligned}$$

where $\Delta t = \frac{1}{250}$.

From the Euler discretization of the process X_t we infer that the unconditional expected value and variance can be derived as:

$$\begin{aligned} \mathbb{E}[X_{t+\Delta t}] &= X_t + \kappa X_t \Delta t \\ Var(X_{t+\Delta t}) &= \sigma^2\, X_t \Delta t \end{aligned}$$

To estimate the best possible set of parameters $\Theta = (\kappa, \sigma)$ using Maximum Likelihood Estimation (MLE), given the sample of independent identically distributed (i.i.d.) of $\{X_t\}_{t=1}^T$, we define the likelihood function as:

$$L(\Theta; X) := \prod_{t=1}^{T} f_X(x_i;\Theta)$$

Minimizing the negative of log-likelihood function, $logL(\Theta; X)$, with respect to Θ we obtain as a result the MLE estimators $\hat{\kappa}$ and $\hat{\sigma}$:

$$\begin{aligned} \hat{\kappa} &= \text{argmax}_\kappa\, \ell(\kappa, \sigma; X) \\ \hat{\sigma} &= \text{argmax}_\sigma\, \ell(\kappa, \sigma; X) \end{aligned}$$

1.4 Problem 4

Question: Estimate π_0^E using Monte Carlo techniques.

Answer: Our goal is to estimate the price of a plain-vanilla European call on the underlying using Monte Carlo techniques. Formally, we estimate π_0^E where

$$\pi_0^E = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[(X_T - K)^+]$$

Firstly, we construct a Euler discretization $\hat{X}_t$ of the process X_t on the set of monitoring dates $\{t_i\}_{i=0,1\ldots,n}$. Setting $\Delta t = \frac{1}{250}$, taking a unit measurement of time of one year and accounting for daily observations, we obtain the number of steps $n = \frac{T}{\Delta t} = 150$. We define the Euler discretization $\hat{X}_{t_i}$ of the process X_t as in 1.3 by

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + \kappa \hat{X}_{t_i}\,\Delta t + \sigma\sqrt{X_{t_i}}\,\sqrt{\Delta t}\mathcal{N}(0,1)$$

for $i = 0, 1, \ldots, n$ with $\hat{X}_0 = X_0$. We repeat this process $n_{sim} = 20000$ times and obtain simulations of the path of the stochastic process of the underlying X_t. We plot 50 paths of the simulations in Figure 4 for simplicity.

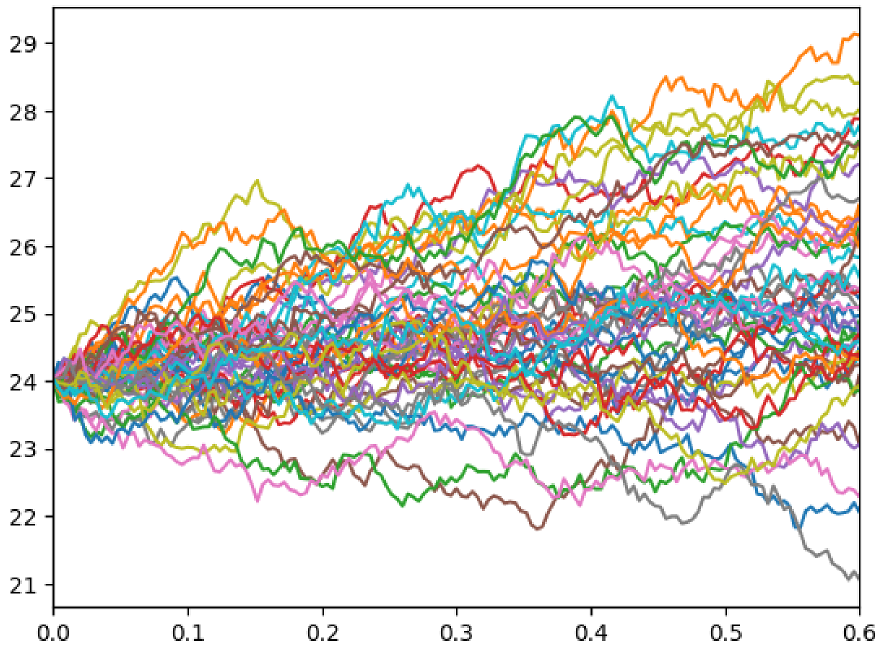

Figure 4: Simulated paths of the underlying process

Taking the final values $\hat{X}_T$ we determine the expected value of the European call payoff, which we discount to the present moment. Finally, we obtain the no-arbitrage price of the European call using Monte Carlo simulations:

$$\pi_0^E = 1.5834$$

1.5 Problem 5

Question: Compute π_0^E by backward recursion along a computationally feasible lattice discretization of X.

Answer: We compute the no-arbitrage price of the European call 1.7 using a backward recursion along a computationally feasible lattice discretization of process X_t. Due to the non-constant diffusive coefficient of the process X_t, we apply a (invertible) transformation $f(X_t)$ and build its lattice. The goal is to obtain a transformation $f(X_t)$ that has a constant diffusive coefficient. We explicitly solve the following problem:

$$f(x) = \int \frac{1}{b(x)} dx$$

where $b(x) = \sigma\sqrt{x}$ and we obtain

$$f(x) = \frac{2}{\sigma}\sqrt{x}.$$

Computing the stochastic differential of $f(X_t) = \frac{2}{\sigma}\sqrt{X_t}$ using Itô's formula, we get

$$\begin{aligned} df(X_t) &= a_f(t, X_t)dt + b_f(t, X_t)dW_t \\ &= \left(\frac{\kappa\sqrt{X_t}}{\sigma} - \frac{\sigma}{4\sqrt{X_t}}\right) dt + dW_t \end{aligned}$$

which entails a constant diffusive coefficient. Therefore, we build the lattice for $f(X_t) = \frac{2}{\sigma}\sqrt{X_t}$ as

$$\begin{cases} f(\widehat{X_{t+\Delta t}}) = f(\widehat{X_t}) \pm \Delta f \\ f(\widehat{X_0}) = f(x_N) \end{cases}$$

setting the increment in the function $f(X_t)$ for every step as

$$\Delta f = \sqrt{\Delta t}$$

and the probability of an upward movement as

$$q(X_t) = \max\left\{0, \min\left\{1, \frac{1}{2} + \frac{\left(\frac{\kappa\sqrt{X_t}}{\sigma} - \frac{\sigma}{4\sqrt{X_t}}\right)\sqrt{\Delta t}}{2}\right\}\right\}$$

We keep the parameter $dt = \frac{1}{250}$, therefore $n = 150$, as in 1.4 for consistency. We map the former lattice into the one for X_t, applying f^{-1} and we obtain the following result:

$$
\begin{cases}
\widehat{X}_{t+\Delta t} = \begin{cases} f^{-1}\left(\widehat{f(X_t)} + \Delta f\right) & \text{with probability} \quad \max\left\{0, \min\left\{1, \frac{1}{2} + \frac{\left(\frac{\kappa\sqrt{X_t}}{\sigma} - \frac{\sigma}{4\sqrt{X_t}}\right)\sqrt{\Delta t}}{2}\right\}\right\} \\ f^{-1}\left(\widehat{f(X_t)} - \Delta f\right) & \text{with probability} \quad \max\left\{0, \min\left\{1, \frac{1}{2} - \frac{\left(\frac{\kappa\sqrt{X_t}}{\sigma} - \frac{\sigma}{4\sqrt{X_t}}\right)\sqrt{\Delta t}}{2}\right\}\right\} \end{cases} \\
\widehat{X}_0 = x_N
\end{cases}
$$

Finally, we apply a backward recursion loop to obtain the estimate of the price of the European call at each step in time. We obtain the no-arbitrage price at time zero of:

$$\pi_0^E = 1.5816$$

1.6 Problem 6

Question: Compute π_0^E solving the PDE-representation of the (discounted) expected value in (2) by means of a finite difference explicit scheme.

Answer: To calculate the price of an european derivative using the explicit finite difference scheme for a general stochastic process, we need to numerically solve the following PDE:

$$
\begin{cases}
\frac{\partial f}{\partial t} + a(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}b^2(x,t)\frac{\partial^2 f}{\partial x^2} = rf & \text{for } (t,x) \in [0,T) \times \mathbb{R}^+ \\
f(T,x) = F(x) & \text{for } x \in \mathbb{R}^+
\end{cases}
$$

As for our process, the PDE becomes the following:

$$
\begin{cases}
\frac{\partial f}{\partial t} + \kappa x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x\frac{\partial^2 f}{\partial x^2} = rf & \text{for } (t,x) \in [0,T) \times \mathbb{R}^+ \\
f(T,x) = F(x) & \text{for } x \in \mathbb{R}^+
\end{cases}
$$

We discretize the domain of the PDE: $[0,T] \times \mathbb{R}^+$. Firstly, we limit the second dimension with $\overline{S} = 2x_N$, where x_N is the final observation in the sample. We set a uniform grid with:

$$G = \{(i\Delta t, j\Delta x) : i = 0, 1, \ldots, m_t, j = 0, 1, \ldots, m_x\}$$

where Δt is fixed as in the previous problems for consistency, therefore $m_t = n = 150$, we set $m_x << m_t$ for stability as $m_x = 100$ (additionally we take into account m_x needs to be an even number to include x_N), finally we set $\Delta x = \frac{\overline{S}}{m_x}$.

We approximate the partial derivatives with finite differences: the partial derivative with respect to time is approximated with a backward difference, whereas the first and second partial derivatives with respect to the value of the underlying are approximated with central differences in the following way:

$$\frac{\partial f}{\partial t} = \frac{f^{i,j} - f^{i-1,j}}{\Delta t}$$

$$\frac{\partial f}{\partial x} = \frac{f^{i+1,j} - f^{i-1,j}}{2\Delta x}$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{f^{i,j+1} - 2f^{i,j} + f^{i,j-1}}{(\Delta x)^2}$$

Plugging those formulas into the PDE, we obtain a formula that allows us to calculate the value of the derivative at a previous time node knowing the values at subsequent grid points. This method works by calculating the value of the derivative backward in time, starting with the final value at maturity, where the value is known because it coincides with the option payoff. Next, we iterate backward by applying the discrete formula until we obtain the option price at the initial time.

$$f^{i,j} = \Delta t \left[\frac{f^{i+1,j-1}}{2} \left(\frac{\sigma^2 j}{\Delta x} - \kappa j \right) + f^{i+1,j} \left(\frac{1}{\Delta t} - r - \frac{\sigma^2 j}{\Delta x} \right) + \frac{f^{i+1,j+1}}{2} \left(\frac{\sigma^2 j}{\Delta x} + \kappa j \right) \right]$$

The price obtained from the pricing technique just described is:

$$\pi_0^E = 1.5733$$

To properly run the simulation, we also need to define the boundary conditions. When the price of the underlying is equal to 0, the value of the call is zero, since it will end up out of the money (OTM).

$$\text{Call: } f(0, t) = 0$$

However, if the underlying is at the maximum level of the underlying properly defined, the call takes on a value close to the discounted value of the difference between the price and the strike price, since it will end up in the money (ITM).

$$\text{Call: } f(X, t) = (X - K)e^{-r(T-t)}$$

1.7 Problem 7

Question: Using a numerical technique of your choice, compute the price at inception π_0^A of the American version of the call option previously analysed. Namely, compute

$$\pi_0^A = \sup_{\tau \in [0,T]} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-\tau)} (X_\tau - K)^+ \right]$$

Compute also the early exercise premium $\pi_0^A - \pi_0^E$, if any.

Answer: Our aim is to estimate the price of the American version of the call option previously analysed. Namely, we estimate π_0^A where

$$\pi_0^A = \sup_{\tau \in [0,T]} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-\tau)} (X_\tau - K)^+ \right]$$

The price of this American call option has been computed using a backward recursion along the lattice discretization of X_t found in *point* 5. At each time step, we compare:

- The **continuation value**, representing the expected discounted value of holding the option.
- The **immediate exercise value**, given by the intrinsic value of the option.

We apply a backward recursion loop to obtain the estimate of the price of the American call at each step in time. We obtain the no-arbitrage price at time zero of:

$$\pi_0^A = 1.5816$$

From the results, we observe that the American call option has the same price as the European call option. This implies that early exercise is never optimal, and therefore, there is no early exercise premium at $t = 0$.

$$\pi_0^A - \pi_0^E = 0$$

1.8 Problem 8

Question: Given $B > X(0)$, the payoff at maturity T of the up-and-out barrier version of the (European) call option previously analysed is

$$\begin{cases} (X_T - K)^+ & \text{if } \max_{t\in[0,T]} S(t) \leq B \\ 0 & \text{else} \end{cases} = (X_T - K)^+ \mathbb{I}_{\{\max_{t\in[0,T]} S(t) \leq B\}}$$

Using a numerical technique of your choice, compute the price at inception $\pi_0^{UO,E}$ of the up-and-out barrier version of the (European) call option previously analysed. Namely, compute

$$\pi_0^{UO,E} = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(X_T - K)^+ \mathbb{I}_{\{\max_{t\in[0,T]} S(t) \leq B\}}\right]$$

with $B = 28$.

Answer: In order to evaluate the up-and-out barrier option, a Monte Carlo simulation was performed. Given the pay-off of the option at maturity:

$$\text{Payoff} = \begin{cases} (X_T - K)^+ & \text{if } \max_{t\in[0,T]} S(t) \leq B, \\ 0 & \text{otherwise} \end{cases}$$

If the price of the underlying exceeds the barrier math B at any time before expiration, the option loses its value and is not exercised.

After performing a Monte Carlo simulation, the pay-off at maturity is calculated as a European plain vanilla call option. To capture the path-dependence nature of the barrier option, a restriction is applied to the pay-off vector: any in the money option is excluded if the price trajectory of the underlying has exceeded the predetermined barrier level, B, during the lifetime of the derivative. From the procedure just described, we obtain price:

$$\pi_0^{UO,E} = 1.2272$$

In addition, exploiting the property of barrier options that the price of an up-knock-in call and the price of an up-knock-out summed must return the price of a plain vanilla call option with the same contractual characteristics, the corresponding up-knock-in 'twin' was also priced to verify the correct pricing of the exotic derivative.

$$\pi_0^{UI,E} = 0.3538$$

$$\pi_0^{E} = \pi_0^{UO,E} + \pi_0^{UI,E} = 1.5810$$