Matrix A, B: A*B 不一定等于 B*A, matrix 乘法 direction matters

Identity Matrix: I*A = A: 1 in diagonal position.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}*\begin{bmatrix}a & b\\c & d\end{bmatrix}=\begin{bmatrix}a & b\\c & d\end{bmatrix}$$

Why it works? Row information from indentity matrix 的 row and column information from A 的 column，比如[1,0] 乘以 $\begin{bmatrix}a\\c\end{bmatrix}$ or $\begin{bmatrix}b\\d\end{bmatrix}$, 0 cancel out every elements 除了 first term (a,b) in the column vector，第二行[0,1] cancel out every elemts 除了 second term(c, d)

Inverse 2*2:

$$\mathrm{A}^{-1}*A=I,\mathrm{A}*\mathrm{A}^{-1}=I,\mathrm{A}\text{ is also inverse of }\mathrm{A}^{-1}$$

Calculate the inverse:

$$\mathrm{A}=\begin{bmatrix}a & b\\c & d\end{bmatrix},\ A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}=\frac{1}{|A|}\begin{bmatrix}d & -b\\-c & a\end{bmatrix},\quad ab-bc\ \ is\ determinant\ of\ A$$

Determinant: $|\mathrm{A}|=ad-bc$

Indentiy matrix: I*A = A, A*I = A，两个都满足的只有当 A 是 square matrix 的时候

Inverse 3*3: Gauss Jordan elimination (augment the matrix, operation: elementary row operation)

Perform some operation 在 left side and same operation on right side，当 have indentity matrix 在 left-hand side(变成 indentiy matrix 的形式 叫做 reduced row echelon form), right-hand side 就是原来的 invers

$$\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\0 & 2 & 1 & 0 & 1 & 0\\1 & 1 & 1 & 0 & 0 & 1\end{array}$$

Row3 = row3 – row1

$$\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\0 & 2 & 1 & 0 & 1 & 0\\0 & 1 & 0 & -1 & 0 & 1\end{array}$$

Swap row3 and row2

$$\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\0 & 1 & 0 & -1 & 0 & 1\\0 & 2 & 1 & 0 & 1 & 0\end{array}$$

Row3 = row3 – 2*row2

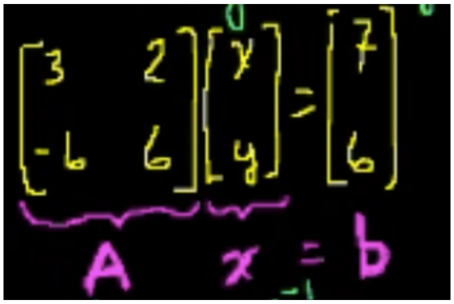
$$\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\0 & 1 & 0 & -1 & 0 & 1\\0 & 0 & 1 & 2 & 1 & -2\end{array}$$

Row1 = row1 – row3

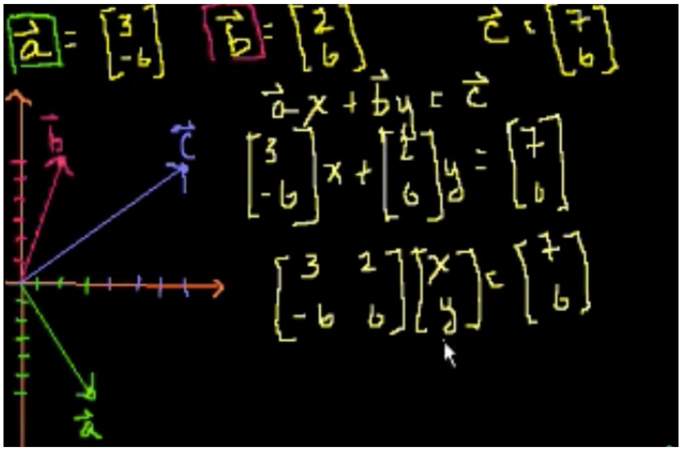
$$\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & -1 & 2\\0 & 1 & 0 & -1 & 0 & 1\\0 & 0 & 1 & 2 & 1 & 0\end{array}$$

Hint why this work: 当对左面 matrix 进行操作可以想成乘以多个 matrices，so we multiply matrix 得到 indentity matrix，乘以的多个 matrices 就是A^{-1}，而我们知道 identity matrix 乘以任何 matrix 就是 matrix itself

Solve system of equations (2*2):



Ax = b → $A^{-1}Ax = A^{-1}b$ → $Ix = A^{-1}b$ → $x = A^{-1}b$



Matrices to solve vector combination: 可以想成 matrix multiplication problem 把两个 vector 合成一个 vector

比如 vector a = $\begin{bmatrix}3\\-6\end{bmatrix}$, b = $\begin{bmatrix}2\\6\end{bmatrix}$, 得到 c = $\begin{bmatrix}7\\6\end{bmatrix}$

A = $\begin{bmatrix}3 & 2\\-6 & 6\end{bmatrix}$ * $\begin{bmatrix}x\\y\end{bmatrix}$ = $\begin{bmatrix}7\\6\end{bmatrix}$

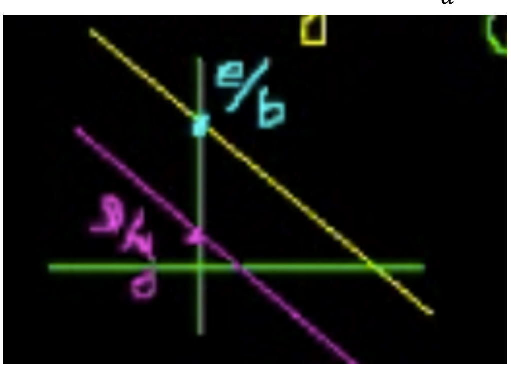
$A^{-1} = \frac{1}{30}\begin{bmatrix}6 & -2\\6 & 3\end{bmatrix}$, $A^{-1} * \begin{bmatrix}7\\6\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$, 需要 1 个 a,两个 b

Singular matrices: 没有 inverse 的 matrix

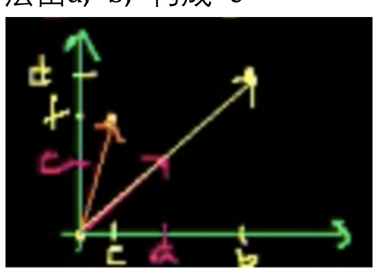
$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}, \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}, A^{-1} \text{ is undefined iff } |A| = ad - bc = 0 \ \ (\text{or } \frac{a}{b} = \frac{c}{d} \text{ or } \frac{a}{c} = \frac{b}{d})$$

Prove: 比如 $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}e\\f\end{bmatrix}$

得到 $\begin{matrix}ax + by = e\\cx + dy = f\end{matrix}$ => $\begin{matrix}y = -\frac{a}{b}x + \frac{e}{b}\\y = -\frac{c}{d}x + \frac{f}{d}\end{matrix}$ 如果$\frac{a}{b} = \frac{c}{d}$, 两条线平行，不会相交没有解



如果从 vector 角度考虑，如下图 $\begin{bmatrix}a\\c\end{bmatrix}$, $\begin{bmatrix}b\\d\end{bmatrix}$ 是重合的 if $\frac{a}{c} = \frac{b}{d}$, 无法由$\vec{a}$, $\vec{b}$, 构成 $\vec{c}$



Solve system of equations (3*3):

$$\begin{aligned}-x+2y-z&=9\\3x-7y-2&=-20\\2x+2y+z&=2\end{aligned} \quad => \left[\begin{array}{ccc|c}-1&2&-1&9\\3&-7&-2&-20\\2&2&1&2\end{array}\right]$$

通过 row operation 变成

$$\left[\begin{array}{ccc|c}-1&2&-1&9\\0&-1&-5&7\\0&0&1&-2\end{array}\right] => \begin{aligned}x&=-1\\y&=3\\z&=-2\end{aligned}$$

Vectors and Spans:

Set Colinear vectors: {c S = {c $\vec{v}$ | c ∈ R} 比如 vector 在一条线上(slope), $\vec{v}=\begin{bmatrix}2\\1\end{bmatrix}$

Linear Combination: $c_1v_1+c_2v_2+\cdots+c_nv_n$, $c_1 \to c_n \in R$; We can fill Any point in R^2 with combination of vector a and b: we can write span $(\vec{a},\vec{b})=R^2$, we can represent any vector in R^2 with some linear combination of a and b where a and b cannot be collinear (a,b 不能共线, 换种思维考虑: 如果共线了,组成的 matrix 没有 inverse A*c = b, A 没有 inverse). span $(\vec{0})$ = $\vec{0}$ (c$\begin{bmatrix}0\\0\end{bmatrix}$ = $\begin{bmatrix}0\\0\end{bmatrix}$)

比如 unit vector $\hat{\imath}=\begin{bmatrix}1\\0\end{bmatrix}, \hat{\jmath}=\begin{bmatrix}0\\1\end{bmatrix}$, 可以构建任何 vector in R^2 by using these unit vectors

span $(v_1,v_2,v_3)=\{c_1v_1+c_2v_2+\cdots+c_nv_n \mid c_i \in R\ 1\le i\le n\}$: The space of all of the combination of vectors $v_1, v_2 \ldots . v_n$

Linearly Dependent set: some vector in the set can be represented by some combinations of other vectors in the set, 比如$\begin{bmatrix}2\\3\end{bmatrix}$, $\begin{bmatrix}4\\6\end{bmatrix}$是 linearly dependent, 再比如 $\begin{bmatrix}2\\3\end{bmatrix}$ $\begin{bmatrix}7\\2\end{bmatrix}$ $\begin{bmatrix}9\\5\end{bmatrix}$是 linearly dependent,因为其中一个可以由另外两个构成构成

R^n : $\left\{\begin{bmatrix}x_1\\x_2\\x_n\end{bmatrix} x_i \in R,\ 1\le i\le n\right\}$

V is Subspace of R^n (me vector from R^n) 必须满足：

1. V contains 0 vector
2. If $\vec{x}$ in V then any scaler c: $c\vec{x}$ also in V (closure under scaler multiplication)
3. If $\vec{a}$ in V and $\vec{b}$ in V, $\vec{a}$ + $\vec{b}$ also in V (closure under addition)

同样如果满足这三个条件的也是 subspace

e.g. v = {0} = $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$: v 只有 vector 0, v is subspace of R^3

1. 满足条件 1, vector 0 在 v 中
2. 满足条件 2: c$\begin{bmatrix}0\\0\\0\end{bmatrix}$ = $\begin{bmatrix}0\\0\\0\end{bmatrix}$
3. 满足条件 3: $\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

e.g. S =$\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix} \in R^2 \mid x\ge 0\right\}$: v is not subspace of R^2

1. 满足条件 1, vector 0 在 v 中
2. 不满足条件 2: -1$\begin{bmatrix}a\\b\end{bmatrix}$, -1*a 为负数
3. 满足条件 3: $\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}a+c\\b+d\end{bmatrix}$ a+c 是正数 given a>=0 and c>=0

e.g V = span (v_1,v_2,v_3). is valid subspace of R^n

1. 满足条件 1: $\vec{0}=0\vec{v_1}+0\vec{v_2}+0\vec{v_3}$
2. 满足条件 2: $\vec{x}=c_1\vec{v_1}+c_2\vec{v_2}+c_3\vec{v_3}$; then $a\vec{x}=ac_1\vec{v_1}+ac_2\vec{v_2}+ac_3\vec{v_3}$; ac_1, ac_2, ac_3 can be arbitrary constant, 因为 span 是 all linear combination 所以新的也在 span 当中

3. 满足条件 3： $\vec{x} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + c_3\overrightarrow{v_3}$; $\vec{y} = d_1\overrightarrow{v_1} + d_2\overrightarrow{v_2} + d_3\overrightarrow{v_3}$, then $\vec{x} + \vec{y} = (c_1 + d_1)\overrightarrow{v_1} + (c_2 + d_2)\overrightarrow{v_2} + (c_3 + d_3)\overrightarrow{v_3}$, it also in span

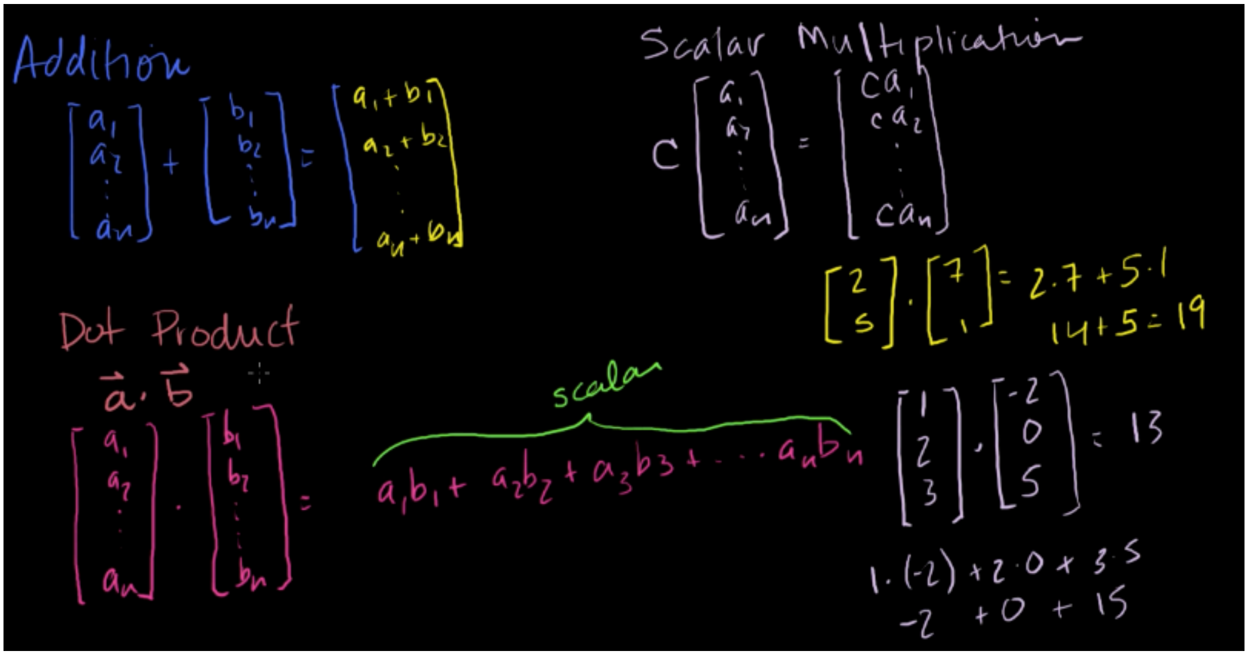
e.g V = span ($\begin{bmatrix}1\\1\end{bmatrix}$). is valid subspace of R^2

1. 满足条件 1: $\vec{0} = 0\begin{bmatrix}1\\1\end{bmatrix}$
2. 满足条件 2: c $\begin{bmatrix}1\\1\end{bmatrix}$ 就是 span itself (combination of vector)
3. 满足条件 3： $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix}; = (c_1 + c_2)\begin{bmatrix}1\\1\end{bmatrix}$ 在 span 当中

Span($\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$) of any vector is valid subspace,
Basis S= {$\boldsymbol{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}}$ }: 1. Span($\boldsymbol{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}}$ all those $\boldsymbol{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}}$ linearly independent 2. $\mathbf{c_1}\boldsymbol{v_1} + \boldsymbol{c_2 v_2} + \cdots + \mathbf{c_n v_n}$ = 0 when $\mathbf{c_1} = \boldsymbol{c_2} = \cdots = \boldsymbol{c_n} = \mathbf{0}$
Basis(minimum set of vectors that spans the subspace): 如果用 any vector in S 可以 construct any vector in subspace V

e.g. T = $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}, \overrightarrow{v_s}(= \mathrm{c} + \vec{2})\}$, the span of T is still going to be subspace V but T is linearly dependent -> T is not basis for V ( )
Basis：比如，需要两个 non-redundant vector
Standard Basis for R^2 $T = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$
Advantage of Basis: represent any vector in subspace by some unique combination of vectors in basis 比如 Basis $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ $\vec{a} \in U, \vec{a} = \mathrm{c}_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_n\overrightarrow{v_n}$, $\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_\mathrm{n}$是 unique 的

Vectors Dot Product



Length : $||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ $\quad\quad \vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + \cdots + a_n^2 = ||\vec{a}||^2$
Communicative: $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$
Distributive: $(\vec{v} + \vec{w})\cdot\vec{x} = \vec{w}\vec{x}\cdot\vec{v}\vec{x} \quad = (v_1 + w_1)x_1 + (v_2 + w_2)x_2 + \cdots + (v_1 + w_1)x_1 = v_1x_1 + w_1x_1 + v_2x_2 + w_2x_2 \ldots + v_nx_n + w_nx_n$
Associative: $(c\vec{v})\cdot\vec{w} = c(\vec{v}\cdot\vec{w})$

Cauchy Schwarz Inequality: If $\vec{x}, \vec{y} \in R^n$, $|\vec{x}\cdot\vec{y}| \le ||\vec{x}||\,||\vec{y}||$, $\quad |\vec{x}\cdot\vec{y}| = ||\vec{x}||\,||\vec{y}||$ only if two vector colinear, one vector 是另一个 vector 乘以的倍数 ($|\vec{x}\cdot\vec{y}|$ *absolute value of dot product*)
Prove: suppose $\vec{x}$, $\vec{y}$ is non-zero vector

$$\text{suppose } p(t) = ||t\,\vec{y} - \vec{x}\,||^2 \ge 0$$
$$p(t) = (t\,\vec{y} - \vec{x})(t\,\vec{y} - \vec{x})$$
$$p(t) = t\,\vec{y}\cdot t\,\vec{y} - \vec{x}\cdot t\,\vec{y} - t\,\vec{y}\cdot\vec{x} + \vec{x}\cdot\vec{x}$$
$$p(t) = t^2(\vec{y}\;\vec{y}) - 2\,(\vec{x}\cdot\vec{y})t + \vec{x}\cdot\vec{x} \ge 0$$

此时设$\vec{y}\;\vec{y} = a,\ 2(\vec{x}\cdot\vec{y}) = b, \vec{x}\cdot\vec{x} = c$

$$p(t) = at^2 - bt + c \ge 0$$
$$p\left(\frac{b}{2a}\right) = a\frac{b^2}{4a^2} - b\,\frac{b}{2a} + c \ge 0$$
$$p\left(\frac{b}{2a}\right) = \frac{b^2}{4a} - \frac{2b^2}{4a} + c$$
$$p\left(\frac{b}{2a}\right) = -\frac{b^2}{4a} + c \ge 0 \quad \Rightarrow c \ge \frac{b^2}{4a} \Rightarrow 4ac \ge b^2$$
$$4ac \ge b^2 = 4(\,||\vec{y}\,||^2\ ||\vec{x}||^2) \ge \left(2\ (\vec{x}\cdot\vec{y})\right)^2$$
$$(\,||\vec{y}\,||^2\ ||\vec{x}||^2) \ge (\vec{x}\cdot\vec{y})^2$$

Take square root

$$||\vec{y}\,||\ ||\vec{x}|| \ge |\vec{x}\cdot\vec{y}|$$

When $\vec{x} = c\vec{y}$

$$|\vec{x} \cdot \vec{y}| = |\, c\,\vec{x} \cdot \vec{y}| = c\,|\vec{y} \cdot \vec{y}| = c\left|\|\vec{y}\|\right|^2 = \|c\vec{y}\| \cdot \|\vec{y}\|$$

Triangle Inequality:

$$\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\cdot\,\vec{y})$$
$$= \vec{x}(\vec{x}+\vec{y}) + \vec{y}(\vec{x}+\vec{y})$$
$$= \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y}$$
$$= \left|\|\vec{x}\|^2 + 2(\vec{x}\cdot\vec{y}) + \|\vec{y}\|\right|^2$$

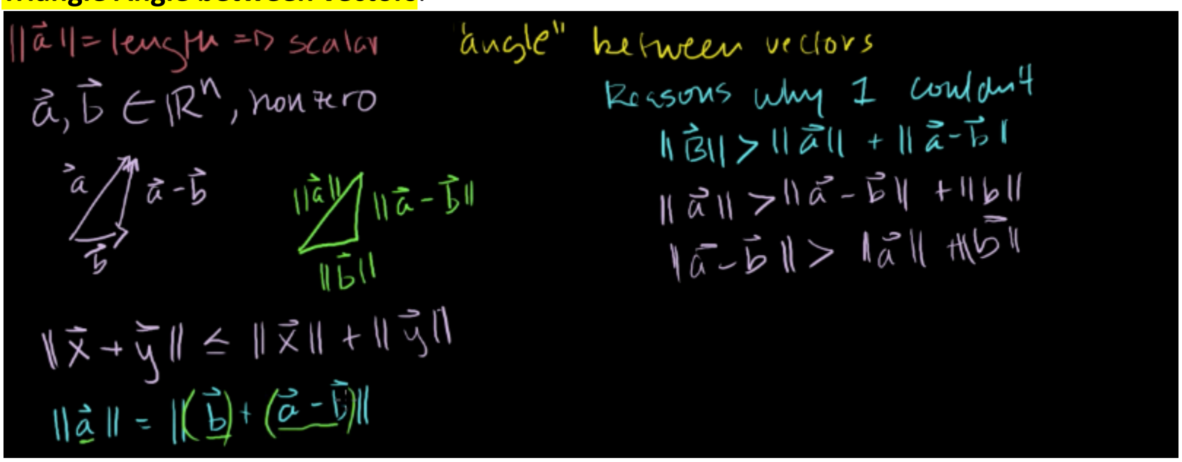
根据 cauchy Schwarz inequality: $\vec{x}\cdot\vec{y} \le |\vec{x}\cdot\vec{y}| = \|\vec{x}\|\|\vec{y}\|$, dot product 可以是负数

$$\|\vec{x}+\vec{y}\|^2 \le \left|\|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|\right|^2 = (\|\vec{x}\| + \|\vec{y}\|)^2$$
$$\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$$

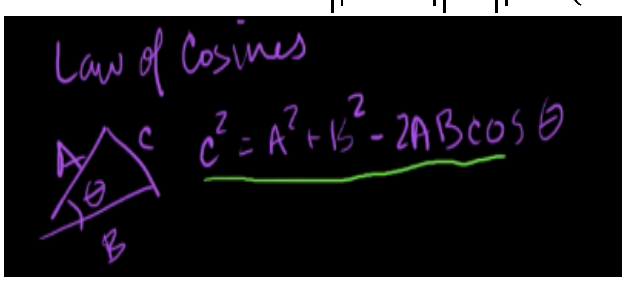
当 $\vec{x} = c\,\vec{y}$, c 是 positive 的时候$(c > 0)$, $\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$

$\vec{x}$ doesn't need to only 2-dimensional, 可以是 n dimension

Triangle Angle between vectors:



$$\|\vec{a}\| = \left\|\vec{b} + (\vec{a} - \vec{b}\,)\right\| \le \left\|\vec{b}\right\| + \|\vec{a} - \vec{b}\|$$
$$\|\vec{b}\| = \left\|\vec{a} + (\vec{b} - \vec{a}\,)\right\| \le \|\vec{a}\| + \|\vec{b} - \vec{a}\|$$
$$\left\|\vec{a} - \vec{b}\right\| = \left\|\vec{a} + (-\vec{b})\right\| \le \|\vec{a}\| + \left\|-\vec{b}\right\| = \|\vec{a}\| + \left\|\vec{b}\right\|$$



Law of Cosine

$$\left\|\vec{a} - \vec{b}\right|^2 = \left|\|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\|\vec{a}\|\left\|\vec{b}\right\| cos\theta$$

Left-hand side

$$(\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$
$$= \|\vec{a}\|^2 + \|b\|^2 - 2(\vec{a}\cdot\vec{b})$$

Left-hand side = right-hand side

$$\|\vec{a}\|^2 + \|b\|^2 - 2(\vec{a}\cdot\vec{b}) = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\|\vec{a}\|\left\|\vec{b}\right\| cos\theta$$

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\left\|\vec{b}\right\| cos\theta$$

If $\vec{a} = c\,\vec{b}$; $c > 0 \Rightarrow \theta = 0$; $c < 0 \Rightarrow \theta = 180^o$;

Perpendicular $\theta = 90^o$, $\vec{a}\cdot\vec{b} = 0$

但是如果 dot product = 0 不意味着垂直，比如 $\vec{0}\cdot\vec{b} = 0$

但是当 a, b 都是 nonzero vector, dot product 意味着垂直(perpendicular)

$\vec{a} \cdot \vec{b} = 0$ => orthogonal, zero vector is orthogonal to everything; perpendicular is orthogonal，但是 othogonal 不是 perpendicular

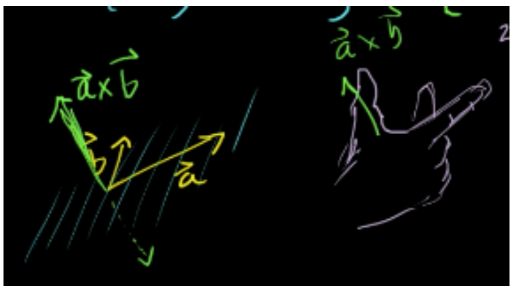
Dot Product: $\vec{a}, \vec{b} \in \mathrm{R}^n$ => 得到 scalar

Cross Product: only defined in R^3， 得到 vector

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{a} \times \vec{b} = \begin{bmatrix} -7 * 4 - 1 * 2 \\ 1 * 5 - 1 * 4 \\ 1 * 2 - (-7 * 5) \end{bmatrix} = \begin{bmatrix} -30 \\ 1 \\ 37 \end{bmatrix}$$

Cross product 乘积是 orthogonal to $\vec{a}$ and $\vec{b}$



判断 cross product 的方向可以用 right hand rule，食指指向 a 的方向，中指指向 b，大拇哥的方向是 a 和 b 的 cross product

Prove Orthogonal for $\vec{a}$ and $\vec{a} \times \vec{b}$: (Same for $\vec{b}$)

$$\begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= a_2 b_3 \mathrm{a}_1 - a_3 b_2 a_1 + a_3 b_1 a_2 - a_1 b_3 a_2 + a_1 b_2 a_3 - a_2 b_1 a_3$$

$$= a_2 b_3 \mathrm{a}_1 - a_1 b_3 a_2 + a_1 b_2 a_3 - a_3 b_2 a_1 + a_3 b_1 a_2 - a_2 b_1 a_3 = 0$$

$$\begin{cases} \vec{a} \cdot \vec{b} = ||\vec{a}|| \left|\left|\vec{b}\right|\right| cos\theta \\ \vec{a} \times \vec{b} = ||\vec{a}|| \left|\left|\vec{b}\right|\right| sin\theta \end{cases}$$

Prove:

$$\left|\left|\vec{a} \times \vec{b}\right|\right|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= \mathrm{a}_2^2 b_3^2 - 2 a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + \mathrm{a}_3^2 b_1^2 - 2 a_1 a_3 b_1 b_3 + a_1^2 b_3^2 + \mathrm{a}_1^2 b_2^2 - 2 a_1 a_2 b_1 b_2 + a_2^2 b_1^2$$

$$= \mathrm{a}_1^2 (b_2^2 + b_3^3) + \mathrm{a}_2^2 (b_1^2 + b_3^2) + \mathrm{a}_3^2 (b_1^2 + b_2^3) - 2(a_2 a_3 b_2 b_3 + a_1 a_3 b_1 b_3 + 2 a_1 a_2 b_1 b_2)$$

$$||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \cos^2\theta = \left(\vec{a} \cdot \vec{b}\right)^2 = (a_1 b_1 + a_2 b_2 + \mathrm{a}_3 b_3)^2$$

$$= \mathrm{a}_1^2 b_1^2 + a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + \mathrm{a}_2^2 b_2^2 + a_1 a_2 b_1 b_2 + a_2 a_3 b_2 b_3 + \mathrm{a}_3^2 b_3^2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3$$

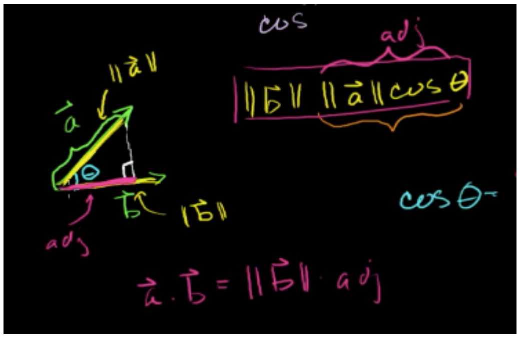
$$= \mathrm{a}_1^2 b_1^2 + \mathrm{a}_2^2 b_2^2 + \mathrm{a}_3^2 b_3^2 + 2(a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3)$$

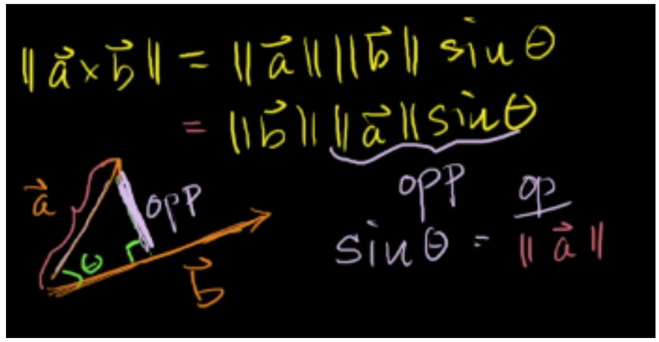
$$\left|\left|\vec{a} \times \vec{b}\right|\right|^2 + ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \cos^2\theta = \mathrm{a}_1^2 (b_1^2 + b_2^2 + b_3^3) + \mathrm{a}_2^2 (b_1^2 + b_2^2 + b_3^3) + \mathrm{a}_3^2 (b_1^2 + b_2^2 + b_3^3)$$

$$= (\mathrm{a}_1^2 + \mathrm{a}_2^2 + \mathrm{a}_3^2)(b_1^2 + b_2^2 + b_3^3) = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2$$

$$\left|\left|\vec{a} \times \vec{b}\right|\right|^2 = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 - ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \cos^2\theta = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 (1 + \cos^2\theta) = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \sin^2\theta$$

$$\vec{a}\cdot\vec{b} = ||\vec{a}||\,||\vec{b}||\,cos\theta = ||\vec{b}||\,adj$$
$$\vec{a}\times\vec{b} = ||\vec{a}||\,||\vec{b}||\,sin\theta = ||\vec{b}||\,opp$$



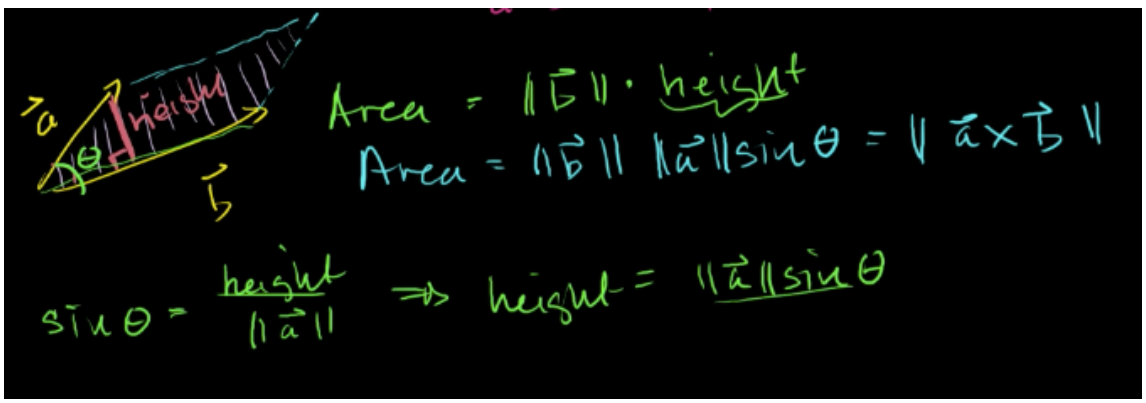


$||\vec{a}||cos\theta$ 构成直角三角形的 a 的 projection 是 adj

$||\vec{a}||sin\,\theta$ 构成直角三角形的高

Dot product tells: product of lengths of vectors move together at same direction with b. When $\vec{a}\cdot\vec{b}=0$, perpendicular, $\vec{a}$ onto $\vec{b}$ is zero

Cross product tells: product of lengths of vectors move perpendicular direction with b. When $\vec{a}\cdot\vec{b} = ||\vec{a}||\,||\vec{b}||$, perpendicular, 获得最大值，当 a 和 b colinear, $\vec{a}\cdot\vec{b}=0$ no perpendicular vector



Cross product 还可以算平行四边形的面积

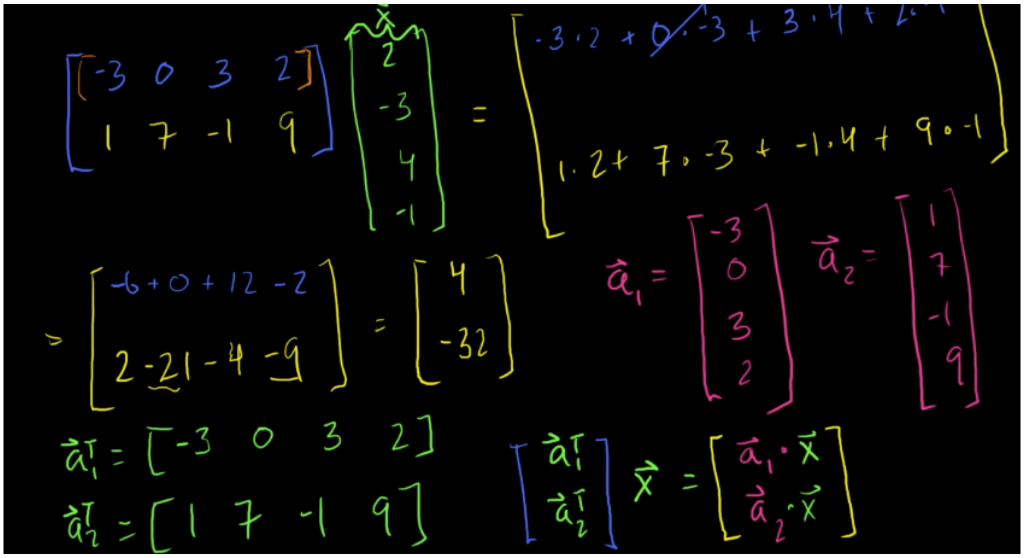
Rowe chelon Form:

Pivot entry: 那个 column 只能它不是 0，且那行前面没有数

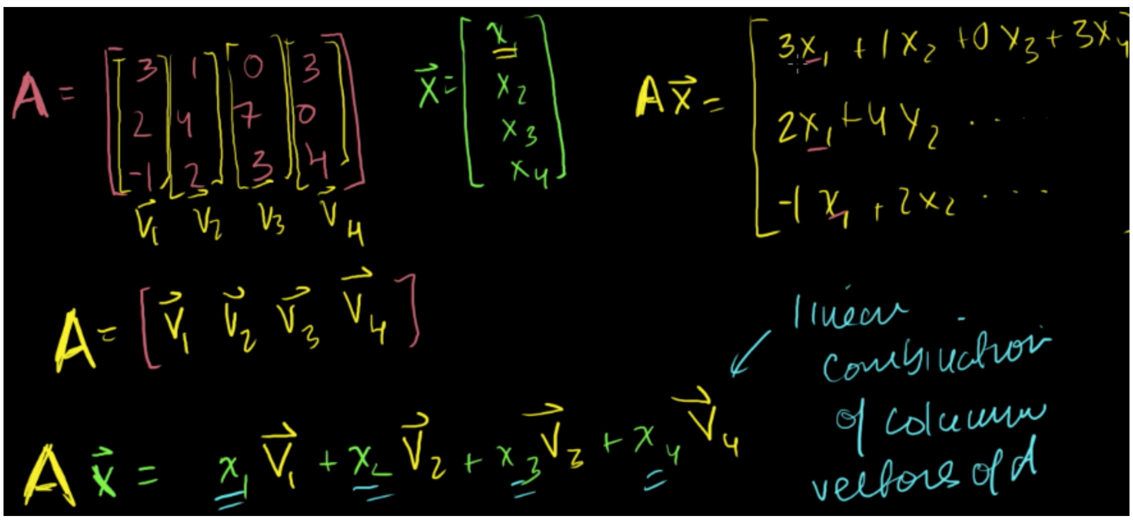
Free-variable: row 中在 pivot 后面的 variable

Matrix Vector Product

1. As row vector and x dot product



2. As column vector and x linear combination



Null Space

$$N = \{ \vec{x} \in R^n \mid A\vec{x} = \vec{0} \}$$

满足 1. $\vec{0}$ in this subspace; 2. If $\vec{v_1}, \vec{v_2} \in N, then\ A(\vec{v_1} + \vec{v_2}) = A\vec{v_1} + A\vec{v_2} = \vec{0} \in N$; 3. $\vec{v} \in N, A(c\vec{v}) = c(A\vec{v}) = \vec{0} \in N$; N is valid subspace

e.g 求 $A\vec{x} = 0$

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$
$$4x_1 + 3x_2 + 2x_3 + x_4 = 0$$

得到 augmented matrix in row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & -2 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$= N(A) = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

Original problem $A\vec{x} = 0$ can be transformed to $(\mathrm{ref}(A))\vec{x} = 0$, (ref(A): null space of reduced row echelon form of matrix A)

Relationship to linear Independent

Matrix A (M × N); Null space $N(A) = \{ \vec{x} \in R^n \mid A\vec{x} = \vec{0} \}$, $\vec{0} \in R^m$; 把 A 用 column vector 来表示

$$A = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}] \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n}$$

如果$\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ 都是 linear independent, 唯一的解是$x_1, x_2, \ldots, x_n$ 都是 0 $N(A) = N(\mathrm{rref}(A)) = \{\vec{0}\}$

$\mathrm{rref}(A)\,\vec{x} = \vec{0}$, which means no free variable

$N(A) = N(\mathrm{rref}(A)) = \{\vec{0}\}$ if only if column vectors of A linear independent (only do if A is N × N matrix)

Column Space

$$A = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}], C(A) = span\,(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$$

满足 1. $\vec{0}$ in this subspace (用每个 vector 乘以 0); 2. If $\vec{b}, \vec{c} \in C(A), then\ (\vec{b} + \vec{c}) = (b_1 + c_1)\vec{v_1} + (b_2 + c_2)\vec{v_2} + \cdots + (b_n + c_n)\vec{v_n} = \vec{0} \in N$; 3. $\vec{v} \in N, A(c\vec{v}) = c(A\vec{v}) \in C(A)$; C(A) is valid subspace

e.g.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \text{column space} = C(A) = span\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right)$$

$$\text{Null Space } N(A) = N(rref(A))$$

Row echelon form: $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Null Space:}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \mathrm{x}_3\begin{bmatrix}-3\\2\\1\\0\end{bmatrix} + \mathrm{x}_4\begin{bmatrix}-2\\1\\0\\1\end{bmatrix} = \mathrm{span}\left(\begin{bmatrix}-3\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}\right)$$

Is $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}, \begin{bmatrix}1\\4\\1\end{bmatrix}, \begin{bmatrix}1\\3\\2\end{bmatrix}$ linear independent? 因为 null space contain $\{\vec{0}\}$.,所以是 linear depenent set

因为是 linear dependent （后两个是 redundant 的）

$\mathrm{C(A)} = span\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}\right)$, $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}$ is a basis for C(A)，跟 row-reduced echelon form pivot 所在的 column 到原来的 matrix 中选 basis

求 column space 的 function:

我们知道 cross product 垂直于$\vec{a}$ and $\vec{b}$, normal vector $\vec{n} = \vec{a} \times \vec{b} = \begin{bmatrix}1\\2\\3\end{bmatrix} \times \begin{bmatrix}1\\1\\4\end{bmatrix} = \begin{bmatrix}8-3 = 5\\3-4 = -1\\1-2 = -1\end{bmatrix}$

$$\vec{n}\cdot\left(\vec{x} - \begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}5\\-1\\-1\end{bmatrix}\cdot\left(\vec{x} - \begin{bmatrix}1\\2\\3\end{bmatrix}\right)$$

$$5\mathrm{x} - \mathrm{y} - \mathrm{z} = 0$$

另一种方法: what kind of B will give valid solution $\{\vec{b} \mid A\vec{x} = \vec{b} \;\&\; \vec{x} \in R^n\}$

$$\mathrm{A} = \left[\begin{array}{cccc|c}1 & 1 & 1 & 1 & x\\2 & 1 & 4 & 3 & y\\3 & 4 & 1 & 2 & z\end{array}\right]$$

化成 row echelon form:

$$\mathrm{A} = \left[\begin{array}{cccc|c}1 & 1 & 1 & 1 & x\\0 & 1 & -2 & -1 & y\\0 & 0 & 0 & 0 & 2x - y - z + 3x\end{array}\right]$$

为了让 system 有解 2x – y – z + 3x = 5x – y -z = 0

Dimension

Dim(V) : the number (cardinality) of a basis of V (比如 $A = \{a_1, a_2, \ldots, a_n\}$ is a basis of V, Dim(V) = n)

All basis of the same subspace must have the same number of elements

Dimension of Null space: Dim(N(B)) is the **Nullity** = number of **free variables** (non-pivot) in reduced echelon form in Matrix A

e.g.

$$A\vec{x} = \vec{0}: \begin{bmatrix} 1 & 0 & -1 & -2 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$= N(A) = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

两个 free variable x_3, x_4, Dim(N(B)) = nullity = 2

Dimension of Column space: Dim(C(A)) is the **Rank** = number of **pivot variables** in reduced echelon form in Matrix A (rank of A number of linear independent column vector you have)

e.g.

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix}$$

To reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

第 1，2，4 列 linearly independent, column space 的 basis $span\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}\right)$

Dim(C(A)) = 3

Linear Transformation

When function map to R (一维的) called scaler value / Real valued function
When function map to R^2, R^3 (多维的) called vector value
Transformation: function operating on vectors (linear algebra)
Linear Transformation:

$$T: R^n \to R^m \text{ if only if}$$
$$\vec{a}, \vec{b} \in R^n,\ \ 1.\ T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$$
$$2.\ \ T(c\vec{a}) = cT(\vec{a})$$

如果看 T 是不是 linear transformation 需要证明是不是符合上面的两个条件

Matrix vector products is linear transformation

$$T: R^n \to R^m:\ \ T(\vec{x}) = A\vec{x}, \qquad A \text{ 的 dimension } m \times n$$

Prove it is linear transformation:

$$A = [\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}] \qquad A \cdot (\vec{a} + \vec{b}) = A \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \dots \\ a_n + b_n \end{bmatrix}$$
$$= (a_1 + b_1)\vec{v_1} + (a_2 + b_2)\vec{v_2} + \dots + (a_n + b_n)\vec{v_n}$$
$$= a_1\vec{v_1} + b_1\vec{v_1} + a_2\vec{v_2} + b_2\vec{v_2} + \dots + a_n\vec{v_n} + b_n\vec{v_n}$$
$$= A \cdot \vec{a} + A \cdot \vec{b}$$

$$A \cdot (c\vec{a}) = ca_1\vec{v_1} + ca_2\vec{v_2} + \dots + ca_n\vec{v_n}$$
$$= c(a_1\vec{v_1} + a_2\vec{v_2} + \dots + a_n\vec{v_n})$$

Any linear matrix transformation can be viewed as matrix product

Standard basis for R^n

$$\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}\ \vec{e_i} \text{ 是} n \times n \text{ indentity matrix 的 ith column}$$

Image: transformation from one set into another set $T(L_0)$ *is image of* L_0 under T

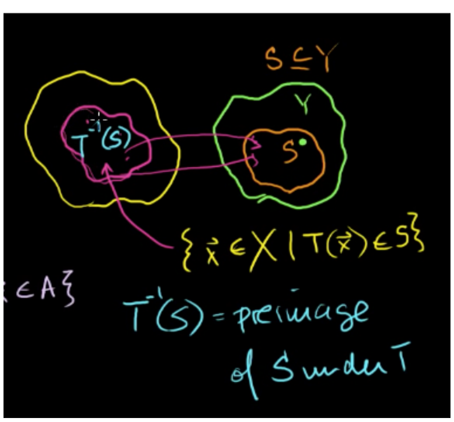
$$T: R^n \to R^m:\ T(V): \textit{image of } V \textit{ under } T$$

$T(V)$ is valid subspace:

Prove: 1. $T(\vec{a}), T(\vec{b}) \in T(V)$ 因为是 linear transformation $T(\vec{a}) + T(\vec{b})$ in V, 所以 $T(\vec{a}) + T(\vec{b}) = T(\vec{a} + \vec{b}) \in T(V)$

2. c $T(\vec{a}) = T(c\vec{a})$ 因为 $T(c\vec{a})$ in V, c $T(\vec{a})$ also in V

Image of T: $T(\vec{x}) = A\vec{x} = \text{column space of A } (C(A)) = span(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n})$



Preimage： $T^{-1}(S)\ \{\vec{x} \in X\ |T(\vec{x}) \in S\}$, given co-domain, what subset of domain map into co-domain, (不是每个 S 都需要$\vec{x}$ map 到) $T(T^{-1}(S)) \subseteq S$

Kernel of T: $Ker(T) = \{x \in R^2\ |T(\vec{x}) = \{\vec{0}\}\}$: A vector v is in the kernel of a linear transformation if and only if T(v) = 0. It is the same things as null space

Sums and scalar multiples of linear transformation

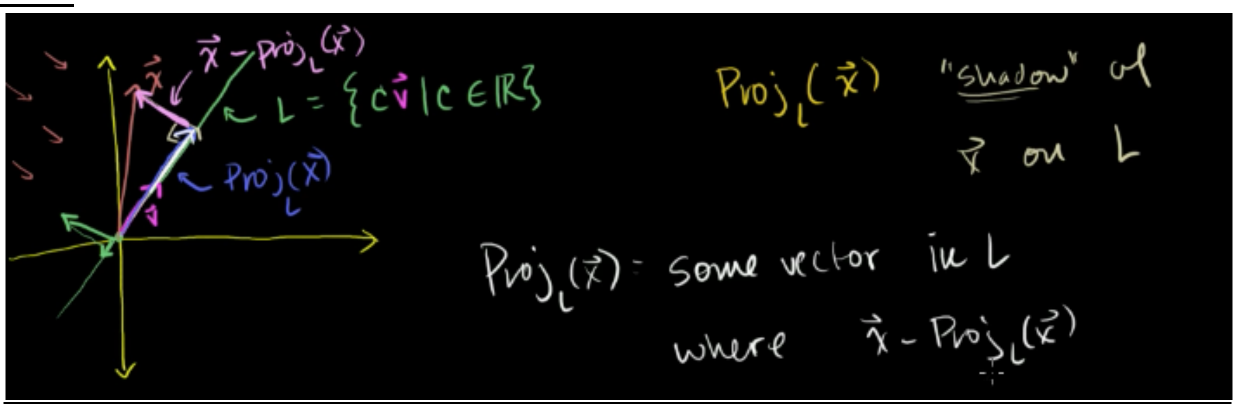
$$S: \mathrm{R}^{\mathrm{n}} \to R^m \quad T: \mathrm{R}^{\mathrm{n}} \to R^m$$
$$\text{Def: } (\mathrm{S}+\mathrm{T})(\vec{x}) = S(\vec{x}) + T(\vec{x}) \qquad (S+T): \mathrm{R}^{\mathrm{n}} \to R^m$$
$$\text{Def: } (\mathrm{cS})(\vec{x}) = c\big(\mathrm{S}(\vec{x})\big): \quad cS: \ \mathrm{R}^{\mathrm{n}} \to R^m$$

Linear Transformation example:
让所有 x 变成负-x，所有 y 乘以 2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

A 是 **diagonal matrix**：只有对角线有值，剩下都是 0;
Unit vector: vector has length of 1 $\|\vec{u}\| = \|\frac{1}{\|\vec{v}\|}\vec{v}\| = \frac{1}{\|\vec{v}\|}\vec{v}$
e.g. $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{u} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Projection



$\mathrm{Proj}_{\mathrm{L}}(\vec{x})$ 把$\vec{x}$ 到 L 做垂线，$\vec{x}$ 在 L 的射影是 projection，垂线是 $\vec{x} - \mathrm{Proj}_{\mathrm{L}}(\vec{x})$，因为垂直 dot product = 0, $(\vec{x} - \mathrm{Proj}_{\mathrm{L}}(\vec{x})) \cdot \mathrm{Proj}_{\mathrm{L}}(\vec{x}) = 0$
$\mathrm{Proj}_{\mathrm{L}}(\vec{x})$: some vector in L where $\vec{x} - \mathrm{Proj}_{\mathrm{L}}(\vec{x})$ is orthogonal to L

$$\mathrm{L} = \{\mathrm{c}\vec{v} \mid \mathrm{c} \in \mathrm{R}\} \quad (\vec{x} - c\vec{v}) \cdot \vec{v} = 0$$
$$\vec{x} \cdot \vec{v} - c\vec{v} \cdot \vec{v} = 0$$
$$\vec{x} \cdot \vec{v} = c\vec{v} \cdot \vec{v}$$
$$c = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$
$$\mathrm{Proj}_{\mathrm{L}}(\vec{x}) = \mathrm{c}\vec{v} = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}$$

If $\vec{v}$ is unit vector, $\mathrm{Proj}_{\mathrm{L}}(\vec{x}) = (\vec{x} \cdot \vec{v})\,\vec{v}$
Prove: projection is linear transformation ($\vec{u}$ is unit vector)

$$1.\ \mathrm{Proj}_{\mathrm{L}}(\vec{a} + \vec{b}) = ((\vec{a} + \vec{b}) \cdot \vec{u})\vec{u}$$
$$= (\vec{a} \cdot \vec{u} + \vec{b} \cdot \vec{u})\,\vec{u}$$
$$= (\vec{a} \cdot \vec{u})\vec{u} + (\vec{b} \cdot \vec{u})\vec{u} = \mathrm{Proj}_{\mathrm{L}}(\vec{a}) + \mathrm{Proj}_{\mathrm{L}}(\vec{b})$$

$$2.\ \mathrm{Proj}_{\mathrm{L}}(c\vec{a}) = (c\vec{a} \cdot \vec{u})\vec{u}$$
$$= c(\vec{a} \cdot \vec{u})\vec{u}$$
$$= c\mathrm{Proj}_{\mathrm{L}}(\vec{a})$$

So $\mathrm{Proj}_{\mathrm{L}}(\vec{x}) = A\vec{x}$

$$A = [\left(\begin{bmatrix}1\\0\end{bmatrix} \cdot \begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix}, \ \left(\begin{bmatrix}0\\1\end{bmatrix} \cdot \begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix}]$$

$$A = \left[\mathrm{u1}\begin{bmatrix}u_1\\u_2\end{bmatrix}, \ \mathrm{u2}\begin{bmatrix}u_1\\u_2\end{bmatrix}\right] = \begin{bmatrix}u_1^2 & u_2u_1\\u_1u_2 & u_2^2\end{bmatrix}$$

Composition

Composition: transformation of transformation

Composition is linear transformation (given S, T is linear transformation)

Prove:

$$\mathrm{T} \diamond \mathrm{S}(\vec{x} + \vec{y}) = T\big(S(\vec{x} + \vec{y})\big) = T\big(S(\vec{x}) + S(\vec{y})\big) = T\big(S(\vec{x})\big) + T\big(S(\vec{y})\big) = \mathrm{T} \diamond \mathrm{S}(\vec{x}) + \mathrm{T} \diamond \mathrm{S}(\vec{y})$$

$$\mathrm{T} \diamond \mathrm{S}(c\vec{x}) = \mathrm{T}\big(\mathrm{S}(c\vec{x})\big) = \mathrm{T}\big(c\mathrm{S}(\vec{x})\big) = c\mathrm{T}\big(\mathrm{S}(\vec{x})\big) = c(\mathrm{T} \diamond \mathrm{S})(\vec{x})$$

因为 composition is linear transformation, 可以把$\mathrm{T} \diamond \mathrm{S}(\vec{x} + \vec{y})$ 写成 $A\vec{x}$

$$C = \left[\mathrm{B}\left(\mathrm{A}\begin{bmatrix}1\\0\\\cdots\\0\end{bmatrix}\right), \mathrm{B}\left(\mathrm{A}\begin{bmatrix}0\\1\\\cdots\\0\end{bmatrix}\right), \ldots, \mathrm{B}\left(\mathrm{A}\begin{bmatrix}0\\0\\\cdots\\1\end{bmatrix}\right)\right]$$

$$C = [\mathrm{B}(\vec{a_1}), \mathrm{B}(\vec{a_2}), \ldots, \mathrm{B}(\vec{a_n})]$$

$$\mathrm{T} \diamond \mathrm{S}(\vec{x}) = B\big(A(\vec{x})\big) = B\,A\,\vec{x}$$

$$AB = [A\,\vec{b_1}, A\,\vec{b_2}, \ldots, A\,\vec{b_n}\,]$$

Associative $\big((\mathrm{H} \diamond \mathrm{G}) \diamond \mathrm{F}\big)(\vec{x}) = (\mathrm{H} \diamond \mathrm{G})(F\vec{x}) = H\big(G\big(F(\vec{x})\big) = H((G \diamond \mathrm{F})\vec{x})$

e.g. $A = \begin{bmatrix}1 & -1 & 2\\0 & -2 & 1\end{bmatrix}, \ B = \begin{bmatrix}1 & 0 & 1 & 1\\2 & 0 & 1 & -1\\3 & 1 & 0 & 2\end{bmatrix}$

$$AB = [A\begin{bmatrix}1\\2\\3\end{bmatrix} \ A\begin{bmatrix}0\\0\\1\end{bmatrix} \ A\begin{bmatrix}1\\1\\0\end{bmatrix} \ A\begin{bmatrix}1\\-1\\2\end{bmatrix}]$$

Matrix product properties

Associative: $(AB)C = A(BC)$ *Doesn't matter where to put 括号*

Not Communicative: $AB \neq BA$

Distributive: $A(B + C) = AB + BC$, $(B + C)A = BA + CA$

Prove Distributive:

$$B = [\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}], \ C = [\vec{c_1}, \vec{c_2}, \ldots, \vec{c_n}]$$

$$A(B + C) = A[\vec{b_1} + \vec{c_1}, \vec{b_2} + \vec{c_2}, \ldots, \vec{b_n} + \vec{c_n}]$$

$$= A[\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}] + A[\vec{c_1}, \vec{c_2}, \ldots, \vec{c_n}] = AB + AC$$

$$(B + C)A = (B + C)\,[\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}] = [(B + C)\vec{a_1}, (B + C)\vec{a_2}, \ldots, (B + C)\,\vec{a_n}]$$

$$= [B\vec{a_1}, B\,\vec{a_2}, \ldots, B\vec{a_n}] + [C\vec{a_1}, C\vec{a_2}, \ldots, C\vec{a_n}]$$

$$= BA + CA$$

Inverse

f (function X -> Y) is **Invertible** if and only if there exist a function f^{-1} (Y->X) such that $f^{-1} \circ f = I_x$ and $f \circ f^{-1} = I_y$

$$f^{-1} \circ f(a) = I_x(a) = a\,; \ f \circ f^{-1}(y) = y$$

Every function if has inverse, its inverse must be unique

Invertibility implies a unique solution to f(x) = y

Prove: If f is invertible, for every $y \in Y$, there is unique solution $x \in X$ such that f(x) = y

$$f(x) = y; \quad f^{-1}(y) = f^{-1}(f(x)) == (f^{-1} \circ f)(x) = \mathrm{I_x}(x) = x$$

For every $y \in Y$ f(x) = y has a unique solution, then f is invertibility

$$\mathrm{S: Y \rightarrow X;} \ \text{因为 } \mathrm{f(x)} = y \ \textit{has unique solution.} \ S(y){:}\, \textit{well defined}$$
$$S(b) \textit{ is the unique solution to } f(x) = b$$
$$f(S(b)) = b \rightarrow \ (f \circ s)(b) = I_y(b) = \ b\,; \ f \circ s = I_y$$
$$S(f(a)) = \textit{the unqiue solution to the equation} = a; \ (s \circ f) = I_x$$
$$\textit{因为} \ (s \circ f) = I_x, f \circ s = I_y, \textit{by definition, function is invertible}$$

Inverse is linear transformation

Prove:

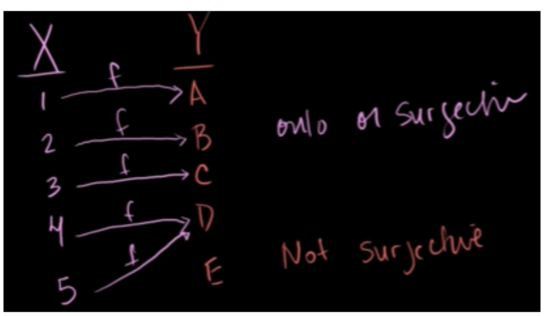
$$(f^{-1} \circ f)(\vec{a} + \vec{b}) = \vec{a} + \vec{b} = (f^{-1} \circ f)(\vec{a}) + \ (f^{-1} \circ f)(\vec{b})$$
$$f\left(f^{-1}(\vec{a} + \vec{b})\right) = f(f^{-1}(\vec{a})) + f\left(f^{-1}(\vec{b})\right) = f\left(f^{-1}(\vec{a}) + f^{-1}(\vec{b})\right) \textit{因为f是linear transformation}$$
$$f^{-1}(f\left(f^{-1}(\vec{a} + \vec{b})\right)) = f^{-1}(f(f^{-1}(\vec{a}))) + f^{-1}(f\left(f^{-1}(\vec{b})\right))$$
$$(f^{-1} \circ f)\left(f^{-1}(\vec{a} + \vec{b})\right) = \ (f^{-1} \circ f)(f^{-1}(\vec{a})) + (f^{-1} \circ f)(f^{-1}(b))$$
$$f^{-1}(\vec{a} + \vec{b}) = \ f^{-1}(\vec{a}) + \ f^{-1}(b)$$

$$(f \circ f^{-1})(c\vec{a}) = c\vec{a} = c\,((f \circ f^{-1})(\vec{a}) = f(cf^{-1}(\vec{a}))$$
$$f^{-1}(f(f^{-1}(c\vec{a}))) = f^{-1}(\,f(cf^{-1}(\vec{a})))$$
$$(f^{-1} \circ f)(f^{-1}(c\vec{a})) = \ (f^{-1} \circ f)(cf^{-1}(\vec{a}))$$
$$f^{-1}(c\vec{a}) = \ cf^{-1}(\vec{a})$$

Onto & One-to-One

Onto (surjective): every elements in co-domain $y \in Y$, there exist at least one $x \in X$ such that f(x) = y.

Every y in co-domain at least 一个 x map to



F onto then image(f) = Y

左侧是 Not onto example

One-to-one (injective): for every value that map to there at most at most one x map to it.

每个一个 y 只有一个 x map，每个 x map to unqiue y: f(x) = y

上面不是 onto 的例子，也不符合 one-to-one， 假如 5 不指向 D, 5 改指向 E，表示 onto 和 one-to-one

$\mathbf{f: x \rightarrow y}$ is invertible if and only if f is onto and one-to-one

Invertible means For every $y \in Y$ f(x) = y has a unique solution, that means one-to-one，如果有$y \in Y$ 但是没有相应的 x 对应， 就不是 invertible 了，所以 invertible means onto

T is onto iff C(A) = R^m, its reduced echelon form has a pivot entry in every row (m pivot entry rank = M) : T is onto if and only if Rank(A) = m

Rank(A) = dim(C(A)) = # of basis vectors for C(A)

$$T: \mathrm{R}^n \to R^m \quad T(\vec{x}) = A\vec{x}$$

Onto => for any $\vec{b} \in \mathrm{R}^m$, at east one solution $A\vec{x} = \vec{b}$ where $\vec{x} \in \mathrm{R}^n$

$$A\vec{x} = [x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots + x_n\overrightarrow{a_n}]$$

For T to be onto $\mathrm{span}(\overrightarrow{a_1}, \overrightarrow{a_2}, .., \overrightarrow{a_n}) = R^m$ which is column space, column space is R^m

e.g $S: \mathrm{R}^2 \to R^3, S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\vec{x}$

row reduced echelon form: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, rank = 2, S is not onto, S is not invertible

e.g. $T: \mathrm{R}^2 \to R^3, T(\vec{x}) = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \end{bmatrix}$

ow reduced echelon form $\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ -1 & 3 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 0 & b_1 + b_2 \end{array}\right]$

only member $\vec{b} \in R^m$ that has solution are the ones $b_1 + b_2 = 0$

solution set = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, when $T(\vec{x}) = \vec{0}$, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the null space of T

从上面可以看出: Assuming $A(\vec{x}) = \vec{b}$ has a solution, the solution set = $\{\overrightarrow{x_p}\} U\ N(A)\ null\ space$, some particular vector union null space; if one-to-one, at most 1 solution => N(A) has just zero vector(trival)

Any solution to the inhomogeneous system ($A(\vec{x}) = \vec{b}\ give\ \vec{b} \neq 0$) system will take the form $x_p + x_h$ (particular solution + homogeneous solution)

Prove: $A(x_p + x_h) = Ax_p + Ax_h = \vec{b} + \vec{0}$

Prove any solution $\vec{x}$ to $A\vec{x} = \vec{b}$ take the form $\vec{x} = x_p + x_h$:

$$A(\vec{x} - x_p) = A\vec{x} - Ax_p = \vec{0}$$

$$\vec{x} - x_p \text{ is a solution } A\vec{x} = \vec{0}, \vec{x} - x_p\, is\ a\ member\ of\ null\ space\ N(A)$$

$$\vec{x} - x_p = x_h \to \quad \vec{x} = x_p + x_h$$

如果是 one-to-one: $x_p + x_h$ 只能是 one-solution so x_h has to be $\vec{0}$ null space has to be $\{\vec{0}\}$ so $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}$ are linearly independent; C(A) = $\mathrm{span}(\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n})$, $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}\}$ are basis for column space, dim (column space) = n; rank (A) = N

Invertible: 1. onto: rank(A) = m; 2. One-to-one： rank(A) = n; in order to let transformation to be invertible, rank(A) = m = n: **matrix has to be square matrix （n by n matrix），变成 reduced echelon form 每一行每一列又有 pivot entry (n by n indentity matrix)（linearly independent pivot colum**

$$T: \mathrm{R}^n \to R^n(\text{不考虑}\mathrm{R}^n \to \mathrm{R}^m): \quad T(\vec{x}) = A\vec{x}\ only\ invertible\ if\ row\ reduced\ echelon\ form\ is\ I_n$$

对 matrix 进行 row operation 等于进行 linear transformation, **linear transformation 的矩阵是等同于 identity matrix 进行一样的 row operation**

比如 $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ 等同于 $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \to \begin{bmatrix} a_1 \\ a_2 + a_1 \\ a_3 - a_1 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\mathrm{T}(\vec{x}) = S\vec{x} = \left[\mathrm{S}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathrm{S}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \mathrm{S}\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

Determinant

3*3 determinant:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

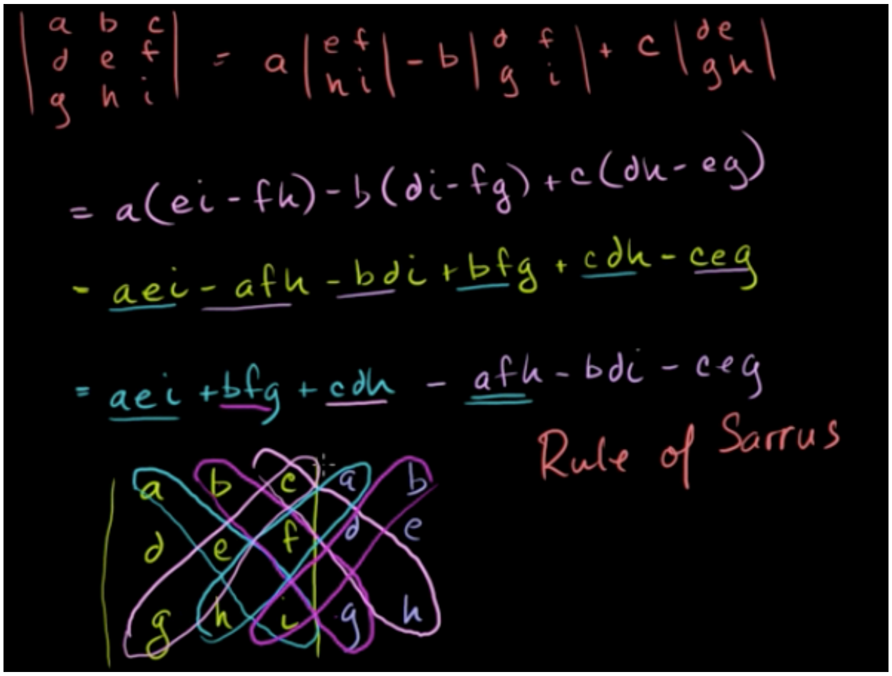
e.g.

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

$$Det(C) = 1\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$$

$$Det(C) = 1 * (-1 * 1 - 0 * 3) - 2(2 * 1 - 4 * 3) + 4\big(2 * 0 - (-1 * 4)\big) = 35$$

Quick way: Rule of Sarrus



n*n determinant:

A = $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, Define $\mathrm{A_{ij}} = (\mathrm{n}-1) \times (\mathrm{n}-1)$ matrix by ignore i-th row and j-th column

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\,\mathrm{def}(A_{12}) + a_{13}\det(A_{13}) + \cdots + (-1)^{i+j}\mathrm{a_{ij}}\det(A_{ij})$$

$$\det(A) = (-1)^{i+1}\mathrm{a_{i1}}\det(A_{ij}) + (-1)^{i+2}\mathrm{a_{i2}}\det(A_{i2}) + .. + (-1)^{i+n}\mathrm{a_{in}}\det(A_{in})$$

$$\det(A) = \sum_{j=1}^{j=n} (-1)^{i+j}\mathrm{a_{ij}}\det(A_{ij})$$

Row 乘以 scaler 的 determinant:

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = ad - bc$$

如果 row2 乘以 k

$$\mathrm{A} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \det(A) = kad - kbc = k * \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

如果 row1 也乘以 k

$$\mathrm{A} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}, \det(A) = k^2ad - k^2bc = k^2 * \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

如果是 3*3 matrix row 乘以 k

$$\mathrm{A} = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{kd} & \mathrm{ke} & \mathrm{kf} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{bmatrix}, \det(A) = -kd\begin{vmatrix} b & c \\ h & i \end{vmatrix} + ke\begin{vmatrix} a & c \\ g & i \end{vmatrix} - kf\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

如果是 n*n matrix row 乘以 k

$$\det(A) = (-1)^{i+1}\mathrm{ka_{i1}}\det(A_{ij}) + (-1)^{i+2}\mathrm{ka_{i2}}\det(A_{i2}) + .. + (-1)^{i+n}\mathrm{ka_{in}}\det(A_{in})$$

$$\det(A) = k\sum_{j=1}^{j=n} (-1)^{i+j}\mathrm{a_{ij}}\det(A_{ij}) = k * det(A)$$

如果 matrix 每行都乘以 k

$$\det(A) = k^n * \det(A)$$

When row is added 的 determinant:

2*2 matrix:

$$X = \begin{bmatrix} a & b \\ x_1 & x_2 \end{bmatrix}, Y = \begin{bmatrix} a & b \\ y_1 & y_2 \end{bmatrix}, Z = \begin{bmatrix} a & b \\ x_1 + y_1 & x_2 + y_2 \end{bmatrix}$$

$$\det(X) = ax_2 - bx_1, \quad \det(Y) = ay_2 - by_1, \det(Z) = a(x_2 + y_2) - b(x_1 + y_1) = \det(X) + \det(y)$$

3*3 matrix:

$$X = \begin{bmatrix} a & b & c \\ x_1 & x_2 & x_3 \\ d & e & f \end{bmatrix}, \det(X) = -x_1 \begin{vmatrix} b & c \\ e & f \end{vmatrix} + x_2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} - x_3 \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$Y = \begin{bmatrix} a & b & c \\ y_1 & y_2 & y_3 \\ d & e & f \end{bmatrix}, \det(Y) = -y_1 \begin{vmatrix} b & c \\ e & f \end{vmatrix} + y_2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} - y_3 \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$Z = \begin{bmatrix} a & b & c \\ x_1 + y_1 & x_2 + y_2 & y_1 + y_3 \\ d & e & f \end{bmatrix}, \det(Z) = -(x_1 + y_1) \begin{vmatrix} b & c \\ e & f \end{vmatrix} + (x_2 + y_2) \begin{vmatrix} a & c \\ d & f \end{vmatrix} - (x_3 + y_3) \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$\det(Z) = \det(X) + \det(Y)$$

n*n matrix

$$\det(X) = k \sum_{j=1}^{j=n} (-1)^{i+j} \mathrm{x_{ij}} \det(A_{ij}), \quad \det(Y) = k \sum_{j=1}^{j=n} (-1)^{i+j} \mathrm{Y_{ij}} \det(A_{ij}),$$

$$\det(Z) = k \sum_{j=1}^{j=n} (-1)^{i+j} (\mathrm{x_{ij}} + y_{ij}) \det(A_{ij}) = \det(X) + \det(Y)$$

Determinant operations are not linear on matrix addition

Swap Row determinant: 比如第 i 行和第 j 行互换了

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j \\ \cdots \end{bmatrix}, swap\ i\ 和 j 行, \quad A_{ij} = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_j \\ \cdots \\ a_i \\ \cdots \end{bmatrix}, \det(A) = -\det(A_{ij})$$

假如第 i 行 = 第 j 行，$\det(A) = \det(A_{ij})$,根据上面的定理： $\det(A) = -\det(A_{ij}) = \det(A_{ij})$, $\det(A) = 0$

Duplication row determinant = 0，因为 duplicate row never get reduced echelon form to be invertible => det = 0

Determinant of row operation: row j = row j – c*rowi

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j \\ \cdots \end{bmatrix}, B = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j - c * a_i \\ \cdots \end{bmatrix}$$

因为 $\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ -c * a_i \\ \cdots \end{bmatrix}$ 不是 linear independent，第 j 行可以由第 i 行乘以-c 得到，所以 $\det\left(\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ -c * a_i \\ \cdots \end{bmatrix}\right) = 0$

$$\det(B) = \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j \\ \cdots \end{bmatrix}\right) + \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ -c * a_i \\ \cdots \end{bmatrix}\right) = \det(A)$$

Determinant of upper triangular: diagonal 所有数的乘积： $\det(A) = a_{11}a_{22}\ldots.a_{nn}$

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \det(A) = ad, B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \det(B) = adf$$

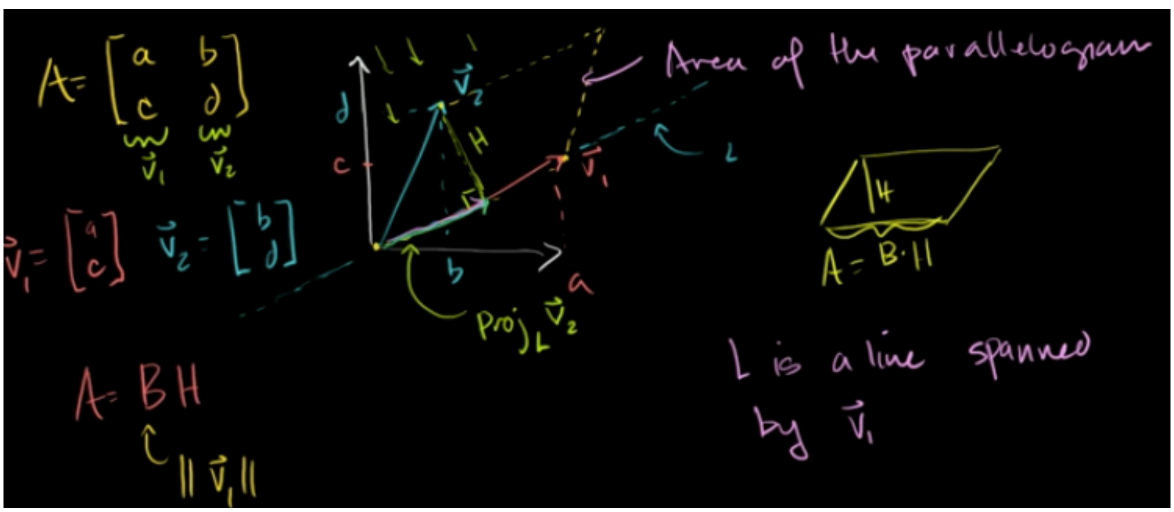
Simple 4*4 determinant: 利用 row operation 不 change determinant 和 upper triangular determinant 的性质，将 4*4 matrix 变成 diagonal matrix

e.g

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{bmatrix} \rightarrow \begin{matrix} \text{第二行}-= \text{第一行} \\ \text{第三行}-= 2 * \text{第一行} \\ \text{第四行}+= \text{第一行} \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 6 & -4 & 4 \end{bmatrix} \rightarrow \begin{matrix} \text{第二三行互换 } \det *= -1 \\ \text{第四行}+= \text{第二行} * 2 + 3 * \text{第三行} \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \rightarrow \det = -1 * 3 * 2 * 7 = -42$$

Determinant of area of a parallelogram: 平行四边形(parallelogram) 边长 vector 组成 matrix 的 abs(determinant) = 它们的面积



$$\overrightarrow{v_1} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \overrightarrow{v_2} = \begin{bmatrix} b \\ d \end{bmatrix}, \quad \mathrm{Proj}_L(\overrightarrow{v_2}) \text{ 为}\overrightarrow{v_2}\text{ 映射到}\overrightarrow{v_1}\text{长}$$

$$H^2 + ||\mathrm{Proj}_L(\overrightarrow{v_2})||^2 = ||\overrightarrow{v_2}||^2$$

$$H^2 = ||\overrightarrow{v_2}||^2 - ||\mathrm{Proj}_L(\overrightarrow{v_2})||^2$$

$$H^2 = \overrightarrow{v_2} \cdot \overrightarrow{v_2} - \left|\left|\frac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\overrightarrow{v_1}\right|\right|^2$$

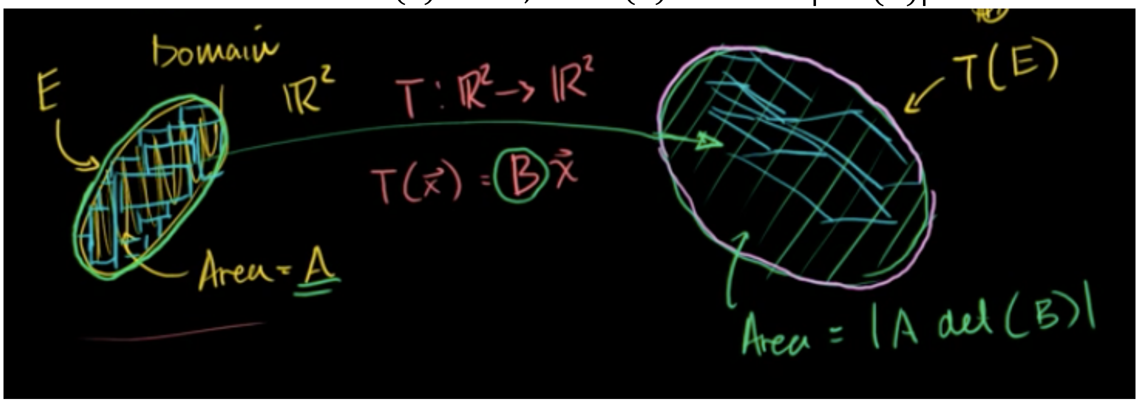
$$H^2 = \overrightarrow{v_2} \cdot \overrightarrow{v_2} - \frac{(\overrightarrow{v_1} \cdot \overrightarrow{v_2})(\overrightarrow{v_1} \cdot \overrightarrow{v_2})}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}$$

$$area^2 = B^2H^2 = \vec{v_1}\cdot\vec{v_1}\left(\vec{v_2}\cdot\vec{v_2} - \frac{(\vec{v_1}\cdot\vec{v_2})(\vec{v_1}\cdot\vec{v_2})}{\vec{v_1}\cdot\vec{v_1}}\right) = (\vec{v_1}\cdot\vec{v_1})(\vec{v_2}\cdot\vec{v_2}) - (\vec{v_1}\cdot\vec{v_2})(\vec{v_1}\cdot\vec{v_2})$$
$$area^2 = (a^2+c^2)(b^2+d^2) - (ab+cd)^2$$
$$area^2 = a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2 - a^2b^2 - 2abcd - c^2d^2$$
$$area^2 = a^2d^2 + c^2b^2 - 2abcd = (ad-bc)^2 = \det(\mathrm{A})^2$$

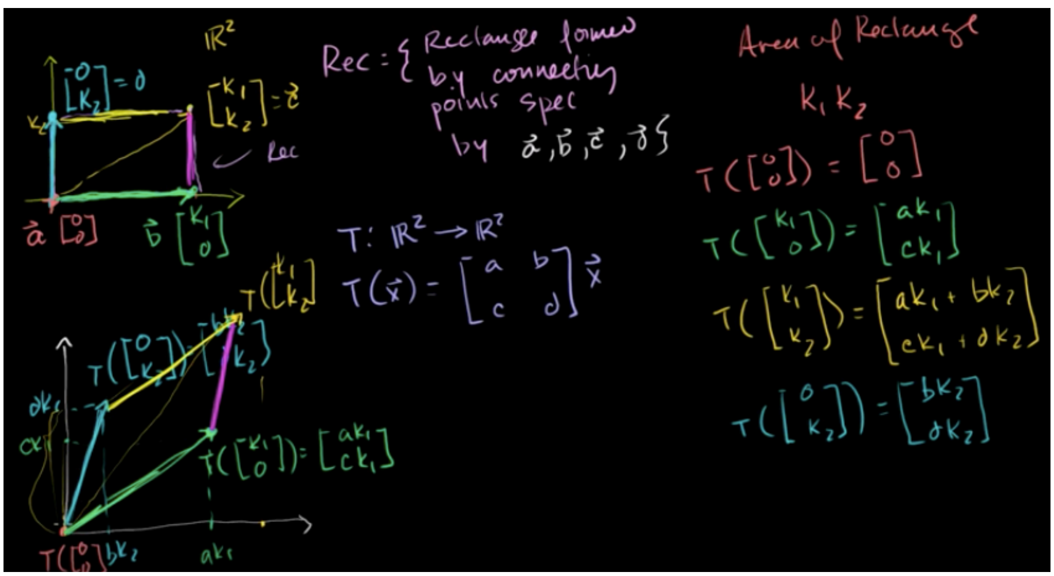
Determinant as scaling factor:

如果我们 transform 从一个 area A 到另外一个 area B

$$\mathrm{T}: \mathrm{R}^2 \to \mathrm{R}^2:\ \mathrm{T}(\vec{x}) = \mathrm{B}\vec{x}\,;\quad \mathrm{Area(B)} = Area * |\det(B)|$$



Prove： 长方形 transform 到平行



我们有边长$\vec{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} k_1 \\ 0 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 0 \\ k_2 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$, $\mathrm{T}: \mathrm{R}^2 \to \mathrm{R}^2, \mathrm{T}(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\vec{x}$

Transformation 后的点为$\vec{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} ak_1 \\ ck_1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} bk_2 \\ dk_2 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} ak_1+bk_2 \\ ck_1+dk_2 \end{bmatrix}$

根据上面我们知道新的平行四边形面积是 $\det\left(\begin{bmatrix} ak_1 & bk_2 \\ ck_1 & dk_2 \end{bmatrix}\right) = |k_1k_2ad - k_1k_2bc| = k1k2|ad-bc| = area(A) * |\det(T)|$

Transpose

Properties:

1. $(C^T)^T = C$
2. $\det(A^T) = \det(A) \quad for\ A : n * n\ matrix$
3. $(AB)^T = B^T A^T, \ (XYZ)^T = Z^T Y^T X^T$
4. $(A+B)^T = A^T + B^T$
5. $(A^{-1})^T = (A^T)^{-1}$; $<= \ (AA^{-1})^T = (A^{-1})^T A^T = I_n^T = (A^{-1}A)^T = A^T (A^{-1})^T =>$
6. $for\ vector\ \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$, for A $m \times n$, $\vec{x} \in n \times 1, \vec{y}\ n \times 1 \in R^m$; $(A\vec{x}) \cdot \vec{y} = (A\vec{x})^T \vec{y} = \vec{x}^T A^T \vec{y} = \vec{x}^T (A^T \vec{y}) = \vec{x} \cdot (A^T \vec{y})$ *因为matrix product is associative*

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}, \quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \quad a = [a_1, a_2, \ldots, a_n], \quad \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$

Orthogonal Complements:

Orthogonal complements of V: for some V, $V^{\perp} = \{\vec{x} \in R^n | \vec{x} \cdot \vec{v} = 0\ for\ every\ \vec{v} \in V\}$

Prove orthogonal complements: 1. $\vec{a} \cdot \vec{v} = 0$, so $\vec{a}$ *can be equal to* $\vec{0}, \vec{0}$ *is in subspace* 2.. $\vec{a} \cdot \vec{v} = 0$ for any $\vec{v} \in V, \vec{b} \cdot \vec{v} = 0$ for any $\vec{b} \in V$, $(\vec{a} + \vec{b}) \cdot \vec{v} = \vec{a} \cdot \vec{v} + \vec{b} \cdot \vec{v} = 0$; 3. $c\vec{a} \cdot \vec{v} = c(\vec{a} \cdot \vec{v}) = 0$

N(A) is orthogonal complements of the rowspace of A(is the same as column space of A transpose)

$$N(A) = (C(A^T))^{\perp}$$

Prove:

$$A = \begin{bmatrix} ---\overrightarrow{a_1}^T--- \\ ---\overrightarrow{a_2}^T--- \\ \cdots \\ ---\overrightarrow{a_n}^T--- \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} = \begin{bmatrix} \overrightarrow{a_1}^T \cdot \vec{x} \\ \overrightarrow{a_2}^T \cdot \vec{x} \\ \cdots \\ \overrightarrow{a_n}^T \cdot \vec{x} \end{bmatrix} \rightarrow \vec{x}\text{来自 } V, \vec{x} \in N(A), \vec{x}\ is\ orthogonal\ to\ \overrightarrow{a_1}^T, \overrightarrow{a_2}^T \ldots \overrightarrow{a_n}^T$$

N(A) is orthogonal to A

N(A) is othogonal to A, and also orthogonal to any lienar combination of A, orthogonal to row space of A,