

Matrix A, B: $A*B$ 不一定等于 $B*A$, matrix 乘法 direction matters

Identity Matrix: $I*A = A$: 1 in diagonal position.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Why it works? Row information from identity matrix 的 row and column information from A 的 column, 比如 $[1,0]$ 乘以 $\begin{bmatrix} a \\ c \end{bmatrix}$ or $\begin{bmatrix} b \\ d \end{bmatrix}$, 0 cancel out every elements 除了 first term (a,b) in the column vector, 第二行 $[0,1]$ cancel out every elements 除了 second term (c, d)

Inverse 2*2:

$$A^{-1} * A = I, A * A^{-1} = I, A \text{ is also inverse of } A^{-1}$$

Calculate the inverse:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad ad-bc \text{ is determinant of } A$$

Determinant: $|A| = ad - bc$

Identity matrix: $I*A = A, A*I = A$, 两个都满足的只有当 A 是 square matrix 的时候

Inverse 3*3: Gauss Jordan elimination (augment the matrix, operation: elementary row operation)

Perform some operation 在 left side and same operation on right side, 当 have identity matrix 在 left-hand side (变成 identity matrix 的形式 叫做 reduced row echelon form), right-hand side 就是原来的 invers

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

Row3 = row3 - row1

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}$$

Swap row3 and row2

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}$$

Row3 = row3 - 2*row2

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}$$

Row1 = row1 - row3

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}$$

Hint why this work: 当对左面 matrix 进行操作可以想成乘以多个 matrices, so we multiply matrix 得到 identity matrix, 乘以的多个 matrices 就是 A^{-1} , 而我们知道 identity matrix 乘以任何 matrix 就是 matrix itself

Solve system of equations (2*2):

$$Ax = b \rightarrow A^{-1}Ax = A^{-1}b \rightarrow Ix = A^{-1}b \rightarrow x = A^{-1}b$$

Matrices to solve vector combination: 可以想成 matrix multiplication problem 把两个 vector 合成一个 vector

比如 vector $a = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, 得到 $c = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 \\ -6 & 6 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

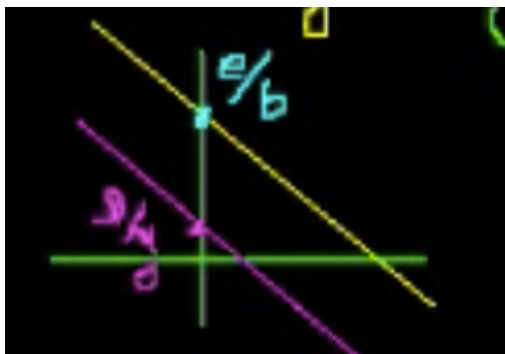
$$A^{-1} = \frac{1}{30} \begin{bmatrix} 6 & -2 \\ 6 & 3 \end{bmatrix}, A^{-1} * \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{需要 1 个 } a, \text{两个 } b$$

Singular matrices: 没有 inverse 的 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, A^{-1} \text{ is undefined iff } |A| = ad - bc = 0 \text{ (or } \frac{a}{b} = \frac{c}{d} \text{ or } \frac{a}{c} = \frac{b}{d} \text{)}$$

Prove: 比如 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

得到 $\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \Rightarrow \begin{cases} y = -\frac{a}{b}x + \frac{e}{b} \\ y = -\frac{c}{d}x + \frac{f}{d} \end{cases}$ 如果 $\frac{a}{b} = \frac{c}{d}$, 两条线平行, 不会相交没有解



如果从 vector 角度考虑, 如下图 $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ 是重合的 if $\frac{a}{c} = \frac{b}{d}$, 无法由 \vec{a}, \vec{b} 构成 \vec{c}



Solve system of equations (3*3):

$$\begin{array}{rcl} -x + 2y - z & = & 9 \\ 3x - 7y - 2z & = & -20 \\ 2x + 2y + z & = & 2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} -1 & 2 & -1 & 9 \\ 3 & -7 & -2 & -20 \\ 2 & 2 & 1 & 2 \end{array} \right]$$

通过 row operation 变成

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 9 \\ 0 & -1 & -5 & 7 \\ 0 & 0 & 1 & -2 \end{array} \right] \Rightarrow \begin{array}{l} x = -1 \\ y = 3 \\ z = -2 \end{array}$$

Vectors and Spans:

Set Colinear vectors: $\{c \vec{v} \mid c \in \mathbb{R}\}$ 比如 vector 在一条线上(slope), $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Linear Combination: $c_1 v_1 + c_2 v_2 + \dots + c_n v_n$, $c_1 \rightarrow c_n \in \mathbb{R}$; We can fill Any point in \mathbb{R}^2 with combination of vector a and b: we can write $\text{span}(\vec{a}, \vec{b}) = \mathbb{R}^2$, we can represent any vector in \mathbb{R}^2 with some linear combination of a and b where a and b cannot be collinear (a,b 不能共线, 换种思维考虑: 如果共线了, 组成的 matrix 没有 inverse $A^*c = b$, A 没有 inverse). $\text{span}(\vec{0}) = \vec{0}$ ($c \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

比如 unit vector $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, 可以构建任何 vector in \mathbb{R}^2 by using these unit vectors

$\text{span}(v_1, v_2, v_3) = \{c_1 v_1 + c_2 v_2 + \dots + c_n v_n \mid c_i \in \mathbb{R} \ 1 \leq i \leq n\}$: The space of all of the combination of vectors v_1, v_2, \dots, v_n

Linearly Dependent set: some vector in the set can be represented by some combinations of other vectors in the set, 比如 $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ 是 linearly dependent, 再比如 $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 9 \\ 5 \end{bmatrix}$ 是 linearly dependent, 因为其中一个可以由另外两个构成构成

$$\mathbb{R}^n : \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_i \in \mathbb{R}, 1 \leq i \leq n \right\}$$

V is Subspace of \mathbb{R}^n (me vector from \mathbb{R}^n) 必须满足:

1. V contains 0 vector
2. If \vec{x} in V then any scalar c: $c\vec{x}$ also in V (closure under scalar multiplication)
3. If \vec{a} in V and \vec{b} in V, $\vec{a} + \vec{b}$ also in V (closure under addition)

同样如果满足这三个条件的也是 subspace

e.g. $v = \{0\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$: v 只有 vector 0, v is subspace

of \mathbb{R}^3

1. 满足条件 1, vector 0 在 v 中

2. 满足条件 2: $c \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3. 满足条件 3: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

e.g. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x \geq 0 \right\}$: v is not subspace of \mathbb{R}^2

1. 满足条件 1, vector 0 在 v 中
2. 不满足条件 2: $-1 \begin{bmatrix} a \\ b \end{bmatrix}$, $-1*a$ 为负数
3. 满足条件 3: $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ $a+c$ 是正数 given $a \geq 0$ and $c \geq 0$

e.g. $V = \text{span}(v_1, v_2, v_3)$. is valid subspace of \mathbb{R}^n

1. 满足条件 1: $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$
2. 满足条件 2: $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$; then $a\vec{x} = ac_1 \vec{v}_1 + ac_2 \vec{v}_2 + ac_3 \vec{v}_3$; ac_1, ac_2, ac_3 can be arbitrary constant, 因为 span 是 all linear combination 所以新的也在 span 当中

3. 满足条件 3 : $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$; $\vec{y} = d_1\vec{v}_1 + d_2\vec{v}_2 + d_3\vec{v}_3$, then $\vec{x} + \vec{y} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + (c_3 + d_3)\vec{v}_3$, it also in span

e.g $V = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$. is valid subspace of \mathbb{R}^2

1. 满足条件 1: $\vec{0} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2. 满足条件 2: $c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 就是 span itself (combination of vector)
3. 满足条件 3 : $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 在 span 当中

Span($\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$) of any vector is valid subspace,

Basis $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$: 1. Span($\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$) all those $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ linearly independent 2. $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$ when $c_1 = c_2 = \dots = c_n = 0$

Basis(minimum set of vectors that spans the subspace): 如果用 any vector in S 可以 construct any vector in subspace V

e.g. $T = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}_s (= c + \vec{2})\}$, the span of T is still going to be subspace V but T is linearly dependent -> T is not basis for V ()

Basis: 比如, 需要两个 non-redundant vector

Standard Basis for \mathbb{R}^2 $T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Advantage of Basis: represent any vector in subspace by some unique combination of vectors in basis 比如 Basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ $\vec{a} \in U, \vec{a} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n, c_1, c_2, \dots, c_n$ 是 unique 的

Vectors Dot Product

Addition

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Scalar Multiplication

$$c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

Dot Product

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n$$

Example 1: $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 2 \cdot 7 + 5 \cdot 1 = 14 + 5 = 19$

Example 2: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = 1 \cdot (-2) + 2 \cdot 0 + 3 \cdot 5 = -2 + 0 + 15 = 13$

Length: $||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$ $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + \dots + a_n^2 = ||\vec{a}||^2$

Communicative: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

Distributive: $(\vec{v} + \vec{w}) \cdot \vec{x} = \vec{w} \cdot \vec{x} + \vec{v} \cdot \vec{x} = (v_1 + w_1)x_1 + (v_2 + w_2)x_2 + \dots + (v_n + w_n)x_n = v_1 x_1 + w_1 x_1 + v_2 x_2 + w_2 x_2 + \dots + v_n x_n + w_n x_n$

Associative over scalar multiplication: $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$

Dot product self is the length square: $\vec{v} \cdot \vec{v} = ||\vec{v}||^2 = v_1^2 + v_2^2 + \dots + v_n^2$

Cauchy Schwarz Inequality: If $\vec{x}, \vec{y} \in R^n$, $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| ||\vec{y}||$, $|\vec{x} \cdot \vec{y}| = ||\vec{x}|| ||\vec{y}||$ only if two vector colinear, one vector 是另一个 vector 乘以的倍数 ($|\vec{x} \cdot \vec{y}|$ absolute value of dot product)

Prove: suppose \vec{x}, \vec{y} is non-zero vector

$$\text{suppose } p(t) = ||t\vec{y} - \vec{x}||^2 \geq 0$$

$$p(t) = (t\vec{y} - \vec{x})(t\vec{y} - \vec{x})$$

$$p(t) = t\vec{y} \cdot t\vec{y} - \vec{x} \cdot t\vec{y} - t\vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{x}$$

$$p(t) = t^2(\vec{y} \cdot \vec{y}) - 2(\vec{x} \cdot \vec{y})t + \vec{x} \cdot \vec{x} \geq 0$$

此时设 $\vec{y} \cdot \vec{y} = a$, $2(\vec{x} \cdot \vec{y}) = b$, $\vec{x} \cdot \vec{x} = c$

$$p(t) = at^2 - bt + c \geq 0$$

$$p\left(\frac{b}{2a}\right) = a \frac{b^2}{4a^2} - b \frac{b}{2a} + c \geq 0$$

$$p\left(\frac{b}{2a}\right) = \frac{b^2}{4a} - \frac{2b^2}{4a} + c$$

$$p\left(\frac{b}{2a}\right) = -\frac{b^2}{4a} + c \geq 0 \Rightarrow c \geq \frac{b^2}{4a} \Rightarrow 4ac \geq b^2$$

$$4ac \geq b^2 = 4(||\vec{y}||^2 ||\vec{x}||^2) \geq (2(\vec{x} \cdot \vec{y}))^2$$

$$(||\vec{y}||^2 ||\vec{x}||^2) \geq (\vec{x} \cdot \vec{y})^2$$

Take square root

$$||\vec{y}|| ||\vec{x}|| \geq |\vec{x} \cdot \vec{y}|$$

When $\vec{x} = c\vec{y}$

$$|\vec{x} \cdot \vec{y}| = |c \vec{x} \cdot \vec{y}| = c |\vec{y} \cdot \vec{y}| = c |\vec{y}|^2 = ||c\vec{y}|| \cdot ||\vec{y}||$$

Triangle Inequality:

$$\begin{aligned} ||\vec{x} + \vec{y}||^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= \vec{x}(\vec{x} + \vec{y}) + \vec{y}(\vec{x} + \vec{y}) \\ &= \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{x} + \vec{y} \cdot \vec{y} \\ &= ||\vec{x}||^2 + 2(\vec{x} \cdot \vec{y}) + ||\vec{y}||^2 \end{aligned}$$

根据 cauchy Schwarz inequality: $\vec{x} \cdot \vec{y} \leq |\vec{x}| \cdot |\vec{y}| = ||\vec{x}|| ||\vec{y}||$, dot product 可以是负数

$$\begin{aligned} ||\vec{x} + \vec{y}||^2 &\leq ||\vec{x}||^2 + 2||\vec{x}|| ||\vec{y}|| + ||\vec{y}||^2 = (||\vec{x}|| + ||\vec{y}||)^2 \\ ||\vec{x} + \vec{y}|| &\leq ||\vec{x}|| + ||\vec{y}|| \end{aligned}$$

当 $\vec{x} = c\vec{y}$, c 是 positive 的时候 ($c > 0$), $||\vec{x} + \vec{y}|| = ||\vec{x}|| + ||\vec{y}||$

\vec{x} doesn't need to only 2-dimensional, 可以是 n dimension

Triangle Angle between vectors:

$||\vec{a}|| = \text{length} \Rightarrow \text{scalar}$ "angle" between vectors

$\vec{a}, \vec{b} \in \mathbb{R}^n, \text{non zero}$

Reasons why I couldn't

$$\begin{aligned} ||\vec{b}|| &> ||\vec{a}|| + ||\vec{a} - \vec{b}|| \\ ||\vec{a}|| &> ||\vec{a} - \vec{b}|| + ||\vec{b}|| \\ ||\vec{a} - \vec{b}|| &> ||\vec{a}|| + ||\vec{b}|| \end{aligned}$$

$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$

$||\vec{a}|| = ||\vec{b} + (\vec{a} - \vec{b})||$

$$||\vec{a}|| = ||\vec{b} + (\vec{a} - \vec{b})|| \leq ||\vec{b}|| + ||\vec{a} - \vec{b}||$$

$$||\vec{b}|| = ||\vec{a} + (\vec{b} - \vec{a})|| \leq ||\vec{a}|| + ||\vec{b} - \vec{a}||$$

$$||\vec{a} - \vec{b}|| = ||\vec{a} + (-\vec{b})|| \leq ||\vec{a}|| + ||-\vec{b}|| = ||\vec{a}|| + ||\vec{b}||$$

Law of Cosines

$C^2 = A^2 + B^2 - 2AB \cos \theta$

Law of Cosine

$$||\vec{a} - \vec{b}||^2 = ||\vec{b}||^2 + ||\vec{a}||^2 - 2||\vec{a}|| ||\vec{b}|| \cos \theta$$

Left-hand side

$$\begin{aligned} (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) &= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\ &= ||\vec{a}||^2 + ||\vec{b}||^2 - 2(\vec{a} \cdot \vec{b}) \end{aligned}$$

Left-hand side = right-hand side

$$||\vec{a}||^2 + ||\vec{b}||^2 - 2(\vec{a} \cdot \vec{b}) = ||\vec{b}||^2 + ||\vec{a}||^2 - 2||\vec{a}|| ||\vec{b}|| \cos \theta$$

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos \theta$$

If $\vec{a} = c\vec{b}$; $c > 0 \Rightarrow \theta = 0^\circ$; $c < 0 \Rightarrow \theta = 180^\circ$;

Perpendicular $\theta = 90^\circ$, $\vec{a} \cdot \vec{b} = 0$ 但是如果 dot product = 0 不意味着垂直, 比如 $\vec{0} \cdot \vec{b} = 0$

但是当 a, b 都是 nonzero vector, dot product 意味着垂直(perpendicular)

$\vec{a} \cdot \vec{b} = 0 \Rightarrow$ orthogonal, zero vector is orthogonal to everything; perpendicular is orthogonal, 但是 orthogonal 不一定是 perpendicular

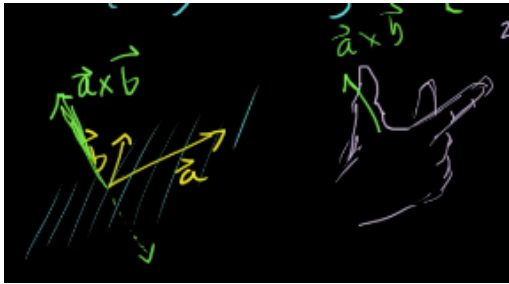
Dot Product: $\vec{a}, \vec{b} \in \mathbb{R}^n \Rightarrow$ 得到 scalar

Cross Product: only defined in \mathbb{R}^3 , 得到 vector

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{a} \times \vec{b} = \begin{bmatrix} -7 * 4 - 1 * 2 \\ 1 * 5 - 1 * 4 \\ 1 * 2 - (-7 * 5) \end{bmatrix} = \begin{bmatrix} -30 \\ 1 \\ 37 \end{bmatrix}$$

Cross product 乘积是 orthogonal to \vec{a} and \vec{b}



判断 cross product 的方向可以用 right hand rule, 食指指向 a 的方向, 中指指向 b, 大拇指的方向是 a 和 b 的 cross product

Prove Orthogonal for \vec{a} and $\vec{a} \times \vec{b}$: (Same for \vec{b})

$$\begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= a_2 b_3 a_1 - a_3 b_2 a_1 + a_3 b_1 a_2 - a_1 b_3 a_2 + a_1 b_2 a_3 - a_2 b_1 a_3$$

$$= a_2 b_3 a_1 - a_1 b_3 a_2 + a_1 b_2 a_3 - a_3 b_2 a_1 + a_3 b_1 a_2 - a_2 b_1 a_3 = 0$$

$$\begin{cases} \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \\ |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta \end{cases}$$

Prove:

$$\begin{aligned} |\vec{a} \times \vec{b}|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\ &= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_3^2 + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ &= a_1^2 (b_2^2 + b_3^2) + a_2^2 (b_1^2 + b_3^2) + a_3^2 (b_1^2 + b_2^2) - 2(a_2 a_3 b_2 b_3 + a_1 a_3 b_1 b_3 + 2a_1 a_2 b_1 b_2) \end{aligned}$$

$$\begin{aligned} |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta &= (\vec{a} \cdot \vec{b})^2 = (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= a_1^2 b_1^2 + a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + a_2^2 b_2^2 + a_1 a_2 b_1 b_2 + a_2 a_3 b_2 b_3 + a_3^2 b_3^2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3 \\ &= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2(a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3) \end{aligned}$$

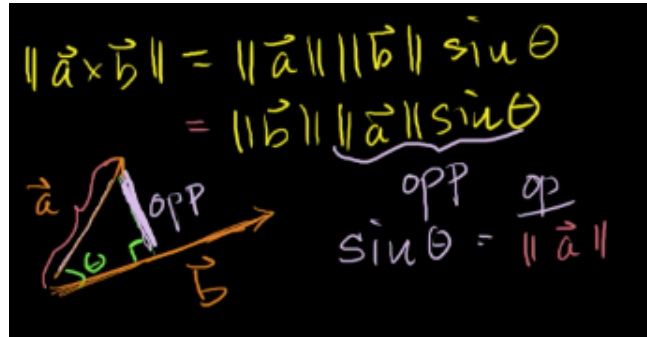
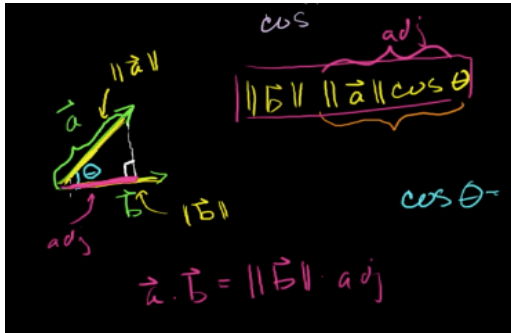
$$\begin{aligned} |\vec{a} \times \vec{b}|^2 + |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta &= a_1^2 (b_1^2 + b_2^2 + b_3^2) + a_2^2 (b_1^2 + b_2^2 + b_3^2) + a_3^2 (b_1^2 + b_2^2 + b_3^2) \\ &= (a_1^2 + a_2^2 + a_3^2) (b_1^2 + b_2^2 + b_3^2) = |\vec{a}|^2 |\vec{b}|^2 \end{aligned}$$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta = |\vec{a}|^2 |\vec{b}|^2 (1 + \cos^2 \theta) = |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta = \|\vec{b}\| \text{adj}$$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta = \|\vec{b}\| \text{opp}$$

$$(\text{Optional}) \quad \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

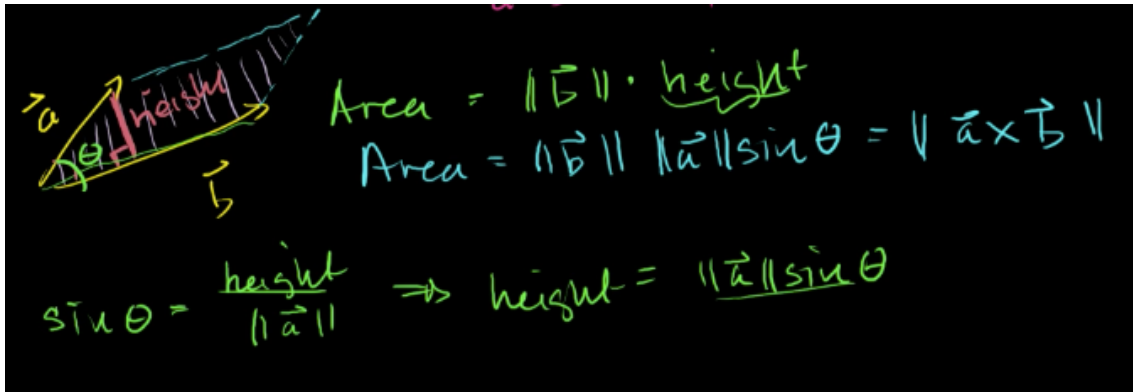


$\|\vec{a}\| \cos\theta$ 构

成直角三角形的 a 的 projection 是 adj
 $\|\vec{a}\| \sin\theta$ 构成直角三角形的高

Dot product tells: product of lengths of vectors move together at same direction with b. When $\vec{a} \cdot \vec{b} = 0$, perpendicular, \vec{a} onto \vec{b} is zero

Cross product tells: product of lengths of vectors move perpendicular direction with b. When $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\|$, perpendicular, 获得最大值, 当 a 和 b colinear, $\vec{a} \cdot \vec{b} = 0$ no perpendicular vector



Cross product 还可以算平行四边形的面积

Row echelon Form:

Pivot entry: 那个 column 只能它不是 0, 且那行前面没有数

Free-variable: row 中在 pivot 后面的 variable

Matrix Vector Product

1. As row vector and x dot product

$$\begin{aligned}
 & \begin{bmatrix} -3 & 0 & 3 & 2 \\ 1 & 7 & -1 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \cdot 2 + 0 \cdot (-3) + 3 \cdot 4 + 2 \cdot (-1) \\ 1 \cdot 2 + 7 \cdot (-3) + (-1) \cdot 4 + 9 \cdot (-1) \end{bmatrix} \\
 & = \begin{bmatrix} -6 + 0 + 12 - 2 \\ 2 - 21 - 4 - 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -32 \end{bmatrix} \quad \vec{a}_1 = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 2 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 7 \\ -1 \\ 9 \end{bmatrix} \\
 & \vec{a}_1^T = [-3 \ 0 \ 3 \ 2] \quad \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \end{bmatrix} \\
 & \vec{a}_2^T = [1 \ 7 \ -1 \ 9]
 \end{aligned}$$

2. As column vector and x linear combination

$$\begin{aligned}
 A &= \begin{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad A\vec{x} = \begin{bmatrix} 3x_1 + 1x_2 + 0x_3 + 3x_4 \\ 2x_1 + 4x_2 + \dots \\ -1x_1 + 2x_2 + \dots \end{bmatrix} \\
 & \quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \\
 A &= [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4] \\
 A\vec{x} &= \underline{x_1} \vec{v}_1 + \underline{x_2} \vec{v}_2 + \underline{x_3} \vec{v}_3 + \underline{x_4} \vec{v}_4 \quad \leftarrow \text{Linear combination of column vectors of } A
 \end{aligned}$$

Null Space

$$N = \{ \vec{x} \in \mathbb{R}^n \mid A \vec{x} = \vec{0} \}$$

满足 1. $\vec{0}$ in this subspace; 2. If $\vec{v}_1, \vec{v}_2 \in N$, then $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \vec{0} \in N$; 3. $\vec{v} \in N, A(c\vec{v}) = c(A\vec{v}) = \vec{0} \in N$; N is valid subspace

e.g 求 $A\vec{x} = 0$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

得到 augmented matrix in row echelon form:

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & -1 & -2 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \\ &\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \\ &= N(A) = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Original problem $A\vec{x} = 0$ can be transformed to $(\text{ref}(A))\vec{x} = 0$, $(\text{ref}(A))$: null space of reduced row echelon form of matrix A

Relationship to linear Independent

Matrix A ($M \times N$); Null space $N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A \vec{x} = \vec{0} \}$, $\vec{0} \in \mathbb{R}^m$; 把 A 用 column vector 来表示

$$A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n] \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n$$

如果 $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ 都是 linear independent, 唯一的解是 x_1, x_2, \dots, x_n 都是 0 $N(A) = N(\text{rref}(A)) = \{\vec{0}\}$
 $\text{rref}(A) \vec{x} = \vec{0}$, which means no free variable

$N(A) = N(\text{rref}(A)) = \{\vec{0}\}$ if only if column vectors of A linear independent (only do if A is $N \times N$ matrix)

Column Space

$$A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n], C(A) = \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$$

满足 1. $\vec{0}$ in this subspace (用每个 vector 乘以 0); 2. If $\vec{b}, \vec{c} \in C(A)$, then $(\vec{b} + \vec{c}) = (b_1 + c_1)\vec{v}_1 + (b_2 + c_2)\vec{v}_2 + \dots + (b_n + c_n)\vec{v}_n = \vec{0} \in N$; 3. $\vec{v} \in N, A(c\vec{v}) = c(A\vec{v}) \in C(A)$; $C(A)$ is valid subspace

e.g.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \text{column space} = C(A) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right)$$

$$\text{Null Space } N(A) = N(\text{rref}(A))$$

$$\text{Row echelon form: } \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Null Space: } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \text{span} \left(\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ linear independent? 因为 null space contain $\{\vec{0}\}$, 所以是 linear dependent set

因为是 linear dependent (后两个是 redundant 的)

$C(A) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right), \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ is a basis for $C(A)$, 跟 row-reduced echelon form pivot 所在的 column 到

原来的 matrix 中选 basis

求 column space 的 function:

我们知道 cross product 垂直于 \vec{a} and \vec{b} , normal vector $\vec{n} = \vec{a} \times \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 - 3 = 5 \\ 3 - 4 = -1 \\ 1 - 2 = -1 \end{bmatrix}$

$$\vec{n} \cdot \left(\vec{x} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} \cdot \left(\vec{x} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$$

$$5x - y - z = 0$$

另一种方法: what kind of B will give valid solution $\{\vec{b} \mid A\vec{x} = \vec{b} \text{ \& } \vec{x} \in R^n\}$

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & x \\ 2 & 1 & 4 & 3 & y \\ 3 & 4 & 1 & 2 & z \end{array} \right]$$

化成 row echelon form:

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & x \\ 0 & 1 & -2 & -1 & y \\ 0 & 0 & 0 & 0 & 2x - y - z + 3x \end{array} \right]$$

为了让 system 有解 $2x - y - z + 3x = 5x - y - z = 0$

Dimension

$\text{Dim}(V)$: the number (cardinality) of a basis of V (比如 $A = \{a_1, a_2, \dots, a_n\}$ is a basis of V , $\text{Dim}(V) = n$)

All basis of the same subspace must have the same number of elements

Dimension of Null space: $\text{Dim}(N(B))$ is the **Nullity** = number of **free variables** (non-pivot) in reduced echelon form in Matrix A

e.g.

$$\begin{aligned} A\vec{x} &= \vec{0}: \begin{bmatrix} 1 & 0 & -1 & -2 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \\ &= N(A) = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

两个 free variable x_3, x_4 , $\text{Dim}(N(B)) = \text{nullity} = 2$

Dimension of Column space: $\text{Dim}(C(A))$ is the **Rank** = number of **pivot variables** in reduced echelon form in Matrix A (rank of A number of linear independent column vector you have)

e.g.

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix}$$

To reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

第 1, 2, 4 列 linearly independent, column space 的 basis $\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right)$

$\text{Dim}(C(A)) = 3$

Linear Transformation

When function map to \mathbb{R} (一维的) called **scaler value / Real valued function**

When function map to $\mathbb{R}^2, \mathbb{R}^3$ (多维的) called **vector value**

Transformation: function operating on vectors (linear algebra)

Linear Transformation:

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ if only if

$$\vec{a}, \vec{b} \in \mathbb{R}^n, \quad 1. T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$$

$$2. T(c\vec{a}) = cT(\vec{a})$$

如果看 T 是不是 linear transformation 需要证明是不是符合上面的两个条件

Matrix vector products is linear transformation

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m: T(\vec{x}) = A\vec{x}, \quad A \text{ 的 dimension } m \times n$$

Prove it is linear transformation:

$$\begin{aligned} A &= [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n] & A \cdot (\vec{a} + \vec{b}) &= A \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \\ & & &= (a_1 + b_1)\vec{v}_1 + (a_2 + b_2)\vec{v}_2 + \dots + (a_n + b_n)\vec{v}_n \\ & & &= a_1\vec{v}_1 + b_1\vec{v}_1 + a_2\vec{v}_2 + b_2\vec{v}_2 + \dots + a_n\vec{v}_n + b_n\vec{v}_n \\ & & &= A \cdot \vec{a} + A \cdot \vec{b} \end{aligned}$$

$$\begin{aligned} A \cdot (c\vec{a}) &= ca_1\vec{v}_1 + ca_2\vec{v}_2 + \dots + ca_n\vec{v}_n \\ &= c(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n) \end{aligned}$$

Any linear matrix transformation can be viewed as matrix product

Standard basis for \mathbb{R}^n

$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ \vec{e}_i 是 $n \times n$ identity matrix 的 i th column

Image: transformation from one set into another set $T(L_0)$ is image of L_0 under T

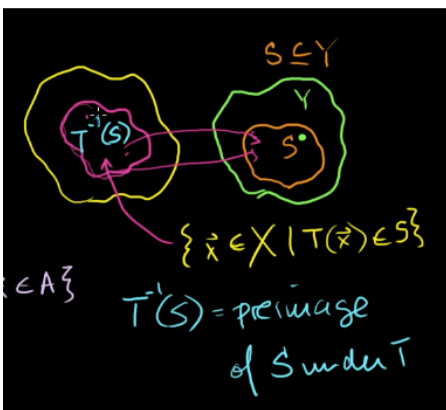
$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m: T(V): \text{image of } V \text{ under } T$$

$T(V)$ is valid subspace:

Prove: 1. $T(\vec{a}), T(\vec{b}) \in T(V)$ 因为是 linear transformation $T(\vec{a}) + T(\vec{b})$ in V , 所以 $T(\vec{a}) + T(\vec{b}) = T(\vec{a} + \vec{b}) \in T(V)$

2. $cT(\vec{a}) = T(c\vec{a})$ 因为 $T(c\vec{a})$ in V , $cT(\vec{a})$ also in V

Image of T : $T(\vec{x}) = A\vec{x} = \text{column space of } A = \text{span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$



Preimage: $T^{-1}(S) = \{\vec{x} \in X \mid T(\vec{x}) \in S\}$, given co-domain, what subset of domain map into co-domain, (不是每个 S 都需要 \vec{x} map 到)
 $T(T^{-1}(S)) \subseteq S$

Kernel of T : $\text{Ker}(T) = \{x \in \mathbb{R}^2 \mid T(\vec{x}) = \vec{0}\}$: A vector v is in the kernel of a linear transformation if and only if $T(v) = 0$. It is the same things as null space

Sums and scalar multiples of linear transformation

$$S: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\text{Def: } (S + T)(\vec{x}) = S(\vec{x}) + T(\vec{x}) \quad (S + T): \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\text{Def: } (cS)(\vec{x}) = c(S(\vec{x})): \quad cS: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Linear Transformation example:

让所有 x 变成负- x , 所有 y 乘以 2

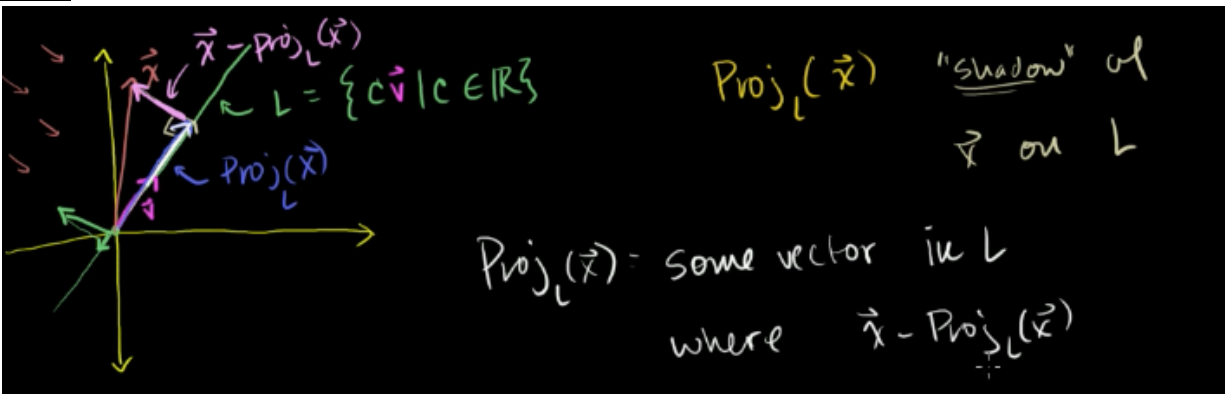
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

A 是 **diagonal matrix**: 只有对角线有值, 剩下都是 0;

Unit vector: vector has length of 1 $||\vec{u}|| = ||\frac{1}{||\vec{v}||} \vec{v}|| = \frac{1}{||\vec{v}||} ||\vec{v}||$

$$\text{e.g. } \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{u} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Projection



Proj_L(\vec{x}) 把 \vec{x} 到 L 做垂线, \vec{x} 在 L 的射影是 projection, 垂线是 $\vec{x} - \text{Proj}_L(\vec{x})$, 因为垂直 dot product = 0, $(\vec{x} - \text{Proj}_L(\vec{x})) \cdot \text{Proj}_L(\vec{x}) = 0$

$\text{Proj}_L(\vec{x})$: some vector in L where $\vec{x} - \text{Proj}_L(\vec{x})$ is orthogonal to L

$$L = \{c\vec{v} \mid c \in \mathbb{R}\} \quad (\vec{x} - c\vec{v}) \cdot \vec{v} = 0$$

$$\vec{x} \cdot \vec{v} - c\vec{v} \cdot \vec{v} = 0$$

$$\vec{x} \cdot \vec{v} = c\vec{v} \cdot \vec{v}$$

$$c = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

$$\text{Proj}_L(\vec{x}) = c\vec{v} = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

If \vec{v} is unit vector, $\text{Proj}_L(\vec{x}) = (\vec{x} \cdot \vec{v}) \vec{v}$

Prove: projection is linear transformation (\vec{u} is unit vector)

$$\begin{aligned} 1. \text{Proj}_L(\vec{a} + \vec{b}) &= ((\vec{a} + \vec{b}) \cdot \vec{u}) \vec{u} \\ &= (\vec{a} \cdot \vec{u} + \vec{b} \cdot \vec{u}) \vec{u} \\ &= (\vec{a} \cdot \vec{u}) \vec{u} + (\vec{b} \cdot \vec{u}) \vec{u} = \text{Proj}_L(\vec{a}) + \text{Proj}_L(\vec{b}) \end{aligned}$$

$$\begin{aligned} 2. \text{Proj}_L(c\vec{a}) &= (c\vec{a} \cdot \vec{u}) \vec{u} \\ &= c(\vec{a} \cdot \vec{u}) \vec{u} \\ &= c\text{Proj}_L(\vec{a}) \end{aligned}$$

So $\text{Proj}_L(\vec{x}) = A\vec{x}$

$$A = [([1] \cdot [u_1]) [u_1], ([0] \cdot [u_1]) [u_1]]$$

$$A = \begin{bmatrix} u_1 [u_1], & u_2 [u_1] \\ u_1 [u_2], & u_2 [u_2] \end{bmatrix} = \begin{bmatrix} u_1^2 & u_2 u_1 \\ u_1 u_2 & u_2^2 \end{bmatrix}$$

Composition

Composition: transformation of transformation

Composition is linear transformation (given S, T is linear transformation)

Prove:

$$T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x} + \vec{y})) = T(S(\vec{x}) + S(\vec{y})) = T(S(\vec{x})) + T(S(\vec{y})) = T \circ S(\vec{x}) + T \circ S(\vec{y})$$

$$T \circ S(c\vec{x}) = T(S(c\vec{x})) = T(cS(\vec{x})) = cT(S(\vec{x})) = c(T \circ S)(\vec{x})$$

因为 composition is linear transformation, 可以把 $T \circ S(\vec{x} + \vec{y})$ 写成 $A\vec{x}$

$$C = \left[B \left(A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right), B \left(A \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \right), \dots, B \left(A \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right) \right]$$

$$C = [B(\vec{a}_1), B(\vec{a}_2), \dots, B(\vec{a}_n)]$$

$$T \circ S(\vec{x}) = B(A(\vec{x})) = B A \vec{x}$$

$$AB = [A \vec{b}_1, A \vec{b}_2, \dots, A \vec{b}_n]$$

Associative $((H \circ G) \circ F)(\vec{x}) = (H \circ G)(F\vec{x}) = H(G(F\vec{x})) = H((G \circ F)\vec{x})$

e.g. $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & -1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$

$$AB = [A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, A \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}]$$

Matrix product properties

Associative: $(AB)C = A(BC)$ Doesn't matter where to put 括号

Not Commutative: $AB \neq BA$

Distributive: $A(B + C) = AB + AC, (B + C)A = BA + CA$

Prove Distributive:

$$B = [\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n], C = [\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n]$$

$$A(B + C) = A[\vec{b}_1 + \vec{c}_1, \vec{b}_2 + \vec{c}_2, \dots, \vec{b}_n + \vec{c}_n]$$

$$= A[\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n] + A[\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n] = AB + AC$$

$$(B + C)A = (B + C) [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n] = [(B + C)\vec{a}_1, (B + C)\vec{a}_2, \dots, (B + C)\vec{a}_n]$$

$$= [B\vec{a}_1, B\vec{a}_2, \dots, B\vec{a}_n] + [C\vec{a}_1, C\vec{a}_2, \dots, C\vec{a}_n]$$

$$= BA + CA$$

Inverse

f (function $X \rightarrow Y$) is **Invertible** if and only if there exist a function $f^{-1} (Y \rightarrow X)$ such that $f^{-1} \circ f = I_x$ and $f \circ f^{-1} = I_y$

$$f^{-1} \circ f(a) = I_x(a) = a; f \circ f^{-1}(y) = y$$

Every function if has inverse, its inverse must be unique

Invertibility implies a unique solution to $f(x) = y$

Prove: If f is invertible, for every $y \in Y$, there is unique solution $x \in X$ such that $f(x) = y$

$$f(x) = y; \quad f^{-1}(y) = f^{-1}(f(x)) = (f^{-1} \circ f)(x) = I_x(x) = x$$

For every $y \in Y$ $f(x) = y$ has a unique solution, then f is invertibility

$S: Y \rightarrow X$; 因为 $f(x) = y$ has unique solution. $S(y)$: well defined

$S(b)$ is the unique solution to $f(x) = b$

$$f(S(b)) = b \rightarrow (f \circ S)(b) = I_y(b) = b; \quad f \circ S = I_y$$

$S(f(a)) =$ the unique solution to the equation $= a$; $(S \circ f) = I_x$

因为 $(S \circ f) = I_x, f \circ S = I_y$, by definition, function is invertible

Inverse is linear transformation

Prove:

$$(f^{-1} \circ f)(\vec{a} + \vec{b}) = \vec{a} + \vec{b} = (f^{-1} \circ f)(\vec{a}) + (f^{-1} \circ f)(\vec{b})$$

$$f(f^{-1}(\vec{a} + \vec{b})) = f(f^{-1}(\vec{a})) + f(f^{-1}(\vec{b})) = f(f^{-1}(\vec{a}) + f^{-1}(\vec{b})) \text{ 因为 } f \text{ 是 linear transformation}$$

$$f^{-1}(f(f^{-1}(\vec{a} + \vec{b}))) = f^{-1}(f(f^{-1}(\vec{a}))) + f^{-1}(f(f^{-1}(\vec{b})))$$

$$(f^{-1} \circ f)(f^{-1}(\vec{a} + \vec{b})) = (f^{-1} \circ f)(f^{-1}(\vec{a})) + (f^{-1} \circ f)(f^{-1}(\vec{b}))$$

$$f^{-1}(\vec{a} + \vec{b}) = f^{-1}(\vec{a}) + f^{-1}(\vec{b})$$

$$(f \circ f^{-1})(c\vec{a}) = c\vec{a} = c((f \circ f^{-1})(\vec{a})) = f(cf^{-1}(\vec{a}))$$

$$f^{-1}(f(f^{-1}(c\vec{a}))) = f^{-1}(f(cf^{-1}(\vec{a})))$$

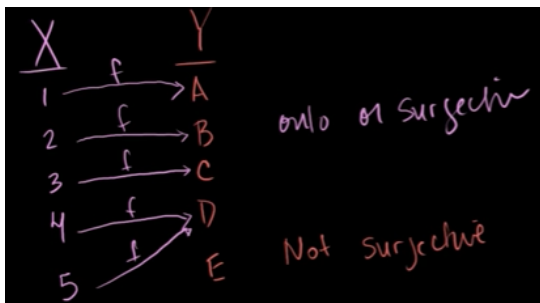
$$(f^{-1} \circ f)(f^{-1}(c\vec{a})) = (f^{-1} \circ f)(cf^{-1}(\vec{a}))$$

$$f^{-1}(c\vec{a}) = cf^{-1}(\vec{a})$$

Onto & One-to-One

Onto (surjective): every elements in co-domain $y \in Y$, there exist at least one $x \in X$ such that $f(x) = y$.

Every y in co-domain at least 一个 x map to



f onto then $\text{image}(f) = Y$

左侧是 Not onto example

One-to-one (injective): for every value that map to there at most at most one x map to it.

每个一个 y 只有一个 x map, 每个 x map to unique y : $f(x) = y$

上面不是 onto 的例子, 也不符合 one-to-one, 假如 5 不指向 D, 5 改指向 E, 表示 onto 和 one-to-one

$f: x \rightarrow y$ is invertible if and only if f is onto and one-to-one

Invertible means For every $y \in Y$ $f(x) = y$ has a unique solution, that means one-to-one, 如果有 $y \in Y$ 但是没有相应的 x 对应, 就不是 invertible 了, 所以 invertible means onto

T is onto iff $C(A) = R^m$, its reduced echelon form has a pivot entry in every row (m pivot entry rank = M): T is onto if and only if $\text{Rank}(A) = m$

$\text{Rank}(A) = \dim(C(A)) = \# \text{ of basis vectors for } C(A)$

$$T: R^n \rightarrow R^m \quad T(\vec{x}) = A\vec{x}$$

Onto \Rightarrow for any $\vec{b} \in R^m$, at least one solution $A\vec{x} = \vec{b}$ where $\vec{x} \in R^n$

$$A\vec{x} = [x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n]$$

For T to be onto $\text{span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n) = R^m$ which is column space, column space is R^m

e.g. $S: R^2 \rightarrow R^3, S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \vec{x}$

row reduced echelon form: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, rank = 2, S is not onto, S is not invertible

e.g. $T: R^2 \rightarrow R^3, T(\vec{x}) = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

row reduced echelon form $\begin{bmatrix} 1 & -3 & | & b_1 \\ -1 & 3 & | & b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & | & b_1 \\ 0 & 0 & | & b_1 + b_2 \end{bmatrix}$

only member $\vec{b} \in R^m$ that has solution are the ones $b_1 + b_2 = 0$

solution set = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, when $T(\vec{x}) = \vec{0}$, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the null space of T

从上面可以看出: Assuming $A(\vec{x}) = \vec{b}$ has a solution, the solution set = $\{\vec{x}_p\} \cup N(A)$ null space, some particular vector union null space; if one-to-one, at most 1 solution $\Rightarrow N(A)$ has just zero vector(trivial)

Any solution to the inhomogeneous system ($A(\vec{x}) = \vec{b}$ give $\vec{b} \neq \vec{0}$) system will take the form $x_p + x_h$ (particular solution + homogeneous solution)

Prove: $A(x_p + x_h) = Ax_p + Ax_h = \vec{b} + \vec{0}$

Prove any solution \vec{x} to $A\vec{x} = \vec{b}$ take the form $\vec{x} = x_p + x_h$:

$$A(\vec{x} - x_p) = A\vec{x} - Ax_p = \vec{0}$$

$\vec{x} - x_p$ is a solution $A\vec{x} = \vec{0}$, $\vec{x} - x_p$ is a member of null space $N(A)$

$$\vec{x} - x_p = x_h \rightarrow \vec{x} = x_p + x_h$$

如果是 one-to-one: $x_p + x_h$ 只能是 one-solution so x_h has to be $\vec{0}$ null space has to be $\{\vec{0}\}$ so $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly independent; $C(A) = \text{span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$, $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ are basis for column space, $\dim(\text{column space}) = n$; rank $(A) = n$

Invertible: 1. onto: rank(A) = m; 2. One-to-one: rank(A) = n; in order to let transformation to be invertible, rank(A) = m = n: matrix has to be square matrix (n by n matrix), 变成 reduced echelon form 每一行每一列又有 pivot entry (n by n identity matrix) (linearly independent pivot column)

$T: R^n \rightarrow R^n$ (不考虑 $R^n \rightarrow R^m$): $T(\vec{x}) = A\vec{x}$ only invertible if row reduced echelon form is I_n

对 matrix 进行 row operation 等于进行 linear transformation, linear transformation 的矩阵是等同于 identity matrix 进行一样的 row operation

比如 $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ 等同于 $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ a_2 + a_1 \\ a_3 - a_1 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$T(\vec{x}) = S\vec{x} = \left[S \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, S \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, S \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

Determinant

3*3 determinant:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Det}(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

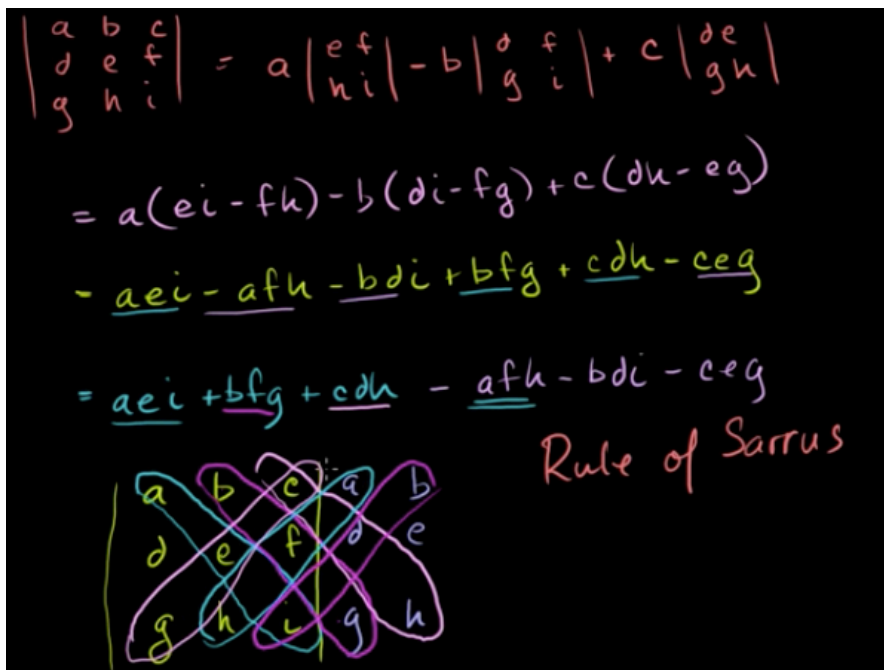
e.g.

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\text{Det}(C) = 1 \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$$

$$\text{Det}(C) = 1 * (-1 * 1 - 0 * 3) - 2(2 * 1 - 4 * 3) + 4(2 * 0 - (-1 * 4)) = 35$$

Quick way: Rule of Sarrus



$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= \underline{a} \underline{e} \underline{i} - \underline{a} \underline{f} \underline{h} - \underline{b} \underline{d} \underline{i} + \underline{b} \underline{f} \underline{g} + \underline{c} \underline{d} \underline{h} - \underline{c} \underline{e} \underline{g}$$

$$= \underline{a} \underline{e} \underline{i} + \underline{b} \underline{f} \underline{g} + \underline{c} \underline{d} \underline{h} - \underline{a} \underline{f} \underline{h} - \underline{b} \underline{d} \underline{i} - \underline{c} \underline{e} \underline{g}$$

Rule of Sarrus

n*n determinant:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \text{ Define } A_{ij} = (n-1) \times (n-1) \text{ matrix by ignore } i\text{-th row and } j\text{-th column}$$

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) + \cdots + (-1)^{i+j} a_{ij} \det(A_{ij})$$

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in})$$

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

Row 乘以 scalar 的 determinant:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = ad - bc$$

如果 row2 乘以 k

$$A = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \det(A) = kad - kbc = k * \det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

如果 row1 也乘以 k

$$A = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}, \det(A) = k^2 ad - k^2 bc = k^2 * \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

如果是 3*3 matrix row 乘以 k

$$A = \begin{bmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{bmatrix}, \det(A) = -kd \begin{vmatrix} b & c \\ h & i \end{vmatrix} + ke \begin{vmatrix} a & c \\ g & i \end{vmatrix} - kf \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

如果是 n*n matrix row 乘以 k

$$\det(A) = (-1)^{i+1}ka_{i1} \det(A_{ij}) + (-1)^{i+2}ka_{i2} \det(A_{i2}) + \dots + (-1)^{i+n}ka_{in} \det(A_{in})$$

$$\det(A) = k \sum_{j=1}^{j=n} (-1)^{i+j} a_{ij} \det(A_{ij}) = k * \det(A)$$

如果 matrix 每行都乘以 k

$$\det(A) = k^n * \det(A)$$

When row is added 的 determinant:

2*2 matrix:

$$X = \begin{bmatrix} a & b \\ x_1 & x_2 \end{bmatrix}, Y = \begin{bmatrix} a & b \\ y_1 & y_2 \end{bmatrix}, Z = \begin{bmatrix} a & b \\ x_1 + y_1 & x_2 + y_2 \end{bmatrix}$$

$$\det(X) = ax_2 - bx_1, \det(Y) = ay_2 - by_1, \det(Z) = a(x_2 + y_2) - b(x_1 + y_1) = \det(X) + \det(Y)$$

3*3 matrix:

$$X = \begin{bmatrix} a & b & c \\ x_1 & x_2 & x_3 \\ d & e & f \end{bmatrix}, \det(X) = -x_1 \begin{vmatrix} b & c \\ e & f \end{vmatrix} + x_2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} - x_3 \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$Y = \begin{bmatrix} a & b & c \\ y_1 & y_2 & y_3 \\ d & e & f \end{bmatrix}, \det(Y) = -y_1 \begin{vmatrix} b & c \\ e & f \end{vmatrix} + y_2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} - y_3 \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$Z = \begin{bmatrix} a & b & c \\ x_1 + y_1 & x_2 + y_2 & y_1 + y_3 \\ d & e & f \end{bmatrix}, \det(Z) = -(x_1 + y_1) \begin{vmatrix} b & c \\ e & f \end{vmatrix} + (x_2 + y_2) \begin{vmatrix} a & c \\ d & f \end{vmatrix} - (x_3 + y_3) \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$\det(Z) = \det(X) + \det(Y)$$

n*n matrix

$$\det(X) = k \sum_{j=1}^{j=n} (-1)^{i+j} x_{ij} \det(A_{ij}), \det(Y) = k \sum_{j=1}^{j=n} (-1)^{i+j} y_{ij} \det(A_{ij}),$$

$$\det(Z) = k \sum_{j=1}^{j=n} (-1)^{i+j} (x_{ij} + y_{ij}) \det(A_{ij}) = \det(X) + \det(Y)$$

Determinant operations are not linear on matrix addition

Swap Row determinant: 比如第 i 行和第 j 行互换了

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ a_j \\ \dots \end{bmatrix}, \text{swap } i \text{ 和 } j \text{ 行}, \quad A_{ij} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_j \\ \dots \\ a_i \\ \dots \end{bmatrix}, \det(A) = -\det(A_{ij})$$

假如第 i 行 = 第 j 行, $\det(A) = \det(A_{ij})$, 根据上面的定理: $\det(A) = -\det(A_{ij}) = \det(A_{ij}), \det(A) = 0$

Duplication row determinant = 0, 因为 duplicate row never get reduced echelon form to be invertible => det = 0

Determinant of row operation: row j = row j - c*rowi

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ a_j \\ \dots \end{bmatrix}, B = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ a_j - c * a_i \\ \dots \end{bmatrix}$$

因为 $\begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ -c * a_i \\ \dots \end{bmatrix}$ 不是 linear independent, 第 j 行可以由第 i 行乘以 -c 得到, 所以 $\det \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ -c * a_i \\ \dots \end{pmatrix} = 0$

$$\det(B) = \det \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ a_j \\ \dots \end{pmatrix} + \det \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ -c * a_i \\ \dots \end{pmatrix} = \det(A)$$

Determinant of upper triangular: diagonal 所有数的乘积: $\det(A) = a_{11}a_{22} \dots a_{nn}$

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \det(A) = ad, B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \det(B) = adf$$

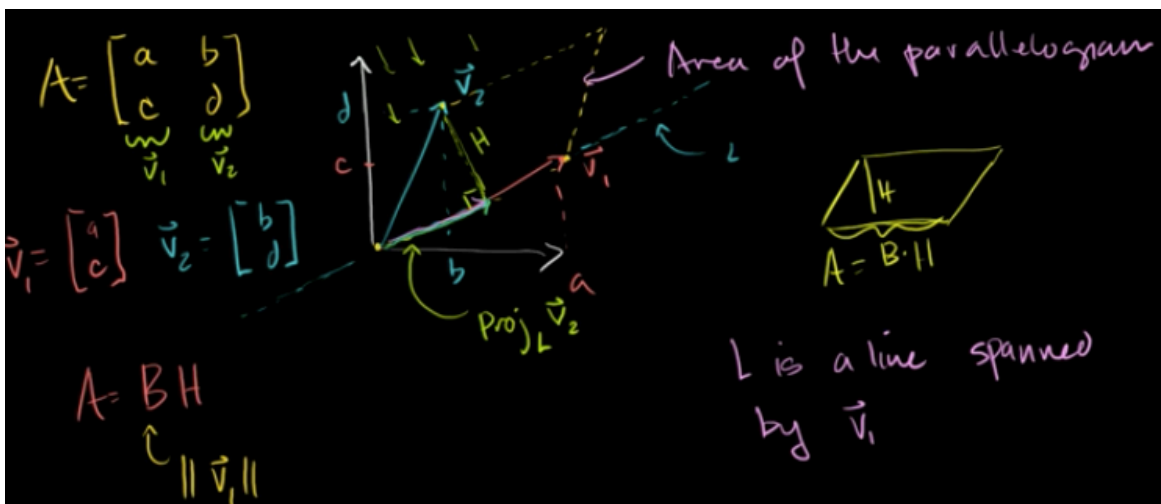
Simple 4*4 determinant: 利用 row operation 不 change determinant 和 upper triangular determinant 的性质, 将 4*4 matrix 变成 diagonal matrix

e.g

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{bmatrix} \rightarrow \begin{array}{l} \text{第二行} - \text{第一行} \\ \text{第三行} - 2 * \text{第一行} \\ \text{第四行} + \text{第一行} \end{array} \quad \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 6 & -4 & 4 \end{bmatrix} \rightarrow \begin{array}{l} \text{第二三行互换 } \det *= -1 \\ \text{第四行} + \text{第二行} * 2 + 3 * \text{第三行} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \rightarrow \det = -1 * 3 * 2 * 7 = -42$$

Determinant of area of a parallelogram: 平行四边形 (parallelogram) 边长 vector 组成 matrix 的 $\text{abs}(\text{determinant}) =$ 它们的面积



$$\vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}, \quad \text{Proj}_L(\vec{v}_2) \text{ 为 } \vec{v}_2 \text{ 映射到 } \vec{v}_1 \text{ 长}$$

$$H^2 + ||\text{Proj}_L(\vec{v}_2)||^2 = ||\vec{v}_2||^2$$

$$H^2 = ||\vec{v}_2||^2 - ||\text{Proj}_L(\vec{v}_2)||^2$$

$$H^2 = \vec{v}_2 \cdot \vec{v}_2 - \left| \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 \right|^2$$

$$H^2 = \vec{v}_2 \cdot \vec{v}_2 - \frac{(\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{v}_2)}{\vec{v}_1 \cdot \vec{v}_1}$$

$$\text{area}^2 = B^2 H^2 = \vec{v}_1 \cdot \vec{v}_1 \left(\vec{v}_2 \cdot \vec{v}_2 - \frac{(\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{v}_2)}{\vec{v}_1 \cdot \vec{v}_1} \right) = (\vec{v}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{v}_2) - (\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{v}_2)$$

$$\text{area}^2 = (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2$$

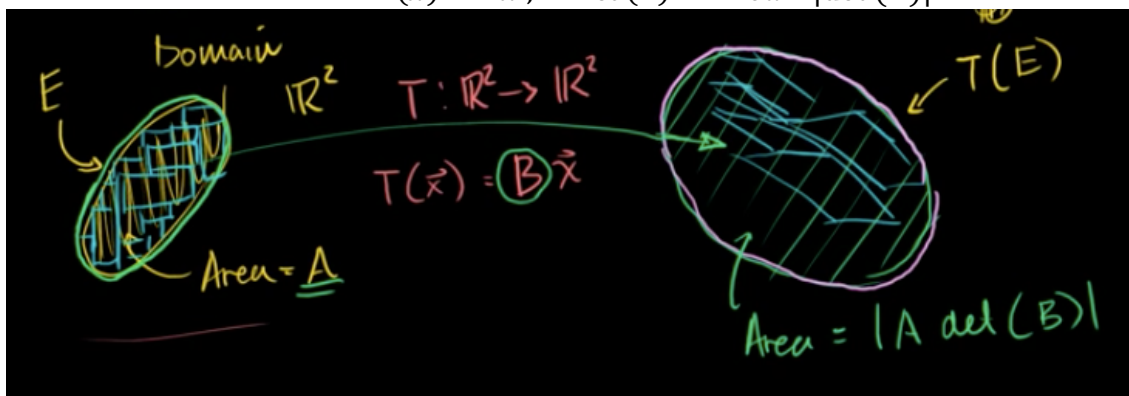
$$\text{area}^2 = a^2 b^2 + a^2 d^2 + c^2 b^2 + c^2 d^2 - a^2 b^2 - 2abcd - c^2 d^2$$

$$\text{area}^2 = a^2 d^2 + c^2 b^2 - 2abcd = (ad - bc)^2 = \det(A)^2$$

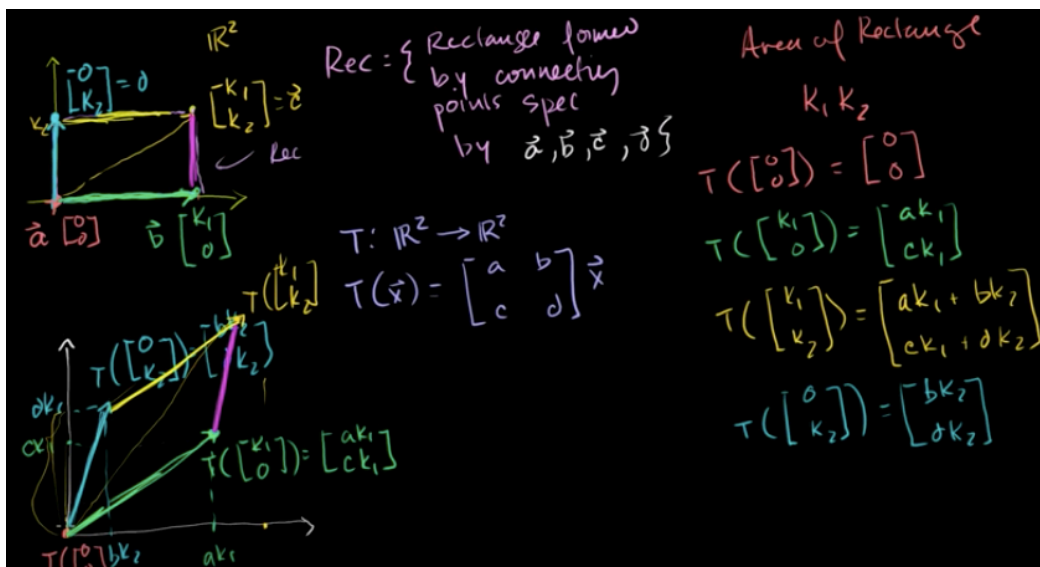
Determinant as scaling factor:

如果我们 transform 从一个 area A 到另外一个 area B

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2: T(\vec{x}) = B\vec{x}; \quad \text{Area}(B) = \text{Area} * |\det(B)|$$



Prove: 长方形 transform 到平行



我们有边长 $\vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$

Transformation 后的点为 $\vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} ak_1 \\ ck_1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} bk_2 \\ dk_2 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} ak_1 + bk_2 \\ ck_1 + dk_2 \end{bmatrix}$

根据上面我们知道新的平行四边形面积是 $\det \begin{pmatrix} ak_1 & bk_2 \\ ck_1 & dk_2 \end{pmatrix} = |k_1 k_2 ad - k_1 k_2 bc| = k_1 k_2 |ad - bc| = \text{area}(A) * |\det(T)|$

Transpose

Properties:

1. $(C^T)^T = C$
2. $\det(A^T) = \det(A)$ for $A: n \times n$ matrix
3. $(AB)^T = B^T A^T$, $(XYZ)^T = Z^T Y^T X^T$
4. $(A+B)^T = A^T + B^T$
5. $(A^{-1})^T = (A^T)^{-1}$; $\leq (AA^{-1})^T = (A^{-1})^T A^T = I_n^T = (A^{-1}A)^T = A^T (A^{-1})^T = >$
6. for vector $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$, for $A: m \times n$, $\vec{x} \in n \times 1$, $\vec{y} \in n \times 1 \in R^n$; $(A\vec{x}) \cdot \vec{y} = (A\vec{x})^T \vec{y} = \vec{x}^T A^T \vec{y} = \vec{x}^T (A^T \vec{y}) = \vec{x} \cdot (A^T \vec{y})$ 因为 matrix product is associative
7. $\text{Rank}(A) = \text{Rank}(A^T)$, 根据 definition, $\text{Rank}(A^T) = \dim(C(A^T)) = \#$ of basis of for row space of A : $C(A^T) = \#$ of pivot entry in reduced row echelon form = $\text{Rank}(A)$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \quad a = [a_1, a_2, \dots, a_n], \quad \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$

Orthogonal Complements:

Orthogonal complements of V : for some V , $V^\perp = \{\vec{x} \in R^n | \vec{x} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V\}$

Prove orthogonal complements: 1. $\vec{a} \cdot \vec{v} = 0$, so \vec{a} can be equal to $\vec{0}$, $\vec{0}$ is in subspace 2.. $\vec{a} \cdot \vec{v} = 0$ for any $\vec{v} \in V$, $\vec{b} \cdot \vec{v} = 0$ for any $\vec{b} \in V$, $(\vec{a} + \vec{b}) \cdot \vec{v} = \vec{a} \cdot \vec{v} + \vec{b} \cdot \vec{v} = 0$; 3. $c\vec{a} \cdot \vec{v} = c(\vec{a} \cdot \vec{v}) = 0$

$N(A)$ is orthogonal complements of the row space of A (is the same as column space of A transpose)

Null space is orthogonal complement of row space

$$N(A) = (C(A^T))^\perp$$

Left Null space is orthogonal of the complement of column space

$$N(A^T) = (C((A^T)^T))^\perp = (C(A))^\perp, \quad (N(A^T))^\perp = C(A)$$

Prove:

$$A = \begin{bmatrix} - & - & \vec{a}_1^T & - & - \\ - & - & \vec{a}_2^T & - & - \\ & & \vdots & & \\ - & - & \vec{a}_n^T & - & - \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T \cdot \vec{x} \\ \vec{a}_2^T \cdot \vec{x} \\ \vdots \\ \vec{a}_n^T \cdot \vec{x} \end{bmatrix} \rightarrow \vec{x} \text{ 来自 } V, \vec{x} \in N(A), \vec{x} \text{ is orthogonal to } \vec{a}_1^T, \vec{a}_2^T \dots \vec{a}_n^T$$

$N(A)$ is orthogonal to A

$N(A)$ is orthogonal to A , and also orthogonal to any linear combination of A , orthogonal to row space of A ,

$$\vec{w} = c_1 \vec{a}_1^T + c_2 \vec{a}_2^T + \cdots + c_n \vec{a}_n^T; \quad \vec{v} \cdot \vec{w} = \vec{w} = c_1 \vec{v}_1 \cdot \vec{a}_1^T + c_2 \vec{v}_2 \cdot \vec{a}_2^T + \cdots + c_n \vec{v}_n \cdot \vec{a}_n^T$$

$\dim(V) + \dim(\text{orthogonal complement of } v) = n$ (# columns)

Prove:

$$\text{Rank}(A^T) + \text{Nullity}(A^T) = n \rightarrow \text{因为 } \text{rank}(A) = \text{rank}(A^T) \quad \text{Rank}(A) + \text{Nullity}(A^T) = n$$

$$\text{根据 rank 的定义} \rightarrow \dim(C(A)) + \dim(N(A^T)) = n$$

$$\text{因为 } C(A) = (N(A^T))^\perp \quad \dim(C(A)) + \dim(C(A)^\perp) = n$$

$$\dim(V) + \dim(V^\perp) = n$$

若 $V \in R^n$, $V^\perp \in R^n$, $\dim(V) + \dim(V^\perp) = n$, V 的 rank 是 k , 则 R^n 中所有的点可以表示成 $\vec{a} = \vec{v} + \vec{x}$, $\vec{v} \in V$, $\vec{x} \in V^\perp$, 且 \vec{v} 和 \vec{x} 是 unique

Prove: 是 unique 的, $\vec{a} = \vec{v}_1 + \vec{x}_1 = \vec{v}_2 + \vec{x}_2$, 假设 \vec{v}_1 和 \vec{v}_2 不等, \vec{x}_1 和 \vec{x}_2 不等, $\vec{z} = \vec{v}_1 - \vec{v}_2 =$

$\vec{x}_1 - \vec{x}_2$, 因为 \vec{v}_1, \vec{v}_2 来自 V , $\vec{v}_1 - \vec{v}_2$ 在 V 中, $\vec{x}_1 - \vec{x}_2$ 在 V^\perp 中, 因为只有 $\vec{0}$ 既在 V 中, 也在 V^\perp 中, 所以 $\vec{v}_1 - \vec{v}_2 = \vec{0}, \vec{v}_1 = \vec{v}_2$; $\vec{x}_1 - \vec{x}_2 = \vec{0}, \vec{x}_1 = \vec{x}_2$

Orthogonal complement of the orthogonal complement of V is V

$$V = ((V)^\perp)^\perp$$

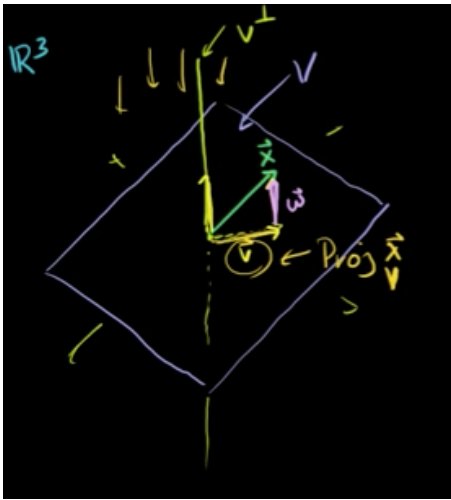
$A^T A$ is invertible given A is $k \times n$ matrix and each columns in A is linearly independent

Prove: $\vec{v} \in N(A^T A)$, 根据定义 $(A^T A) \vec{v} = \vec{0} \rightarrow \vec{v}^T (A^T A) \vec{v} = \vec{v}^T \vec{0} = (\vec{v}^T A^T)(A \vec{v}) = (A \vec{v})^T A \vec{v} = 0$, 则 $\|A \vec{v}\|^2 = 0, A \vec{v} = \vec{0}$, so if $\vec{v} \in N(A^T A)$, then $\vec{v} \in N(A)$, 因为 A 是 linearly independent column 的, $N(A)$ 只包括了 $\vec{v} = \vec{0}$, then only solution for $N(A^T A)$ is $\vec{0}$, $A^T A$ is invertible

Projection on a plane:

$\vec{x} = \vec{v} + \vec{w}$, \vec{w} is orthogonal to everything in V , \vec{v}, \vec{w} 相当于直角三角形的两个边

$$\vec{x} = \text{Proj}_V \vec{x} + \vec{w} = \text{Proj}_V \vec{x} + \text{Proj}_{V^\perp} \vec{x}$$



$\text{Proj}_V \vec{x}$ = the unique vector \vec{v} such that $\vec{x} = \vec{v} + \vec{w}$ where \vec{w} is a unique member of V^\perp

$\text{Proj}_V \vec{x}$ = some unique vector in V such that $\vec{x} - \text{Proj}_V \vec{x}$ is orthogonal to every member of V

如果 A is matrix consists of basis of V :

$$\text{Proj}_V \vec{x} = A (A^T A)^{-1} A^T \vec{x}$$

Prove: $\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k\}$ is basis for V , 如果 $\vec{a} \in V \Rightarrow \vec{a} = y_1 \vec{b}_1 + y_2 \vec{b}_2 + \dots + y_k \vec{b}_k$, 如果由 basis 组成 matrix, 是 $n \times k$ 维的, $A \vec{y} = [\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k] * \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = y_1 \vec{b}_1 + y_2 \vec{b}_2 + \dots + y_k \vec{b}_k$, for some $\vec{y} \in R^k$

$\text{Proj}_V \vec{x} = A \vec{y}$ for 某个 \vec{y} , $\vec{x} = \text{Proj}_V \vec{x} + \vec{w}$ where \vec{w} is the member of $V^\perp = C(A)^\perp = N(A^T)$, $\vec{x} - \text{Proj}_V \vec{x} = \vec{w} \in N(A^T)$

根据 null space 的定义, $A^T (\vec{x} - \text{Proj}_V \vec{x}) = A^T \vec{x} - A^T \text{Proj}_V \vec{x} = A^T \vec{x} - A^T A \vec{y}$

$$A^T \vec{x} = A^T A \vec{y}$$

$\vec{y} = (A^T A)^{-1} A^T \vec{x}$, 根据上面定义我们知道 $A^T A$ is invertible

$$\text{Proj}_V \vec{x} = A (A^T A)^{-1} A^T \vec{x}$$

e.g. $V = \{\text{all the } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ that satisfy } x_1 + x_2 + x_3 = 0\}$ find projection matrix of V , $x_1 = -x_2 - x_3$

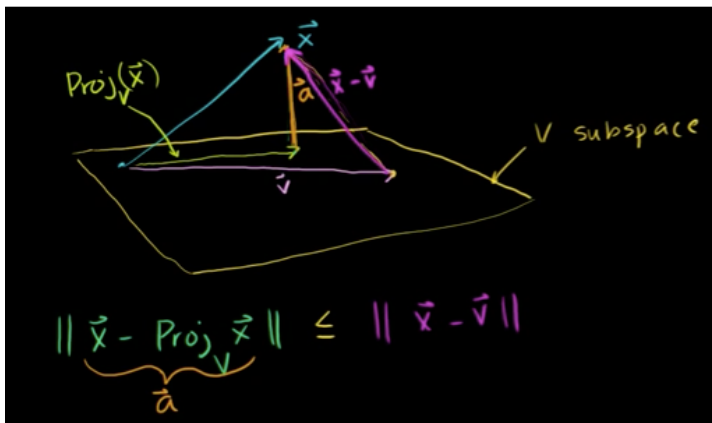
$$V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, V = \text{span} \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$A (A^T A)^{-1} A^T = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

另一种思路： $\vec{x} = \text{Proj}_V \vec{x} + \vec{w} = B\vec{x} + C\vec{x}$, $I_x \vec{x} = (B + C)\vec{x}$

因为 $x_1 + x_2 + x_3 = 0$, then $V = N([1,1,1])$, $V^\perp = N([1,1,1])^\perp = C \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

$$C = C (C^T C)^{-1} C^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = I - C$$



Vector 到 plane 做 projection, projection 的高是最短的 distance from vector 到 plane

Least Square

Least square for $A\vec{x} = \vec{b}$, $A \text{ } n \times k$, $\vec{x} \in R^k$, $\vec{b} \in R^n$

$$\vec{x} = (A^T A)^{-1} A^T \vec{b}$$

What if no solution for x , we want to find closest solution

Minimize $||\vec{b} - A\vec{x}||$, $A\vec{x}$ is column space of A , the close between \vec{b} and plane is the **projection**. $\vec{b} - A\vec{x}$ is orthogonal to $A\vec{x}$ (projection), $A\vec{x} - \vec{b} = \text{proj}_{C(A)} \vec{b} - \vec{b} \in N(A^T) =$

$$A^T(A\vec{x} - \vec{b}) = \vec{0} \\ A^T A \vec{x} - A^T \vec{b} = \vec{0} \quad A^T A \vec{x} = A^T \vec{b}, \quad \vec{x} = (A^T A)^{-1} A^T \vec{b}$$

Basis transformation

V is subspace of R^n , $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ if $\vec{a} \in V \rightarrow \vec{a} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$, $k \leq n$

Call c_1, c_2, \dots, c_k the coordinates of \vec{a} with respect to B , $[\vec{a}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$, even if \vec{a} in R^n , only give k coordinates

$\vec{a} = C[\vec{a}]_B$, C is the basis matrix is called change of basis matrix

直角坐标系中的 coordinate 是对应 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 的坐标

Change of basis matrix is the matrix with the basis vectors as columns

If change of basis matrix is invertible: C is square, $k = n \Rightarrow n$ basis vectors (columns are linear independent) $\Rightarrow \text{span}(C) = R^n$; 如果反过来 if $\text{span}(C) = R^n$, then C is invertible

$$C[\vec{a}]_B = \vec{a} \\ C^{-1}C[\vec{a}]_B = C^{-1}\vec{a}; \quad [\vec{a}]_B = C^{-1}\vec{a}$$

e.g.

if $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $\vec{a} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$, what is $[\vec{a}]_B$?

$$[\vec{a}]_B = C^{-1}\vec{a} = -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 19/5 \end{bmatrix}$$

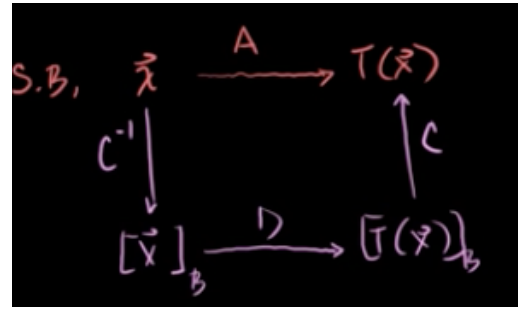
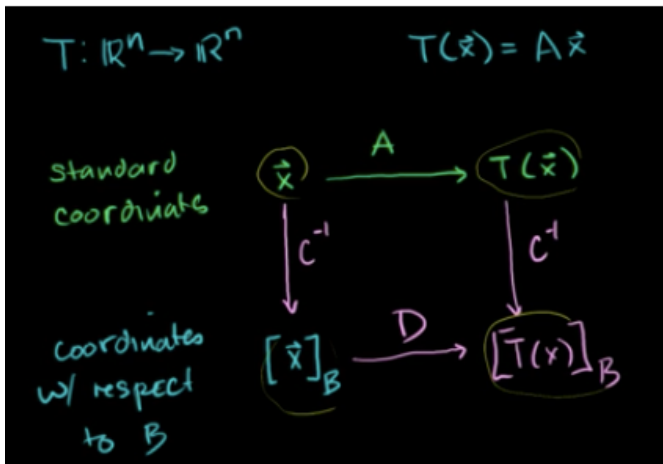
If D is the transformation matrix for T with respect to the basis B , $[T(\vec{x})]_B = D[\vec{x}]_B$

and C is the change of basis for B , $C[\vec{x}]_B = \vec{x}$

and A is the transformation matrix for T with respect to the standard basis

$$D = C^{-1}AC \\ A = CDC^{-1}$$

Prove: $C[\vec{x}]_B = \vec{x}$, $[\vec{x}]_B = C^{-1}\vec{x}$, $D[\vec{x}]_B = [T(\vec{x})]_B = [A\vec{x}]_B = C^{-1}A\vec{x} = C^{-1}AC[\vec{x}]_B$



Orthonormal Basis

$B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$, 1. $\|\vec{v}_i\| = 1$, each vector has length = 1 2. Each vector is orthogonal to each other
 $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$. Then **B is orthonormal set for V**

Orthonormal basis: 由 orthonormal set 构成的 basis

性质:

1. B is linearly independent

Prove: if \vec{v}_i, \vec{v}_j are linear dependent, then $\vec{v}_i = c \vec{v}_j$ for $c \neq 0$, $\vec{v}_i \cdot \vec{v}_j = 0 = c \vec{v}_j \cdot \vec{v}_j \rightarrow \|\vec{v}_j\| = 0$, contradict assumption

2. If $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$, then $\vec{v}_i \cdot \vec{x} = c_1 \vec{v}_i \cdot \vec{v}_1 + c_2 \vec{v}_i \cdot \vec{v}_2 + \dots + c_i \vec{v}_i \cdot \vec{v}_i + \dots + c_k \vec{v}_i \cdot \vec{v}_k = c_i \vec{v}_i \cdot \vec{v}_i = c_i$

$$3. [\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vdots \\ \vec{v}_k \cdot \vec{x} \end{bmatrix}$$

4. If orthonormal basis 组成 matrix A, 则 $A^T A = I_k$ the identity matrix

$$\text{Prove: } A^T A = \begin{bmatrix} - & - & \vec{v}_1^T & - & - \\ - & - & \vec{v}_2^T & - & - \\ - & - & \vdots & - & - \\ - & - & \vec{v}_k^T & - & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_k \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

5. C is $n \times n$ matrix whose columns form an orthonormal set, $C^{-1} = C^T$, 因为 C is $n \times n$ 所以 C 是 invertible $C^{-1}C = I$, 而 from 4, 已知 $C^T C = I$

Good for coordinate system, 比如 standard basis for $\mathbb{R}^3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

e.g. under orthonormal basis $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}, \vec{x} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$, calculate the coordinates,

$$[\vec{x}]_B = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} \frac{27}{5} + -\frac{8}{5} \\ -\frac{36}{5} + -\frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{19}{5} \\ -\frac{42}{5} \end{bmatrix}$$

If $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ for V , $V \in \mathbb{R}^n$, 因为 $\vec{x} = \vec{v} + \vec{w} = Proj_V \vec{x} + \vec{w}$, \vec{w} 是 V 的 orthogonal complement

$$Proj_V \vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k = (\vec{v}_1 \cdot \vec{x}) \vec{v}_1 + (\vec{v}_2 \cdot \vec{x}) \vec{v}_2 + \dots + (\vec{v}_k \cdot \vec{x}) \vec{v}_k$$

$$Proj_V \vec{x} = A(A^T A)^{-1} A^T \vec{x} = AA^T \vec{x}$$

If $C = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is basis for \mathbb{R}^n , C ($n \times n$ matrix is invertible) which means $C^{-1}C = I$, and above we know $C^T C = I$

If transformation matrix is orthogonal matrix (由 orthonormal set 组成), it will preserve length and angle for transformation

Prove:

$$||C\vec{x}||^2 = C\vec{x} \cdot C\vec{x} = (C\vec{x})^T C\vec{x} = \vec{x}^T C^T C \vec{x} = \vec{x}^T \vec{x} = ||\vec{x}||^2$$

$$\cos\theta = \frac{C\vec{v} \cdot C\vec{w}}{||C\vec{v}|| ||C\vec{w}||} = \frac{(C\vec{v})^T C\vec{w}}{||\vec{v}|| ||\vec{w}||} = \frac{\vec{v}^T C^T C \vec{w}}{||\vec{v}|| ||\vec{w}||} = \frac{\vec{v}^T \vec{w}}{||\vec{v}|| ||\vec{w}||}$$

Gram-Schmidt process

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ a basis for V , To find orthonormal basis

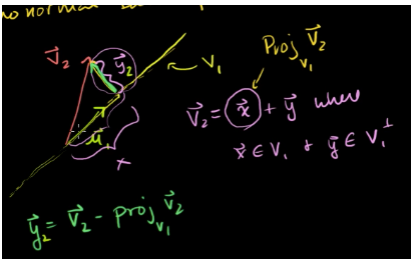
$$\text{One dimensional subspace } V_1 = \text{span}(\vec{v}_1), \quad u_1 = \frac{\vec{v}_1}{||\vec{v}_1||}$$

$$V_2 = \text{span}(\vec{u}_1, \vec{v}_2) = \text{span}(\vec{u}_1, \vec{y}_2) = \text{span}(\vec{u}_1, \vec{u}_2), \quad \vec{y}_2 = \vec{v}_2 - Proj_{V_1} \vec{v}_2$$

$$\vec{y}_2 = \vec{v}_2 - Proj_{V_1} \vec{v}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1, \quad u_2 = \frac{\vec{y}_2}{||\vec{y}_2||}$$

$$V_3 = \text{span}(\vec{u}_1, \vec{u}_2, \vec{v}_3) = \text{span}(\vec{u}_1, \vec{u}_2, \vec{y}_3) = \text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3), \quad \vec{y}_3 = \vec{v}_3 - Proj_{V_2} \vec{v}_3$$

$$\vec{y}_3 = \vec{v}_3 - Proj_{V_2} \vec{v}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2, \quad u_3 = \frac{\vec{y}_3}{||\vec{y}_3||}$$



e.g. find orthonormal basis for $x_1 + x_2 + x_3 = 0$

$$V = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, u_2 = \frac{\vec{y}_2}{\|\vec{y}_2\|} = \sqrt{\frac{3}{2}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$\{\vec{u}_1, \vec{u}_2\}$ is orthonormal basis

Eigenvector and Eigenvalue

$T(\vec{v}) = \lambda \vec{v}$, for non-zero vector \vec{v} is **eigenvector**, λ is **eigenvalue**.

$$A\vec{v} = \lambda \vec{v}$$

$$A\vec{v} - \lambda I_n \vec{v} = \vec{0}$$

$$(A - \lambda I_n) \vec{v} = \vec{0}$$

因为 \vec{v} is non-zero vector, $(A - \lambda I_n)$ is linear dependent matrix 否则 \vec{v} 必须等于 zero vector, so $\det(A - \lambda I_n) = 0$

Eigenspace E_λ : the space of vector correspond to eigenvalue $E_\lambda = \text{null space}(A - \lambda I_n)$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $\det(\lambda I_n - A) = \det\left(\begin{bmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{bmatrix}\right) \Rightarrow \lambda^2 - 4\lambda + 5 = 0, \lambda = -1, 5$

$$E_5 = N\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\right) = N\left(\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right), E_5 = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

$$E_{-1} = N\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\right) = N\left(\begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right),$$

When having n 个 linear-independent eigenvectors for \mathbb{R}^n for transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n, T(\vec{x}) = \lambda \vec{x}$, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^n , called **eigenbasis** (linear independent 肯定能是 basis, show this basis is good basis)

$$T(\vec{v}_1) = A\vec{v}_1 = \lambda_1 \vec{v}_1 = \lambda_1 \vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_n$$

$$T(\vec{v}_2) = A\vec{v}_2 = \lambda_2 \vec{v}_2 = 0\vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + 0\vec{v}_n$$

$$T(\vec{v}_n) = A\vec{v}_n = \lambda_n \vec{v}_n = 0\vec{v}_1 + 0\vec{v}_2 + \dots + \lambda_n \vec{v}_n$$