Matrix A, B: A*B 不一定等于 B*A, matrix 乘法 direction matters

Identity Matrix: I*A = A: 1 in diagonal position.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} * \begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

Why it works? Row information from indentity matrix 的 row and column information from A 的 column, 比如[1,0] 乘以 $\begin{bmatrix}a\\c\end{bmatrix}$ or $\begin{bmatrix}b\\d\end{bmatrix}$, 0 cancel out every elements 除了 first term (a,b) in the column vector, 第二行[0,1] cancel out every elemts 除了 second term(c, d)

**Inverse 2*2:**

$$\mathrm{A}^{-1} * A = I, \mathrm{A} * \mathrm{A}^{-1} = I, \mathrm{A\ is\ also\ inverse\ of\ } \mathrm{A}^{-1}$$

Calculate the inverse:

$$\mathrm{A} = \begin{bmatrix}a & b\\c & d\end{bmatrix},\ A^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix} = \frac{1}{|A|}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}, \quad ab - bc\ \ is\ determinant\ of\ A$$

Determinant: $|\mathrm{A}| = ad - bc$

Indentiy matrix: I*A = A, A*I = A, 两个都满足的只有当 A 是 square matrix 的时候

**Inverse 3*3: Gauss Jordan elimination (augment the matrix, operation: elementary row operation)**

Perform some operation 在 left side and same operation on right side, 当 have indentity matrix 在 left-hand side(变成 indentiy matrix 的形式 叫做 reduced row echelon form ), right-hand side 就是原来的 invers

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0\\ 0 & 2 & 1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

Row3 = row3 – row1

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0\\ 0 & 2 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}$$

Swap row3 and row2

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}$$

Row3 = row3 – 2*row2

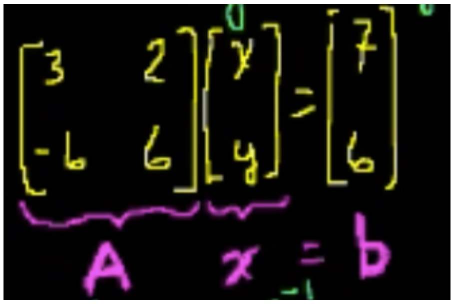
$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}$$

Row1 = row1 – row3

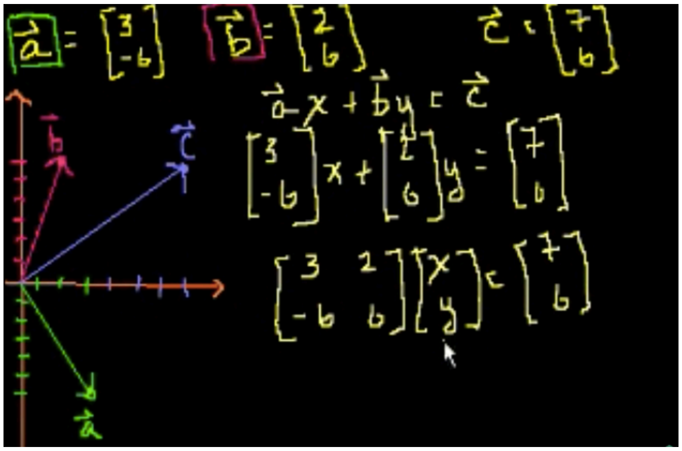
$$\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}$$

Hint why this work: 当对左面 matrix 进行操作可以想成乘以多个 matrices, so we multiply matrix 得到 indentity matrix, 乘以的多个 matrices 就是$\mathrm{A}^{-1}$, 而我们知道 identity matrix 乘以任何 matrix 就是 matrix itself

## Solve system of equations (2*2):




Ax = b → $A^{-1}Ax = A^{-1}b$ → $Ix = A^{-1}b$ → $x = A^{-1}b$



Matrices to solve vector combination: 可以想成 matrix multiplication problem 把两个 vector 合成一个 vector

比如 vector a = $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$, b = $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, 得到 c = $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$

A = $\begin{bmatrix} 3 & 2 \\ -6 & 6 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

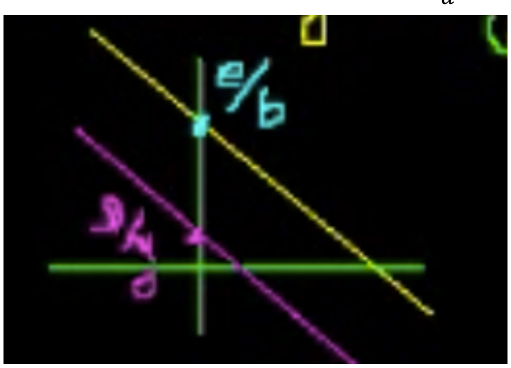
$A^{-1} = \frac{1}{30}\begin{bmatrix} 6 & -2 \\ 6 & 3 \end{bmatrix}$, $A^{-1} * \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, 需要 1 个 a,两个 b

Singular matrices: 没有 inverse 的 matrix

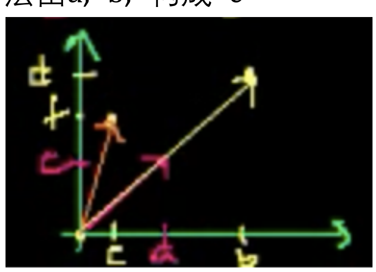
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, A^{-1} \text{ is undefined iff } |A| = ad - bc = 0 \ \left(\text{or } \frac{a}{b} = \frac{c}{d} \text{ or } \frac{a}{c} = \frac{b}{d}\right)$$

Prove: 比如 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

得到 $\begin{matrix} ax + by = e \\ cx + dy = f \end{matrix}$ => $\begin{matrix} y = -\frac{a}{b}x + \frac{e}{b} \\ y = -\frac{c}{d}x + \frac{f}{d} \end{matrix}$ 如果$\frac{a}{b} = \frac{c}{d}$, 两条线平行，不会相交没有解




如果从 vector 角度考虑，如下图 $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ 是重合的 if $\frac{a}{c} = \frac{b}{d}$, 无法由$\vec{a}$, $\vec{b}$, 构成 $\vec{c}$



## Solve system of equations (3*3):

$$\begin{aligned} -x+2y-z&=9 \\ 3x-7y-2&=-20 \\ 2x+2y+z&=2 \end{aligned} \quad => \left[\begin{array}{ccc|c} -1 & 2 & -1 & 9 \\ 3 & -7 & -2 & -20 \\ 2 & 2 & 1 & 2 \end{array}\right]$$

通过 row operation 变成

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 9 \\ 0 & -1 & -5 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right] => \begin{aligned} x&=-1 \\ y&=3 \\ z&=-2 \end{aligned}$$

## Vectors and Spans:

Set Colinear vectors: {c $S=\{c\,\vec{v} \mid c \in R\}$ 比如 vector 在一条线上(slope), $\vec{v}=\begin{bmatrix}2\\1\end{bmatrix}$

Linear Combination: $c_1v_1+c_2v_2+\cdots+c_nv_n$, $c_1 \to c_n \in R$; We can fill Any point in $R^2$ with combination of vector a and b: we can write span $(\vec{a},\vec{b})=R^2$, we can represent any vector in $R^2$ with some linear combination of a and b where a and b cannot be collinear (a,b 不能共线, 换种思维考虑: 如果共线了,组成的 matrix 没有 inverse A*c = b, A 没有 inverse). span $(\vec{0})$ = $\vec{0}$ (c$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$)

比如 unit vector $\hat{\imath}=\begin{bmatrix}1\\0\end{bmatrix}, \hat{\jmath}=\begin{bmatrix}0\\1\end{bmatrix}$, 可以构建任何 vector in $R^2$ by using these unit vectors

**span $(v_1, v_2, v_3)=\{c_1v_1+c_2v_2+\cdots+c_nv_n \mid c_i \in R\ 1 \le i \le n\}$**: The space of all of the combination of vectors $v_1, v_2 \ldots . v_n$

**Linearly Dependent set**: some vector in the set can be represented by some combinations of other vectors in the set, 比如$\begin{bmatrix}2\\3\end{bmatrix}$, $\begin{bmatrix}4\\6\end{bmatrix}$是 linearly dependent, 再比如 $\begin{bmatrix}2\\3\end{bmatrix}$ $\begin{bmatrix}7\\2\end{bmatrix}$ $\begin{bmatrix}9\\5\end{bmatrix}$是 linearly dependent,因为其中一个可以由另外两个构成构成

$R^n$ : $\left\{\begin{bmatrix}x_1\\x_2\\x_n\end{bmatrix} x_i \in R,\ 1 \le i \le n\right\}$

**V is Subspace of $R^n$** ( me vector from $R^n$) 必须满足：

1. V contains 0 vector
2. If $\vec{x}$ in V then any scaler c: $c\vec{x}$ also in V (closure under scaler multiplication)
3. If $\vec{a}$ in V and $\vec{b}$ in V, $\vec{a}+\vec{b}$ also in V (closure under addition)

同样如果满足这三个条件的也是 subspace

e.g. v = {0} = $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$: v 只有 vector 0, v is subspace of $R^3$

1. 满足条件 1, vector 0 在 v 中
2. 满足条件 2: c$\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$
3. 满足条件 3: $\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

e.g. S =$\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix} \in R^2 \mid x \ge 0\right\}$: v is not subspace of $R^2$

1. 满足条件 1, vector 0 在 v 中
2. 不满足条件 2: -1$\begin{bmatrix}a\\b\end{bmatrix}$, -1*a 为负数
3. 满足条件 3: $\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}a+c\\b+d\end{bmatrix}$ a+c 是正数 given a>=0 and c>=0

e.g V = span $(v_1, v_2, v_3)$. is valid subspace of $R^n$

1. 满足条件 1: $\vec{0}=0\vec{v_1}+0\vec{v_2}+0\vec{v_3}$
2. 满足条件 2: $\vec{x}=c_1\vec{v_1}+c_2\vec{v_2}+c_3\vec{v_3}$; then $a\vec{x}=ac_1\vec{v_1}+ac_2\vec{v_2}+ac_3\vec{v_3}$; $ac_1, ac_2, ac_3$ can be arbitrary constant, 因为 span 是 all linear combination 所以新的也在 span 当中

3. 满足条件 3： $\vec{x} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + c_3\overrightarrow{v_3}$; $\vec{y} = d_1\overrightarrow{v_1} + d_2\overrightarrow{v_2} + d_3\overrightarrow{v_3}$, then $\vec{x} + \vec{y} = (c_1 + d_1)\overrightarrow{v_1} + (c_2 + d_2)\overrightarrow{v_2} + (c_3 + d_3)\overrightarrow{v_3}$, it also in span

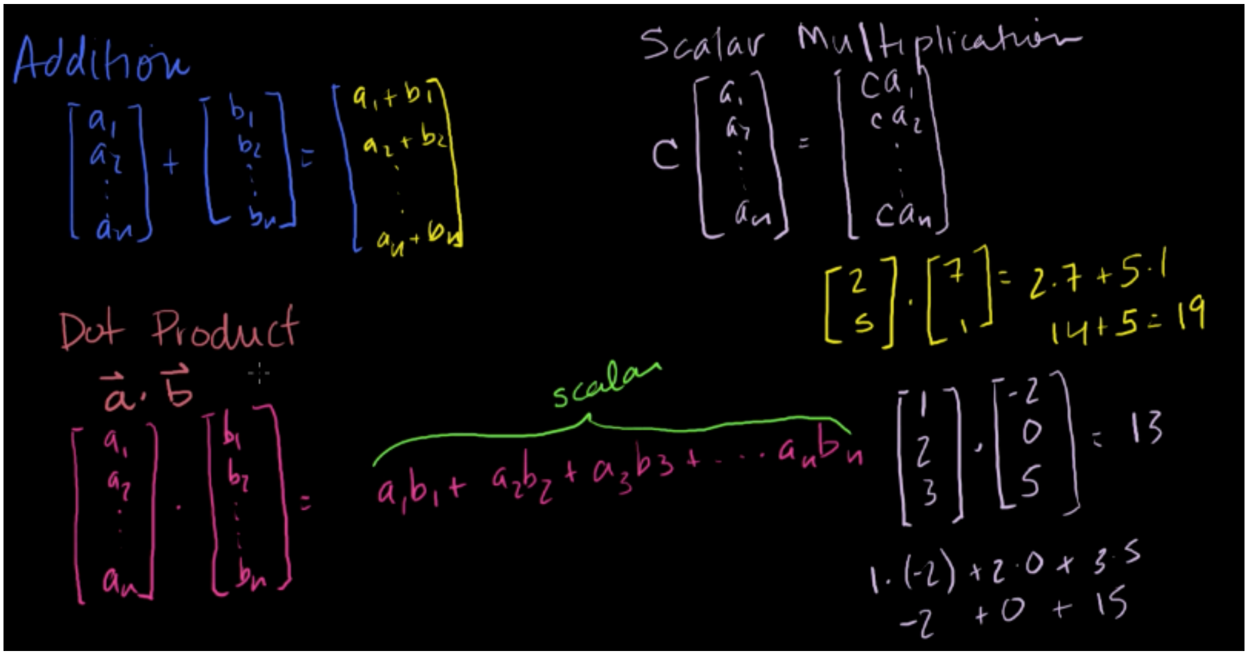
e.g V = span ( $\begin{bmatrix}1\\1\end{bmatrix}$). is valid subspace of $\mathrm{R}^2$

1. 满足条件 1: $\vec{0} = 0\begin{bmatrix}1\\1\end{bmatrix}$
2. 满足条件 2: c $\begin{bmatrix}1\\1\end{bmatrix}$ 就是 span itself (combination of vector)
3. 满足条件 3： $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix}; = (c_1 + c_2)\begin{bmatrix}1\\1\end{bmatrix}$ 在 span 当中

**Span($\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ ) of any vector is valid subspace,**
**Basis S= {$\overrightarrow{\boldsymbol{v_1}}, \overrightarrow{\boldsymbol{v_2}}, \dots, \overrightarrow{\boldsymbol{v_n}}$ }: 1. Span($\overrightarrow{\boldsymbol{v_1}}, \overrightarrow{\boldsymbol{v_2}}, \dots, \overrightarrow{\boldsymbol{v_n}}$ all those $\overrightarrow{\boldsymbol{v_1}}, \overrightarrow{\boldsymbol{v_2}}, \dots, \overrightarrow{\boldsymbol{v_n}}$ linearly independent 2. $\mathbf{c_1}\boldsymbol{v_1} + \boldsymbol{c_2}\boldsymbol{v_2} + \cdots + \mathbf{c_n}\mathbf{v_n}$ = 0 when $\mathbf{c_1} = \boldsymbol{c_2} = \cdots = \boldsymbol{c_n} = \mathbf{0}$**
Basis(minimum set of vectors that spans the subspace): 如果用 any vector in S 可以 construct any vector in subspace V

e.g. T = {$\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}, \overrightarrow{v_s}(= c + \vec{2})$}, the span of T is still going to be subspace V but T is linearly dependent -> T is not basis for V ( )
Basis: 比如，需要两个 non-redundant vector
Standard Basis for $\mathrm{R}^2$ $T = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$
Advantage of Basis: represent any vector in subspace by some unique combination of vectors in basis 比如 Basis $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ $\vec{a} \in U, \vec{a} = \mathrm{c}_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_n\overrightarrow{v_n}$, $\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_n$是 unique 的

# Vectors Dot Product




Length : $||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ $\quad \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + \cdots + a_n^2 = ||\vec{a}||^2$

Communicative: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

Distributive: $(\vec{v} + \vec{w}) \cdot \vec{x} = \vec{w}\vec{x} + \vec{v}\vec{x} = (v_1 + w_1)x_1 + (v_2 + w_2)x_2 + \cdots + (v_1 + w_1)x_1 = v_1x_1 + w_1x_1 + v_2x_2 + w_2x_2 \ldots + v_nx_n + w_nx_n$

Associative over scaler multiplication: $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$

Dot product self is the length square: $\vec{v} \cdot \vec{v} = ||\vec{v}||^2 = v_1^2 + v_2^2 + \cdots + v_n^2$

**Cauchy Schwarz Inequality**: If $\vec{x}, \vec{y} \in R^n, |\vec{x} \cdot \vec{y}| \leq ||\vec{x}||\,||\vec{y}||$, $\quad |\vec{x} \cdot \vec{y}| = ||\vec{x}||\,||\vec{y}||$ only if two vector colinear, one vector 是另一个 vector 乘以的倍数 ($|\vec{x} \cdot \vec{y}|$ *absolute value of dot product*)

Prove: suppose $\vec{x}$, $\vec{y}$ is non-zero vector

$$\text{suppose } p(t) = ||t\,\vec{y} - \vec{x}\,||^2 \geq 0$$
$$p(t) = (t\,\vec{y} - \vec{x})(t\,\vec{y} - \vec{x})$$
$$p(t) = t\,\vec{y} \cdot t\,\vec{y} - \vec{x} \cdot t\,\vec{y} - t\,\vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{x}$$
$$p(t) = t^2(\vec{y}\;\vec{y}) - 2\,(\vec{x} \cdot \vec{y})t + \vec{x} \cdot \vec{x} \geq 0$$

此时设$\vec{y}$ $\vec{y} = a,\ 2(\vec{x} \cdot \vec{y}) = b, \vec{x} \cdot \vec{x} = c$

$$p(t) = at^2 - bt + c \geq 0$$
$$p\left(\frac{b}{2a}\right) = a\frac{b^2}{4a^2} - b\,\frac{b}{2a} + c \geq 0$$
$$p\left(\frac{b}{2a}\right) = \frac{b^2}{4a} - \frac{2b^2}{4a} + c$$
$$p\left(\frac{b}{2a}\right) = -\frac{b^2}{4a} + c \geq 0 \quad => c \geq \frac{b^2}{4a} \quad => 4ac \geq b^2$$
$$4ac \geq b^2 = 4(\,||\vec{y}\,||^2\ ||\vec{x}||^2) \geq (2\ (\vec{x} \cdot \vec{y}))^2$$
$$(\,||\vec{y}\,||^2\ ||\vec{x}||^2) \geq (\vec{x} \cdot \vec{y})^2$$

Take square root

$$||\vec{y}\,||\ ||\vec{x}|| \geq |\vec{x} \cdot \vec{y}|$$

When $\vec{x} = c\vec{y}$

$$|\vec{x} \cdot \vec{y}| = |\, c\,\vec{x} \cdot \vec{y}| = c\,|\vec{y} \cdot \vec{y}| = c\big||\vec{y}|\big|^2 = ||c\vec{y}|| \cdot ||\vec{y}||$$

**Triangle Inequality**:

$$\big||\vec{x} + \vec{y}|\big|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} +\cdot \vec{y})$$
$$= \vec{x}(\vec{x} + \vec{y}) + \vec{y}(\vec{x} + \vec{y})$$
$$= \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{x} + \vec{y} \cdot \vec{y}$$
$$= \Big||\vec{x}||^2 + 2(\vec{x} \cdot \vec{y}) + ||\vec{y}|\Big|^2$$

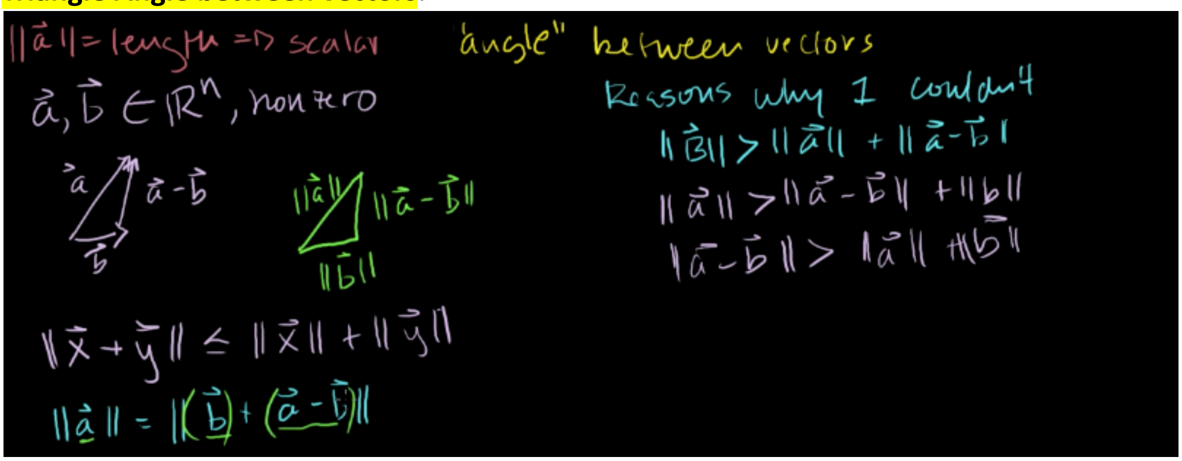
根据 cauchy Schwarz inequality: $\vec{x} \cdot \vec{y} \leq |\vec{x} \cdot \vec{y}| = \big||\vec{x}|\big|\big||\vec{y}|\big|$, dot product 可以是负数

$$\big||\vec{x} + \vec{y}|\big|^2 \leq \Big||\vec{x}||^2 + 2||\vec{x}||||\vec{y}|| + ||\vec{y}|\Big|^2 = \big(||\vec{x}|| + ||\vec{y}||\big)^2$$
$$\big||\vec{x} + \vec{y}|\big| \leq \big||\vec{x}|\big| + \big||\vec{y}|\big|$$

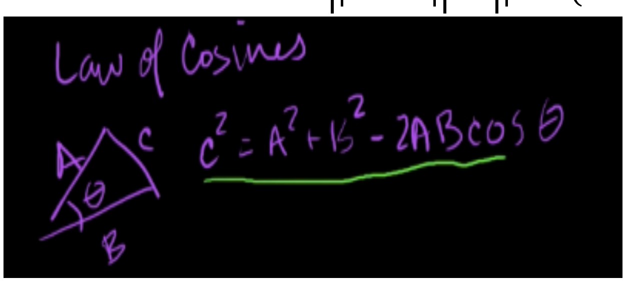
当 $\vec{x} = c\,\vec{y}$, c 是 positive 的时候$(c > 0)$, $\big||\vec{x} + \vec{y}|\big| = \big||\vec{x}|\big| + \big||\vec{y}|\big|$

$\vec{x}$ doesn't need to only 2-dimensional, 可以是 n dimension

**Triangle Angle between vectors**:




$$\big||\vec{a}|\big| = \Big||\vec{b} + (\vec{a} - \vec{b}\,)|\Big| \leq \Big||\vec{b}|\Big| + ||\vec{a} - \vec{b}||$$
$$\big||\vec{b}|\big| = \Big||\vec{a} + (\vec{b} - \vec{a}\,)|\Big| \leq \big||\vec{a}|\big| + ||\vec{b} - \vec{a}||$$
$$\Big||\vec{a} - \vec{b}|\Big| = \Big||\vec{a} + (-\vec{b})|\Big| \leq \big||\vec{a}|\big| + \Big||-\vec{b}|\Big| = \big||\vec{a}|\big| + \Big||\,\vec{b}|\Big|$$




Law of Cosine

$$\Big||\vec{a} - \vec{b}|\Big|^2 = \Big||\vec{b}||^2 + ||\vec{a}|\Big|^2 - 2\big||\vec{a}|\big|\Big||\vec{b}|\Big| \cos\theta$$

Left-hand side

$$(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$
$$= \big||\vec{a}|\big|^2 + \big||b|\big|^2 - 2(\vec{a} \cdot \vec{b})$$

Left-hand side = right-hand side

$$\big||\vec{a}|\big|^2 + \big||b|\big|^2 - 2(\vec{a} \cdot \vec{b}) = \Big||\vec{b}||^2 + ||\vec{a}|\Big|^2 - 2\big||\vec{a}|\big|\Big||\vec{b}|\Big| \cos\theta$$

$$\vec{a} \cdot \vec{b} = \big||\vec{a}|\big|\Big||\vec{b}|\Big| \cos\theta$$

If $\vec{a} = c\,\vec{b}$; $c > 0 \Rightarrow \theta = 0$; $c < 0 \Rightarrow \theta = 180^o$;

Perpendicular $\theta = 90^o$, $\vec{a} \cdot \vec{b} = 0$但是如果 dot product = 0 不意味着垂直，比如 $\vec{0} \cdot \vec{b} = 0$

但是当 a, b 都是 nonzero vector, dot product 意味着垂直(perpendicular)

$\vec{a} \cdot \vec{b} = 0$ => orthogonal, zero vector is orthogonal to everything; perpendicular is orthogonal，但是 othogonal 不一定是 perpendicular

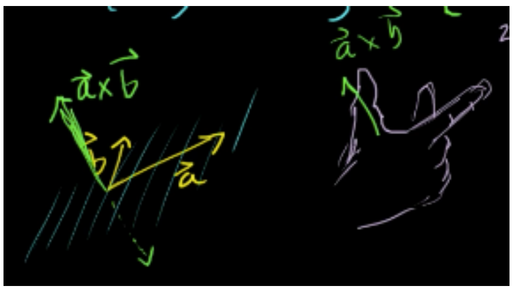
Dot Product: $\vec{a}, \vec{b} \in \mathrm{R^n}$ => 得到 scalar

Cross Product: only defined in $\mathrm{R}^3$， 得到 vector

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{a} \times \vec{b} = \begin{bmatrix} -7*4-1*2 \\ 1*5-1*4 \\ 1*2-(-7*5) \end{bmatrix} = \begin{bmatrix} -30 \\ 1 \\ 37 \end{bmatrix}$$

Cross product 乘积是 orthogonal to $\vec{a}$ and $\vec{b}$




判断 cross product 的方向可以用 right hand rule，食指指向 a 的方向，中指指向 b，大拇哥的方向是 a 和 b 的 cross product

Prove Orthogonal for $\vec{a}$ and $\vec{a} \times \vec{b}$: (Same for $\vec{b}$)

$$\begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= a_2 b_3 \mathrm{a}_1 - a_3 b_2 a_1 + a_3 b_1 a_2 - a_1 b_3 a_2 + a_1 b_2 a_3 - a_2 b_1 a_3$$

$$= a_2 b_3 \mathrm{a}_1 - a_1 b_3 a_2 + a_1 b_2 a_3 - a_3 b_2 a_1 + a_3 b_1 a_2 - a_2 b_1 a_3 = 0$$

$$\begin{cases} \vec{a} \cdot \vec{b} = ||\vec{a}|| \left|\left|\vec{b}\right|\right| cos\theta \\ ||\vec{a} \times \vec{b}|| = ||\vec{a}|| \left|\left|\vec{b}\right|\right| sin\theta \end{cases}$$

Prove:

$$\left|\left|\vec{a} \times \vec{b}\right|\right|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= \mathrm{a}_2^2 b_3^2 - 2 a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + \mathrm{a}_3^2 b_1^2 - 2 a_1 a_3 b_1 b_3 + a_1^2 b_3^2 + \mathrm{a}_1^2 b_2^2 - 2 a_1 a_2 b_1 b_2 + a_2^2 b_1^2$$

$$= \mathrm{a}_1^2 (b_2^2 + b_3^3) + \mathrm{a}_2^2 (b_1^2 + b_3^2) + \mathrm{a}_3^2 (b_1^2 + b_2^3) - 2(a_2 a_3 b_2 b_3 + a_1 a_3 b_1 b_3 + 2 a_1 a_2 b_1 b_2)$$

$$||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \cos^2\theta = \left(\vec{a} \cdot \vec{b}\right)^2 = (a_1 b_1 + a_2 b_2 + \mathrm{a}_3 b_3)^2$$

$$= \mathrm{a}_1^2 b_1^2 + a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + \mathrm{a}_2^2 b_2^2 + a_1 a_2 b_1 b_2 + a_2 a_3 b_2 b_3 + \mathrm{a}_3^2 b_3^2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3$$

$$= \mathrm{a}_1^2 b_1^2 + \mathrm{a}_2^2 b_2^2 + \mathrm{a}_3^2 b_3^2 + 2(a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3)$$

$$\left|\left|\vec{a} \times \vec{b}\right|\right|^2 + ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \cos^2\theta = \mathrm{a}_1^2 (b_1^2 + b_2^2 + b_3^3) + \mathrm{a}_2^2 (b_1^2 + b_2^2 + b_3^3) + \mathrm{a}_3^2 (b_1^2 + b_2^2 + b_3^3)$$

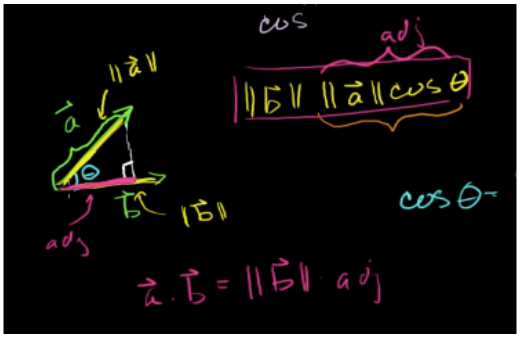
$$= (\mathrm{a}_1^2 + \mathrm{a}_2^2 + \mathrm{a}_3^2)(b_1^2 + b_2^2 + b_3^3) = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2$$

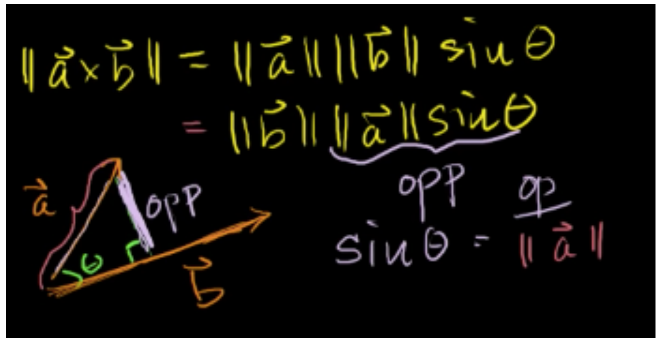
$$\left|\left|\vec{a} \times \vec{b}\right|\right|^2 = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 - ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \cos^2\theta = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 (1 + \cos^2\theta) = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \sin^2\theta$$

$$\vec{a}\cdot\vec{b} = ||\vec{a}||\,||\vec{b}||\,cos\theta = ||\vec{b}||\,adj$$

$$||\vec{a}\times\vec{b}|| = ||\vec{a}||\,||\vec{b}||\,sin\theta = ||\vec{b}||\,opp$$

$$(Optional)\quad \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$



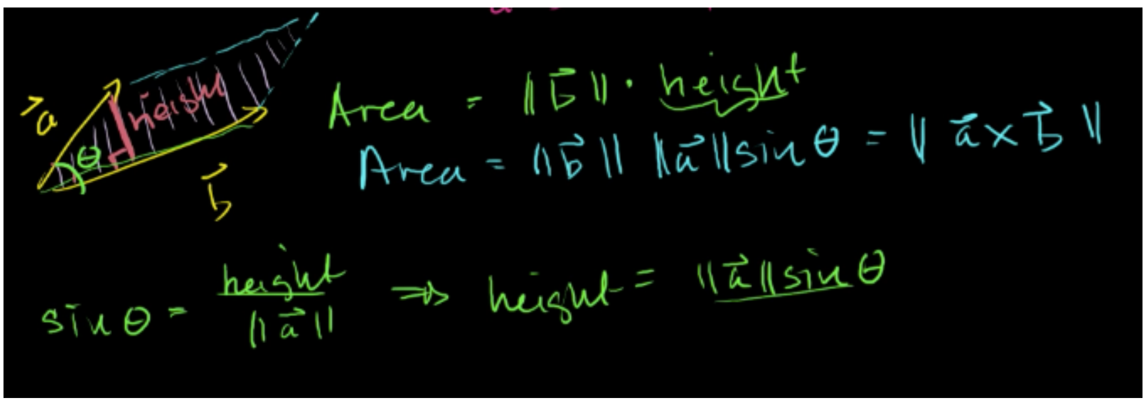




$||\vec{a}||cos\theta$ 构成直角三角形的 a 的 projection 是 adj

$||\vec{a}||sin\,\theta$ 构成直角三角形的高

Dot product tells: product of lengths of vectors move together at same direction with b. When $\vec{a}\cdot\vec{b}=0$, perpendicular, $\vec{a}$ onto $\vec{b}$ is zero

Cross product tells: product of lengths of vectors move perpendicular direction with b. When $\vec{a}\cdot\vec{b}=||\vec{a}||\,||\vec{b}||$, perpendicular, 获得最大值，当 a 和 b colinear, $\vec{a}\cdot\vec{b}=0$ no perpendicular vector




Cross product 还可以算平行四边形的面积

Rowe chelon Form:

Pivot entry: 那个 column 只能它不是 0，且那行前面没有数

Free-variable: row 中在 pivot 后面的 variable

# Matrix Vector Product

1. As row vector and x dot product

$$\begin{bmatrix} -3 & 0 & 3 & 2 \\ 1 & 7 & -1 & 9 \end{bmatrix} \overset{\vec{x}}{\begin{bmatrix} 2 \\ -3 \\ 4 \\ -1 \end{bmatrix}} = \begin{bmatrix} -3\cdot 2 + \cancel{0}\cdot -3 + 3\cdot 4 + 2\cdot \\ 1\cdot 2 + 7\cdot -3 + -1\cdot 4 + 9\cdot -1 \end{bmatrix}$$

$$= \begin{bmatrix} -6+0+12-2 \\ 2-21-4-9 \end{bmatrix} = \begin{bmatrix} 4 \\ -32 \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 2 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 7 \\ -1 \\ 9 \end{bmatrix}$$

$$\vec{a}_1^T = \begin{bmatrix} -3 & 0 & 3 & 2 \end{bmatrix}$$

$$\vec{a}_2^T = \begin{bmatrix} 1 & 7 & -1 & 9 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \end{bmatrix}$$

2. As column vector and x linear combination

$$A = \begin{array}{cccc} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 3x_1 + 1x_2 + 0x_3 + 3x_4 \\ 2x_1 + 4x_2 \cdots \\ -1x_1 + 2x_2 \cdots \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}$$

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4$$

linear combination of column vectors of A

# Null Space

$$N = \{\vec{x} \in R^n \mid A\vec{x} = \vec{0}\}$$

满足 1. $\vec{0}$ in this subspace; 2. If $\vec{v_1}, \vec{v_2} \in N, then\ A(\vec{v_1} + \vec{v_2}) = A\vec{v_1} + A\vec{v_2} = \vec{0} \in N$; 3. $\vec{v} \in N, A(c\vec{v}) = c(A\vec{v}) = \vec{0} \in N$; N is valid subspace

e.g 求 $A\vec{x} = 0$

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$
$$4x_1 + 3x_2 + 2x_3 + x_4 = 0$$

得到 augmented matrix in row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & -2 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$= N(A) = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

Original problem $A\vec{x} = 0$ can be transformed to $(\mathrm{ref}(A))\vec{x} = 0$, (ref(A): null space of reduced row echelon form of matrix A)

**Relationship to linear Independent**

Matrix A (M $\times$ N); Null space $N(A) = \{\vec{x} \in R^n \mid A\vec{x} = \vec{0}\}$, $\vec{0} \in R^m$; 把 A 用 column vector 来表示

$$A = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n}$$

如果$\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ 都是 linear independent，唯一的解是$x_1, x_2, \ldots, x_n$ 都是 0　$N(A) = N(\mathrm{rref}(A)) = \{\vec{0}\}$

$\mathrm{rref}(A)\,\vec{x} = \vec{0}$, which means no free variable

$N(A) = N(\mathrm{rref}(A)) = \{\vec{0}\}$ if only if column vectors of A linear independent (only do if A is N $\times$ N matrix)

# Column Space

$$A = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}], C(A) = span\,(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$$

满足 1. $\vec{0}$ in this subspace (用每个 vector 乘以 0); 2. If $\vec{b}, \vec{c} \in C(A), then\ (\vec{b} + \vec{c}) = (b_1 + c_1)\vec{v_1} + (b_2 + c_2)\vec{v_2} + \cdots + (b_n + c_n)\vec{v_n} = \vec{0} \in N$; 3. $\vec{v} \in N, A(c\vec{v}) = c(A\vec{v}) \in C(A)$; C(A) is valid subspace

e.g.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \text{column space} = C(A) = span\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)$$

$$\text{Null Space } N(A) = N(rref(A))$$

Row echelon form: $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Null Space:} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathrm{x}_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \mathrm{x}_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathrm{span}\left( \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ linear independent? 因为 null space contain $\{\vec{0}\}$.,所以是 linear depenent set

因为是 linear dependent （后两个是 redundant 的）

$C(A) = span\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right)$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ is a basis for C(A), 跟 row-reduced echelon form pivot 所在的 column 到原来的 matrix 中选 basis

求 column space 的 function:

我们知道 cross product 垂直于$\vec{a}$ and $\vec{b}$, normal vector $\vec{n} = \vec{a} \times \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8-3= & 5 \\ 3-4= & -1 \\ 1-2= & -1 \end{bmatrix}$

$$\vec{n} \cdot \left(\vec{x} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} \cdot \left(\vec{x} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

$$5\mathrm{x} - \mathrm{y} - \mathrm{z} = 0$$

另一种方法: what kind of B will give valid solution $\{\vec{b} \mid A\vec{x} = \vec{b}\ \&\ \vec{x} \in R^n\}$

$$\mathrm{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & x \\ 2 & 1 & 4 & 3 & y \\ 3 & 4 & 1 & 2 & z \end{array}\right]$$

化成 row echelon form:

$$\mathrm{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & x \\ 0 & 1 & -2 & -1 & y \\ 0 & 0 & 0 & 0 & 2x-y-z+3x \end{array}\right]$$

为了让 system 有解 2x – y – z + 3x = 5x – y -z = 0

# Dimension

Dim(V) : the number (cardinality) of a basis of V (比如 $\mathrm{A} = \{\mathrm{a}_1, \mathrm{a}_2, \ldots., \mathrm{a}_\mathrm{n}\}$ is a basis of V, Dim(V) = n )

All basis of the same subspace must have the same number of elements

Dimension of Null space: Dim(N(B)) is the **Nullity** = number of **free variables** (non-pivot) in reduced echelon form in Matrix A

e.g.

$$A\vec{x} = \vec{0}: \begin{bmatrix} 1 & 0 & -1 & -2 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$= N(A) = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

两个 free variable $\mathrm{x}_3, \mathrm{x}_4$, Dim(N(B)) = nullity = 2

Dimension of Column space: Dim(C(A)) is the **Rank** = number of **pivot variables** in reduced echelon form in Matrix A (rank of A number of linear independent column vector you have)

e.g.

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix}$$

To reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

第 1，2，4 列 linearly independent, column space 的 basis $\mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}\right)$

Dim(C(A)) = 3

# Linear Transformation

When function map to R (一维的)   called scaler value / Real valued function

When function map to $R^2, R^3$ (多维的) called vector value

Transformation: function operating on vectors (linear algebra)

Linear Transformation:

$$T: R^n \to R^m \text{ if only if}$$
$$\vec{a}, \vec{b} \in R^n, \quad 1.\ T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$$
$$2.\ T(c\vec{a}) = cT(\vec{a})$$

如果看 T 是不是 linear transformation 需要证明是不是符合上面的两个条件

Matrix vector products is linear transformation

$$T: R^n \to R^m: \ T(\vec{x}) = A\vec{x}, \qquad A \text{ 的 dimension } m \times n$$

Prove it is linear transformation:

$$A = [\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}] \qquad A \cdot (\vec{a}+\vec{b}) = A \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \dots \\ a_n+b_n \end{bmatrix}$$
$$= (a_1+b_1)\vec{v_1} + (a_2+b_2)\vec{v_2} + \cdots + (a_n+b_n)\vec{v_n}$$
$$= a_1\vec{v_1} + b_1\vec{v_1} + a_2\vec{v_2} + b_2\vec{v_2} + \cdots + a_n\vec{v_n} + b_n\vec{v_n}$$
$$= A \cdot \vec{a} + A \cdot \vec{b}$$

$$A \cdot (c\vec{a}) = ca_1\vec{v_1} + ca_2\vec{v_2} + \cdots + ca_n\vec{v_n}$$
$$= c(a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n})$$

Any linear matrix transformation can be viewed as matrix product

Standard basis for $R^n$

$$\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}\ \vec{e_i} \text{ 是} n \times n \text{ indentity matrix 的 ith column}$$

Image: transformation from one set into another set $T(L_0)$ *is image of* $L_0$ under T

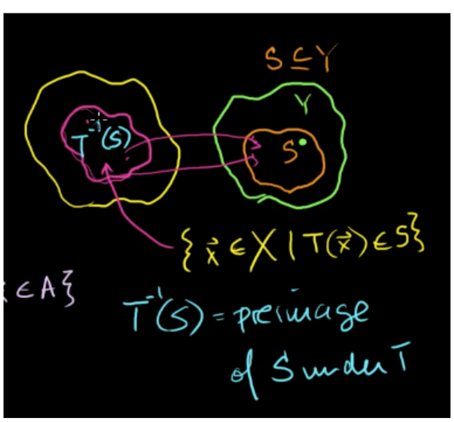
$$T: R^n \to R^m: \ T(V): \textit{image of } V \textit{ under } T$$

$T(V)$ is valid subspace:

Prove: 1. $T(\vec{a}), T(\vec{b}) \in T(V)$ 因为是 linear transformation $T(\vec{a}) + T(\vec{b})$ in V, 所以 $T(\vec{a}) + T(\vec{b}) = T(\vec{a}+\vec{b}) \in T(V)$

2. c $T(\vec{a}) = T(c\vec{a})$ 因为 $T(c\vec{a})$ in V, c $T(\vec{a})$ also in V

Image of T: $T(\vec{x}) = A\vec{x} = \text{column space of } A\ (C(A)) = span(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n})$




Preimage : $T^{-1}(S)\ \{\vec{x} \in X \mid T(\vec{x}) \in S\}$, given co-domain, what subset of domain map into co-domain, (不是每个 S 都需要 $\vec{x}$ map 到) $T(T^{-1}(S)) \subseteq S$

**Kernel** of T: $Ker(T) = \{x \in R^2 \mid T(\vec{x}) = \{\vec{0}\}\}$: A vector v is in the kernel of a linear transformation if and only if T(v) = 0. It is the same things as null space

## Sums and scalar multiples of linear transformation

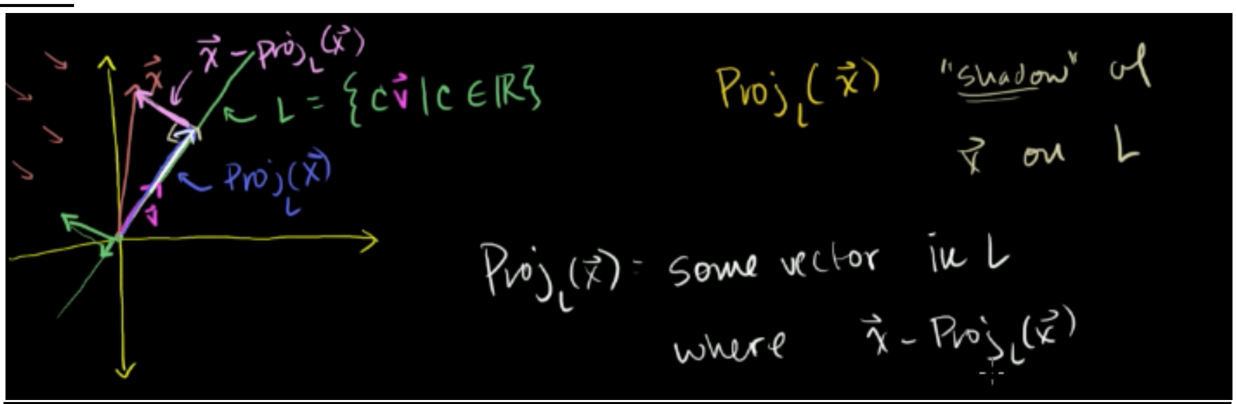
$$S: R^n \to R^m \quad T: R^n \to R^m$$
$$\text{Def:}\ (S+T)(\vec{x}) = S(\vec{x}) + T(\vec{x}) \qquad (S+T): R^n \to R^m$$
$$\text{Def:}\ (cS)(\vec{x}) = c\big(S(\vec{x})\big): \quad cS:\ R^n \to R^m$$

Linear Transformation example:
让所有 x 变成负-x，所有 y 乘以 2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

A 是 **diagonal matrix**：只有对角线有值，剩下都是 0;
**Unit vector**: vector has length of 1 $\|\vec{u}\| = \|\frac{1}{\|\vec{v}\|}\vec{v}\| = \frac{1}{\|\vec{v}\|}\vec{v}$
e.g. $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{u} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

## Projection




**$\mathrm{Proj}_L(\vec{x})$** 把$\vec{x}$ 到 L 做垂线，$\vec{x}$ 在 L 的射影是 projection，垂线是 $\vec{x} - \mathrm{Proj}_L(\vec{x})$，因为垂直 dot product = 0, $(\vec{x} - \mathrm{Proj}_L(\vec{x})) \cdot \mathrm{Proj}_L(\vec{x}) = 0$
$\mathrm{Proj}_L(\vec{x})$: some vector in L where $\vec{x} - \mathrm{Proj}_L(\vec{x})$ is orthogonal to L

$$L = \{c\vec{v}\ |c \in R\} \quad (\vec{x} - c\vec{v}) \cdot \vec{v} = 0$$
$$\vec{x} \cdot \vec{v} - c\vec{v} \cdot \vec{v} = 0$$
$$\vec{x} \cdot \vec{v} = c\vec{v} \cdot \vec{v}$$
$$c = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$
$$\mathrm{Proj}_L(\vec{x}) = c\vec{v} = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}$$

If $\vec{v}$ is unit vector, $\mathrm{Proj}_L(\vec{x}) = (\vec{x} \cdot \vec{v})\,\vec{v}$
**Prove: projection is linear transformation** ($\vec{u}$ is unit vector)

$$1.\ \mathrm{Proj}_L(\vec{a} + \vec{b}) = ((\vec{a} + \vec{b}) \cdot \vec{u})\vec{u}$$
$$= (\vec{a} \cdot \vec{u} + \vec{b} \cdot \vec{u})\,\vec{u}$$
$$= (\vec{a} \cdot \vec{u})\vec{u} + (\vec{b} \cdot \vec{u})\vec{u} = \mathrm{Proj}_L(\vec{a}) + \mathrm{Proj}_L(\vec{b})$$

$$2.\ \mathrm{Proj}_L(c\vec{a}) = (c\vec{a} \cdot \vec{u})\vec{u}$$
$$= c(\vec{a} \cdot \vec{u})\vec{u}$$
$$= c\mathrm{Proj}_L(\vec{a})$$

So $\mathrm{Proj}_L(\vec{x}) = A\vec{x}$

$$A = [\left(\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix},\ \left(\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix}]$$

$$A = \left[\mathrm{u1}\begin{bmatrix}u_1\\u_2\end{bmatrix},\ \mathrm{u2}\begin{bmatrix}u_1\\u_2\end{bmatrix}\right] = \begin{bmatrix}u_1^2 & u_2u_1\\ u_1u_2 & u_2^2\end{bmatrix}$$

## Composition

Composition: transformation of transformation

Composition is linear transformation (given S, T is linear transformation)

Prove:

$$\mathrm{T}\diamond\mathrm{S}(\vec{x}+\vec{y}) = T(S(\vec{x}+\vec{y})) = T(S(\vec{x})+S(\vec{y})) = T(S(\vec{x}))+T(S(\vec{y})) = \mathrm{T}\diamond\mathrm{S}(\vec{x})+\mathrm{T}\diamond\mathrm{S}(\vec{y})$$

$$\mathrm{T}\diamond\mathrm{S}(c\vec{x}) = \mathrm{T}(\mathrm{S}(c\vec{x})) = \mathrm{T}(c\mathrm{S}(\vec{x})) = c\mathrm{T}(\mathrm{S}(\vec{x})) = c(\mathrm{T}\diamond\mathrm{S})(\vec{x})$$

因为 composition is linear transformation，可以把$\mathrm{T}\diamond\mathrm{S}(\vec{x}+\vec{y})$ 写成 $A\vec{x}$

$$C = \left[\mathrm{B}\left(\mathrm{A}\begin{bmatrix}1\\0\\\cdots\\0\end{bmatrix}\right),\mathrm{B}\left(\mathrm{A}\begin{bmatrix}0\\1\\\cdots\\0\end{bmatrix}\right),\dots,\mathrm{B}\left(\mathrm{A}\begin{bmatrix}0\\0\\\cdots\\1\end{bmatrix}\right)\right]$$

$$C = [\mathrm{B}(\overrightarrow{a_1}),\mathrm{B}(\overrightarrow{a_2}),\dots,\mathrm{B}(\overrightarrow{a_n})]$$

$$\mathrm{T}\diamond\mathrm{S}(\vec{x}) = B(A(\vec{x})) = B\,A\,\vec{x}$$

$$AB = [A\,\overrightarrow{b_1},A\,\overrightarrow{b_2},\dots,A\,\overrightarrow{b_n}\,]$$

Associative $((\mathrm{H}\diamond\mathrm{G})\diamond\mathrm{F})(\vec{x}) = (\mathrm{H}\diamond\mathrm{G})(F\vec{x}) = H(G(F(\vec{x})) = H((G\diamond\mathrm{F})\vec{x})$

e.g. $A = \begin{bmatrix}1 & -1 & 2\\0 & -2 & 1\end{bmatrix},\ B = \begin{bmatrix}1 & 0 & 1 & 1\\2 & 0 & 1 & -1\\3 & 1 & 0 & 2\end{bmatrix}$

$$AB = [A\begin{bmatrix}1\\2\\3\end{bmatrix}\ A\begin{bmatrix}0\\0\\1\end{bmatrix}\ A\begin{bmatrix}1\\1\\0\end{bmatrix}\ A\begin{bmatrix}1\\-1\\2\end{bmatrix}]$$

## Matrix product properties

**Associative**: $(AB)C = A(BC)$ *Doesn't matter where to put 括号*

**Not Communicative**: $AB \neq BA$

**Distributive**: $A(B+C) = AB+BC,\ (B+C)A = BA+CA$

Prove Distributive:

$$B = [\overrightarrow{b_1},\overrightarrow{b_2},\dots,\overrightarrow{b_n}],\ C = [\overrightarrow{c_1},\overrightarrow{c_2},\dots,\overrightarrow{c_n}]$$

$$A(B+C) = A[\overrightarrow{b_1}+\overrightarrow{c_1},\overrightarrow{b_2}+\overrightarrow{c_2},\dots,\overrightarrow{b_n}+\overrightarrow{c_n}]$$

$$= A[\overrightarrow{b_1},\overrightarrow{b_2},\dots,\overrightarrow{b_n}] + A[\overrightarrow{c_1},\overrightarrow{c_2},\dots,\overrightarrow{c_n}] = AB+AC$$

$$(B+C)A = (B+C)\,[\overrightarrow{a_1},\overrightarrow{a_2},\dots,\overrightarrow{a_n}] = [(B+C)\overrightarrow{a_1},(B+C)\overrightarrow{a_2},\dots,(B+C)\,\overrightarrow{a_n}]$$

$$= [B\overrightarrow{a_1},B\,\overrightarrow{a_2},\dots,B\overrightarrow{a_n}] + [C\overrightarrow{a_1},C\overrightarrow{a_2},\dots,C\overrightarrow{a_n}]$$

$$= BA+CA$$

## Inverse

f (function X -> Y) is **Invertible** if and only if there exist a function $f^{-1}$ ( Y->X) such that $f^{-1}\circ f = I_x$ and $f\circ f^{-1} = I_y$

$$f^{-1}\circ f(a) = I_x(a) = a\,;\ f\circ f^{-1}(y) = y$$

Every function if has inverse, its inverse must be unique

Invertibility implies a unique solution to f(x) = y

Prove: If f is invertible, for every $y \in Y$, there is unique solution $x \in X$ such that f(x) = y

$$f(x) = y;\quad f^{-1}(y) = f^{-1}\big(f(x)\big) == (f^{-1}\circ f)(x) = \mathrm{I_x}(x) = x$$

For every $y \in Y$ f(x) = y has a unique solution, then f is invertibility

$$\mathrm{S{:}\,Y \to X}\text{；因为 f(x)} = y\ \textit{has unique solution.}\ S(y){:}\ \textit{well defined}$$
$$S(b)\ \textit{is the unique solution to}\ f(x) = b$$
$$f\big(S(b)\big) = b \to\ (f\circ s)(b) = I_y(b) =\ b\ ;\ f\circ s = I_y$$
$$S\big(f(a)\big) = \textit{the unqiue solution to the equation} = a;\ (s\circ f) = I_x$$
$$\text{因为}\ (s\circ f) = I_x, f\circ s = I_y, \textit{by definition, function is invertible}$$

**Inverse is linear transformation**

Prove:

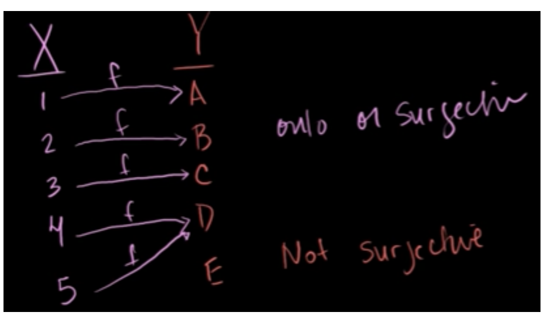
$$(f^{-1}\circ f)\big(\vec{a}+\vec{b}\big) = \vec{a}+\vec{b} = (f^{-1}\circ f)(\vec{a}) +\ (f^{-1}\circ f)\big(\vec{b}\big)$$
$$f\left(f^{-1}\big(\vec{a}+\vec{b}\big)\right) = f\big(f^{-1}(\vec{a})\big) + f\left(f^{-1}\big(\vec{b}\big)\right) = f\left(f^{-1}(\vec{a}) + f^{-1}\big(\vec{b}\big)\right)\ \text{因为}f\text{是}\textit{linear transformation}$$
$$f^{-1}(f\left(f^{-1}\big(\vec{a}+\vec{b}\big)\right)) = f^{-1}(f\big(f^{-1}(\vec{a})\big)) + f^{-1}(f\left(f^{-1}\big(\vec{b}\big)\right))$$
$$(f^{-1}\circ f)\left(f^{-1}\big(\vec{a}+\vec{b}\big)\right) =\ (f^{-1}\circ f)\big(f^{-1}(\vec{a})\big) + (f^{-1}\circ f)\big(f^{-1}(b)\big)$$
$$f^{-1}\big(\vec{a}+\vec{b}\big) =\ f^{-1}(\vec{a}) +\ f^{-1}(b)$$

$$(f\circ f^{-1})(c\vec{a}) = c\vec{a} = c\,((f\circ f^{-1})(\vec{a}) = f(cf^{-1}(\vec{a}))$$
$$f^{-1}(f(f^{-1}(c\vec{a}))) = f^{-1}(\,f(cf^{-1}(\vec{a})))$$
$$(f^{-1}\circ f)\big(f^{-1}(c\vec{a})\big) =\ (f^{-1}\circ f)(cf^{-1}(\vec{a}))$$
$$f^{-1}(c\vec{a}) =\ cf^{-1}(\vec{a})$$

## Onto & One-to-One

**Onto (surjective)**: every elements in co-domain $y \in Y$, there exist at least one $x \in X$ such that f(x) = y.

Every y in co-domain at least 一个 x map to




F onto then image(f) = Y

左侧是 Not onto example

**One-to-one (injective)**: for every value that map to there at most at most one x map to it.

每个一个 y 只有一个 x map，每个 x map to unqiue y: f(x) = y

上面不是 onto 的例子，也不符合 one-to-one， 假如 5 不指向 D, 5 改指向 E，表示 onto 和 one-to-one

**$f{:}\,x \to y$ is invertible if and only if f is onto and one-to-one**

Invertible means For every $y \in Y$ f(x) = y has a unique solution, that means one-to-one，如果有$y \in Y$ 但是没有相应的 x 对应，就不是 invertible 了，所以 invertible means onto

**T is onto iff C(A) = $\mathrm{R^m}$, its reduced echelon form has a pivot entry in every row (m pivot entry rank = M) : T is onto if and only if Rank(A) = m**

**Rank(A) = dim(C(A)) = # of basis vectors for C(A)**

$$\mathrm{T}: \mathrm{R^n} \to R^m \quad T(\vec{x}) = A\vec{x}$$

Onto => for any $\vec{b} \in \mathrm{R^m}$, at east one solution $A\vec{x} = \vec{b}$ where $\vec{x} \in \mathrm{R^n}$

$$A\vec{x} = [\mathrm{x_1}\overrightarrow{a_1} + \mathrm{x_2}\overrightarrow{a_2} + \cdots + \mathrm{x_n}\overrightarrow{a_n}]$$

For T to be onto $\mathrm{span}(\overrightarrow{a_1}, \overrightarrow{a_2}, .., \overrightarrow{a_n}) = R^m$ which is column space, column space is $\mathrm{R^m}$

e.g $\mathrm{S}: \mathrm{R^2} \to R^3, \mathrm{S}(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \vec{x}$

row reduced echelon form: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, rank = 2, S is not onto, S is not invertible

e.g. $\mathrm{T}: \mathrm{R^2} \to R^3, \mathrm{T}(\vec{x}) = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \end{bmatrix}$

ow reduced echelon form $\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ -1 & 3 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 0 & b_1 + b_2 \end{array}\right]$

only member $\vec{b} \in R^m$ that has solution are the ones $b_1 + b_2 = 0$

solution set = $\begin{bmatrix} x_1 \\ \mathrm{x_2} \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + \mathrm{x_2} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, when $\mathrm{T}(\vec{x}) = \vec{0}$, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the null space of T

从上面可以看出: Assuming $A(\vec{x}) = \vec{b}$ has a solution, the solution set = $\{\overrightarrow{x_p}\} U\ N(A)\ null\ space$ , some particular vector union null space; if one-to-one, at most 1 solution => N(A) has just zero vector(trival)

Any solution to the inhomogeneous system ($\mathrm{A}(\vec{x}) = \vec{b}\ give\ \vec{b} \neq 0$) system will take the form $\mathrm{x_p} + x_h$ (particular solution + homogeneous solution)

Prove: $\mathrm{A}(\mathrm{x_p} + \mathrm{x_h}) = Ax_p + Ax_h = \vec{b} + \vec{0}$

Prove any solution $\vec{x}$ to $\mathrm{A}\vec{x} = \vec{b}$ take the form $\vec{x} = \mathrm{x_p} + \mathrm{x_h}$:

$$\mathrm{A}(\vec{x} - \mathrm{x_p}) = A\vec{x} - \mathrm{Ax_p} = \vec{0}$$

$$\vec{x} - \mathrm{x_p} \text{ is a solution } A\vec{x} = \vec{0}, \vec{x} - \mathrm{x_p} is\ a\ member\ of\ null\ space\ N(A)$$

$$\vec{x} - \mathrm{x_p} = x_h \to \vec{x} = \mathrm{x_p} + x_h$$

**如果是 one-to-one: $\mathrm{x_p} + x_h$ 只能是 one-solution so $x_h$ has to be $\vec{0}$ null space has to be $\{\vec{0}\}$ so $\overrightarrow{a_1}, \overrightarrow{a_2}, \dots, \overrightarrow{a_n}$ are linearly independent; C(A) = $\mathrm{span}(\overrightarrow{a_1}, \overrightarrow{a_2}, \dots, \overrightarrow{a_n})$, $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \dots, \overrightarrow{a_n}\}$ are basis for column space, dim (column space) = n; rank (A) = N**

Invertible: 1. onto: rank(A) = m; 2. One-to-one： rank(A) = n; in order to let transformation to be invertible, rank(A) = m = n: **matrix has to be square matrix （n by n matrix）, 变成 reduced echelon form 每一行每一列又有 pivot entry (n by n indentity matrix)（linearly independent pivot colum**

$$\mathrm{T}: \mathrm{R^n} \to R^n (\text{不考虑} \mathrm{R^n} \to \mathrm{R^m}): \quad \mathrm{T}(\vec{x}) = A\vec{x}\ only\ invertible\ if\ row\ reduced\ echelon\ form\ is\ I_n$$

对 matrix 进行 row operation 等于进行 linear transformation, **linear transformation 的矩阵是等同于 identity matrix 进行一样的 row operation**

比如 $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ 等同于 $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \to \begin{bmatrix} a_1 \\ a_2 + a_1 \\ a_3 - a_1 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\mathrm{T}(\vec{x}) = S\vec{x} = \left[ \mathrm{S}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathrm{S}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \mathrm{S}\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

# Determinant

**3*3 determinant:**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

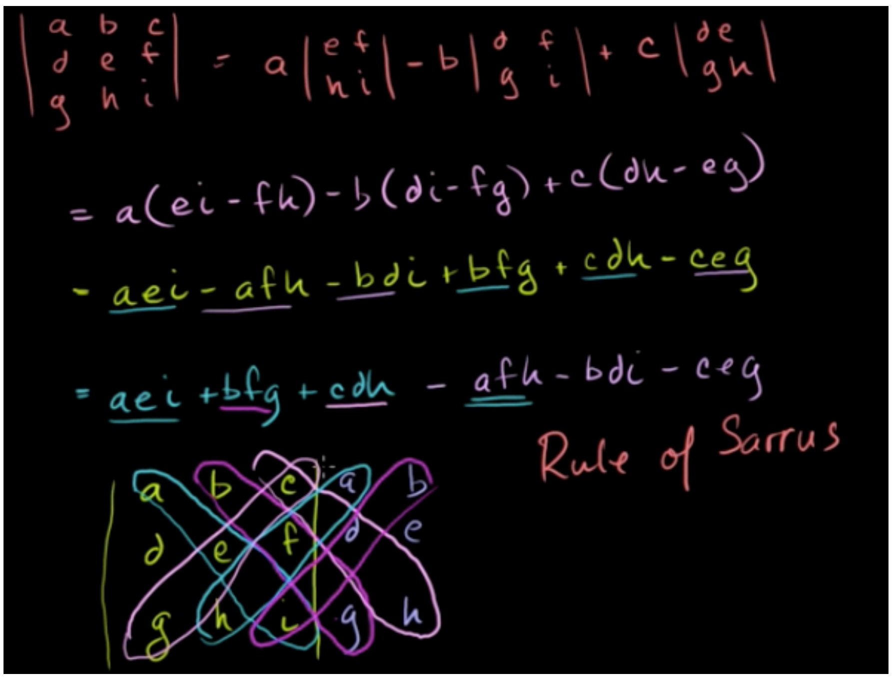
e.g.

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

$$Det(C) = 1\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$$

$$Det(C) = 1 * (-1 * 1 - 0 * 3\ ) - 2(2 * 1 - 4 * 3) + 4\big(2 * 0 - (-1 * 4)\big) = 35$$

Quick way: Rule of Sarrus




**n*n determinant:**

A = $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ , Define $\mathrm{A_{ij}} = (\mathrm{n}-1) \times (\mathrm{n}-1)$ matrix by ignore i-th row and j-th column

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\,\mathrm{def}(A_{12}) + a_{13}\det(A_{13}) + \cdots + (-1)^{i+j}\mathrm{a_{ij}}\det(A_{ij})$$

$$\det(A) = \ (-1)^{i+1}\mathrm{a_{i1}}\det\big(A_{ij}\big) + \ (-1)^{i+2}\mathrm{a_{i2}}\det(A_{i2}) +..+ (-1)^{i+n}\mathrm{a_{in}}\det(A_{in})$$

$$\det(A) = \sum_{j=1}^{j=n}(-1)^{i+j}\mathrm{a_{ij}}\det\big(A_{ij}\big)$$

**Row 乘以 scaler 的 determinant:**

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = ad - bc$$

如果 row2 乘以 k

$$\mathrm{A} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \det(A) = kad - kbc = k * \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

如果 row1 也乘以 k

$$A = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}, \det(A) = k^2ad - k^2bc = k^2 * \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

如果是 3*3 matrix row 乘以 k

$$A = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{kd} & \mathrm{ke} & \mathrm{kf} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{bmatrix}, \det(A) = -kd\begin{vmatrix} b & c \\ h & i \end{vmatrix} + ke\begin{vmatrix} a & c \\ g & i \end{vmatrix} - kf\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

如果是 n*n matrix row 乘以 k

$$\det(A) = (-1)^{i+1}\mathrm{ka_{i1}}\det(A_{ij}) + (-1)^{i+2}\mathrm{ka_{i2}}\det(A_{i2}) + .. + (-1)^{i+n}\mathrm{ka_{in}}\det(A_{in})$$

$$\det(A) = k\sum_{j=1}^{j=n}(-1)^{i+j}\mathrm{a_{ij}}\det(A_{ij}) = k * det(A)$$

如果 matrix 每行都乘以 k

$$\det(A) = k^n * \det(A)$$

**When row is added 的 determinant:**

2*2 matrix:

$$\mathrm{X} = \begin{bmatrix} a & b \\ x_1 & x_2 \end{bmatrix}, \mathrm{Y} = \begin{bmatrix} a & b \\ y_1 & y_2 \end{bmatrix}, \mathrm{Z} = \begin{bmatrix} a & b \\ x_1 + y_1 & x_2 + y_2 \end{bmatrix}$$

$$\det(X) = ax_2 - bx_1, \quad \det(Y) = ay_2 - by_1, \det(Z) = a(x_2 + y_2) - b(x_1 + y_1) = \det(X) + \det(y)$$

3*3 matrix:

$$\mathrm{X} = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{x_1} & \mathrm{x_2} & \mathrm{x_3} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \end{bmatrix}, \det(X) = -x_1\begin{vmatrix} b & c \\ e & f \end{vmatrix} + x_2\begin{vmatrix} a & c \\ d & f \end{vmatrix} - x_3\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$\mathrm{Y} = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{y_1} & \mathrm{y_2} & \mathrm{y_3} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \end{bmatrix}, \det(Y) = -y_1\begin{vmatrix} b & c \\ e & f \end{vmatrix} + y_2\begin{vmatrix} a & c \\ d & f \end{vmatrix} - y_3\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$Z = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{x_1 + y_1} & \mathrm{x_2 + y_2} & \mathrm{y_1 + y_3} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \end{bmatrix}, \det(Z) = -(x_1 + y_1)\begin{vmatrix} b & c \\ e & f \end{vmatrix} + (x_2 + y_2)\begin{vmatrix} a & c \\ d & f \end{vmatrix} - (x_3 + y_3)\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$\det(Z) = \det(X) + \det(Y)$$

n*n matrix

$$\det(X) = k\sum_{j=1}^{j=n}(-1)^{i+j}\mathrm{x_{ij}}\det(A_{ij}), \quad \det(Y) = k\sum_{j=1}^{j=n}(-1)^{i+j}\mathrm{Y_{ij}}\det(A_{ij}),$$

$$\det(Z) = k\sum_{j=1}^{j=n}(-1)^{i+j}(\mathrm{x_{ij}} + y_{ij})\det(A_{ij}) = \det(X) + \det(Y)$$

Determinant operations are not linear on matrix addition

**Swap Row determinant:** 比如第 i 行和第 j 行互换了

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j \\ \cdots \end{bmatrix}, swap\ i\ 和 j 行, \qquad A_{ij} = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_j \\ \cdots \\ a_i \\ \cdots \end{bmatrix}, \det(A) = -\det(A_{ij})$$

假如第 i 行 = 第 j 行，$\det(A) = \det(A_{ij})$,根据上面的定理： $\det(A) = -\det(A_{ij}) = \det(A_{ij}), \det(A) = 0$

**Duplication row determinant = 0**, 因为 duplicate row never get reduced echelon form to be invertible => det = 0

**Determinant of row operation**: row j = row j – c*rowi

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j \\ \cdots \end{bmatrix}, B = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j - c * a_i \\ \cdots \end{bmatrix}$$

因为$\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ -c * a_i \\ \cdots \end{bmatrix}$不是 linear independent，第 j 行可以由第 i 行乘以-c 得到，所以$\det\left(\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ -c * a_i \\ \cdots \end{bmatrix}\right) = 0$

$$\det(B) = \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ a_j \\ \cdots \end{bmatrix}\right) + \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_i \\ \cdots \\ -c * a_i \\ \cdots \end{bmatrix}\right) = \det(A)$$

**Determinant of upper triangular**: diagonal 所有数的乘积： $\det(A) = a_{11} a_{22} \ldots . a_{nn}$

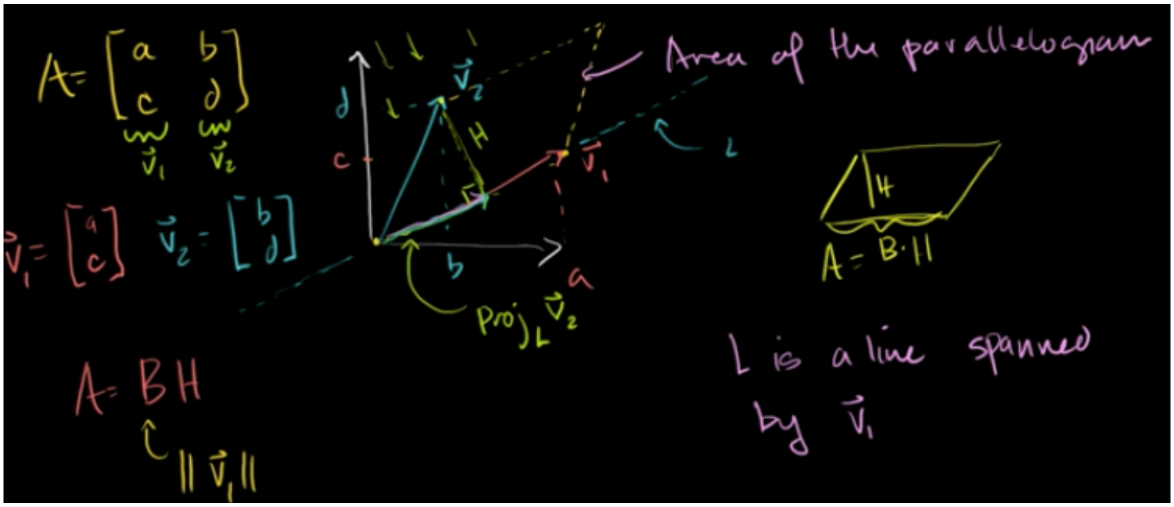
$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \det(A) = ad, B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \det(B) = adf$$

**Simple 4*4 determinant:** 利用 row operation 不 change determinant 和 upper triangular determinant 的性质，将 4*4 matrix 变成 diagonal matrix
e.g

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{bmatrix} \rightarrow \begin{matrix} \text{第二行}-=\text{第一行} \\ \text{第三行}-= 2 * \text{第一行} \\ \text{第四行}+=\text{第一行} \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 6 & -4 & 4 \end{bmatrix} \rightarrow \begin{matrix} \text{第二三行互换 } \det *= -1 \\ \\ \text{第四行}+=\text{第二行} * 2 + 3 * \text{第三行} \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \rightarrow \det = -1 * 3 * 2 * 7 = -42$$

**Determinant of area of a paralleogram:** 平行四边形(parallelogram) 边长 vector 组成 matrix 的 abs(determinant) = 它们的面积




$$\vec{v_1}=\begin{bmatrix}a\\c\end{bmatrix},\qquad \vec{v_2}=\begin{bmatrix}b\\d\end{bmatrix},\quad \mathrm{Proj_L}(\vec{v_2})\ 为\vec{v_2}\ 映射到\vec{v_1}长$$

$$\mathrm{H}^2+\left||\mathrm{Proj_L}(\vec{v_2})\right||^2=||\vec{v_2}||^2$$

$$\mathrm{H}^2=\left||\vec{v_2}|\right|^2-||\mathrm{Proj_L}(\vec{v_2})||^2$$

$$\mathrm{H}^2=\vec{v_2}\cdot\vec{v_2}-\left|\left|\frac{\vec{v_1}\cdot\vec{v_2}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1}\right|\right|^2$$

$$\mathrm{H}^2=\vec{v_2}\cdot\vec{v_2}-\frac{(\vec{v_1}\cdot\vec{v_2})(\vec{v_1}\cdot\vec{v_2})}{\vec{v_1}\cdot\vec{v_1}}$$

$$area^2=B^2H^2=\vec{v_1}\cdot\vec{v_1}\left(\vec{v_2}\cdot\vec{v_2}-\frac{(\vec{v_1}\cdot\vec{v_2})(\vec{v_1}\cdot\vec{v_2})}{\vec{v_1}\cdot\vec{v_1}}\right)=(\vec{v_1}\cdot\vec{v_1})(\vec{v_2}\cdot\vec{v_2})-(\vec{v_1}\cdot\vec{v_2})(\vec{v_1}\cdot\vec{v_2})$$

$$area^2=(a^2+c^2)(b^2+d^2)-(ab+cd)^2$$

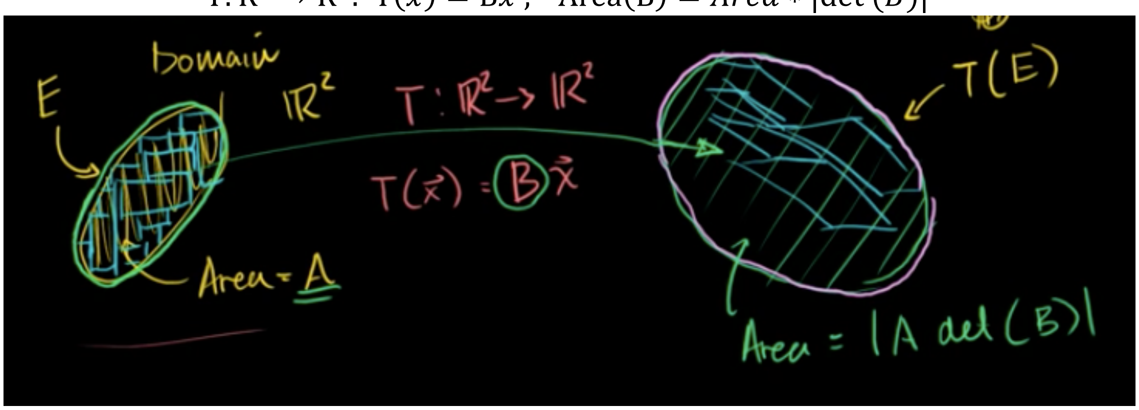
$$area^2=a^2b^2+a^2d^2+c^2b^2+c^2d^2-a^2b^2-2abcd-c^2d^2$$

$$area^2=\ a^2d^2+c^2b^2-\ 2abcd=(ad-bc)^2=\det(\mathrm{A})^2$$

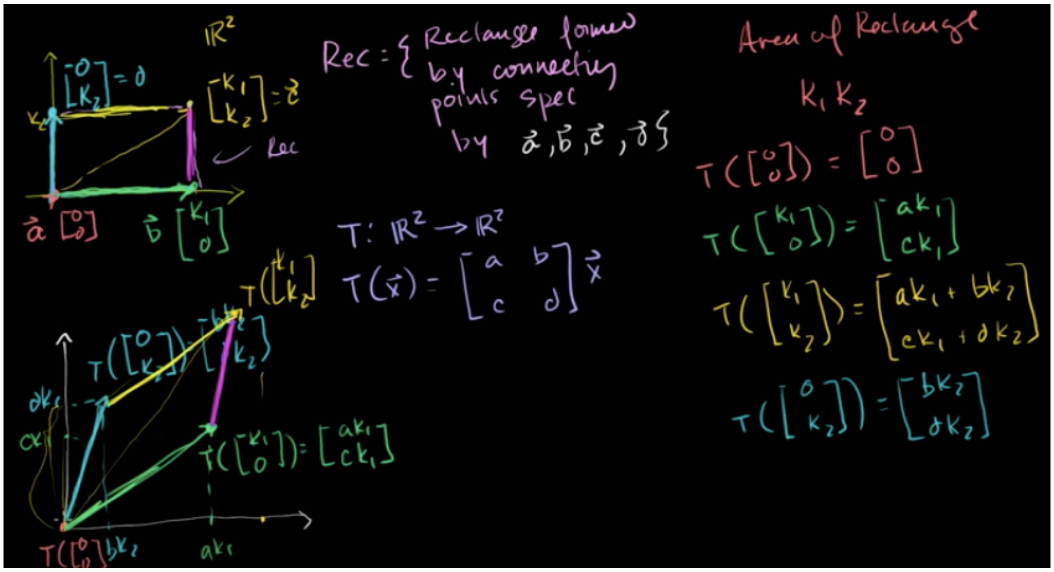
**Determinant as scaling factor:**

如果我们 transform 从一个 area A 到另外一个 area B

$$\mathrm{T}:\mathrm{R}^2\to\mathrm{R}^2:\ \mathrm{T}(\vec{x})=\mathrm{B}\vec{x}\,;\quad \mathrm{Area}(\mathrm{B})=Area*|\det(B)|$$




Prove： 长方形 transform 到平行




我们有边长$\vec{a} = \begin{bmatrix}0\\0\end{bmatrix}$，$\vec{b} = \begin{bmatrix}k_1\\0\end{bmatrix}$，$\vec{c} = \begin{bmatrix}0\\k_2\end{bmatrix}$，$\vec{d} = \begin{bmatrix}k_1\\k_2\end{bmatrix}$， $\mathrm{T}: \mathrm{R}^2 \to \mathrm{R}^2, \mathrm{T}(\vec{x}) = \begin{bmatrix}a & b\\c & d\end{bmatrix}\vec{x}$

Transformation 后的点为$\vec{a} = \begin{bmatrix}0\\0\end{bmatrix}$，$\vec{b} = \begin{bmatrix}ak_1\\ck_1\end{bmatrix}$，$\vec{c} = \begin{bmatrix}bk_2\\dk_2\end{bmatrix}$，$\vec{d} = \begin{bmatrix}ak_1 + bk_2\\ck_1 + dk_2\end{bmatrix}$

根据上面我们知道新的平行四边形面积是 $\det\left(\begin{bmatrix}ak_1 & bk_2\\ck_1 & dk_2\end{bmatrix}\right) = |k_1k_2ad - k_1k_2bc| = k1k2|ad - bc| = area(A) * |\det(T)|$

## Transpose

**Properties:**

1. $(C^T)^T = C$
2. $\det(A^T) = \det(A) \quad for\ A: n * n\ matrix$
3. $(AB)^T = B^T A^T$, $(XYZ)^T = Z^T Y^T X^T$
4. $(A+B)^T = A^T + B^T$
5. $(A^{-1})^T = (A^T)^{-1}$; $<= (AA^{-1})^T = (A^{-1})^T A^T = I_n^T = (A^{-1}A)^T = A^T (A^{-1})^T =>$
6. $for\ vector\ \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$, for A m × n, $\vec{x} \in n \times 1, \vec{y}\ n \times 1 \in R^m$; $(A\vec{x}) \cdot \vec{y} = (A\vec{x})^T \vec{y} = \vec{x}^T A^T \vec{y} = \vec{x}^T (A^T \vec{y}) = \vec{x} \cdot (A^T \vec{y})$ *因为matrix product is associative*
7. Rank(A) = Rank($A^T$), 根据 definition, Rank($A^T$) = dim(C($A^T$)) = # of basis of for rowspace of A: C($A^T$) = # of pivot entry in reduced row echelon form = Rank(A)

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}, \quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \quad a = [a_1, a_2, \ldots, a_n], \quad \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$

## Orthogonal Complements:

**Orthogonal complements of V**: for some V, $V^\perp = \{\vec{x} \in R^n | \vec{x} \cdot \vec{v} = 0\ for\ every\ \vec{v} \in V\}$

Prove orthogonal complements: 1. $\vec{a} \cdot \vec{v} = 0$, so $\vec{a}$ *can be equal to* $\vec{0}, \vec{0}$ *is in subspace* 2.. $\vec{a} \cdot \vec{v} = 0$ for any $\vec{v} \in V, \vec{b} \cdot \vec{v} = 0$ for any $\vec{b} \in V$, $(\vec{a} + \vec{b}) \cdot \vec{v} = \vec{a} \cdot \vec{v} + \vec{b} \cdot \vec{v} = 0$; 3. $c\vec{a} \cdot \vec{v} = c(\vec{a} \cdot \vec{v}) = 0$

**N(A) is orthogonal complements of the rowspace of A**(is the same as column space of A transpose)

Null space is orthogonal complement of row space

$$N(A) = (C(A^T))^\perp$$

Left Null space is orthogonal of the complement of column space

$$N(A^T) = (C((A^T)^T))^\perp = (C(A))^\perp, \quad (N(A^T))^\perp = C(A)$$

Prove:

$$A = \begin{bmatrix} ---\vec{a_1}^T --- \\ ---\vec{a_2}^T --- \\ \cdots \\ ---\vec{a_n}^T --- \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{a_1}^T \cdot \vec{x} \\ \vec{a_2}^T \cdot \vec{x} \\ \cdots \\ \vec{a_n}^T \cdot \vec{x} \end{bmatrix} \rightarrow \vec{x}\text{来自 } V, \vec{x} \in N(A), \vec{x}\ is\ orthogonal\ to\ \vec{a_1}^T, \vec{a_2}^T \ldots \vec{a_n}^T$$

N(A) is orthogonal to A

$$\text{N(A) is othogonal to A, and also orthogonal to any lienar combination of A, orthogonal to row space of A,}$$

$$\vec{w} = c_1 \vec{a_1}^T + c_2 \vec{a_2}^T + \cdots + c_n \vec{a_n}^T; \quad \vec{v} \cdot \vec{w} = \vec{w} = c_1 \vec{v_1} \cdot \vec{a_1}^T + c_2 \vec{v_2} \cdot \vec{a_2}^T + \cdots + c_n \vec{v_n} \cdot \vec{a_n}^T$$

**Dim(V) + Dim (orthogonal complement of v) = n (# columns)**

Prove:

$$\text{Rank}(A^T) + \text{Nullity}(A^T) = n \rightarrow \text{因为}\ rank(A) = rank(A^T) \quad \text{Rank}(A) + \text{Nullity}(A^T) = n$$

$$\text{根据}rank\text{的定义} \rightarrow \dim(C(A)) + \dim(N(A^T)) = n$$

$$\text{因为 } C(A) = (N(A^T))^\perp \quad \dim(C(A)) + \dim(C(A)^\perp) = n$$

$$\dim(V) + \dim(V^\perp) = n$$

**若 $V \in R^N$, $V^\perp \in R^n$, $\dim(V) + \dim(V^\perp) = n$, $V$的rank是$k$, 则$R^N$中所有的点可以表示成 $\vec{a} = \vec{v} + \vec{x}$, $\vec{v} \in V$, $\vec{x} \in V^\perp$, 且$\vec{v}$和$\vec{x}$ 是 unique**

Prove: 是 unique 的, $\vec{a} = \vec{v_1} + \vec{x_1} = \vec{v_2} + \vec{x_2}$, 假设$\vec{v_1}$ 和 $\vec{v_2}$ 不等, $\vec{x_1}$ 和 $\vec{x_2}$ 不等, $\vec{z} = \vec{v_1} - \vec{v_2} =$

$\vec{x_1} - \vec{x_2}$, 因为$\vec{v_1}, \vec{v_2}$ 来自 V， $\vec{v_1} - \vec{v_2}$ 在 V 中， $\vec{x_1} - \vec{x_2}$在$V^{\perp}$中，因为只有$\vec{0}$ 既在 V 中， 也在$V^{\perp}$中，所以 $\vec{v_1} - \vec{v_2} = \vec{0}, \vec{v_1} = \vec{v_2}$； $\vec{x_1} - \vec{x_2} = \vec{0}$，$\vec{x_1} = \vec{x_2}$

**Orthogonal complement of the orthogonal complement of V is V**

$$V = ((V)^{\perp})^{\perp}$$

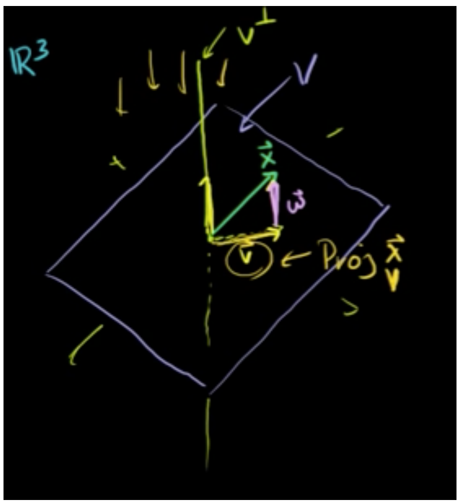
**$A^T A$ is invertible given $A$ is k*n matrix and each columns in A is linearly independent**

Prove: $\vec{v} \in N(A^T A)$, *根据定义*$(A^T A) \Rightarrow A^T A\vec{v} = \vec{0} \rightarrow \vec{v}^T (A^T A)\vec{v} = \vec{v}^T\vec{0} = (\vec{v}^T A^T)(A\vec{v}) = (A\vec{v})^T A\vec{v} = 0$， 则 $||A\vec{v}||^2 = 0, A\vec{v} = 0$，so if $\vec{v} \in N(A^T A)$，then $\vec{v} \in N(A)$，因为 A 是 linearly independent column 的, N(A) 只包括了$\vec{v} = \vec{0}$， then only solution for $N(A^T A)$ is $\vec{0}$， $A^T A$ is invertible

## Projection on a plane:

$\vec{x} = \vec{v} + \vec{w}$, $\vec{w}$ is orthogonal to everything in $\vec{v}$, $\vec{v}, \vec{w}$ 相当于直角三角形的两个边

$$\vec{x} = \mathrm{Proj}_v\vec{x} + \vec{w} = \mathrm{Proj}_v\vec{x} + \mathrm{Proj}_{v^{\perp}}\vec{x}$$




$\mathrm{Proj}_v\vec{x}$ = the unique vector $\vec{v}$ such that $\vec{x} = \vec{v} + \vec{w}$ *where $\vec{w}$ is a unique member of $V^{\perp}$*

$\mathrm{Proj}_v\vec{x}$ = some unique vector in V such that $\vec{x} - \mathrm{Proj}_v\vec{x}$ is orthogonal to every member of V

如果 A 是 matrix consists of basis of V:

$$\mathbf{Proj_v\vec{x} = A\,(A^TA)^{-1}\,A^T\vec{x}}$$

Prove: $\{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_k}\}$ *is basis for V*， 如果$\vec{a} \in V \Rightarrow \vec{a} = y_1\vec{b_1} + y_2\vec{b_2} + \cdots + y_k\vec{b_k}$， 如果由 basis 组成 matrix，是 n*k 维的，$A\,\vec{y} = [\vec{b_1}, \vec{b_2}, \ldots, \vec{b_k}] * \begin{bmatrix} y_1 \\ \cdots \\ y_n \end{bmatrix} = y_1\vec{b_1} + y_2\vec{b_2} + \cdots + y_k\vec{b_k}, \textit{for some } \vec{y} \in R^k$

$\mathrm{Proj}_v\vec{x} = A\,\vec{y}$ *for 某个*$\vec{y}$， $\vec{x} = \mathrm{Proj}_v\vec{x} + \vec{w}$ where $\vec{w}$ is the member of $V^{\perp} = C(A)^{\perp} = N(A^T)$， $\vec{x} - \mathrm{Proj}_v\vec{x} = \vec{w} \in N(A^T)$

根据 null space 的定义，$A^T(\vec{x} - \mathrm{Proj}_v\vec{x}) = A^T\vec{x} - A^T\mathrm{Proj}_v\vec{x} = A^T\vec{x} - A^TA\vec{y}$

$$A^T\vec{x} = A^TA\vec{y}$$

$$\vec{y} = (A^TA)^{-1}A^T\vec{x}, \textit{根据上面定义我们知道}A^TA \text{ is invertible}$$

$$\mathrm{Proj}_v\vec{x} = A\,(A^TA)^{-1}A^T\vec{x}$$

e.g. V = {all the $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *that satisfy* $x_1 + x_2 + x_3 = 0$} find projection matrix of V, $x_1 = -x_2 - x_3$

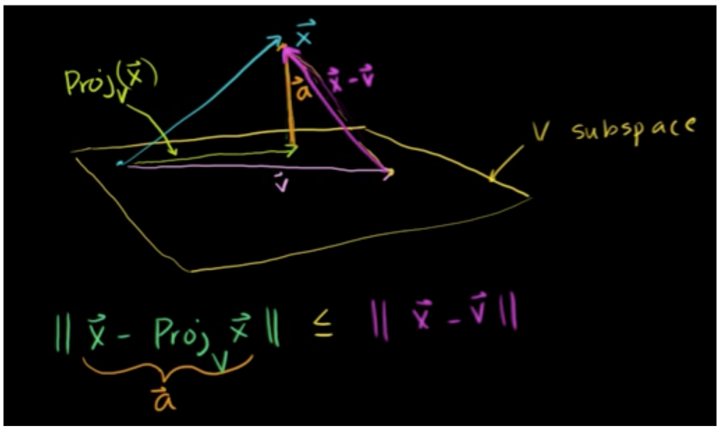
$$V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, V = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$A\,(A^TA)^{-1}\,A^T = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

另一种思路： $\vec{x} = \mathrm{Proj}_{\mathrm{v}}\vec{x} + \vec{w} = B\vec{x} + C\vec{x},\ I_x\vec{x} = (B + C)\vec{x}$

因为$x_1 + x_2 + x_3 = 0$, then V = N([1,1,1]), $\mathrm{V}^{\perp} = N([1,1,1])^{\perp} = C\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = span\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$

$$C = C\,(C^TC)^{-1}\,C^T = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix},\quad B = I - C$$




Vector 到 plane 做 projection, projection 的高是最短的 distance from vector 到 plane

## Least Square

Least square for $A\vec{x} = \vec{b}, A\ n*k, \vec{x} \in R^k, \vec{b} \in R^n$

$$\boldsymbol{\vec{x} = (A^T A)^{-1} A^T \vec{b}}$$

What if no solution for x, we want to find closest solution

Minimize $||\vec{b} - A\vec{x}||$, $A\vec{x}$ 是column space 的linear combination , the close between $\vec{b}$ and plane is the projection. $\vec{b} - A\vec{x}$ is orthogonal to $A\vec{x}$(projection), $A\vec{x} - \vec{b} = proj_{C(A)}\,\vec{b} - \vec{b} \in N(A^T)$ =

$$A^T\left(A\vec{x} - \vec{b}\right) = \vec{0}$$

$$A^T A\vec{x} - A^T\vec{b} = \vec{0}\ \ A^T A\vec{x} = A^T\vec{b}, \qquad \vec{x} = (A^T A)^{-1} A^T\vec{b}$$

## Basis transformation

V is subspace of $\mathrm{R}^{\mathrm{n}}, B = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}\ if\ \vec{a} \in V \rightarrow\ \vec{a} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$ , $\mathrm{k} \leq \mathrm{n}$

Call $c_1, c_2, \dots, c_k$ the coordinates of $\vec{a}$ with respect to B, $[\vec{a}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_k \end{bmatrix}$, even if $\vec{a}$ in $\mathrm{R}^{\mathrm{n}}$, only give k coordinates

$$\vec{a} = \mathrm{C}[\vec{a}]_B\ , C\ is\ the\ basis\ matrix\ is\ called\ change\ of\ basis\ matrix$$

直角坐标系中的 coordinate 是对应$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$的坐标

**Change of basis matrix is the matrix with the basis vectors as columns**

If change of basis matrix is invertible: C is square, k = n => n basis vectors ( columns are linear independent) => $span(C) = \mathrm{R}^{\mathrm{n}}$; 如果反过来 if $span(C) = \mathrm{R}^{\mathrm{n}}$, then C is invertible

$$C[\vec{a}]_B = \vec{a}$$

$$\mathrm{C}^{-1}C[\vec{a}]_B = C^{-1}\vec{a}\ ; \quad [\vec{a}]_B = C^{-1}\vec{a}$$

e.g.

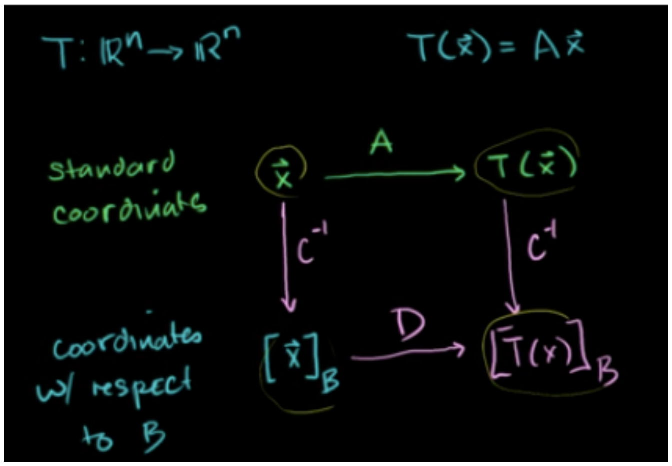
if $\mathrm{C} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $\vec{a} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$, $what\ is\ [\vec{a}]_B$?

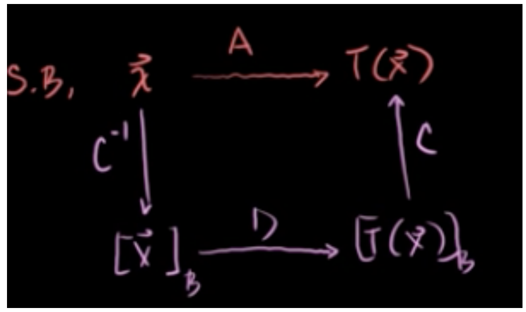
$$[\vec{a}]_B = \mathrm{C}^{-1}\vec{a} = -\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 19/5 \end{bmatrix}$$

If D is the transformation matrix for T with respect to the basis B, $[T(\vec{x})]_B = D[\vec{x}]_B$
and C is the change of basis for B, $C[\vec{x}]_B = \vec{x}$
and A is the transformation matrix for T with respect to the standard basis

$$\mathrm{D} = \mathrm{C}^{-1}AC$$
$$\mathrm{A} = CDC^{-1}$$

Prove: $C[\vec{x}]_B = \vec{x}$, $[\vec{x}]_B = C^{-1}\vec{x}$, $D[\vec{x}]_B = [T(\vec{x})]_B = [A\vec{x}]_B = C^{-1}A\vec{x} = C^{-1}AC\,[\vec{x}]_B$




## Orthonormal Basis

$B = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$, 1. $||\vec{v_i}|| = 1$, each vector has length = 1  2. Each vector is orthogonal to each other $\vec{v_i} \cdot \vec{v_j} = \begin{cases} 0 \; for \; i \neq j \\ 1 \; for \; i = j \end{cases}$. Then **B is orthonormal set for V**

**Orthonormal basis**: 由 orthonormal set 构成的 basis

性质:

1. B is linearly independent

   Prove: if $\vec{v_i}\,\vec{v_j}$ are linear dependent, then $\vec{v_i} = c\,\vec{v_j}$ for $c \neq 0$, $\vec{v_i} \cdot \vec{v_j} = 0 = c\vec{v_j} \cdot \vec{v_j} \to ||\vec{v_j}|| = 0$, contradict assumption

2. If $\vec{x} = c_1\vec{v_1} + c_2\,\vec{v_2} + \dots + c_k\,\vec{v_k}$, $then\; \vec{v_i} \cdot \vec{x} = c_1\vec{v_i}\,\vec{v_1} + c_2\vec{v_i}\,\vec{v_2} + \dots + c_i\vec{v_i}\,\vec{v_i} + \dots + c_k\,\vec{v_k} = c_i\vec{v_i}\,\vec{v_i} = c_i$

3. $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_k \end{bmatrix} = \begin{bmatrix} \vec{v_1} \cdot \vec{x} \\ \vec{v_2} \cdot \vec{x} \\ \cdots \\ \vec{v_k} \cdot \vec{x} \end{bmatrix}$

4. If orthonormal basis 组成 matrix A, 则$\mathrm{A^T}A = I_k$ the identity matrix

   Prove: $\mathrm{A^T A} = \begin{bmatrix} - - \vec{v_1}^T - - \\ - - \vec{v_2}^T - - \\ \cdots \\ - - \vec{v_k}^T - - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_k} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

5. C is n*n matrix whose columns form an orthonormal set, $\mathrm{C}^{-1} = C^T$, 因为 C is n*n 所以 C 是 invertible $\mathrm{C}^{-1}C = I$, 而 from 4, 已知$\mathrm{C^T C = I}$

Good for coordinate system, 比如 standard basis for $\mathrm{R}^3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

e.g. under orthonormal basis $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}, \vec{x} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$, calculate the coordinates,

$$[\vec{x}]_B = \begin{bmatrix} \overrightarrow{v_1} \cdot \vec{x} \\ \overrightarrow{v_2} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} \frac{27}{5} + -\frac{8}{5} \\ -\frac{36}{5} + -\frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{19}{5} \\ \frac{-42}{5} \end{bmatrix}$$

If $B = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}$ for V, V ∈ $R^n$, 因为$\vec{x} = \vec{v} + \vec{w} = Proj_V\vec{x} + \vec{w}$, $\vec{w}$ 是 V 的 orthogonal complement

$$Proj_V\vec{x} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_k\overrightarrow{v_k} = (\overrightarrow{v_1} \cdot \vec{x})\overrightarrow{v_1} + (\overrightarrow{v_2} \cdot \vec{x})\overrightarrow{v_2} + \cdots + (\overrightarrow{v_k} \cdot \vec{x})\overrightarrow{v_k}$$
$$Proj_V\vec{x} = A(A^TA)^{-1}A\vec{x} = AA\vec{x}$$

If $C = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ is basis for $R^n$, C (n*n matrix is invertible) which means $C^{-1}C = I$, and above we know $C^TC = I$

If transformation matrix is orthogonal matrix ( 由 orthonormal set 组成), it will preserve length and angle for transformation

Prove:

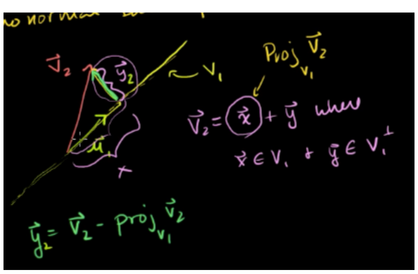
$$||C\vec{x}||^2 = C\vec{x} \cdot C\vec{x} = (C\vec{x})^TC\vec{x} = \vec{x}^TC^TC\,\vec{x} = \vec{x}^T\vec{x} = ||\vec{x}||^2$$
$$cos\theta = \frac{C\vec{v} \cdot C\vec{w}}{||C\vec{v}||\,||C\vec{w}||} = \frac{(C\vec{v})^TC\vec{w}}{||\vec{v}||\,||\vec{w}||} = \frac{\vec{v}^TC^TC\vec{w}}{||\vec{v}||\,||\vec{w}||} = \frac{\vec{v}^T\vec{w}}{||\vec{v}||\,||\vec{w}||}$$

## Gram-Schmidt process

$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}$ a basis for V, To find orthonormal basis

$$One\ dimensional\ subspace\ V_1 = span(\overrightarrow{v_1}), \qquad u_1 = \frac{\vec{v}_1}{||\vec{v}_1||}$$
$$V_2 = span(\overrightarrow{u_1}, \overrightarrow{v_2}) = span(\overrightarrow{u_1}, \overrightarrow{y_2}) = span(\overrightarrow{u_1}, \overrightarrow{u_2}), \qquad \overrightarrow{y_2} = \overrightarrow{v_2} - Proj_{V_1}\overrightarrow{v_2}$$
$$\overrightarrow{y_2} = \overrightarrow{v_2} - Proj_{V_1}\overrightarrow{v_2} = \overrightarrow{v_2} - (\overrightarrow{v_2} \cdot \overrightarrow{u_1})\overrightarrow{u_1}, \ \ u_2 = \frac{\vec{y}_2}{||\vec{y}_2||}$$

$$V_3 = span(\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{v_3}) = span(\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{y_3}) = span(\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}), \qquad \overrightarrow{y_3} = \overrightarrow{v_3} - Proj_{V_2}\overrightarrow{v_3}$$
$$\overrightarrow{y_3} = \overrightarrow{v_3} - Proj_{V_2}\overrightarrow{v_3} = \overrightarrow{v_3} - (\overrightarrow{v_3} \cdot \overrightarrow{u_1})\overrightarrow{u_1} - (\overrightarrow{v_3} \cdot \overrightarrow{u_2})\overrightarrow{u_2}, \ , \ u_3 = \frac{\vec{y}_3}{||\vec{y}_3||}$$




e.g. find orthonormal basis for $x_1 + x_2 + x_3 = 0$

$$x_1 = -x_2 - x_3$$
$$V = span\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}, u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$y_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} - \left(\begin{bmatrix}-1\\0\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}, u_2 = \frac{\vec{y}_2}{||\vec{y}_2||} = \sqrt{\frac{3}{2}}\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}$$

$$\{\overrightarrow{u_1}, \overrightarrow{u_2}\}\ is\ orthonormal\ basis$$

## Eigenvector and Eigenvalue

$\boldsymbol{T(\vec{v}) = \lambda\vec{v}}$, for non-zero vector $\vec{v}$ is **eigenvector**, $\lambda$ is **eigenvalue**.

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda I_n\vec{v} = \vec{0}$$
$$(A - \lambda I_n)\vec{v} = \vec{0}$$

因为 $\vec{v}$ is non-zero vector, $(A - \lambda I_n)$ is linear dependent matrix 否则 $\vec{v}$ 必须等于 zero vector, so $\det(A - \lambda I_n) = 0$

Eigenspace $E_\lambda$: the space of vector correspond to eigenvalue $E_\lambda = null\ space(A - \lambda I_n)$

e.g. $A = \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}, \det(\lambda I_n - A) = \det\left(\begin{bmatrix}\lambda - 1 & -2\\-4 & \lambda - 3\end{bmatrix}\right) => \lambda^2 - 4\lambda + 5 = 0, \lambda = -1, 5$

$$E_5 = N\left(\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix} - \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}\right) = N\left(\begin{bmatrix}4 & -2\\-4 & 2\end{bmatrix}\right) = span\left(\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}\right), E_5 = \left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} = t\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}, t \in R\right\}$$

$$E_{-1} = N\left(\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} - \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}\right) = N\left(\begin{bmatrix}-2 & -2\\-4 & -4\end{bmatrix}\right) = span\left(\begin{bmatrix}-1\\1\end{bmatrix}\right),$$

When having n 个 linear-independent eigenvectors for $R^n$ for transformation T: $R^n \to R^n, T(\vec{x}) = \lambda\vec{x}, \{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ form a basis for $R^n$, called eigenbasis (linear independent 肯定能是 basis, show this basis is good basis)

$$T(\overrightarrow{v_1}) = A\overrightarrow{v_1} = \lambda_1\overrightarrow{v_1} = \lambda_1\overrightarrow{v_1} + 0\overrightarrow{v_2} + \cdots + 0\overrightarrow{v_n}$$
$$T(\overrightarrow{v_2}) = A\overrightarrow{v_2} = \lambda_2\overrightarrow{v_2} = 0\overrightarrow{v_1} + \lambda_2\overrightarrow{v_2} + \cdots + 0\overrightarrow{v_n}$$
$$T(\overrightarrow{v_n}) = A\overrightarrow{v_n} = \lambda_n\overrightarrow{v_n} = 0\overrightarrow{v_1} + 0\overrightarrow{v_2} + \cdots + \lambda_n\overrightarrow{v_n}$$