Mathematics Institute - UFRJ

Graduation in Mathematics

Real Analysis Exam/MAA 740 - 08/22/2018

Question 1: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function, with $n \geq 2$. Suppose that there is $c \in \mathbb{R}$ such that

$$f^{-1}(c) = \{x \in \mathbb{R}^n \ ; \ f(x) = c\}$$

is bounded.

(i) Prove that one of the following sets

$$C^- = \{x \in \mathbb{R}^n \ ; \ f(x) \leq c\} \qquad \text{or} \qquad C^+ = \{x \in \mathbb{R}^n \ ; \ f(x) \geq c\}$$

is bounded.

(ii) Prove that f attains a maximum or minimum value in $\mathbb{R}^n$.

Question 2: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function. Suppose that $\frac{\partial f}{\partial u}(u) > 0$ for all $u \in \mathbb{S}^{n-1}$, prove that f admits a critical point in $\mathbb{R}^n$.

Question 3: Show that every manifold of class C^k in $\mathbb{R}^n$ is locally the graphic of an application of class C^k.

Question 4: Let A be a J-mensurable set and ε be a positive real number. Prove that there exists a compact J-mensurable set K_ϵ contained in A such that

$$\int_A \chi_{A \setminus K_\epsilon} dx < \varepsilon.$$

Question 5: The electric potential of an electric field of a point charge located at origin is given by the function $f(x, y, z) = 1/(x^2 + y^2 + z^2)^{1/2}$.

(i) Show that the flow through a closed surface which does not contain the origin is 0.

(ii) Calculate the flow[1] through the sphere with ratio r centered at the origin.

(iii) Calculate the flow[1] through any closed surface which contains the origin.

[1]Remember that the flow of a vector field F through a surface M is given by the integral

$$\int_M \langle F, n \rangle d\sigma,$$

where n is the exterior unit normal to surface and $d\sigma$ represents the area element.