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A formula for the core of an ideal*

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Abstract. The core of an ideal is the intersection of all its reductions. For large classes of ideals *I* we explicitly describe the core as a colon ideal of a power of a single reduction and a power of *I*.

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1. Introduction

The purpose of this paper is to prove a formula for the core of an ideal that had been conjectured by the authors and A. Corso in [3]. For ideals $J \subset I$ in a Noetherian ring one says that J is a reduction of I, or I is integral over J, if $I^{n+1} = JI^n$ for some $n \ge 0$. The core of I, $\operatorname{core}(I)$, defined as the intersection of all reductions of I, is a somewhat mysterious subideal of I that encodes information about the possible reductions of I. The concept was introduced by Rees and Sally ([18]), and has been studied further by Huneke and Swanson and by Corso and the authors ([11], [2], [3], [4]). It has a close relation to Briançon-Skoda type theorems and to coefficient, adjoint and multiplier ideals ([11], [13], [14]). Moreover, Hyry and Smith recently discovered an unexpected connection with Kawamata's conjecture on the non-vanishing of sections of line bundles. They showed that Kawamata's conjecture would follow from a formula that essentially amounts to a graded analogue of the above conjecture on the core. They were also able to prove our original conjecture under additional assumptions that arise naturally in the geometric context ([14]).

If R is a Noetherian local ring with infinite residue field k, then core(I) is the intersection of *minimal* reductions of I, i.e., of reductions minimal with respect to inclusion. The minimal number of generators of every minimal reduction of I is the *analytic spread* of I, which can also be defined as $\ell(I) = \dim \operatorname{gr}_I(R) \otimes_R k$. Given a reduction J of I we write $r_J(I)$ for the least integer $n \geq 0$ such that $I^{n+1} = JI^n$, and we define the *reduction number* r(I) of I to be $\min\{r_J(I)\}$, where J ranges over all minimal reductions of I. In this paper we prove the following result:

Let (R, \mathfrak{m}) be a local Gorenstein ring with infinite residue field k, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. Assume I satisfies G_ℓ , depth $R/I^j \ge \dim R/I - j + 1$ for $1 \le j \le \ell - g$, and either char k = 0 or char $k > r - \ell + g$. Then

$$core(I) = J^{n+1} : I^n$$

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for every $n \ge \max\{r - \ell + g, 0\}$.

We are now going to discuss the assumptions in this theorem. The condition on the characteristic is vacuous if $r \le \ell - g + 1$, in which case the result has been shown in [3]. The G_ℓ property is a rather weak requirement on the local number of generators of I, that is always satisfied if $I_\mathfrak{p}$ can be generated by $\dim R_\mathfrak{p}$ elements for every prime ideal $\mathfrak{p} \ne \mathfrak{m}$ containing I. Both assumptions, the G_ℓ condition and the inequalities $\operatorname{depth} R/I^j \ge \dim R/I - j + 1$ for $1 \le j \le \ell - g$, are automatically satisfied if I is \mathfrak{m} -primary, or more generally, if I is equimultiple, i.e., $\ell = g$. They also hold for one-dimensional generic complete intersection ideals, or more generally, for Cohen-Macaulay generic complete intersections with analytic deviation one, i.e., $\ell = g + 1$. In the presence of the G_ℓ property, the depth inequalities for the powers obtain if I is perfect with g = 2, I is perfect Gorenstein with g = 3, or more generally, if I is in the linkage class of a complete intersection.

Although most of the paper is devoted to proving the main theorem, Theorem 4.5, we show some of the auxiliary results in greater generality than needed since they might be interesting in their own right. For instance, to prove the inclusions $J^{n+1}: I^n \subset \operatorname{core}(I)$, we identify the first ideals with graded components of the canonical module of the extended Rees algebra $R[It,t^{-1}] \subset R[t,t^{-1}]$ of I. The computation of the canonical module is the content of Proposition 2.1. It relies on residual intersection theory and earlier results from [21], where graded components of canonical modules have been used to identify colon ideals of powers of reductions that are independent of the chosen reduction. The computation of graded components of canonical modules has also played an important role in [13], [3] and [14]. To prove the reverse containment $core(I) \subset J^{n+1}: I^n$ we first consider the case $\ell = g = 1$. In this case Theorem 3.4 provides a general formula for core(I) that holds in a Cohen-Macaulay ring of arbitrary characteristic and specializes to the desired equality $core(I) = J^{n+1}$: I^n under suitable assumptions on the characteristic. Important ingredients in this proof are ideas from [11] as well as a use of characteristic zero that was inspired to us by K. Smith. In Lemma 4.2 and Theorem 4.4 we lift the containment obtained for $\ell = g = 1$ to establish an inclusion for the core of ideals of arbitrary height and arbitrary analytic spread. Once Lemma 4.2, the main new technical result, is in place, the proof of Theorem 4.4 essentially follows the lines of [3]. For both results we need to use residual intersection theory since we are not restricting ourselves to the case of equimultiple ideals. We give various classes of examples showing that the main theorem may fail if $0 < \operatorname{char} k \le r - \ell + g$ or if any of the other assumptions are dropped.

The formula $core(I) = J^{n+1}: I^n$ has been proved independently by Huneke and Trung under the assumption that R is a local Cohen-Macaulay ring whose residue field has characteristic zero and I is an equimultiple ideal ([12]). Both papers, the present one and [12], were preceded by the work of Hyry and Smith who proved the same formula if in addition the Rees ring of I is Cohen-Macaulay ([14]). We would like to thank Karen Smith for sharing her ideas with us before [14] was completed.

Preliminaries

We begin by reviewing some definitions and basic facts. Let R be a Noetherian ring, I an R-ideal and j,s integers. We set $I^j=R$ whenever $j\leq 0$. The ideal I satisfies condition G_s if for every prime ideal $\mathfrak p$ containing I with $\dim R_{\mathfrak p}\leq s-1$, the minimal number of generators $\mu(I_{\mathfrak p})$ of $I_{\mathfrak p}$ is at most $\dim R_{\mathfrak p}$. One says that I satisfies G_{∞} in case G_s holds for every s. When writing that $\mathfrak a:I$ is a geometric s-residual intersection of I we mean that $\mathfrak a$ is an s-generated R-ideal properly contained in I and that $\mathfrak a:I\geq s$, $\operatorname{ht}(I,\mathfrak a:I)\geq s+1$. The ideal I is said to be of linear type if the natural map from the symmetric algebra to the Rees algebra of I is an isomorphism; in this case I has no proper reductions.

Now assume in addition that R is a local Cohen-Macaulay ring, and let H_{\bullet} denote the homology of the Koszul complex on a generating sequence f_1, \ldots, f_n of I. One says that I satisfies sliding depth if depth $H_i \geq \dim R - n + i$ for every $i \geq 0$, and that I is strongly Cohen-Macaulay if H_{\bullet} is Cohen-Macaulay. These notions fit into the following sequence of implications: In case I is a perfect ideal of grade 2 or a perfect Gorenstein ideal of grade 3, then I is in the linkage class of a complete intersection, which in turn implies that I is strongly Cohen-Macaulay ([10, 1.11]). Strong Cohen-Macaulayness obviously implies the sliding depth property. If I satisfies sliding depth and G_{∞} , then I is of linear type and the associated graded ring $gr_I(R)$ is Cohen-Macaulay ([6, 9.1]). If on the other hand I is strongly Cohen-Macaulay of height g and satisfies G_s , then depth $R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq s - g + 1$ ([6, the proof of 5.1]). The latter condition in turn implies that I is Cohen-Macaulay for every (geometric) I is residual intersection of I, at least if I is Gorenstein and I satisfies I is Gorenstein and I satisfies I is I is Gorenstein and I satisfies I is I

2. The canonical module

We begin by computing the graded canonical module of certain extended Rees rings.

Proposition 2.1. Let R be a Noetherian local ring, I an R-ideal with $\operatorname{ht} I > 0$, and J a reduction of I with $r = r_J(I)$. Write $A = R[Jt, t^{-1}] \subset B = R[It, t^{-1}]$. Assume that A is Cohen-Macaulay with graded canonical module $\omega_A \cong (1, It)^s A(a)$ for some integers s and a. Then for every integer $n \geq \max\{r - s, 0\}$,

$$\omega_B \cong (A:_{R[t,t^{-1}]} I^n)(a+n).$$

Proof. First notice that R is Cohen-Macaulay. We write $K = \operatorname{Quot}(R)$ and make the identification $\omega_A = (1, It)^s A t^{-a} \subset R[t, t^{-1}]$. For $L = (1, (1, It)^s I^n t^n) A \subset R[t, t^{-1}]$ consider the exact sequence of A-modules,

$$0 \to L \longrightarrow B \longrightarrow C \to 0$$
.

Since $s + n \ge r_J(I)$ it follows that C is concentrated in finitely many degrees, and hence has grade ≥ 2 . Thus, dualizing the above exact sequence into ω_A we obtain

$$\omega_{B} \cong \operatorname{Hom}_{A}(B, \omega_{A}) \cong \operatorname{Hom}_{A}(L, \omega_{A})
\cong \omega_{A} :_{K(t)} L
= \omega_{A} \cap (\omega_{A} :_{K(t)} (1, It)^{s} I^{n} t^{n} A)
= \omega_{A} \cap (\omega_{A} :_{K(t)} \omega_{A} I^{n} t^{a+n})
= \omega_{A} \cap ((\omega_{A} :_{K(t)} \omega_{A}) :_{K(t)} I^{n} t^{a+n})
= \omega_{A} \cap (A :_{K(t)} I^{n} t^{a+n})
= A :_{\omega_{A}} I^{n} t^{a+n}.$$

Finally, for every integer i one has

$$[A:_{R[t,t^{-1}]} I^n t^{a+n}]_i = (J^{i+a+n}:_R I^n) t^i$$

$$\subset (J^{i+a+n}:_R J^n) t^i$$

$$= J^{i+a} t^i,$$

where the last equality holds because $gr_J(R)$ is Cohen-Macaulay, ht J > 0, and $n \ge 0$. However,

$$J^{i+a}t^i=[At^{-a}]_i\subset [\omega_A]_i.$$

Therefore $A:_{R[t,t^{-1}]}I^nt^{a+n}\subset \omega_A\subset R[t,t^{-1}],$ or equivalently,

$$A:_{\omega_A} I^n t^{a+n} = A:_{R[t,t^{-1}]} I^n t^{a+n}.$$

Thus by the above,

$$\omega_B \cong (A:_{R[t,t^{-1}]} I^n)(a+n)$$

as claimed.

Remark 2.2.

- 1. In Proposition 2.1, the condition that A is Cohen-Macaulay can be replaced by the weaker assumption that A satisfies S_2 and R is universally catenary.
- 2. Let R be a local Gorenstein ring with infinite residue field, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and assume that I satisfies G_{ℓ} and depth $R/I^{j} \ge \dim R/I j + 1$ for $1 \le j \le \ell g$. According to [21, the proof of 2.1], every minimal reduction I of I satisfies the assumptions of Proposition 2.1 with I with I and I and I are I and I are I and I are I are I and I are I are I and I are I and I are I and I are I and I are I are I and I are I and I are I and I are I are I and I are I are I and I are I are I and I are I and I are I and I are I and I are I are I and I are I and I are I and I are I and I are I are

$$\omega_B = (A :_{R[t,t^{-1}]} I^n) t^{g-n-1}$$

is a graded canonical module of B.

In particular, for every i and every $n \ge \max\{r_I(I) - \ell + g, 0\}$,

$$[\omega_B]_{i+g-1}t^{-i-g+1} = J^{i+n} :_R I^n.$$

Corollary 2.3. Let R be a local Gorenstein ring with infinite residue field, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and assume that I satisfies G_{ℓ} and $\operatorname{depth} R/I^{j} \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. Then for every fixed integer i, the ideal

$$J^{i+n}:_R I^n$$

is independent of J and n, as long as J is a minimal reduction of I and $n \ge \max\{r_J(I) - \ell + g, 0\}$.

Proof. By Remark 2.2(2) one has $\omega_B = (A :_{R[t,t^{-1}]} I^n) t^{g-n-1}$ and it suffices to show that this submodule of $R[t,t^{-1}]$ is uniquely determined by B. Notice that ω_B is a graded canonical module of B and a graded submodule of $R[t,t^{-1}]$, and that $[\omega_B]_i = Rt^i$ for $i \ll 0$. Now the first two properties determine this submodule up to multiplication with a unit u in Quot(R), and the last property then forces u to be a unit in R.

The next result has been shown in [13, 3.2 and 3.4] under the assumption that R[It] is Cohen-Macaulay.

Remark 2.4. In addition to the assumptions of Corollary 2.3 assume that $gr_I(R)$ is Cohen-Macaulay. Then for every $i \ge 0$ and every $n \ge \max\{r_I(I) - \ell + g, 0\}$,

$$J^{i+n}: I^n = J^i(J^n: I^n) = I^i(J^n: I^n).$$

In particular J^n : I^n is the coefficient ideal of I with respect to J in the sense of [1, 2.1].

Indeed, the claim about the coefficient ideal \mathfrak{a} follows from the second asserted equality because it gives $J^n \colon I^n \subset \mathfrak{a}$, whereas the reverse inclusion is always true. To prove the equalities we may

assume that R is complete. By Remark 2.2(2) it suffices to show that ω_B as a graded A-module is generated in degrees $\leq g-1$. Let T be a regular local ring mapping onto R, set $c=\dim T-\dim R$, and consider the polynomial ring $S=T[X_1,\ldots,X_\ell]$. Mapping S homogeneously onto $\operatorname{gr}_J(R)$, the ring $\operatorname{gr}_I(R)$ becomes a finite graded S-module whose homogeneous minimal free resolution F_{\bullet} has length $c+\ell$. As $\operatorname{pd}_T(I^j/I^{j+1}) \leq c+g+j < c+\ell$ for every $0 \leq j \leq \ell-g-1$, it follows that $F_{c+\ell}$ is generated in degrees $\geq \ell-g$. Therefore the canonical module of $\operatorname{gr}_I(R)$ is generated in degrees $\leq g$ as a graded module over S, hence over $\operatorname{gr}_J(R)$. Thus indeed ω_B is generated in degrees $\leq g-1$ as a graded A-module.

3. The case of analytic spread one

The next lemma is a minor modification of [3, 2.2], which in turn was based on [11, the proof of 3.8]. To simplify notation we write x: y instead of (x): $_R(y)$ for elements x, y of a ring R.

Lemma 3.1. Let (R, \mathfrak{m}, k) be a Noetherian local ring, let K be an R-ideal, and let x, y be elements of R such that $yK \subset xK$ and x is a non-zerodivisor. Let $c > \dim_k(K \colon \mathfrak{m}/K)$ and let u_1, \ldots, u_c be units in R that are not all congruent modulo \mathfrak{m} . Then for every $j \geq 0$,

$$x(K \colon \mathfrak{m}) \cap \bigcap_{i=1}^{c} (x + u_i y)(K \colon \mathfrak{m}) \subset x(x^j \colon y^j) \cap \bigcap_{i=1}^{c} (x + u_i y)(x^j \colon y^j).$$

Proof. Let α be an element of the intersection on the left hand side. Write

$$\alpha = xs = (x + u_1 y)s_1 = \dots = (x + u_c y)s_c$$

where s and all s_i belong to K: m. We are going to prove by induction on j that s and all s_i are in $x^j : y^j$.

The assertion being trivial for j = 0 we may assume that j > 0. We first show that $s_i \in x^j : y^j$ for $1 \le i \le c$. By our induction hypothesis,

$$s_i(x + u_i y) = \alpha \in x(x^{j-1} : y^{j-1}).$$

Hence

$$s_{i} \in (x(x^{j-1} : y^{j-1})) : (x + u_{i}y)$$

$$\subset (x^{j} : y^{j-1}) : (x + u_{i}y)$$

$$= x^{j} : (x + u_{i}y)y^{j-1}$$

$$= x^{j} : (xy^{j-1} + u_{i}y^{j}).$$

Since $s_i \in x^{j-1}$: $y^{j-1} = x^j$: xy^{j-1} by induction hypothesis, it follows that $s_i \in x^j$: $u_i y^j$. Therefore $s_i \in x^j$: y^j , as asserted.

Next we prove that $s \in x^j : y^j$. As $yK \subset xK$ one has $K \subset x^j : y^j$. If $s_i \in K$ for some i, then

$$xs = (x + u_i y)s_i \in xK + yK = xK \subset x(x^j : y^j).$$

Thus $s \in x^j : y^j$ since x is a non-zerodivisor. So we may assume that $s_i \notin K$ for $1 \le i \le c$. Let '-' denote images in $\overline{R} = R/K$. Now $\overline{s}_1, \ldots, \overline{s}_c$, or equivalently, $\overline{u}_1 \overline{s}_1, \ldots, \overline{u}_c \overline{s}_c$ are c nonzero elements of the k-vector space $\overline{K} : m$. They are linearly dependent over k, and after shrinking c if needed we

may assume that c is minimal with respect to this property and that $\overline{u}_1, \ldots, \overline{u}_c$ are still not all equal. Obviously $c \ge 2$. Now there exist units $\lambda_1, \ldots, \lambda_c$ in R so that $\sum_{i=1}^c \overline{\lambda}_i \overline{u}_i \overline{s}_i = \overline{0}$. Notice that $\sum_{i=1}^c \overline{\lambda}_i \overline{s}_i \ne \overline{0}$

because $\overline{u}_1, \dots, \overline{u}_c$ are not all equal. Since $\sum_{i=1}^c \lambda_i u_i s_i \in K$, the element $\sum_{i=1}^c y \lambda_i u_i s_i$ belongs to $yK \subset xK$,

hence can be written as $x\xi$ for some $\xi \in K$. Set $\lambda = \sum_{i=1}^{c} \lambda_i$ and multiply both sides by α . Rewriting α by means of the above equations, cancelling x, and using the containments $s_i \in x^j$: y^j , we obtain

$$\lambda s = \sum_{i=1}^{c} \lambda_i s_i + \xi \in (x^j : y^j) + K = x^j : y^j.$$

If $\lambda \in \mathfrak{m}$ then $\lambda s \in K$ since $s \in K$: \mathfrak{m} , and we conclude that $\overline{0} = \sum_{i=1}^{c} \overline{\lambda}_{i} \overline{s}_{i}$, which is impossible. Thus λ is a unit, and the desired inclusion $s \in x^{j}$: y^{j} follows.

Lemma 3.2. Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with $\ell(I) = \operatorname{ht} I = 1$ and r = r(I), let J and H be minimal reductions of I, and let $n \ge r$ and i be integers.

- (a) $H^{i}(J^{n}:I^{n})=I^{i}(J^{n}:I^{n})$, and this ideal is independent of J, H and n.
- (b) $I(J^n:I^n) \subset \operatorname{core}(I)$.

Proof. To prove (a) write $K = \operatorname{Quot}(R)$ and consider $S = \bigcup_{j \geq 0} (I^j :_K I^j)$, the blowup ring of I. Let x be a generator of J and notice that $r_J(I) = r$ (see, e.g., [9, 2.1]). One has $S = I^n :_K I^n \subset I^n :_K J^n = I^n \frac{1}{x^n} \subset R[\frac{I}{x}] \subset S$, in particular $I^n \frac{1}{x^n} = S$ (see also [15, 1.1 and its proof]). Hence $J^n :_R I^n = J^n :_K I^n = R :_K I^n \frac{1}{x^n} = R :_K S$ is independent of J and n. Furthermore IS is principal, hence IS = HS. As $J^n :_R I^n = R :_K S$ is an S-ideal we conclude that

$$H^{i}(J^{n}:_{R}I^{n}) = H^{i}S(J^{n}:_{R}I^{n})$$

= $I^{i}S(J^{n}:_{R}I^{n})$
= $I^{i}(J^{n}:_{R}I^{n})$.

This proves (a). Part (b) follows because (a) implies that $I(J^n :_R I^n) \subset H$ for every choice of H. \square

We are now ready to prove our formula for the core of equimultiple height one ideals. We first remark that in this case the core is the intersection of a specific (finite) number of general principal reductions. Here we denote the Hilbert-Samuel multiplicity by e(-) and length by $\lambda(-)$.

Remark 3.3. Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with $\ell(I) = \operatorname{ht} I = 1$, and write $t = \max\{e(I_{\mathfrak{p}}) | \mathfrak{p} \in \operatorname{Min}(I)\}$. Then $\operatorname{core}(I)$ is the intersection of t general principal ideals in I.

Indeed, [2, 4.9 and 4.5] gives that $\operatorname{core}(I)$ is the intersection of $\max\{\operatorname{type}(R_{\mathfrak{p}}/\operatorname{core}(I_{\mathfrak{p}})) | \mathfrak{p} \in \operatorname{Min}(I)\}$ general principal ideals in I. On the other hand let $\mathfrak{p} \in \operatorname{Min}(I)$ and let J be a minimal reduction of $I_{\mathfrak{p}}$. Write $\operatorname{core}(I_{\mathfrak{p}}) = JL$ for some $R_{\mathfrak{p}}$ -ideal L. As $\operatorname{core}(I_{\mathfrak{p}}) : \mathfrak{p}_{\mathfrak{p}} = JL : \mathfrak{p}_{\mathfrak{p}} \subset JL : J = L$, it follows that $\operatorname{type}(R_{\mathfrak{p}}/\operatorname{core}(I_{\mathfrak{p}})) \leq \lambda(L/JL) = \lambda(R_{\mathfrak{p}}/J) = e(I_{\mathfrak{p}})$.

Theorem 3.4. Let R be a local Cohen-Macaulay ring with infinite residue field k, let I be an R-ideal with $\ell(I) = \operatorname{ht} I = 1$ and r = r(I), and let J be a minimal reduction of I. Let $(y_1), \ldots, (y_t)$ be minimal reductions of I so that $\operatorname{core}(I) = (y_1) \cap \cdots \cap (y_t)$ and write $s = \max\{r((J, y_i)) | 1 \le i \le t\}$.

(a)
$$\operatorname{core}(I) = J^{n+1} : \sum_{y \in I} (J, y)^n = J(J^n : \sum_{y \in I} (J, y)^n)$$

$$= J^{n+1} : \sum_{i=1}^t (J, y_i)^n = J(J^n : \sum_{i=1}^t (J, y_i)^n)$$
for every $n > s$.

(b) If $\operatorname{char} k = 0$ or $\operatorname{char} k > r$, then

$$core(I) = J^{n+1} : I^n = J(J^n : I^n)$$

for every $n \ge r$.

Proof. First notice that the second and fourth equality in (a) and the second equality in (b) are obvious because J is generated by a single regular element.

We now prove part (a). Since (J, y) is a reduction of I and (y_i) is a reduction of (J, y_i) , it follows that

$$\operatorname{core}(I) \subset \bigcap_{y \in I} \operatorname{core}((J, y)) \subset \bigcap_{i=1}^{t} \operatorname{core}((J, y_i))$$

$$\subset \bigcap_{i=1}^{t} (y_i) = \operatorname{core}(I).$$

Therefore

$$\operatorname{core}(I) = \bigcap_{y \in I} \operatorname{core}((J, y)) = \bigcap_{i=1}^{t} \operatorname{core}((J, y_i)).$$

On the other hand,

$$J^{n+1}: \sum_{y \in I} (J, y)^n \subset J^{n+1}: \sum_{i=1}^t (J, y_i)^n \\ \| \\ \bigcap_{y \in I} (J^{n+1}: (J, y)^n) \subset \bigcap_{i=1}^t (J^{n+1}: (J, y_i)^n) \subset \bigcap_{i=1}^t \operatorname{core}((J, y_i)),$$

where the last containment follows from Lemma 3.2(*b*), applied to the ideals (J, y_i) . Thus it suffices to prove that $core((J, y)) \subset J^{n+1} : (J, y)^n$.

To this end we may assume that $n \ge r((J,y))$, because $J^{n+1}: (J,y)^n$ form a decreasing sequence of ideals. Furthermore for every associated prime $\mathfrak p$ of J, $\dim R_{\mathfrak p} = 1$ and $(\operatorname{core}((J,y)))_{\mathfrak p} = \operatorname{core}((J,y)_{\mathfrak p})$ by [2,4.8]. Thus after localizing at $\mathfrak p$ we may suppose that $\dim R = 1$. Write $\mathfrak m$ for the maximal ideal of R, J = (x) and $K = J^n: (J,y)^n$. Lemma 3.2(a), applied to the ideal (J,y), shows that $yK \subset xK$. We use the notation of Lemma 3.1, assuming in addition that $z_i = x + u_i y$ generate minimal reductions of (J,y). Now $z_iK = xK$ by Lemma 3.2(a), and hence

$$(z_i) \cap (xK : \mathfrak{m}) = (z_i) \cap (z_iK : \mathfrak{m}) = z_i((z_iK : \mathfrak{m}) : z_i)$$

= $z_i(z_iK : z_i\mathfrak{m}) = z_i(K : \mathfrak{m}).$

Likewise

$$(x) \cap (xK : \mathfrak{m}) = x(K : \mathfrak{m}).$$

Using these facts and Lemma 3.1 we deduce

$$(x) \cap (z_1) \cap \ldots \cap (z_c) \cap (xK : \mathfrak{m}) = x(K : \mathfrak{m}) \cap z_1(K : \mathfrak{m}) \cap \ldots \cap z_c(K : \mathfrak{m})$$
$$\subset x(\bigcap_{j=1}^n (x^j : y^j)) = x(x^n : (x, y)^n) = xK.$$

As xK is an m-primary ideal it follows that $(x) \cap (z_1) \cap ... \cap (z_c) \subset xK$, hence $core((J,y)) \subset (x) \cap (z_1) \cap ... \cap (z_c) \subset J^{n+1} : (J,y)^n$. This completes the proof of (a).

To prove part (b) notice that our assumption on the characteristic gives $I^r = \sum_{y \in I} (J, y)^r$. Since $r = r_J(I)$ (see, e.g., [9, 2.1]) we obtain $I^j = \sum_{y \in I} (J, y)^j$ for $j \gg 0$. Thus by part (a), core $(I) = J^{j+1}$: I^j . However $J^{j+1}: I^j = J^{n+1}: I^n$ for every $n \ge r$.

4. The proof of the main Theorem

The next lemma, though elementary, plays an important role in the proof of Lemma 4.2. Its use was inspired to us by K. Smith.

Lemma 4.1. Let R be a ring and let x_1, \ldots, x_n be elements in R such that x_i, x_j form a regular sequence for all $1 \le i < j \le n$. Then $(x_1) \cap \ldots \cap (x_n) = (x_1 \cdot \ldots \cdot x_n)$.

Lemma 4.2. Let R be a local Cohen-Macaulay ring with infinite residue field and assume that R has a canonical module. Let J be an R-ideal with $\ell = \mu(J) > 0$ satisfying G_{∞} and sliding depth, and write

$$\mathcal{A} = \mathcal{A}(J) = \{ \mathfrak{a} \mid \mathfrak{a} : J \text{ is a geometric } (\ell - 1) \text{-residual intersection and } \mu(J/\mathfrak{a}) = 1 \}.$$

Let t be a positive integer and let H be an R-ideal satisfying $ht(J, J^t : H) \ge \ell$. Then

$$H\cap \bigcap_{\mathfrak{a}\in\mathcal{A}}(\mathfrak{a},J^t)\subset J^t.$$

Proof. We induct on ℓ . If $\ell = 1$ then $\mathcal{A} = \{0\}$ and the assertion is clear. Hence we may assume $\ell \geq 2$. Let $b \in H$ and suppose that $b \in J^{j-1} \setminus J^j$ for some j with $1 \leq j \leq t$. We are going to prove that there exists an ideal $\mathfrak{a} \in \mathcal{A}$ with $b \notin (\mathfrak{a}, J^t)$. For this we may assume that $b \in J$.

Let '-' denote images in $\overline{R}=R/0$: J^{∞} . Notice that $J\cap(0:J^{\infty})=0$ since J satisfies G_1 . Thus the canonical epimorphism $R\twoheadrightarrow \overline{R}$ induces isomorphisms $J^n\cong \overline{J}^n$ for every $n\geq 1$. Therefore $\overline{b}\in \overline{J}^{j-1}\setminus \overline{J}^j$, $\mu(\overline{J})=\mu(J)=\ell$, and every ideal in $\mathcal{A}(\overline{J})$ is of the form $\overline{\mathfrak{a}}$ for some $\mathfrak{a}\in \mathcal{A}(J)$. One trivially has that $\operatorname{ht}(\overline{J},\overline{J}^i:\overline{H})\geq \ell$ and \overline{J} satisfies G_{∞} . Finally, \overline{R} is Cohen-Macaulay and \overline{J} has the sliding depth property by [7,3.6]. Thus we may replace $J\subset R$ by $\overline{J}\subset \overline{R}$ to assume that $\operatorname{ht} J>0$.

Write $G = \operatorname{gr}_J(R)$, $G_+ = \bigoplus_{i>0} G_i$, and $b^* = b + J^j \in [G]_{j-1}$. By [6, 9.1], J is of linear type and G is Cohen-Macaulay. Also notice that $b^* \neq 0$. Thus $b^* \notin Q$ for some primary component Q of 0 in G. Write $P = \sqrt{Q}$ and let \mathfrak{p} be the preimage of P in R. We claim that

$$\operatorname{ht}(G_+, Q)/Q \ge 2. \tag{4.1}$$

By Cohen-Macaulayness P is a minimal prime of G, therefore $\ell(J_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$. If $\dim R_{\mathfrak{p}} < \ell$ then $H_{\mathfrak{p}} \subset J_{\mathfrak{p}}^{t}$. Since $b \in H$, it would follow that $\frac{b^{*}}{1} = \frac{0}{1}$ in $G_{\mathfrak{p}}$, hence $b^{*} \in Q$, contradicting the choice of Q. Therefore $\dim R_{\mathfrak{p}} \geq \ell$, which gives $\ell(J_{\mathfrak{p}}) = \dim R_{\mathfrak{p}} \geq \ell \geq 2$. On the other hand as G/P is a positively graded domain, $(G_{+}, P)/P$ is a prime ideal and we obtain $\operatorname{ht}(G_{+}, P)/P = \operatorname{ht}(G_{+}, P)_{\mathfrak{p}}/P_{\mathfrak{p}}$. Since J is of linear type, $G \otimes_{R} R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}$ is a domain as well. Hence $P_{\mathfrak{p}} = \mathfrak{p} G_{\mathfrak{p}}$ because $P_{\mathfrak{p}}$ is a minimal prime ideal. Now we conclude that

$$\begin{aligned} \operatorname{ht}(G_+,Q)/Q &= \operatorname{ht}(G_+,P)/P \\ &= \operatorname{ht}(G_+,P)_{\mathfrak{p}}/P_{\mathfrak{p}} \\ &= \operatorname{ht}(G_+,\mathfrak{p}G)_{\mathfrak{p}}/\mathfrak{p}G_{\mathfrak{p}} \\ &= \ell(J_{\mathfrak{p}}) \geq 2. \end{aligned}$$

This completes the proof of (4.1).

Write M for the homogeneous maximal ideal of G, $A = (G/Q)_M$, $N = ((G_+,Q)/Q)_M$, and $B = \operatorname{End}_A(\omega_A)$ for the S_2 -ification of A. Notice that $A \hookrightarrow B$ since A is unmixed. Furthermore $\operatorname{ht} NB = \operatorname{ht} N \geq 2$, where the equality follows from [8, 3.5(b)] and the inequality is implied by (4.1). Therefore grade $NB \geq 2$. Since $0 \neq b^*A \subset A \subset B$ and $N \subset \operatorname{Rad}(B)$, by Krull's intersection theorem there exists an integer n so that $b^*A \not\subset (NB)^n$. Let x_1, \ldots, x_n be n general elements of J and write $x_i^* = x_i + J^2 \in [G]_1$. As grade $NB \geq 2$, Lemma 4.1 implies that

$$x_1^*A \cap \dots \cap x_n^*A \subset x_1^*B \cap \dots \cap x_n^*B$$
$$= x_1^* \cdot \dots \cdot x_n^*B$$
$$\subset (NB)^n.$$

Therefore $b^*A \not\subset x_i^*A$ for some i. Write $x = x_i$ and let '-' denote images in $\overline{R} = R/(x)$. Notice that $b^* \not\in x^*G$. By the general choice of x and since $\operatorname{ht} J > 0$, it follows that x is R-regular. Thus the Cohen-Macaulayness of G and the genericity of x imply $\operatorname{gr}_{\overline{J}}(\overline{R}) = G/x^*G$. Therefore $\overline{b} \not\in \overline{J}^j$. Again by the general choice of x, $\mu(\overline{J}) = \ell - 1$ and \overline{J} satisfies G_{∞} . Moreover \overline{J} has the sliding depth property according to [7, 3.5], $\operatorname{ht}(\overline{J}, \overline{J}^i : \overline{H}) \ge \ell - 1$ and every ideal of $\mathcal{A}(\overline{J})$ is of the form $\overline{\mathfrak{a}}$ for some $\mathfrak{a} \in \mathcal{A}(J)$. Now the induction hypothesis shows that $\overline{b} \not\in (\overline{\mathfrak{a}}, \overline{J}^i)$ for some $\mathfrak{a} \in \mathcal{A}(J)$, hence $b \not\in (\mathfrak{a}, J^i)$.

Remark 4.3. Let R be a local Gorenstein ring with infinite residue field and let I be an R-ideal with $g=\operatorname{ht} I$ and $\ell=\ell(I)$. Assume that I satisfies G_ℓ and $\operatorname{depth} R/I^j\geq \dim R/I-j+1$ for $1\leq j\leq \ell-g$. Then any minimal reduction I of I satisfies G_∞ and sliding depth, as required in Lemma 4.2. Furthermore I is of linear type, $\operatorname{gr}_I(R)$ is Cohen-Macaulay, and $\operatorname{ht} I:I\geq \ell$.

Indeed by [20, 2.9(a) and 1.11], ht $J: I \ge \ell$. Hence J satisfies G_∞ . Now according to [20, 2.9(a), 1.12 and 1.8(c)] J has the sliding depth property. Therefore J is of linear type and $\operatorname{gr}_J(R)$ is Cohen-Macaulay by [6, 9.1].

Theorem 4.4. Let R be a local Gorenstein ring with infinite residue field, let I be an R-ideal with $g = \operatorname{ht} I$ and $\ell = \ell(I)$, and let J be a minimal reduction of I. Assume I satisfies G_{ℓ} and $\operatorname{depth} R/I^{j} \geq \operatorname{dim} R/I - j + 1$ for $1 \leq j \leq \ell - g$. Then

$$core(I) \subset J^{n+1} : \sum_{y \in I} (J, y)^n$$

for every $n \ge 0$.

Proof. We may assume that $\ell > 0$ and n > 0. We use the notation of Lemma 4.2 with t = n + 1 and H the intersection of all primary components of J^{n+1} of height $< \ell$. Notice that the assumptions of Lemma 4.2 are satisfied by Remark 4.3. Write $L = \sum_{y \in I} (J, y)^n$.

We first prove

$$core(I) \subset H: L,$$
 (4.2)

or equivalently, $(\operatorname{core}(I))_{\mathfrak{p}} \subset (H : L)_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} with $\dim R_{\mathfrak{p}} < \ell$. Indeed by Remark 4.3, $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ and hence $L_{\mathfrak{p}} = J_{\mathfrak{p}}^n$. Thus $(\operatorname{core}(I))_{\mathfrak{p}} \subset J_{\mathfrak{p}} \subset J_{\mathfrak{p}}^{n+1} : J_{\mathfrak{p}}^n = H_{\mathfrak{p}} : L_{\mathfrak{p}}$, which shows (4.2).

Next let $\mathfrak{a} \in \mathcal{A}$. We prove that

$$\operatorname{core}(I) \subset (J^{n+1}, \mathfrak{a}) : L.$$
 (4.3)

Let '-' denote images in $\overline{R}=R/\mathfrak{a}$: I. According to Remark 4.3, $\operatorname{ht} J\colon I\geq \ell$ and hence $\mathfrak{a}\colon I$ is a geometric $(\ell-1)$ -residual intersection of I. Thus by $[20,\,2.9(a)$ and $1.7(a,c)],\,(\mathfrak{a}\colon I)\cap I=\mathfrak{a},\,\operatorname{ht}\overline{I}\geq 1$ and \overline{R} is Cohen-Macaulay. Furthermore $\ell(\overline{I})\leq 1$. Hence $\ell(\overline{I})=\operatorname{ht}\overline{I}=1$ and \overline{J} is a minimal reduction of \overline{I} . As \overline{J} is generated by a single regular element, $\overline{J}^{j+1}\colon \sum_{\overline{v}\in\overline{I}}(\overline{J},\overline{v})^j$ form a decreasing

sequence of ideals. Thus Theorem 3.4(a) implies that $\operatorname{core}(\overline{I}) \subset \overline{J}^{n+1} : \overline{L}$. On the other hand by [2, 4.5], $\operatorname{core}(\overline{I}) = (\overline{\alpha}_1) \cap \ldots \cap (\overline{\alpha}_t)$ for some integer t and t general principal ideals $(\alpha_1), \ldots, (\alpha_t)$ in I.

Notice that (\mathfrak{a}, α_i) are reductions of I, hence $\operatorname{core}(I) \subset \bigcap_{i=1}^t (\mathfrak{a}, \alpha_i)$. Therefore $\overline{\operatorname{core}(I)} \subset \bigcap_{i=1}^t (\mathfrak{a}, \alpha_i) \subset I$

$$\bigcap_{i=1}^t (\overline{\alpha}_i) = \operatorname{core}(\overline{I}). \text{ As } \operatorname{core}(\overline{I}) \subset \overline{J}^{n+1} : \overline{L} \text{ we obtain }$$

$$core(I) \subset (J^{n+1}, \mathfrak{a} : I) : L$$

$$= (J^{n+1}, (\mathfrak{a} : I) \cap I) : L$$

$$= (J^{n+1}, \mathfrak{a}) : L,$$

which proves (4.3).

Finally, combining (4.2), (4.3) and Lemma 4.2 we deduce

$$\operatorname{core}(I) \subset (H \cap \bigcap_{\mathfrak{a} \in \mathcal{A}} (J^{n+1}, \mathfrak{a})) : L$$
$$\subset J^{n+1} : L = J^{n+1} : \sum_{y \in I} (J, y)^n,$$

as claimed.

We are now ready to assemble the proof of the main theorem.

Theorem 4.5. Let R be a local Gorenstein ring with infinite residue field k, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. Assume I satisfies G_ℓ , depth $R/I^j \ge \dim R/I - j + 1$ for $1 \le j \le \ell - g$, and either $\operatorname{char} k = 0$ or $\operatorname{char} k > r - \ell + g$. Then

$$core(I) = J^{n+1} : I^n$$

for every $n > \max\{r - \ell + g, 0\}$.

Proof. To prove the containment $J^{n+1}: I^n \subset \operatorname{core}(I)$ we show that if K is an arbitrary minimal reduction of I then $J^{n+1}: I^n \subset K$. By Corollary 2.3, $J^{n+1}: I^n = K^{j+1}: I^j$ for $j \gg 0$. However, $K^{j+1}: I^j \subset K^{j+1}: K^j = K$, since g > 0 and $\operatorname{gr}_K(R)$ is Cohen-Macaulay by Remark 4.3. Thus indeed $J^{n+1}: I^n \subset K$.

To show the inclusion $\operatorname{core}(I) \subset J^{n+1} : I^n$, we may assume $n = \max\{r - \ell + g, 0\}$ again by Corollary 2.3. But then $I^n = \sum_{y \in I} (J, y)^n$ according to our assumption on the characteristic, and the asserted containment follows from Theorem 4.4.

In the next corollary we prove that the core of I is integrally closed whenever R[It] is regular in codimension one. Similar results can be found in [11, 3.12], [3, 2.11], [14, 5.5.3], where essentially the normality of the ideal and the Cohen-Macaulayness of the Rees ring were required.

Corollary 4.6. If in addition to the assumptions of Theorem 4.5, R[It] satisfies Serre's condition R_1 , then core(I) is integrally closed.

Proof. Let $A = R[It, t^{-1}] \subset B = R[It, t^{-1}]$, and for $n \ge \max\{r - \ell + g, 0\}$ write $L = \bigoplus_i (J^{n+i+1} : I^n)t^i$. Notice that $L \subset A \subset B$ by the proof of Proposition 2.1, $[L]_0 = \operatorname{core}(I)$ by Theorem 4.5, and $\omega_B(g) \cong L$ by Remark 2.2(2). Therefore L is a B-module satisfying S_2 , and hence an unmixed B-ideal of height 1. As B satisfies R_1 , L is an integrally closed B-ideal, which forces $[L]_0$ to be integrally closed as an R-ideal. □

The next result has been shown in [14, 5.3.1] for equimultiple ideals.

Remark 4.7. If in addition to the assumptions of Theorem 4.5, R is a regular local ring essentially of finite type over a field of characteristic zero and R[It] has only rational singularities, then

$$core(I) = adj(I^g) = I adj(I^{g-1}),$$

where adj denotes adjoint ideals in the sense of [16, 1.1].

Indeed, notice that $core(I) = [\omega_B]_g t^{-g} = I[\omega_B]_{g-1} t^{-g+1}$ by Theorem 4.5 and Remarks 2.2(2) and 2.4, whereas $[\omega_B]_i t^{-i} = adj(I^i)$ for every $i \ge 0$ according to [13, the proof of 3.5].

Remark 4.8. Even without the assumption on the characteristic of the residue field in Theorem 4.5, Theorem 4.4 and the proof of Theorem 4.5 still show that

$$J^{n+1}$$
: $I^n \subset \operatorname{core}(I) \subset J^{n+1}$: $\sum_{y \in I} (J, y)^n$

for $n \ge \max\{r - \ell + g, 0\}$. In particular if $\mu(I) \le \ell + 1$ then

$$core(I) = J^{n+1} : I^n.$$

In general however, the formula of Theorem 4.5 is no longer valid if $\mu(I) > \ell + 1$ and $0 < \text{char } k \le r - \ell + g$. We illustrate this with a class of examples in which the ambient ring is a domain:

Example 4.9. Let k be an infinite field of characteristic p > 0, let q > p be an integer not divisible by p, consider the numerical semigroup ring $R = k[[t^{p^2}, t^{pq}, t^{pq+q}]] \subset k[[t]]$, and let $I = \mathfrak{m}$ be the maximal ideal of R. Now R is a one-dimensional local Gorenstein domain, and one has the *proper*

containment $core(I) \supseteq J^{n+1}$: I^n for any minimal reduction J of I and any $n \ge r(I)$. In fact $core(I) = (t^{p^3}, \mathfrak{m}^{2p-1})$, whereas J^{n+1} : $I^n = \mathfrak{m}^{2p-1}$.

To prove these claims consider the presentation $R \cong k[\![X,Y,Z]\!]/(Y^p-X^q,Z^p-X^qY)$ where X,Y,Z are mapped to $x=t^{p^2},y=t^{pq},z=t^{pq+q}$, respectively. Clearly R is Gorenstein. Comparing multiplicities one sees that $\operatorname{gr}_{\mathfrak{m}}(R)\cong k[X,Y,Z]/(Y^p,Z^p)$. Thus $r(\mathfrak{m})=2(p-1)$. Furthermore the leading form x^* of x in $\operatorname{gr}_{\mathfrak{m}}(R)$ is a regular element, in particular x generates a minimal reduction of \mathfrak{m} .

We apply Theorem 3.4(a) taking (x) as the minimal reduction of I and using the definition of y_i and s as in that theorem. Since $(y^p, z^p) = (x^q, x^q y) \subset (x^p)$ and $\operatorname{char} k = p$, one has $y_i^p \in (x^p)$ for every i, and therefore $s \le p - 1 < r(\mathfrak{m})$. As $p - 1 < \operatorname{char} k$, Theorem 3.4(a) implies

$$core(\mathfrak{m}) = (x^p) : \mathfrak{m}^{p-1} = (x^{n+1}) : x^{n+1-p} \mathfrak{m}^{p-1}.$$

On the other hand, J^{n+1} : $\mathfrak{m}^n = (x^{n+1})$: \mathfrak{m}^n by Lemma 3.2(a), and

$$(x^{n+1})$$
: $\mathfrak{m}^n \subseteq (x^{n+1})$: $x^{n+1-p}\mathfrak{m}^{p-1}$

because *R* is Gorenstein and $n \ge r(\mathfrak{m}) > p-1$. Thus $core(\mathfrak{m}) \supseteq J^{n+1} : \mathfrak{m}^n$.

Now, to compute these ideals let us write '-' for images in $\overline{R} = R/(x^p)$. One has $\operatorname{gr}_{\overline{\mathfrak{m}}}(\overline{R}) \cong \operatorname{gr}_{\mathfrak{m}}(R)/(x^{*p}) \cong k[X,Y,Z]/(X^p,Y^p,Z^p)$ by the regularity of x^* , hence

$$core(\mathfrak{m}) = (x^p) \colon \mathfrak{m}^{p-1} = (x^p, \mathfrak{m}^{2p-1}) = (t^{p^3}, \mathfrak{m}^{2p-1}).$$

Likewise one sees that

$$J^{n+1}$$
: $\mathfrak{m}^n = (x^{n+1})$: $\mathfrak{m}^n = (x^{n+1}, \mathfrak{m}^{2p-1}) = \mathfrak{m}^{2p-1}$.

In the next example we show that the G_{ℓ} condition cannot be removed from Theorem 4.5.

Example 4.10. Let k be an infinite field, write $R = k[X,Y,Z,W]/(X^2 + Y^2 + Z^2,ZW)$, let x,y,z,w denote the images of X,Y,Z,W in R, and consider the R-ideal I = (x,y,z). Notice that R is a local Gorenstein ring, ht I = 1, $\ell(I) = 2$, R/I is Cohen-Macaulay, but I does not satisfy G_2 . Let J = (x,y). The ideal J is a minimal reduction of I with $r_J(I) = 1$. One has $core(I) = I^2 \subsetneq J^2$: I. The same holds if one replaces J by a general minimal reduction of I.

Indeed, the special fiber ring $\operatorname{gr}_I(R) \otimes_R k$ is defined by a single quadric. Hence $\ell(I) = 2$, and $r_K(I) = 1$ for every minimal reduction K of I, which gives $I^2 \subset \operatorname{core}(I)$. On the other hand, (x,y), (x,z) and (y,z) are minimal reductions of I, thus $\operatorname{core}(I) \subset (x,y) \cap (x,z) \cap (y,z) = I^2$. Therefore $\operatorname{core}(I) = I^2$. To conclude notice that $I^2 \subsetneq (I^2, xw, yw) = J^2 \colon I$.

Finally, the formula of Theorem 4.5 does not hold for g = 0 even if $\ell > 0$:

Example 4.11. Let k be an infinite field, let $\Delta_i \in k[X,Y,Z]$ be the maximal minor of the matrix

$$\begin{pmatrix}
X & Y & 0 & Z \\
Y & 0 & Z & X \\
0 & Z & X & Y
\end{pmatrix}$$

obtained by deleting the i^{th} column, set $R = k[X, Y, Z]/(\Delta_1, \Delta_2)$ and define $J = \Delta_3 R$, $I = (\Delta_3, \Delta_4)R$. Then R is a local Gorenstein ring, ht I = 0, $\ell(I) = 1$, I satisfies G_1 , R/I is Cohen-Macaulay, and *J* is a minimal reduction of *I* with $r = r_J(I) = 2$. However, $core(I) \subsetneq J^{n+1}$: I^n for every $n \ge 1 = \max\{r - \ell + g, 0\}$.

Indeed, [21, 5.1] and [17, 3.6] show that J is a minimal reduction of I with r = 2. Writing $\mathfrak{m} = (X,Y,Z)R$ one has $J:I=\mathfrak{m}$. As $r=2=\ell-g+1$, [3, 2.6(2)] then gives $\operatorname{core}(I)=\mathfrak{m}J=\mathfrak{m}I$. On the other hand, a computation shows that $\mathfrak{m}J\subsetneq J^2\colon I=J^3\colon I^2$. The assertion now follows since $J^{n+1}\colon I^n$ form an increasing sequence of ideals for $n\geq r=2$.

Remark 4.12. If in addition to the assumptions of Theorem 4.5, $gr_I(R)$ is Cohen-Macaulay, then

$$core(I) = J(J^n : I^n) = I(J^n : I^n)$$

for every $n \ge \max\{r - \ell + g, 0\}$.

This assertion follows from Theorem 4.5 and Remark 2.4.

In the light of Theorem 4.5 and Remark 2.2(2) the formula of Remark 4.12 is equivalent to saying that the canonical module of $B = R[It, t^{-1}]$ as a graded module over $A = R[Jt, t^{-1}]$ has no homogeneous minimal generators in degree g. In fact one could ask whether the A-module ω_B is generated in degrees $\leq g - 1$, or equivalently, whether

$$J^{i+n}: I^n = J^i(J^n: I^n) = I^i(J^n: I^n)$$

for every $i \ge 0$. This stronger statement still holds if $\operatorname{gr}_I(R)$ is Cohen-Macaulay, according to Remark 2.4. For I an equimultiple ideal it also holds under the weaker condition that the S_2 -ification of B is Cohen-Macaulay (i.e., ω_B is Cohen-Macaulay), or under the assumption that R is a domain essentially of finite type over a field of characteristic zero and $\operatorname{Proj}(R[It])$ is smooth. For the latter case one uses a Kodaira type vanishing theorem due to Cutkosky ([16, A2]).

We finish with an example showing that the equality of Remark 4.12 does not hold in general, even for m-primary ideals of reduction number two in two-dimensional regular local rings. This example was constructed by Heinzer, Johnston and Lantz for a slightly different purpose ([5, 5.4]).

Example 4.13. Let k be an infinite field with char $k \neq 2$, let R = k[X,Y], and $I = (X^7,Y^7,X^3Y^5 + X^5Y^3,X^4Y^6)$. One has $r = r_J(I) = 2$, but $J(J^n : I^n) \subset I(J^n : I^n) \subsetneq J^{n+1} : I^n = \operatorname{core}(I)$ for every minimal reduction J of I and every $n \geq 2 = \max\{r - \ell + g, 0\}$.

Indeed, a computation shows that depth $\operatorname{gr}_I(R) = 1$ and r(I) = 2, see [5, 5.4], and then $r_J(I) = 2$ for every minimal reduction J of I according to [9, 2.1] or [19, 1.2]. Now Remark 2.2(2) shows that $I(J^n:I^n)$ is independent of J and $n \geq 2$, and Theorem 4.5 gives $J^{n+1}:I^n = \operatorname{core}(I)$. Finally, taking $J = (X^7,Y^7)$ one easily computes that $I(J^2:I^2) \subsetneq J^3:I^2$.

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