A proof of the Brouwer Fixed Point Theorem through Differential Topology pathway

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It will be proved on this article one of the most famous fixed points theorems: The Brouwer Fixed Point Theorem, which says says that every continuous mapping of the closed ball into itself admits a fixed point. During the developing of the topology, several different proofs for this theorem were created and here it will be showed the proof due Hirsch. Before we start, let's remember the definition of fixed point.

Definition 1: Let X be a set and $f : X \longrightarrow X$ be a function. A element $x \in X$ is called fixed point of f if $f(x) = x$.

In order to perform the proof, we use tools and notions of differential topology. We will use the following theorem.

Theorem 2: Every compact, connected, one-dimensional smooth[1] manifold with boundary M is diffeomorphic to $[0, 1]$ or $\mathbb{S}^1$.

Proof: Consult the Appendix of [1]. $\square$

As an immediate corollary of this theorem, we obtain the following result.

Corollary 3: The boundary of a compact, one-dimensional smooth manifold with boundary M contains an even number of elements.

Proof: in fact, let $\{M_\lambda\}_{\lambda \in \Gamma}$ be the family of all connected components of M. Since M is a

[1]Smooth here means that M admits a C^∞-compatible maximal atlas

manifolds, each M_λ is an open subset. Since M is a compact manifold and

$$M = \bigcup_{\lambda \in \Gamma} M_\lambda,$$

we conclude that $\{M_\lambda\}_{\lambda \in \Gamma}$ is finite, that is, $M = M_1 \cup \cdots \cup M_n$. As $M_i \cap M_j = \emptyset$ if $i \neq j$, $\partial M = \partial M_1 \cup \cdots \cup \partial M_n$ and each M_i has zero or two elements on its boundary, we conclude that ∂M has an even number of elements. $\square$

Now it will proved a lemma which will be essential for the proof of smooth version of Brouwer Fixed Point Theorem. Although it be a lemma here, this result is itself very interesting.

Lemma 4: Let M be any compact manifold with boundary, then there exists no smooth map $g : M \longrightarrow \partial M$ such that $\partial g := g|_{\partial M} : \partial M \longrightarrow \partial M$ is the identity. That is, there is no ´´retraction" of a manifold M onto its boundary.

Proof: Suppose that such a smooth map $g : M \longrightarrow \partial M$ exists. By Sard's Theorem, we can find $z \in \partial M$, which is a regular value of g. Then $g^{-1}(z)$ is a submanifold of M with boundary. As the codimension of $g^{-1}(z)$ in M is the codimension of $\{z\}$ in ∂M, we conclude that

$$\dim(M) - \dim(g^{-1}(z)) = \dim(M) - 1$$

that is, $\dim(g^{-1}(z)) = 1$. Since g is continuous, $g^{-1}(z)$ is a closed subset of M and, since M is compact, we conclude that $g^{-1}(z)$ is compact. Finally, since $\partial g = 1_{\partial M}$ by hypothesis, we conclude that

$$\partial g^{-1}(z) = g^{-1}(z) \cap \partial M = \{z\}.$$

Hence we found a compact one-dimensional smooth manifold with boundary such that its boundary contains an odd number of points, which contradicts the corollary 3. Thus such smooth map g does not exist. $\square$

Theorem 5: (Smooth Brouwer Fixed Point Theorem). Any smooth mapping

$$f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$$

must have a fixed point.

Proof: Let $f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ be a smooth mapping. Suppose, by contradiction that f does not admit fixed point. We will construct a smooth retraction $g : \mathbb{B}^n \longrightarrow \partial \mathbb{B}^n = \mathbb{S}^{n-1}$. Given $x \in \mathbb{B}^n$, since

$f(x) \neq x$, there is an unique line passing by x and $f(x)$. Let $g(x)$ be the intersection between the unitary sphere $\mathbb{S}^{n-1}$ and the straight segment starting in $f(x)$ and passing by x. Note that, if $x \in \mathbb{S}^{n-1}$, then $g(x) = x$, so g is the identity in $\mathbb{S}^{n-1}$. Since there is no smooth retraction of a manifold in its boundary, it is enough to show that g is smooth, getting a contradiction.

Given $x \in \mathbb{B}^n$, we have that

$$g(x) - f(x) = t(x)(x - f(x)),$$

where $t(x) \in [1, \infty)$ for all $x \in \mathbb{B}^n$. So

$$g(x) = f(x) + t(x)(x - f(x)).$$

If we show that $t : \mathbb{B}^n \longrightarrow \mathbb{R}$ is smooth, then we prove that g is smooth and we are done. Note that $\|g(x)\| = 1$ for all $x \in \mathbb{B}^n$, then

$$\begin{aligned} 1 = \langle g(x), g(x)\rangle &= \langle f(x) + t(x)(x - f(x)), f(x) + t(x)(x - f(x))\rangle \\ &= t(x)^2\|x - f(x)\|^2 + 2t(x)\langle f(x), x - f(x)\rangle + \|f(x)\|^2 \end{aligned}$$

Hence

$$t(x)^2\|x - f(x)\|^2 + 2t(x)\langle f(x), x - f(x)\rangle + (\ \|f(x)\|^2 - 1\ ) = 0$$

Finally, by quadratic formula, we conclude that

$$t(x) = \frac{-\langle f(x), x - f(x)\rangle + \sqrt{\langle f(x), x - f(x)\rangle^2 - \|x - f(x)\|^2(\|f(x)\|^2 - 1)}}{\|x - f(x)\|^2},$$

which is a smooth function. Thus $g : \mathbb{B}^n \longrightarrow \partial\mathbb{B}^n$ is a smooth retraction, which is an absurd, so f admits a fixed point. $\square$

Using the Stone-Wierstrass Theorem, we are able to switch the smoothness condition by the continuity and so proving the Brouwer Fixed Point Theorem.

Theorem 6: (Brouwer Fixed Point Theorem). Any continuous mapping

$$f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$$

must have a fixed point.

Proof: Let $f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ be a continuous mapping. Suppose, by contradiction that f does not admit fixed point. Since $\mathbb{B}^n$ is a compact metric space, there is $c > 0$ such that

$$\|f(x) - x\| \geq c$$

for all $x \in \mathbb{B}^n$. Using again that K is compact, by Stone-Wierstrass Theorem, there is a polynomial mapping $\phi : \mathbb{B}^n \longrightarrow \mathbb{R}^n$ such that

$$\|\phi(x) - f(x)\| < \frac{c}{4} := \delta$$

for all $x \in \mathbb{B}^n$. Note that $\|\phi(x)\| \leq \|\phi(x) - f(x)\| + \|f(x)\| < 1 + \delta$. So define the polynomial mapping

$$p : \mathbb{B}^n \longrightarrow \mathbb{B}^n$$

$$x \longmapsto \tfrac{\phi(x)}{1+\delta}.$$

It is clear that p is well-defined and is smooth, because p is polynomial. Moreover, note that

$$\|f(x) - p(x)\| = \frac{1}{1+\delta}\Big\|(1+\delta)f(x) - \phi(x)\Big\| = \frac{1}{1+\delta}\left(\Big\|f(x) - \phi(x)\Big\| + \Big\|\delta f(x)\Big\|\right) < \frac{2\delta}{1+\delta} < 2\delta = c/2.$$

Finally, note that

$$\|p(x) - x\| = \|p(x) - f(x) + f(x) - x\| \geq \|f(x) - x\| - \|p(x) - f(x)\| > c - c/2 = c/2 > 0.$$

That is, $p : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ is a smooth map from unitary ball into itself which does not admit fixed point, a contradiction. Thus f admits a fixed point. $\square$

Example 7: The Brouwer Fixed Point Theorem fails if we drop the closure condition the unitary ball, that is, if we consider continuous functions on open ball. In fact, consider the function $f : (-1, 1) \longrightarrow (-1, 1)$ such that

$$f(x) = \begin{cases} 1/2, & \text{if } x \in (-1, 0], \\ (1+x)/2, & \text{if } x \in [0, 1). \end{cases}$$

Note that f is continuous, $f((-1, 1)) \subseteq (-1, 1)$, however $f(x) \neq x$ for all $x \in (-1, 1)$.

As we saw, the proof of Brauwer fixed point theorem is nothing trivial. However, on dimension 1, there is a simple prove using the intermediary value theorem.

Theorem 8: (Brouwer Fixed Point Theorem in dimension 1). Any continuous mapping

$$f : [-1, 1] \longrightarrow [-1, 1]$$

must have a fixed point.

Proof: Let $f : [-1, 1] \longrightarrow [-1, 1]$ be a continuous mapping. Consider the function

$$\begin{aligned} h : [-1, 1] &\longrightarrow \mathbb{R} \\ x &\longmapsto x - f(x) \end{aligned}$$

It is immediate that h is continuous. Moreover, we have

$$h(-1) = -1 - f(-1) \leq 0$$
$$h(1) = 1 - f(1) \geq 0.$$

By intermediary value theorem, we conclude that there is $c \in [-1, 1]$ such that $0 = h(c) = c - f(c)$. So

$$f(c) = c.$$

Thus f admits a fixed point. $\square$

References

[1] GUILLEMIN, V; POLLACK, A. Differential topology. American Mathematical Soc., 2010.