Polar and Hessian Curves

Kevin Alves Vasconcellos

November, 2020

1 Introduction

In this text, it will be worked two tools of Algebraic Curves Theory: the Polar curves and the Hessian curves. These tools will allow us to find points in a projective plane curve C with certain properties.

Given a curve $C = \mathrm{V}(F)$ and $P = (p_1 : p_2 : p_3) \in \mathbb{P}^2$, it'll be defined the Polar of C as

$$\mathrm{V}(\Delta_P(F)) := \mathrm{V}\left(p_1 \frac{\partial F}{\partial X_1} + p_2 \frac{\partial F}{\partial X_2} + p_3 \frac{\partial F}{\partial X_3}\right).$$

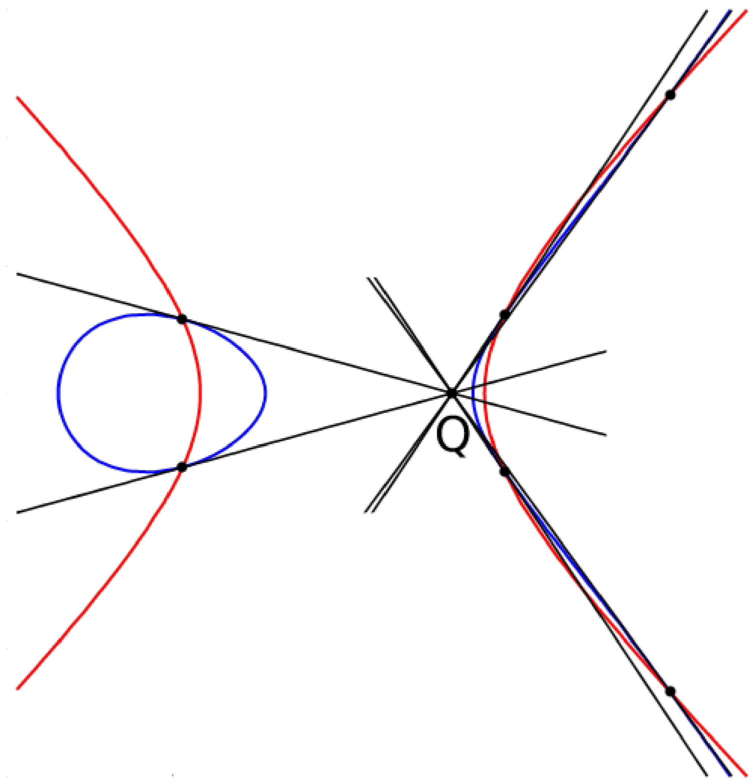
The main result we will prove about this topic is that, given k a field and $F \in k[X_1, X_2, X_3]$ a square-free homogeneous polynomial with $\deg(F) \geq 2$, if $\mathrm{V}(\Delta_P(F)) \neq 0$, then the intersection $\mathrm{V}(\Delta_P(F)) \cap \mathrm{V}(F)$ is exactly constituted by the singularities of $\mathrm{V}(F)$ and the smooth points $Q \in \mathrm{V}(F)$ whose tangent line also passes by P.

Example 1.1. *Consider the eliptic cubic, represented at Figure 1 in blue, given by* $F(X_0, X_1, X_2) = 4X_1X_2^2 - X_0^3 - X_0X_2^2$ *and the point* $Q = (0 : 9 : 0)$*. Note that* $Q \notin \mathrm{V}(F)$ *and calculating the polar curve in* Q*, we obtain the following Polar Curve*

$$\mathrm{V}(\Delta_Q(F)) = \mathrm{V}(4X_1^2 - 2.7X_0^2 + 2X_0X_2 + 0.9X_2^2),$$

which is represented on the Figure 1 in red. Since this a smooth cubic, the points of intersection represented in black are exactly the points of $\mathrm{V}(F)$ *whose tangent tangent line also passes by* Q*.*

Figure 1: Points of intersection of $\mathrm{V}(F)$ and $\mathrm{V}(\Delta_Q(F))$.



The other topic it'll be present in this work is the Hessian Curves. Given a homogeneous polynomial $F \in k[X_0, X_1, X_2]$, it'll be defined the Hessian of F by

$$H_F = \det \left[\frac{\partial^2 F}{\partial X_i \partial X_j} \right]_{i,j=0,1,2}$$

and the Hessian curve of F by $\mathrm{V}(H_F)$. The importance of studies of this curve is if F is square free with $\deg(F) \geq 3$, and $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > d$, then

- $H_F \equiv 0 \pmod{F}$ if and only if $\mathrm{V}(F)$ is union of lines;
- If $H_F \not\equiv 0 \pmod{F}$, then $\mathrm{V}(F) \cap \mathrm{V}(H_F)$ consists of singularities of $\mathrm{V}(F)$ and the flexes of $\mathrm{V}(F)$.

In the section 2, it'll be proved the Study Lemma and it'll be defined the concept of minimal polynomial. In section 3, it'll defined the polar curves, their basic properties and the main theorem about this topic and, finally, in section 4, it'll be present the Hessian curves, their basic properties and the main theorem about this topic.

2 Preliminaries

In this section, it'll proved a simple, however, important Lemma that is the Study Lemma. Also it'll defined the concept of minimal polynomial.

Proposition 2.1 (Study lemma). *Let k be an algebraically closed field and* $\mathrm{V}(F), \mathrm{V}(G)$ *hyperfaces, with* $F, G \in k[X_1, \ldots, X_n]$. *If F is irreducible in* $k[X_1, \ldots, X_n]$ *and* $\mathrm{V}(F) \subseteq \mathrm{V}(G)$, *then F divides G.*

Proof: Since $\mathrm{V}(F) \subseteq \mathrm{V}(G)$, by Hilbert Nullstellensatz, we have that $(G) \subseteq \sqrt{(G)} \subseteq \sqrt{(F)}$. Thus $G \in \sqrt{(F)}$ and so there is $n \in \mathbb{N}$ such that $G^n = HF$. Since F is irreducible, we conclude that F divides G. $\square$

Let $C = \mathrm{V}(F)$ be an algebraic curve in $\mathbb{A}^2$, where $F \in k[X_1, X_2]$. Since $k[X_1, X_2]$ is an UFD, we can decompose F as the product of irreducible polynomials

$$F = F_1^{r_1} \ldots F_n^{r_n}.$$

If C is also given by $\mathrm{V}(G)$ for some other polynomial $G \in k[X_1, X_2]$, then we have by Study Lemma

$$G = \lambda F_1^{l_1} \ldots F_n^{l_n},$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and $l_i \in \mathbb{N} \setminus \{0\}$ for all $1 \leq i \leq n$. Indeed, since $\mathrm{V}(F_i) \subseteq \mathrm{V}(F) = \mathrm{V}(G)$, we conclude that each F_i divides G using the Study Lemma. Thus we have $G = F_1 F_2 \ldots F_n H$, where $H \in k[X_1, X_2]$. On the other hand, if H contained some irreducible polynomial $F' \neq F_i$ for each $i = 1, 2, \ldots, n$, we have

$$\mathrm{V}(F') \subseteq \mathrm{V}(G) = \mathrm{V}(F).$$

So, using Study Lemma again, we would have that F' divides F, which is a contradiction, because $k[X_1, X_2]$ is an UFD. Thus H is product of non-negative powers of $F_1, \ldots, F_n$ up to a unity in $k[X_1, X_2]$ and so G is of of form $G = \lambda F_1^{l_1} \ldots F_n^{l_n}$.

With this observation in mind, we define the *minimal polynomial* of the curve C by

$$\hat{F} = F_1 \ldots F_n.$$

The same definition is applied in projective hyperfaces. In next section, we'll define the first of the two tools worked of this text.

3 Polar curves

In this section, it'll introduced the polar curve of a given plane projective curve C, some elementary results and the main result which is the Theorem 3.3.

Definition 3.1. *Let k be a field and F a homogeneous polynomial in $k[X_0, X_1, X_2]$. Let $C = \mathrm{V}(F) \subseteq \mathbb{P}^2$ and $P = (x_0 : x_1 : x_2)$ be an arbitrary point of $\mathbb{P}^2$. Define*

$$\Delta_P(F) := x_0 \frac{\partial F}{\partial X_0} + x_1 \frac{\partial F}{\partial X_1} + x_2 \frac{\partial F}{\partial X_2}.$$

If $\deg(\Delta_P(F)) \neq 0$, then $\mathrm{V}(\Delta_P(F))$ is called the polar curve of C with respect the pole P.

Now It'll proved some general properties of polars curves.

Proposition 3.2. *Let k be a field, $P = (x_0 : x_1 : x_2) \in \mathbb{P}^2$ and F a homogeneous polynomial in $k[X_0, X_1, X_2]$ with $\deg(F) = d$. Thus*

(i) The polar curve $\mathrm{V}(\Delta_P(F))$ is independent of the choice of coordinates;

(ii) If $\Delta_P(F) \neq 0$, then $\deg(\Delta_P(F)) = d - 1$.

Proof: (i): Consult the page 82 of [1].
(ii): Since F is a homogeneous polynomial of degree d, we have write F as

$$F(X_0, X_1, X_2) = \sum_{i_0+i_1+i_2=d} a_{i_1,i_2,i_3} X_0^{i_0} X_1^{i_1} X_2^{i_2}.$$

Thus, since $\Delta_P(F) \neq 0$, the polar of F with respect with P is defined and $\Delta_P(F)$ is

$$x_0 \sum_{i_0+i_1+i_2=d} i_0 a_{i_1,i_2,i_3} X_0^{i_0-1} X_1^{i_1} X_2^{i_2} + x_1 \sum_{i_0+i_1+i_2=d} i_1 a_{i_1,i_2,i_3} X_0^{i_0} X_1^{i_1-1} X_2^{i_2} +$$
$$x_2 \sum_{i_0+i_1+i_2=d} i_2 a_{i_1,i_2,i_3} X_0^{i_0} X_1^{i_1} X_2^{i_2-1}.$$

Thus it's easy to see that $\Delta_P(F)$ is a homogeneous polynomial of degree $d - 1$. □

The geometrical significance of the polars curves is given by the following Theorem, which is the main result of this section.

Theorem 3.3. *Let k be a field, $P = (x_0 : x_1 : x_2) \in \mathbb{P}^2$ and C a projective curve with polynomial F. Suppose that $\Delta_P(F)$ is defined, that is, $\Delta_P(F) \neq 0$. Then $\mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))$ consists of*

(i) The singularities of F;

(ii) The point of contact of all tangent lines to F that pass through P.

Also, if $\mathrm{char}(k)$ *doesn't divide* $\deg(F)$*, then we have* $P \in \mathrm{V}(\Delta_P(F))$ *if and only if* $P \in \mathrm{V}(F)$*.*

Proof: Let $Q \in \mathrm{V}(F)$. If Q is singularity of F, thus by Jacobian criterion we have that

$$\frac{\partial F}{\partial X_0}(Q) = \frac{\partial F}{\partial X_1}(Q) = \frac{\partial F}{\partial X_2}(Q) = 0.$$

Then it's clear that $Q \in \mathrm{V}(\Delta_P(F))$. Thus all singular points of C are in $\mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))$. Now let Q be a regular point of C whose the tangent line passes through P. Since the equation of tangent line of C at Q is

$$\ell : \ X_0 \frac{\partial F}{\partial X_0}(Q) + X_1 \frac{\partial F}{\partial X_1}(Q) + X_2 \frac{\partial F}{\partial X_2}(Q) = 0.$$

we have $Q \in \mathrm{V}(\Delta_P(F))$ and then $Q \in \mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))$. On the other hand, let $Q \in \mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))$ and let

$$X_0 \frac{\partial F}{\partial X_0}(Q) + X_1 \frac{\partial F}{\partial X_1}(Q) + X_2 \frac{\partial F}{\partial X_2}(Q) = 0.$$

be the tangent line of C at Q. Since $Q \in \mathrm{V}(\Delta_P(F))$, we conclude this line passes through P, proving the first affirmation.

Finally suppose that $\deg(F)$ doesn't divide $\mathrm{char}(k)$, by Euler relation, we have that

$$X_0 \frac{\partial F}{\partial X_0} + X_1 \frac{\partial F}{\partial X_1} + X_2 \frac{\partial F}{\partial X_2} = (\deg(F))F$$

Then

$$F(P) = 0 \quad \text{if and only if} \quad x_0 \frac{\partial F}{\partial X_0}(P) + x_1 \frac{\partial F}{\partial X_1}(P) + x_2 \frac{\partial F}{\partial X_2}(P) = 0$$

that is, $P \in \mathrm{V}(F)$ if and only if $P \in \mathrm{V}(\Delta_P(F))$. □

Corollary 3.4. *Let* k *be a field and* $C = \mathrm{V}(F)$ *be a smooth curve with* $\deg(F) = d > 1$*. Suppose that* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > d$*. Then, for each* $P \in \mathbb{P}^2$*, there are at most* $d(d-1)$ *tangent lines to* F *passing through* P*.*

Proof: Observe that $\mathrm{V}(F)$ and $\Delta_P(F)$ don't have common component. Since C is a smooth curve, all points of $\mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))$ are points of contact of tangent lines to C that pass through P. Thus, by Bézoult Theorem, we have

$$\sum_{Q \in \mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))} I(Q, \mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))) = d(d-1).$$

Then there are at most $d(d-1)$ points tangent points to C passing through P. □

Example 3.5. *Consider the smooth conic* $C = \mathrm{V}(X_0^2 - X_1X_2)$ *in* $\mathbb{P}^2$ *and* $P = (1:0:1) \notin C$. *Calculating the polar with respect* P*, we obtain*

$$\Delta_P(F) = 2X_0 - X_1.$$

Solving the system of equations

$$\begin{cases} X_0^2 - X_1X_2 = 0 \\ 2X_0 - X_1 = 0 \end{cases}$$

we'll find $\mathrm{V}(F) \cap \mathrm{V}(\Delta_P(F))$. *The solution of this system is* $\mathrm{V}(X_0, X_1) \cup \mathrm{V}(X_0 - 2X_2, X_1 - 2X_0)$. *Taking, for example,* $Q = (2:4:1) \in \mathrm{V}(F)$. *The tangent line at this point is*

$$\mathrm{V}(4X_0 - X_1 - 4X_3).$$

Finally observe that $P \in \mathrm{V}(4X_0 - X_1 - 4X_2)$.

4 Hessian curves

In this section we'll define the second main object of this article: the Hessian curves, their elementary properties and the main result about this topic which is the Theorem 4.13. However, before to give the definition of Hessian curves, let's remember the definition of flex points and prove a result about them which will be useful in the prove of Theorem 4.13.

Definition 4.1. *Let* k *be a field. A point* $P \in \mathbb{P}^2$ *is called flex or inflection point of* $C = \mathrm{V}(F)$ *if the following two conditions are satisfied*

(i) P *is a smooth point of* F*;*

(ii) If G *is the tangent line to* C *at* P*, then* $I(P, F \cap G) \geq 3$.

A tangent at an inflection point is called a flex tangent.

A flex point where the flex tangent is not a component of F is called a proper flex. Otherwise, it's called improper flex.

Proposition 4.2. *Let* k *be an algebraically closed field,* F *a minimal homogeneous polynomial in* $k[X_0, X_1, X_2]$ *with* $\deg(F) \geq 3$. *Let* P *be a smooth point of* $C = \mathrm{V}(F)$ *and* G *be the tangent line at* P. *Suppose, after projective change of coordinates, that* $P = (1:0:0)$ *and* $G = \mathrm{V}(X_2)$. *Thus* *(i) P is an improper flex point of C if and only if X_2 is a factor of F.*

(ii) If P is proper flex point of F then, denoting the dehomonization F_* of F with respect X_0, we have F_* is of form

$$f = X_1^{\mu}\phi(X_1) + X_2\psi(X_1, X_2),$$

where ϕ and ψ are units in $\mathcal{O}_{(0,0)}(\mathbb{A}^2)$ and $\mu > 2$.

Proof (i): Suppose that X_2 is a factor of F. In order to prove that P is a improper flex point of C, it's enough to prove $I(P, F \cap G) \geq 3$ and G is a component of F. Since X_2 divides F, it's clear that G is component of F, because, being $F = X_2H$ for some polynomial $H \in k[X_0, X_1, X_2]$, we obtain

$$\mathrm{V}(F) = \mathrm{V}(X_2) \cup \mathrm{V}(H) = G \cup \mathrm{V}(H).$$

Now consider F_* the dehomogenization of F with respect of X_0. Since X_2 divides F, $d \geq 3$ and F is minimal homogeneous polynomial, $\mathfrak{m}_P(F) = \mathfrak{m}_{(0,0)}(F_*) \geq 2$. Finally, since G is a tangent line of C in P, we conclude that $I(P, F \cap G) > \mathfrak{m}_P(F) \geq 2$, that is, $I(P, F \cap G) \geq 3$. Hence P is a improper flex point of F.

Conversely, if P is an improper flex point of C, then we have that G is a component of C, that is, considering the decomposition of $\mathrm{V}(F)$ in irreducible varieties, we have

$$\mathrm{V}(F) = G \cup V_1 \cup \cdots \cup \mathrm{V}_m\,.$$

Thus $G = \mathrm{V}(X_2) \subseteq \mathrm{V}(F)$. Since X_2 is an irreducible polynomial in $k[X_0, X_1, X_2]$, using the Study Lemma, we conclude that X_2 divides F. This proves the first affirmation.

(ii): Suppose P is a proper flex point. By item (i), we have that X_2 doesn't divide F, and so X_2 doesn't divides F_*. Separating F_* in two parts: one part which contains only monomials in X_1 and other part which contains monomials in both variables, we can write F_* as

$$F_*(X_1, X_2) = X_1^{\mu}\phi(X_1) + X_2\psi(X_1, X_2),$$

where $\mu \in \mathbb{N}$, ϕ is a polynomial in X_1 alone with $\phi(0) \neq 0$ and ψ is a polynomial in X_1, X_2 with $\psi(0,0) \neq 0$.

Since this last property is valid, we have that ϕ and ψ are unity in $\mathcal{O}_{(0,0)}(\mathbb{A}^2)$. Also we have that

$$I(P, F \cap G) = \dim_k(\mathcal{O}_{(0,0)}(\mathbb{A}^2)/(F_*, X_2)) = \dim_k k[X_2]_{(X_2)}/(X_2^{\mu}) = \mu.$$

Thus P is proper flex point of C if and only if $\mu > 2$. $\square$

Now we'll remember the Hessian determinant, a tool also very used in differential calculus to determine local maximum or minimum of C^2-functions $f : U \longrightarrow R$, where U is an open in $\mathbb{R}^3$. Given F a homogeneous polynomial, we define the Hessian determinant of F by

$$H_F = \det \left[\frac{\partial^2 F}{\partial X_i \partial X_j} \right]_{i,j=0,1,2} = \det \begin{bmatrix} \frac{\partial^2 F}{\partial X_0 \partial X_0} & \frac{\partial^2 F}{\partial X_0 \partial X_1} & \frac{\partial^2 F}{\partial X_0 \partial X_2} \\ \frac{\partial^2 F}{\partial X_1 \partial X_0} & \frac{\partial^2 F}{\partial X_1 \partial X_1} & \frac{\partial^2 F}{\partial X_1 \partial X_2} \\ \frac{\partial^2 F}{\partial X_3 \partial X_0} & \frac{\partial^2 F}{\partial X_3 \partial X_1} & \frac{\partial^2 F}{\partial X_2 \partial X_2} \end{bmatrix}.$$

Example 4.3. *Consider $F = X_0(X_0^2 + X_1^2 + X_2^2) \in k[X_0, X_1, X_2]$ and $C = \mathrm{V}(F)$. Calculating the Hessian of F, we obtain*

$$H_F = \det \begin{bmatrix} 6X_0 & 2X_1 & 2X_2 \\ 2X_1 & 2X_0 & 0 \\ 2X_2 & 0 & 2X_0 \end{bmatrix} = 8X_0(3x_0^2 - X_1^2 - X_2^2)$$

Thus $C = L \cup C_1$ and $H_F = L \cup C_2$, where $L = \mathrm{V}(X_0)$, $C_1 = \mathrm{V}(X_0^2 + x_1^2 + X_2^2)$ and $C_2 = \mathrm{V}(3X_0 - X_1^2 - X_2^2)$.

The flexes points of a given homogeneous polynomial F can be determined with help of the Hessian of F. Now it'll be defined the Hessian curve which is the other main object of this article.

Definition 4.4. *Let k be a field and $F \in k[X_0, X_1, X_2]$ a homogeneous polynomial. If $H_F \neq 0$, then the hyperface $\mathrm{V}(H_F) \subseteq \mathbb{P}^2$ is said the Hessian curve of F or, simply, the Hessian of F.*

Before we prove the main result about Hessian curves, it'll proved some elementary facts about the Hessian of a given polynomial..

Proposition 4.5. *Let k be a field and $F \in k[X_0, X_1, X_2]$ an homogeneous polynomial with $\deg(F) = d$, thus*

(i) If $H_F \neq 0$, then $\deg(H_F) = 3(d-2)$;

(ii) The Hessian curve of F is independent of change of coordinates.

Proof: (i) In fact, calculating each double partial derivative, we obtain a polynomial homogeneous of degree $d-2$. Note that the sum of polynomials homogeneous with the same degree is homogeneous. Thus, calculating the determinant of Hessian matrix by traditional method, we have that H_F is sum of homogeneous polynomial of degree $3(d-2)$. Finally, since $H_F \neq 0$, we conclude $\deg(H_F) = 3(d-2)$.
(ii) Consult the page 84 of [1]. $\square$

Definition 4.6. *Let k be a field, $P \in \mathbb{P}^2$ and $\mathcal{O}_P = \mathcal{O}_P(\mathbb{P}^2)$ the local ring in P, we call*

$$I(F)_P = \left\{ \frac{\phi}{\psi} \in \mathcal{O}_P \ ; \ \phi \in (F) \right\}$$

the ideal of F in $\mathcal{O}_P$.

This definition motivates the following important definition

Definition 4.7. *Let k be a field and $P \in \mathbb{P}^2$. For curves $C_1 = \mathrm{V}(F_1), \ldots, C_n = \mathrm{V}(F_n)$ in $\mathbb{P}^2$, we set*

$$\mathcal{O}_{F_1 \cap \cdots \cap F_n, P} = \mathcal{O}_P/(I(F_1)_P + \cdots + I(F_n)_P).$$

The pair $(\mathrm{V}(F_1) \cap \cdots \cap \mathrm{V}(F_n), \mathcal{O}_{F_1 \cap \cdots \cap F_n, P})$ is called intersection scheme $F_1 \cap \cdots \cap F_n$ of the curves $C_1 = \mathrm{V}(F_1), \ldots, C_n = \mathrm{V}(F_n)$.

It's easy to prove that $P \in \mathrm{V}(F_1) \cap \cdots \cap \mathrm{V}(F_n)$ if and only if $\mathcal{O}_{F_1 \cap \cdots \cap F_n, P} = 0$.

Proposition 4.8. *Let k be a field and $P = (x_0 : x_1 : x_2) \in \mathbb{P}^2$. The mapping*

$$\begin{aligned} \eta : k_h(\mathbb{P}^2) &\longrightarrow k(X_1, X_2) \\ \tfrac{\phi}{\psi} &\longmapsto \tfrac{\phi(1, X_1, X_2)}{\psi(1, X_1, X_2)} \end{aligned}$$

is a k-isomorphism and η induces the k-isomorphism

$$\overline{\eta} : \mathcal{O}_P \longrightarrow k[X_1, X_2]_{(X_1 - x_1, X_2 - x_2)}.$$

Proof: Consult the Lemma 4.1 of [1]. □

Corollary 4.9. *Let k be a field and $P = (x_0 : x_1 : x_2) \in \mathbb{P}^2$. Given homogeneous polynomials $F_1, \ldots, F_n \in k[X_0, X_1, X_2]$, we have*

$$\mathcal{O}_{F_1 \cap \cdots \cap F_n, P} \cong (k[X_1, X_2]/((F_1)_*, \ldots, (F_n)_*))_{\overline{(X_1 - x_1, X_2 - x_2)}},$$

where $(F_1)_*, \ldots, (F_n)_*$ are dehomogenization of $F_1, \ldots, F_n$ with respect to X_0 respectively.

Proof: Calling $\mathfrak{m}_P = (X_1 - x_1, X_2 - x_2)$, by Proposition 4.8, we have the isomorphism $\hat{\eta} : \mathcal{O}_P \longrightarrow k[X_1, X_2]_{\mathfrak{m}_P}$. This isomorphism will map each $I(F_j)_P$ to the principal ideal $((F_j)_*)$ in $k[X_1, X_2]_{\mathfrak{m}_P}$. Thus, this isomorphism induces the isomorphism

$$\zeta : \mathcal{O}_{F_1 \cap \cdots \cap F_n, P} \longrightarrow k[X_1, X_2]_{\mathfrak{m}_P}/((F_1)_*), \ldots, ((F_n)_*))_{\mathfrak{m}_P} \cong (k[X_1, X_2]/((F_1)_*), \ldots, ((F_n)_*))_{\overline{\mathfrak{m}_P}}$$

$\square$

Before to prove the main result of this section, we'll prove the Lemma 4.10. In order to simplificate the notation, we'll use the following notation: given $F \in k[X_0, X_2, X_2]$, we'll denote

$$F_{X_i} = \frac{\partial F}{\partial X_i} \qquad \text{and} \qquad F_{X_iX_j} = \frac{\partial^2 F}{\partial X_i \partial X_j}.$$

Lemma 4.10. *Let k be a field and $F \in k[X_0, X_1, X_2]$ a homogeneous polynomial with* $\deg(F) = d$. *It's true that*

$$X_0^2 H_F = \det \begin{bmatrix} d(d-1)F & (d-1)F_{X_1} & (d-1)F_{X_2} \\ (d-1)F_{X_1} & F_{X_1X_1} & F_{X_1X_2} \\ (d-1)F_{X_2} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix}.$$

Proof: It's just application of Euler relation with theory of determinants. Firstly note that

$$X_0^2 H_F = X_0 \det \begin{bmatrix} X_0F_{X_0X_0} & X_0F_{X_0X_1} & X_0F_{X_0X_2} \\ F_{X_1X_0} & F_{X_1X_1} & F_{X_1X_2} \\ F_{X_2X_0} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix}.$$

Using the Euler relation in the polynomial F_{X_i}, we obtain for each $0 \leq i \leq 2$

$$(d-1)F_{X_i} = \sum_{j=0}^{2} F_{X_iX_j} X_j.$$

If we add X_1 times the second row to the first row and X_2 times the third row to the first row, by determinant rules and using the relations above, we obtain

$$X_0^2 H_F = X_0 \det \begin{bmatrix} (d-1)F_{X_0} & (d-1)F_{X_1} & (d-1)F_{X_2} \\ F_{X_1X_0} & F_{X_1X_1} & F_{X_1X_2} \\ F_{X_2X_0} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix}.$$

By determinant rule, we have that

$$X_0 \det \begin{bmatrix} (d-1)F_{X_0} & (d-1)F_{X_1} & (d-1)F_{X_2} \\ F_{X_1X_0} & F_{X_1X_1} & F_{X_1X_2} \\ F_{X_2X_0} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix} = \begin{bmatrix} (d-1)X_0F_{X_0} & (d-1)F_{X_1} & (d-1)F_{X_2} \\ X_0F_{X_1X_0} & F_{X_1X_1} & F_{X_1X_2} \\ X_0F_{X_2X_0} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix}.$$

Thus, if we add X_1 times the second column to first column and X_2 times the third column to the first column, using the Euler relation, then we obtain the wished result. $\square$

Corollary 4.11. *Let k be a field and $F \in k[X_0, X_1, X_2]$ a homogeneous polynomial with* $\deg(F) = d$. *Also it's true that.*

$$X_1^2 H_F = \det \begin{bmatrix} F_{X_0X_0} & (d-1)F_{X_0} & F_{X_0X_2} \\ (d-1)F_{X_0} & d(d-1)F & (d-1)F_{X_2} \\ F_{X_2X_0} & (d-1)F_{X_2} & F_{X_2X_2} \end{bmatrix},$$

$$X_2^2 H_F = \det \begin{bmatrix} F_{X_0X_0} & F_{X_0X_1} & (d-1)F_{X_0} \\ F_{X_1X_0} & F_{X_1X_1} & (d-1)F_{X_1} \\ (d-1)F_{X_0} & (d-1)F_{X_1} & d(d-1)F \end{bmatrix}.$$

Proof: Proceed similarly to the proof of Lemma 4.10. □

Corollary 4.12. *Let k be a field and $F \in k[X_0, X_1, X_2]$ a homogeneous polynomial with* $\deg(F) = d$. *If P is a singular point of $C = \mathrm{V}(F)$, then $H_F(P) = 0$. Also, $H_F = 0$ if* $\mathrm{char}(k)$ *divides $d-1$.*

Proof: Let $P = (x_0 : x_1 : x_2)$ a singularity of C. Note that there is $0 \leq i \leq 2$ such that $x_i \neq 0$. Since P is a singularity, by Jacobian Criterion, we have that $x_i^2 H_F(P) = 0$, then $H_F(P) = 0$. If $\mathrm{char}(k)$ divides $d-1$, then $d-1 = 0$ in k, thus, again we have $x_i^2 H_F(P) = 0$, then $H_F = 0$. □

Finally we are ready to enunciate and prove the main result of this section, whose says that if F doesn't divide H_F, then $\mathrm{V}(F) \cap \mathrm{V}(H_F)$ consists of singular points of $C = \mathrm{V}(F)$ and flexes of C.

Theorem 4.13. *Let k be an algebraically closed field and F a homogeneous and minimal polynomial in $k[X_0, X_1, X_2]$ with* $\deg(F) = d \geq 3$. *Suppose* $\mathrm{char}(k) = p$ *and assume that $p = 0$ or $p > d$. Then*

(i) $H_F \equiv 0 \pmod{F}$ if and only if F is union of lines;

(ii) If $H_F \not\equiv 0 \pmod{F}$, then $\mathrm{V}(F) \cap \mathrm{V}(H_F)$ consists of singular points of $C := \mathrm{V}(F)$ and the flexes of C;

(iii) For every regular point of P of F whose tangent line G at P isn't a component of C, we have

$$I(P, F \cap G) = I(P, F \cap H_F) + 2.$$

Proof: By Corollary 4.12, if P is a singular point of $\mathrm{V}(F)$, then $P \in \mathrm{V}(F) \cap \mathrm{V}(H_F)$, hence it'll only worked with smooth points on the following proof.
(i): Let P be a smooth point of C and let G be the tangent to $C = \mathrm{V}(F)$ at P. To determine whether P is a flex point of C and whether $H_F(P) = 0$, under a projective change of coordinate, we can suppose that $P = (1 : 0 : 0)$ and that $G = \mathrm{V}(X_1)$. Let F_* the dehomogenization with respect X_0. By Lemma 4.10, we have that $H_F(P)$ is the value of the following determinant

applied in $(0,0)$

$$\Delta := \det \begin{bmatrix} d(d-1)F_* & (d-1)(F_*)_{X_1} & (d-1)(F_*)_{X_2} \\ (d-1)(F_*)_{X_1} & (F_*)_{X_1X_1} & (F_*)_{X_1X_2} \\ (d-1)(F_*)_{X_2} & (F_*)_{X_2X_1} & (F_*)_{X_2X_2} \end{bmatrix},$$

which is equal to

$$(n(n-1))(F_*)((F_*)_{X_1X_1}(F_*)_{X_2X_2} - (F_*)^2_{X_1X_2}) - (n-1)^2((F_*)^2_{X_1}(F_*)_{X_2X_2} + (F_*)^2_{X_2}(F_*)_{X_1X_1} - \\ 2(F_*)_{X_1}(F_*)_{X_2}(F_*)_{X_1X_2}).$$

In particular, if P is a improper flex point of C, we already saw that X_2 divides F and, so F_*. In this case, since $\deg(F) \geq 3$, we have

$$(F_*)((0,0)) = (F_*)_{X_1}((0,0)) = (F_*)_{X_1X_1}((0,0)) = 0,$$

we conclude from calculation of this determinant that $H_F(P) = 0$. Suppose now that P is a proper flex of C. In this case, we can write F_* as

$$F_* = X_1^{\mu}\phi(X_1) + X_2\psi(X_1, X_2).$$

With μ, ϕ, ψ satisfying the conditions of Proposition 4.2. A calculation of the partial derivatives gives

$$(F_*)_{X_1} = \mu X_1^{\mu-1}\phi + X_1^{\mu}\phi' + X_2\psi_{X_1};$$
$$(F_*)_{X_1X_1} = \mu(\mu-1)X_1^{\mu-2}\phi + 2\mu X_1^{\mu-1}\phi' + X_1^{\mu}\phi'' + X_2\psi_{X_1X_1};$$
$$(F_*)_{X_2} = \psi + X_2\psi_{X_2};$$
$$(F_*)_{X_2X_2} = 2\psi_{X_2} + X_2\psi_{X_2X_2};$$
$$(F_*)_{X_1X_2} = f_{X_2X_1} = \psi_{X_1} + X_2\psi_{X_2X_1}.$$

Now we'll consider the image of Δ in $\mathcal{O}_{F,P} \cong k[X_1, X_2]_{(X_1,X_2)}/(F_*)$. Since ϕ and ψ are units, the congruence $X_2\psi(X_1, X_2) \equiv -X_1^{\mu}\phi(X_1) \pmod{F_*}$ implies that

$$X_2 = -X_1^{\mu}\phi(X_1)(\psi(X_1, X_2))^{-1} \text{ in } \mathcal{O}_{F,P}.$$

Then the maximal ideal of $\mathcal{O}_{F,P}$ is principal and it's generated by the image of X_1 in $\mathcal{O}_{F,P}$ and, so, this ring is a discrete valuation ring. Thus, considering the valuation of $\mathcal{O}_{F,P}$, observe the image of X_2 in $\mathcal{O}_{F,P}$ has value μ, because $\phi(0) \neq 0$ and $\psi(0,0) \neq 0$. We see now that in the above expression for Δ the image of $f_{X_2}^2 f_{X_1X_1}$ has value $\mu - 2$ while the remaing terms have higher order. This will allow to conclude that Δ has order $\mu - 2$.

In particular, it shows that $\Delta \not\equiv 0 \,(\mathrm{mod}\, F_*)$, because the value of F_* is ∞. Thus $H_F \not\equiv 0 \,(\mathrm{mod}\, F)$, provided there exists a regular point $P = (1 : 0 : 0)$ whose tangent is not a component of F. This is the case when F is not union of lines.

On the other hand, if F is union of lines, then we can directly calculate that $H_F \equiv 0 \,(\mathrm{mod}\, F)$. In fact, suppose WLOG that $F = X_0G$, where G product $d-1$ of linear homogeneous polynomials, then the Lemma 4.10 shows that

$$X_0^2 H_F \equiv \left(\frac{d-1}{d-2}\right)^2 X_0^5 H_G \,(\mathrm{mod}\, F). \tag{1}$$

Indeed, we know that

$$X_0^2 H_G = \det \begin{bmatrix} (d-1)(d-2)G & (d-2)G_{X_1} & (d-2)G_{X_2} \\ (d-2)G_{X_1} & G_{X_1X_1} & G_{X_1X_2} \\ (d-2)G_{X_2} & G_{X_2X_1} & G_{X_2X_2} \end{bmatrix}.$$

Since $F = X_0G$ and $F_{X_iX_j} = X_0G_{X_iX_j}$ for all $i, j \in \{1, 2\}$, multiplying each line of X_0H_G by X_0 and using determinant rules, we obtain

$$X_0^5 H_G = \det \begin{bmatrix} (d-1)(d-2)F & (d-2)F_{X_1} & (d-2)F_{X_2} \\ (d-2)F_{X_1} & F_{X_1X_1} & F_{X_1X_2} \\ (d-2)F_{X_2} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix}.$$

Multiplying the first column and the first row by $d - 1$, we have

$$(d-1)^2 X_0^5 H_G = \det \begin{bmatrix} (d-1)^2(d-2)F & (d-1)(d-2)F_{X_1} & (d-1)(d-2)F_{X_2} \\ (d-1)(d-2)F_{X_1} & F_{X_1X_1} & F_{X_1X_2} \\ (d-1)(d-2)F_{X_2} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix}.$$

On the other hand, using again the Lemma 4.10 for F and multiplying the first column and the first row by $d - 2$, we obtain

$$(d-2)^2 X_0^2 H_F = \det \begin{bmatrix} d(d-1)(d-2)^2F & (d-1)(d-2)F_{X_1} & (d-1)(d-2)F_{X_2} \\ (d-1)(d-2)F_{X_1} & F_{X_1X_1} & F_{X_1X_2} \\ (d-1)(d-2)F_{X_2} & F_{X_2X_1} & F_{X_2X_2} \end{bmatrix}.$$

Now it's clear that $X_0^2 H_F - [(d-1)^2/(d-2)^2] X_0^5 H_G$ is divisible by F. By principle of induction, we can assume that $H_G \equiv 0 \text{ mod}(G)$. So, using the equivalence (1), we conclude that

F divides $X_0^2 H_F$. Finally it can be concluded that $H_F \equiv 0 \,(\mathrm{mod}\, F)$. Indeed, if F divides H_F, there is nothing to prove. Otherwise, since F is a minimal polynomial, we necessarily would have that X_0 doesn't divide H_F. However, calculating H_F using that $F = X_0 G$, we obtain

$$H_F = \det \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & X_0 G_{X_1X_1} & X_0 G_{X_1X_2} \\ \bullet & X_0 G_{X_2X_1} & X_0 G_{X_2X_2} \end{bmatrix},$$

which is clearly a multiple of X_0, a contradiction. This prove the affirmation (i).
(iii) Now observe that

$$I(P, F \cap H_F) = \dim_k \left(\frac{\mathcal{O}_{(0,0)}(\mathbb{A}^2)}{(F_*, (H_F)_*)} \right) = \dim_k \left(\frac{k[X_1, X_2]_{(X_1,X_2)}}{(F_*, \Delta)} \right) = \mu - 2, \tag{2}$$

where $\mu = I(P, F \cap G)$ by virtue the Proposition 4.2.
(ii) The formula (2) says that $P \in \mathrm{V}(F) \cap \mathrm{V}(H_F)$ precisely when $\mu - 2 \geq 1$, that is, when $\mu \geq 3$, i.e. when P is a flex point of C. Since $\mathrm{V}(F) \cap \mathrm{V}(H_F)$ contains all singular points of C by Corollary 4.12, we conclude the proof of theorem. □

Now it'll be derived some imediate corollaries from this powerful theorem.

Corollary 4.14. *Under the assumptions of Theorem 4.13, let $C := \mathrm{V}(F)$ be an irreducible curve in $\mathbb{P}^2$ and s be the number of singularities of C. Then C has at most $3d(d-2) - s$ flex points.*

Proof: Since F is an irreducible polynomial and $\deg(F) \geq 3$, C cannot be union of lines. Thus, by Theorem 4.13 (i), we have $H_F \not\equiv 0 \bmod(F)$. Also, by Theorem 4.13 (iii), $I(P, F \cap H_F) < \infty$ for all regular point P of C. In particular, F isn't a divisor of H_F, then, applying the Bézout Theorem, we have

$$\sum_{P \in F \cap H_F} I(P, F \cap H_F) = \deg(F) \cdot \deg(H_F) = 3d(d-2).$$

Thus $\mathrm{V}(C) \cap \mathrm{V}(H_F)$ contains at most $3d(d-2)$ points. Since C has s singularities, we conclude C contains at most $3d(d-2) - s$ flex points. □

Corollary 4.15. *Under the assumptions of Theorem 4.13, let $F \in k[X_0, X_1, X_2]$ and $C = \mathrm{V}(F)$ be a smooth curve. Then C has at least one flex point. Also, if $\{P_1, \ldots, P_n\}$ is the set of all flexes of F, and $\{G_1, \ldots, G_n\}$ the set of the corresponding flex tangents, then*

$$\sum_{i=1}^{n} [I(P_i, F \cap G_i) - 2] = 3d(d-2).$$

Proof: Since all smooth curve in $\mathbb{P}^2$ is irreducible, we can proceed as the Corollary 4.14 to conclude that

$$\sum_{P\in F\cap H_F} I(P, F\cap H_F) = 3d(d-2).$$

In particular, since the C has no singularities and $\deg(F) = d \geq 3$, we conclude that C has at least one flex point. Thus, if $\{P_1, \ldots, P_n\}$ is the set of all flex points of F, then

$$\sum_{k=1}^{n} I(P_k, F\cap H_F) = 3d(d-2).$$

Finally, since $\mathrm{V}(G_i)$ isn't common component of $\mathrm{V}(F)$, because $\mathrm{V}(F)$ is irreducible and $\deg(F) \geq 3 > 1 = \deg(G_i)$ for each $i = 1, 2 \ldots, n$, we have by part (iii) of Theorem 4.13

$$\sum_{i=1}^{n} [I(P_i, F\cap G_i) - 2] = 3d(d-2).$$

□

Now it'll be seen a direct application of this Theorem.

Example 4.16. *Consider $k = \mathbb{C}$ and the folium of Descartes given by $C = \mathrm{V}(F) = \mathrm{V}(X_1^3 + X_2^3 - X_0X_1X_2)$. Determining the Hessian of C, we obtain $H_F = -2(3X_1^2 + 3X_2^2 + X_0X_1X_2)$. Note that $H_F \not\equiv 0 \pmod F$. Thus now we'll determine $\mathrm{V}(F) \cap \mathrm{V}(H_F)$, that is, solve the system of equations*

$$\begin{cases} X_1^3 + X_2^3 - X_0X_1X_2 = 0 & (i) \\ 3X_1^3 + 3X_2^3 + X_0X_1X_2 = 0 & (ii) \end{cases}$$

Let $P = (x_0 : x_1 : x_2)$ be a solution this system and suppose $x_0 \neq 0$. In this case, making $(ii) - 3(i)$, we obtain $4X_0X_1X_2 = 0$, then $X_1X_2 = 0$. Thus $X_1 = 0$ or $X_2 = 0$. In both cases, we conclude that the other variable is also 0, then $P = (1 : 0 : 0)$. On the other hand, if $x_0 = 0$, we obtain that $X_1^3 + X_3^3 = 0$. Thus, assuming $X_2 = -\lambda \in \mathbb{C}$, we have three possibilities for P

$$P = (0 : \lambda : -\lambda) = (0, 1, -1), \quad P = (0 : \zeta\lambda, -\lambda) = (0, \zeta, -1), \quad P = (0 : \lambda\zeta^2 : -\lambda) = (0 : \zeta^2 : -1),$$

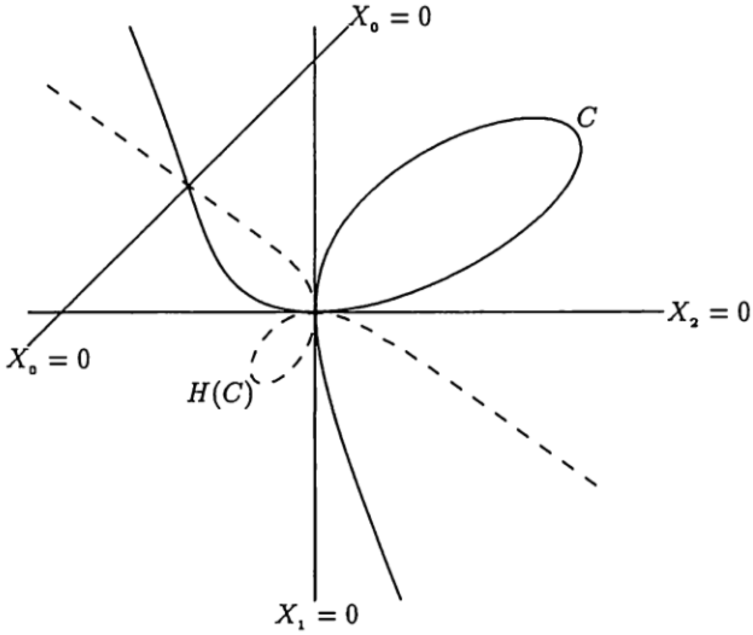
where ζ is a primitive cubic root of unity. Calculating the partial derivatives, we obtain

$$\begin{cases} F_{X_0} = -X_1X_2 \\ F_{X_1} = 3X_1^2 - X_0X_2 \\ F_{X_2} = 3X_2^2 - X_0X_1 \end{cases}$$

So it's easy to see that these four points are smooth, then by Theorem 4.13, these four points are flex points.

On Figure 2, you'll see the representation of C and H_F. and $\mathrm{V}(H_F)$

Figure 2: Folium de Descartes.



References

[1] KUNZ, Ernst; BELSHOFF, Richard G. Introduction to plane algebraic curves. Boston: Birkhäuser, 2005.

[2] FULTON, William. Algebraic curves. An Introduction to Algebraic Geom, p. 114, 2008.

[3] FISCHER, Gerd. Plane algebraic curves. American Mathematical Soc., 2001.