Geometric Hahn-Banach Theorem

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Abstract

The goals of this work are to present a direct proof of Geometric Hahn-Banach Theorem, which can be found at [2], as well as to show, from an example, how can we solve the Geometric Hahn-Banach Theorem problem using the Analytic Hahn-Banach Theorem.

1 Introduction

The history of Hahn-Banach Theorems closely parallels the history of functional analysis. The road was paved in late of nineteenth century and the first precursor of the Hahn-Banach Theorem is found in a 1907 paper by Riesz [4], using essentially a trick that had already been used by Helly in a proof of the so called Hamburger problem. In 1927, Hahn [3] presented a proof of the first version of Hahn-Banach Theorem.

Theorem 1.1 (Hahn-Banach Theorem - First version)**.** *Let E be a linear space over $\mathbb{R}$, p be a norm in E and M be a vector subspace of E. Let f_0 be a continuous linear functional in M. Then there exists a continuous linear functional f in E that extends f_0 such that*

$$\|f_0\| = \|f\|.$$

The second version of the Hahn-Banach Theorem first appears in Banach [5]. In 1932, Banach's classic came out [6]. Banach credits Helly and Hahn for their roles in the first version of Hahn-Banach Theorem, however Hahn does not refer to Helly's contribution.

Theorem 1.2 (Hahn-Banach Theorem - Second version)**.** *Let E be a linear space of $\mathbb{R}$ and p be a sublinear function, that is, a map which satisfies the following two properties*

- $p(x+y) \le p(x) + p(y)$ *for all* x, $y \in E$*;*

- $p(\alpha x) = \alpha p(x)$ *for all* $x \in E$ *and* $\alpha \in \mathbb{R}^+$.

Let M *be a linear subspace of* E *and* f_0 *be a linear functional in* M *for which*

$$f_0(x) \leq p(x) \qquad \text{for all } x \in M.$$

Then there exists a linear functional f *in* E *that extends* f_0 *and for which*

$$f(x) \leq p(x) \qquad \text{for all } x \in E.$$

Hochstadt calls Helly the father of the Hahn-Banach theorem. It is more accurate to call Helly the father of the first version Hahn-Banach Theorem and contribute the second version of the Hahn-Banach Theorem. However, these theorems are known by Hahn-Banach theorems, because Bohnenblust and Sobczyk [7] in 1938 coined the term *Hahn-Banach Theorem.* During the time, several versions of Hahn-Banach Theorem was created, always looking the theorem from a different view point. One way to eye the Hahn-Banach Theorem is interpreting it in a geometric way, pointed by Mazur.

Theorem 1.3 (Mazur)**.** *Let* C *be a convex subset of a* $\mathbb{R}$*-linear space* E *for which* $\operatorname{int}(C) \neq \varnothing$*. Let* V *be a linear variety (that is, a subset of form* $x + M$*, where* M *is a linear subspace of* E*) for which* $\operatorname{int}(C) \neq V = \varnothing$*. Then there is a closed hyperplane* H *of* E *such that*

- $\operatorname{int}(C) \cap H = \varnothing$*;*
- $V \subseteq H$.

There are many variations on this theorem. The goal of this work is to prove the geometric version of Hahn-Banach Theorem which can be found at paper [2] due Fernández and Fernández-Arroyo. This version holds in topological linear spaces (TLS). However, before we state the theorem, let's remember the definition of TLS.

Definition 1.4. *Let* E *be a linear space over* $\mathbb{K}$ *and* τ *be a topology in* E*. We say that* (E, τ) *is a topological linear space (TLS) if the operations*

$$\begin{aligned} S : E \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y \end{aligned} \qquad \text{and} \qquad \begin{aligned} P : \mathbb{K} \times E &\longrightarrow E \\ (\lambda, x) &\longmapsto \lambda x \end{aligned}$$

are continuous.

Theorem 1.5 (Geometric Hahn-Banach Theorem)**.** *Let E be a TLS over $\mathbb{R}$, A be an open and convex subset of E and M be a subspace of E such that $S \cap A = \varnothing$. Then there exists a hyperplane H of E verifying $M \subseteq H$ and $A \cap H = \varnothing$.*

Also in [2], they proved the equivalence between this geometric version of Hahn-Banach Theorem and the second of form of Hahn-Banach Theorem, also known as Analytic Form of Hahn-Banach Theorem.

2 Preliminaries

In this section, we will prove some elementaries results which will be used during the proof of Geometric Hahn-Banach Theorem. The goal is to turn the proof of the this theorem as clean and clear as possible. Firstly we will prove that, given a proper subspace M of a linear space E, always it is possible to find a hyperplane W such that $M \subseteq W$.

Proposition 2.1. *Let E be a linear space over $\mathbb{K}$. Given M a linear subspace of E, there is hyperplane W such that $M \subseteq W$.*

Proof: Indeed, the Zorn's lemma ensures that there is a basis $\{v_\lambda\}_{\lambda \in I}$ for M and that this basis can be extended to a basis $\{v_\lambda\}_{\lambda \in I} \cup \{u_\gamma\}_{\gamma \in J}$ for E. Let $\gamma_0 \in J$. By linearity, define the following linear functional $\phi : E \longrightarrow \mathbb{K}$ such that

$$\phi(x) = \begin{cases} 0 & \text{if } x = v_\lambda \text{ for every } \lambda \in I; \\ 0 & \text{if } x = u_\gamma \text{ for every } \gamma \in J \setminus \{\gamma_0\}; \\ 1 & \text{if } x = u_{\gamma_0}. \end{cases}$$

Note that ϕ is a nonzero linear functional such that $M \subseteq \ker(\phi)$. Thus $W := \ker(\phi)$ is a hyperplane containing M. □

Now we will prove that, given x and y elements of a topological linear space E, then convex combination function is continuous.

Proposition 2.2. *Let E be a topological linear space over $\mathbb{R}$. Given $x, y \in E$, the mapping*

$$f : [0,1] \longrightarrow E$$
$$t \longmapsto tx + (1-t)y$$

is continuous.

Proof: Consider the mapping

$$g_1 : [0,1] \longrightarrow \mathbb{R} \times E$$
$$t \longmapsto (t, x)$$

Note that g_1 is continuous. In fact, let V be an arbitrary open subset of $\mathbb{R} \times E$ and $s \in g_1^{-1}(V)$. Thus $(s, x) \in V$. By definition of product topology, there are open sets $W_1 \subseteq \mathbb{R}$, $W_2 \subseteq E$ such that $(s, x) \in W_1 \times W_2 \subseteq V$, which implies that $s \in W_1 \cap [0,1] \subseteq g_1^{-1}(V)$, which implies that $s \in \operatorname{int}(g_1^{-1}(V))$ and so $g_1^{-1}(V)$ is open. Since V is an arbitrary open of $\mathbb{R} \times E$, we conclude that g_1 is continuous. Defining $h_1 = P \circ g_1$, we conclude

$$h_1 : [0,1] \longrightarrow E$$
$$t \longmapsto tx$$

is continuous. Similarly, defining $\zeta : [0,1] \longrightarrow [0,1]$ such that $\zeta(t) = 1 - t$ and

$$g_2 : [0,1] \longrightarrow \mathbb{R} \times E$$
$$t \longmapsto (t, y)$$

,

we conclude that the mapping

$$h_2 := P \circ g_2 \circ \zeta : \ [0,1] \longrightarrow E$$
$$t \longmapsto (1 - t)y$$

is also continuous. Finally, consider the function

$$\eta : [0,1] \longrightarrow E \times E$$
$$t \longmapsto (h_1(t), h_2(t))$$

Since $h_1, h_2 : [0,1] \longrightarrow E$ are continuous, it is classical fact from general topology that η is continuous, which implies that

$$S \circ \eta : [0,1] \longrightarrow E$$
$$t \longmapsto tx + (1 - t)y$$

is also continuous. $\square$

Now we will prove that, for any collection $\{L_\lambda\}_{\lambda \in I}$ of linear subspaces of E which is totally ordered by inclusion, the set $\bigcup_{\lambda \in I} L_\lambda$ is also a linear subspace of E.

Proposition 2.3. *Let E be a topological vector space over $\mathbb{K}$ and $\{L_\lambda \ ; \ \lambda \in I\}$ be a collection of linear subspaces of E, where $I \neq \varnothing$. If $\{L_\lambda \ ; \ \lambda \in I\}$ is totally ordered by inclusion, then*

$$\bigcup_{\lambda \in I} L_\lambda$$

is a linear subspace of E.

Proof: Indeed, since $I \neq \varnothing$, let $\lambda_0 \in I$, so $0 \in L_{\lambda_0} \subseteq \bigcup_{\lambda \in I} L_\lambda$. Now let v_1, $v_2 \in \bigcup_{\lambda \in I} L_\lambda$ and $\alpha \in \mathbb{K}$. Thus there are $\lambda_1, \lambda_2 \in I$ such that $v_1 \in L_{\lambda_1}$ and $v_2 \in L_{\lambda_2}$. Since $\{L_\lambda \ ; \ \lambda \in I\}$ is totally ordered by inclusion, we can assume without lost of generality that $L_{\lambda_1} \subseteq L_{\lambda_2}$. Thus

$$v_1 + \alpha v_2 \in L_{\lambda_2} \subseteq \bigcup_{\lambda \in I} L_\lambda.$$

Then $v_1 + \alpha v_2 \in \bigcup_{\lambda \in I} L_\lambda$, which implies that $\bigcup_{\lambda \in I} L_\lambda$ is a linear subspace of E. □

Proposition 2.4. *Let E be a topological vector space over $\mathbb{R}$, A be a nonempty subset of E and M be a linear subspace of E such that $M \cap A = \varnothing$. Consider $N := M + \bigcup_{\alpha > 0}(\alpha A)$, then*

(i) If A is open, then N is an open set of E.

(ii) The set $-N := \{x \in E \ ; \ -x \in N\}$ can be written as

$$-N = M + \bigcup_{\alpha < 0} (\alpha A).$$

(iii) $(-N) \cap M = N \cap M = \varnothing$. Moreover, if A is convex, then $N \cap (-N) = \varnothing$.

Proof: (i) : In fact, since E is topological vector space, the mapping

$$\begin{aligned} \phi : E &\longrightarrow E \\ x &\longmapsto \alpha x + v \end{aligned}$$

is an homeomorphism for any $\alpha \neq 0$ and $v \in E$. Thus, since A is open, αA is open for any $\alpha > 0$, which implies that

$$\bigcup_{\alpha > 0} (\alpha A)$$

is open. Now, given $v \in M$, using the homeomorphism ϕ again, we conclude that

$$v + \bigcup_{\alpha > 0} (\alpha A)$$

is open. Finally, note that

$$M + \bigcup_{\alpha > 0} (\alpha A) = \bigcup_{v \in M} \big(v + \bigcup_{\alpha > 0} (\alpha A)\big)$$

is open in E, which proves the part (i) of the proposition.

(ii) : Let $x \in -N$, then $x = -y$, where $y \in N$, thus there are $u \in M$, $\beta > 0$ and $v \in A$ such that $y = u + \beta v$. Thus

$$x = -y = -(u + \beta v) = (-u) + (-\beta)v \in M + (-\beta)A \subseteq M + \bigcup_{\alpha<0}(\alpha A).$$

Conversely, let $x \in M + \bigcup_{\alpha<0}(\alpha A)$, then there are $u \in M$, $\beta < 0$ and $v \in A$ such that $x = u + \beta v$. Then $x = -\big((-u) + (-\beta)v\big)$. Since

$$(-u) + (-\beta)v \in M + \bigcup_{\alpha>0}(\alpha A) = N,$$

we conclude that $x \in -N$.

(iii) : Suppose that $N \cap M \neq \varnothing$, so let $z \in N \cap M$. By definition, there are $m_1, m_2 \in M$, $\beta > 0$ and $v \in A$ such that

$$m_1 = z = m_2 + \beta v$$

Thus $v = \beta^{-1}(m_1 - m_2) \in A \cap M$, which contradicts the fact that $A \cap M = \varnothing$. The case $(-N) \cap M$ is proved similarly.

Now suppose that A is convex and $(-N) \cap N \neq \varnothing$, so there are m_1, $m_2 \in M$, $\beta_1 > 0$, $\beta_2 < 0$, v_1, $v_2 \in A$ such that

$$m_1 + \beta_1 v_1 = m_2 + \beta_2 v_2$$

Thus

$$m_2 - m_1 = \beta_1 v_1 - \beta_2 v_2 = (\beta_1)v_1 + (-\beta_2)v_2.$$

Note that $\beta_1 \neq \beta_2$, since $\beta_2 < 0 < \beta_1$. Multiplying the equation above by $(\beta_1 - \beta_2)^{-1}$, we have that

$$\frac{1}{\beta_1 - \beta_2} \cdot (m_2 - m_1) = \frac{\beta_1}{\beta_1 - \beta_2} \cdot v_1 + \frac{-\beta_2}{\beta_1 - \beta_2} v_2.$$

Then, setting $t := \beta_1(\beta_1 - \beta_2)^{-1}$, we conclude that $t \in [0, 1]$ and

$$tv_1 + (1 - t)v_2 = \frac{1}{\beta_1 - \beta_2} \cdot (m_2 - m_1) \in M.$$

However, since A is a convex subset of E, we have that $tv_1 + (1-t)v_2 \in A \cap M$, which contradicts the hypothesis that $A \cap M = \varnothing$. $\square$

3 Direct proof of the Geometric of Hahn-Banach Theorem

As in proof of the analytic version, the direct proof of Geometric Hahn-Banach Theorem also invokes the Zorn's Lemma as we will see soon.

Theorem 3.1. *Let E be a TLS over $\mathbb{R}$, A be an open and convex subset of E and M be a subspace of E such that $S \cap A = \varnothing$. Then there exists a hyperplane G of E verifying $M \subseteq G$ and $A \cap G = \varnothing$.*

Proof: Indeed, if $A = \varnothing$, the Proposition 2.1 ensures that there is a hyperplane W containing M such that $W \cap A = W \cap \varnothing = \varnothing$. Thus assume that $A \neq \varnothing$ and consider the following family

$$\mathcal{F} = \{L \subseteq E \ ; \ L \text{ linear subspace of } E \text{ containing } M \text{ and } L \cap A = \varnothing\}.$$

Note that $\mathcal{F}$ is nonempty since $M \in \mathcal{F}$. Consider in $\mathcal{F}$ the inclusion order. We will show that $(\mathcal{F}, \subseteq)$ has a maximal element. In fact, given $\{L_\lambda\}_{\lambda \in L}$ an arbitrary chain[1] in $\mathcal{F}$. we have that

$$\bigcup_{\lambda \in L} L_\lambda$$

is an upper bound for $\{L_\lambda\}_{\lambda \in L}$. Indeed, by Proposition 2.3, we have that is $\bigcup_{\lambda \in L} L_\lambda$ a linear subspace of E. It is clear that $M \subseteq \bigcup_{\lambda \in L} L_\lambda$ since each L_λ contains M. Moreover, if

$$A \cap \left(\bigcup_{\lambda \in L} L_\lambda \right) \neq \varnothing,$$

then it would exists $\lambda \in L$ such that $L_\lambda \cap A \neq \varnothing$, which is a contradiction. Thus $\bigcup_{\lambda \in L} L_\lambda \in \mathcal{F}$ and it is an upper bound for $\{L_\lambda\}_{\lambda \in L}$. Thus, by Zorn's Lemma, $(\mathcal{F}, \subseteq)$ contains a maximal element G. Let

$$N = G + \bigcup_{\alpha > 0} \alpha A.$$

Since $A \subseteq \bigcup_{\alpha > 0} \alpha A$, it is clear that $A \subseteq N$. Moreover, by the Proposition 4.1, the following properties hold:

- N is open in E;
- $-N = N + \bigcup_{\alpha < 0} (\alpha A)$;
- $N \cap (-N) = N \cap G = (-M) \cap G = \varnothing$.

[1] Totally ordered subset of $(\mathcal{F}, \subseteq)$

Now we will prove the following claim: If $E = G \cup N \cup (-N)$, then G is a hyperplane of E. In fact, suppose by contradiction $E = G \cup N \cup (-N)$ and that G is a hyperplane of E. Since $A \neq \varnothing$, let $a \in A$, thus $G \oplus [a] \neq E$. Let $y \in E \setminus G \oplus [a]$. Since $E = G \cup N \cup (-N)$, then we can suppose without lost of generality that $y \in -N$. Define the following mapping

$$f : [0,1] \longrightarrow E$$
$$t \longmapsto ta + (1-t)y.$$

Follows from proposition 2.2 that f is continuous. Note that $f^{-1}(G) = \varnothing$. Indeed, if $f^{-1}(G) \neq \varnothing$, let $t \in f^{-1}(G)$. Thus

$$ta + (1-t)y \in G.$$

Thus, we have two possible cases

- If $t \neq 1$, from the fact that $ta + (1-t)y \in G$, we conclude that $(1-t)y \in G \oplus [a]$, which implies that $y \in G \oplus [a]$, which is a contradiction.
- If $t = 1$, from the fact that $ta + (1-t)y \in G$, we conclude that $a \in G$, which is impossible since $G \cap A = \varnothing$.

Thus $f^{-1}(G) = \varnothing$. Then, since we have $E = G \cup N \cup (-N)$, we have that

$$[0,1] = f^{-1}(E) = f^{-1}\big(G \cup N \cup (-N)\big) = f^{-1}(G) \cup f^{-1}(N) \cup f^{-1}\big((-N)\big) = f^{-1}(N) \cup f^{-1}\big((-N)\big).$$

Since $f(1) = a \in N$ and $f(0) = y \in -N$, then $f^{-1}(N) \neq \varnothing \neq f^{-1}\big((-N)\big)$. Moreover, since $f^{-1}(N) \cap f^{-1}\big((-N)\big) = \varnothing$, we conclude that $[0,1]$ is disconnected, which is a contradiction, proving the claim.

Finally, if the maximal subspace G found by Zorn's lemma was not a hyperplane, then $E \neq G \cup N \cup (-N)$. Let $z \in E \setminus \big(G \cup N \cup (-N)\big)$. Note that $A \cap (G \oplus [z]) = \varnothing$, because, if not, given $w = g_1 + \alpha z = a \in A \cap (G \oplus [z])$, where $g_1 \in G$, $\alpha \neq 0$, $a \in A$, we would get that $z \in N$ or $z \in (-N)$, which is a contradiction. This fact implies that $G \oplus [z] \in \mathcal{F}$ contains properly G , which contradicts the maximality of G in $\mathcal{F}$. Thus G is a hyperplane, $M \subseteq G$ and $A \cap G = \varnothing$, which concludes the proof. $\square$

4 Example

We conclude this work showing, from an example, the method which is used at paper [2] to solve the problem in finding the hyperplane of Geometric Hahn-Banach Theorem using the

Analytic Hahn-Banach Theorem.

Lemma 4.1. *Let the normed linear space* $\mathbb{R}^3$ *with the usual topology,* $S = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x = y = 0\}$ *and* $A = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x > 0\}$. *Then*

(i) S *is a linear subspace of* $\mathbb{R}^3$.

(ii) A *is an open convex subset of* $\mathbb{R}^3$ *disjoint from* S.

(iii) If we define

$$B := \bigcup_{\alpha>0} \alpha A,$$

then $B = A$.

(iv) Given $\overline{x} = (1,-3,0)$, *if we define*

$$D := (A - \overline{x}) \cap (\overline{x} - A),$$

then $D = \{(x,y,z) \in \mathbb{R}^3 \ ; \ |x| < 1\}$.

Proof: (i) : In fact, note that $S \neq \varnothing$ since $(0,0,0) \in S$. Given $(0,0,a)$, $(0,0,b) \in S$ and $\lambda \in \mathbb{R}$, we have that

$$(0,0,a) + \lambda(0,0,b) = (0,0,a+\lambda b) \in S.$$

Thus we conclude that S is a linear subspace of $\mathbb{R}^3$.

(ii) : Firstly we prove that A is open. Consider the function

$$\begin{aligned} \phi : \quad \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x,y,z) &\longmapsto x \end{aligned}$$

It is clear that ϕ is continuous. Since $A = \phi^{-1}\big((0,+\infty)\big)$, we conclude that A is open in $\mathbb{R}^3$. In order to prove the convexicity, let (x_1,y_1,z_1), $(x_2,y_2,z_2) \in A$ and let $t \in [0,1]$. Note that

$$t(x_1,y_1,z_1) + (1-t)(x_2,y_2,z_2) = \big(tx_1 + (1-t)x_2,\ ty_1 + (1-t)y_2,\ tz_1 + (1-t)z_2\big)$$

Since $x_1 > 0$ and $x_2 > 0$, we conclude that $tx_1 + (1-t)x_2 > 0$ for all $t \in [0,1]$, which implies that A is a convex subset of $\mathbb{R}^3$. Finally, it is easy to see that $A \cap S = \varnothing$.

(iii) : Taking $\alpha = 1 > 0$, we have that $\alpha A = A$, thus

$$A \subseteq \bigcup_{\alpha>0} \alpha A = B.$$

On the other hand, given $\alpha > 0$ and $(x,y,z) \in A$, we have that $\alpha(x,y,z) \in A$ because $\alpha x > 0$. Thus $\alpha A \subseteq A$ for any $\alpha > 0$, then

$$B = \bigcup_{\alpha>0} \alpha A \subseteq A.$$

(iv) : Note that $A - \overline{x} = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x > -1\}$ and that $\overline{x} - A = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x < 1\}$. Thus

$$D = (A - \overline{x}) \cap (\overline{x} - A) = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x > -1 \text{ and } x < 1\} = \{(x,y,z) \in \mathbb{R}^3 \ ; \ |x| < 1\}.$$

□

Lemma 4.2. *Let the normed linear space $\mathbb{R}^3$ with the usual topology and $D = \{(x,y,z) \,;\, |x| < 1\}$. Consider*

$$\begin{array}{rccc} \Psi : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & \inf\{\lambda \in \mathbb{R}_+ \ ; \ (x,y,z) \in \lambda D\} \end{array}$$

the Minkowsky functional associated to D. Then $\Psi(x,y,z) = |x|$ for all $(x,y,z) \in \mathbb{R}^3$.

Proof: Indeed, let $(x,y,z) \in \mathbb{R}^3$. If $x = 0$, then

$$\frac{1}{\epsilon}(x,y,z) \in D$$

for all $\epsilon > 0$, which implies that $\Psi(x,y,z) = 0$. On the other hand, if $x \neq 0$, for all $\epsilon > 0$, we have that

$$\frac{1}{|x| + \epsilon}(x,y,z) \in D \qquad \text{and} \qquad \frac{1}{|x|} \cdot (x,y,z) \notin D.$$

This fact implies that $\Psi\big((x,y,z)\big) = \inf\{\lambda \in \mathbb{R}_+ \ ; \ (x,y,z) \in \lambda D\} = |x|$. □

Lemma 4.3. *Let the normed linear space $\mathbb{R}^3$ with the usual topology, $S = \{(x,y,z) \ ; \ 3x + y = 0\}$ be a linear subspace of $\mathbb{R}^3$. If $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a linear functional satisfying the following properties*

(i) $|g(x,y,z)| \leq |x|$ *for all* $(x,y,z) \in \mathbb{R}^3$;

(ii) $g(x,y,z) = x$ *for all* $(x,y,z) \in S$;

Then $g(x,y,z) = x$ for all $(x,y,z) \in \mathbb{R}^3$.

Proof: Let $\mathfrak{A} = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. We know that, given $(x,y,z) \in \mathbb{R}^3$, then we can write $(x,y,z) = xe_1 + ye_2 + ze_3$. Thus, using the linearity of g, we have that

$$g(x,y,z) = g(e_1)x + g(e_2)y + g(e_3)z.$$

Our task is to find the coefficients $g(e_1)$, $g(e_1)$ and $g(e_3)$. Since $(0,0,1) \in S$, by property (ii), we have that $0 = g(0,0,1) = g(e_3)$, thus $g(x,y,z) = g(e_1)x + g(e_2)y$. Using the property (ii) again, since $(1,-3,0) \in S$, we have that

$$g(e_1) = 1 + 3g(e_2),$$

which implies that $g(x,y,z) = (1+3g(e_2))x + g(e_2)y$ for all $(x,y,z) \in \mathbb{R}^3$. Taking $(0,1,0) \in \mathbb{R}^3$ and using the property (i), we have that $|g(0,1,0)| = |g(e_2)| \leq 0$, which implies that $g(e_2) = 0$. Then we conclude that $g(e_1) = 1$ and

$$g(x,y,z) = x$$

for all $(x,y,z) \in \mathbb{R}^3$. □

Example 4.4. *Let the normed linear space $\mathbb{R}^3$ with the usual topology, consider $S = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x = y = 0\}$ and $A = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x > 0\}$. Find a hyperplane H containing S and disjoint from A.*

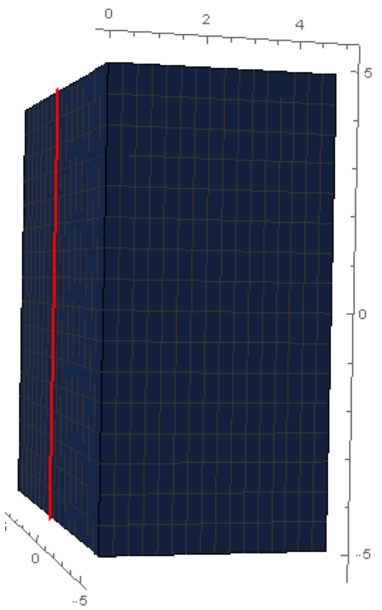

Figure 1: The subspace S in red and A in blue.

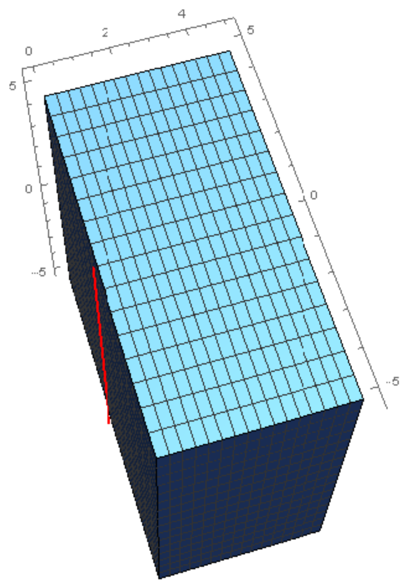

Figure 2: Top view

Proof: By Lemma 4.1, S is a linear subspace of E disjoint of A and A is an open convex subset of E. Moreover, setting

$$B = \bigcup_{\alpha > 0} \alpha A,$$

we have that $B = A$. Let $\overline{x} = (1,-3,0) \in B$ and set $D = (B - \overline{x}) \cap (B - \overline{x})$. Also by Lemma 4.1, we have that

$$D = (B - \overline{x}) \cap (B - \overline{x}) = \{(x,y,z) \in \mathbb{R}^3 \ ; \ |x| < 1\}.$$

Consider $\Psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ the Minkowsky functional associated to D. By Lemma 4.2, we have that Ψ is given by $\Psi(x,y,z) = |x|$. Consider the linear subspace

$$L = S \oplus [\overline{x}] = \big[\{(0,0,1),(1,-3,0)\}\big] = \{(x,y,z) \in \mathbb{R}^3 \ ; \ 3x + y = 0\}$$

and define the following function

$$\begin{aligned} f : \qquad\qquad L \qquad\qquad &\longrightarrow \mathbb{R} \\ s(0,0,1) + t(1,-3,0) &\longmapsto t \end{aligned}$$

Note that, in cartesian coordinates, we have that $f(x,y,z) = x = -y/3$ for all $(x,y,z) \in L$. Observe that $|f(x,y,z)| = |x| \leq |x| = \Psi(x,y,z)$ for all $(x,y,z) \in L$. Since Ψ is a seminorm, by Analytic Hahn-Banach Theorem, there exists a linear functional $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that

$$|g(x,y,z)| \leq \Psi(x,y,z) = |x|,$$

and extends f, that is, $g(x,y,z) = x$ whenever $x = -\frac{1}{3}y$. By Lemma 4.3, we have that $g(x,y,z) = x$ for all $(x,y,z) \in \mathbb{R}^3$. Let

$$H = \mathrm{Ker}(g) = \{(x,y,z) \in \mathbb{R}^3 \ ; \ g(x,y,z) = 0\} = \{(x,y,z) \in \mathbb{R}^3 \ ; \ x = 0\}$$

I claim that H is a hyperplane containing S and disjoint from A. Indeed, since H is kernel of a nonzero linear functional, we have that

$$\dim_{\mathbb{R}}\left(\frac{\mathbb{R}^3}{H}\right) = \dim_{\mathbb{R}}\left(\frac{\mathbb{R}^3}{\mathrm{Ker}(g)}\right) = \dim_{\mathbb{R}}(\mathbb{R}) = 1.$$

Thus, H is a hyperplane. Moreover, given $(x_0,y_0,z_0) \in S$, we have $x_0 = 0$, thus $g\big((x_0,y_0,z_0)\big) = 0$, which implies that $S \subseteq H$. Finally, given $(x_1,y_1,z_1) \in A$, we have that $x_1 > 0$ and so $g\big((x_1,y_1,z_1)\big) \neq 0$, which implies that $H \cap A = \varnothing$. $\square$

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