Advanced Linear Algebra

Compiled of all exercises and their solutions

1 Homework I: Fundamentals

Exercise 1.1. *Prove that the zero element of a vector space is unique.*

Solution: In fact, let 0 and $0'$ be zero elements of a linear space V, so

$$0 = 0 + 0' = 0',$$

where the first equality is because we are considering that $0'$ is a zero of V and the second equality is because we are considering that 0 is a zero of V. $\square$

Exercise 1.2. *Let X be a linear space over $\mathbb{K}$ and Y, Z be linear subspaces. Prove that*

$$Y + Z := \{y + z \ ; \ y \in Y, z \in Z\}$$

is a linear subspace of X.

Solution: Indeed, first note that $Y + Z \neq \emptyset$, because $0 + 0 = 0 \in Y + Z$. Now, let $y + z$ and $y' + z' \in Y + Z$. Thus

$$(y + z) + (y' + z') = (y + y') + (z + z') \in Y + Z.$$

Finally, given $y + z \in Y + Z$ and $\lambda \in \mathbb{K}$, we have that

$$\lambda(y + z) = \lambda y + \lambda z \in Y + Z.$$

Thus $Y + Z$ is a linear subspace of X. $\square$

Exercise 1.3. *Let X be a linear space over $\mathbb{K}$ and Y, Z be linear subspaces. Prove that $Y \cap Z$ is a linear subspace of X.*

Solution: Indeed, first note that $Y \cap Z \neq \emptyset$, because $0 \in Y$ and $0 \in Z$, so $0 \in Y \cap Z$. Now, let $x \in Y \cap Z$ and $x' \in Y \cap Z$. Thus both $x, x' \in Y$ and both $x, x' \in Z$, so $x + x' \in Y$ and $x + x' \in Z$, which implies that $x + x' \in Y \cap Z$

Finally, given $x \in Y \cap Z$ and $\lambda \in \mathbb{K}$, we have that $x \in Y$ and $x \in Z$, thus $\lambda x \in Y$ and $\lambda x \in Z$, so $\lambda x \in Y \cap Z$. Thus $Y \cap Z$ is a linear subspace of X. $\square$

Exercise 1.4. *Let X be a linear space over $\mathbb{K}$ and $x_1, \ldots, x_j \in X$. Show that*

$$\mathrm{Span}(\{x_1, \ldots, x_j\}) = \left\{ \sum_{k=1}^{j} \lambda_k x_k \ ; \ \lambda_1, \ldots, \lambda_j \in \mathbb{K} \right\}$$

is a linear subspace of X. Moreover, show that it is the smallest linear subspace of X which contains of $x_1, \ldots, x_j$.

Solution: The proof of the first part is similar to Exercise 1.2. Now let Y be a linear subspace of X containing $x_1, \ldots, x_j$, so, by linearity, Y contains every linear combinations of $x_1, \ldots, x_j$, thst is

$$\mathrm{Span}(\{x_1, \ldots, x_j\}) \subseteq Y$$

In other words, this means that $\mathrm{Span}(\{x_1, \ldots, x_j\})$ is the smallest linear subspace of X which contains of $x_1, \ldots, x_j$. $\square$

Exercise 1.5. *Let X be a linear space over $\mathbb{K}$ and $x_1, \ldots, x_j \in X$. Show that if $x_1, \ldots, x_j$ is linearly independent, then none of x_i is the zero vector.*

Solution: In fact, by contradiction, suppose that one of x_i is the zero vector, say $x_1 = 0$. Then

$$1 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_j = \sum_{k=1}^{j} 0 = 0,$$

which contradicts the fact that $x_1, \ldots, x_j$ is linearly independent. $\square$

Exercise 1.6. *Let X be a finite-dimensional linear space over $\mathbb{K}$. Prove that there is $n \in \mathbb{N}_{\geq 0}$ such that X is isomorphic to $\mathbb{K}^n$. Show that this isomoophism is not unique when $n > 1$.*

Solution: Since X is a finite-dimensional linear space, let $\{v_1, \ldots, v_n\}$ be a basis for X. Define by linearity the following map

$$\phi : X \longrightarrow \mathbb{K}^n$$

$$v_i \longmapsto e_i$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{K}^n$. Since $\{v_1, \ldots, v_n\}$ is a basis, it is easy to see that ϕ is well-defined. Moreover, it is straightforward to check that ϕ is an isomorphism. Suppose that $\dim(X) = n > 1$, it is easy to see that $\{v_1, v_1 + v_2, v_3, \ldots, v_n\}$ is another basis for X. Now define by linearity this isomorphism

$$\begin{aligned} \psi : X &\longrightarrow \mathbb{K}^n \\ v_i &\longmapsto e_i \text{ if } i \neq 2 \\ v_1 + v_2 &\longmapsto e_2 \end{aligned}$$

It is easy to see that $\psi \neq \phi$. $\square$

Exercise 1.7. *Let X be the linear space of all polynomials with coefficients in $\mathbb{K}$ and degree less than n, and Y be the set of polynomials which are zero at $t_1, \cdots, t_j$, with $j < n$*

(i) Show that Y is a linear subspace of X.

(ii) Determine $\dim_{\mathbb{K}}(Y)$.

(iii) Determine $\dim_{\mathbb{K}}(X/Y)$.

Solution: **(i)**: Firstly note that $Y \neq \emptyset$, because the zero polynomial belongs to Y. Now, given $p_1(t), p_2(t) \in Y$, we have that $p_1(t_k) = p_2(t_k) = 0$ for all $k = 1, \ldots, j$, so

$$p_1(t_k) + p_2(t_k) = 0 + 0 = 0$$

for all $k = 1, \ldots, j$. Finally, given $p(t) \in Y$ and $\lambda \in \mathbb{K}$, we have

$$(\lambda \cdot p)(t_k) = \lambda \cdot 0 = 0$$

for all $k = 1, \ldots, j$, so $(\lambda \cdot p)(t) \in Y$. Thus we conclude that Y is a linear subspace of X.

(ii): By basic algebra, we know that if $t_1, \ldots, t_j$ are roots of a polynomial $p(t) \in \mathbb{K}[t]$, then we can write

$$p(t) = g(t) \prod_{k=1}^{j} (t - t_k),$$

where, $\deg(g) = \deg(p) - j$. Call

$$L(t) = \prod_{k=1}^{j} (t - t_k)$$

Note that, given $p(t) \in Y$, then we can write

$$p(t) = L(t)(a_0 + a_1 t + \cdots + a_{(n-1)-j} t^{(n-1)-j}) = \sum_{k=0}^{(n-1)-j} (a_k L(t) t^t)$$

Thus $\mathcal{B} = \{L(t), L(t)t, \ldots, L(t)t^{(n-1)-j}\}$ is a set of generators of Y. Moreover, since $\deg(L(t)t^i) \neq \deg(L(t)t^j)$ if $i \neq j$, we conclude that $\mathcal{B}$ is a basis for Y, whence we have $\dim_{\mathbb{K}}(Y) = n - j$.

(iii): Just use the formula

$$\dim_{\mathbb{K}}(X) = \dim_{\mathbb{K}}(Y) + \dim_{\mathbb{K}}(X/Y)$$

and conclude that $\dim_{\mathbb{K}}(X/Y) = n - (n - j) = j$. □

Exercise 1.8. *Which of the following sets of vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are a subspace of $\mathbb{R}^n$.*

(a) All x such that $x_1 \geq 0$.

(b) All x such that $x_1 + x_2 = 0$.

(c) All x such that $x_1 + x_2 + 1 = 0$.

(d) All x such that $x_1 = 0$.

(e) All x such that $x_1 \in \mathbb{Z}$.

Solution: Yes: (b) and (d). No: (a) (multiply by negative scalar); (c) and (e) (Multiply by an irrational number). □

Exercise 1.9.

(1) Give an example of a set $V \subseteq \mathbb{R}^n$ such that V is closed under the usual sum, but V is not closed under the scalar mutiplication

(2) Give an example of a set $V \subseteq \mathbb{R}^n$ such that V is closed under the scalar multiplication, but V is not closed under the usual sum.

Solution: **(1)**: Consider $V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \ ; \ a_1, \ldots, a_n \in \mathbb{Z}\}$. V is closed under addition, but

$$\frac{1}{2} \cdot (1, \ldots, 1) = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \notin V$$

(2): Consider $V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \ ; \ 2a_1^2 = a_2^2\}$. V is closed scalar multiplication. However

$$(1, \sqrt{2}, 0, \ldots, 0), (1, -\sqrt{2}, 0, \ldots, 0) \in V,$$

but

$$(1, \sqrt{2}, 0, \ldots, 0) + (1, -\sqrt{2}, 0, \ldots, 0) = (2, 0, \ldots, 0) \notin V.$$

□

Exercise 1.10. *Let $Q = \{(x, y) \in \mathbb{R}^2 \ ; \ x, y > 0\}$. Prove that Q is a linear space over $\mathbb{R}$ under the operations*

$$+ : Q \times Q \longrightarrow Q \qquad\qquad \cdot : \mathbb{R} \times Q \longrightarrow Q$$

$$((x_1, y_1), (x_2, y_2)) \longmapsto (x_1 x_2, y_1 y_2) \qquad\qquad (\lambda, (x, y)) \longmapsto (x^\lambda, y^\lambda)$$

Solution: Firstly, note that these operations are well-defined. Observe that $(Q, +)$ is an Abelian group with zero element $0 = (1, 1)$ and inverse $-z = (x^{-1}, y^{-1})$ for a given $z = (x, y) \in Q$. Moreover, given $z = (x, y)$, $z' = (a, b) \in Q$, $\lambda, \gamma \in \mathbb{R}$, then

$$(\gamma\lambda) \cdot z = (\gamma\lambda) \cdot (x, y) = (x^{\gamma\lambda}, y^{\gamma\lambda}) = \lambda \cdot (x^\gamma, y^\gamma) = \lambda \cdot (\gamma \cdot z)$$
$$1 \cdot z = 1 \cdot (x, y) = (x^1, y^1) = (x, y) = z$$
$$(\gamma + \lambda) \cdot z = (\gamma + \lambda) \cdot (x, y) = (x^{\gamma+\lambda}, y^{\gamma+\lambda}) = (x^\gamma x^\lambda, y^\gamma y^\lambda) = (x^\gamma, y^\gamma) + (x^\lambda, y^\lambda) = \lambda z + \gamma z$$
$$\lambda(z + z') = \lambda \cdot (xa, yb) = ((xa)^\lambda, (yb)^\lambda) = (x^\lambda a^\lambda, y^\lambda b^\lambda) = (x^\lambda, y^\lambda) + (a^\lambda, b^\lambda) = \lambda \cdot z + \lambda \cdot z'$$

Thus $(Q, \mathbb{R}, +, \cdot)$ is a linear space over $\mathbb{R}$. □

Exercise 1.11. *Consider the linear spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ over the fields $\mathbb{R}$ and $\mathbb{C}$, respectively.*

(i) For which $\alpha \in \mathbb{C}$ are the vectors $(1 + \alpha, 1 - \alpha)$ and $(1 - \alpha, 1 + \alpha)$ linearly independent in $\mathbb{C}^2$?

(ii) For which $\beta \in \mathbb{R}$ are the vectors $(\beta, 1, 0)$, $(1, \beta, 1)$ and $(0, 1, \beta)$ linearly independent in $\mathbb{R}^3$?

(iii) For which β, $\eta \in \mathbb{C}$ are the vectors $(1, \beta)$, $(1, \eta)$ linearly dependent in $\mathbb{C}^2$?

Solution: Just apply the determinant test. Remember that the vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ are linearly independent if and only if $\det(v_1, \ldots, v_n) \neq 0$.

(i) Calculating the determinant, we obtain

$$\begin{vmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{vmatrix} = (1+\alpha)^2 - (1-\alpha)^2 = 4\alpha.$$

Thus $(1+\alpha, 1-\alpha)$ and $(1-\alpha, 1+\alpha)$ are linearly independent if and only if $\alpha \neq 0$.

(ii) Calculating the determinant, we obtain

$$\begin{vmatrix} \beta & 1 & 0 \\ 1 & \beta & 1 \\ 0 & 1 & \beta \end{vmatrix} = \beta^3 - 2\beta.$$

Thus $(\beta, 1, 0)$, $(1, \beta, 1)$ and $(0, 1, \beta)$ are linearly independent if and only if $\beta \notin \{-\sqrt{2}, 0, \sqrt{2}\}$.

(iii) Calculating the determinant, we obtain

$$\begin{vmatrix} 1 & 1 \\ \beta & \eta \end{vmatrix} = \eta - \beta.$$

Thus $(1, \beta)$, $(1, \eta)$ are linearly dependent if and only if $\alpha = \eta$. $\square$

Exercise 1.12. *Let X be a linear space and V_1 and V_2 be linear subspace of X. Prove that $V_1 \cup V_2$ is not necessarily a linear subspace of X. Give a necessary and enough condition for that the union of linear subspaces be a linear subspace.*

Solution: Consider $X = \mathbb{R}^2$ and the linear subspaces

$$V_1 = \{(x_1, x_2) \in \mathbb{R}^2 \ ; \ x_1 = 0\}$$
$$V_2 = \{(x_1, x_2) \in \mathbb{R}^2 \ ; \ x_2 = 0\}$$

If $V_1 \cup V_2$ was a linear subspace, then $(1, 1)$ would belong to $V_1 \cup V_2$, but it is impossible because $(1, 1) \notin V_1$ and $(1, 1) \notin V_2$.

A necessary and enough condition for that the union of two linear subspaces W_1 and W_2 be a linear subspace is that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. In fact, the sufficiency is trivial, that is, if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2$ is a linear subspace of X. Now suppose that $W_1 \cup W_2$ is a linear subspace and that $W_1 \not\subset W_2$ and $v_1 \in W_1 \setminus W_2$. We will prove that $W_2 \subseteq W_1$. Let $v_2 \in W_2$. Since $W_1 \cup W_2$ is a linear subspace, we have that $v_1 + v_2 \in W_1 \cup W_2$. Note that $v_1 + v_2$ necessarily belongs to W_1, so

$$v_2 = (v_1 + v_2) - v_1 \in W_1,$$

which implies that $W_2 \subseteq W_1$. $\square$

Exercise 1.13. *Let $C^0([0,1],\mathbb{R})$ be the space of real continuous functions defined in $[0,1]$ with the usual operations of sum and scalar multiplication. Prove that $\dim_{\mathbb{R}}(C^0([0,1],\mathbb{R})) = \infty$.*

Solution: Just observe that the linear space $\mathcal{P}([0,1])$ of polynomials functions defined in $[0,1]$ has infinite dimension and that $\mathcal{P}([0,1]) \subseteq C^0([0,1],\mathbb{R})$. Then it is impossible that $C^0([0,1],\mathbb{R})$ has finite dimension. $\square$

Exercise 1.14. *Let Y be a finite set with n elements. Prove that $\mathcal{F}_n(\mathbb{R}) = \mathbb{R}^Y$ is isomorphic to $\mathbb{R}^n$*

Solution: Indeed, it is easy to see that $(\mathcal{F}_n(\mathbb{R}), \mathbb{R}, +, \cdot)$ has structure of $\mathbb{R}$-linear space defining $(f+g)(t) = f(t) + g(t)$ and $(\lambda \cdot f)(t) = \lambda \cdot f(t)$. Now let $Y = \{y_1, \dots, y_n\}$ and, for each $i = 1, \dots, n$, define the map

$$f_i : Y \longrightarrow \mathbb{R}$$

$$y_j \longmapsto \delta_{ij}$$

It is easy to see that $\{f_1, \dots, f_n\}$ is a linearly independent subset of $\mathcal{F}_n(\mathbb{R})$. Moreover $\{f_1, \dots, f_n\}$ generates $\mathcal{F}_n(\mathbb{R})$, because, given $f \in \mathcal{F}_n(\mathbb{R})$, we have

$$f = \sum_{k=1}^{n} f(y_k) f_k$$

Thus, $\dim_{\mathbb{R}}(\mathcal{F}_n(\mathbb{R})) = n$, so $\mathcal{F}_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^n$. $\square$

Exercise 1.15. *Let $\mathbb{P}_3(\mathbb{K})$ be the linear space of all polynomials with coefficients in $\mathbb{K}$ and degree less or equal to 3. Prove that there is a basis of $\mathbb{P}_3(\mathbb{K})$ without polynomials of degree two.*

Solution: Indeed, consider $\mathcal{B} = \{1, t, t^3 + t^2, t^3\} \subseteq \mathbb{P}_3(\mathbb{K})$. It is easy to see that this set is linearly independent. Moreover, given $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \in \mathbb{P}_3(\mathbb{K})$, then

$$p(t) = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot (t^3 + t^2) + (a_3 - a_2) \cdot t^3$$

Thus $\mathcal{B}$ is a basis for $\mathbb{P}_3(\mathbb{K})$ which does not contain polynomials of degree 2. $\square$

2 Homework II: Duality

Exercise 2.1. *Let X be a linear space over K and $x \in X$ be a non-zero vector. Prove that there is a linear functional ℓ in X such that*

$$\ell(x) \neq 0.$$

Solution: Let $\{x_i\}_{i \in I}$ be a basis of X such that $x_j = x$ for some $j \in I$. Define the following linear functional by linearity such that

$$\ell(x_i) = \begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$$

ℓ is a linear functional and $\ell(x) = \ell(x_j) = 1 \neq 0$. So ℓ satisfies the wished condition. $\square$

Exercise 2.2. *Let X be a linear space over K and Y be a linear subspace of X. Define*

$$Y^\perp = \{\ell \in X' \ ; \ \ell(y) = 0 \text{ for all } y \in Y\}.$$

Prove that $Y^\perp$ is a linear subspace of X.

Solution: Indeed, observe that $Y^\perp \neq \emptyset$, because the 0 functional belongs to $Y^\perp$. Now, let $f_1, f_2 \in Y^\perp$ and $\lambda \in K$. So, given $y \in Y$, we have

$$(f_1 + \lambda \cdot f_2)(y) = f_1(y) + \lambda \cdot f_2(y) = 0$$

Since $y \in Y$ is arbitrary, we conclude that $f_1 + \lambda \cdot f_2 \in Y^\perp$, so $Y^\perp$ is a linear subspace of X'. $\square$

Exercise 2.3. *Let X be a linear space over K and S be a subset of X. Define*

$$S^\perp = \{f \in X' \ ; \ f(x) = 0 \text{ for all } x \in S\}.$$

Prove that $S^\perp = (\mathrm{Span}(S))^\perp$

Solution: Indeed, since $S \subseteq \mathrm{Span}(S)$, given $f \in (\mathrm{Span}(S))^\perp$, we have that $f(y) = 0$ for all $y \in \mathrm{Span}(S)$. In particular, $f(y) = 0$ for all $y \in S$, so $f \in S^\perp$, then

$$(\mathrm{Span}(S))^\perp \subseteq S^\perp.$$

On the other hand, let $f \in S^\perp$. Given $x \in \mathrm{Span}(S)$, we know that there are $c_1, \ldots, c_n \in K$ and $x_1, \ldots, x_k \in S$ such that

$$x = \sum_{i=1}^{k} c_i x_i.$$

Thus

$$f(x) = f\left(\sum_{i=1}^{k} c_i x_i\right) = \sum_{i=1}^{k} c_i f(x_i) = \sum_{i=1}^{k} c_i \cdot 0 = 0.$$

Since $x \in \operatorname{Span}(S)$ is arbitrary, we conclude that $f \in (\operatorname{Span}(S))^{\perp}$, which implies that $S^{\perp} \subseteq (\operatorname{Span}(S))^{\perp}$. Then

$$S^{\perp} = (\operatorname{Span}(S))^{\perp}.$$

$\square$

Exercise 2.4. *Let $\mathcal{P}_2(\mathbb{R})$ be the linear space of all polynomials with coefficients in $\mathbb{R}$ and degree less or equal to 2. Let ξ_1, ξ_2, ξ_3 be three distinct real numbers, and then define, for each $j = 1, 2, 3$, the following functional*

$$\ell_j : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$p \longmapsto p(\xi_j)$$

(i) Show that ℓ_1, ℓ_2, ℓ_3 are linearly independent linear functions in $\mathcal{P}_2(\mathbb{R})$.

(ii) Show that $\{\ell_1, \ell_2, \ell_3\}$ is a basis for the dual space $\mathcal{P}_2(\mathbb{R})'$

(iii) (a) Suppose that $\{e_1, \ldots, e_n\}$ is a basis for a vector space V, Show that there exist linear functions $\{f_1, \ldots, f_n\}$ in the dual space V' defined by

$$f_i(e_j) = \begin{cases} 1; & \text{if } i = j; \\ 0; & \text{otherwise.} \end{cases}$$

Show that $\{f_1, \ldots, f_n\}$ is a basis for V'.

(b) Find the polynomials $p_1(x), p_2(x), p_3(x) \in \mathcal{P}_2(\mathbb{R})$ for which $\{\ell_1, \ell_2, \ell_3\}$ is the dual basis in $\mathcal{P}_2(\mathbb{R})'$.

Solution: **(i):** It is easy to see that $\xi_i : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$ is a linear functional for each $i = 1, 2, 3$. Let $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\ell_1 + c_2\ell_2 + c_3\ell_3 = 0.$$

Considering the polynomial $p_1(x) := (x - \xi_2)(x - \xi_3)$ and applying p_1 in the relation above, we get that

$$(\xi_1 - \xi_2)(\xi_1 - \xi_3)c_1 = 0,$$

so $c_1 = 0$. Proceeding similarly we are able to conclude that $c_2 = c_3 = 0$, so $\{\ell_1, \ell_2, \ell_3\}$ is a linearly independent set.

(ii): Since

$$\dim_{\mathbb{R}}(\mathcal{P}_2(\mathbb{R})') = \dim_{\mathbb{R}}(\mathcal{P}_2(\mathbb{R})) = 3$$

and ℓ_1, ℓ_2, ℓ_3 are linearly independent linear functions, it is immediate that $\{\ell_1, \ell_2, \ell_3\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

(iii)-(a): Since $\{e_1, \ldots, e_n\}$ is a basis for V, we can construct by linearity a well-defined linear functional $f_i : V \longrightarrow \mathbb{R}$ such that

$$f_i(e_j) = \begin{cases} 1; & \text{if } i = j; \\ 0; & \text{otherwise.} \end{cases}$$

Observe that $\{f_1 \ldots f_n\}$ is a linearly independent subset of V'. Indeed, let $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\sum_{k=1}^{n} c_k \cdot f_k = 0.$$

Applying in the equation above the vector e_i, we conclude that

$$0 = 0(e_i) = c_1 \cdot f_1(e_i) + \cdots + c_i \cdot f_i(e_i) + \cdots + c_n \cdot f_n(e_i) = c_1 \cdot 0 + \cdots + c_i \cdot 1 + \cdots + c_n \cdot 0 = c_i.$$

Proceeding similarly, we are able to conclude that $c_1 = \cdots = c_n = 0$, so $\{f_1 \ldots f_n\}$ is a linearly independent subset of V'. Moreover, given $f \in V'$, it is easy to see that

$$f = \sum_{k=1}^{n} f(e_k) \cdot f_k,$$

so $\mathrm{Span}(\{f_1, \ldots, f_n\}) = V'$, then $\{f_1 \ldots f_n\}$ is a basis of V'.

(iii)-(b): Consider the polynomials

$$p_1(x) = \frac{(x - \xi_2)(x - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} \qquad p_2(x) = \frac{(x - \xi_1)(x - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} \qquad p_3(x) = \frac{(x - \xi_1)(x - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)}$$

For each $1 \leq i, j \leq 3$, note that

$$\ell_i(p_j(x)) = \begin{cases} 1; & \text{if } i = j; \\ 0; & \text{otherwise.} \end{cases}$$

so $\{\ell_1, \ell_2, \ell_3\}$ is the dual basis of $\{p_1(x), p_2(x), p_3(x)\}$. □

Exercise 2.5. *Let W be the linear subspace of $\mathbb{R}^4$ generated by $(1,0,-1,2)$ and $(2,3,1,1)$. Which linear functions $\ell(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator of W.*

Solution: We are looking for $c_1, x_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1 + 0c_2 - 2c_3 + 3c_4 = 0$$

$$2c_1 + 3c_1 + c_3 + c_4 = 0$$

Using the matrix language, we are trying to solve the following linear system

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solving this system by Gauss elimination method, we obtain the $W^{\perp}$ is the space generated by

$$\ell_1(x,y,z,w) = x - y + z \qquad \text{and} \qquad \ell_2(x,y,z,w) = -2x + y + w$$

□

Exercise 2.6. *Let $I \subseteq R$ be an interval and $t_1, \ldots, t_n \in I$ with $t_i \neq t_j$ if $i \neq j$. Prove that there are $m_1, \ldots, m_n \in \mathbb{R}$ such that for all $p \in \mathcal{P}_{n-1}(\mathbb{R})$*

$$\int_I p(x)dx = \sum_{k=1}^{n} m_k p(t_k). \tag{1}$$

Solution: Since $\mathcal{P}_{n-1}(\mathbb{R})$ is isomorphic to $\mathbb{R}^n$, we get that $\dim_{\mathbb{R}}(\mathbb{P}_{n-1}(\mathbb{R})) = n$. For each $j = 1, \ldots, n$, define the functional linear

$$\begin{aligned} f_j : \mathcal{P}_{n-1}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ p &\longmapsto p(t_i) \end{aligned}$$

Proceeding as in Exercise 2.4, we conclude that $\{f_1, \ldots, f_n\}$ is linearly independent subset of $\mathcal{P}_{n-1}(\mathbb{R})'$ and so it is a basis for $\mathcal{P}_{n-1}(\mathbb{R})'$. Since

$$\begin{aligned} g : \mathcal{P}_{n-1}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ p &\longmapsto \textstyle\int_I p(x)dx \end{aligned}$$

is a linear functional, there are $m_1, \ldots, m_n \in \mathbb{R}$ such that

$$g = \sum_{k=1}^{n} m_k f_k$$

Thus, for all $p \in \mathcal{P}_{n-1}(\mathbb{R})$, we have

$$\int_I p(x)dx = g(p(x)) = \sum_{k=1}^{n} m_k f_k(p(x)) = \sum_{k=1}^{n} m_k p(t_k).$$

$\square$

Exercise 2.7. *Taking $I = [-1, 1]$, $n = 3$ and the points $t_1 = -a$, $t_2 = 0$ and $t_3 = a$.*

(i) Determine the weights m_1, m_2, m_3 such that the equation (1) holds for all polynomial of degree < 3.

(ii) Show that for $a > \sqrt{1/3}$, all three weights are positive.

(iii) Show that for $a = \sqrt{3/5}$ the equality holds for all polynomials of degree < 6.

Solution: **(i):** By Exercise 2.6, there are $m_1, m_2, m_3 \in \mathbb{R}$ such that

$$f(p) := \int_{-1}^{1} p(x)dx = m_1 \cdot p(-a) + m_2 \cdot p(0) + m_3 \cdot p(a)$$

for all polynomial with degree less than 2. Taking $p_1(x) = x(x-a)$, we have that

$$(2a^2)m_1 = \int_{-1}^{1} x(x-a)dx = \frac{2}{3}.$$

Thus $m_1 = 1/(3a^2)$. Taking $p_2(x) = (x+a)(x-a)$, we have that

$$(-a^2)m_2 = \int_{-1}^{1} (x+a)(x-a)dx = \frac{2}{3} - 2a^2 = \frac{2-6a^2}{3}.$$

Thus $m_2 = (6a^2-2)/a^2$. Taking $p_3(x) = (x+a)x$, we have that

$$(2a^2)m_3 = \int_{-1}^{1} (x+a)xdx = 2/3.$$

Thus $m_3 = 1/(3a^2)$. So we conclude that

$$\int_{-1}^{1} p(x)dx = \frac{1}{3a^2} \cdot p(-a) + \frac{6a^2-2}{a^2} \cdot p(0) + \frac{1}{3a^2} \cdot p(a).$$

(ii): If $a > \sqrt{1/3}$, then $m_1 = m_3 > 0$ trivially and $m_2 = (6a^2-2)/a^2 > 0$, so all weights are positive.

(iii): Suppose $a = \sqrt{3/5}$. We will prove that

$$\int_{-1}^{1} p(x)dx = \frac{5}{9} \cdot p(-\sqrt{5/9}) + \frac{8}{3}p(0) + \frac{5}{9} \cdot p(\sqrt{5/9}).$$

for all polynomial of degree less than 6. Call

$$\ell(p) = \frac{5}{9} \cdot p(-\sqrt{5/9}) + \frac{8}{3}p(0) + \frac{5}{9} \cdot p(\sqrt{5/9}).$$

Since ℓ and f are linear functionals in $\mathcal{P}_5(\mathbb{R})$, it is enough to show that $\ell(x^i) = f(x^i)$ for all $i = 0, \ldots, 5$. For $i = 0, 1, 2$, it is not necessary. For $i = 3$, we have

$$f(x^3) = 0 = \frac{5}{9} \cdot (-\sqrt{3/5})^3 + \frac{8}{3} \cdot 0^3 + \frac{5}{9} \cdot (\sqrt{3/5})^3 = g(x^3).$$

For $i = 4$, we have

$$f(x^4) = 2/5 = \frac{5}{9} \cdot (-\sqrt{3/5})^4 + \frac{8}{3} \cdot 0^4 + \frac{5}{9} \cdot (\sqrt{3/5})^4 = g(x^3).$$

For $i = 5$, we have

$$f(x^5) = 0 = \frac{5}{9} \cdot (-\sqrt{3/5})^5 + \frac{8}{3} \cdot 0^5 + \frac{5}{9} \cdot (\sqrt{3/5})^5 = g(x^5).$$

$\square$

Exercise 2.8. *Taking* $I = [-1, 1]$, $n = 4$ *and the points* $t_1 = -a$, $t_2 = -b$, $t_3 = b$ *and* $t_4 = a$.

(i) Determine the weights m_1, m_2, m_3 *and* m_4 *such that the equation* (1) *holds for all polynomial of degree* < 4.

(ii) For what values of a *and* b *are the weights positive?*

Solution: **(i):** By Exercise 2.6, there are $m_1, m_2, m_3, m_4 \in \mathbb{R}$ such that

$$f(p) := \int_{-1}^{1} p(x)dx = m_1 \cdot p(-a) + m_2 \cdot p(-b) + m_3 \cdot p(b) + m_4 p(a)$$

for all polynomial with degree less than 4. Taking $p_1(x) = (x - b)(x - a)(x + b)$, we have that

$$2a(b + a)(b - a)m_1 = \int_{-1}^{1} (x - b)(x - a)(x + b)dx = \frac{2a(3b^2 - 1)}{3}$$

Thus $m_1 = (3b^2 - 1)/(3(b + a)(b - a))$. Taking $p_2(x) = (x - b)(x - a)(x + a)$, we have that

$$2b(a + b)(a - b)m_2 = \int_{-1}^{1} (x - b)(x - a)(x + a)dx = \frac{2b(3a^2 - 1)}{3}$$

Thus $m_2 = (3a^2 - 1)/(3(a + b)(a - b))$. Taking $p_3(x) = (x + b)(x + a)(x - a)$, we have that

$$2b(b + a)(b - a)m_3 = \int_{-1}^{1} (x + b)(x + a)(x - a)dx = \frac{2b(1 - 3a^2)}{3}$$

Thus $m_3 = (1-3a^2)/(3(b+a)(b-a))$. Taking $p_4(x) = (x+b)(x+a)(x-b)$, we have that

$$2a(a+b)(a-b)m_4 = \int_{-1}^{1} (x+b)(x+a)(x-b)dx = \frac{2a(1-3b^2)}{3}$$

Thus $m_4 = (1-3b^2)/(3(a+b)(a-b))$ So we conclude that

$$\int_{-1}^{1} p(x)dx = \frac{3b^2-1}{3(b+a)(b-a)}p(-a) + \frac{3a^2-1}{3(a+b)(a-b)}p(-b) + \frac{1-3a^2}{3(b+a)(b-a))}p(b) + \frac{1-3b^2}{3(a+b)(a-b))}p(a).$$

(ii): We want to find $(a,b) \in \mathbb{R}^2$ such that

$$\begin{cases} 3b^2 - 1 \leq 0 \\ 3a^2 - 1 \geq 0 \\ 1 - 3a^2 \leq 0 \\ 1 - 3b^2 \geq 0 \end{cases}$$

The solution of this system is

$$S = \{(a,b) \in \mathbb{R}^2 \ ; \ a \in (-\infty, -\sqrt{3}/3] \cup [\sqrt{3}/3, \infty) \text{ and } b \in [-\sqrt{3}/3, \sqrt{3}/3]\}.$$

□

3 Homework III: Matrices and linear mappings

Exercise 3.1. *Verify if the following affirmations are true or false.*

(i) *The vector* $w = (-1, 2, 3)^t$ *is on the linear subspace of* $\mathbb{R}^3$ *generated by* $v_1 = (2, -1, 2)^t$ *and* $v_2 = (5, -4, 1)^t$.

(ii) *The vector* $w = (1, -2, -3)^t$ *is on the linear subspace of* $\mathbb{R}^3$ *generated by* $v_1 = (1, 1, 0)^t$ *and* $v_2 = (0, 1, 1)^t$.

(iii) *The vector* $w = (1, -2, -1)^t$ *is on the linear subspace of* $\mathbb{R}^3$ *generated by* $v_1 = (1, 2, 2)^t$ *and* $v_2 = (1, -2, 0)^t$ *and* $v_3 = (0, 3, 4)^t$.

Solution: **(i):** Just check the determinant

$$D = \det \begin{vmatrix} -1 & 2 & 5 \\ 2 & -1 & -4 \\ 3 & 2 & 1 \end{vmatrix} = 0.$$

Since $D = 0$, $\{w, v_1, v_2\}$ is linearly dependent. Moreover, since $\{v_1, v_2\}$ is linearly independent, we conclude that it is possible to obtain w as linear combination of v_1 and v_2. Hence the assertion is true.

(ii): Just check the determinant

$$D = \det \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ -3 & 0 & 1 \end{vmatrix} = 0.$$

Since $D = 0$, $\{w, v_1, v_2\}$ is linearly dependent. Moreover, since $\{v_1, v_2\}$ is linearly independent, we conclude that it is possible to obtain w as linear combination of v_1 and v_2. Hence the assertion is true.

(iii): Just check the determinant

$$D = \det \begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 3 \\ 2 & 0 & 4 \end{vmatrix} = -10.$$

Since $D \neq 0$, $\{v_1, v_2, v_3\}$ is linearly dependent subset of $\mathbb{R}^3$. Moreover, since $\dim_{\mathbb{R}}(\mathbb{R}^3) = 3$, we conclude that $\{v_1, v_2, v_3\}$ is a basis. Whence we can express w as linear combination of v_1, v_2 and v_3. $\square$

Exercise 3.2. *Let X be a finite-dimensional K-linear space. If $z \in X$ is linear combination of x, y and w, is it true that w is linear combination of x, y and z?*

Solution: No. suppose that $\{x, y, w\}$ is a linearly independent subset of X. Take $z = x + y$. Since

$$z = 1x + 1y + 0w,$$

then z is linear combination of x, y and w. However, if we would have that w were linear combination of x, y and z, then there would be $c_1, c_2, c_3 \in K$ such that

$$w = c_1 x + c_2 y + c_3(x + y) = (c_1 + c_3)x + (c_2 + c_3)y,$$

which contradicts the fact that $\{x, y, w\}$ is a linearly independent subset of X. $\square$

Exercise 3.3. *Verify if the following affirmations are true or false*

(i) $\{(0, 1, 1)^t, (1, -1, 0)^t, (3, 1, 2)^t\}$ *is a linearly independent subset of* $\mathbb{R}^3$.

(ii) $\{(1, 1, 1, 0)^t, (1, 1, -1, 0)^t, (1, -1, 0, 1)^t, (1, -1, 0, -1)^t\}$ *is a linearly independent subset of* $\mathbb{R}^4$

Solution: **(i):** Just check the determinant

$$D = \det \begin{vmatrix} 0 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2.$$

Since $D \neq 0$, $\{(0, 1, 1)^t, (1, -1, 0)^t, (3, 1, 2)^t\}$ is a linearly independent subset of $\mathbb{R}^3$, so the assertion is true.

(ii): Just check the determinant

$$D = \det \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = 8.$$

Since $D \neq 0$, $\{(1, 1, 1, 0)^t, (1, 1, -1, 0)^t, (1, -1, 0, 1)^t, (1, -1, 0, -1)^t\}$ is a linearly independent subset of $\mathbb{R}^4$, so the assertion is true. $\square$

Exercise 3.4. *Consider the linear spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ over the fields $\mathbb{R}$ and $\mathbb{C}$, respectively.*

(i) For which $\alpha \in \mathbb{C}$ are the vectors $(1+\alpha, 1-\alpha)$ and $(1-\alpha, 1+\alpha)$ linearly independent in $\mathbb{C}^2$?

(ii) For which $\beta \in \mathbb{R}$ are the vectors $(\beta, 1, 0)$, $(1, \beta, 1)$ and $(0, 1, \beta)$ linearly independent in $\mathbb{R}^3$?

(iii) For which β, $\eta \in \mathbb{C}$ are the vectors $(1, \beta)$, $(1, \eta)$ linearly dependent in $\mathbb{C}^2$?

Solution: Just apply the determinant test. Remember that the vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ are linearly independent if and only if $\det(v_1, \ldots, v_n) \neq 0$.
(i) Calculating the determinant, we obtain

$$\begin{vmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{vmatrix} = (1+\alpha)^2 - (1-\alpha)^2 = 4\alpha.$$

Thus $(1+\alpha, 1-\alpha)$ and $(1-\alpha, 1+\alpha)$ are linearly independent if and only if $\alpha \neq 0$.
(ii) Calculating the determinant, we obtain

$$\begin{vmatrix} \beta & 1 & 0 \\ 1 & \beta & 1 \\ 0 & 1 & \beta \end{vmatrix} = \beta^3 - 2\beta.$$

Thus $(\beta, 1, 0)$, $(1, \beta, 1)$ and $(0, 1, \beta)$ are linearly independent if and only if $\beta \notin \{-\sqrt{2}, 0, \sqrt{2}\}$.
(iii) Calculating the determinant, we obtain

$$\begin{vmatrix} 1 & 1 \\ \beta & \eta \end{vmatrix} = \eta - \beta.$$

Thus $(1, \beta)$, $(1, \eta)$ are linearly dependent if and only if $\alpha = \eta$. □

Exercise 3.5. *Determine two basis $\mathcal{B}_1$ and $\mathcal{B}_2$ which generates the plane*

$$\pi = \{(x, y, z) \in \mathbb{R}^3 \ ; \ x - 2y - 4z = 0\}$$

such that none element of $\mathcal{B}_1$ be collinear with none element of $\mathcal{B}_2$.

Solution: Note that $\mathcal{B}_1 = \{(2, 1, 0), (4, 0, 1)\}$ is a basis for π, since both vectors satisfy the plane equation and they linearly independent. Now note that

$$v_1 = 1 \cdot (2, 1, 0) + 1 \cdot (4, 0, 1) = (6, 1, 1) \qquad \text{and} \qquad v_2 = 2 \cdot (2, 1, 0) + 1 \cdot (4, 0, 1) = (8, 2, 1)$$

are two linearly independent elements of π and so constitute another basis $\mathcal{B}_2$ for π. It is easy to see that none element of $\mathcal{B}_1$ is collinear with none element of $\mathcal{B}_2$. □

Exercise 3.6. *Construct a basis $\mathcal{B}$ for $\mathcal{P}_2(\mathbb{C})$ such that $\mathcal{B}$ does not contain polynomial of degree 0 and 1.*

Solution: Consider $\mathcal{B} = \{t^2,\, t^2+t,\, t^2+t+1\}$. Note that, given $P(t) = a_0 + a_1 t + a_2 t^2 \in \mathcal{P}_2(\mathbb{C})$, we have

$$P(t) = a_0(1+t+t^2) + (a_1 - a_0)(t+t^2) + (a_2 - a_1)t^2$$

Thus $\mathcal{P}_2(\mathbb{C}) = \mathrm{Span}_{\mathbb{C}}(\mathcal{B})$. Since $\mathcal{B}$ contains $\dim_{\mathbb{C}}(\mathcal{P}_2(\mathbb{C}))$ elements, we conclude that $\mathcal{B}$ is a basis for $\mathcal{P}_2(\mathbb{C})$. $\square$

Exercise 3.7. *Determine the dimension and a basis for the real linear space of symmetric matrices of order n.*

$$S_n(\mathbb{R}) = \{M \in \mathcal{M}_n(\mathbb{R}) \;;\; M = M^t\}$$

Solution: Given $1 \leq j \leq i \leq n$, let $E_{ij} = [z_{ij}]_{(i,j)\in I_n \times I_n}$ be the matrix such that

$$z_{rs} = \begin{cases} 1, & \text{if } r = i \text{ and } s = j; \\ 1, & \text{if } r = j \text{ and } s = i; \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to see that $\mathcal{B} = \{E_{ij} \;;\; 0 \leq j \leq i \leq n\}$ is basis for $S_n(\mathbb{R})$. Finally observe that $\mathcal{B}$ contains

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \text{ elements.}$$

Hence we conclude that $\dim_{\mathbb{R}}(S_n(\mathbb{R})) = n(n+1)/2$. $\square$

Exercise 3.8. *Let $\mathcal{P}_3(\mathbb{R})$ the linear space of polynomial of degree less or equal to 3.*

(i) Prove that $S = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \;;\; p(1-x) = p(1+x)\ \forall\, x \in \mathbb{R}\}$ is a linear subspace of $\mathcal{P}_3(\mathbb{R})$.

(ii) Determine $\dim_{\mathbb{R}}(S)$.

Proof: **(i):** Note that $S \neq \emptyset$, because the zero polynomial belongs to S. Now let $p(x), g(x) \in S$ and $\lambda \in \mathbb{R}$, thus, given $x \in \mathbb{R}$, we have

$$(p + \lambda g)(1-x) = p(1-x) + \lambda g(1-x) = p(1+x) + \lambda g(1+x) = (p+\lambda g)(1+x).$$

Thus $p + \lambda g \in S$, so S is a linear subspace of $\mathcal{P}_3(\mathbb{R})$.

(ii): Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \in S$. By hypothesis, we have that

$$a_3(1-x)^3 + a_2(1-x)^2 + a_1(1-x) + a_0 = a_3(1+x)^3 + a_2(1+x)^2 + a_1(1+x) + a_0$$

for all $x \in \mathbb{R}$. This implies that $a_3 = a_2 = a_1 = 0$. So $S = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \ ; \ p(x) \text{ is constant}\} = \operatorname{Span}_{\mathbb{R}}(\{1\})$. Hence $\dim_{\mathbb{R}}(S) = 1$. $\square$

Exercise 3.9. *Let V be a n-dimensional linear space. Prove the following assertions.*

(i) Every set $\mathcal{B}$ linearly independent with n elements is a basis for V.

(ii) Every set $\mathcal{B}$ of generators with n elements is a basis for V.

Solution: **(i):** Suppose, by contradiction, that $\mathcal{B}$ is not a basis for V. So $\mathcal{B}$ does not generate V. Let $v \in V \setminus \operatorname{Span}_K(\mathcal{B})$. It is not hard to prove that $\mathcal{B} \cup \{v\}$ is still a linearly independent subset of V. Since every set linearly independent can be extended to a basis for V, we can obtain a basis for V with more than $n+1$ elements. This fact contradicts the dimension of V.

(ii): I claim that $\mathcal{B}$ is a linearly independent subset of V. In fact, if not, we can exclude elements of $\mathcal{B}$, obtaining a subset $\mathcal{B}'$ such that

$$\operatorname{Span}_K(\mathcal{B}') = \operatorname{Span}_K(\mathcal{B}) = V.$$

However, we again contradict the definition of dimension. So $\mathcal{B}$ is a linearly independent subset of V. Since, it generates V, we conclude that $\mathcal{B}$ is a basis for V. $\square$

Exercise 3.10. *Determine a basis for linear space of homogeneous solutions of the following linear systems, whose matrix are*

(i) $\begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ 4 & 0 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 6 & -4 \\ -1 & -3 & 2 \end{pmatrix}$

Solution: **(i):** Calculating the linear space N_A of solutions of the homogeneous system associated, we obtain that

$$N_A = \operatorname{Span}_{\mathbb{R}}(\{(10, 7, -8)^t\}).$$

(ii): Calculating the linear space N_A of solutions of the homogeneous system associated, we obtain that

$$N_A = \text{Span}_{\mathbb{R}}(\{(2,0,1)^t, (-3,1,0)^t\}).$$

$\square$

Exercise 3.11. *If* $(1,2,3)^t$ *is a particular solution of*

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{x} = \mathbf{b},$$

determine $\mathbf{b}$ *and the space of solutions of this system.*

Solution: Since $(1,2,3)^t$ is a particular solution, we have that

$$\mathbf{b} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

We know that the space of solutions is the affine space $(1,2,3)^t + V$, where V is the linear space of solutions of the homogeneous system associated. Calculating V, we obtain that $V = \text{Span}_{\mathbb{R}}(\{(1,1,1)^t\})$. Hence

$$S = \{(1,2,3)^t + s(1,1,1)^t \ ; \ s \in \mathbb{R}\}.$$

$\square$

Exercise 3.12. *Consider the following linear operators in* $\mathcal{P}_n(\mathbb{R})$

$$\begin{array}{ccc} \mathcal{D} : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathcal{P}_n(\mathbb{R}) & \qquad & \mathcal{A} : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathcal{P}_n(\mathbb{R}) \\ \sum_{k=0}^n a_k x^k \longmapsto \sum_{k=1}^n k a_k x^{k-1} & & p(x) \longmapsto p(x+1) \end{array}$$

Prove that

$$I + \frac{\mathcal{D}}{1!} + \frac{\mathcal{D}^2}{2!} + \cdots + \frac{\mathcal{D}^n}{n!} = \mathcal{A}.$$

Solution: Consider the linear operator $T : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathcal{P}_n(\mathbb{R})$ such that

$$T = I + \frac{\mathcal{D}}{1!} + \frac{\mathcal{D}^2}{2!} + \cdots + \frac{\mathcal{D}^n}{n!}.$$

Since T and $\mathcal{A}$ are linear operators, in order to show that $T = \mathcal{A}$, it is enough to show that $T(x^i) = \mathcal{A}(x^i)$ for all $i = 0, \dots, n$. However, it is easy, because, given $0 \le i \le n$, then

$$\begin{aligned} T(x^i) &= \sum_{k=0}^{n} \frac{\mathcal{D}^k(x^i)}{k!} = \sum_{k=0}^{i} \frac{\mathcal{D}^k(x^i)}{k!} = \sum_{k=0}^{i} \frac{i(i-1)(i-2)\dots(i-k+1)x^{i-k}}{k!} \\ &= \sum_{k=0}^{i} \frac{i(i-1)(i-2)\dots(i-k+1)}{k!} x^{i-k} = \sum_{k=0}^{i} \binom{i}{k} x^{i-k} = (x+1)^i = \mathcal{A}(x^i). \end{aligned}$$

Hence

$$I + \frac{\mathcal{D}}{1!} + \frac{\mathcal{D}^2}{2!} + \dots + \frac{\mathcal{D}^n}{n!} = T = \mathcal{A}.$$

□

Exercise 3.13. *Let V be an n-dimensional linear space over K. Give two linear mappings $A : V \longrightarrow V$ and polynomials $p(X) \in \mathcal{P}_{n^2}(\mathbb{R})$ such that $p(A) = 0$.*

Solution: Let $\{e_1, \dots, e_n\}$ be a basis for V.

Example 1: Consider the linear mapping

$$\begin{aligned} A : V &\longrightarrow V \\ e_1 &\longmapsto e_1 \\ e_2 &\longmapsto e_2 \\ e_i &\longmapsto 0 \quad \text{for } 3 \le i \le n \end{aligned}$$

Note that A is a projection over the linear subspace generated by $\mathrm{Span}_{\mathbb{R}}(\{e_1, e_2\})$. Thus we have that $A^2 = A$, that is, $A^2 - A = 0$. Hence $p(X) = X^2 - X$ is a polynomial such that $p(A) = 0$.

Example 2: Consider the linear mapping

$$\begin{aligned} A : V &\longrightarrow V \\ e_1 &\longmapsto e_2 \\ e_2 &\longmapsto e_3 \\ &\vdots \\ e_{n-1} &\longmapsto e_n \\ e_n &\longmapsto 0 \end{aligned}$$

Note that A^n is the zero map, because $A^n(e_i) = 0$ for all $1 \le i \le n$. Thus we have that $A^n = 0$. Hence $p(X) = X^n$ is a polynomial such that $p(A) = 0$. □

Exercise 3.14. *Let $\mathcal{L} : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathcal{P}_n(\mathbb{R})$ such that*

$$\mathcal{L}(p(x)) = p(x+1)$$

Determine the matrix of $\mathcal{L}$ associate with basis $\mathcal{B} = \{1, x, \dots, x^n\}$. Determine $\mathcal{R}(\mathcal{L})$ and $\mathcal{N}(\mathcal{L})$.

Solution: It is easy to see that $\mathcal{L}$ is a linear operator. Now note that

$$\begin{aligned}
\mathcal{L}(1) &= 1 \\
\mathcal{L}(x) &= 1 + x \\
&\vdots \\
\mathcal{L}(x^i) &= \sum_{k=0}^{i} \binom{i}{k} x^k \\
&\vdots \\
\mathcal{L}(x^n) &= \sum_{k=0}^{n} \binom{n}{k} x^k.
\end{aligned}$$

Thus, we have that

$$A = [\mathcal{L}]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 1 & \cdots & \binom{i}{0} & \cdots & \binom{n}{0} \\ 0 & 1 & \cdots & \binom{i}{1} & \cdots & \binom{n}{1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \binom{i}{i} & \cdots & \binom{n}{i} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \binom{n}{n} \end{pmatrix}$$

Since A has rank $n+1$, we conclude that A has nullity zero, so $\ker(\mathcal{L}) = 0$. Since $\operatorname{rank}(A) = \dim(\mathcal{P}_n(\mathbb{R})) = n+1$, we conclude that $\mathcal{R}(\mathcal{L}) = \mathcal{P}_n(\mathbb{R})$. $\square$

Exercise 3.15. *Let a and r be non-zero real numbers. Define*

$$A = \begin{pmatrix} ar & ar^2 & ar^3 \\ ar^4 & ar^5 & ar^6 \\ ar^7 & ar^8 & ar^9 \end{pmatrix}.$$

Calculate N_A and $\mathcal{R}_A$.

Solution: By Gauss' elimination method, we have

$$\begin{pmatrix} ar & ar^2 & ar^3 \\ ar^4 & ar^5 & ar^6 \\ ar^7 & ar^8 & ar^9 \end{pmatrix} \xrightarrow{\text{Dividing by } a} \begin{pmatrix} r & r^2 & r^3 \\ r^4 & r^5 & r^6 \\ r^7 & r^8 & r^9 \end{pmatrix} \xrightarrow[\text{to the second line}]{\text{Summing } (-r^3)\times \text{ first line}} \begin{pmatrix} r & r^2 & r^3 \\ 0 & 0 & 0 \\ r^7 & r^8 & r^9 \end{pmatrix}$$

$$\xrightarrow[\text{to the third line}]{\text{Summing } (-r^6)\times \text{ first line}} \begin{pmatrix} r & r^2 & r^3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus we have that

$$N_A = \mathrm{Span}_{\mathbb{R}}\{(r,-1,0)^t, (r^2,0,-1)^t\}.$$

Since $\dim_{\mathbb{R}}(\mathcal{R}_A) = 1$ and $(1, r^3, r^6)^t \in \mathcal{R}_A$, we conclude that $\mathcal{R}_A = \mathrm{Span}_{\mathbb{R}}\{(1, r^3, r^6)^t\}$. □

Exercise 3.16. *Generalize the Exercise 3.15 for a matrix $A \in \mathcal{M}_n(\mathbb{R})$.*

Solution: Proceeding similarly by using the Gauss' elimination method, we obtain the following matrix

$$A = \begin{pmatrix} r & r^2 & \cdots & r^n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus linear space N_A is the space

$$N_A = \mathrm{Span}_{\mathbb{R}}\{(r,-1,0,\ldots,0)^t, (r^2,0,-1,\ldots,0)^t, \ldots, (r^{n-1},0,0,\ldots,-1)^t\}.$$

Since $\dim_{\mathbb{R}}(\mathcal{R}_A) = n - \dim_{\mathbb{R}}(N_A) = 1$ and $(1, r^n, r^{2n}, \ldots, r^{n(n-1)})^t \in \mathcal{R}_A$, we conclude that $\mathcal{R}_A = \mathrm{Span}_{\mathbb{R}}\{(1, r^n, r^{2n}, \ldots, r^{n(n-1)})^t\}$. □

4 Homework IV: Traces and determinants

Exercise 4.1. *Prove that function*

$$D : K^n \times \cdots \times K^n \longrightarrow K$$

$$(a_1, \ldots, a_n) \longmapsto \sum \sigma(p) a_{p_1 1} \ldots a_{p_n n}$$

is a n-linear alternating functional such that $D(e_1, \ldots, e_n) = 1$.

Solution: It is easy to see that this function is n-linear. Suppose that $a_i = a_j$ for some $1 \leq i \neq j \leq n$. Without lost of generality, suppose that $i = 1$, $j = 2$. Note that, given a permutation $p : I_n \longrightarrow I_n$ such that $p(1) = s$, $p(2) = r$, we can get another permutation q such that $q(m) = p(m)$ if $m \neq 1, 2$, $q(1) = r$ and $q(2) = s$. Observe that $\sigma(p) = -\sigma(q)$ and

$$a_{p_1 1} \ldots a_{p_n n} = a_{q_1 1} \ldots a_{q_n n}.$$

So, for each summand of this sum, we can find another with the opposite signal, which implies that

$$D(a_1, \ldots, a_n) = \sum \sigma(p) a_{p_1 1} \ldots a_{p_n n} = 0.$$

That is, this function is alternating n-linear. Finally, if $a_i = e_i$ for each $1 \leq i \leq n$, then

$$D(e_1, \ldots, e_n) = \sum \sigma(p) a_{p_1 1} \ldots a_{p_n n} = \sigma(I) a_{I(1)1} \ldots a_{I(n)n} = 1,$$

because the product $a_{p_1 1} \ldots a_{p_n n}$ will be non-zero only if p be identity. $\square$

Exercise 4.2. *Let A be a matrix whose jth column is e_i. Then*

$$\det(A) = (-1)^{i+j} \det(A_{ij}),$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by striking out the ith row and jth column of A.

Solution: We know that the determinant function is alternating functional n-linear on the columns of the matrix, as well it is alternating functional n-linear on the rows of the matrix. If

$$A = \begin{bmatrix} v_1 & v_2 & \ldots v_{j-1} & e_i & v_{j+1} \ldots v_n \end{bmatrix}$$

, then

$$\det\left(\begin{bmatrix} e_i & v_1 & v_2 & \ldots & v_n \end{bmatrix}\right) = (-1)^j \det(A).$$

Using the now the alternating property on the rows, we conclude that

$$\det\left(\begin{bmatrix} e_i & v_1 & v_2 & \dots & v_n \end{bmatrix}\right) = (-1)^i \det \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A_{ij} \end{pmatrix}$$

Thus we conclude that

$$\det(A) = (-1)^j \det\left(\begin{bmatrix} e_i & v_1 & v_2 & \dots & v_n \end{bmatrix}\right) = (-1)^j(-1)^i \det \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A_{ij} \end{pmatrix} = (-1)^{i+j} \det \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A_{ij} \end{pmatrix}$$

Finally, using the Lemma 4, we conclude that

$$\det(A) = (-1)^{i+j} \det(A_{ij}).$$

□

Exercise 4.3. *Let A be a square matrix. Prove that* $\det(A) = \det(A^T)$.

Solution: In fact, let $A = [\, a_1 \;\; a_2 \;\; \dots \;\; a_n \,]$. By definition, we have that

$$\det(A) = \sum \sigma(p) a_{p_1 1} \dots a_{p_n n}.$$

On the other hand, if $A^t = [\, b_1 \;\; b_2 \;\; \dots \;\; b_n \,]$, we have that $b_{ij} = a_{ji}$ by definition of transpose matrix and

$$\begin{aligned} \det(A^T) = \sum \sigma(p) b_{p_1 1} \dots b_{p_n n} = \sum \sigma(p) a_{1 p_1} \dots a_{n p_n} &= \sum \sigma(p) a_{p^{-1}(p(1))p(1)} \dots a_{p^{-1}(p(n))p(n)} \\ &= \sum \sigma(p^{-1}) a_{p^{-1}(p(1))p(1)} \dots a_{p^{-1}(p(n))p(n)} \end{aligned}$$

where the last equality is because $\sigma(p) = \sigma(p^{-1})$. Now note that,

$$a_{p^{-1}(p(1))p(1)} \dots a_{p^{-1}(p(n))p(n)} = a_{p^{-1}(1)1} \dots a_{p^{-1}(n)n}$$

Thus

$$\det(A^T) = \sum \sigma(p^{-1}) a_{p^{-1}(p(1))p(1)} \dots a_{p^{-1}(p(n))p(n)} = \sum \sigma(p^{-1}) a_{p^{-1}(1)1} \dots a_{p^{-1}(n)n} = \det(A),$$

where the last equality is because

$$\begin{aligned} \psi : S_n &\longrightarrow S_n \\ p &\longmapsto p^{-1} \end{aligned}$$

is a group automorphism. □

Exercise 4.4. *Given a permutation $p \in S_n$, we define an associated matrix called permutation matrix $P = [P_{ij}]_{1\leq i,j\leq n}$ as follows*

$$P_{ij} = \begin{cases} 1, & \text{if } j = p(i) \\ 0, & \text{otherwise.} \end{cases}$$

Show that the action of P on any vector $x \in K^n$ performs the permutation p on the components of x. Show that if p and q are two permutations and P, Q are the associated permutation matrices, then the permutation matrix associated to $p \circ q$ is the product PQ.

Solution: Let $x = (x_1, x_2, \ldots, x_n) \in K^n$ and $y = Px = (y_1, \ldots, y_n)$. A simple computation shows that

$$y_i = \sum_{k=1}^{n} P_{ik} x_k = P_{ip(i)} x_{p(i)} = x_{p(i)}.$$

so P performs the permutation p on the components of x. Now, let p, q permutations in S_n and $P = [P_{ij}]_{1\leq i,j\leq n}$, $Q = [Q_{ij}]_{1\leq i,j\leq n}$ be the associated permutation matrices, respectively. I claim that PQ is the associated permutation matrix of $p \circ q$. Indeed, let $PQ = [a_{ij}]_{1\leq i,j\leq n}$. Given $1 \leq i \leq n$, by definition, we have that

$$a_{ij} = \sum_{k=1}^{n} P_{ik} Q_{kj}$$

Observe that

$$a_{ij} \neq 0 \Longleftrightarrow P_{ik} = 1 \text{ and } Q_{kj} = 1$$

This implies that $a_{ij} \neq 0$ if and only if $k = p(i)$ and $j = q(p(i))$ and, in this case $a_{iq(p(i))} = 1$. So PQ is the associated permutation matrix of $p \circ q$. ☐

Exercise 4.5. *Let A be a matrix $m \times n$, B an $n \times m$ matrix. Show that*

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

Solution: Denote $A = [A_{ij}]$ and $B = [B_{ij}]$. We know that AB is an $m \times m$ matrix and that

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Similarly, We know that BA is an $n \times n$ matrix and that

$$(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}.$$

Finally, observe that

$$\operatorname{tr}(AB) = \sum_{k=1}^{n}(AB)_{kk} = \sum_{k=1}^{m}\left(\sum_{t=1}^{n} A_{kt}B_{tk}\right) = \sum_{t=1}^{n}\left(\sum_{k=1}^{m} A_{kt}B_{tk}\right) = \sum_{t=1}^{n}\left(\sum_{k=1}^{m} B_{tk}A_{kt}\right)$$
$$= \sum_{t=1}^{n}(BA)_{tt} = \operatorname{tr}(BA).$$

□

Exercise 4.6. *Let A be an $n \times n$ matrix and A^T its transpose. Show that*

$$\operatorname{tr}(AA^t) = \sum_{1\leq i,j\leq n} A_{ij}^2.$$

Solution: Indeed, observe that for any $1 \leq i, j \leq n$.

$$(AA^T)_{ij} = \sum_{t=1}^{n} A_{it}(A^T)_{tj} = \sum_{t=1}^{n} A_{it}A_{jt}$$

$$\operatorname{tr}(AA^T) = \sum_{k=1}^{n}(AA^T)_{kk} = \sum_{k=1}^{n}\left(\sum_{t=1}^{n} A_{kt}A_{kt}\right) = \sum_{k=1}^{n}\left(\sum_{t=1}^{n} A_{kt}^2\right) = \sum_{1\leq i,j\leq n} A_{ij}^2.$$

□

Exercise 4.7. *Show that the determinant of the 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $\det(A) = ad - bc$.

Solution: In fact, defining $v_1 = (a, c)^t = ae_1 + ce_2$ and $v_2 = (b, d)^t = be_1 + de_2$, then

$$\det(A) = D(v_1, v_2) = D(ae_1 + ce_2, be_1 + de_2) = abD(e_1, e_1) + adD(e_1, e_2) + cbD(e_2, e_1) + cdD(e_2, e_2)$$
$$= ab \cdot 0 + ad \cdot 1 + cb \cdot (-1) + cd \cdot 0 = ad - bc.$$

□

Lemma 4.8. *Let $p \in S_n$ be a permutation of n elements. If there exists $k \in I_n$ such that $p(k) < k$, then there exists $j \in I_n$ such that $p(j) > j$.*

Proof: Indeed, since p is a permutation, we have that

$$\sum_{k=1}^{n} p(k) = \sum_{k=1}^{n} k.$$

Let p be a permutation such that there exists $k \in I_n$ such that $p(k) < k$. If $p(j) \leq j$ for all $j \in I_n$, we would have that

$$\sum_{k=1}^{n} p(k) < \sum_{k=1}^{n} k,$$

which is an absurd. So there is $i \in I_n$ such that $p(i) > i$. $\square$

Exercise 4.9. *Show that the determinant of an upper triangular matrix equals the product of its elements along the diagonal.*

Solution: Let $A = [\, a_1 \;\; a_2 \;\; \cdots \;\; a_n]$ be an $n \times n$ upper triangular matrix. Note that $a_{ij} = 0$ whenever $i > j$. By definition of determinant, we have that.

$$\det(A) = \sum \sigma(p) a_{p_1 1} \ldots a_{p_n n}.$$

By Lemma 4.8, except when the permutation p is the identity, we always have that $\sigma(p) a_{p_1 1} \ldots a_{p_n n} = 0$, thus

$$\det(A) = \sum \sigma(p) a_{p_1 1} \ldots a_{p_n n} = a_{11} \ldots a_{nn}.$$

$\square$

Exercise 4.10. *How many multiplications does it take to evaluate* $\det(A)$ *by using the Gauss elimination to bring into upper triangular matrix from.*

Solution: Let A be an $n \times n$ matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

In order to obtain the upper triangular matrix from A using the Gauss elimination method, we have to do

$$(n-1) + (n-2) + \cdots + 1 = \frac{(n-1)n}{2}$$

multiplications. Now, in order to calculation the determinant of the upper triangular matrix, we have to do $n-1$ multiplications. So we have to do

$$\frac{(n-1)n}{2} + n - 1 = \frac{(n-1)(n+2)}{2}$$

multiplications. $\square$

Exercise 4.11. *How many multiplications does it take to evaluate* $\det(A)$ *by using the formula*

$$\det(A) = \sum_{p \in S_n} \sigma(p) a_{p_1 1} \dots a_{p_n n}.$$

Solution: Firstly, observe that group of permutations S_n contains $n!$ elements. For each permutation, in order to calculate $\sigma(p)a_{p_1 1} \dots a_{p_n n}$, we need to do n multiplications. Then in order to calculate $\det(A)$ by the formula above, we need to do $n \cdot n!$ products. □

Exercise 4.12. *Let* $M \in M_n(\mathbb{C})$ *be an* $n \times n$ *matrix and* M_{ij} *be the matrix in* $M_{n-1}(\mathbb{C})$ *obtained from* M *by eliminating the row* i *and column* j *of* M. *The cofactors matrix of* M *is the matrix* $\hat{M}$ *whose its elements are* $(\hat{M})_{ij} = \det(M_{ij})$

(i) Show that $\det(M)$ *can be expressed in function the entries of* $\hat{M}$.

(ii) Defining the adjoint matrix of M *by* $\operatorname{adj}(M) = (\hat{M})^t$, *show that*

$$M \operatorname{adj}(M) = \operatorname{adj}(M) M = \det(M) I_n.$$

*Solution:***(i)**: Denoting $M = [a_{ij}]_{1 \le i,j \le n}$, by Laplace formula, we know that

$$\det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \hat{M}_{1k} = \sum_{k=1}^{n} (-1)^{2+k} a_{2k} \hat{M}_{2k} = \dots = \sum_{k=1}^{n} (-1)^{n+k} a_{nk} \hat{M}_{nk}.$$

(ii): Observe that for $i \neq j$, the sum

$$\sum_{k=1}^{n} (-1)^{i+k} a_{ik} \hat{M}_{jk} = 0,$$

because this sum is the determinant of a matrix whose the lines i and j are equal to the line i of M. Similarly, for $i \neq j$, the sum

$$\sum_{k=1}^{n} (-1)^{i+k} a_{ki} \hat{M}_{kj} = 0,$$

because this sum is the determinant of a matrix whose the columns i and j are equal to the columns i of M.

Now call $[c_{ij}]_{1 \le i,j \le n}$ the product $M \operatorname{adj}(M)$. For $i \neq j$, by observation above we have that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} (\hat{M}^t)_{kj} = \sum_{k=1}^{n} a_{ik} \hat{M}_{jk} = 0.$$

If $i = j$, we have

$$c_{ij} = c_{ii} = \sum_{k=1}^{n} a_{ik} (\hat{M}^t)_{ki} = \sum_{k=1}^{n} a_{ik} \hat{M}_{ik} = \det(M),$$

where the last equality is due the part (i). Thus we have that $M\,\mathrm{adj}(M) = \det(M)I_n$.

Similarly, call $[d_{ij}]_{1\le i,j\le n}$ the product $\mathrm{adj}(M)M$. For $i \neq j$, by observation above we have that

$$d_{ij} = \sum_{k=1}^{n} (\hat{M}^t)_{ik} a_{kj} = \sum_{k=1}^{n} \hat{M}_{ki} a_{kj} = 0.$$

and if $i = j$, we have

$$c_{ij} = c_{ii} = \sum_{k=1}^{n} (\hat{M}^t)_{ik} a_{ki} = \sum_{k=1}^{n} a_{ki}\hat{M}_{ki} = \det(M),$$

where the last equality is due the part (i). Thus we have that $\mathrm{adj}(M)M = \det(M)I_n$. $\square$

Exercise 4.13. *Prove that there is a matrix* $M = [a_{ij}]_{1\le i,j\le n}$ *whose entries are* 0 *or* 1 *and* $\det(M) \neq 0$, 1 *or* -1.

Solution: Consider the matrix M

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\det(M) = 2$. $\square$

Exercise 4.14. *Let* $M = [a_{ij}]_{1\le i,j\le n} \in M_n(\mathbb{R})$ *such that* $a_{ij} = i + j$. *Determine* $\det(M)$.

Solution: Indeed, let $M' = [a'_{ij}]_{1\le i,j\le n}$ obtained from M by subtraction the first row from the second row. By determinant properties, we know that $\det(M') = \det(M)$. Note that the second row of M' is $[\ 1\ 1\ \dots\ 1\ ]$.

Similarly, let $M'' = [a''_{ij}]_{1\le i,j\le n}$ obtained from M' by subtraction the first row from the third row. By determinant properties, we know that $\det(M'') = \det(M') = \det(M)$ Note that the third row of M'' is $[\ 2\ 2\ \dots\ 2\ ]$.

Since M'' has two rows linearly dependent, by determinant properties, we have that

$$\det(M) = \det(M'') = 0.$$

$\square$

Exercise 4.15. *Prove or give a counter-example for the following assertions, with* A, $B \in M_n(K)$, *for* $K = \mathbb{R}$, $\mathbb{C}$.

(i) $\det(2A) = 2\det(A)$.

(ii) $\det(A+B) = \det(A) + \det(B)$.

(iii) $\det((A+B)(A-B)) = \det(A^2 - B^2)$.

(iv) $\det(cA) = c^n \det(A)$, *where* $c \in K$.

Solution: **(i):** False. Consider $A = I_2$. We have that $\det(2A) = 4$ and $2\det(A) = 2$.

(ii): False. Consider $A = I_2$, $B = -I_2$. Note that $\det(A) = \det(B) = 1$, while $\det(A+B) = \det(0) = 0$.

(iii): False. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Note that

$$A - B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A + B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $\det((A+B)(A-B)) = \det(A+B)\det(A-B) = 1$.

On the other hand, we have that

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A^2 - B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $\det(A^2 - B^2) = 0$.

(iv): True. Consider $A = [\ v_1\ v_2\ \ldots\ v_n\ ]$, where $v_i \in K^n$ for each $1 \leq i \leq n$. By definition, we have

$$\det(cA) = D(cv_1, cv_2, \ldots, cv_n) = c^n D(v_1, \ldots, v_n) = c^n \det(A).$$

□

Exercise 4.16. *For each values of* a, b, $c \in \mathbb{R}$, *the matrix*

$$M = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

is non-singular. What is the geometrical interpretation of $v = Mx$.

Solution: Calculating the determinant of M, we obtain that $\det(M) = a^2 + b^2 + c^2$. Thus

$$\Sigma := \{(a, b, c) \in \mathbb{R}^3 \ ; \ M \text{ is non-singular}\} = \mathbb{R}^3 \setminus \{0\}.$$

Note that M is a skew-symmetric matrix. The action of an $n \times n$ skew-symmetric matrix in $\mathbb{R}^n$ can be thought as an infinitesimal rotation. $\square$

Exercise 4.17. *Calculate the determinant of the Vandermonde matrix $V(a_1, \ldots, a_n) \in \mathcal{M}_n(\mathbb{R})$. Give a condition over the coefficients $a_1, \ldots, a_n$ such that $V(a_1, \ldots, a_n)$ be non-singular.*

$$V(a_1, \ldots, a_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{pmatrix}$$

Solution: By operations between rows, we can obtain the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^{n-2}(a_2 - a_1) & a_3^{n-2}(a_3 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{pmatrix}.$$

Repeating the proceeding, we obtain the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ 0 & a_2(a_2 - a_1) & a_3(a_3 - a_1) & \cdots & a_n(a_n - a_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^{n-2}(a_2 - a_1) & a_3^{n-2}(a_3 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{pmatrix}.$$

Using the Laplace formula, we obtain that

$$\det(A) = (a_2 - a_1) \cdots (a_n - a_1) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_n \\ a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \end{pmatrix}.$$

Thus, note that we can use the induction and conclude that

$$\det(V(a_1,\dots,a_n)) = \prod_{1\le i<j\le n} (a_j - a_i).$$

Finally note that

$$\begin{aligned} V(a_1,\dots,a_n) \text{ is non-singular} \quad &\iff \quad \det(V(a_1,\dots,a_n)) = \prod_{1\le i<j\le n} (a_j - a_i) \neq 0 \\ &\iff \quad a_i \neq a_j \text{ if } i \neq j. \end{aligned}$$

□

Exercise 4.18. *Let $n > 1$ and $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $a_{ij} = \pm 1$ for all $1 \le i,j \le n$. Prove that $\det(A)$ is even.*

Solution: We will prove by induction on n that $\det(A)$ is divisible by 2^{n-1} whenever all the entries of A are in $\{-1,0,1\}$. For $n = 2$, note that, after the application of Gauss elimination on first column, we obtain the following matrix

$$\begin{pmatrix} 1 & * \\ 0 & a \end{pmatrix}$$

where $a \in \{0,2,-2\}$. Thus

$$\det(A) = \det \begin{pmatrix} 1 & * \\ 0 & a \end{pmatrix} = a,$$

which is divisible by 2. Suppose that this result holds for $n\times n$ matrices and let $A \in M_{(n+1)\times(n+1)}(\mathbb{R})$ whose entries are is $\{-1,0,1\}$. Applying again the Gauss elimination on first column of A, we conclude that

$$\det(A) = \det \begin{pmatrix} 1 & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix},$$

where $\mathbf{0}$ is the zero $n\times 1$-matrix, $\mathbf{B}$ is an $1\times n$ matrix and $\mathbf{C}$ is an $n\times n$ matrix whose all entries are in $\{-2,0,2\}$. Note that $\mathbf{C} = 2\mathbf{D}$, where $\mathbf{D}$ is an $n\times n$ matrix, whose all entries are in $\{-1,0,1\}$. Using the Laplace formula and the induction hypothesis, we conclude that

$$\det(A) = \det(\mathbf{C}) = \det(2\mathbf{D}) = 2^n \det(\mathbf{D}) = 2^{(n+1)-1}\det(\mathbf{D}).$$

□

Exercise 4.19. *Consider the matrices*

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}.$$

Let the linear map $T : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathcal{M}_2(\mathbb{R})$ *such that* $T(X) = AXB$. *Determine* $\mathrm{tr}(T)$ *and* $\det(T)$.

Solution: Consider $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ the ordered canonical basis of $\mathcal{M}_2(\mathbb{R})$. Let determine the matrix of T with respect this basis.

$$\begin{aligned} T(E_{11}) &= \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} = 2E_{11} + E_{12} - 2E_{21} - E_{22}, \\ T(E_{12}) &= \begin{pmatrix} 0 & 4 \\ 0 & -4 \end{pmatrix} = 0E_{11} + 4E_{12} + 0E_{21} - 4E_{22}, \\ T(E_{21}) &= \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix} = 4E_{11} + 2E_{12} + 6E_{21} + 3E_{22}, \\ T(E_{22}) &= \begin{pmatrix} 0 & 8 \\ 0 & 12 \end{pmatrix} = 0E_{11} + 8E_{12} + 0E_{21} + 12E_{22}. \end{aligned}$$

Thus associated matrix of T with the respect the ordered basis $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 4 & 0 \\ 1 & 4 & 2 & 8 \\ -2 & 0 & 6 & 0 \\ -1 & -4 & 3 & 12 \end{pmatrix}$$

So $\mathrm{tr}(T) = \mathrm{tr}([T]_{\mathcal{B}}) = 24$ and $\det(T) = \det([T]_{\mathcal{B}}) = 1600$. $\square$

5 Homework V: Spectral Theory

Exercise 5.1. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Suppose that $(A-\lambda I)^2 f = 0$ and set $h = (A-\lambda I)f$. Prove that*

$$A^n(f) = \lambda^n f + n\lambda^{n-1}h$$

for all $n \in \mathbb{N}$.

Proof: We will proceed by induction. For $n = 1$, it is simply the fact that $(A-\lambda I)(f) = h$, because, as $h = A(f) - \lambda f$, we have that

$$A(f) = \lambda f + h = \lambda^1 f + 1\lambda^0 h.$$

Suppose that this identity is true for $k \leq n$, that is, $A^k(f) = \lambda^k f + k\lambda^{k-1}h$ for all $1 \leq k \leq n$. By hypothesis, we have that $(A-\lambda I)^2(f) = 0$, thus

$$(A-\lambda I)h = (A-\lambda I)\big((A-\lambda I)f\big) = (A-\lambda I)^2 f = 0,$$

which implies that h is an eigenvector of A. Thus, using the induction hypothesis, we conclude

$$A^{n+1}(f) = A^n(A(f)) = A^n(\lambda f + h) = \lambda A^n(f) + \lambda^n h = \lambda(\lambda^n f + n\lambda^{n-1}h) + \lambda^n h =$$
$$\lambda^{n+1} f + (n+1)\lambda^{(n+1)-1}h,$$

which completes the proof. □

Exercise 5.2. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Suppose that $(A-\lambda I)^2 f = 0$ and set $h = (A-\lambda I)f$. Given $q(t) \in \mathcal{P}(\mathbb{R})$, prove that*

$$q(A)(f) = q(\lambda)f + q'(\lambda)h,$$

where q' is the derivative of q.

Proof: In fact, denote

$$q(x) = \sum_{k=0}^{n} a_k x^k.$$

Thus, using the Exercise 5.1, we conclude that

$$q(A)(f) = \left(\sum_{k=0}^{n} a_k A^k\right)(f) = \sum_{k=0}^{n} a_k A^k(f) = \sum_{k=0}^{n} a_k(\lambda^k f + k\lambda^{k-1}h) = \sum_{k=0}^{n} a_k \lambda^k f + \sum_{k=0}^{n} a_k k\lambda^{k-1}h$$
$$= \left(\sum_{k=0}^{n} a_k \lambda^k\right) f + \left(\sum_{k=0}^{n} a_k k\lambda^{k-1}\right) h = q(\lambda)f + q'(\lambda)h.$$

□

Exercise 5.3. *Let A be an $n \times n$-matrix, denote its distinct eigenvalues by $a_1, \dots, a_k$, and denote the index of a_i by d_i. Prove that the minimal polynomial of A is*

$$m_A(s) = \prod_{j=1}^{k} (s - a_j)^{d_j}.$$

Solution: Call

$$p(s) = \prod_{j=1}^{k} (s - a_j)^{d_j}$$

Note that $p(A) = 0$. In fact, let $x \in \mathbb{C}^n$. Denoting the generalized eigenspace of a_i by N_{a_i}, by spectral theorem, we know that there are $v_1 \in N_{a_1}, \dots, v_k \in N_{a_k}$ such that

$$x = \sum_{j=1}^{n} v_j$$

Since $(A - a_i I_n)^{d_i}(v_i) = 0$ and

$$p(A)(v_i) = \left(\prod_{j=1}^{k} (A - a_j I_n)^{d_j} \right)(v_i) = \left(\prod_{j=1, j \neq i}^{k} (A - a_j I_n)^{d_j} \right)(A - a_i I_n)^{d_i}(v_i)$$
$$= \left(\prod_{j=1, j \neq i}^{k} (A - a_j I_n)^{d_j} \right)(0) = 0.$$

By linearity, we conclude that $p(A)(x) = 0$ and so $p(A) = 0$. Thus, by definition of minimal polynomial, we have that

$$m_A(x) = \prod_{j=1}^{k} (s - a_j)^{c_j},$$

where $1 \leq c_i \leq d_i$ for each $i = 1, \dots, k$. Suppose that $1 \leq c_i < d_i$ for some $1 \leq i \leq k$. Without lost of generality, we can suppose that $i = k$. Thus we know that there is $v_0 \in N_k$ such that $(A - a_k I_n)^{c_k}(v_0) \neq 0$. Moreover, since the polynomials

$$p_1(s) = (s - a_1)^{c_1} \qquad p_2(s) = (s - a_2)^{c_2} \qquad \cdots \qquad p_k(s) = (s - a_k)^{c_k}$$

are pairwise without common zero, we conclude that

$$m_A(A)(v_0) = \left(\prod_{j=1}^{k} (A - a_j I_n)^{c_j} \right)(v_0) \neq 0,$$

which is a contradiction, so $d_j = c_j$ for all $1 \leq j \leq k$ and then

$$m_A(s) = p(s) = \prod_{j=1}^{k} (s - a_j)^{d_j}.$$

□

Exercise 5.4. *Consider the sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_0 = 0$, $x_1 = 1$ and $x_{n+2} = x_n + x_{n+1}$ for all $n \in \mathbb{N}$.*

(i) Calculate x_{359}.

(ii) If it exists, determine

$$\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right).$$

Proof: firstly note that

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$$

for all $n \in \mathbb{N}$. Call M the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

By induction in $n \in \mathbb{N}$, we can prove that

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = M^n \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$$

Hence, in order to discover a general formula for the nth term of Fibonacci sequence, it is enough to know the powers of M. If $\mathbb{R}^2$ admits a basis of eigenvectors of M, this calculus becomes easy. Let's calculate the characteristic polynomial of M.

$$p_M(t) = \det\begin{pmatrix} t-1 & -1 \\ -1 & t \end{pmatrix} = \left(t - \frac{1+\sqrt{5}}{2}\right)\left(t - \frac{1-\sqrt{5}}{2}\right)$$

Thus the eigenvalues of M are $a_1 = (1+\sqrt{5})/2$ and $a_2 = (1-\sqrt{5})/2$. Now we will calculate the eigenvectors of M.

- For $a_1 = (1+\sqrt{5})/2$, we have the following eigenvector

$$v_1 = \begin{pmatrix} a_1 \\ 1 \end{pmatrix}.$$

- For $a_2 = (1-\sqrt{5})/2$, we have the following eigenvector

$$v_2 = \begin{pmatrix} a_2 \\ 1 \end{pmatrix}.$$

Since $\mathcal{B}$ is a basis of $\mathbb{R}^2$, we can write $(x_1, x_0) = (1, 0) = e_1$ as linear combination of v_1 and v_2, obtaining

$$e_1 = 1/(\sqrt{5})v_1 - 1/(\sqrt{5})v_2.$$

So

$$(x_{n+1}, x_n) = M^n(e_1) = M^n\left(\frac{1}{\sqrt{5}}v_1 - \frac{1}{\sqrt{5}}v_2\right) = \frac{1}{\sqrt{5}}M^n(v_1) - \frac{1}{\sqrt{5}}M^n(v_2) = \frac{a_1^n v_1 - a_2^n v_2}{\sqrt{5}}.$$

Looking at the second coordinate, we conclude that

$$x_n = \frac{a_1^n - a_2^n}{\sqrt{5}}.$$

One interesting observation is that, as $a_2^n/\sqrt{5}$ is always less than $1/2$ and x_n is always integer, then x_n is the integer number nearnest than $a_1^n/\sqrt{5}$.

(ii): In fact

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) &= \lim_{n\to\infty}\left(\frac{a_1^{n+1} - a_2^{n+1}}{a_1^n - a_2^n}\right) = \lim_{n\to\infty}\left(\frac{a_1^{n+1}(1-(a_2/a_1)^{n+1})}{a_1^n(1-(a_2/a_1)^n)}\right) \\ &= \lim_{n\to\infty}\left(a_1\frac{1-(a_2/a_1)^{n+1}}{1-(a_2/a_1)^n}\right) = a_1,\end{aligned}$$

where the last equality is because $|a_2/a_1| < 1$. □

Exercise 5.5 (Cayley-Hamilton Theorem)**.** *Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a matrix. Prove that $p_A(A) = 0$, where $p_A(t)$ is the characteristic polynomial of A.*

(i) Prove the Cayley-Hamilton theorem on case of all eigenvalues are distinct.

(ii) Prove the Caylay-Hamilton Theorem on general case.

Proof: **(i):** Since all eigenvalues $a_1, \ldots, a_n$ are distinct, $\mathbb{R}^n$ admits a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of eigenvectors of A. Let $v \in \mathbb{R}^n$, we know that there are $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$v = \sum_{k=1}^n c_k v_k$$

Thus

$$p_A(A)(v) = p_A(A)\left(\sum_{k=1}^n c_k v_k\right) = \sum_{k=1}^n c_k p_A(A)(v_k) = \sum_{k=1}^n c_k p_A(a_k) v_k = \sum_{k=1}^n c_k \cdot 0 \cdot v_k = 0.$$

Since $v \in \mathbb{R}^n$ is arbitrary, we conclude that the operator $p_A(A) = 0$.

(ii): Consider the polynomials with matrix coefficients $R(s) = sI_n - A$ and let $S(s) = \operatorname{adj}(R(s))$ be its adjoint matrix. By identity of adjoint matrix, we have

$$R(s)S(s) = (sI_n - A)\operatorname{adj}(sI_n - A) = p_A(s)I_n$$

Since $S(A) = AI_n - A = 0$, then we have that $p_A(A) = p_A(A)I_n = 0$. $\square$

Exercise 5.6. *Determine the eigenvalues, eigenvectors and generalized eigenvectors of the following complex matrices*

(i) $M = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(ii) $M = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

(iii) $M = \begin{pmatrix} i & 1 \\ 0 & -1+i \end{pmatrix}$

Proof: **(i)** Calculating the characteristic polynomial of M, we obtain

$$p_M(t) = \det(tI_3 - M) = \det \begin{pmatrix} t & 1 & -1 \\ -1 & t-2 & -1 \\ -1 & -1 & t-2 \end{pmatrix} = t(t-1)(t-3).$$

Thus the eigenvalues of M are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 3$. Now it is enough to calculate the eigenvectors. Since each eigenvalue has multiplicity, then the eigenvectors and generalized eigenvectors coincide.

(ii): Calculating the characteristic polynomial of M, we obtain

$$p_M(t) = \det(tI_2 - M) = \det \begin{pmatrix} t-3 & -1 \\ -1 & t-3 \end{pmatrix} = (t-2)(t-4).$$

Thus the eigenvalues of M are $\lambda_1 = 2$ and $\lambda_2 = 4$. Now it is enough to calculate the eigenvectors. Since each eigenvalue has multiplicity, then the eigenvectors and generalized eigenvectors coincide.

(iii): Calculating the characteristic polynomial of M, we obtain

$$p_M(t) = \det(tI_2 - M) = \det \begin{pmatrix} t-i & -1 \\ 0 & t-(-1+i) \end{pmatrix} = (t-i)(t-(-1+i)).$$

Thus the eigenvalues of M are $\lambda_1 = i$ and $\lambda_2 = -1+i$. Now it is enough to calculate the eigenvectors. Since each eigenvalue has multiplicity, then the eigenvectors and generalized eigenvectors coincide. $\square$

Exercise 5.7. *Determine the eigenvalues and eigenvectors of the matrix*

$$M = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. What is the interpretation of $v = Mx$?

Proof: Calculating the characteristic polynomial of M, we obtain

$$p_M(t) = \det(tI_3 - M) = \det \begin{pmatrix} t & -c & b \\ c & t & -a \\ -b & a & t \end{pmatrix} = t(t^2 + (a^2 + b^2 + c^2)).$$

Thus the eigenvalues of M are $\lambda_1 = 0$, $\lambda_2 = (a^2 + b^2 + c^2)i$ and $\lambda_3 = -(a^2 + b^2 + c^2)i$. Now it is enough to calculate the eigenvectors. $\square$

Exercise 5.8. *A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is nilpotent if there is $k > 0$ such that $A^k = 0$. Prove that 0 is the unique eigenvalue of A. We*

Solution: Since the $\mathbb{C}$ is an algebraically closed field, we know that A admits an eigenvalue. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathbb{C}$ and $v \in \mathbb{C}^n$ be an eigenvector associated. Then we have that

$$A(v) = \lambda v.$$

Since $A^k = 0$, we have that $0 = A^k(v) = \lambda^k v$. Finally, since $v \neq 0$, we conclude $\lambda^k = 0$, which implies that $\lambda = 0$. $\square$

Exercise 5.9. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, with $n > 1$. Prove that there exists a linear subspace $M \subseteq \mathbb{R}^n$, with $\dim_{\mathbb{R}}(M) = 2$, such that $T(M) \subseteq M$.*

Solution: We know that $\dim_{\mathbb{R}}(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)) = n^2$. Consider the set $\{I, T, T^2, \dots, t^{n^2}\} \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ Since this set has $n^2 + 1$ elements, it is linearly dependent, so there are $a_0, \dots, a_{n^2} \in \mathbb{R}$ not all zeros such that

$$a_0 + a_1 T + a_2 T^2 + \cdots + a_{n^2} T^{n^2} = 0.$$

Considering the polynomial $p(t) = a_0 + a_1 t + \cdots + a_{n^2} t^{n^2}$, we conclude that there is a nonzero polynomiam $f(t) \in \mathbb{R}[t]$ such that $f(T) = 0$. It is well known that $f(t)$ can be factored as

$$f(t) = \prod_{k=1}^{s} g_k(t),$$

where $g_k(t)$ is a polynomial with degree 1 or 2 for each $1 \leq k \leq s$. Thus

$$\prod_{k=1}^{s} g_k(A) = g_1(A) g_2(A) \cdots g_s(A) = 0$$

This fact implies that there is $1 \leq i \leq s$ such that $g_i(T)$ is not invertible. Now we have two possible cases: $\deg(g_i) = 2$ for at least one i, or $\deg(g_i) = 1$ for all i.

- $\deg(g_i) = 2$ for at least one i: In this case, denoting $g(t) = t^2 + a_1 t + a_0$, we have that

  $$T^2 + a_1 T + a_2 I = 0$$

  If $v \neq 0 \in \mathbb{R}^n$ is such that $T^2(v) + a_1 T(v) + a_2 v = 0$, then it can be easily proved that $\{T(v), v\}$ are linearly independent vectors and, denoting $W = \mathrm{Span}_{\mathbb{R}}(\{v, T(v)\})$, we have that $T(W) \subseteq W$.

- Observe that if $g(t) = t - \lambda$ is a polynomial such that

  $$g(T) = T - \lambda I = 0,$$

  then, since there is $v \neq 0 \in \mathbb{R}^n$ such that $0 = g(T) = T(x) - \lambda I(x) = 0$, we conclude that v is eigenvector of T. Since $n > 1$ and all polynomials of factorization are linear, we conclude that T has two linearly independent eigenvectors w_1 and w_2. Finally, denoting $W = \mathrm{Span}_{\mathbb{R}}(\{w_1, w_2\})$, we have that $\dim(W) = 2$ and $T(W) \subseteq W$.

$\square$

Exercise 5.10. *Let $\pi \in \mathcal{P}_n(\mathbb{C})$ and $D : \mathcal{P}_n(\mathbb{C}) \longrightarrow \mathcal{P}_n(\mathbb{C})$ such that*

$$D(p(x)) = \frac{dp(x)}{dx}.$$

Determine the minimal polynomial of $\pi(D)$.

Solution: Consider $\pi(x) = a_m X^m + \cdots + a_1 x^1 + a_0$, thus

$$\pi(D) = a_m D^m + \cdots + a_1 D + a_0 I.$$

Let $\{1, x, \dots, x^n\}$ the canonical basis of $\mathcal{P}_n(\mathbb{R})$. Thus

$$\pi(D)(1) = a_0;$$
$$\pi(D)(x) = a_0 x + a_1;$$
$$\vdots$$
$$\pi(D)(x^n) = a_0 x^n + a_1 n x^{n-1} + a_m n(n-1)\cdots(n-m+1)x^{n-m}.$$

Note that, if A is the matrix of $\pi(D)$, then A is an upper triangular matrix whose all entries on diagonal are a_0. This fact implies that $p_{\pi(D)} = (x - a_0)^{n+1}$. So we conclude that the minimal polynomial of $\pi(D)$ is a power $g(t) = (t - a_0)^r$, with $1 \leq r \leq n+1$. However, note that

$$g(A) = (a_m D^m + \cdots + a_1 D)^r.$$

Let $k = \min\{t \in \{1, \dots, m\} \ ; \ a_k \neq 0\}$. Thus

$$g(A) = (a_m D^m + \cdots + a_k D^k)^r.$$

Let $r_0 = \max\{t \in \{1, \dots, n\} \ ; \ kt \leq n\}$. It can be proved easily that $(a_m D^m + \cdots + a_k D^k)^{r_0} \neq 0$, thus

$$g(t) = (t - a_0)^{r_0+1}$$

is the minimal polynomial of $\pi(D)$. $\square$

Exercise 5.11. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that*

$$A^2 + 2A + 5I_n = 0.$$

Prove that n is even.

Solution: Consider the polynomial $f(t) = t^2 + 2t + 5 \in \mathbb{R}[t]$. Note that this polynomial is irreducible in $\mathbb{R}[t]$, because $f(t)$ has no real roots in $\mathbb{R}$. Since $f(T) = 0$ and f is irreducible in $\mathbb{R}[t]$, we conclude that $f(t)$ is the minimal polynomial of A. Let λ and $\overline{\lambda}$ be the roots of f in $\mathbb{C}$. Since the roots of f are the same that the roots of the characteristic polynomial $p_A(t)$ up to their multiplicities, then we get that

$$p_A(t) = (x - \lambda)^r (x - \overline{\lambda})^s.$$

Finally, as the coefficients of $p_A(t)$ are in $\mathbb{R}$, it is necessary that $r = s$, then

$$p_A(t) = (x - \lambda)^r (x - \overline{\lambda})^r = (x^2 - |\lambda|^2)^r,$$

which is a polynomial with degree $2r$. Since $n = \deg(p_A) = 2r$, we conclude that n is even. $\square$

Exercise 5.12. *Let $I_n, J_n \in \mathcal{M}_n(\mathbb{R})$, where I_n is the identity matrix and $[J_n]_{ij} = 1$ for all $1 \leq i, j \leq n$.*

(i) Calculate the determinant, eigenvalues and eigenvectors of I_n and J_n.

(ii) Calculate the eigenvalues, eigenvectors of $A_{a,b} = aI_n + bJ_n$, onde $a, b \in \mathbb{R}$.

(iii) Calculate the determinant of $A_{a,b}$

Let $\mathcal{X} = \{x_1, \ldots, x_m\}$ be a set and $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a family of subsets of X. We define the incidence of family $\mathcal{A}$ as the matrix $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ such that

$$B_{ij} = \begin{cases} 1, & \text{if } x_j \in A_i; \\ 0, & \text{if } x_j \notin A_i. \end{cases}$$

What is the meaning of BB^t, where B^t is the transpose of B? Calculate the eigenvalues of BB^t.

Solution: **(i):** By definition of determinant, $\det(I_n) = 1$. Moreover, since $p_{I_n}(t) = (t-1)^n$, we conclude that $\lambda_1 = 1$ is the unique eigenvalue of I_n and it has geometric multiplicity n. Since $I_n(e_i) = e_i$ for all $1 \leq i \leq n$, we conclude $\{e_1, \ldots, e_n\}$ are eigenvectors of I_n.

Since J_n has two repeated rows, we have that $\det(J_n) = 0$. Now note $\mathcal{N}_{J_n}$ is a linear space with dimension $n-1$, which implies that $\lambda_1 = 0$ is an eigenvalue of J_n with multiplicity algebraic $\geq n-1$. However, since $J_n^2 - nJ_n = 0$ and $J_n \neq 0$, we conclude that $\lambda_2 = n$ is also an eigenvalue of J_n, whence the eigenvalues of J_n are $\lambda_1 = 0$ and $\lambda_2 = n$.

- For $\lambda_1 = 0$, it is enough to calculate a basis for $\mathcal{N}_{J_n}$ and we can see easily that

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad v_{n-1} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

  constitute a basis.

- For $\lambda_1 = n$, it is easy to see that $\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}^t$ is an eigenvector associated.

(ii): Firstly observe that

$$aI_n + bJ_n = \begin{pmatrix} a+b & b & b & \cdots & b \\ b & a+b & b & \cdots & b \\ b & b & a+b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a+b \end{pmatrix}$$

I claim that $\lambda_1 = a + nb$ is an eigenvalue with multiplicity 1 and $\lambda_2 = a$ is an eigenvalue with multiplicity $n-1$. Indeed

- For $\lambda_1 = a + nb$, note that

$$\begin{pmatrix} a+b & b & b & \cdots & b \\ b & a+b & b & \cdots & b \\ b & b & a+b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a+b \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a+nb \\ a+nb \\ a+nb \\ \vdots \\ a+nb \end{pmatrix} = (a+nb) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- For $\lambda_1 = a$, note that

$$\begin{pmatrix} a+b & b & b & \cdots & b \\ b & a+b & b & \cdots & b \\ b & b & a+b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a+b \end{pmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -a \\ a \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} a+b & b & b & \cdots & b \\ b & a+b & b & \cdots & b \\ b & b & a+b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a+b \end{pmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \\ a \\ \vdots \\ 0 \end{bmatrix} = a \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vdots$$

$$\begin{pmatrix} a+b & b & b & \cdots & b \\ b & a+b & b & \cdots & b \\ b & b & a+b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a+b \end{pmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \\ 0 \\ \vdots \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Thus the following vectors are eigenvectors of $aI_n + bJ_n$ associated to $\lambda_2 = a$.

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad v_{n-1} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

(iii): We know that the determinant of a matrix whose characteristic polynomial splits can be given by the product of all eigenvalues. Thus

$$\det(A_{a,b}) = a^{n-1}(a + nb).$$

Let $C := B^t = [c_{ij}]_{1 \le i,j \le n}$. By definition, we have that

$$[BB^t]_{ij} = \sum_{k=1}^{m} b_{ik} c_{kj} = \sum_{k=1}^{m} b_{ik} b_{jk}.$$

Note that

$$b_{ik} = 1 = b_{jk} \iff x_k \in A_i \cap A_j.$$

Thus

$$[BB^t]_{ij} = \sum_{k=1}^{m} b_{ik} b_{jk} = \operatorname{card}(A_i \cap A_j).$$

Hence BB^t is an $n \times n$-matrix such that $[BB^t]_{ij} = \operatorname{card}(A_i \cap A_j)$. $\square$

Exercise 5.13 (Fisher's inequality)**.** *Let $\mathcal{X} = \{x_1, \ldots, x_m\}$ be a set and $A_1, \ldots, A_n$ be subsets of $\mathcal{X}$ such that $\operatorname{card}(A_i) = k$ for all $1 \le i \le n$. If every intersection $A_i \cap A_j$, $1 \le i \ne j \le n$, contains λ elements, prove that $n \le m$.*

Solution: In fact, consider B the incidence matrix of family $\mathcal{A} = \{A_1, \ldots, A_n\}$. Since all subsets has k elements, every element of diagonal of BB^t is k. Moreover, since every intersection $A_i \cap A_j$ has λ elements, then every element out of the diagonal of BB^t is λ. Hence, using the notation of the Exercise 5.12, we conclude that

$$BB^t = (k - \lambda)I_n + \lambda J_n.$$

Using the Exercise 5.12 again, we conclude that the eigenvalues of BB^t are $\lambda_1 = k - \lambda$ with multiplicity $n - 1$ and $\lambda_2 = (k - \lambda) + n\lambda = k + (n - 1)\lambda$ with multiplicity 1. Since the sets are distinct, we have that $k > \lambda$, so every eigenvalue is nonzero and so $\text{rank}(BB^t) = n$. Since B^t has m columns and the rank of the product does not exceed the rank of the factors, we conclude that

$$m \geq \text{rank}(B^t) \geq \text{rank}(BB^t) = n,$$

so $n \leq m$. $\square$

6 Homework VI: Euclidean Structures I

Exercise 6.1. *Let* $(X, \langle \ , \ \rangle_1)$ *and* $(U, \langle \ , \ \rangle_2)$ *be a real finite-dimensional linear spaces with inner dot. Given* $A : X \longrightarrow U$ *a linear transformation, let denote* $A^* : U \longrightarrow A$ *the adjoint operator of* A*, prove that*

(i) If A *and* B *are linear operators from* X *to* U*, then*

$$(A + B)^* = A^* + B^*.$$

(ii) If $A : X \longrightarrow U$ *and* $C : U \longrightarrow V$ *are linear operators, where* $(C, \langle \ , \ \rangle_3)$ *is a liner space with inner dot, then*

$$(BA)^* = A^* B^*.$$

(iii) If $A : X \longrightarrow U$ *is an isomorphism, then* A^* *is also an isomorphism and*

$$(A^*)^{-1} = (A^{-1})^*$$

(iv) Given $A : X \longrightarrow U$ *and* $C : U \longrightarrow V$ *a linear operator, then*

$$(A^*)^* = A.$$

Solution: **(i):** Let $x \in X$ and $y \in U$, then, by definition, we have that

$$\langle A(x), y \rangle_2 = \langle x, A^*(y) \rangle_1 \qquad \text{and} \qquad \langle B(x), y \rangle_2 = \langle x, B^*(y) \rangle_1.$$

Thus

$$\begin{aligned} \langle x, (A + B)^*(y) \rangle_1 = \langle (A + B)(x), y \rangle_2 = \langle A(x) + B(x), y \rangle_2 = \langle A(x), y \rangle_2 + \langle B(x), y \rangle_2 \\ = \langle x, A^*(y) \rangle_1 + \langle x, B^*(y) \rangle_1 = \langle x, (A^* + B^*)(y) \rangle_1 \end{aligned}$$

Since $x \in X$ and $y \in U$ are chosen arbitrarily, we conclude that $(A + B)^* = A^* + B^*$.

(ii): Let $x \in X$, $y \in U$ and $z \in V$. Then, by definition, we have

$$\langle A(x), y \rangle_2 = \langle x, A^*(y) \rangle_1 \qquad \text{and} \qquad \langle C(y), z \rangle_3 = \langle y, C^*(z) \rangle_2.$$

Thus

$$\begin{aligned} \langle x, (AC)^*(z) \rangle_1 = \langle (CA)(x), z \rangle_3 = \langle C(A(x)), z \rangle_3 = \langle A(x), C^*(z) \rangle_2 = \langle x, A^*(C^*(z)) \rangle_1 \\ = \langle x, (A^* C^*)(z) \rangle_1. \end{aligned}$$

Since $x \in X$ and $z \in V$ are chosen arbitrarily, we conclude that $(CA)^* = A^*C^*$.

(iii): It is easy to see that the adjoint of identity operator is the identity itself, that is, $I_X^* = I_X$. Thus, since $A^{-1}A = I_X$ and $AA^{-1} = I_U$, we have that

$$A^*(A^{-1})^* = (A^{-1}A)^* = (I_X)^* = I_X \quad \text{and} \quad (A^{-1})^*A^* = (AA^{-1})^* = (I_U)^* = I_U.$$

Thus A^* is an isomorphism and $(A^*)^{-1} = (A^{-1})^*$.

(iv): Let $x \in X$ and $y \in U$. Then, by definition, we have

$$\langle A(x), y\rangle_2 = \langle x, A^*(y)\rangle_1 = \langle A^*(y), x\rangle_1 = \langle y, ((A^*)^*)(x)\rangle_2 = \langle((A^*)^*)(x), y\rangle_2$$

Since $x \in X$ and $z \in V$ are chosen arbitrarily, we conclude that $((A^*)^*)(x) = A(x)$ for all $x \in X$, so $(A^*)^* = A$. $\square$

Exercise 6.2. *Let $(X, \langle \ , \ \rangle)$ be a real finite-dimensional linear spaces with inner dot and $Y \subseteq X$ be a linear subspace. Prove that orthogonal projection on Y, denoted by p_Y, is such that $p_Y = (p_Y)^*$.*

Solution: Let $x = u + u^\perp \in X$ be an arbitrary element of X. We have to prove that $(p_Y)^*(x) = u = (p_Y)(x)$. So let $y = v + v^\perp \in X$, note that

$$\begin{aligned}\langle (p_Y)^*(x) - u, y\rangle &= \langle (p_Y)^*(x), y\rangle - \langle u, y\rangle = \langle x, p_Y(y)\rangle - \langle u, y\rangle = \langle u + u^\perp, v\rangle - \langle u, v + v^\perp\rangle \\ &= \langle u, v\rangle - \langle u, v\rangle = 0.\end{aligned}$$

Since $y \in X$ was chose arbitrarily, we conclude that $(p_Y)^*(x) = u = p_Y(x)$. Finally, since $x \in X$ was chosen arbitrarily, we conclude that $p_Y = (p_Y)^*$. $\square$

Exercise 6.3. *Let V be a real finite-dimensional linear space with inner dot. Prove that for all x, $y \in V$, we have*

(i) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

(ii) $4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2$

Solution: **(i):** In fact

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y\rangle + \langle x - y, x - y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \\ \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle = 2(\langle x, x\rangle + \langle x, x\rangle) = 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

(ii): In fact

$$\begin{aligned}\|x+y\|^2 - \|x-y\|^2 &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - (\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle)\\ &= 2\langle x,y\rangle + 2\langle y,x\rangle = 2\langle x,y\rangle + 2\langle x,y\rangle = 4\langle x,y\rangle.\end{aligned}$$

□

Exercise 6.4. *Considering the linear space $\mathbb{R}^4$ with the usual inner dot, find an orthonormal basis for the subspace generated by the vectors*

$$v_1 = (1,1,0,0) \qquad v_2 = (1,1,1,1) \qquad v_3 = (-1,0,2,1)$$

Solution: We will use the Gram-schmidt method. Consider $w_1 = v_1 = (1,1,0,0)$. Thus

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (1,1,1,1) - \frac{2}{2}(1,1,0,0) = (0,0,1,1).$$

and

$$\begin{aligned} w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 &= (-1,0,2,1) - \frac{-1}{2}(1,1,0,0) - \frac{3}{2}(0,0,1,1)\\ &= (-1/2, 1/2, 1/2, -1/2).\end{aligned}$$

Normalizing, we conclude that

$$u_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0\right) \qquad u_2 = \left(0, 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \qquad u_3 = (-1/2, 1/2, 1/2, -1/2).$$

is an orthonormal basis for this subspace. □

Exercise 6.5. *Let V be a real finite-dimensional linear space with inner dot. Consider $\{v_1, v_2, v_3\}$ be a basis for V. Is it true that the Gram-Schmidt method produces the same bases when applied to $\{v_1, v_2, v_3\}$ and $\{v_2, v_1, v_3\}$?*

Solution: No. Indeed, consider $V = \mathbb{R}^3$ and $V_1 = (1,0,0)$, $v_2 = (1,1,0)$ and $v_3 = (1,1,1)$. Considering $\mathcal{B} = \{v_1, v_2, v_3\}$ and $\mathcal{B}' = \{v_2, v_1, v_3\}$, the Gram-Schmidt method applied to these bases gives us the orthonormal bases

$$\begin{aligned}\mathcal{B} = \{v_1, v_2, v_3\} &\longrightarrow \{(1,0,0), (0,1,0), (0,0,1)\},\\ \mathcal{B}' = \{v_2, v_1, v_3\} &\longrightarrow \left\{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right), (0,0,1)\right\}.\end{aligned}$$

□

Exercise 6.6. *The set* $\{(1,1),(2,-1)\}$ *is an orthonormal basis of* $\mathbb{R}^2$ *with the respect the inner dot* $\langle\ ,\ \rangle$. *Describe this inner dot.*

Solution: We know that an inner dot is a positive, symmetric $\mathbb{R}$-bilinear functional. Thus, in order to decribe $\langle\ ,\ \rangle$, it is enough to know $\langle e_i, e_j\rangle$ for all $1 \le i, j \le 2$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. Since this basis is orthonormal, we have that

$$\langle e_1 + e_2, e_1 + e_2\rangle = \|(1,1)\|^2 = 1.$$

Thus

$$\langle e_1, e_1\rangle + 2\langle e_1, e_2\rangle + \langle e_2, e_2\rangle = 1.$$

Similarly, since $\langle 2e_1 - e_2, 2e_1 - e_2\rangle = \|(2,-1)\|^2 = 1$, we have

$$4\langle e_1, e_1\rangle - 4\langle e_1, e_2\rangle + \langle e_2, e_2\rangle = 1.$$

Finally, since $\langle (1,1),(2,-1)\rangle = 0$, we have

$$2\langle e_1, e_1\rangle + \langle e_1, e_2\rangle - \langle e_2, e_2\rangle = 0.$$

Hence we obtain the following linear system

$$\begin{cases} \langle e_1, e_1\rangle + 2\langle e_1, e_2\rangle + \langle e_2, e_2\rangle = 1 \\ 4\langle e_1, e_1\rangle - 4\langle e_1, e_2\rangle + \langle e_2, e_2\rangle = 1 \\ 2\langle e_1, e_1\rangle + \langle e_1, e_2\rangle - \langle e_2, e_2\rangle = 0 \end{cases}$$

Solving this linear system, we conclude that

$$\begin{cases} \langle e_1, e_1\rangle = 2/9 \\ \langle e_1, e_2\rangle = 1/9 \\ \langle e_2, e_2\rangle = 5/9 \end{cases}$$

Thus

$$\langle (x,y),(x',y')\rangle = \frac{2xx'}{9} + \frac{xy' + x'y}{9} + \frac{5yy'}{9}.$$

$\square$

Exercise 6.7. *Consider* $V = C^0([a,b])$ *the linear space of the real continuous functions defined on compact* $[a,b]$.

(i) Prove that the function $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{R}$ *such that*

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx$$

is an inner dot and that it induces the norm $\|.\| : V \longrightarrow \mathbb{R}$ *such that*

$$\|f\|_{L^2(a,b)} = \int_a^b |f(x)|^2 dx.$$

(ii) Determine all polynomials with degree 2 *orthogonal to* $p_0(x) = 1$ *and* $p_1(x) = x$ *in* $L^2(-1, 1)$.

(iii) Calculate an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$ *with respect to inner dot defined above and starting from the basis* $\{1, t, t^2\}$[1].

Solution: **(i):** In fact, given $f,\ g\ h \in C^0([a, b])$ and $\lambda \in \mathbb{R}$, then, by integral properties we have that

$$\begin{aligned}\langle \lambda f + g, h\rangle &= \int_a^b (\lambda f(x) + g(x))h(x)dx = \int_a^b \lambda f(x)h(x) + g(x)h(x)dx\\ &= \lambda \int_a^b f(x)h(x)dx + \int_a^b g(x)h(x)dx = \lambda\langle f, h\rangle + \langle g, h\rangle.\end{aligned}$$

Similarly, we can prove that $\langle f, \lambda g + h\rangle = \lambda\langle f, g\rangle + \langle f, h\rangle$. Moreover

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = \langle g, f\rangle$$

Thus we conclude that $\langle\ ,\ \rangle$ is a symmetric $\mathbb{R}$-bilinear functional. Finally, given $f \in C^0([a, b])$, we have that

$$\|f\| = \langle f, f\rangle = \int_a^b f(x)^2 dx = \int_a^b |f(x)|^2 dx$$

and this integral is zero if and only if f vanishes everywhere (Classical exercise of real analysis). Hence, $\langle\ ,\ \rangle$ is a positive, symmetric $\mathbb{R}$-bilinear functional, that is, an inner dot.

(ii): We want to find polynomials $p(x) = a_0 + a_1x + a_2x^2$ such that

$$\begin{aligned}0 = \langle p_0, p\rangle &= \int_{-1}^1 p_0(x)p(x)dx = \int_{-1}^1 a_0 + a_1x + a_2x^2 dx = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3\Big|_{-1}^1 = 2a_0 + \frac{2a_2}{3}\\ 0 = \langle p_1, p\rangle &= \int_{-1}^1 p_1(x)p(x)dx = \int_{-1}^1 a_0x + a_1x^2 + a_2x^3 dx = \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_2}{4}x^4\Big|_{-1}^1 = \frac{2a_1}{3}\end{aligned}$$

Thus the family of polynomials with degree 2 and simultaneously orthonormal to p_0 and p_1 is

$$\mathcal{S} = \{p(x) = -a + (3a)x^2\ ;\ a \in \mathbb{R}\}.$$

[1]The obtained polynomials are called Legendre's polynomials.

(iii): We will use the Gram-schmidt method. Consider $w_1 = 1$. Thus

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} 1 = x,$$

and

$$\begin{aligned} w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 &= x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} 1 - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} x \\ &= x^2 - \frac{1}{3} \end{aligned}$$

Normalizing these vectors, we conclude that

$$u_1(x) = \frac{\sqrt{2}}{2} \qquad u_2(x) = \frac{\sqrt{6}}{2} x \qquad u_3(x) = \frac{3\sqrt{10}}{4} x^2 - \frac{\sqrt{10}}{4}$$

is an orthonormal basis of $\mathcal{P}_2((-1,1))$. □

Exercise 6.8. *A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is said definite positive if the quadratic form $q(x) = x^t M x$ is such that $q(x) > 0$ for all $x \neq 0 \in \mathbb{R}^n$. Consider the Matrix G such that*

$$[G]_{ij} = \langle v_i, v_j\rangle \quad 1 \leq i, j \leq n,$$

where $\langle\ ,\ \rangle$ is the usual inner dot of $\mathbb{R}^n$. Prove that G is definite positive if and only if $\{v_1, \ldots, v_n\}$ are linearly independent.

Solution: Suppose that G is definite positive. Let $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

Performing the scalar product with each v_i, we conclude that

$$a_1\langle v_1, v_i\rangle + a_2\langle v_2, v_i\rangle + \cdots + a_n\langle v_n, v_i\rangle = 0 \quad \text{for each } i = 1, \ldots, n.$$

Putting all these n equations together, we get

$$G \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence

$$q(a_1,\ldots,a_n) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} G \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0.$$

Since G is definite positive, we conclude that $a_1 = a_2 = \cdots = a_n = 0$, so $\{v_1, \ldots v_n\}$ is linearly independent.

Conversely, suppose that $\{v_1, \ldots, v_n\}$ is linearly independent. It is not hard to see that, if we denote $A = [\ v_1\ v_2\ \cdots\ v_n\ ]$, then $G = A^t A$. Thus

$$q(x) = x^t G x = x^t A^t A x = (Ax)^t A x = \langle Ax, Ax\rangle.$$

Thus, since $\{v_1, \ldots, v_n\}$ is linearly independent, for all $x \neq 0 \in \mathbb{R}^n$, we have $A(x) \neq 0$, which implies that $q(x) = \langle A(x), A(x)\rangle > 0$, whence G is definite positive. □

Exercise 6.9. *Determine the QR factorization of the following matrices*

(i) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$

Solution: **(i):** Using the Gram-Schmidt method, let's find the Q matrix. Set $w_1 = (2,0,-1)$. Calculating w_2, we obtain

$$w_2 = (1,1,-1) - \frac{\langle (1,1,-1),(2,0,-1)\rangle}{\langle (2,0,-1),(2,0,-1)\rangle}(2,0,-1) = (1,1,-1) - \frac{3}{5}(2,0,-1) = \Big(-\frac{1}{5}, 1, -\frac{2}{5}\Big).$$

For calculating effects, we can change $w_2 = (-1,5,-2)$. Calculating w_3, we obtain

$$\begin{aligned} w_3 &= (-1,3,1) - \frac{\langle (-1,3,1),(2,0,-1)\rangle}{\langle (2,0,-1),(2,0,-1)\rangle}(2,0,-1) - \frac{\langle (-1,3,1),(-1,5,-2)\rangle}{\langle (-1,5,-2),(-1,5,-2)\rangle}(-1,5,-2) \\ &= (-1,3,1) - \frac{-3}{5}(2,0,-1) - \frac{14}{30}(-1,5,-2) = \Big(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\Big) \end{aligned}$$

Normalizing w_1, w_2 and w_3 , we conclude that

$$Q = \begin{pmatrix} 2\sqrt{5}/5 & -\sqrt{30}/30 & \sqrt{6}/6 \\ 0 & \sqrt{30}/6 & \sqrt{6}/6 \\ -\sqrt{5}/5 & -\sqrt{30}/15 & \sqrt{6}/3 \end{pmatrix}.$$

Calling e_1, e_2 and e_3 these unitary vectors, the matrix R is given by

$$R = \begin{pmatrix} \langle e_1, a_1\rangle & \langle e_1, a_2\rangle & \langle e_1, a_3\rangle \\ 0 & \langle e_2, a_2\rangle & \langle e_2, a_3\rangle \\ 0 & 0 & \langle e_3, a_3\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 3\sqrt{5}/5 & -3\sqrt{5}/5 \\ 0 & \sqrt{30}/6 & 7\sqrt{30}/15 \\ 0 & 0 & 2\sqrt{6}/3 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{5}/5 & -\sqrt{30}/30 & \sqrt{6}/6 \\ 0 & \sqrt{30}/6 & \sqrt{6}/6 \\ -\sqrt{5}/5 & -\sqrt{30}/15 & \sqrt{6}/3 \end{pmatrix} \begin{pmatrix} \sqrt{5} & 3\sqrt{5}/5 & -3\sqrt{5}/5 \\ 0 & \sqrt{30}/6 & 7\sqrt{30}/15 \\ 0 & 0 & 2\sqrt{6}/3 \end{pmatrix}.$$

(ii): Using the Gram-Schmidt method, let's find the Q matrix. Set $w_1 = (1, 2)$. Calculating w_2, we obtain

$$w_2 = (-3, 1) - \frac{\langle(-3, 1), (1, 2)\rangle}{\langle(1, 2), (1, 2)\rangle}(1, 2) = (-3, 1) - \frac{-1}{5}(1, 2) = \left(\frac{-14}{5}, \frac{7}{5}\right).$$

Normalizing w_1 and w_2, we conclude that

$$Q = \begin{pmatrix} \sqrt{5}/5 & -2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{pmatrix}.$$

Calling e_1 and e_2 these unitary vectors, the matrix R is given by

$$R = \begin{pmatrix} \langle e_1, a_1\rangle & \langle e_1, a_2\rangle \\ 0 & \langle e_2, a_2\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{5} & -\sqrt{5}/5 \\ 0 & 7\sqrt{5}/5 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5}/5 & -2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{pmatrix} \begin{pmatrix} \sqrt{5} & -\sqrt{5}/5 \\ 0 & 7\sqrt{5}/5 \end{pmatrix}.$$

Exercise 6.10. *Let V be a real finite-dimensional linear space. Suppose that there exists $\omega : V \times V \longrightarrow \mathbb{R}$ such that*

- *ω is bilinear*
- *ω is skew-symmetric*
- *ω is non-degenerated.*

Then we say that (V, ω) is a sympletic linear space.

(i) Prove that $(\mathbb{R}^{2n}, \omega_0)$ *is a sympletic linear space with*

$$\omega_0(v_1, v_2) = -\sum_{k=1}^{n} y_{n+k}x_k + \sum_{k=1}^{n} y_k x_{n+k},$$

where $v_1 = (x_1, x_2, \dots, x_{2n})$ *and* $v_2 = (y_1, y_2, \dots, y_{2n})$.

(ii) Let (V, ω) *be a sympletic linear space. Given* W *a linear subspace of* V*, we define its orthogonal sympletic space* W^ω *as*

$$W^\omega = \{v \in V \ ; \ \omega(v, u) = 0 \text{ for all } u \in W\}.$$

Prove that W^ω *is a linear subspace of* V *and that* $\dim(W) + \dim(W^\omega) = \dim(V)$.

(iii) Let X *be a linear subspace of* V*.* X *is said sympletic subspace if the restriction* $\omega|_{X\times X}$ *is non-degenerate. Prove that* X *is a sympletic subspace if and only if* $X^\omega \cap X = \{0\}$ *if and only if* $V = X \oplus X^\omega$.

(iv) Prove that $\dim_{\mathbb{R}}(V)$ *is even.*

(v) Prove that every sympletic linear space (V, ω) *is isosympletictomorphic to* $(\mathbb{R}^{2n}, \omega_0)$*, that is, there exists an* $\mathbb{R}$*-isomorphism* $i : V \longrightarrow \mathbb{R}^{2n}$ *such that*

$$\omega = \omega_0 \circ i.$$

Solution: **(i):** In fact, let $u_1 = (x_1, x_2, \dots, x_{2n})$, $u_2 = (x'_1, x'_2, \dots, x'_{2n})$, $v_1 = (y_1, y_2, \dots, y_{2n})$ and $v_2 = (y'_1, y'_2, \dots, y'_{2n})$ be vectors of $\mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}$. Thus

$$\begin{aligned}
\omega_0(u_1 + \lambda u_2, v_1) &= \omega_0((x_1 + \lambda x'_1, x_2 + \lambda x'_2, \dots, x_{2n} + \lambda x'_{2n}), (y_1, y_2, \dots, y_{2n})) \\
&= -\sum_{k=1}^{n} y_{n+k}(x_k + \lambda x'_k) + \sum_{k=1}^{n} y_k(x_{n+k} + \lambda x'_{n+k}) \\
&= -\sum_{k=1}^{n} y_{n+k}x_k + \sum_{k=1}^{n} y_k x_{n+k} - \sum_{k=1}^{n} y_{n+k}\lambda x'_k + \sum_{k=1}^{n} y_k \lambda x'_{n+k} \\
&= -\sum_{k=1}^{n} y_{n+k}x_k + \sum_{k=1}^{n} y_k x_{n+k} - \lambda\sum_{k=1}^{n} y_{n+k}x'_k + \lambda\sum_{k=1}^{n} y_k x'_{n+k} = \omega_0(u_1, v_1) + \lambda\omega_0(u_2, v_1).
\end{aligned}$$

The equality $\omega_0(u_1, v_1 + \lambda v_2) = \omega_0(u_1, v_1) + \lambda\omega_0(u_1, v_2)$ can be proved similarly, whence ω_0 is $\mathbb{R}$-bilinear. Moreover, if $u = (x_1, \dots, x_{2n})$ and $v = (y_1, \dots, y_{2n})$, then

$$\omega_0(u, v) = -\sum_{k=1}^{n} y_{n+k}x_k + \sum_{k=1}^{n} y_k x_{n+k} = -\Big(-\sum_{k=1}^{n} x_{n+k}y_k + \sum_{k=1}^{n} x_k y_{n+k}\Big) = -\omega_0(v, u)$$

Thus ω_0 is skew-symmetric. Finally, fix $v = (y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n}$ and suppose that $\omega_0(u, v) = 0$ for all $u \in \mathbb{R}^{2n}$. In particular, considering $u = (-y_{n+1}, -y_{n+2}, \ldots, -y_{2n}, y_1, \ldots, y_n)$, we conclude that

$$0 = \omega_0(u, v) = -\sum_{k=1}^{n} y_{n+k}(-y_{n+k}) + \sum_{k=1}^{n} y_k y_k = \sum_{k=1}^{2n} y_k^2 = \|u\|^2,$$

which implies that $u = 0$ and so ω_0 is non-degenerate. Since ω_0 satisfies these three conditions, we conclude that (V, ω_0) is a simpletic linear space.

(ii): Note that $0 \in W^{\omega}$, because, since ω is $\mathbb{R}$-bilinear, we have that $\omega(0, w) = 0$ for all $w \in W$. Now let $u, v \in W^{\omega}$ and $\lambda \in \mathbb{R}$. Thus

$$\omega(u + \lambda v, w) = \omega(u, w) + \lambda\omega(v, w) = 0$$

for all $W \in W$, thus $u + \lambda v \in W^{\omega}$. Hence W^{ω} is a linear subspace of V.

Consider the operator

$$\begin{aligned} \xi : V &\longrightarrow V^* \\ v &\longmapsto \omega(v,\ \ ) : V \longrightarrow \quad \mathbb{R} \\ &\qquad\qquad\qquad u \longrightarrow \ \ \omega(u, v) \end{aligned}$$

Note that ξ is linear and that $W^{\omega} = \xi^{-1}(\mathrm{Ann}(W))$. Since ξ is an $\mathbb{R}$-isomorphism, we conclude that $\dim_{\mathbb{R}}(W^{\omega}) = \dim_{\mathbb{R}}(\mathrm{Ann}(W))$. However

$$\mathrm{Ann}(W) \cong \left(\frac{V}{W}\right)^*$$

Thus

$$\dim_{\mathbb{R}}(W^{\omega}) = \dim_{\mathbb{R}}(\mathrm{Ann}(W)) = \dim_{\mathbb{R}}\left(\left(\frac{V}{W}\right)^*\right) = \dim_{\mathbb{R}}\left(\frac{V}{W}\right) = \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(W).$$

Hence $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W) + \dim_{\mathbb{R}}(W^{\omega})$.

(iii): Suppose that X is a sympletic space. Let $v \in X \cap X^{\omega}$. So we have that $\omega(v, u) = 0$ for all $u \in X$. Since $\omega|_{X \times X}$ is non-degenerate, we conclude that $v = 0$. So $X \cap X^{\omega} = \{0\}$.

Suppose that $X \cap X^{\omega} = \{0\}$. Since $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(X) + \dim_{\mathbb{R}}(X^{\omega})$, then $V = X + X^{\omega}$. Moreover, since $X \cap X^{\omega} = \{0\}$, then $V = X \oplus X^{\omega}$.

Suppose that $V = X \oplus X^{\omega}$. Let $x \in X$ and suppose that $\omega(x, u) = 0$ for all $u \in X$. Let $w \in V$. Since $V = X \oplus X^{\omega}$, there are $v \in X$, $v^{\omega} \in X^{\omega}$ such that

$$w = v + v^{\omega}.$$

Hence

$$\omega(x, w) = \omega(x, v + v^{\omega}) = \omega(x, v) + \omega(x, v^{\omega}) = 0.$$

Since $w \in V$ is arbitrary and ω is non-degenerate, we conclude that $x = 0$. Thus $\omega|_{X\times X}$ is non-degenerate.

(iv): Consider Σ the following family of linear subspaces of V.

$$\Sigma = \{X \subseteq V \ ; \ X \subseteq X^{\omega}\}.$$

Note that $\Sigma \neq \emptyset$, because $\{0\} \in \Sigma$. Since $\dim_{\mathbb{R}}(V) < \infty$, Σ has maximal elements with respect the inclusion order. Let W be a maximal element of Σ. I claim that $W = W^{\omega}$. In fact, if not, let $v \in W^{\omega} \smallsetminus W$. It is easy to see that

$$W + \mathrm{Span}_{\mathbb{R}}(\{v\}) \subseteq \big(W + \mathrm{Span}_{\mathbb{R}}(\{v\})\big)^{\omega}$$

This fact contradicts the maximality of W in Σ. Thus $W = W^{\omega}$. Using the part (ii), we conclude that

$$\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W) + \dim_{\mathbb{R}}(W^{\omega}) = \dim_{\mathbb{R}}(W) + \dim_{\mathbb{R}}(W) = 2\dim_{\mathbb{R}}(W) = 2m,$$

where $m := \dim_{\mathbb{R}}(W)$.

(v): Using the Corollary 6.12, let $\mathcal{B} = \{u_1, \dots, u_n, v_1, \dots, v_n\}$ be a sympletic basis for V. Define the following linear mapping

$$\phi : V \longrightarrow \mathbb{R}^{2n}$$

$$v_i \longmapsto e_i$$

$$u_i \longmapsto e_{n+i}$$

It is clear that ϕ is an $\mathbb{R}$-isomorphism. In order to prove that $\omega(x, y) = \omega_0(\phi(x), \phi(y))$ for all $x, y \in V$, it is enough to show that this equality holds for every ordered pair $(x, y) \in \mathcal{B} \times \mathcal{B}$. However, it is easy because for all $1 \leq i, j \leq n$, we have

$$\omega(u_i, u_j) = 0 = \omega_0(e_{n+i}, e_{n+j}) = \omega_0(\phi(u_i), \phi(u_j));$$

$$\omega(v_i, v_j) = 0 = \omega_0(e_i, e_j) = \omega_0(\phi(v_i), \phi(v_j));$$

$$\omega(u_i, v_j) = \delta_{ij} = \omega_0(e_{n+i}, e_j) = \omega_0(\phi(u_i), \phi(v_j));$$

$$\omega(v_i, u_j) = -\delta_{ij} = \omega_0(e_i, e_{n+j}) = \omega_0(\phi(v_i), \phi(u_j)).$$

Thus $\omega = \omega_0 \circ \psi$, where

$$\psi : V \times V \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}$$
$$(x, y) \longmapsto (\phi(x), \phi(y)).$$

□

Lemma 6.11. *Let V be a finite-dimensional $\mathbb{R}$-linear space and $\omega : V \times V \longrightarrow \mathbb{R}$ be a skew-symmetric $\mathbb{R}$-bilinear functional. There is a basis $\mathcal{B} = \{u_1, \ldots, u_k, e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ for V such that*

$$\omega(u_i, v) = 0 \textit{ for all } i = 1, \ldots, k \textit{ and } v \in V \quad \textit{and} \quad \omega(e_i, e_j) = 0 = \omega(f_i, f_j) \quad \textit{and} \quad \omega(e_i, f_j) = \delta_{ij}.$$

Proof: This proof consists in an induction process based in a skew-symmetric Gram-Schmidt method. Consider

$$U := V^{\omega} = \{u \in V \ ; \ \omega(u, v) = 0 \text{ for all } v \in V\}.$$

Choose $\{u_1, \ldots, u_k\}$ a basis for U and let W be the complement of U in V, that is, $W \subseteq V$ is such that $V = U \oplus W$. Let $e_1 \in W$, so, since $e_1 \notin U$, there is $f_1 \in V$ such that $\omega(e_1, f_1) = 1$. Now define

$$W_1 = \mathrm{Span}_{\mathbb{R}}(\{e_1, f_1\}) \qquad \textit{and} \qquad W_1^{\omega} = \{v \in W \ ; \ \omega(v, u) = 0 \text{ for all } u \in W_1\}$$

I claim that $W = W_1 \oplus W_1^{\omega}$. In fact, let $u = ae_1 + bf_1 \in W_1 \cap W_1^{\omega}$. Thus

$$0 = \omega(ae_1 + bf_1, f_1) = a\omega(e_1, f_1) + b\omega(f_1, f_1) = a\omega(e_1, e_1) = a$$
$$0 = \omega(ae_1 + bf_1, e_1) = a\omega(e_1, e_1) + b\omega(f_1, e_1) = -b\omega(e_1, f_1) = -b,$$

which implies that $u = 0$, that is $W_1 \cap W_1^{\omega} = \{0\}$. Moreover, given $v \in W$, consider $w = v - \omega(v, f_1)e_1 + \omega(v, e_1)f_1$. Note that

$$\omega(w, e_1) = \omega(v - \omega(v, f_1)e_1 + \omega(v, e_1)f_1, e_1) = \omega(v, e_1) - \omega(v, f_1)\omega(e_1, e_1) + \omega(v, e_1)\omega(f_1, e_1) = 0$$
$$\omega(w, f_1) = \omega(v - \omega(v, f_1)e_1 + \omega(v, e_1)f_1, f_1) = \omega(v, e_1) - \omega(v, f_1)\omega(e_1, f_1) + \omega(v, e_1)\omega(f_1, f_1) = 0.$$

Thus $v - \omega(v, f_1)e_1 + \omega(v, e_1)f_1 \in W_1^{\omega}$, which implies that $v \in W_1 \oplus W_1^{\omega}$. That is, $W = W_1 \oplus W_1^{\oplus}$. Hence

$$V = U \oplus W_1 \oplus W_1^{\omega}.$$

We can repeat the same proceeding now with W_1^ω, obtaining that $W_1^\omega = W_2 \oplus W_2^\omega$, where $W_2 = \text{Span}_{\mathbb{R}}(\{e_2, f_2\})$ and $\omega(e_2, f_2) = 1$. Note that, since $e_2, f_2 \in W_1^\omega$, we have that $\omega(e_2, e_1) = 0$ $\omega(f_2, e_1) = 0$ and $\omega(e_1, f_2) = \omega(f_1, e_2) = 0$ and so we obtain

$$V = U \oplus W_1 \oplus W_2 \oplus W_2^\omega.$$

Since $\dim(V) < \infty$, there will be $n \in \mathbb{N}$ such that $W_n^\omega = \{0\}$ and so we will conclude that

$$V = U \oplus \bigoplus_{k=1}^{n} W_k.$$

$\square$

Corollary 6.12. *Let (V, ω) be a sympletic linear space, where V is a finite-dimensional $\mathbb{R}$-linear space. There is a basis $\mathcal{B} = \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ for V such that*

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j) \quad \textit{and} \quad \omega(e_i, f_j) = \delta_{ij}.$$

This basis is called a sympletic basis for (V, ω). In particular, every finite-dimensional sympletic linear space has even dimension.

Proof: Since ω is non-degenerate, the subspace U constructed on the previous proof is the null space. $\square$

Exercise 6.13. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $t > 0$. Determine $z^* \in \mathbb{C}^n$ where*

$$\min_{z\in\mathbb{C}^n} \|Az - y\|^2 + t\|z\|^2$$

attains.

Solution: I'm not sure about the solution. The idea is to try find $z^* \in \mathbb{C}^n$ where we minimize both $\|Az - y\|^2$ and $\|z\|^2$. In order to minimize $\|Az - y\|^2$, let $y_0 \in \text{Im}(A)$ such that $y_0 - y$ is orthogonal to $\text{Im}(A)$. Note that, for every $u \in \text{Im}(A)$, the Pythagorean theorem give us that

$$\|u - y\|^2 = \|u - y_0 + y_0 - y\|^2 = \|u - y_0\|^2 + \|y_0 - y\|^2 \geq \|y_0 - y\|^2.$$

So our initial candidate must be $z_0 \in \mathbb{C}^n$ such that $Az_0 = y_0$. However, we need to find $z_0 \in \mathbb{C}^n$ the nearest to origin as possible, because we also need to minimize $\|z_0\|^2$. We know that every $z \in \mathbb{C}^n$ such that $Az = y_0$ is of form $z_0 + u$, where $u \in \ker(A)$. If x_0 is the orthogonal projection of z_0 over $\ker(A)$, then I claim that $z_0 - x_0$ is the wished element, because

$$\|z_0 + u\|^2 = \|z_0 - x_0 + x_0 + u\|^2 = \|z_0 - x_0\|^2 + \|x_0 + u\|^2 \geq \|z_0 - x_0\|^2.$$

So $z^* = z_0 - x_0$ minimizes the function $f : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that $f(z) = \|Az - y\|^2 + t\|z\|^2$.

7 Homework VII: Euclidean Structures II

Exercise 7.1. *Let $A \in \mathrm{SO}(n)$. Is it true that A always has $\lambda = 1$ as eigenvalue.*

Solution: It is false. Consider $n = 2$ and

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

A is an orthogonal matrix and $\lambda = -1$ is the unique eigenvalue of A. It has multiplicity 2. $\square$

Exercise 7.2. *Let $\mathcal{X} = \{x_1, \dots, x_n\}$ be an orthonormal basis of $(V, \langle\ ,\ \rangle)$ and $U : V \longrightarrow V$ be a linear transformation. Prove that U is an isometry if and only if the image of $\mathcal{X}$ under U is an orthonormal basis of V.*

Solution: Suppose that U is an isometry. Since U is linear, we have that

$$\langle U(x), U(y)\rangle = \langle x, y\rangle \qquad \text{for all } x, y \in V.$$

Since $\mathcal{X}$ is an orthonormal basis, we have that $\langle U(x_i), U(x_j)\rangle = \delta_{ij}$. Thus $U(\mathcal{X}) = \{U(x_1), \dots, U(x_n)\}$ is an orthonormal set and so linearly independent. Since $\dim_{\mathbb{R}}(V) = n$, then $U(\mathcal{X})$ is an orthonormal basis for V.

Conversely, Suppose that $U(\mathcal{X}) = \{U(x_1), \dots, U(x_n)\}$ is an orthonormal basis for V. Let x and y in V, so

$$x = \sum_{k=1}^{n} \langle x_k, x\rangle x_k$$
$$y = \sum_{k=1}^{n} \langle x_k, y\rangle x_k$$

Thus

$$\begin{aligned} \langle U(x), U(y)\rangle = \left\langle \sum_{k=1}^{n} \langle x_k, x\rangle U(x_k), \sum_{j=1}^{n} \langle x_j, y\rangle U(x_j) \right\rangle &= \sum_{k=1}^{n}\sum_{k=1}^{n} \langle x_k, x\rangle \langle x_j, y\rangle \langle U(x_k), U(x_j)\rangle \\ &= \sum_{k=1}^{n} \langle x_k, x\rangle \langle x_k, y\rangle = \langle x, y\rangle. \end{aligned}$$

In particular, $\|U(x)\| = \|x\|$. Since U is linear, we have that

$$\|U(x) - U(y)\| = \|x - y\|,$$

whence U is an isometry. $\square$

Exercise 7.3. *Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear isometry com positive determinant. Prove that T is a rotation around the origin*

Solution: Since T is a linear isometry, we know that there is a matrix $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$ such that

$$T(x) = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x.$$

Note that $\det(A) = 1$, so

$$\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = A^{-1} = A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}.$$

Thus there are $\alpha, \beta \in \mathbb{R}$ such that

$$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

Since A is orthogonal, we know that $\alpha^2 + \beta^2 = 1$, so there is an unique $\theta \in [0, 2\pi)$ such that $\alpha = \cos(\theta)$ and $\beta = \sin(\theta)$. Thus

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

However this matrix is the matrix of rotation of θ radian counter-clockwise around the origin. $\square$

Exercise 7.4. *Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear isometry com positive determinant. Prove that T is a rotation R_ℓ^t around some line ℓ and angle θ.*

Solution: If $\det(T) = -1$, let $S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation such that

$$[S]_{\{e_1,e_2,e_3\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that S is a reflection with respect the plane π_{XY} and $R := TS : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is an isometry with positive determinant. I will prove that R is of form R_θ^ℓ. In fact, note that

$$\det(R - I_3) = \det(R - RR^t) = \det(R(I_3 - R^t)) = \det(R)\det(I_3 - R^t) = \det(I_3 - R^t)$$
$$= -\det(R - I_3).$$

So $\det(R - I_3) = 0$. Thus $R - I_3$ is singular, which implies that there is a non-zero unitary vector $u \in \mathbb{R}^3$ such that $R(u) = u$. Let $W = \mathrm{Span}_{\mathbb{R}}(\{u\})^\perp$. Since R is an ortogonal linear mapping, we have that W is R-invariant, so $R|_W : W \longrightarrow W$ is an orthogonal mapping on plane W with $\det(R|_W) = 1$ and so $R|_W$ is a rotation. Thus there is a orthonormal basis $\mathcal{B} = \{u, v_1, v_2\}$ such that

$$[R]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some $\theta \in [0, 2\pi)$. Thus $T = R \circ S$, where S is the reflection with respect the plane π_{XY} and R is a rotation around $\langle u \rangle$ of θ radians counter-clockwise.

Exercise 7.5. *consider a quadratic form in* $\mathbb{R}^2$ *given by*

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where $b \neq 0$ *and* $a^2 + c^2 \neq 0$. *Prove that there is a basis of* $\mathbb{R}^2$ *such that with the new system of coordinates this quadratic form has form*

$$AX^2 + BY^2 + CX + DY + E = 0.$$

Proof: In fact, note that

$$ax^2 + bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Call by A the 2×2 matrix above. Note the A is symmetric, so by Spectral theorem, A admits a pair $\mathcal{B} = \{v_1, v_2\}$ of orthonormal eigenvectors. Let λ_1 and λ_2 be the associated eigenvalues and consider M the matrix of change of canonical basis from basis $\mathcal{B}$. Note that M is the matrix $[\ v_1\ \ v_2\ ]$, so M is an orthogonal matrix

$$A = M^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} M = M^t \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} M.$$

Moreover, in the system of coordinates with respect the basis $\mathcal{B}$, we have that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus

$$0 = ax^2 + bxy + cy^2 + dx + ey + f = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} M^t \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} = \left(M \begin{bmatrix} x \\ y \end{bmatrix} \right)^t \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \left(M \begin{bmatrix} x \\ y \end{bmatrix} \right) + \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f \end{bmatrix}$$

$$= \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix} M^t \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} f \end{bmatrix}.$$

Setting

$$\begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} d & e \end{bmatrix} M^t,$$

then

$$ax^2 + bxy + cy^2 + dx + ey + f = \lambda_1 X^2 + \lambda_2 Y^2 + CX + DY + f.$$

□

Exercise 7.6. *True or false*

(i) A square matrix $A \in \mathcal{M}_n(\mathbb{R})$ whose the columns constitute an orthogonal basis of $\mathbb{R}^n$ is an orthogonal matrix.

(ii) A square matrix $A \in \mathcal{M}_n(\mathbb{R})$ whose the rows constitute an orthogonal basis of $\mathbb{R}^n$ is an orthogonal matrix.

(iii) A matrix $A \in O(n)$ is symmetric if and only if it is diagonal.

Solution: **(i):** False. Consider

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2.$$

If A was ortogonal, then $A^{-1} = A^t$, however $(2I_2)^t = 2I_2$ and

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii): True. Consider $A = [a_{ij}]_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{R})$ a matrix whose the rows $v_1^t, \ldots, v_n^t \in \mathbb{R}^n$

constitute an orthogonal basis of $\mathbb{R}^n$. In order to show that A is orthogonal, it is enough to show that $AA^t = I_n$. Let $AA^t = [c_{ij}]_{1\leq i,j\leq n}$. Observe that

$$c_{ij} = \sum_{k=1}^{n} a_{ik}a_{jk} = \langle v_i, v_j \rangle = \delta_{ij}$$

Hence $AA^t = [\delta_{ij}]_{1\leq i,j\leq n} = I_n$, so A is orthogonal.

(iii): False. Consider

$$A = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -\sqrt{3}/2 \end{bmatrix}$$

A is orthogonal and symmetric, however A is not diagonal.

Exercise 7.7. *Let $(V, \langle\ ,\ \rangle)$ be a complex linear space with inner dot. Prove that for all x, $y \in V$*

$$\|x + y\|^2 = \|x\|^2 + 2\operatorname{Re}(\langle x, y\rangle) + \|y\|^2$$

Solution: In fact

$$\begin{aligned}\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \\ = \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 = \|x\|^2 + 2\operatorname{Re}(\langle x, y\rangle) + \|y\|^2.\end{aligned}$$

□

Exercise 7.8. *Let $(X, \langle\ ,\ \rangle)$ be a complex finite-dimensional linear space with inner dot. Prove that*

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

for all x, $y \in X$.

Solution: If $\langle x, y\rangle = 0$, the inequality is obvious. So we can suppose without lost of generality that $\langle x, y\rangle \neq 0$. Since $\langle x, y\rangle \neq 0$, in particular, we have that $y \neq 0$, then, there is $\alpha \in \mathbb{S}^1$ such that that, defining $z = \alpha y$, we obtain

$$\langle x, z\rangle = \langle z, x\rangle = |\langle x, y\rangle|.$$

For all $t \in \mathbb{R}$, we have

$$0 \leq \langle x - tz, x - tz\rangle = \|x\|^2 - 2t\langle x, z\rangle + t^2\|z\|^2 = \|x\|^2 - 2t|\langle x, y\rangle| + t^2\|y\|^2.$$

This implies that

$$4|\langle x, y\rangle|^2 - 4\|y\|^2\|x\|^2 \leq 0,$$

which implies that $|\langle x, y\rangle| \leq \|y\|\|x\|$

Exercise 7.9. *Let $(X, \langle\ ,\ \rangle)$ be a complex finite-dimensional linear space and Y be a linear subspace of X. Defining*

$$Y^\perp = \{x \in X\ ;\ \langle x, y\rangle = 0 \text{ for all } x \in Y\},$$

prove that $X = Y \oplus Y^\perp$.

Solution: It is clear that $Y^\perp$ is a linear subspace of X. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for Y and let $v \in X$, Note that

$$\left\langle v - \sum_{k=1}^{n} \langle v, u_k\rangle u_k, u_i \right\rangle = 0$$

for all $i = 1, \ldots, n$. Thus $v - \sum_{k=1}^{n} \langle v, u_k\rangle u_k \in Y^\perp$, which implies that

$$v = \sum_{k=1}^{n} \langle v, u_k\rangle u_k + \left(v - \sum_{k=1}^{n} \langle v, u_k\rangle u_k\right) \in Y + Y^\perp,$$

that is, $V = Y + Y^\perp$. Now, if $y \in Y \cap Y^\perp$, observe that

$$\|y\|^2 = \langle y, y\rangle = 0,$$

so $y = 0$, which implies that $Y \cap Y^\perp = \{0\}$. These two facts allow us to conclude that $X = Y \oplus Y^\perp$. □

Exercise 7.10. *Let $(X, \langle\ ,\ \rangle)$ be a complex finite-dimensional linear space and Y be a linear subspace of X. Define*

$$\pi_Y : X \longrightarrow X$$

$$x \longmapsto x_Y$$

where x_Y is the component of x in Y.

(i) Prove that π_Y is linear;

(ii) Prove that $\pi_Y^2 = \pi_Y$ is linear

Solution: **(i):** Let $x = u + u^{\perp} \in X$ and $y = v + v^{\perp} \in X$. Given $\lambda \in \mathbb{C}$, then

$$x + y = (u + \lambda v) + (u^{\perp} + \lambda v^{\perp}).$$

So

$$\pi_Y(x + \lambda y) = u + \lambda v = \pi_Y(x) + \lambda \pi_Y(y)$$

Hence π_Y is a linear mapping.

(ii): Let $x = u + u^{\perp} \in X$. So

$$\pi_Y^2(x) = \pi_Y(\pi_Y(x)) = \pi_Y(u) = u = \pi_Y(x).$$

Since $x \in X$ is arbitrary, then $\pi_Y^2 = \pi_Y$. □

Exercise 7.11. *Let $(X, \langle\ ,\ \rangle)$ be a complex finite-dimensional linear space and Y be a linear subspace of X. Given $x \in X$, show that, among all $y \in Y$, the one closest in Euclidean distance to x is $\pi_Y(x)$.*

Solution: In fact, let $y \in Y$, then

$$\|x - y\|^2 = \|x - \pi_Y(x) + \pi_Y(x) - y\|^2 = \|x - \pi_Y(x)\|^2 + \|\pi_Y(x) - y\|^2 \geq \|x - \pi_Y(x)\|^2,$$

where the second equality is due $x - \pi_Y(x) \in Y^{\perp}$ and $\pi_Y(x) - y \in Y$ are orthogonal. □

Exercise 7.12. *Let $(V, \langle\ ,\ \rangle)$ be a complex finite-dimensional linear space with inner dot and $M : V \longrightarrow V$ linear mapping such that $\|M(x)\| = \|x\|$ for all $x \in V$. A linear mapping with this property is called unitary operator,*

(i) Prove that $M^*M = I_V$;

(ii) Prove that M^{-1} and M^* are unitary;

(iii) Prove that $|\det(M)| = 1$.

Solution: **(i):** Since $\|M(x)\| = \|x\|$ for all $x \in V$, by polarization identity, we conclude that

$$\langle M(x), M(y)\rangle = \langle x, y\rangle$$

for all $x, y \in V$. So

$$\langle x, (M^*M)(y)\rangle = \langle x, y\rangle.$$

Using the linearity, we conclude that

$$\langle x, (M^*M)(y) - y\rangle = 0$$

for all $x, y \in V$ and so $(M^*M)(y) = 0$ for all $y \in V$, whence $M^*M = I_V$.

(ii): By item (i), M is invertible and $M^{-1} = M^*$. Note that

$$\begin{aligned}\|x\|^2 = \langle x, x\rangle = \langle (MM^*)(x), (MM^*)(x)\rangle = \langle M(M^*(x)), M(M^*(x))\rangle = \langle M^*(x), M^*(M(M^*(x)))\rangle \\ = \langle M^*(x), (M^*M)((M^*(x)))\rangle = \langle M^*(x), M^*(x)\rangle = \|M^*(x)\|^2.\end{aligned}$$

Then M^* is unitary. Since $M^* = M^{-1}$, so M^{-1} is also unitary.

(iii): Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V and $A = [M]_{\mathcal{B}}$ the matrix of M with respect $\mathcal{B}$. We know that

$$[M^{-1}]_{\mathcal{B}} = [M^*]_{\mathcal{B}} = [M]_{\mathcal{B}}^H$$

Thus

$$1 = \det(I) = \det([M]_{\mathcal{B}}[M^{-1}]_{\mathcal{B}}) = \det([M]_{\mathcal{B}}[M]_{\mathcal{B}}^H) = \det([M]_{\mathcal{B}})\overline{\det([M]_{\mathcal{B}})} = |\det([M]_{\mathcal{B}})|^2$$

Thus $|\det(M)| = 1$. $\square$

8 Homework VIII: Spectral Theory for Self-Adjoint and Normal Operators

Exercise 8.1. *Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional complex linear space with inner dot and $T : V \longrightarrow V$ be a isometry. Prove that the following assertions are equivalent:*

(i) T is a isometry;

(ii) $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x,\ y \in V$;

(iii) T takes orthonomal basis in orthonormal basis;

(iv) T^* is an isometry;

(v) $\langle T^*(x), T^*(y)\rangle = \langle x, y\rangle$ for all $x,\ y \in V$;

(vi) T^* takes orthonomal basis in orthonormal basis;

Proof: **(i)$\Longrightarrow$ (ii)**: Using the polarization identity, we have that

$$\langle T(x), T(y)\rangle = \langle x, y\rangle$$

(ii)$\Longrightarrow$ (iii): Let $\mathcal{B} = \{u_1, \ldots, u_n\}$ be an orthonormal basis of V. Since

$$\langle T(u_i), T(u_j)\rangle = \langle u_i, u_j\rangle = \delta_{ij},$$

we have that $S := \{T(u_1), \ldots, T(u_n)\}$ is an orthonormal set. Since S contains $\dim(V) = n$ elements, we have that S is also an orthonormal basis for V.

(iii)$\Longrightarrow$ (iv): Let $\mathcal{B} = \{u_1, \ldots, u_n\}$ be an orthonormal basis of V. Given $x \in V$, we have

$$x = \sum_{k=1}^{n} \langle x, u_k\rangle u_k.$$

Thus

$$\|T(x)\|^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} \langle x, u_k\rangle\langle x, u_j\rangle\langle T(u_k), T(u_j)\rangle = \sum_{k=1}^{n} \langle x, u_k\rangle^2,$$

where the last equality is due that $\{T(u_1), \ldots, T(u_n)\}$ is also orthonormal. So

$$\|T(x)\|^2 = \sum_{k=1}^{n} \langle x, u_k\rangle^2 = \langle x, x\rangle = \|x\|^2$$

That is, T is an isometry. Since T is an isometry, then $T^* = T^{-1}$ is also an isometry.

(iv)$\Longrightarrow$ (v): By $(i) \Longrightarrow (ii)$, since T^* is an isometry, we have that

$$\langle T^*(x), T^*(y)\rangle = \langle x, y\rangle.$$

(v)$\Longrightarrow$ **(vi)**: By $(ii) \Longrightarrow (iii)$, since

$$\langle T^*(x), T^*(y)\rangle = \langle x, y\rangle.$$

for all x, $y \in V$, we have that T^* takes orthonomal basis in orthonormal basis

(vi)$\Longrightarrow$ **(i)**: By argument of $(iii) \Longrightarrow (iv)$, since T^* takes orthonomal basis in orthonormal basis, we have that $T = (T^*)^*$ is an isometry. □

Lemma 8.2. *Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional complex linear space with inner dot and $T : V \longrightarrow V$ be a self-adjoint operator. If $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = 0$.*

Proof: Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V constituted by eigenvectors of T and let λ_i be the eigenvalue associated to v_i. By hypothesis, we have

$$\lambda_i = \lambda_i\langle u_i, u_i\rangle = \langle \lambda_i u_i, u_i\rangle = \langle T(u_i), u_i\rangle = 0.$$

Since all eigenvalues of T are zero, then T is the zero operator. □

Exercise 8.3. *Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional complex linear space with inner dot and $T : V \longrightarrow V$ be a linear mapping*

1. *Prove that T is normal if and only if $\|T(x)\| = \|T^*(x)\|$ for all $x \in X$.*
2. *Prove that if T is normal and v is an eigenvector of T with associated eigenvalue λ, then v is a eigenvector of T^* with associated eigenvalue $\overline{\lambda}$*

Solution: **(i)**: Suppose that T is normal, then $TT^* = T^*T$. Thus, given $x \in V$, we have

$$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle = \langle x, TT^*(x)\rangle = \langle T^*(x), T^*(x)\rangle = \|T^*(x)\|^2,$$

that is, $\|T(x)\| = \|T^*(x)\|$.

Conversely, suppose that T is such that $\|T(x)\| = \|T^*(x)\|$ for all $x \in X$, then $\|T(x)\|^2 = \|T^*(x)\|^2$, which implies that

$$\langle T(x), T(x)\rangle = \langle T^*(x), T^*(x)\rangle$$

This implies that

$$\langle (T^*T - TT^*)x, x\rangle = 0$$

for all $x \in X$. Since $T^*T - TT^*$ is a self-adjoint operator, by Lemma 8.2, we conclude that $T^*T = TT^*$, that is, T is normal.

(ii): It is easy to check that if T is normal, then $T - \lambda I$ is also normal. Thus

$$\|(T^* - \overline{\lambda} I)v\| = \|(T - \lambda I)^* v\| = \|(T - \lambda I)v\| = \|T(v) - \lambda v\| = 0,$$

which implies that $T^*(v) = \overline{\lambda} v$. Hence $\overline{\lambda}$ is an eigenvalue of T^*. □

Exercise 8.4.

(i) Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional complex linear space with inner dot and $T : V \longrightarrow V$ be a linear transformation. Prove that T is normal if and only if there is an orthonormal basis of V constituted by eigenvectors of T.

(ii) Give an example of not normal linear mapping and verify that V does not admit a basis constituted by eigenvectors.

Solution: **(i):** Without lost of generality, we can suppose $V = \mathbb{C}^n$ for some $n \in \mathbb{N}$. Suppose that T is normal and let $\mathfrak{a}$ be the matrix of T with respect the canonical basis. We know that there is an orthonormal basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V such that the matrix $\mathfrak{t} = [T]_{\mathcal{B}}$ of T with respect the basis $\mathcal{B}$ is upper-triangular. Thus

$$\mathfrak{t} = \mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u}$$

and $\mathfrak{u}$ is an unitary matrix. Since T is normal and

$$\begin{aligned} \mathfrak{t}\mathfrak{t}^* &= \left(\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u}\right)\left(\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u}\right)^* = \mathfrak{u}^{-1}\mathfrak{a}\mathfrak{a}^*\mathfrak{u} \\ \mathfrak{t}^*\mathfrak{t} &= \left(\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u}\right)^*\left(\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u}\right) = \mathfrak{u}^{-1}\mathfrak{a}^*\mathfrak{a}\mathfrak{u}, \end{aligned}$$

we have that $\mathfrak{t}\mathfrak{t}^* = \mathfrak{t}^*\mathfrak{t}$, which implies that $\mathfrak{t}$ is triangular. However, triangular and normal matrices are diagonal, which implies that the columns of is a basis of V constituted by eigenvectors of T.

Conversely, if V admits a basis constituted by eigenvectors of T, then there is an unitary matrix $\mathfrak{u}$ such that $\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u}$ is a diagonal matrix and so normal. This fact implies that

$$\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{a}^*\mathfrak{u} = \mathfrak{t}\mathfrak{t}^* = \mathfrak{t}^*\mathfrak{t} = \mathfrak{u}^{-1}\mathfrak{a}^*\mathfrak{a}\mathfrak{u}$$

Multiplying on left by $\mathfrak{u}$ and on right by $\mathfrak{u}^{-1}$, we conclude that $\mathfrak{a}\mathfrak{a}^* = \mathfrak{a}^*\mathfrak{a}$, so $\mathfrak{a}$ is normal.

(ii): Consider $V = \mathbb{C}^2$, define the following linear mapping

$$T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$e_1 \longmapsto e_1 + 2e_2$$

$$e_2 \longmapsto (1/2)e_1 + e_2$$

The matrix A of T with respect the canonical basis is

$$\begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}$$

Thus the matrix of T^* with respect the canonical basis is

$$\begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}$$

Now note that

$$\begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}$$

Then T is not normal. Calculating the eigenvalues and eigenvectors, we obtain that the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$ and that the eigenspaces are $E_1 = \langle (1, -2) \rangle$ and $E_2 = \langle (1, 2) \rangle$. Observe that $\{(1-2), (1, 2)\}$ is a basis for V, but it is not orthonormal. $\square$

Lemma 8.5. *Let $(V, \langle, \rangle)$ be a finite-dimensional real linear space with inner dot and $T : V \longrightarrow V$ be a linear transformation. If $W \subseteq V$ is a T-invariant subspace of V, so $W^\perp$ is a T^*-invariant subspace of V.*

Proof: Trivial. $\square$

Exercise 8.6.

(i) Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional real linear space with inner dot and $T : V \longrightarrow V$ be a linear transformation. Prove that T is self-adjoint if and only if there is an orthonormal basis of V constituted by eigenvectors of T.

(ii) Give an example of not self-adjoint linear mapping and verify that V does not admit a basis constituted by eigenvectors.

Solution: **(i):** Suppose that T is a self-adjoint linear mapping. We will proceed by induction in $n = \dim_{\mathbb{R}}(V)$. This fact holds trivially for $n = 1$ and it is classical verification that it holds for two-dimensional linear spaces. Now suppose that this fact holds for n-dimensional linear spaces and let V be an $(n+1)$-dimensional linear space. We know that T has an invariant subspace W with $1 \leq \dim_{\mathbb{R}}(W)) \leq 2$. Since $T|_W : W \longrightarrow W$ is still self-adjoint, we conclude W has an orthonormal basis constituted by eigenvectors of T. Since T is self-adjoint, by Lemma above, $W^{\perp}$ is also T-invariant subspace with dimension less than $n+1$. By induction hypothesis, $W^{\perp}$ also has an orthonormal basis constituted by eigenvectors of T. Since $V = W \oplus W\perp$, we conclude that V admits an orthonormal basis constituted by eigenvectors of T.

Conversely suppose that there is an orthonormal basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V constituted by eigenvectors of T. Let λ_i be the eigenvalue associated to v_i. We will prove that $T = T^*$. Note that it is enough to show that $T^*(v_i) = T(v_i) = \lambda_i v_i$ for all $1 \leq i \leq n$. However, since is an orthonormal basis, we have

$$T^*(v_i) = \sum_{k=1}^{n} \langle T^*(v_i), v_k \rangle v_k = \sum_{k=1}^{n} \langle v_i, T(v_k) \rangle v_k = \sum_{k=1}^{n} \langle v_i, \lambda_k v_k \rangle v_k = \sum_{k=1}^{n} \lambda_k \langle v_i, v_k \rangle v_k = \lambda_i v_i.$$

Thus T is a self-adjoint operator.

(ii): Consider $V = \mathbb{R}^2$, define the following linear mapping

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ e_1 &\longmapsto e_1 + 2e_2 \\ e_2 &\longmapsto (1/2)e_1 + e_2 \end{aligned}$$

Since the matrix of T with respect the canonical basis is not symmetric, then T is not self-adjoint. Note that the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$ and that the eigenspaces are $E_1 = \langle (1, -2) \rangle$ and $E_2 = \langle (1, 2) \rangle$. Observe that $\{(1-2), (1, 2)\}$ is a basis for V, but it is not orthonormal. $\square$

Exercise 8.7. *Let $(V, \langle\ ,\ \rangle)$ be a two-dimensional real linear space with inner dot. Give an example of $N : V \longrightarrow V$ such that N is normal, but not self-adjoint operator. Give the matrix of N with respect to an arbitrary orthonormal basis.*

Solution: Let $\mathcal{B} = \{e_1, e_2\}$ be an orthonormal basis of V and let $N : V \longrightarrow V$ be a linear operator such that $N(e_1) = e_2$ and $N(e_2) = -e_1$. Note that the matrix M of N with respect $\mathcal{B}$ is

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since M is not symmetric, we have that N is not self-adjoint. Note that the matrix o N^* is

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus $N^*(e_1) = -e_2$ and $N^*(e_2) = e_1$. Note that

$$(N^*N)(e_1) = e_1 = (NN^*)(e_1)$$
$$(N^*N)(e_2) = e_2 = (NN^*)(e_2)$$

So $N^*N = NN^*$, so N is normal. Let $\mathcal{C} = \{v_1, v_2\}$ be an orthonormal basis of V. By basis change identity, we have that

$$[N]_{\mathcal{C}} = [I_V]^{\mathcal{B}}_{\mathcal{C}}[N]_{\mathcal{B}}[I_V]^{\mathcal{C}}_{\mathcal{B}} = [I_V]^{\mathcal{B}}_{\mathcal{C}} M [I_V]^{\mathcal{C}}_{\mathcal{B}}$$

Exercise 8.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$. If $A + I$, $A^2 + I$ and $A^3 + I$ are unitary matrices. Prove that $A = 0$.*

Solution: Note that, since $A + I$ is unitary, we have

$$(A + I)(A + I)^* = (A + I)(A^* + I) = AA^* + A + A^* + I = I.$$

So $A^*A = -(A + A^*)$. Calculating $(A + I)^*(A + I)$, we conclude that A is a normal matrix. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V constituted by eigenvectors of A and let λ_i be eigenvalue associated to v_i. It is enough to show that $\lambda_i = 0$ for all $1 \leq i \leq n$. Since $A+I$, A^2+I and $A^3 + I$ are unitary matrices, given $\lambda := \lambda_i$, we have

$$(\lambda + 1)(\overline{\lambda} + 1) = (\lambda^2 + 1)(\overline{\lambda}^2 + 1) = (\lambda^3 + 1)(\overline{\lambda}^3 + 1) = 1$$

Thus

$$|\lambda|^2 = -2\,\mathrm{Re}(\lambda) \qquad |\lambda|^4 = -2\,\mathrm{Re}(\lambda^2) \qquad |\lambda|^6 = -2\,\mathrm{Re}(\lambda^3)$$

In particular, $\mathrm{Re}(\lambda) \leq 0$, $\mathrm{Re}(\lambda^2) \leq 0$ and $\mathrm{Re}(\lambda^3) \leq 0$. So $\arg(\lambda)$ is such that

$$\arg(\lambda) \in [\pi/2, 3\pi/2]$$
$$2\arg(\lambda) \in [\pi/2 + 2\pi k, 3\pi/2 + 2\pi k]$$
$$3\arg(\lambda) \in [\pi/2 + 2\pi n, 3\pi/2 + 2\pi n]$$

The unique $\arg(\lambda)$ which satisfies this property is when λ is imaginary pure. So $\mathrm{Re}(\lambda) = 0$, which implies that $|\lambda| = 0$. Then $\lambda = 0$. $\square$

Exercise 8.9. *Let $(V, \langle \ , \ \rangle)$ be a finite-dimensional complex linear space with inner dot and $T : V \longrightarrow V$ be a normal operator. Prove that if T is unitary if and only if all eigenvalues of T are in $\mathbb{S}^1$.*

Solution: Suppose that T is unitary. Since T is normal, let $\mathcal{B} = \{v_1, \dots, v_n\}$ be an orthonormal basis of eigenvectors of T and let λ_i be the associated eigenvalue to v_i. Thus

$$|\lambda_i| = |\lambda_i| \|v_i\| = \|\lambda_i v_i\| = \|T(v_i)\| = \|v_i\| = 1$$

Conversely, suppose that all eigenvalues of T are in $\mathbb{S}^1$. Since T is normal, let $\mathcal{B} = \{v_1, \dots, v_n\}$ be an orthonormal basis of eigenvectors of T and let λ_i be the associated eigenvalue to v_i. Given $x \in V$, then

$$x = \sum_{k=1}^{n} \langle x, v_k \rangle v_k.$$

Thus

$$\|T(x)\|^2 = \langle T(x), T(x) \rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} \langle x, v_k \rangle \langle x, v_j \rangle \langle T(v_k), T(v_j) \rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} \langle x, v_k \rangle \langle x, v_j \rangle \lambda_k \overline{\lambda_j} \langle v_k, v_j \rangle$$
$$= \sum_{k=1}^{n} \langle x, v_k \rangle \langle x, v_k \rangle |\lambda_k|^2 = \sum_{k=1}^{n} \langle x, v_k \rangle \langle x, v_k \rangle = \|x\|^2.$$

Thus $\|T(x)\| = \|x\|$ and T is unitary. □

Exercise 8.10. *Let $A = [a_{ij}]_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix such that $a_{ij} \ge 0$ for all $1 \le i, j \le n$. Prove that A admits an eigenvector $x = (x_1, \dots, x_n)$ such that $x_i \ge 0$ for all $1 \le i \le n$.*

Solution: Indeed, let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be the eigenvalues of A. By spectral theorem, they exists and belong to $\mathbb{R}$. Note that

$$\sum_{k=1}^{n} \lambda_k = \operatorname{tr}(A) = \sum_{k=1}^{n} a_{kk} \ge 0.$$

Thus, calling $\lambda := \lambda_n$ the greatest eigenvalue of A, we have that $\lambda \ge 0$. Let $v = (v_1, \dots, v_n)$ be the unitary eigenvalue associated to λ. Thus, observe that

$$v_i \lambda = \sum_{j=1}^{n} a_{ij} v_j$$

for all $1 \le i \le n$. Since $\|v\| = 1$, we have that

$$\lambda = \lambda \cdot 1 = \lambda \sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} \lambda v_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} v_i v_j \le \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |v_i| |v_j| = \langle Ax, x \rangle,$$

where $x = (|v_1|, \dots, |v_n|)$. Note that we also have that $\|x\| = 1$. Thus, denoting by R_A the Rayleigh-Ritz quotient of A, we have that

$$\lambda = \langle Ax, x\rangle = R_A(x)\langle x, x\rangle = R_A(x) \leq \lambda,$$

where the last inequality is due the Min-max Theorem. Thus $R_A(x) = \lambda$. Thus x is an eigenvector of A and so A admits an eigenvector whose entries are all non-negative.

Exercise 8.11. *Let $M \in \mathcal{M}_n(\mathbb{C})$ be a symmetric matrix. Prove that, if for $\epsilon > 0$, there are $\lambda \in \mathbb{C}$ and v with $\|v\| = 1$ such that*

$$\|M(v) - \lambda v\| < \epsilon$$

Then M has an eigenvalue λ' such that $|\lambda' - \lambda| < \epsilon$.

Solution: Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be an orthonormal basis of V constituted by eigenvectors of M. Suppose that all eigenvalues λ_i of M are just that $|\lambda_i - \lambda| \geq \epsilon$. Thus note that

$$\|M(v_i) - \lambda v_i\|^2 = \|\lambda_i v_i - \lambda v_i\|^2 = |\lambda_i - \lambda|^2\|v_i\|^2 = |\lambda_i - \lambda|^2 \geq \epsilon^2.$$

Now let $v \in \mathbb{S}(0, 1)$, thus there are $c_1, \dots, c_n \in \mathbb{C}$ such that

$$v = \sum_{k=1}^{n} c_k v_k,$$

where $\sum_{k=1}^{n} |c_k|^2 = 1$. Thus

$$\begin{aligned}\|M(v) - \lambda v\|^2 = \left\|\sum_{k=1}^{n}(c_k\lambda_k v_k - \lambda c_k v_k)\right\|^2 &= \left\|\sum_{k=1}^{n} c_k(\lambda_k - \lambda)v_k\right\|^2 = \sum_{k=1}^{n} \|c_k(\lambda_k - \lambda)v_k\|^2 \\ &= \sum_{k=1}^{n} \left(|c_k|^2|\lambda_k - \lambda|^2\|v_k\|^2\right) \geq \epsilon^2 \sum_{k=1}^{n} |c_k|^2 = \epsilon^2.\end{aligned}$$

Thus $\|M(v) - \lambda v\| \geq \epsilon$ for all $v \in \mathbb{S}(0, 1)$, which contradicts the hypothesis. So there exists an eigenvalue λ_i such that $|\lambda_i - \lambda| < \epsilon$. □

Exercise 8.12. *Let $(V, \langle\, , \,\rangle)$ be a finite-dimensional complex linear space with inner dot. Given $u \in V$, $u \neq 0$, define the following linear mapping*

$$\begin{aligned} H_u : V &\longrightarrow V \\ v &\longmapsto v - 2\langle v, u\rangle u \end{aligned}$$

(i) Determine $u \in V$ such that H_u is unitary;

(ii) Prove that H_u is self-adjoint;

(iii) If $V = \mathbb{C}^n$, find the matrix of H_u with respect the canonical basis;

(iv) What condition $w \in \mathbb{C}^n$ must satisfy in order that $U = Id - 2ww^t$ be unitary.

Solution: **(i):** Firstly we must find the adjoint of T. The non-trivial part is to find the adjoint of

$$\phi : V \longrightarrow V$$
$$v \longmapsto \langle v, u\rangle u$$

Well, given $x,\ y \in V$, we have that

$$\langle x, \phi^*(y)\rangle = \langle \phi(x), y\rangle = \langle \langle x, u\rangle u, y\rangle = \langle x, u\rangle\langle u, y\rangle = \langle x, \overline{\langle u, y\rangle} u\rangle$$

Since $x \in V$ is arbitrary, then $\phi^*(y) = \overline{\langle u, y\rangle} u$. So

$$H_u^* : V \longrightarrow V$$
$$v \longmapsto v - 2\overline{\langle u, v\rangle} u$$

Returning to the question, if $u \in V$ is such that H_u is unitary, then

$$\begin{aligned} v = H^*H(v) = H^*(v - 2\langle v, u\rangle u) &= v - 2\overline{\langle u, v\rangle} u - 2\langle v, u\rangle(u - 2\overline{\langle u, u\rangle} u) \\ &= v - 2\langle v, u\rangle u - 2\langle v, u\rangle u + 4\langle v, u\rangle\langle u, u\rangle u \end{aligned}$$

for all $v \in V$, which implies that $\|u\| = 1$.

(ii): In fact, for all $v \in V$, we have

$$H_u^*(v) = v - 2\overline{\langle u, v\rangle} u = v - 2\langle v, u\rangle u = H_u(v).$$

Thus H_u is self-adjoint.

(iii): Denoting $u = (x_1, x_2 \ldots, x_n)$. we have

$$H_u(e_i) = e_i - 2x_i u = (-2x_i x_1, \ldots, -2x_i x_{i-1}, 1 - 2x_i^2, -2x_i x_{i+1}, \ldots, -2x_i x_n).$$

Thus we can easily construct the matrix.

(iv): It has problem. $\square$

Exercise 8.13. *Let $e \in \mathbb{R}^3$ such that $\|e\| = 1$, where $\|.\|$ is the usual norm. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the rotation by π radians around the axis determined by the vector e. Write the matrix of T in the canonical basis $\mathcal{C} = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.*

Solution: Set $u_1 := e$ and $W := \langle w \rangle$. Let $u_2 \in W^\perp$ such that $\|u_2\| = 1$ and let $u_3 = u_1 \wedge u_2$ the vector product. Note that $u_3 \in W^\perp$ and that $\mathcal{B} = \{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbb{R}^3$. It is easy to see that the matrix of T with respect the basis $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now it is enough to find the matrix of T with respect the canonical basis $\mathcal{C}$ by the formula

$$[T]_{\mathcal{C}} = [I]_{\mathcal{C}}^{\mathcal{B}} [T]_{\mathcal{B}} [I]_{\mathcal{B}}^{\mathcal{C}}.$$

Note that

$$[I]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} \langle e_1, u_1 \rangle & \langle e_2, u_1 \rangle & \langle e_3, u_1 \rangle \\ \langle e_1, u_2 \rangle & \langle e_2, u_2 \rangle & \langle e_3, u_2 \rangle \\ \langle e_1, u_3 \rangle & \langle e_2, u_3 \rangle & \langle e_3, u_3 \rangle \end{bmatrix}.$$

and

$$[I]_{\mathcal{C}}^{\mathcal{B}} = \begin{bmatrix} \langle e_1, u_1 \rangle & \langle e_1, u_2 \rangle & \langle e_1, u_3 \rangle \\ \langle e_2, u_1 \rangle & \langle e_2, u_2 \rangle & \langle e_2, u_3 \rangle \\ \langle e_3, u_1 \rangle & \langle e_3, u_2 \rangle & \langle e_3, u_3 \rangle \end{bmatrix}.$$

Hence

$$[T]_{\mathcal{C}} = \begin{bmatrix} \langle e_1, u_1 \rangle & \langle e_1, u_2 \rangle & \langle e_1, u_3 \rangle \\ \langle e_2, u_1 \rangle & \langle e_2, u_2 \rangle & \langle e_2, u_3 \rangle \\ \langle e_3, u_1 \rangle & \langle e_3, u_2 \rangle & \langle e_3, u_3 \rangle \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \langle e_1, u_1 \rangle & \langle e_2, u_1 \rangle & \langle e_3, u_1 \rangle \\ \langle e_1, u_2 \rangle & \langle e_2, u_2 \rangle & \langle e_3, u_2 \rangle \\ \langle e_1, u_3 \rangle & \langle e_2, u_3 \rangle & \langle e_3, u_3 \rangle \end{bmatrix}.$$

$\square$

Exercise 8.14. *Prove that the matrix*

$$A = \begin{bmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

has two positive eigenvalues and two negatives eigenvalues.

Solution: Since A is symmetric, A admits four eigenvalues and all they are reals. Let λ_1, λ_2, λ_3 and λ_4 be the eigenvalues of A. Calculating the determinant and the trace of A, we conclude that

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$$
$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 625.$$

Since the determinant is positive, then all eigenvalues are non-zero. Moreover. They satisfies one of the these three conditions.

- All they are positive;
- All they are negative;
- Two of these are positive and two of these are negative.

Since the $\mathrm{tr}(A) = 0$, the unique possibility is the last: Two of these are positive and two of these are negative. $\square$.

9 Homework IX: Singular Values Decomposition

Definiton 9.1. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $\lambda_1, \lambda_2, \dots, \lambda_r$ be the non-negative eigenvalues of the hermitian matrix A^tA. The values*

$$\sigma_i = \sqrt{\lambda_i}$$

for $i = 1, \dots, r$ are called the singular values of A. The set of singular values of A is denoted by $\mathrm{SV}(A)$.

Observation 9.2. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. If even yet we have that $n \neq m$, that is, if A is not a square matrix, we still have that A^*A is a square matrix.*

Proposition 9.3. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. If A is a symmetric matrix, then the set of singular values of A are*

$$\mathrm{SV}(A) = \{|\lambda| \ ; \ \lambda \textit{ is eigenvalue of } A\}$$

Proof: Note that $A^*A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is a symmetric matrix, thus, by Spectral Theorem, there is an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ constituted by eigenvectors of A. Let λ_i be eigenvalue associated to v_i. By Spectral Theorem, we know that $\lambda_i \in \mathbb{R}$. Now note that

$$(A^*A)(v_i) = A^*(A(v_i)) = A^*(\lambda_i v_i) = \lambda_i A^*(v_i) = \lambda_i A(v_i) = \lambda_i^2 v_i$$

Observe that if λ and $-\lambda$ are eigenvalues of A, with associated eigenspaces E_λ and $E_{-\lambda}$, respectively, then the $E_\lambda \oplus E_{-\lambda}$ is the eigenspace associated to the eigenvalue $|\lambda|$ of A^*A. This observation allows us to conclude that the set of eigenvalues of A^*A is $\{\lambda^2 \,;\, \lambda \text{ is eigenvalue of } A\}$. Thus

$$\mathrm{SV}(A) = \{|\lambda| \ ; \ \lambda \text{ is eigenvalue of } A\}.$$

□

Lemma 9.4. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. A non-zero real number λ is eigenvalue of A^*A if and only if λ is eigenvalue of AA^*.*

Proof: In fact, let λ be a non-zero eigenvalue of A^*A and v be an eigenvector associated to λ. Thus $(A^*A)(v) = \lambda v$. Applying the operator A in both sides of this equation, we obtain

$$(AA^*)(A(v)) = A(A^*A)(v) = A(\lambda v) = \lambda A(v).$$

Since $\lambda \neq 0$, then $A(v) \neq 0$. Thus λ is an eigenvalue of AA^*. Switching A by A^* and proceeding in similar way, we can conclude the converse. □

Theorem 9.5 (Singular Value Decomposition Theorem). *Let $A \in \mathcal{M}_{n\times m}(\mathbb{C})$ with $\operatorname{rank}(A) = r$. Calling $q = \min\{m, n\}$, then there are unitary matrices $V \in U(n)$ and $W \in U(m)$ and a diagonal matrix*

$$E_q = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_q \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_q = 0$, where $\sigma_1, \ldots, \sigma_r$ are the singular values of A, such that

$$A = UEW^*$$

where

- *If $n = m$, then $E = E_q$;*
- *If $n < m$, then*

$$E = \begin{bmatrix} E_q & \mathbf{0} \end{bmatrix}$$

- *If $n > m$, then*

$$E = \begin{bmatrix} E_q \\ \mathbf{0} \end{bmatrix}$$

Proof: **Case: $\mathbf{n = m}$:** Since the eigenvalues of AA^* and A^*A are the same, these matrices are similar, thus there are $V \in U(n)$ (Spectral Theorem) such that

$$A^*A = UAA^*U^*$$

Note that UA is a normal matrix, because

$$(UA)^*(UA) = (A^*U^*)UA = A^*(U^*U)A = A^*A = UAA^*U^* = UA(UA)^*.$$

Thus, by Spectral Theorem, there exists $X \in U(n)$ such that $UA = X\Delta X^t$, where

$$\Delta = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and $\lambda_1, \ldots, \lambda_n$ eigenvalues of UA. Writting

$$\lambda_1 = |\lambda_1| e^{i\theta_1}, \quad \lambda_2 = |\lambda_2| e^{i\theta_2}, \quad \cdots \quad \lambda_n = |\lambda_n| e^{i\theta_n}$$

with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, since rank$(A) = r$, we have that

$$\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$$

Now let the matrices

$$D = \begin{bmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_n} \end{bmatrix} \quad \text{and} \quad E_q = \begin{bmatrix} |\lambda_1| & 0 & \cdots & 0 \\ 0 & |\lambda_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\lambda_n| \end{bmatrix}.$$

Then

$$A = U^*(UA) = U^*(X\Delta X^*) = (U^t X)\Delta(X^*) = (U^* X)E_q D(X^*) = (U^* X)E_q(DX^*)$$

Note that $U^*X \in U(n)$ and $DX^* \in U(n)$. Now it is enough to check that $|\lambda_i| = \sigma_i$. However note that

$$A^*A = ((U^*X)E_q(DX^*))^*(U^*X)E_q(DX^*) = (XD)E_q(X^*U)(U^*X)E_q(DX^*) = XE_q^2D^2X^*$$

Thus $\sigma_i = |\lambda_i|$ for all $1 \leq i \leq n$.

Case: $\mathbf{n < m}$: In this case, we have rank$(A) = r \leq n = q = \min\{m, n\}$. Thus $\dim(\ker(A)) = m - r \geq m - n$. Let $\{x_1, \ldots, x_{m-n}\}$ be an orthonormal set in $\ker(A)$ and complete this set to a basis $\{x_1, \ldots, x_{m-n}, x_{m-n+1}, \ldots, x_m\}$ of $\mathbb{C}^m$. Consider the following matrix

$$X = \begin{bmatrix} x_{m-n+1} & \ldots & x_{m-n} & x_1 & x_2 & \cdots & x_{m-n} \end{bmatrix} := \begin{bmatrix} X_1 & | & x_1 & x_2 & \cdots & x_{m-n} \end{bmatrix} \in U(m)$$

Note that $AX = \begin{bmatrix} AX_1 & | & 0 \end{bmatrix}$ with $AX_1 \in U(m)$. By case (i), we have

$$Ax_1 = VE_qW^t$$

Thus

$$A = (AX)X^t = \begin{bmatrix} AX_1 & | & 0 \end{bmatrix} X^t = \begin{bmatrix} VE_qW^t & | & 0 \end{bmatrix} X^t = V \begin{bmatrix} E_q & | & 0 \end{bmatrix} \left(\begin{bmatrix} W^t & 0 \\ 0 & I_{m-n} \end{bmatrix} X^t \right),$$

which proves the result.

Case: $\mathbf{n > m}$: Switch A by A^t and proceed similarly to case (ii). $\square$

Exercise 9.6. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Prove that the singular values of A are the same that the singular values of A^*.*

Solution: Indeed, the singular values of A are the square root of the non-negatives eigenvalues of A^*A. Note that

$$(A^*)^*A^* = AA^*.$$

Since A^*A and AA^* have the same positive eigenvalues, the result follows. $\square$

Exercise 9.7. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and B be a matrix similar to A. Prove that the singular values of B are not necessarily the same that the singular values of A.*

Solution: Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Note that A and B are similar because

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

However we have that

$$A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B^*B = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus the singular values of A is $\sigma_1^A = 0$ and $\sigma_2^A = 1$ and the singular values of B is $\sigma_1^B = 0$ and $\sigma_2^B = 2$ $\square$

Exercise 9.8. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Prove that singular values of A^2 are not necessarily the square of the singular values of A.*

Solution: Indeed, consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Note that A^2 is the zero matrix, thus

$$A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad (A^2)^*(A^2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, while $\sigma(A) = \{0, 1\}$, we have $\sigma(A^2) = \{0\}$. $\square$

Exercise 9.9. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Prove that* $\operatorname{rank}(A)$ *is the number of non-zero singular values of A.*

Solution: Indeed, note that $\operatorname{rank}(A) = \operatorname{rank}(A^*A)$. Note that A^*A is diagonalizable. Since the rank is a property invariant under similarity and the rank of a diagonal matrix is the number of non-zero elements at diagonal, we conclude that $\operatorname{rank}(A)$ is the number of non-zero singular values of A. □

Exercise 9.10. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and singular values $\sigma_1, \ldots, \sigma_r$. Prove that*

$$\max\{|\lambda_1|, \ldots, |\lambda_n|\} \leq \max\{\sigma_1, \ldots, \sigma_n\}.$$

Solution: Suppose without lost of generality that $|\lambda_n| = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$. Let v be the unitary eigenvector of A associated to λ_n. Note that

$$|\lambda_n|^2 = \lambda_n \overline{\lambda_n} = \langle \lambda_n v, \lambda_n v\rangle = \langle A(v), A(v)\rangle = \|A(v)\|^2.$$

On the other hand A^*A is a symmetric matrix, so by spectral theorem, A^*A can be diagonalized. Moreover, for all $x \in \mathbb{R}^n$, we have

$$\langle x, A^*A(x)\rangle \geq 0.$$

Let $\{\gamma_1, \ldots, \gamma_n\}$ be the eigenvalues of A^*A and suppose without lost of generality that $\gamma_n = \max\{\gamma_1, \ldots, \gamma_n\}$. Thus

$$|\lambda_n|^2 = \langle A(v), A(v)\rangle = \langle v, A^*A(v)\rangle \leq \max_{\|x\|=1} \langle x, A^*A(x)\rangle \leq \gamma_n.$$

Thus

$$\max\{|\lambda_1|, \ldots, |\lambda_n|\} = |\lambda_n| \leq \sqrt{\gamma_n} = \sigma_n = \max\{\sigma_1, \ldots, \sigma_n\}.$$

□

10 Homework X: Jordan Canonical Form

Exercise 10.1. *Let V be a finite-dimensional linear space over a field k and $T : V \longrightarrow V$ be a linear transformation. Let $v \in V$ such that $T^k(v) = 0$ and $T^{k-1}(v) = 0$. Prove that*

(i) Prove that the set $S = \{v, T(v), \ldots, T^{(k-1)}(v)\}$ is linearly independent;

(ii) If $W = \mathrm{Span}(S)$, prove that $T(W) \subseteq W$;

(iii) Prove that $T|_W : W \longrightarrow W$ is well-defined and that T is nilpotent

(iv) Write the matrix of $T|_W$ with respect the basis S.

Solution: **(i):** In fact, let $a_1, \ldots, a_k \in \mathbb{K}$ such that

$$a_1 v + a_2 T(v) + \cdots + a_k T^{(k-1)}(v) = 0$$

Applying $T^{(k-1)}$ on this equation, we obtain that $a_1 T^{(k-1)}(v) = 0$. Thus we get that $a_1 = 0$. Proceeding similarly we conclude that $a_1 = a_2 = \cdots = a_n = 0$.

(ii): It is enough to show that $T(T^i(v)) \in W$ for all $i = 0, \ldots, k-1$. However

$$T(T^i(v)) = T^{i+1}(v) \in W$$

for all $i = 0, \ldots, k-1$ and $T(T^{k-1}(v)) = T^k(v) = 0 \in W$. So $T(W) \subseteq W$.

(iii): Since $T(W) \subseteq W$, T is certainly well-defined. Moreover, given $x \in W$, then there are $a_1, \ldots, a_k \in \mathbb{K}$ such that

$$x = \sum_{i=1}^{k} a_i T^{i-1}(v).$$

Since $T^j(v) = 0$ for all $j \geq k$, we get that

$$T^k(x) = \sum_{i=1}^{k} a_i T^{k+i-1}(v) = 0.$$

Thus $T^k(W) = 0$.

(iv): It is easy to see that $T|_W$ on the basis S is

$$[T|_W]_S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Exercise 10.2. *Write all possible Jordan canonical forms and its minimal polynomials for $M \in \mathcal{M}_{4\times 4}(\mathbb{C})$ if the unique eigenvalue of M is $\lambda = 2$.*

Proof: Note that the characteristic polynomial of M is $p(t) = (t-2)^4$. Consider the subspace

$$E = \{x \in \mathbb{C}^4 \ ; \ (M - 2I)x = 0\}.$$

We have the following possible cases:

- If $\dim_{\mathbb{R}}(E) = 4$, then M is diagonalizable, its Jordan Canonical form is

$$J_M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

  and its minimal polynomial is $m_M(t) = t - 2$.

- If $\dim_{\mathbb{R}}(E) = 3$, then M has three eigenvalues linearly independent, its Jordan canonical form is

$$J_M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

  and its minimal polynomial is $m_M(t) = (t-2)^2$.

- If $\dim_{\mathbb{R}}(E) = 2$, then M has two eigenvalues linearly independent, thus the Jordan canonical form is composed by two Jordan blocks 2×2. That is

$$J_M = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

  and its minimal polynomial is $m_M(t) = (t-2)^2$.

- If $\dim_{\mathbb{R}}(E) = 1$, then the Jordan Canonical form of M is composed by an unique Jordan block 4×4. That is

$$J_M = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and its minimal polynomial is $m_M(t) = (t-2)^4$.

$\square$

Exercise 10.3. *Let V be an n-dimensional linear space over $\mathbb{C}$. Construct all linear mappings $T : V \longrightarrow V$ such that $T^2 = I$.*

Solution: Let $T : V \longrightarrow V$ such that $T^2 = I$. Thus, defining $p(t) = (t-1)(t+1)$, we have that

$$p(T) = (T - I)(T + I) = T^2 - I = 0.$$

Thus we have three possibilities for the minimal polynomial of T:

$$m_1(t) = t - 1 \qquad m_2(t) = t + 1 \qquad m_3(t) = (t-1)(t-2).$$

Thus

- If the polynomial minimal of T is m_1, then $T - I = 0$, which implies that $T = I$.
- If the polynomial minimal of T is m_2, then $T + I = 0$, which implies that $T = -I$.
- If the polynomial minimal of T is m_3, then the matrix of T with respect any basis is diagonalizable. Moreover, the diagonal entries are constituted by elements 1 or -1, and 1 and -1 appear at least one time. Counting every possibilities, we conclude that there are $\binom{n-1}{1}$ distinct linear mappings with this property.

$\square$

Exercise 10.4. *Prove that for all $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, A and A^t are similar.*

Solution Firstly note that

$$p_A(t) = \det(tI - A) = \det((t^tI - A^t)^t) = \det(tI - A^t) = p_{A^t}(t)$$

Thus A and A^t have the same eigenvalues and, for each eigenvalue, the multiplicities in A and in A^t are the same. Let λ be an eigenvalue of A. If we prove that

$$\dim_{\mathbb{R}}(\ker(\lambda I - A)^m) = \dim_{\mathbb{R}}(\ker(\lambda I - A^t)^m)$$

for all $m \in \mathbb{N}$, we get that A and A^t have the same Jordan canonical form and so they are similar. However, this is trivial, because, since the rank of a matrix is invariant under transposition, we have

$$\begin{aligned}\dim_{\mathbb{R}}(\ker((\lambda I - A))^m) &= n - \operatorname{rank}((\lambda I - A)^m) = n - \operatorname{rank}(((\lambda I^t - A^t)^t)^m) \\ = n - \operatorname{rank}(((\lambda I - A^t)^m)^t) &= n - \operatorname{rank}((\lambda I - A^t)^m) = \dim_{\mathbb{R}}(\ker((\lambda I - A^t)^m)).\end{aligned}$$

□

Exercise 10.5. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and let p_A and m_A be its characteristic and minimal polynomials, respectively. If*

$$p_A(t) = m_A(t)(t-i) \qquad \text{and} \qquad m_A^2(t) = p_A(t)(t^2+1),$$

determine the Jordan canonical form of A.

Solution: Note that

$$m_A(t) = \left(\frac{p_A(t)}{m_A(t)}\right)(t^2+1) = (t-i)^2(t+i)$$

and so

$$p_A(t) = m_A(t)(t-i) = (t-i)^3(t+i).$$

Thus A is 4×4 matrix. Since its minimal polynomial is $m_A(t) = (t-i)^2(t+i)$, we have that

$$J_A = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{bmatrix},$$

where a prior $X \in \{i, -i\}$. However, since the algebraic multiplicity of $\lambda = -i$ is one, necessarily we have that $X = i$, that is

$$J_A = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

□

Exercise 10.6. *Let* $T : \mathcal{M}_{2\times 2}(\mathbb{C}) \longrightarrow \mathcal{M}_{2\times 2}(\mathbb{C})$ *given by* $T(X) = XA - AX$*, where*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Determine the Jordan canonical form of A*.*

Solution: Consider $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ the ordered canonical basis of $\mathcal{M}_2(\mathbb{R})$. Let determine the matrix of T with respect this basis.

$$\begin{aligned}
T(E_{11}) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0E_{11} + 1E_{12} + 0E_{21} + 0E_{22}, \\
T(E_{12}) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0E_{11} + 0E_{12} + 0E_{21} + 0E_{22}, \\
T(E_{21}) &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = (-1)E_{11} + 0E_{12} + 0E_{21} + 1E_{22}, \\
T(E_{22}) &= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = 0E_{11} + (-1)E_{12} + 0E_{21} + 0E_{22}.
\end{aligned}$$

Thus the matrix of T with respect the basis $\mathcal{B}$ is

$$A := [T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of A is $p_A(t) = t^4$. Denoting E the eigenspace of A in $t = 0$, we have that

$$\dim(E_0) = 4 - \operatorname{rank}(A) = 4 - 2 = 2.$$

Note that

$$\dim_{\mathbb{R}}(\ker((A - 0I)^2)) = 4 - \operatorname{rank}(A^2) = 4 - 1 = 3.$$

This fact implies that the Jordan canonical form has the following form

$$J_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$