Secondary Representation Theory: The dual of Primary Decomposition Theory

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1 Introduction

When we start our studies in Commutative Algebra, one of the first very important theorems we learn is the Primary Decomposition Theorem, which can be stated as following

Theorem 1.1 (Primary Decomposition Theorem)**.** *Let R be a Noetherian ring, M be an R-module Noetherian and N be a proper submodule of M. Then*

1. *The submodule N can be expressed as finite intersection of primary R-submodules of M*

2. *If $N = N_1 \cap N_2 \cap \cdots \cap N_k$ is a decomposition primary of M, where N_i are $\mathfrak{p}_i$-primary submodules of M, then*

$$\mathrm{Ass}\left(\frac{M}{N}\right) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}.$$

Proof: Consult the Theorem 6.8 of [2]. $\square$

Just for remember that, given a Noetherian R-module M, a prime ideal $\mathfrak{p}$ is said an associated prime of M if there is non-zero $x \in M$ such that $\mathfrak{p} = \mathrm{Ann}(x)$, the family of all associated prime of M is denote by $\mathrm{Ass}(M)$ and that an R-submodule N of M is said primary if $\mathrm{Ass}(M/N)$ contains only one element $\mathfrak{p}$. In this case, the associated prime is given by $\mathfrak{p} = \sqrt{N :_R M}$ and N is called $\mathfrak{p}$-primary submodule of M.

Note that this theorem does not hold a prior for Artinian R-modules. However, there is a theory for decomposing Artinian R-modules and it works as it was a dual in some sense of the Primary Decomposition Theory. Having in mind this ”duality”, this theory is called of “Secundary Representation Theory”.

2 Secondary Representation

We start defining the secondary R-modules, which we can think it as the dual of the idea "primary submodule".

Definition 2.1. *Let R be a ring and S be an R-module. We say that S is secondary precisely when $S \neq 0$ and, for each $r \in R$, either $rS = S$ or $r^n S = 0$ for some $n \in \mathbb{N}$.*

Example 2.2. *Let R be an integral domain. Then the field of fractions* $\operatorname{Quot}(R)$ *is a secondary R-module.*

Proposition 2.3. *Let R be a ring and S be an R-module. Then $\mathfrak{p} := \sqrt{\operatorname{Ann}_R(S)}$ is a prime ideal of R*

Proof: Since $S \neq \{0\}$, then $1 \notin \mathfrak{p}$, so $\mathfrak{p}$ is a proper ideal of R. Let $x, y \in R$ and suppose that $xy \in \mathfrak{p}$. Thus $x^n y^n S = 0$ for some $n \in \mathbb{N}$. Since S is a secondary R-module, we have two possibilities:

- If $S = yS$, then $y^n S = S$, so $x^n S = 0$, which implies that $x^n \in \operatorname{Ann}_R(S)$ and so $x \in \mathfrak{p}$.
- If $y^m S = 0$ for some $m \in \mathbb{N}$, then $y^m \in \operatorname{Ann}_R(S)$ and so $y \in \mathfrak{p}$.

Thus $\mathfrak{p}$ is a prime ideal of R. □

With this proposition in mind, we can say that S is a $\mathfrak{p}$-secondary R-module. Note that, since S is secondary, then

$$R \setminus \mathfrak{p} = \{x \in R \ ; \ xS = S\}.$$

On Example 2.2, note that $\mathfrak{p} = \sqrt{\operatorname{Ann}_R(\operatorname{Quot}(R))} = 0$, thus we can say that $\operatorname{Quot}(R)$ is a (0)-secondary R-module.

Example 2.4. *Let $(R, \mathfrak{m})$ be a local ring. If every element $x \in \mathfrak{m}$ is nilpotent, then R is a $\mathfrak{m}$-secondary R-module. Indeed, let $x \in R$. If $x \in \mathfrak{m}$, then $x^n = x^n R = 0$ for some $n \in \mathbb{N}$. If $x \notin \mathfrak{m}$, then x is unit, so $xR = R$.*

Example 2.5. *Let R be a ring and $\mathfrak{m}$ be a maximal ideal of R. Then $S = R/\mathfrak{m}^n$ is a $\mathfrak{m}$-secondary R-module for all $n \in \mathbb{N}$. Indeed, let $a \in R$. If $a \in \mathfrak{m}$, then $a^n S = 0$. If $a \notin \mathfrak{m}$, then, there are $x \in R$ and $m \in \mathfrak{m}^n$ such that $xa + m = 1$. Thus*

$$S = 1S = (xa + m)S = axS \subseteq aS \subseteq S \quad \Longrightarrow \quad S = aS.$$

Proposition 2.6. *Let R be a ring and S be a $\mathfrak{p}$-secondary R-module. Given a non-zero R-module M, if M is homomorphic image of S, then M is also a $\mathfrak{p}$-secondary R-module.*

Proof: Firstly we will prove that M is a secondary R-module. Since M is homomorphic image of S, there is a surjective R-module homomorphism $\phi : S \longrightarrow M$. Let $r \in R$:

- If $S = rS$, then

$$rM = r\phi(S) = \phi(rS) = \phi(S) = M.$$

- If there exists $n \in \mathbb{N}$ such that $r^n S = 0$, then

$$r^n M = r^n \phi(S) = \phi(r^n S) = \phi(0) = 0.$$

Thus M is also an secondary R-module. Now we will prove that

$$\sqrt{\mathrm{Ann}_R(M)} = \sqrt{\mathrm{Ann}_R(S)} = \mathfrak{p}.$$

Let $x \in \sqrt{\mathrm{Ann}_R(S)}$, so $x^n S = 0$ for some $n \in \mathbb{N}$. Thus

$$x^n M = x^n \phi(S) = \phi(x^n S) = \phi(0) = 0,$$

which implies that $x \in \mathrm{Ann}_R(M)$. Conversely, let $x \in \mathrm{Ann}_R(M)$. Since S is secondary, then eiher $xS = S$ or $x^n S = 0$ for some $n \in \mathbb{N}$, however it is not possible that $xS = S$, because, if it was true, we would have

$$0 = x^n M = x^n \phi(S) = \phi(x^n S) = \phi(S) = M,$$

which is a contradiction. So $x^n S = 0$, that is, $x \in \sqrt{\mathrm{Ann}_R(S)}$. □

Corollary 2.7. *Let R be a ring and M be an R-module. If M is a $\mathfrak{p}$-secondary R-module and N is a submodule of M, then M/N is a $\mathfrak{p}$-secondary R-module.*

Proof: In fact, just consider the natural epimorphism $\pi : M \longrightarrow M/N$. □

In Primary Decomposition Theory, given an R-module M and $\mathfrak{p}$-primary submodules $N_1, \ldots, N_n$, we can prove that the intersection $\cap_{k=1}^n N_k$ is also a $\mathfrak{p}$-primary submodule of M. In Secondary Decomposition Theory, we have

Proposition 2.8. *Let R be a ring and M be an R-module. If $S_1, S_2, \ldots, S_n$ are $\mathfrak{p}$-secondary submodules of M, then so is $\sum_{k=1}^n S_k$.*

Proof: In fact, since $S_1 \neq 0$ and $S_1 \subseteq \sum_{k=1}^{n} S_k$, then $\sum_{k=1}^{n} S_k \neq 0$. Denote $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(S_i)}$. Let $x \in R$. We have two possibilities:

- $x \in \mathfrak{p}$: In this case, for each $1 \leq i \leq n$, there is $k_i \in \mathbb{N}$ such that $x^{k_i} S_i = 0$. So, setting $m = k_1 + \cdots + k_n$, we have
$$x^m \left(\sum_{k=1}^{n} S_k \right) = \sum_{k=1}^{n} x^m S_k = 0,$$
which implies that $x \in \sqrt{\mathrm{Ann}_R(\sum_{k=1}^{n} S_k)}$.
- $x \notin \mathfrak{p}$: On this case, it is true that $xS_k = S_k$ for each $1 \leq k \leq n$. Thus
$$x \left(\sum_{k=1}^{n} S_k \right) = \sum_{k=1}^{n} xS_k = \sum_{k=1}^{n} S_k.$$

This implies that $\sum_{k=1}^{n} S_k$ is a secondary submodule of M. Now we will prove that
$$\sqrt{\mathrm{Ann}_R \left(\sum_{k=1}^{n} S_k \right)} = \mathfrak{p} = \sqrt{\mathrm{Ann}_R(S_k)}.$$
Let $x \in \sqrt{\mathrm{Ann}_R(\sum_{k=1}^{n} S_k)}$. Thus there exists $m \in \mathbb{N}$ such that
$$x^m S_k \subseteq x^m \left(\sum_{k=1}^{n} S_k \right) = 0.$$
Thus $x \in \mathfrak{p} = \sqrt{\mathrm{Ann}_R(S_k)}$. Conversely, suppose that $x \in \mathfrak{p} = \sqrt{\mathrm{Ann}_R(S_k)}$ for each $1 \leq k \leq n$, thus there is $m \in \mathbb{N}$ such that
$$x^m S_1 = x^m S_2 = \cdots = x^m S_n = 0,$$
Thus
$$x^m \left(\sum_{k=1}^{n} S_k \right) = 0,$$
which implies that $x \in \sqrt{\mathrm{Ann}_R(\sum_{k=1}^{n} S_k)}$. □

Definition 2.9. *Let R be a ring and M an R-module. A secondary representation of M is an expression for M as a sum of finitely many secondary submodules of M. Such a secondary representation*
$$M = S_1 + S_2 + \cdots + S_n \qquad \textit{with } S_i \ \mathfrak{p}_i\textit{- secondary } (1 \leq i \leq n)$$
of M is said to be minimal precisely when

(i) $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *are different prime ideals of* R

(ii) for $j = 1, \ldots, n$*, we have that*

$$S_j \not\subseteq \sum_{k=1,\ k\neq j}^{n} S_k$$

Definition 2.10. *Let* R *be a ring. An* R*-module* M *is said representable if* M *admits a secondary representation.*

Proposition 2.11. *Let* R *be a ring and* M *be a representable* R*-module. Then* M *has a minimal secondary representation.*

Proof: Since M is representable, let

$$M = S_1 + S_2 + \cdots + S_n$$

be a secondary representation of M. Descarding the submodules S_j such that

$$S_j \subseteq \sum_{k=1,\ k\neq j}^{n} S_k,$$

we can suppose that the condition (ii) of Definition 2.9 holds. Using the Proposition 2.8, we can suppose that all prime ideals are distinct. Then we have a minimal representation

$$M = S_1 + S_2 + \cdots + S_n$$

of M. $\square$

In Secondary Representation Theory, the dual ideal of set of associated prime ideals is the set of attained primes ideal.

Definition 2.12. *Let* R *be a ring and* M *be an* R*-module. A prime ideal* $\mathfrak{p}$ *is said attached prime ideal of* M *if* M *has a* $\mathfrak{p}$*-secondary quotient. The family of all attached prime ideals of* M *is denoted by* $\mathrm{Att}(M)$*.*

Theorem 2.13 (**The First Uniqueness Theorem**)**.** *Let* R *be a ring and* M *be a representable* R*-module. Consider*

$$M = S_1 + S_2 + \cdots + S_n \qquad \textit{with } S_i \ \mathfrak{p}_i\textit{- secondary } (1 \leq i \leq n)$$

and

$$M = S'_1 + S'_2 + \cdots + S'_m \qquad \textit{with } S'_i \ \mathfrak{p}'_i\textit{- secondary } (1 \leq i \leq m)$$

be two minimal secondary representations of M. Then $n = m$ and

$$\mathrm{Att}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_m\}.$$

Proof: For each $1 \leq i \leq n$, consider the natural epimorphism

$$\pi : M \longrightarrow \frac{M}{S_1 + \cdots + S_{i-1} + S_{i+1} + \cdots + S_n}.$$

Note that $N_i := M/(S_1 + \ldots S_{i-1} + S_{i+1} + \cdots + S_n)$ is a non-zero quotient of S_i, so N_i is a $\mathfrak{p}$-secondary R-module and a quotient of M, so $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Att}(M)$. On other hand, let $\mathfrak{p} \in \mathrm{Att}(M)$, so there is an R-submodule W of M such that M/W is $\mathfrak{p}$-secondary. Since

$$\frac{M}{W} = \frac{S_1 + S_2 + \cdots + S_n}{W} = \overline{S_1} + \overline{S_2} + \cdots + \overline{S_n}.$$

Taking the minimal representation, we conclude that

$$\frac{M}{W} = \overline{S_{i_1}} + \cdots + \overline{S_{i_k}}$$

and then we conclude that $\{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_k}\} \subseteq \mathrm{Att}(M/W)$. On the other hand, $\mathrm{Att}(M/W) = \{\mathfrak{p}\}$, since M/W is $\mathfrak{p}$-secondary. So $\mathfrak{p} = \mathfrak{p}_{i_j}$ for some $1 \leq j \leq k$. That is, $\mathrm{Att}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Thus

$$\mathrm{Att}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

Applying the same proceeding on the representation $M = S'_1 + S'_2 + \cdots + S'_m$, we conclude that $\mathrm{Att}(M) = \{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_m\}$. Thus $m = n$ and

$$\mathrm{Att}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_m\}.$$

$\square$

It is well known that if we have a short exact sequence of R-modules

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \longrightarrow 0,$$

then

$$\mathrm{Ass}(L) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(L) \cup \mathrm{Ass}(N)$$

On the case of the attained primes, reverse situation occurs.

Proposition 2.14. *Let R be a ring and L, M and N be representable R-modules. Suppose that there exists a exact sequence*

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \longrightarrow 0.$$

Then

$$\operatorname{Att}(N) \subseteq \operatorname{Att}(M) \subseteq \operatorname{Att}(L) \cup \operatorname{Att}(N).$$

Proof: In fact, since N is isomorphic to M/L, any quotient of N will be isomorphic to a quotient of M, so $\operatorname{Att}(N) \subseteq \operatorname{Att}(M)$. Now let $\mathfrak{p} \in \operatorname{Att}(M)$, then there exists an R-submodule W of M such that M/W is $\mathfrak{p}$-secondary.

- If $W + L = M$, then M/W is a non-trivial quotient of L, so $\mathfrak{p} \in \operatorname{Att}(L)$.
- If $W + L \neq M$, then $M/(W + L)$ is a non-trivial quotient of N as well of M/W, so $\mathfrak{p} \in \operatorname{Att}(N)$.

□

On the Primary Decomposition Theorem, we can show that, given $N \subseteq M$ R-modules, if the component $\mathfrak{p}$-primary N_i of M/N is such that $\mathfrak{p}$ is minimal in $\operatorname{Ass}(M/N)$, then we can write N_i as

$$N_i = (\phi_{\mathfrak{p}})^{-1}(N_{\mathfrak{p}}),$$

where $\phi_{\mathfrak{p}} : M \longrightarrow M_{\mathfrak{p}}$ is the natural map. Once again, a similar behaviour happens in Secondary representation.

Theorem 2.15 (The Second Uniqueness Theorem). *Let R be a ring and M be a representable R-module. Let*

$$M = S_1 + S_2 + \cdots + S_n \qquad \text{with } S_i \quad \mathfrak{p}_i\text{-secondary} \quad (1 \leq i \leq n).$$

be a minimal representation of M. Suppose that $\mathfrak{p}_j$ is a minimal member of $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ with respect the inclusion order. Then

$$S_j = \bigcap_{r \in R \setminus \mathfrak{p}_j} rM.$$

Proof: Let $r \in R \setminus \mathfrak{p}_j$, so we have that $r^n S_j \neq 0$ for all $n \in \mathbb{N}$. Since S_j is a secondary R-module, then $rS_j = S_j$, which implies that

$$S_j = rS_j \subseteq rM.$$

Since $r \in R \smallsetminus \mathfrak{p}_j$ is arbitrary, then

$$S_j \subseteq \bigcap_{r \in R \smallsetminus \mathfrak{p}_j} rM.$$

Conversely, since $\mathfrak{p}_j$ is minimal in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, for each $i = 1, \ldots, n$, $i \neq j$, there exists $x_i \in \mathfrak{p}_i \smallsetminus \mathfrak{p}_j$. Thus there exists $m \in \mathbb{N}$ such that

$$x_1^m S_1 = x_2^m S_2 = \cdots = x_{j-1}^m S_{j-1} = x_{j+1}^m S_{j+1} = \cdots x_n^m S_n = 0.$$

Consider $z = x_1^m \cdot x_2^m \cdots x_{j-1}^m \cdot x_{j+1}^m \cdot \cdots \cdot x_n^m$. Note that $z \notin \mathfrak{p}_j$, so $zS_j = S_j$. Moreover $zS_i = 0$ if $i \neq j$. Thus

$$\bigcap_{r \in R \smallsetminus \mathfrak{p}_j} rM \subseteq zM = z(S_1 + S_2 + \cdots + S_n) = zS_j = S_j$$

Hence

$$S_j = \bigcap_{r \in R \smallsetminus \mathfrak{p}_j} rM$$

$\square$

Definition 2.16. *Let R be a ring and M be an R-module. We say that M is sum-irreducible precisely when M is non-zero and M cannot be expressed by sum of two proper submodules of itself.*

Proposition 2.17. *Let R be a ring and M be an R-module. If M is an Artinian sum-irreducible R-module, then M is secondary.*

Proof: Suppose by contradiction that M is not secondary, so there is $r \in R$ such that $M \neq rM$ and $r^n M \neq 0$ for all $n \in \mathbb{N}$. Since M is Artinian, the chain

$$M \supset rM \supseteq r^2 M \supseteq \cdots \supseteq r^n M \supseteq \cdots$$

stops, so there is $n \in \mathbb{N}$ such that $r^n M = r^{n+1} M$.

Now, let $x \in M$, thus $r^n x \in r^n M = r^{n+1} M$, so there is $y \in M$ such that $r^n x = r^{n+1} y$. Thus

$$r^n (x - ry) = 0.$$

Defining $K = \{m \in M \ ; \ r^n m = 0\}$, then

$$x = (x - ry) + ry \in K + rM.$$

Then $M = K + rM$. Since r was chosen such that $rM \neq M$ and $r^n M \neq 0$ for all $n \in \mathbb{N}$, we have both K and rM are proper submodules of M, so we contradict the hypothesis of M be sum-irreducible. $\square$

Finally, we can show that, in the category of Artinian R-modules, there is o Secondadry representation theorem, a dual of the Primary Decomposition Theorem.

Theorem 2.18. *Let R be a ring and M be an Artinian R-module. Every submodule of M is finite sum of sum-irreducible R-submodules. In particular, M is finite sum of sum-irreducible R-submodules and M is representable.*

Proof: Consider Σ the collection of all R-submodules N of M such that N is not finite sum of sum-irreducible submodules, that is

$$\Sigma = \{N \subseteq M \ ; \ N \text{ is not finite sum of sum-irreducible submodules}\}.$$

I claim that Σ is empty. In fact, if $\Sigma \neq \emptyset$, let $W \in \Sigma$ be a minimal element of Σ with respect the inclusion order. This element exists, because M is Artinian. Note W is necessarily sum-reducible, that is, there are proper submodules W_1 and W_2 such that

$$W = W_1 + W_2.$$

Since $W_1,\ W_2 \notin \Sigma$, we conclude that

$$W_1 = S_1 + \cdots + S_n$$
$$W_2 = S_1' + \cdots + S_m'$$

where S_i and S_j' are sum-irreducible submodules for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus

$$W = S_1 + \cdots + S_n + S_1' + \cdots + S_m',$$

which contradicts the hypothesis of W do not be finite sum of sum-irreducible submodules. So $\Sigma = \emptyset$. In particular, M is finite sum of sum-irreducible submodules. Since each Artinian sum-irreducible R-module is secondary, we conclude that every Artinian R-module is representable. $\square$

Corollary 2.19. *Let R be a ring and M be an Artinian R-module. Then $\mathrm{Att}(M)$ is finite and $\mathrm{Att}(M) = \emptyset$ if and only if $M = 0$.*

Proof: We have already seen that M can be expressed as finite sum of secondary R-submodules

$$M = S_1 + \cdots + S_n.$$

Supposing this representation minimal and denoting $\mathfrak{p}_i = \sqrt{\mathrm{Ann}(S_i)}$, then $\mathrm{Att}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is finite. $\square$

Proposition 2.20. *Let R be a ring, M be an Artinian R-module and $r \in R$. Then*

(i) $M = rM$ if and only if $r \in R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}$.

(ii) $\sqrt{\mathrm{Ann}(M)} = \bigcap_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}$.

Proof: **(i):** Since $\mathrm{Att}(0) = \emptyset$ and the equalities above are trivially satisfied on this case, we can suppose without lost of generality that M is a non-zero Artinian, let

$$M = S_1 + S_2 + \cdots + S_n \qquad \text{with } S_i \quad \mathfrak{p}_i\text{-secondary} \quad (\leq i \leq n).$$

be a minimal secondary representation of M. Suppose that $r \in \bigcup_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}$, thus there is $1 \leq i \leq n$ such that $r \in \mathfrak{p}_i = \sqrt{\mathrm{Ann}(S_i)}$. Thus there is $m \in \mathbb{N}$ such that $r^m S_i = 0$ and then

$$r^m M = r^m(S_1 + S_2 + \cdots + S_n) \subseteq S_1 + S_2 + \cdots S_{i-1} + S_{i+1} + \cdots + S_n \subsetneq M$$

Then $M \neq rM$, that is, if $M = rM$, then $r \in R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}$.

Conversely, if $r \in R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}$, then $r \notin \mathfrak{p}_i = \sqrt{\mathrm{Ann}(S_i)}$ for all $1 \leq i \leq n$. Since each S_i is a secondary R-module, then necessarily we have that $S_i = rS_i$ for all $1 \leq i \leq n$. Thus

$$M = S_1 + S_2 + \cdots + S_n = rS_1 + rS_2 + \cdots + rS_n = r(S_1 + S_2 + \cdots + S_n) = rM.$$

(ii): Let $r \in \bigcap_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}$, thus there is $m \in \mathbb{N}$ such that

$$r^m S_1 = r^m S_2 = \cdots = r^m S_n = 0.$$

So

$$r^m M = r^m(S_1 + S_2 + \cdots + S_n) = r^m S_1 + r^m S_2 + \cdots + r^m S_n = 0,$$

whence $r \in \sqrt{\mathrm{Ann}(M)}$.

Conversely, if $r \in \sqrt{\mathrm{Ann}(M)}$, then $r^m M = 0$ for some $m \in \mathbb{N}$. In particular, $r^m S_k = 0$ for all $1 \leq k \leq n$. Thus $r \in \sqrt{\mathrm{Ann}(S_k)} = \mathfrak{p}_k$ for all $1 \leq k \leq n$. Hence

$$r \in \bigcap_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}.$$

$\square$

Corollary 2.21. *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring and* M *be an Artinian* R*-module. Then* M *is finitely generated, and so of finite length, if and only if* $\mathrm{Att}(M) \subseteq \{m\}$

Proof: Since the case $M = 0$ is trivial, we can consider $M \neq 0$. If M is finitely generated, then M is a Noetherian R-module. Since M is also Artinian, then we conclude that M has finite length, so there is $n \in \mathbb{N}$ such that $\mathfrak{m}^n M = 0$. So

$$\mathfrak{m} \subseteq \sqrt{\mathrm{Ann}_R(M)} = \bigcap_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p}.$$

Since $\mathrm{Att}(M)$ is finite, we conclude $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Att}(M)$ and actually it is the unique prime ideal attained M.

Conversely, if $\mathrm{Att}(M) \subseteq \{m\}$, then, since $M \neq 0$, we actually have that $\mathrm{Att}(M) = \{\mathfrak{m}\}$. Also

$$\sqrt{\mathrm{Ann}_R(M)} = \bigcap_{\mathfrak{p} \in \mathrm{Att}(M)} \mathfrak{p} = \mathfrak{m}.$$

Since R is Notherian ring, there is $n \in \mathbb{N}$ such that $\mathfrak{m}^n M = 0$. Since M is an Artinian R-module, we conclude that M is also Noetherian and so finitely generated R-module. □

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