Closed graph Theorems

Kevin Alves Vasconcellos

January, 2021

In mathematics, there are results that are very similar in different categories. These theorems usually receive the same name. That's the case of Closed Graph Theorem. Given the sets X, Y and a mapping $f : X \longrightarrow Y$, we define the graph of f, denoting by $\mathrm{G}(f)$, the set

$$\mathrm{G}(f) := \{(x, y) \in X \times Y \;\; ; \;\; y = f(x), x \in X\}.$$

The first version of Closed Graph Theorem that will be proved is of nature purely topological as we'll see in following.

Proposition 1. *Let X and Y be topological spaces, with Y Hausdorff. Consider $f : X \longrightarrow Y$ a mapping.*

(i) If f is continuous, then $\mathrm{G}(f)$ is closed in $X \times Y$;

(ii) Conversely, if Y is compact and $\mathrm{G}(f)$ is closed, then f is continuous.

Proof: (i): In fact, it's enough to prove that $\mathrm{G}(f)^c$ is open in $X \times Y$. Let $(x, y) \in \mathrm{G}(f)^c$, then we have that $y \neq f(x)$. Since Y is Hausdorff, there are disjoint open sets U and V such that $y \in U$ and $f(x) \in V$.

Since f is a continuous function, the set $W := f^{-1}(V)$ is open in X. Thus, consider the open sets $W \times V$ and $W \times U$ in $X \times Y$. It's easy to see that $(W \times V) \cap (W \times U) = \varnothing$, then we conclude that $W \times U \subseteq \mathrm{G}(f)^c$, then $\mathrm{G}(f)^c$ is open in $X \times Y$.

(ii): Let V be an open set in Y. Note that $\mathrm{G}(f) \cap (X \times (Y \smallsetminus V))$ is closed in $X \times Y$. Consider the mapping

$$\pi_1 : X \times Y \longrightarrow X$$

$$(x, y) \longmapsto x$$

Since Y is compact, it's easy to prove that π is closed, then

$$\pi(\mathrm{G}(f) \cap X \times (Y \setminus V)) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$$

is closed in X. Hence we have that $f^{-1}(V)$ is open in X. So f is continuous. $\square$

The second version of Closed Graph Theorem that will be proved is one of pillars of functional analysis. This version requires both normed vector spaces are Banach, and uses strongly the Open Mapping Theorem.

Proposition 2. *Let X and Y be Banach spaces over $\mathbb{C}$ or $\mathbb{R}$ and $f : X \longrightarrow Y$ be a linear mapping. Thus f is continuous if and only if $\mathrm{G}(f)$ is closed in $X \times Y$*

Proof: Suppose that f is a continuous function. Consider the function

$$g : X \times Y \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto \|f(x) - y\|$$

It's easy to see that g is a continuous function, then we conclude that $\mathrm{G}(f) = f^{-1}(\{0\})$ is closed in $X \times Y$.

On the other hand suppose that $\mathrm{G}(f)$ is closed in $X \times Y$. Consider the mapping

$$h : \mathrm{G}(f) \longrightarrow X$$
$$(x, f(x)) \longmapsto x$$

Since $\mathrm{G}(f)$ is closed, $\mathrm{G}(f)$ is also a Banach space. Moreover, it's easy to see that h is a bijective, continuous function. By Open Mapping Theorem, h has a continuous inverse. Thus

$$h^{-1} : X \longrightarrow \mathrm{G}(f)$$
$$x \longmapsto (x, f(x))$$

is continuous. Since the projection mapping in the second coordinate π_2 is continuous, then

$$f = \pi_2 \circ h^{-1} : X \longrightarrow Y$$
$$x \longmapsto f(x)$$

is continuous. $\square$

In the categories of algebraic objects, Also there are versions of the Closed Graph Theorem. We'll prove the version for the Modules.

Proposition 3. *Let R be a ring, M and N be R-modules and $f : M \longrightarrow N$ a mapping between modules. Thus f is a homomorphism of R-modules if and only if $\mathrm{G}(f)$ is R-submodule of $M \oplus N$.*

Proof: Suppose that f is a homomorphism of R-modules. Given $(x, f(x)), (y, f(y)) \in \mathrm{G}(f)$, we have that

$$(x, f(x)) + (y, f(y)) = (x + y, f(x) + f(y)) = (x + y, f(x + y)) \in \mathrm{G}(f).$$

Then $\mathrm{G}(f)$ is closed under addition. Moreover, given $r \in R$ and $(x, f(x)) \in \mathrm{G}(f)$, we have

$$r(x, f(x)) = (rx, rf(x)) = (rx, f(rx)) \in \mathrm{G}(f).$$

Thus $\mathrm{G}(f)$ is closed under multiplication by scalar and then $\mathrm{G}(f)$ is a R-submodule of $M \oplus N$.

Now suppose that $\mathrm{G}(f)$ is a R-submodule of $M \oplus N$. Let $x, Y \in M$. Since $\mathrm{G}(f)$ is a R-submodule, we have that

$$(x, f(x)) + (y, f(y)) = (x + y, f(x) + f(y)) \in \mathrm{G}(f).$$

Thus $f(x + y) = f(x) + f(y)$. Moreover, given $r \in R$ and $x \in M$, we have

$$r(x, f(x)) = (rx, rf(x)) \in \mathrm{G}(f).$$

Thus $rf(x) = f(rx)$ and then $f : M \longrightarrow N$ is a homomorphism of R-modules. $\square$

Observation 4. *Imitating the proof of Proposition 3, we can prove similar versions of Closed Mapping Theorem for groups and rings.*

The Closed Graph Theorem also has a version in Order Theory. Before state the result, let's remember the product order: Given $(X, \leq_1)$ and $(Y, \leq_2)$ two totally ordered sets, we define in $X \times Y$ the following order relation

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{if and and only if} \quad x_1 \leq_1 x_2 \text{ and } y_1 \leq_2 y_2.$$

Proposition 5. *Let $(X, \leq_1)$ and $(Y, \leq_2)$ be two sets totally ordered and $f : X \longrightarrow Y$ be a function. Thus f is monotone increasing function if and only if $\mathrm{G}(f)$ is totally ordered with the product order.*

Proof: Suppose that f is monotone increasing function. Consider $(x, f(x))$ and $(y, f(y))$ in $\mathrm{G}(f)$. If $x \leq_1 y$, then, since f monotone increasing, we have $f(x) \leq_2 f(y)$, so $(x, f(x)) \leq (y, f(y))$. Similarly, if $y \leq_1 x$, we have $f(y) \leq_2 f(x)$ and then $(y, f(y)) \leq (x, f(x))$, then $\mathrm{G}(f)$ is totally ordered.

On other hand, if $\mathrm{G}(f)$ is totally ordered with product order, given $x, y \in X$, we have two possibilities

$$(x, f(x)) \leq (y, f(y)) \quad \text{or} \quad (x, f(x)) \geq (y, f(y)).$$

- If $(x, f(x)) \leq (y, f(y))$, then $x \leq y$ and $f(x) \leq f(y)$;
- If $(x, f(x)) \geq (y, f(y))$, then $x \geq y$ and $f(x) \geq f(y)$.

Hence f a monotone increasing function. $\square$