Tensors and Forms

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1 Motivation

The most known theorem of Calculus, commonly called by The Fundamental Theorem of Calculus, says that, if $f : [a, b] \longrightarrow \mathbb{R}$ is an integrable function which admits a primitive function, that is, there is $F : [a, b] \longrightarrow \mathbb{R}$ differentiable such that $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

In particular, if $f : [a, b] \longrightarrow \mathbb{R}$ is a differentiable function with integrable derivative, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Note that, seeing the interval $I = [a, b]$ as a manifold, we have that $\partial I = \{a, b\}$. The equality above say that, roughly speaking, there is a way on how to establish a relation between the integral of f' over the manifold I and the integral of f over the boundary of I. A natural question is if it is possible to generalize this result for differentiable manifolds of arbitrary dimension. The answer is yes and is contained on the Stokes Theorem.

However this theorem requires a non-trivial tool called differential forms. In order to learn this tool, it is necessary a reasonable algebraic machinery: the tensors and forms. The purpose of this article is to present the basic theory of tensors and forms. All linear spaces here defined will be assumed to have finite dimension and over the $\mathbb{R}$, even though there will be results, whose validity still holds on infinite dimension and over arbitrary fields.

2 Exterior Algebra

Definition 2.1. *Let V be a linear space and n a positive integer. A n-tensor is a functional*

$$T : \underbrace{V \times V \times \cdots \times V}_{n \text{ times}} \longrightarrow \mathbb{R}$$

such that T is separately linear in each variable. That is, for each $k \in \{1, \ldots, n\}$, we have that T satisfies the linearity in the following sense.

$$T(u_1, \ldots, u_k + \lambda v_k, \ldots, u_n) = T(u_1, \ldots, , u_k, \ldots, u_n) + \lambda T(u_1, \ldots, v_k, \ldots, u_n)$$

In particular, when $n = 1$, the 1-tensors are simply the linear functionals defined in V.

Example 2.2. *Let V be a linear space. The inner dot on V is a classic example of 2-tensor in V.*

Note that set of all n-tensors defined in V, which we will denote by $\mathfrak{J}^n(V^*)$, is closed by sum and scalar multiples.

Proposition 2.3. *Let V be a linear space and n a positive integer. The set $\mathfrak{J}^n(V^*)$ of all n-tensors defined in V constitutes a linear space.*

Proof: We will prove that $\mathfrak{J}^n(V^*)$ is closed by sums and scalar multiplications. The checking of the other linear space's axioms is straightforward. Let f, $g \in \mathfrak{J}^n(V^*)$ and $\gamma \in \mathbb{R}$. Note that

$$\begin{aligned}
&(f + \gamma g)(u_1, \ldots, u_k + \lambda v_k, \ldots, u_n) = f(u_1, \ldots, u_k + \lambda v_k, \ldots, u_n) + \gamma g(u_1, \ldots, u_k + \lambda v_k, \ldots, u_n) \\
&= f(u_1, \ldots, u_k, \ldots, u_n) + \lambda f(u_1, \ldots, v_k, \ldots, u_n) + \gamma(g(u_1, \ldots, u_k, \ldots, u_n) + \lambda g(u_1, \ldots, v_k, \ldots, u_n)) \\
&= (f(u_1, \ldots, u_k, \ldots, u_n) + \gamma g(u_1, \ldots, u_k, \ldots, u_n)) + \lambda(f(u_1, \ldots, v_k, \ldots, u_n) + \gamma g(u_1, \ldots, v_k, \ldots, u_n)) \\
&= (f + \gamma g)(u_1, \ldots, u_k, \ldots, u_n) + \lambda(f + \gamma g)(u_1, \ldots, v_k, \ldots, u_n).
\end{aligned}$$

Thus $f + \gamma g \in \mathfrak{J}^n(V^*)$, so $\mathfrak{J}^n(V^*)$ is closed by sum and scalar multiplication. □

Let V be a linear space and n, m positive integers. Given $f \in \mathfrak{J}^n(V^*)$ and $g \in \mathfrak{J}^m(V^*)$, we can construct an $(n + m)$-tensor $f \otimes g : V^{n+m} \longrightarrow \mathbb{R}$ in V such that

$$(f \otimes g)(v_1, \ldots, v_n, u_1, \ldots, u_m) = f(v_1, \ldots, v_n)g(u_1, \ldots, u_m).$$

for all $(v_1, \ldots, v_n, u_1, \ldots, u_m) \in V^{n+m}$. We call this product by tensor product of f and g.

Remark 2.4. *Note that this product is not commutative. Indeed, consider the following* 2*-tensors*

$$\begin{aligned} f : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (e_i, e_j) &\longmapsto 1 \end{aligned} \qquad\qquad \begin{aligned} g : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (e_i, e_j) &\longmapsto \delta_{ij} \end{aligned}$$

By definition, we have that

$$\begin{aligned} (f \otimes g)(e_1, e_2, e_1, e_1) &= f(e_1, e_2) g(e_1, e_1) = 1, \\ (g \otimes f)(e_1, e_2, e_1, e_1) &= g(e_1, e_2) f(e_1, e_1) = 0. \end{aligned}$$

Thus $f \otimes g \neq g \otimes f$.

Definition 2.5. *Let* V *be a linear space. The tensor algebra of* V^* *is the graded linear space*

$$T(V^*) = \bigoplus_{n=0}^{\infty} \mathfrak{J}^n(V^*),$$

where the multiplication is defined by the tensor product on its homogeneous coordinates, where we define $\mathfrak{J}^0(V^*) = \mathbb{R}$.

Thus we can easily check that tensor algebra of V^* is a graded non-commutative $\mathbb{R}$-algebra. A natural question is about the dimension of $\mathfrak{J}^n(V^*)$ and how we can obtain a natural basis for $\mathfrak{J}^n(V^*)$ for a given positive integer n.

Theorem 2.6. *Let* V *be an* m*-dimensional linear space and* $\{\phi_1, \phi_2, \ldots, \phi_m\}$ *a basis for the dual space* V^**. Given a positive integer* n*, then the family*

$$\mathcal{S} = \{\phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_n} \ ; \ 1 \leq i_1, i_2, \ldots, i_n \leq m\}$$

constitutes a basis for $\mathfrak{J}^n(V^*)$.

Proof: Firstly we will prove that $\mathfrak{J}^n(V^*) = \mathrm{Span}_{\mathbb{R}}(\mathcal{S})$. Indeed, by a basic argument of linear algebra, we can construct a basis $\{v_1, \ldots, v_m\}$ for V such that

$$\phi_i(v_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore note that, given $(u_1, \ldots, u_n) \in V^n$, where

$$u_i = \sum_{k=1}^{m} a_{ik} v_k$$

for each $k = 1, \ldots, n$, and $f \in \mathfrak{J}^n(V^*)$, the n-linearity of f implies that

$$f(u_1, u_2, \ldots, u_n) = \sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \cdots \sum_{k_n=1}^{m} a_{1k_1} a_{2k_2} \cdots a_{nk_n} f(v_{k_1}, v_{k_2}, \ldots, v_{k_n}),$$

which means that f is totally determined by its values on the set

$$\mathcal{R} = \{(v_{k_1}, v_{k_2}, \ldots, v_{k_n}) \in V^n \ ; \ 1 \leq k_1, \ldots, k_n \leq m\}.$$

This means that if we find an n-tensor g such that

$$f(v_{k_1}, v_{k_2}, \ldots, v_{k_n}) = g(v_{k_1}, v_{k_2}, \ldots, v_{k_n})$$

for all $(v_{k_1}, v_{k_2}, \ldots, v_{k_n})$ with $1 \leq k_1, \ldots, k_n \leq m$, then $g = f$. Thus, defining

$$g = \sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \cdots \sum_{k_n=1}^{m} f(v_{k_1}, v_{k_2}, \ldots, v_{k_n}) \phi_{k_1} \otimes \phi_{k_2} \otimes \cdots \otimes \phi_{k_n},$$

we have that g satisfies exactly this property. Then we have that $\mathfrak{J}^n(V^*) = \mathrm{Span}_{\mathbb{R}}(\mathcal{S})$.

Let's prove now that $\mathcal{S}$ is a linearly independent subset of $\mathfrak{J}^n(V^*)$. Consider a family of scalars $a_{k_1,k_2,\ldots,k_n}$ with $1 \leq k_1, \ldots, k_n \leq m$ such that

$$\sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \cdots \sum_{k_n=1}^{m} a_{k_1,k_2,\ldots,k_n} (\phi_{k_1} \otimes \phi_{k_2} \otimes \cdots \otimes \phi_{k_n}) = 0.$$

Applying $(v_{k_1}, v_{k_2}, \ldots, v_{k_n})$ in both sides of this equation, we get that $a_{k_1,k_2,\ldots,a_{k_n}} = 0$. Hence $\mathcal{S}$ is a basis for $\mathfrak{J}^n(V^*)$. $\square$

Corollary 2.7. *Let V be an m-dimensional linear space and n a positive integer. Then*

(i) $\dim_{\mathbb{R}}(\mathfrak{J}^n(V^*)) = m^n$*;*

(ii) $\dim_{\mathbb{R}}(T(V^*)) = \infty$.

Proof: **(i):** It a trivial combinatorial argument.

(ii): Firstly suppose $m \geq 2$. Since $\mathfrak{J}^n(V^*) \subseteq T(M)$ for all $n \geq 1$ and

$$\lim_{n\to\infty} \big(\dim_{\mathbb{R}}(\mathfrak{J}^n(V^*))\big) = \lim_{n\to\infty} m^n = \infty$$

then its clear that $\dim_{\mathbb{R}}(T(V^*)) = \infty$. Now, if $\dim_{\mathbb{R}}(V) = 1$. Even though $\dim_(\mathbb{R})(\mathfrak{J}^n(V^*)) = 1$, we have that

$$\dim_{\mathbb{R}} \big(\mathfrak{J}^0(V^*) \oplus \mathfrak{J}^1(V^*) \oplus \cdots \oplus \mathfrak{J}^n(V^*)\big) = n + 1.$$

Thus, since $\mathfrak{I}^0(V^*) \oplus \mathfrak{I}^1(V^*) \oplus \cdots \oplus \mathfrak{I}^n(V^*) \subseteq T(V^*)$ for all $n \in \mathbb{N}$, we conclude that

$$\dim_{\mathbb{R}}(T(V^*)) = \infty.$$

$\square$

A special class of n-tensor is that whose sign is reversed whenever two variables are transposed.

Definition 2.8. *Let V be a linear space and n a positive integer. An n-tensor $f \in \mathfrak{I}^n(V^*)$ is said alternating if, for any $1 \leq i < j \leq n$ and $(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \in V^n$, it is true that*

$$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n).$$

Example 2.9. *Let V be a linear space. Any 1-tensor in V is an alternating tensor by vacuous argument.*

Example 2.10. *The determinant function* $\det : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$ *is an alternating n-tensor.*

Given $n \in \mathbb{N}$, set $I_n := \{1, \ldots, n\}$. Consider the set S_n of all permutations on I_n

$$S_n = \{\sigma : I_n \longrightarrow I_n \ ; \quad \sigma \text{ is bijection}\}.$$

It is a well known from group theory that every permutation $\sigma \in S_n$ can be written as finite composition of transpositions [1] and that it is said even or odd depending on whether it expressible as a product of an even or odd number of transpositions. Let $(-1)^\sigma$ be $+1$ or -1, depending on whether σ is even or odd. Given n-tensor $f \in \mathfrak{I}^n(V^*)$ and $\sigma \in S_n$, we define the n-tensor f^σ as

$$\begin{aligned} f^\sigma : V^n &\longrightarrow \mathbb{R} \\ (v_1, \ldots, v_n) &\longmapsto f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}). \end{aligned}$$

By a simple induction on the number of transpositions, we can prove that

$$f^\sigma = (-1)^\sigma f$$

for all f alternating n-tensor on V and $\sigma \in S_n$. Moreover, given σ, $\pi \in S_n$, then $(f^\pi)^\sigma = f^{\pi \circ \sigma}$ and $(-1)^\pi (-1)^\sigma = (-1)^{\pi \circ \sigma}$ as we will see on the following proposition.

Proposition 2.11. *Let V be a linear space, n a positive integer and $f \in \mathfrak{I}^n(V^*)$. Given σ, $\pi \in S_n$, then*

[1] Remember that a transposition is the most simple nontrivial permutation. It takes i to j, takes j to i and all others elements remain unchanged.

(i) $(f^\pi)^\sigma = f^{\pi\circ\sigma}$*;*

(ii) $(-1)^\pi(-1)^\sigma = (-1)^{\pi\circ\sigma}$.

Proof: **(i):** In fact, let $(v_1, \ldots, v_n) \in V^n$, then

$$(f^{\pi\circ\sigma})(v_1, \ldots, v_n) = f(v_{(\pi\circ\sigma)(1)}, \ldots, v_{(\pi\circ\sigma)(n)}) = f(v_{\pi(\sigma(1))}, \ldots, v_{\pi(\sigma(n))}) = (f^\pi)(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$
$$= ((f^\pi)^\sigma)(v_1, \ldots, v_n).$$

Since this equality holds for all $(v_1, \ldots, v_n) \in V^n$, we conclude that $(f^\pi)^\sigma = f^{\pi\circ\sigma}$.

(ii): Since every permutation $\sigma \in S_n$ can be written as finite product of transpositions, we can assume without lost of generality that σ is a transposition. Being σ a transposition, we observe that $\pi \circ \sigma$ is a permutation with opposite parity than π, thus

$$(-1)^{\pi\circ\sigma} = -(-1)^\pi = (-1)^\pi(-1)^\sigma.$$

□

There is a standard procedure for making alternating tensors out of arbitrary ones. Indeed let f be an n-tensor in V, we define the new one $\mathrm{Alt}(f)$ by

$$\mathrm{Alt}(f) = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma f^\sigma.$$

Proposition 2.12. *Let V be a linear space and n a positive integer. Given $f \in \mathfrak{J}^n(V^*)$, then* $\mathrm{Alt}(f)$ *is an alternating n-tensor.*

Proof: Indeed let $\pi \in S_n$, then

$$[\mathrm{Alt}(f)]^\pi = \frac{1}{n!}\sum_{\sigma\in S_n}[(-1)^\sigma f^\sigma]^\pi = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma f^{\sigma\circ\pi}$$

Since $1 = (-1)^\pi(-1)^\pi$, then

$$[\mathrm{Alt}(f)]^\pi = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\cdot 1\cdot f^{\sigma\circ\pi} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma(-1)^\pi(-1)^\pi f^{\sigma\circ\pi} = \frac{(-1)^\pi}{n!}\sum_{\sigma\in S_n}(-1)^{\sigma\circ\pi}f^{\sigma\circ\pi}.$$

However, by group theory, we have that

$$\phi : S_n \longrightarrow S_n$$
$$\sigma \longmapsto \sigma\circ\pi$$

is an automorphism. This means that as σ ranges in S_n, so does $\sigma\circ\pi$, which implies that

$$[\mathrm{Alt}(f)]^\pi = \frac{(-1)^\pi}{n!}\sum_{\sigma\in S_n}(-1)^\sigma f^\sigma = (-1)^\pi\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma f^\sigma = (-1)^\pi\,\mathrm{Alt}(f),$$

which means that $\mathrm{Alt}(f)$ is an alternating n-tensor. □

Remark 2.13. *If f is already an alternating n-tensor, then* $\mathrm{Alt}(f) = f$. *Indeed*

$$\mathrm{Alt}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma f^\sigma = \frac{1}{n!} \sum_{\sigma \in S_n} f.$$

Since $\mathrm{card}(S_n) = n!$, *we conclude that*

$$\mathrm{Alt}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f = \frac{1}{n!}(n! f) = f.$$

The sum and scalar multiplication of alternating n-forms are closed, whence the family of all alternating n-tensors on V, denoted by $\Lambda^n(V^*)$, is a linear subspace of $\mathfrak{J}^n(V^*)$.

Proposition 2.14. *Let V be a linear space and n a positive integer. The family of all alternating n-tensors on V is a linear subspace of $\mathfrak{J}^n(V^*)$.*

Proof: Indeed it is enough to show that $\Lambda^n(V^*)$ is closed by addition and multiplication by scalar. Let $f, g \in \Lambda^n(V^*)$ and $\lambda \in \mathbb{R}$, given $1 \leq i < j \leq n$ and $(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \in V^n$, we have

$$\begin{aligned}(f + \lambda g)(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) &= f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) + \lambda g(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \\ &= -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n) - \lambda g(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n) \\ &= -(f + \lambda g)(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n).\end{aligned}$$

Since it holds for each $(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \in V^n$, we conclude that $f + \lambda g \in \Lambda^n(V^*)$. □

The tensor product of alternating tensors is not necessarily an alternating tensor as we will see on the next example

Example 2.15. *Consider the following alternating 2-tensor*

$$\begin{aligned} f : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (e_1, e_2) &\longmapsto 1; \\ (e_2, e_1) &\longmapsto -1; \\ (e_i, e_i) &\longmapsto 0. \end{aligned}$$

Consider $f^{\otimes 2} := f \otimes f$ *and observe that*

$$f^{\otimes 2}(e_1, e_2, e_1, e_2) \neq -f^{\otimes 2}(e_1, e_1, e_2, e_2).$$

Thus $f^{\otimes 2}$ is not an alternating tensor.

Fortunately here the Alt operator defined early will be userful to produce an alternating tensor. Given $f \in \Lambda^n(V^*)$ and $g \in \Lambda^m(V^*)$, the *wedge product* of f and g, denoted by $f \wedge g$, is defined by

$$f \wedge g = \mathrm{Alt}(f \otimes g) \in \Lambda^{m+n}(V^*).$$

The wedge product clearly distributes over the addition and scalar multiplication, because the operator Alt is linear. Next we will prove that this product is associative, but the proof will require the following technical lemma

Lemma 2.16. *Let V be a linear space and n, m positive integers. Given $f \in \mathfrak{J}^n(V^*)$, if* $\mathrm{Alt}(f) = 0$, *then $f \wedge g = g \wedge f = 0$ for any $g \in \mathfrak{J}^m(V^*)$.*

Proof: Note that S_{n+m} has a subgroup isomorphic to S_n. Indeed, given $\sigma \in S_n$, consider the function

$$\begin{aligned} \sigma' : I_{n+m} &\longrightarrow I_{n+m} \\ i &\longmapsto \begin{cases} \sigma(i), & \text{if } i \in I_n, \\ i, & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly σ' is a permutation in S_{n+m} and that

$$\begin{aligned} \phi : S_n &\longrightarrow S_{n+m} \\ \sigma &\longmapsto \sigma' \end{aligned}$$

is a group monomorphism. In other words, S_{n+m} has a subgroup G isomorphic to S_n via ϕ. Denoting by $\sigma^* = \phi^{-1}(\sigma)$, where $\sigma \in G = \mathrm{Im}(\phi)$, it is straightforward to verify that $(f \otimes g)^\sigma = f^{\sigma^*} \otimes g$ and $(-1)^\sigma = (-1)^{\sigma^*}$ for any $\sigma \in G$. Thus

$$\sum_{\sigma \in G} (-1)^\sigma (f \otimes g)^\sigma = \sum_{\sigma \in G} (-1)^{\sigma^*} (f^{\sigma^*} \otimes g) = \left(\sum_{\sigma \in S_n} (-1)^\sigma f^\sigma \right) \otimes g = \mathrm{Alt}(f) \otimes g = 0$$

Now, by group theory, we know that G induces in S_{n+m} a partition in right cosets, that is

$$S_{n+m} = \bigcup_{\sigma \in S_{n+m}} G \circ \sigma,$$

where $(G \circ \sigma) \cap (G \circ \pi) = \emptyset$ or $(G \circ \sigma) = (G \circ \pi)$ for each σ, $\pi \in S_{n+m}$. However for each such right coset, we have

$$\sum_{\sigma \in G} (-1)^{\sigma \circ \pi} (f \otimes g)^{\sigma \circ \pi} = (-1)^\pi \left(\sum_{\sigma \in G} (-1)^\sigma \left((f \otimes g)^\sigma \right)^\pi \right) = (-1)^\pi \left(\sum_{\sigma \in G} (-1)^\sigma \left(f \otimes g \right)^\sigma \right)^\pi$$

$$= (-1)^\pi \left(\sum_{\sigma \in G} (-1)^{\sigma^*} \left(f^{\sigma^*} \otimes g \right) \right)^\pi = (-1)^\pi \left(\left(\sum_{\sigma \in S_n} (-1)^\sigma f^\sigma \right) \otimes g \right)^\pi = (-1)^\pi (\mathrm{Alt}(f) \otimes g)^\pi = 0.$$

Since $f \wedge g = \mathrm{Alt}(f \otimes g)$ is the sum of these partial summations over the right cosets of G, then $f \wedge g = 0$. Using similar argument, we prove that $g \wedge f = 0$. $\square$

Theorem 2.17. *Let V be a linear space. The wedge product of tensors in V is associative, that is, given $f \in \Lambda^n(V^*)$, $g \in \Lambda^m(V^*)$ and $h \in \Lambda^p(V^*)$, then*

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Proof: I claim that $(f \wedge g) \wedge h = \mathrm{Alt}(f \otimes g \otimes h)$. In fact, by definition

$$(f \wedge g) \wedge h = \mathrm{Alt}((f \wedge g) \otimes h).$$

So, by linearity of Alt, we have

$$\begin{aligned}
(f \wedge g) \wedge h - \mathrm{Alt}(f \otimes g \otimes h) &= \mathrm{Alt}((f \wedge g) \otimes h) - \mathrm{Alt}(f \otimes g \otimes h) \\
= \frac{1}{(n+m+p)!} \sum_{\sigma \in S_{n+m+p}} (-1)^\sigma ((f \wedge g) \otimes h)^\sigma &- \frac{1}{(n+m+p)!} \sum_{\sigma \in S_{n+m+p}} (-1)^\sigma (f \otimes g \otimes h)^\sigma \\
&= \frac{1}{(n+m+p)!} \sum_{\sigma \in S_{n+m+p}} (-1)^\sigma ((f \wedge g) \otimes h - f \otimes g \otimes h)^\sigma \\
= \frac{1}{(n+m+p)!} \sum_{\sigma \in S_{n+m+p}} (-1)^\sigma [(f \wedge g) - f \otimes g) \otimes h]^\sigma &= \mathrm{Alt}\big((f \wedge g) - f \otimes g) \otimes h\big).
\end{aligned}$$

Since $f \wedge g$ is an alternating tensor, we have that

$$\mathrm{Alt}(f \wedge g - f \otimes g) = \mathrm{Alt}(f \wedge g) - \mathrm{Alt}(f \otimes g) = f \wedge g - f \wedge g = 0.$$

Thus, by Lemma 2.16, we have that

$$(f \wedge g) \wedge h - \mathrm{Alt}(f \otimes g \otimes h) = \mathrm{Alt}\big((f \wedge g) - f \otimes g) \otimes h\big) = \big(f \wedge g - f \otimes g\big) \wedge h = 0.$$

Hence $(f \wedge g) \wedge h = \mathrm{Alt}(f \otimes g \otimes h)$. A similar argument also shows that $f \wedge (g \wedge h) = \mathrm{Alt}(f \otimes g \otimes h)$, thus

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

$\square$

The associativity of wedge product allows us to write $(f \wedge g) \wedge h$ as $f \wedge g \wedge h$ without ambiguity. Moreover the claim proved on the theorem above can be easily generalized. A fundamental property of wedge product is that one satisfy the following anticommutative relation

$$f \wedge g = (-1)^{mn} g \wedge f.$$

for every $f \in \Lambda^n(V^*)$ and $g \in \Lambda^m(V^*)$. However the proof of this property is not so evident, whence we are going to prove by parts.

Lemma 2.18. *Let V be a linear space and $f,\ g \in V^*$ linear functionals on V. Then*

$$f \wedge g = -g \wedge f.$$

In particular, if $g = f$, then $g \wedge f = 0$.

Proof: In fact note that

$$f \wedge g = \mathrm{Alt}(f \otimes g) = \frac{1}{2!} \sum_{\sigma \in S_2} (-1)^\sigma (f \otimes g)^\sigma = \frac{1}{2!}(f \otimes g - g \otimes f),$$

because, if $\sigma \in S_2$ is the non-trivial permutation, a simple calculation shows that $(f \otimes g)^\sigma = g \otimes f$. Thus

$$f \wedge g = \frac{1}{2!}(f \otimes g - g \otimes f) = -\left[\frac{1}{2!}(g \otimes f - f \otimes g)\right] = -g \wedge f.$$

In particular, if $g = f$, we have that $f \wedge f = -f \wedge f$, which implies that $2(f \wedge f) = 0$. Since the characteristic of $\mathbb{R}$ is 0, we conclude that $f \wedge f = 0$. □

Corollary 2.19. *Let V be a linear space, $f_1, \ldots, f_k$ k linear functionals on V and $I = (i_1, \ldots, i_p)$ a sequence of indices for which each index is between 1 and k. If J is another sequence which differs from I only by ordering by a permutation $\sigma \in S_k$, then*

$$f_I := \bigwedge_{t=1}^{p} f_{i_t} = \pm \bigwedge_{t=1}^{p} f_{\sigma(i_t)} = f_J$$

Moreover, if any of the indices of I are equal, then $f_I = 0$

Proof: The proof based on the iterated use of anticommutativity and associativity of the wedge product. We left as exercise for the reader. □

Theorem 2.20. *Let V be an m-dimensional linear space and $\{\phi_1, \ldots, \phi_m\}$ a basis for V^*. Then*

$$\mathcal{B} = \left\{ \phi_I = \bigwedge_{j=1}^{n} \phi_{i_j} \ ;\ 1 \leq i_1 < i_2 < \cdots < i_n \leq m \right\}$$

is a basis for $\Lambda^n(V^*)$.

Proof: First I will prove that $\mathcal{B}$ generates $\Lambda^n(V^*)$. Let f be an alternating n-tensor. By Theorem 2.6, we can write f as

$$f = \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_n=1}^{m} a_{i_1, i_2, \ldots, a_{i_n}} (\phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_n}),$$

where $a_{i_1,i_2,\dots,i_n} \in \mathbb{R}$. Since f is an alternating n-tensor, we have that $\mathrm{Alt}(f) = f$. Thus Using the linearity of Alt operator, we get that

$$\begin{aligned} f = \mathrm{Alt}(f) &= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} a_{i_1,i_2,\dots,i_n}\,\mathrm{Alt}(\phi_{i_1}\otimes\phi_{i_2}\otimes\cdots\otimes\phi_{i_n}) \\ &= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} a_{i_1,i_2,\dots,i_n}(\phi_{i_1}\wedge\phi_{i_2}\wedge\cdots\wedge\phi_{i_n}). \end{aligned}$$

Since $\phi_I = 0$ when any of the indices of I are equal and $\phi_J = \pm\phi_I$ when J is a reordering of elements of I in increasing order, then we can write f as

$$f = \sum_{1\le i_1<\cdots<i_n\le m} b_{i_1,i_2,\dots,i_n}(\phi_{i_1}\wedge\phi_{i_2}\wedge\cdots\wedge\phi_{i_n}),$$

where $b_{i_1,i_2,\dots,i_n} \in \mathbb{R}$. Thus $\mathcal{B}$ generates $\Lambda^n(V^*)$.

Now we will prove that $\mathcal{B}$ is linearly independent. Consider $\{v_1,\dots,v_m\}$ the basis of V dual of $\{\phi_1,\dots,\phi_m\}$. For an increasing index sequence $I = (i_1,\dots,i_n)$, denote $v_I = (v_{i_1},\dots,v_{i_n})$. Note that

$$\phi_I(v_I) = \bigwedge_{k=1}^{n}\phi_{i_k}(v_I) = \mathrm{Alt}(\phi_{i_1}\otimes\cdots\otimes\phi_{i_n}) = \left[\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\sigma}(\phi_{i_1}\otimes\cdots\otimes\phi_{i_n})^{\sigma}\right](v_I) = \frac{1}{n!},$$

because, for all $\sigma \neq \mathrm{Id}$, we have $(\phi_{i_1}\otimes\cdots\otimes\phi_{i_n})^{\sigma}(v_I) = 0$. Moreover J is another increasing sequence with n elements different from I, it is easy to see that $\phi_I(v_J) = 0$.

Finally, for each sequence $1 \le i_1 < \cdots < i_n \le m$, consider $a_{i_1 i_2\dots i_n} \in \mathbb{R}$ such that

$$\sum_{1\le i_1<\cdots<i_n=m} a_{i_1,i_2,\dots,i_n}(\phi_{i_1}\wedge\phi_{i_2}\wedge\cdots\wedge\phi_{i_n}) = 0.$$

Given an increasing index sequence $I = (j_1,\dots,j_n)$, applying v_J in both sides of this equation, we get that

$$\frac{1}{n!}a_J = \left[\sum_{1\le i_1<\cdots<i_n=m} a_{i_1,i_2,\dots,i_n}(\phi_{i_1}\wedge\phi_{i_2}\wedge\cdots\wedge\phi_{i_n})\right](v_J) = 0(v_J) = 0,$$

which implies that $a_J = 0$ for all increasing index sequence J. Then $\mathcal{B}$ is a basis for $\Lambda^n(V^*)$. □

Corollary 2.21. *Let V be an m-dimensional linear space. Then*

$$\dim_{\mathbb{R}}(\Lambda^n(V^*)) = \binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

Proof: It is enough to count how many increasing index sequences

$$I = (1 \le i_1 < i_2 < \cdots < i_n \le m)$$

there are. By combinatorial analysis, we know that there are $\binom{m}{n}$ such sequences. □

Corollary 2.22. *Let V be a k-dimensional linear space. The wedge product satisfy the following anticommutative relation: for all $f \in \Lambda^n(V^*)$ and $g \in \Lambda^m(V^*)$, we have*

$$f \wedge g = (-1)^{mn} g \wedge f$$

Proof: Let $\{\phi_1, \dots, \phi_k\} \subseteq V^*$ be a basis for the dual V^*, $I = (1 \leq i_1 < \cdots < i_n \leq k)$ and $J = (1 \leq j_1 < \cdots < j_m \leq k)$ be increasing index sequences. It is easy to see that

$$\phi_I \wedge \phi_J = (-1)^{nm} \phi_J \wedge \phi_I.$$

Now given $f \in \Lambda^n(V^*)$ and $g \in \Lambda^m(V^*)$, we can write

$$\begin{aligned} f &= \sum_{1 \leq i_1 < \cdots < i_n \leq k} a_{i_1 i_2 \dots i_n} \phi_{i_1} \wedge \cdots \wedge \phi_{i_n}, \\ g &= \sum_{1 \leq j_1 < \cdots < j_m \leq k} b_{j_1 j_2 \dots j_n} \phi_{j_1} \wedge \cdots \wedge \phi_{j_m}. \end{aligned}$$

Thus by distributivity property of wedge product, we have

$$\begin{aligned} f \wedge g &= \sum_{1 \leq i_1 < \cdots < i_n \leq k} \sum_{1 \leq j_1 < \cdots < j_m \leq k} a_{i_1 i_2 \dots i_n} b_{j_1 j_2 \dots j_m} \phi_{j_1} (\phi_{i_1} \wedge \cdots \wedge \phi_{i_n}) \wedge (\phi_{j_1} \wedge \cdots \wedge \phi_{j_m}) \\ &= \sum_{1 \leq i_1 < \cdots < i_n \leq k} \sum_{1 \leq j_1 < \cdots < j_m \leq k} a_{i_1 i_2 \dots i_n} b_{j_1 j_2 \dots j_m} (-1)^{mn} (\phi_{j_1} \wedge \cdots \wedge \phi_{j_m}) \wedge (\phi_{i_1} \wedge \cdots \wedge \phi_{i_n}) \\ &= (-1)^{mn} \left[\sum_{1 \leq j_1 < \cdots < j_m \leq k} b_{j_1 j_2 \dots j_m} \phi_{j_1} \wedge \cdots \wedge \phi_{j_m} \right] \wedge \left[\sum_{1 \leq i_1 < \cdots < i_n \leq k} a_{i_1 i_2 \dots i_n} \phi_{i_1} \wedge \cdots \wedge \phi_{i_n} \right] \\ &= (-1)^{mn} g \wedge f. \end{aligned}$$

□

The Corollary 2.21 says that the if V is an n-dimensional $\mathbb{R}$-linear space, then the space of alternating n-tensors is unidimensional. In particular, if $V = \mathbb{R}^n$, there is only one alternating n-tensor f in $\mathbb{R}^n$ such that $f(e_1, e_2, \dots, e_n) = 1$ and this tensor is very familiar: the determinant function. Another interesting fact we can extract from Corollary 2.21 is that if the lenght of the index sequence I is greater than the dimension of V, then I must repeat at least one integer, which implies that $\phi_I = 0$, thus $\Lambda^k(V^*) = 0$ whenever $k > \dim(V)$.

It is useful to define $\Lambda^0(V^*) = \mathbb{R}$ and we extend the wedge product by simply letting the wedge product any element of $\mathbb{R}$ with any tensor in $\Lambda^n(V^*)$ be the usual multiplication. Setting $k = \dim(V)$, the wedge product then makes the direct sum

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^k(V^*)$$

a noncommutative algebra with identity called exterior algebra of V^*, whose identity is $1 \in \Lambda^0(V^*)$.

Now let V and W be linear spaces and $T : V \longrightarrow W$ a linear transformation. By elementary linear algebra, T induces a transpose map

$$T^* : W^* \longrightarrow V^*$$
$$f \longmapsto f \circ T.$$

The transpose map can be naturally extended to a linear mapping between the exterior algebras $\Lambda(T) : \Lambda^n(W^*) \longrightarrow \Lambda^n(V^*)$ such that

$$\Lambda(T)(f)(v_1, \ldots, v_n) = f(T(v_1), \ldots, T(v_n))$$

for all $f \in \Lambda^n(W^*)$ and $(v_1, \ldots, v_n) \in V^n$.

Proposition 2.23. *Let V, W and U be linear spaces, n a positive integer and $T : V \longrightarrow W$ a linear transformation. Then*

(i) The induced map $\Lambda(T) : \Lambda^n(W^*) \longrightarrow \Lambda^n(V^*)$ is linear;

(ii) For all f, $g \in \Lambda^n(W^*)$, $\Lambda(T)(f \wedge g) = \Lambda(T)(f) \wedge \Lambda(T)(g)$;

(iii) If $S : W \longrightarrow U$ is a linear transformation, then $\Lambda(S \circ T) = \Lambda(T) \circ \Lambda(S)$.

Proof: **(i):** Let f, $g \in \Lambda^n(W^*)$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}\Lambda(T)(f + \lambda g)(v_1, \ldots, v_n) = (f + \lambda g)(T(v_1), \ldots, T(v_n)) &= f(T(v_1), \ldots, T(v_n)) + \lambda g(T(v_1), \ldots, T(v_n)) \\ &= (\Lambda(T)(f) + \lambda\Lambda(T)(g))(v_1, \ldots, v_n).\end{aligned}$$

for all $(v_1, \ldots, v_n) \in V^n$. So $\Lambda(T)(f + \lambda g) = \Lambda(T)(f) + \lambda\Lambda(T)(g)$, which means that $\Lambda(T)$ is linear.

(ii): Using the Theorem 2.20, it is enough to prove for $n = 1$, that is, when f, g are linear functionals in W. Then

$$\begin{aligned}\Lambda(T)(f \wedge g)(v_1, v_2) = (f \wedge g)(T(v_1), T(v_2)) &= \frac{1}{2!} \sum_{\sigma \in S_2} \left[(-1)^\sigma (f \otimes g)^\sigma (T(v_1), T(v_2))\right] \\ &= \frac{1}{2!}(f(T(v_1))g(T(v_2)) - f(T(v_2))g(T(v_1))) \\ &= \frac{1}{2!}((\Lambda(T)(f))(v_1)(\Lambda(T)(g))(v_2) - (\Lambda(T)(f))(v_2)(\Lambda(T)(g))(v_1) \\ &= \frac{1}{2!} \sum_{\sigma \in S_2} \left[(-1)^\sigma (\Lambda(T)(f) \otimes \Lambda(T)(g))^\sigma (v_1, v_2)\right] = (\Lambda(T)(f) \wedge \Lambda(T)(g))(v_1, v_2).\end{aligned}$$

for all $(v_1, v_2) \in V^2$. Thus we conclude that $\Lambda(T)(f \wedge g) = \Lambda(T)(f) \wedge \Lambda(T)(g)$

(iii): Given $f \in \Lambda^n(U^*)$ and $(v_1, \ldots, v_n) \in V^n$, then

$$\Lambda(S \circ T)(f)(v_1, \ldots, v_n) = f((S \circ T)(v_1), \ldots, (S \circ T)(v_n)) = f(S(T(v_1)), \ldots, S(T(v_n)))$$
$$= \Lambda(S)(f)(T(v_1), \ldots, T(v_n)) = (\Lambda(T) \circ \Lambda(S))(f)(v_1, \ldots, v_n).$$

Since $(v_1, \ldots, v_n) \in V^n$ and $f \in \Lambda^n(U^*)$ are chosen arbitrarily, then $\Lambda(S \circ T) = \Lambda(T) \circ \Lambda(S)$. $\square$

We finish this section with an interesting an important theorem commonly known as Determinant Theorem.

Theorem 2.24. *Let V be an n-dimensional linear space and $T : V \longrightarrow V$ a linear map. Considering the map $\Lambda(T) : \Lambda^n(V^*) \longrightarrow \Lambda^n(V^*)$, then*

$$\Lambda^n(T)(\omega) = \det(T)\omega.$$

for every $\omega \in \Lambda^n(V)$. In particular, if $\phi_1, \ldots, \phi_n \in \Lambda^1(V^*)$, then

$$\Lambda(T)(\phi_1) \wedge \cdots \wedge \Lambda(T)(\phi_n) = (\det(T))\phi_1 \wedge \cdots \wedge \phi_n.$$

Proof: By Corollary 2.21, we have that $\Lambda^n(V^*)$ is an 1-dimensional linear space. Thus the map $\Lambda(T) : \Lambda^n(V^*) \longrightarrow \Lambda^n(V^*)$ is multiplication by a constant $\lambda \in \mathbb{R}$, that is, $T(\omega) = \lambda\omega$ for all $\omega \in \Lambda^n(V^*)$. I claim that λ is the determinant of T. Indeed, we know that $\det \in \Lambda^n((R^n)^*)$. So choose any isomorphism $S : V \longrightarrow \mathbb{R}^n$, and consider $\omega = \Lambda(S)(\det) \in \Lambda^n(V^*)$. Then

$$\Lambda(T)(\Lambda(S)(\det)) = \lambda(\Lambda(S)(\det)),$$

which implies that

$$\Lambda(S \circ T \circ S^{-1})(\det) = \Lambda(S^{-1})\big(\Lambda(T)(\Lambda(S)(\det))\big) = \lambda\Lambda(S^{-1})\big((\Lambda(S)(\det)\big) = \lambda\Lambda(S \circ S^{-1})(\det)$$
$$= \lambda \det.$$

Evaluating both sides of this equation on the standard ordered basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$, we get that

$$\lambda = \lambda \det(e_1, \ldots, e_n) = \Lambda(S \circ T \circ S^{-1})(\det)(e_1, \ldots, e_n) = \det((STS^{-1})(e_1), \ldots, (STS^{-1})(e_n))$$
$$= \det(STS^{-1}) \det(e_1, \ldots, e_n) = \det(STS^{-1}) = \det(S) \det(T) \det(S^{-1}) = \det(T).$$

In particular, if $\phi_1, \ldots, \phi_n \in \Lambda^1(V^*)$, then $\phi_1 \wedge \cdots \wedge \phi_n \in \Lambda^n(V^*)$, so

$$\Lambda(T)(\phi_1) \wedge \cdots \wedge \Lambda(T)(\phi_n) = \Lambda(T)(\phi_1 \wedge \cdots \wedge \phi_n) = \det(T)\phi_1 \wedge \cdots \wedge \phi_n.$$

$\square$

3 Differential Forms

Definition 3.1. *Let M be a smooth manifold and n a positive integer. An n-form on M is a map ω that assigns to each $x \in M$ an alternating n-tensor on $T_x(M)$, that is, $\omega(x) \in \Lambda^n((T_x(M))^*)$. The family of all n-forms on manifold M is denoted by $\Omega^n(M)$.*

Since $\Lambda^n((T_x(M))^*)$ is a linear space, $\Omega^n(M)$ constitutes a linear space with the following operations

$$+ : \Omega^n(M) \times \Omega^n(M) \longrightarrow \Omega^n(M) \qquad\qquad \cdot : \mathbb{R} \times \Omega^n(M) \longrightarrow \Omega^n(M)$$

$$(\omega, \theta) \longmapsto \omega + \theta : \, x \longmapsto \quad \omega(x) + \theta(x). \qquad\qquad (\lambda, \omega) \longmapsto \lambda\omega : \quad x \longmapsto \quad \lambda\omega(x).$$

Similarly the wedge product of forms is defined as

$$\wedge : \Omega^n(M) \times \Omega^m(M) \longrightarrow \Omega^{n+m}(M)$$

$$(\omega, \theta) \longmapsto \omega \wedge \theta : \qquad x \longmapsto \quad \omega(x) \wedge \theta(x).$$

The wedge product of differential forms inherits the anticommutative relation as we will see on the next proposition.

Proposition 3.2. *Let M be a smooth manifold. Given $\omega \in \Omega^n(M)$ and $\theta \in \Omega^m(M)$, then*

$$\omega \wedge \theta = (-1)^{mn} \theta \wedge \omega$$

Proof: Given $x \in M$, then

$$(\omega \wedge \theta)(x) = \omega(x) \wedge \theta(x) = (-1)^{mn} \theta(x) \wedge \omega(x) = (-1)^{mn} (\theta \wedge \omega)(x).$$

Since $x \in M$ is arbitrary, then $\omega \wedge \theta = (-1)^{mn} \theta \wedge \omega$. $\square$

Remember that we defined the $\Lambda^0(V^*) = \mathbb{R}$. So the 0-forms on a manifold M are exactly the real-valued functions on M.

Example 3.3. *Let M be a smooth manifold, $\phi : M \longrightarrow \mathbb{R}$ a smooth function and $x \in M$, then $d\phi(x) : T_x M \longrightarrow \mathbb{R}$ is a linear functional on $T_x(M)$. Thus the mapping*

$$d\phi : x \longrightarrow d\phi(x)$$

is an 1-form on M and it is called by differential of ϕ.

In particular, given $k \in \mathbb{N}$ and $1 \leq i \leq k$, the coordinate function

$$\begin{aligned} x_i : \quad \mathbb{R}^k \quad &\longrightarrow \mathbb{R} \\ (x_1, \ldots, x_k) &\longmapsto x_i \end{aligned}$$

yields the 1-form dx_i. It is simple to check that, given $z \in \mathbb{R}^k$, we have $dx_i(z)(a_1, \ldots, a_k) = a_i$. Moreover, we will prove that, for each $z \in \mathbb{R}^k$, the linear functionals $\{dx_1(z), \ldots, dx_k(z)\}$ constitutes a basis for $(\mathbb{R}^k)^*$.

Lemma 3.4. *Let k be an integer positive. For each $z \in \mathbb{R}^k$, the family of linear functionals $\{dx_1(z), \ldots, dx_k(z)\}$ constitutes a basis for $(\mathbb{R}^k)^*$*

Proof: Since $\dim_{\mathbb{R}}((\mathbb{R}^k)^*) = \dim_{\mathbb{R}}(\mathbb{R}^k) = k$, it is enough to show that $\{dx_1(z), \ldots, dx_k(z)\}$ is a linearly independent subset of $(\mathbb{R}^k)^*$. Indeed, let $c_1, \ldots, c_k \in \mathbb{R}$ such that

$$\sum_{j=1}^{n} c_j dx_j(z) = 0.$$

Applying e_i in both sides of the equation above, we get that

$$a_i = \sum_{j=1}^{n} c_j dx_j(z)(e_i) = 0(e_i) = 0.$$

Thus $\{dx_1(z), \ldots, dx_k(z)\}$ is linearly independent and so a basis for $(\mathbb{R}^k)^*$ □

For each increasing index sequence of integers between 1 and k, let $I = (i_1, \ldots, i_n)$ and define

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}$$

be the n-form associated to I. An interesting question is about some characterization of the k-forms on an open submanifold U of $\mathbb{R}^k$. Now we are going to answer it.

Proposition 3.5. *Let U be an open subset of $\mathbb{R}^k$. Every n-form on U may be uniquely expressed as a sum*

$$\sum_{I} f_I dx_I$$

over increasing index sequences $I = (1 \leq i_1, \ldots, i_n \leq k)$, and f_I being functions on U.

Proof: Let ω be an n-form on U. By definition, for each $z \in U$, we have that

$$\omega(z) \in \Lambda^n((T_z(U))^*).$$

Since $(T_z(U))^* = (\mathbb{R}^k)^*$ for each $z \in U$, then $\omega(z) \in \Lambda^n((\mathbb{R}^k)^*)$ for each $z \in U$. However we know, by Theorem 2.20, that $\{dx_I(z) \;;\; I \text{ increasing index sequence}\}$ is a basis for $\Lambda^n((\mathbb{R}^k)^*)$. Thus, for each increasing index sequence I, there is $a_I(z) \in \mathbb{R}$ such that

$$\omega(z) = \sum_I a_I(z) dx_I(z).$$

Defining for each increasing index sequence I

$$\begin{aligned} f_I : U &\longrightarrow \mathbb{R} \\ y &\longmapsto a_I(y), \end{aligned}$$

we get

$$\omega(z) = \sum_I f_I(z) dx_I(z),$$

which implies that $\omega = \sum_I f_I dx_I$. Now let $\sum_I g_I dx_I$ be another representation for ω. Given $z \in U$, we have

$$\sum_I f_I(z) dx_I(z) = \sum_I g_I(z) dx_I(z)$$

So, given an increasing index sequence I, if we apply e_I in both sides of the equality above we get that $f_I(z) = g_I(z)$, which that functions coefficients are unique. □

Example 3.6. *Let U be an open subset of $\mathbb{R}^n$ and consider the coordinates functions $x_1, x_2, \ldots, x_n$ on U. An important n-form on U is*

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

It is called volume form on U and it plays an important role on theory of integration in manifolds.

Given $\phi : U \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}$ a smooth mapping, the written of the differential of ϕ in the terms of coordinates function is very simple.

Proposition 3.7. *Let U be an open subset of $\mathbb{R}^k$ and $f : U \longrightarrow \mathbb{R}$ a smooth function. Then*

$$d\phi = \sum_{i=1}^{k} \frac{\partial \phi}{\partial x_i} dx_i.$$

Proof: Indeed, we have already know that $d\phi$ is an 1-form, so, by Proposition 3.5, there are functions $f_1, \ldots, f_k : U \longrightarrow \mathbb{R}$ such that

$$d\phi = \sum_{i=1}^{k} f_i dx_i.$$

Given $x \in U$, then $d\phi(x) = \sum_{i=1}^{k} f_i(x) dx_i(x)$. Thus applying e_j in both sides, we get

$$\frac{\partial \phi}{\partial x_j}(x) = (d\phi(x))(e_j) = \sum_{i=1}^{k} f_i(x)(dx_i(x)(e_j)) = f_j(x).$$

Since x is arbitrary, then $f_j = \partial \phi / \partial x_j$ for each $j = 1, \ldots, k$, whence

$$d\phi = \sum_{i=1}^{k} \frac{\partial \phi}{\partial x_i} dx_i.$$

$\square$

One of the most features of forms is that they pullback naturally under smooth mappings. Let M and N be smooth manifolds, $f : M \longrightarrow N$ a smooth map and $\omega \in \Omega^n(N)$. The pullback of ω by f, denoted by $f^*\omega$, is defined as follows: If $y = f(x)$, then f induces $df(x) : T_x(M) \longrightarrow T_y(N)$. Since $\omega(y)$ is an alternating n-tensor on $T_y(N)$, then we can pull it back to $T_x(M)$ using the linear map $\Lambda(df(x)) : \Lambda^n(T_y(N)) \longrightarrow \Lambda^n(T_x(M))$ and obtaining

$$(f^*\omega)(x) = \Lambda(df(x))[\omega(f(x))].$$

Then $(f^*\omega)(x)$ is an alternating n-tensor on $T_x(M)$ for each $x \in M$, which implies that $f^*\omega \in \Omega^n(M)$.

When ω is a 0-form on N — ω is real-valued function on N — then $f^*\omega = \omega \circ f$ is a real-valued function on M. Before we unravel the definition of f^*, let's check some basic properties which pullback satisfies.

Proposition 3.8. *Let M and N be smooth manifolds, $f : M \longrightarrow N$ a smooth map.*

(i) Given the n-forms ω_1 and ω_2 on N, then $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$;

(ii) Given an n-form ω on N and an m-form θ on N, then then $f^*(\omega \wedge \theta) = f^*(\omega) \wedge f^*(\theta)$;

(iii) Given a manifold P and $g : N \longrightarrow P$ a smooth map and ω an n-form in P, then $(f \circ h)^*\omega = h^* f^* \omega$.

Proof: **(i):** By definition, given $x \in M$, we have

$$f^*(\omega_1 + \omega_2)(x) = \Lambda(df(x))((\omega_1 + \omega_2)(f(x))) = \Lambda(df(x))(\omega_1(f(x)) + \omega_2(f(x)))$$
$$= \Lambda(df(x))(\omega_1(f(x))) + \Lambda(df(x))(\omega_2(f(x))) = f^*(\omega_1)(x) + f^*(\omega_2)(x) = (f^*(\omega_1) + f^*(\omega_2))(x).$$

Thus $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$.

(ii): By definition, given $x \in M$, we have

$$f^*(\omega \wedge \theta)(x) = \Lambda(df(x))((\omega \wedge \theta)(f(x))) = \Lambda(df(x))((\omega(f(x)) \wedge \theta(f(x))$$
$$= \Lambda(df(x))(\omega(f(x))) \wedge \Lambda(df(x))(\theta(f(x))) = f^*(\omega)(x) \wedge f^*(\theta)(x) = (f^*(\omega) \wedge f^*(\theta))(x).$$

Thus $f^*(\omega \wedge \theta) = f^*(\omega) \wedge f^*(\theta)$.

(iii): By definition, given $x \in M$, we have

$$(g \circ f)^*(\omega)(x) = \Lambda(d(g \circ f)(x))\omega((g \circ f)(x)) = \Lambda\big(dg(f(x)) \circ df(x)\big)\omega((g \circ f)(x))$$
$$= \Lambda(d(f(x)))\Lambda(dg(f(x)))(\omega(g(f(x)))) = \Lambda(d(f(x)))(g^*\omega(f(x))) = f^*(g^*(\omega(x))) = (f^* \circ g^*)(\omega(x)).$$

Thus $(g \circ f)^*(\omega) = (f^* \circ g^*)(\omega)$. $\square$

Now we are going to see explicitly what f^* does on Euclidean space. Let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^l$ be open subsets, and $f : V \longrightarrow U$ be a smooth map. Use $x_1, \ldots, x_k$ for the coordinates functions on $\mathbb{R}^k$ and $y_1, \ldots, y_l$ on $\mathbb{R}^l$. Write f concretely as $f = (f_1, \ldots, f_k)$, where each $f_i : V \longrightarrow \mathbb{R}$ is a smooth function on V. Given $y \in V$, observe that

$$f^*(dx_j)(y) = \Lambda(df(y))(dx_j(f(y)) = dx_j(f(y))(df(y)) = df_j(y).$$

Now we know the behaviour of f^* on the 0-forms and on the basic 1-form dx_i, we are able to know the behaviour of f^* in any n-form.

Proposition 3.9. *Let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^l$ be open subsets, and $f : V \longrightarrow U$ be a smooth map. Using $x_1, \ldots, x_k$ for the coordinates functions on $\mathbb{R}^k$, $y_1, \ldots, y_l$ on $\mathbb{R}^l$ and writing f concretely as $f = (f_1, \ldots, f_k)$, so, for a given an n-form ω on U, we have*

$$f^*\omega = \sum_I (a_I \circ f) df_I,$$

where f_I denote $f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge f_{i_n}$ for $I = (i_1, i_2, \ldots, i_n)$.

Proof: Indeed, given ω an arbitrary n-form on U, we know the it may be written uniquely as

$$\omega = \sum_I a_I dx_I,$$

where $a_I : U \longrightarrow \mathbb{R}$ are functions defined on U and so 0-forms. The application of properties proved on Proposition 3.8 gives us

$$f^*\omega = \sum_I f^* a_I df_I,$$

where $f^*a_I = a_I \circ f$ is the pullback of the 0-form a_I from U to V and f_I is the pullback of n-form $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}$ from U to V. $\square$

Example 3.10 (Determinant Theorem for forms)**.** *Suppose that $f : V \longrightarrow U$ is a diffeomorphism between two open subsets of $\mathbb{R}^k$ and ω be the volume form on U. If $f(y) = x$, both T_xV and T_yU are equal $\mathbb{R}^k$. Although in coordinates notation we have been using, the standard basis of linear functions on $\mathbb{R}^k$ is written as $dy_1(y), \dots, dy_k(y)$ for $T_y(V)$ but $dx_1(x), \dots, dx_k(x)$ for $T_x(U)$, the determinant theorem on Section 2 gives us the following formula*

$$f^*\omega(y) = \big(\Lambda(df(x))dx_1(x)\big) \wedge \cdots \wedge \big(\Lambda(df(x))dx_k(x)\big) = \det(df(y))dy_1(y) \wedge \cdots \wedge dy_k(y).$$

More succinctly,

$$f^*(dx_1 \wedge \cdots \wedge dx_k) = \det(df)dy_1 \wedge \cdots \wedge dy_k.$$

Definition 3.11. *Let U be an open in $\mathbb{R}^k$ and ω an n-form in U. Writing ω as*

$$\omega = \sum_I a_I dx_I,$$

we say that ω is a smooth if $a_I : U \longrightarrow \mathbb{R}$ is smooth function for each increasing index sequence $I = (1 \leq i_1 < \cdots < i_n \leq k)$.

Since composition, product and sum of smooth functions are smooth, it follows from Proposition 3.9 that, when $f : V \longrightarrow U$ is a smooth map between open subsets of Euclidean spaces, the pullback $f^*\omega$ of any smooth form ω on U is a smooth form on V.

Returning to forms in manifolds, we define smoothness for a form ω on a manifold $M \subseteq \mathbb{R}^n$ to mean that for every parametrization $\phi : U \subseteq \mathbb{R}^k \longrightarrow V \subseteq M$, $\phi^*\omega$ is a smooth form on the open set $U \subseteq \mathbb{R}^k$. We don't need really check this property for all parametrization. It is enough to show that this property holds for a cover of M, that is, if $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in L}$ is an atlas for M, that is

$$M = \bigcup_{\alpha \in L} \phi_\alpha(U_\alpha),$$

then the form ω is smooth provided each $\phi_\alpha^*\omega$ is smooth.

This is the basic theory of tensors and differential forms. Now the reader is ready to study the theory of integration on manifolds and the famous Stokes Theorem.