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Functional Analysis I

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These notes were written during the course of Functional Analysis I at the first semester of 2021 at Federal University of Rio de Janeiro. They contain the resolution of almost all exercises which were proposed during the classes as well the solution of all homework lists of the course. I hope this material may be useful for someone. If you find some error or some important misprint, please feel free to contact me.

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Chapter 1

Class exercises

1.1 Class 1: Banach Spaces - Part 1

Question 1: Let (M, d) be a metric space and $\{G_\lambda\}_{\lambda \in L}$ be a family of open sets. Prove that $\bigcup_{\lambda \in L} G_\lambda$ is an open set and, for any finite subset $L' \subseteq L$, $\bigcap_{\lambda \in L'} G_\lambda$ is an open set.

Proof: Let $x \in \bigcup_{\lambda \in L} G_\lambda$, thus there is $\lambda \in L$ such that $x \in G_\lambda$. Since G_λ is open, there is $r > 0$ such that

$$\mathrm{B}(x, r) \subseteq G_\lambda \subseteq \bigcup_{\lambda \in L} G_\lambda.$$

Thus $x \in \operatorname{int}(\bigcup_{\lambda \in L} G_\lambda)$, then $\bigcup_{\lambda \in L} G_\lambda$ is open.

Now let $L' = \{\lambda_1, \ldots, \lambda_n\} \subseteq L$ and $x \in \bigcap_{i=1}^n G_{\lambda_i}$. Thus $x \in G_{\lambda_i}$ for all $1 \leq i \leq n$. Since each G_{λ_i} is open, there is $r_i > 0$ such that

$$x \in \mathrm{B}(x, r_i) \subseteq G_{\lambda_i}$$

Take $r = \min\{r_1, \ldots, r_n\}$. Thus we have that

$$\mathrm{B}(x, r) \subseteq \mathrm{B}(x, r_i) \subseteq G_{\lambda_i}$$

for every $i = 1, \ldots, n$. Then

$$\mathrm{B}(x, r) \subseteq \bigcap_{i=1}^n G_{\lambda_i}.$$

Thus $x \in \operatorname{int}(\bigcap_{i=1}^n G_{\lambda_i})$, then $\bigcap_{i=1}^n G_{\lambda_i}$ is open. $\square$

Question 2: Let (M, d) be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in M. Prove that if M is a convergent sequence, then $(x_n)_{n \in \mathbb{N}}$ is Cauchy. Prove that the converse isn't necessarily convergent.

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence and x be its limit in M. Let $\epsilon > 0$, then there is $N \in \mathbb{N}$ such that for every $n, m > N$, we have

$$d(x_n, x) < \epsilon/2 \qquad \text{and} \qquad d(x_m, x) < \epsilon/2$$

Thus, using the triangular inequality, we conclude that for every $n, m > N$

$$d(x_n, x_m) \le d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence.

Now consider the metric space $((0,1), |.|)$. Consider the Cauchy sequence $x_n = 1/(2n)$. Note that $(x_n)_{n\in\mathbb{N}}$ converges to 0, when considered a sequence in $(\mathbb{R}, |.|)$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, |.|)$. In particular, it's a Cauchy sequence $(\mathbb{R}, |.|)$ and so a Cauchy sequence in $((0,1), |.|)$. By uniqueness limit property, we conclude that this sequence cannot converge in $(0,1)$, because $0 \notin (0,1)$. $\square$

Question 3: Let $(E, \mathbb{K}, +, *, ||.||)$ be a normed linear space. Prove that its norm induces in E a metric d defined by

$$d : E \times E \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto ||x - y||$$

Proof: in fact, it's enough to verify the metric's axioms.

- It's clear that $d(x, y) = ||x - y|| \ge 0$ for every $x, y \in E$.
- If $d(x, y) = ||x - y|| = 0$, then, by norm's definition, $x - y = 0$, thus $x = y$. Moreover, for every $x \in E$, we have $d(x, x) = ||x - x|| = ||0|| = 0$.
- Given $x, y \in E$, we have that $d(x, y) = ||x - y|| = ||(-1)(y - x)|| = |-1|.||y - x|| = d(y, x)$.
- Given $x, y, z \in E$, we have

$$d(x, z) = ||x - z|| = ||(x - y) + (y - z)|| \le ||x - y|| + ||y - z|| = d(x, y) + d(y, z).$$

Thus d is a metric. $\square$

Question 4: Let $\mathbb{K}$ be a field and consider $E = \mathbb{K}^n$. Prove that the following functions

$$||.||_E : E \longrightarrow \mathbb{R} \qquad ||.||_m : E \longrightarrow \mathbb{R} \qquad ||.||_s : E \longrightarrow \mathbb{R}$$

$$(x_1 \ldots, x_n) \longmapsto \sqrt{\textstyle\sum_{k=1}^n |x_k|^2} \qquad (x_1 \ldots, x_n) \longmapsto \max_{1\le k\le n}\{|x_k|\} \qquad (x_1 \ldots, x_n) \longmapsto \textstyle\sum_{k=1}^n |x_k|$$

are norms in E

Proof: It's enough to check the norm's axioms. For $||.||_m$

- It's clear that $||(x_1,\dots,x_n)||_m = \max_{1\le i\le n}\{|x_i|\} \ge 0$ for every $(x_1,\dots,x_n) \in E$;
- If $||(x_1,\dots,x_n)||_m = 0$, then $|x_i| = 0$ for every $1 \le i \le n$, which implies that $x_i = 0$ for every $1 \le i \le n$. Then $(x_1,\dots,x_n) = (0,\dots,0)$. On the other hand, $||(0,\dots,0)||_m = 0$ trivially;
- Given $\lambda \in \mathbb{K}$ and $(x_1,\dots,x_n) \in \mathbb{K}^n$, we have that

$$\begin{aligned}||\lambda(x_1,\dots,x_n)||_m = ||(\lambda x_1,\dots,\lambda x_n))||_m = \max_{1\le i\le n}\{|\lambda x_i|\} = \max_{1\le i\le n}\{|\lambda||x_i|\} = |\lambda| \max_{1\le i\le n}\{|x_i|\}\\ = |\lambda|||(x_1,\dots,x_n)||_m.\end{aligned}$$

- Let $(x_1,\dots,x_n),(y_1,\dots,y_n) \in \mathbb{K}^n$, we have

$$\begin{aligned}||(x_1,\dots,x_n)+(y_1,\dots,y_n)||_m = ||(x_1+y_1,\dots,x_n+y_n)||_m = \max_{1\le i\le n}\{|x_i+y_i|\} \le \max_{1\le i\le n}\{|x_i|+|y_i|\}\\ \le \max_{1\le i\le n}\{|x_i|\} + \max_{1\le i\le n}\{|y_i|\} = ||(x_1,\dots,x_n)||_m + ||(y_1,\dots,y_n)||_m\end{aligned}$$

For the norm $||.||_s$

- $||(x_1,\dots,x_n)||_s = \sum_{k=1}^n |x_k| \ge 0$ for every $(x_1,\dots,x_n) \in E$.
- Given $(x_1,\dots,x_n) \in E$, if $||(x_1,\dots,x_n)||_s = \sum_{k=1}^n |x_k| = 0$, then $x_1 = \cdots = x_n = 0$, thus $(x_1,\dots,x_n) = (0,\dots,0)$. On the other hand, $||(0,\dots,0)||_s = 0$ trivially.
- Given $\lambda \in \mathbb{K}$ and $(x_1,\dots,x_n) \in \mathbb{K}^n$, we have that

$$||\lambda(x_1,\dots,x_n)||_s = ||(\lambda x_1,\dots,\lambda x_n))||_s = \sum_{k=1^n} |\lambda x_k| = |\lambda| \sum_{k=1^n} |x_k| = |\lambda|||(x_1,\dots,x_n)||_s$$

- Let $(x_1,\dots,x_n),(y_1,\dots,y_n) \in \mathbb{K}^n$, we have

$$\begin{aligned}||(x_1,\dots,x_n)+(y_1,\dots,y_n)||_s = ||(x_1+y_1,\dots,x_n+y_n)||_s = \sum_{k=1}^n |x_k+y_k| \le \sum_{k=1}^n (|x_k|+|y_k|) =\\ \sum_{k=1}^n |x_k| + \sum_{k=1}^n (|y_k|) = ||(x_1,\dots,x_n)||_s + ||(y_1,\dots,y_n)||_s\end{aligned}$$

For $||.||_E$

- It's clear that $||(x_1,\dots,x_n)||_E = \sqrt{\sum_{k=1}^n |x_k|^2} \ge 0$ for every $(x_1,\dots,x_n) \in \mathbb{K}^n$;

- Given $(x_1,\dots,x_n) \in E$, if $\|(x_1,\dots,x_n)\|_E = 0$, then $\sum_{k=1}^n |x_k|^2 = 0$, which implies that $x_1 = \cdots = x_n = 0$. Thus $(x_1\dots,x_n) = (0,\dots,0)$. On the other hand, it's clear that $\|(0,\dots,0)\| = 0$.

- Given $\lambda \in \mathbb{K}$ and $(x_1,\dots,x_n) \in \mathbb{K}^n$, we have that

$$\|\lambda(x_1,\dots,x_n)\|_E = \|(\lambda x_1,\dots,\lambda x_n)\|_E = \sqrt{\sum_{k=1}^n |\lambda x_k|^2} = \sqrt{\sum_{k=1}^n |\lambda|^2 |x_k|^2} = |\lambda| \sqrt{\sum_{k=1}^n |x_k|^2}$$
$$= |\lambda| \|(x_1,\dots,x_n)\|_E$$

$\square$

Question 5: Let X be a non-empty set. A function $f : X \longrightarrow \mathbb{K}$ is said bounded if there is $M > 0$ such that for $|f(x)| \le M$ for every $x \in X$. Prove that

$$B(X) = \{f \in \mathbb{K}^X \ ; \ f \text{ is bounded}\}$$

is a normed linear space with the norm

$$\begin{aligned} \|.\|_\infty : B(X) &\longrightarrow \mathbb{R} \\ f &\longmapsto \sup_{x\in X}\{|f(x)|\} \end{aligned}$$

Proof: Since $\mathbb{K}^X$ is a $\mathbb{K}$-linear space, in order to show that $B(X)$ is a linear space, it's enough to prove that $B(X)$ is a linear subspace of $\mathbb{K}^X$. To show it, let $f, g \in B(X)$, thus there are $M_1, M_2 > 0$ such that $|f(x)| \le M_1$ for every $x \in X$ and $|g(x)| \le M_2$ for every $x \in X$. Thus, using the triangular inequality, we conclude

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le M_1 + M_2$$

for every $x \in X$. Similarly, given $\lambda \in \mathbb{K}$, we have that

$$|(\lambda f)(x)| = |\lambda||f(x)| \le \lambda M_1$$

Thus $B(X)$ is a linear subspace of $\mathbb{K}^X$. Now it remains to prove that $\|.\|_\infty$ is a norm.

- It's evident that for every $f \in B(X)$, we have that $\|f\|_\infty = \sup_{x\in X}\{|f(x)|\} \ge 0$.

- Let $f \in B(X)$. If $\|f\|_\infty = 0$, then $\sup_{x\in X}\{|f(x)|\} = 0$, which implies that $f(x) = 0$ for every $x \in X$, that is, $f = 0$. On the other hand, if $f = 0$, it's evident that $\|f\|_\infty = 0$.

- Let $f \in B(X)$ and $\lambda \in \mathbb{K}$, thus

$$||\lambda f||_\infty = \sup_{x\in X}\{|(\lambda f)(x)|\} = \sup_{x\in X}\{|\lambda||f(x)|\} = |\lambda|\sup_{x\in X}\{|f(x)|\} = |\lambda|||f||_\infty$$

- Let $f, g \in B(X)$, thus

$$||f+g||_\infty = \sup_{x\in X}\{|(f+g)(x)|\} \leq \sup_{x\in X}\{|f(x)| + |g(x)|\} \leq \sup_{x,y\in X}\{|f(x)| + |g(y)|\}$$

$$\sup_{x\in X}\{|f(x)|\} + \sup_{y\in X}\{|g(y)|\} = ||f||_\infty + ||g||_\infty$$

□

Question 6: Let X be a non-empty set and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $B(X)$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges in $B(X)$ if and only if $(f_n)_{n\in\mathbb{N}}$ converges uniformly.

Proof: Suppose that $(f_n)_{n\in\mathbb{N}}$ converges in $B(X)$ to f, thus given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for every $n > N$, we have

$$|f_n(x) - f(x)| \leq ||f_n - f||_\infty < \epsilon$$

for every $x \in X$. Thus $f_n \to f$ uniformly. Suppose that f_n converges uniformy to f, then there, given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for every $n > N$, we have

$$|f_n(x) - f(x)| < \epsilon/2$$

Thus

$$||f_n - f||_\infty \leq \epsilon/2 < \epsilon$$

□

Question 7: Let X be a compact topological space, consider $C^0(X) = \{f \in \mathbb{K}^X; f \text{ is continuous}\} \subseteq B(X)$. Prove that $C^0(X)$ is a Banach space.

Proof: It's clear that $C^0(X)$ is a linear space. Let's prove that $C^0(X)$ is a complete normed linear space. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C^0(X)$, thus given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for every $n, m > N$ and $x \in X$, we have

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty < \epsilon/2$$

Thus $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy sequence. Since $\mathbb{K}$ is complete, let $f(x) := \lim_{n\to\infty} f_n(x)$. I claim that f is bounded. In fact, Using the fact that

$$|f_n(x)| \leq |f_{m_0}(x)| + ||f_n - f_{m_0}||_\infty < ||f_{m_0}|| + \epsilon/2$$

for all $n > N$, $m_0 > N$ fixed and $x \in X$ and letting $n \to \infty$, we conclude that f is bounded. I claim that $(f_n)_{n\in\mathbb{N}}$ converges to f in $B(X)$. Using the fact that

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty < \epsilon/2$$

for all $n, m > N$ and $x \in X$ and letting $m \to \infty$,

$$|f_n(x) - f(x)| \leq \lim_{m\to\infty} (||f_n - f_m||_\infty) \leq \epsilon/2 < \epsilon$$

Thus $(f_n)_{n\in\mathbb{N}}$ converges to f in $B(X)$. It's enough now to prove that f is continuous, but it's easy, because, given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for every $n > N$ and $x \in X$, we have

$$|f_n(x) - f(x)| < ||f_n - f||_\infty < \epsilon/3$$

Fixing $m > N$, given $x \in X$, there is a neighborhood of U of x such that $|f_m(y) - f_m(x)| < \epsilon/3$ for every $y \in U$. Thu,s for every $y \in U$, we get

$$|f(y) - f(x)| \leq |f(y) - f_m(y)| + |f_m(y) - f_m(x)| + |f_m(x) - f(x)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

Thus f is continuous, $f \in C^0(X)$ and then $C^0(X)$ is a Banach space. $\square$

Question 8: Let k be a positive integer and consider

$$C^k([a,b]) = \{f \in C^0([a,b]) \; ; \; f \text{ é de classe } C^k\}.$$

Show that $C^k([a,b])$ is a linear subspace of $C^0([a,b])$ and, defining the following norm

$$||.|| : C^k([a,b]) \longrightarrow \mathbb{R}$$

$$f \longmapsto \textstyle\sum_{i=0}^{n} ||f^{(i)}||_\infty$$

Prove that $(C^k([a,b]), ||.||)$ is a Banach space.

Question 9: Consider

$$\ell_0 = \{f \in \mathbb{K}^{\mathbb{N}} \; ; \; \lim_{n\to\infty} f(n) = 0\} = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} \; ; \; \lim_{n\to\infty} x_n = 0\}.$$

and define in ℓ_0 the following norm

$$||.||_\infty : \ell_0 \longrightarrow \mathbb{K}$$

$$(x_n)_{n\in\mathbb{N}} \longmapsto \sup_{n\in\mathbb{N}}\{|f(n)|\}$$

Prove that ℓ_0 is a Banach space

Proof: Since $\lim_{n\to\infty} x_n = 0$, every sequence em ℓ_0 is bounded, so this function is well-defined. It's easy to check this function is a norm in ℓ_0. Now it remains to prove that ℓ_0 is complete. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ_0. Thus, given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that, for any $n, m > N$ and $k \in \mathbb{N}$, we have

$$|f_n(k) - f_m(k)| \leq \|f_n - f_m\|_\infty < \epsilon/2$$

Thus, observe that the sequence $(f_n(k))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ for every $k \in \mathbb{N}$. Since $\mathbb{K}$ is complete, define for each $k \in \mathbb{N}$:

$$f(k) := \lim_{n\to\infty} f_n(k).$$

Consider the sequence $g := (g_n)_{n\in\mathbb{N}} = (f(n))_{n\in\mathbb{N}}$. I claim that $g \in \ell_0$. Indeed, making $m \to \infty$ in the inequality

$$|f_n(k) - f_m(k)| \leq \|f_n - f_m\|_\infty < \epsilon/2$$

we get

$$|f_n(k) - f(k)| \leq \lim_{m\to\infty} \|f_n - f_m\|_\infty \leq \epsilon/2 < \epsilon$$

Thus

$$|f(k)| \leq |f_n(k)| + \epsilon$$

Since $f_n \in \ell_0$, making $k \to \infty$, we conclude that $\lim_{k\to\infty} f(k) = 0$. Hence $g \in \ell_0$. Now I claim that $g = \lim_{n\to\infty} f_n$. In fact, making $m \to \infty$ in the inequality

$$|f_n(k) - f_m(k)| \leq \|f_n - f_m\|_\infty < \epsilon/2$$

we get

$$|f_n(k) - f(k)| \leq \lim_{m\to\infty} \|f_n - f_m\|_\infty \leq \epsilon/2 < \epsilon$$

for every $k \in \mathbb{N}$. Thus

$$\|f_n - g\|_\infty \leq \epsilon/2 < \epsilon,$$

which implies that $(f_n)_{n\in\mathbb{N}}$ is convergent, so ℓ_0 is a Banach space. □

Question 10: Consider the following subspace of ℓ_0

$$\ell_{00} = \{f \in \ell_0 \;;\; \text{There is } N \in \mathbb{N} \text{ such that } f(k) = 0 \text{ for all } k \geq N\}$$

Prove that $(\ell_{00}, ||.||_\infty)$ isn't a Banach space.

Proof: In order to show that $(\ell_{00}, ||.||_\infty)$ isn't a Banach space, it's enough to show that ℓ_{00} isn't closed in ℓ_0. For each $n \in \mathbb{N}$, consider

$$f_n = \left(\frac{1}{1}, \dots, \frac{1}{n}, 0, 0, \dots\right)$$

Note that $f_n \in \ell_{00}$ for every $n \in \mathbb{N}$. I claim that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ_{00}. Indeed, given $n, m \in \mathbb{N}$, we have that

$$||f_n - f_m||_\infty \leq \frac{1}{\min\{n, m\} + 1}$$

Thus, given ϵ, take N such that $1/N < \epsilon$. Then, for every $n, m > N$, we have

$$||f_n - f_m||_\infty \leq \frac{1}{\min\{n, m\} + 1} < \frac{1}{N + 1} < \epsilon.$$

However, this sequence doesn't converge in ℓ_{00}, because its limit in ℓ_0 is the sequence

$$g : \mathbb{N} \longrightarrow \mathbb{K}$$

$$n \longmapsto \tfrac{1}{n}$$

because $||f_n - g||_\infty = \frac{1}{n+1} < \epsilon$ for every enoughly great. Since $g \in \ell_0 \setminus \ell_{00}$, we conclude that ℓ_{00} isn't complete. □

1.2 Class 2: Banach Spaces - Part 2

Question 1: Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : X \longrightarrow Y$ be a continuous function. Suppose that X is compact, then prove that $f(X)$ is compact.

Proof: In fact, let $\{f(X) \cap V_\lambda\}_{\lambda\in L}$ be an open cover of $f(X)$. Since f is continuous, the family $\{f^{-1}(f(X) \cap V_\lambda)\}$ is an open cover of X. Since X is compact, there are $\lambda_1, \dots, \lambda_n \in L$ such that

$$X \subseteq (f^{-1}(f(X) \cap V_{\lambda_1})) \cup \dots \cup (f^{-1}(f(X) \cap V_{\lambda_n}))$$

Thus

$$f(X) \subseteq (f(f^{-1}(f(X) \cap V_{\lambda_1})) \cup \dots \cup (f(f^{-1}(f(X) \cap V_{\lambda_n})))) \subseteq (f(X) \cap V_{\lambda_n}) \cup \dots \cup (f(X) \cap V_{\lambda_n})$$

Thus $\{f(X) \cap V_\lambda\}_{\lambda \in L}$ admits open subcover, then $f(X)$ is compact. $\square$

Question 2: Show that every compact metric space M is totally bounded.

Proof: Let $\epsilon > 0$. Consider the open cover $\mathcal{B} = \{B(x,\epsilon)\}_{x \in M}$ of M. Since M is compact metric space, there are $x_1, \dots, x_n \in M$ such that

$$M = B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$$

Then M is totally bounded. $\square$

Question 3: Let (M,d) be a metric space. Prove that the following affirmations are equivalent

(i) M is compact;

(ii) Every infinite subset of M has an accumulation point;

(iii) Every sequence $(x_n)_{n \in \mathbb{N}}$ in M has a subsequence which converges to some element $x \in M$;

(iv) M is complete and totally bounded.

Proof: $(i) \to (ii)$: Let X be an infinite set of M. Suppose that X has no accumulation point, then there for any $x \in K$, there are $r_x > 0$ such that $B(x, r_x) \cap X = \{x\}$ or $B(x, r_x) \cap X = \varnothing$. Thus the family $\{B(x, r_x)\}$ is an open cover of K. Since K is compact, this implies that

$$K = B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_n})$$

Thus

$$X = X \cap K = (X \cap B(x_1, r_{x_1})) \cup \cdots \cup (X \cap B(x_n, r_{x_n}))$$

is certainly finite, which is a contradiction, because X is infinite by hypothesis.

$(ii) \to (iii)$: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in M. Consider $X = \{x_n \; ; \; n \in \mathbb{N}\}$. If X is finite, certainly there exists $n_0 \in \mathbb{N}$ such that $\mathbb{N}' = \{n \in \mathbb{N} \; ; \; x_n = x_{n_0}\}$ is infinite, so clearly $(x_n)_{n \in \mathbb{N}'}$ is convergent. Now suppose that X is infinite, so X has an accumulation point, that is, there is a sequence $(b_k)_{k \in \mathbb{N}}$ which converges to $x \in X$, so it's possible to build the wished subsequence.

$(iii) \to (iv)$: It's well known that, if a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence, then $(x_n)_{n \in \mathbb{N}}$ is convergent. Thus (M,d) is complete. Now suppose that M is not totally bounded, thus there is $\epsilon > 0$ with the property that there is no finite set of ϵ-balls covering M.

Thus, let $x \in M$. Since $B(x, \epsilon)$ does not cover M, let $x_1 \in M \setminus B(x_1, \epsilon)$. Inductively, suppose we have built $x_1, \dots, x_n$ such that $x_{i+1} \notin M \setminus (B(x_1, \epsilon) \cup \dots \cup B(x_i, \epsilon))$ for all $i = 1, \dots, n-1$. Since

$$M \setminus (B(x_1, \epsilon) \cup \dots \cup B(x_n, \epsilon)) \neq \varnothing,$$

let $x_{n+1} \in M \setminus (B(x_1, \epsilon) \cup \dots \cup B(x_n, \epsilon))$. Considering the sequence $(x_n)_{n \in \mathbb{N}}$, I claim this sequence has no convergente subsequence. In fact, if $(x_n)_{n \in \mathbb{N}}$ admitted convergent subsequence, say $(x_{n_k})_{k \in \mathbb{N}}$, then given there would be $k_0 \in \mathbb{N}$ such that for all $k > t > k_0$, we would have

$$d(x_{n_k}, x_{n_t}) < \epsilon/2$$

However $x_{n_k} \notin B(x_1, \epsilon) \cup \dots \cup B(x_{n_k - 1}, \epsilon)$. In particular, $x_{n_k} \notin B(x_{n_t}, \epsilon)$, which is a contradiction.

□

Question 4: Let (M, d) be a metric space and $N \subseteq M$ be a subset. If M is totally bounded, prove that (N, d) is totally bounded. In particular, any subset of a totally bounded complete metric space is relatively compact.

Proof: Indeed, let $\epsilon > 0$. Since M is totally bounded, there are $x_1, \dots, x_n \in M$ such that

$$M = B(x_1, \epsilon/2) \cup \dots \cup B(x_n, \epsilon/2).$$

Let $X_i = B(x_i, \epsilon/2) \cap N$. If $X_i \neq \varnothing$, let $y_i \in X_i$. Considering $i_1, \dots, i_m$ the indexes i where $X_i \neq \varnothing$, I claim that

$$N \subseteq B(y_{i_1}, \epsilon) \cup \dots \cup B(y_{i_m}, \epsilon).$$

In fact, let $x \in N$, then $x \in B(x_i, \epsilon/2)$ for some $i \in \{1, \dots, n\}$. By definition, we have that $y_i \in N \cap B(x_i, \epsilon/2)$. Thus $z \in B(y_i, \epsilon)$. In fact,

$$d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Now let M be a totally bounded complete metric space and $X \subseteq M$. Note that $\overline{X}$ is totally bounded. Also, note that, since $\overline{X}$ is closed and M is complete, $\overline{X}$ is complete. So, by Question 3 of this class, we have $\overline{X}$ is compact, that is, X is relatively compact. □

1.3 Class 3: Banach spaces - Part 3

Question 1: Let (M, d) be a metric space and $N \subseteq M$ be a subset of M. If M is separable space, prove that (N, d) is also separable.

Proof: Let $E = \{x_n \in M \;;\; n \in \mathbb{N}\}$ be countable and dense in M. Given $n \in \mathbb{N}$ and $X_m \in E$, consider $X_{m,n} = B_M(x_m, 1/n) \cap N$. If $X_{m,n} \neq \varnothing$, choose $x_{m,n} \in X_{m,n}$. Consider

$$E' = \{x_{m,n} \in M \in \;;\; \text{when } X_{m,n} \neq \varnothing\}.$$

It's clear that $E' \subseteq N$ is countable. Moreover, given $z \in N$ and $\epsilon > 0$, there is $x_n \in E$ such that $d(x_n, z) < \epsilon/2$. Taking $m \in \mathbb{N}$ such that $1/m < \epsilon/2$, note that $X_{n,m} = B_M(x_n, 1/m) \cap N \neq \varnothing$, because $d(x_n, z) < \epsilon/2$. Now

$$d(x_{n,m}, z) \leq d(x_{n,m}, x_n) + d(x_n, z) < 1/m + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon.$$

So E' is dense in N, then (N, d) is separable $\square$

Question 2: Let (M, d) be a metric space. If M is compact, then M is separable.

Proof: In fact, given $n \in \mathbb{N}$, consider $\mathcal{B}_n = \{B(x, 1/n)\}_{x \in M}$ the open cover of M. Since M is compact, there are $x_1^n, \ldots, x_{m_n}^n \in M$ such that

$$M = B(x_1^n, 1/n) \cup \cdots \cup B(x_{m_n}^n, 1/n).$$

Consider $X_n = \{x_1^n, \ldots x_{m_n}^n\}$ and $E = \bigcup_{n \in \mathbb{N}} X_n$. Evidently, E is countable. Now I claim that E is dense in M. Indeed, given $x \in M$ and $\epsilon > 0$, let $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon$. Since $\mathcal{B}_{n_0}$ is an open cover of M, there is $i \in \{1, \ldots, m_{n_0}\}$ such that $x \in B(x_i^{n_0}, 1/n_0)$. Thus $d(x, x_i^{n_0}) < 1/n_0 < \epsilon$. Then E is dense in M, so M is a separable metric space. $\square$

Question 3: Prove that $(C^0([a,b]), \|.\|)$ is a separable metric space, where

$$\begin{aligned} \|.\| : C^0([a,b]) &\longrightarrow \mathbb{R} \\ f &\longmapsto \max_{x \in [a,b]} |f(x)|. \end{aligned}$$

Proof: In fact, consider the the functions

$$\begin{aligned} f_n : [a,b] &\longrightarrow \mathbb{R} \\ x &\longmapsto x^n. \end{aligned}$$

Let $E = \{f_1, \ldots, f_n, \ldots\}$. Clearly E is countable subset of $C^0([a,b])$. Moreover

$$[E] := \mathrm{Span}_K(E) = \{f \in C^0([a,b]) \;;\; f \text{ is polynomial}\}.$$

Since $[E]$ is dense in $C^0([a,b])$ by Wierstrass theorem, we conclude that $C^0([a,b])$ is separable.

1.4 Class 4: Linear bounded operators

Question 1: Let K be a field, E, F be K-vector spaces and $T \in \mathrm{Hom}_K(E, F)$. Prove that $T(0) = 0$.

Proof: In fact, using the linearity of T, we have

$$T(0) = T(0 + 0) = T(0) + T(0)$$

Thus

$$T(0) = T(0) + 0 = T(0) + (T(0) + (-T(0))) = (T(0) + T(0)) + (-T(0)) = T(0) + (-T(0)) = 0$$

$\square$

Question 2: Let E and F be normed vector spaces. Define

$$\mathcal{L}(E, F) = \{T \in F^E \ ; \ T \text{ is linear and continuous}\}$$

Define in $\mathcal{L}(E, F)$ the following function

$$\begin{aligned} ||.|| : \mathcal{L}(E, F) &\longrightarrow \mathbb{R} \\ T &\longmapsto \sup_{x \in B[0,1]} ||T(x)|| \end{aligned}$$

Prove that $||.||$ is a norm in $\mathcal{L}(E, F)$.

Proof: It's enough to check the norm's axioms.

(i) Note that for all $x \in B[0, 1]$, we have $||T(X)|| \geq 0$, then $||T|| = \sup_{x \in B[0,1]} ||T(x)|| \geq 0$;

(ii) If $||T|| = 0$, then $\sup_{x \in B[0,1]} ||T(x)|| = 0$. This implies that $||T(x)|| = 0$ for all $x \in B[0, 1]$. Thus, given $x \neq 0 \in E$

$$T\left(\frac{x}{||x||}\right) = 0,$$

which implies that $T(x) = ||x||0 = 0$. So $T = 0$.

(iii) Let $\lambda \in \mathbb{K}$, $T \in \mathcal{L}(E, F)$, then

$$||\lambda T|| = \sup_{x \in B[0,1]} ||(\lambda T)(x)|| = \sup_{x \in B[0,1]} (|\lambda|||(T(x)||) = |\lambda| \sup_{x \in B[0,1]} (||(T(x)||) = |\lambda|||T||.$$

(iv) Let $T, S \in \mathcal{L}(E,F)$, then

$$\|T+S\| = \sup_{x\in B[0,1]} \|(T+S)(x)\| = \sup_{x\in B[0,1]} \|T(x)+S(x)\| \leq \sup_{x\in B[0,1]} (\|T(x)\| + \|S(x)\|)$$

$$\leq \sup_{x,y\in B[0,1]} (\|T(x)\| + \|S(y)\|) = \sup_{x\in B[0,1]} (\|T(x)\|) + \sup_{y\in B[0,1]} (\|S(y)\|) = \|T\| + \|S\|$$

Thus $\|.\|$ is a norm in $\mathcal{L}(E,F)$. □

Question 3: Let E and F be normed vector spaces. Denoting $S(0,1) = \{z \in E \ ; \ \|x\| = 1\}$, prove that, for all $T \in \mathcal{L}(E,F)$, we have

$$\|T\| = \sup_{x\in B[0,1]} \|T(x)\| = \sup_{x\in S(0,1)} \|T(x)\| = \sup_{x\in E, x\neq 0} \left(\frac{\|T(x)\|}{\|x\|} \right)$$

Proof: Proof of

$$\sup_{x\in S(0,1)} \|T(x)\| = \sup_{x\in E, x\neq 0} \left(\frac{\|T(x)\|}{\|x\|} \right)$$

Note that

$$\left\{ \frac{\|T(x)\|}{\|x\|} \ ; \ x \in E, x \neq 0 \right\} = \{\|T(x)\| \ ; \ x \in S(0,1)\}$$

In fact, given $x \in S(0,1)$, we have $\|T(x)\| = \|T(x)\|/1 = \|T(x)\|/\|x\|$, thus

$$\{\|T(x)\| \ ; \ x \in S(0,1)\} \subseteq \left\{ \frac{\|T(x)\|}{\|x\|} \ ; \ x \in E, x \neq 0 \right\}$$

On the other hand, given $x \in E, x \neq 0$, we have

$$\frac{\|T(x)\|}{\|x\|} = \left\| T\left(\frac{x}{\|x\|} \right) \right\|$$

and $x/\|x\| \in S(0,1)$, thus

$$\left\{ \frac{\|T(x)\|}{\|x\|} \ ; \ x \in E, x \neq 0 \right\} \subseteq \{\|T(x)\| \ ; \ x \in S(0,1)\}$$

Then

$$\sup_{x\in S(0,1)} \|T(x)\| = \sup\{\|T(x)\| \ ; \ x \in S(0,1)\} = \sup \left\{ \frac{\|T(x)\|}{\|x\|} \ ; \ x \in E, x \neq 0 \right\} = \sup_{x\in E, x\neq 0} \left(\frac{\|T(x)\|}{\|x\|} \right)$$

Proof of

$$\sup_{x\in B[0,1]} \|T(x)\| = \sup_{x\in S(0,1)} \|T(x)\|.$$

Since $S(0,1) \subseteq B[0,1]$, it is clear that

$$\sup_{x \in B[0,1]} \|T(x)\| \geq \sup_{x \in S(0,1)} \|T(x)\|.$$

If we prove that for any $x \in B[0,1]$, we can find $y \in S(0,1)$ such that $\|T(x)\| \leq \|T(y)\|$, then we can conclude that

$$\sup_{x \in B[0,1]} \|T(x)\| \leq \sup_{x \in S(0,1)} \|T(x)\|.$$

and the proof is finished. Let's prove it. Let $x \in B[0,1]$. If $x = 0$, then for any $y \in S[0,1]$, we have $\|T(0)\| \leq \|T(x)\|$. So we can assume without lost of generality that $x \neq 0$. Consider $y = x/\|x\|$. Thus

$$\|T(y)\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\| = \frac{\|T(x)\|}{\|x\|} \geq \|T(x)\|,$$

where the last inequality is because $\|x\| \leq 1$. Hence the wished equality is proved. □

Question 4: Let E and F be normed linear spaces and $T : E \longrightarrow F$ be a linear transformation. If $\dim_K(E) < \infty$, prove that T is bounded.

Proof: In fact, let $\{v_1, \ldots, v_n\}$ be a basis of E and define in E the following auxiliary norm

$$\|.\|_0 : E \longrightarrow \mathbb{R}$$

$$\textstyle\sum_{i=1}^n \alpha_i v_i \longmapsto \sum_{i=1}^n |\alpha_i|$$

Thus, given $v = \sum_{i=1}^n \alpha_i v_i \in E$, we have

$$\|T(v)\| = \left\|T\left(\sum_{i=1}^n \alpha_i v_i\right)\right\| \leq \sum_{i=1}^n |\alpha_i| \|T(v_i)\| \leq \big(\max_{i=1,\ldots,n} \|T(v_i)\|\big) \sum_{i=1}^n |\alpha_i| = \big(\max_{i=1,\ldots,n} \|T(v_i)\|\big) \|v\|_0$$

Call $L = \max_{i=1,\ldots,n} \|T(v_i)\|$. Since $\dim_K(E) < \infty$, all norms are equivalent, thus there is $C > 0$ such that $\|v\|_0 \leq C\|v\|$, then

$$\|T(v)\| \leq \big(\max_{i=1,\ldots,n} \|T(v_i)\|\big) \|v\|_0 \leq (LC)\|v\|$$

Then T is bounded. □

1.5 Class 5: The Hahn-Banach theorems

Question 1: Prove that every linear space V has a Hamel Basis.

Proof: If $V = 0$, then $\varnothing$ is a basis of V. Then, suppose that $V \neq 0$. Define

$$\Sigma = \{B \in \mathcal{P}(V)\ ;\ B \text{ is linearly independent}\}.$$

Note that $\Sigma \neq \varnothing$, because, given $v \neq 0 \in V$, the set $\{v\} \in \Sigma$. Considering the inclusion order, the set $(\Sigma, \leq)$ is partially ordered. Let $\{B_\lambda\}_{\lambda \in L}$ be a chain in Σ. Consider

$$B' = \bigcup_{\lambda \in L} B_\lambda.$$

I claim that $B' \subseteq \Sigma$ is linearly independent. In fact, let $v_1, \ldots, v_n \in B$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ and suppose

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

Since $\{B_\lambda\}_{\lambda \in L}$ is chain, there is $\lambda_0 \in L$ such that $v_1, \ldots, v_n \in B_{\lambda_0}$ and, as B_{λ_0} is linearly independent, we conclude that $\alpha_1 = \cdots = \alpha_n = 0$. Then B' is linearly independent and is an upper bound of $\{B_\lambda\}_{\lambda \in L}$, because $B_\lambda \subseteq B'$ for every $\lambda \in L$. By Zorn's Lemma, we conclude that Σ has a maximal element B with respect the inclusion. I claim that B is a Hamel basis of V. In fact, if not, let

$$v \in V \setminus \operatorname{Span}_{\mathbb{K}}(B).$$

It's easy to see that $B \cup \{v\}$ is linearly independent and, as B is properly contained in $B \cup \{v\}$, we get a contradiction. So

$$V = \operatorname{Span}_{\mathbb{K}}(B).$$

and so B is a Hamel Basis of V. $\square$

1.6 Class 6: Applications of Hahn-Banach Theorems

Question 1: Let E and F be normed linear spaces, $f : E \longrightarrow F$ be continuous mapping and let $A \subseteq E$ be subset of E dense in E.

(i) Prove that the norm function is continuous.[1]

(ii) Prove that, if $\|f(x)\| \leq \|x\|$ for all $x \in A$, then $\|f(x)\| \leq \|x\|$ for all $x \in E$.

[1]Actually, looking the proof, we see that this result is true for semi-norm more generally.

Proof: (i): Consider $||.||$ a norm at linear space E. Observe that, given $x, y \in E$, using the triangular inequality, we obtain

$$||x|| \leq ||x - y|| + ||y||,$$

that is, $||x|| - ||y|| \leq ||x - y||$. Switching x and y, we also obtain that $||y|| - ||x|| \leq ||y - x|| = ||x - y||$. Thus, we conclude

$$\Big| ||x|| - ||y|| \Big| = ||x - y||.$$

Then $||.|| : E \longrightarrow \mathbb{R}$ is continuous.

(ii): Let $x \in E$. Since A is dense in E, there is a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that

$$\lim_{n \to \infty} x_n = x.$$

Using the monotonicity of limit and the continuity of f and of the norm function, we conclude that

$$||f(x)|| = ||f\big(\lim_{n \to \infty} x_n\big)|| = ||\lim_{n \to \infty} f(x_n)|| = \lim_{n \to \infty} ||f(x_n)|| \leq \lim_{n \to \infty} ||x_n|| = ||\lim_{n \to \infty} x_n|| = ||x||.$$

Thus $||f(x)|| \leq ||x||$ for all $x \in E$. $\square$

Question 2: Let (M, d) and (N, d') be metric spaces. A mapping $f : M \longrightarrow N$ is said an isometry if, for all $x, y \in M$, we have $d'(f(x), f(y)) = d(x, y)$. Prove that every isometry is injective.

Proof: In fact, let (M, d) and (N, d') be metric spaces and $f : M \longrightarrow N$ be an isometry. Let $x, y \in M$ such that $f(x) = f(y)$, then

$$0 = d'(f(x), f(y)) = d(x, y).$$

Since d is a metric, we conclude that $x = y$, then f is injective. $\square$

1.7 Class 7: Hahn-Banach Separation Theorems

Definição: Let V be a linear space and W be a subspace of V. W is said a hyperplane if $W \neq V$ and, if W_1 is another subspace of V such that $W \subseteq W_1 \subseteq V$, then $W_1 = W$ or $W_1 = V$.

Question 1: Let V be a linear space and W be a linear subspace. Prove that W is a hyperplane in V if and only if there is $f \in V^*$ such that $f \neq 0$ and $W = \mathrm{Ker}(f) := \{x \in V \ ; \ f(x) = 0\}$.

Proof: Suppose that W is a hyperplane of V. Let $\beta = \{v_\lambda\}_{\lambda \in L}$ be a basis of W. Since $W \neq V$, let $v \in V \smallsetminus \mathrm{Span}_{\mathbb{K}}(\beta)$. It is a classical fact of Linear Algebra that $\beta' = \beta \cup \{v\}$ is a linearly independent. Moreover,since W is a maximal proper subspace of V, it is clear that β' is a basis of V. So define $f : V \longrightarrow \mathbb{K}$ such that $f(v_\lambda) = 0$ for all $\lambda \in L$ and $f(v) = 1$. I claim that $W = \mathrm{Ker}(f)$. Since $f(v_\lambda) = 0$ for all $\lambda \in L$ then

$$f(W) = f(\mathrm{Span}_{\mathbb{K}}(\beta)) = 0.$$

Thus $W \subseteq \mathrm{Ker}(f)$. On the other hand, given $x = w + tv \in \mathrm{Ker}(f)$, where $w \in W$ and $t \in \mathbb{K}$, we have that

$$0 = f(x) = f(w + tv) = f(w) + tf(v) = t.$$

Thus $x = w \in W$, so $\mathrm{Ker}(f) \subseteq W$ and then $\mathrm{Ker}(f) = W$. Now suppose that $W := \mathrm{Ker}(f)$ for some $f \neq 0 \in V^*$. It is simple to verify that W is a subspace of V. Since $f \neq 0$, W is necessarily a proper subspace of V. Now let $v_0 \in V$ such that $f(v_0) \neq 0$ and $\beta = \{v_\lambda\}_{\lambda \in L}$ be a basis of W. I claim that $\beta \cup \{v_0\}$ is basis of V. In fact, given $x \in V$, then

$$x - \frac{f(x)}{f(v_0)} v \in W$$

because $f(x) = 0$, thus $x \in W + \mathrm{Span}_{\mathbb{K}}\{v_0\}$. Moreover, it is easy to show that $\beta \cup \{v_0\}$ is linearly independent. Thus $\beta \cup \{v_0\}$ is a basis of V. Thus W is a maximal subspace, thus W is a hyperplane. □

Question 2: Let V be a normed linear space and W be a linear subspace. Prove that W is a hyperplane closed in V if and only if there is $f \in V'$ such that $f \neq 0$ and $W = \mathrm{Ker}(f) := \{x \in V \ ; \ f(x) = 0\}$.

Proof: Suppose that $W = \mathrm{Ker}(f)$ for some $f \neq 0 \in V'$. Since f é continuous and $\{0\}$ is closed in E, by Question 1, $W = \mathrm{Ker}(f) = f^{-1}(\{0\})$ is a closed hyperplane of V. Conversely let W be a closed hyperplane of V. Since W is a proper closed subspace, we have that $W \neq V$, so let $x_0 \in V \smallsetminus W$. It has already shown that there is $f \in E'$ such that $f(x) = 0$ for all $x \in W$ and $f(x_0) > 0$. Finally note that $W \subseteq \mathrm{Ker}(f) \subseteq V$. Since $\mathrm{Ker}(f) \neq V$ and W is a hyperplane of V, we conclude that $W = \mathrm{Ker}(f)$. □

Question 3: Let V be a normed linear space and C be an open, convex, non-empty subset of

V containing the origin. Consider the Minkowski functional of C

$$\begin{aligned} p_C : V &\longrightarrow \mathbb{R} \\ x &\longmapsto \inf\{a \in \mathbb{R}_{>0} \ ; \ x/a \in C\}. \end{aligned}$$

Then

(i) $p_C(bx) = bp_C(x)$ for all $b > 0, x \in V$;

(ii) $C = \{x \in V \ ; \ p_C(x) < 1\}$;

Proof: (i): Indeed, given $x \in V$, $b > 0$, we have

$$p_C(bx) = \inf\left\{a > 0 \ ; \ \frac{bx}{a} \in C\right\} = \inf\left\{a > 0 \ ; \ \frac{x}{a/b} \in C\right\} = \inf\left\{b \cdot \frac{a}{b} > 0 \ ; \ \frac{x}{a/b} \in C\right\}.$$

Since the function

$$\begin{aligned} f : \mathbb{R}_{>0} &\longrightarrow \mathbb{R}_{>0} \\ x &\longmapsto x/b \end{aligned}$$

is surjective, then

$$p_C(bx) = \inf\left\{b \cdot \frac{a}{b} > 0 \ ; \ \frac{x}{a/b} \in C\right\} = \inf\left\{b \cdot a > 0 \ ; \ \frac{x}{x} \in C\right\} = b \inf\left\{a > 0 \ ; \ \frac{x}{a} \in C\right\} = bp_C(x).$$

(ii): Indeed, if $x \in C$, then $x/1 = x \in C$. Since C is open, there is $\epsilon > 0$ such that $B(x, \epsilon) \subseteq C$. Thus, it is clear that

$$x + \frac{\epsilon}{2||x||} x \in C,$$

because

$$\left\| x + \frac{\epsilon}{2||x||} x - x \right\| = \epsilon/2 < \epsilon.$$

Since

$$x + \frac{\epsilon}{2||x||} x = \left(1 + \frac{\epsilon}{2||x||}\right) x = \frac{x}{\left(1 + \frac{\epsilon}{2||x||}\right)^{-1}}$$

and

$$\ell := \left(1 + \frac{\epsilon}{2||x||}\right)^{-1} < 1,$$

we have $p_C(x) \le \ell < 1$. So $x \in \{y \in E \ ; \ p_C(y) < 1\}$. On the other hand, suppose $p_C(x) = \ell < 1$, then $x/\ell \in C$. Setting $a = 1/\ell$, we have that $a > 1$ and $ax \in C$. Since C is convex and contains 0, C contains $\{tax \ ; \ t \in [0,1]\}$. In particular, C contains $\ell ax = (\ell \cdot a)x = 1.x = x$. Then

$$C = \{x \in V \ ; \ p_C(x) < 1\}.$$

□

Question 3: Let E, F be linear spaces and $A \subseteq E$ a nonempty convex subset. Given a linear mapping $T : E \longrightarrow F$, prove that $T(A)$ is a convex subset of F.

Proof: Indeed, let $y_1, y_2 \in T(A)$, then there are $x_1, x_2 \in A$ such that $y_1 = T(x_1)$ and $y_2 = T(x_2)$. Since A is convex, for all $t \in [0,1]$, we have

$$tx_1 + (1-t)x_2 \in A.$$

Applying T at expression above, we have that

$$ty_1 + (1-t)y_2 = tT(x_1) + (1-t)T(x_2) = T(tx_1 + (1-t)x_2) \in T(A).$$

Thus $T(A)$ is a convex subset of F □

Question 4: Let E be linear spaces and $A \subseteq E$ a nonempty convex subset. Given $\epsilon > 0$, prove that

$$A + B(0,\epsilon) = \{x + z \in E \ ; \ x \in A, \ z \in B(0,\epsilon)\}.$$

is convex.

Proof: Let $x_1 + z_1 \in A + B(0,\epsilon)$ and $x_2 + z_2 \in A + B(0,\epsilon)$. Given $t \in [0,1]$, we have

$$t(x_1 + z_1) + (1-t)(x_2 + z_2) = [tx_1 + (1-t)x_2] + tz_1 + (1-t)z_2.$$

Since A is convex, we have that $tx_1 + (1-t)x_2 \in A$. Now, it is enough to prove that $tz_1 + (1-t)z_2 \in B(0,\epsilon)$. However

$$||tz_1 + (1-t)z_2|| = |t|.||z_1|| + |(1-t)|.||z_2|| = t||z_1|| + (1-t)||z_2|| < t\epsilon + (1-t)\epsilon = \epsilon.$$

Thus $tz_1 + (1-t)z_2 \in B(0,\epsilon)$ and so $A + B(0,\epsilon)$ is a convex subset of E. □

1.8 Class 8: The Banach-Steinhaus Theorem

Question 1: The Banach-Steinhaus Theorem says that:

Let E be a Banach space and F be a normed linear space. Given a family $\{T_\lambda\}_{\lambda \in L} \subseteq \mathcal{L}(E,F)$ of bounded linear operators, if, for each $x \in E$, there is $M_x < \infty$ such that

$$\sup_{\lambda \in L} \|T_\lambda(x)\| \leq M_x,$$

then there is $M < \infty$ such that

$$\sup_{\lambda \in L} \|T_\lambda\| \leq M.$$

Prove that the hypothesis of E be a Banach space is indispensable.

Proof: Consider the following space

$$c_{00} = \{(x_n)_{n \in \mathbb{N}} \in \ell^\infty \ ; \ \text{there is } N > 0 \text{ such that } x_n = 0 \text{ for all } n \geq N\}$$

with the induced norm. It is well known that c_{00} is not a Banach space. Define the sequence $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}(c_{00}, \mathbb{R})$ of continuous linear operator, where

$$T_n : c_{00} \longrightarrow \mathbb{R}$$

$$(x_k)_{k \in \mathbb{N}} \longmapsto n \cdot x_n.$$

Note that, in fact, each T_n is continuous functional, because

$$|T_n((x_k)_{k \in \mathbb{N}})| = |n \cdot x_n| = n|x_n| \leq n\|(x_k)_{k \in \mathbb{N}}\|_\infty.$$

Moreover, given $(x_k)_{k \in \mathbb{N}} \in c_{00}$, there is $N \in \mathbb{N}$ such that $x_k = 0$ for all $k > N$, thus we have that

$$|T_n((x_k)_{k \in \mathbb{N}})| = |n \cdot x_n| \leq \max\{|x_1|, 2|x_2|, \dots, N|x_N|\} < \infty$$

for all $n \in \mathbb{N}$. Thus

$$\sup_{n \in \mathbb{N}} |T_n((x_k)_{k \in \mathbb{N}})| \leq \max\{|x_1|, 2|x_2|, \dots, N|x_N|\} < \infty.$$

Except of the fact of c_{00} not be Banach space, the sequence $(T_n)_{n \in \mathbb{N}}$ satisfies all the conditions of the Banach-Steinhaus Theorem's statement. However, we see that

$$\|T_n\| = n$$

for all $n \in \mathbb{N}$ and then

$$\sup_{n \in \mathbb{N}} \|T_n\| = \infty,$$

which contradicts the thesis of Banach-Steinhaus theorem. $\square$

1.9 Class 9: The Open Mapping Theorem

Question 1: Let E_1, E_2 and F be linear normed spaces and let

$$B : E_1 \times E_2 \longrightarrow F$$
$$(x, y) \longmapsto B(x, y)$$

be a bilinear mapping. Prove that, if there is $c > 0$ such that $||B(x, y)|| \leq c||x||||y||$, then B is continuous.

Proof: In fact, suppose without lost of generality that the norm in $E_1 \times E_2$ is given by

$$||.|| : E_1 \times E_2 \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto ||x||_{E_1} + ||y||_{E_2}.$$

Now let $(x_0, y_0) \in E_1 \times E_2$ and let $0 < \epsilon < 1$. Observe that

$$B(x, y) - B(x_0, y_0) = B(x, y) - B(x_0, y) + B(x_0, y) - B(x_0, y_0) = B(x - x_0, y) + B(x_0, y - y_0).$$

Thus

$$||B(x, y) - B(x_0, y_0)|| \leq ||B(x - x_0, y) + B(x_0, y - y_0)|| \leq ||B(x - x_0, y)|| + ||B(x_0, y - y_0)||$$
$$\leq c||x - x_0||.||y|| + c||x_0||.||y - y_0||.$$

If $||(x, y) - (x_0, y_0)|| < \epsilon/c$, then $||x - x_0|| < \epsilon/c$ and $||y - y_0|| < \epsilon/c$. Moreover $||y|| \leq \epsilon/c + ||y_0||$, so

$$||B(x, y) - B(x_0, y_0)|| \leq \epsilon(\epsilon + ||y_0||) + ||x_0||\epsilon = \epsilon(\epsilon + ||y_0|| + ||x_0||) \leq \epsilon(1 + ||y_0|| + ||x_0||).$$

Thus B is continuous at (x_0, y_0). Since (x_0, y_0) is arbitrary, we conclude that B is continuous. $\square$

Question 2: Define the shift operator

$$\sigma : \ell_2 \longrightarrow \ell_2$$
$$(x_n)_{n\in\mathbb{N}} \longmapsto (x_{n+1})_{n\in\mathbb{N}}$$

Now let $(T_n)_{n\in\mathbb{N}}$ the following sequence of linear operators

$$T_n = \sigma^n : \ell_2 \longrightarrow \ell_2$$
$$(x_k)_{k\in\mathbb{N}} \longmapsto (x_{k+n})_{k\in\mathbb{N}}$$

Prove that T_n is a bounded linear operator, determine $\|T_n(x)\|_{n\in\mathbb{N}}$ and the limit operator guaranteed by Banach-Steinhaus.

Proof: In fact, given $(x_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}} \in \ell_2$ and $\lambda \in \mathbb{C}$, we have

$$T_n((x_k)_{k\in\mathbb{N}} + \lambda(y_k)_{k\in\mathbb{N}}) = T_n((x_k + \lambda y_k)_{k\in\mathbb{N}}) = (x_{k+n} + \lambda y_{k+n})_{k\in\mathbb{N}} = (x_{k+n})_{k\in\mathbb{N}} + \lambda(y_{k+n})_{k\in\mathbb{N}}$$
$$= T_n((x_k)_{k\in\mathbb{N}}) + \lambda T_n((y_k)_{k\in\mathbb{N}}).$$

Thus T_n is a linear map. I claim that $\|T_n\| = 1$. In fact, given $(x_k)_{k\in\mathbb{N}} \in \ell_2$ is such that $\|(x_k)_{k\in\mathbb{N}}\|_2 \le 1$, then

$$\|T_n((x_k)_{k\in\mathbb{N}})\|_2 = \sqrt{\sum_{k=n+1}^{\infty} |x_k|^2} \le \sqrt{\sum_{k=1}^{\infty} |x_k|^2} \le 1.$$

On the other hand, if $(y_k)_{k\in\mathbb{N}}$ is such that

$$y_k = \begin{cases} 0 & \text{if } k \neq n+1; \\ 1 & \text{if } k = n+1. \end{cases}$$

Then $\|(y_k)_{k\in\mathbb{N}}\|_2 = 1$ and $\|T_n((y_k)_{k\in\mathbb{N}})\| = 1$, thus $\|T_n\| = 1$. Then T_n is a bounded linear operator for any $n \in \mathbb{N}$. Now let $(x_k)_{k\in\mathbb{N}} \in \ell_2$. Since $\|T_n\| = 1$ for any $n \in \mathbb{N}$, we have that

$$\|T_n((x_k)_{k\in\mathbb{N}})\|_2 \le \|(x_k)_{k\in\mathbb{N}}\|_2$$

for $n \in \mathbb{N}$. Then

$$\sup_{n\in\mathbb{N}} \|T_n((x_k)_{k\in\mathbb{N}})\|_2 \le \|(x_k)_{k\in\mathbb{N}}\|_2.$$

Since ℓ_2 is a Banach space, we are in conditions to use the Banach-Steinhauss Theorem to conclude that the limit operator exists, that is, the mapping

$$\lim(T_n) : \ell_2 \longrightarrow \ell_2$$
$$(x_k)_{k\in\mathbb{N}} \longmapsto \lim_{n\to\infty} T_n((x_k)_{k\in\mathbb{N}}).$$

is well-defined. I claim that $\lim(T_n)$ is the null map. Let $(x_k)_{k\in\mathbb{N}} \in \ell_2$. Given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\sum_{k=n}^{\infty} |x_k - 0|^2 = \sum_{k=n_0}^{\infty} |x_k|^2 < \epsilon^2$$

Thus for all $n \ge n_0$, we have

$$\|T_n((x_k)_{k\in\mathbb{N}}) - (0)_{k\in\mathbb{N}}\|_2 = \|((x_{k+n} - 0)_{k\in\mathbb{N}})\|_2 = \sqrt{\sum_{k=n}^{\infty} |x_k - 0|^2} = \sqrt{\sum_{k=n_0}^{\infty} |x_k|^2} < \epsilon.$$

Thus we conclude that $\lim_{n\to\infty} T_n((x_k)_{k\in\mathbb{N}}) = 0$ and so $\lim(T_n)$ is the zero map. $\square$

Question 3: Let E be a normed linear space and M be a subset of E. Given $z \in \mathbb{C}$, prove that $\overline{zM} = z\overline{M}$.

Proof: In fact, if $z = 0$, then $zM = \{0\}$, then $\overline{zM} = \overline{\{0\}} = \{0\}$ and $z\overline{M} = 0\overline{M} = \{0\}$. Thus

$$\overline{zM} = z\overline{M}.$$

Thus suppose without lost of generality that $z \neq 0$. Let $x \in \overline{zM}$, thus there is a sequence $(zy_n)_{n\in\mathbb{N}}$ such that $y_n \in M$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(zy_n) = x$. This means that, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$||zy_n - x|| < |z|\epsilon.$$

Thus, dividing by $|z|$, we obtain

$$\left|\left|y_n - \frac{x}{z}\right|\right| < \epsilon.$$

This means that y_n converges to $\frac{1}{z}x$, which implies that $\frac{1}{z}x \in \overline{M}$. So

$$x \in z\overline{M}$$

and then $\overline{zM} \subseteq z\overline{M}$. Conversely, let $x \in z\overline{M}$, thus $x = zy$, where $y \in \overline{M}$. Note that there is a sequence $(y_n)_{n\in\mathbb{N}}$ in M such that $\lim_{n\to\infty} y_n = y$, which implies that, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$||y_n - y|| < \frac{\epsilon}{|z|}$$

Thus

$$||zy_n - x|| = ||zy_n - zy|| = \big|\big||z|(y_n - y)\big|\big| = |z|.||y_n - y|| < \epsilon,$$

which implies that $\lim_{n\to\infty}(zy_n) = x$, then $x \in \overline{zM}$ and $z\overline{M} \subseteq \overline{zM}$. □

Question 4: Let E and F be normed linear spaces and $T : E \longrightarrow F$ be linear, open operator. Prove that T is surjective. In particular, given E and F Banach spaces and $T : E \longrightarrow F$ a continuous linear operator, then T is surjective if and only if T is open.

Proof: In fact, let $B_E(0,1)$ the open ball centered in 0. Since T is open, then $T(B(0,1))$ is an open set containing 0. Since $T(B(0,1))$ is open, there is $r > 0$ such that $B_F[0,r] \subseteq T(B(0,1))$. Let $y \neq 0$, $y \in F$, then $ry/||y|| \in B_F[0,r]$, thus there is $x \in B(0,1)$ such that

$$T(x) = ry/||y||.$$

Then

$$T\left(\frac{\|y\|}{r}x\right) = y.$$

So $F \subseteq \mathrm{Im}(T)$, then T is surjective. $\square$

Question 5: Let E and F be normed linear spaces. Prove that the condition of E and F be Banach spaces is indispensable for the Open Mapping Theorem

Proof: Consider the space

$$c_{00} = \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \ ; \ \text{there is } N > 0 \text{ such that } x_n = 0 \text{ for all } n \geq N\}.$$

It is well known that c_{00} isn't a Banach space. Define the following linear operator

$$\phi : c_{00} \longrightarrow c_{00}$$

$$(x_n)_{n\in\mathbb{N}} \longmapsto (x_n/n)_{n\in\mathbb{N}}.$$

Note that T is linear, because given $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in c_{00}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\phi((x_n)_{n\in\mathbb{N}} + \lambda(y_n)_{n\in\mathbb{N}}) = \phi((x_n + \lambda y_n)_{n\in\mathbb{N}}) = ((x_n + \lambda y_n)/n)_{n\in\mathbb{N}}) &= (x_n/n)_{n\in\mathbb{N}} + \lambda(y_n/n)_{n\in\mathbb{N}} \\ &= \phi((x_n)_{n\in\mathbb{N}}) + \lambda\phi((y_n)_{n\in\mathbb{N}}).\end{aligned}$$

Moreover, T is continuous, because, given $(x_n)_{n\in\mathbb{N}} \in c_{00}$, we have

$$\|\phi((x_n)_{n\in\mathbb{N}})\|_\infty = \|(x_n/n)_{n\in\mathbb{N}}\|_\infty \leq \|(x_n)_{n\in\mathbb{N}}\|_\infty,$$

because $|x_n/n| \leq |x_n|$ for all $n \in \mathbb{N}$. Finally, ϕ is a bijective, because, given $(y_n)_{n\in\mathbb{N}} \in c_{00}$, take $(x_n)_{n\in\mathbb{N}} = (ny_n)_{n\in\mathbb{N}}$. Thus

$$\phi((x_n)_{n\in\mathbb{N}}) = \phi((ny_n)_{n\in\mathbb{N}}) = (y_n)_{n\in\mathbb{N}}.$$

Then ϕ is surjective. Moreover, if $\phi((z_n)_{n\in\mathbb{N}}) = 0$, then $z_n/n = 0$ for all $n \in \mathbb{N}$ and and so $(z_n)_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}$, then ϕ is injective. Consider

$$\phi^{-1} : c_{00} \longrightarrow c_{00}$$

$$(x_n)_{n\in\mathbb{N}} \longmapsto (nx_n)_{n\in\mathbb{N}}.$$

I claim that ϕ^{-1} is not bounded, so ϕ^{-1} can not be continuous. In fact, give $n \in \mathbb{N}$ arbitrary, consider the sequence $(y_n)_{n\in\mathbb{N}} \in c_{00}$ such that

$$y_k = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

We have that $\|(y_k)_{k\in\mathbb{N}}\| = 1$ and $\|\phi^{-1}((y_k)_{k\in\mathbb{N}})\| = n$, which implies that ϕ^{-1} is not bounded. $\square$

1.10 Class 10: The Closed Graph Theorem and Duality

Question 1: Let $(E, ||.||_E)$ and $(F, ||.||_F)$ be normed linear spaces. Prove that the following functions

$$||.||_e : E \times F \longrightarrow \mathbb{R} \qquad ||.||_m : E \times F \longrightarrow \mathbb{R} \qquad ||.||_s : E \times F \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto \sqrt{||x||_E + ||y||_F} \qquad (x,y) \longmapsto \max\{||x||_E, ||y||_F\} \qquad (x,y) \longmapsto ||x||_E + ||y||_F$$

are norms in E

Proof: It's enough to check the norm's axioms. For $||.||_m$

- It's clear that $||(x,y)||_m = \max\{||x||_E, ||y||_F\} \geq 0$ for every $(x,y) \in E \times F$;
- If $||(x,y)||_m = 0$, then $||x||_E = ||y||_F = 0$, which implies that $x = y = 0$. Then $(x,y) = (0,0)$. On the other hand, $||(0,0)||_m = 0$ trivially;
- Given $\lambda \in \mathbb{K}$ and $(x,y) \in E \times F$, we have that

$$||\lambda(x,y)||_m = ||(\lambda x, \lambda y))||_m = \max\{||\lambda x||_E, ||\lambda y||_F\} = |\lambda| \max\{||x||_E, ||y||F\} = |\lambda|.||(x,y)||_m.$$

- Given $(x_1,y_1), (x_2,y_2) \in E \times F$, we have

$$||(x_1,y_1) + (x_2,y_2)||_m = ||(x_1+x_2, y_1+y_2)||_m = \max\{||x_1+x_2||_E, ||y_1+y_2||_F\} \leq$$
$$\max\{||x_1||_E + ||x_2||_E, ||y_1||_F + ||y_2||_F\} \leq \max\{||x_1||_E, ||y_1||F\} + \max\{||x_2||_E, ||y_2||F\}$$
$$= ||(x_1,y_1)||_m + ||(x_2,y_2)||_m.$$

For the norm $||.||_s$

- $||(x,y)||_s = ||x||_E + ||y||_F \geq 0$ for every $(x,y) \in E \times F$.
- Given $(x,y) \in E \times F$, if $||(x,y)||_s = ||x||_E + ||y||_F = 0$, then $x = y = 0$, thus $(x,y) = (0,0)$. On the other hand, $||(0,0)||_s = ||0||_E + ||0||_F = 0$ trivially.
- Given $\lambda \in \mathbb{K}$ and $(x,y) \in E \times F$, we have that

$$||\lambda(x,y)||_s = ||(\lambda x, \lambda y)||_s = ||\lambda x||_E + ||\lambda y||_F = |\lambda|.||x||_E + |\lambda|.||y||_F = |\lambda|.||(x,y)||_s.$$

- Given $(x_1,y_1), (x_2,y_2) \in E \times F$, we have

$$||(x_1,y_1) + (x_2,y_2)||_s = ||(x_1+x_2, y_1+y_2)||_s = ||x_1+x_2||_E + ||y_1+y_2||_F$$
$$\leq (||x_1||_E + ||x_2||_E) + (||y_1||_F + ||y_2||_F) = ||(x_1,y_1)||_s + ||(x_2,y_2)||_s.$$

For $||.||_e$

- It's clear that $||(x,y)||_e = \sqrt{||x||_E^2 + ||y||_F^2} \geq 0$ for every $(x,y) \in E \times F$;
- Given $(x,y) \in E \times F$, if $||(x,y)||_e = 0$, then $||x||_E^2 + ||y||_F^2 = 0$, which implies that $x = y = 0$. Thus $(x,y) = (0,0)$. On the other hand, it's clear that $||(0,0)||_e = 0$.
- Given $\lambda \in \mathbb{K}$ and $(x,y) \in E \times F$, we have that

$$\begin{aligned}||\lambda(x,y)||_e = ||(\lambda x, \lambda y)||_e = \sqrt{||\lambda x||_E^2 + ||\lambda y||_F^2} = \sqrt{|\lambda|^2(||x||_E^2 + ||y||_F^2)} = |\lambda|\sqrt{||x||_E^2 + ||y||_F^2} \\ = |\lambda|.||(x,y)||_e.\end{aligned}$$

- Given $(x_1,y_1),(x_2,y_2) \in E \times F$, we have

$$\begin{aligned}||(x_1,y_1) + (x_2,y_2)||_e = ||(x_1 + x_2, y_1 + y_2)||_e = \sqrt{||x_1 + x_2||_E^2 + ||y_1 + y_2||_F^2} \\ \leq \sqrt{||x_1||_E^2 + ||y_1||_F^2} + \sqrt{||x_2||_E^2 + ||y_2||_F^2} = ||(x_1,y_1)||_e + ||(x_2,y_2)||_e.\end{aligned}$$

□

Question 2: Let $(E,||.||_E)$ and $(F,||.||_F)$ be normed linear spaces. Prove that the norms defined above are all equivalent.

Proof: In fact, given $(x,y) \in E \times F$, we have

$$||(x,y)||_m \leq ||(x,y)||_e \leq ||(x,y)||_s \leq 2||(x,y)||_m,$$

where these inequalities are trivial. □

Question 3: Let X and Y and be topological spaces and $f : X \longrightarrow Y$ be a continuous mapping. If Y is Hausdorff space (T_2), then

$$\text{gr}(f) = \{(x,y) \in X \times Y \ ; \ y = f(x)\}$$

is a closed subset of $X \times Y$. In particular, if X and Y are normed linear spaces and $T : X \longrightarrow Y$ is a continuous linear mapping, then $\text{gr}(T)$ is closed in $X \times Y$.

Proof: In fact, let $(x,y) \notin \text{gr}(f)$. Thus $y \neq f(x)$. Since Y is a Hausdorff space, there are non-empty disjoint open sets U and V of Y such that

$$y \in U \qquad \text{and} \qquad f(x) \in V$$

Now, since f is continuous, there $f^{-1}(V)$ is an open in X containing x. So consider the open $W := f^{-1}(V) \times U \subseteq X \times Y$. Note that $(x, y) \in W$ and $W \cap \mathrm{gr}(f) = \varnothing$. Thus $X \times Y \setminus \mathrm{gr}(f)$ is open, then $\mathrm{gr}(f)$ is closed in $X \times Y$.

Question 4: Let $1 < p < \infty$ and $q \in \mathbb{R}$ such that $1/p + 1/q = 1$. Prove that the mapping

$$\begin{aligned} \Phi : \qquad\qquad & \ell_q \longrightarrow \ell_p' \\ \mathbf{a} = (a_j)_{j\in\mathbb{N}} & \longmapsto \left[\phi_{\mathbf{a}} : (b_j)_{j\in\mathbb{N}} \longmapsto \textstyle\sum_{k=1}^{\infty} a_k b_k\right] \end{aligned}$$

is an isometric isomorphism.

Proof: Firstly we have to prove that Φ is well-defined, that is, given $\mathbf{a} = (a_j)_{j\in\mathbb{N}} \in \ell_q$, the mapping $\phi_{\mathbf{a}} \in \ell_p'$. Note that, given $(b_j)_{j\in\mathbb{N}} \in \ell_p$, the sum

$$\sum_{j=1}^{\infty} a_j b_j$$

converges. In fact, consider the sequence

$$\begin{aligned} s : \mathbb{N} &\longrightarrow \mathbb{K} \\ n &\longmapsto \textstyle\sum_k^n a_k b_k \end{aligned}$$

Note that, given $n \geq m$, we have

$$\left|\sum_{k=m+1}^{n} a_k b_k\right| \leq \sum_{k=m+1}^{n} |a_k b_k| \leq \left(\sum_{k=m+1}^{n} |a_k|^q\right)^{\frac{1}{q}} \left(\sum_{k=m+1}^{n} |b_k|^p\right)^{\frac{1}{p}}$$

Since $\|(a_k)_{k\in\mathbb{N}}\|_q < \infty$ and $\|(b_k)_{k\in\mathbb{N}}\|_p < \infty$, there is $n_0 \in \mathbb{N}$ for all $n \geq m \geq n_0$, we have

$$\left(\sum_{k=m+1}^{\infty} |a_k|^q\right)^{\frac{1}{q}} < 1 \qquad \text{and} \qquad \left(\sum_{k=m+1}^{n} |b_k|^p\right)^{\frac{1}{p}} < \epsilon$$

Thus, for all $n \geq m \geq n_0$, we have

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} a_k b_k\right| \leq \sum_{k=m+1}^{n} |a_k b_k| \leq \left(\sum_{k=m+1}^{n} |a_k|^q\right)^{\frac{1}{q}} \left(\sum_{k=m+1}^{n} |b_k|^p\right)^{\frac{1}{p}} < \epsilon$$

Since ℓ_1 is complete, we conclude that

$$\sum_{k=1}^{\infty} a_k b_k$$

converges. Now, we will show that $\phi_{\mathbf{a}}$ is linear. In fact, let $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}} \in \ell_p$ and $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} \phi_{\mathbf{a}}((b_k)_{k\in\mathbb{N}} + \lambda(c_k)_{k\in\mathbb{N}}) = \phi_{\mathbf{a}}((b_k + \lambda c_k)_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty}(a_k(b_k + \lambda c_k)) = \sum_{k=1}^{\infty} a_k b_k + \lambda \sum_{k=1}^{\infty}(a_k c_k) \\ = \phi_{\mathbf{a}}((b_k)_{k\in\mathbb{N}}) + \lambda\phi_{\mathbf{a}}((c_k)_{k\in\mathbb{N}}). \end{aligned}$$

Thus $\phi_{\mathbf{a}}$ is linear. Now we will prove that $\phi_{\mathbf{a}}$ is continuous. We will use the Hölder inequality in order to do it. It is enough to note that

$$|\phi_{\mathbf{a}}((b_k)_{k\in\mathbb{N}})| = \left|\sum_{k=1}^{\infty} a_k b_k\right| \leq \left(\sum_{k=1}^{\infty} |a_k|^q\right)^{\frac{1}{q}} \left(\sum_{k=1}^{\infty} |b_k|^p\right)^{\frac{1}{p}} = \|(a_k)_{k\in\mathbb{N}}\|_q \|(b_k)_{k\in\mathbb{N}}\|_p.$$

So $\phi_{\mathbf{a}}$ is continuous, thus Φ is well-defined. Now we will prove that Ψ is linear. Let $\mathbf{a} = (a_k)_{k\in\mathbb{N}}$, $\mathbf{b} = (b_k)_{k\in\mathbb{N}} \in \ell_q$ and $\lambda \in \mathbb{K}$. Given $(c_k)_{k\in\mathbb{N}} \in \ell_p$, we have

$$\begin{aligned}\phi_{\mathbf{a}+\lambda\mathbf{b}}((c_k)_{k\in\mathbb{N}}) &= \sum_{k=1}^{\infty}((a_k + \lambda b_k)c_k) = \sum_{k=1}^{\infty} a_k c_k + \lambda \sum_{k=1}^{\infty} b_k c_k = \phi_{\mathbf{a}}((c_k)_{k\in\mathbb{N}}) + \lambda\phi_{\mathbf{b}}((c_k)_{k\in\mathbb{N}}) \\ &= (\phi_{\mathbf{a}} + \lambda\phi_{\mathbf{b}})((c_k)_{k\in\mathbb{N}}).\end{aligned}$$

for all $(c_k)_{k\in\mathbb{N}} \in \ell_p$, so

$$\Phi(\mathbf{a} + \lambda\mathbf{b}) = \phi_{\mathbf{a}+\lambda\mathbf{b}} = \phi_{\mathbf{a}} + \lambda\phi_{\mathbf{b}} = \Phi(\mathbf{a}) + \lambda\Phi(\mathbf{b}).$$

Now we will prove that Φ is an isometric linear mapping. In fact, given $\mathbf{a} = (a_k)_{k\in\mathbb{N}} \in \ell_q$, we have to prove that

$$\|(a_k)_{k\in\mathbb{N}}\|_q = \|\Phi((a_k)_{k\in\mathbb{N}})\| = \sup_{\|(b_k)_{k\in\mathbb{N}}\|_p \leq 1} |\Phi((a_k)_{k\in\mathbb{N}})((b_k)_{k\in\mathbb{N}})|.$$

Note that, if $(b_k)_{k\in\mathbb{N}} \in \ell_p$ is such that $\|(b_k)_{k\in\mathbb{N}}\|_p \leq 1$, by Hölder inequality, we have

$$|\Psi((a_k)_{k\in\mathbb{N}})((b_k)_{k\in\mathbb{N}})| = \left|\sum_{k=1}^{\infty} a_k b_k\right| \leq \|(a_k)_{k\in\mathbb{N}}\|_q,$$

which implies that

$$\|\Phi((a_k)_{k\in\mathbb{N}})\| \leq \|(a_k)_{k\in\mathbb{N}}\|_q.$$

In particular, Φ is continuous. Now we need to prove that $\|\Phi((a_k)_{k\in\mathbb{N}})\| \geq \|(a_k)_{k\in\mathbb{N}}\|_q$ and that Φ is surjective.

1.11 Class 11: Reflexive Spaces - Part I

Lemma: Let E be a normed linear space. If E is finite-dimensional linear space, then E' also has finite dimension. Moreover, we have that $\dim_{\mathbb{K}} E = \dim_{\mathbb{K}} E'$. In particular, we have that $\dim_{\mathbb{K}} E = \dim_{\mathbb{K}} E''$.

Proof: In fact, let $\{e_1, \ldots, e_n\}$ be a basis of E. Define, using the linearity, a family $\{f_1, \ldots, f_n\}$

of linear functional as follows

$$f_i : E \longrightarrow \mathbb{K}$$

$$e_j \longmapsto \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq \text{j} \end{cases}$$

is the Kronecker's delta symbol. Since every linear functional is continuous in finite dimensional space, we have that $\{f_1, \ldots, f_n\} \subseteq E'$. Now I will prove that $\{f_1, \ldots, f_n\}$ is linearly independent. Indeed, let $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ such that

$$\sum_{k=1}^{n} \alpha_k f_k = 0.$$

Evaluating both sides the equation above by e_i, we conclude that $\alpha_i = 0$ for each $i = 1, \ldots, n$. Then $\{f_1, \ldots, f_n\}$ is linearly independent. Now I claim that $\{f_1, \ldots, f_n\}$ is a basis for E'. In fact, given $f \in E'$, then it is easy to see that

$$f = \sum_{k=1}^{n} f(e_k) f_k.$$

Thus $\{f_1, \ldots, f_n\}$ is a basis for E' and then $\dim_{\mathbb{K}} E' = n = \dim_{\mathbb{K}} E$. □

Question 1: Let E be a normed linear space. Prove that, if E is a finite-dimensional linear space, then E is reflexive.

Proof: In fact, consider the the standard immersion

$$\begin{aligned} J_E : E &\longrightarrow E'' \\ x &\longmapsto \phi_x : E' \longrightarrow \mathbb{K} \\ &\qquad\quad f \;\longmapsto f(x) \end{aligned}$$

Using the Image-Kernel Theorem and the fact that J_E is injective, we conclude that

$$\dim_{\mathbb{K}} E'' = \dim_{\mathbb{K}} E = \dim_{\mathbb{K}} \mathrm{Ker}(J_E) + \dim_{\mathbb{K}} \mathrm{Im}(J_E) = \dim_{\mathbb{K}} \mathrm{Im}(J_E)$$

Thus, since $J_E(E) \subseteq E''$ and $\dim_{\mathbb{K}} \mathrm{Im}(J_E) = \dim_{\mathbb{K}} E'' < \infty$. we conclude that E is reflexive. □

Question 2: Consider the following space

$$c_{00} = \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \ ; \ \text{There exists } N \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n \geq N\}.$$

Prove that c_{00} is not reflexive.

Proof: Indeed, we have already seen that c_{00} is not a Banach space. Since all reflexive space is Banach, we conclude that c_{00} can not be a reflexive space. $\square$

Question 3: Prove that c_0 is not a reflexive space.

Question 4: Let $\mathbf{Vet}_N$ be the category of normed linear spaces over a field $\mathbb{K}$.. Define the following association

$$H : \text{obj}(\mathbf{Vet}_N) \longrightarrow \text{obj}(\mathbf{Vet}_N) \qquad E \longmapsto E'$$

$$\begin{aligned} H : \mathcal{L}_{\mathfrak{K}}(E,F) &\longrightarrow \mathcal{L}_{\mathfrak{K}}(F',E') \\ T &\longmapsto \begin{array}{rcl} T^* : F' & \longrightarrow & E' \\ f & \longmapsto & f \circ T. \end{array} \end{aligned}$$

Prove that this association is a contravariant functor between the categories $\mathbf{Vet}_N$.

Proof: In fact, by definition, H takes objects of $\mathbf{Vet}_N$ to objects of $\mathbf{Vet}_N$ and, given $E, F \in \text{obj}(\mathbf{Vet}_N)$ and $f \in \mathcal{L}_{\mathbb{K}}(E,F)$, we have that H takes f to $H(f) \in \mathcal{L}_{\mathbb{K}}(H(F), H(E))$. Now, observe that, given $E \in \mathbf{Vet}_N$ and $1_E \in \mathcal{L}_{\mathbb{K}}(E,E)$, we have that $H(1_E) = 1_{E'}$. Indeed

$$\begin{aligned} (1_E)' : E' &\longrightarrow E' \\ f &\longmapsto f \circ 1_E = f. \end{aligned}$$

Thus $H(1_E) = (1_E)' = 1_{E'}$. Moreover, let E, F and $V \in \text{obj}(\mathbf{Vet}_N)$, $T \in \mathcal{L}_{\mathbb{K}}(E,F)$, $G \in \mathcal{L}_{\mathbb{K}}(F,V)$ and $f \in V'$. We have that

$$(G \circ T)'(f) = f \circ (G \circ T) = (f \circ G) \circ T = G'(f) \circ T.$$

Note that $G'(f) = f \circ G \in F'$, thus

$$(G \circ T)'(f) = G'(f) \circ T = T'(G'(f)) = (T' \circ G')(f).$$

Since $f \in V'$ is arbitrary, we conclude that

$$H(G \circ T) = (G \circ T)' = T' \circ G' = H(T') \circ H(G'),$$

which implies that H is contravariant functor. $\square$

1.12 Class 12: Reflexive Spaces - Part II

Question 1: Prove that, for every $1 < p < \infty$, the Banach space ℓ^p is a reflexive space.

1.13 Class 13: Weak Topology

Question 1: Let X be a nonempty set and $\{(Y_i,\tau_i)\}_{i\in I}$ be a collection of topological spaces. Consider $\mathcal{F} = \{\phi_i : X \longrightarrow Y_i\}_{i\in I}$ be a collection of mappings. Prove that $\sigma(X,\mathcal{F})$ is the coarsest topology in X such that each function ϕ_i is continuous.

Proof: indeed, we will prove the following: If τ is a topology in X such that each ϕ_i is continuous, then $\sigma(X,\mathcal{F}) \subseteq \tau$. In fact, since $\phi_i : (X,\tau) \longrightarrow (Y_i,\tau_i)$ is continuous for each $i \in I$, then $(\phi_i)^{-1}(U) \in \tau$ for every $U \in \tau_i$ and $i \in I$. Since $\sigma(X,\mathcal{F})$ is the topology generated by the subbasis

$$B = \{(\phi_i)^{-1}(U_i) \ ; \ U_i \in \tau_i,\ i \in I\},$$

we conclude that $\sigma(X,\mathcal{F}) \subseteq \tau$. □

Question 2: Let X be a nonempty set and $\{(Y_i,\tau_i)\}_{i\in I}$ be a collection of topological spaces. Consider $\mathcal{F} = \{\phi_i : X \longrightarrow Y_i\}_{i\in I}$ be a collection of mappings. Given $x \in X$, prove that

$$B(x) = \left\{ \bigcap_{j\in J\subseteq I} (\phi_j)^{-1}(U) \ ; \ J \text{ finite and } U \text{ neighborhood of } \phi_j(x) \right\}$$

is a system of neighborhood of x in $\sigma(X,\mathcal{F})$.

Proof: indeed, we know that the topology $\sigma(X,\mathcal{F})$ is generated by the subbasis

$$B = \{(\phi_i)^{-1}(U_i) \ ; \ U_i \in \tau_i,\ i \in I\}.$$

Let V be an open set containing x, then, by definition of subbasis, there is $J \subseteq I$ finite such that

$$x \in \bigcap_{j\in J\subseteq I} (\phi_j)^{-1}(U_j) \subseteq V.$$

where U_j is an open subset of Y_j containing $\phi_j(x)$. Since each

$$\bigcap_{j\in J\subseteq I} (\phi_j)^{-1}(U_j),$$

with $J \subseteq I$ finite and $U_j \in \tau_j$ containing $\phi_j(x)$, is an open subset of X containing x, we conclude that $B(x)$ is a system of neighborhood of x in $\sigma(X,\mathcal{F})$. □

Question 3: Let X be a nonempty set and $\{(Y_i,\tau_i)\}_{i\in I}$ be a collection of Hausdorff topological spaces. Consider $\mathcal{F} = \{\phi_i : X \longrightarrow Y_i\}_{i\in I}$ be a collection of mappings. Prove that $\sigma(X,\mathcal{F})$ is Hausdorff if and only if $\mathcal{F}$ is a family which separates points, that is, given $x,y \in X$, there is $\phi \in \mathcal{F}$

such that $\phi(x) \neq \phi(y)$.

Proof: Suppose that $\sigma(X, \mathcal{F})$ is Hausdorff. Given $x \neq y \in X$, there are disjoint open sets U and V such that such that $x \in U$ and $y \in V$. By definition of initial topology, there are $J_1, J_2 \subseteq I$ finite such that

$$x \in \bigcap_{j \in J_1 \subseteq I} (\phi_j)^{-1}(U_j) \subseteq U \qquad\qquad y \in \bigcap_{j \in J_2 \subseteq I} (\phi_j)^{-1}(V_j) \subseteq V$$

I claim that

$$y \notin \bigcap_{j \in J_1 \subseteq I} (\phi_j)^{-1}(U_j)$$

In fact, if it was true, then $y \in U$ and so $U \cap V \neq \varnothing$. Since $y \notin \bigcap_{j \in J_1 \subseteq I} (\phi_j)^{-1}(U_j)$, there is $j \in J_1$ such that $y \notin (\phi_j)^{-1}(U_j)$, which implies that $\phi_j(y) \notin U_j$. On the other hand, it is clear that $\phi_j(x) \in U_j$, so $\phi_j(x) \neq \phi_j(y)$, thus $\mathcal{F}$ is a collection which separates points.

Now suppose that $\mathcal{F}$ is a collection which separates points. Let $x \neq y \in X$. Since $\mathcal{F}$ is a collection which separates points, there is $\phi_i \in \mathcal{F}$ such that $\phi_i(x) \neq \phi_i(y)$. Since Y_i is Hausdorff, there are disjoint open sets U, V in Y_i such that $\phi_i(x) \in U$ and $\phi_i(y) \in V$, then

$$x \in (\phi_i^{-1})(U) \subseteq X \qquad \text{and} \qquad y \in (\phi_i^{-1})(V) \subseteq Y$$

and $(\phi_i^{-1})(U) \cap (\phi_i^{-1})(V) = \varnothing$ since $U \cap V = \varnothing$. $\square$

Question 4: Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. Show that the product topology in

$$\prod_{i \in I} X_i$$

is the coarsest topology such that every projection

$$\begin{aligned} \pi_j : \textstyle\prod_{i \in I} X_i &\longrightarrow X_j \\ (x_i)_{i \in I} &\longmapsto x_j \end{aligned}$$

is continuous.

Proof: In fact, remember that the product topology has as basis the following set

$$B = \left\{ \prod_{i\in I} U_i \ ; \ U_i = X_i \text{ for all except a finite number of indices} \right\}$$
$$= \left\{ \bigcap_{i\in I} (\pi_i)^{-1}(V_i) \ ; \ V_i = Y_i \text{ for all except a finite number of indices} \right\}$$
$$= \left\{ \bigcap_{j\in J\subseteq I} (\pi_i)^{-1}(V_i) \ ; \ J \subseteq I \text{ finite, } V_i \in \tau_i \text{ for all } i \in J \right\}.$$

Thus we have that the product topology is exactly

$$\sigma\left(\prod_{i\in I} X_i, \{\pi_i\}_{i\in I} \right),$$

so the topology product is the coarsest topology such that the projection mappings are continuous.

Question 5: Let X be a nonempty set and $\{(Y_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces. Consider $\mathcal{F} = \{\phi_i : X \longrightarrow Y_i\}_{i\in I}$ be a collection of mappings. Given a sequence $(x_n)_{n\in\mathbb{N}}$ in X, prove that x_n converges in $(X, \sigma(X, \mathcal{F}))$ to x if and only if $\phi_i(x_n)$ converges to $\phi_i(x)$ for all $i \in I$.

Proof: Suppose x_n converges in $(X, \sigma(X, \mathcal{F}))$ to x. Let $i \in I$ and V be a neighborhood of $\phi_i(x)$. Since, by definition of initial topology, ϕ_i is continuous, so $(\phi_i)^{-1}(V)$ is a neighborhood of x. Since x_n converges in $(X, \sigma(X, \mathcal{F}))$ to x, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $x_n \in (\phi_i)^{-1}(V)$, which implies that $\phi(x_n) \in V$ for every $n \geq n_0$. This means that $\phi_i(x_n)$ converges to $\phi_i(x)$ for all $i \in I$.

Now suppose that $\phi_i(x_n)$ converges to $\phi_i(x)$ for all $i \in I$. Let U be a neighborhood of x. By definition of initial topology, there is $J = \{j_1, \ldots, j_t\} \subseteq I$ finite such that

$$x \in \bigcap_{j\in J} (\phi)^{-1}(V_j) \subseteq U,$$

where $V_j \in \tau_j$ for every $j \in J$. Since $\phi_j(x_n)$ converges to $\phi_j(x)$ for all $j \in J$, there is $n_{j_k} \in \mathbb{N}$ such that, for every $n \geq n_{j_k}$, we have $\phi_{j_k}(x_n) \in V_{j_k}$. Let $n_0 = \max\{n_1, \ldots, n_t\}$. Thus, for every $n \geq n_0$, we have that $\phi_{j_k}(x_n) \in V_{j_k}$ for each $k \in \{1, \ldots, t\}$, that is

$$x_n \in \bigcap_{j\in J} (\phi_j)^{-1}(V_j) \subseteq U$$

for all $n \geq n_0$, so x_n converges to x. □

Question 6: Let Z be a topological space, X be a nonempty set and $\{(Y_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces. Consider $\mathcal{F} = \{\phi_i : X \longrightarrow Y_i\}_{i\in I}$ be a collection of mappings. Let $\psi : Z \longrightarrow X$.

Prove that ψ is continuous if and only if $\phi_i \circ \psi$ is continuous for every $i \in I$.

Proof: Suppose that ψ is continuous. Since each ϕ_i is continuous by definition of initial topology and the composition of continuous mappings is continuous, then we have that $\phi_i \circ \psi$ is continuous for every $i \in I$.

Now suppose that $\phi_i \circ \psi$ is continuous for every $i \in I$ Let V be an open set of X, we will prove that $\psi^{-1}(V)$ is open in Z. In fact, if $\psi^{-1}(V) = \varnothing$, there is nothing to prove. Suppose that $\psi^{-1}(V) \neq \varnothing$. Let $a \in \psi^{-1}(V)$. Observe that $\phi(a) \in V$, thus, by definition of initial topology, there is $J \subseteq I$ finite such that

$$\psi(a) \in \bigcap_{j \in J} (\phi_j)^{-1}(V_j) \subseteq V$$

On the other hand, since $\phi_j \circ \psi$ is continuous for each $j \in J$, we have that

$$(\phi_j \circ \psi)^{-1}(V_j) = \psi^{-1}((\phi_j)^{-1}(V_j))$$

is open in Z. Thus, since finite intersection of open sets is open, we have

$$\bigcap_{j \in J} (\phi_j \circ \psi)^{-1}(V_j) = \psi^{-1}\left(\bigcap_{j \in J} (\phi_j)^{-1}(V_j) \right) \subseteq \psi^{-1}(V)$$

is an open subset of Z containing a. That is, $a \in \operatorname{int}(\psi^{-1}(V))$, so $\psi^{-1}(V)$ is open and then ψ is continuous. $\square$

Question 7: Let E be a normed linear space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in E. Suppose that $x_n \xrightarrow{w} x$ in E.

(a) Prove that this sequence is bounded, that is, there is $M \in \mathbb{R}$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Proof: (a) Consider the canonical immersion

$$\begin{aligned} J_E : E &\longrightarrow E'' \\ x &\longmapsto \phi_x : E' \longrightarrow \mathbb{K} \\ &\qquad\qquad f \longmapsto f(x) \end{aligned}$$

we know that J_E is isometric isomorphism, thus

$$\|x\| = \|J_E(x)\| = \|\phi_x\|$$

for all $x \in E$. If x_n converges to x weakly, then $f(x_n)$ converges to $f(x)$, this implies that the sequence $(f(x_n))_{n \in \mathbb{N}}$ is bounded and and so $\sup_{n \in \mathbb{N}} |f(x_n)| < \infty$. This implies that

$$\sup_{n \in \mathbb{N}} |\phi_{x_n}(f)| = \sup_{n \in \mathbb{N}} |f(x_n)| < \infty$$

Since E'' is a Banach space, by Banach-Steinhaus Theorem, we have that

$$\sup_{n\in\mathbb{N}} \|\phi_{x_n}\| = M < \infty.$$

Then we conclude that $\|x_n\| = \|J_E(x_n)\| \leq \|\phi_{x_n}\| \leq M$ for all $n \in \mathbb{N}$. $\square$

1.14 Class 14: Weak Topology and Weak* Topology

Question 1: Let E be a normed linear space and let $\sigma(E, E')$ the weak topology in E. Prove that

$$E' = \{f \in E^* \ ; \ f \text{ continuous in } (E, \sigma(E, E'))\}.$$

Proof: In fact, let $f \in E'$, by definition of initial topology and weak topology, we have that f is continuous in $(E, \sigma(E, E'))$, then $f \in \{f \in E^* \ ; \ f \text{ continuous in } (E, \sigma(E, E'))\}$, then

$$E' \subseteq \{f \in E^* \ ; \ f \text{ continuous in } (E, \sigma(E, E'))\}.$$

Suppose that $f \in E^* \smallsetminus \{f \in E^* \ ; \ f \text{ continuous in } (E, \sigma(E, E'))\}$, then f is a linear functional which is not continuous in $(E, \sigma(E, E'))$, thus there is V open in $\mathbb{K}$ such that $f^{-1}(V) \notin \sigma(E, E')$. In particular, $f^{-1}(V) \notin \tau_E$, then $f \notin E'$, thus

$$\{f \in E^* \ ; \ f \text{ continuous in } (E, \sigma(E, E'))\} \subseteq E'.$$

$\square$

Question 2: Let X and Y be topological spaces, $A \subseteq X$ and $x \in X$.

(a) $x \in \overline{A}$ if and only if there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in A converging to x.

Proof: (a) Suppose that $x \in \overline{A}$, then, given a open subset U of X containing x, we have that $U \cap A \neq \varnothing$. Let Λ the collection of all neighborhoods of x. Now define the following binary relation.

$$U \leq V \qquad \text{If and only if} \qquad V \subseteq U$$

It is easy to see that $(\Lambda, \leq)$ is a direct set. Given $U \in \Lambda$, let $x_U \in A \cap U$ and consider the net $(x_\lambda)_{\Lambda\in L}$ in A. I claim that $(x_\lambda)_{\Lambda\in L}$ converges to x. In fact, given U be a neighborhood of x, then, for $V \geq U$, we have that

$$x_V \in V \cap A \subseteq V \subseteq U.$$

Thus $(x_\lambda)_{\Lambda\in L}$ converges to x in X.

Now suppose that there is a net $(x_\lambda)_{\lambda\in L}$ in A converging to x. Let U be a neighborhood of x, since $(x_\lambda)_{\lambda\in L}$ converges to x, there is $\lambda\in L$ such that for all $\gamma\geq\lambda$, we have $x_\gamma\in U$. Since L is a direct set, certainly there exists $\gamma\geq\lambda$, so $x_\gamma\in A\cap U$, which implies that $A\cap U\neq\varnothing$, so $x\in\overline{A}$ □

1.15 Class 15: The Weak* Topology

There were not exercises on this class.

1.16 Class 16: The Banach-Alaoglu-Bourbaki Theorem

There were not exercises on this class.

1.17 Class 17: Kakutani's Theorem

Question 1: Let E be a linear normed space and F be a linear subspace of E. Prove that the weak topology $\sigma(F,F')$ in F is the induced topology $\sigma(E,E')$ in F.

Proof: In fact, it is enough to prove that every open basis of $\sigma(F,F')$ is an open basis of $\sigma(E,E')$ induced in F and every open basis of $\sigma(E,E')$ induced in F is an open basis of $\sigma(F,F')$. Let $x_0\in F$ and $\{x\in F\,;\,\|\phi_i(x)-\phi_i(x_0)\|<r,\ i=1,\dots,n\}$ be an open basis of F, where $\phi_1,\dots,\phi_n\in F'$. By Hahn-Banach theorem, there are $\psi_1,\dots,\psi_n\in E'$ such that $\psi_i|_F=\phi_i$. Thus note that

$$\{x\in F\ ;\ \|\phi_i(x)-\phi_i(x_0)\|<r,\ \ i=1,\dots,n\}=F\cap\{x\in E\ ;\ \|\psi_i(x)-\psi_i(x_0)\|<r,\ \ i=1,\dots,n\}.$$

So every open basis of $\sigma(F,F')$ is an open basis of $\sigma(E,E')$ induced in F. Conversely, if $\{x\in E\ ;\ \|\psi_i(x)-\psi_i(x_0)\|<r,\ \ i=1,\dots,n\}$ is an open basis of $\sigma(E,E')$, then

$$\{x\in E\ ;\ \|\psi_i(x)-\psi_i(x_0)\|<r,\ i=1,\dots,n\}\cap F=\{x\in F\ ;\ \|(\psi_i|_F)(x)-(\psi_i|_F)(x_0)\|<r,\ i=1,\dots,n\}$$

where $\psi_1|_F,\dots,\psi_n|_F\in F'$. Thus $\{x\in E\ ;\ \|\psi_i(x)-\psi_i(x_0)\|<r,\ \ i=1,\dots,n\}\cap F$ is an open basis of F.

1.18 Class 18: Uniformly Convex Spaces and Hilbert Spaces

Question 1: Let E be a linear space with inner dot. Prove the following assertions.

(i) $\langle 0, x\rangle = \langle x, 0\rangle = 0$ for all $x \in E$;

(ii) If $\langle x_0, y\rangle = 0$ for all $y \in E$, then $x_0 = 0$;

(iii) $\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$ for all $x, y_1, y_2 \in E$

(iv) $\langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$ for all $x, y \in E$ and $\lambda \in \mathbb{C}$.

Proof: (i): Note that

$$\langle 0+0, x\rangle = \langle 0, x\rangle + \langle 0, x\rangle.$$

Thus $\langle 0, x\rangle = 0$. Moreover, since $\langle 0, x\rangle = \overline{\langle x, 0\rangle}$, we conclude that $\langle 0, x\rangle = 0$.

(ii): If $\langle x_0, y\rangle = 0$ for all $y \in E$, in particular, $\langle x_0, x_0\rangle = 0$. Since $\langle\cdot,\cdot\rangle$ is positive, we conclude that $x_0 = 0$.

(iii): Note that

$$\langle x, y_1+y_2\rangle = \overline{\langle y_1+y_2, x\rangle} = \overline{\langle y_1, x\rangle + \langle y_2, x\rangle} = \overline{\langle y_1, x\rangle} + \overline{\langle y_2, x\rangle} = \langle x, y_1\rangle + \langle x, y_2\rangle.$$

(iv): Note that

$$\langle x, \lambda y\rangle = \overline{\langle \lambda y, x\rangle} = \overline{\lambda\langle y, x\rangle} = \overline{\lambda}\overline{\langle y, x\rangle} = \overline{\lambda}\langle x, y\rangle$$

□

Question 2: Let E be a linear space with inner dot. Prove that the function

$$\begin{aligned} \|\cdot\| : E &\longrightarrow \mathbb{R} \\ x &\longmapsto \sqrt{\langle x, x\rangle} \end{aligned}$$

is a norm in E.

Proof: In fact, it is enough to check the norm's axioms.

- If $\|x\| = 0$, then $\langle x, x\rangle = 0$, thus $x = 0$;
- It is clear that $\|0\| = 0$;
- Given $x \in E$ and $\lambda \in \mathbb{C}$, then

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{\lambda\overline{\lambda}\langle x, x\rangle} = \sqrt{|\lambda|^2\langle x, x\rangle} = |\lambda|\sqrt{\langle x, x\rangle} = |\lambda|\|x\|;$$

- Given $x, y \in E$, we have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle = \|x\|^2 + 2\operatorname{Re}(\langle x,y\rangle) + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

  Then

$$\|x+y\| \leq \|x\| + \|y\|.$$

Thus $\|.\|$ is a norm in E. □

Question 3: Let E be a linear space with inner dot. Prove the parallelogram's law: Given $x, y \in E$, then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof: In fact, given $x, y \in E$, we have

$$\|x+y\|^2 + \|x-y\|^2 = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle = 2(\langle x,x\rangle + \langle y,y\rangle) =$$
$$2(\|x\|^2 + \|y\|^2).$$

□

Question 4: Let E be a linear space over $\mathbb{R}$ with inner dot. Prove the Polarization formula: Given $x, y \in E$, then

$$\langle x,y\rangle = \frac{1}{4}\cdot(\|x+y\|^2 - \|x-y\|^2)$$

Proof: In fact, given $x, y \in E$, we have

$$\|x+y\|^2 - \|x-y\|^2 = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle - (\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle) = 4\langle x,y\rangle$$

Thus

$$\langle x,y\rangle = \frac{1}{4}\cdot(\|x+y\|^2 - \|x-y\|^2).$$

□

Question 5: Let E be a linear space over $\mathbb{C}$ with inner dot. Prove the Polarization formula: Given $x, y \in E$, then

$$\langle x,y\rangle = \frac{1}{4}\cdot(\|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)).$$

Proof: In fact, given $x, y \in E$, note that

$$\|x+y\|^2 - \|x-y\|^2 = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - (\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle) =$$
$$2(\langle x,y\rangle + \langle y,x\rangle).$$

Similarly, we have that

$$\|x+iy\|^2 - \|x-iy\|^2 = \langle x,x\rangle - i\langle x,y\rangle + i\langle y,x\rangle + \langle y,y\rangle - (\langle x,x\rangle + i\langle x,y\rangle - i\langle y,x\rangle + \langle y,y\rangle) =$$
$$2i(\langle y,x\rangle - \langle x,y\rangle)$$

Thus, we have

$$(\|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)) = 2(\langle x,y\rangle + \langle y,x\rangle) - 2(\langle y,x\rangle - \langle x,y\rangle) = 4\langle x,y\rangle$$

Finally

$$\langle x,y\rangle = \frac{1}{4}\cdot(\|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)).$$

□

1.19 Class 19: Hilbert spaces: Part II

Question 1: Let E be a linear space with inner dot. Given x, $y \in E$, if x and y are orthogonal vectors, prove that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof: In fact, since x and y are orthogonal vectors, then $\langle x,y\rangle = \langle y,x\rangle = 0$. Thus

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle + \langle x,y\langle + \langle y,x\rangle = \|x\|^2 + \|y\|^2.$$

□

Question 1: Let E be a linear space with inner dot and $A \subseteq E$ be a subset. Prove the following assertions.

(i) $A \subseteq (A^\perp)^\perp$;

(ii) $E^\perp = \{0\}$ and $\{0\}^\perp = E$;

(iii) $A^\perp$ is a closed subspace of E;

(iv)

$$A \cap A^{\perp} = \begin{cases} \{0\} & \text{if } 0 \in A; \\ \varnothing & \text{if } 0 \notin A. \end{cases}$$

Proof: (i) By definition, we have that

$$A^{\perp} = \{x \in E \ ; \ \langle x, y \rangle = 0 \text{ for all } y \in A\}.$$

Thus, given $x \in A$, we have that $\langle x, y \rangle = 0$ for all $y \in A^{\perp}$, so $x \in (A^{\perp})^{\perp}$. Then $A \subseteq (A^{\perp})^{\perp}$.

(ii): Let $x \in E^{\perp}$. In particular, we have $\|x\|^2 = \langle x, x \rangle = 0$, which implies that $x = 0$, thus $E^{\perp} = \{0\}$. Similarly, given any $x \in E$, we have $\langle x, 0 \rangle = 0$, which implies that $E \subseteq \{0\}^{\perp}$. Thus $\{0\}^{\perp} = E$.

(iii): $A^{\perp}$ is a linear subspace of E. In fact, given x, $y \in A^{\perp}$ and $\lambda \in \mathbb{K}$, we have that

$$\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle = 0 + \lambda \cdot 0 = 0$$

for all $z \in A$, thus $x + \lambda y \in A^{\perp}$ and so $A^{\perp}$ is a linear subspace of E.

Now we will prove that $A^{\perp}$ is a closed subspace of E. In fact, let $z \in \overline{A^{\perp}}$, then there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $A^{\perp}$ such that $x_n \to z$. Since $x_n \in A^{\perp}$ for all $n \in \mathbb{N}$, then $\langle x_n, y \rangle = 0$ for all $n \in \mathbb{N}$ and $y \in A$. Using the continuity of the function

$$\begin{aligned} \phi_y : E &\longrightarrow \mathbb{K} \\ x &\longmapsto \langle x, y \rangle, \end{aligned}$$

we conclude that, for any $y \in A$, we have

$$\langle z, y \rangle = \lim_{n \to \infty} \langle x_n, y \rangle = \lim_{n \to \infty} 0 = 0.$$

Then $z \in A^{\perp}$ and so $\overline{A^{\perp}} = A^{\perp}$.

(iv): If $A \cap A^{\perp} = \varnothing$, then, in particular, $0 \notin A$. Suppose now that $A \cap A^{\perp} \neq \varnothing$ and let $x \in A \cap A^{\perp}$. By definition, we have that

$$\|x\|^2 = \langle x, x \rangle = 0.$$

Thus $x = 0$. So

$$A \cap A^{\perp} = \begin{cases} \{0\} & \text{if } 0 \in A; \\ \varnothing & \text{if } 0 \notin A. \end{cases}$$

□

1.20 Class 20: Hilbert Spaces: Part III

Question 1: Let E be a normed linear space and $\sum_{n=1}^{\infty} x_n$ be an inconditionally convergent series. If $\sigma_1, \sigma_2 : \mathbb{N} \longrightarrow \mathbb{N}$ are bijections, prove that

$$\sum_{n=1}^{\infty} x_{\sigma_1(n)} = \sum_{n=1}^{\infty} x_{\sigma_2(n)}.$$

Proof: In fact, let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection and $\phi \in E'$. Denoting

$$x_\sigma = \sum_{n=1}^{\infty} x_{\sigma(k)},$$

we have that

$$\phi(x_\sigma) = \phi\left(\lim_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \phi(x_{\sigma(k)}) = \sum_{k=1}^{\infty} \phi(x_{\sigma(k)}).$$

Thus the series $\sum_{k=1}^{\infty} \phi(x_{\sigma(k)})$ converges. Since σ is an arbitrary bijection in $\mathbb{N}$, we conclude that $\sum_{n=1}^{\infty} \phi(x_n)$ is a numeric inconditionally convergent series. Since numeric inconditionally convergent series converges always to the same limit, we conclude that, given $\sigma_1, \sigma_2 : \mathbb{N} \longrightarrow \mathbb{N}$ bijections

$$\phi\left(\sum_{k=1}^{\infty} x_{\sigma_1(k)}\right) = \sum_{k=1}^{\infty} \phi(x_{\sigma_1(k)}) = \sum_{k=1}^{\infty} \phi(x_{\sigma_2(k)}) = \phi\left(\sum_{k=1}^{\infty} x_{\sigma_2(k)}\right).$$

Thus

$$\phi\left(\sum_{k=1}^{\infty} x_{\sigma_1(k)} - \sum_{k=1}^{\infty} x_{\sigma_2(k)}\right) = 0.$$

Set $z = \sum_{k=1}^{\infty} x_{\sigma_1(k)} - \sum_{k=1}^{\infty} x_{\sigma_2(k)}$. Since $\phi \in E'$ is an arbitrary bounded linear functional, we conclude by Hahn-Banach Theorem

$$\|z\| = \sup_{\|\phi\|\le 1} \left\| \phi\left(\sum_{k=1}^{\infty} x_{\sigma_1(k)} - \sum_{k=1}^{\infty} x_{\sigma_2(k)}\right)\right\| = \sup_{\|\phi\|\le 1} 0 = 0.$$

Then $z = 0$ and

$$\sum_{n=1}^{\infty} x_{\sigma_1(n)} = \sum_{n=1}^{\infty} x_{\sigma_2(n)}.$$

$\square$

Question 2: Let H be a Hilbert space and $S = \{x_i \ ; \ i \in I\}$ be an orthonormal subset of H. The following assertions are equivalent.

(i) For each $x \in H$, $x = \sum_{i\in I} \langle x, x_i\rangle x_i$;

(ii) S is a complete orthonormal system;

(iii) $\overline{[S]} = H$;

(iv) For each $x \in H$, we have $\|x\|^2 = \sum_{i\in I} |\langle x, x_i\rangle|^2$;

(v) Given $x, y \in H$, we have $\langle x, y\rangle = \sum_{i\in I} \langle x, x_i\rangle \overline{\langle y, x_i\rangle}$.

Proof: $(i) \longrightarrow (ii)$: In fact, given $x \in S^{\perp}$, we have that $\langle x, x_i\rangle = 0$ for all $i \in I$. Since, by hypothesis, we have that $x = \sum_{i\in I} \langle x, x_i\rangle x_i$, we conclude that $x = \sum_{i\in I} 0 \cdot x_i = 0$.

$(ii) \longrightarrow (iii)$: Let $M = \overline{[S]}$. By Orthogonal projection theorem, we have that

$$H = M \oplus M^{\perp}.$$

Since S is a complete orthonormal system, we have that $S^{\perp} = M^{\perp} = \{0\}$, thus $H = M = \overline{[S]}$.

$(iii) \longrightarrow (iv)$: Let $x \in H$ and $\epsilon > 0$. By (iii), there is $y_\epsilon \in [S]$ such that $\|x - y_\epsilon\| < \epsilon$. Thus there is $J_\epsilon \subseteq I$ finite and scalars $\{a_i\}_{i\in J_\epsilon}$ such that

$$x_\epsilon = \sum_{i\in J_\epsilon} a_i x_i.$$

Since $\|x - \sum_{i\in J_\epsilon} \langle x, x_i\rangle x_i\| = d(x, [x_i \ ; \ i \in J_\epsilon])$, then

$$\left\| x - \sum_{i\in J_\epsilon} \langle x, x_i\rangle x_i \right\| \leq \left\| x - \sum_{i\in J_\epsilon} a_i x_i \right\| = \|x - x_\epsilon\| < \epsilon$$

As J_ϵ is orthonormal, we have

$$\|x\|^2 - \sum_{i\in J_{j_\epsilon}} |\langle x, x_i\rangle|^2 = \left\langle x - \sum_{i\in J_{j_\epsilon}} \langle x, x_i\rangle x_i, x - \sum_{i\in J_{j_\epsilon}} \langle x, x_i\rangle x_i \right\rangle = \left\| x - \sum_{i\in J_{j_\epsilon}} \langle x, x_i\rangle x_i \right\|^2 \leq \epsilon^2.$$

By Bessel inequality, we conclude that

$$\|x\|^2 \leq \sum_{i\in J_{j_\epsilon}} |\langle x, x_i\rangle|^2 + \epsilon^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 + \epsilon^2 \leq \|x\|^2 + \epsilon^2,$$

Since ϵ is arbitrary, we conclude that

$$\|x\|^2 = \sum_{i\in I} |\langle x, x_i\rangle|^2.$$

1.21 Class 21: Hilbert Spaces: Part IV

Question 1: Let I be an arbitrary set. Call $\mathcal{F}$ the collection of all finite subsets of I. Given a collection of scalars $(a_j)_{j\in I}$ indexed by I, we write

$$\|(a_j)_{j\in I}\|_2 = \sup_{A\in\mathcal{F}} \left(\sum_{j\in A} |a_j|^2 \right)^{1/2}.$$

It is a classic exercise to show that $(\ell_2(I), \|.\|_2)$ is a Banach Space and that $(\ell_2(I))'$ is isometrically isomorphic to $\ell_2(I)$

(a) Let $\{e_i \ ; \ i \in I\}$ be a complete orthonormal system for a Hilbert space H. Prove that H is isometrically isomorphic to $\ell_2(I)$.

(b) Prove that every Hilbert space is isometrically isomorphic to its dual.

Proof: (a): Define the following mapping

$$\phi : H \longrightarrow \ell_2(I)$$

$$x \longmapsto (\langle x, e_i \rangle)_{i \in I}$$

It is easy to see that ϕ is a linear mapping. Note that ϕ is isometry. In fact, given $x \in H$, we have

$$\|\phi(x)\|_2^2 = \sup_{A \in \mathcal{F}} \sum_{j \in A} |\langle x, e_i \rangle|^2$$

However, we already know that $K = \{i \in I \ ; \ \langle x, e_i \rangle \neq 0\}$ is at most countable, which implies that, if

$$K := \bigcup_{n=1}^{\infty} \{i_1, \ldots, i_n\} := \bigcup_{n=1}^{\infty} K_n$$

Thus

$$\sup_{A \in \mathcal{F}} \sum_{j \in A} |\langle x, e_i \rangle|^2 = \sup_{A \in \mathcal{F}, A \subseteq K} \sum_{j \in A} |\langle x, e_i \rangle|^2 = \lim_{n \to \infty} \sum_{j=1}^{n} |\langle x, e_{i_k} \rangle|^2 = \sum_{i \in I} |\langle x, e_{i_k} \rangle|^2 = \|x\|^2$$

Thus $\|\phi(x)\|_2 = \|x\|$ for all $x \in H$. Thus ϕ is continuous, is injective and is isometry. It remains to prove that ϕ is surjective. Let $(a_i)_{i \in I} \in \ell_2(I)$ with $\|(a_i)_{i \in I}\|_2 = L$. I claim that $K = \{i \in I \ ; \ a_i \neq 0\}$ is at most countable. Indeed, note that

$$K = \bigcup_{n=1}^{\infty} K_n,$$

where $K_n = \{i \in I \ ; \ |a_i| \geq 1/n\}$. Suppose that K_n contains a finite subset J of cardinality m, thus we have that

$$L = \|(a_i)_{i \in I}\|_2 \geq \sum_{j \in J} |a_j|^2 \geq m/n^2,$$

which implies that $m \leq n^2 L$. Thus K_n is finite for all $n \in \mathbb{N}$, which implies that K is at most countable. Let

$$x = \sum_{i \in I} a_i e_i = \sum_{i \in K} a_i e_i$$

It is not hard to show that $x \in H$, so we have that $\phi(x) = (a_i)_{i \in I}$ and thus we conclude that ϕ is surjective. Then H is isometrically isomorphic to $\ell_2(I)$.

(b): Since $(\ell_2(I))' \cong \ell_2(I)$, then, given a Hilbert Space H with complete orthonormal system $\{e_i \ ; \ i \in I\}$, we have that

$$H' \cong (\ell_2(I))' \cong \ell_2(I) \cong H.$$

Since each isomorphism is isometric, we conclude that H' is isometrically isomorphic to H. $\square$

1.22 Class 22: Hilbert Spaces and Elements of Spectral Theory

Question 1: Given a Hilbert space H, prove that H' is also a Hilbert space.

Proof: In fact, by Riesz-Fréchet Theorem, we know that, given $\phi \in H'$, there is a unique $y \in H$ such that $\phi(x) = \langle x, y \rangle$ for all $x \in H$ and $||y|| = ||\phi||$. Thus we can represent ϕ as ϕ_y. Define the following map

$$B : H' \times H' \longrightarrow \mathbb{K}$$

$$(\phi_x, \phi_y) \longmapsto \langle y, x \rangle$$

I claim that B is an inner dot in H'. In fact

- Given ϕ_x, ϕ_y, $\phi_z \in H'$, we have

$$B(\phi_x + \phi_y, \phi_z) = B(\phi_{x+y}, \phi_z) = \langle z, x + y \rangle = \langle z, x \rangle + \langle z, y \rangle = B(\phi_x, \phi_z) + B(\phi_y, \phi_z)$$

- Given ϕ_x, $\phi_y \in H'$ and $\lambda \in \mathbb{K}$, we have

$$B(\lambda \phi_x, \phi_y) = B(\phi_{\overline{\lambda} x}, \phi_y) = \langle y, \overline{\lambda} x \rangle = \lambda \langle y, x \rangle = \lambda B(x, y).$$

- Given ϕ_x, $\phi_y \in H'$, we have

$$B(\phi_x, \phi_y) = \langle y, x \rangle = \overline{\langle x, y \rangle} = \overline{B(\phi_y, \phi_x)}.$$

- Given $\phi_x \in H'$, we have

$$B(\phi_x, \phi_x) = \langle x, x \rangle \geq 0.$$

  Moreover, $B(\phi_x, \phi_x) = 0$ if and only if $x = 0$. Thus, $B(\phi, \phi) = 0$ if and only if $\phi = 0$.

Finally, note that $\|\phi_x\|^2 = \|x\|^2 = \langle x, x\rangle = B(\phi_x, \phi_x)$. Thus

$$\|\phi_x\| = \sqrt{B(\phi_x, \phi_x)},$$

which implies that the natural norm in H' is the norm induced by inner dot B. Since the H' is a Banach space, we conclude that H' is a Hilbert space. □

Question 2: Let E be a Banach Space and $T \in \mathcal{L}(E,E)$ a continuous linear mapping. Prove that, given $\lambda \in \mathbb{K}^*$, if $1 \in \rho\big(\frac{1}{\lambda}T\big)$, then $\lambda \in \rho(T)$.

Proof: In fact, if $1 \in \rho\big(\frac{1}{\lambda}T\big)$, then

$$\left(\frac{1}{\lambda}\right)T - I$$

is an homeomorphism. That is, there is $G \in \mathcal{L}(E,E)$ such that

$$\left[\left(\frac{1}{\lambda}\right)T - I\right] \circ G = G \circ \left[\left(\frac{1}{\lambda}\right)T - I\right] = I.$$

Multiplying these equalities by λ/λ, we conclude that

$$(T - \lambda I) \circ \left[\left(\frac{1}{\lambda}\right)G\right] = \left[\left(\frac{1}{\lambda}\right)G\right] \circ (T - \lambda I) = I.$$

Since the $\big(\frac{1}{\lambda}\big)G$ is continuous, we conclude that $(T - \lambda I)$ is bijective, so $\lambda \in \rho(T)$. □

1.23 Class 23: Compact Operators

There weren't exercises on this class.

1.24 Class 24: Compact Operators: Part II

There weren't exercises on this class.

1.25 Class 25: Compact Operators: Part III

Question 1: Let A be a nonempty set and $f : A \longrightarrow A$ be an injective, but not surjective mapping. Prove that $f(f(A)) \subsetneq f(A)$

Proof: Indeed, since f is not surjective, $f(A)$ is a proper subset of A. Let $z \in A \setminus f(A)$ and $y := f(z) \in A$. Note that $y \in f(A)$. It is clear that $(f(A)) \subseteq f(A)$. If we have $f(f(A)) = f(A)$, then there would exist $x \in f(A)$ such that $f(x) = y$. This would imply that $f(x) = y = f(z)$, where $z \in A \smallsetminus f(A)$ and $x \in f(A)$ and so $x \neq z$, which contradicts the one-to-oneness of f. So $f(f(A)) \subsetneq f(A)$. $\square$

Question 2: Let E, F be normed linear spaces and $T : E \longrightarrow F$ be a compact operator. Given $M \subseteq E$ a linear subspace of E, prove that $G = T|_M : M \longrightarrow F$ is also compact

Proof: Consider the unitary balls

$$B_M = \{x \in M \ ; \ \|x\| \leq 1\}$$
$$B_E = \{x \in E \ ; \ \|x\| \leq 1\}.$$

Note that $G(B_M) = T(B_M) \subseteq T(B_E)$. Since the closure operation preserves inclusion, we conclude that $\overline{G(B_M)} = \overline{T(B_M)} \subseteq \overline{T(B_E)}$. As T is a compact operator, we have that $\overline{G(B_M)}$ is a closed subset of a compact space $\overline{T(B_E)}$. Thus $\overline{G(B_M)}$ is compact and so G is a compact operator. $\square$

Question 3: Let E and F be normed linear spaces and $T \in \mathcal{L}(E, F)$. Prove that

(a) $\mathrm{Ker}(T) = \{x \in E \ ; \ \phi(x) = 0 \text{ for all } \phi \in T'(F')\} := (T'(F'))^{\perp}$;

(b) $\mathrm{Ker}(T') = \{\phi \in F' \ ;: \ \phi(y) = 0 \text{ for all } y \in T(E)\} := (T(E))^{\perp}$.

Proof: (a): Suppose that $\phi(x) = 0$ for all $\phi \in T'(F')$, thus for all $\psi \in F'$, we have that $0 = \psi(T(x))$. If $T(x) \neq 0$, then, by Hahn-Banach Theorem, we would able to get a $\psi \in F'$ such that $\psi(T(x)) \neq 0$. Thus we have necessarily that $x \in \mathrm{Ker}(T)$. If $x \in \mathrm{Ker}(T)$, then, for all $\phi \in F'$, we have that $T'(\phi)(x) = \phi(T(x)) = 0$.

(b): Suppose that $\phi \in \mathrm{Ker}(T')$. Let $y \in T(E)$, then $y = T(x)$ for some $x \in E$, thus $\phi(y) = \phi(T(x)) = T'(\phi(x)) = 0(x) = 0$, so $\phi(y) = 0$ for all $y \in T(E)$. Suppose that $\phi(y) = 0$ for all $y \in T(E)$, thus $0 = \phi(T(x)) = T'(\phi)(x)$ for all $x \in E$, which implies that $T'(\phi) = 0$ and so $\phi \in \mathrm{Ker}(T')$. $\square$

1.26 Class 26: Self-adjoint operators

Chapter 2

Homework Lists

2.1 Banach Spaces, Bounded Linear Operators and Hahn-Banach Theorems

Question 1: Consider the sets $E = \{f \in C^0[0,1] \ ; \ f(0) = 0\}$ and $F = \{f \in E \ ; \ \int_0^1 f(t)dt = 0\}$

(a) Prove that E is a closed subspace of $C^0[0,1]$.

(b) Prove that F is a closed subspace of E.

(c) [1] Show that there is no $g \in E$ such that $||g|| = 1$ and $||g - f|| \geq 1$ for all $f \in F$.

Resolution: (a): Firstly we have to prove that E is, in fact, a subspace of $C^0[0,1]$. However, this is simple, because the null-function belongs to E and, given $f, g \in E$ and $\lambda \in \mathbb{R}$, then

$$(f + \lambda g)(0) = f(0) + \lambda g(0) = 0 + \lambda 0 = 0.$$

Thus $f + \lambda g \in E$. Now, we'll prove that it's closed. Indeed, given f in the topological closure $\overline{E}$ of E in $C^0[0,1]$, there is a sequence $(f_n)_{n \in \mathbb{N}}$ in E such that $\lim f_n = f$. This means that, given $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that for all $n > N_0$, we have

$$\max_{x \in [0,1]} |f_n(x) - f(x)| = ||f_n - f|| < \epsilon.$$

[1]This exercise shows the limitation of Riesz Lemma, in other words, it shows that not always we can find an element at the unitary sphere whose distance to a closed proper subspace is greater or equal to 1.

In particular, we have

$$|f(0)| = |f(0) - f_n(0)| \leq \max_{x \in [0,1]} |f_n(x) - f(x)| = ||f_n - f|| < \epsilon.$$

Since ϵ is arbitrary, we conclude that $f(0) = 0$ and that $f \in E$. So $\overline{E} = E$ and then E is closed in $C^0[0,1]$.

(b) Firstly we have to prove that F is, in fact, a subspace of E. However, this is simple, because the null-function belongs to F and, given $f, g \in F$ and $\lambda \in \mathbb{R}$, then, by elementary rules of integration, we have

$$\int_0^1 (f(t) + \lambda g(t))dt = \int_0^1 (f(t))dt + \lambda \int_0^1 (g(t))dt = 0 + \lambda 0 = 0.$$

Thus $f + \lambda g \in F$. Now, we'll prove that F it's closed. Indeed, given f in the topological closure $\overline{F}^E$ of F in E, there is a sequence $(f_n)_{n \in \mathbb{N}}$ in F such that $\lim f_n = f$. This means that, given $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that, for all $n > N_0$ and $x \in [0,1]$, we have

$$|f_n(x) - f(x)| \leq \max_{x \in [0,1]} |f_n(x) - f(x)| = ||f_n - f|| < \epsilon.$$

In particular, for all $x \in [0,1]$ and $n > N_0$

$$-\epsilon \leq -|f_n(x) - f(x)| \leq f_n(x) - f(x) \leq |f_n(x) - f(x)| < \epsilon.$$

By monotone property of integration, we have

$$-\epsilon = \int_0^1 (-\epsilon)dt \leq \int_0^1 (f_n(t) - f(t))dt \leq \int_0^1 \epsilon dt = \epsilon,$$

which implies that

$$-\epsilon \leq \int_0^1 f(t)dt - \int_0^1 f_n(t)dt = \int_0^1 f(t)dt \leq \epsilon,$$

that is

$$-\epsilon \leq \int_0^1 f(t)dt \leq \epsilon.$$

Since ϵ is arbitrary, we conclude that $f \in F$ and $F = \overline{F}^E$. Thus F is closed in E.

(c) Suppose that there is $g \in E$ such that $||g|| = 1$ and $||g - f|| \geq 1$ for every $f \in F$. Since the inclusion function $i : [0,1] \longrightarrow \mathbb{R}$ is in E and

$$\int_0^1 i(s)ds \neq 0,$$

we have $F \neq E$. Let $h \in E \smallsetminus F$. Choose $\lambda \in \mathbb{R}$ such that

$$\int_0^1 g(s)ds = \lambda \int_0^1 h(s)ds$$

and consider the function $f - \lambda g$. Note that $g - \lambda h \in F$, because $(g - \lambda h)(0) = g(0) - \lambda h(0) = 0$ and

$$\int_0^1 (g(s) - \lambda h(s))ds = \int_0^1 g(s)ds - \lambda \int_0^1 h(s)ds = 0.$$

Since $g - \lambda h \in F$, we have that $1 \leq ||g - (g - \lambda.h)|| = |\lambda|.||h||$ by hypothesis. Now, for each $n \in \mathbb{N}$, consider the following continuous functions

$$\begin{aligned} h_n : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto x^{1/n}. \end{aligned}$$

We have that $h(0) = 0$ and

$$\int_0^1 h_n(s)ds = \int_0^1 s^{1/n}ds = \left.\frac{ns^{(n+1)/n}}{n+1}\right|_0^1 = \frac{n}{n+1} \neq 0.$$

Thus $h_n \in E \smallsetminus F$. So, since $1 \leq ||g - (g - \lambda_n.h_n)|| = |\lambda_n|.||h_n||$ and $||h_n|| = 1$ for every $n \in \mathbb{N}$, we conclude $\lim_{n\in\mathbb{N}} \inf |\lambda_n| \geq 1$. Finally since $\int_0^1 (g(s) - \lambda_n h_n(s))ds = 0$, we conclude that

$$\left|\int_0^1 (g(s))ds\right| = |\lambda_n|\left|\int_0^1 h_n(s)ds\right|$$

for each $n \in \mathbb{N}$, which implies that

$$\left|\int_0^1 (g(s))ds\right| = \liminf_{n\in\mathbb{N}} \left[|\lambda_n|\left|\int_0^1 h_n(s)ds\right|\right] = \liminf_{n\in\mathbb{N}} |\lambda_n| \cdot \lim_{n\to\infty} \left|\int_0^1 h_n(s)ds\right| \geq 1.$$

However, since $||g|| = 1$, we have that

$$\left|\int_0^1 g(s)ds\right| \leq 1$$

and, since g is continuous, the definite integral of g just would be 1 if and only if g was the constant functions g_1 or g_2, where

$$\begin{aligned} g_1 : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto 1 \end{aligned} \qquad\qquad \begin{aligned} g_2 : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto -1. \end{aligned}$$

and it generates a contradiction, because $g(0) = 0$. □

Question 2: Show that $C^0[a,b]$ with the norm

$$\begin{aligned} ||.|| : C^0[a,b] &\longrightarrow \mathbb{R} \\ f &\longmapsto \textstyle\int_a^b |f(t)|dt \end{aligned}$$

is not a Banach space.

Resolution: In fact, consider $0 < \epsilon < (b-a)/2$ fixed and the sequence $(f_n)_{n\in\mathbb{N}}$ in $C^0[a,b]$, where f_n is defined as

$$f_n(x) = \begin{cases} 0 & a \le x \le (a+b)/2 - \epsilon/n; \\ (n/\epsilon)[x - ((a+b)/2 - \epsilon/n)] & (a+b)/2 - \epsilon/n \le x \le (b+a)/2; \\ 1 & (a+b)/2 \le x \le b. \end{cases}$$

It's clear that each f_n is continuous. Moreover, for each $n \le m \in \mathbb{N}$, we have

$$||f_n - f_m|| = \int_a^b |f_n(t) - f_m(t)|dt = \int_{(a+b)/2-\epsilon/n}^{(a+b)/2} |f_n(t) - f_m(t)|dt \le \int_{(a+b)/2-\epsilon/n}^{(a+b)/2} dt = \epsilon/n.$$

Then, it's clear that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0[a,b]$. I claim that its limits is $f : [a,b] \longrightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 0 & a \le x < (a+b)/2; \\ 1 & (a+b)/2 \le x \le b. \end{cases}$$

Indeed, firstly observe that f is integrable, because the set of discontinuity of f has null Lebesgue measure. Moreover, note that, given $n \in \mathbb{N}$, we have

$$||f_n - f|| = \int_a^b |f_n(t) - f(t)|dt = \int_{(a+b)/2-\epsilon/n}^{(a+b)/2} |f_n(t) - f(t)|dt \le \int_{(a+b)/2-\epsilon/n}^{(a+b)/2} dt = \epsilon/n.$$

Making $n \to \infty$, we conclude $(f_n)_{n\in\mathbb{N}}$ converges to f, but $f \notin C^0[a,b]$, because f is not continuous, then we conclude $(C^0[a,b], ||.||)$ is not a Banach space. $\square$

Question 3: Let E be a normed linear space, F be a Banach space and $T : D(T) \subseteq E \longrightarrow F$ be a bounded linear operator. Show that T admits a bounded linear extension $\hat{T}$, whose domain is $\overline{D(T)}$ and $||\hat{T}|| = ||T||$.

Resolution: Indeed, given $x \in \overline{D(T)}$, there is a convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $D(T)$ such that $\lim_{n\to\infty} x_n = x$. Since T is a linear and bounded, then T is uniformly continuous, so $(T(x_n))_{n\in\mathbb{N}}$

is a Cauchy sequence in F. As F is a Banach space, there is $\lim_{n\to\infty} T(x_n)$. Define the following mapping

$$\hat{T} : \overline{D(T)} \longrightarrow F$$

$$x \longmapsto \lim_{n\to\infty} T(x_n),$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence in $D(T)$ converging to x. Well, firstly we have to verify that $\hat{T}$ is well-defined. In fact, let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{\in\mathbb{N}}$ be sequences in $D(T)$ converging to $x \in \overline{D(T)}$. Suppose that

$$\lim_{n\to\infty} T(x_n) = z_1$$

$$\lim_{n\to\infty} T(y_n) = z_2$$

Now define the sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$u_n = \begin{cases} x_n & \text{if } n \text{ is even;} \\ y_n & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to see that $(u_n)_{n\in\mathbf{N}}$ also converges to x. Let $z = \lim_{n\to\infty} T(u_n)$. Since $(T(u_n))_{n\in\mathbb{N}}$ converges, each subsequence of this sequence converges too, and to the same limit. Thus

$$z_1 = \lim_{n\to\infty} T(u_{2n}) = z = \lim_{n\to\infty} T(u_{2n+1}) = z_2.$$

Then $\hat{T}$ is well-defined. Moreover, $\hat{T}$ is linear. Indeed, given $x = \lim_{n\to\infty} x_n, y = \lim_{n\to\infty} y_n \in \overline{D(T)}$ and $\lambda \in \mathbb{K}$, we have

$$\hat{T}(x + \lambda y) = \lim_{n\to\infty} T(x_n + \lambda y_n) = \lim_{n\to\infty} T(x_n) + \lambda \lim_{n\to\infty} T(y_n) = \hat{T}(x) + \lambda\hat{T}(y).$$

Also it's clear that $\hat{T}$ extends T, because, given $x \in D(T)$, consider the constant sequence $(x)_{n\in\mathbb{N}}$. Thus

$$\hat{T}(x) = \lim_{n\to\infty} T(x) = T(x).$$

Finally note that $\|\hat{T}\| = \|T\|$. In fact, note that

$$\|\hat{T}\| = \sup\{\|T(x)\| \ ; \ x \in \overline{D(T)}\,, \|x\| = 1\} \geq \sup\{\|T(x)\| \ ; \ x \in D(T)\,, \|x\| = 1\} = \|T\|,$$

and, given $x = \lim_{n\to\infty} x_n \in \overline{D(T)}$ with $\|x\| = 1$, we have

$$\|\hat{T}(x)\| = \Big\| \lim_{n\to\infty} T(x_n) \Big\| = \lim_{n\to\infty} \|T(x_n)\| \leq \lim_{n\to\infty} (\|T\|\|x_n\|) = \|T\|\|x\|,$$

which implies that $\|\hat{T}\| \leq \|T\|$. Then $\|\hat{T}\| = \|T\| < \infty$. Hence $\hat{T}$ is a bounded linear extension of T which preserves the norm of operator. $\square$

Question 4: Let $(E, \|.\|)$ be a finite-dimensional normed $\mathbb{K}$-linear space. Show that every linear functional in E is bounded.

Resolution: In fact, let $\{v_1, \ldots, v_n\}$ be a basis of E and define in E the following auxiliary norm

$$\begin{aligned} \|.\|_0 : E &\longrightarrow \mathbb{R} \\ \textstyle\sum_{i=1}^n \alpha_i v_i &\longmapsto \textstyle\sum_{i=1}^n |\alpha_i|. \end{aligned}$$

Now, let $f \in E^*$, given $v = \sum_{i=1}^n \alpha_i v_i \in E$, we have

$$|f(v)| = \left| f\left(\sum_{i=1}^n \alpha_i v_i \right) \right| \leq \sum_{i=1}^n |\alpha_i||f(v_i)| \leq \big(\max_{i=1,\ldots,n} |f(v_i)| \big) \sum_{i=1}^n |\alpha_i| = \big(\max_{i=1,\ldots,n} |f(v_i)| \big) \|v\|_0.$$

Calling $L = \max_{i=1,\ldots,n} |f(v_i)|$, we have

$$|f(v)| \leq L\|v\|_0.$$

Since $\dim_{\mathbb{K}}(E) < \infty$, all norms in E are equivalent, thus there is $C > 0$ such that $\|v\|_0 \leq C\|v\|$ for all $v \in E$, thus

$$|f(v)| \leq (LC)\|v\|.$$

Then f is bounded. $\square$

Question 5: Show that the space

$$\ell_p(\mathbb{C}) = \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ ; \ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

equipped with norm

$$\begin{aligned} \|.\|_p : \ \ell_p(\mathbb{C}) &\longrightarrow \mathbb{R} \\ (x_n)_{n\in\mathbb{N}} &\longmapsto \left(\textstyle\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \end{aligned}$$

is a Banach space.

Proof: Considering a sequence $(x_n)_{n\in\mathbb{N}} \in \ell_p(\mathbb{C})$ as a function

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{C} \\ n &\longmapsto x_n, \end{aligned}$$

let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell_p(\mathbb{C})$, thus, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, for all $m, n \geq n_0$, we have

$$||f_n - f_m||_p^p = \sum_{i=1}^{\infty} |f_n(i) - f_m(i)|^p < \epsilon^p.$$

Note that, for each $k \in \mathbb{N}$, and $n, m \geq n_0$, we have

$$|f_n(k) - f_m(k)| < \epsilon.$$

Thus the sequence $(f_n(k))_{n\in\mathbb{N}}$ is Cauchy for each $k \in \mathbb{N}$. Since $\mathbb{C}$ is a complete metric space, let

$$y_k = \lim_{n\to\infty} f_n(k).$$

I claim that the sequence

$$f : \mathbb{N} \longrightarrow \mathbb{C}$$

$$n \longmapsto y_n$$

belongs to $\ell_p(\mathbb{C})$ and it is the limit of the Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\ell_p(\mathbb{C})$. In fact, note that

$$\sum_{i=1}^{k} |f_n(i) - f_m(i)|^p \leq \sum_{i=1}^{\infty} |f_n(i) - f_m(i)|^p < \epsilon^p.$$

Making $m \to \infty$, we obtain the following inequality

$$\sum_{i=1}^{k} |f_n(i) - f(i)|^p \leq \sum_{i=1}^{\infty} |f_n(i) - f_m(i)|^p < \epsilon^p$$

for each $k \in \mathbb{N}$. Thus we have for all $n \geq n_0$

$$\sum_{i=1}^{\infty} |f_n(i) - f(i)|^p \leq \sum_{i=1}^{\infty} |f_n(i) - f_m(i)|^p < \epsilon^p.$$

On the other hand, using the Minkowski inequality, we obtain that

$$\left(\sum_{i=1}^{k} |f(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{k} |f_n(i) - f(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{k} |f_n(i)|^p\right)^{1/p}.$$

Making $k \to \infty$, we obtain for all $n \geq n_0$

$$||f||_p^p \leq ||f - f_n||_p^p + ||f_n||_p^p < \epsilon^p + ||f_n||_p^p < \infty.$$

Then $f \in \ell_p(\mathbb{C})$. Moreover, since, for all $n \geq n_0$, we have that

$$||f_n - f||_p^p = \sum_{i=1}^{\infty} |f_n(i) - f(i)|^p < \epsilon^p,$$

we conclude that $(f_n)_{n\in\mathbb{N}}$ converges to f in $\ell_p(\mathbb{C})$. Since $(f_n)_{n\in\mathbb{N}}$ is an arbitrary Cauchy sequence, we conclude that $(\ell(\mathbb{C}), \mathbb{C}, +, *, ||.||_p)$ is a Banach space. $\square$

Question 6: Let E be a Banach space. Show that the following assertions are equivalent

(a) E is separable;

(b) The unitary ball $B_E[0,1] = \{x \in E \ ; \ \|x\| \leq 1\}$ is separable;

(c) The unitary sphere $S_E(0,1) = \{x \in E \ ; \ \|x\| = 1\}$ is separable.

Resolution:

Lemma: Let (M,d) be a metric space and $N \subseteq M$ be a subset of M. If M is a separable metric space, prove that (N,d) is also separable metric space.

Proof: Let $E = \{x_n \in M \ ; \ n \in \mathbb{N}\}$ be countable and dense in M. Given $n \in \mathbb{N}$ and $x_m \in E$, define $X_{m,n} = B_M(x_m, 1/n) \cap N$. If $X_{m,n} \neq \varnothing$, choose $x_{m,n} \in X_{m,n}$. Consider

$$E' = \{x_{m,n} \in M \ ; \ \text{when } X_{m,n} \neq \varnothing\}.$$

It's clear that $E' \subseteq N$ is countable. Moreover, given $z \in N$ and $\epsilon > 0$, there is $x_n \in E$ such that $d(x_n, z) < \epsilon/2$. Taking $m \in \mathbb{N}$ such that $1/m < \epsilon/2$, note that $X_{n,m} = B_M(x_n, 1/m) \cap N \neq \varnothing$, because $d(x_n, z) < \epsilon/2$. Now

$$d(x_{n,m}, z) \leq d(x_{n,m}, x_n) + d(x_n, z) < 1/m + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon.$$

So E' is dense in N, then (N,d) is separable. Now we are ready to prove the question 6.

$(a) \to (b)$: Since $B_E[0,1] \subseteq E$ and E is a separable metric space, it is enough to apply the previous Lemma.

$(b) \to (c)$: Since $S_E(0,1) \subseteq B_E[0,1]$ and $B_E[0,1]$ is a separable metric space, it is enough to apply the previous Lemma.

$(c) \to (a)$: Suppose tha $S_E(0,1)$ is separable and let $A = \{x_n \ ; \ n \in \mathbb{N}\}$ be a countable subset dense in $S_E(0,1)$. Consider

$$A' = \{\alpha x_n \ ; \ \alpha \in \mathbb{Q}_{>0}, \ x_n \in A\} \cup \{0\}$$

Since $\mathbb{Q}_{>0}$ and A are countable sets, it is easy to see that A' is a countable subset of E. To prove that E is separable, it's enough to prove that

$$E = \overline{A'}.$$

In fact, let $x \in E \setminus \{0\}$, then $x/\|x\| \in S_E(0,1)$. Since A is dense in $S(0,1)$, there $y \in A$ such that

$$\left\| y - \frac{x}{\|x\|} \right\| < \frac{\epsilon}{2\|x\|}.$$

Then

$$\|(\|x\|y - x)\| < \epsilon/2.$$

Now, since $\mathbb{Q}_{>0}$ is dense in $\mathbb{R}_{>0}$, there is $\alpha \in \mathbb{Q}_{>0}$ such that $|\alpha - \|x\|| < \epsilon/(2\|y\|)$. Then

$$\|(\alpha y - \|x\|y)\| = |(\alpha - \|x\|)|.\|y\| < \frac{\epsilon}{2}.$$

Finally

$$\|\alpha y - x\| = \|(\alpha y - \|x\|y + \|x\|y - x)\| \leq \|(\alpha y - \|x\|y)\| + \|(\|x\|y - x)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus A' is dense in E, so E is a separable metric space. □

Question 7: Let E be normed linear space over $\mathbb{C}$. Given a discontinuous linear functional $\varphi : E \longrightarrow \mathbb{C}$, show that

$$\mathbb{C} = \{\varphi(x) \ ; \ x \in E \, , \, \|x\| \leq 1\}.$$

Resolution: In fact, let $z \in \mathbb{C}$. Since $\|\varphi\| = \infty$, then there is $x \in B[0,1]$ such that $|\varphi(x)| > |z|$. Considering $x' = |z|x/|\varphi(x)|$, we have that

$$\|x'\| = \left(\frac{|z|}{|\varphi(x)|}\right)\|x\| \leq \|x\| \leq 1$$

and

$$|\varphi(x')| = \left|\varphi\left(\frac{|z|}{|\varphi(x)|}x\right)\right| = \left(\frac{|\varphi(x)|}{|\varphi(x)|}\right)|z| = |z|.$$

Since $\varphi(x')$ and z are in the same circle in complex plane, there is $u \in \mathbb{C}$ such that $|u| = 1$ and $u\varphi(x') = z$. Then, considering $x'' = ux' \in E$, we have that $\|x''\| = |u|.\|x'\| = \|x'\| \leq 1$ and

$$\varphi(x'') = \varphi(ux') = u\varphi(x') = z.$$

Thus $\mathbb{C} \subseteq \{\varphi(x) \ ; \ x \in E \, , \, \|x\| \leq 1\}$. On the other hand, it's clear that $\{\varphi(x) \ ; \ x \in E \, , \, \|x\| \leq 1\} \subseteq \mathbb{C}$. Then

$$\mathbb{C} = \{\varphi(x) \ ; \ x \in E \, , \, \|x\| \leq 1\}.$$

□

Question 8: Let E be a linear normed space and M be a linear subspace of E. Given $x, y \in E$, we say that $x \sim y$ if $x - y \in M$. Define $[x] = \{y \in E \ ; \ x \sim y\}$ and the **quotient space** $E/M = \{[x] \ ; \ x \in E\}$.

(a) If M is a closed subspace of E, show that the function

$$\begin{aligned} \|.\| : E/M &\longrightarrow \mathbb{R} \\ [x] &\longmapsto \inf\{\|x-y\| \ ; \ y \in M\} \end{aligned}$$

defines a norm in E.

(b) If E is a Banach space and M is a closed subspace of E, show that $(E/M, \mathbb{K}, +, *, \|.\|)$ is a Banach space.

(c) From now, consider that M is a closed subspace of E. Defining the natural projection

$$\begin{aligned} \pi : E &\longrightarrow E/M \\ x &\longmapsto [x], \end{aligned}$$

prove that $\|[x]\| \leq \|x\|$ for every $x \in E$.

(d) [2] Given $x \in E$ and $\epsilon > 0$, show that there is $y \in E$ such that $\pi(x) = \pi(y)$ and $\|y\| \leq \|\pi(x)\| + \epsilon$.

Resolution: (a): We have to check the norm's axioms.

- Let $x \in E$. Since $\|x-y\| \geq 0$ for all $y \in M$, then $\|[x]\| = \inf\{\|x-y\| \ ; \ y \in M\} \geq 0$ for all $[x] \in E/M$.

- If $[x] = [0]$, then $x \in M$. Thus $\|[0]\| = \|[x]\| = \inf\{\|x-y\| \ ; \ y \in M\} \leq \|x-x\| = 0$. Thus $\|[0]\| = 0$. On the other hand, if $\|[x]\| = 0$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in M such that $\|x - x_n\| < 1/n$, that is, $x_n \to x$ as $n \to \infty$. Since M is a closed subspace of E, we conclude that $x \in M$, thus $[x] = [0]$.

- Let $\lambda \in \mathbb{K}$ and $[x] \in E/M$. If $\lambda = 0$, then

$$\|\lambda[x]\| = \|0[x]\| = \|[0]\| = 0 = 0\|[x]\| = |\lambda|.\|[x]\|.$$

So suppose without lost of generality that $\lambda \neq 0$. Thus

$$\begin{aligned} \|\lambda[x]\| = \|[\lambda x]\| = \inf\{\|\lambda x - y\| \ ; \ y \in M\} &= \inf\{|\lambda|.\|x - (y/|\lambda|)\| \ ; \ y \in M\} \\ &= |\lambda| \inf\{\|x - (y/\lambda)\| \ ; \ y \in M\}. \end{aligned}$$

[2] This exercise suggests that we can reformulate equivalently the definition of the norm in quotient space as $\|[x]\| = \inf\{\|y\| \ ; \ y \sim x\}$.

However the mapping

$$\varphi : (M,+) \longrightarrow (M,+)$$
$$y \longmapsto y/\lambda,$$

is a group isomorphism, which implies that

$$\{\|x-(y/\lambda)\| \ ; \ y \in M\} = \{\|x-y\| \ ; \ y \in M\}.$$

Then we have

$$\|\lambda[x]\| = |\lambda| \inf\{\|x-(y/\lambda)\| \ ; \ y \in M\} = |\lambda| \inf\{\|x-y\| \ ; \ y \in M\} = |\lambda|.\|[x]\|.$$

- Let x and $y \in E$. Thus

$$\|[x]+[y]\| = \|[x+y]\| = \inf\{\|x+y-v\| \ ; \ v \in M\} \overset{*}{=} \inf\{\|x+y-u-v\| \ ; \ u,v \in M\}$$
$$\leq \inf\{\|x-u\|+\|y-v\| \ ; \ u,v \in M\} = \inf\{\|x-u\| \ ; \ u \in M\} + \inf\{\|y-v\| \ ; \ v \in M\} = \|[x]\| + \|[y]\|,$$

where the equality with asterisk is true simply because M is a linear subspace of E

(b): The first step we have to do is to prove that the defined relation is, in fact, an equivalence relation and that we can define in E/M two operations in order to equip it with linear space structure. Inclusive, we should do this before to answer the latter (a).

Claim 1: The relation defined above is an equivalence relation.
In fact, we have to check the properties: reflexivity, symmetry and transitivity.

- Reflexivity: given $x \in E$, then $x - x = 0 \in M$, thus $x \sim x$;

- Symmetry: given $x, y \in E$, if $x \sim y$, then $x - y \in M$. Since M is a linear subspace of E, then $y - x = (-1)(x-y) \in M$, thus $y \sim x$;

- Transitivity: given $x, y, z \in E$, if $x \sim y$ and $y \sim z$, then

$$x - y \in M \qquad \text{and} \qquad y - z \in M$$

Thus, since M is a linear subspace of E, we have that $x - z = (x-y) + (y-z) \in M$. Thus $x \sim z$.

Thus we conclude that $\sim$ is, in fact, an equivalence relation.

Claim 2: Defining in E/M the following binary operations

$$+ : E/M \times E/M \longrightarrow E/M \qquad\qquad * : \mathbb{K} \times E/M \longrightarrow E/M$$

$$([x],[y]) \longmapsto [x+y] \qquad\qquad (\lambda,[x]) \longmapsto [\lambda x],$$

then $(E/M, \mathbb{K}, +, *)$ is a linear space.

Firstly we have to prove that this operations are well-defined. In fact

- If $[x] = [x']$ and $[y] = [y']$, then $x - x' \in M$ and $y - y' \in M$. Thus $(x+y) - (x'+y') = (x-x') + (y-y') \in M$, so $[x+y] = [x'+y']$.
- Given $[x] = [x']$ and $\lambda \in \mathbb{K}$, thus $x - x' \in M$. Since M is a linear subspace of E, then $\lambda(x-x') = \lambda x - \lambda x' \in M$, so $[\lambda x] = [\lambda x']$.

It is straightforward to verify that $(E/M, \mathbb{K}, +, *)$ has linear space structure, where $0 = [0]$ and, given $z = [x] \in E/M$, its inverse is $-z = [-x]$. Since it is simple to verify, although pedant, I will not do here.

Since $||.||$ is a norm in E/M, we conclude that $(E/M, \mathbb{K}, +, *, ||.||)$ is normed linear space over $\mathbb{K}$. Now it remains to prove that it is a Banach space. However we will need to prove the following interesting Lemma before.

Lemma: Let (M,d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in M. Then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $d(x_{n_{k+1}}, x_{n_k}) < 2^{-k}$ for all $k \in \mathbb{N}$.

Proof: The wished subsequence will be construct by induction. Let $n_1 \in \mathbb{N}$ such that $d(x_{n_1}, x_m) < 2^{-1}$ for all $m \geq n_1$. Now let $n_2 \in \mathbb{N}$ such that $n_2 \geq n_1$ and $d(x_{n_2}, x_m) < 2^{-2}$ for all $m \geq n_2$. This choose is possible because $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Note that, by construction, $d(x_{n_2}, x_{n_1}) < 2^{-1}$. Now suppose that we have constructed $x_{n_1}, \ldots, x_{n_m}$ such that $n_1 \leq n_2 \leq \cdots \leq n_m$ and $d(x_{n_{i+1}}, x_{n_i}) < 2^{-i}$ for all $i = 1, \ldots, m-1$ and $d(x_{n_m}, x_t) < 2^{-m}$ for all $t \geq n_m$. Again, since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there is $n_{m+1} \geq n_m$ such that for all $t \geq n_{m+1}$, we have $d(x_{n_{m+1}}, x_t) < 2^{-(m+1)}$. By construction, we have $d(x_{n_{m+1}}, x_m) < 2^{-m}$. Using the induction principle, we can construct a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $d(x_{n_{k+1}}, x_{n_k}) < 2^{-k}$ for all $k \in \mathbb{N}$, finishing the proof of the Lemma.

Finally we are ready to prove that $(E/M, \mathbb{K}, +, *, ||.||)$ is a Banach space. In fact, let $(z_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in E/M. By Lemma above, there is a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ such that $||z_{n_{k+1}} - z_{n_k}|| < 2^{-k}$ for all $k \in \mathbb{N}$. If we represent z_{n_k} by $\overline{x_{n_k}}$, after we choose the correct representative for z_{n_k}, we can assume without lost of generality that $\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}$ for all $k \in \mathbb{N}$. Note that, given $k_0 \in \mathbb{N}$, for every $t \geq l \geq k_0$, we have

$$\|x_{n_t} - x_{n_l}\| \leq \sum_{k=l}^{t-1} \|x_{n_{k+1}} - x_{n_k}\| \leq \sum_{k=l}^{t-1} 2^{-k} \leq \sum_{k=l}^{\infty} 2^{-k} = 2^{-(l-1)}.$$

Making $k_0 \to \infty$, we conclude that $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in E. Thus $(x_{n_k})_{k\in\mathbb{N}}$ converges to some $x \in E$. I claim that

$$\lim_{k\to\infty} z_{n_k} = \overline{x}$$

In fact, as $n \to \infty$, we have

$$\|z_{n_k} - \overline{x}\| = \|\overline{x_{n_k}} - \overline{x}\| \leq \|x_{n_k} - x\| \to 0.$$

Since $(z_n)_{n\in\mathbb{N}}$ is Cauchy sequence which admits a subsequence converging to $\overline{x}$, it is a classical exercise to show that $(z_n)_{n\in\mathbb{N}}$ also converges to $\overline{x}$. Since $(z_n)_{n\in\mathbb{N}}$ is an arbitrary Cauchy sequence, we conclude that $(E/M, \mathbb{K}, +, *, \|.\|)$ is a Banach space. □

(c): In fact, since $0 \in M$, we have

$$\|[x]\| = \inf\{\|x - y\| \ ; \ y \in M\} \leq \|x - 0\| = \|x\|.$$

In particular, since π is a linear mapping, we conclude that π is a weak contraction.

(d): Let $x \in E$ and $\epsilon > 0$. By the definitions of the norm of E/M and the infimum, we can find $z \in M$ such that

$$\|[x]\| \leq \|x - z\| < \|[x]\| + \epsilon.$$

Setting $y = x - z$, we have that $\|y\| < \|[x]\| + \epsilon$. Moreover, since $x - y = z \in M$, we have that $[x] = [y]$, thus

$$\pi(x) = [x] = [y] = \pi(y).$$

Although this last question can seem a little useless, it suggests that we can reformulate the definition of the norm in E/M defining

$$\begin{aligned} \|.\| : E/M &\longrightarrow \mathbb{R} \\ [x] &\longmapsto \inf\{\|y\| \ ; \ [y] = [x]\}. \end{aligned}$$

□

Question 9: Let E be a normed linear space and F be a linear subspace of E. If $\overline{F} \neq E$, then there is $f \neq 0 \in E'$ such that $f(x) = 0$ for all $x \in F$.

Proof: In fact, consider $M = E/\overline{F}$. Since M is a non-zero normed linear space with the norm

$$\begin{aligned} ||.|| : M &\longrightarrow \mathbb{R} \\ [x] &\longmapsto \inf\{||x - y|| \ ; \ y \in \overline{F}\}, \end{aligned}$$

one of the classic consequences of Hahn-Banach Theorem says that M admits a non-zero continuous linear functional $g : M \longrightarrow \mathbb{K}$. Consider $\pi : E \longrightarrow M$ be the natural projection. By question 8(c), π is a bounded linear mapping. Thus, since the composition of bounded linear mappings is bounded and linear, we conclude that $f = g \circ \pi : E \longrightarrow \mathbb{K}$ is a bounded linear functional. I claim that f satisfies the wished conditions. Indeed, since $g \neq 0$, let $[x] \in M$ such that $g([x]) \neq 0$. Thus

$$f(x) = (g \circ \pi)(x) = g(\pi(x)) = g([x]) \neq 0.$$

Moreover, given $x \in F$, so $x \in \overline{F}$, then $[x] = [0]$. Hence

$$f(x) = (g \circ \pi)(x) = g(\pi(x)) = g([0]) = 0$$

where the last equality is true since g is linear. Thus $f(x) = 0$ for all $x \in F$. Then f satisfies the wished conditions. $\square$

Question 10: Let E be a normed linear space over $\mathbb{K}$ and u be a element of E. If $f(u) = 0$ for all $f \in E'$, then $u = 0$.

Resolution: Suppose that $u \neq 0$. Define the following linear functional

$$\begin{aligned} f : \mathrm{Span}_{\mathbb{K}}(\{u\}) &\longrightarrow \mathbb{K} \\ tu &\longmapsto t. \end{aligned}$$

It's clear that f is a bounded linear functional, because, given $x = tu \in [u]$ arbitrary, then

$$|f(x)| = |f(tu)| = |t| = |t|\frac{||u||}{||u||} = \left(\frac{1}{||u||}\right)||tu|| = \left(\frac{1}{||u||}\right)||x||.$$

Since the function

$$\begin{aligned} p : E &\longrightarrow \mathbb{R} \\ x &\longmapsto (\tfrac{1}{||x||})||x|| \end{aligned}$$

is sublinear and $|f(x)| \leq p(x)$ for all $x \in [u]$, by Hahn-Banach Theorem, there is a linear functional $\varphi : E \longrightarrow \mathbb{K}$ which extends f and

$$|\varphi(x)| \leq p(x) = \left(\frac{1}{||u||}\right)||x||.$$

Thus $\varphi \in E'$ and $\varphi(u) = 1$. Then, if $u \in E$ is such that $f(u) = 0$ for every $f \in E'$, we conclude that $u = 0$ necessarily. $\square$

Question 11: Let E be a normed linear space over $\mathbb{K}$ and F be a linear subspace of E. Suppose that the following property holds: If $f \in E'$ is such that $f(x) = 0$ for every $x \in F$, then $f(x) = 0$ for every $x \in E$. Prove that F is dense in E, that is, $\overline{F} = E$.

Resolution: Suppose by contradiction that F is not dense in E, that is, $\overline{F} \neq E$. Then, applying the question 9, we can find a nonzero bounded linear functional f such that $f(x) = 0$ for every $x \in F$. This fact contradicts directly the property of enunciate, so we conclude that $\overline{F} = E$ and then F is dense in E. $\square$

Question 12: Prove that the operator

$$T : \ell_\infty \longrightarrow \mathcal{L}(\ell_2, \ell_2)$$
$$(a_n)_{n\in\mathbb{N}} \longmapsto [(b_n)_{n\in\mathbb{N}} \longmapsto (a_n b_n)_{n\in\mathbb{N}}]$$

is a linear isometry.

Resolution: Firstly we have to prove that T is well-defined. In fact, given a sequence $z = (a_n)_{n\in\mathbb{N}}$ in ℓ_∞, consider the operator

$$T((a_n)_{n\in\mathbb{N}}) := T_z : \ell_2 \longrightarrow \ell_2$$
$$(b_n)_{n\in\mathbb{N}} \longmapsto (a_n b_n)_{n\in\mathbb{N}}.$$

I claim that T_z is an well-defined bounded linear mapping. Indeed, since $(a_n)_{n\in\mathbb{N}} \in \ell_\infty$, there is $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Thus, given $(b_n)_{n\in\mathbb{N}} \in \ell_2$, we have

$$\sum_{n=1}^{\infty} |b_n|^2 < \infty$$

and so

$$\sum_{n=1}^{\infty} |a_n b_n|^2 = \sum_{n=1}^{\infty} |a_n|^2.|b_n|^2 \leq \sum_{n=1}^{\infty} M^2 |b_n|^2 = M^2 \sum_{n=1}^{\infty} |b_n|^2 < \infty,$$

that is, $T_z((b_n)_{n\in\mathbb{N}}) \in \ell_2$. Then T_z is well-defined. Now, given $(b_n)_{n\in\mathbb{N}}$, we have that

$$\|T_z((b_n)_{n\in\mathbb{N}})\|_2 = \|(a_nb_n)_{n\in\mathbb{N}}\|_2 = \sum_{n=1}^{\infty}|a_nb_n|^2 \leq \sum_{n=1}^{\infty}\|(a_k)_{k\in\mathbb{N}}\|_\infty|b_n|^2 = \|(a_k)_{k\in\mathbb{N}}\|_\infty\sum_{n=1}^{\infty}|b_n|^2$$
$$= \|(a_k)_{k\in\mathbb{N}}\|_\infty\|(b_n)_{n\in\mathbb{N}}\|_2 = \|z\|_\infty\|(b_n)_{n\in\mathbb{N}}\|_2$$

for all $(b_n)_{n\in\mathbb{N}} \in \ell^2$, thus T_z bounded. Now let $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}} \in \ell_2$ and $\lambda \in \mathbb{K}$. Then

$$T_z((b_n)_{n\in\mathbb{N}} + \lambda(c_n)_{n\in\mathbb{N}}) = T_z((b_n + \lambda c_n)_{n\in\mathbb{N}}) = (a_n(b_n + \lambda c_n))_{n\in\mathbb{N}} = (a_nb_n + \lambda a_nc_n))_{n\in\mathbb{N}} =$$
$$(a_nb_n)_{n\in\mathbb{N}} + \lambda(a_nc_n)_{n\in\mathbb{N}} = T_z((b_n)_{n\in\mathbb{N}}) + \lambda T_z((c_n)_{n\in\mathbb{N}}).$$

Thus T_z is linear and so $T((a_n)_{n\in\mathbb{N}}) \in \mathcal{L}(\ell_2, \ell_2)$, that is, T is well-defined. Now I claim that T is a linear operator. Indeed, let $z = (a_n)_{n\in\mathbb{N}}$, $z' = (a'_n)_{n\in\mathbb{N}} \in \ell_\infty$ and $\lambda \in \mathbb{K}$. Then, for any $(b_n)_{n\in\mathbb{N}} \in \ell_2$, we have

$$(T(z + \lambda z'))((b_n)_{n\in\mathbb{N}}) = T_{z+\lambda z'}((b_n)_{n\in\mathbb{N}}) = ((a_n + \lambda a'_n)b_n)_{n\in\mathbb{N}} = (a_nb_n + \lambda a'_nb_n)_{n\in\mathbb{N}}$$
$$= (a_nb_n)_{n\in\mathbb{N}} + \lambda(a'_nb_n)_{n\in\mathbb{N}} = T_z((b_n)_{n\in\mathbb{N}}) + \lambda T_{z'}((b_n)_{n\in\mathbb{N}}) = (T_z + \lambda T_z)((b_n)_{n\in\mathbb{N}})$$
$$= (T(z) + \lambda T(z'))((b_n)_{n\in\mathbb{N}}).$$

Thus $T(z+\lambda z') = T(z)+\lambda T(z')$ and T is a linear operator. Now we'll prove that T is an isometry, that is, $\|T(z)\| = \|z\|_\infty$ for any $z \in \ell_\infty$. In fact, let $z = (a_n)_{n\in\mathbb{N}} \in \ell_\infty$, let's calculate $\|T(z)\| = \|T_z\|$ in $\mathcal{L}(\ell_2, \ell_2)$. Let $L = \|(a_n)_{n\in\mathbb{N}}\|_\infty$, thus, given $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $L - \epsilon < |a_{n_\epsilon}| \leq L$. Consider $(b_n)_{n\in\mathbb{N}} \in \ell_2$, where

$$b_n = \begin{cases} 0 & \text{if } n \neq n_\epsilon; \\ 1 & \text{if } n = n_\epsilon. \end{cases}$$

Thus $\|(b_n)_{n\in\mathbb{N}}\|_2 = 1$ and

$$\|T(z)((b_n)_{n\in\mathbb{N}})\| = \|T_z((b_n)_{n\in\mathbb{N}})\| = \sqrt{\sum_{n=1}^{\infty}|a_nb_n|^2} = |a_\epsilon| > L - \epsilon = \|(a_n)_{n\in\mathbb{N}}\|_\infty - \epsilon = \|z\|_\infty - \epsilon.$$

Thus since $\epsilon > 0$ is arbitrary, we have

$$\|T(z)\| = \sup\{\|T_z((b_n)_{n\in\mathbb{N}})\|_2 \ ; \ (b_n)_{n\in\mathbb{N}} \in \ell_2, \|(b_n)_{n\in\mathbb{N}}\|_2 \leq 1\} \geq \|(a_n)_{n\in\mathbb{N}}\|_\infty = \|z\|_\infty.$$

On the other hand, given $(b_n)_{n\in\mathbb{N}} \in \ell_2$ such that $\|(b_n)_{n\in\mathbb{N}}\|_2 = 1$, we have that

$$\|T(z)((b_n)_{n\in\mathbb{N}})\|_2 = \|(a_nb_n)_{n\in\mathbb{N}}\|_2 = \sqrt{\sum_{n=1}^{\infty}|a_nb_n|^2} \leq \sqrt{\sum_{n=1}^{\infty}\|(a_k)_{k\in\mathbb{N}}\|_\infty^2|b_n|^2}$$
$$= \|(a_k)_{k\in\mathbb{N}}\|_\infty\sqrt{\sum_{n=1}^{\infty}|b_n|^2} = \|(a_k)_{k\in\mathbb{N}}\|_\infty\|(b_k)_{k\in\mathbb{N}}\|_2 = \|(a_k)_{k\in\mathbb{N}}\|_\infty = \|z\|_\infty.$$

Thus, remembering that $z = (a_n)_{n\in\mathbb{N}}$, we conclude

$$\|T((a_n)_{n\in\mathbb{N}})\| = \|T(z)\| = \sup\{\|T(z)((b_n)_{n\in\mathbb{N}})\|_2 \ ; \ (b_n)_{n\in\mathbb{N}} \in \ell_2, \|(b_n)_{n\in\mathbb{N}}\|_2 = 1\} = \|(a_n)_{n\in\mathbb{N}}\|_\infty = \|z\|_\infty.$$

After a bunch of calculations, we conclude that $\|T(z)\| = \|z\|_\infty$ for all $z \in \ell_\infty$. Since T is linear we conclude that, given $z, w \in \ell_\infty$, we have

$$\|T(z) - T(w)\| = \|T(z - w)\| = \|z - w\|_\infty.$$

Thus T is a linear isometry. Since T is a linear isometry, $\mathcal{L}(\ell_2, \ell_2)$ has a linear subspace isometrically isomorphic [3] to ℓ_∞, which is not separable. However this fact implies that $\mathcal{L}(\ell_2, \ell_2)$ is not separable, because the property of to be separable in metric spaces is hereditary, that is, if $(N, d|_{N\times N})$ is metric subspace of a separable metric space (M, d), then $(N, d|_{N\times N})$ is separable. $\square$

Question 13: Let $(E, \mathbb{R}, +, *, \|.\|)$ be a normed linear space over $\mathbb{R}$. Prove that the Minkowski functional of the open ball $B(0,1) := \{x \in E \ ; \ \|x\| < 1\}$ coincides with the norm $\|.\|$.

Resolution: Let $p_{B(0,1)}$ be the Minkowski functional of $B(0,1)$. We will prove that $p_{B(0,1)}(x) = \|x\|$ for every $x \in B(0,1)$. Let $x \in B(0,1)$. If $x = 0$, then $0/\lambda = 0 \in B(0,1)$ for every $\lambda > 0$, thus

$$p_{B(0,1)}(0) = \inf\{\lambda \in \mathbb{R}_{>0} \ ; \ 0/\lambda \in B(0,1)\} = 0 = \|0\|.$$

Now suppose that $x \neq 0$. Since

$$\left\| \frac{x}{\|x\|} \right\| = 1,$$

we have that $x/\|x\| \notin B(0,1)$ and then $\|x\| \notin \{\lambda \in \mathbb{R}_{>0} \ ; \ x/\lambda \in B(0,1)\}$ On the other hand, given $\epsilon > 0$, we have that

$$\left\| \frac{x}{\|x\| + \epsilon} \right\| = \frac{\|x\|}{\|x\| + \epsilon} < 1,$$

which implies that

$$\frac{x}{\|x\| + \epsilon} \in B(0,1)$$

and so $\|x\| + \epsilon \in \{\lambda \in \mathbb{R}_{>0} \ ; \ x/\lambda \in B(0,1)\}$ for every $\epsilon > 0$. So we conclude that

$$p_{B(0,1)}(x) = \inf\{\lambda \in \mathbb{R}_{>0} \ ; \ x/\lambda \in B(0,1)\} = \|x\|.$$

[3] The condition of to be an isometric isomorphism is essential, that is, in general, I could not claim the same if T was only an isomorphism.

Then the Minkowski functional of $B(0,1)$ coincides with the norm $||.||$. $\square$

Question 14: Let $(E,\mathbb{C})$ be a complex normed linear space and A,B be nonempty, disjoint and convex subsets of E.

a) If A is open, then there are a continuous functional linear $\phi \in E'$ and $a \in \mathbb{R}$ such that

$$\operatorname{Re}(\phi(x)) < a \le \operatorname{Re}(\phi(y))$$

for all $x \in A$ and $y \in B$.

b) If A is closed and B is compact, then there are a continuous functional linear $\phi \in E'$ and $a,b \in \mathbb{R}$ such that

$$\operatorname{Re}(\phi(x)) \le a < b \le \operatorname{Re}(\phi(y))$$

for all $x \in A$ and $y \in B$.

Resolution: a): In fact, doing the descomplexification of E, that is, restricting the field of scalars to real numbers, we obtain a normed vector space $E_{\mathbb{R}}$ with norm

$$\begin{aligned} ||.||_{\mathbb{R}} : E_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ x &\longmapsto ||x|| \end{aligned}$$

Since the norm is essentially the same, we have that A is a open subset of E. Moreover, the descomplexification preserves the convexicity, because, being convex in complex sense means that, given x,y in a convex subset, then, for all $(a,b) \in \mathbb{C}^2$ with $|a|+|b| = 1$, we have $ax+by$ belongs to that set. In particular, such fact is true for $(t,1-t)$ with $t \in [0,1]$. Then A,B remain being convex in $\mathbb{E}_{\mathbb{R}}$. Since A,B are non-empty and disjoint, by Hahn-Banach geometric theorem, there are $\phi \in E'_{\mathbb{R}}$ and $a \in \mathbb{R}$ such that

$$\phi(x) < a \le \phi(y)$$

for all $x \in A$ and $y \in B$. Now, we want to construct a continuous linear functional $\psi : E \longrightarrow \mathbb{C}$ whose $\operatorname{Re}(\psi) = \phi$. Observe that, if $f = u+iv = \psi$ is a linear functional in E', then, given $x \in E$

$$u(ix) + iv(ix) = f(ix) = if(x) = u(x) + iv(x) = i(\operatorname{Re}(\psi(x)) + i\operatorname{Im}(\psi(x))) = \\ -\operatorname{Im}(\psi(x)) + i\operatorname{Re}(\psi(x)).$$

Thus

$$v(ix) = \phi(x),$$

so

$$v(x) = \phi(x/i) = \phi(-ix) = -\phi(ix).$$

This last equality suggests how to construct ψ. Formally, define

$$\begin{aligned} \psi : E &\longrightarrow \mathbb{C} \\ x &\longmapsto \phi(x) - i\phi(ix). \end{aligned}$$

I claim that ψ is the wished mapping. In fact, ψ linear, because given $x, y \in E$, we have

$$\begin{aligned} \psi(x+y) = \phi(x+y) - i\phi(i(x+y)) = \phi(x) + \phi(y) - i(\phi(ix) + \phi(iy)) &= (\phi(x) - \phi(ix)) + (\phi(y) - i\phi(iy)) \\ &= \psi(x) + \psi(y) \end{aligned}$$

and, given $x \in E$ and $a + ib \in \mathbb{C}$, we have

$$\begin{aligned} \psi((a+ib)x) = \psi(ax + ibx) = \psi(ax) + \psi(ibx) &= a\phi(x) - ia\phi(ix) + b\phi(ix) + ib\phi(x) \\ &= (a+bi)(\phi(x) - i\phi(ix)) = (a+bi)\psi(x). \end{aligned}$$

Moreover, since ϕ is bounded, we also have ψ is a bounded linear functional, because

$$|\psi(x)| = |\phi(x) - i\phi(ix)| \leq |\phi(x)| + |\phi(ix)| \leq ||\phi||.||x|| + ||\phi||.||ix|| = 2||\phi||.||x||$$

for all $x \in E$. Finally, note that $\mathrm{Re}(\psi) = \phi$ and, by construction, there is $a \in \mathbb{R}$ such that

$$\mathrm{Re}(\psi(x)) = \phi(x) < a \leq \phi(y) = \mathrm{Re}(\psi(y))$$

for all $x \in A$ and $y \in B$.

b): The proof is similar. Doing the descomplexification of E, we obtain a real normed vector space $E_{\mathbb{R}}$ with the same norm defined above. It is easy to see that topological properties as compactness and closure remain invariant. It was showed above that the convexicity of A and B also is preserved. Since they are non-empty, disjoint, the Hahn-Banach Geometric Theorem (2° form) says that there are a continuous functional linear $\phi \in E'_{\mathbb{R}}$ and $a, b \in \mathbb{R}$ such that

$$\phi(x) \leq a < b \leq \phi(y)$$

for all $x \in A$ and $y \in B$. Defining ψ as above, that is

$$\begin{aligned} \psi : E &\longrightarrow \mathbb{C} \\ x &\longmapsto \phi(x) - i\phi(ix), \end{aligned}$$

we have that $\psi \in E'$ is a continuous functional linear with

$$\mathrm{Re}(\psi(x)) = \phi(x) \leq a < b \leq \phi(y) = \mathrm{Re}(\psi(y))$$

for all $x \in A$ and $y \in B$. $\square$

2.2 The Banach-Steinhaus Theorem, The Open Mapping Theorem, The Closed Graph Theorem. Duality and Reflexive Spaces. Weak and Weak* Topologies

Question 1: Let E and F be Banach spaces and T, T_2, $T_3, \dots$ be linear operators in $\mathcal{L}(E, F)$ such that $T_n(x) \to T(x)$ for all $x \in E$. Show that for all compact subset $K \subseteq E$, we have

$$\lim_{n\to\infty} \left(\sup_{x\in K} ||T_n(x) - T(x)|| \right) = 0.$$

Proof: Firstly I will prove that $\sup\{||T_1 - T||, ||T_2 - T||, \dots\} < \infty$. Indeed, given $x \in E$, define the sequence $\{||T_n(x) - T(x)||\}_{n\in\mathbb{N}}$. Since this sequence converges to 0 by hypothesis, this sequence is bounded, that is, there is $C_x > 0$ such that

$$\sup_{n\in\mathbb{N}} ||T_n(x) - T(x)|| \le C_x$$

Since each $T_n - T$ is a bounded operator and E is a Banach space, by Banach-Steinhaus Theorem, there is $C > 0$ such that

$$\sup_{n\in\mathbb{N}} ||T_n - T|| \le C,$$

that is, $||T_n - T|| \le C$ for each $n \in \mathbb{N}$. Suppose, by contradiction, that there is a compact subset $K \subseteq E$ such that

$$\sup_{x\in K} ||T_n(x) - T(x)|| \not\to 0 \text{ as } n \to \infty.$$

Thus there are $r > 0$, a subsequence $(T_{n_k})_{k\in\mathbb{N}}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in K such that

$$||T_{n_k}(x_k) - T(x_k)|| \ge r.$$

Since K is a compact subset of E, $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence converging to $x \in K$ and we can assume without lost of generality that $(x_k)_{k\in\mathbb{N}}$ converges to x. Finally, observe

$$\begin{aligned} r \le ||T_{n_k}(x_k) - T(x_k)|| &= ||T_{n_k}(x_k) - T_{n_k}(x) + T_{n_k}(x) + -T(x) - T(x) - T(x_k)|| \\ &= ||(T_{n_k} - T)(x) + (T_{n_k} - T)(x_k - x)|| \le ||(T_{n_k} - T)(x)|| + ||(T_{n_k} - T)(x_k - x)|| \\ &\le ||T_{n_k}(x) - T(x)|| + ||T_{n_k} - T|| \cdot ||x_k - x||. \end{aligned}$$

Since $||T_{n_k} - T|| \le C$ for all $k \in \mathbb{N}$ and $||x_k - x||$ converges to 0 as $k \to \infty$, we conclude that

$$\lim_{k\to\infty} (||T_{n_k} - T|| \cdot ||x_k - x||) = 0.$$

Moreover, by hypothesis we have that

$$\lim_{k\to\infty} \|T_{n_k}(x) - T(x)\| = 0.$$

Thus, as $k \to \infty$, we conclude that $r \le 0$, which is a contradiction. $\square$

Question 2: Let E, F be linear normed spaces and $T : E \longrightarrow F$ be a bounded linear operator. Consider the quotient operation

$$\pi : E \longrightarrow E/\operatorname{Ker}(T)$$

$$x \longmapsto \overline{x}$$

Prove that there exists an unique bounded linear operator $\hat{T} : E/\operatorname{Ker}(T) \longrightarrow F$ satisfying $T = \hat{T} \circ \pi$. Moreover $\|T\| = \|\hat{T}\|$. Using this result, show the Isomorphism Theorem: Given E and F Banach spaces and $T \in \mathcal{L}(E, F)$, if $T(E)$ is closed in F, then $T(E)$ is isomorphic to $E/\operatorname{Ker}(T)$.

Solution: **Claim 1:** Existence and uniqueness of $\hat{T}$

In fact, consider the following mapping

$$g : E/\operatorname{Ker}(T) \longrightarrow F$$

$$\overline{x} \longmapsto T(x)$$

I claim that g is well-defined. Indeed, if $\overline{x} = \overline{y}$, then $x - y \in \operatorname{Ker}(T)$, thus $T(x) = T(y)$, that is

$$g(\overline{x}) = T(x) = T(y) = g(\overline{y}).$$

I claim that g is a linear mapping. Indeed, given $\overline{x}, \overline{y} \in E/\operatorname{Ker}(T)$ and $\lambda \in \mathbb{K}$, we have

$$g(\overline{x} + \lambda\overline{y}) = g\big(\overline{x + \lambda y}\big) = T(x + \lambda y) = T(x) + \lambda T(y) = g(\overline{x}) + \lambda g(\overline{y}).$$

Moreover, given $x \in E$, we have $(g \circ \pi)(x) = g(\overline{x}) = T(x)$, thus the following diagram

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[dr, "T"'] & E/\operatorname{Ker}(T) \arrow[d, "g"] \\ & F \end{tikzcd}$$

commutes. Now it is enough to show that g is bounded. In fact, let $z = \overline{x} \in E/\operatorname{Ker}(T)$. By definition of norm in $E/\operatorname{Ker}(T)$, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in $\operatorname{Ker}(T)$ such that

$$\|z\| \le \|x - y_n\| \le \|z\| + \frac{1}{n}.$$

Thus

$$\|g(z)\| = \|g(\overline{x - y_n})\| = \|T(x - y_n)\| \leq \|T\| \cdot \left(\|z\| + \frac{1}{n}\right).$$

for all $n \in \mathbb{N}$. Since this inequality holds for all $n \in \mathbb{N}$, we conclude that

$$\|g(z)\| \leq \|T\| \cdot \|z\|$$

and thus g is bounded as $\|T\| < \infty$.

Now let $h : E/\operatorname{Ker}(T) \longrightarrow F$ be another bounded linear mapping which makes the following diagram

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[dr, "T"'] & E/\operatorname{Ker}(T) \arrow[d, "h"] \\ & F \end{tikzcd}$$

commute. Thus given, $\overline{x} \in E/\operatorname{Ker}(T)$, we have $h(\overline{x}) = h \circ \pi(x) = T(x) = g(\overline{x})$. Thus there exists an unique linear mapping which satisfies the whished property, so we will denote it by $\hat{T}$.

Claim 2: $\|T\| = \|\hat{T}\|$

Indeed, we have that

$$\|T\| = \sup_{\|x\|<1} \|T(x)\| \leq \sup_{\|\overline{x}\|<1} \|\hat{T}(\overline{x})\| = \|\hat{T}\|,$$

because $\|\overline{x}\| \leq \|x\|$ for every $x \in E$. In order to prove that $\|T\| \geq \|\hat{T}\|$, it is enough to prove the following: Given $\overline{x} \in E/\operatorname{Ker}(T)$ with $\|\overline{x}\| < 1$, there $y \in E$ such that $\|y\| < 1$ and $\|T(y)\| = \|T(x)\| = \|\hat{T}(x)\|$. However, it is easy, because, given $\overline{x} \in E/\operatorname{Ker}(T)$ with $\|\overline{x}\| < 1$, there is $y \in \operatorname{Ker}(T)$ such that $\|x - y\| < 1$. Thus

$$\|T\| = \sup_{\|z\|<1} \|T(z)\| \geq \|T(x - y)\| = \|T(x)\| = \|\hat{T}(x)\|.$$

Then $\|T\| = \|\hat{T}\|$. In particular, T is bounded if and only if $\hat{T}$ is bounded.

Claim 3: Isomorphism Theorem: Given E and F Banach spaces and $T \in \mathcal{L}(E, F)$, if $T(E)$ is closed in F, then $T(E)$ is isomorphic to $E/\operatorname{Ker}(T)$.

Indeed, consider the linear mapping

$$\begin{aligned} \hat{T} : E/\operatorname{Ker}(T) &\longrightarrow T(E) \\ \overline{x} &\longmapsto T(x) \end{aligned}$$

Clearly T is a surjective mapping. Let $\overline{x} \in \mathrm{Ker}(\hat{T})$, then

$$\hat{T}(\overline{x}) = T(x) = 0$$

Thus $x \in \mathrm{Ker}(T)$, then $\overline{x} = 0$. Since $\|T\| = \|\hat{T}\|$ and T is continuous, we also have that $\hat{T}$ is continuous. Finally, since F is Banach and $T(E) \subseteq F$ is closed, we have that $T(E)$ is a Banach space, thus by the Open Mapping Theorem, $\hat{T}$ is an open continuous linear bijection mapping and so $\hat{T}$ is an isomorphism, that is, $T(E)$ is isomorphic to $E/\mathrm{Ker}(T)$. □

Question 3: Show that an open mapping is not a closed mapping necessarily.

Solution: In fact, consider the projection

$$\pi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto y$$

Note that π is an open mapping. Indeed, let U be an open in $\mathbb{R} \times \mathbb{R}$. I will prove that $\pi(U)$ is a open subset of $\mathbb{R}$. In fact, given $y \in \pi(U)$, then $(a, y) \in U$ for some $a \in \mathbb{R}$. By definition of product topology, there exists open sets W_1, W_2 in $\mathbb{R}$ such that

$$(a, y) \in W_1 \times W_2 \subseteq U.$$

Thus $y \in W_2 \subseteq \pi(U)$, which implies $y \in \mathrm{int}(\pi(U))$. Then $\pi(U)$ is open in $\mathbb{R}$. Now remember that the product topology product coincides with the metric topology of $\mathbb{R}^2$. Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto e^x$$

Note that f is a continuous mapping and, since $\mathbb{R}$ is Hausdorff, by the Closed Mapping Theorem (from general topology), we have that

$$\mathrm{gr}(f) = \{(x, e^x) \in \mathbb{R} \times \mathbb{R} \ ; \ x \in \mathbb{R}\}$$

is a closed subset of $\mathbb{R} \times \mathbb{R}$. However, it is easy to see that

$$\pi(\mathrm{gr}(f)) = (0, \infty),$$

which is not closed in $\mathbb{R}$, because $\overline{(0, \infty)} = [0, \infty)$. □

Question 4: Let E be a Banach space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in E such that

$$\sum_{n=1}^{\infty} |f(x_n)| < \infty$$

for all $f \in E'$. Show that

$$\sup_{\|f\|\leq 1} \sum_{n=1}^{\infty} |f(x_n)| < \infty.$$

Proof: In fact, define the mapping

$$\begin{aligned} \psi : E' &\longrightarrow \ell_1 \\ f &\longmapsto (f(x_n))_{n\in\mathbb{N}} \end{aligned}$$

Observe that ψ is well-defined, because, given $f \in E'$, we have that $\psi(f) \in \ell_1$. In fact, by hypothesis, for all $f \in E'$, we have

$$\sum_{n=1}^{\infty} |f(x_n)| < \infty.$$

Evidently ψ is a linear mapping, because given $f_1,\ f_2 \in E'$ and $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} \psi(f_1 + \lambda f_2) = ((f_1 + \lambda f_2)(x_n))_{n\in\mathbb{N}} = (f_1(x_n) + \lambda f_2(x_n))_{n\in\mathbb{N}} &= (f_1(x_n))_{n\in\mathbb{N}} + \lambda(f_2(x_n))_{n\in\mathbb{N}} \\ &= \psi(f_1) + \lambda\psi(f_2). \end{aligned}$$

Moreover ψ is bounded. In order to show it, we will use the Closed Graph Theorem. In fact, let

$$\mathrm{gr}(\psi) = \{(f, (f(x_n))_{n\in\mathbb{N}})\ ;\ f \in E'\}$$

be the graph of ψ and $(f, y) \in \overline{\mathrm{gr}(\psi)}$, then there is a sequence $(f_k, (f_k(x_n))_{n\in\mathbb{N}})$ in $\mathrm{gr}(\psi)$ such that $(f_k, (f_k(x_n))_{n\in\mathbb{N}})$ converges to (f, y) in the normed space $E' \times \ell_1$. Thus f_k converges to f in $\|.\|_{E'}$ and $(f_k(x_n))_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ converges to $y = (y_k)_{k\in\mathbb{N}}$ in $\|.\|_1$.

Since $((f_k(x_n))_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ converges to $y = (y_n)_{n\in\mathbb{N}}$ in $\|.\|_1$, given $\epsilon > 0$, there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we have

$$\|(f_k(x_n))_{n\in\mathbb{N}} - (y_n)_{n\in\mathbb{N}}\|_1 = \sum_{n=1}^{\infty} |f_k(x_n) - y_n| < \epsilon.$$

So $|f_k(x_n) - y_n| < \epsilon$ for all $n \in \mathbb{N}$ and $k \geq k_0$, which allows us to conclude that

$$\lim_{k\to\infty} f_k(x_n) = y_n$$

for all $n \in \mathbb{N}$. On the other hand, since f_k converges to f in $\|\cdot\|_{E'}$, given $n \in \mathbb{N}$ and $\epsilon > 0$, let $n_0 \in \mathbb{N}$ such that for all $m \geq n_0$, we have $\|f_m - f\|_{E'} < \epsilon/\|x_n\|$. Thus

$$|f_m(x_n) - f(x_n)| = \|f_m - f\|_{E'} \cdot \|x_n\| = \frac{\epsilon}{\|x_n\|} \cdot \|x_n\| = \epsilon$$

for all $n \in \mathbb{N}$. By uniqueness of limit in Hausdorff spaces, we conclude that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then

$$(f, y) = (f, (y_n)_{n\in\mathbb{N}}) = (f, f(x_n)_{n\in\mathbb{N}}) = (f, \psi(f)).$$

Since E' and ℓ_1 are Banach spaces, by Closed Graph Theorem, we conclude that ψ is continuous. Thus, given $f \in B_{E'}[0,1]$, we have that

$$\sum_{k=1}^{\infty} |f(x_k)| = \|\psi(f)\| \leq \|\psi\| \cdot \|f\|_{E'} \leq \|\psi\| < \infty.$$

Since $f \in B_{E'}[0,1]$ is an arbitrary element, we conclude that

$$\sup_{\|f\|\leq 1} \sum_{k=1}^{\infty} |f(x_k)| \leq \sup_{\|f\|\leq 1} \|\psi(f)\| \leq \sup_{\|f\|\leq 1} \|\psi\| = \|\psi\| < \infty.$$

□

Question 5: Prove that all normed linear space which is isomorphic to a reflexive space is also reflexive.

Solution: Let E be a reflexive space which isomorphic to a normed linear space F and let $T : E \longrightarrow F$ be an isomorphism between E and F. It is well known that $T' : F' \longrightarrow E'$ is also an isomorphism and so $T'' : E'' \longrightarrow F''$ is an isomorphism. Consider the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "J_E"'] & F \arrow[d, "T''\circ J_E \circ T^{-1}"] \\ E'' \arrow[r, "T''"] & F'' \end{tikzcd}$$

Note that $T'' \circ J_E \circ T^{-1} : F \longrightarrow F''$ is an isomorphism since it is a composition of isomorphisms. Before we proceed, we will use the following notation. For a given normed linear space V, we define

$$J_V : V \longrightarrow V''$$

$$x \longmapsto \phi_x : V' \longrightarrow \mathbb{K}$$

$$f \longmapsto f(x)$$

Finally, it is enough to prove that F is a reflexive space. Since $T'' \circ J_E \circ T^{-1}$ is already an isomorphism, it is enough to show that

$$T'' \circ J_E \circ T^{-1} = J_F.$$

In fact, let $x \in F$ and $f \in F''$ be arbitrary elements, then

$$\big((T'' \circ J_E \circ T^{-1})(x)\big)(f) = \big((T'' \circ J_E \circ (T^{-1}(x))\big)(f) = (T'' \circ \phi_{T^{-1}(x)})(f) = (\phi_{T^{-1}(x)} \circ T')(f)$$
$$= \phi_{T^{-1}(x)} \circ T'(f) = \phi_{T^{-1}(x)} \circ (f \circ T) = (f \circ T)(T^{-1}(x)) = f(x) = \big(J_F(x)\big)(f).$$

Since $f \in F''$ is arbitrary, we conclude that

$$T'' \circ J_E \circ T^{-1}(x) = J_F(x)$$

for all $x \in F$, then $J_F = T'' \circ J_E \circ T^{-1}$ is an isomorphism. Then F is a reflexive space. □

Question 6: Let E and F be normed linear spaces. Prove that the mapping

$$G : \mathcal{L}(E, F) \longrightarrow \mathcal{L}(F', E')$$
$$T \longmapsto T'$$

is a isometric isomorphism over its image.

Solution: Indeed, firstly note that G is a linear mapping, because, given T_1, $T_2 \in \mathcal{L}(E, F)$, $\lambda \in \mathbb{C}$ and an arbitrary $\phi \in F'$, we have

$$G(T_1 + \lambda T_2)(\phi) = (T_1 + \lambda T_2)'(\phi) = \phi \circ (T_1 + \lambda T_2) = \phi \circ T_1 + \phi \circ (\lambda T_2) = \phi \circ T_1 + \lambda(\phi \circ (T_2))$$
$$= T_1'(\phi) + \lambda T_2'(\phi) = G(T_1)(\phi) + \lambda G(T_2)(\phi) = (G(T_1) + \lambda G(T_2))(\phi)$$

Since ϕ is arbitrary in $\mathcal{L}(F', E')$, we conclude that

$$G(T_1 + \lambda T_2) = G(T_1) + \lambda G(T_2),$$

so G is linear. Now we will prove that $||T|| = ||T'||$. In fact, let $x \in E$ with $||x|| \leq 1$. By Hahn-Banach Theorem, we that

$$||T(x)|| = \sup_{||\phi|| \leq 1} |(\phi \circ T)(x)| = \sup_{||\phi|| \leq 1} |(T'(\phi))(x)| \overset{*}{\leq} \sup_{||\phi|| \leq 1} ||(T'(\phi))||.||x|| \leq \sup_{||\phi|| \leq 1} ||(T'(\phi))|| := ||T'||,$$

where the marked inequality is because $T'(\phi) \in E'$. Thus

$$||T|| = \sup_{||x|| \leq 1} ||T(x)|| \leq ||T'||.$$

On the other hand, let $\phi \in F'$ with $||\phi|| \leq 1$. We have

$$||T'(\phi)|| = ||\phi \circ T|| = \sup_{||x|| \leq 1} |(\phi \circ T)(x)| \leq \sup_{||x|| \leq 1} (||\phi||.||T(x)||) \leq \sup_{||x|| \leq 1} (||T(x)||) := ||T||.$$

Thus

$$\|T'\| = \sup_{\|\phi\|\leq 1} \|T'(\phi)\| \leq \|T\|.$$

Then $\|T\| = \|T'\|$ and so, given $T \in \mathcal{L}(L, F)$, we have

$$\|G(T)\| = \|T'\| = \|T\|.$$

Then G is a isometric monomorphism and an isometric linear bijection between E and $G(E)$. Since the inverse of an isometry is an isometry and every isometry is continuous, we conclude that G is an isometric isomorphism over its image. $\square$

Question 7: Let E and F be Banach spaces and $T \in \mathcal{L}(E, F)$. Show that $T'' := (T')' \in \mathcal{E}'', \mathcal{F}''$ is an extension of T to E'', that is

$$T'' \circ J_E = J_F \circ T.$$

Use this fact in order to prove that, if E is a reflexive space, then $T'' = T$.

Solution: Given a normed linear space V, we will use the following notation for the canonical immersion

$$\begin{aligned} J_V : V &\longrightarrow V'' \\ x &\longmapsto \phi_x : V' \longrightarrow \mathbb{K} \\ &\qquad\quad f \ \longmapsto f(x) \end{aligned}$$

Let $x \in E$ and $f \in F'$ be arbitrary elements. Note that

$$\begin{aligned} \Big((T'' \circ J_E)(x)\Big)(f) &= (T'' \circ \phi_x)(f) = (\phi_x \circ T')(f) = \phi_x \circ (T'(f)) = \phi_x \circ (f \circ T) = (f \circ T)(x) \\ &= f(T(x)) = \phi_{T(x)}(f) = \Big((J_F \circ T)(x)\Big)(f). \end{aligned}$$

Since $f \in F'$ is arbitrary, we have that

$$(T'' \circ J_E)(x) = (J_F \circ T)(x).$$

Since the equality above holds for all $x \in E$, we conclude that

$$T'' \circ J_E = J_F \circ T.$$

Remember that, given a normed linear space V, we can see V inside of V'' by associating each $x \in V$ to ϕ_x since the canonical immersion is isometric monomorphism. Suppose that E is a reflexive space and let $\sigma \in E''$. Since J_E is surjective, then $\sigma = \phi_z$ for some $z \in E$, thus, making the identification commented above, we obtain

$$T''(z) = T''(\phi_z) = T''(J_E(z)) = (T'' \circ J_E)(z) = (J_F \circ T)(z) = J_F(T(z)) = \phi_{T(z)} = T(z),$$

where the last equality is because $\phi_{T(z)} \in J_F(F)$ and so we can associate it to $T(z) \in F$. Since $z \in E$ is arbitrary, we conclude that $T'' = T$. $\square$

Question 8: A convex combination of elements of a subset A of a linear space E is a element of form

$$\sum_{k=1}^{n} a_k v_k, \ n \in \mathbb{N}, \ a_k \geq 0, \ \forall\, k \in \{1, \ldots, n\}, \ \sum_{k=1}^{n} a_k = 1.$$

Consider $(x_n)_{n\in\mathbb{N}}$ a sequence in a Banach space such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to $x \in E$. Prove that there is a sequence $(y_n)_{n\in\mathbb{N}}$, where each y_n is a convex combination of elements of $X = \{x_n \ ; \ n \in \mathbb{N}\}$, such that $(y_n)_{n\in\mathbb{N}}$ converges strongly to x.

Solution: Before to solve this question, we need some definitions and lemmas.

Definition: Let E be a linear space and A be a subset of E. The convex hull of A, denoted by A_c, is the smallest convex set containing A, that is, if C is a convex subset of E containing A, then $A_c \subseteq C$.

Lemma 1: Let E be a linear space.

(i) Given $\mathcal{C} = \{C_\lambda\}_{\lambda\in L}$ a collection of convex subsets of E, then

$$\bigcap_{\lambda\in L} C_\lambda$$

is convex.

(ii) Given $A \subseteq E$ and $\mathcal{C} = \{C_\lambda\}_{\lambda\in L}$ the collection of all convex subsets of E containing A, then

$$\bigcap_{\lambda\in L} C_\lambda$$

is the convex hull of A. In particular, the convex hull of any set always exists.

Proof: (i): If $\bigcap_{\lambda\in L} C_\lambda = \varnothing$, then, by vacuously, $\bigcap_{\lambda\in L} C_\lambda$ is trivially a convex set. Suppose that $\bigcap_{\lambda\in L} C_\lambda \neq \varnothing$ and let x, $y \in \bigcap_{\lambda\in L} C_\lambda$. Thus x, $y \in C_\lambda$ for all $\lambda \in L$. Since each C_λ is a convex set, for all $t \in [0, 1]$, we have

$$tx + (1 - t)y \in C_\lambda.$$

Then

$$tx + (1-t)y \in \bigcap_{\lambda \in L} C_\lambda$$

for all $t \in [0,1]$. Since x, $y \in \bigcap_{\lambda \in L} C_\lambda$ are arbitrary elements, we conclude that $\bigcap_{\lambda \in L} C_\lambda$ is a convex subset.

(ii): Firstly note that $\mathcal{C} \neq \varnothing$ since $E \in \mathcal{C}$. As all element of $\mathcal{C}$ contains A, then it is clear that

$$A \subseteq \bigcap_{\lambda \in L} C_\lambda.$$

Note that $\bigcap_{\lambda \in L} C_\lambda$ is a convex subset of E containing A. On the other hand, if C is a convex subset of E containing A, then $C \in \mathcal{C}$, thus

$$\bigcap_{\lambda \in L} C_\lambda \subseteq C,$$

which implies that $\bigcap_{\lambda \in L} C_\lambda$ is the convex hull of A. □

Lemma 2: Let E be a linear space and A be a subset of E. The set

$$B = \left\{ \sum_{k=1}^{n} a_k v_k \in E \ ; \ n \in \mathbb{N}, \ v_k \in A, \ a_k \geq 0, \ \forall\, k \in \{1, \ldots, n\}, \ \sum_{k=1}^{n} a_k = 1. \right\}$$

is a convex set. Moreover, B is the convex hull of A.

Proof: B is a convex subset of A. In fact, given $x = \sum_{k=1}^{m} a_k v_k$ and $y = \sum_{k=1}^{n} b_k u_k$, I claim that, for all $t \in [0,1]$, we have that $tx + (1-t)y \in B$. Indeed, given $t \in [0,1]$, we have

$$tx + (1-t)y = t \sum_{k=1}^{m} a_k v_k + (1-t) \sum_{k=1}^{n} b_k u_k = \sum_{k=1}^{m} (ta_k) v_k + \sum_{k=1}^{n} [(1-t)b_k] u_k.$$

Since

$$\sum_{k=1}^{m} ta_k + \sum_{k=1}^{n} (1-t) b_k = t\left(\sum_{k=1}^{m} a_k \right) + (1-t)\left(\sum_{k=1}^{n} b_k \right) = t \cdot 1 + (1-t) \cdot 1 = 1$$

and $ta_i \geq 0$, $(1-t)b_j \geq 0$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, we conclude that B is a convex set.

In order to prove that B is the convex hull of A, it is enough to prove that $B \subseteq C$ for all convex subset C of E containing A, that is, it is enough to prove that C contains all convex combinations of elements of A. In order to prove it, I will proceed by induction on the number of elements N on the convex combination.

- If $N = 1$, the result is trivially true.

- If $N = 2$, given $x, y \in A$, it is clear that $ax + by \in C$, because, setting $a = t \in [0,1]$, so $b = 1 - a = 1 - t$ and we have that $tx + (1-t)y \in C$ since C is convex.

Finally, suppose that this result is true for $N = n$ and let $x_1, \ldots, x_n, x_{n+1} \in A$ and $a_1, \ldots, a_n, a_{n+1} \in [0, \infty)$ such that $\sum_{k=1}^{n+1} a_k = 1$. If $a_{n+1} = 1$, then $a_1 = \cdots = a_n = 0$, then

$$\sum_{k=1}^{n+1} a_k x_k = x_{n+1} \in C.$$

If not, $1 - a_{n+1} > 0$, then

$$\sum_{k=1}^{n+1} a_k x_k = \sum_{k=1}^{n} a_k x_k + a_{n+1} x_{n+1} = (1 - a_{n+1}) \left(\sum_{k=1}^{n} \frac{a_k}{1 - a_{n+1}} x_k \right) + a_{n+1} x_{n+1}$$

Now observe that

$$\sum_{k=1}^{n} \frac{a_k}{1 - a_{n+1}} = \frac{\sum_{k=1}^{n} a_k}{1 - a_{n+1}} = \frac{1 - a_{n+1}}{1 - a_{n+1}} = 1$$

Thus, by induction hypothesis, we have that

$$\sum_{k=1}^{n} \frac{a_k}{1 - a_{n+1}} x_k \in C.$$

Finally, by the case $N = 2$, we conclude that

$$\sum_{k=1}^{n+1} a_k x_k \in C.$$

$\square$

Now let's solve the wished question: Let X_c be the convex hull of X and $\overline{X_c}^w$ be the topological closure of X_c in the weak topology $\sigma(E, E')$. Since $(x_n)_{n \in \mathbb{N}}$ is a sequence in $X \subseteq X_c$ which converges weakly to x, then $x \in \overline{X_c}^w$. Since X_c is convex, by Mazur's Theorem, we have that

$$x \in \overline{X_c}^w = \overline{X_c}.$$

Since all elements of X_c are convex combinations of elements of X and $x \in \overline{X_c}$, we conclude that there is sequence $(y_n)_{n \in \mathbb{N}}$ in E such that each y_n is convex combination of elements of X and

$$\lim_{n \to \infty} y_n = x.$$

$\square$

Question 9: Let E be an infinite-dimensional normed linear space. Prove that every nonempty open subset in weak topology is unbounded. Conclude that, in infinite-dimensional normed linear spaces, the strong and weak topologies do not coincide.

Solution: Since every nonempty open set contains an open basis subset, it is enough to prove that every open basis subset is unbounded. Given $x_0 \in E$, $\phi_1, \dots, \phi_n \in E'$ and $\epsilon > 0$, let

$$V = V(x_0; \phi_1, \dots, \phi_n; \epsilon) = \{x \in E \ ; \ |\phi_i(x) - \phi_i(x_0)| < \epsilon, \ \forall \, i \in \{1, \dots, n\}\}$$

be an arbitrary open basis in E. I claim that there exists $y \neq 0$ in E such that $\phi_1(y) = \cdots = \phi_n(y) = 0$. In fact, suppose that it does not hold, thus the following linear mapping

$$\begin{aligned} \psi : E &\longrightarrow \mathbb{K}^n \\ x &\longmapsto (\phi_1(x), \dots, \phi_n(x)). \end{aligned}$$

is injective. However, this fact is impossible, because $\dim_{\mathbb{K}} E = \infty$, thus we can not inject E inside of $\mathbb{K}^n$. Then let $y \neq 0$ with this property and consider the family of vectors

$$\mathcal{F} := \{x_0 + ty \in E \ ; \ t \in \mathbb{R}\}.$$

Observe that $\mathcal{F} \subseteq V$, because, given $x_0 + ty \in \mathcal{F}$, we have that

$$|\phi_i(x_0 + ty) - \phi_i(x_0)| = |t\phi_i(y_0)| = |t \cdot 0| = 0 < \epsilon$$

for all $i \in \{1, \dots, n\}$ and $t \in \mathbb{R}$. Finally observe that

$$||z_0 + ty|| \geq |t|||y|| - ||x_0||,$$

thus, doing $t \to \infty$, we conclude that

$$\lim_{t \to \infty} ||z_0 + ty|| = \infty.$$

Then V is unbounded. Thus we conclude that every open subset is unbounded in an infinite-dimensional normed linear space equipped with weak topology. Since $B(0,1) = \{x \in E \ ; \ ||x|| < 1\}$ is a nonempty bounded open subset of E in the strong topology, we conclude that the strong and weak topologies do not coincide in infinite-dimensional normed linear spaces. □

Question 10: Solve the following questions.

(a) Consider the canonical vectors $e_n = (\delta_{ni})_{i \in \mathbb{N}}$ in ℓ_p, with $1 < p \leq \infty$. Show that $(e_n)_{n \in \mathbb{N}}$ converges weakly to 0.

(b) Show that, considering $(e_n)_{n \in \mathbb{N}}$ in ℓ_1, there is no subsequence of $(e_n)_{n \in \mathbb{N}}$ which converges weakly in the topology $\sigma(\ell_1, \ell_\infty)$.

(c) Construct a Banach Space E and a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded linear functionals in E with $\|f_n\| = 1$ for all $n \in \mathbb{N}$ such that $(f_n)_{n\in\mathbb{N}}$ has not any convergent subsequence in $\sigma(E', E)$. Is there contradiction with the compactness of $B_{E'}$ in weak* topology?

Solution: (a): Indeed, firstly suppose that $1 < p < \infty$. By duality relation, we know that the dual of ℓ_p is

$$(\ell_p)' = \ell_{p^*},$$

where p^* is the Hölder conjugate of p and we can establish the following duality relation: Given $\phi \in (\ell_p)'$, there is $(b_n)_{n\in\mathbb{N}} \in \ell_{p^*}$ such that, for all $(a_n)_{n\in\mathbb{N}} \in \ell_p$, we have

$$\phi((a_n)_{n\in\mathbb{N}}) = \sum_{k=1}^{\infty} b_k a_k.$$

With this fact in mind, let $\phi \in (\ell_p)'$ and let $(b_n)_{n\in\mathbb{N}} \in \ell_{p^*}$ its associated sequence. Given $n \in \mathbb{N}$, we have that

$$\phi(e_n) = \sum_{k=1}^{\infty} b_k \delta_{nk} = b_n.$$

Since the sequence $\sum_{k=1}^{\infty} |b_k|^{p^*}$ converges, it is clear that b_n converges to 0 as $n \to \infty$. Thus, considering the sequence $(e_n)_{n\in\mathbb{N}}$, we have that

$$\lim_{n\to\infty} \phi(e_n) = \lim_{n\to\infty} b_n = 0 = \phi(0),$$

which implies that $(e_n)_{n\in\mathbb{N}}$ converges weakly to 0 since $\phi \in E'$ is arbitrary. Now suppose that $p = \infty$. In order to prove it, we will use the following lemma.

Lemma: Given $\phi \in (\ell_\infty)'$, the series $\sum_{n=1}^{\infty} |\phi(e_n)|$ converges. Moreover, this sum is less than $\|\phi\|$.

With this result in mind, note that

$$\lim_{n\to\infty} \phi(e_n) = 0 = \phi(0)$$

for all $\phi \in (\ell_\infty)'$. So, by definition of weak topology, we conclude that $(e_n)_{n\in\mathbb{N}}$ converges weakly to 0.

(b): By duality relation, we know that the dual of ℓ_1 is

$$(\ell_1)' = \ell_\infty,$$

where we can establish the following duality relation: Given $\phi \in (\ell_1)'$, there is $(b_n)_{n\in\mathbb{N}} \in \ell_\infty$ such that, for all $(a_n)_{n\in\mathbb{N}} \in \ell_1$, we have

$$\phi((a_n)_{n\in\mathbb{N}}) = \sum_{k=1}^{\infty} b_k a_k.$$

With this fact in mind, let $(e_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence of $(e_n)_{n\in\mathbb{N}}$. Consider the bounded linear functional $\phi \in (\ell_\infty)'$ with associated sequence $(b_k)_{k\in\mathbb{N}} \in \ell^\infty$ such that

$$b_k = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

Thus

$$\phi(e_{n_k}) = \sum_{j=1}^{\infty} b_j \delta_{n_k j} = b_k$$

Then

$$\phi(e_{n_k}) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

Then the alternate sequence $(\phi(e_k))_{k\in\mathbb{N}}$ does not converge and then the subsequence $(e_{n_k})_{k\in\mathbb{N}}$ does not converge in the topology $\sigma(E, E')$.

(c): In fact, consider the Banach space $E = (\ell_\infty, ||.||_\infty)$ and the sequence of linear functionals $(f_n)_{n\in\mathbb{N}}$ in ℓ'_∞, where

$$f_n : \ell^\infty \longrightarrow \mathbb{K}$$

$$(x_k)_{k\in\mathbb{N}} \longmapsto x_n$$

It is clear that f_n is linear for each $n \in \mathbb{N}$. Moreover, since $||e_n||_\infty = 1$, $|f_n(e_n)| = 1$ and

$$||f_n|| = \sup_{||(a_k)_{k\in\mathbb{N}}||_\infty \le 1} |f_n((a_k)_{k\in\mathbb{N}})| = \sup_{||(a_k)_{k\in\mathbb{N}}||_\infty \le 1} |a_n| \le \sup_{||(a_k)_{k\in\mathbb{N}}||_\infty \le 1} ||(a_k)_{k\in\mathbb{N}}||_\infty = 1,$$

we conclude that $||f_n|| = 1$ for all $n \in \mathbb{N}$. Now let $(f_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(f_n)_{n\in\mathbb{N}}$. If $(f_{n_k})_{k\in\mathbb{N}}$ converges to some $f \in \ell'_\infty$ in $\sigma(\ell'_\infty, \ell_\infty)$, then, by definition of initial topology, we have that

$$\lim_{k\to\infty} f_{n_k}((a_k)_{k\in\mathbb{N}}) = f((a_k)_{k\in\mathbb{N}})$$

for all $(a_k)_{k\in\mathbb{N}} \in \ell_\infty$. However, if we consider the sequence $(c_n)_{n\in\mathbb{N}} \in \ell_\infty$ such that

$$c_n = \begin{cases} 1 & \text{if } n = n_k \text{ for some even } k \in \mathbb{N}; \\ -1 & \text{if } n = n_k \text{ for some odd } k \in \mathbb{N}; \\ 0 & \text{Otherwise.} \end{cases}$$

we have that

$$f((c_k)_{k\in\mathbb{N}}) = \lim_{k\to\infty} f_{n_k}((c_k)_{k\in\mathbb{N}}) = \lim_{k\to\infty} (-1)^k.$$

which clearly is an absurd, since the sequence $((-1)^n)_{n\in\mathbb{N}}$ does not converges. Thus $(f_n)_{n\in\mathbb{N}}$ does not admit convergent subsequence. In particular, $B_{\ell'_\infty}$ is compact subset of $(\ell'_\infty, \sigma(\ell'_\infty, \ell_\infty))$ (Banach-Alaoglu-Bourbaki Theorem) which is not sequentially compact. This fact does not contradict the compactness of $B_{\ell'_\infty}$, because the notions of compactness and sequentially compactness does not always coincide. In particular, we conclude that the weak* topology is not metrizable in general.

2.3 Hilbert Spaces and General Aspects of Spectral Theory:

Question 1: Let H be a Hilbert space and $(x_n)_{n\in\mathbb{N}}$ be an orthonormal sequence. Prove that $(x_n)_{n\in\mathbb{N}}$ converges weakly to 0 in $\sigma(H, H')$

Proof: It is enough to prove that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(0) = 0$ for every $f \in H'$. Since H is a Hilbert space, by Riesz-Fréchet Theorem, it is enough to prove that

$$\lim_{n\to\infty} \langle x_n, y\rangle = 0.$$

for every $y \in H$. Well, since $S = \{x_n \ ; \ n \in \mathbb{N}\}$ is an orthonormal set, by Bessel inequality, we get

$$S_y = \sum_{n=1}^{\infty} |\langle x_n, y\rangle|^2 \leq \|y\|^2 < \infty.$$

for every $y \in H$. Thus, since the series S_y converges, we conclude that its general term $|\langle x_n, y\rangle|^2$ converges to 0, so we conclude that the sequence $(\langle x_n, y\rangle)_{n\in\mathbb{N}}$ converges to 0 for every $n \in \mathbb{N}$. □

Question 2: Let H be a Hilbert space and $T : H \longrightarrow H$ be a linear mapping such that $\langle T(x), y\rangle = \langle x, T(y)\rangle$ for all x, $y \in H$. Prove that T is continuous.

Proof: In fact, as H is an Hilbert space and T is a linear operator, we are in condition to use the Closed Graph Theorem. So it is enough to show that $\mathrm{gr}(T)$ is closed. Let $(x, y) \in \overline{\mathrm{gr}(T)}$, so there is a sequence $\{(x_n, T(x_n))\}_{n\in\mathbb{N}}$ in $\mathrm{gr}(T)$ such that

$$\lim_{n\to\infty} x_n = x \qquad \text{and} \qquad \lim_{n\to\infty} T(x_n) = y$$

in strong topology. In order to prove that $\mathrm{gr}(T)$ is closed, we have to prove that $(x, y) \in \mathrm{gr}(T)$, that is, $y = T(x)$. However, it is equivalent to prove that $\langle y, z\rangle = \langle T(x), z\rangle$ for all $z \in H$. Well, using the fact that inner dot is separately continuous and that $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in H$, we conclude that

$$\langle y, z\rangle = \left\langle \lim_{n\to\infty} T(x_n), z\right\rangle = \lim_{n\to\infty} \langle T(x_n), z\rangle = \lim_{n\to\infty} \langle x_n, T(z)\rangle = \langle x, T(z)\rangle = \langle T(x), z\rangle.$$

Since $z \in H$ was chosen arbitrarily, we conclude that $y = T(z)$, so $\mathrm{gr}(T)$ is closed and, by Closed Graph Theorem, we conclude that T is continuous. □

Question 3: Let H be a Hilbert space, F be a subspace of H and $\phi \in F'$. Prove that there is an unique $\overline{\phi} \in H'$, which extends ϕ and $\|\phi\| = \|\overline{\phi}\|$.

Proof: **Lemma:** Let E be a normed linear space, F be a Banach space and $T : D(T) \subseteq E \longrightarrow F$

be a bounded linear operator. Show that T admits a bounded linear extension $\hat{T}$, whose domain is $\overline{D(T)}$ and $\|\hat{T}\| = \|T\|$.

Proof: Consult the Question 3 of List I. Now, let's get to the problem in question. By the Lemma, we can extend ϕ to $\hat{\phi} : \overline{F} \longrightarrow \mathbb{K}$ such that $\|\hat{\phi}\| = \|\phi\|$. Since H is a Hilbert space and $\overline{F}$ is closed in H, then $\overline{F}$ is also a Hilbert space with the induced inner dot. By Riesz-Fréchet Theorem, there is $y \in \overline{F}$ such that

$$\begin{aligned} \hat{\phi} : \overline{F} &\longrightarrow \mathbb{K} \\ x &\longmapsto \langle x, y \rangle. \end{aligned}$$

So define

$$\begin{aligned} \overline{\phi} : H &\longrightarrow \mathbb{K} \\ x &\longmapsto \langle x, y \rangle. \end{aligned}$$

Clearly we have that $\overline{\phi}$ extends $\hat{\phi}$, whose extends ϕ. Moreover, by Riesz-Fréchet Theorem, we also have that

$$\|\overline{\phi}\|_{H'} = \|y\|_H = \|\hat{\phi}\|_{\overline{F}'} = \|\phi\|_{F'}.$$

Note that the y chosen above is an element of $\overline{F}$. Now let $\psi : H \longrightarrow \mathbb{K}$ be another continuous linear extension of ϕ preserving the norm. By Riesz-Fréchet Theorem, we have that there is $z \in \mathbb{K}$ such that $\psi(\cdot) = \langle \cdot, z \rangle$. Now note that $\overline{\phi}|_{\overline{F}} - \psi|_{\overline{F}} = 0$. This implies that

$$\langle x, y - z \rangle = 0 \qquad \text{for all } x \in \overline{F}$$

This implies that $y - z \in \overline{F}^{\perp}$. Now, using the Pithagoras Theorem, we conclude that

$$\|y\|_H = \|y\|_{\overline{F}} = \|\phi\|_{F'} = \|\psi\|_H = \|z\|_H = \|(z - y) + y\|_H = \sqrt{\|z - y\|_H^2 + \|y\|_H^2}.$$

Then

$$\|y\|_H^2 = \|z - y\|_H^2 + \|y\|_H^2,$$

which implies that $\|z - y\|_H^2 = 0$ and so $z = y$, that allows us to conclude the uniqueness of extension. $\square$

Question 4: Let I be any set and $1 \leq p \leq \infty$. Define $\mathcal{F}$ the collection of all finite subsets of I. Given a collection of scalars $(a_j)_{j \in I}$, we write

$$\|(a_j)_{j\in I}\|_p = \sup_{A \in \mathcal{F}} \left(\sum_{j \in A} |a_j|^p \right)^{1/p} \quad \text{if } 1 \leq p < \infty \quad \text{and} \quad \|(a_j)_{j\in I}\|_\infty = \sup_{i \in I} |a_i|$$

(a) Prove that

$$\ell_p(I) = \{(a_j)_{j\in I} \ ; \ \|(a_j)_{j\in I}\|_p < \infty\}$$

is a normed linear space, whose norm is $\|\cdot\|_p$.

(b) Prove that $(\ell_p(I), \|\cdot\|_p)$ is a Banach space.

(c) Prove that, for $1 \leq p < \infty$, $(\ell_p)'$ is isometrically isomorphic to $\ell_{p'}(I)$, where

$$\frac{1}{p} + \frac{1}{p'} = 1$$

and $1' = \infty$.

(d) Let $S = \{x_i \ ; \ i \in I\}$ be a complete orthonormal system in a Hilbert space H. Prove that H is isometrically isomorphic to $\ell_p(I)$.

(e) Prove that every Hilbert space is isometrically isomorphic to its dual.

Proof: (a): **Case** $p = \infty$**:** In order to show that $\ell_\infty(I)$ is a linear space, it is enough to show that $\ell_\infty(I)$ is a linear subspace of space of sequences of scalars indexed in I. Indeed, given $(a_j)_{j\in I}$, $(b_j)_{j\in I} \in \ell_\infty(I)$ and $\lambda \in \mathbb{K}$, we have that

$$\left\|(a_j)_{j\in I} + \lambda(b_j)_{j\in I}\right\|_\infty = \left\|(a_j + \lambda b_j)_{j\in I}\right\|_\infty = \sup_{j\in I}|a_j + \lambda b_j| \leq \sup_{j\in I}\left(|a_j| + |\lambda||b_j|\right) \leq$$

$$\sup_{j\in I}|a_j| + |\lambda|\sup_{j\in I}|b_j| = \left\|(a_j)_{j\in I}\right\|_\infty + |\lambda|\cdot\left\|(b_j)_{j\in I}\right\|_\infty < \infty.$$

Thus $(a_j)_{j\in I} + \lambda(b_j)_{j\in I} \in \ell_\infty(I)$ and so $\ell_\infty(I)$ is a linear subspace of space of sequences of scalars indexed in I. Now we will show that $(\ell_\infty(I), \|\cdot\|_\infty)$ is a normed linear space. Thus we have to check the norm's axioms

- Given $(a_j)_{j\in I} \in \ell_\infty(I)$, we have that $\|(a_j)_{j\in I}\|_\infty = \sup_{j\in I}|a_j| \geq 0$.
- Given $(a_j)_{j\in I} \in \ell_\infty(I)$, if $\|(a_j)_{j\in I}\|_\infty = \sup_{i\in I}|a_i| = 0$, then $|a_i| = 0$ for all $i \in I$, thus $a_i = 0$ for all $i \in I$, then $(x_n)_{n\in\mathbb{N}} = 0$.
- Given $(a_i)_{i\in I} \in \ell_\infty(I)$ and $\lambda \in \mathbb{K}$, we have that

$$\|\lambda(a_j)_{j\in I}\|_\infty = \|(\lambda a_j)_{j\in I}\|_\infty = \sup_{i\in I}|\lambda a_i| = |\lambda|\cdot\sup_{i\in I}|a_i| = |\lambda|\cdot\|(a_j)_{j\in I}\|_\infty.$$

- Given $(a_i)_{i\in I}$, $(b_i)_{i\in I} \in \ell_\infty(I)$, we have

$$\left\|(a_i)_{i\in I} + (b_i)_{i\in I}\right\|_\infty = \left\|(a_i + b_i)_{i\in I}\right\|_\infty = \sup_{i\in I}|a_i + b_i| \leq \sup_{i\in I}\left(|a_i| + |b_i|\right) \leq \sup_{i\in I}|a_i| + \sup_{i\in I}|b_i| =$$

$$\left\|(a_i)_{i\in I}\right\|_\infty + \left\|(b_i)_{i\in I}\right\|_\infty.$$

Thus $(\ell_\infty(I), \|\cdot\|_\infty)$ is a normed linear space.

Case $1 \le p < \infty$: In order to show that $\ell_p(I)$ is a linear space, it is enough to show that $\ell_p(I)$ is a linear subspace of space of sequences of scalars indexed in I. Indeed, given $(a_j)_{j\in I}$, $(b_j)_{j\in I} \in \ell_p(I)$, $\lambda \in \mathbb{K}$ and $A \in \mathcal{F}$, we have that

$$\left(\sum_{j\in A} |a_j|^p\right)^{1/p} \le \|(a_j)_{j\in I}\|_p \quad \text{and} \quad \left(\sum_{j\in A} |\lambda b_j|^p\right)^{1/p} \le \|(\lambda b_j)_{j\in I}\|_p = |\lambda| \|(b_j)_{j\in I}\|_p.$$

Using the Minkowsky inequality we conclude that

$$\left(\sum_{j\in A} |a_j + \lambda b_j|^p\right)^{1/p} \le \left(\sum_{j\in A} |a_j|^p\right)^{1/p} + \left(\sum_{j\in A} |\lambda b_j|^p\right)^{1/p} \le \|(a_j)_{j\in I}\|_p + |\lambda| \|(b_j)_{j\in I}\|_p.$$

Since $A \in \mathcal{F}$ is arbitrary, we conclude that

$$\|(a_j)_{j\in I} + \lambda (b_j)_{j\in I}\|_p = \sup_{A\in\mathcal{F}} \left(\sum_{j\in A} |a_j + \lambda b_j|^p\right)^{1/p} \le \|(a_j)_{j\in I}\|_p + |\lambda| \|(b_j)_{j\in I}\|_p < \infty.$$

Thus $(a_j)_{j\in I} + \lambda (b_j)_{j\in I} \in \ell_p(I)$ and so $\ell_p(I)$ is a linear subspace of space of sequences of scalars indexed in I. Now we will show that $(\ell_p(I), \|\cdot\|_p)$ is a normed linear space. Thus we have to check the norm's axioms

- Given $(a_j)_{j\in I} \in \ell_p(I)$ and $A \in \mathcal{F}$, we have that

  $$\left(\sum_{j\in A} |a_j|^p\right)^{1/p} \ge 0.$$

  Thus

  $$\|(a_j)_{j\in I}\|_p = \sup_{A\in\mathcal{F}} \left(\sum_{j\in A} |a_j|^p\right)^{1/p} \ge 0.$$

- Given $(a_j)_{j\in I} \in \ell_p(I)$, if $(a_j)_{j\in I} = 0$ in $\ell_p(I)$, then for all $A \in \mathcal{F}$, we have that

  $$\left(\sum_{j\in A} |a_j|^p\right)^{1/p} = 0.$$

  Thus

  $$\|(a_j)_{j\in I}\|_p = \sup_{A\in\mathcal{F}} \left(\sum_{j\in A} |a_j|^p\right)^{1/p} = 0.$$

  On the other hand, if $(a_j)_{j\in I} \ne 0$ in $\ell_p(I)$, there are $j_0 \in I$ such that $a_{j_0} \ne 0$, thus, considering $A_0 = \{j_0\}$, we get

  $$\left(\sum_{j\in A_0} |a_j|^p\right)^{1/p} = |a_{j_0}| > 0.$$

Thus

$$\|(a_j)_{j\in I}\|_p = \sup_{A\in\mathcal{F}}\left(\sum_{j\in A}|a_j|^p\right)^{1/p} \geq |a_{j_0}| > 0.$$

- Given $(a_j)_{j\in I} \in \ell_p(I)$, $\lambda \in \mathbb{K}$ and $A \in \mathcal{F}$, we have that

$$\left(\sum_{j\in A}|\lambda a_j|^p\right)^{1/p} = \left(\sum_{j\in A}|\lambda|^p|a_j|^p\right)^{1/p} = |\lambda|\left(\sum_{j\in A}|a_j|^p\right)^{1/p}.$$

Thus

$$\|\lambda(a_j)_{j\in I}\|_p = \|(\lambda a_j)_{j\in I}\|_p = \sup_{A\in\mathcal{F}}\left(\sum_{j\in A}|\lambda a_j|^p\right)^{1/p} = \sup_{A\in\mathcal{F}}\left[|\lambda|\left(\sum_{j\in A}|a_j|^p\right)^{1/p}\right] = |\lambda|\sup_{A\in\mathcal{F}}\left(\sum_{j\in A}|a_j|^p\right)^{1/p} =$$

$$|\lambda|\|(a_j)_{j\in I}\|_p.$$

- Given $(a_j)_{j\in I}$ and $(b_j)_{j\in I} \in \ell_p(I)$, by Minkowsky inequality, for all $A \in \mathcal{F}$, we have that

$$\left(\sum_{j\in A}|a_j+b_j|^p\right)^{1/p} \leq \left(\sum_{j\in A}|a_j|^p\right)^{1/p} + \left(\sum_{j\in A}|b_j|^p\right)^{1/p}.$$

Thus

$$\|(a_j)_{j\in I} + (b_j)_{j\in I}\|_p = \|(a_j+b_j)_{j\in I}\|_p = \sup_{A\in\mathcal{F}}\left(\sum_{j\in A}|a_j+b_j|^p\right)^{1/p} \leq$$

$$\sup_{A\in\mathcal{F}}\left[\left(\sum_{j\in A}|a_j|^p\right)^{1/p} + \left[\left(\sum_{j\in A}|b_j|^p\right)^{1/p}\right] \leq \sup_{A,B\in\mathcal{F}}\left[\left(\sum_{j\in A}|a_j|^p\right)^{1/p} + \left(\sum_{j\in B}|b_j|^p\right)^{1/p}\right] =$$

$$\sup_{A\in\mathcal{F}}\left(\sum_{j\in A}|a_j|^p\right)^{1/p} + \sup_{B\in\mathcal{F}}\left(\sum_{j\in B}|b_j|^p\right)^{1/p} = \|(a_j)_{j\in I}\|_p + \|(b_j)_{j\in I}\|_p.$$

Thus $(\ell_p(I), \|\cdot\|_p)$ is a normed linear space.

(b): **Case** $p = \infty$**:** In order to show that $(\ell_\infty(I), \|\cdot\|_\infty)$ is a Banach space, it is enough to show that every Cauchy sequence in $(\ell_\infty(I), \|\cdot\|_\infty)$ is convergent. Considering $(x_i)_{i\in I}$ as a function $f \in \mathbb{K}^I$, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\ell_\infty(I), \|\cdot\|_\infty)$. Thus, given $\epsilon > 0$, there $N \in \mathbb{N}$ such that for every n, $m > N$ and $i \in I$, we have

$$|f_n(i) - f_m(i)| \leq \|f_n - f_m\|_\infty < \epsilon.$$

Since $\mathbb{K}$ is complete, the sequence $(f_n(i))_{n\in\mathbb{N}}$ converges to some $y := f(i) \in \mathbb{K}$. I claim that $f : I \longrightarrow \mathbb{K}$ is an element of $(\ell_\infty(I), \|\cdot\|_\infty)$ and that

$$\lim_{n\to\infty} f_n = f.$$

Firstly, note that, since, for every $n, m > N$ and $i \in I$, we have that $|f_n(i) - f_m(i)| \leq \|f_n - f_m\|_\infty < \epsilon$. Doing $m \to \infty$, we have

$$|f_n(i) - f(i)| \leq \|f_n - f_m\|_\infty < \epsilon$$

for all $i \in I$. Thus for some $n > N$, we have

$$|f(i)| \leq |f(i) - f_n(i) + f_n(i)| \leq |f(i) - f_n(i)| + |f_n(i)| \leq \|f_n\|_\infty + \epsilon$$

for all $i \in I$, which implies that $f \in \ell_\infty(I)$. Similarly, on inequality $|f_n(i) - f_m(i)| \leq \|f_n - f_m\|_\infty < \epsilon$, doing $m \to \infty$, we obtain that

$$|f_n(i) - f(i)| \leq \|f_n - f_m\|_\infty < \epsilon$$

for all $i \in I$ and $n > N$, which implies that $\|f_n - f\|_\infty < \epsilon$ for every $n > N$. Thus f_n converges to f and $(\ell_\infty(I), \|\cdot\|_\infty)$ is a Banach space.

Case $1 \leq p < \infty$**:** In order to show that $(\ell_p(I), \|\cdot\|_p)$ is a Banach space, it is enough to show that every Cauchy sequence in $(\ell_p(I), \|\cdot\|_p)$ is convergent. Considering $(x_i)_{i\in I}$ as a function $f \in \mathbb{K}^I$, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\ell_p(I), \|\cdot\|_p)$. Thus, given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m < n \in \mathbb{N}$, we have that

$$\|f_n - f_m\|_p < \epsilon.$$

In particular, given $j \in I$ and considering the set $A_j = \{j\} \in \mathcal{F}$, we have that for all $n > m \in \mathbb{N}$, we have

$$|f_n(j) - f_m(j)| \leq \sup_{A\in\mathcal{F}} \left(\sum_{i\in A} |f_n(i) - f_m(i)|^p \right)^{1/p} = \|f_n - f_m\|_p < \epsilon.$$

Since $\mathbb{K}$ is complete, the sequence $(f_n(j))_{n\in\mathbb{N}}$ converges to some $y := f(j) \in \mathbb{K}$. I claim that $f : I \longrightarrow \mathbb{K}$ is an element of $(\ell_p(I), \|\cdot\|_p)$ and that

$$\lim_{n\to\infty} f_n = f.$$

Indeed, given $A \in \mathcal{F}$ and $m > N$ fixed, by Minkowsky inequality, we have

$$\left(\sum_{i\in A} |f(i)|^p\right)^{1/p} = \left(\sum_{i\in A} |f(i) - f_m(i) + f_m(i)|^p\right)^{1/p} \leq \left(\sum_{i\in A} |f(i) - f_m(i)|^p\right)^{1/p} + \left(\sum_{i\in A} |f_m(i)|^p\right)^{1/p} \leq$$

$$\epsilon + \|f_m\|_p < \infty.$$

Thus, since $A \in \mathcal{F}$ is arbitrary, we conclude

$$\|f\|_p = \sup_{A\in\mathcal{F}} \left(\sum_{i\in A} |f(i)|^p\right)^{1/p} \leq \epsilon + \|f_m\|_p < \infty$$

and thus $f \in (\ell_p(I), \|\cdot\|_p)$. Now, we are going to prove that $f_n \to f$. In fact, given $A \in \mathcal{F}$, we have that

$$\left(\sum_{i\in A} |f_n(i) - f_m(i)|^p\right)^{1/p} \leq \|f_n - f_m\|_p < \epsilon.$$

Doing $n \to \infty$, for all $m > N$ we have that

$$\left(\sum_{i\in A} |f(i) - f_m(i)|^p\right)^{1/p} < \epsilon.$$

Thus, since $A \in \mathcal{F}$ is arbitrary, we conclude

$$\|f_n - f\|_p = \sup_{A\in\mathcal{F}} \left(\sum_{i\in A} |f(i) - f_m(i)|^p\right)^{1/p} \leq \epsilon$$

for all $m > N$, which implies that f_n converges to f.

(c): **Case $p' = \infty$:** Define the following mapping

$$\begin{aligned} \psi : \ell_\infty(I) &\longrightarrow (\ell_1(I))' \\ (a_i)_{i\in I} &\longmapsto \phi_{(a_i)_{i\in I}} : \ell_1(I) \longrightarrow \mathbb{K} \\ &\qquad (b_i)_{i\in I} \longmapsto \sup_{A\in\mathcal{F}} \left(\textstyle\sum_{i\in A} a_i b_i\right) \end{aligned}$$

Note that ψ is well-defined, because given $(a_i)_{i\in I} \in \ell_\infty$, then $\phi_{(a_i)_{i\in I}} \in (\ell_1(I))'$. Indeed, given $(b_i)_{i\in I} \in \ell_1(I)$ and $A \in \mathcal{F}$, we have

$$\left|\sum_{i\in A} a_i b_i\right| \leq \sum_{i\in A} |a_i||b_i| \leq \sum_{i\in A} \|(a_i)_{i\in I}\|_\infty |b_i| \leq \|(a_i)_{i\in I}\|_\infty \cdot \sum_{i\in A} |b_i| \leq \|(a_i)_{i\in I}\|_\infty \|(b_i)_{i\in I}\|_1 < \infty$$

Thus, since $A \in \mathcal{F}$ is arbitrary, we have

$$\left|\phi_{(a_i)_{i\in I}}((b_i)_{i\in I})\right| = \left|\sup_{A\in\mathcal{F}}\left(\sum_{i\in A} a_i b_i\right)\right| \le \sup_{A\in\mathcal{F}}\left|\sum_{i\in A} a_i b_i\right| \le \|(a_i)_{i\in I}\|_\infty \|(b_i)_{i\in I}\|_1 < \infty. \tag{2.1}$$

Moreover $\phi_{(a_i)_{i\in I}}$ is a continuous functional mapping. Indeed, let $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \ell_1(I)$ and $\lambda \in \mathbb{K}$, then

$$\phi_{(a_i)_{i\in I}}((b_i)_{i\in I} + \lambda(c_i)_{i\in I}) = \phi_{(a_i)_{i\in I}}((b_i + \lambda c_i)_{i\in I}) = \sup_{A\in\mathcal{F}}\left(\sum_{i\in A} a_i(b_i + \lambda c_i)\right) = \sup_{A\in\mathcal{F}}\left(\sum_{i\in A}(a_i b_i + \lambda a_i c_i)\right) =$$
$$\sup_{A\in\mathcal{F}}\left(\sum_{i\in A} a_i b_i\right) + \lambda \sup_{A\in\mathcal{F}}\left(\sum_{i\in A}(a_i c_i)\right) = \phi_{(a_i)_{i\in I}}((b_i)_{i\in I}) + \lambda\phi_{(a_i)_{i\in I}}((c_i)_{i\in I}).$$

Thus $\phi_{(a_i)_{i\in I}}$ is a linear mapping. Since the inequality (1) holds, we conclude that $\phi_{(a_i)_{i\in I}}$ is a continuous linear functional and so ψ is well-defined. Now we will prove that ψ is a linear mapping. In fact, given $(a_j)_{j\in I}$ and $(\alpha_j)_{j\in I}$ in $\ell_\infty(I)$ and $\lambda \in \mathbb{K}$, we have that

$$\psi((a_j)_{j\in I} + \lambda(\alpha_j)_{j\in I}) = \psi((a_j + \lambda\alpha)_{j\in I}) = \phi_{(a_j+\lambda\alpha_j)_{j\in I}} = \phi_{(a_j)_{j\in I}+\lambda(\alpha_j)_{j\in I}} = \phi_{(a_j)_{j\in I}} + \lambda\phi_{(\alpha_j)_{j\in I}} =$$
$$\psi((a_j)_{j\in I}) + \lambda\psi((\alpha_j)_{j\in I}).$$

given $(a_j)_{j\in I} \in \ell_\infty$ and $(b_i)_{i\in I} \in \ell_1$, by inequality (1), we have

$$\|\psi((a_i)_{i\in I})\|_{(\ell_1(I))'} = \|\phi_{(a_j)_{j\in I}}\|_{(\ell_1(I))'} = \sup_{\|(b_i)_{i\in I}\|\le 1} |\phi_{(a_j)_{j\in I}}((b_i)_{i\in I})| \le \sup_{\|(b_i)_{i\in I}\|\le 1} \left|\|(a_i)_{i\in I}\|_\infty \|(b_i)_{i\in I}\|_1\right| \le$$
$$\sup_{\|(b_i)_{i\in I}\|\le 1} \left|\|(a_i)_{i\in I}\|_\infty\right| = \|(a_i)_{i\in I}\|_\infty.$$

Thus ψ is continuous. We can prove that ψ is an isometry and surjective, so ψ is an isometric isomorphism.

Case $1 \le p' < \infty$**:** Define the following mapping

$$\begin{aligned}
\psi : \ell_{p'}(I) &\longrightarrow (\ell_p(I))' \\
(a_i)_{i\in I} &\longmapsto \phi_{(a_i)_{i\in I}} : \ell_p(I) \longrightarrow \mathbb{K} \\
&\qquad\qquad (b_i)_{i\in I} \longmapsto \sup_{A\in\mathcal{F}}\left(\textstyle\sum_{i\in A} a_i b_i\right)
\end{aligned}$$

Note that ψ is well-defined, because given $(a_i)_{i\in I} \in \ell_{p'}$, then $\phi_{(a_i)_{i\in I}} \in (\ell_p(I))'$. Indeed, given $(b_i)_{i\in I} \in \ell_p(I)$ and $A \in \mathcal{F}$, by Hölder inequality

$$\left|\sum_{i\in A} a_i b_i\right| \le \sum_{i\in A} |a_i b_i| \le \left(\sum_{i\in A} |a_i|^{p'}\right)^{\frac{1}{p'}} \left(\sum_{i\in A} |b_i|^p\right)^{\frac{1}{p}} \le \|(a_i)_{i\in I}\|_{p'} \|(b_i)_{i\in I}\|_p$$

Thus, since $A \in \mathcal{F}$ is arbitrary, we have

$$\left|\phi_{(a_i)_{i\in I}}((b_i)_{i\in I})\right| = \left|\sup_{A\in\mathcal{F}}\left(\sum_{i\in A} a_i b_i\right)\right| \le \sup_{A\in\mathcal{F}}\left|\sum_{i\in A} a_i b_i\right| \le \|(a_i)_{i\in I}\|_{p'}\|(b_i)_{i\in I}\|_p < \infty. \tag{2.2}$$

Moreover $\phi_{(a_i)_{i\in I}}$ is a continuous functional mapping. Indeed, let $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \ell_p(I)$ and $\lambda \in \mathbb{K}$, then

$$\phi_{(a_i)_{i\in I}}((b_i)_{i\in I} + \lambda(c_i)_{i\in I}) = \phi_{(a_i)_{i\in I}}((b_i + \lambda c_i)_{i\in I}) = \sup_{A\in\mathcal{F}}\left(\sum_{i\in A} a_i(b_i + \lambda c_i)\right) = \sup_{A\in\mathcal{F}}\left(\sum_{i\in A}(a_i b_i + \lambda a_i c_i)\right) =$$

$$\sup_{A\in\mathcal{F}}\left(\sum_{i\in A} a_i b_i\right) + \lambda\sup_{A\in\mathcal{F}}\left(\sum_{i\in A}(a_i c_i)\right) = \phi_{(a_i)_{i\in I}}((b_i)_{i\in I}) + \lambda\phi_{(a_i)_{i\in I}}((c_i)_{i\in I}).$$

Thus $\phi_{(a_i)_{i\in I}}$ is a linear functional. Since the inequality (2) holds, we conclude that $\phi_{(a_i)_{i\in I}}$ is a continuous linear functional and so ψ is well-defined. Now we will prove that ψ is a linear mapping. In fact, given $(a_j)_{j\in I}$ and $(\alpha_j)_{j\in I}$ in $\ell_{p'}(I)$ and $\lambda \in \mathbb{K}$, we have that

$$\psi((a_j)_{j\in I} + \lambda(\alpha_j)_{j\in I}) = \psi((a_j + \lambda\alpha_j)_{j\in I}) = \phi_{(a_j+\lambda\alpha_j)_{j\in I}} = \phi_{(a_j)_{j\in I}+\lambda(\alpha_j)_{j\in I}} = \phi_{(a_j)_{j\in I}} + \lambda\phi_{(\alpha_j)_{j\in I}} =$$

$$\psi((a_j)_{j\in I}) + \lambda\psi((\alpha_j)_{j\in I}).$$

Given $(a_j)_{j\in I} \in \ell_{p'}(I)$ and $(b_j)_{j\in I} \in \ell_p(I)$, by inequality (2), we have

$$\|\psi((a_i)_{i\in I})\|_{(\ell_p(I))'} = \|\phi_{(a_j)_{j\in I}}\|_{(\ell_p(I))'} = \sup_{\|(b_i)_{i\in I}\|\le 1}|\phi_{(a_j)_{j\in I}}((b_i)_{i\in I})| \le \sup_{\|(b_i)_{i\in I}\|\le 1}\left|\|(a_i)_{i\in I}\|_{p'}\|(b_i)_{i\in I}\|_p\right| \le$$

$$\sup_{\|(b_i)_{i\in I}\|\le 1}\left|\|(a_i)_{i\in I}\|_{p'}\right| = \|(a_i)_{i\in I}\|_{p'}.$$

Thus ψ is continuous. We also can prove that ψ is an isometry and surjective, so ψ is an isometric isomorphism.

(d): Define the following mapping

$$\phi : H \longrightarrow \ell_2(I)$$

$$x \longmapsto (\langle x, x_i\rangle)_{i\in I}$$

Observe that ϕ is a linear mapping. In fact, given x, $y \in H$ and $\lambda \in \mathbb{K}$, we have that

$$\phi(x + \lambda y) = (\langle x + \lambda y, x_i\rangle)_{i\in I} = (\langle x, x_i\rangle + \lambda\langle y, x_i\rangle)_{i\in I} = (\langle x, x_i\rangle)_{i\in I} + \lambda(\langle y, x_i\rangle)_{i\in I} = \phi(x) + \lambda\phi(y).$$

Thus ϕ is a linear mapping. Moreover, note that ϕ is an isometry. In fact, given $x \in H$, we have

$$\|\phi(x)\|_2^2 = \sup_{A\in\mathcal{F}}\sum_{j\in A}|\langle x, x_i\rangle|^2.$$

However, we already know that $K = \{i \in I\ ;\ \langle x, e_i \rangle \neq 0\}$ is at most countable, which implies that, if

$$K := \bigcup_{n=1}^{\infty} \{i_1, \dots, i_n\} := \bigcup_{n=1}^{\infty} K_n.$$

Then

$$\sup_{A \in \mathcal{F}} \sum_{j \in A} |\langle x, x_i \rangle|^2 = \sup_{A \in \mathcal{F}, A \subseteq K} \sum_{j \in A} |\langle x, x_i \rangle|^2 = \lim_{n \to \infty} \sum_{j=1}^{n} |\langle x, x_{i_k} \rangle|^2 = \sum_{i \in I} |\langle x, x_{i_k} \rangle|^2 = \|x\|_2^2.$$

where the last equality is true because S is a complete orthonormal system. Thus, given $x \in H$, we have

$$\|\phi(x)\|_2^2 = \sup_{A \in \mathcal{F}} \sum_{j \in A} |\langle x, x_i \rangle|^2 = \sum_{i \in I} |\langle x, x_{i_k} \rangle|^2 = \|x\|_2^2,$$

which implies that $\|\phi(x)\|_2 = \|x\|_2$ for all $x \in H$ Thus ϕ is continuous, is injective and is an isometry. It remains to prove that ϕ is surjective. Let $(a_i)_{i \in I} \in \ell_2(I)$ with $\|(a_i)_{i \in I}\|_2 = L$. I claim that $B = \{i \in I\ ;\ a_i \neq 0\}$ is at most countable. Indeed, note that

$$B = \bigcup_{n=1}^{\infty} B_n,$$

where $B_n = \{i \in I\ ;\ |a_i| \geq 1/n\}$. Suppose that B_n contains a finite subset J of cardinality m, thus we have that

$$L = \|(a_i)_{i \in I}\|_2 \geq \sum_{j \in J} |a_j|^2 \geq m/n^2,$$

which implies that $m \leq n^2 L$. Thus B_n is finite for all $n \in \mathbb{N}$, which implies that B is at most countable. Let $\{i_n\ ;\ n \in \mathbb{N}\}$ be an enumeration of B and consider Let

$$x = \sum_{i \in I} a_i x_i = \sum_{i \in B} a_i x_i = \sum_{n=1}^{\infty} a_{i_n} x_{i_n}.$$

Note that $x \in H$. In fact, consider the sequence $(s_n)_{n \in \mathbb{N}}$ of partial sums, that is

$$s_m = \sum_{n=1}^{m} a_{i_n} x_{i_n}$$

Since H is complete space, it is enough to show that $(s_n)_{n \in \mathbb{N}}$ is Cauchy. However it is easy, because for every $n > m$, we have

$$\|s_n - s_m\|^2 = \left\| \sum_{k=m+1}^{n} a_{i_k} x_{i_k} \right\|^2 = \sum_{k=m+1}^{n} |a_{i_k}|^2 \to 0 \quad \text{as } m \to \infty$$

since $\|(a_i)_{i \in I}\|_2 < \infty$. Thus $x \in H$. So we have that $\phi(x) = (a_i)_{i \in I}$ and thus we conclude that ϕ is surjective. Then H is isometrically isomorphic to $\ell_2(I)$.

(e): Since $(\ell_2(I))'$ is isometrically isomorphic to $\ell_2(I)$, then, given a Hilbert Space H with complete orthonormal system $S = \{x_i \ ; \ i \in I\}$, we have that

$$H' \cong (\ell_2(I))' \cong \ell_2(I) \cong H.$$

Since each isomorphism is isometric, we conclude that H' is isometrically isomorphic to H. $\square$

Question 5: Let $T : \ell_2 \longrightarrow \ell_2$ be the operator defined by

$$\begin{array}{rccc} T: & \ell_2 & \longrightarrow & \ell_2 \\ & (a_n)_{n\in\mathbb{N}} & \longmapsto & (\frac{a_n}{2^n})_{n\in\mathbb{N}} \end{array}$$

Show that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{K}$ such that $|\lambda| > 1/2$ and that $(T - \lambda I)^{-1} \in \mathcal{L}(\ell_2, \ell_2)$.

Proof: Firstly note that T is a well-defined continuous linear mapping with $\|T\| \leq 1/2$. Indeed, given $(a_n)_{n\in\mathbb{N}} \in \ell_2$, we have that

$$\|(a_n)_{n\in\mathbb{N}}\|_2^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Thus, since $0 \leq |a_n|^2/2^{2n} \leq |a_n|^2$ for all $n \in \mathbb{N}$, we have that

$$\|T\big((a_n)_{n\in\mathbb{N}}\big)\|_2^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{2^{2n}} \leq \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Thus $T\big((a_n)_{n\in\mathbb{N}}\big) \in \ell_2$, so T is well-defined. Now, given $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}} \in \ell_2$ and $\lambda \in \mathbb{K}$, we have that

$$T\big((a_n)_{n\in\mathbb{N}} + \lambda(b_n)_{n\in\mathbb{N}}\big) = T\big((a_n + \lambda b_n)_{n\in\mathbb{N}}\big) = \big((a_n + \lambda b_n)/2^n)\big)_{n\in\mathbb{N}} = \big((a_n/2^n)\big)_{n\in\mathbb{N}} + \lambda\big((b_n/2^n)\big)_{n\in\mathbb{N}} = T\big((a_n)_{n\in\mathbb{N}}\big) + \lambda T\big((b_n)_{n\in\mathbb{N}}\big).$$

Thus T is linear. Now, given $(a_n)_{n\in\mathbb{N}} \in \ell_2$, we have that

$$\left\|T\big((a_n)_{n\in\mathbb{N}}\big)\right\|_2^2 = \sum_{n=1}^{\infty} \left|\frac{a_n}{2^n}\right|^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{2^{2n}} = \sum_{n=1}^{\infty} |a_n|^2 \left(\frac{1}{2^{2n}}\right).$$

Since for every $n \in \mathbb{N}$, we have that

$$|a_n|^2 \left(\frac{1}{2^{2n}}\right) \leq |a_n|^2 \left(\frac{1}{4}\right)$$

Then

$$\left\|T\big((a_n)_{n\in\mathbb{N}}\big)\right\|_2^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{2^{2n}} \leq \frac{1}{4} \sum_{n=1}^{\infty} |a_n|^2 = \frac{1}{4}\|(a_n)_{n\in\mathbb{N}}\|_2^2,$$

which implies that $\left\|T\big((a_n)_{n\in\mathbb{N}}\big)\right\|_2 \le (1/2)\|(a_n)_{n\in\mathbb{N}}\|_2$. Thus T is continuous and $\|T\| \le 1/2$.

Finally, let $\lambda \in \mathbb{K}$ with $|\lambda| > 1/2$, we have that

$$\left\|\frac{T}{\lambda}\right\| < \frac{1/2}{1/2} = 1.$$

Thus, we have already seen that $I - (T/\lambda)$ is invertible and so $\lambda I - T$. Since $\lambda I - T$ is continuous and ℓ_2 is a Banach space, by Open Mapping Theorem, we conclude that $(\lambda I - T)^{-1}$ is also continuous. $\square$

Question 6: Let E be an infinite-dimensional normed linear space and $T : E \longrightarrow E$ be a compact linear operator. If T is a bijection, prove that E is not a Banach space.

Proof: **Lemma:** Let E be an infinite-dimensional normed linear space and $T : E \longrightarrow E$ be a compact linear operator. If T is a bijection, then T^{-1} is not continuous.

Proof: Suppose by contradiction that T^{-1} is continuous. Since E is an infinite-dimensional space, by Riesz contruction, we can create a sequence $(x_n)_{n\in\mathbb{N}}$ in E which does not admit convergent subsequence. Since T is compact, the sequence $(T(x_n))_{n\in\mathbb{N}}$ has a convergent subsequence $(T(x_{n_k}))_{k\in\mathbb{N}}$. Suppose that $(T(x_{n_k}))_{k\in\mathbb{N}}$ converges to y. Since T^{-1} is continuous, we have that

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} T^{-1}(T(x_{n_k})) = T^{-1}\left(\lim_{k\to\infty} T(x_{n_k})\right) = T^{-1}(y) \in E,$$

implying in a contradiction since $(x_n)_{n\in\mathbb{N}}$ has not convergent subsequence. Now Let's get to the problem at hand. The normed linear space E can not be a Banach space simply because, if it was, the Open Mapping Theorem would imply that T^{-1} is also continuous, what we have already seen that it is impossible. $\square$

Question 7: Let E be a reflexive normed linear space. Denoting by $\mathcal{K}(E, \ell_1)$ the set of compact operators from E to ℓ_1, prove that $\mathcal{K}(E, \ell_1) = \mathcal{L}(E, \ell_1)$.

Proof: Let $T \in \mathcal{K}(E, \ell_1)$. Since

$$T(B_E) \subseteq \overline{T(B_E)}$$

and $\overline{T(B_E)}$ is compact, we conclude that $T(B_E)$ is bounded, so there is $M > 0$ such that $\|T(x)\| \le M$ for all $x \in B_E$. Thus

$$\|T\| = \sup_{\|x\|\le 1} \|T(x)\| \le \sup_{\|x\|\le 1} M = M < \infty.$$

So T is linear and bounded, hence T is continuous, that is, $T \in \mathcal{L}(E, \ell_1)$. On the other hand, let $T \in \mathcal{L}(E, \ell_1)$ and $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Since E is reflexive, $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let x be its limit. Since T is continuous, T is also weakly continuous, so we have that $(T(x_{n_k}))_{k\in\mathbb{N}}$ converges weakly to $T(x)$. Now, since ℓ_1 has the Schur property, we have that $(T(x_{n_k}))_{k\in\mathbb{N}}$ converges strongly to $T(x)$ and so we conclude that T is a compact operator. $\square$

Question 8: Let H be a Hilbert space and $T \in \mathcal{L}(H, H)$ be a self-adjoint operator. For every $n \in \mathbb{N}$

(i) Prove T^n is a self-adjoint operator.

(ii) Prove that $\|T^n\| = \|T\|^n$.

Proof: (i) : We will proceed with induction in $n \in \mathbb{N}$. Since T is self-adjoint, it is clear that $T^1 = T$ is self-adjoint. Suppose that this result is true for $n \in \mathbb{N}$, we will prove now that T^{n+1} is self-adjoint. Indeed, given x, $y \in H$, we have that

$$\langle T^{n+1}(x), y\rangle = \langle T^n(T(x)), y\rangle = \langle T(x), (T^n)^*(y)\rangle = \langle T(x), T^n(y)\rangle = \langle x, (T^*(T^n)(y))\rangle$$
$$= \langle x, T^{n+1}(y)\rangle.$$

So T^{n+1} is a self-adjoint operator. Hence, by induction principle, we conclude that $\langle T^n(x), y\rangle = \langle x, T^n(y)\rangle$ for all $n \in \mathbb{N}$. Since $x, y \in H$ were chosen arbitrarily, by uniqueness of adjoint operator, we conclude that T^n is self-adjoint.

(ii): **Lemma:** Let H be a Hilbert space and $T \in \mathcal{L}(H, H)$ be a self-adjoint operator. Prove that Then $\|T^{2^n}\| = \|T\|^{2^n}$ for all $n \in \mathbb{N}$.

In fact, we will prove by induction in n. Since T^2 is self-adjoint, for $n = 1$, we have

$$\|T^2\| = \sup_{\|x\|\le 1} |\langle T^2(x), x\rangle| = \sup_{\|x\|\le 1} |\langle T(x), T(x)\rangle| = \sup_{\|x\|\le 1} \|T(x)\|^2 = \left(\sup_{\|x\|\le 1} \|T(x)\|\right)^2 = \|T\|^2.$$

Suppose that $\|T^{2^n}\| = \|T\|^{2^n}$ for some $n \in \mathbb{N}$. Thus, since T^{2^n} is also self-adjoint, we have

$$\|T^{2^{n+1}}\| = \|T^{2^n}T^{2^n}\| = \|T^{2^n}\|^2 = \left(\|T\|^{2^n}\right)^2 = \|T\|^{2^{n+1}}.$$

So $\left\|T^{2^n}\right\| = \|T\|^{2^n}$ for every $n \in \mathbb{N}$. Let's get to the problem at hand. Firstly, note that, since $\|S_1 S_2\| \le \|S_1\|\|S_2\|$ for any pair of continuous linear operators S_1, $S_2 \in \mathcal{L}(H, H)$, by a simple induction process, we prove that

$$\|T^n\| \le \|T\|^n.$$

for any $n \in \mathbb{N}$. Suppose by contradiction that we had $\|T^n\| < \|T\|^n$ for some $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ such that $2^k > n$, thus $2^k = n + m$, where $m \geq 1$. So

$$\|T^{2^k}\| = \|T^{n+m}\| \leq \|T^n\|\|T^m\| < \|T\|^n\|T^m\| \leq \|T\|^n\|T\|^m = \|T\|^{n+m} = \|T\|^{2^k},$$

which implies that $\|T^{2^k}\| < \|T\|^{2^k}$, a contradiction, so $\|T^n\| \geq \|T\|^n$ for every $n \in \mathbb{N}$ and then $\|T^n\| = \|T\|^n$ for every $n \in \mathbb{N}$. □

Question 9: Let H be a separable Hilbert space and $T \in \mathcal{L}(H, H)$ be a compact and self-adjoint operator. Prove that $\sigma(T)$ is the topological closure of the set of eigenvalues of T.

Proof: Firstly suppose that $\dim_{\mathbb{K}}(H) < \infty$, since the $\sigma(T) = \{\lambda \in \mathbb{K}; \lambda \text{ is eigenvector of } T\}$ (basic linear algebra) and the spectrum of an operator is always compact, we have that

$$\sigma(T) = \overline{\sigma(T)} = \overline{\{\lambda \in \mathbb{K} \ ; \ \lambda \text{ is eigenvector of } T\}}.$$

Then we can suppose without lost of generality that $\dim_{\mathbb{K}}(H) = \infty$. In this case, since T is compact, we have

$$\sigma(T) = \{0\} \cup \{\lambda \in \mathbb{K} \ ; \ \lambda \text{ is eigenvector of } T\},$$

which implies that

$$\sigma(T) = \overline{\sigma(T)} = \overline{\{0\} \cup \{\lambda \in \mathbb{K} \ ; \ \lambda \text{ is eigenvector of } T\}} = \{0\} \cup \overline{\{\lambda \in \mathbb{K} \ ; \ \lambda \text{ is eigenvector of } T\}}.$$

Thus, in order to show that $\sigma(T) = \overline{\{\lambda \in \mathbb{K} \ ; \ \lambda \text{ is eigenvector of } T\}}$, it is enough to show that $0 \in \overline{\{\lambda \in \mathbb{K} \ ; \ \lambda \text{ is eigenvector of } T\}}$, that is, it is enough to show that there is a sequence of distinct eigenvalues of T converging to 0. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of distinct eigenvalues of T. Without lost of generality, we can assume that $\lambda_n \neq 0$ for all $n \in \mathbb{N}$. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence of eigenvectors of T associated with the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. Since T is self-adjoint, we have that $\langle x_n, x_m\rangle = 0$ whenever $n \neq m$, so we have that $\{x_n\}_{n\in\mathbb{N}}$ is linearly independent. Define the n-dimensional subspace

$$M_n = [x_1, \ldots, x_n].$$

It is clear that $M_n \subsetneq M_{n+1}$, because $x_{n+1} \notin M_n$. Since each M_n is a proper closed subspace of M_{n+1}, by Riesz lemma, we can find $y_{n+1} \in M_{n+1}$ such that $\|y_{n+1}\| = 1$ and $d(y_{n+1}, M_n) > 1/2$. If we write

$$y_{n+1} = \sum_{k=1}^{n+1} \alpha_k x_k,$$

then

$$(T-\lambda_{n+1}I)(y_{n+1}) = \left(T-\lambda_{n+1}I\right)\left(\sum_{k=1}^{n+1}\alpha_k x_k\right) = \sum_{k=1}^{n+1}\lambda_k\alpha_k x_k - \sum_{k=1}^{n+1}\lambda_n\alpha_k x_k = \sum_{k=1}^{n}(\lambda_k-\lambda_{n+1})\alpha_k x_k \in M_n$$

Since $\lambda_n \neq 0$ for all $n \in \mathbb{N}$, for any pair of integers $1 \leq m \leq n$, we set

$$y = y_m - \lambda_m^{-1}(\lambda_m I - T)y_m + \lambda_n^{-1}(\lambda_n I - T)y_n,$$

so that $T(\lambda_m^{-1}y_m) - T(\lambda_n^{-1}y_n) = y - y_n$. Since $y \in M_{n-1}$, we have that

$$\frac{1}{2} < \|y - y_n\| = \|T(\lambda_m^{-1}y_m) - T(\lambda_n^{-1}y_n)\|.$$

Thus the sequence $(T(\lambda_n^{-1}y_n))_{n\in\mathbb{N}}$ has no convergent subsequence. Then, since T is compact, $(\lambda_n^{-1}y_n)_{n\in\mathbb{N}}$ is unbounded. However, since $\|y_n\| = 1$ for all $n \in \mathbb{N}$, we conclude

$$\sup_{k\in\mathbb{N}}|\lambda_k|^{-1} = \sup_{k\in\mathbb{N}}\|\lambda_k^{-1}y_k\| = \infty.$$

Thus

$$\inf_{k\in\mathbb{N}}|\lambda_k| = 0,$$

which implies that $(\lambda_k)_{k\in\mathbb{N}}$ has a subsequence converging to 0, so $0 \in \overline{\{\lambda \in \mathbb{K} \ ; \ \lambda \text{ is eigenvector of } T\}}$.

□

Question 10: Let H be a Hilbert space and $T \in \mathcal{L}(H,H)$. Prove that the following assertions are equivalent

(i) $T^* \circ T = T \circ T^*$.

(ii) $\langle T(x), T(y)\rangle = \langle T^*(x), T^*(y)\rangle$ for all $x,\ y \in H$.

Proof: $(i) \to (ii)$: Given $x,\ y \in H$, then

$$\begin{aligned}\langle T(x), T(y)\rangle = \langle T^*(T(x)), y\rangle = \langle (T^*\circ T)(x), y\rangle = \langle (T\circ T^*)(x), y\rangle = \langle T(T^*(x)), y\rangle \\ = \langle T^*(x), T^*(y)\rangle.\end{aligned}$$

$(ii) \to (i)$: Fix $y \in H$, thus given $x \in H$ we have

$$\langle x, (T^*\circ T)(y)\rangle = \langle T(x), T(y)\rangle = \langle T^*(x), T^*(y)\rangle = \langle x, (T\circ T^*)(y)\rangle.$$

Thus

$$\begin{aligned}\langle x, (T^*\circ T - T\circ T^*)(y)\rangle = \langle x, (T^*\circ T)(y) - (T\circ T^*)(y)\rangle = \langle x, (T^*\circ T)(y)\rangle - \langle x, (T\circ T^*)(y)\rangle \\ = 0.\end{aligned}$$

Since x can be chosen arbitrarily, we conclude that $(T^* \circ T)(y) = (T \circ T^*)(y)$. Since y can be chosen arbitrarily, we conclude that

$$T^* \circ T = T \circ T^*.$$

$\square$

Chapter 3

Some Lists of Summer course at UFF on 2020

3.1 Normed Linear Spaces

Question 1.1: Let E be a normed linear space. Prove that

$$\big| \|x\| - \|y\| \big| \leq \|x - y\|$$

for all $x, y \in E$. In particular, prove that the norm function is continuous.

Proof: Indeed, given $x, y \in E$, we have

$$\|x\| = \|x + (-y + y)\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|.$$

Thus

$$\|x\| - \|y\| \leq \|x - y\|.$$

Switching x with y, we also obtain

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|.$$

Then we conclude that $\big| \|x\| - \|y\| \big| \leq \|x - y\|$. Now consider the following function

$$\begin{aligned} \varphi : E &\longrightarrow \mathbb{R} \\ x &\longmapsto \|x\|. \end{aligned}$$

I claim that φ is continuous. In fact, let $x_0 \in E$. Given $\epsilon > 0$, let $\delta = \epsilon$. Thus, for all $x \in B(x_0, \delta)$, we have

$$|\varphi(x) - \varphi(x_0)| = \big| \|x\| - \|x_0\| \big| \leq \|x - x_0\| < \epsilon.$$

Thus φ is continuous. $\square$

Question 1.2: Let E be a normed linear space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. Prove that if $x_n \to x$ in E, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$.

Proof: Indeed, since $x_n \to x$ in E, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $\|x_n - x\| < \epsilon$. Thus, using the inequality of the question above, we conclude that

$$\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\| < \epsilon$$

for all $n \geq n_0$. Thus $(\|x_n\|)_{n\in\mathbb{N}}$ converges to $\|x\|$. $\square$

Question 1.3: A series $\sum_{n=1}^{\infty} x_n$ in a normed linear space E is said to be **absolutely convergent** if the series $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.

a) If E is a Banach space, prove that if the series is absolutely convergent, then it is convergent.

b) Suppose that E is a Banach space. If there a sequence $(M_n)_{n\in\mathbb{N}} \subseteq [0, \infty)$ such that

$$\|x_n\| \leq M_n, \ \ \forall n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} M_n < \infty$$

Then the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. This is known as **Wierstrass M-test**.

Proof: a): Consider the sequence

$$s : \mathbb{N} \longrightarrow E$$
$$n \longmapsto \textstyle\sum_{k=1}^{n} x_k.$$

Since $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq m \geq n_0$, we have

$$\|x_m\| + \|x_{m+1}\| + \cdots + \|x_n\| < \epsilon$$

In particular, for all $n \geq m \geq n_0$, have that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| < \epsilon.$$

Thus $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. Since E is a Banach space, we conclude that $(s_n)_{n\in\mathbb{N}}$ converges in E.

b): Suppose that there is a sequence $(M_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ such that $\|x_n\| \leq M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$. Since $\sum_{n=1}^{\infty} M_n < \infty$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq m \geq n_0$

$$M_m + M_{m+1} + \cdots + M_n < \epsilon.$$

Thus

$$\|x_m\| + \|x_{m+1}\| + \cdots + \|x_n\| \leq M_m + M_{m+1} + \cdots + M_n < \epsilon.$$

Then $\sum_{n=1}^{\infty} \|x_n\|$ is a convergent sequence, since $\mathbb{R}$ is a Banach Space. Then the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. □

Question 1.4: Let E be a normed linear space. Let F and K be subsets of E such that F is closed in E and K is compact in E. If $F \cap K = \varnothing$, prove that there exist $\epsilon > 0$ such that

$$\operatorname{dist}(F,K) := \inf\{\|x-y\| \ ; \ x \in F \text{ and } y \in K\} > \epsilon.$$

Proof: Suppose by contradiction that $\operatorname{dist}(F,K) = 0$. Thus there are sequences $(x_n)_{n\in\mathbb{N}} \subseteq F$ and $(y_n)_{n\in\mathbb{N}} \subseteq K$ such that $\|x_n - y_n\| < 1/n$. Since K is a compact metric space, there is a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} y_{n_k} = y \in K.$$

It is easy to see that

$$\lim_{k\to\infty} x_{n_k} = y.$$

Since F is a closed subset of E, we also have $y \in F$. Thus $F \cap K \neq \varnothing$, which is a contradiction. Then we have that there is $\epsilon > 0$ such that

$$\operatorname{dist}(F,K) := \inf\{\|x-y\| \ ; \ x \in F \text{ and } y \in K\} > \epsilon.$$

□

Question 1.5: If M is a linear subspace of a normed linear space E, prove that $\overline{M}$ is also a linear subspace of E.

Proof: In fact, $\overline{M} \neq \varnothing$, because $0 \in \overline{M}$. Let x and $y \in \overline{M}$. Thus there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in M such that

$$x = \lim_{n\to\infty} x_n \qquad y = \lim_{n\to\infty} y_n.$$

Then we have that

$$x + y = \lim_{n\to\infty} (x_n + y_n),$$

where $x_n + y_n \in M$ for all $n \in \mathbb{N}$, thus $x + y \in \overline{M}$. Now, given $\lambda \in \mathbb{K}$, we have that

$$\lambda x = \lim_{n\to\infty} \lambda x,$$

where $\lambda x \in M$ for all $n \in \mathbb{N}$, thus $\lambda x \in \overline{M}$ and so $\overline{M}$ is a linear subspace of M. □

Question 1.6: [Banach Fixed Point Theorem]. Let (X, d) be a complete metric space. Consider a map $f : X \longrightarrow X$ and suppose that there is $\lambda \in (0, 1)$ such that

$$d(f(x), f(y)) \le \lambda d(x, y)$$

for all $x, y \in X$. Consider $x_0 \in X$ and, for each $n \in \mathbb{N}$, define $x_n = f^{(n)}(x_0)$, where

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

a) Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

b) Show that $a = \lim_{n\to\infty} f^n(x_0)$ is the unique fixed point of f.

Proof: a) Firstly I claim that $d(f^{n+1}(x_0), f^n(x_0)) \le \lambda^n d(f(x_0), x_0)$ for all $n \in \mathbb{N}$. In fact, we prove it by induction. For $n = 0$, the inequality is true trivially. Suppose that

$$d(f^{n+1}(x_0), f^n(x_0)) \le \lambda^n d(f(x_0), x_0)$$

Thus

$$\begin{aligned} d(f^{n+2}(x_0), f^{n+1}(x_0)) = d(f(f^{n+1}(x_0)), f(f^n(x_0))) \le \lambda d(f^{n+1}(x_0), f^n(x_0)) \le \lambda(\lambda^n d(f(x_0), x_0)) \\ = \lambda^{n+1} d(f(x_0), x_0). \end{aligned}$$

Set $A \coloneqq d(f(x_0), x_0)$. Now, given $n \ge m$, we have the following

$$d(f^n(x_0), f^m(x_0)) \le \sum_{k=m}^{n-1} d(f^{k+1}(x_0), f^k(x_0)) \le \sum_{k=m}^{n-1} \lambda^k A \le \sum_{k=m}^{\infty} \lambda^k A = \frac{\lambda^{m-1} A}{1-\lambda}$$

Making $m \to \infty$, we conclude that $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence, because

$$\lim_{m\to\infty} \left(\frac{\lambda^{m-1} A}{1-\lambda} \right) = 0.$$

b) Since f is a continuous mapping, a is fixed point of f, because

$$f(a) = f\left(\lim_{n\to\infty} f^n(x_0) \right) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{n+1}(x_0) = a.$$

Finally, let $b \in X$ another fixed point of f. Since $d(b, a) = d(f(b), f(a)) \le \lambda d(b, a)$ and $\lambda \in (0, 1)$, we conclude that $d(b, a) = 0$, then $b = a$. Then f has a unique fixed point. □

3.2 Bounded Linear operators and Hahn-Banach Theorem

Question 2.1: Let E be a normed linear space over $\mathbb{R}$ and $\varphi : E \longrightarrow \mathbb{R}$ be a linear functional. Suppose that there exists $C > 0$ such that $\varphi(x) \leq C$ for all $x \in E$. Prove that $\varphi = 0$.

Proof: In fact, suppose by contradiction that $\varphi \neq 0$ and let $x \in E$ such that $\varphi(x) \neq 0$. Thus, setting

$$\lambda = \frac{C+1}{\varphi(x)},$$

we obtain that

$$\varphi(\lambda x) = \frac{C+1}{\varphi(x)}.\varphi(x) = C + 1 > C,$$

which is a contradiction. Then $\varphi = 0$. □

Question 2.2: Let $\varphi : E \longrightarrow F$ be a linear operator. If there exists $C > 0$ such that

$$||\varphi(x)|| \leq C \ \ \forall\, x \in E \text{ with } ||x|| < 1$$

Prove that $||\varphi(x)|| \leq C$ for all $x \in E$ with $||x|| = 1$. Conclude that $\varphi \in \mathcal{L}(E, F)$ and that

$$\sup_{||x||<1} ||\varphi(x)|| = ||\varphi||.$$

Proof: Indeed, suppose by contradiction that there exists $x \in E$ with $||x|| = 1$ such that $||\varphi(x)|| = D > C$. Note that, for all $0 < \epsilon < 1$, $(1-\epsilon)x$ is such that $||(1-\epsilon)x|| = (1-\epsilon) < 1$. Thus, setting

$$\lambda = \frac{C+D}{2D},$$

we have $\lambda x \in E$ is such that $||\lambda x|| = \lambda < 1$ and

$$\varphi(\lambda x) = \lambda\varphi(x) = \frac{C+D}{2D}.D = \frac{C+D}{2} > C,$$

which is a contradiction, so $||\varphi(x)|| \leq C$ for all $x \in E$ with $||x|| = 1$. □

Question 2.3: Let E, F be linear spaces and $T : E \longrightarrow F$ be a bijective linear mapping. Prove that T^{-1} is also linear.

Proof: In fact, let $y_1 = T(x_1), y_2 = T(x_2) \in F$ and $\lambda \in \mathbb{K}$. Thus

$$T^{-1}(y_1 + \lambda y_2) = T^{-1}(T(x_1) + \lambda T(x_2)) = T^{-1}(T(x_1 + \lambda x_2)) = x_1 + \lambda x_2 = T^{-1}(y_1) + \lambda T(y_2).$$

So T^{-1} is also a linear mapping. $\square$

Question 2.4: Let $(E_1, ||.||_1)$ and $(E_2, ||.||_2)$ be two normed linear spaces. Prove that the mapping

$$||.|| : E_1 \times E_2 \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto ||x||_1 + ||y||_2$$

is a norm on $E_1 \times E_2$. Moreover, prove that $E_1 \times E_2$ is a Banach space if and only if E_1 and E_2 are Banach spaces.

Proof: Let's check the norm's axioms.

- $||(x, y)|| = ||x||_1 + ||y||_2 \geq 0$ for every $(x, y) \in E_1 \times E_2$.
- Given $(x, y) \in E_1 \times E_2$, if $||(x, y)|| = ||x||_1 + ||y||_2 = 0$, then $x = y = 0$, thus $(x, y) = (0, 0)$. On the other hand, $||(0, 0)|| = ||0||_1 + ||0||_1 = 0$ trivially.
- Given $\lambda \in \mathbb{K}$ and $(x, y) \in E_1 \times E_2$, we have that

$$||\lambda(x, y)|| = ||(\lambda x, \lambda y)|| = ||\lambda x||_1 + ||\lambda y||_2 = |\lambda|.||x||_1 + |\lambda|.||y||_2 = |\lambda|.||(x, y)||.$$

- Given $(x_1, y_1), (x_2, y_2) \in E_1 \times E_2$, we have

$$\begin{aligned} ||(x_1, y_1) + (x_2, y_2)|| &= ||(x_1 + x_2, y_1 + y_2)|| = ||x_1 + x_2||_1 + ||y_1 + y_2||_2 \\ &\leq (||x_1||_1 + ||x_2||_1) + (||y_1||_2 + ||y_2||_2) = ||(x_1, y_1)|| + ||(x_2, y_2)||. \end{aligned}$$

Now suppose that $E_1 \times E_2$ is a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in E_1 and consider the sequence $((x_n, 0))_{n\in\mathbb{N}}$ in $E_1 \times E_2$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $||x_n - x_m||_1 < \epsilon$. Thus, we obtain that

$$||(x_n, 0) - (x_m, 0)|| = ||(x_n - x_m, 0)|| = ||x_n - x_m||_1 < \epsilon$$

for all $n, m \geq n_0$. Thus $((x_n, 0))_{n\in\mathbb{N}}$ is Cauchy sequence in $E_1 \times E_2$. Since $E_1 \times E_2$ is a Banach space, we conclude $((x_n, 0))_{n\in\mathbb{N}}$ converges to $(x, 0)$ in $E_1 \times E_2$. Now it is easy to see that $(x_n)_{n\in\mathbb{N}}$ converges to x in E_1. Since $(x_n)_{n\in\mathbb{N}}$ is an arbitrary Cauchy sequence in E_1, we conclude that E_1 is a Banach space. Similarly, we prove that E_2 is also a Banach space.

Now suppose that E_1, E_2 are Banach spaces and let $((x_n, y_n))_{n\in\mathbb{N}}$ be a Cauchy sequence in $E_1 \times E_2$. Thus, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have

$$||(x_n, y_n) - (x_m, y_m)|| = ||(x_n - x_m, y_n - y_m)|| = ||x_n - x_m||_1 + ||y_n - y_m||_2 < \epsilon$$

Thus

$$\|x_n - x_m\|_1 < \epsilon \qquad\qquad \|y_n - y_m\|_2 < \epsilon$$

for all $n, m \geq n_0$. Thus $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in E_1 and E_2, respectively. Since E_1, E_2 are Banach spaces, there are $x \in E_1$, $y \in E_2$ such that $x_n \to x$ and $y_n \to y$. Thus, given $\epsilon > 0$, there are $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\|x_n - x\|_1 < \epsilon/2 \qquad\qquad \|y_n - y\|_2 < \epsilon/2$$

So

$$\|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\|_1 + \|y_n - y\|_2 < \epsilon/2 + \epsilon/2 = \epsilon.$$

for all $n \geq n_0$, so $(x_n, y_n) \to (x, y)$. Since $((x_n, y_n))_{n\in\mathbb{N}}$ is an arbitrary Cauchy sequence, we conclude that $E_1 \times E_2$ is a Banach space. □

Question 2.5: Let $(E, \|.\|)$ be a normed linear space.

a) Prove that the mappings

$$\begin{aligned} \psi : E \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y \end{aligned} \qquad\qquad \begin{aligned} \phi : \mathbb{R} \times E &\longrightarrow E \\ (\lambda, x) &\longmapsto \lambda x \end{aligned}$$

are continuous.

b) Fixed $x_0 \in E$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Prove that the mapping.

$$\begin{aligned} \zeta : E &\longrightarrow E \\ x &\longmapsto \lambda x + x_0 \end{aligned}$$

is an homeomorphism.

Proof: a) In fact, given $(x_0, y_0) \in E \times E$, note that

$$\|\psi(x,y) - \psi(x_0,y_0)\| = \|x + y - (x_0 + y_0)\| \leq \|x - x_0\| + \|y - y_0\| = \|(x - x_0, y - y_0)\| = \|(x, y) - (x_0, y_0)\|.$$

Thus the function ψ is Lipschtzian, so ψ is continuous. Now, let $\lambda_0 \in \mathbb{R}$ and $x_0 \in E$. Note that

$$\begin{aligned} \|\phi(\lambda, x) - \phi(\lambda_0, x_0)\| &= \|\lambda x - \lambda_0 x_0\| = \|(\lambda - \lambda_0)(x - x_0) + \lambda_0(x - x_0) + (\lambda - \lambda_0)x_0\| \\ &\leq |\lambda - \lambda_0|.\|x - x_0\| + |\lambda_0|.\|x - x_0\| + |\lambda - \lambda_0|.\|x_0\| \end{aligned}$$

Thus, given $\epsilon > 0$, let $\delta = \min\{1, \epsilon, 1/|\lambda_0|, 1/||x_0||\}$. Then, given $x \in B((\lambda_0, x_0), \lambda/3)$, we have

$$||\phi(\lambda, x) - \phi(\lambda_0, x_0)|| \leq |\lambda - \lambda_0|.||x - x_0|| + |\lambda_0|.||x - x_0|| + |\lambda - \lambda_0|.||x_0|| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Thus ϕ is continuous in (x_0, y_0). Since (x_0, y_0) is arbitrary, we conclude that ϕ is continuous.

b) Fixed $x_0 \in E$ and $\lambda \in \mathbb{R} \setminus \{0\}$, note that the mapping

$$\zeta : E \longrightarrow E$$

$$x \longmapsto \lambda x + x_0$$

is continuous, because the mappings

$$\psi_{x_0} : E \longrightarrow E \qquad\qquad \phi_{\lambda_0} : E \longrightarrow E$$

$$x \longmapsto x + x_0 \qquad\qquad x \longmapsto \lambda_0 x$$

are continuous and $\zeta = \psi_{x_0} \circ \phi_{\lambda_0}$. Finally note that ζ^{-1} exists and

$$\zeta^{-1} : E \longrightarrow E$$

$$x \longmapsto (\lambda)^{-1}x - \lambda^{-1}x_0$$

which has the same form that ζ, so ζ^{-1} is continuous and so ζ is a homeomorphism. □

Question 2.6: Let $T : E \longrightarrow F$ be a linear operator. Prove that $T \in \mathcal{L}(E, F)$ if and only if T maps bounded sets in bounded sets.

Proof: Suppose that T is a continuous mapping, thus T is a bounded function, that is, there exists $M > 0$ such that $||T(x)|| \leq M||x||$ for all $x \in E$. Let A be a bounded subset of E, so there is $R > 0$ such that $A \subseteq B(0, R)$. Since $||T(x)|| \leq M||x||$ for all $x \in E$, then, given $x \in A$, we have that $x \in B(0, R)$, then $||T(x)|| \leq MR$. Thus we conclude that

$$T(A) \subseteq B(0, MR),$$

which is bounded. Then T maps maps bounded sets in bounded sets. Conversely, suppose that T maps maps bounded sets in bounded sets and consider $A = B[0, 1]$ the closed unit ball. Since A is bounded, $T(A)$ is also bounded, so there is $M > 0$ such that $||T(x)|| \leq M$ for all $x \in B[0, 1]$. Then

$$||T|| = \sup_{x \in B[0,1]} ||T(x)|| \leq M < \infty.$$

Hence T is a continuous mapping, since it is bounded. $\square$

Question 2.7: Consider the normed linear space

$$c_0 = \{(x_n)_{n\in\mathbb{N}} \ ; \ \lim_{n\to\infty} x_n = 0\}$$

with the usual norm. Given $u = (u_n)_{n\in\mathbb{N}} \in c_0$, define $f : c_0 \longrightarrow \mathbb{K}$ such that

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n.$$

Prove that f is a continuous linear functional and find its norm.

Proof: Firstly note that f is well-defined, because, since $u \in c_0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $|u_n| \leq 1$. Thus the series

$$\sum_{k=n_0}^{\infty} \frac{1}{2^n} u_n$$

converges, because $|\frac{1}{2^n} u_n| \leq \frac{1}{2^n}$ for all $n \geq n_0$ and

$$\sum_{k=n_0}^{\infty} \frac{1}{2^n}$$

converges. Note now that f is linear, because, given $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in c_0$ and $\lambda \in \mathbb{C}$, we have

$$f(x + \lambda y) = f((x_n + \lambda y_n)_{n\in\mathbb{N}}) = f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n}(x_n + \lambda y_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n + \lambda \sum_{n=1}^{\infty} \frac{1}{2^n} y_n = f(x) + \lambda f(y).$$

Now I claim that f is continuous. Indeed, let $x_0 = (x_n)_{n\in\mathbb{N}} \in c_0$. Given $\epsilon > 0$, take $\delta = \epsilon$. So, given $y = (y_n)_{n\in\mathbb{N}} \in B(x_0, \delta/2)$, we have that $|y_n - x_n| \leq \delta/2 = \epsilon/2$ for all $n \in \mathbb{N}$ and so

$$|f(y) - f(x_0)| = \left| \sum_{n=1}^{\infty} \frac{1}{2^n} y_n - \sum_{n=1}^{\infty} \frac{1}{2^n} x_n \right| = \left| \sum_{n=1}^{\infty} \frac{1}{2^n}(y_n - x_n) \right| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |y_n - x_n| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} . \frac{\epsilon}{2} = \epsilon/2 < \epsilon.$$

Thus f is continuous. Now, given $x = (x_n)_{n\in\mathbb{N}} \in c_0$ with $||x|| \leq 1$, we have that

$$|f(x)| = \left| \sum_{n=1}^{\infty} \frac{1}{2^n} x_n \right| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus

$$||f|| = \sup_{x\in B[0,1]} |f(x)| \leq 1.$$

On the other hand, considering the sequence $x^m : \mathbb{N} \longrightarrow \mathbb{K}$ such that

$$(x^m)_n = \begin{cases} 1, & \text{if } 1 \leq n \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

We have that $\|x^m\| = 1$ and

$$f(x^m) = \sum_{k=1}^{m} \frac{1}{2^k} = 1 - (1/2)^m.$$

Making $m \to \infty$, we conclude that

$$\|f\| = \sup_{x \in B[0,1]} |f(x)| \geq 1.$$

Then f is bounded and $\|f\| = 1$. $\square$

Question 2.8: Let E be an infinite dimensional normed linear space. Prove that that there exists a linear functional $f : E \longrightarrow \mathbb{K}$ that is not continuous.

Proof: Let $\{v_\lambda\}_{\lambda \in L}$ be a basis of E such that $\|v_\lambda\| = 1$ for all $\lambda \in L$. Since $\dim(E) = \infty$, we have that $\mathrm{Card}(L) \geq \aleph_0$. Let L' be an infinite countable subset of L and consider the corresponding linearly subset $\{v_{\lambda_n}\}_{\lambda_n \in L'} \subseteq \{v_\lambda\}_{\lambda \in L}$. Define, by linearity, the linear functional $f : E \longrightarrow \mathbb{K}$ such that

$$f(v_\lambda) = \begin{cases} n, & \text{if } \lambda = \lambda_n \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f is linear functional, however f is not continuous because $|f(v_{\lambda_n})| = n$ for each $n \in \mathbb{N}$. Thus

$$\|f\| = \sup_{x \in B[0,1]} |f(v)| \geq n$$

for all $n \in \mathbb{N}$, implying that $\|f\| = \infty$, thus f is not continuous. $\square$

Question 2.9: For each $n \in \mathbb{N}$, define

$$e_n = (0, \ldots, 0, \underbrace{1}_{n^{th} \text{ term}}, 0, \ldots)$$

Prove that $(e_n)_{n \in \mathbb{N}}$ is not a Cauchy sequence in in ℓ^p, where $p \in [1, \infty]$, in particular, conclude that thoses spaces are infinite dimensional.

Proof: For $p \in [1, \infty)$, observe that for all $n > m \geq 1$, we have

$$e_n - e_m = (0, \ldots, 0, \underbrace{-1}_{m^{th} \text{ term}}, 0, \ldots, 0, \underbrace{1}_{n^{th} \text{ term}}, 0, \ldots).$$

Thus

$$\|e_n - e_m\|_p = \left(2\right)^{\frac{1}{p}} \geq 1.$$

Then $(e_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence in in ℓ^p for $p \in [1, \infty)$. Similarly, for $p = \infty$, for all $n > m \geq 1$, we have

$$e_n - e_m = (0, \ldots, 0, \underbrace{-1}_{m^{th} \text{ term}}, 0, \ldots, 0, \underbrace{1}_{n^{th} \text{ term}}, 0, \ldots).$$

Thus

$$||e_n - e_m||_\infty = 1.$$

Then $(e_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence in in ℓ^∞. □

Question 2.10: Let E be a normed linear space and suppose that $\{x_1, \ldots, x_n\}$ is a linearly independent subset of E. Prove that there are $\phi_1, \ldots, \phi_n \in E'$ such that $\phi_i(x_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$.

Proof: Consider the finite dimensional linear space

$$M = \text{Span}_{\mathbb{K}}(\{x_1, \ldots, x_n\}).$$

For each $i \in \{1, \ldots, n\}$, define by linearity the following linear functional

$$\begin{aligned} \psi_i : M &\longrightarrow \mathbb{K} \\ x_j &\longmapsto \delta_{ij} \end{aligned}$$

Since $\{x_1, \ldots, x_n\}$ is linearly independent, each ψ_i is well-defined. Moreover, as M is a finite dimensional normed linear space, we have that ψ_i is continuous for each $i \in \{1, \ldots, n\}$, that is, there is $||\psi_i|| < \infty$ such that

$$|\psi_i(x)| \leq ||\psi_i||.||x|| \;\; \forall \, x \in M$$

Since the function

$$\begin{aligned} p_i : E &\longrightarrow \mathbb{R} \\ v &\longmapsto ||\psi_i||.||x|| \end{aligned}$$

is sublinear and $|\psi_i(x)| \leq p_i(x)$ for all $x \in M$, by Hahn-Banach Theorem, we conclude that there is $\phi_i : E \longrightarrow \mathbb{K}$, which extends ψ_i and

$$|\phi_i(x)| \leq ||\psi_i||.||x|| \;\; \forall \, x \in E$$

Thus ϕ_i is continuous for each $i \in \{1, \ldots, n\}$ and $\phi_i(x_j) = \psi_i(x_j) = \delta_{ij}$. □

Question 2.11: Let E be a normed linear space. Prove that, if $\phi(u) = 0$ for all $\phi \in E'$, then $u = 0$.

Proof: Suppose that $u \neq 0$. Define the following linear functional

$$f : \mathrm{Span}_{\mathbb{K}}(\{u\}) \longrightarrow \mathbb{K}$$

$$tu \longmapsto t.$$

It's clear that f is a bounded linear functional, because, given $x = tu \in [u]$ arbitrary, then

$$|f(x)| = |f(tu)| = |t| = |t|\frac{||u||}{||u||} = \left(\frac{1}{||u||}\right)||tu|| = \left(\frac{1}{||u||}\right)||x||.$$

Since the function

$$p : E \longrightarrow \mathbb{R}$$

$$x \longmapsto (\tfrac{1}{||x||})||x||$$

is sublinear and $|f(x)| \leq p(x)$ for all $x \in [u]$, by Hahn-Banach Theorem, there is a linear functional $\phi : E \longrightarrow \mathbb{K}$ which extends f and

$$|\phi(x)| \leq p(x) = \left(\frac{1}{||u||}\right)||x||.$$

Thus $\phi \in E'$ and $\phi(u) = 1$. Then, if $u \in E$ is such that $\phi(u) = 0$ for every $\phi \in E'$, we conclude that $u = 0$ necessarily. □

Question 2.12: Let E be a Banach space and $J_E : E \longrightarrow E''$ the canonical injection. Prove that if $X \subseteq E$ is closed, then $J_E(X)$ is closed. In particular, $J_E(E)$ is a closed subspace of E'', where we are identifying E with the subspace of E'' using the canonical injection.

Proof: In fact, let $X \subseteq E$ be a closed subset of E and let $y \in \overline{J_E(X)}$, thus

$$y = \lim_{n\to\infty} y_n, \qquad y_n = J_E(x_n) \in J_E(X),$$

where $x_n \in X$ for all $n \in \mathbb{N}$. Since $(J(x_n))_{n\in\mathbb{N}}$ converges, $(J(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover, since J_E is isometric monomorphism, we conclude that $(x_n \in \mathbb{N})$ is also a Cauchy sequence in X. Since E is a Banach space and X is a closed subset of E, we conclude that x_n converges to $x \in X$. Thus

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} J_E(x_n) = J_E\big(\lim_{n\to\infty} x_n\big) = J_E(x).$$

Thus $y \in J_E(X)$ and so $J_E(X)$ is closed in E''. Since E is closed in E, we have that $E \cong J_E(E) \subseteq E''$ is closed in E''. □

Question 2.13 (Adapted) Let E be a normed linear space and M be a subspace of E, we set

$$M^{\perp} = \{\phi \in E' \ ; \ \phi(x) = 0 \ \forall \, x \in M\}.$$

Prove that

a) $M^{\perp}$ is a closed subspace of E'.

b) If M is a subspace of E, then M' is isometrically isomorphic to $E'/M^{\perp}$.

c) If M is a proper closed subspace of E, then $M^{\perp} \neq \{0\}$.

Proof: a) Note that $M^{\perp} \neq \varnothing$, since $0 \in M^{\perp}$. Given $\phi, \psi \in M^{\perp}$ and $\lambda \in \mathbb{K}$, we have that

$$(\phi + \lambda\psi)(x) = \phi(x) + \lambda\psi(x) = 0 + 0 = 0$$

for all $x \in M$, thus $\phi + \lambda\psi \in M^{\perp}$. Then $M^{\perp}$ is a linear subspace of E'. Moreover, let $\phi \in \overline{M^{\perp}}$, thus

$$\phi = \lim_{n\to\infty} \phi_n, \qquad \phi_n \in M^{\perp}, \ \forall \, n \in \mathbb{N}.$$

Thus, given $x \in M$ and $\epsilon > 0$, there is $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, we have

$$||\phi_n - \phi|| < \frac{\epsilon}{||x|},$$

which implies that, for $n \geq n_0$

$$|\phi(x)| = |\phi_n(x) - \phi(x)| \leq ||\phi_n - \phi||.||x|| \leq \frac{\epsilon}{||x|}||x|| = \epsilon,$$

implying that $\phi(x) = 0$ for all $x \in M$, so $\phi \in M^{\perp}$ and then $\overline{M^{\perp}} = M^{\perp}$.

b) Define the following mapping

$$\zeta : E' \longrightarrow M'$$

$$f \longmapsto f|_M$$

Clearly ζ is a linear mapping. Note that ζ is surjective, because, given $\phi \in M'$, we have that

$$|\phi(x)| \leq ||\phi||.||x|| \ \ \forall \, x \in M$$

Defining the sublinear function

$$p : E \longrightarrow \mathbb{R}$$

$$x \longmapsto ||\phi||.||x||$$

we have that $|\phi| \le p(x)$ for all $x \in M$. Thus, By Hahn-Banach Theorem, there are $\psi \in E'$ such that $\psi|_M = \phi$, that is, $\zeta(\psi) = \phi$. Now, observe that

$$\mathrm{Ker}(\zeta) = \{\phi \in E' \ ; \ \zeta(\phi) = 0\} = \{\phi \in E' \ ; \ \phi|_M = 0\} = \{\phi \in E' \ ; \ \phi(x) = 0 \, \forall \, x \in M\} = M^{\perp}$$

By First Isomorphism Theorem, we conclude that

$$\zeta : \frac{E'}{M^{\perp}} \longrightarrow M'$$

is a linear bijection. Now, let $\phi + M' \in E'/M^{\perp}$, we have that

$$\|\zeta(\phi + M')\| = \|\phi|_M\| = \sup_{x \in B_M[0,1]} |\phi(x)| = \sup_{x \in B_M[0,1]} \{|\phi(x) + \psi(x)| \ ; \ \psi \in M^{\perp}\} = |||\phi|||$$

So ζ is an isometric isomorphism. $\square$

c) Applying the second form of geometrical Hahn-Banach theorem, consider $F = M$ the closed subset and $K = \{x_0\}$ the compact subset, where $x_0 \notin M$. The Theorem says that there exist $\phi \in E'$, $c \in \mathbb{R}$ such that

$$\phi(y) < c < \phi(x_0)$$

for all $y \in M$. Since F is a linear subspace of E, $\dim(\mathbb{R}) = 1 = \dim(\mathrm{Im}(\phi))$ and $\phi(x) \le c$ for all $x \in F$, we conclude that $\phi(x) = 0$ for all $x \in F = M$ and $\phi(x_0) > c > 0$, thus $\phi \in M^{\perp}$ and $\phi \ne 0$. $\square$

Question 2.15 Let E be a normed linear space and M be a closed linear subspace of E. We say that $x \in E$ and $y \in E$ are **equivalent modulo** M if $x - y \in M$. In this case, we denote

$$x \equiv y (\mathrm{mod}\, M).$$

a) Show that this is an equivalence relation. Let us denote by $[x]$ the equivalence class of each element $x \in E$.

b) Consider the **quotient space** of E **modulo** M, defined by

$$E/M = \{[x] \ ; \ x \in E\}$$

Show that the mappings

$$\begin{aligned} + : E/M \times E/M &\longrightarrow E/M \\ ([x],[y]) &\longmapsto [x+y] \end{aligned} \qquad\qquad \begin{aligned} \cdot : \mathbb{K} \times E/M &\longrightarrow E/M \\ (\lambda,[y]) &\longmapsto [\lambda y] \end{aligned}$$

are well defined. Prove that E/M is a linear space with these operations.

c) Let F be a normed linear space and consider the linear mapping $T : E \longrightarrow F$. Show that there is a linear bijection between $E/\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$.

Proof: a) The relation defined is an equivalence relation.

In fact, we have to check the properties: reflexivity, symmetry and transitivity.

- Reflexivity: given $x \in E$, then $x - x = 0 \in M$, thus $x \equiv x$;
- Symmetry: given $x, y \in E$, if $x \equiv y$, then $x - y \in M$. Since M is a linear subspace of E, then $y - x = (-1)(x - y) \in M$, thus $y \equiv x$;
- Transitivity: given $x, y, z \in E$, if $x \equiv y$ and $y \equiv z$, then

$$x - y \in M \qquad\qquad \text{and} \qquad\qquad y - z \in M$$

  Thus, since M is a linear subspace of E, we have that $x - z = (x - y) + (y - z) \in M$. Thus $x \equiv z$.

Thus we conclude that $\equiv$ is, in fact, an equivalence relation.

b) In fact

- If $[x] = [x']$ and $[y] = [y']$, then $x - x' \in M$ and $y - y' \in M$. Thus $(x + y) - (x' + y') = (x - x') + (y - y') \in M$, so $[x + y] = [x' + y']$.
- Given $[x] = [x']$ and $\lambda \in \mathbb{K}$, thus $x - x' \in M$. Since M is a linear subspace of E, then $\lambda(x - x') = \lambda x - \lambda x' \in M$, so $[\lambda x] = [\lambda x']$.

It is straightforward to verify that $(E/M, \mathbb{K}, +, *)$ has linear space structure, where $0 = [0]$ and, given $z = [x] \in E/M$, its inverse is $-z = [-x]$. Since it is simple to verify, although pedant, I will not do here.

c) Define

$$\phi : E/\ker(T) \longrightarrow F$$

$$[x] \longmapsto T(x)$$

Note that ϕ is well-defined, because, if $[x] = [y]$, then $x - y \in \operatorname{Ker}(T)$, so $T(x) = T(y)$, that is

$$\phi([x]) = T(x) = T(y) = \phi([y]).$$

Note that ϕ is linear. Indeed, given $[x]$, $[y] \in E/\operatorname{Ker}(T)$ and $\lambda \in \mathbb{K}$, we have that

$$\phi([x] + \lambda[y]) = \phi([x + \lambda y]) = T(x + \lambda y) = T(x) + \lambda T(y) = \phi([x]) + \lambda\phi([y]).$$

Evidently we have that ϕ is surjective, because, given $y = T(x) \in \mathrm{Im}(T)$, then $\phi([x]) = T(x) = y$. Finally, ϕ is one-to-one, if $[z] \in \mathrm{Ker}(\phi)$, then

$$\phi([z]) = T(z) = 0.$$

Thus, $z \in \mathrm{Ker}(T)$, implying that $[z] = [0]$. Then ϕ is an linear isomorphism of linear spaces.