Weakly Associated Primes

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Abstract

In this paper I'll present a brief introduction to the concept of weakly associated primes and their main results. This concept several times is not mentioned in Commutative Algebra classes, because we'll see that, in Noetherian rings which are the more interesting class of rings for us commutative algebrists, this concept coincides with the associated primes one. I'll assume the reader has a basic knowledge of Commutative Algebra and its classical notation.

As it is a topic of commutative algebra, it'll be assumed that all rings are commutative and have unity. Let's look the main definition of this text.

Definition 1. *Let R be a ring and M be an R-module. A prime ideal $\mathfrak{p}$ is called a weakly prime associated of M if there is $m \in M$ such that*

$$\mathfrak{p} \in \mathrm{MinV}(\mathrm{Ann}(m)) = \mathrm{MinV}(0 :_R m)$$

The set of all weakly associated prime of M will be denoted by $\mathrm{WeakAss}(M)$.

Given $\mathfrak{p}$ an associated prime of an R-module M, we know that there is $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}(m)$. Thus, it's clear that $\mathfrak{p} \in \mathrm{MinV}(\mathrm{Ann}(m))$, then $\mathfrak{p}$ is weakly associated prime of M, obtaining the following inclusion

$$\mathrm{Ass}(M) \subseteq \mathrm{WeakAss}(M).$$

Now it'll be seen the behavior of the localization of a R-module M in some weakly associated prime $\mathfrak{p} \in \mathrm{WeakAss}(M)$.

Proposition 2. *Let R be a ring, M be an R-module and $\mathfrak{p}$ a prime ideal of R. The following assertions are equivalent* *(i)* $\mathfrak{p}$ *is a weakly associated prime of* M*;*

(i) $\mathfrak{p}R_{\mathfrak{p}}$ *is a weakly associated prime of* $M_{\mathfrak{p}}$*;*

(i) $M_{\mathfrak{p}}$ *contains an element whose annihilator has radical equal to* $\mathfrak{p}R_{\mathfrak{p}}$*.*

Proof: $(i) \to (ii)$: Let $\mathfrak{p} \in \text{WeakAss}_R(M)$, then there is $m \in M$ such that $\mathfrak{p} \in \text{MinV}_R(\text{Ann}(m))$. Since (m) is a finite R-module, we have that

$$(\text{Ann}_R(m))_{\mathfrak{p}} = (0 :_R (m))_{\mathfrak{p}} = 0 :_{R_{\mathfrak{p}}} (m/1) = \text{Ann}_{R_{\mathfrak{p}}}(m/1)$$

Thus, by inclusion correspondence, we have that $\mathfrak{p}R_{\mathfrak{p}}$ is a prime ideal containing $\text{Ann}_{\mathfrak{p}}(m)$ and, again, by inclusion correspondence, $\mathfrak{p}R_{\mathfrak{p}} \in \text{MinV}_{R_{\mathfrak{p}}}(\text{Ann}_{R_{\mathfrak{p}}}(m))$, then $\mathfrak{p}R_{\mathfrak{p}} \in \text{WeakAss}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$.
$(ii) \to (iii)$: Suppose that $\mathfrak{p}R_{\mathfrak{p}}$ is a weakly associated prime of $M_{\mathfrak{p}}$, then there exists $m/s \in M_{\mathfrak{p}}$ such that

$$\mathfrak{p}R_{\mathfrak{p}} \in \text{MinV}(\text{Ann}_{R_{\mathfrak{p}}}(m/s))$$

That is, $\mathfrak{p}R_{\mathfrak{p}}$ contains minimally $\text{Ann}_{R_{\mathfrak{p}}}(m)$. Since the radical of any ideal is the intersection of all prime ideals which contain it and $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$, we conclude that $\sqrt{Ann_{R_{\mathfrak{p}}}(m)} = \mathfrak{p}R_{\mathfrak{p}}$.
$(iii) \to (i)$: If $M_{\mathfrak{p}}$ contains an element m whose annihilator has radical equal to $\mathfrak{p}R_{\mathfrak{p}}$, then $\text{Ann}_{R_{\mathfrak{p}}}(m)$ is contained minimally in $R_{\mathfrak{p}}$, then, by inclusion correspondence, we have that $\text{Ann}_R(m)$ is minimally contained in $\mathfrak{p}$, Hence $\mathfrak{p} \in \text{WeakAss}_R(M)$. □

In particular, this proposition says that if $\mathfrak{p} \in \text{WeakAss}_R(M)$, we can suppose without lost of generality that R is a local ring, because it's enough to locate in $\mathfrak{p}$.

Proposition 3. *For any ring* R*, the minimal primes are weakly associated primes of* R*.*

Proof: Let $\mathfrak{p}$ a minimal prime of R. We have two cases.

- If $\mathfrak{p} = 0$, then $R_{\mathfrak{p}}$ is field. Thus, given $x/1 \neq 0 \in R_{\mathfrak{p}}$, we have $0 :_{R_{\mathfrak{p}}} (x/1) = 0 = \mathfrak{p}R_{\mathfrak{p}}$. Since

$$0 :_{R_{\mathfrak{p}}} (x/1) = (0 :_R x)_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$$

  we conclude that $(0 :_R x) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is already a minimal prime of R, we conclude $\mathfrak{p} \in \text{WeakAss}(R)$

- Suppose now $\mathfrak{p} \neq 0$. Take $x/1 \neq 0 \in R_{\mathfrak{p}}$. It's clear that $0 :_{R_{\mathfrak{p}}} (x/1) \subseteq \mathfrak{p}R_{\mathfrak{p}}$, because, if not, there would be z unit such that $z(x/1) = 0$, which would imply that $(x/1) = 0$, a contradiction. Since $0 :_{R_\mathfrak{p}} (x/1) \subseteq \mathfrak{p}R_\mathfrak{p}$, we again have that $0 :_R x \subseteq \mathfrak{p}$ and, since $\mathfrak{p}$ is minimal prime, $\mathfrak{p} \in \mathrm{WeakAss}(R)$.

□

On the next proposition, we'll prove that, if R is reduced, then the set of weakly associated primes of R is exactly the set of its minimal primes. Remember that a ring is said reduced if $\mathrm{nil}(R) = 0$.

Proposition 4. *For a reduced ring R, the set of weakly associated primes of R is the set of minimal primes.*

Proof: Let $\mathfrak{p} \in \mathrm{WeakAss}_R(R)$. Let $x \in R$ such that $0 :_R x$ is minimally contained in $\mathfrak{p}$. Localizing in $\mathfrak{p}$, we have that $0 :_{R_\mathfrak{p}} (x/1)$ has radical $\mathfrak{p}R_\mathfrak{p}$. Observe that $R_\mathfrak{p}$ is even a reduced ring. Suppose by contradiction that $\mathfrak{p}R_\mathfrak{p} \neq 0$. Since $\sqrt{0 :_{R_\mathfrak{p}} (x/1)} = \mathfrak{p}R_\mathfrak{p}$, $x/1$ cannot be unit in $R_\mathfrak{p}$, then $x/1 \in \mathfrak{p}R_\mathfrak{p}$, so there $n \in \mathbb{N}$ such that $(x/1)^n \in 0 :_{R_\mathfrak{p}} (x/1)$ and, hence, $(x/1)^{n+1} = 0$. Since $R_\mathfrak{p}$ is reduced and $x/1 \in \mathrm{nil}(R_\mathfrak{p})$, then $x/1 = 0$. This fact implies that $0 :_{R_\mathfrak{p}} (x/1) = R_\mathfrak{p}$, which is a contradiction. Then $\mathfrak{p}R_\mathfrak{p} = 0$. This means that $\mathfrak{p}$ is minimal ideal of R, because, if not, it would contradict the Inclusion Correspondence Theorem. □

On the next proposition, we'll prove, as the associated primes of an R-module M, we also have that $\mathrm{WeakAss}_R(M) \subseteq \mathrm{Supp}_R(M)$.

Proposition 5. *Let R be a ring and M be an R-module. Then*

$$\mathrm{Ass}_R(M) \subseteq \mathrm{WeakAss}_R(M) \subseteq \mathrm{Supp}_R(M)$$

Proof: The first inclusion was already justified previously and follows direct from the definitions. Now let $\mathfrak{p} \in \mathrm{WeakAss}(M)$, then there exists $m \in M$ such that $0 :_R m$ is minimally contained in $\mathfrak{p}$. Thus we have that $m/1 \neq 0$ in $M_\mathfrak{p}$, because there is not any element x of $R \setminus \mathfrak{p}$ such that $xm = 0$. Hence $M_\mathfrak{p} \neq 0$ and $\mathfrak{p} \in \mathrm{Supp}(M)$. □

On other hand, the next Proposition shows that, given an R-module M, every prime ideal $\mathfrak{p} \in \mathrm{MinSupp}(M)$ is also a weakly associated prime of M.

Proposition 6. *Let R be a ring and M be an R-module. Then*

$$\mathrm{MinSupp}(M) \subseteq \mathrm{WeakAss}(M).$$

Proof: In fact, let $\mathfrak{p} \in \mathrm{MinSupp}(M)$. We know that there is $m/s \neq 0 \in M_\mathfrak{p}$. I claim that $0 :_R m \subseteq \mathfrak{p}$. Indeed, since $m/s \neq 0$ in $M_\mathfrak{p}$, for every $t \in R \setminus \mathfrak{p}$, we have $tm \neq 0$, thus $0 :_R m \subseteq \mathfrak{p}$. Moreover, if $\mathfrak{q}$

is a prime ideal such that

$$0 :_R m \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$$

then $m/s \neq 0$ in $M_{\mathfrak{q}}$, which implies that $M_{\mathfrak{q}} \neq 0$. However, this fact contradicts the minimality of $\mathfrak{p}$ in $\mathrm{Supp}(M)$. Thus $0 :_R m$ is minimally contained in $\mathfrak{p}$ and then $\mathfrak{p} \in \mathrm{WeakAss}(M)$. □

The next proposition shows the behaviour of weakly associated primes in a short exact sequence of R-modules.

Proposition 7. *Let R be a ring and*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of R-modules. Then

(i) $\mathrm{WeakAss}(M') \subseteq \mathrm{WeakAss}(M)$;

(ii) $\mathrm{WeakAss}(M) \subseteq \mathrm{WeakAss}(M') \cup \mathrm{WeakAss}(M'')$.

Proof: (i): In fact, let $\mathfrak{p} \in \mathrm{WeakAss}(M)$, then there is $m \in M$ such that $0 :_R m$ is minimally contained in $\mathfrak{p}$. Observe that $0 :_R f(m)$ is contained in $\mathfrak{p}$. Indeed, if $x \in 0 :_R f(m)$, then $xf(m) = f(xm) = 0$. Since f is a monomorphism, we conclude that $xm = 0$ and then $x \in 0 :_R m \subseteq \mathfrak{p}$. Moreover, I claim that $0 :_R f(m)$ is minimally contained in $\mathfrak{p}$. If not, suppose there a prime ideal $\mathfrak{q}$ such that

$$0 :_R f(m) \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$$

Thus, given $x \in 0 :_R m$, we have that $0 = f(0) = f(xm) = xf(m)$, then we would conclude that

$$0 :_R m \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$$

which is a contradiction, since $0 :_R m$ is minimally contained in $\mathfrak{p}$.
(ii) Let $\mathfrak{p} \in \mathrm{WeakAss}(M)$. Since the localization functor is exact, we have the following exact sequence

$$0 \longrightarrow M'_{\mathfrak{p}} \xrightarrow{f'} M_{\mathfrak{p}} \xrightarrow{g'} M''_{\mathfrak{p}} \longrightarrow 0$$

Since $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{WeakAss}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, the Proposition 2 tell us that there exists $m/1 \in M_{\mathfrak{p}}$ such that $\sqrt{0 :_{R_{\mathfrak{p}}} m/1} = \mathfrak{p}R_{\mathfrak{p}}$. Now we have two cases:

- If $m/1$ is image of f'. Then $g'(m/1) = 0$ since all exact sequence is, a prior, a complex of R-modules. In this case, if $m/1 = f'(m'/s)$ for some $m'/s \in M'_{\mathfrak{p}}$, we have

$$0 :_{R_{\mathfrak{p}}} (m'/s) \subseteq \mathfrak{p}R_{\mathfrak{p}}$$

Indeed, if $z \in 0 :_{R_{\mathfrak{p}}} (m'/s)$, then $zm'/s = 0$, so $zf'(m'/s) = zm/1 = 0$, which implies $z \in 0 :_{R_{\mathfrak{p}}} (m/1) \subseteq \mathfrak{p}$. Moreover, we have that $0 :_{R_{\mathfrak{p}}} (m'/s)$ is minimally contained in $\mathfrak{p}R_{\mathfrak{p}}$. If not, that is, if $0 :_{R_{\mathfrak{p}}} (m'/s) \subseteq \mathfrak{q}R_{\mathfrak{p}} \subsetneq \mathfrak{p}R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{q}R_{\mathfrak{p}}$, then given $z \in 0 :_{R_{\mathfrak{p}}} (m/1)$, then $zm/1 = 0$, so $zf'(m'/s) = f'(zm'/s) = 0$, which implies $z(m'/s) = 0$, since f' is injective, then

$$z \in 0 :_{R_{\mathfrak{p}}} (m'/s) \subseteq \mathfrak{q}R_{\mathfrak{p}} \subsetneq \mathfrak{p}R_{\mathfrak{p}}$$

Thus $0 :_{R_{\mathfrak{p}}} (m/1) \subseteq \mathfrak{q}R_{\mathfrak{p}} \subsetneq \mathfrak{p}R_{\mathfrak{p}}$, which contradicts the minimally of $\mathfrak{p}R_{\mathfrak{p}}$. Thus, by Proposition 2, we have $\mathfrak{p} \in \operatorname{WeakAss}(M')$, thus

$$\operatorname{WeakAss}(M) \subseteq \operatorname{WeakAss}(M').$$

- If $m/1$ isn't image of f', $g'(m/1) \neq 0$. In this case, I claim that

$$\sqrt{0 :_{R_{\mathfrak{p}}} g'(m/1)} = \mathfrak{p}R_{\mathfrak{p}}.$$

Indeed, if $z \in \mathfrak{p}R_{\mathfrak{p}} = \sqrt{0 :_{R_{\mathfrak{p}}} (m/1)}$, then $z^n(m/1) = 0$ for some $n \in \mathbb{N}$, so $z^n g'(m/1) = 0$ and then $z \in \sqrt{0 :_{R_{\mathfrak{p}}} g'(m/1)}$; Conversely, if $z \in \sqrt{0 :_{R_{\mathfrak{p}}} g'(m/1)}$, then $z^r g'(m/1) = 0$ for some $r \in \mathbb{N}$. Since $g'(m/1) \neq 0$ and z cannot be unity element, we conclude that $z \in \mathfrak{p}R_{\mathfrak{p}}$ and so $\sqrt{0 :_{R_{\mathfrak{p}}} g'(m/1)} = \mathfrak{p}R_{\mathfrak{p}}$. By Proposition 2, we conclude $\mathfrak{p} \in \operatorname{WeakAss}(M")$, thus

$$\operatorname{WeakAss}(M) \subseteq \operatorname{WeakAss}(M")$$

Finally, we conclude $\operatorname{WeakAss}(M) \subseteq \operatorname{WeakAss}(M') \cup \operatorname{WeakAss}(M")$. □

The next Proposition will show that every nonzero R-module has a weakly associated prime

Proposition 8. *Let R be a ring and M be an R-module. Then*

$$M = 0 \Longleftrightarrow \operatorname{WeakAss}(M) = \varnothing.$$

Proof: If $M = 0$, then, for all $m \in M$, we have $0 :_R m = R$, thus it's clear that $\operatorname{WeakAss}_R(M) = \varnothing$. Conversely, suppose $M \neq 0$ and let $m \neq 0 \in M$. Consider $I = 0 :_R m$ and define the map

$$\phi : R \longrightarrow M$$

$$r \longmapsto rm$$

This map has kernel equal to I, so there is the following exact sequence

$$0 \longrightarrow R/I \xrightarrow{\phi} M \longrightarrow \operatorname{Coker}(\phi) \longrightarrow 0$$

Since $m \neq 0$, I is a proper ideal of R. Moreover $I = 0 :_R \overline{1}$. Thus any prime ideal $\mathfrak{p} \in \operatorname{MinV}(I) \neq \varnothing$ belongs to $\operatorname{WeakAss}(R/I)$. Using the Proposition 7, we conclude that $\operatorname{WeakAss}(M) \neq \varnothing$. □

Proposition 9. *Let R be a ring, M be an R-module. Then the set of zerodivisors of M, denoted by $\mathcal{Z}(M)$, is exactly*

$$\bigcup_{\mathfrak{p}\in\operatorname{WeakAss}(M)} \mathfrak{p}.$$

Proof: Let r be a zerodivisor of M, then there is $m \in M$ such that $r \in 0 :_R m \neq R$. Let $\mathfrak{q}$ be a prime ideal which contains minimally $0 :_R m$, thus, by definition, $\mathfrak{q} \in \operatorname{WeakAss}(M)$. Thus $z \in 0 :_R m \subseteq \mathfrak{q}$, that is

$$\mathcal{Z}(M) \subseteq \bigcup_{\mathfrak{p}\in\operatorname{WeakAss}(M)} \mathfrak{p}.$$

Conversely, let $f \in \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{WeakAss}(M)$. Let $m \in M$ such that $0 :_R m$ is minimally contained in $\mathfrak{q}$. Localizing R in $\mathfrak{q}$, we obtain that $0 :_R m$ is minimally contained in $\mathfrak{q}R_{\mathfrak{q}}$. Since

$$\sqrt{0 :_{R_{\mathfrak{q}}} m} = \mathfrak{q}R_{\mathfrak{q}},$$

there is $n \in \mathbb{N}$ such that $f^n m/1 = 0$, thus there is $g \in R \smallsetminus \mathfrak{q}$ such that $gf^n gm = 0$ in R. Thus, summing up, we can choose $g \in R \smallsetminus \mathfrak{q}$ and $n \in \mathbb{N}$ such that $gf^n m = 0$. Note that $gm \neq 0$ because $g \notin 0 :_R m$. Choose n such that n is the minimum natural number such that $gf^n m = 0$. Thus $gf^{n-1}m \neq 0$ and $f(gf^{n-1}m) = 0$, implying that f is a zerodivisor of M. Then

$$\bigcup_{\mathfrak{p}\in\operatorname{WeakAss}(M)} \mathfrak{p} \subseteq \mathcal{Z}(M).$$

□

Since the more interesting rings of Commutative Algebra are always Noetherian, the next Proposition may explain why the concept of Weakly associated primes isn't so well known and studied in Commutative Algebra classes.

Proposition 10. *Let R be a ring, M be an R-module and $\mathfrak{p}$ a finitely generated prime ideal of R. Then*

$$\mathfrak{p} \in \operatorname{Ass}(M) \Longleftrightarrow \mathfrak{p} \in \operatorname{WeakAss}(M).$$

In particular, if R is a Noetherian ring, then $\operatorname{Ass}(M) = \operatorname{WeakAss}(M)$.

Proof: If $\mathfrak{p}$ is an associated prime of M, follows from Proposition 5 that $\mathfrak{p} \in \operatorname{WeakAss}(M)$. Conversely, write $\mathfrak{p} = (g_1, \ldots, g_n)$. By Proposition 2, there is $m \in M_{\mathfrak{p}}$ such that

$$\sqrt{0 :_{R_{\mathfrak{p}}} m} = \mathfrak{p}R_{\mathfrak{p}}.$$

Thus, for each $i \in \{1, 2 \ldots, n\}$, there the minimum n_i such that $g_i^{n_i} m = 0$. If $e_i > 1$ for some $i \in \{1, 2 \ldots, n\}$, then we we can replace m by $g_i^{e_i - 1} m \neq 0$ and decrease the following sum

$$\sum_{i=1}^{n} e_i.$$

Since this sum is always lower bounded by 0, we can repeat this process finitely many times and suppose that $\mathfrak{p} R_{\mathfrak{p}} = 0 :_{R_{\mathfrak{p}}} m$, then we conclude that $\mathfrak{p} \in \operatorname{Ass}(M)$. □

It's well known that if $\phi : R \longrightarrow S$ is a ring homomorphism, then this map induces a map on the spectrum of these rings:

$$\begin{aligned} \operatorname{Spec}(\phi) : \operatorname{Spec}(S) &\longrightarrow \operatorname{Spec}(R) \\ \mathfrak{p} &\longmapsto \phi^{-1}(\mathfrak{p}) \end{aligned}$$

Moreover, if W is multiplicatively closed, then ϕ induces

$$\begin{aligned} \phi' : R_W &\longrightarrow S_{\phi(W)} \\ x &\longmapsto \phi(x)/1 \end{aligned}$$

Having these facts in mind, we have the next Proposition.

Proposition 11. *Let $\phi : R \longrightarrow S$ be a ring homomorphism and M be an S-module. Then*

$$\operatorname{WeakAss}_R(M) \subseteq \operatorname{Spec}(\phi)(\operatorname{WeakAss}_S(M)).$$

Proof: Let $\mathfrak{p} \in \operatorname{WeakAss}_R(M)$. We know there is $m \in M_{\mathfrak{p}}$ such that $0 :_{R_{\mathfrak{p}}} m$ is minimally contained in $R_{\mathfrak{p}}$ and this prime ideal is the unique prime ideal containing $0 :_{R_{\mathfrak{p}}} m$. Consider $T = R \smallsetminus \mathfrak{p}$, it's well known that T and $W := \phi(T)$ are multiplicatively closed sets. Denote $S_{\mathfrak{p}}$ the localization of S in T. Consider now the ideal $0 :_{S_{\mathfrak{p}}} m$. As $(0 :_{R_{\mathfrak{p}}} m) S \subseteq 0 :_{S_{\mathfrak{p}}} m$ and $\mathfrak{p} R_{\mathfrak{p}}$ is the unique prime ideal of $R_{\mathfrak{p}}$ containing $0 :_{R_{\mathfrak{p}}} m$, any prime ideal $\mathfrak{q} S_{\mathfrak{p}}$ of S containing minimally $0 :_{S_{\mathfrak{p}}} m$ is such that $\mathfrak{p} R_{\mathfrak{p}} \subseteq \operatorname{Spec}(\phi')(\mathfrak{q} S_{\mathfrak{p}})$. Moreover, since $\mathfrak{p} R_{\mathfrak{p}}$ is maximal ideal, we have

$$\mathfrak{p} R_{\mathfrak{p}} = \operatorname{Spec}(\phi')(\mathfrak{q} S_{\mathfrak{p}})$$

Thus by correspondence theorem, $\mathfrak{q} S_{\mathfrak{p}}$ contains minimally $0 :_{S_{\mathfrak{p}}} m$, so $\mathfrak{q} S_{\mathfrak{p}} \in \operatorname{WeakAss}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}})$ and thus, returning to the original rings, we obtain that

$$\mathfrak{q} \in \operatorname{WeakAss}_S(M).$$

Thus returning to the original ring, we have that

$$\mathfrak{p} = \operatorname{Spec}(\phi)(\mathfrak{q}), \text{ where } \mathfrak{q} \in \operatorname{WeakAss}_S(M)$$

Then

$$\mathrm{WeakAss}_R(M) \subseteq \mathrm{Spec}(\phi)(\mathrm{WeakAss}_S(M)).$$

$\square$

Given an R-module M and S a multiplicatively closed set of R, it's known that M_S has a natural structure of R-module under the ring homomorphism

$$\psi : R \longrightarrow R_S$$

$$x \longmapsto x/1$$

Proposition 12. *Let R be a ring, M be an R-module and S a multiplicatively closed set. Assume that every element of S is a nonzerodivisor of M. Then*

$$\mathrm{WeakAss}_R(M) = \mathrm{WeakAss}_R(M_S).$$

Proof: Since every element of S is a nonzerodivisor of M, we have that the R-module homomorphism $\phi : M \longrightarrow M_S$ is injective. By Proposition 7, we have

$$\mathrm{WeakAss}_R(M) \subseteq \mathrm{WeakAss}_R(M_S).$$

Conversely, let $\mathfrak{p} \in \mathrm{WeakAss}_R(M_S)$ and $m/s \in M_S$ such that $0 :_R (n/s)$ is minimally contained in $\mathfrak{p}$. Using th e hypothesis that every element of S is nonzerodivisor of M, it's easy to prove that $0 :_R (n/s) = 0 :_R n$, thus $0 :_R n$ is minimally contained in $\mathfrak{p}$, then $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$, that is

$$\mathrm{WeakAss}_R(M_S) \subseteq \mathrm{WeakAss}_R(M).$$

Then

$$\mathrm{WeakAss}_R(M_S) = \mathrm{WeakAss}_R(M).$$

$\square$

Proposition 13. *Let R be a ring and M be an R-module. Then the natural map*

$$\phi : M \longrightarrow \prod_{\mathfrak{q} \in \mathrm{WeakAss}(M)} M_{\mathfrak{q}}$$

is injective.

Proof: Let $m \in \ker(\phi)$ and consider an R-module $N := Rm$. Since $N \subseteq M$, we already saw that if $\mathfrak{p} \in \mathrm{WeakAss}(N)$, then $\mathfrak{p} \in \mathrm{WeakAss}(M)$. But $N_{\mathfrak{p}} \neq 0$, since $m/1 \neq 0$. As the localization functor is exact, we conclude that $x/1 \neq 0 \in M_{\mathfrak{p}}$ and that $\phi(m) \neq 0$, which is a contradiction, because $\phi(m) = 0$ by hypothesis. $\square$