A Quick Tour in the Theory of F-Regular Rings

Kevin Alves Vasconcellos

In this article we shall exclusively consider commutative rings with units. The letter p will reserved to represent a prime number, while q will be defined as p^e, where e is a positive integer. Furthermore, when we refer to "for $q \gg 0$," we mean for $q = p^e$ with $e \gg 0$. For a given ideal I, we shall denote $I^{[q]}$ as the ideal generated by taking the q-th powers of the elements belonging to I.

1 Frobenius Homomorphism

Definition 1.1. *Let R be a ring. One says that R has characteristic p if the kernel of ring homomorphism $\mathbb{Z} \longrightarrow R$ is the ideal generated by p.*

Definition 1.2. *Let R be a ring of characteristic p. The map*

$$\mathrm{F} : R \longrightarrow R$$

$$x \longmapsto x^p$$

is called the Frobenius homomorphism.

Indeed F is a ring homomorphism. Thus it is possible to equip R with an R-algebra structure induced by the map F and we denote it by $\mathrm{F}_*(R)$. The action of Frobenius on a ring of prime characteristic has a long history of being used to characterize the singularities of the associated varieties. Work of Kunz shows that a Noetherian ring R of characteristic p is regular if and only if $\mathrm{F}_*(R)$ is a flat R-algebra. This powerful Theorem is known as Frobenius Flatness.

Theorem 1.3 ([Kun76])**.** *Let R be a Noetherian ring of characteristic p. Then R is regular if and only if the Frobenius homomorphism $\mathrm{F} : R \longrightarrow R$ is flat.*

Singularities that are characterized by the behavior of the Frobenius homomorphism are commonly referred to *F-singularities*, and rings of characteristic p can are categorized based on the type of F-singularity they exhibit. In the existing literature, numerous well-established F-singularities are recognized, including:

$$F-\begin{cases} Injective \\ Finite \\ Pure \\ Split \\ Regular \\ Rational \end{cases} \qquad rings$$

In this lecture, we will define the class of F-regular rings. However, before delving into F-regularity, it is essential to first explore the concept of the tight closure of ideals.

2 Tight closure of ideals

In this section we will assume that all rings are Noetherian and of characteristic p. Let R be a ring, we will denote by R^o the set of all elements $x \in R$ such that x does not belong to any minimal prime of R, that is

$$R^o = R \setminus \bigcup_{\mathfrak{p} \in \operatorname{MinSpec}(R)} \mathfrak{p}.$$

Notice that R^o is a multiplicatively closed subset of R and $R^o = R \setminus \{0\}$ when R is an integral domain.

Definition 2.1. *Let R be a Noetherian ring of characteristic p and I an ideal of R. The Tight closure I^* of I is the set of all elements $x \in R$ such that there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for $q \gg 0$. The ideal I is said tightly closed if $I = I^*$.*

From definition, we can see that if $cx^{q_0} \in I^{[q_0]}$ for some $q_0 = p^{e_0}$, then $cx^q \in I^{[q]}$ for all $q \geq q_0$. Indeed if $cx^{q_0} \in I^{[q_0]}$, then

$$(cx^{p^{e_0}})^p = c^p x^{p^{e_0+1}} \in (I^{[q_0]})^{[p]} = I^{[p^{e_0+1}]}.$$

It is straightforward to show that I^* is an ideal of R and that I^* contains I. Moreover the tight closure behaves as expected for an operation of closure.

Proposition 2.2. *Let R be a Noetherian ring of characteristic p and I, J ideals of R. Then the following holds*

(i): *If $I \subseteq J$, then $I^* \subseteq J^*$;*

(ii): *There exists $c \in R^o$ such that $c(I^*)^{[q]} \subseteq I^{[q]}$ for $q \gg 0$;*

(iii): $(I^*)^* = I^*$

(iv): *If I is tightly closed, then so is $I :_R J$;*

(v): *$x \in I^*$ if and only if $\overline{x} \in ((I + \mathfrak{p})/\mathfrak{p})^*$ for all $\mathfrak{p} \in \operatorname{MinSpec}(R)$;*

(vi): *$(I \cap J)^* \subseteq I^* \cap J^*$. In particular if I and J are tightly closed, then so is $I \cap J$;*

(vii): $(I + J)^* = (I^* + J^*)^*$;

(viii): $(IJ)^* = (I^* J^*)^*$;

(ix): $(0)^* = \sqrt{(0)}$.

Proof: The proofs of these properties are straightforward, with some of them documented in Proposition 10.1.2 of [BH93]. □

Example 2.3. *Let $R = \mathbb{F}_3[x, y]/(x^2 - y^3)$. Then $x \in (y)^*$; Indeed observe that $x \in R^0$. Thus*

$$x \cdot x^{3^e} = (x^2)^{\frac{3^e+1}{2}} = (y^3)^{\frac{3^e+1}{2}} = y^{3^e} \cdot (y^3)^{\frac{3^e+1}{2}} \in (y^{3^e}) = (y)^{[3^e]}.$$

Thus (y) is not tightly closed.

The following theorem demonstrates a good connection between the ideal of q-powers and quotient ideals when the ring is regular. Furthermore it establishes that the concept of tight closure is trivial within this context.

Theorem 2.4. *Let R be a regular ring of characteristic p. Then*

(i): *$I^{[q]} :_R J^{[q]} = (I :_R J)^{[q]}$ for all ideals I and J of R;*

(ii): *For every ideal I of R, I is tightly closed closed;*

Proof: (i) : Since R is a regular ring, then $F : R \longrightarrow F_*(R)$ is flat ring homomorphism and so does $F^{(e)} : R \longrightarrow F_*^e(R)$ for every $e \geq 1$. Thus, given ideals I and J of R, we have

$$(I :_R J)^{[q]} = (I :_R J)F_*^e(R) = (IF_*^e(R)) :_{F_*^e(R)} (JF_*^e(R)) = (I^{[q]}) :_{F_*^e(R)} (J^{[q]}) = I^{[q]} :_R J^{[q]}.$$

(ii) : Let I be an ideal of R. Suppose by contradiction that I is not tightly closed, thus there exists $x \in I^*$ such that $x \notin I$. Since $x \in I^*$, there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$. By hypothesis, we have that $I :_R x \neq R$. Considering $I :_R x \subseteq \mathfrak{p}$ a prime ideal, observe x/1 still is an element of $I_{\mathfrak{p}}^*$ such that $x \notin I_{\mathfrak{p}}$, thus we can assume without lost of generality that $R = (R, \mathfrak{m})$ is a regular local ring. Observe that

$$c \in I^{[q]} :_R x^q = (I :_R x)^{[q]} \subseteq (I :_R x)^q$$

for all $q \gg 0$, which implies that

$$c \in \bigcap_{e=1}^{\infty} (I :_R x)^{p^e} = 0,$$

which is a contradiction. Thus $I = I^*$ is tightly closed. □

Recall that, in a local ring R, a sequence $\mathbf{x} : x_1, \ldots, x_n$ contained in maximal ideal is said R-regular if

$$(x_1, \ldots, x_i) :_R x_{i+1} = (x_1, \ldots, x_i)$$

for all $i = 0, \ldots, n-1$. The next result due Hochester and Huneke reveals us that, under certain conditions, every system of parameters of R is a regular sequence up to its tight closure.

Theorem 2.5 (Hochster-Huneke)**.** *Let R be an equidimensional Noetherian ring of characteristic p. Suppose that R is homomorphic image of a Cohen-Macaulay local ring A. Let $x_1, \ldots, x_d$ be a system of parameters of R. Then*

$$(x_1, \ldots, x_{i-1}) :_R x_i \subseteq (x_1, \ldots, x_{i-1})^*$$

for all $i = 1, \ldots, d$.

Proof: Consult the Theorem 10.1.9 of [BH93]. □

3 F-regular rings

We have already seen that regular rings have the property that every ideal is tightly closed. However the converse is not true. Hochester and Huneke proved in [HH94] that any generic determinantal ring shares this property.

Definition 3.1. *Let R be a Noetherian ring of characteristic p.*

- *One says that R is a weakly F-regular ring if every ideal of R is tightly closed;*
- *One says that R is a F-regular if R_S is multiplicatively closed for every multiplicative subset $S \subseteq R$.*

One of Huneke's frustrations stemmed from the difficulty in proving the commutativity of Tight Closure with localization. In chapter 12 of [Hun96], he says

"This chapter is devoted to the most frustrating problem in the Theory of Tight Closure. From the first day it was clearly an important problem to know that tight closure commutes with localization."

Unfortunately this problem became real when H. Brenner and P. Monsky gave the first counterexample of a weakly F-regular ring, which is not F-regular in 2010 in [BM10]. The following lemma unveils a profound relationship between tightly closed ideals and ideals that are primary to maximal ideals.

Lemma 3.2. *Let R be a Noetherian ring of characteristic p and I an ideal of R.*

(i): If I is an $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$, then $(IR_{\mathfrak{m}})^* = I^* R_{\mathfrak{m}}$;

(ii): If every ideal primary to a maximal ideal of R is tightly closed, then R is weakly F-regular.

Proof: Consult the Proposition 10.1.12 of [BH93]. □

Proposition 3.3. *Let R be a Noetherian ring of characteristic p.*

(i): R is weakly F-regular if and only if $R_{\mathfrak{m}}$ is weakly F-regular for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$;

(ii): A weakly F-regular is normal;

(iii): If R is weakly F-regular and class ring of a Cohen-Macaulay ring, then R is Cohen-Macaulay.

Proof: (i) : Suppose that R is weakly F-regular and let $\mathfrak{m}$ be a maximal ideal of R. Let $IR_{\mathfrak{m}}$ be an $\mathfrak{m}R_{\mathfrak{m}}$-primary ideal of $R_{\mathfrak{m}}$. Since R is weakly F-regular, the previous lemma gives us that

$$(IR_{\mathfrak{m}})^* = I^* R_{\mathfrak{m}} = IR_{\mathfrak{m}}.$$

Thus $IR_{\mathfrak{m}}$ is tightly closed. Hence the previous Lemma gives that $R_{\mathfrak{m}}$ is weakly F-regular ring. Now suppose that $R_{\mathfrak{m}}$ is weakly F-regular for all $\mathfrak{m} \in \operatorname{MaxSpec}(R)$. Let I be an $\mathfrak{m}$-primary ideal of R, thus $IR_{\mathfrak{m}}$ is an $\mathfrak{m}R_{\mathfrak{m}}$-primary ideal of $R_{\mathfrak{m}}$. Since $R_{\mathfrak{m}}$ is weakly F-regular, applying the previous lemma, one obtains

$$I^* R_{\mathfrak{m}} = (IR_{\mathfrak{m}})^* = IR_{\mathfrak{m}}$$

Since $\operatorname{ass}(I) \subseteq \{\mathfrak{m}\}$, one concludes that $I^* = I$.

(ii) : It is enough to use the criterion of normality: Let R be a Noetherian ring. R is a Normal ring if and only if R satisfies the following two conditions

- $R_{\mathfrak{p}}$ is a field for all $\mathfrak{p} \in \operatorname{MinSpec}(R) \cap \operatorname{MaxSpec}(R)$;
- Every ideal of form (x) with $x \in R^o$ is integrally closed.

It is possible to prove that, for every principal ideal I, one has $\overline{I} = I^*$. Thus, if R is weakly F-regular, then $I = I^* = \overline{I}$ for every principal ideal. Furthermore, given $\mathfrak{q} \in \operatorname{MinSpec}(R) \cap \operatorname{MaxSpec}(R)$, $R_{\mathfrak{q}}$ is a weakly F-regular 0-dimensional local reduced ring, and so a field.

(iii) : It is enough to show that $R_{\mathfrak{m}}$ is Cohen-Macaulay for all $\mathfrak{m} \in \operatorname{MaxSpec}(R)$. Since R is weakly F-regular, then $R_{\mathfrak{m}}$ is weakly F-regular for all $\mathfrak{m} \in \operatorname{MaxSpec}(R)$, which implies that $R_{\mathfrak{m}}$ is a local normal domain, and so equidimensional. Observe that $R_{\mathfrak{m}}$ still residual ring of a Cohen-Macaulay ring, so we are conditions to use the Hochster-Huneke Theorem. Now let $x_1, \ldots, x_d$ be a system of parameters of $R_{\mathfrak{m}}$. Using the F-regularity of $R_{\mathfrak{m}}$ and the Hochster-Huneke Theorem, one concludes that

$$(x_1, \ldots, x_i) = (x_1, \ldots, x_i) :_{R_{\mathfrak{m}}} (x_{i+1}) \subseteq (x_1, \ldots, x_i)^* = (x_1, \ldots, x_i)$$

for all $i = 0, 1, \ldots, d-1$, which implies that $x_1, \ldots, x_d$ is a regular sequence in $R_{\mathfrak{m}}$ and so $R_{\mathfrak{m}}$ is Cohen-Macaulay. $\square$

References

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[BM10] Holger Brenner and Paul Monsky. Tight closure does not commute with localization. *Annals of mathematics*, pages 571–588, 2010.

[HH94] Melvin Hochster and Craig Huneke. Tight closure of parameter ideals and splitting in module-finite extensions. *J. Algebraic Geom.*, 3(4):599–670, 1994.

[Hun96] Craig Huneke. *Tight closure and its applications*. Number 88. American Mathematical Soc., 1996.

[Kun76] Ernst Kunz. On noetherian rings of characteristic p. *American Journal of Mathematics*, 98(4):999–1013, 1976.