Занятие 15. Обыкновенные дифференциальные уравнения

9.24 $yy' + x = 0$

$yy' = -x$

$y\frac{dy}{dx} = -x \quad | \times dx$

$y\,dy = (-x)\,dx$

$\int y\,dy = -\int x\,dx + C$

$\frac{y^2}{2} = -\frac{x^2}{2} + C$

$y^2 = -x^2 + 2C$

Ответ: $y^2 = -x^2 + 2C$

9.25 $xy' = 2y$

$y' = \frac{2y}{x}$

$\frac{dy}{dx} = \frac{2y}{x} \quad | \times \frac{1}{y}\,dx$

$\frac{dy}{y} = \frac{2\,dx}{x}$

$\int \frac{dy}{y} = \int \frac{2\,dx}{x} + C$

$\ln|y| = 2\ln|x| + C$

$\ln|y| = \ln(e^c \cdot x^2)$

$y = C_1 x^2$

Ответ: $y = C_1 x^2$

~~9.26 $(x+1)y' + xy = 0$~~

9.26 $(x+1)y' + xy = 0$

$$(x+1)\frac{dy}{dx} = -xy$$

$$\frac{dy}{dx} = -\frac{xy}{x+1} \quad \Big| \times \frac{1}{y}dx$$

$$\frac{dy}{y} = -\frac{x}{x+1}dx$$

$$\int \frac{dy}{y} = -\int \frac{x+1-1}{x+1}dx + C$$

$$\ln|y| = -\left(\int 1\,dx - \int \frac{1}{x+1}d(x+1)\right) + C$$

$$\ln|y| = -x + \ln|x+1| + C$$

$$\ln|y| = \ln(e^{-x}|x+1|e^{c})$$

$$y = C_1 e^{-x}(x+1)$$

Ответ: $y = C_1 e^{-x}(x+1)$

9.28 $y' = e^{x+y}$

$$\frac{dy}{dx} = e^{x}\cdot e^{y} \quad \Big| \times \frac{1}{e^{y}}dx$$

$$\frac{dy}{e^{y}} = e^{x}dx$$

$$\int \frac{dy}{e^{y}} = \int e^{x}dx + C$$

$$\int e^{-y}dy = e^{x} + C$$

$$-\int e^{-y}d(-y) = e^{x} + C$$

9.28 (продолжение)

$$-e^{-y}=e^{x}+C$$

$$-e^{x}-e^{-y}=C$$

$$e^{x}+e^{-y}=C_1$$

Ответ: $e^{x}+e^{-y}=C_1$

9.37 $\quad y'=\cos(x+y)$

Пусть $u(x)=x+y$

$$\frac{du(x)}{dx}=1+\frac{dy}{dx}$$

$$\frac{dy}{dx}=\cancel{1}\frac{du(x)}{dx}-1$$

Тогда

$$\frac{du(x)}{dx}-1=\cos(u(x))$$

$$\frac{du(x)}{dx}=\cos(u(x))+1 \quad \Big| \times \frac{1}{\cos(u(x))+1}dx \quad \text{или} \quad \cos(x+y)=-1$$

$$x+y=\pi+2\pi k;\ k\in\mathbb{Z}$$

$$x+y=(2k+1)\pi,\ k\in\mathbb{Z}$$

$$\frac{du(x)}{\cos(u(x))+1}=1\,dx$$

$$\int\frac{du(x)}{\cos(u(x))+1}=\int 1\,dx+C$$

$$\int\frac{du(x)}{\cos(u(x))+1}=\left|t=u(x)\right|=\int\frac{dt}{\cos t+1}=\left|\begin{array}{l} k=\operatorname{tg}\frac{t}{2} \\ t=2\operatorname{arctg}k \\ dt=\frac{2\,dk}{1+k^2} \\ \cos t=\frac{1-k^2}{1+k^2}\end{array}\right|=$$

$$=\int\frac{2\,dk}{(1+k^2)\left(\frac{1-k^2}{1+k^2}+1\right)}=\cancel{2\int\frac{dk}{2+k^2}}=2\int\frac{dk}{2}=\int 1\,dk=k+C_1=$$

$$\cancel{=\operatorname{tg}(u(x))+C_1}=\operatorname{tg}\frac{u(x)}{2}+C_1$$

$\cancel{tg\,(u(x)) = x + C}$

$$tg\,\frac{u(x)}{2} = x + C$$

$$tg\,\frac{x+y}{2} - x = C$$

Ответ: $tg\,\frac{x+y}{2} - x = C$, $x + y = (2k+1)\pi$, $k \in \mathbb{Z}$

9.38 $y' = \frac{1}{2x+y}$

Пусть $2x + y = u$

$$\frac{du}{dx} = 2 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{du}{dx} - 2$$

$$\frac{du}{dx} - 2 = \frac{1}{u}$$

$$\frac{du}{dx} = \frac{1}{u} + 2 \quad \Big| : \left(\frac{1}{u} + 2\right)\frac{1}{dx}$$

или $\frac{1}{u} + 2 = 0$

$\frac{1+2u}{u} = 0$

$2u + 1 = 0$

$4x + 2y + 1 = 0$ (*)

$$\frac{du}{\frac{1}{u} + 2} = 1\,dx$$

$$\frac{u\,du}{1+2u} = 1\,dx$$

$$\int \frac{u\,du}{1+2u} = \int 1\,dx + C$$

$$\cancel{\int \frac{u\,du}{1+2u} = \int \frac{u\,du}{2\left(\frac{1}{2}+u\right)} = \frac{1}{2}\int \frac{u + \frac{1}{2} - \frac{1}{2}}{u + \frac{1}{2}}\,du = \frac{1}{2}\left(\int 1\,du - \int \frac{1}{u+\frac{1}{2}}\,d\left(u+\tfrac{1}{2}\right)\right) =}$$

$$\cancel{= \frac{1}{2}\left(u - \ln\left|u + \frac{1}{2}\right|\right.}$$

9.38 (продолжение)

$$\int \frac{u\,du}{1+2u} = \int \frac{u\,du}{2\left(\frac{1}{2}+u\right)} = \frac{1}{2}\int \frac{u+\frac{1}{2}-\frac{1}{2}}{u+\frac{1}{2}}\,du = \frac{1}{2}\left(\int 1\,du - \int \frac{1}{2(u+\frac{1}{2})}\,du\right) =$$

$$= \frac{1}{2}\left(u - \int \frac{1}{2u+1}\,du\right) = \frac{u}{2} - \frac{1}{4}\int \frac{1}{2u+1}\,d(2u+1) =$$

$$= \frac{u}{2} - \frac{1}{4}\ln|2u+1| + C_1$$

$$\frac{u}{2} - \frac{1}{4}\ln|2u+1| = x + C$$

$$\frac{2x+y}{2} - \frac{1}{4}\ln|4x+2y+1| = x + C$$

$$\cancel{x} + \frac{y}{2} - \frac{1}{4}\ln|4x+2y+1| = \cancel{x} + C$$

$$-\frac{1}{4}\ln|4x+2y+1| = -\frac{y}{2} + C \quad |\times(-4)$$

$$\ln|4x+2y+1| = 2y + C_2$$

$$\ln|4x+2y+1| = \ln e^{2y} + \ln e^{C_2}$$

$$\ln|4x+2y+1| = \ln\left(e^{C_2}e^{2y}\right)$$

$$4x+2y+1 = C_3 e^{2y} \qquad (*) \text{ входит } (C_3 = 0)$$

Ответ: $4x+2y+1 = C_3 e^{2y}$

9.47 $y' = \frac{y}{x} + \sin\frac{y}{x}$

Замена:

$y = x \cdot z(x) \qquad y' = z(x) + x \cdot z'(x)$

9.47 (продолжение)

$$z + x \cdot z' = z + \sin z$$

$$x \cdot z' = \sin z$$

$$x \frac{dz}{dx} = \sin z \;\Big|:\left(\sin z \cdot \frac{1}{dx}\right) \quad \text{или} \quad \sin z = 0$$

$$\sin \frac{y}{x} = 0$$

$$\frac{y}{x} = \pi k \;;\; k \in \mathbb{Z}$$

$$y = \pi k x \quad k \in \mathbb{Z}$$

$$\frac{x\,dz}{\sin z} = dx \;\Big|:x$$

$$\frac{dz}{\sin z} = \frac{dx}{x}$$

$$\ln\left|tg \frac{z}{2}\right| = \ln|x| + C$$

$$\left|tg \frac{z}{2}\right| = e^c |x|$$

$$tg \frac{y}{2x} = C_1 x$$

$$\frac{y}{2x} = arctg(C_1 x) + \pi k$$

$$y = 2x\,(arctg(C_1 x) + \pi k) \quad k \in \mathbb{Z}$$

Ответ: $2x\,(arctg(C_1 x) + \pi k)$, $y = \pi k x$; $k \in \mathbb{Z}$

9.48 $y' = \dfrac{x-y}{x+y}$

Замена:

$$y = x \cdot z(x) \qquad y' = z(x) + x \cdot z'(x)$$

$$z + x \cdot z' = \frac{1-z}{1+z}$$

$$x \cdot z' = \frac{1-z}{1+z} - z$$

$$x \cdot z' = \frac{1 - z - z - z^2}{1+z}$$

$$x \cdot z' = \frac{1 - 2z - z^2}{1+z}$$

9.43 (продолжение)

$$x\frac{dz}{dx} = \frac{1-2z-z^2}{1+z} \quad \Big| \cdot \frac{dx}{x}$$

$$dz = \frac{1-2z-z^2}{1+z}\,\frac{dx}{x} \quad \Big| : \frac{1-2z-z^2}{1+z} \neq 0 \quad \text{или} \quad 1-2z-z^2=0 \quad (*)$$

$$\frac{(1+z)\,dz}{1-2z-z^2} = \frac{dx}{x}$$

$$\int \frac{(1+z)\,dz}{1-2z-z^2} = \int \frac{dx}{x} + C$$

$$\int \frac{(1+z)\,dz}{1-2z-z^2} = \left| \begin{array}{l} t = 1-2z-z^2 \\ dt = -2z-2 \end{array} \right| = -\frac{1}{2}\int \frac{dt}{t} = -\frac{1}{2}\ln|t| + C_1 =$$

$$= -\frac{1}{2}\ln|1-2z-z^2| + C_1$$

$$-\frac{1}{2}\ln|1-2z-z^2| = \ln|x| + C \quad \Big| \cdot (-2)$$

$$\ln|1-2z-z^2| = -2\ln|x| + C$$

$$|1-2z-z^2| = e^C \cdot x^{-2}$$

$$1-2z-z^2 = C_2 x^{-2} \qquad (*) \text{ входит } (C_2 = 0)$$

$$1 - 2\frac{y}{x} - \frac{y^2}{x^2} = C_2 x^{-2} \quad \Big| \times x^2$$

$$x^2 - 2xy - y^2 = C_2$$

Ответ: $x^2 - 2xy - y^2 = C_2$

9.54

$xy' = y + x\,tg\,\frac{y}{x}$ $(x \neq 0)$

Замена:

$y = x \cdot z(x)$ $y' = z(x) + x \cdot z'(x)$

$x(z + xz') = x \cdot z + x\,tg\,z \quad |:x$

$z + xz' = z + tg\,z$

$xz' = tg\,z$

$x\frac{dz}{dx} = tg\,z \; \Big| \cdot \frac{dx}{x\,tg\,z}$ или $tg\,z = 0$

$z = \pi k;\; k \in \mathbb{Z}$

$\frac{dz}{tg\,z} = \frac{dx}{x}$

$\frac{y}{x} = \pi k;\; k \in \mathbb{Z}$

$y = \pi k x$

$\int \frac{dz}{tg\,z} = \int \frac{dx}{x} + C$

$\int ctg\,z\,dz = \ln|x| + C$

$\ln|\sin z| = \ln|x| + C$

$\sin z = C_2 x$

$\sin\frac{y}{x} = C_2 x$

Ответ: $\sin\frac{y}{x} = C_2 x$, $y = \pi k x$; $k \in \mathbb{Z}$