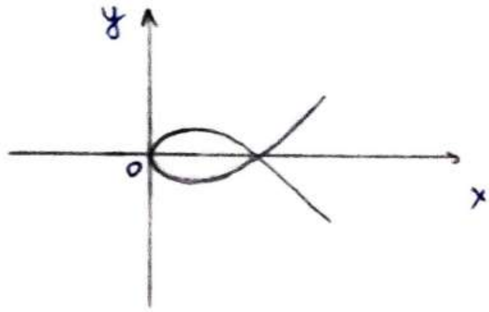
Занятие 13

6.504

$$\begin{cases} x = t^2 \\ y = t\left(\frac{1}{3} - t^2\right) \end{cases}$$




Точки самопересечения:

$$\begin{cases} x(t_1) = x(t_2) \\ y(t_1) = y(t_2) \\ t_1 \neq t_2 \end{cases}; \quad \begin{cases} t_1^2 = t_2^2 \\ t_1\left(\frac{1}{3} - t_1^2\right) = t_2\left(\frac{1}{3} - t_2^2\right) \\ t_1 \neq t_2 \end{cases};$$

$$\begin{cases} t_1 = \pm t_2 \\ t_1\left(\frac{1}{3} - t_1^2\right) = t_2\left(\frac{1}{3} - t_2^2\right) \\ t_1 \neq t_2 \end{cases} \Rightarrow \begin{cases} t_1 = -t_2 \\ t_1\left(\frac{1}{3} - t_1^2\right) = t_2\left(\frac{1}{3} - t_2^2\right) \\ t_1 \neq t_2 \end{cases} \Rightarrow$$

$$\Rightarrow -t_2\left(\frac{1}{3} - t_2^2\right) = t_2\left(\frac{1}{3} - t_2^2\right) \;\Big|: t_2 \neq 0 \;(\text{т.к } t_1 \neq t_2)$$

$$-\frac{1}{3} + t_2^2 = \frac{1}{3} - t_2^2$$

$$2t_2^2 = \frac{2}{3}$$

$$t_2^2 = \frac{1}{3}$$

~~Пусть~~

Пусть $t_1 = -\frac{1}{\sqrt{3}}$, $t_2 = \frac{1}{\sqrt{3}}$

Тогда

$$l = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \sqrt{(2t)^2 + \left(\frac{1}{3} - 3t^2\right)^2}\, dt =$$

$$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \sqrt{4t^2 + \frac{1}{9} - 2\cdot\frac{1}{\cancel{3}}\cdot \cancel{3}t^2 + 9t^4}\, dt = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \sqrt{9t^4 + 2t^2 + \frac{1}{9}}\, dt =$$

$$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \sqrt{\left(3t^2+\frac{1}{3}\right)^2}\, dt = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(3t^2+\frac{1}{3}\right) dt = \left(\frac{3t^3}{3}+\frac{t}{3}\right)\Bigg|_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} =$$

$$= \left(t^3+\frac{t}{3}\right)\Bigg|_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} = \left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{3\sqrt{3}} - \left(\left(-\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{3\sqrt{3}}\right) =$$

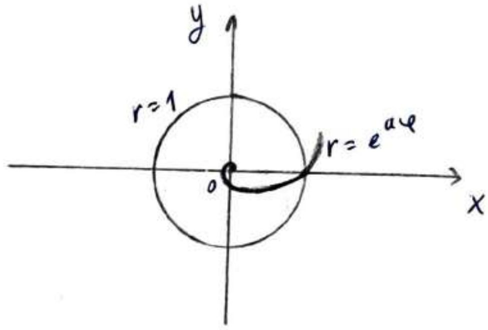
$$= \frac{1}{3^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} = \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9}$$

Ответ: $\frac{4\sqrt{3}}{9}$

6.508

$r = e^{a\varphi}$

$r = 1$




Точка пересечения:

$e^{a\varphi} = 1$

$a\varphi = 0$

$\varphi = 0$

Тогда:

$$l = \int_{-\infty}^{0} \sqrt{r^2 + (r')^2}\, d\varphi = \int_{-\infty}^{0} \sqrt{e^{2a\varphi} + a^2 e^{2a\varphi}}\, d\varphi =$$

$$= \int_{-\infty}^{0} \sqrt{e^{2a\varphi}(1+a^2)}\, d\varphi = \sqrt{1+a^2} \int_{-\infty}^{0} e^{a\varphi}\, d\varphi =$$

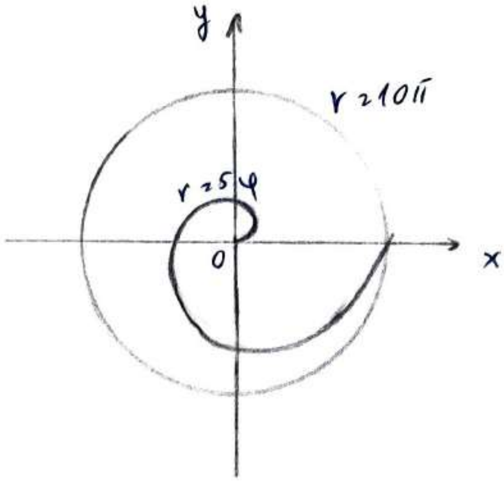
$$= \frac{\sqrt{1+a^2}}{a} \int_{-\infty}^{0} e^{a\varphi}\, d(a\varphi) = \frac{\sqrt{1+a^2}}{a}\, e^{a\varphi}\Big|_{-\infty}^{0} = \frac{\sqrt{1+a^2}}{a}$$

Ответ: $\frac{\sqrt{1+a^2}}{a}$

6.5 11

$r = 5\varphi$

$r = 10\pi$




Точка пересечения:

$5\varphi = 10\pi$

$\varphi = 2\pi$

Тогда:

$$l = \int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\varphi = \int_0^{2\pi} \sqrt{(5\varphi)^2 + 5^2}\, d\varphi = \int_0^{2\pi} \sqrt{25\varphi^2 + 25}\, d\varphi =$$

$$= 5 \int_0^{2\pi} \sqrt{\varphi^2 + 1}\, d\varphi \ominus$$

$$\int \sqrt{\varphi^2 + 1}\, d\varphi = |\varphi = tg\, t| = \int \sqrt{tg^2 t + 1}\, \frac{1}{\cos^2 t}\, dt =$$

$$= \int \frac{1}{\cos^3 t}\, dt = \frac{1}{2} \ln \left| tg\left(\frac{t}{2} + \frac{\pi}{4}\right) \right| + \frac{\sin t}{2\cos^2 t} + C =$$

$$= \frac{1}{2} \ln \left| \frac{\varphi - 1 + \sqrt{1+\varphi^2}}{\varphi + 1 - \sqrt{1+\varphi^2}} \right| + \frac{\varphi(1+\varphi^2)}{2\sqrt{1+\varphi^2}} + C =$$

$$= \frac{1}{2} \left( \ln \left| \varphi + \sqrt{1+\varphi^2} \right| + \varphi\sqrt{1+\varphi^2} \right) + C$$

$$\ominus \frac{5}{2} \cdot \left( \ln \left| \varphi + \sqrt{1+\varphi^2} \right| + \varphi\sqrt{1+\varphi^2} \right) \Big|_0^{2\pi} =$$

$$= \frac{5}{2} \left( \ln\left(2\pi + \sqrt{1+4\pi^2}\right) + 2\pi\sqrt{1+4\pi^2} \right) =$$

$$= 5\pi\sqrt{1+4\pi^2} + \frac{5}{2} \ln\left(2\pi + \sqrt{1+4\pi^2}\right)$$