Занятие 17. ОДУ высших порядков.
Уравнения, допускающие понижение порядка.

9.211

$$y'' = \frac{1}{1+x^2}$$

$$y' = \int \frac{1}{1+x^2}\,dx + C_1$$

$$y' = \operatorname{arctg} x + C_1$$

$$y = \int y'\,dx = \int (\operatorname{arctg} x + C_1)\,dx = x \operatorname{arctg} x - \ln\sqrt{x^2+1} + C_1 x + C_2$$

Ответ: $y = x\operatorname{arctg} x - \ln\sqrt{x^2+1} + C_1 x + C_2$

9.212

$$y'' = x + \sin x$$

$$y' = \int (x + \sin x)\,dx + C_1$$

$$y' = \frac{x^2}{2} - \cos x + C_1$$

$$y = \int y'\,dx = \int \left(\frac{x^2}{2} - \cos x + C_1\right) dx = \frac{x^3}{6} - \sin x + C_1 x + C_2$$

Ответ: $y = \frac{x^3}{6} - \sin x + C_1 x + C_2$

9.217

$$y'' + y' \operatorname{tg} x = \sin 2x$$

Замена $p(x) = y'$; $p'(x) = y''$

$$p' + p \operatorname{tg} x = \sin 2x$$

$$p = uv \ ; \ p' = u'v + v'u$$

$$u'v + v'u = -uv \operatorname{tg} x + \sin 2x$$

$$v(u' + u \operatorname{tg} x) + v'u = \sin 2x$$

$$\begin{cases} u' + u \operatorname{tg} x = 0 \\ v'u = \sin 2x \end{cases}$$

Найдем u:

$u' + u\,tg x = 0$; $\frac{du}{dx} = -u\,tg x$; $\frac{du}{u} = -tg x\,dx$; $\int \frac{du}{u} = -\int tg x\,dx$;

$\ln u = \ln \cos x$; $u = \cos x$

Найдем v:

$v' \cos x = \sin 2x$; $v' = \frac{2\sin x \cos x}{\cos x}$, $v' = 2\sin x$; $v = \int 2\sin x\,dx + C_1$

$v = -2\cos x + C_1$

$p = \cos x\,(-2\cos x + C_1)$

$p = -2\cos^2 x + C_1 \cos x$

$y' = -2\cos^2 x + C_1 \cos x$

$y = \int(-2\cos^2 x + C_1\cos x)\,dx = \int(-2\cos^2 x)\,dx + \int C_1 \cos x\,dx =$

$= -2\int \cos^2 x\,dx + C_1 \sin x = -2\int \frac{1+\cos 2x}{2}\,dx + C_1 \sin x =$

$= -(x + \frac{1}{2}\sin 2x) + C_1 \sin x + C_2 = -x - \sin x \cos x + C_1 \sin x + C_2$

Ответ: $y = -x - \sin x \cos x + C_1 \sin x + C_2$

9.212

$x y'' - y' = e^x x^2$

Замена $p = y'$; $p' = y''$

$x p' - p = e^x x^2$

$p = uv$; $p' = u'v + v'u$

$x u'v + x v'u - uv = e^x x^2$

$v(u'x - u) + x v'u = e^x x^2$

$$\begin{cases} u'x - u = 0 \\ x v' u = e^x x^2 \end{cases}$$

Найдем u:

$u'x - u = 0$; $\frac{du}{dx} x = u$; $\frac{du}{u} = \frac{dx}{x}$; $\int \frac{du}{u} = \int \frac{dx}{x}$; $\ln u = \ln x$;

$u = x$

Найдём v:

$x v' x = e^x x^2$; $v' = e^x$; $v = \int e^x dx + C_1$; $v = e^x + C_1$

$p = uv = x(e^x + C_1)$

$y' = xe^x + xC_1$

$y = \int (xe^x + C_1 x)\,dx = \int xe^x dx + C_1 \frac{x^2}{2} = xe^x - e^x + C_1 x^2 + C_2 =$

$= e^x(x-1) + C_1 x^2 + C_2$

Ответ: $y = e^x(x-1) + C_1 x^2 + C_2$

<u>9.219</u>

$2yy'' = 1 + y'^2$

Введём новую независимую ~~не~~ переменную y и сделаем замену

$z(y) = y'$, $y'' = z'_y \cdot z$

Получаем

$2yz'z = 1 + z^2$

$z' = \frac{1+z^2}{2yz}$

$\frac{dz}{dy} = \frac{1+z^2}{2yz}$

$\frac{2z\,dz}{1+z^2} = \frac{dy}{y}$; $\int \frac{2z\,dz}{1+z^2} = \int \frac{dy}{y} + C_1$; $2 \cdot \frac{1}{2} \ln|1+z^2| = \ln|y| + C_1$

$\ln|1+z^2| = \ln|y| + C_1$; $1+z^2 = e^{C_1} y$; $z^2 = C_1 y - 1$

$z = \pm\sqrt{C_1 y - 1}$

$y' = \pm\sqrt{C_1 y - 1}$; $\frac{dy}{dx} = \pm\sqrt{C_1 y - 1}$, $\frac{dy}{\pm\sqrt{C_1 y - 1}} = dx$;

$\cancel{y \neq \pm 1}$ $\int \frac{dy}{\pm\sqrt{C_1 y - 1}} = x + C_2$, $\pm \int \frac{dy}{\sqrt{C_1 y - 1}} = x + C_2$;

$\pm \frac{1}{C_1} \int \frac{d(C_1 y - 1)}{\sqrt{C_1 y - 1}} = x + C_2$; $2\sqrt{C_1 y - 1} = C_1 x + C_2$

$$4(C_1 y - 1) = (C_1 x + C_2)^2$$

$$4C_1 y = 4 + (C_1 x + C_2)^2$$

Ответ: $4C_1 y = 4 + (C_1 x + C_2)^2$

9.220

$$y y'' + y'^3 = y'^2$$

$$z(y) = y', \quad y'' = z'_y \cdot z$$

$$y z' z + z^3 = z^2$$

$$z(y z' + z^2 - z) = 0$$

$z = 0$ или $y z' + z^2 - z = 0$

$y' = 0$ $\qquad y\dfrac{dz}{dy} = z - z^2$

$y = C$ $\qquad \dfrac{dz}{z - z^2} = \dfrac{dy}{y}$

$$\int \frac{dz}{z - z^2} = \int \frac{dy}{y} + C_1$$

$$\int \frac{dz}{z(1-z)} = \left| \begin{array}{l} t = \frac{1}{z} \\ z = \frac{1}{t} \\ dz = -\frac{1}{t^2} \end{array} \right| = \int \frac{-t\,dt}{t^2\left(1 - \frac{1}{t}\right)} = -\int \frac{dt}{t-1} = -\ln|t-1| + C =$$

$$= -\ln\left|\frac{1}{z} - 1\right| + C = -\ln\left|\frac{1-z}{z}\right| + C = \ln\left|\frac{z}{1-z}\right| + C$$

$$\ln\left|\frac{z}{1-z}\right| = \ln|y| + C_1; \quad \frac{z}{1-z} = C_1 y; \quad \frac{1}{\frac{1}{z} - 1} = C_1 y;$$

$$\frac{1}{z} - 1 = \frac{1}{C_1 y}; \quad \frac{1}{z} = \frac{1}{C_1 y} + 1; \quad z = \frac{C_1 y}{1 + C_1 y}; \quad z = \frac{y}{C_1 + y}$$

$$y' = \frac{y}{C_1 + y}; \quad \frac{dy}{dx} = \frac{y}{C_1 + y}; \quad \frac{(C_1 + y)\,dy}{y} = dx; \quad \int\left(\frac{C_1}{y} + 1\right)dy = x + C_2$$

$$C_1 \ln|y| + y = x + C_2; \quad y = x + C_1 \ln|y| + C_2$$

Ответ: $y = x + C_1 \ln|y| + C_2$; $y = C$.