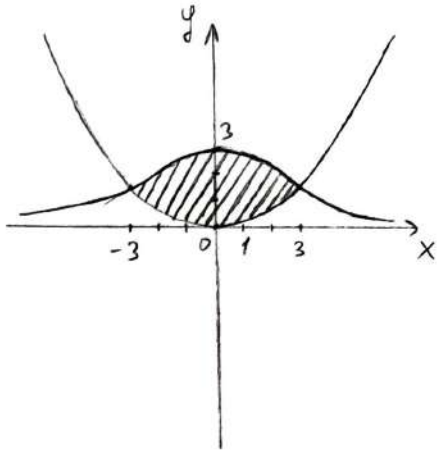
„Вычисление площади плоской фигуры".

6.457 $y = \dfrac{27}{x^2+9}$ $y = \dfrac{x^2}{6}$



Найдем точки пересечения кривых:

$$\begin{cases} y = \dfrac{27}{x^2+9} \\ y = \dfrac{x^2}{6} \end{cases} \Rightarrow \dfrac{27}{x^2+9} = \dfrac{x^2}{6}$$

$$x^4 + 9x^2 - 162 = 0$$

$$\left[\begin{array}{l} x^2 = -18 \\ x^2 = 9 \end{array}\right. \Rightarrow x^2 = 9 \Rightarrow \left[\begin{array}{l} x = 3 \\ x = -3 \end{array}\right.$$

$$S = \int_{-3}^{3} \left(\frac{27}{x^2+9} - \frac{x^2}{6}\right) dx = \int_{-3}^{3} \frac{27}{x^2+9}\, dx - \int_{-3}^{3} \frac{x^2}{6}\, dx =$$

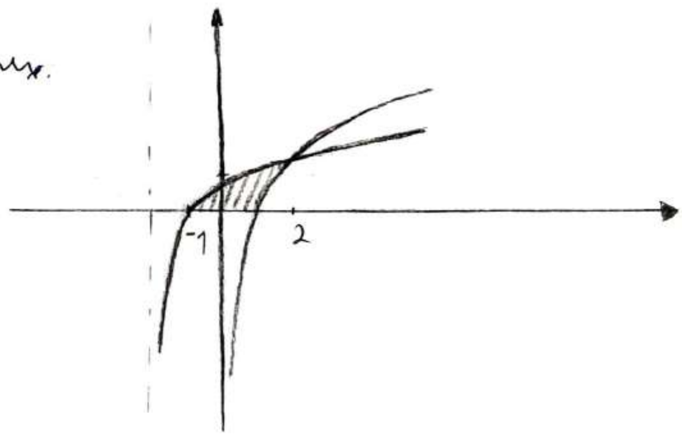
$$= \frac{27}{3} \operatorname{arctg} \frac{x}{3}\Big|_{-3}^{3} - \frac{1}{6}\,\frac{x^3}{3}\Big|_{-3}^{3} = 9 \operatorname{arctg} \frac{x}{3}\Big|_{-3}^{3} - \frac{x^3}{18}\Big|_{-3}^{3} =$$

$$= 9 \operatorname{arctg} \frac{3}{3} - 9 \operatorname{arctg} \frac{-3}{3} - \left(\frac{3^3}{18} - \frac{(-3)^3}{18}\right) = 9\frac{\pi}{4} + 9\frac{\pi}{4} - \frac{27}{18} - \frac{27}{18} =$$

$$= \frac{9\pi}{2} - 3$$

6.468 $y = \ln(x+2)$; $y = 2\ln x$; $y = 0$.

Найдем точки пересечения кривых.



$$\begin{cases} y = \ln(x+2) \\ y = 2\ln x \end{cases} \Rightarrow \ln(x+2) = 2\ln x$$

$$\ln(x+2) = \ln x^2$$

$$x + 2 = x^2$$

$$x^2 - x - 2 = 0$$

$\left[\begin{array}{l} x = 2 \\ x = -1 \end{array}\right.$ не удв условию $x > 0$

$$x = 2$$

$$S = \int_{-1}^{1} \ln(x+2)\, dx + \int_{1}^{2} (\ln(x+2) - 2\ln x)\, dx = \int_{-1}^{2} (\ln(x+2))\, dx - \int_{1}^{2} 2\ln(x)\, dx =$$

$$= \left((x+2)\ln(x+2) - (x+2)\right)\Big|_{-1}^{2} - 2\left(x\ln x - x\right)\Big|_{1}^{2} =$$

$$= 4\ln 4 - 4 + 1 - 2\left(2\ln 2 - 2 + 1\right) = 4\ln 4 - 3 - 2\ln 4 + 2 =$$

$$= 2\ln 4 - 1 = 4\ln 2 - 1$$

<u>6.482</u> $\begin{cases} x = 2t - t^2 \\ y = 2t^2 - t^3 \end{cases}$

Найдем точку самопересечения (пусть $t_1 < t_2$)

$$\begin{cases} x(t_1) = x(t_2) \\ y(t_1) = y(t_2) \\ t_1 \neq t_2 \end{cases} \Rightarrow \begin{cases} 2t_1 - t_1^2 = 2t_2 - t_2^2 \\ 2t_1^2 - t_1^3 = 2t_2^2 - t_2^3 \\ t_1 \neq t_2 \end{cases} \Rightarrow \begin{cases} (t_1 - t_2)(2 - t_1 - t_2) = 0 \\ (t_1 - t_2)(2t_1 + 2t_2 - t_1^2 - t_1t_2 - t_2^2) = 0 \\ t_1 \neq t_2 \end{cases}$$

$$\Rightarrow \begin{cases} t_1 + t_2 = 2 \\ 2(t_1 + t_2) - (t_1 + t_2)^2 + t_1t_2 = 0 \end{cases} \Rightarrow \begin{cases} t_1 + t_2 = 2 \\ t_1 t_2 = 0 \end{cases} \Rightarrow$$

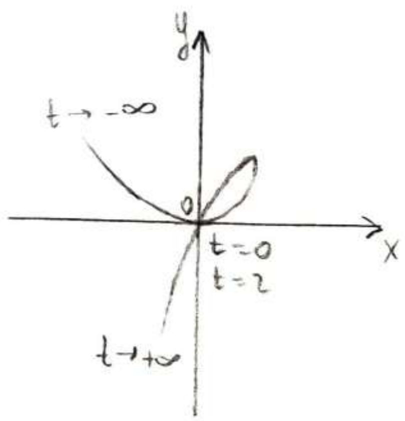
$$\Rightarrow \begin{cases} t_1 = 0 \\ t_2 = 2 \end{cases}$$

$$S = \int_2^0 (2t^2 - t^3)(2 - 2t)\,dt = 2\int_2^0 (2t^2 - t^3)(1 - t)\,dt =$$

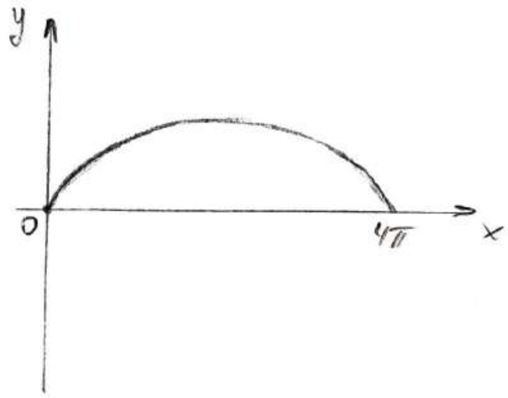
$$= 2\int_2^0 (2t^2 - 2t^3 - t^3 + t^4)\,dt = 2\int_2^0 (t^4 - 3t^3 + 2t^2)\,dt =$$

$$= 2\left(\frac{t^5}{5} - \frac{3t^4}{4} + \frac{2t^3}{3}\right)\Big|_2^0 = -2\left(\frac{2^5}{5} - \frac{3\cdot 2^4}{4} + \frac{2\cdot 2^3}{3}\right) =$$

$$= -2\left(\frac{32}{5} - 12 + \frac{16}{3}\right) = -2\left(-\frac{4}{15}\right) = \frac{8}{15}$$



6.480 $\begin{cases} x = 2(t-\sin t) \\ y = 2(1-\cos t) \end{cases}$



Точки пересечение с осью Ox:

$2-2\cos t = 0$

$\cos t = 1$

$t = 2\pi k,\ k\in\mathbb{Z}$

Возьмем две соседние точки пересечение с осью Ox:

$t_1 = 0$ и $t_2 = 2\pi$

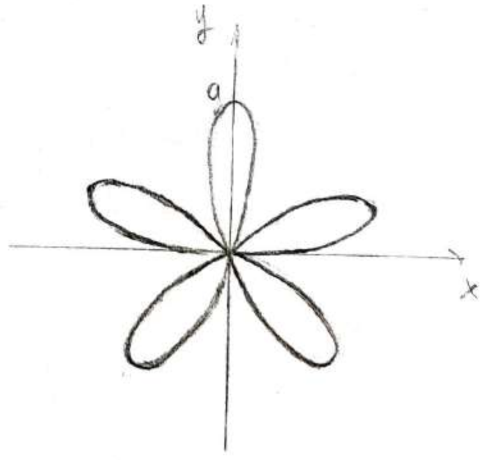
$$S = \int_0^{2\pi} 2(1-\cos t)\, d(2t-2\sin t) = 4\int_0^{2\pi}(1-\cos t)\, d(t-\sin t) =$$

$$= 4\int_0^{2\pi}(1-\cos t)^2\, dt = 4\int_0^{2\pi}(1-2\cos t+\cos^2 t)\, dt =$$

$$= 4\left(t - 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = (6t - 8\sin t + \sin 2t)\Big|_0^{2\pi} =$$

$= 12\pi$

6.485 $r = a\sin 5\varphi$



$a\sin 5\varphi = 0$

$\sin 5\varphi = 0$

$\varphi = \frac{\pi k}{5},\ k\in\mathbb{Z}$

$\varphi_1 = 0 \quad \varphi_2 = \frac{\pi}{5}$

$$S = 5\cdot\frac{1}{2}\int_0^{\frac{\pi}{5}} r^2\, d\varphi = 5\cdot\frac{1}{2}\int_0^{\frac{\pi}{5}} a^2\sin^2 5\varphi\, d\varphi =$$

$$= \frac{5a^2}{2\cdot 5}\int_0^{\frac{\pi}{5}}\sin^2 5\varphi\, d(5\varphi) = \frac{a^2}{2\cdot 2}\int_0^{\frac{\pi}{5}}\frac{1-\cos 10\varphi}{2}\, d(10\varphi) =$$

$$= \frac{a^2}{4}\left(\frac{10\varphi}{2} - \frac{\sin 10\varphi}{2}\right)\Big|_0^{\frac{\pi}{5}} = \frac{\pi a^2}{4}$$

6.488 $r = e^{\varphi}$

$$S = \frac{1}{2}\int_{2\pi}^{4\pi}(e^{\varphi})^2 d\varphi - \frac{1}{2}\int_0^{2\pi}(e^{\varphi})^2 d\varphi = \frac{1}{4}\left(e^{2\varphi}\Big|_{2\pi}^{4\pi} - e^{2\varphi}\Big|_0^{2\pi}\right) =$$

$$= \frac{1}{4}\left(e^{8\pi} - e^{4\pi} - e^{4\pi} + 1\right) = \frac{1}{4}\left(e^{8\pi} - 2e^{4\pi} + 1\right) = \frac{1}{4}\left(e^{4\pi} - 1\right)^2$$

„Несобственные интегралы и их сходимость".

6.412 $$\int_e^{+\infty}\frac{dx}{x\sqrt{\ln x}} = \int_e^{+\infty}\frac{d(\ln x)}{\sqrt{\ln x}} = \left(2\sqrt{\ln x}\right)\Big|_e^{+\infty} = 2\lim_{x\to+\infty}\sqrt{\ln x} - 2$$

Расходится.

6.413 $$\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+11} = \int_{-\infty}^{0}\frac{dx}{x^2+6x+11} + \int_0^{+\infty}\frac{dx}{x^2+6x+11} =$$

$$= \int_{-\infty}^{0}\frac{dx}{(x+3)^2+2} + \int_0^{+\infty}\frac{dx}{(x+3)^2+2} = \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x+3}{\sqrt{2}}\Big|_{-\infty}^{0} + \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x+3}{\sqrt{2}}\Big|_0^{+\infty} =$$

$$= -\frac{1}{\sqrt{2}}\lim_{x\to-\infty}\operatorname{arctg}\frac{x+3}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{3}{\sqrt{2}}} + \frac{1}{\sqrt{2}}\lim_{x\to+\infty}\operatorname{arctg}\frac{x+3}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{3}{\sqrt{2}}} =$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{\sqrt{2}}$$

6.414 $$\int_0^{+\infty} e^{-2x}\cos x\, dx \ominus$$

$$\int e^{-2x}\cos x\, dx = -\frac{1}{2}\int e^{-2x}\cos x\, d(-2x) =$$

$$= -\frac{1}{2}\int \cos x\, de^{-2x} = -\frac{1}{2}\left(e^{-2x}\cos x - \int e^{-2x}\, d\cos x\right) =$$

$$= -\frac{1}{2}\left(e^{-2x}\cos x + \int e^{-2x}\sin x\, dx\right) = -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}\int \sin x\, de^{-2x} =$$

$$= -\frac{1}{2}e^{-2x}\cos x + \frac{1}{4}\left(e^{-2x}\sin x - \int e^{-2x}\cos x\, dx\right) =$$

$$= -\frac{1}{2} e^{-2x} \cos x + \frac{1}{4} e^{-2x} \sin x - \frac{1}{4}\int e^{-2x} \cos x \, dx$$

$$I = -\frac{1}{2} e^{-2x} \cos x + \frac{1}{4} e^{-2x} \sin x - \frac{1}{4} I$$

$$\frac{5}{4} I = -\frac{1}{2} e^{-2x} \cos x + \frac{1}{4} e^{-2x} \sin x \quad | \times 4$$

$$5I = -2 e^{-2x} \cos x + e^{-2x} \sin x$$

$$I = \frac{-2 e^{-2x} \cos x + e^{-2x} \sin x}{5}$$

$$I = \frac{-2 \cos x + \sin x}{5 e^{2x}}$$

$$= \left.\frac{-2\cos x + \sin x}{5e^{2x}}\right|_0^{+\infty} = \lim_{x \to +\infty} \underbrace{\frac{-2\cos x + \sin x}{5e^{2x}}}_{\text{б.м на огр}} + \frac{2}{5} = \frac{2}{5}$$

6.421 $\int\limits_1^{+\infty} \frac{x^2+1}{x^3} dx = \int\limits_1^{+\infty} \left(\frac{1}{x} + \frac{1}{x^3}\right) dx = \left.\left(\ln|x| - \frac{1}{2x^2}\right)\right|_1^{+\infty} =$

$$= \left.\left(\ln x - \frac{1}{2x^2}\right)\right|_1^{+\infty} = \lim_{x \to +\infty} \left(\ln x - \frac{1}{2x^2}\right) + \frac{1}{2}.$$

Расходится

6.422 $\int\limits_0^{+\infty} \frac{x\,dx}{\sqrt{1+x^3}}$ расходится

При $x \to +\infty$ $\frac{x}{\sqrt{1+x^3}} = \frac{1}{\sqrt{\frac{1}{x^2} + x}} \sim \frac{1}{x^{\frac{1}{2}}}$

Т.к интеграл $\int\limits_0^{+\infty} \frac{dx}{x^{\frac{1}{2}}}$ расходится $\left(\frac{1}{2} < 1\right)$, то и

$\int\limits_0^{+\infty} \frac{x\,dx}{\sqrt{1+x^3}}$ расходится

6.423 $\displaystyle\int_2^{+\infty} \frac{dx}{\sqrt[3]{x^3-1}}$

При $x \to +\infty$ $\quad \dfrac{1}{\sqrt[3]{x^3-1}} \sim \dfrac{1}{x}$

Так как интеграл $\displaystyle\int_2^{+\infty} \frac{dx}{x}$ расходится, то и $\displaystyle\int_2^{+\infty} \frac{dx}{\sqrt[3]{x^3-1}}$ расходится.

6.427 $\displaystyle\int_1^{+\infty} \frac{3x^2+\sqrt{(x+1)^3}}{2x^3+\sqrt[3]{x^5}+1}\,dx$

$$\frac{3x^2+\sqrt{(x+1)^3}}{2x^3+\sqrt[3]{x^5}+1} = \frac{3x^2+\sqrt{x^3+3x^2+3x+1}}{2x^3+\sqrt[3]{x^5}+1} = \frac{x^2\left(3+\sqrt{\frac{1}{x}+\frac{3}{x^2}+\frac{3}{x^3}+\frac{1}{x^4}}\right)}{2x^3+\sqrt[3]{x^5}+1} \sim$$

$\sim \dfrac{x^2}{x^3} = \dfrac{1}{x}$; интеграл расходится

6.429 $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x(x+1)(x+2)}}$

$\dfrac{1}{\sqrt{x(x+1)(x+2)}} = \dfrac{1}{\sqrt{x^3+3x^2+2x}} \sim \dfrac{1}{\sqrt{x^3}} = \dfrac{1}{x^{\frac{3}{2}}}$; интеграл сходится.

6.431 $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x}+\cos^2 x}$

$\dfrac{1}{\sqrt{x}+\cos^2 x} \sim \dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{\frac{1}{2}}}$; интеграл расходится