Занятие 16 Интегрирование линейных ОДУ первого порядка и уравнений Бернулли.

9.68 $y' = \frac{3y}{x} + x$

$y = uv$; $y' = u'v + v'u$

Тогда

$u'v + v'u = \frac{3}{x} uv + x$

$v\left(u' - \frac{3}{x}u\right) + v'u = x$

$$\begin{cases} u' - \frac{3}{x}u = 0 \\ v'u = x \end{cases}$$

Найдем u:

$u' - \frac{3}{x}u = 0$; $\frac{du}{dx} = \frac{3}{x}u$; $\frac{du}{u} = \frac{3dx}{x}$; $\int \frac{du}{u} = \int \frac{3dx}{x}$;

$\ln u = 3\ln x$; $u = x^3$

Найдем v:

$v' x^3 = x$; $v' = \frac{x}{x^3}$; $v' = \frac{1}{x^2}$; $v = \int \frac{1}{x^2} dx + C$;

$v = -\frac{1}{x} + C$

$y = uv = x^3\left(-\frac{1}{x} + C\right) = Cx^3 - x^2$

Ответ: $y = Cx^3 - x^2$

9.69 $y' + y\,\mathrm{tg}\,x = \frac{1}{\cos x}$

$y = uv$; $y' = u'v + v'u$

Тогда

$u'v + v'u + uv\,\mathrm{tg}\,x = \frac{1}{\cos x}$

$v\left(u' + u\,\mathrm{tg}\,x\right) + v'u = \frac{1}{\cos x}$

$$\begin{cases} u' + u\,tg\,x = 0 \\ v'u = \dfrac{1}{\cos x} \end{cases}$$

Найдем u:

$u' + u\,tg\,x = 0$; $\dfrac{du}{dx} = -u\,tg\,x$; $\dfrac{du}{u} = -tg\,x\,dx$; $\int \dfrac{du}{u} = -\int tg\,x\,dx$;

$\ln u = -(-\ln(\cos x))$; $\ln u = \ln\cos x$; $u = \cos x$

Найдем v:

$v' \cdot \cos x = \dfrac{1}{\cos x}$; $v' = \dfrac{1}{\cos^2 x}$; $v = \int \dfrac{1}{\cos^2 x} dx + C$;

$v = tg\,x + C$

$y = uv = \cos x\,(tg\,x + C) = \sin x + C\cos x$

Ответ: $y = \sin x + C\cos x$

<u>9.70</u> $(1+x^2)y' = 2xy + (1+x^2)^2 \quad | : (1+x^2)$

$$y' = \frac{2xy}{1+x^2} + (1+x^2)$$

$y = uv$; $y' = u'v + v'u$

$$u'v + v'u = \frac{2xuv}{1+x^2} + (1+x^2)$$

$$v\left(u' - \frac{2xu}{1+x^2}\right) + v'u = 1+x^2$$

$$\begin{cases} u' - \dfrac{2xu}{1+x^2} = 0 \\ v'u = 1+x^2 \end{cases}$$

Найдем u:

$u' - \dfrac{2xu}{1+x^2} = 0$; $\dfrac{du}{dx} = \dfrac{2xu}{1+x^2}$; $\dfrac{du}{u} = \dfrac{2x\,dx}{1+x^2}$; $\int \dfrac{du}{u} = \int \dfrac{2x\,dx}{1+x^2}$;

$\ln u = \int \dfrac{d(x^2+1)}{x^2+1}$; $\ln u = \ln(x^2+1)$; $\cancel{u = (x^2+1)^2}$ $u = x^2+1$

Найдем v:

$v'(x^2+1)=x^2+1$; $v'=1$; $v=\int 1\,dx+C$; $v=x+C$

$y=uv=(x^2+1)(x+C)$

Ответ: $y=(x^2+1)(x+C)$

9.87 $dy=(y^2e^x-y)\,dx \mid : dx$

$y'=y^2e^x-y$

Если $y=C$, решением будет $y=0$.

$y=uv$; $y=u'v+v'u$

$u'v+v'u=u^2v^2e^x-uv$

$v(u'+u)+v'u=u^2v^2e^x$

$$\begin{cases} u'+u=0 \\ v'u=u^2v^2e^x \end{cases}$$

Найдем u:

$u'+u=0$; $\frac{du}{dx}=-u$; $\frac{du}{u}=-dx$; $\int\frac{du}{u}=-\int 1\,dx$;

$\ln u=-x$; $u=e^{-x}$

Найдем v:

$v'e^{-x}=e^{-2x}v^2e^x$; $v'=\frac{e^{-2x}\cdot e^x}{e^{-x}}v^2$; $v'=v^2$; $\frac{dv}{dx}=v^2$;

$\frac{dv}{v^2}=dx$; $\int\frac{dv}{v^2}=\int 1\,dx+C$; $-\frac{1}{v}=x+C$; $\frac{1}{v}=C-x$;

$$v=\frac{1}{C-x}$$

$$y=uv=e^{-x}\cdot\frac{1}{C-x}$$

Ответ: $y=\frac{e^{-x}}{C-x}$; $y=0$

9.88 $y' = y(y^3\cos x + \operatorname{tg} x)$

Если $y = C$, решением будет $y = 0$

$y = uv, \quad y' = u'v + v'u$

~~u'v+v'u~~ $u'v + v'u = u^4 v^4 \cos x + uv\operatorname{tg} x$

$v(u' - u\operatorname{tg} x) + v'u = u^4 v^4 \cos x$

$$\begin{cases} u' - u\operatorname{tg} x = 0 \\ v'u = u^4 v^4 \cos x \end{cases}$$

Найдем u:

$u' - u\operatorname{tg} x = 0; \quad u' = u\operatorname{tg} x; \quad \frac{du}{dx} = u\operatorname{tg} x; \quad \frac{du}{u} = \operatorname{tg} x\,dx;$

$\int \frac{du}{u} = \int \operatorname{tg} x\,dx; \quad \ln u = -\ln\cos x; \quad u = \frac{1}{\cos x}$

Найдем v:

$v' \frac{1}{\cos x} = \frac{1}{\cos^4 x} v^4 \cos x$

$v' = \frac{1}{\cos^2 x} v^4; \quad \frac{dv}{dx} = \frac{1}{\cos^2 x} v^4; \quad \frac{dv}{v^4} = \frac{1}{\cos^2 x}dx;$

$\int \frac{dv}{v^4} = \int \frac{1}{\cos^2 x}dx + C; \quad -\frac{1}{3v^3} = \operatorname{tg} x + C; \; |\times(-3)$

$\frac{1}{v^3} = -3\operatorname{tg} x + C; \quad v = \left(\sqrt[3]{C - 3\operatorname{tg} x}\right)^{-1}$

$y = \left(\cos x \cdot \sqrt[3]{C - 3\operatorname{tg} x}\right)^{-1}$

Ответ: $y = \left(\cos x \cdot \sqrt[3]{C - 3\operatorname{tg} x}\right)^{-1}, \; y = 0$

9.91 $y' = \dfrac{x(x^2 + y^2 - 1)}{2y(x^2 - 1)}$

$y' = \dfrac{x\cancel{(x^2-1)}}{2y\cancel{(x^2-1)}} + \dfrac{y^2 x}{2y(x^2-1)}$

$y' = \dfrac{x}{2y} + \dfrac{yx}{2(x^2-1)}$

$$y' = \frac{x}{2} y^{-1} + \frac{yx}{2(x^2-1)} \Big| : y^{-1}$$

$$y'y = \frac{x}{2} + \frac{y^2 x}{2(x^2-1)}$$

Замена $z = y^2$

$z' = 2yy'$

$$\frac{z'}{2} = \frac{x}{2} + \frac{zx}{2(x^2-1)} \Big| \times 2$$

$$z' = x + \frac{zx}{x^2-1}$$

$z = uv; \quad z' = u'v + v'u$

$$u'v + v'u = x + \frac{uvx}{x^2-1}$$

$$v\left(u' - \frac{ux}{x^2-1}\right) + v'u = x$$

$$\begin{cases} u' - \frac{ux}{x^2-1} = 0 \\ v'u = x \end{cases}$$

Найдем u:

$u' - \frac{ux}{x^2-1} = 0; \quad u' = \frac{ux}{x^2-1}; \quad \frac{du}{dx} = \frac{ux}{x^2-1}; \quad \frac{du}{u} = \frac{x\,dx}{x^2-1};$

$\int \frac{du}{u} = \int \frac{x\,dx}{x^2-1}; \quad \ln u = \frac{1}{2} \ln|x^2-1|; \quad u = \sqrt{|x^2-1|}$

Найдем v

$v'u = x; \quad v'\sqrt{|x^2-1|} = x; \quad \frac{dv}{dx} = \frac{x}{\sqrt{|x^2-1|}}; \quad dv = \frac{x\,dx}{\sqrt{|x^2-1|}}$

$\int dv = \int \frac{x\,dx}{\sqrt{|x^2-1|}} + C; \quad v = \sqrt{|x^2-1|} + C$

$z = y^2 = uv = \sqrt{|x^2-1|} * \left(\sqrt{|x^2-1|} + C\right) = |x^2-1| + C\sqrt{|x^2-1|}$

Ответ: $y^2 = |x^2-1| + C\sqrt{|x^2-1|}$