Math Formulas

A Comprehensive Reference

December 01, 2025

Syllabus

1. Basic Mathematics
   - Sets and Functions
   - Two dimensional and three-dimensional Coordinate Geometry
2. Algebra
   - Polynomials
   - Complex numbers
   - Sequence and series
   - Permutation and combination
   - Equations and inequalities
   - Matrices and Determinants
   - Eigen values and Eigen vectors, Diagonalization of matrix
   - Linear Programming
3. Vector Analysis
   - Vector Algebra: Vectors and Scalars, product of two, three and four vectors, reciprocal system
   - Vector Calculus: Gradient, Curl and Divergence, line integral, surface integral and volume integral.
4. Calculus
   - Limits and Continuity, Ordinary and Partial Differentiation
   - Indefinite and definite Integration
   - Application of Derivatives and Anti-derivatives
   - Ordinary Differential Equations.
5. Elementary Statistics and Probability
6. Elementary Trigonometry, Logarithm
7. Transforms: Laplace transform, Fourier series

Table of Contents

Set Theory and Functions

Set Operations and Properties

Basic Set Operations

Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Difference: $A - B = \{x : x \in A \text{ and } x \notin B\}$

Symmetric Difference: $A \Delta B = (A - B) \cup (B - A)$

Complement: $A' = U - A = \{x : x \in U \text{ and } x \notin A\}$

Cartesian Product: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Identity Laws

$$A \cup A = A$$

(Idempotent)

$$A \cap A = A$$

(Idempotent)

$$A \cup \emptyset = A$$

(Identity for union)

$$A \cap U = A$$

(Identity for intersection)

$$A \cup U = U$$

(Domination)

$$A \cap \emptyset = \emptyset$$

(Domination)

Commutative Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$A \Delta B = B \Delta A$$

Note:

$$A - B \neq B - A$$

(Not commutative)

$$A \times B \neq B \times A$$

(Not commutative)

Associative Laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B - C) = (A \times B) - (A \times C)$$

De Morgan's Laws

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Generalized:

$$\left(\bigcup_{i=1}^{n} A_i \right)' = \bigcap_{i=1}^{n} A_{i'}$$

$$\left(\bigcap_{i=1}^{n} A_i \right)' = \bigcup_{i=1}^{n} A_{i'}$$

Difference Laws

$$A - (B \cap C) = (A - B) \cup (A - C)$$

$$A - (B \cup C) = (A - B) \cap (A - C)$$

$$A - B = A \cap B'$$

$$B - A = B \cap A'$$

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

Symmetric Difference Laws

$$A \Delta B = (A - B) \cup (B - A)$$

$$A \Delta B = (A \cup B) - (A \cap B)$$

$$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$$

$$A \Delta \emptyset = A$$

$$A \Delta A = \emptyset$$

$$A \Delta U = A'$$

Complement Laws

$$A \cup A' = U$$

$$A \cap A' = \emptyset$$

$$(A')' = A$$

(Involution)

$$U' = \emptyset$$

$$\emptyset' = U$$

Additional Properties

$$A \subseteq A \cup B$$

$$A \cap B \subseteq A$$

$$A \subseteq B \Leftrightarrow A \cup B = B$$

$$A \subseteq B \Leftrightarrow A \cap B = A$$

$$A - B = A - (A \cap B)$$

$$A \cup (A \cap B) = A$$

(Absorption)

$$A \cap (A \cup B) = A$$

(Absorption)

Relations

Types of Relations

Reflexive: $(a, a) \in R$ for all $a \in A$

Symmetric: $(a, b) \in R \Rightarrow (b, a) \in R$

Transitive: $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

Anti-symmetric: $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$

Equivalence Relations

A relation is an **equivalence relation** if it is:

- Reflexive
- Symmetric
- Transitive

Equivalence Class: $[a] = \{x \in A : (a, x) \in R\}$

Partial Order Relations

A relation is a **partial order** if it is:

- Reflexive
- Anti-symmetric
- Transitive

Functions

Types of Functions

One-to-one (Injective):

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Onto (Surjective): For every $y \in B$, there exists $x \in A$ such that $f(x) = y$

Bijective: Both one-to-one and onto

Function Composition

$$(g \circ f)(x) = g(f(x))$$

Properties:

- Generally not commutative: $g \circ f \neq f \circ g$
- Associative: $(h \circ g) \circ f = h \circ (g \circ f)$

Inverse Function

If $f : A \to B$ is bijective, then $f^{-1} : B \to A$ exists.

$$f(f^{-1}(x)) = x$$

and

$$f^{-1}(f(x)) = x$$

Domain and Range of Standard Functions

Trigonometric Functions

S.N.	Function	Domain	Range
1	$y = \sin x$	$\mathbb{R}$	$[-1, 1]$
2	$y = \cos x$	$\mathbb{R}$	$[-1, 1]$
3	$y = \tan x$	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}$	$\mathbb{R}$
4	$y = \cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$	$\mathbb{R}$
5	$y = \sec x$	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}$	$(-\infty, -1] \cup [1, \infty)$
6	$y = \csc x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$	$(-\infty, -1] \cup [1, \infty)$

Inverse Trigonometric Functions

S.N.	Function	Domain	Range
1	$y = \sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
2	$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3	$y = \tan^{-1} x$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
4	$y = \cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
5	$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\frac{\pi}{2}\}$
6	$y = \csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

Other Standard Functions

S.N.	Function	Domain	Range
1	$y = a^x \ (a > 0, a \neq 1)$	$\mathbb{R}$	$(0, \infty)$
2	$y = \log_a x \ (a > 0, a \neq 1)$	$(0, \infty)$	$\mathbb{R}$
3	$y = e^x$	$\mathbb{R}$	$(0, \infty)$
4	$y = \ln x$	$(0, \infty)$	$\mathbb{R}$
5	$y = \|x\|$	$\mathbb{R}$	$[0, \infty)$
6	$y = x^2$	$\mathbb{R}$	$[0, \infty)$
7	$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
8	$y = \frac{1}{x}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$
9	$y = \lfloor x \rfloor$ (floor)	$\mathbb{R}$	$\mathbb{Z}$
10	$y = \lceil x \rceil$ (ceiling)	$\mathbb{R}$	$\mathbb{Z}$
11	$y = \{x\}$ (fractional)	$\mathbb{R}$	$[0, 1)$

Polynomial and Rational Functions

Polynomial: $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

- Domain: $\mathbb{R}$
- Range: Depends on degree and leading coefficient

Rational: $f(x) = \frac{P(x)}{Q(x)}$

- Domain: $\mathbb{R}$ excluding zeros of $Q(x)$
- Range: Varies based on function

Hyperbolic Functions

S.N.	Function	Domain	Range
1	$y = \sinh x = \frac{e^x - e^{-x}}{2}$	$\mathbb{R}$	$\mathbb{R}$
2	$y = \cosh x = \frac{e^x + e^{-x}}{2}$	$\mathbb{R}$	$[1, \infty)$
3	$y = \tanh x = \frac{\sinh x}{\cosh x}$	$\mathbb{R}$	$(-1, 1)$
4	$y = \coth x$	$\mathbb{R} - \{0\}$	$(-\infty, -1) \cup (1, \infty)$
5	$y = \text{sech } x = \frac{1}{\cosh x}$	$\mathbb{R}$	$(0, 1]$
6	$y = \text{csch } x = \frac{1}{\sinh x}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$

Cardinality

Finite Sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \times B| = |A| \times |B|$$

$$|A - B| = |A| - |A \cap B|$$

For three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Power Set

Number of subsets of a set with n elements:

$$|\mathcal{P}(A)| = 2^n$$

Number of proper subsets: $2^n - 1$

Algebra and Complex Numbers

Complex Numbers

Representation

Cartesian form:

$$z = x + iy$$

where $x = \mathrm{Re}(z)$ (real part) and $y = \mathrm{Im}(z)$ (imaginary part)

Polar form:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where:

$$r = |z| = \sqrt{x^2 + y^2}$$

(modulus)

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

(argument)

Conversions:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

From these:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

Multiplication and Division

Multiplication:

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Roots of Complex Numbers

nth root:

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

for $k = 0, 1, 2, ..., n - 1$

Square root:

$$\sqrt{x + iy} = \pm\left[\sqrt{\frac{r + x}{2}} + i\ \text{sgn}(y)\sqrt{\frac{r - x}{2}}\right]$$

where $r = \sqrt{x^2 + y^2}$

Cube Roots of Unity

$$z^3 = 1 \Rightarrow z^3 - 1 = 0$$

Roots:

$$z = 1, \omega, \omega^2$$

where:

$$\omega = \frac{-1 + i\sqrt{3}}{2} = e^{i2\frac{\pi}{3}} = \cos\left(2\frac{\pi}{3}\right) + i\sin\left(2\frac{\pi}{3}\right)$$

$$\omega^2 = \frac{-1 - i\sqrt{3}}{2} = e^{i4\frac{\pi}{3}} = \cos\left(4\frac{\pi}{3}\right) + i\sin\left(4\frac{\pi}{3}\right)$$

Properties:

$$1 + \omega + \omega^2 = 0$$

$$\omega^3 = 1$$

$$\omega^2 = \overline{\omega}$$

$$(\omega)^{3n} = 1$$

$$(\omega)^{3n+1} = \omega$$

$$(\omega)^{3n+2} = \omega^2$$

nth Roots of Unity

$$z^n = 1$$

Roots:

$$z_k = e^{i2\pi\frac{k}{n}} = \cos\left(2\pi\frac{k}{n}\right) + i\sin\left(2\pi\frac{k}{n}\right)$$

for $k = 0, 1, 2, ..., n-1$

Sum of all nth roots:

$$\sum_{k=0}^{n-1} z_k = 0$$

Complex Conjugate

Definition

$$\overline{z} = \overline{x+iy} = x - iy$$

$$\overline{re^{i\theta}} = re^{-i\theta}$$

Properties

$$z + \overline{z} = 2\ \mathrm{Re}(z) = 2x$$

$$z - \overline{z} = 2i\ \mathrm{Im}(z) = 2iy$$

$$z\overline{z} = |z|^2 = x^2 + y^2 = r^2$$

$$\overline{\overline{z}} = z$$

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\overline{z^n} = (\overline{z})^n$$

Real and imaginary parts:

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}$$

$$\mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$$

z is real iff: $z = \overline{z}$

z is purely imaginary iff: $z = -\overline{z}$

Modulus (Absolute Value)

Definition

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}} = r$$

Properties

$$|z|^2 = z\overline{z}$$

$$|z| = |\overline{z}| = |-z|$$

$$|z| = 0 \Leftrightarrow z = 0$$

$$|z| \geq 0$$

$$|\mathrm{Re}(z)| \leq |z|$$

$$|\mathrm{Im}(z)| \leq |z|$$

$$|z_1 z_2| = |z_1||z_2|$$

$$|z^n| = (|z|)^n$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0$$

Triangle Inequalities

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

Generalized triangle inequality:

$$|z_1 + z_2 + ... + z_n| \leq |z_1| + |z_2| + ... + |z_n|$$

Distance

$$|z_1 - z_2| = \text{distance between } z_1 \text{ and } z_2$$

Argument

Definition

$$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Principal argument:

$$\operatorname{Arg}(z) = \theta, \quad -\pi < \theta \le \pi$$

Properties

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2\pi k$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi k$$

$$\arg(z^n) = n \arg(z) + 2\pi k$$

$$\arg(\overline{z}) = -\arg(z)$$

$$\arg(-z) = \arg(z) + \pi$$

Logarithm of Complex Numbers

Definition

If $z = re^{i\theta}$, then:

$$\ln z = \ln r + i(\theta + 2\pi k), \quad k \in \mathbb{Z}$$

Principal value:

$$\log z = \ln r + i\theta, \quad -\pi < \theta \le \pi$$

$$\log z = \ln|z| + i \arg(z)$$

Properties

$$\log(z_1 z_2) = \log z_1 + \log z_2 + 2\pi i k$$

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 + 2\pi i k$$

$$\log(z^n) = n \log z + 2\pi i k$$

Complex Numbers in Geometry

Distance and Midpoint

Distance between z_1 and z_2:

$$d = |z_1 - z_2|$$

Midpoint:

$$z_m = \frac{z_1 + z_2}{2}$$

Section formula (internal division in ratio $m : n$):

$$z = \frac{mz_2 + nz_1}{m+n}$$

Equation of a Line

Parametric form through z_1 in direction of $(z_2 - z_1)$:

$$z = z_1 + t(z_2 - z_1), \quad t \in \mathbb{R}$$

Two-point form:

$$\frac{z - z_1}{z_2 - z_1} = t, \quad t \in \mathbb{R}$$

Determinant form:

$$\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0$$

General form:

$$az + \overline{az} + b = 0$$

where a is complex and b is real.

Collinearity

Three points z_1, z_2, z_3 are collinear iff:

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$$

Or equivalently:

$$\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}$$

Equation of a Circle

Centre z_0, radius r:

$$|z - z_0| = r$$

$$(z - z_0)(\overline{z} - \overline{z_0}) = r^2$$

General form:

$$z\overline{z} + \overline{a}z + a\overline{z} + b = 0$$

where b is real, centre $= -a$, radius $= \sqrt{|a|^2 - b}$

Diameter form with endpoints z_1 and z_2:

$$\frac{z - z_1}{z - z_2} + \frac{\overline{z} - \overline{z_1}}{\overline{z} - \overline{z_2}} = 0$$

Angle Between Two Lines

If two lines have directions z_1 and z_2:

$$\theta = \arg\left(\frac{z_2}{z_1}\right)$$

Perpendicular lines:

$$\mathrm{Re}(z_1\overline{z_2}) = 0$$

$$\frac{z_1}{z_2} \text{ is purely imaginary}$$

Parallel lines:

$$\mathrm{Im}(z_1\overline{z_2}) = 0$$

$$\frac{z_1}{z_2} \text{ is real}$$

Important Loci

Circle: $|z - z_0| = r$

Perpendicular bisector: $|z - z_1| = |z - z_2|$

Ellipse: $|z - z_1| + |z - z_2| = 2a$ (where $2a > |z_1 - z_2|$)

Hyperbola: $||z - z_1| - |z - z_2|| = 2a$ (where $2a < |z_1 - z_2|$)

Ray from origin: $\arg(z) = \theta$

Half-plane: $\mathrm{Re}(z) > a$ or $\mathrm{Im}(z) > b$

Permutations, Combinations and Statistics

Fundamental Principles

Addition Principle

If an event can occur in m ways and another event can occur in n ways, and both cannot occur together, then either event can occur in $m + n$ ways.

Multiplication Principle

If an event can occur in m ways and for each of these, another event can occur in n ways, then both events can occur in $m \times n$ ways.

Factorial

Definition

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

$$0! = 1$$

Properties

$$n! = n \times (n-1)!$$

$$\frac{n!}{(n-r)!} = n \times (n-1) \times ... \times (n-r+1)$$

Permutations

Basic Permutation

Number of arrangements of n objects taken r at a time:

$$P(n,r) = {}^nP_r = \frac{n!}{(n-r)!}$$

When $r = n$:

$$P(n,n) = n!$$

Permutation with Repetition

Permutations of n objects where p are of one kind, q of another kind, etc.:

$$\frac{n!}{p!q!r!...}$$

Circular Permutations

Circular arrangements of n distinct objects:

$$(n-1)!$$

Circular arrangements with clockwise and anticlockwise same:

$$\frac{(n-1)!}{2}$$

Properties

$$^{n}P_{r} = n \times {}^{n-1}P_{r-1}$$

$$^{n}P_{r} = {}^{n}P_{n-r} \times \frac{n!}{(n-r)!}$$

Combinations

Basic Combination

Number of selections of n objects taken r at a time:

$$C(n,r) = {}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Properties

$$^{n}C_{r} = {}^{n}C_{n-r}$$

$$^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$$

(Pascal's Identity)

$$^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + ... + {}^{n}C_{n} = 2^{n}$$

$$^{n}C_{0} - {}^{n}C_{1} + {}^{n}C_{2} - ... + (-1)^{n}{}^{n}C_{n} = 0$$

$$^{n}C_{r} = \frac{n}{r}{}^{n-1}C_{r-1}$$

Selection with Repetition

Number of ways to select r objects from n types with repetition:

$$^{n+r-1}C_{r} = {}^{n+r-1}C_{n-1}$$

Binomial Theorem

Expansion

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$$

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + ... + {}^nC_n y^n$$

General Term

$(r+1)$th term:

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

Middle Term

If n is even: middle term is $T_{\frac{n}{2}+1}$ If n is odd: middle terms are $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

Greatest Term

If $\frac{(n+1)|y|}{|x|+|y|}$ is an integer m, then T_m and T_{m+1} are equal and greatest.

Otherwise, greatest term is T_{m+1} where $m = \left\lfloor \frac{(n+1)|y|}{|x|+|y|} \right\rfloor$

Special Cases

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

$$(1-x)^n = \sum_{r=0}^{n} {}^nC_r (-1)^r x^r$$

Multinomial Theorem

$$(x_1 + x_2 + ... + x_k)^n = \sum \frac{n!}{r_{1!} r_{2!} ... r_{k!}} x_1^{r_1} x_2^{r_2} ... x_k^{r_k}$$

where $r_1 + r_2 + ... + r_k = n$

Statistics - Measures of Central Tendency

Mean

Arithmetic Mean (ungrouped):

$$\overline{x} = \frac{\sum x_i}{n} = \frac{x_1 + v x_2 + ... + x_n}{n}$$

Arithmetic Mean (grouped):

$$\overline{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{\sum f_i x_i}{N}$$

Weighted Mean:

$$\overline{x}_w = \frac{\sum w_i x_i}{\sum w_i}$$

Properties:

$$\sum (x_i - \overline{x}) = 0$$

If $y_i = ax_i + b$, then $\overline{y} = a\overline{x} + b$

Median

For ungrouped data (ordered):

- If n is odd: Median $= x_{\frac{n+1}{2}}$
- If n is even: Median $= \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$

For grouped data:

$$\text{Median} = L + \frac{\frac{N}{2} - F}{f} \times h$$

where:
L = lower boundary of median class
N = total frequency
F = cumulative frequency before median class
f = frequency of median class
h = class width

Mode

For grouped data:

$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

where:
L = lower boundary of modal class
f_1 = frequency of modal class
f_0 = frequency of class before modal class
f_2 = frequency of class after modal class
h = class width

Relationship

For moderately asymmetric distribution:

$$\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$$

Measures of Dispersion

Range

$$\text{Range} = \text{Maximum value} - \text{Minimum value}$$

Mean Deviation

Mean Deviation about Mean:

$$\text{M.D.}(\overline{x}) = \frac{\sum |x_i - \overline{x}|}{n}$$

Mean Deviation about Median:

$$\text{M.D.}(\text{Median}) = \frac{\sum |x_i - \text{Median}|}{n}$$

For grouped data:

$$\text{M.D.} = \frac{\sum f_i |x_i - A|}{N}$$

where A is mean or median.

Variance

Population Variance:

$$\sigma^2 = \frac{\sum (x_i - \mu)^2}{N} = \frac{\sum x_i^2}{N} - \mu^2$$

Sample Variance:

$$s^2 = \frac{\sum (x_i - \overline{x})^2}{n-1} = \frac{\sum x_i^2 - n\overline{x}^2}{n-1}$$

For grouped data:

$$\sigma^2 = \frac{\sum f_i (x_i - \overline{x})^2}{N} = \frac{\sum f_i x_i^2}{N} - \overline{x}^2$$

Standard Deviation

Coefficient of Variation

$$\text{C.V.} = \frac{\sigma}{\overline{x}} \times 100\%$$

Used to compare variability of two or more datasets.

Probability Basics

Classical Definition

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Properties

$$0 \leq P(A) \leq 1$$

$$P(\text{certain event}) = 1$$

$$P(\text{impossible event}) = 0$$

$$P(A') = 1 - P(A)$$

(Complement)

Addition Rules

For mutually exclusive events:

$$P(A \cup B) = P(A) + P(B)$$

For any two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For three events:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rules

For independent events:

$$P(A \cap B) = P(A) \times P(B)$$

For dependent events:

$$P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$$

Bayes' Theorem

$$P(A_i|B) = \frac{P(A_i) \times P(B|A_i)}{\sum_{j=1}^{n} P(A_j) \times P(B|A_j)}$$

Probability Distributions

Binomial Distribution

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

where $q = 1 - p$

Mean:

$$\mu = np$$

Variance:

$$\sigma^2 = npq$$

Standard Deviation:

$$\sigma = \sqrt{npq}$$

Poisson Distribution

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

Mean:

$$\mu = \lambda$$

Variance:

$$\sigma^2 = \lambda$$

Normal Distribution

Probability density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Standard Normal Distribution ($\mu = 0, \sigma = 1$):

$$z = \frac{x - \mu}{\sigma}$$

Properties:

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.6827$$

(68.27%)

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.9545$$

(95.45%)

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.9973$$

(99.73%)

Correlation and Regression

Correlation Coefficient (Pearson's r)

$$r = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum\left(x_i - \overline{x}\right)^2 \sum\left(y_i - \overline{y}\right)^2}}$$

$$r = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{\left[N\sum x_i^2 - \left(\sum x_i\right)^2\right]\left[N\sum y_i^2 - \left(\sum y_i\right)^2\right]}}$$

Properties:

$$-1 \leq r \leq 1$$

Regression Lines

Regression line of y on x:

$$y - \overline{y} = \frac{r\sigma_y}{\sigma_x}(x - \overline{x})$$

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

Regression line of x on y:

$$x - \overline{x} = \frac{r\sigma_x}{\sigma_y}(y - \overline{y})$$

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

Relationship:

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$$r^2 = b_{yx} \times b_{xy}$$

(if both slopes have same sign)

Vector Algebra

Vector Basics

Representation

Vector in component form:

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

Position vector:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Magnitude

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Unit Vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Direction Cosines

If vector makes angles α, β, γ with axes:

$$\cos\alpha = \frac{a_x}{|\vec{a}|}, \cos\beta = \frac{a_y}{|\vec{a}|}, \cos\gamma = \frac{a_z}{|\vec{a}|}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Vector Operations

Addition and Subtraction

$$\vec{a} \pm \vec{b} = (a_x \pm b_x)\hat{\imath} + \left(a_y \pm b_y\right)\hat{\jmath} + (a_z \pm b_z)\hat{k}$$

Triangle law:

$$\vec{a} + \vec{b} = \vec{c}$$

Parallelogram law:

$$\vec{a} + \vec{b} = \text{diagonal}$$

Scalar Multiplication

$$k\vec{a} = ka_x\hat{\imath} + ka_y\hat{\jmath} + ka_z\hat{k}$$

Properties

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(Commutative)

$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$$

(Associative)

$$k\left(\vec{a} + \vec{b}\right) = k\vec{a} + k\vec{b}$$

(Distributive)

Dot Product (Scalar Product)

Definition

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Properties

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

(Commutative)

$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

(Distributive)

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$$

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$$

Angle Between Vectors

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\left|\vec{b}\right|}$$

Perpendicular if:

$$\vec{a} \cdot \vec{b} = 0$$

Parallel if:

$$\vec{a} = k\vec{b}$$

Projection

Projection of $\vec{a}$ on $\vec{b}$:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|}$$

Vector projection:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|^2}\vec{b}$$

Cross Product (Vector Product)

Definition

$$\vec{a} \times \vec{b} = |\vec{a}|\left|\vec{b}\right| \sin\theta \hat{n}$$

where $\hat{n}$ is unit vector perpendicular to both.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \left(a_y b_z - a_z b_y\right)\hat{\imath} + \left(a_z b_x - a_x b_z\right)\hat{\jmath} + \left(a_x b_y - a_y b_x\right)\hat{k}$$

Properties

$$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

(Anti-commutative)

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

(Distributive)

$$\vec{a} \times \vec{a} = \vec{0}$$

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath}, \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

Magnitude

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin\theta$$

Area of parallelogram:

$$\left|\vec{a} \times \vec{b}\right|$$

Area of triangle:

$$\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

Parallel if:

$$\vec{a} \times \vec{b} = \vec{0}$$

Scalar Triple Product

Definition

$$\left[\vec{a}\vec{b}\vec{c}\right] = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$$

$$\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Properties

$$\left[\vec{a}\vec{b}\vec{c}\right] = \left[\vec{b}\vec{c}\vec{a}\right] = \left[\vec{c}\vec{a}\vec{b}\right]$$

(Cyclic)

$$\left[\vec{a}\vec{b}\vec{c}\right] = -\left[\vec{b}\vec{a}\vec{c}\right]$$

(Anti-cyclic)

$$\left[\vec{a}\vec{b}\vec{c}\right] = 0$$

(Coplanar vectors)

Volume

Volume of parallelepiped:

$$\left|\left[\vec{a}\vec{b}\vec{c}\right]\right|$$

Volume of tetrahedron:

$$\frac{1}{6}\left|\left[\vec{a}\vec{b}\vec{c}\right]\right|$$

Vector Triple Product

BAC-CAB Rule

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = (\vec{a} \cdot \vec{c})\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$$

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a}$$

Properties

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$$

(Not associative)

Calculus

Differentiation

Basic Rules

Constant Rule:

$$\frac{d}{dx}(c) = 0$$

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Constant Multiple Rule:

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

Sum Rule:

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

Product Rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Chain Rule:

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Table of Derivatives

S.N.	$f(x)$	$f'(x)$
1	c	0
2	x^n	nx^{n-1}
3	$\frac{1}{x}$	$-\frac{1}{x^2}$
4	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
5	e^x	e^x

6	a^x	$a^x \ln(a)$
7	$\ln x$	$\dfrac{1}{x}$
8	$\log_a x$	$\dfrac{1}{x \ln a}$
9	$\sin x$	$\cos x$
10	$\cos x$	$-\sin x$
11	$\tan x$	$\sec^2 x$
12	$\cot x$	$-\csc^2 x$
13	$\sec x$	$\sec x \tan x$
14	$\csc x$	$-\csc x \cot x$
15	$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
16	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
17	$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
18	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
19	$\sec^{-1} x$	$\dfrac{1}{\|x\|\sqrt{x^2-1}}$
20	$\csc^{-1} x$	$-\dfrac{1}{\|x\|\sqrt{x^2-1}}$
21	$\sinh x$	$\cosh x$
22	$\cosh x$	$\sinh x$
23	$\tanh x$	$\text{sech}^2 x$
24	$\coth x$	$-\text{csch}^2 x$
25	$\text{sech }x$	$-\text{sech }x \tanh x$
26	$\text{csch }x$	$-\text{csch }x \coth x$
27	$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$
28	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
29	$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$
30	$\coth^{-1} x$	$\dfrac{1}{1-x^2}$
31	$\text{sech}^{-1} x$	$-\dfrac{1}{x\sqrt{1-x^2}}$

32	$\operatorname{csch}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{1+x^2}}$

Higher Order Derivatives

Second derivative:

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

nth derivative:

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

Leibniz Rule:

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k u}{dx^k} \frac{d^{n-k} v}{dx^{n-k}}$$

Implicit Differentiation

For equation $F(x, y) = 0$:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Parametric Differentiation

If $x = f(t)$ and $y = g(t)$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dt} \frac{\frac{dy}{dx}}{\frac{dx}{dt}}$$

Logarithmic Differentiation

For $y = f(x)$:

$$\frac{d}{dx}(\ln y) = \frac{1}{y} \frac{dy}{dx}$$

Useful for products, quotients, and powers.

Integration

Basic Rules

Constant Rule:

$$\int c dx = cx + C$$

Power Rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Sum Rule:

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

Constant Multiple:

$$\int c f(x) dx = c \int f(x) dx$$

Table of Integrals

S.N.	$f(x)$	$\int f(x)dx$
1	c	$cx + C$
2	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
3	$\frac{1}{x}$	$\ln\lvert x\rvert + C$
4	e^x	$e^x + C$
5	a^x	$\frac{a^x}{\ln a} + C$
6	$\ln x$	$x \ln x - x + C$
7	$\log_a x$	$x \log_a x - \frac{x}{\ln a} + C$
8	$\sin x$	$-\cos x + C$
9	$\cos x$	$\sin x + C$
10	$\tan x$	$-\ln\lvert\cos x\rvert + C = \ln\lvert\sec x\rvert + C$
11	$\cot x$	$\ln\lvert\sin x\rvert + C$
12	$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$
13	$\csc x$	$-\ln\lvert\csc x + \cot x\rvert + C$
14	$\sec^2 x$	$\tan x + C$
15	$\csc^2 x$	$-\cot x + C$
16	$\sec x \tan x$	$\sec x + C$
17	$\csc x \cot x$	$-\csc x + C$
18	$\sin^{-1} x$	$x \sin^{-1} x + \sqrt{1 - x^2} + C$

19	$\cos^{-1} x$	$x \cos^{-1} x - \sqrt{1-x^2} + C$
20	$\tan^{-1} x$	$x \tan^{-1} x - \frac{\ln(1+x^2)}{2} + C$
21	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + C$
22	$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C$
23	$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C$
24	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
25	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| + C$
26	$\sinh x$	$\cosh x + C$
27	$\cosh x$	$\sinh x + C$
28	$\tanh x$	$\ln(\cosh x) + C$
29	$\coth x$	$\ln\|\sinh x\| + C$
30	$\text{sech } x$	$\tan^{-1}(\sinh x) + C$
31	$\text{csch } x$	$\ln\left\|\tanh\left(\frac{x}{2}\right)\right\| + C$
32	$\sinh^{-1} x$	$x \sinh^{-1} x - \sqrt{x^2+1} + C$
33	$\cosh^{-1} x$	$x \cosh^{-1} x - \sqrt{x^2-1} + C$
34	$\tanh^{-1} x$	$x \tanh^{-1} x + \frac{\ln(1-x^2)}{2} + C$
35	$f'\frac{x}{f(x)}$	$\ln\|f(x)\| + C$
36	$f'(x)(f(x))^n$	$\frac{(f(x))^{n+1}}{n+1} + C, \quad n \neq -1$

Integration Techniques

Integration by Parts

$$\int u dv = uv - \int v du$$

Or:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

LIATE Rule (priority for choosing u):

- L: Logarithmic functions

- I: Inverse trigonometric functions
- A: Algebraic functions
- T: Trigonometric functions
- E: Exponential functions

Integration by Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du, \quad \text{where } u = g(x)$$

Partial Fractions

For rational functions $\frac{P(x)}{Q(x)}$ where $\deg(P) < \deg(Q)$:

Linear factors: $\frac{A}{x-a}$

Repeated linear: $\frac{A}{x-a} + \frac{B}{(x-a)^2} + ... + \frac{C}{(x-a)^n}$

Quadratic factors: $\frac{Ax+B}{x^2+bx+c}$

Trigonometric Substitution

- For $\sqrt{a^2 - x^2}$: use $x = a\sin\theta$
- For $\sqrt{a^2 + x^2}$: use $x = a\tan\theta$
- For $\sqrt{x^2 - a^2}$: use $x = a\sec\theta$

Definite Integrals

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

Properties:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Symmetric intervals:

- If $f(-x) = f(x)$ (even):

  $$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$
- If $f(-x) = -f(x)$ (odd):

  $$\int_{-a}^a f(x)dx = 0$$

Periodic functions:

$$\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx$$

where T is the period.

Important Definite Integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{(n-1)!!}{n!!} & \text{n even} \\ \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & \text{n odd} \end{cases}$$

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Applications of Derivatives

Tangent and Normal

At point (x_0, y_0):

Tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

Normal line:

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$

Rate of Change

$$\text{Rate of change} = \frac{dy}{dx}$$

Related rates: Use chain rule to relate rates of different quantities.

Maxima and Minima

Critical points: $f'(x) = 0$ or $f'(x)$ does not exist

First derivative test:

- f' changes from + to -: local maximum
- f' changes from - to +: local minimum

Second derivative test:

- $f''(x_0) > 0$: local minimum at x_0
- $f''(x_0) < 0$: local maximum at x_0
- $f''(x_0) = 0$: test inconclusive

Concavity and Inflection Points

Concave up: $f''(x) > 0$

Concave down: $f''(x) < 0$

Inflection point: $f''(x)$ changes sign

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b):

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem: If additionally $f(a) = f(b)$, then $\exists c : f'(c) = 0$

Applications of Integrals

Area Under Curve

$$A = \int_a^b f(x)dx$$

Area between curves:

$$A = \int_a^b |f(x) - g(x)|dx$$

Volume of Solids of Revolution

Disk method (about x-axis):

$$V = \pi \int_a^b (f(x))^2 dx$$

Washer method:

$$V = \pi \int_a^b ((f(x))^2 - (g(x))^2)dx$$

Shell method (about y-axis):

$$V = 2\pi \int_a^b x f(x) dx$$

Arc Length

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

For parametric: $x = f(t), y = g(t)$:

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Surface Area of Revolution

About x-axis:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

Series and Taylor Expansions

Common Taylor Series

S.N.	Function	Series Expansion
1	e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
2	$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
3	$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
4	$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad \lvert x\rvert < 1$
5	$(1+x)^\alpha$	$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots$
6	$\tan^{-1} x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, \quad \lvert x\rvert \le 1$
7	$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
8	$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

Taylor's Theorem

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N$$

where R_N is the remainder term.

Maclaurin series (expansion around $a = 0$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Partial Derivatives

Basic Definitions

For $f(x, y)$:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Mixed Partial Derivatives

If continuous:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Chain Rule for Partial Derivatives

If $z = f(x, y)$, $x = g(t)$, $y = h(t)$:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Directional Derivative

In direction of unit vector $\vec{u} = (u_1, u_2)$:

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

Gradient Vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Points in direction of maximum increase of f.

Euler's Theorem for Homogeneous Functions

If $f(tx, ty, tz) = t^n f(x, y, z)$ (homogeneous of degree n):

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

Vector Calculus

Vector Differentiation

$$\frac{d}{dt}\left(\vec{a} + \vec{b}\right) = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$$

$$\frac{d}{dt}(\varphi\vec{a}) = \frac{d\varphi}{dt}\vec{a} + \varphi\frac{d\vec{a}}{dt}$$

$$\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right) = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$$

Velocity and Acceleration

Position vector:

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

Velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

Acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Speed:

$$|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Vector Integration

$$\int\left(\vec{a}+\vec{b}\right)dt = \int\vec{a}dt + \int\vec{b}dt$$

$$\int\varphi\vec{a}dt = \varphi\int\vec{a}dt$$

(if φ is constant)

Del Operator (Nabla)

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient

For scalar field φ:

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

Properties:

- Points in direction of maximum increase
- Magnitude gives rate of maximum increase
- Perpendicular to level surfaces

Divergence

For vector field $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$:

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Measures “outflow” from a point (scalar).

Curl

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

Measures “rotation” of field (vector).

Laplacian

Scalar Laplacian:

$$\nabla^2\varphi = \nabla \cdot \nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Vector Laplacian:

$$\nabla^2\vec{F} = (\nabla^2 F_x)\hat{\imath} + \left(\nabla^2 F_y\right)\hat{\jmath} + (\nabla^2 F_z)\hat{k}$$

Important Identities

$$\nabla \times (\nabla\varphi) = \vec{0}$$

(Curl of gradient is zero)

$$\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$$

(Divergence of curl is zero)

$$\nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla^2\vec{F}$$

$$\nabla \cdot \left(\varphi\vec{F}\right) = \varphi\nabla \cdot \vec{F} + \vec{F} \cdot \nabla\varphi$$

$$\nabla \times \left(\varphi\vec{F}\right) = \varphi\nabla \times \vec{F} + (\nabla\varphi) \times \vec{F}$$

$$\nabla\left(\vec{F} \cdot \vec{G}\right) = \left(\vec{F} \cdot \nabla\right)\vec{G} + \left(\vec{G} \cdot \nabla\right)\vec{F} + \vec{F} \times \left(\nabla \times \vec{G}\right) + \vec{G} \times \left(\nabla \times \vec{F}\right)$$

Line Integrals

Scalar line integral:

$$\int_C f ds = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt$$

Vector line integral:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

Surface Integrals

Scalar surface integral:

$$\oiint_S f dS$$

Vector surface integral (flux):

$$\oiint_S \vec{F} \cdot d\vec{S} = \oiint_S \vec{F} \cdot \hat{n} dS$$

where $\hat{n}$ is unit normal vector.

Integral Theorems

Green's Theorem (2D):

$$\oint_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Stokes' Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \oiint_S \left(\nabla \times \vec{F} \right) \cdot \hat{n} dS$$

Divergence Theorem (Gauss's Theorem):

$$\oiint_S \vec{F} \cdot \hat{n} dS = \oiiint_V \left(\nabla \cdot \vec{F} \right) dV$$

Special Functions

Gamma Function

Definition:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$$

Properties:

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}$$

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Duplication formula:

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma\left(n + \frac{1}{2}\right)$$

Beta Function

Definition:

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m, n > 0$$

Properties:

$$\beta(m, n) = \beta(n, m)$$

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Alternate forms:

$$\beta(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$$

$$\beta(m, n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}dt$$

Important Integrals Using Gamma and Beta

$$\int_0^{\infty} e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}, \quad a > 0$$

$$\int_0^{\infty} e^{-x^2}x^{2n-1}dx = \frac{\Gamma(n)}{2}$$

$$\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Fourier Transform

Definition

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Inverse Fourier transform:

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

Properties

Linearity:

$$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$$

Time shifting:

$$\mathcal{F}\{f(t - t_0)\} = e^{-i\omega t_0}F(\omega)$$

Frequency shifting:

$$\mathcal{F}\left\{e^{i\omega_0 t}f(t)\right\} = F(\omega - \omega_0)$$

Scaling:

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

Differentiation:

$$\mathcal{F}\{f'(t)\} = i\omega F(\omega)$$

$$\mathcal{F}\left\{f^{(n)}(t)\right\} = (i\omega)^n F(\omega)$$

Convolution:

$$\mathcal{F}\{f(t) * g(t)\} = F(\omega)G(\omega)$$

Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Laplace Transforms

Definition

The Laplace transform of a function $f(t)$ is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \lim_{A \to \infty} \int_0^A e^{-st} f(t) dt$$

The inverse Laplace transform of $F(s)$ is defined as:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds$$

where c is a real constant chosen so that the contour path is in the region of convergence of $F(s)$.

Properties of Laplace Transform

Linearity

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

First Translation (Shifting) Theorem

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

Second Translation Theorem

$$\mathcal{L}\{u(t - c)f(t - c)\} = e^{-cs} F(s)$$

where $u(t - c)$ is the unit step function.

Change of Scale

$$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right)$$

Derivative of Transform

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} = (-1)^n F^{(n)}(s)$$

Transform of Derivative

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

Transform of Integral

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Division by t

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$$

Convolution Theorem

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} = F(s)G(s)$$

Transform of Periodic Function

If $f(t+T) = f(t)$ (periodic with period T):

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$$

Table of Laplace Transforms

S.N.	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}, \quad s > 0$
2	t	$\frac{1}{s^2}$
3	$t^n, \quad n = 1, 2, 3, ...$	$\frac{n!}{s^{n+1}}$
4	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
6	$t^{-\frac{1}{2}}$	$\sqrt{\frac{\pi}{s}}$
7	e^{at}	$\frac{1}{s-a}, \quad s > a$
8	te^{at}	$\frac{1}{(s-a)^2}$

9	$t^n e^{at}, \quad n = 1, 2, 3, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
10	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
11	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
12	$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$
13	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
14	$\sin(at) - at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$
15	$\sin(at) + at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
16	$\sinh(at)$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$
17	$\cosh(at)$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$
18	$t\sinh(at)$	$\dfrac{2as}{(s^2-a^2)^2}$
19	$t\cosh(at)$	$\dfrac{s^2+a^2}{(s^2-a^2)^2}$
20	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
21	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
22	$e^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$
23	$e^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
24	$\sin(at)\sinh(at)$	$\dfrac{2a^2s}{s^4+4a^4}$
25	$\sin(at)\cosh(at)$	$\dfrac{a(s^2+2a^2)}{s^4+4a^4}$
26	$\cos(at)\sinh(at)$	$\dfrac{a(s^2-2a^2)}{s^4+4a^4}$
27	$\cos(at)\cosh(at)$	$\dfrac{s^3}{s^4+4a^4}$
28	$u(t-c)$	$\dfrac{e^{-cs}}{s}$

29	$\delta(t-c)$	e^{-cs}
30	$u(t-c)f(t-c)$	$e^{-cs}F(s)$
31	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
32	$e^{ct}f(t)$	$F(s-c)$
33	$tf(t)$	$-F'(s)$
34	$t^n f(t), \quad n=1,2,3,...$	$(-1)^n F^{(n)}(s)$
35	$\frac{f(t)}{t}$	$\int_s^\infty F(u)du$
36	$f'(t)$	$sF(s)-f(0)$
37	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
38	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-...-f^{(n-1)}(0)$
39	$\int_0^t f(v)dv$	$\frac{F(s)}{s}$
40	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
41	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$

Special Functions in Laplace Transform

Unit Step Function (Heaviside Function)

$$u(t-c)=\begin{cases}0 \text{ if } t<c\\ 1 \text{ if } t\geq c\end{cases}$$

$$\mathcal{L}\{u(t-c)\}=\frac{e^{-cs}}{s}$$

Dirac Delta Function

$$\delta(t-c)=\begin{cases}\infty \text{ if } t=c\\ 0 \quad \text{if } t\neq c\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(t-c)dt=1$$

$$\mathcal{L}\{\delta(t-c)\}=e^{-cs}$$

Rectangular Pulse

$$f(t) = u(t-a) - u(t-b) = \begin{cases} 1 \text{ if } a \leq t < b \\ 0 \text{ otherwise} \end{cases}$$

$$\mathcal{L}\{u(t-a) - u(t-b)\} = \frac{e^{-as} - e^{-bs}}{s}$$

Applications of Laplace Transform

Solving Differential Equations

For differential equation with initial conditions:
1. Take Laplace transform of both sides
2. Use properties to convert derivatives
3. Solve algebraic equation for $F(s)$
4. Take inverse Laplace transform to get $f(t)$

Transfer Functions

For linear system with input $f(t)$ and output $g(t)$:

$$H(s) = \frac{G(s)}{F(s)}$$

where $H(s)$ is the transfer function.

Fourier Series

Definition

A periodic function $f(x)$ with period 2π can be represented as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Fourier Coefficients

For Period 2π (interval $[\alpha, \alpha + 2\pi]$)

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos(nx) dx, \quad n = 1, 2, 3, ...$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin(nx) dx, \quad n = 1, 2, 3, ...$$

Common choice: $\alpha = -\pi$ (interval $[-\pi, \pi]$)

For Arbitrary Period $2c$ (interval $[-c, c]$)

If $f(x)$ has period $2c$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right]$$

Coefficients:

$$a_0 = \frac{1}{c} \int_{-c}^{c} f(x) dx$$

$$a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos\left(\frac{n\pi x}{c}\right) dx, \quad n = 1, 2, 3, ...$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx, \quad n = 1, 2, 3, ...$$

Special Cases

Case I: Even Function

If $f(-x) = f(x)$ (even function):

$$a_0 = \frac{2}{c}\int_0^c f(x)dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

$$b_n = 0$$

Fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right)$$

Case II: Odd Function

If $f(-x) = -f(x)$ (odd function):

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right)$$

Half-Range Expansions

Half-Range Cosine Series

For function defined on $[0, c]$, extend as even function:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right)$$

$$a_0 = \frac{2}{c}\int_0^c f(x)dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Half-Range Sine Series

For function defined on $[0, c]$, extend as odd function:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right)$$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

Complex Form of Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

For period $2c$:

$$c_n = \frac{1}{2c} \int_{-c}^{c} f(x) e^{-in\pi x/c} dx$$

Relation with real coefficients:

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{a_n - ib_n}{2}, \quad n > 0$$

$$c_{-n} = \frac{a_n + ib_n}{2}, \quad n > 0$$

Parseval's Theorem

For Fourier Series

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

For period $2c$:

$$\frac{1}{c} \int_{-c}^{c} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Convergence of Fourier Series

Dirichlet Conditions

Fourier series of $f(x)$ converges if:
1. $f(x)$ is periodic
2. $f(x)$ is single-valued and finite
3. $f(x)$ has finite number of discontinuities in one period
4. $f(x)$ has finite number of maxima and minima in one period

At point of continuity: Series converges to $f(x)$

At point of discontinuity: Series converges to $\frac{f(x^+)+f(x^-)}{2}$

Common Fourier Series Examples

Square Wave

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n} \sin(nx) = \frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + ... \right]$$

Sawtooth Wave

$$f(x) = x, \quad -\pi < x < \pi$$

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - ... \right]$$

Triangular Wave

$$f(x) = |x|, \quad -\pi < x < \pi$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2} \cos(nx)$$

Full-Wave Rectifier

$$f(x) = |\sin x|, \quad -\pi < x < \pi$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx)$$

Geometry

Two Dimensional Coordinate Geometry

https://www.scribd.com/document/155282071/Two-Dimensional-Co-Ordinate-Geometry

Lines

Introduction

Angle between lines:

$$\tan\theta = \frac{m_1 + m_2}{1 + m_1 m_2}$$

If point cuts on m_1:m_2

$$x = \frac{x_1 + rx_2}{1 + r}, y = \frac{y_1 + ry_2}{1 + r}$$

Area of triangle =

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Area of n-sided polygon=

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & ... & x_n & x_1 \\ y_1 & y_2 & y_3 & ... & y_n & y_1 \end{vmatrix}$$

Change of axis

Translation of axis:P' is new .

$$x = x' + h, y = y' + k$$

Rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Straight Lines

Point slope form: $y - y_1 = m(x - x_1)$

Parametric:

$$x = x_1 + t\cos\theta$$

$$y = y_1 + t\sin\theta$$

Slope intercept: $y = mx + c$

Two point form:

$$\frac{x - x_1}{y - y_1} = \frac{x_1 - x_2}{y_1 - y_2}$$

Parametric:

$$x = tx_1 + (1 - t)x_2$$

$$y = ty_1 + (1 - t)y_2$$

Intercept Form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Normal form: $x \cos \alpha + y \sin \alpha = p$

for $Ax + By + C = 0$,

$$\frac{A}{\pm\sqrt{A^2 + B^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2}} = 0$$

Distance between line and points:

For $x \cos \alpha + y \sin \alpha - p = 0$

$$d = x_0 \cos \alpha + y_0 \sin \alpha - p$$

For $Ax + By + C = 0$,

$$d = \left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|$$

Concurrancy,

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0$$

Equation of line pairs

For $ax^2 + 2hxy + by^2 = 0$,
1. real and distinct, $h^2 > ab$
2. real and coincident, $h^2 \geq ab$
3. imaginary, $h^2 < ab$

Equation of bisector: $hx^2 - (a - b)xy - hy^2 = 0$

General eqution $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, shall represent line if,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Circles

General: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
Condition: a=b & h = 0
Centre:$\left(-\frac{g}{a}, -\frac{f}{a}\right)$
radius:

$$\sqrt{\frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}}$$

Equation with diameter points, (x_1, y_1) ,(x_2, y_2)

$$\left(\frac{y - y_1}{x - x_2}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

Equation at tangent:(x_1, y_1)

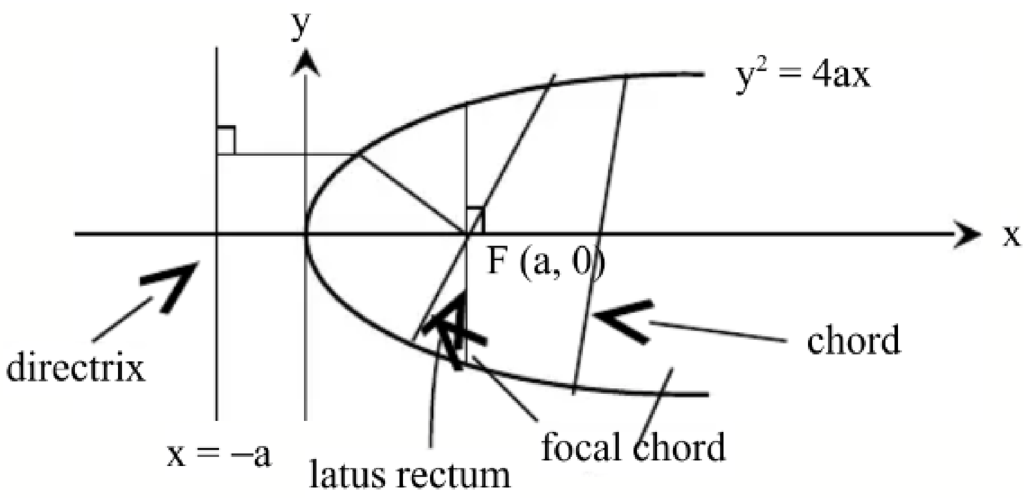
$$x_1x + y_1y + g(x_1 + x) + f(y_1 + y) + c = 0$$

* Same formula gives chord if point is outside.

Tangent having slope m:

$$y = mx \pm r\sqrt{1 + m^2}$$

Parabola



Parametric:

$$x = at^2$$
$$y = 2at$$

Chord of parabola with ends $(x_1, y_1), (x_2, y_2)$:

$$y(y_1 - y_2) = 4ax + y_1 * y_2$$

Remark: If $y_1 + y_2 = 0, x = x_1 = x_2$

Equation of tangent (x_1, y_1)

$$y_1 y = 2a(x + x_1)$$

In parametric form. $(at^2, 2at)$

$$y = \frac{x}{t} + at$$

Ellipse

Standard form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Eccentricity (e):

$$e^2 = 1 - \frac{b^2}{a^2}, \text{for } a > b$$

Parametric: $x = a\cos\theta$, $x = b\sin\theta$

Tangent:

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Parametric:

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Hyperbola

Standard form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Eccentricity (e): $e^2 = 1 + \frac{b^2}{a^2} > 1$

Asymptotes:

$$\frac{x}{a} \pm \frac{y}{b} = 1$$

Parametric: $x = a\sec`\theta$, $x = b\tan\theta$ Chord: (x_1, y_1),(x_2, y_2)

$$\frac{x}{a^2}(x_1 + x_2) - \frac{y}{b^2}(y_1 + y_2) = \frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} + 1$$

General Equation of second degree

$ax^2 + 2hxy + by^2 + 2gx + 2gy + c = 0$
Tangent: (x_1, y_1)

$$ax_1x^2 + 2h(xy_1 + yx_1) + by_1y^2 + 2g(x + x_1) + 2f(y + y_1) + c = 0$$

If $ab - h^2$

- =0. parabola
- >0, Ellipse
- <0, hyperbola

Pair of lines:

$$\tan 2\theta = \frac{2h}{a - b}$$

Three Dimensional Coordinate Geometry

Distance and Section Formula

Distance Formula

Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Section Formula

Point dividing line in ratio $m : n$ internally:

$$x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}, z = \frac{mz_2 + nz_1}{m+n}$$

Midpoint:

$$x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}, z = \frac{z_1 + z_2}{2}$$

Direction Cosines and Direction Ratios

Direction Cosines

If line makes angles α, β, γ with axes:

$$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$$

$$l^2 + m^2 + n^2 = 1$$

Direction cosines between two points:

$$l = \frac{x_2 - x_1}{d}, m = \frac{y_2 - y_1}{d}, n = \frac{z_2 - z_1}{d}$$

Direction Ratios

If a, b, c are direction ratios:

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Angle Between Two Lines

If direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2):

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

If direction ratios are (a_1, b_1, c_1) and (a_2, b_2, c_2):

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Perpendicular if:

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Parallel if:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Straight Lines in 3D

Equation of a Line

Cartesian form through (x_1, y_1, z_1) with direction ratios (a, b, c):

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Two point form through (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Parametric form:

$$x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c$$

Planes

Equation of a Plane

General form:

$$Ax + By + Cz + D = 0$$

where (A, B, C) are direction ratios of normal.

Intercept form:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Normal form:

$$lx + my + nz = p$$

where (l, m, n) are direction cosines of normal and p is distance from origin.

Plane through point (x_1, y_1, z_1) with normal (A, B, C):

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

Plane through three points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Distance from Point to Plane

Distance from (x_0, y_0, z_0) to plane $Ax + By + Cz + D = 0$:

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Angle Between Planes

For planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$:

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Perpendicular if:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0$$

Parallel if:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

Angle Between Line and Plane

For line with direction ratios (a, b, c) and plane $Ax + By + Cz + D = 0$:

$$\sin\theta = \cos(90 - \theta) = \frac{|Aa + Bb + Cc|}{\sqrt{A^2 + B^2 + C^2}\sqrt{a^2 + b^2 + c^2}}$$

Line parallel to plane if:

$$Aa + Bb + Cc = 0$$

Line perpendicular to plane if:

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

Sphere

Standard Forms

General equation:

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Centre: $(-u, -v, -w)$
Radius:

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

Centre-radius form with centre (h, k, l) and radius r:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Diameter form with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

Tangent Plane

Tangent plane at point (x_1, y_1, z_1) on sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$:

$$xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$$

Sphere through Four Points

Sphere through $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$:

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Vector Forms

Position Vector

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Equation of Line

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

where $\vec{a}$ is position vector of point on line and $\vec{b}$ is direction vector.

Equation of Plane

$$\vec{r} \cdot \vec{n} = d$$

where $\vec{n}$ is normal vector and d is constant.

Plane through point $\vec{a}$ with normal $\vec{n}$:

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

Distance from Point to Plane

Distance from point $\vec{a}$ to plane $\vec{r} \cdot \vec{n} = d$:

$$\text{distance} = \frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$$