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Math 024
03D
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4.4 / 21, 26, 34, 46, 55

21 $y'' + 4y' = 1$

$r^2 + 4r = 0$

$r(r+4) = 0$

$r = 0, -4$

$y_h(t) = C_1 + C_2 e^{-4t}$

$y_p(t) = At$

$y_p' = A$

$y_p'' = 0$

$0 + 4A = 1$

$A = \frac{1}{4}$

$y_p(t) = \frac{1}{4}t$

$$\boxed{y(t) = C_1 + C_2 e^{-4t} + \frac{1}{4}t}$$

26 $y'' - y' - 2y = 6e^t$

$r^2 - r - 2 = 0$

$(r+1)(r-2) = 0$

$r = -1, 2$

$y(t) = C_1 e^{-t} + C_2 e^{2t}$

$y_p(t) = Ae^t$

$y_p' = Ae^t$

$y_p'' = Ae^t$

$Ae^t - Ae^t - 2Ae^t = 6e^t$

$-2Ae^t = 6e^t$

$-2A = 6$

$A = -3$

$y_p(t) = -3e^t$

$$\boxed{y(t) = C_1 e^{-t} + C_2 e^{2t} - 3e^t}$$

34. $y'' - 4y' + 3y = 20\cos t$

$r^2 - 4r + 3 = 0$

$(r-1)(r-3) = 0$

$r = 1, 3$

$y_h(t) = C_1 e^t + C_2 e^{3t}$

$y_p(t) = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t - 4(-A\sin t + B\cos t) + 3(A\cos t + B\sin t) = 20\cos t$

$2A\cos t + 2B\sin t + 4A\sin t - 4B\cos t = 20\cos t$

$(2A + 2B)\sin t + (2A - 4B)\cos t = 20\cos t$

$4A + 2B = 0$ $\quad A = 2$

$2A - 4B = 20$ $\quad B = -4$

$y_p(t) = 2\cos t - 4\sin t$

$\boxed{y(t) = C_1 e^t + C_2 e^{3t} + 2\cos t - 4\sin t}$

4b $y'' + 9y = \cos 3t$

$r^2 + 9 = 0$

$r = \pm 3i$

$y_h(t) = C_1 \cos 3t + C_2 \sin 3t$

$y_p(t) = A\cos 3t + B\sin 3t$

$y_p(t) = (At+B)\cos 3t + (Ct+D)\sin 3t$

$y'_p = A\cos 3t - 3(At+B)\sin 3t + C\sin 3t + 3(Ct+D)\cos 3t$

$y''_p = -6A\sin 3t - 9(At+B)\cos 3t + 6C\cos 3t - 9(Ct+D)\sin 3t$

$-6A\sin 3t - 9(At+B)\cos 3t + 6C\cos 3t - 9(Ct+D)\sin 3t + 9[(At+B)\cos 3t + (Ct+D)\sin 3t] = \cos 3t$

$-6A\sin 3t + 6C\cos 3t = \cos 3t$

$-6A = 0 \quad D = 0 \quad A = 0 \qquad y_p(t) = \frac{1}{6} t \sin 3t$

$B = 0 \quad 6C = 1 \quad C = \frac{1}{6}$

$y(t) = y_h(t) + y_p(t)$

$= C_1 \cos 3t + C_2 \sin 3t + \frac{1}{6} t \sin 3t.$

$y'(t) = -3C_1 \sin 3t + 3C_2 \cos 3t + \frac{1}{6}\sin 3t + \frac{1}{2} t \cos 3t$

$y(0) = 1$

$C_1 \cos 3(0) + C_2 \sin 3(0) + \frac{1}{6}(0)\sin 3(0) = 1$

$C_1 = 1 \qquad y'(0) = -1$

$-3C_1 \sin 3(0) + 3C_2 \cos 3(0) + \frac{1}{6}\sin 3(0) + \frac{1}{2}(0)\cos 3(0) = -1$

$3C_2 = -1 \qquad C_2 = -\frac{1}{3} \qquad y(t) = \cos 3t - \frac{1}{3}\sin 3t + \frac{1}{6} t \sin 3t.$

55 $y'' + 5y' + 6y = \cos t - te^t$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3$

$y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$y_p(t) = A\cos t + B\sin t + (Ct + D)e^t$

Particular solution are linealy dependent on the homogeneous solution

4.5 / 7, 8, 12, 14,

7 $y'' - 3y' + 2y = \frac{1}{1+e^{-t}}$

$y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = (r-1)(r-2)$

$r = 1, 2$

$y_h(t) = C_1 e^t + C_2 e^{2t}$

$y_1 = e^t \quad y_2 = e^{2t}$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}$

$= (e^t)(2e^{2t}) - (e^{2t})(e^t)$

$= e^{3t}$

$V_1' = -\frac{y_2 f}{W}$

$= -\frac{e^{2t}}{e^{3t}(1+e^{-t})}$

$= \frac{e^{-t}}{1+e^{-t}}$

$V_2' = \frac{y_1 f}{W}$

$= \frac{e^t}{e^{2t}(1+e^{-t})}$

$= \frac{e^{-2t}}{1+e^{-t}}$

$V_1 = \int v_1'$

$= \int -\frac{e^{-t}}{(1+e^{-t})} dt$

$= \ln(e^t + 1) - t$

$V_2 = \int v_2'$

$= \int \frac{e^{-2t}}{1+e^{-t}} dt$

$= \ln(e^{-t} + 1) - e^{-t}$

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$y_p = v_1 y_1 + y_2 v_2$

$y_p = (\ln(e^{t}+1) - t)e^{t} + (\ln(e^{-t}+1) - e^{-t})e^{2t}$

$= (\ln(\frac{1+e^{-t}}{e^{-t}}) - t)e^{t} + (\ln(e^{-t}+1) - e^{-t})e^{2t}$

$= (\ln(1+e^{-t}) + t - t)e^{-t} + (\ln(e^{-t}+1) - e^{-t})e^{2t}$

$= (e^{t} + e^{2t})\ln(1+e^{-t}) - e^{t}$

$y(t) = y_h(t) + y_p(t)$

$= C_1 e^{t} + C_2 e^{2t} + (e^{t}+e^{2t})\ln(1+e^{-t}) - e^{t}$

$= (C_1 - 1)e^{t} + C_2 e^{2t} + (e^{t}+e^{2t})\ln(1+e^{-t})$

$y(t) = C_1 e^{t} + C_2 e^{2t} + (e^{t} + e^{2t})\ln(1+e^{-t})$

8. $y'' + 2y' + y = e^{-t}\ln t$

$r^2 + 2r + 1 = 0$

$(r+1)(r+1) = 0$

$r = -1, -1$

$y_h(t) = C_1 e^{-t} + C_2 t e^{-t}$

$y_1 = e^{-t} \quad y_2 = te^{-t}$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix}$

$= (e^{-t})(1-t)e^{-t} - (te^{-t})(-e^{-t})$

$= e^{-2t}$

$v_1' = -\frac{y_2 f}{W}$

$= -\frac{(te^{-t})(e^{-t}\ln t)}{e^{-2t}}$

$v_1' = -t\ln t$

$v_2' = \frac{y_1 f}{W}$

$= \frac{e^{-t}e^{-t}\ln t}{e^{-2t}}$

$v_2' = \ln t$

$v_1 = \int v_1'$

$= \int -t\ln t\,dt$

$v_1 = -\frac{1}{2}t^2\ln t + \frac{1}{4}t^2$

$v_2 = \int v_2'$

$= \int \ln t\,dt$

$v_2 = t\ln t - t$

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$y_p = v_1 y_1 + y_2 v_2$

$y_p = (e^{-t})(-\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2) + (te^{-t})(t\ln t - t)$

$= -\frac{1}{2}t^2 e^{-t}\ln t + \frac{1}{4}r^2 e^{-t} + t^2 e^{-t}\ln t - t^2 e^{-t}$

$= \frac{1}{2}t^2 e^{-t}\ln t - \frac{3}{4}t^2 e^{-t}$

$y(t) = y_h(t) + y_p(t)$

$\underline{y(t) = C_1 e^{-t} + C_2 e^{-t} + \frac{1}{2}t^2 e^{-t}\ln t - \frac{3}{4}t^2 e^{-t}}$

12. $y'' - y = \frac{e^t}{t}$

$r^2 - 1 = 0$

$r = \pm 1$

$y_h(t) = C_1 e^t + C_2 e^{-t}$

$y_1 = e^t$

$y_2 = e^{-t}$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_2' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}$

$= -1 - 1 = -2$

$V_1' = -\frac{y_2 f}{W}$

$= \frac{1}{2}e^{-t}\frac{e^t}{t}$

$V_1' = \frac{1}{2t}$

$V_2' = \frac{y_1 f}{W}$

$= -\frac{1}{2}e^t \frac{e^t}{t}$

$V_2' = -\frac{1}{2t}e^{2t}$

$V_1 = \int V_1'$

$= \frac{1}{2}\int \frac{1}{t}\,dt$

$V_1 = \frac{1}{2}\ln|t|$

$V_2 = \int V_2'$

$= -\frac{1}{2}\int \frac{e^{2t}}{t}\,dt$

$V_2 = -\frac{1}{2}\int_{t_0}^{t} \frac{e^{2s}}{s}\,ds$

$y_p = v_1 y_1\ y_2 v_2$

$y_p = \frac{1}{2}e^t \ln|t| - \frac{1}{2}e^{-t}\int_{t_0}^{t}\frac{e^{2s}}{s}\,ds$

$y(t) = y_h(t) + y_p(t)$

$\underline{y(t) = C_1 e^t + C_2 e^{-t} + \frac{1}{2}e^t \ln|t| - \frac{1}{2}e^{-t}\int_{t_0}^{t}\frac{e^{2s}}{s}\,ds}$

14. $y_1(t)=t^2$ $\quad$ $y_2(t)=t^{-2}$

$y_1'(t)=2t$ $\quad$ $y_2'(t)=-2t^{-3}$

$y_1''(t)=2$ $\quad$ $y_2''(t)=6t^{-4}$

$t^2y''+ty'-4y=0$ $\quad$ $t^2y''+ty'-4y=0$

$t^2(2)+t(2t)-4(t^2)=0$ $\quad$ $t^2(6t^{-4})+t(-2t^{-3})-4(t^{-2})=0$

$0=0$ $\quad$ $6t^{-2}-2t^{-2}-4t^{-2}=0 \Rightarrow 0=0.$

$t^2y''+ty'-4y=t^2(1+t^2)$ $\quad$ Method of Variation of Parameters

$y''+\frac{1}{t}y'-\frac{4}{t^2}y=1+t^2$

$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$=\begin{vmatrix} t^2 & t^{-2} \\ 2t & -2t^{-3} \end{vmatrix}$

$=-2t^{-1}-2t^{-1}$

$=-\frac{4}{t}$

$V_1'=-\frac{y_2f}{W}$

$=\frac{t^{-2}t(1+t^2)}{4}$

$\Rightarrow V_1'=\frac{1+t^2}{4t}$

$V_2'=\frac{y_1f}{W}$

$=-\frac{t^2t(1+t^2)}{4}$

$V_2'=-\frac{t^3(1+t^2)}{4}$

$V_1=\int V_1'$

$=\int\frac{1+t^2}{4t}dt$

$V_1=\frac{1}{8}t^2+\frac{1}{4}\ln|t|$

$V_2=\int V_2'$

$=\int-\frac{t^3(1+t^2)}{4}dt$

$V_2=-\frac{1}{24}t^6-\frac{1}{16}t^4$

$y_p=V_1y_1+y_2V_2$

$y_p=(\frac{1}{8}t^2+\frac{1}{4}\ln|t|)t^2+(-\frac{1}{24}t^6-\frac{1}{16}t^4)t^{-2}$

$=\frac{1}{8}t^4+\frac{1}{4}t\ln|t|-\frac{1}{24}t^4-\frac{1}{16}t^2$

$=\frac{1}{12}t^4+\frac{1}{4}t^2(\ln|t|-\frac{1}{4})$

$y(t)=y_h(t)+y_p(t)$

$$\boxed{y(t)=C_1t^2+C_2t^{-2}+\frac{1}{12}t^4+\frac{1}{4}t^2(\ln|t|-\frac{1}{4})}$$

4.5/22

22 $y'' + y = f(t)$

$r^2 + 1 = 0$

$r = \pm i$

$y_h = c_1 \cos t + c_2 \sin t$

$y_p = v_1 \cos t + v_2 \sin t$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$= \cos^2 t + \sin^2 t$

$= \cos^2 t + \sin^2 t = 1$

$v_1' = \int_0^t -(\sin s) f(s) ds$

$v_2' = \int_0^t (\cos s) f(s) ds$

$y_p = \int_0^t -(\sin s) f(s) ds) \cos t + (\int_0^t (\cos s) f(s) ds) \sin t$

$= \int_0^t -(\sin s)(\cos t) f(s) ds + \int_0^t (\cos s)(\sin t) f(s) ds$

$= \int_0^t [-(\sin s)(\cos t) + (\cos s)(\sin t)] f(s) ds$

$= \int_0^t \sin(t-s) f(s) ds.$