4-1

October 13, 2025

1)

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $u = (u_1, u_2)$ and $v = (v_1, v_2)$:

$$u + v = (u_1 + v_1, u_2 + v_2), \quad ku = (0, ku_2)$$

(a) Compute $u + v$ and ku for $u = (-1, 2)$, $v = (3, 4)$, and $k = 3$.
(b) In words, explain why V is closed under addition and scalar multiplication.
(c) Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?
(d) Show that Axioms 7, 8, and 9 hold.
(e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

5) In Exercises 3–12, determine whether each set equipped withthe given operations is a vector space. For those that are not vectorspaces identify the vector space axioms that fail.

The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on R^2.

7)

The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k_2x, k_2y, k_2z)$$

9)

The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

11)

The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \qquad \text{and} \qquad k(1, y) = (1, ky)$$

4-2

October 13, 2025

1. Use Theorem 4.2.1 to determine which of the following are subspaces of R^3. b)

All vectors of the form $(a, 1, 1)$.

c)

All vectors of the form (a, b, c), where $b = a + c$.

3. Use Theorem 4.2.1 to determine which of the following are subspaces of P^3. b)

All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.

c)

All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.

4. Which of the following are subspaces of $F(-\infty, \infty)$? b)

All functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.

c)

All functions f in $F(-\infty, \infty)$ for which $f(-x) = f(x)$.

5. Which of the following are subspaces of R^∞? a)

All sequences v in R^∞ of the form.

$$v = (v, 0, v, 0, v, 0, \ldots).$$

b)

All sequences $\mathbf{v}$ in R^∞ of the form

$$\mathbf{v} = (v, 1, v, 1, v, 1, \ldots)$$

10. In each part express the vector as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$,$\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$. a)

$$-9 - 7x - 15x^2$$

11. In each part, determine whether the vectors span R^3 a)

$$\mathbf{v}_1 = (2, 2, 2),\ \mathbf{v}_2 = (0, 0, 3),\ \mathbf{v}_3 = (0, 1, 1)$$

12. Suppose that $\mathbf{v}_1 = (2, 1, 0, 3)$, $\mathbf{v}_2 = (3, -1, 5, 2)$, and $\mathbf{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? a)

$$(2, 3, -7, 3)$$

c)

$$(1, 1, 1, 1)$$

14. Let f $= \cos^2 x$ **and g** $= \sin^2 x$**. Which of the following lie in the space spanned by f and g? a)**

$$\cos 2x$$

b)

$$(0,\ 0,\ 0,\ 0)$$

October 28, 2025

1 **1. Use the method of Example 3 to show that the following set of vectors forms a basis for R^2.**

[$[(2,1),\ (3,0)]$]

2 **3. Show that the following polynomials form a basis for P_2.**

[x^2 + 1, x^2 - 1, 2x - 1]

3 **5. Show that the following matrices form a basis for M_{22}.**

[
$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$,
$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$,
$\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$,
$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$
]

4 **7. a) In each part, show that the set of vectors is not a basis for R^3.**

$[[(2,-3,1),(4,1,1),(0,-7,1)]]$

5 **9. Show that the following matrices do not form a basis for M_{22}.**

[
$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

$\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$
,
$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$
,
$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$
]

6 11. a) Find the coordinate vector of w relative to the basis $S = [\mathbf{u}_1, \mathbf{u}_2]$ for R^2.

[u_1 = (2, -4), u_2 = (3, 8); w = (1, 1)]

7 13. a) Find the coordinate vector of v relative to the basis $S = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4]$ for R^3.

[v = (2, -1, 3); v_1 = (1, 0, 0), v_2 = (2, 2, 0)]

8 15. In Exercises 15- 16, first show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

[A_1 =
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
, A_2 =
$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
, A_3 =
$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$
,] [A_4 =
$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
; A =
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
]

9 17. In Exercises 17- 18, first show that the set $S = \{p_1, p_2, p_3\}$ is a basis for P_2, then express p as a linear combination of the vectors in S, and then find the coordinate vector of p relative to S.

[p_1 = 1 + x + x^2, p_2 = x + x^2, p_3 = x^2;] [p = 7 - x + 2x^2]

4-4

October 28, 2025

1. Use the method of Example 3 to show that the following set of vectors forms a basis for R^2.

$$[(2,1),\ (3,0)]$$

3. Show that the following polynomials form a basis for P_2.

$$x^2+1, \quad x^2-1, \quad 2x-1$$

5. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

7. a) In each part, show that the set of vectors is not a basis for R^3.

$$[(2,-3,1),(4,1,1),(0,-7,1)]$$

9. Show that the following matrices do not form a basis for M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

11. a) Find the coordinate vector of w relative to the basis $S = [\mathbf{u}_1, \mathbf{u}_2]$ for R^2.

$$\mathbf{u}_1 = (2, -4), \ \mathbf{u}_2 = (3, 8); \quad \mathbf{w} = (1, 1)$$

13. a) Find the coordinate vector of v relative to the basis $S = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4]$ for R^3.

$$\mathbf{v} = (2, -1, 3); \quad \mathbf{v}_1 = (1, 0, 0),\ \mathbf{v}_2 = (2, 2, 0)$$

15. In Exercises 15- 16, first show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

17. In Exercises 17- 18, first show that the set $S = \{p_1, p_2, p_3\}$ is a basis for P_2, then express p as a linear combination of the vectors in S, and then find the coordinate vector of p relative to S.

$$p_1 = 1 + x + x^2, \quad p_2 = x + x^2, \quad p_3 = x^2;$$

$$p = 7 - x + 2x^2$$

4-5

November 1, 2025

In Exercises 1- 6, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

2)

$3x_1 + x_2 + x_3 + x_4 = 0$
$5x_1 - x_2 + x_3 - x_4 = 0$

5)

$x_1 - 3x_2 + x_3 = 0$
$2x_1 - 6x_2 + 2x_3 = 0$
$3x_1 - 9x_2 + 3x_3 = 0$

In each part, find a basis for the given subspace of R^3, and state its dimension.

7 a) The plane $3x - 2y + 5z = 0$.

c) The line $x = 2t$, $y = -t$, $z = 4t$.

d) All vectors of the form (a, b, c), where $b = a + c$.

In each part, find a basis for the given subspace of R^4, and state its dimension.

8 a) All vectors of the form $(a, b, c, 0)$.

c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

10) Find the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

Find a standard basis vector for R^3 that can be added to the set $\{v_1, v_2\}$ to produce a basis for R^3.

12 b)

$$v_1 = (1, -1, 0), \quad v_2 = (3, 1, -2)$$

13) Find standard basis vectors for R^4 that can be added to the set $\{v_1, v_2\}$ to produce R^4.

$$v_1 = (1, -4, 2, -3), \quad v_2 = (-3, 8, -4, 6)$$

15) The vectors $v_1 = (1, -2, 3)$ and $v_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{v_1, v_2\}$ to a basis for R^3.

17) Find a basis for the subspace of R^3 that is spanned by the vectors

$$v_1 = (1, 0, 0), \quad v_2 = (1, 0, 1), \quad v_3 = (2, 0, 1), \quad v_4 = (0, 0, -1)$$

4-6

October 29, 2025

Consider the bases $B = \{u_1, u_2\}$ and $B' = \{u'_1, u'_2\}$ for R^2, where

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad u'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad u'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

1 a)

Find the transition matrix from B' to B.

b)

Find the transition matrix from B to B'.

c)

Compute the coordinate vector $[w]_B$, where

$$w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (12) to compute $[w]'_B$.

d)

Check your work by computing $[w]'_B$ directly.

Consider the bases $B = \{u_1, u_2, u_3\}$ **and** $B' = \{u_1', u_2', u_3'\}$ **for** R^3**, where**

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$u_1' = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad u_2' = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad u_3' = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

3 a)

Find the transition matrix B to B'.

b)

Compute the coordinate vector $[w]_B$, where

$$w = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

and use (12) to compute $[w]'_B$.

c)

Check your work by computing $[w]'_B$ directly.

Let S be the standard basis for R^3, and let $B = \{v_1, v_2, v_3\}$ be the basis in which $v_1 = (1, 2, 1)$, $v_2 = (2, 5, 0)$, and $v_3 = (3, 3, 8)$.

9 a)

Find the transition matrix $P_{B \to S}$ by inspection.

b)

Use Formula (14) to find the transition matrix $P_{S \to B}$.

c)

Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another.

d)

Let $w = (5, -3, 1)$. Find $[w]_B$ and then use Formula (11) to compute $[w]_S$.

e)

Let $w = (3, -5, 0)$. Find $[w]_S$ and then use Formula (12) to compute $[w]_B$.

Formulas

(11)

$$[v]_B = P_{B' \to B}[v]'_B$$

(12)

$$[v]'_B = P_{B \to B'}[v]'_B$$

(14)

$$[\text{new basis} \mid \text{old basis}] \quad \xrightarrow{\text{row operations}} \quad [I \mid \text{transition from old to new}]$$

4-7

November 1, 2025

In Exercises 3- 4, determine whether **b** is in the column space of A , and if so, express **b** as a linear combination of the column vectors of A

3) a)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}; \quad b = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

In Exercises 7–8, find the vector form of the general solution of the linear system $A\mathbf{x} = \mathbf{b}$, and then use that result to find the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

7) b)

$$\begin{array}{rrrr} x_1 + & x_2 + & 2x_3 = & 5 \\ x_1 + & + & x_3 = & -2 \\ 2x_1 + & x_2 + & 3x_3 = & 3 \end{array}$$

In Exercises 9- 10, find bases for the null space and row space of A.

9 a)

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

10 a)

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

In Exercises 11- 12, a matrix in row echelon form is given. By inspection, find a basis for the row space and for the column space of that matrix.

11 b)

$$\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

12 a)

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

13) a) Use the methods of Examples 6 and 7 to find bases for the row space and column space of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

b) Use the method of Example 9 to find a basis for the row space of A that consists entirely of row vectors of A.

In Exericses 16–17, find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.

17) $v_1 = (1, -1, 5, 2)$, $v_2 = (-2, 3, 1, 0)$, $v_3 = (4, -5, 9, 4)$, $v_4 = (0, 4, 2, -3)$, $v_5 = (-7, 18, 2, -8)$

In each part, let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 4 \end{bmatrix}$. For the given vector $\mathbf{b}$, find the general form of all vectors $\mathbf{x}$ in R^3 for which $T_A(\mathbf{x}) = \mathbf{b}$ if such vectors exist.

21 a) $\mathbf{b} = (0, 0)$

c) $\mathbf{b} = (-1, 1)$

24 a) Find a 3×3 matrix whose null space is a point.

b) Find a 3×3 matrix whose null space is a line.