Chapter 4

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4.1 Real Vector Spaces

Definition 4.1. *We call V a vector space if it follows these ten conditions.*

1. *If u and v are object in V, then $u + v$ is in V*
2. $u + (v + w) = (u + v) + w$
3. *There is an object, 0 in V, called a zero vector for V, such that $0 + u = u + 0 = u$ for all u in V.*
4. *For each u in V, there is an object $-u$ in V, called a negative of u, such that $u + (-u) = (-u) + u = 0$.*
5. *if k is any scalar and u is any object in V, then ku is in V.*
6. $k(u + v) = ku + kv$
7. $(k + m)u = ku + mu$
8. $k(mu) = (km)(u)$
9. $1u = u$

Remark. *Something to notice here is that this doesn't state anything about vector operations at, all. This truly does mean that any set with those 10 properties is a vector space. A great example of this is the set of real numbers, $\mathbb{R}$.*

To show a space is a vector space:

1. *Identify the set V of objects that will become vectors.*
2. *Identify the addition and scalar mutliplication operations on V.*
3. *Verify Axioms 1 and 6; that is, adding two vector in V produces a vector in V, and multipliying a vector in V by a scalar also produces a vector in V. Axiom 1 is called closure under addition, and Axiom 6 is called closure under scalar mutliplication.*
4. *Confirm that Axioms 2, 3, 4, 5, 6, 7, 8, 9, and 10 hold.*

Theorem 4.2. *Let V be a vector space, u a vector in V, and k a scalar; then:*

1. $0u = 0$
2. $k0 = 0$
3. $(-1)u = -u$
4. *If $ku = 0$, then $k = 0$ or $u = 0$.*

4.2 Subspaces

Definition 4.3. *A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.*

Written as a set, we are essentially saying,

$$\{W \mid W \subset V\}$$

Theorem 4.4. *If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.*

(a) If u and v are vectors in W, then $u + v$ is in W.

(b) If k is a scalar and u is a vector in W, then ku is in W.