▶ In Exercises 37–38, determine whether A is invertible, and if so, find the inverse. [*Hint:* Solve $AX = I$ for X by equating corresponding entries on the two sides.] ◀

37. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **38.** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

▶ In Exercises 39–40, simplify the expression assuming that A, B, C, and D are invertible. ◀

39. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

40. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

41. Show that if R is a $1 \times n$ matrix and C is an $n \times 1$ matrix, then $RC = \operatorname{tr}(CR)$.

42. If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

43. (a) Show that if A is invertible and $AB = AC$, then $B = C$.

(b) Explain why part (a) and Example 3 do not contradict one another.

44. Show that if A is invertible and k is any nonzero scalar, then $(kA)^n = k^n A^n$ for all integer values of n.

45. (a) Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

(b) What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

46. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

(a) Show that if A is idempotent, then so is $I - A$.

(b) Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

47. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$$

48. Show that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfies the equation

$$A^2 - (a + d)A + (ad - bc)I = 0$$

49. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = C^T$$

50. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$ABC^T DBA^T C = AB^T$$

Working with Proofs

▶ In Exercises 51–58, prove the stated result. ◀

51. Theorem 1.4.1(a)

52. Theorem 1.4.1(b)

53. Theorem 1.4.1(f)

54. Theorem 1.4.1(c)

55. Theorem 1.4.2(c)

56. Theorem 1.4.2(b)

57. Theorem 1.4.8(d)

58. Theorem 1.4.8(e)

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$.

(b) For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

(c) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

(d) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(e) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^T B^T$.

(f) The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

(g) If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$.

(h) If A is an invertible matrix, then so is A^T.

(i) If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$.

(j) A square matrix containing a row or column of zeros cannot be invertible.

(k) The sum of two invertible matrices of the same size must be invertible.