Section 4.3 Linear Independence.

October 1, 2025

Linear independence is when two elements do not change in relation to the other. We can analyze this geometrically. Let's imagine we have:

$$\{u_1, u_2\} \in \mathbb{R}^2$$

Definition 0.1.

$$S = \{u_1, u_2, \ldots, u_m\}$$

If $V_f = hV_f$, thwn these vectors are L.D. (Linearly dependent) to each other, otherwise they are L.I. (Linearly Independent)

Theorem 0.2. *To decide vectors are LI, or LD.*

$$S = \vec{V_1}, \vec{V_2}, \ldots, \vec{V_m} \textit{ in } \boldsymbol{R}^n, \textit{ C-scalar.}$$

1. *For $C_1V_1 + C_2V_2 + \cdots + C_mV_m = 0$.*

$$\begin{array}{ll} \textit{If } C_i \neq 0 & \rightarrow \textit{L.D.} \\ \textit{If } C_1 = C_2 = \cdots = C_m = 0 & \rightarrow \textit{L.I.} \end{array}$$

2. *$V_i = CV_j$ LD (definition)*

3. $$\begin{array}{rl} \vec{0} \in S & \qquad \textit{L.D.} \\ C\vec{0} = 0 & \\ C \in \boldsymbol{R} & \end{array}$$

4. *$S = \{\vec{V}\}$ and $\vec{V}$ is non zero. L.I.*

$$C\vec{V} = 0$$
$$C = 0 \textit{ in } \vec{V} \neq 0 \textit{ L.I.}$$

5. $$\begin{array}{lll} \textit{If } m > n, & \textit{L.D. e.g. } \{\vec{V_1}, \vec{V_2}, \vec{V_3}\} & \boldsymbol{R}^2 \\ \textit{If } m < n, & \{\vec{V_1}, \vec{V_2}\} & \boldsymbol{R}^2 \textit{ No conclusion, use other methods.} \\ \textit{If } m = n, & \textit{use determinent.} & \end{array}$$

6. Wronskia's method.

$$f, g$$

$$w = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \qquad = 0 \qquad L.D$$

$$\neq 0 \qquad L.I$$

Note: To span vectors in the same set, use LD vectors in the set. To span vectors in a space, use LI vectors in the set.

Examples

1 a)

$$u_1 = (-1, 2, 4)$$
$$u_2 = (5, -10, -20)$$
$$u_2 = 5u, \qquad \text{LD}$$

2 b)

4 vectors for $\boldsymbol{R}^3$ LD

4 a)

Determine whether LD or LI in $\boldsymbol{P}_2$

$$2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2 \qquad \text{Use determinent}$$

$$\begin{vmatrix} c & x & x^2 \\ 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{vmatrix} = 39 \neq 0, \quad \text{LI}$$

5 b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_{23}$$

$$
\begin{aligned}
aV_1 + bV_2 + c &= 0 \\
a &= 0 \\
b &= 0 \\
c &= 0 \quad \text{LI}
\end{aligned}
$$

20.

Use Wronskian to show that the functions $f_1(x) = e^x$, $f_2(x) = xe^x$, and $f_3(x) = x^2e^x$ are linearly dependent.

$$
w = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix}
$$

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$$
\begin{aligned}
S = \{v_1, v_2, v_3\} \in V \qquad LD \\
v_4 \in V, \text{ not in } S
\end{aligned}
$$