Chapter2

September 8, 2025

2.1 Preliminary Theroy

2.1.1 IVP - Initial Value Problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) y(x_0) = y_0 \qquad\qquad x_0 \in I, y_0 \in \Re$$

2.1.2 Thm - Existence and Uniqueness Theorem

If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions on I, then there exists a unique sol'n

Ex

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= xy^{\frac{1}{2}} & & & y(0) &= 0 \\ y &= 0 & &\rightarrow & \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ 0 &= x \cdot 0^{\frac{1}{2}} \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} f(x,y) &= xy^{\frac{1}{2}} \\ \frac{\partial f}{\partial y} &= x \cdot \frac{1}{2} y^{-\frac{1}{2}} \\ f(x,y) &\text{ is not continuous on } I \end{aligned}$$

2.2 Seperable Variables

If $g(x)$ is cts, $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is solved by inegration.

$$y = \int g(x)dx + C = G(x) + C$$

Example:

- $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x - \frac{1}{1+x^2} \\ y &= \int (x - \frac{1}{1+x^2}) dx \\ &= \frac{1}{2}x^2 - tan^{-1}x + C \end{aligned}$$

- $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= tanx \\ &= \dots \\ &= |secx|\end{aligned}$$

2.2.1 Definition:

Toolbox:

A seperable DE is one of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)}$. While not all DEs come in this form, if you can manipulate it algebraically to this, you can use this tool:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{g(x)}{h(y)} && \text{Assume that } y = f(x) \text{ is a solution.} \\ h(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= g(x) \\ h(f(x)) \cdot f'(x) &= g(x) \\ \int h(f(x))f'(x)dy &= \int g(x)dx + C \\ h(y)dy &= g(x)dx\end{aligned}$$

We can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a fraction, in this scenario.

Example:

1. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-x}{y}; \quad y(3) = 4 \\ ydy &= -xdx \\ \int ydy &= -\int xdx \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 \\ \dots\end{aligned}$$

2.

$$
\begin{aligned}
(1+x)dy - ydx &= 0 \\
(1+x)dy &= ydx \\
\int \frac{1}{y}dy &= \int \frac{1}{1+x}dx + c_1 \\
ln|y| &= ln|1+x| + c_1 \\
|y| &= |1+x|e^{c_1} \\
y &= c|1+x| \quad c \neq 0
\end{aligned}
$$

3.

$$
\begin{aligned}
xe^{-y} sinxdx - ydy &= 0 \\
\int x \sin x \ dx &= \int e^{-y} ydy \\
\int x \sin x \ dx &= \int e^{-y} ydy \\
u &= xv = -cosx \\
du &= dxdv = sinxdx \\
-xcosx + \int cosx \ dx &= ye^y - \int e^y \ dy + C \\
-xcosx + sinx &= ye^y - e^y + C
\end{aligned}
$$

4.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 - 4; \quad y(0) = -2 \\
\int \frac{dy}{y^2-4} &= \int dx + c_1 \\
\int \frac{dy}{(y-2)(y+2)} &= \int dx + c_1 \\
\int (\frac{\frac{1}{4}}{y-2} + \frac{\frac{1}{4}}{y+2})dy &= x + c_1 \frac{1}{4} ln|y-2| - ln|y+2| && = x + c_1 \\
ln|y-2| - ln|y+2| &= 4x + c_1 \\
|\frac{y-2}{y+2}| &= e^{4x+c_1} \\
|\frac{y-2}{y+2}| &= e^{4x+c_1} \\
&\cdots
\end{aligned}
$$

2.3

2.4 Exact Differential Equations

Given:

$$M(x,y)dx + N(x,y)dy = 0$$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the DE above is an exact DE. In other words $f_x = M$ and $f_y = N$.

So for example:

$$\begin{aligned} f(x,y) &= x^2 - 5xy + y^3 \\ f_x = 2x - 5y &\rightarrow f_{xy} = -5 \\ f_y = 5x + 3y^2 &\rightarrow f_{yx} = -5 \end{aligned}$$

$$(2x - 5y)dx + (-5x + 3y^2)dy = 0$$

This is an exact differential equation.

2.5 Linear Equations

Soln to a standard DE in standard form is:

$$\begin{aligned} y &= \frac{1}{e^{\int P(x)dx}}(\int e^{\int P(x)dx} f(x) + c) \\ \mu &= e^{\int P(x)dx} \\ y &= \frac{1}{\mu(x)}\left(\int \mu(x)f(x) + c\right) \end{aligned}$$

2.5.1

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^6 e^x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} - 4 = x^5 e^x$$
$$P(x) \quad f(x)$$
$$\mu(x) = e^{\int -\frac{4}{x}\ \mathrm{d}x} = e^{ln|x|^{-4} = \frac{1}{x^4}}$$
$$y = \frac{1}{\frac{1}{x^4}}\left[\int \frac{1}{x^4} x^5 e^x \mathrm{d}x\right]$$
$$y = x^4\left[xe^x - e^x + c\right]$$
$$= x^5 e^x - x^4 e^x + cx^4$$

2.5.2

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$$
$$\mu(x) = e^{\int -3\mathrm{d}x} = e^{-3x}$$
$$y = \frac{1}{e^{-3x}}(\int e^{-3x} \cdot 0 \quad \mathrm{d}x + c)$$
$$y = ce^{3x}$$

2.5.3

$$(x^2 + 9)y' + xy = 0$$
$$y' + \frac{x}{x^2 + 9} \cdot y = 0$$
$$y = \frac{1}{\sqrt{x^2 + 9}}(\int \sqrt{x^2 + 9} \cdot 0 \quad \mathrm{d}x + c)$$
$$y = \frac{c}{\sqrt{x^2 + 9}}$$

2.5.4

$$\begin{aligned}
y' + 2xy &= x \qquad\qquad\qquad\qquad\qquad y(0) = -3\\
\mu(x) &= e^{\int 2x\mathrm{d}x} = e^{x^2}\\
y &= \frac{1}{e^{x^2}}(\int xe^{x^2} + c)\\
&= \frac{1}{e^{x^2}}(\frac{1}{2}e^{x^2} + c)\\
&= \frac{1}{2} + \frac{c}{e^{x^2}}\\
-3 &= \frac{1}{2} + \frac{c}{e^0}\\
-3 &= \frac{1}{2} + c\\
-\frac{7}{2} &= c\\
y &= \frac{1}{2} - \frac{7}{2e^{x^2}}
\end{aligned}$$

2.5.5

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x; \quad y(1) = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} \cdot y = 2$$

$$\begin{aligned}
\mu(x) &= e^{\int \frac{1}{x}} \qquad\qquad\qquad\qquad = e^{ln|x|} = |x| = x\\
y &= \frac{1}{x}(\int x \cdot 2\mathrm{d}x + c)\\
y &= \frac{1}{x}(x^2 + c) = x + \frac{c}{x}0 \qquad\qquad\qquad = 1 + \frac{c}{1}\\
c &= -1
\end{aligned}$$

2.5.6

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x+y^2}; \quad y(-2)=0 \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= x+y^2 \\
\frac{\mathrm{d}x}{\mathrm{d}y} - x &= y^2 \\
x &= \frac{1}{\mu}(\int \mu f(y) + \mathrm{d}y + c) \\
\mu(y) &= e^{\int -1\mathrm{d}y} = e^{-y} \\
x &= \frac{1}{e^{-y}}(\int y^2 e^{-y}\mathrm{d}y) \\
u &= y^2 & \mathrm{d}v &= e^{-y} \\
\mathrm{d}u &= 2y & v &= -e^{-y} \\
x &= -y^2 e^{-y} + \int 2ye^{-y}\mathrm{d}y \\
u &= 2y & \mathrm{d}v &= e^{-y} \\
\mathrm{d}u &= 2 & v &= -e^{-y} \\
x &= -y^2 e^{-y} - 2ye^{-y} + \int 2e^{-y}\mathrm{d}x \\
x &= -e^{-y}(y^2+2y+2) \\
&\ldots
\end{aligned}
$$

2.6 Bernoulli's Equation

$$
\begin{aligned}
\frac{\mathrm{d}dy}{\mathrm{d}dx} + P(x)y &= f(x)y^n \\
y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} &= f(x) \\
\text{let} \quad w &= y^{1-n} \\
\frac{\mathrm{d}w}{\mathrm{d}x} &= (1-n)y^{1-n-1}\frac{\mathrm{d}y}{\mathrm{d}x} \\
&= (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{1}{1-n} \cdot \frac{\mathrm{d}w}{\mathrm{d}x} &= y^{-n} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{1}{1-n} \cdot \frac{\mathrm{d}w}{\mathrm{d}x} + P(x)w &= f(x) \\
\frac{\mathrm{d}w}{\mathrm{d}x} + (1-n)P(x)w &= (1-n)f(x) \\
w &= \frac{1}{\mu}(\int \mu \cdot f(x)\mathrm{d}x + c)
\end{aligned}
$$

2.6.1

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y &= xy^2 && \text{Bernoulli, } n = 2 \\
\text{let} \quad w &= y^{1-2} = y^{-1} \\
\frac{\mathrm{d}w}{\mathrm{d}x} &= -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}w}{\mathrm{d}x} &= \frac{1}{-y^2}\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -y^2\frac{\mathrm{d}w}{\mathrm{d}x} \\
w &= \frac{1}{y} \\
y &= \frac{1}{w} \\
y^2 &= \frac{1}{w^2} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{w^2} \cdot \frac{\mathrm{d}w}{\mathrm{d}x} \\
-\frac{1}{w^2}\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{1}{x} \cdot \frac{1}{w} &= (x \cdot \frac{1}{w^2})(-w^2) \\
\frac{\mathrm{d}w}{\mathrm{d}x} - \frac{1}{x} \cdot w &= -x \\
\mu(x) &= e^{\int -\frac{1}{x}\mathrm{d}x} = e^{-ln|x|} = e^{ln|x|^{-1}} = |x|^{-1} = \frac{1}{x} \\
w &= \frac{1}{M(x)}(\int \mu(x)f(x)\mathrm{d}x + c) \\
&= \frac{1}{\frac{1}{x}}(\int \frac{1}{x} \cdot -x\mathrm{d}x + c) = x(-x + c) = -x^2 + cx \\
\frac{1}{y} &= -x^2 + cx \\
y &= \frac{1}{cx - x^2}
\end{aligned}
$$