Chapter 2

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2.1 Determinant

Definition 2.1. *Determinant is a real association with a square matrix.*
Note:

1. *If* $det A = 0$ A *is singular.*
2. *The system of eqs. from* A *has no soln.*

For a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = det\, A = ad - bc$$

For 3x3 or above, use expansion method.

$$3 \times 3 \to 2 \times 2\,(3) \to \text{Use the } 2 \times 2 \text{ formula}$$

Definition 2.2. *Given the matrix:*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \ddots & \vdots \\ a_{31} & \dots & a_{33} \end{bmatrix}$$

Minor: M_{22} *is obtained by deleting the determinent. Correspdoning row 2 and column 2.*

$$M_{22} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} = a_{11}a_{33} - a_{13}a_{31}$$

Definition 2.3. ***Cofactor:*** $C_{22} = M_{22}$ *e.g.* $C_{23} = (2 + 3 = 5 \ odd) = (-)M_{23}$

Example 2.4. 1

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$
$$M_{22} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 4 - (-9) = 13$$
$$C_{22} = 13$$
$$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -8 - 3 = -11$$
$$C_{21} = 11$$

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$$\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$
$$\begin{vmatrix} 4 & 1 \\ 8 & 2 \end{vmatrix} = 8 - 8 = 0 \qquad \textit{Not invertible.}$$

...

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$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$$
$$det(A) = \begin{bmatrix} 3 & -3 & -1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$$
$$= -\begin{vmatrix} 0 & -4 \\ -3 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix}$$

Definition 2.5. *Property of $n \times n$*

$$\begin{vmatrix} a_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ c & \dots & a_{nn} \end{vmatrix} = a_{11}a_{22}\dots a_{nn} \qquad \textit{Multiplication of diagonal (major)}$$

2.2 Determinant Properties

Definition 2.6.

1. $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0$

2. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$

3. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$
   $det(A) = det(A^T)$

4. $\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
   $\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = \begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
   $\left|k(n \times n)\right| = k^n \left|n \times n\right|$

5. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix} = \begin{vmatrix} ka+b & b \\ kd+d & d \end{vmatrix}$

6. $\begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & kd \end{vmatrix} = 0$

Example 2.7. 10

$$\begin{bmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{bmatrix}$$
$$R_2 \leftrightarrow R_3 \rightarrow -\begin{vmatrix} 3 & 6 & -9 \\ -2 & 1 & 5 \\ 0 & 0 & -5 \end{vmatrix}$$
$$R_1 \textit{ factor } 3 \rightarrow \begin{vmatrix} 1 & 2 & -3 \\ -2 & 1 & 5 \\ 0 & 0 & -2 \end{vmatrix}$$
$$\ldots$$

2.3 Properties of derterminet.

First lets start with the matrix

$$A = \begin{bmatrix} -4 & 2 & -2 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}.$$

1.

$$detA = \begin{bmatrix} -4 & 2 & -2 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$R_1(-2) \rightarrow \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix}$$

$$R_1(-2) + R_2 \quad R_1(-3) + R_3 \rightarrow 2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$$R_2(-1) + R_3 \rightarrow 2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= 2(-1)(-1) = 2$$

2. Column operation (+ expansion if needed).

$$\begin{bmatrix} -4 & 2 & -2 \\ 1 & 0 & 1 \\ 3 & -1 & -1 \end{bmatrix} \xrightarrow[C_2(2)+C_1]{C_2+C_3} \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= - \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

1. A : $\quad 2 \times 2 \quad detA = ad - bc$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k^2 det(A)$$

$$n \times n \qquad det[k(n \times n)] = k^n det(n \times n)$$

2. A : $\quad n \times n$ A is invertible if $det(A) \neq 0$

3. $A : n \times n$ $B : n \times n$ $det(AB) = det(A)det(B)$

4. $det(A^{-1}) = \frac{1}{det(A)}$ if A is invertible $\rightarrow A^{-1}$ exists.

$$\begin{aligned}
\rightarrow AA^{-1} =& I \\
det(AA^{-1}) =& det(I) \\
det(A)det(A^{-1}) =& 1 \\
det(A^{-1}) =& \frac{1}{det(A)}
\end{aligned}$$

5. $det(A) = det(A^T)$

Example 2.8. 8

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix} \xrightarrow[R_1+R_3]{} \begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix} = -6$$

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$$A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$$

$$\begin{aligned}
\det A = k^2 - 4 \neq& 0 \\
k^2 \neq& 4 \\
k \neq& \pm 2 \\
&A \textit{ is invertible.}
\end{aligned}$$

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Solve by Cramer's rule, where it applies

$$\begin{aligned} x_1 - \ & 3x_2 + & x_3 & = 2 \\ 2x_1 - \ & x_2 + & x_3 & = 2 \\ 4x_1 \ & & -3x_3 & = 2 \end{aligned}$$

$$D = \begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -3 \end{vmatrix} \xrightarrow[R_1(3)+R_3]{} \begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 7 & -9 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 \\ 7 & -9 \end{vmatrix} = -18 + 7 = -11$$

$$D_{x_1} = \begin{vmatrix} 4 & -3 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{vmatrix}$$

$$\begin{vmatrix} 4 & -3 \\ -2 & -1 \end{vmatrix} = -3(-4-6) = 30$$

$$\vdots$$