Chapter 2

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2.1 Determinant

Definition 2.1. *Determinant is a real association with a square matrix.*
Note:

1. *If $\det A = 0$ A is singular.*

2. *The system of eqs. from A has no soln.*

For a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det A = ad - bc$$

For 3x3 or above, use expansion method.

$$3 \times 3 \to 2 \times 2\,(3) \to \text{Use the } 2 \times 2 \text{ formula}$$

Definition 2.2. *Given the matrix:*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \ddots & \vdots \\ a_{31} & \cdots & a_{33} \end{bmatrix}$$

Minor: M_{22} *is obtained by deleting the determinent. Correspdoning row 2 and column 2.*

$$M_{22} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} = a_{11}a_{33} - a_{13}a_{31}$$

Definition 2.3. **Cofactor:** $C_{22} = M_{22}$ *e.g.* $C_{23} = (2 + 3 = 5 \text{ odd}) = (-)M_{23}$

Example 2.4. 1

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 4 - (-9) = 13$$

$$C_{22} = 13$$

$$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -8 - 3 = -11$$

$$C_{21} = 11$$

$$\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 4 & 1 \\ 8 & 2 \end{vmatrix} = 8 - 8 = 0 \qquad \textit{Not invertible.}$$

$\ldots$

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$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$$

$$det(A) = \begin{bmatrix} 3 & -3 & -1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$$

$$= -\begin{vmatrix} 0 & -4 \\ -3 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix}$$

Definition 2.5. *Property of* $n \times n$

$$\begin{vmatrix} a_{11} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ c & \ldots & a_{nn} \end{vmatrix} = a_{11}a_{22}\ldots a_{nn} \qquad \textit{Multiplication of diagonal (major)}$$

2.2 Determinant Properties

Definition 2.6.

1. $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0$

2. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$

3. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$
   $det(A) = det(A^T)$

4. $\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
   $\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = \begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
   $\begin{vmatrix} k(n \times n) \end{vmatrix} = k^n \begin{vmatrix} n \times n \end{vmatrix}$

5. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix} = \begin{vmatrix} ka+b & b \\ kd+d & d \end{vmatrix}$

6. $\begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & kd \end{vmatrix} = 0$

Example 2.7. 10

$$\begin{bmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \rightarrow - \begin{vmatrix} 3 & 6 & -9 \\ -2 & 1 & 5 \\ 0 & 0 & -5 \end{vmatrix}$$

$$R_1 \ factor \ 3 \rightarrow \begin{vmatrix} 1 & 2 & -3 \\ -2 & 1 & 5 \\ 0 & 0 & -2 \end{vmatrix}$$

$$\dots$$

2.3   Properties of derterminet.

First lets start with the matrix

$$A = \begin{bmatrix} -4 & 2 & -2 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}.$$

1.

$$det A = \begin{bmatrix} -4 & 2 & -2 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$R_1(-2) \rightarrow \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix}$$

$$R_1(-2) + R_2 \quad R_1(-3) + R_3 \rightarrow 2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$$R_2(-1) + R_3 \rightarrow 2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= 2(-1)(-1) = 2$$

2. Column operation (+ expansion if needed).