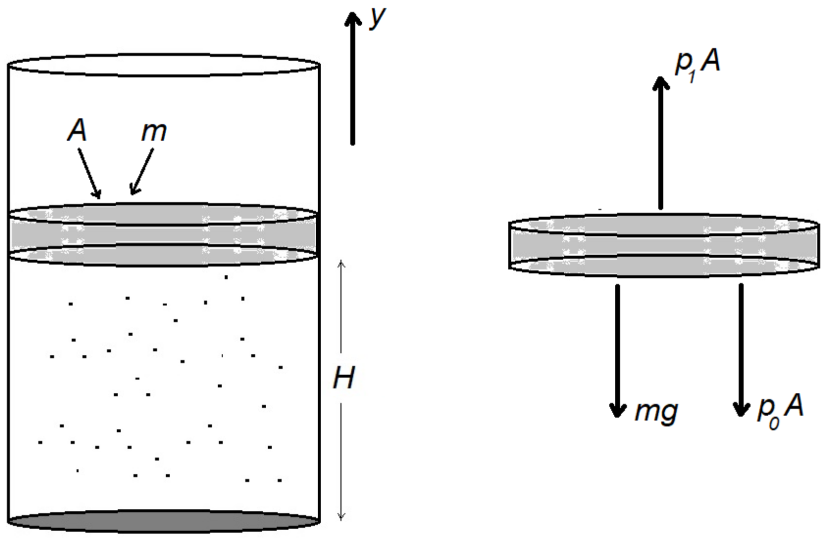
Physics 4C Lab 8: *Derivation of Equation for Adiabatic Oscillations*

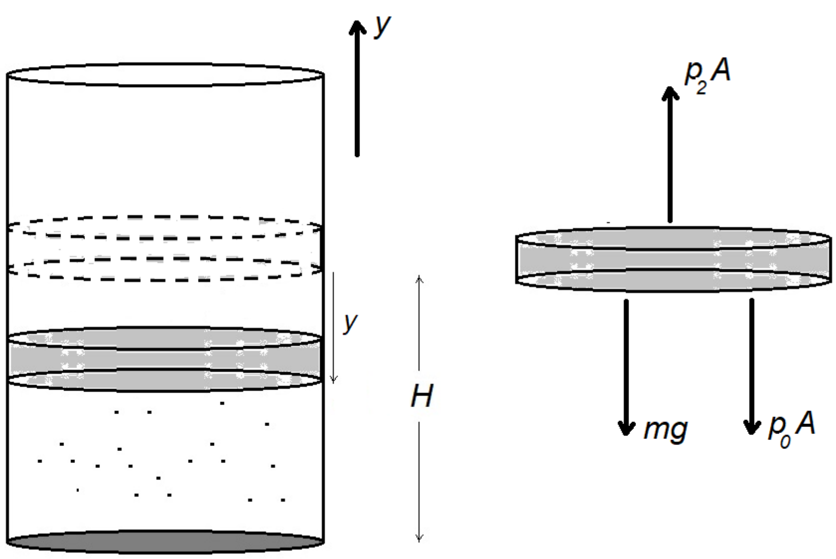
We start with a simple sealed cylinder of air. The cylinder has a movable piston with area *A* and mass *m*. The piston has an *equilibrium position*, at which the forces acting on the piston – gravity, pressure from the air inside the cylinder, pressure from the air outside the cylinder – are balanced. We can also define a *y-axis* for the motion of the pistion:



We can define this state of the air in the cylinder as "State 1", with pressure p_1 and volume V_1. We can then write two equations: one for the volume of the air in the cylinder and one for the forces acting on the piston:

$$V_1 = AH \qquad\qquad p_1 A = mg + p_0 A$$

We can now imagine the piston in motion, moving downward rapidly a distance *y*. This rapid motion compresses the gas in a process that we assume is *adiabatic*. We can define the new state of the gas as "State 2" and we can consider the forces now acting on the piston:



And two new equations, for the volume of the air in the cylinder and the forces acting on the piston:

$$V_2 = V_1 + Ay \qquad\qquad p_2A - mg - p_0A = ma$$

Note that the first equation includes the term $+Ay$ because when the piston moves downward, y is a negative value, so that V_2 is less than V_1. In the second equation, the forces are now imbalanced and the pistion is accelerating.

We can simplify the force equation: $$p_2A - (mg + p_0A) = ma$$

And then use our force equation from State 1 to substitute for the terms in parentheses:

$$p_2A - p_1A = ma$$

We can also use the equation for adiabatic processes to relate State 1 to State 2:

$$p_1 V_1^{\gamma} = p_2 V_2^{\gamma} \qquad \text{or} \qquad p_2 = p_1 \left(\frac{V_1}{V_2}\right)^{\gamma}$$

And then we can use this to sub for p_2 in our force equation:

$$p_1 \left(\frac{V_1}{V_2}\right)^{\gamma} A - p_1A = ma$$

And now a few steps to simplify the algebra. First we factor out $p_1 A$ on the left side, then we invert the fraction of volumes while changing the “gamma” exponent to negative. (This step will make sense in a moment.)

$$p_1 A \left[\left(\frac{V_1}{V_2}\right)^{\gamma} - 1\right] = ma \qquad p_1 A \left[\left(\frac{V_2}{V_1}\right)^{-\gamma} - 1\right] = ma$$

Now we can sub the expressions for V_2 that we created above:

$$p_1 A \left[\left(\frac{V_1 + Ay}{V_1}\right)^{-\gamma} - 1\right] = ma$$

Simplify the fraction:

$$p_1 A \left[\left(1 + \frac{Ay}{V_1}\right)^{-\gamma} - 1\right] = ma$$

We can replace V_1 with the expression we created at the very start, and simplify:

$$p_1\, A \left[\left(1+\frac{Ay}{AH}\right)^{-\gamma} - 1\right] = ma \qquad \text{or} \qquad p_1\, A \left[\left(1+\frac{y}{H}\right)^{-\gamma} - 1\right] = ma$$

Now it is time to employ a sneaky algebra trick. We will focus our attention on just the expression in the parentheses, i.e.

$$\left(1+\frac{y}{H}\right)^{-\gamma}$$

If we have a function of x in the form:

$$f(x) = (1+x)^n$$

we can expand this function as a Taylor Series that will look like this:

$$f(x) = 1 + \frac{n}{1!}x + \frac{n\,(n-1)}{2!}\,x^2 + \frac{n\,(n-1)(n-2)}{3!}\,x^3 + \cdots$$

Of course this series goes on forever, but we can make an important approximation: if “x” is very small, i.e. much smaller than 1, then x^2 x^3 … are all very very small, and can be considered “negligible”. Which means we can approximate the terms with x^2 and beyond as zero. And since *1!* is just equal to one, we get the approximation:

$$(1+x)^n \approx 1+nx \qquad \textit{if } x \textit{ is very small}$$

In our derivation, we will assume that y is very small compared to H, so that the fraction y/H is much smaller than one. We can then use this approximation in our derivation:

$$\left(1+\frac{y}{H}\right)^{-\gamma} \approx 1-\gamma\frac{y}{H}$$

And so our equation becomes:

$$p_1\, A \left[1 - \gamma\,\frac{y}{H} - 1\right] = ma$$

Which simplifies to:

$$-\frac{\gamma\, p_1\, A}{H}\, y = ma$$

Note that this equation includes a lot of constants – m g p_1 A and H – as well as y and a, the position and acceleration of the piston. We can now rearrange this equation into a form that will reveal something very special about the motion of the piston:

$$a + \left(\frac{\gamma\, p_1\, A}{m\, H}\right) y = 0$$

When we studied the motion of our *spring & mass system*, within the topic of oscillatory motion, we derived the equation for the acceleration of the system as:

$$a + \left(\frac{k}{m}\right) y = 0$$

We recognized this as a *differential equation* for which the solution was a sine or cosine function, representing *simple harmonic motion* for the system. We also found that the *angular frequency* and *period* of oscillation were given by:

$$\omega^2 = \frac{k}{m} \qquad \omega\, T = 2\pi$$

So our equation for the *spring & mass system* could be written as: $a + \omega^2\, y = 0$

And the equation for our piston? Note that it has *exactly the same form: acceleration plus a constant times position equals zero!* This means that the equation for our piston is the same differential equation that we found for our spring and mass system, and our piston will have the same simple harmonic motion.

We can write the equation for our piston as:

$$a + \omega^2\, y = 0 \qquad \omega^2 = \frac{\gamma\, p_1\, A}{m\, H}$$

We can now use the simple relationship between period and angular frequency:

$$\left(\frac{2\pi}{T}\right)^2 = \frac{\gamma\, p_1\, A}{m\, H}$$

And then rearrange to solve for T^2 and substitute (from our very first equation) for $p_1\, A$:

$$T^2 = \frac{4\pi^2\, m\, H}{\gamma\, p_1\, A} \qquad \text{or} \qquad T^2 = \frac{4\pi^2\, m}{\gamma\,(\,mg + p_0\, A)}\, H$$

Or, if we divide the numerator and denominator by the mass of the piston:

$$T^2 = \frac{4\pi^2}{\gamma\left(g + \frac{p_0 A}{m}\right)} H$$

Notice that this expression shows that the square of the period of oscillation of the piston is directly related to *H*, the height of the cylinder of air. Everything else in this equation – the mass and area of the pistion, the pressure of the outside air, the value of *gamma* – are constants.

We can combine all of these constants into a new constant, α, and rewrite our equation:

$$T^2 = \alpha H \qquad \text{where} \qquad \alpha = \frac{4\pi^2}{\gamma\left(g + \frac{p_0 A}{m}\right)}$$