

# Chapter 3

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### 3.1 Orthogonal Trajectories

Essentially curves where the tangent lines are orthogonal (perpendicular).

**Consider:**

$$y = cx^3 \quad \rightarrow \quad c = \frac{y}{x^3}$$

Essentially there are an infinite number of solutions.

$$\begin{aligned} \frac{dy}{dx} &= 3cx^2 = 3\left(\frac{y}{x^3}\right)x^2 \\ \text{Substituting } c \text{ back in. } \frac{dy}{dx} &= \frac{3y}{x} \end{aligned}$$

#### Example

Show that the curves  $y = x^3$  and  $x^2 + 3y^2 = 4$  are orthogonal at their points of intersection.

$$\begin{aligned} x^2 + 3y^2 &= 4 \\ \frac{1}{4}x^2 + \frac{3}{4}y^2 &= 1 \\ \frac{1}{2^2}x^2 + \frac{\sqrt{3}^2}{2^2}y^2 &= 1 \end{aligned}$$

$$\begin{aligned} y &= x^3 & x^2 + 3y^2 &= 4 \\ & & x^2 + 3(x^3)^2 &= 4 \\ & & x^2 + 3x^6 &= 4 \\ & & \text{Inspection} & \\ & & x = 1 \quad y = 1 & \\ & & x = -1 \quad y = -1 & \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 & 2x + 6y \frac{dy}{dx} &= 0 \\ \frac{dy(1,1)}{dx} &= 3(1)^2 = 3 & 2(1) + 6(1) \frac{dy}{dx} &= 2/6 \\ \frac{dy(1,1)}{dx} &= 3(-1)^2 = 3 & 2x + 6y \frac{dy}{dx} &= 0 \end{aligned}$$

**Definition:**

When all the curves of one family  $G(x, y, c) = 0$  intersect orthogonally, all the curves of another family  $H(x, y, c) = 0$ , then the families are said to be orthogonal trajectories.

Our goal is given a DE find another family that satisfies the above conditions.

**Example:**

$$y = c_1x \qquad x^2 + y^2 \qquad = c_2$$

To find orthogonal families to other families...

$$\frac{dy}{dx} = f(x, y) \qquad \frac{dy}{dx} \qquad = \frac{-1}{f(x, y)}$$

**3.1.1**

Find the O.T. for  $y = \frac{c_1}{x}$ .

$$\begin{aligned} \frac{dy}{dx} &= -\frac{c_1}{x^2} \\ c_1 &= xy \\ &= -\frac{xy}{x^2} \\ &= -\frac{y}{x} \end{aligned}$$

For O.T.

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y} \\ ydy &= xdx \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + c_2 \\ y^2 &= x^2 + c_2 \\ y^2 - x^2 &= c_2 \end{aligned}$$

### 3.1.2

$$\begin{aligned}
 y &= \frac{c_1 x}{1+x} \\
 \frac{dy}{dx} &= \frac{(1+x)c_1 - (c_1 x)}{(1+x)^2} \\
 &= \frac{c_1(1+x-x)}{(1+x)^2} \\
 &= \frac{c_1(1+x-x)}{(1+x)^2} \\
 c_1 &= \frac{y(1+x)}{x} \\
 &= \frac{\frac{y(1+x)}{x}}{(1+x)^2} \\
 &= \frac{\frac{y}{x}}{(1+x)} \\
 &= \frac{y}{x(1+x)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} &= -\frac{(1+x)x}{y} \\
 &= -\frac{x+x^2}{y} \\
 &= -\frac{x+x^2}{y} \\
 y \, dy &= x+x^2+c_2 \\
 \frac{1}{2}y^2 &= -(\frac{1}{2}x^2 + \frac{1}{3}x^3 + c_2) \\
 \frac{1}{2}y^2 &= -\frac{1}{2}x^2 - \frac{1}{3}x^3 - c_2 \\
 \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 &= c_2 \\
 3y^2 + 3x^2 + 2x^3 &= c_2
 \end{aligned}$$

### 3.1.3

$$\begin{aligned}
 y &= \ln(\tan x + c_1) \\
 y' &= \frac{1}{\tan x + c_1} (\sec^2 x) \\
 &= \frac{\sec^2 x}{\tan x + c_1} (\sec^2 x) \\
 e^y &= \tan x + c_1 \\
 e^y - \tan x &= c_1 \\
 y' &= \frac{\sec^2 x}{\tan x + e^y - \tan x} \\
 &= \frac{\sec^2 x}{e^y}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} &= -\frac{e^y}{\sec^2 x} \\
 &= -(e^y \cos^2 x) \\
 &= -e^y \cos^2 x
 \end{aligned}$$

## 3.2 Applications of Linear Equations

### Growth and Decay

We can of course find the growth and decay of anything, (bacteria, human population, money, radioactive substances, etc...). We call it growth or decay when the rate of change of that thing is dependent on how many there is of that thing now.

$$\begin{aligned}
 \frac{dA(t)}{dt} &= kA(t) & A(t_0) &= A_0 & A(t) > 0 \\
 \int \frac{A(t)}{A} dA &= \int k dt \ln|A(t)| & &= kt + c_1 \\
 A(t) &= e^{kt+c_1} \\
 &= e^{kt} e^{c_1} \\
 &= C e^{kt}
 \end{aligned}$$

### Bacteria Growth

A culture initially has  $N_0$  bacteria at  $t = 1$  hours. The number of bacteria is  $\frac{3}{2}N_0$ . If the growth rate is proportional to the number of bacteria present. How long until the bacteria triples?

$$\begin{aligned}N &= N_0 e^{kt} \\ \frac{3}{2}N_0 &= N_0 e^{k(1)} \\ \frac{3}{2} &= e^k \\ \ln \frac{3}{2} &= |k| & k > 0 \\ k &= \ln \frac{3}{2} \\ N(t) &= N_0 e^{\ln \frac{3}{2} t} = N_0 (e^{\ln \frac{3}{2}})^t \\ &= N_0 \frac{3^t}{2}\end{aligned}$$

$$\begin{aligned}N(t) &= 3N_0 \\ N_0 \frac{3^t}{2} &= 3N_0 \\ \frac{3^t}{2} &= 3 \\ \frac{1^t}{2} &= 1\end{aligned}$$

### Half-Life $\lambda$

#### Example

The half-life of  $U - 238$  is  $4.5 \times 10^9$  years. A breeder reactor converts  $U - 238$  into an isotope of plutonium 239. In 15 years it is determined that 0.043% of the initial amount has disintegrated. Find the half-life.

$$\begin{aligned}
m(t) &= m_0 e^{\lambda t} \\
(\Delta m)m_0 &= m_0 e^{\lambda t} \\
(\Delta m) &= e^{\lambda t} \\
e^{\Delta m} &= \lambda t \\
\frac{e^{\Delta m}}{t} &= \lambda \\
\Delta m &= m_t - m_0
\end{aligned}$$

### Newton's Law of cooling

The rate at which the temp. changes in a cooling body is proportional to the difference in the temp. of the temp of the body and the temp of the surrounding medium (air).

$$T = T_m + C e^{kt}$$

### Bake a cake.

$300^\circ F$  when removed air temp is  $70^\circ F$ . In 3 minutes, it cools off to  $200^\circ F$ . How long until it cools off to  $80^\circ F$ .