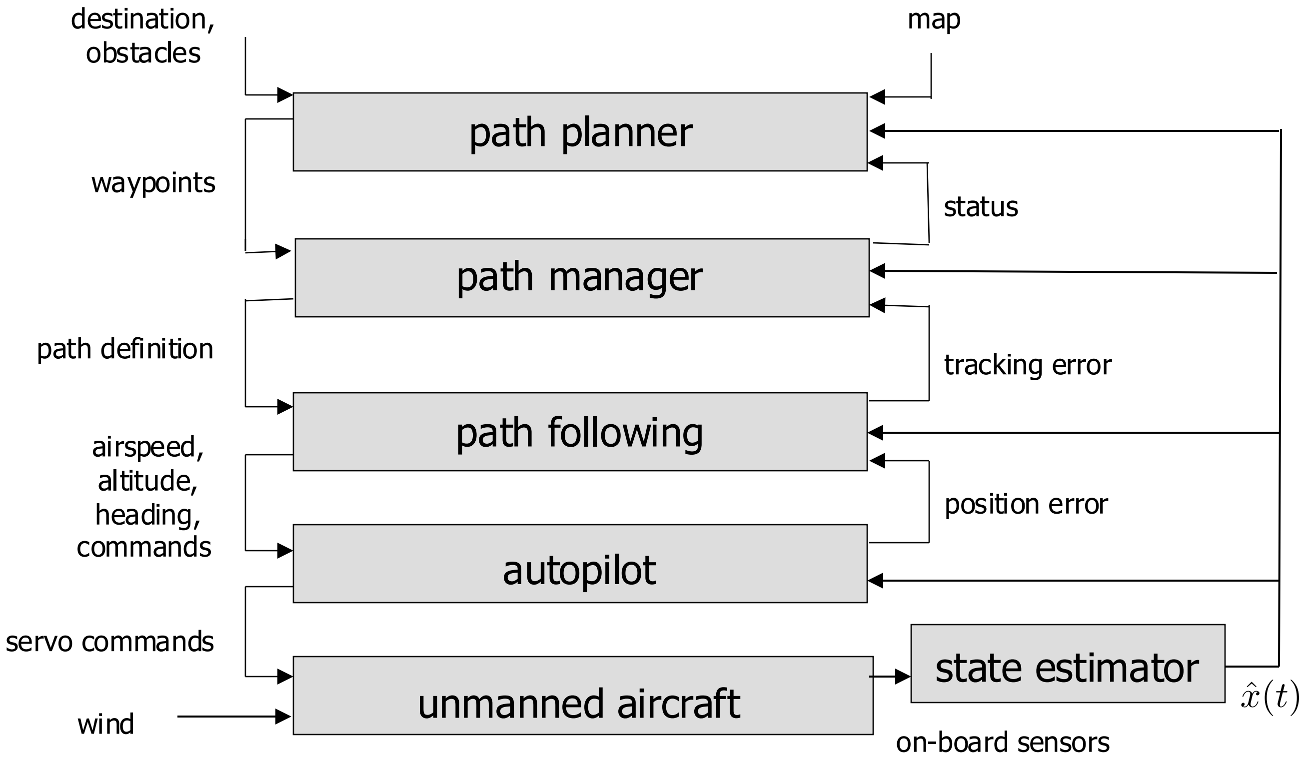
Chapter 10

Path Following

Control Architecture



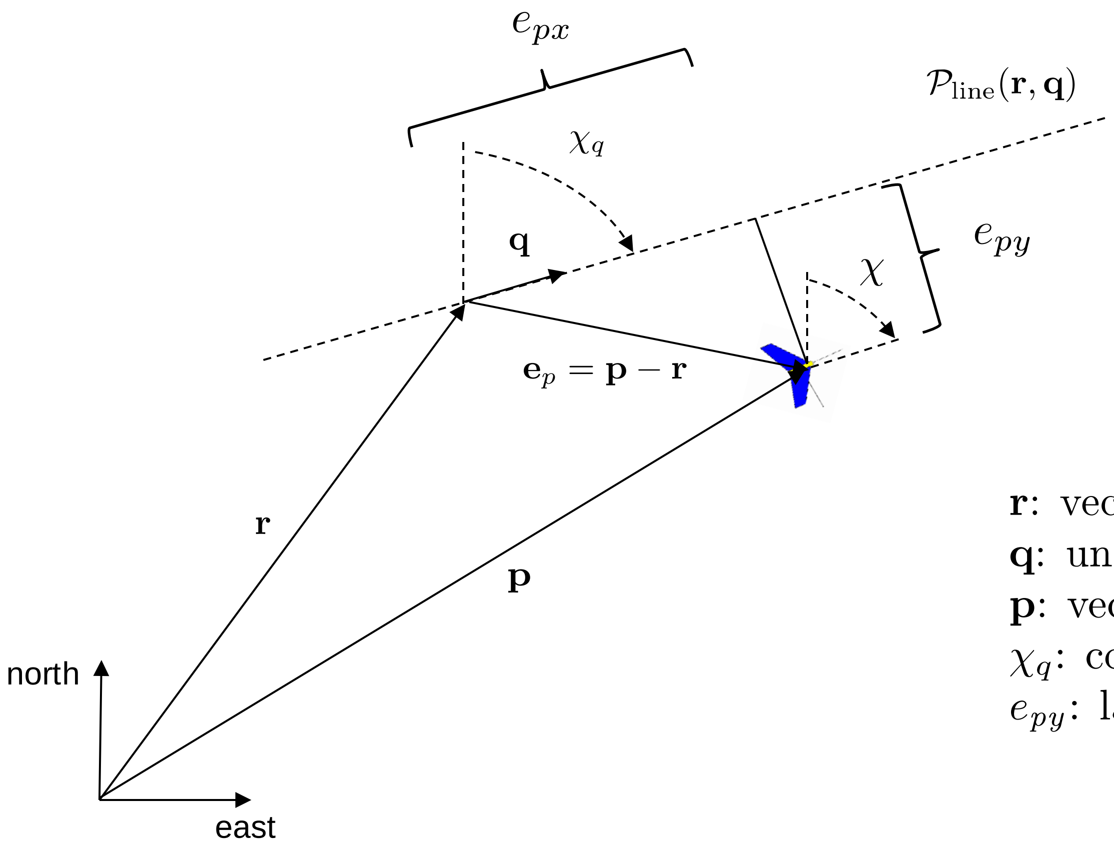
Path Following

- For small UAVs, a major issue is wind
  - Always present to some degree
  - Usually significant with respect to commanded airspeed
- Wind makes traditional trajectory tracking approaches difficult, if not infeasible
  - Have to know the wind precisely at every instant to determine desired airspeed
- Better approach: path following
- Rather than “follow this trajectory”, we control UAV to “stay on this path”

Path Types

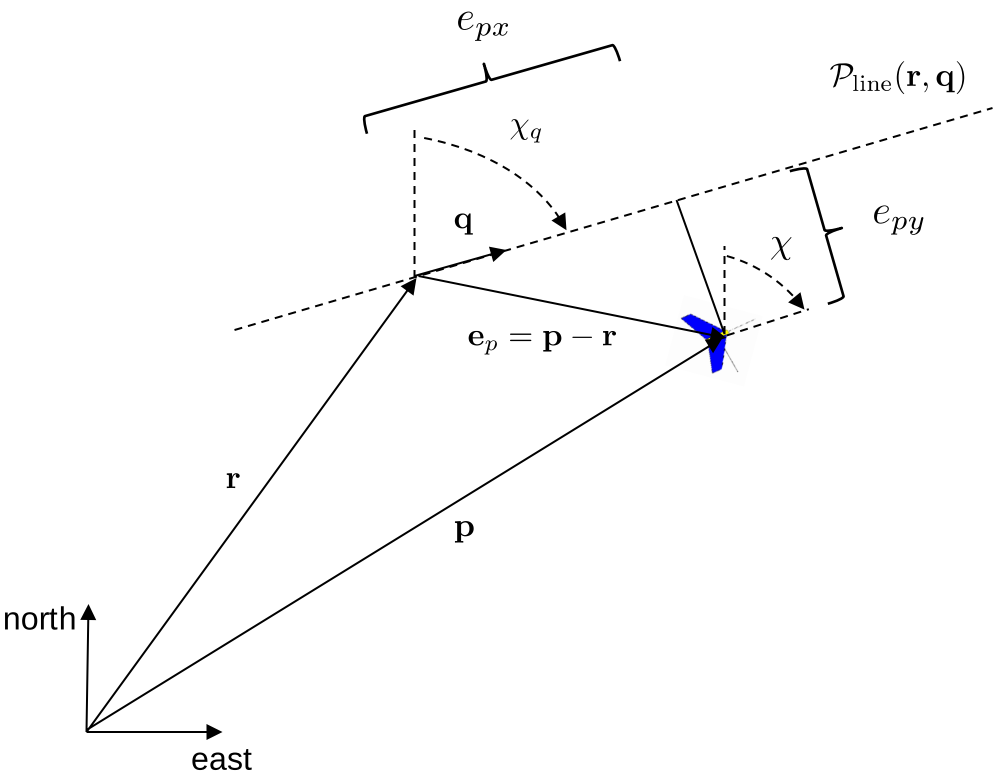
- We will focus on two types of paths to follow:
  - – Straight lines between two points in 3-D
    - Inclination of path within climb capabilities of UAV
  - – Circular orbits or arcs in the horizontal plane
- Paths for common applications can be built up from these path primitives
  - – Methods for following other types of paths found in literature

Straight Line Path Description



$\mathbf{r}$: vector defining initiation of path
$\mathbf{q}$: unit vector defining direction of path
$\mathbf{p}$: vector defining location of MAV
χ_q: course direction of path
e_{py}: lateral tracking error

Lateral Tracking Problem



Path error:

$$\mathbf{e}_p = \begin{pmatrix} e_{px} \\ e_{py} \\ e_{pz} \end{pmatrix} \triangleq \mathcal{R}_i^{\mathcal{P}} \left(\mathbf{p}^i - \mathbf{r}^i\right)$$

where the transformation from inertial frame to path frame is

$$\mathcal{R}_i^{\mathcal{P}} \triangleq \begin{pmatrix} \cos\chi_q & \sin\chi_q & 0 \\ -\sin\chi_q & \cos\chi_q & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lateral Tracking Problem

Relative error dynamics in path frame:

$$\begin{pmatrix} \dot{e}_{px} \\ \dot{e}_{py} \end{pmatrix} = \begin{pmatrix} \cos\chi_q & \sin\chi_q \\ -\sin\chi_q & \cos\chi_q \end{pmatrix} \begin{pmatrix} V_g\cos\chi \\ V_g\sin\chi \end{pmatrix}$$

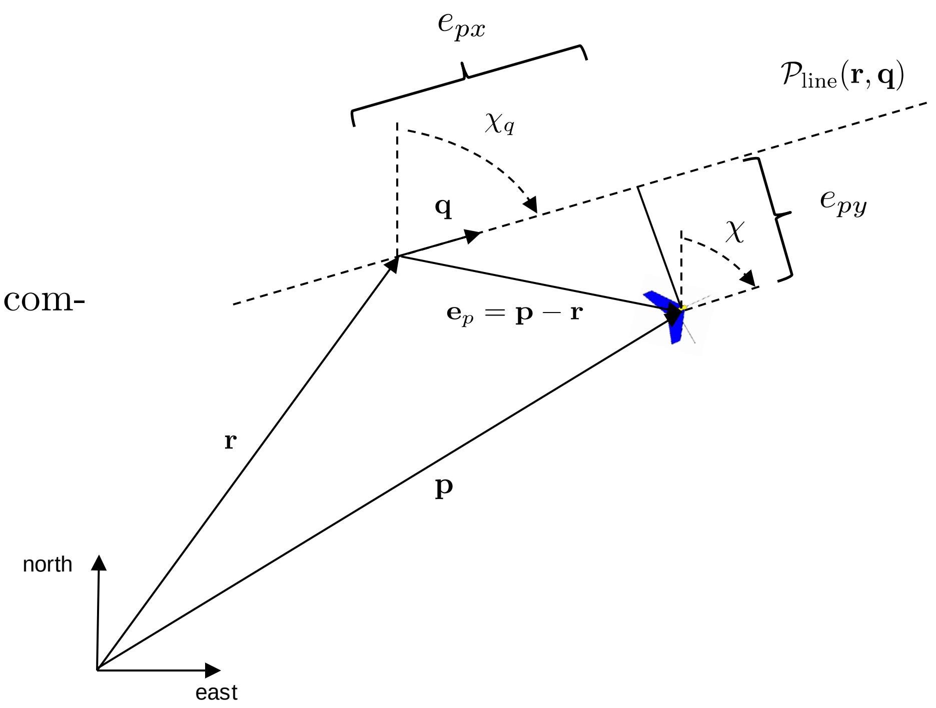
$$= V_g \begin{pmatrix} \cos(\chi - \chi_q) \\ \sin(\chi - \chi_q) \end{pmatrix}$$

Regulate the cross-track error e_{py} to zero by commanding the course angle:

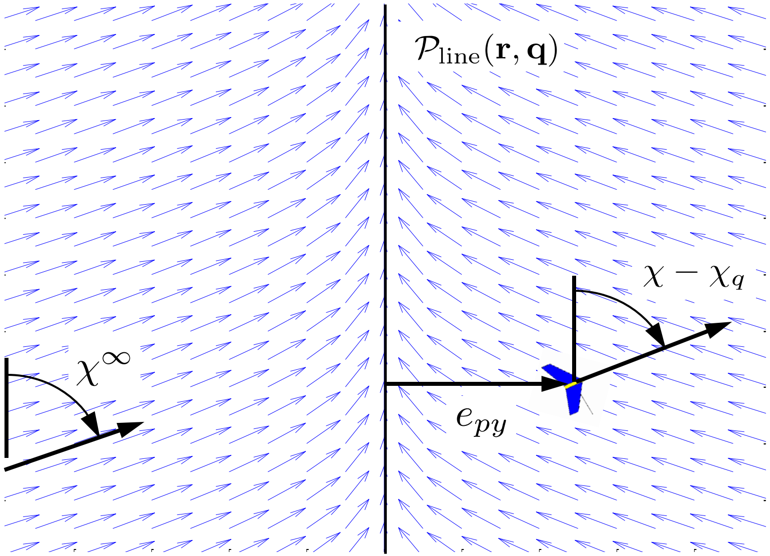
$$\dot{e}_{py} = V_g \sin(\chi - \chi_q)$$

$$\ddot{\chi} = b_{\dot{\chi}}(\dot{\chi}^c - \dot{\chi}) + b_{\chi}(\chi^c - \chi)$$

Select χ^c so that $e_{py} \to 0$

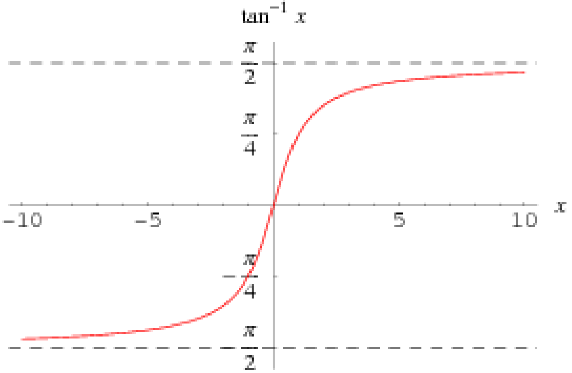


Lateral Tracking - Vector Field Concept



Desired course based on cross-track error:

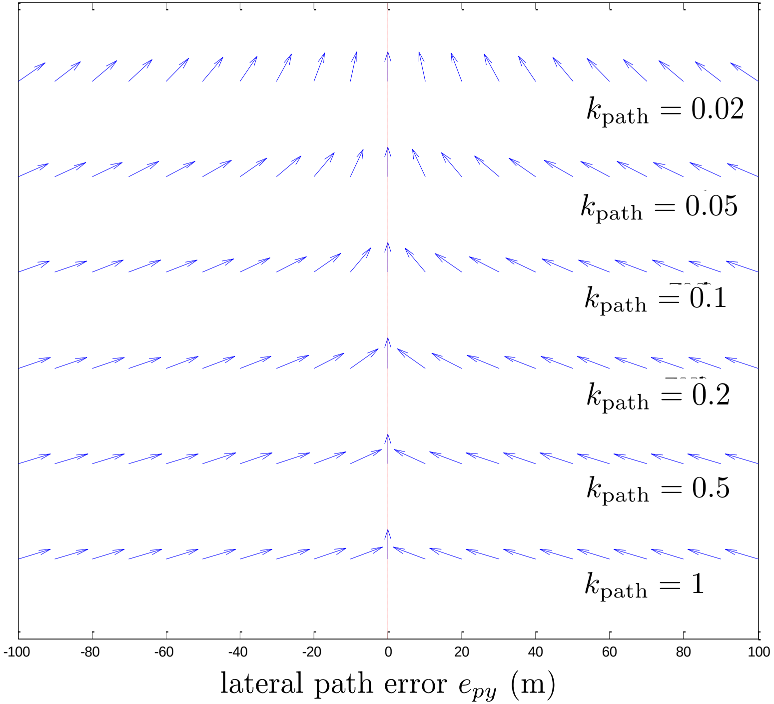
$$\chi_d(e_{py}) = -\chi^\infty \frac{2}{\pi} \tan^{-1}(k_{\text{path}} e_{py})$$



For a general line, the commanded course is

$$\chi^c = \chi_q - \chi^\infty \frac{2}{\pi} \tan^{-1}(k_{\text{path}} e_{py})$$

Vector Field Tuning



- k_{path} is a positive constant that affects the rate of transition of the desired course
- k_{path}-large $\longrightarrow$ short, abrupt transition
- k_{path}-small $\longrightarrow$ long, gradual transition

Rule of thumb:

$$k_{\text{path}} \approx \frac{1}{R_{\min}},$$

where $R_{\min}$ is the minumum turn radius of the aircraft.

Lyapunov's 2nd Method

For a system having a state vector x, consider an energy-like (Lyapunov) function $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$ such that

$$
\begin{aligned}
&V(x) > 0, \forall x \neq 0 \text{ (positive definite)} \\
&V(0) = 0 \\
&\quad \text{and} \\
&\dot{V}(x) < 0, \forall x \neq 0 \text{ (negative definite)} \\
&\dot{V}(0) = 0.
\end{aligned}
$$

If such a function $V(x)$ can be defined, then x goes to zero asymptotically and the system is stable.

Lateral Tracking Stability Analysis

Define the Lyapunov function $W(e_{py}) = \frac{1}{2}e_{py}^2$

Assume that course controller works and $\chi = \chi_q + \chi^d(e_{py})$

Since

$$\begin{aligned}\dot{W} &= e_{py}\dot{e}_{py} \\ &= -V_a e_{py} \sin\left(\chi^\infty \frac{2}{\pi} \tan^{-1}(k_{\text{path}} e_{py})\right) \\ &< 0\end{aligned}$$

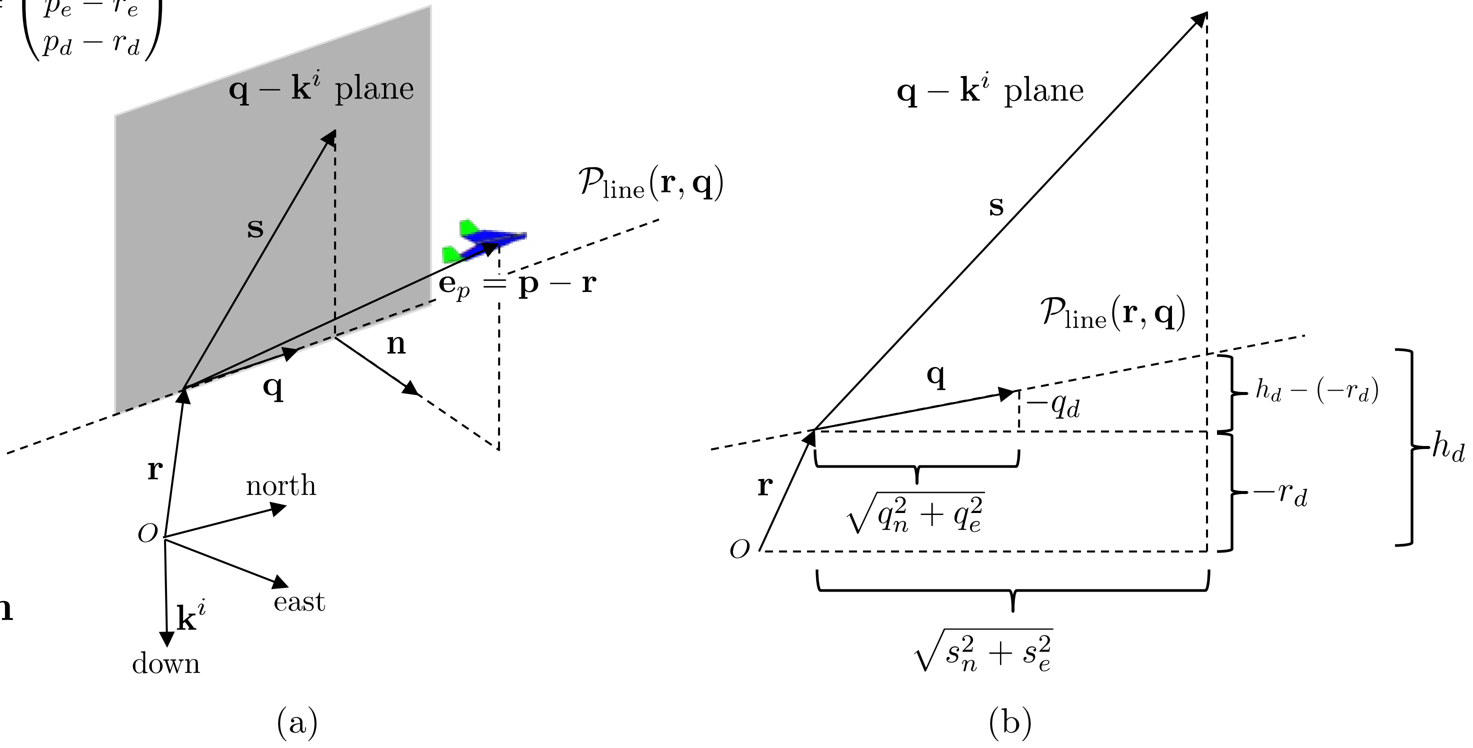
for $e_{py} \neq 0$, then $e_{py} \to 0$ asymptotically

Longitudinal Tracking Problem

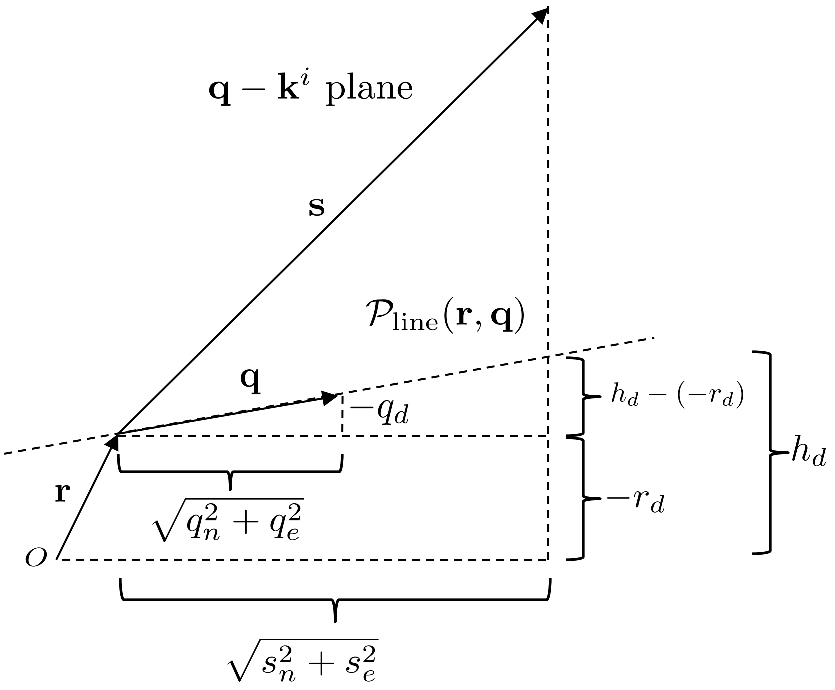
$$\mathbf{e}_p^i = \begin{pmatrix} e_{pn} \\ e_{pe} \\ e_{pd} \end{pmatrix} \triangleq \mathbf{p}^i - \mathbf{r}^i = \begin{pmatrix} p_n - r_n \\ p_e - r_e \\ p_d - r_d \end{pmatrix}$$

$$\mathbf{n} = \frac{\mathbf{k}^i \times \mathbf{q}}{\|\mathbf{k}^i \times \mathbf{q}\|}$$

$$\mathbf{s}^i = \begin{pmatrix} s_n \\ s_e \\ s_d \end{pmatrix} = \mathbf{e}_p^i - (\mathbf{e}_p^i \cdot \mathbf{n})\mathbf{n}$$



Longitudinal Tracking Problem



(b)

By similar triangles

$$\frac{(h_d+r_d)}{\sqrt{s_n^2+s_e^2}}=\frac{-q_d}{\sqrt{q_n^2+q_e^2}}$$

Desired altitude based on current location

$$h_d(\mathbf{r},\mathbf{p},\mathbf{q})=-r_d-\sqrt{s_n^2+s_e^2}\left(\frac{q_d}{\sqrt{q_n^2+q_e^2}}\right)$$

$\boxed{\text{Select } h^c \text{ so that } h \to h_d(\mathbf{r},\mathbf{p},\mathbf{q})}$

Longitudinal Guidance Strategy

Use altitude state machine from Ch. 6.

Closed-loop altitude dynamics:

$$h(s) = \left(\frac{b_{\dot{h}} s + b_h}{s^2 + b_{\dot{h}} s + b_h} \right) h^c(s).$$

Altitude error:

$$e_h \doteq h_d(\mathbf{r}, \mathbf{p}, \mathbf{q}) - h = h^c - h$$

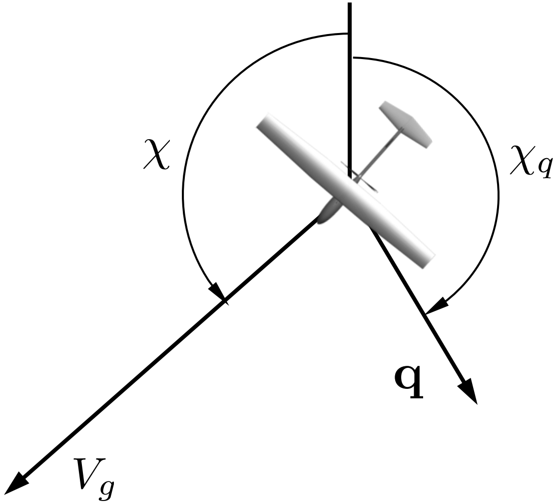
Error dynamics:

$$\begin{aligned} e_h(s) &= (1 - h(s))\, h^c(s) \\ &= \left(\frac{s^2 + b_{\dot{h}} s + b_h}{s^2 + b_{\dot{h}} s + b_h} - \frac{b_{\dot{h}} s + b_h}{s^2 + b_{\dot{h}} s + b_h} \right) h^c(s) \\ &= \left(\frac{s^2}{s^2 + b_{\dot{h}} s + b_h} \right) h^c(s) \end{aligned}$$

Applying FVT:

$$\begin{aligned} e_{h,ss} &= \lim_{s \to 0} s \, \frac{s^2}{s^2 + b_{\dot{h}} s + b_h} \, h^c \\ &= 0, \qquad \text{for } h^c = \frac{H_0}{s}, \frac{H_0}{s^2} \end{aligned}$$

Smallest Angle Turn Logic



$$\chi_q = \mathrm{atan2}(q_e, q_n) + 2\pi m$$

$$m \in \mathcal{N} \text{ is selected so that } -\pi \leq \chi_q - \chi \leq \pi$$

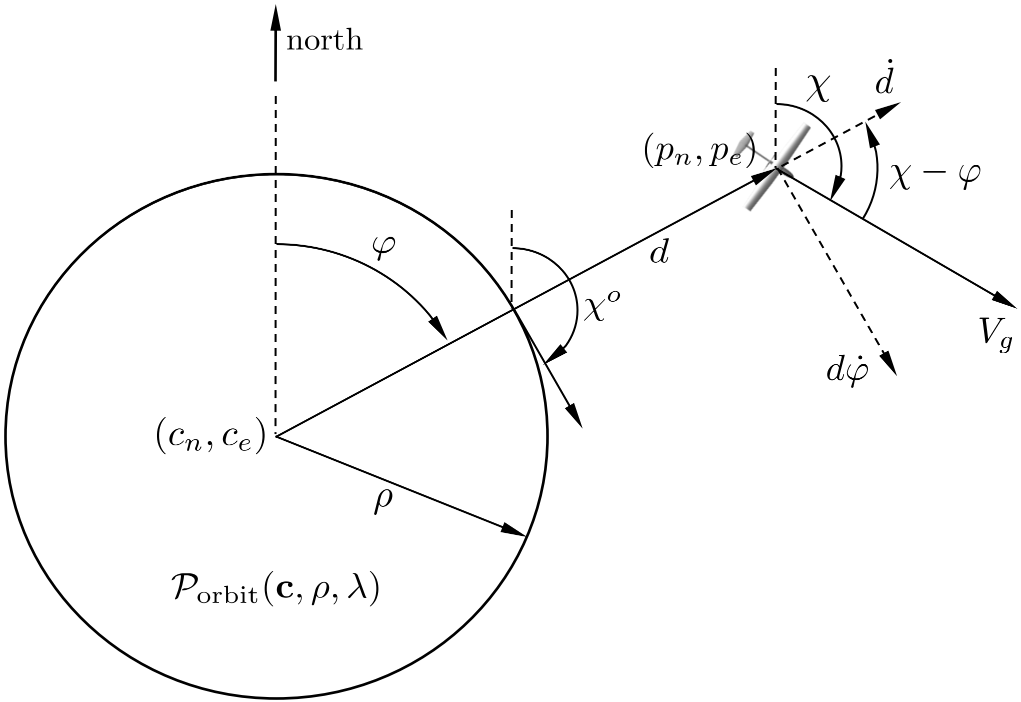
Algorithm 3 Straight-line Following: $[h^c, \chi^c]$ = followStraightLine $(\mathbf{r}, \mathbf{q}, \mathbf{p}, \chi)$

Input: Path definition $\mathbf{r} = (r_n, r_e, r_d)^\top$ and $\mathbf{q} = (q_n, q_e, q_d)^\top$, MAV position $\mathbf{p} = (p_n, p_e, p_d)^\top$, course χ, gains χ_∞, k_{path}, sample rate T_s.

```
Compute commanded altitude using equation (10.5).
χq ← atan2(qe, qn)
while χq − χ < −π do
    χq ← χq + 2π
end while
while χq − χ > π do
    χq ← χq − 2π
end while
epy ← − sin χq(pn − rn) + cos χq(pe − re)
Compute commanded course angle using equation (10.8).
return h^c, χ^c
```

Orbit Following

Orbit definition: $\mathcal{P}_{\text{orbit}}(\mathbf{c}, \rho, \lambda) = \left\{ \mathbf{r} \in \mathbb{R}^3 : \mathbf{r} = \mathbf{c} + \lambda\rho \begin{pmatrix} \cos\varphi, & \sin\varphi & 0 \end{pmatrix}^\top, \varphi \in [0, 2\pi) \right\}$



Center: $\mathbf{c} \in \mathbb{R}^3$
Radius: $\rho \in \mathbb{R}$
Direction: $\lambda = 1$ (CW) or $\lambda = -1$ (CCW)

In polar coordinates, the position of MAV given by:

$$d \doteq \sqrt{(p_n - c_n)^2 + (p_e - c_e)^2}:$$
$$\varphi \doteq \tan^{-1}\left(\frac{p_e - c_e}{p_n - c_n}\right):$$

Orbit Following

Easiest to analyze in polar coordinates. Using

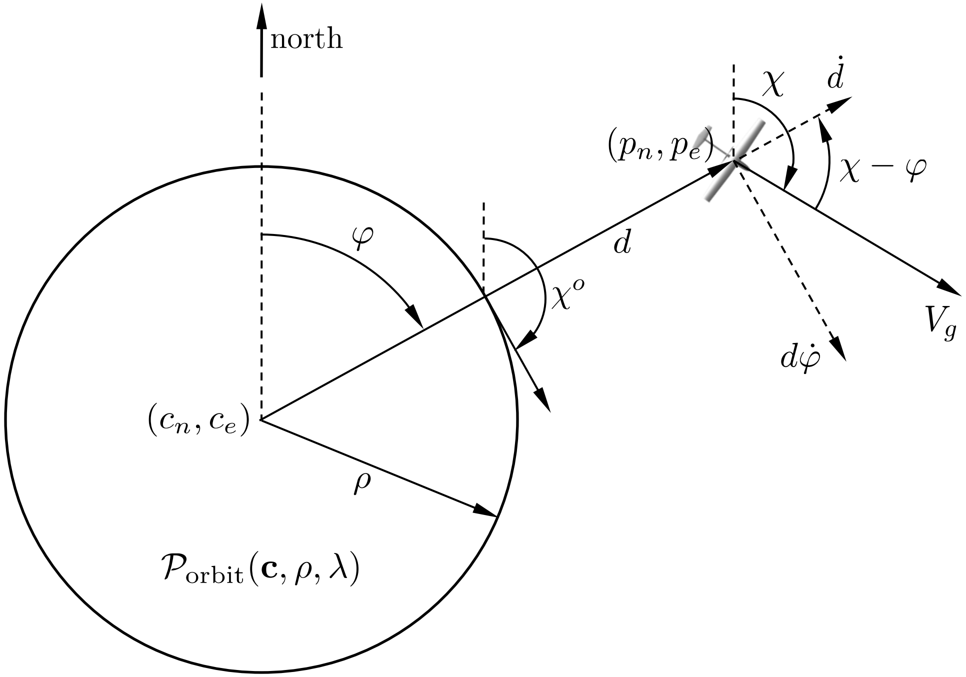
$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \end{pmatrix} = \begin{pmatrix} V_g \cos\chi \\ V_g \sin\chi \end{pmatrix}$$

and converting to polar coordinates gives

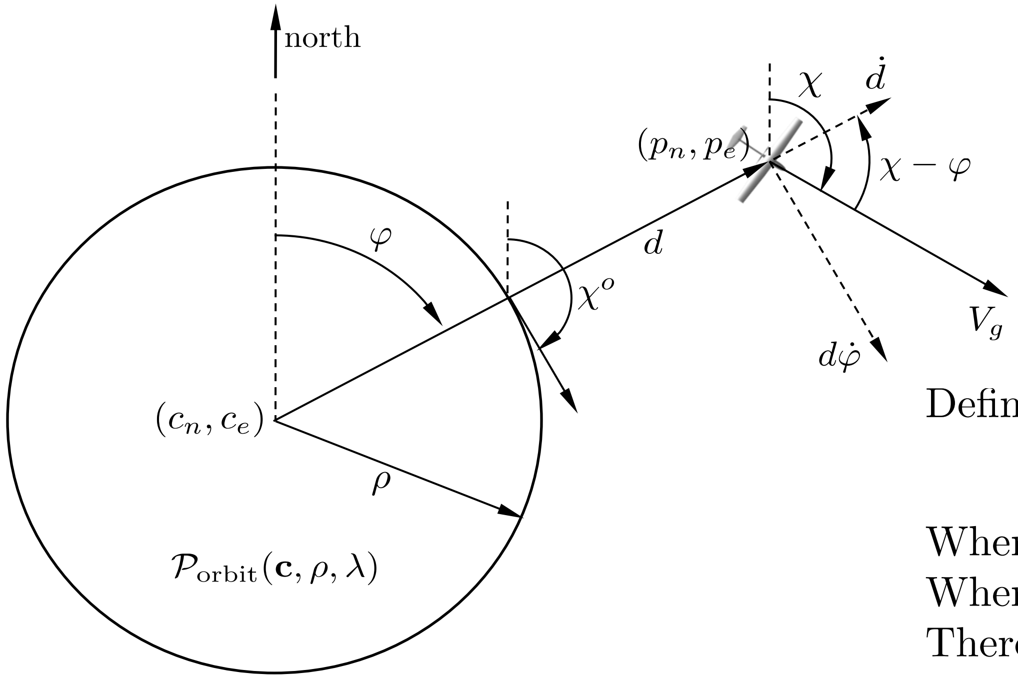
$$\begin{aligned}
\begin{pmatrix} \dot{d} \\ d\dot{\varphi} \end{pmatrix} &= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \dot{p}_n \\ \dot{p}_e \end{pmatrix} \\
&= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} V_g \cos\chi \\ V_g \sin\chi \end{pmatrix} \\
&= \begin{pmatrix} V_g \cos(\chi - \varphi) \\ V_g \sin(\chi - \varphi) \end{pmatrix}
\end{aligned}$$

The result is

$$\boxed{\begin{aligned}
\dot{d} &= V_g \cos(\chi - \varphi) \\
\dot{\varphi} &= \frac{V_g}{d} \sin(\chi - \varphi) \\
\ddot{\chi} &= -b_{\dot{\chi}} \dot{\chi} + b_{\chi}(\chi^c - \chi)
\end{aligned}}$$



Orbit Following



Define

$$\chi^o = \varphi + \lambda \frac{\pi}{2}$$

When $d \gg \rho \quad \rightarrow \quad \chi_d \approx \chi^o + \lambda \frac{\pi}{2}$.
When $d = \rho \quad \rightarrow \quad \chi_d = \chi^o$
Therefore, let the desired course angle be

$$\chi_d(d - \rho, \lambda) = \chi^o + \lambda \tan^{-1}\left(k_{\text{orbit}}\left(\frac{d-\rho}{\rho}\right)\right)$$

Orbit Tracking Stability Analysis

Define the Lyapunov function $W = \frac{1}{2}(d-\rho)^2$

Assume that course controller works and $\chi = \chi^d(d-\rho, \lambda)$

Since

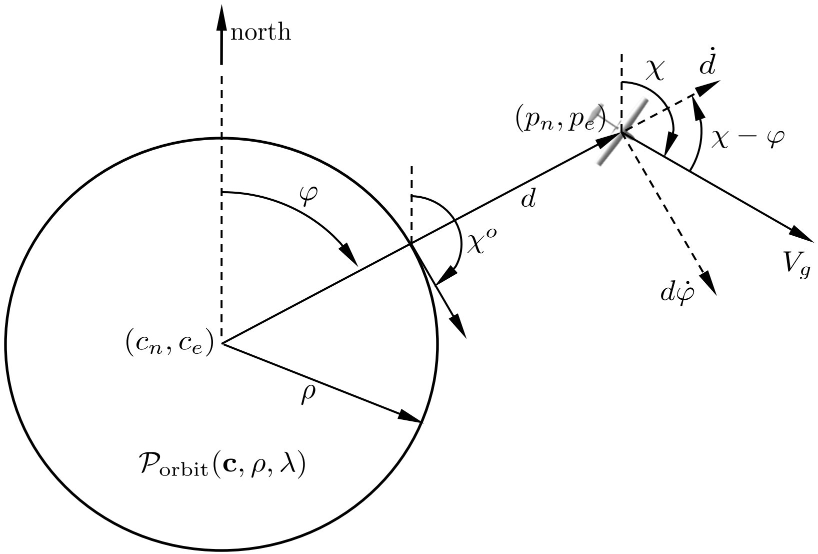
$$\begin{aligned}
\dot{W} &= (d-\rho)\dot{d} \\
&= (d-\rho)(V_g \cos(\chi - \varphi)) \\
&= -V_g(d-\rho)\sin\left(\tan^{-1}\left(k_{\text{orbit}}\left(\frac{d-\rho}{\rho}\right)\right)\right) \\
&< 0
\end{aligned}$$

for $d-\rho \neq 0$, then $d-\rho \to 0$ asymptotically

Orbit Following

The commanded course is: $$\chi^c(t) = \varphi + \lambda\left[\frac{\pi}{2} + \tan^{-1}\left(k_{\text{orbit}}\left(\frac{d-\rho}{\rho}\right)\right)\right]$$

The orbit angle must be wrapped: $$\varphi = \text{atan2}(p_e - c_e, p_n - c_n) + 2\pi m$$



Algorithm 4 Circular Orbit Following: $[h^c, \chi^c]$ = followOrbit $(\mathbf{c}, \rho, \lambda, \mathbf{p}, \chi)$

Input: Orbit center $\mathbf{c} = (c_n, c_e, c_d)^\top$, radius ρ, and direction λ, MAV position $\mathbf{p} = (p_n, p_e, p_d)^\top$, course χ, gains k_{orbit}, sample rate T_s.

```
h^c ← −c_d
d ← sqrt((p_n − c_n)^2 + (p_e − c_e)^2)
φ ← atan2(p_e − c_e, p_n − c_n)
while φ − χ < −π do
    φ ← φ + 2π
end while
while φ − χ > π do
    φ ← φ − 2π
end while
Compute commanded course angle using equation (10.13).
return h^c, χ^c
```

Roll Feedforward: no wind

For orbit following:

$$\chi^c(t) = \varphi + \lambda\left[\frac{\pi}{2} + \tan^{-1}\left(k_{\mathrm{orbit}}\left(\frac{d-\rho}{\rho}\right)\right)\right].$$

Note:

$$\begin{aligned} d-\rho = 0 \quad &\Longrightarrow \quad \chi^c = 0 \\ &\Longrightarrow \quad \phi^c = 0 \\ &\Longrightarrow \quad \text{UAV will immediately deviate from orbit.} \end{aligned}$$

Problem can be fixed by commanding a roll feedforward for when aircraft is on the orbit.

If on the orbit and no wind, then

$$\dot{\psi}^d = \lambda\frac{V_a}{\rho}.$$

Coordinated turn condition:

$$\dot{\psi} = \frac{g}{V_a}\tan\phi.$$

Equating and solving for roll gives

$$\phi_{ff} = \lambda\tan^{-1}\left(\frac{V_a^2}{g\rho}\right). \tag{1}$$

Roll Feedforward: wind

When wind is present we have

$$\dot{\chi}^d(t) = \lambda \frac{V_g(t)}{\rho},$$

where V_g is the time varying ground speed. The coordinated turn condition in wind is

$$\dot{\chi} = \frac{g}{V_g} \tan\phi \cos(\chi - \psi).$$

Equating these expressions and solving for ϕ gives

$$\phi_{ff} = \lambda \tan^{-1}\left(\frac{V_g^2}{g\rho\cos(\chi - \psi)}\right).$$

Dubins Airplane Model

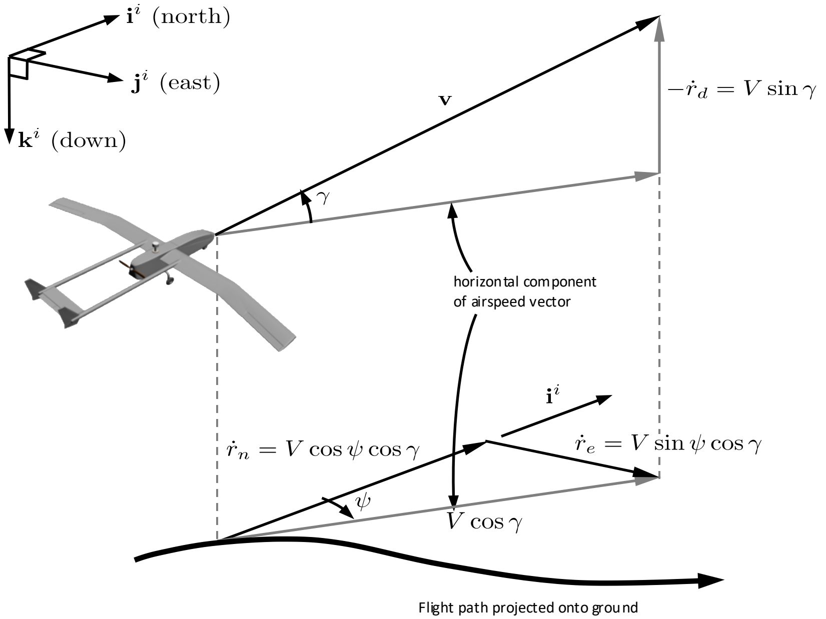
Adapted from: Mark Owen, Randal W. Beard, Timothy W. McLain, "Implementing Dubins Airplane Paths on Fixed-wing UAVs," *Handbook of Unmanned Aerial Vehicles,* ed. Kimon P. Valavanis, George J. Vachtsevanos, Springer Verlag, Section XII, Chapter 68, p. 1677-1702, 2014.

Dubins Airplane model:

$$\begin{aligned}
\dot{r}_n &= V \cos\psi \cos\gamma^c \\
\dot{r}_e &= V \sin\psi \cos\gamma^c \\
\dot{r}_d &= -V \sin\gamma^c \\
\dot{\psi} &= \frac{g}{V} \tan\phi^c
\end{aligned}$$

Where the commanded flight path angle γ^c and the commanded roll angle ϕ^c are constrained by

$$\begin{aligned}
|\phi^c| &\leq \bar{\phi} \\
|\gamma^c| &\leq \bar{\gamma}.
\end{aligned}$$



3D Vector Field Path Following

Adapted from: V. M. Goncalves, L. C. A. Pimenta, C. A. Maia, B. C. O. Durtra, G. A. S. Pereira, B. C. O. Dutra, and G. A. S. Pereira, “Vector Fields for Robot Navigation Along Time-Varying Curves in n-Dimensions,” IEEE Transactions on Robotics, vol. 26, pp. 647–659, Aug 2010.

The path is specified as the intersection of two 2D manifolds given by

$$\alpha_1(\mathbf{r}) = 0$$
$$\alpha_2(\mathbf{r}) = 0$$

$\mathbf{r} \in \mathbb{R}^3$. Define the composite function

$$W(\mathbf{r}) = \frac{1}{2}\alpha_1^2(\mathbf{r}) + \frac{1}{2}\alpha_2^2(\mathbf{r}),$$

Note that the gradient

$$\frac{\partial W}{\partial \mathbf{r}} = \alpha_1(\mathbf{r})\frac{\partial \alpha_1}{\partial \mathbf{r}}(\mathbf{r}) + \alpha_2(\mathbf{r})\frac{\partial \alpha_2}{\partial \mathbf{r}}(\mathbf{r}).$$

points away from the path.

3D Vector Field Path Following

The desired velocity vector can be chosen as

$$\mathbf{u}' = \underbrace{-K_1 \frac{\partial W}{\partial \mathbf{r}}}_{\text{velocity directed toward the path}} + \underbrace{K_2 \frac{\partial \alpha_1}{\partial \mathbf{r}} \times \frac{\partial \alpha_2}{\partial \mathbf{r}}}_{\text{velocity directed along the path}}$$

where $K_1 > 0$ and K_2 are symmetric tuning matrices, and the definiteness of K_2 determines the direction of travel along the path.

Since $\mathbf{u}'$ may not equal V_a, normalize to get

$$\mathbf{u} = V_a \frac{\mathbf{u}'}{\|\mathbf{u}'\|}.$$

3D Vector Field Path Following

Setting the NED components of the velocity of the Dubins airplane model to $\mathbf{u} = (u_1, u_2, u_3)^\top$ gives

$$\begin{aligned} V \cos\psi^d \cos\gamma^c &= u_1 \\ V \sin\psi^d \cos\gamma^c &= u_2 \\ -V \sin\gamma^c &= u_3. \end{aligned}$$

Solving for γ^c, and ψ^d results in

$$\begin{aligned} \gamma^c &= -\mathrm{sat}_{\bar{\gamma}}\left[\sin^{-1}\left(\frac{u_3}{V}\right)\right] \\ \psi^d &= \mathrm{atan2}(u_2, u_1). \end{aligned}$$

Assuming the inner-loop lateral-directional dynamics are accurately modeled by the coordinated-turn equation, the commanded roll angle is

$$\phi^c = \mathrm{sat}_{\bar{\phi}}\left[k_\phi(\psi^d - \psi)\right],$$

where k_ϕ is a positive constant.

3D Vector Field – Straight Line path

The straight line path is given by

$$\mathcal{P}_{\text{line}}(\mathbf{c}_\ell, \psi_\ell, \gamma_\ell) = \left\{\mathbf{r} \in \mathbb{R}^3 : \mathbf{r} = \mathbf{c}_\ell + \sigma \mathbf{q}_\ell, \sigma \in \mathbb{R}\right\},$$

where

$$\mathbf{q}_\ell = \begin{pmatrix} q_n \\ q_e \\ q_d \end{pmatrix} = \begin{pmatrix} \cos\psi_\ell \cos\gamma_\ell \\ \sin\psi_\ell \cos\gamma_\ell \\ -\sin\gamma_\ell \end{pmatrix}.$$

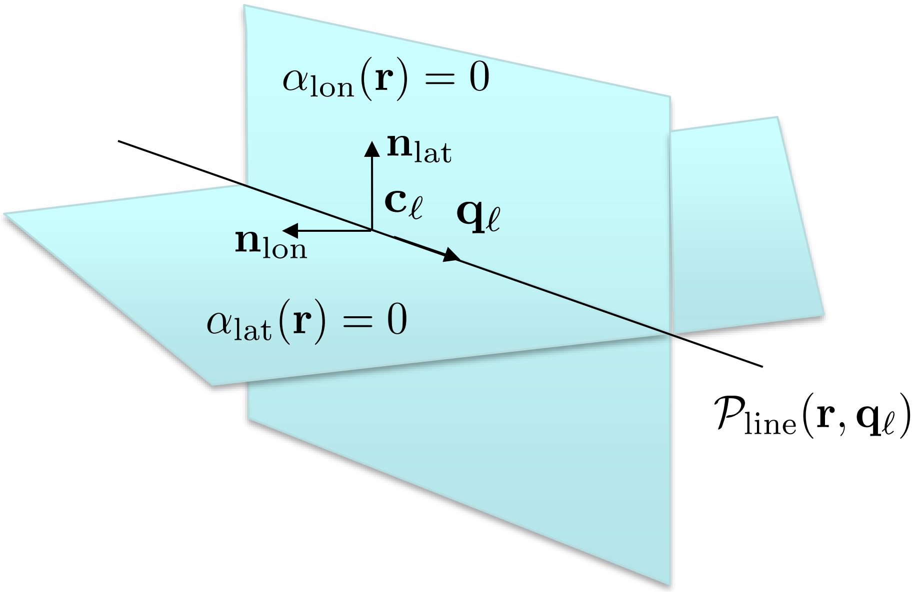
Define

$$\mathbf{n}_{\text{lon}} = \begin{pmatrix} -\sin\psi_\ell \\ \cos\psi_\ell \\ 0 \end{pmatrix}$$

$$\mathbf{n}_{\text{lat}} = \mathbf{n}_{\text{lon}} \times \mathbf{q}_\ell = \begin{pmatrix} -\cos\psi_\ell \sin\gamma_\ell \\ -\sin\psi_\ell \sin\gamma_\ell \\ -\cos\gamma_\ell \end{pmatrix},$$

to get

$$\alpha_{\text{lon}}(\mathbf{r}) = \mathbf{n}_{\text{lon}}^\top (\mathbf{r} - \mathbf{c}_\ell) = 0$$
$$\alpha_{\text{lat}}(\mathbf{r}) = \mathbf{n}_{\text{lat}}^\top (\mathbf{r} - \mathbf{c}_\ell) = 0.$$



3D Vector Field – Helical Path

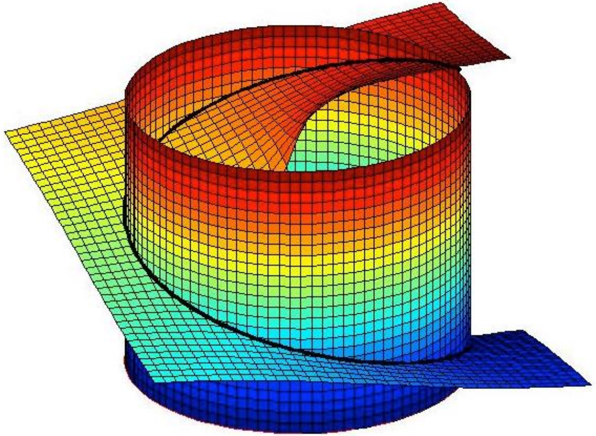
A helical path is then defined as

$$\mathcal{P}_{\text{helix}}(\mathbf{c}_h, \psi_h, \lambda_h, \rho_h, \gamma_h) = \{\mathbf{r} \in \mathbb{R}^3 : \alpha_{\text{cyl}}(\mathbf{r}) = 0 \text{ and } \alpha_{\text{pl}}(\mathbf{r}) = 0\}.$$

where

$$\alpha_{\text{cyl}}(\mathbf{r}) = \left(\frac{r_n - c_n}{\rho_h}\right)^2 + \left(\frac{r_e - c_e}{\rho_h}\right)^2 - 1$$

$$\alpha_{\text{pl}}(\mathbf{r}) = \left(\frac{r_d - c_d}{\rho_h}\right) + \frac{\tan\gamma_h}{\lambda_h}\left(\tan^{-1}\left(\frac{r_e - c_e}{r_n - c_n}\right) - \psi_h\right)$$



where the initial position along the helix is

$$\mathbf{r}(0) = \mathbf{c}_h + \begin{pmatrix} \rho_h \cos\psi_h \\ \rho_h \sin\psi_h \\ 0 \end{pmatrix},$$

and where $\mathbf{c}_h$ is the center of the helix, ρ_h is the radius, γ_h is the climb angle.