



Chapter 2

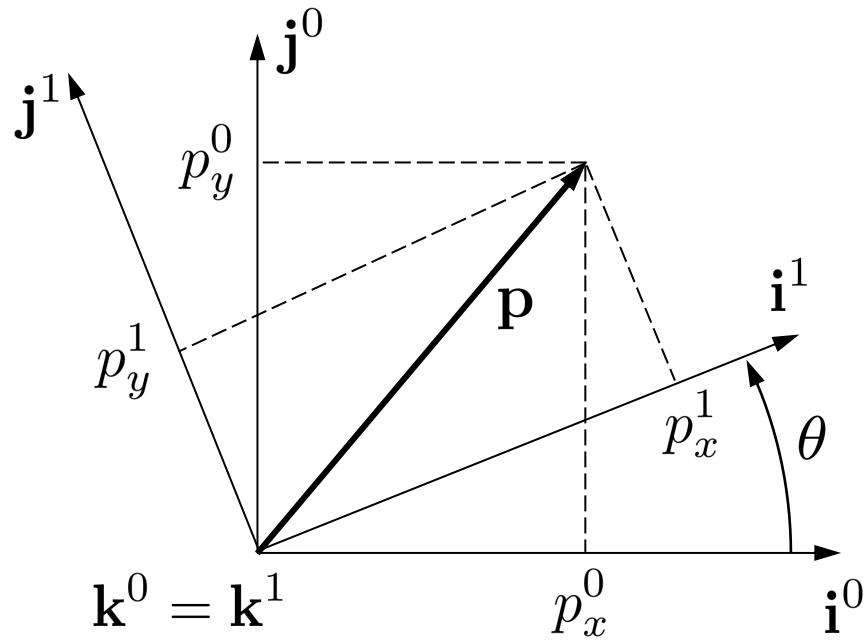
Coordinate Frames

Reference Frames

- In guidance and control of aircraft, reference frames are an essential idea
- Describe relative position and orientation of objects
 - Aircraft relative to direction of wind
 - Camera relative to aircraft
 - Aircraft relative to inertial frame
- Quantities most easily calculated or described in a specific reference frames:
 - Newton's law
 - Aircraft attitude
 - Aerodynamic forces/torques
 - Accelerometers, rate gyros
 - GPS
 - Mission requirements

Must know how to transform
between different reference
frames

Rotation of Reference Frame



Take dot product of both sides –
first with \mathbf{i}^1 , then \mathbf{j}^1 , then \mathbf{k}^1

$$\mathbf{p} = p_x^0 \mathbf{i}^0 + p_y^0 \mathbf{j}^0 + p_z^0 \mathbf{k}^0$$

$$\mathbf{p} = p_x^1 \mathbf{i}^1 + p_y^1 \mathbf{j}^1 + p_z^1 \mathbf{k}^1$$

$$\Rightarrow p_x^1 \mathbf{i}^1 + p_y^1 \mathbf{j}^1 + p_z^1 \mathbf{k}^1 = p_x^0 \mathbf{i}^0 + p_y^0 \mathbf{j}^0 + p_z^0 \mathbf{k}^0$$

$$\Rightarrow \mathbf{p}^1 \triangleq \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} = \begin{pmatrix} \mathbf{i}^1 \cdot \mathbf{i}^0 & \mathbf{i}^1 \cdot \mathbf{j}^0 & \mathbf{i}^1 \cdot \mathbf{k}^0 \\ \mathbf{j}^1 \cdot \mathbf{i}^0 & \mathbf{j}^1 \cdot \mathbf{j}^0 & \mathbf{j}^1 \cdot \mathbf{k}^0 \\ \mathbf{k}^1 \cdot \mathbf{i}^0 & \mathbf{k}^1 \cdot \mathbf{j}^0 & \mathbf{k}^1 \cdot \mathbf{k}^0 \end{pmatrix} \begin{pmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{pmatrix}$$

$$\Rightarrow \mathbf{p}^1 = R_0^1 \mathbf{p}^0 \text{ where } R_0^1 \triangleq \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

R_0^1 is a rotation about the \mathbf{k} axis.

Rotation of Reference Frame

Right-handed rotation about \mathbf{j} axis:

$$R_0^1 \triangleq \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$

Right-handed rotation about \mathbf{i} axis:

$$R_0^1 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

Orthonormal matrix properties:

P.1. $(R_a^b)^{-1} = (R_a^b)^\top = R_b^a$

P.2. $R_b^c R_a^b = R_a^c$

P.3. $\det(R_a^b) = 1$

Rotation of Reference Frame: Alternate View

Note that

$$\mathbf{i}_0^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j}_0^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k}_0^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and that

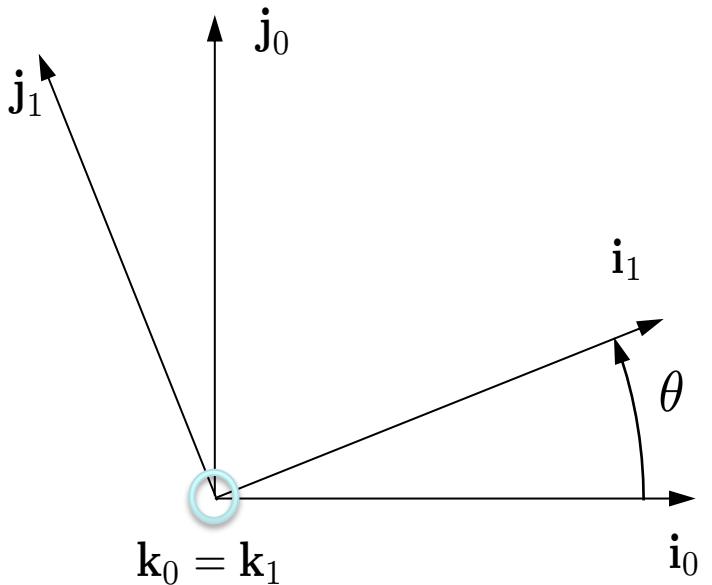
$$\mathbf{i}_1^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j}_1^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k}_1^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now express $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$ in the 1-frame:

$$\mathbf{i}_0^1 = \begin{pmatrix} \cos \theta \\ -\sin \theta \\ 0 \end{pmatrix}, \quad \mathbf{j}_0^1 = \begin{pmatrix} \sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \quad \mathbf{k}_0^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

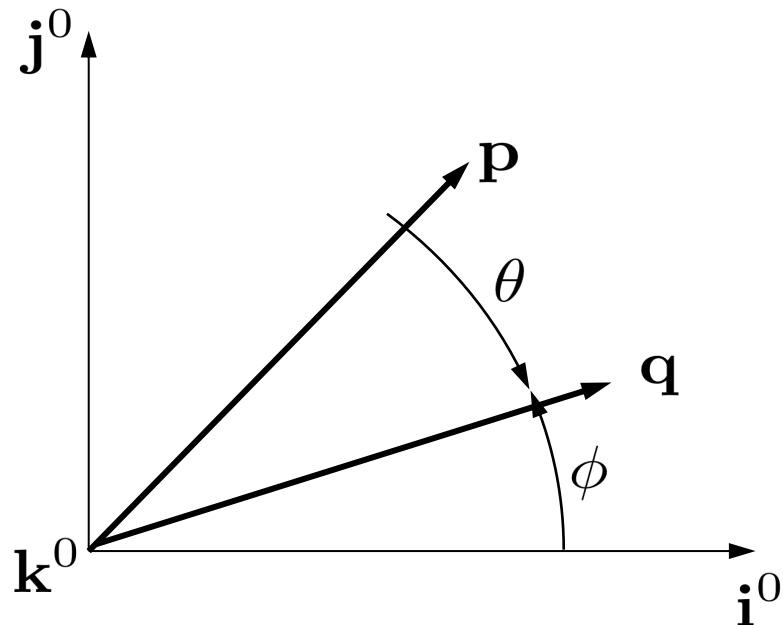
Therefore

$$R_0^1 = (\mathbf{i}_0^1 \quad \mathbf{j}_0^1 \quad \mathbf{k}_0^1) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Rotation of a Vector

Let $p \triangleq |\mathbf{p}| = q \triangleq |\mathbf{q}|$, then



R_0^1 can be interpreted as a left-handed rotation of a vector by an angle θ

$$\begin{aligned}\mathbf{p} &= \begin{pmatrix} p \cos(\theta + \phi) \\ p \sin(\theta + \phi) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p \cos \theta \cos \phi - p \sin \theta \sin \phi \\ p \sin \theta \cos \phi + p \cos \theta \sin \phi \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \cos \phi \\ p \sin \phi \\ 0 \end{pmatrix}\end{aligned}$$

Define

$$\mathbf{q} = \begin{pmatrix} p \cos \phi \\ p \sin \phi \\ 0 \end{pmatrix}$$

then

$$\mathbf{p} = (R_0^1)^\top \mathbf{q} \quad \Rightarrow \quad \mathbf{q} = R_0^1 \mathbf{p}$$

Passive vs. Active Rotation

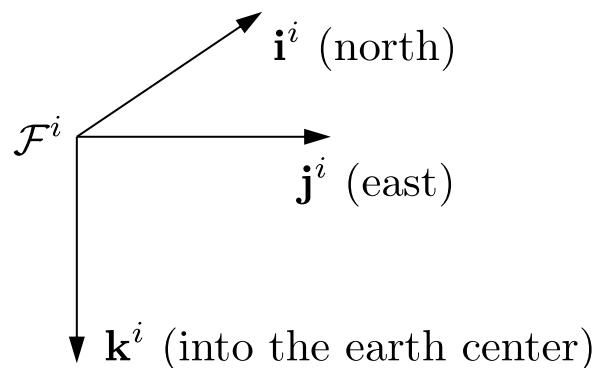
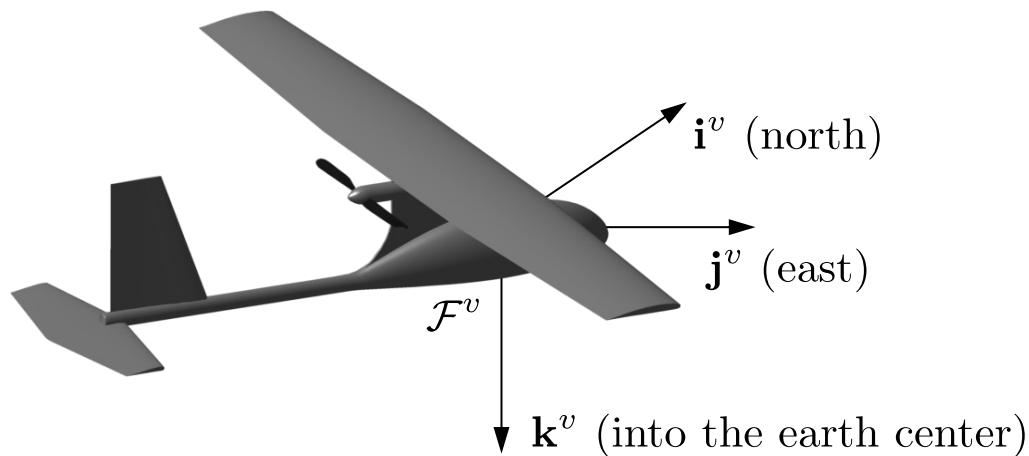
Passive Rotation/Transformation: The vector \mathbf{p} is stationary and the coordinate frame changes

Active Rotation/Transformation: The coordinate frame remains fixed, but the vector is rotated

R_0^1 as defined earlier, represents a right-handed passive transformation, or a left-handed active transformation

If we stick with right-handed rotations, then R_0^1 represents a right-handed passive rotation, and $R_0^{1\top}$ represents a right-handed active rotation

Inertial Frame and Vehicle Frame



- Vehicle frame has same orientation as inertial frame
- Vehicle frame is fixed at cm of aircraft
- Inertial and vehicle frames are referred to as NED frames
- $N \rightarrow x, E \rightarrow y, D \rightarrow z$

Euler Angles

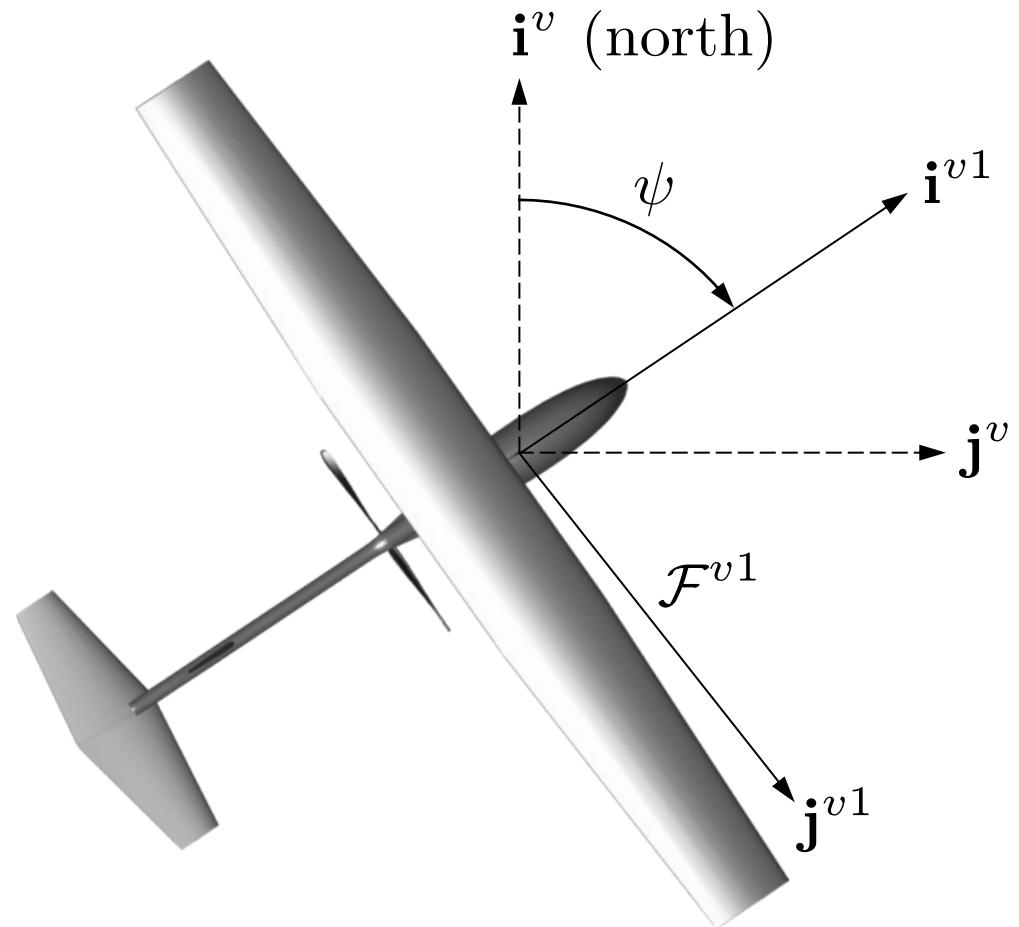
- Need way to describe attitude of aircraft
- Common approach: Euler angles

ψ : heading (yaw)

θ : elevation (pitch)

- Pro: Intuitive ϕ : bank (roll)
- Con: Mathematical singularity
 - Quaternions are alternative for overcoming singularity

Vehicle-1 Frame



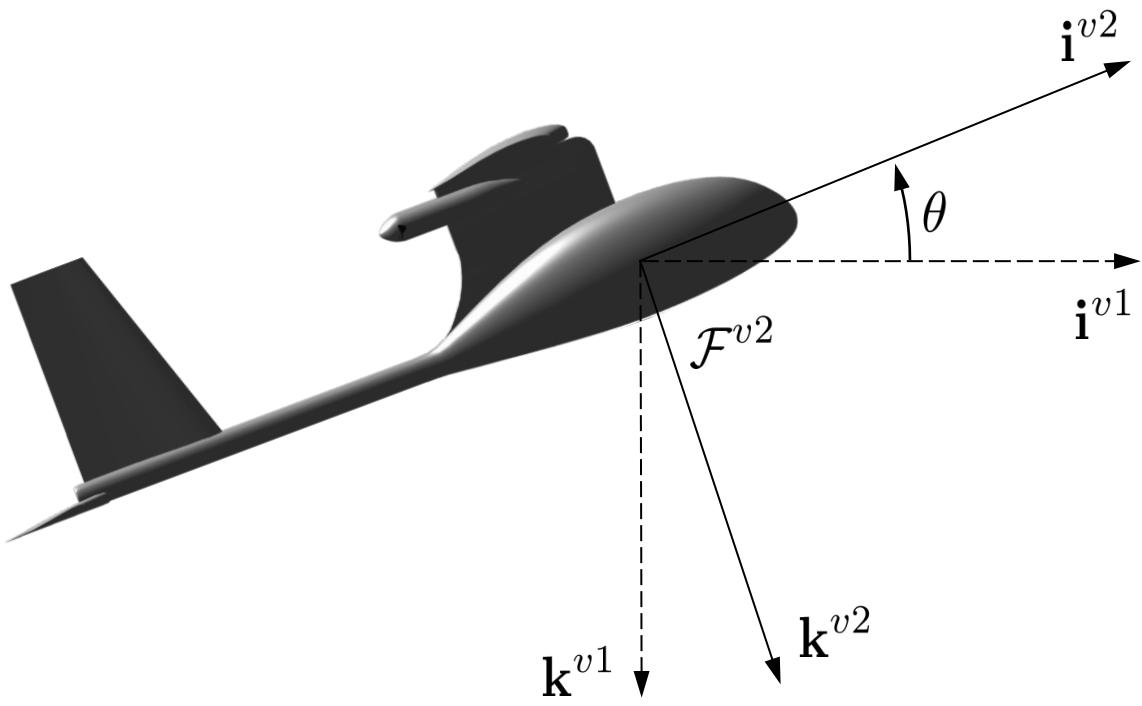
$$R_v^{v1}(\psi) = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives

$$\mathbf{p}^{v1} = R_v^{v1}(\psi)\mathbf{p}^v$$

where ψ is the heading

Vehicle-2 Frame



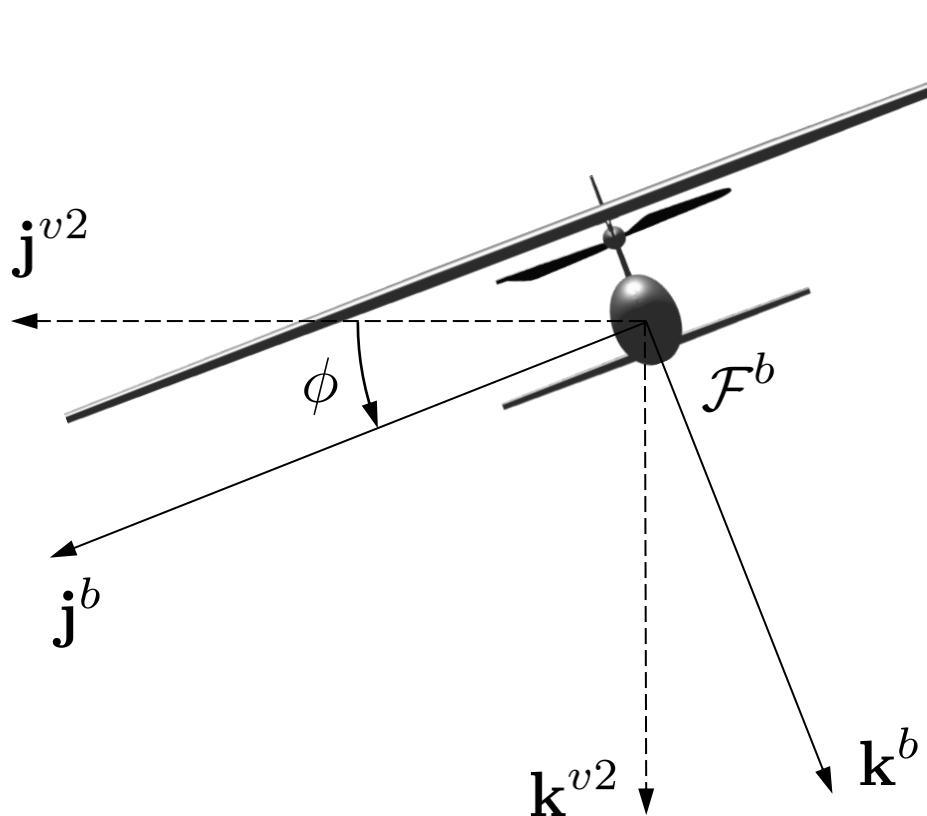
$$R_{v1}^{v2}(\theta) = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$

gives

$$\mathbf{p}^{v2} = R_{v1}^{v2}(\theta)\mathbf{p}^{v1}$$

where θ is the pitch angle

Body Frame



$$R_{v2}^b(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix}$$

gives

$$\mathbf{p}^b = R_{v2}^b(\phi)\mathbf{p}^{v2}$$

where ϕ is the roll (bank) angle

Inertial Frame to Body Frame Transformation

Define

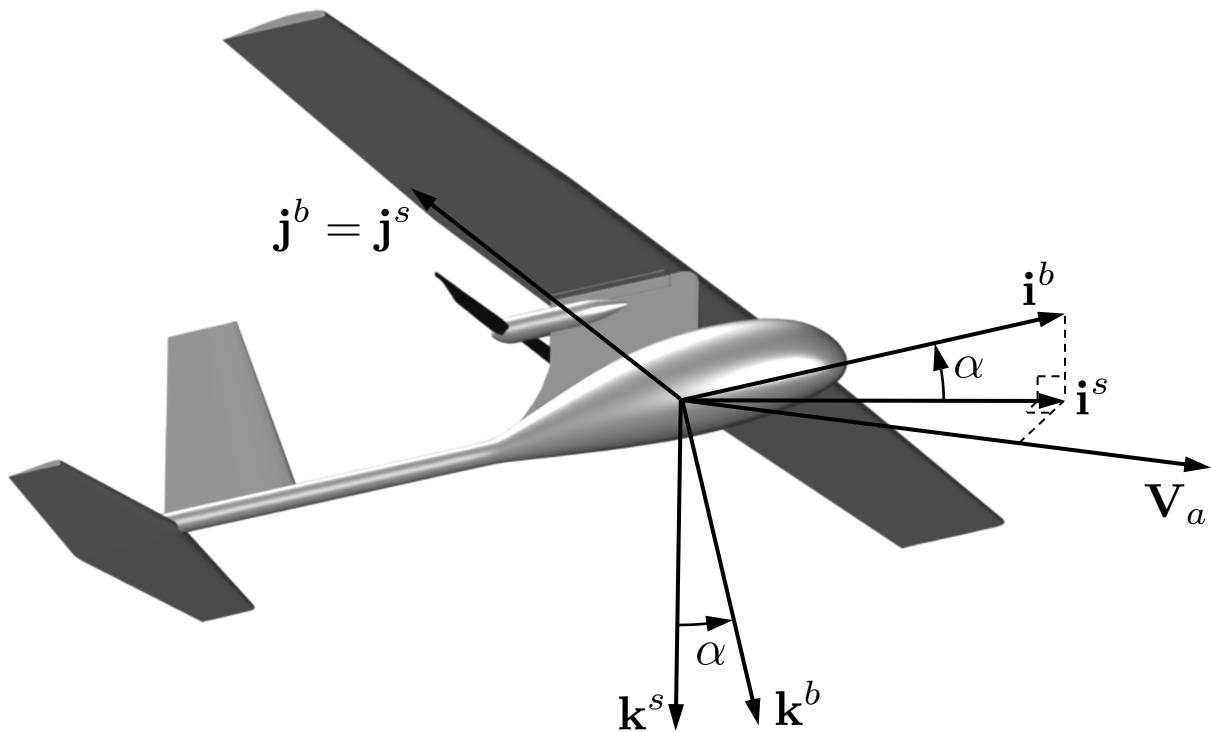
$$\begin{aligned} R_v^b(\phi, \theta, \psi) &= R_{v2}^b(\phi) R_{v1}^{v2}(\theta) R_v^{v1}(\psi) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix} \end{aligned}$$

to give

$$\mathbf{p}^b = R_v^b(\Theta) \mathbf{p}^v$$

where, $\Theta = (\phi, \theta, \psi)^\top$

Stability Frame



Right-handed rotation

$$R_s^b(\alpha) = \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

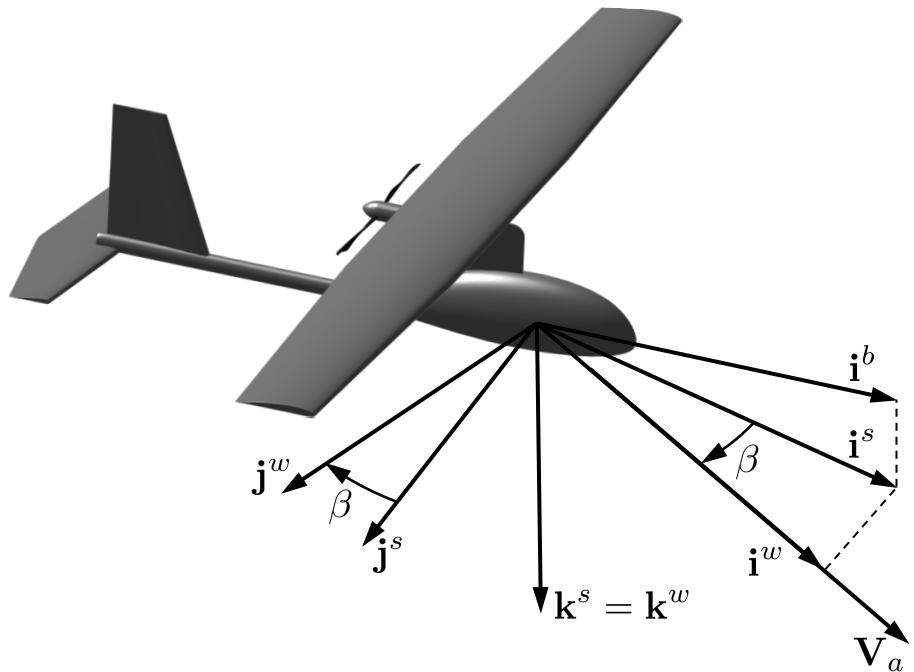
$$\mathbf{p}^b = R_s^b(\alpha)\mathbf{p}^s$$

$$\mathbf{p}^s = R_b^s(\alpha)\mathbf{p}^b = R_s^b(\alpha)^\top \mathbf{p}^b$$

Stability frame helps us rigorously define angle of attack and is useful for analyzing stability of aircraft

Angle of attack defined as a positive RH rotation from stability to body frame

Wind Frame



$$R_s^w(\beta) = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

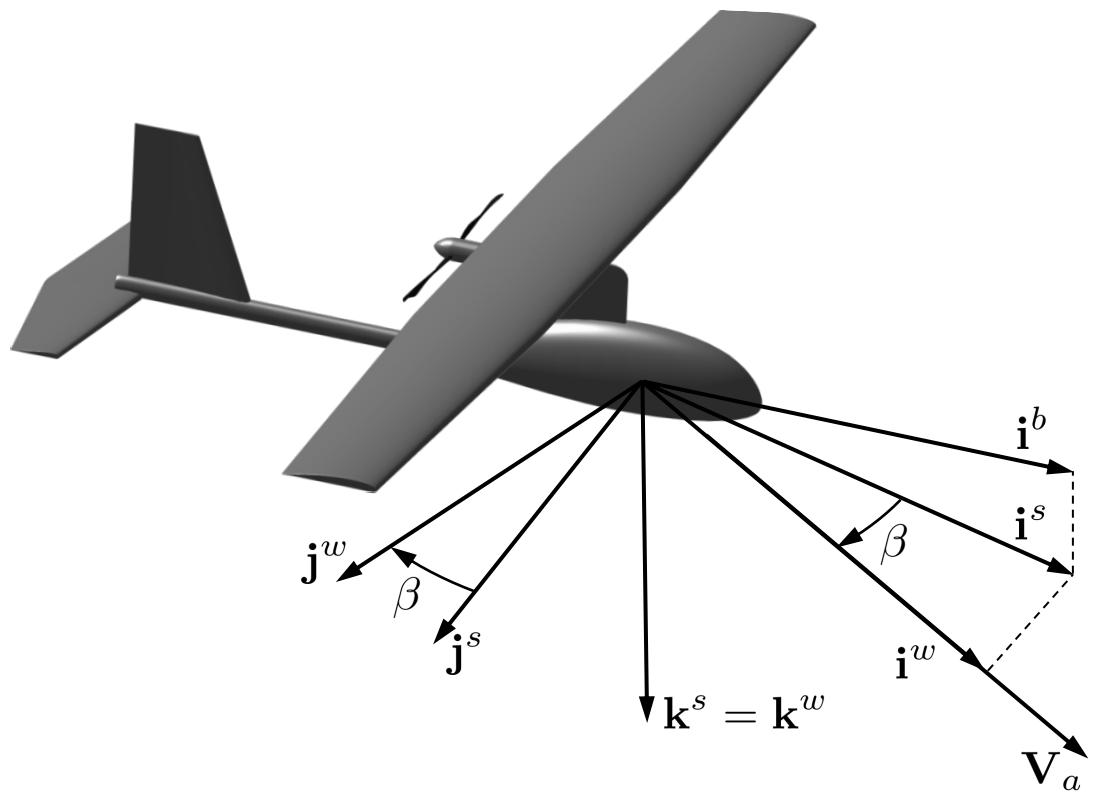
$$\mathbf{p}^w = R_s^w(\beta)\mathbf{p}^s$$

$$R_b^w(\alpha, \beta) = R_s^w(\beta)R_b^s(\alpha)$$

$$= \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos \beta \cos \alpha & \sin \beta & \cos \beta \sin \alpha \\ -\sin \beta \cos \alpha & \cos \beta & -\sin \beta \sin \alpha \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

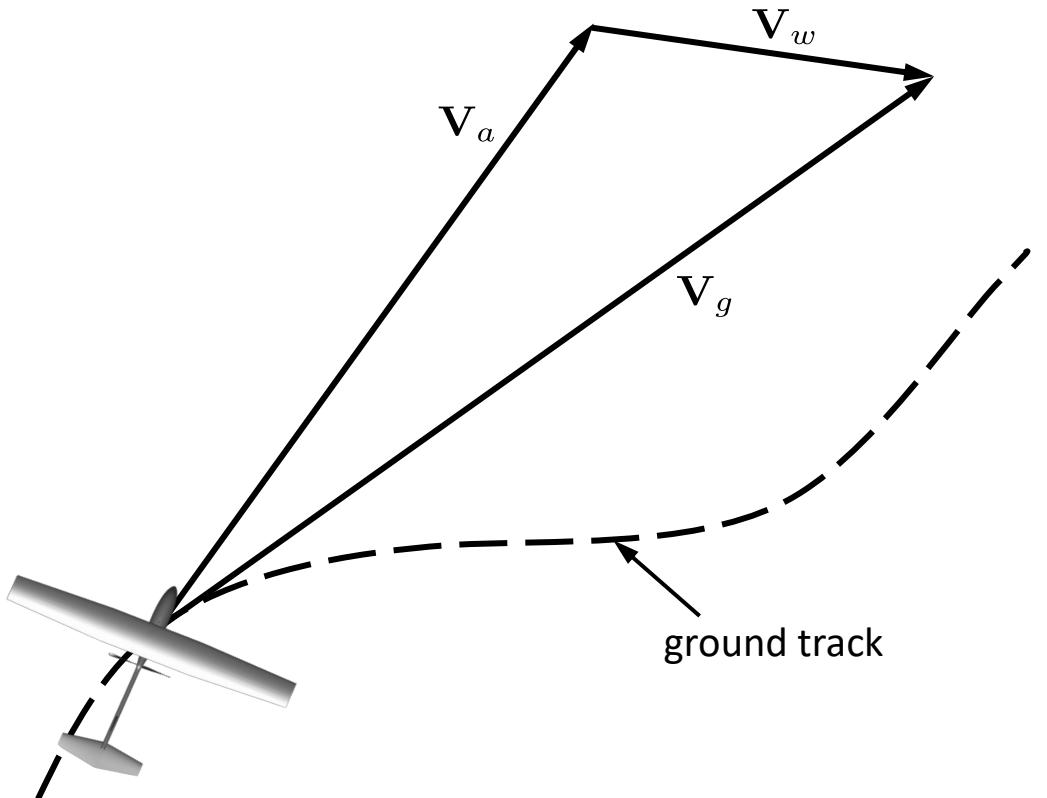
Wind Frame (continued)



- Wind frame helps us rigorously define side-slip angle
- Forces and moments are most naturally defined in wind frame
- Side-slip angle is nominally zero for tailed aircraft

$$R_w^b(\alpha, \beta) = (R_b^w)^\top(\alpha, \beta) = \begin{pmatrix} \cos \beta \cos \alpha & -\sin \beta \cos \alpha & -\sin \alpha \\ \sin \beta & \cos \beta & 0 \\ \cos \beta \sin \alpha & -\sin \beta \sin \alpha & \cos \alpha \end{pmatrix}$$

Airspeed, Wind Speed, Ground Speed



a/c wrt to surrounding air
expressed in body frame

$$\mathbf{V}_a = \mathbf{V}_g - \mathbf{V}_w$$

$$\mathbf{V}_g^b = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

a/c wrt to inertial frame
expressed in body frame

$$\mathbf{V}_w^b = \begin{pmatrix} u_w \\ v_w \\ w_w \end{pmatrix} = R_v^b(\phi, \theta, \psi) \begin{pmatrix} w_n \\ w_e \\ w_d \end{pmatrix}$$

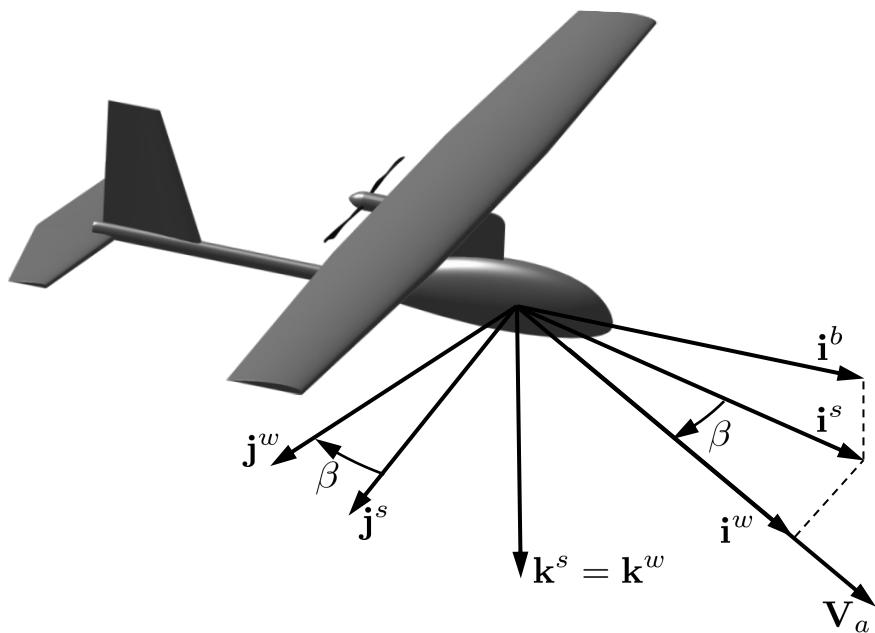
$$\mathbf{V}_a^w = \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{V}_a^b = \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = \begin{pmatrix} u - u_w \\ v - v_w \\ w - w_w \end{pmatrix}$$

wind wrt to inertial frame
expressed in body frame

Airspeed, Angle of Attack, Sideslip Angle

$$\begin{aligned}\mathbf{V}_a^b &= \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = R_w^b \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos \beta \cos \alpha & -\sin \beta \cos \alpha & -\sin \alpha \\ \sin \beta & \cos \beta & -\sin \beta \sin \alpha \\ \cos \beta \sin \alpha & 0 & \cos \alpha \end{pmatrix} \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix} \\ &\quad \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = V_a \begin{pmatrix} \cos \alpha \cos \beta \\ \sin \beta \\ \sin \alpha \cos \beta \end{pmatrix}\end{aligned}$$

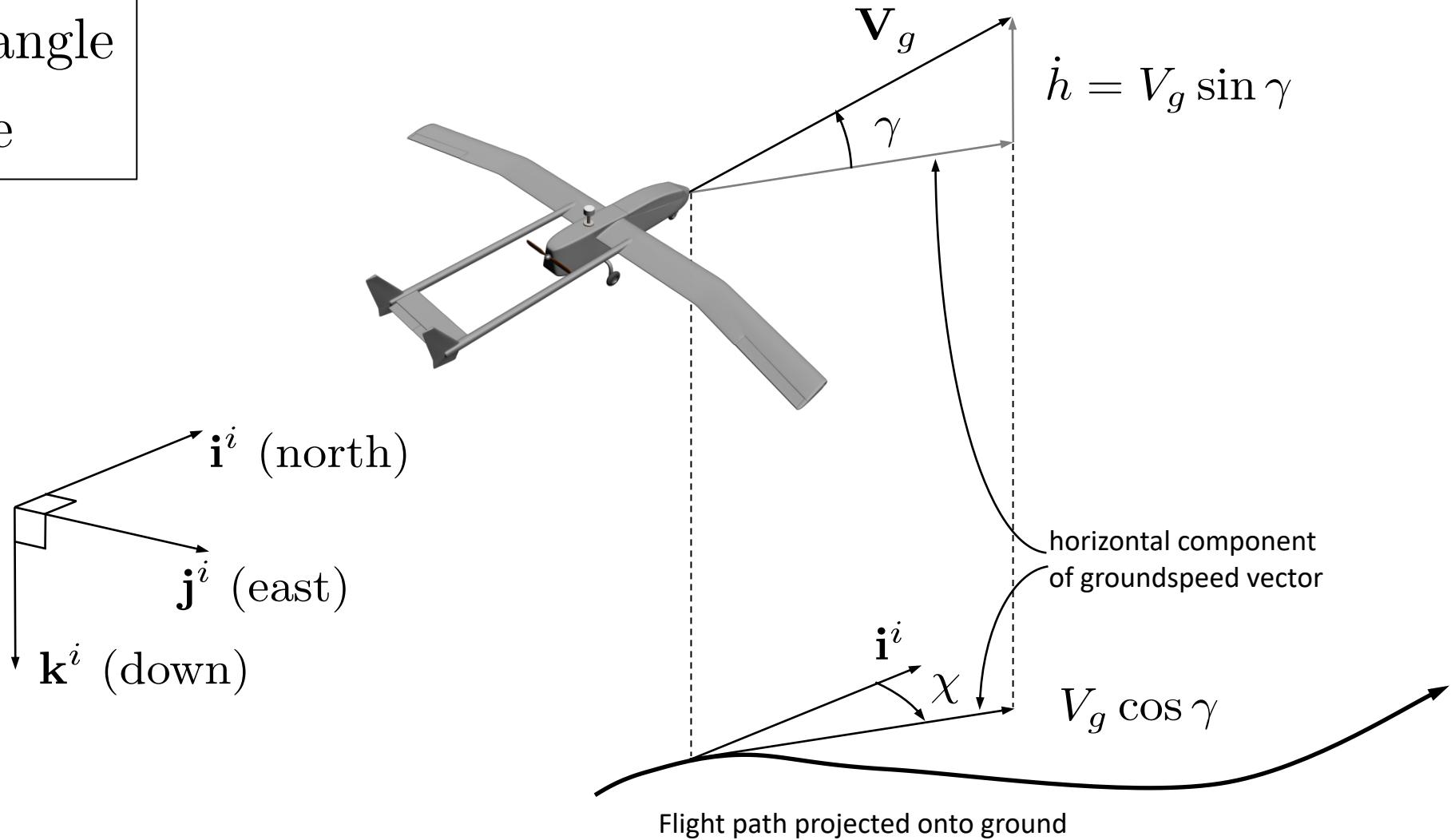


OR

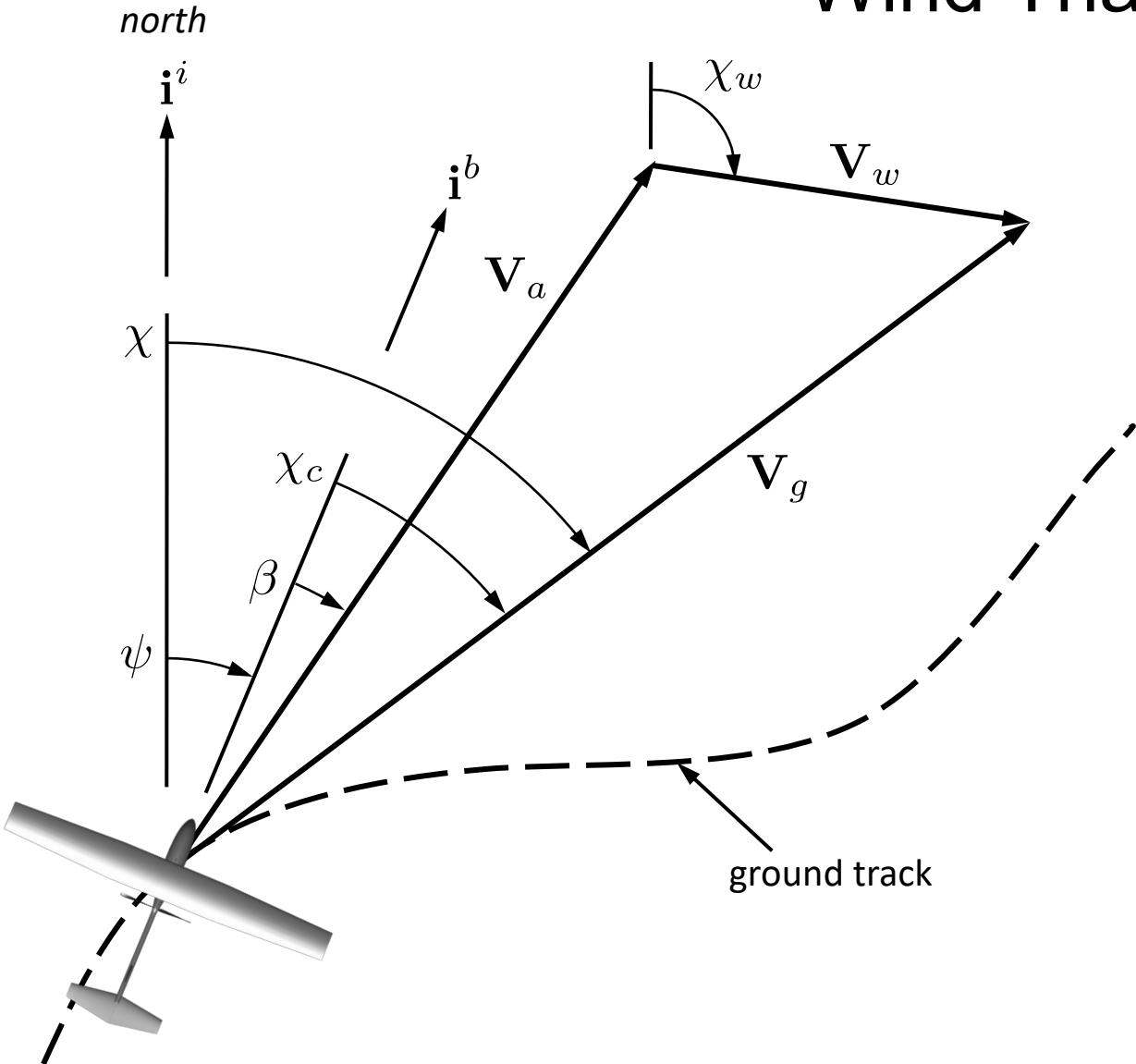
$$\begin{aligned}V_a &= \sqrt{u_r^2 + v_r^2 + w_r^2} \\ \alpha &= \tan^{-1} \left(\frac{w_r}{u_r} \right) \\ \beta &= \tan^{-1} \left(\frac{v_r}{\sqrt{u_r^2 + w_r^2}} \right)\end{aligned}$$

Course and Flight Path Angles

γ : flight path angle
 χ : course angle



Wind Triangle



χ : course angle

ψ : heading angle

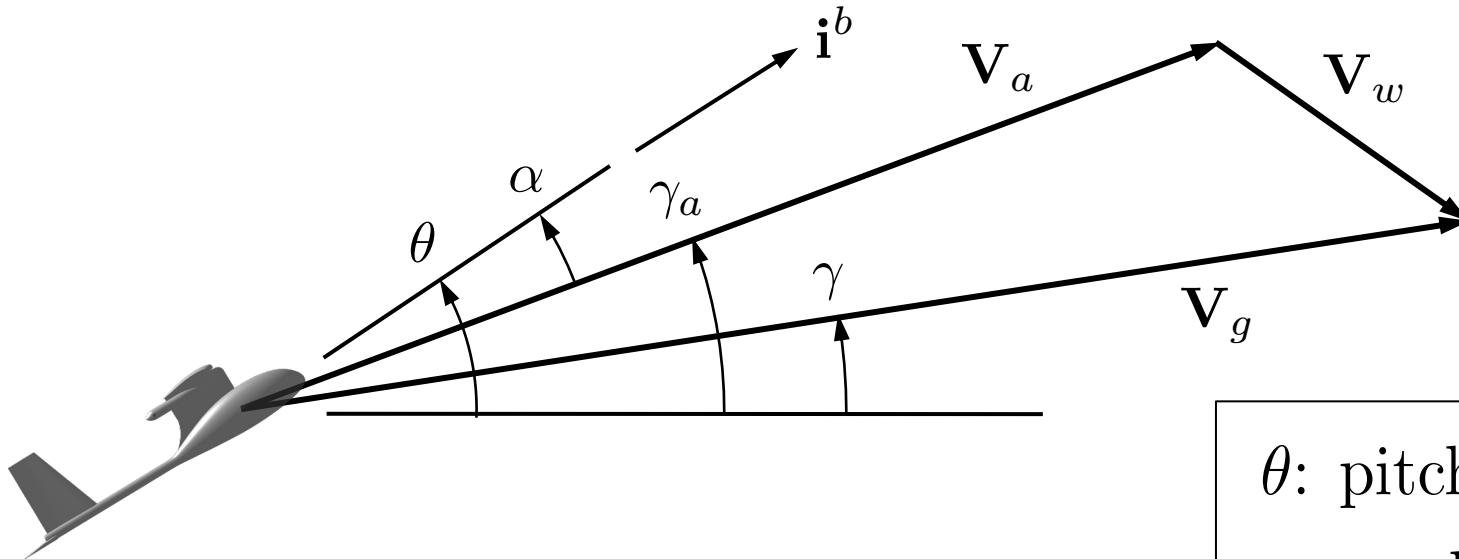
β : side slip angle

χ_c : crab angle

χ_w : wind direction

$$\chi_c \stackrel{\triangle}{=} \chi - \psi$$

Wind Triangle



θ : pitch angle

α : angle of attack

γ : flight path angle

γ_a : air-mass-relative flight path angle

$$\gamma_a = \theta - \alpha$$

When wind speed and sideslip are zero...

If both the windspeed and the sideslip angles are zero, i.e.,

$$V_w = 0$$

$$\beta = 0$$

then we have the following simplifications

$V_a = V_g$ Airspeed equals groundspeed

$u = u_r$ Velocity equals velocity relative to the air mass

$v = v_r$

$w = w_r$

$\psi = \chi$ Heading equals course

$\gamma = \gamma_a$ Flight path angle equals air-mass-referenced flight path angle

Differentiation of a Vector

$$\mathbf{p} = p_x \mathbf{i}^b + p_y \mathbf{j}^b + p_z \mathbf{k}^b$$

$$\frac{d}{dt_i} \mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b + p_x \frac{d}{dt_i} \mathbf{i}^b + p_y \frac{d}{dt_i} \mathbf{j}^b + p_z \frac{d}{dt_i} \mathbf{k}^b$$

$$\frac{d}{dt_b} \mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b$$

$$\dot{\mathbf{i}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{i}^b$$

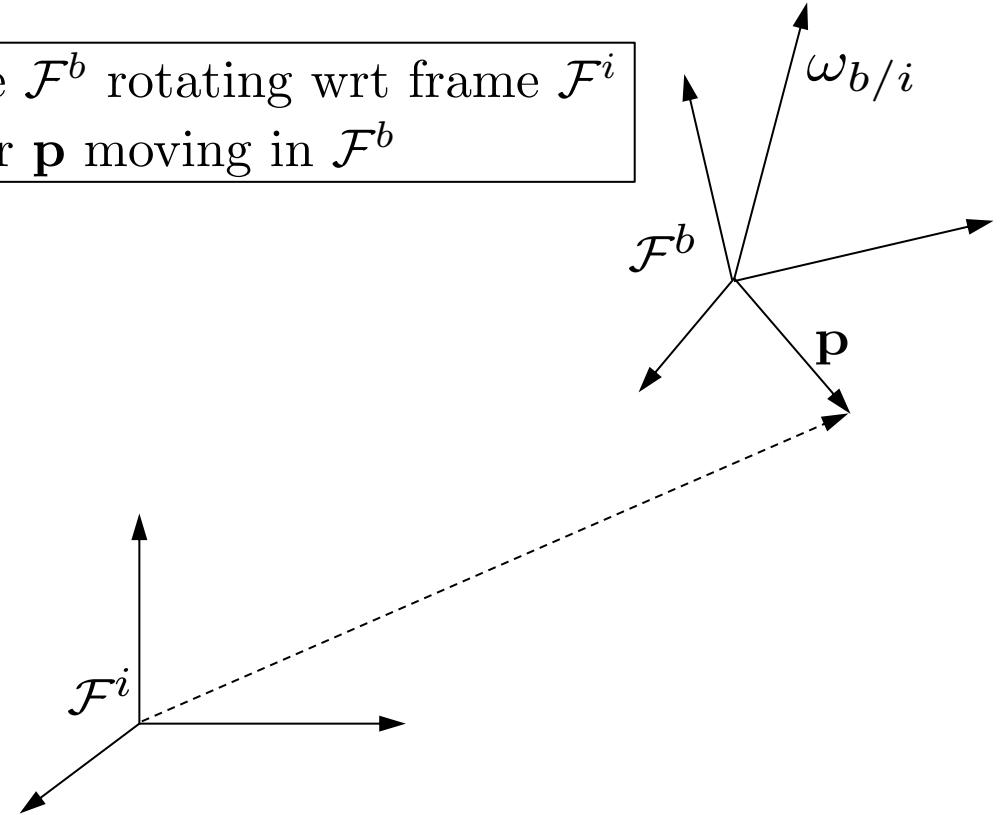
$$\dot{\mathbf{j}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{j}^b$$

$$\dot{\mathbf{k}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{k}^b$$

$$\begin{aligned} p_x \dot{\mathbf{i}}^b + p_y \dot{\mathbf{j}}^b + p_z \dot{\mathbf{k}}^b &= p_x (\boldsymbol{\omega}_{b/i} \times \mathbf{i}^b) + p_y (\boldsymbol{\omega}_{b/i} \times \mathbf{j}^b) + p_z (\boldsymbol{\omega}_{b/i} \times \mathbf{k}^b) \\ &= \boldsymbol{\omega}_{b/i} \times \mathbf{p} \end{aligned}$$

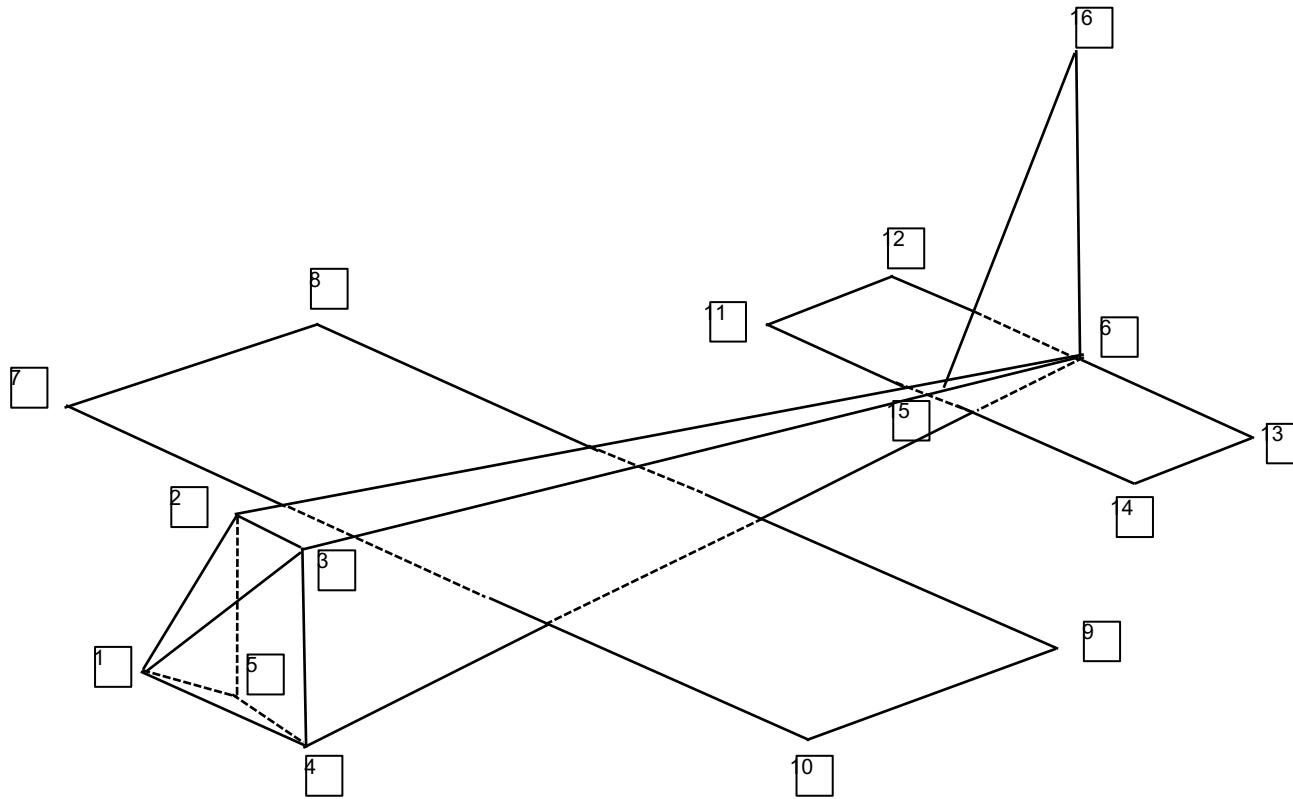
$$\frac{d}{dt_i} \mathbf{p} = \frac{d}{dt_b} \mathbf{p} + \boldsymbol{\omega}_{b/i} \times \mathbf{p}$$

Frame \mathcal{F}^b rotating wrt frame \mathcal{F}^i
Vector \mathbf{p} moving in \mathcal{F}^b

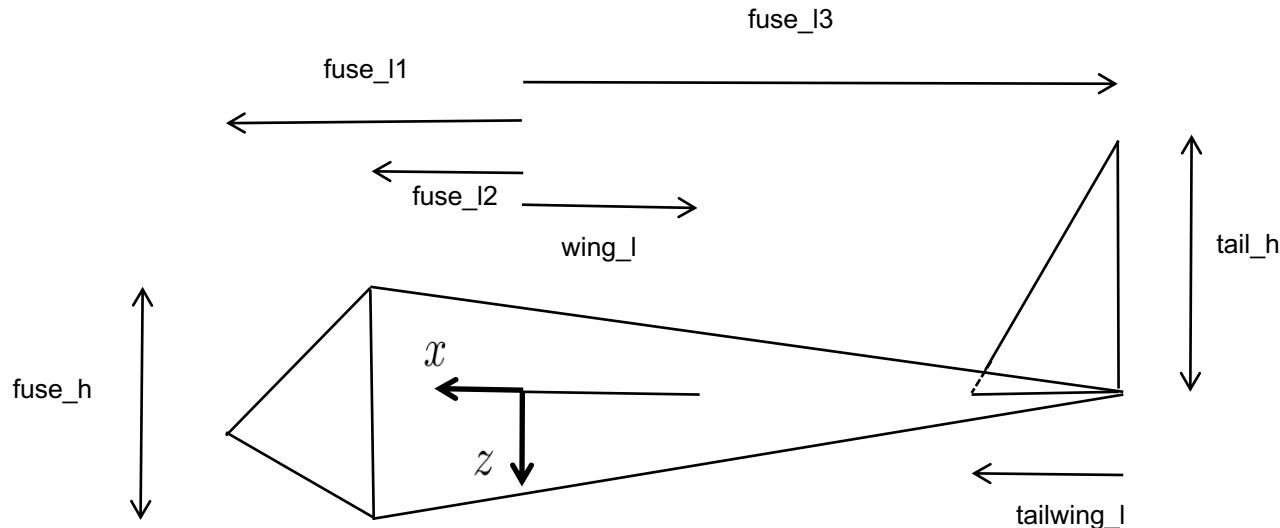


Note: Frame \mathcal{F}^b does not translate
wrt frame \mathcal{F}^i

Project Aircraft



Project Aircraft



Project Aircraft

