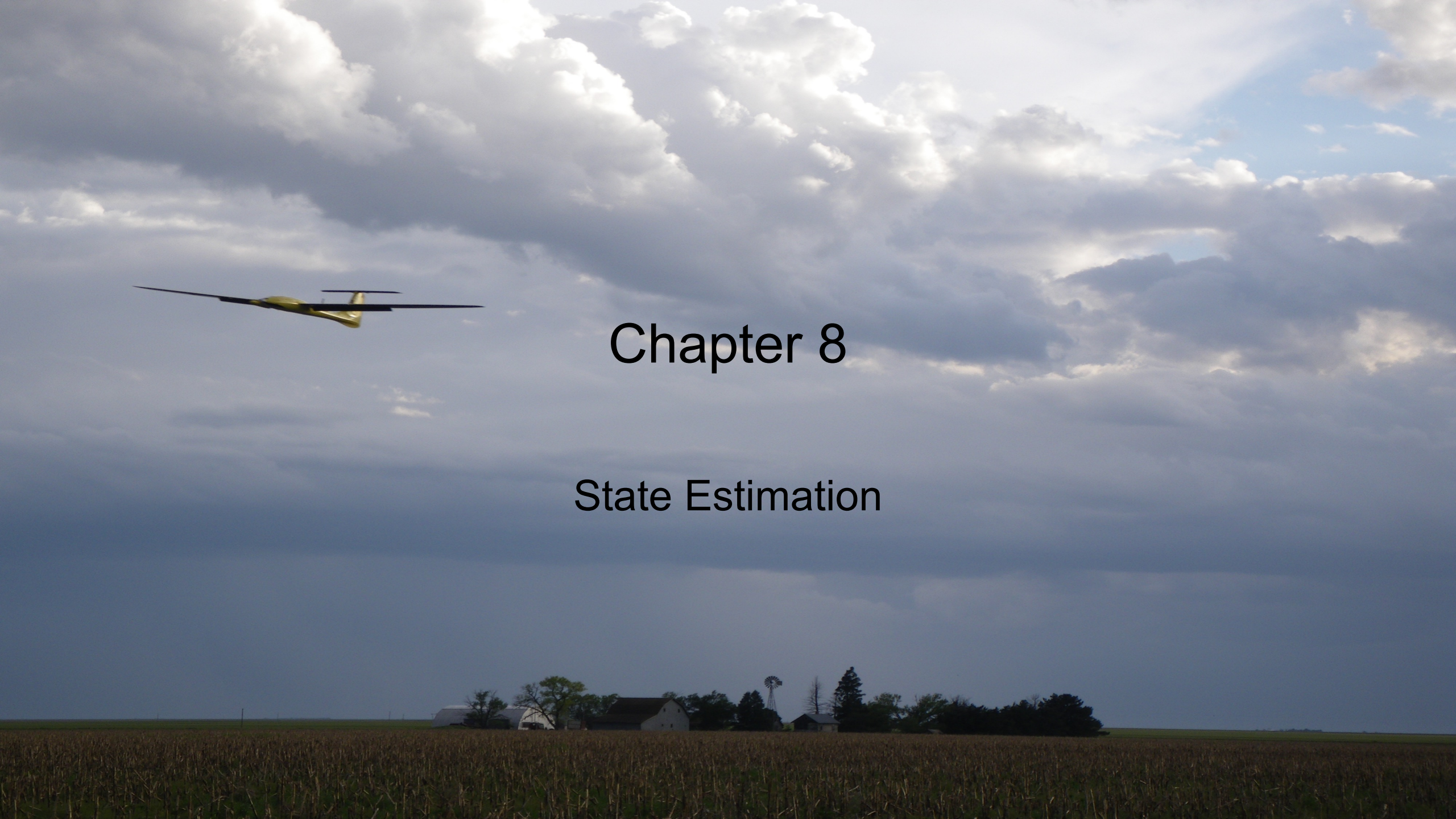
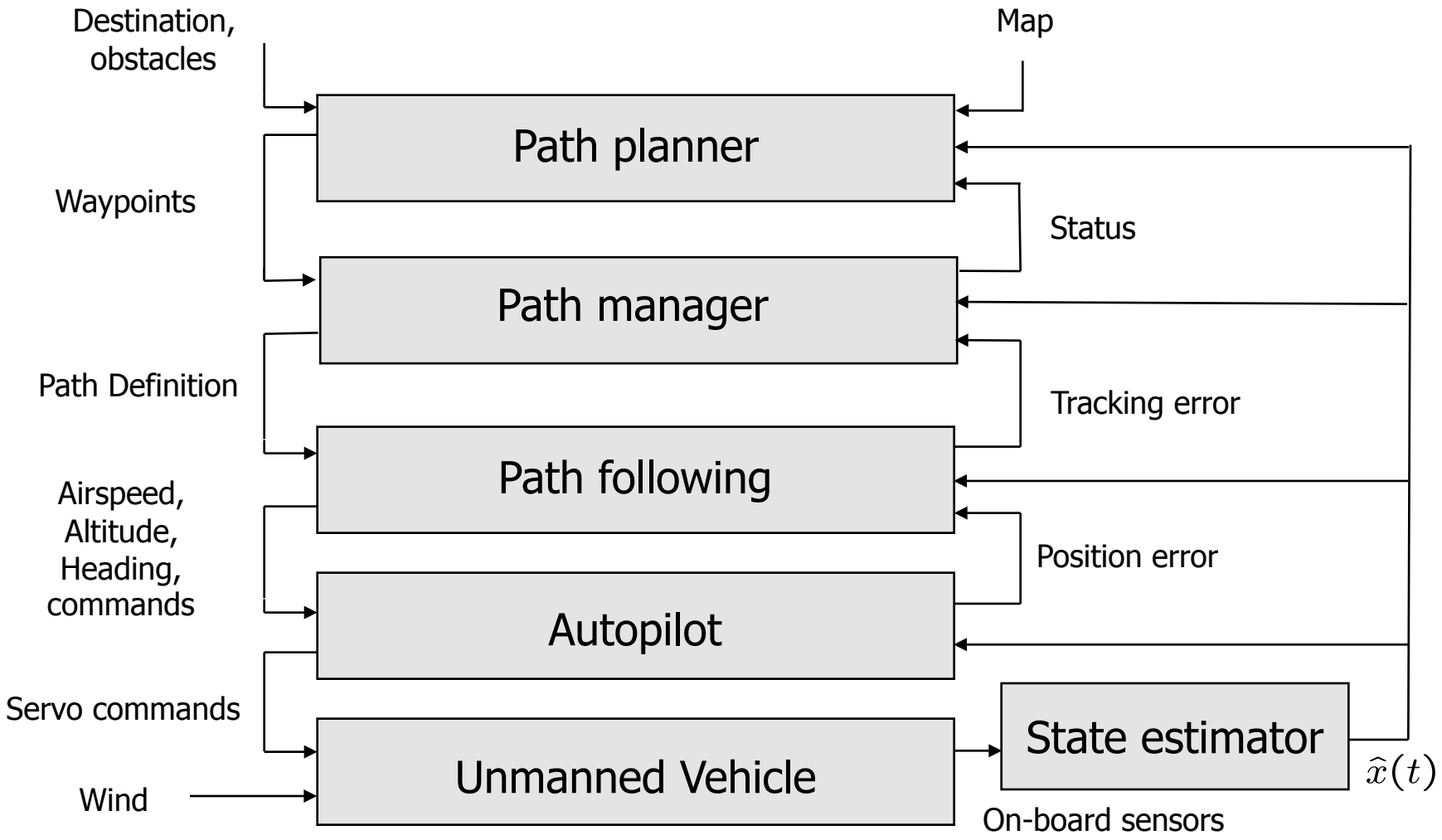


Chapter 8

State Estimation

Architecture

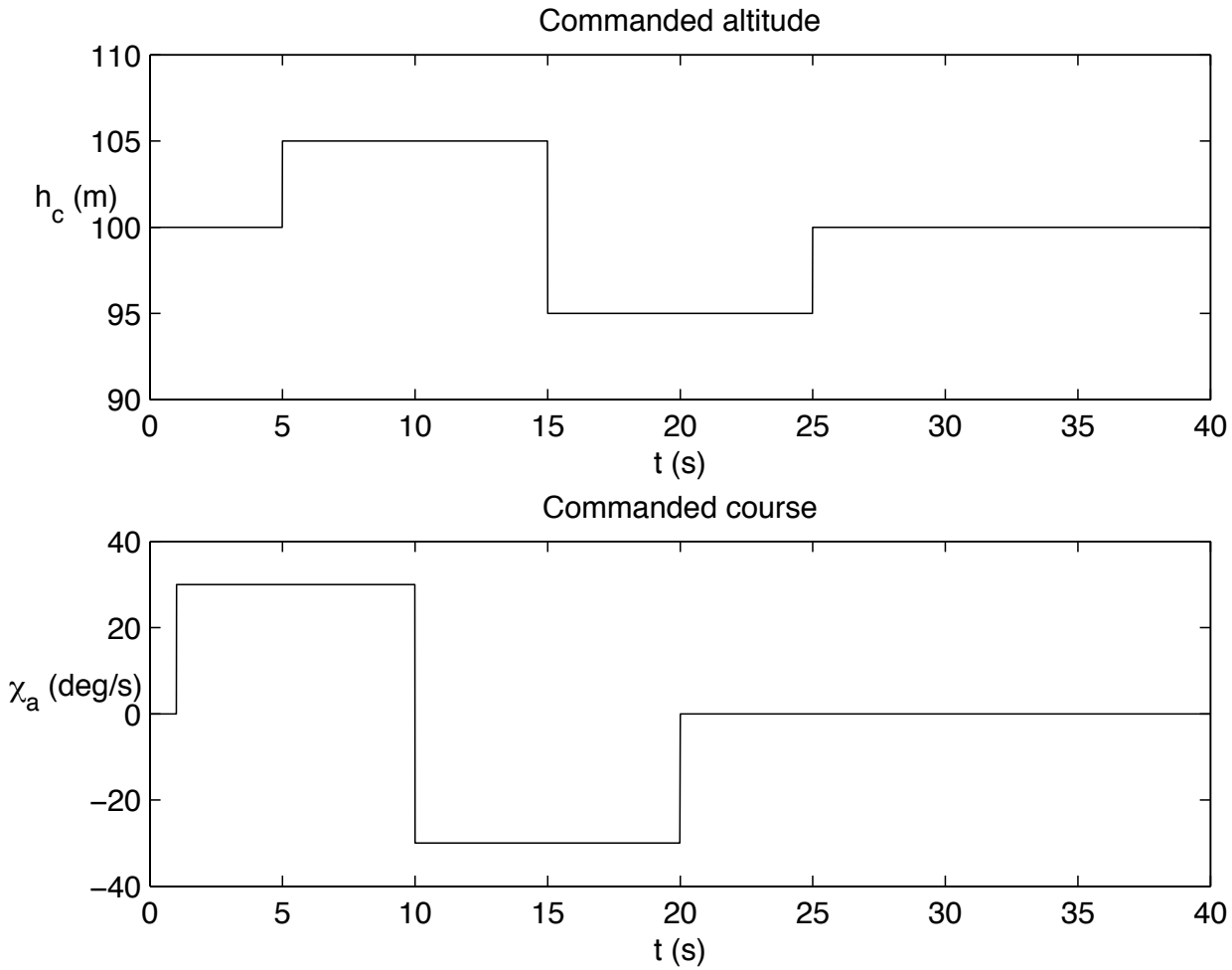


Why estimate states?

- State information needed to control aircraft
- We measure many things
  - accelerations, angular rates, pressure altitude, dynamic pressure (airspeed), magnetic heading (sometimes), GPS position
- Don't have direct measurements of everything we need

State	Measured directly or estimated?
p_n	Measured (GPS) and smoothed
p_e	Measured (GPS) and smoothed
p_d	Measured (absolute pressure, GPS)
u	Estimated with EKF
v	Estimated with EKF
w	Estimated with EKF
ϕ	Estimated with EKF
θ	Estimated with EKF
ψ	Estimated with EKF
p	Measured (rate gyro)
q	Measured (rate gyro)
r	Measured (rate gyro)

Benchmark Maneuver



Inverting Sensor Measurement Model

- Several states can be estimated by inverting sensor measurement model
  - Angular rates, altitude, airspeed
  - *LPF* denotes low pass filter

Mathematical Model

$$y_{\text{gyro,x}} = p + \beta_p + \eta_p$$
$$y_{\text{gyro,y}} = q + \beta_q + \eta_q$$
$$y_{\text{gyro,z}} = r + \beta_r + \eta_r$$
$$y_{\text{abs pres}} = \rho g h + \beta_{\text{abs}} + \eta_{\text{abs}}$$
$$y_{\text{diff pres}} = \frac{1}{2}\rho V_a^2 + \beta_{\text{diff}} + \eta_{\text{diff}}$$

State Estimate

$$\hat{p} = LPF(y_{\text{gyro,x}})$$
$$\hat{q} = LPF(y_{\text{gyro,y}})$$
$$\hat{r} = LPF(y_{\text{gyro,z}})$$
$$\hat{h} = \frac{LPF(y_{\text{static pres}})}{\rho g}$$
$$\hat{V}_a = \sqrt{\frac{2}{\rho} LPF(y_{\text{diff pres}})}$$

The alpha-filter

Suppose that your sensor gives noisy data as shown on the right.

The objective is to process this data to smooth out the noise.

The standard method is to use an alpha-filter:

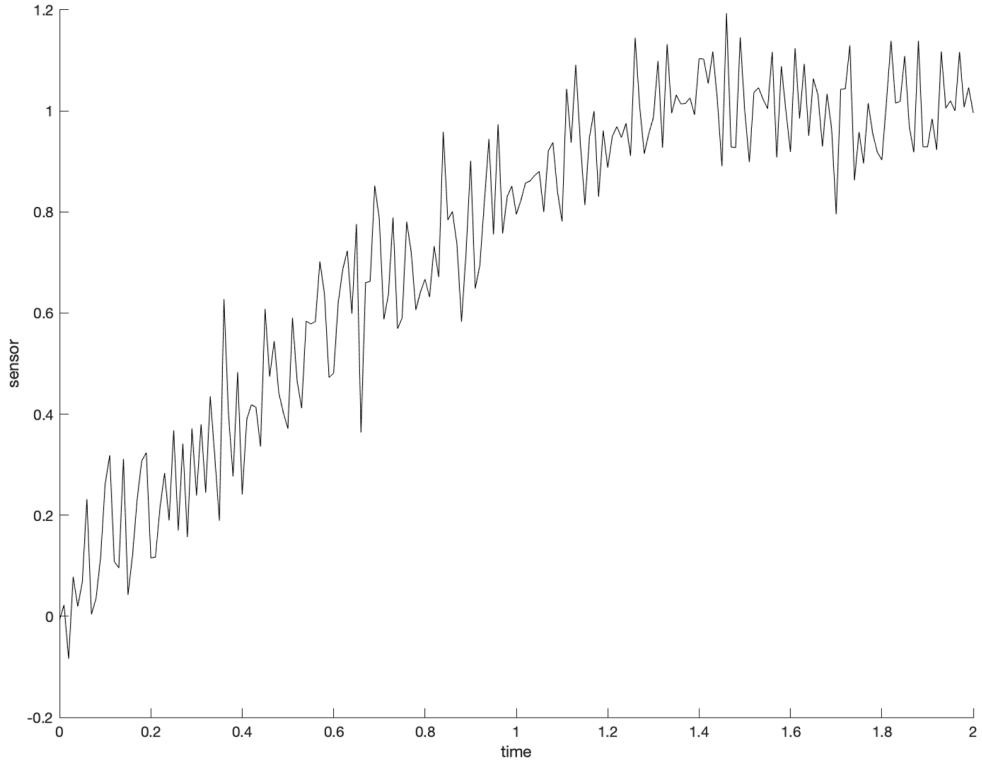
$$z_k = \alpha z_{k-1} + (1-\alpha) y_k$$

where

$0 \leq \alpha \leq 1$

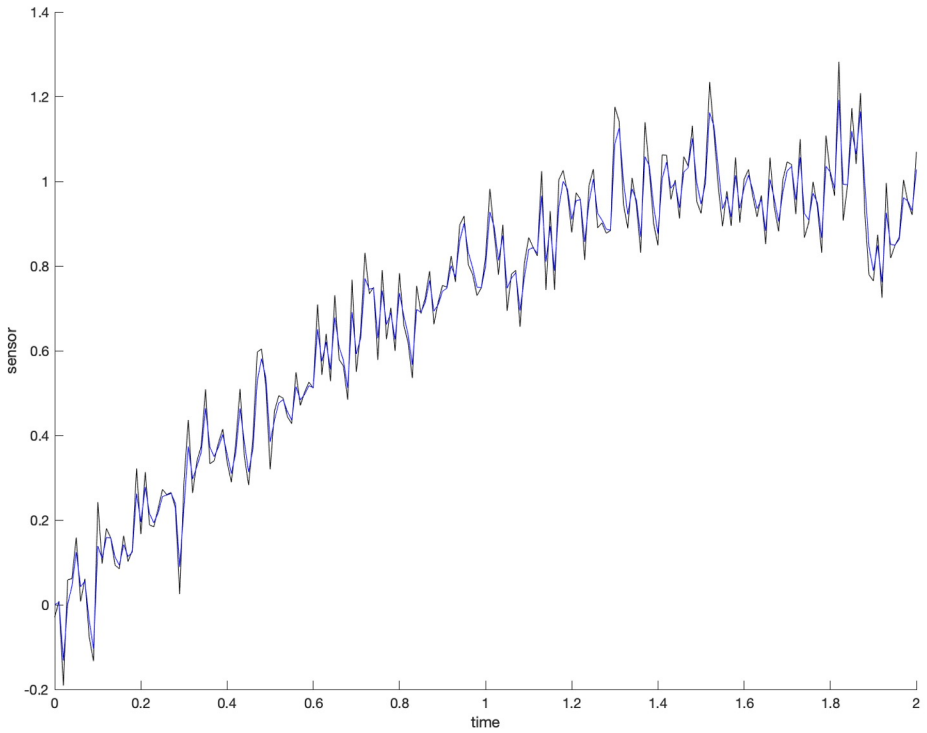
$y_k \equiv$ sensor output at time k

$z_k \equiv$ smoothed sensor output at time k

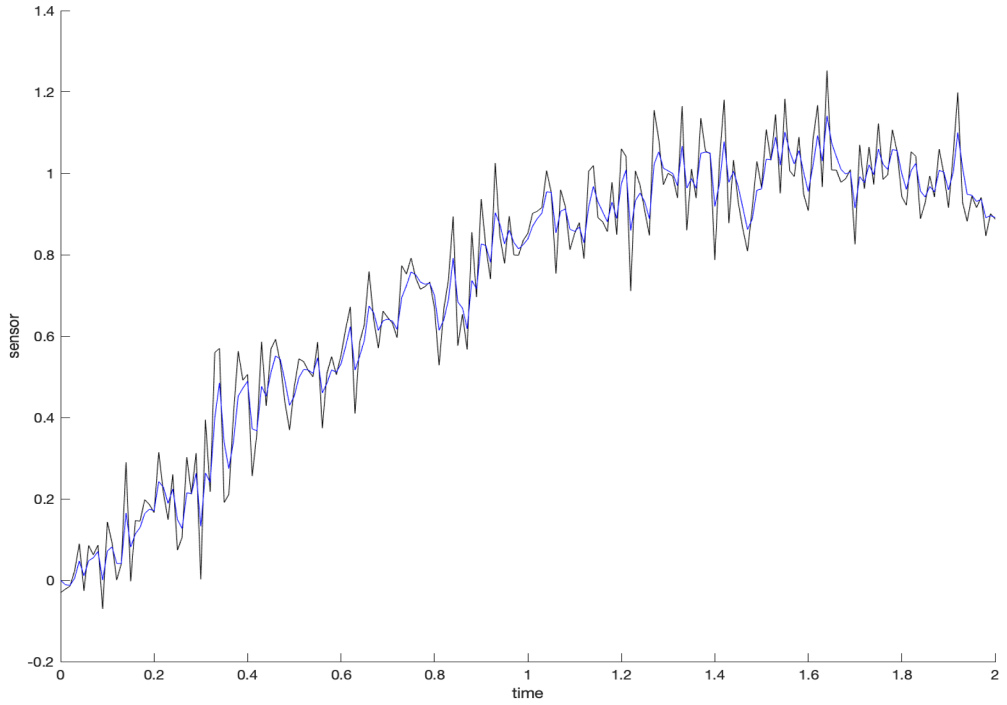


The alpha-filter

$$z_k = \alpha z_{k-1} + (1-\alpha) y_k$$



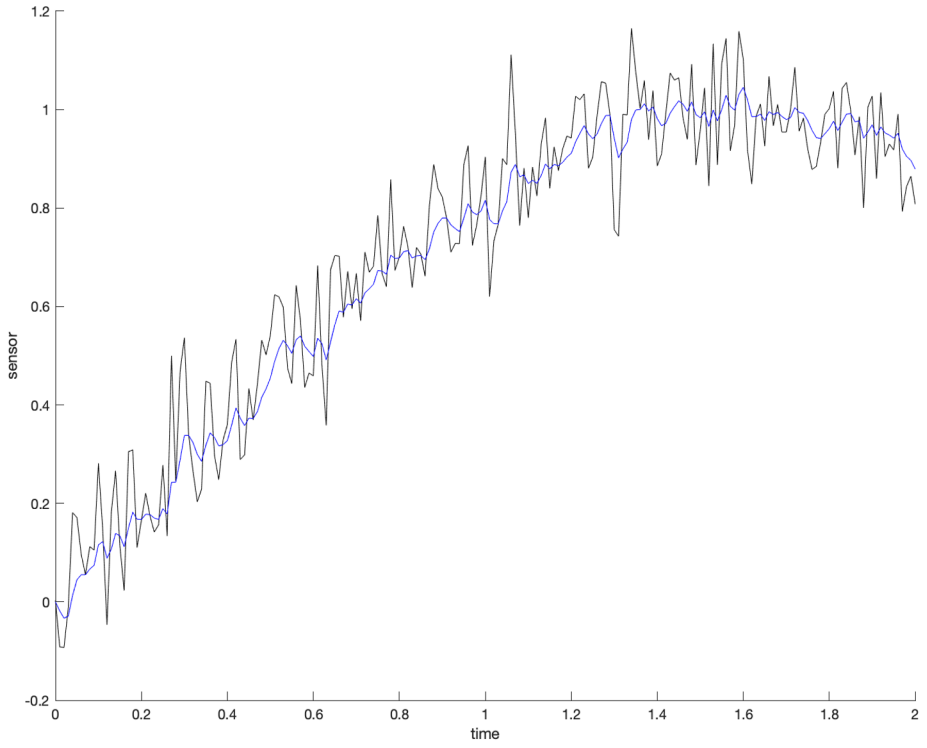
$\alpha = 0.3$



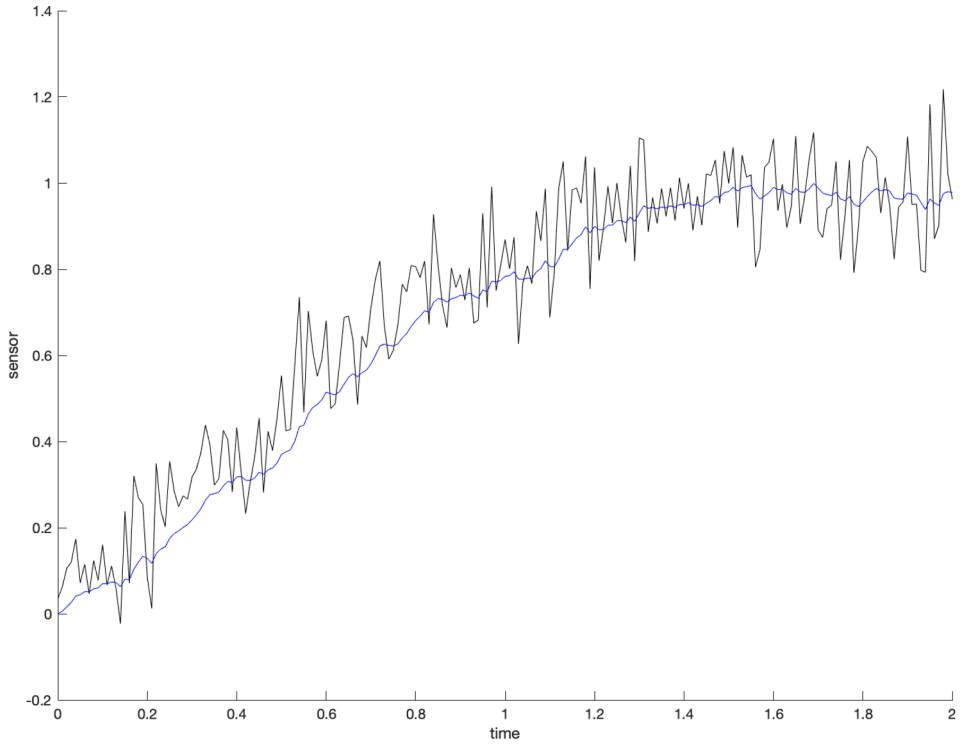
$\alpha = 0.5$

The alpha-filter

$$z_k = \alpha z_{k-1} + (1-\alpha) y_k$$



$\alpha = 0.8$



$\alpha = 0.9$

The alpha-filter: a deeper look

The traditional low-pass filter is given by

$$Z(s) = \frac{a}{s+a} Y(s),$$

where frequencies above a radians/s are "filtered."

The associated differential equation is

$$\dot{z} = -az + ay.$$

Solving the differential equation gives

$$z(t) = e^{-a(t-t_0)} z(t_0) + \int_{t_0}^{t} e^{-a(t-\tau)} a y(\tau)\, d\tau$$

The alpha-filter: a deeper look

Letting $t = t_k$, and $t_0 = t_{k-1}$, and $t_k = t_{k-1} + T_s$, and $\sigma = \tau - t_{k-1}$ gives

$$\begin{aligned} z(t_k) &= e^{-a(t_k - t_{k-1})} z(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{-a(t_k-\tau)} a y(\tau)\, d\tau \\ &= e^{-aT_s} z(t_{k-1}) + a \int_0^{T_s} e^{-a(T_s-\sigma)} y(\sigma + t_{k-1})\, d\sigma \end{aligned}$$

Define $z_k \equiv z(t_k)$, and approximating $y(t)$ as constant between samples gives

$$\begin{aligned} z_k &= e^{-aT_s} z_{k-1} + a \int_0^{T_s} e^{-a(T_s-\sigma)}\, d\sigma\ y_k \\ &= e^{-aT_s} z_{k-1} + (1 - e^{-aT_s})\ y_k \\ &= \alpha z_{k-1} + (1-\alpha) y_k, \end{aligned}$$

where $0 \leq e^{-aT_s} \equiv \alpha \leq 1$.

Therefore the α-filter is just a sampled low-pass filter.

The alpha-filter: Python Implementation

```
class AlphaFilter:
    # alpha filter implements a simple low pass filter
    # y[k] = alpha * y[k-1] + (1-alpha) * u[k]
    def __init__(self, alpha=0.5, y0=0.0):
        self.alpha = alpha  # filter parameter
        self.y = y0  # initial condition

    def update(self, u):
        self.y = self.alpha * self.y + (1-self.alpha) * u
        return self.y

# instantiation
self.lpf_gyro_x = AlphaFilter(alpha=0.7, y0=initial_measurements.gyro_x)
self.lpf_abs = AlphaFilter(alpha=0.9, y0=initial_measurements.abs_pressure)
self.lpf_diff = AlphaFilter(alpha=0.7, y0=initial_measurements.diff_pressure)

# update
# estimates for p, q, r are low pass filter of gyro minus bias estimate
self.estimated_state.p = self.lpf_gyro_x.update(measurement.gyro_x) \
                         - self.estimated_state.bx
abs_pressure = self.lpf_abs.update(measurement.abs_pressure)
diff_pressure = self.lpf_diff.update(measurement.diff_pressure)
```

Inverting Sensor Measurement Model

- Several states can be estimated by inverting sensor measurement model
  - Angular rates, altitude, airspeed
  - *LPF* denotes low pass filter

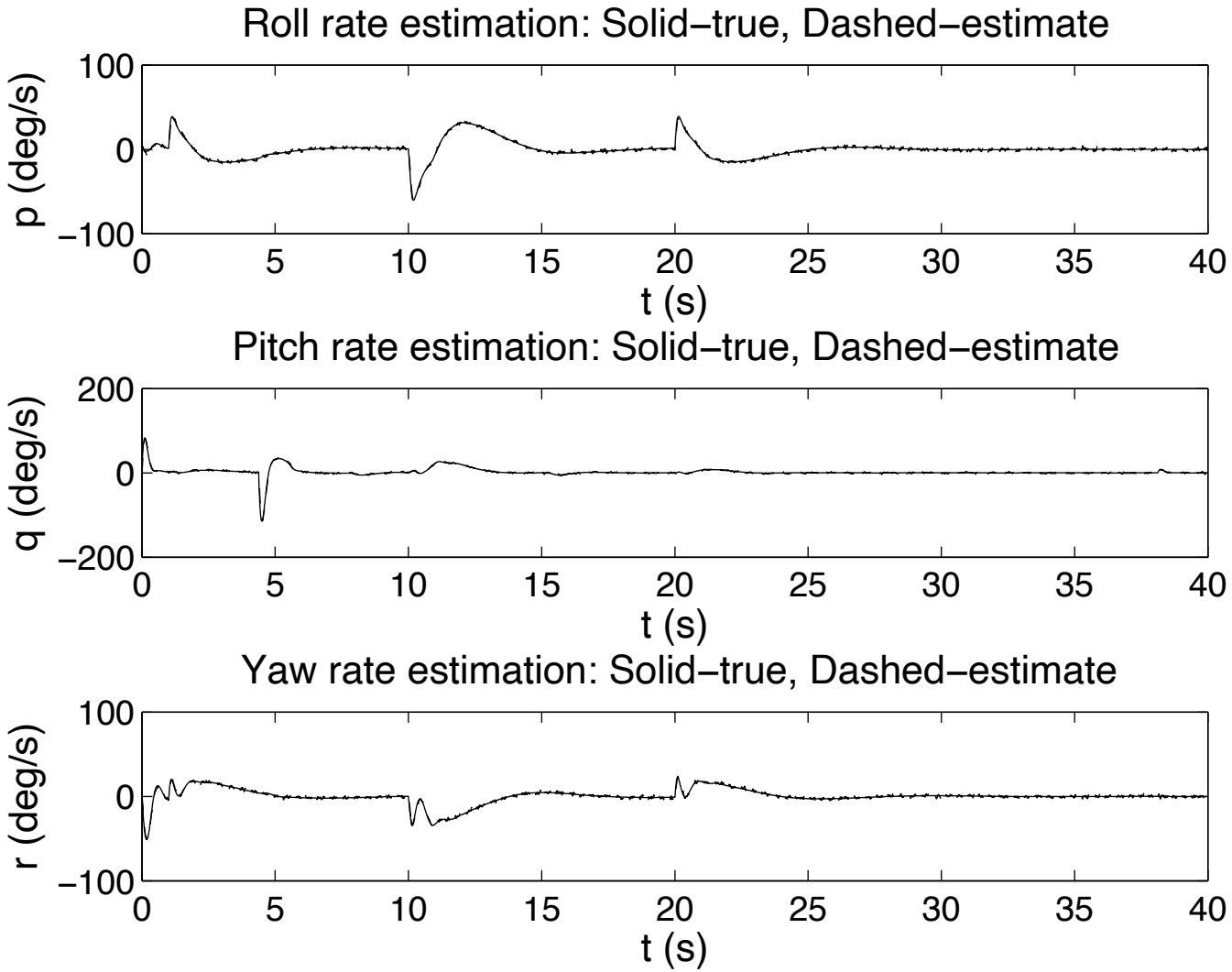
Mathematical Model

$$y_{\text{gyro,x}} = p + \beta_p + \eta_p$$
$$y_{\text{gyro,y}} = q + \beta_q + \eta_q$$
$$y_{\text{gyro,z}} = r + \beta_r + \eta_r$$
$$y_{\text{abs pres}} = \rho g h + \beta_{\text{abs}} + \eta_{\text{abs}}$$
$$y_{\text{diff pres}} = \frac{1}{2}\rho V_a^2 + \beta_{\text{diff}} + \eta_{\text{diff}}$$

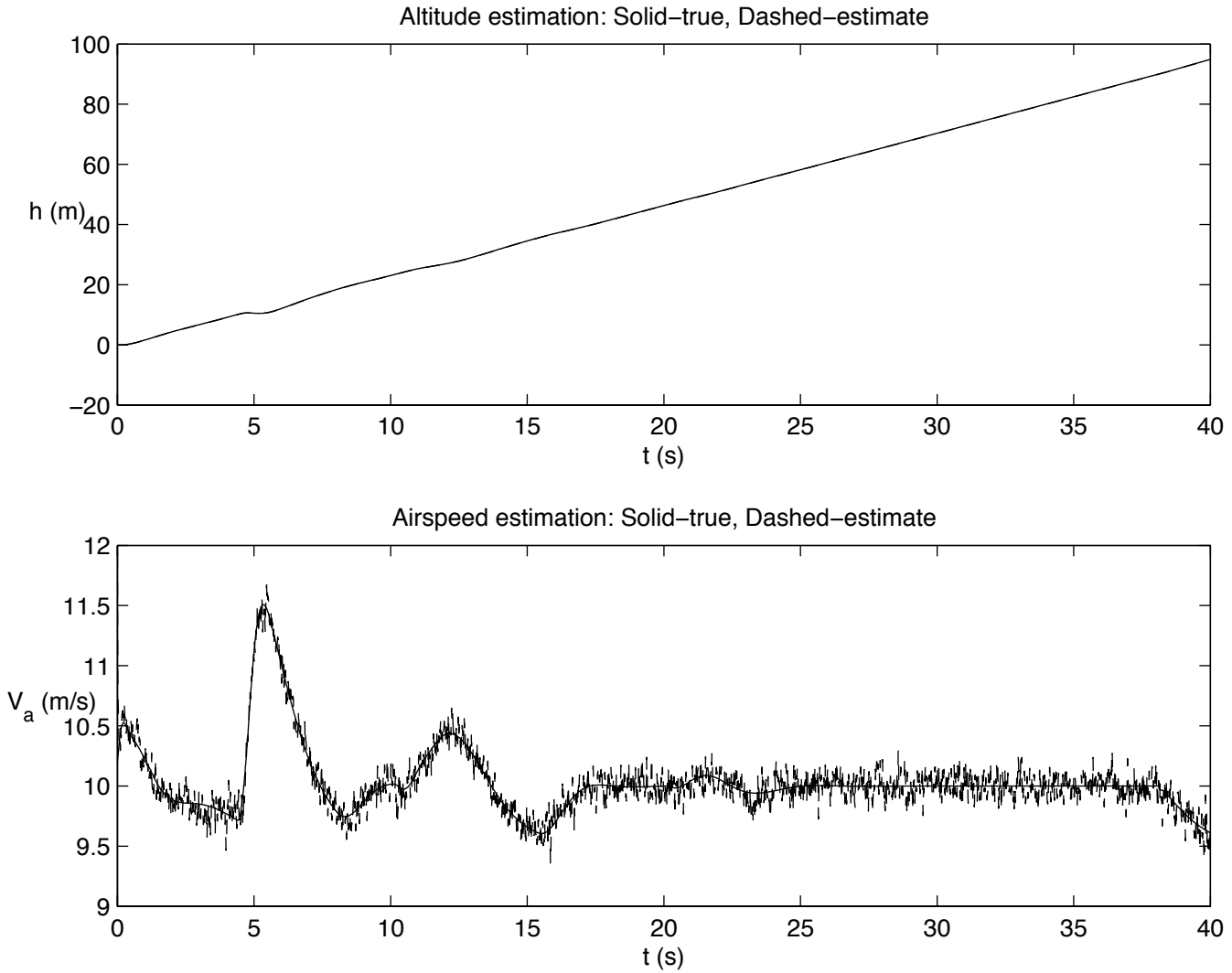
State Estimate

$$\hat{p} = LPF(y_{\text{gyro,x}})$$
$$\hat{q} = LPF(y_{\text{gyro,y}})$$
$$\hat{r} = LPF(y_{\text{gyro,z}})$$
$$\hat{h} = \frac{LPF(y_{\text{static pres}})}{\rho g}$$
$$\hat{V}_a = \sqrt{\frac{2}{\rho} LPF(y_{\text{diff pres}})}$$

Model Inversion - p, q, r (no bias)



Model Inversion - h, V_a



Inverting Sensor Measurement Model

Assuming steady, level flight (no acceleration), we can also invert accelerometer measurements to determine pitch and roll. Accelerometer outputs are

$$
\begin{aligned}
y_{\text{accel,x}} &= \dot{u} + qw - rv + g\sin\theta + \eta_{\text{accel,x}} \\
y_{\text{accel,y}} &= \dot{v} + ru - pw - g\cos\theta\sin\phi + \eta_{\text{accel,y}} \\
y_{\text{accel,z}} &= \dot{w} + pv - qu - g\cos\theta\cos\phi + \eta_{\text{accel,z}}
\end{aligned}
$$

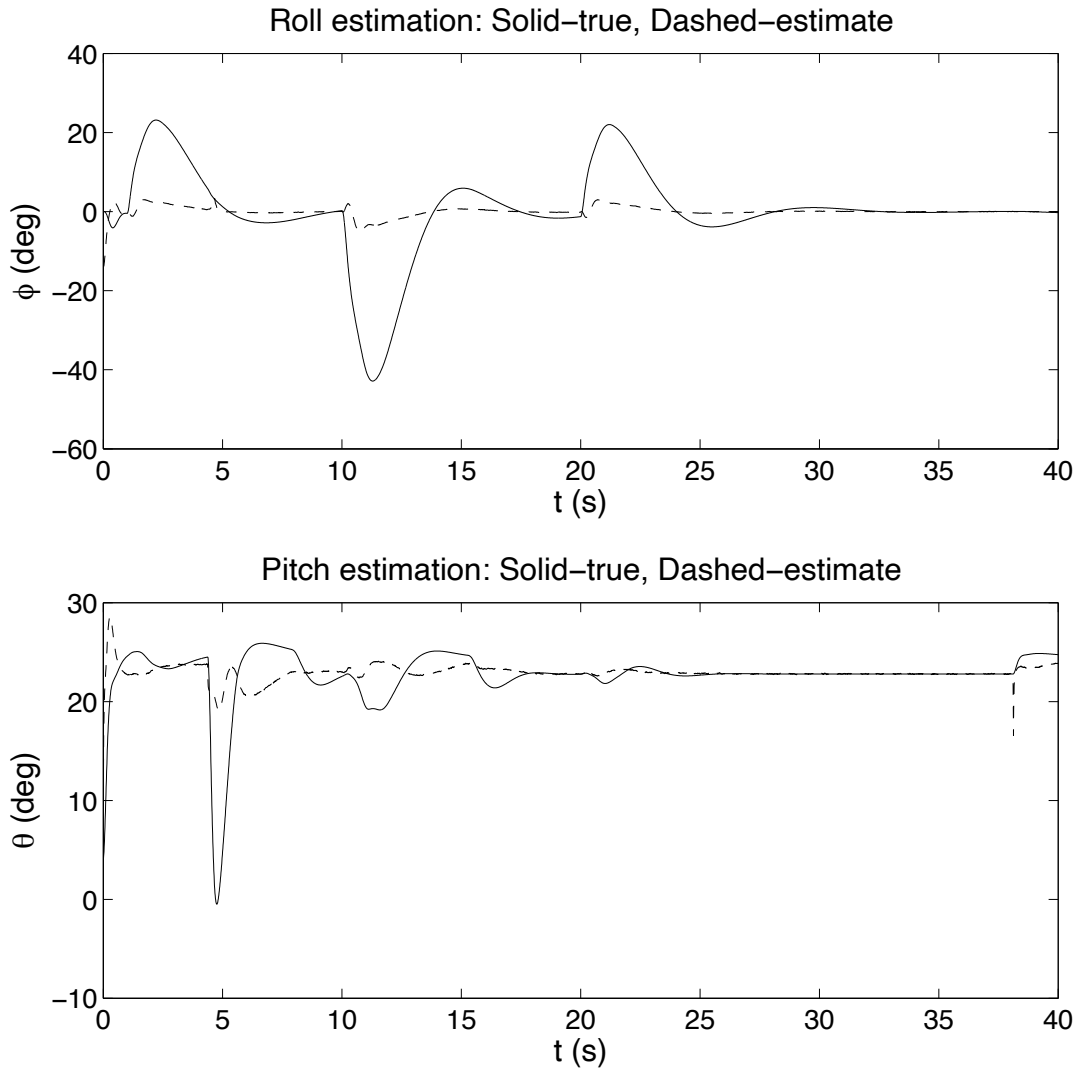
In unaccelerated flight, $\dot{u} = \dot{v} = \dot{w} = p = q = r = 0$, and

$$
\begin{aligned}
LPF(y_{\text{accel,x}}) &= g\sin\theta \\
LPF(y_{\text{accel,y}}) &= -g\cos\theta\sin\phi \\
LPF(y_{\text{accel,z}}) &= -g\cos\theta\cos\phi
\end{aligned}
$$

Solving for ϕ and θ:

$$
\begin{aligned}
\hat{\phi}_{\text{accel}} &= \tan^{-1}\left(\frac{LPF(y_{\text{accel,y}})}{LPF(y_{\text{accel,z}})}\right) \\
\hat{\theta}_{\text{accel}} &= \sin^{-1}\left(\frac{LPF(y_{\text{accel,x}})}{g}\right)
\end{aligned}
$$

Model Inversion - ϕ, θ



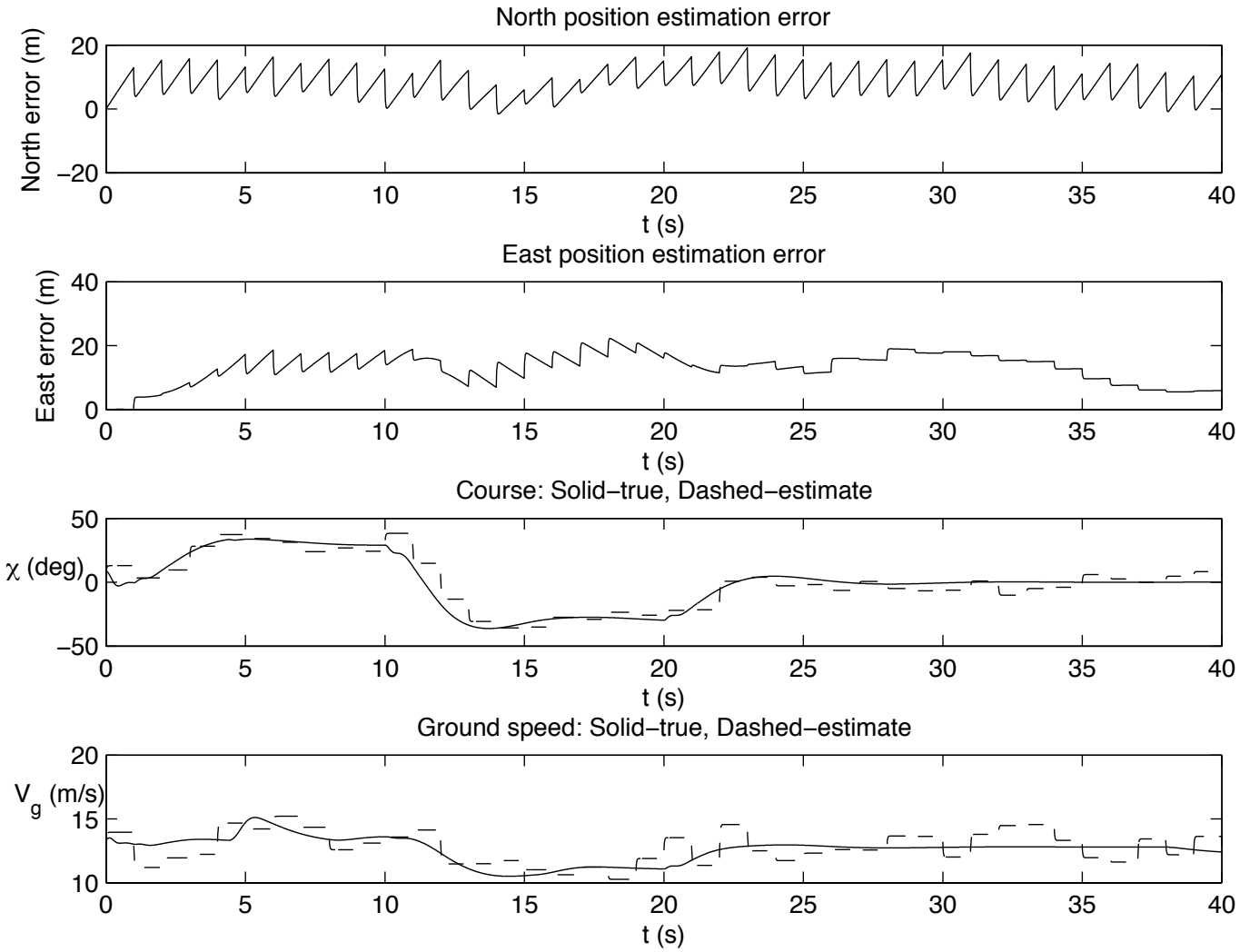
Inverting Sensor Measurement Model

- Can also invert sensor models for GPS signals:

$$\hat{p}_n = LPF(y_{\text{GPS,n}})$$
$$\hat{p}_e = LPF(y_{\text{GPS,e}})$$
$$\hat{\chi} = LPF(y_{\text{GPS},\chi})$$
$$\hat{V}_g = LPF(y_{\text{GPS},V_g})$$

- Update rate too slow (1 Hz)
- Need to fill in samples between measurement updates

Model Inversion - p_n, p_e, χ, V_g



Dynamic Observer Theory

Given linear time-invariant model

$$\dot{x} = Ax + Bu$$
$$y = Cx.$$

A continuous-time observer for this system is given by

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{copy of the model}} + \underbrace{L\,(y - C\hat{x})}_{\text{correction due to sensor reading}},$$

where $\hat{x}$ is the estimated value of x. Defining the observation error as $\tilde{x} = x - \hat{x}$

$$\dot{\tilde{x}} = (A - LC)\tilde{x}.$$

Therefore, observation error $\to 0$ if $\text{eig}(A - LC)$ in LHP.

Predictor-Corrector Structure

For linear systems:
Between Measurements (predictor):

$$\dot{\hat{x}} = A\hat{x} + Bu,$$

At a Measurements (corrector):

$$\hat{x}^+ = \hat{x}^- + L(y(t_n) - C\hat{x}^-),$$

where t_n is the instant in time that the measurement is received and $\hat{x}^-$ is the state estimate produced at time t_n.

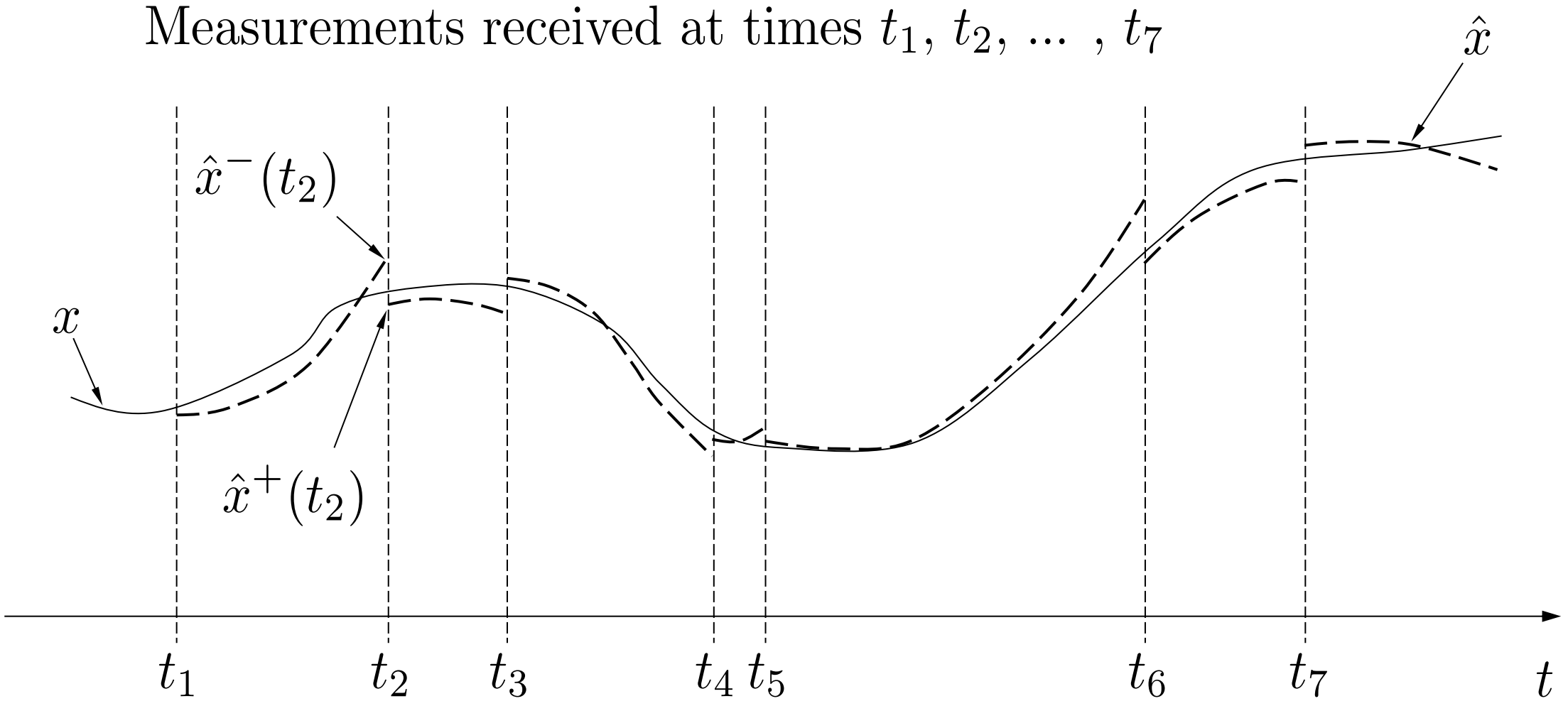
For nonlinear systems:
Between Measurements (predictor):

$$\dot{\hat{x}} = f(\hat{x}, u)$$

At a Measurements (corrector):

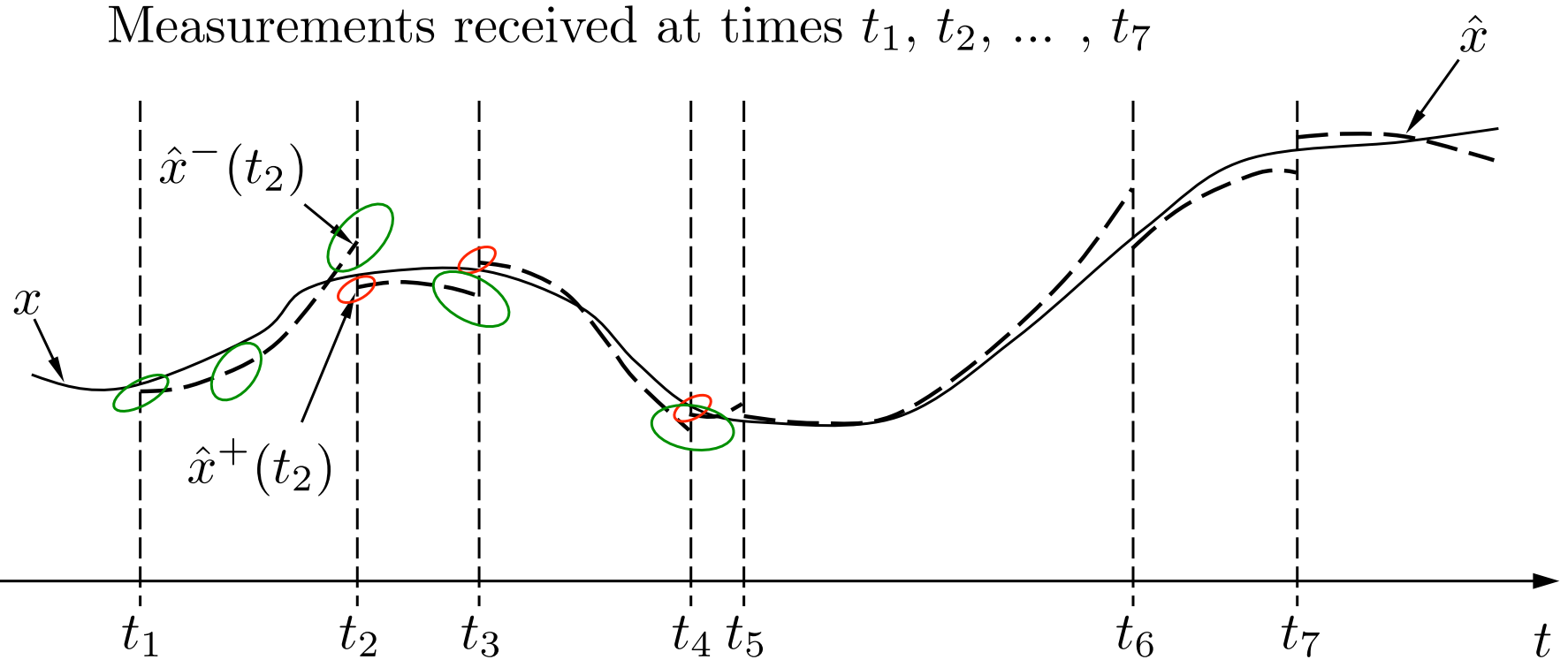
$$\hat{x}^+ = \hat{x}^- + L(y(t_n) - h(\hat{x}^-)).$$

Predictor-Corrector Structure



Kalman Filter

The Kalman filter follows this same process, but also attempts to estimate the quality of the estimate at each time step by propagating an error covariance matrix.

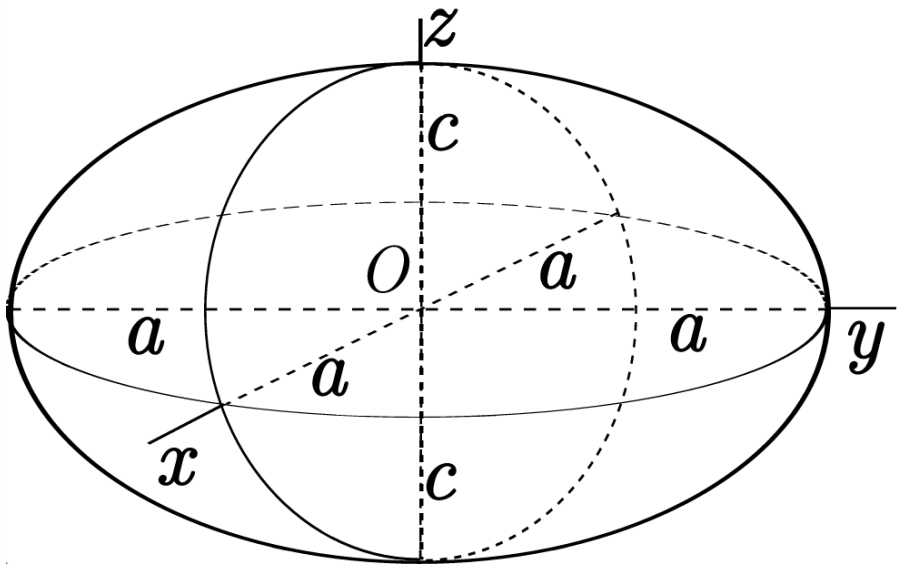


Quadratic Forms

Recall that for a positive definite matrix $P = P^\top > 0$, the set

$$\mathcal{E}(k) = \left\{ y \in \mathbb{R}^n : y^\top P^{-1} y = k^2 \right\}$$

is an ellipsoid whose axes are aligned with the eigenvectors of P, with stretching along each axis given by $k\sqrt{\lambda_i}$ where λ_i is an eigenvalue of P.



(From wikipedia)

The eigenaxes are given by x, y, z. a, b, and c correspond to $k\sqrt{\lambda_i}$.

Multivariate Gaussian

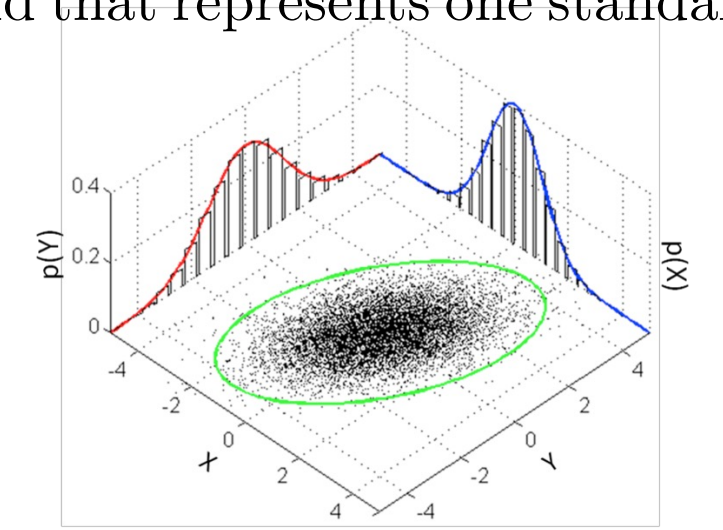
The multivariate Gaussian distribution is given by

$$\mathcal{N}(\mu, P) = \frac{1}{(2\pi)^{k/2}\|P\|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^\top P^{-1}(x-\mu)\right),$$

where μ is the mean and P is the covariance matrix. The $k^{th} - \sigma$ ellipsoid is defined as

$$\mathcal{E}(k) = \left\{x \in \mathbb{R}^k : (x-\mu)^\top P^{-1}(x-\mu) = k^2\right\}$$

where $one-\sigma$ or $k = 1$ corresponds the the ellipsoid that represents one standard deviation from the mean μ.



Kalman Filter

Assume continuous linear system dynamics, with discrete measurement update

$$\dot{x} = Ax + Bu + \xi$$
$$y[n] = Cx[n] + \eta[n],$$

where $\xi \sim \mathcal{N}(0, Q)$ is the process noise, $\nu \sim \mathcal{N}(0, R)$ is the measurement noise.

The measurement covariance R is usually obtained from sensor calibration. The process covariance Q represents all other uncertainties in the system and is the standard tuning parameter for the Kalman filter.

Kalman Filter Derivation

The continuous-discrete Kalman filter has the form

$$\dot{\hat{x}} = A\hat{x} + Bu \qquad \text{(Prediction)}$$
$$\hat{x}^+ = \hat{x}^- + L(y(t_n) - C\hat{x}^-). \qquad \text{(Correction)}$$

Define the estimation error

$$\tilde{x} = x - \hat{x},$$

and the covariance of the estimation error

$$P(t) = E\{\tilde{x}(t)\tilde{x}(t)^\top\},$$

where $P = P^\top \geq 0$.

Estimation Objective: Chose the estimation gain L s.t.

- $E\{\tilde{x}(t)\} = 0$ (unbiased),
- $\text{trace}(P) = \sum \lambda_i$ is minimized.

Kalman Filter Derivation

Between Measurements (prediction):

Differentiate $\tilde{x}$

$$\begin{aligned}\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu + \xi - A\hat{x} - Bu \\ &= A\tilde{x} + \xi.\end{aligned}$$

Solve for $\tilde{x}(t)$:

$$\tilde{x}(t) = e^{At}\tilde{x}_0 + \int_0^t e^{A(t-\tau)}\xi(\tau)\,d\tau.$$

Note that

$$E\{\tilde{x}(t)\} = e^{At}E\{\tilde{x}_0\} + \int_0^t e^{A(t-\tau)}E\{\xi(\tau)\}\,d\tau = 0$$

Kalman Filter Derivation

Between Measurements (prediction):

Compute the evolution of the error covariance P:

$$\begin{aligned}\dot{P} &= \frac{d}{dt}E\{\tilde{x}\tilde{x}^\top\} = E\{\dot{\tilde{x}}\tilde{x}^\top + \tilde{x}\dot{\tilde{x}}^\top\} \\ &= E\left\{A\tilde{x}\tilde{x}^\top + \xi\tilde{x}^\top + \tilde{x}\tilde{x}^\top A^\top + \tilde{x}\xi^\top\right\} \\ &= AP + PA^\top + E\{\xi\tilde{x}^\top\} + E\{\tilde{x}\xi^\top\},\end{aligned}$$

Where

$$\begin{aligned}E\{\tilde{x}\xi^\top\} &= E\{e^{At}\tilde{x}_0\xi^\top(t) + \int_0^t e^{A(t-\tau)}\xi(\tau)\xi^\top(t)\,d\tau\} \\ &= \int_0^t e^{A(t-\tau)}Q\delta(t-\tau)\,d\tau \\ &= \frac{1}{2}Q. \qquad \text{(only used half of area in delta function)}\end{aligned}$$

Therefore

$$\dot{P} = AP + PA^\top + Q.$$

Kalman Filter Derivation

At Measurements (correction):

At a measurement, we have that

$$\begin{aligned}\tilde{x}^+ &= x - \hat{x}^+ \\ &= x - \hat{x}^- - L\left(Cx + \eta - C\hat{x}^-\right) \\ &= \tilde{x}^- - LC\tilde{x}^- - L\eta.\end{aligned}$$

We also have that

$$\begin{aligned}P^+ &= E\{\tilde{x}^+\tilde{x}^{+T}\} \\ &= E\left\{\left(\tilde{x}^- - LC\tilde{x}^- - L\eta\right)\left(\tilde{x}^- - LC\tilde{x}^- - L\eta\right)^\top\right\} \\ &= E\Big\{\tilde{x}^-\tilde{x}^{-\top} - \tilde{x}^-\tilde{x}^{-\top}C^\top L^\top - \tilde{x}^-\eta^\top L^\top - LC\tilde{x}^-\tilde{x}^{-\top} + LC\tilde{x}^-\tilde{x}^{-\top}C^\top L^\top + LC\tilde{x}^-\eta^\top L^\top \\ &\qquad - L\eta\tilde{x}^{-\top} + L\eta\tilde{x}^{-\top}C^\top L^\top + L\eta\eta^\top L^\top\Big\} \\ &= P^- - P^-C^\top L^\top - LCP^- + LCP^-C^\top L^\top + LRL^\top \\ &= (I - LC)P^-(I - LC)^\top + LRL^\top.\end{aligned}$$

Objective: Pick L to minimize $\text{trace}(P^+)$.

Kalman Filter Derivation

At Measurements (correction):

Using

$$\frac{\partial}{\partial A}\text{trace}(BAD) = B^\top D^\top,$$
$$\frac{\partial}{\partial A}\text{trace}(ABA^\top) = 2AB, \text{ if } B = B^\top,$$
$$\frac{\partial}{\partial A}\text{trace}(AB) = B$$

a necessary condition is

$$\frac{\partial}{\partial L}\text{tr}(P^+) = \frac{\partial}{\partial L}\text{tr}\Big((I-LC)P^-(I-LC)^\top + LRL^\top\Big) = -P^-C^\top - P^-C^\top + 2LCP^-C^\top + 2LR = 0$$
$$\implies 2L^*(R + CP^-C^\top) = 2P^-C^\top$$
$$\implies L^* = P^-C^\top(R + CP^-C^\top)^{-1}.$$

Substituting into prior equation for P^+ gives (after algebra)

$$P^+ = (I - L^*C)P^-(I - L^*C)^\top + L^*RL^{*\top}.$$

Kalman Filter Derivation

Between Measurements (prediction):

$$\dot{\hat{x}} = A\hat{x} + Bu$$
$$\dot{P} = AP + PA^\top + Q,$$

At the i^{th} Measurement (correction):

$$L_i = P^- C_i^\top (R_i + C_i P^- C_i^\top)^{-1}$$
$$\hat{x}^+ = \hat{x}^- + L_i(y_i - C_i\hat{x}^-),$$
$$P^+ = (I - L_i C_i)P^-(I - L_i C_i)^\top + L_i R_i L_i^\top$$

where L_i is called the Kalman gain for sensor i.

Covariance Update

Between measurements, the covariance evolves according to the equation

$$\dot{P} = AP + PA^\top + Q.$$

To implement, could use Euler approximation:

$$P_{k+1} = P_k + T_s \left(AP_k + P_k A^\top + Q \right).$$

Disadvantage: Due to numerical error, P may not remain positive definite (i.e., won't represent a covariance matrix).

Covariance Update

Recall that

$$\tilde{x}(t) = e^{A(t-t_0)}\tilde{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\xi(\tau)d\tau.$$

Let $t = kT_s$ and assuming that $\xi(\tau)$ is constant over each time interval, then

$$\tilde{x}_{k+1} = e^{AT_s}\tilde{x}_k + (\int_0^{T_s} e^{A\tau}d\tau)\xi_k.$$

Using the approximation

$$A_d = e^{AT_s} \approx I + AT_s + A^2\frac{T_s^2}{2}$$

$$B_d = (\int_0^{T_s} e^{A\tau}d\tau) \approx \int_0^{T_s} Id\tau = T_sI,$$

gives

$$\tilde{x}_{k+1} = A_d\tilde{x}_k + T_s\xi_k.$$

Covariance Update

Therefore, the covariance update becomes

$$
\begin{aligned}
P_{k+1} &= E\{\tilde{x}_{k+1}\tilde{x}_{k+1}^\top\} \\
&= E\{(A_d\tilde{x}_k + T_s\xi_k)(A_d\tilde{x}_k + T_s\xi_k)^\top\} \\
&= E\{A_d\tilde{x}_k\tilde{x}_k^\top A_d + T_s\xi_k\tilde{x}_k^\top A_d^\top + A_d\tilde{x}_k + T_s\xi_k\xi_k^\top + T_s^2\xi_k\xi_k^\top\} \\
&= A_d E\{\tilde{x}_k\tilde{x}_k^\top\}\ A_d^\top + T_s E\{\xi_k\tilde{x}_k^\top\}A_d^\top + T_s A_d E\{\tilde{x}_k\xi_k^\top\} + T_s^2 E\{\xi_k\xi_k^\top\} \\
&= A_d P_k A_d^\top + T_s^2 Q.
\end{aligned}
$$

Note that the last line ensures that the error covariance remains positive definite since the sum of positive definite matrices is positive definite.

Kalman Filter Derivation

Between Measurements (prediction):

$$\dot{\hat{x}} = A\hat{x} + Bu$$

$$A_d = I + AT_s + A^2\frac{T_s^2}{2}$$

$$P_{k+1} = A_d P_k A_d^\top + T_s^2 Q.$$

At the i^{th} Measurement (correction):

$$L_i = P^- C_i^\top (R_i + C_i P^- C_i^\top)^{-1}$$

$$\hat{x}^+ = \hat{x}^- + L_i(y_i - C_i\hat{x}^-),$$

$$P^+ = (I - L_i C_i)P^-(I - L_i C_i)^\top + L_i R_i L_i^\top$$

where L_i is called the Kalman gain for sensor i.

Extended Kalman Filter

Basic idea of Extended Kalman Filter:

- Propagate multivariate Gaussian distribution that captures probability distribution associated with belief about state
- Mean of distribution is $\hat{x}$ — estimated state
- Covariance quantifies associated estimation error
- Kalman gain minimizes covariance of estimation error given uncertainty in model and measurements

System model given by:

$$\dot{x} = f(x, u) + \xi$$
$$y_i[n] = h_i(x[n], u[n]) + \eta_i[n],$$

where

$$\xi \sim \mathcal{N}(0, Q)$$
$$\eta_i \sim \mathcal{N}(0, R_i)$$

Extended Kalman Filter, cont.

Between Measurements (prediction):

$$\dot{\hat{x}} = f(\hat{x}, u)$$
$$A = \frac{\partial f}{\partial x}(\hat{x}, u)$$
$$A_d = I + AT_s + A^2 \frac{T_s^2}{2}$$
$$P_{k+1} = A_d P_k A_d^\top + T_s^2 Q.$$

At the i^{th} Measurements (correction):

$$C_i = \frac{\partial h_i}{\partial x}(\hat{x}^-)$$
$$L_i = P^- C_i^\top (R_i + C_i P^- C_i^\top)^{-1}$$
$$\hat{x}^+ = \hat{x}^- + L_i(y_i(t_n) - h_i(\hat{x}^-)),$$
$$P^+ = (I - L_i C_i) P^- (I - L_i C_i)^\top + L_i R_i L_i^\top$$

where L_i is called the Kalman gain for sensor i.

EKF Algorithm

Algorithm 1 Continuous-Discrete Extended Kalman Filter

Initialize: $\hat{x} = 0$.
Pick an output sample rate T_{out} which is much less than the sample rates of the sensors.
At each sample time T_{out}:
for $i = 1$ to N **do** {Prediction}
  $T_p = T_{out}/N$
  $\hat{x} \leftarrow \hat{x} + T_p f(\hat{x}, u)$
  $A = \frac{\partial f}{\partial x}(\hat{x}, u)$
  $A_d = I + AT_p + A^2 T_p^2$
  $P \leftarrow A_d P A_d^\top + T_p^2 Q$
end for
if Measurement has been received from sensor i **then** {Correction}
  $C_i = \frac{\partial h_i}{\partial x}(\hat{x}, u[n])$
  $L_i = PC_i^\top (R_i + C_i P C_i^\top)^{-1}$
  $P \leftarrow (I - L_i C_i) P (I - L_i C_i)^\top + L_i R_i L_i^\top$
  $\hat{x} \leftarrow \hat{x} + L_i \left(y_i[n] - h(\hat{x}, u[n])\right)$.
end if

If R is diagonal (uncorrelated sensors), then update one measurement at a time.

Attitude Estimation Using EKF

Use the nonlinear propagation model

$$\dot{\phi} = p + q \sin\phi \tan\theta + r \cos\phi \tan\theta + \xi_\phi$$
$$\dot{\theta} = q \cos\phi - r \sin\phi + \xi_\theta$$

where

$$\xi_\phi \sim \mathcal{N}(0, Q_\phi) \qquad \text{and} \qquad \xi_\theta \sim \mathcal{N}(0, Q_\theta)$$

Use accelerometers as measured outputs:

$$y_{\text{accel}} = \begin{pmatrix} \dot{u} + qw - rv + g\sin\theta \\ \dot{v} + ru - pw - g\cos\theta\sin\phi \\ \dot{w} + pv - qu - g\cos\theta\cos\phi \end{pmatrix} + \eta_{\text{accel}}.$$

Problem: Do not have method for directly measuring $\dot{u}$, $\dot{v}$, $\dot{w}$, u, v, and w.

What do we do?

Attitude Estimation Using EKF

Assume that $\dot{u} = \dot{v} = \dot{w} \approx 0$

Recall that

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \approx V_a \begin{pmatrix} \cos\alpha \cos\beta \\ \sin\beta \\ \sin\alpha \cos\beta \end{pmatrix}.$$

Assuming that $\alpha \approx \theta$ and $\beta \approx 0$ gives

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \approx V_a \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$$

Substituting into original output equation

$$y_{\text{accel}} = \begin{pmatrix} \dot{u} + qw - rv + g\sin\theta \\ \dot{v} + ru - pw - g\cos\theta\sin\phi \\ \dot{w} + pv - qu - g\cos\theta\cos\phi \end{pmatrix} + \eta_{\text{accel}}$$

gives

$$y_{\text{accel}} = \begin{pmatrix} qV_a\sin\theta + g\sin\theta \\ rV_a\cos\theta - pV_a\sin\theta - g\cos\theta\sin\phi \\ -qV_a\cos\theta - g\cos\theta\cos\phi \end{pmatrix} + \eta_{\text{accel}}$$

Attitude Estimation Using EKF

Defining $x = (\phi, \theta)^\top$, $u = (p, q, r, V_a)^\top$, $\xi = (\xi_\phi, \xi_\theta)^\top$, and $\eta = (\eta_\phi, \eta_\theta)^\top$, and noting that there is noise on the gyros and pressure sensor, i.e., $u_{actual} = u + \xi_u$ where $\xi_u = (\xi_p, \xi_q, \xi_r, \xi_{V_a})^\top$, gives

$$\begin{aligned}\dot{x} &= f(x,u) + G(x)\xi_u + \xi \\ y &= h(x,u) + \eta,\end{aligned}$$

where

$$f(x,u) = \begin{pmatrix} p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\ q\cos\phi - r\sin\phi \end{pmatrix}$$

$$h(x,u) = \begin{pmatrix} qV_a\sin\theta + g\sin\theta \\ rV_a\cos\theta - pV_a\sin\theta - g\cos\theta\sin\phi \\ -qV_a\cos\theta - g\cos\theta\cos\phi \end{pmatrix}$$

$$G(x) = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \end{pmatrix}.$$

Note that

$$E\{(G\xi_u + \xi)(G\xi_u + \xi)^\top\} = GE\{\xi_u\xi_u^\top\} + E\{\xi\xi^\top\} = GQ_uG^\top + Q$$

where we have assumed that $E\{\xi_u\xi_u^\top\} = Q_u$.

Attitude Estimation Using EKF

Implementation of Kalman filter requires Jacobians $\frac{\partial f}{\partial x}$ and $\frac{\partial h}{\partial x}$.
Accordingly,

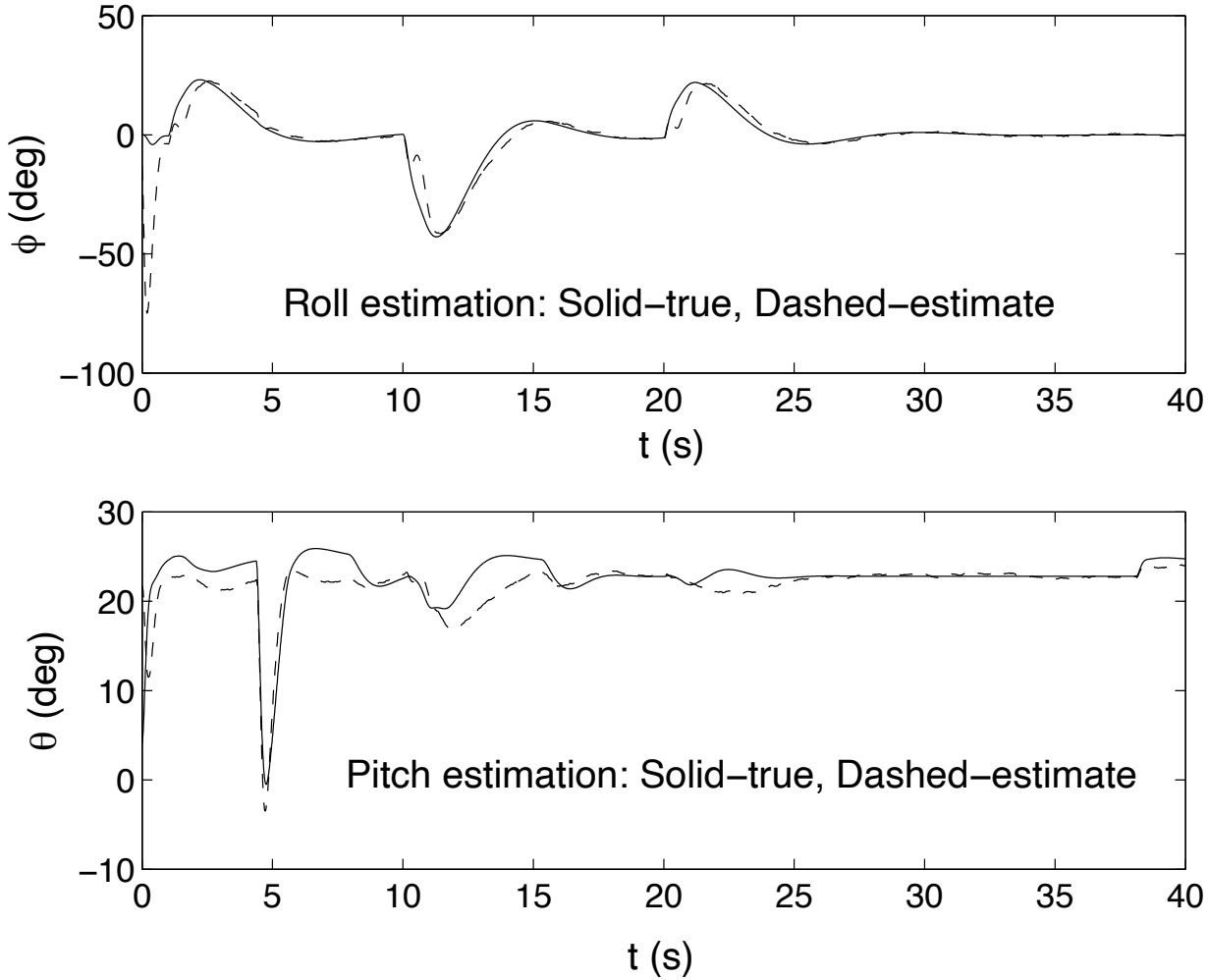
$$\frac{\partial f}{\partial x} = \begin{pmatrix} q\cos\phi\tan\theta - r\sin\phi\tan\theta & \frac{q\sin\phi - r\cos\phi}{\cos^2\theta} \\ -q\sin\phi - r\cos\phi & 0 \end{pmatrix}$$

$$\frac{\partial h}{\partial x} = \begin{pmatrix} 0 & qV_a\cos\theta + g\cos\theta \\ -g\cos\phi\cos\theta & -rV_a\sin\theta - pV_a\cos\theta + g\sin\phi\sin\theta \\ g\sin\phi\cos\theta & (qV_a + g\cos\phi)\sin\theta \end{pmatrix}.$$

At this point, we have everything we need to implement the continuous-discrete EKF.

How well does it work?

Attitude Estimation Results



Not perfect, but significantly better!

GPS Smoothing

- Want to fill in estimates between GPS measurements
- Want to estimate wind

Assuming level flight, evolution of position:

$$\dot{p}_n = V_g \cos\chi$$
$$\dot{p}_e = V_g \sin\chi$$

Evolution of the ground speed:

$$\dot{V}_g = \frac{d}{dt}\sqrt{(V_a\cos\psi + w_n)^2 + (V_a\sin\psi + w_e)^2}$$
$$= \frac{(V_a\cos\psi + w_n)(\dot{V}_a\cos\psi - V_a\dot{\psi}\sin\psi + \dot{w}_n) + (V_a\sin\psi + w_e)(\dot{V}_a\sin\psi + V_a\dot{\psi}\cos\psi + \dot{w}_e)}{V_g}$$

Assuming that wind and airspeed are constant:

$$\dot{V}_g = \frac{(V_a\cos\psi + w_n)(-V_a\dot{\psi}\sin\psi) + (V_a\sin\psi + w_e)(V_a\dot{\psi}\cos\psi)}{V_g}$$

GPS Smoothing

Evolution of χ:

$$\dot{\chi} = \frac{g}{V_g} \tan\phi \cos(\chi - \psi)$$

Assuming that wind is constant:

$$\dot{w}_n = 0$$
$$\dot{w}_e = 0$$

From kinematics, evolution of ψ:

$$\dot{\psi} = q\frac{\sin\phi}{\cos\theta} + r\frac{\cos\phi}{\cos\theta}$$

GPS Smoothing

Define:

$x = (p_n, p_e, V_g, \chi, w_n, w_e, \psi)^\top$

$u = (V_a, q, r, \phi, \theta)^\top$:

$$f(x,u) \triangleq \begin{pmatrix} V_g \cos\chi \\ V_g \sin\chi \\ \frac{(V_a\cos\psi + w_n)(-V_a\dot{\psi}\sin\psi) + (V_a\sin\psi + w_e)(V_a\dot{\psi}\cos\psi)}{V_g} \\ \frac{g}{V_g}\tan\phi\cos(\chi-\psi) \\ 0 \\ 0 \\ q\frac{\sin\phi}{\cos\theta} + r\frac{\cos\phi}{\cos\theta} \end{pmatrix}$$

GPS Smoothing

Define:
$x = (p_n, p_e, V_g, \chi, w_n, w_e, \psi)^\top$
$u = (V_a, q, r, \phi, \theta)^\top$

Jacobian of f:

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 0 & \cos\chi & -V_g \sin\chi & 0 & 0 & 0 \\ 0 & 0 & \sin\chi & V_g \cos\chi & 0 & 0 & 0 \\ 0 & 0 & -\frac{\dot{V}_g}{V_g} & 0 & -\dot{\psi} V_a \sin\psi & \dot{\psi} V_a \cos\psi & \frac{\partial \dot{V}_g}{\partial \psi} \\ 0 & 0 & \frac{\partial \dot{\chi}}{\partial V_g} & \frac{\partial \dot{\chi}}{\partial \chi} & 0 & 0 & \frac{\partial \dot{\chi}}{\partial \psi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\frac{\partial \dot{V}_g}{\partial \psi} = \frac{-\dot{\psi} V_a (w_n \cos\psi + w_e \sin\psi)}{V_g}$$

$$\frac{\partial \dot{\chi}}{\partial V_g} = -\frac{g}{V_g^2} \tan\phi \cos(\chi - \psi)$$

$$\frac{\partial \dot{\chi}}{\partial \chi} = -\frac{g}{V_g} \tan\phi \sin(\chi - \psi)$$

$$\frac{\partial \dot{\chi}}{\partial \psi} = \frac{g}{V_g} \tan\phi \sin(\chi - \psi)$$

GPS Smoothing

Define:
$x = (p_n, p_e, V_g, \chi, w_n, w_e, \psi)^\top$
$u = (V_a, q, r, \phi, \theta)^\top$

For measurements, use GPS signals for north and east position (p_n, p_e), ground speed (V_g), and course (χ)

Notice that states are not independent – related by wind triangle

Use wind triangle relations to introduce two pseudo measurements.

Assume $\gamma = \gamma_a = 0$:

$$V_a \cos\psi + w_n = V_g \cos\chi$$
$$V_a \sin\psi + w_e = V_g \sin\chi$$

From these expressions, define the pseudo measurements

$$y_{\text{windtri,n}} = V_a \cos\psi + w_n - V_g \cos\chi$$
$$y_{\text{windtri,e}} = V_a \sin\psi + w_e - V_g \sin\chi$$

where (pseudo) measurement values are equal to zero.

GPS Smoothing

The resulting measurement model is given by

$$y_{GPS} = h(x,u) + \eta_{GPS}$$

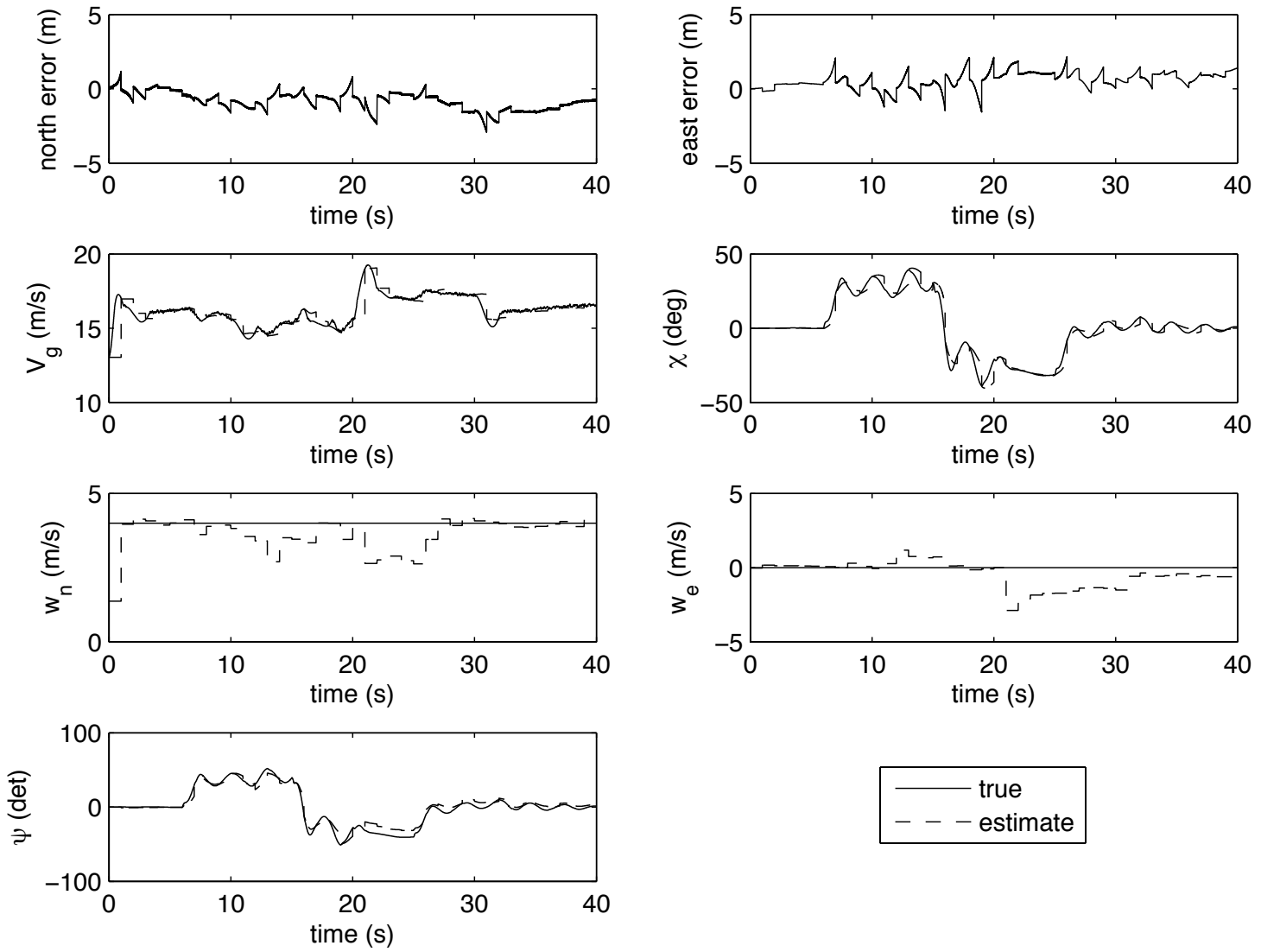
where $y_{GPS} = (y_{GPS,n}, y_{GPS,e}, y_{GPS,V_g}, y_{GPS,\chi}, y_{\text{wind,n}}, y_{\text{wind,e}})$, and

$$h(x,u) = \begin{pmatrix} p_n \\ p_e \\ V_g \\ \chi \\ V_a \cos\psi + w_n - V_g \cos\chi \\ V_a \sin\psi + w_e - V_g \sin\chi \end{pmatrix}$$

and where the Jacobian is given by

$$\frac{\partial h}{\partial x}(\hat{x},u) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\cos\chi & V_g \sin\chi & 1 & 0 & -V_a \sin\psi \\ 0 & 0 & -\sin\chi & -V_g \cos\chi & 0 & 1 & V_a \cos\psi \end{pmatrix}$$

GPS Smoothing Results



Measurement Gating

Similar to least squares, measurement outliers can negatively impact performance of the Kalman Filter.

However, the KF/EKF provides a way to detect outliers.

Define the innovation sequence:

$$v_k = y_k - C\hat{x}_k^- = Cx_k + \eta_k - C\hat{x}_k^- = C\tilde{x}_k^- + \eta_k.$$

The covariance of the innovation sequence is

$$S_k = E\{v_k v_k^\top\} = E\{(\eta_k + C\tilde{x}_k^-)(\eta_k + C\tilde{x}_k^-)^\top\} = R + CP_k^- C^\top.$$

Measurement Gating

The random variable

$$z_k = (y_k - h(x_k))^\top S_k^{-1}(y_k - h(x_k)),$$

is therefore a χ^2 (chi-squared) random variable with m degrees-of-freedom.

Therefore, compute the likelihood that z_k is drawn from a $\chi^2(m)$ distribution, and do a measurement update if the likelihood is about a threshold.

```
import numpy as np
from scipy import stats
self.accel_threshold = stats.chi2.isf(q=0.01, df=3)
def measurement_update(self, measurement, state):
    # measurement updates
    h = self.h(self.xhat, measurement, state)
    C = jacobian(self.h, self.xhat, measurement, state)
    y = np.array([[measurement.accel_x,
                   measurement.accel_y,
                         measurement.accel_z]]).T
    S_inv = np.linalg.inv(self.R_accel + C @ self.P @ C.T)
    if (y - h).T @ S_inv @ (y - h) < self.accel_threshold:
        L = self.P @ C.T @ S_inv
        tmp = np.eye(2) - L @ C
        self.P = tmp@self.P@tmp.T + L@self.R_accel@L.T
        self.xhat = self.xhat + L @ (y - h)
```

Suggestions for Tuning Your EKF

- Implement EKF in steps
  - Attitude estimation
  - GPS smoother
- Test and tune each component independently and thoroughly
  - Attitude estimator first
  - GPS smoother second
- As a first step, make sure your filter estimates track the states when sensors are perfect (no sensor error). This will expose coding errors and show the limits of performance of your filter.
- Keep the wind set to zero until you are confident that your filter is well tuned.
- We know R pretty well typically. Tune filter by changing Q.
- Tune filter by focusing on individual states. Give inputs to those states while keeping other states "quiet". Adjust Q value corresponding to state of interest. Tune by trial and error (use your brain to make a good guess!).
- Don't put extreme inputs into the system when tuning the filter (e.g., large steps). Your filter will have its own dynamic limits – don't make the problem impossibly difficult.
- Tune your controllers so that the response of the aircraft to typical inputs is smooth and graceful. Abrupt and jerky state dynamics are difficult to estimate.
- Check your equations AGAIN!