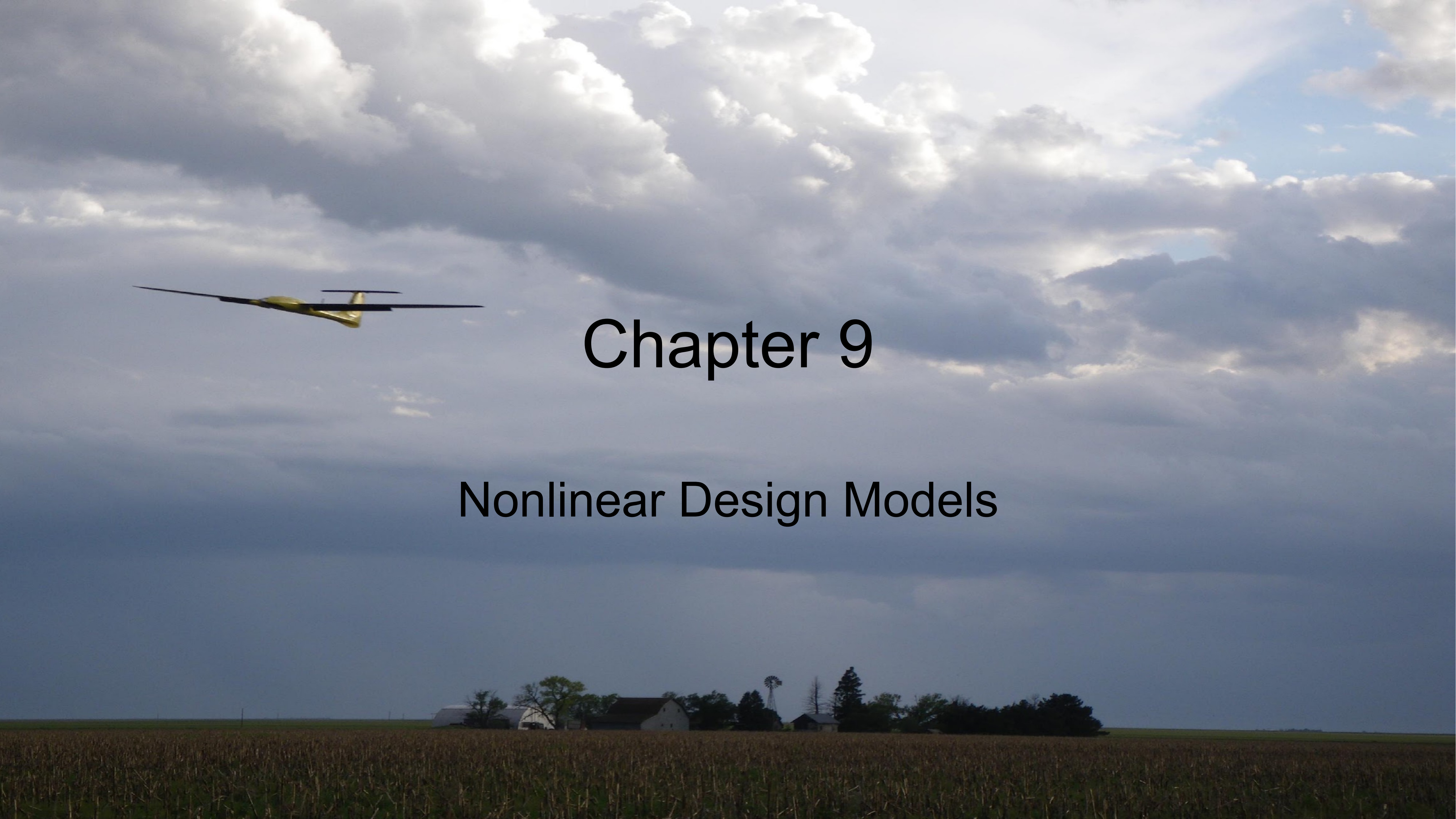
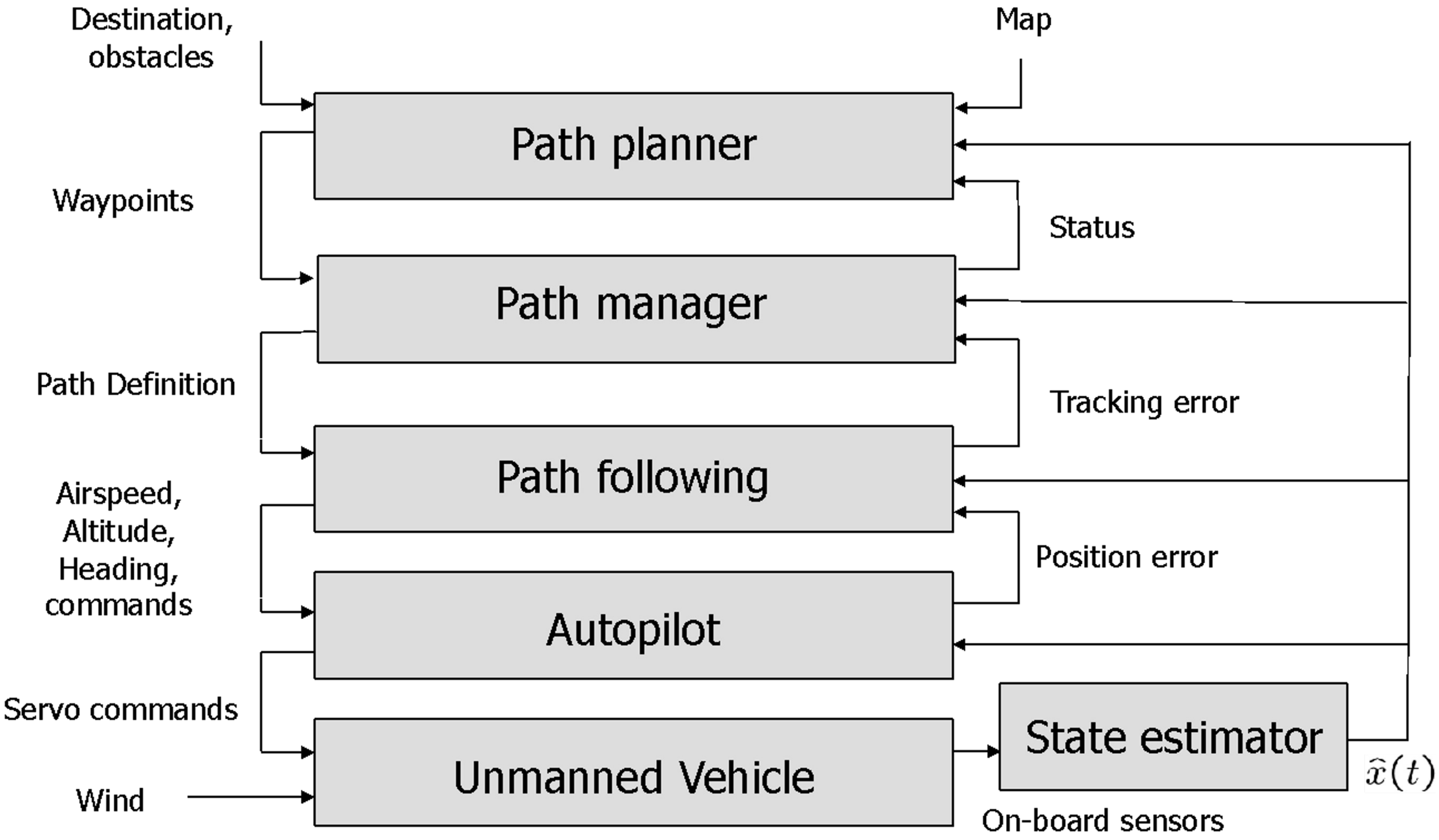


Chapter 9

Nonlinear Design Models

Architecture



Simplifying Dynamic Models

- When designing higher level autopilot functions, we need models that are easier to analyze and simulate
- Models must capture the essential behavior of system
- We will derive reduced-order, reduced-complexity models suitable for design of higher-level guidance strategies
- Two types of guidance models:
  - Kinematic – utilize kinematic relationships, do not consider aerodynamics, forces directly
  - Dynamic – apply force balance relations to point-mass models

Autopilot Models (transfer function)

Airspeed hold and roll hold loops can be modeled as:

$$V_a(s) = \frac{b_{V_a}}{s + b_{V_a}} V_a^c(s) \qquad \phi(s) = \frac{b_\phi}{s + b_\phi} \phi^c(s)$$

Altitude and course hold loops can be modeled as:

$$h(s) = \frac{b_{\dot{h}} s + b_h}{s^2 + b_{\dot{h}} s + b_h} h^c(s) \qquad \chi(s) = \frac{b_{\dot{\chi}} s + b_\chi}{s^2 + b_{\dot{\chi}} s + b_\chi} \chi^c(s)$$

Alternatively, the heading hold loop can be modeled as:

$$\psi(s) = \frac{b_{\dot{\psi}} s + b_\psi}{s^2 + b_{\dot{\psi}} s + b_\psi} \psi^c(s)$$

Flight-path angle and load factor loops can be modeled as:

$$\gamma(s) = \frac{b_\gamma}{s + b_\gamma} \gamma^c(s) \qquad n_{lf}(s) = \frac{b_n}{s + b_n} n_{lf}^c(s)$$

Autopilot Models

Airspeed hold and roll hold loops can be modeled as:

$$\dot{V}_a = b_{V_a}(V_a^c - V_a)$$
$$\dot{\phi} = b_{\phi}(\phi^c - \phi)$$

Altitude and course hold loops can be modeled as:

$$\ddot{h} = b_{\dot{h}}(\dot{h}^c - \dot{h}) + b_h(h^c - h)$$
$$\ddot{\chi} = b_{\dot{\chi}}(\dot{\chi}^c - \dot{\chi}) + b_{\chi}(\chi^c - \chi)$$

Alternatively, the heading hold loop can be modeled as:

$$\ddot{\psi} = b_{\dot{\psi}}(\dot{\psi}^c - \dot{\psi}) + b_{\psi}(\psi^c - \psi)$$

Flight-path angle and load factor loops can be modeled as:

$$\dot{\gamma} = b_{\gamma}(\gamma^c - \gamma)$$
$$\dot{n}_{lf} = b_n(n_{lf}^c - n_{lf})$$

Kinematic Model of Controlled Flight

The velocity vector can be written as

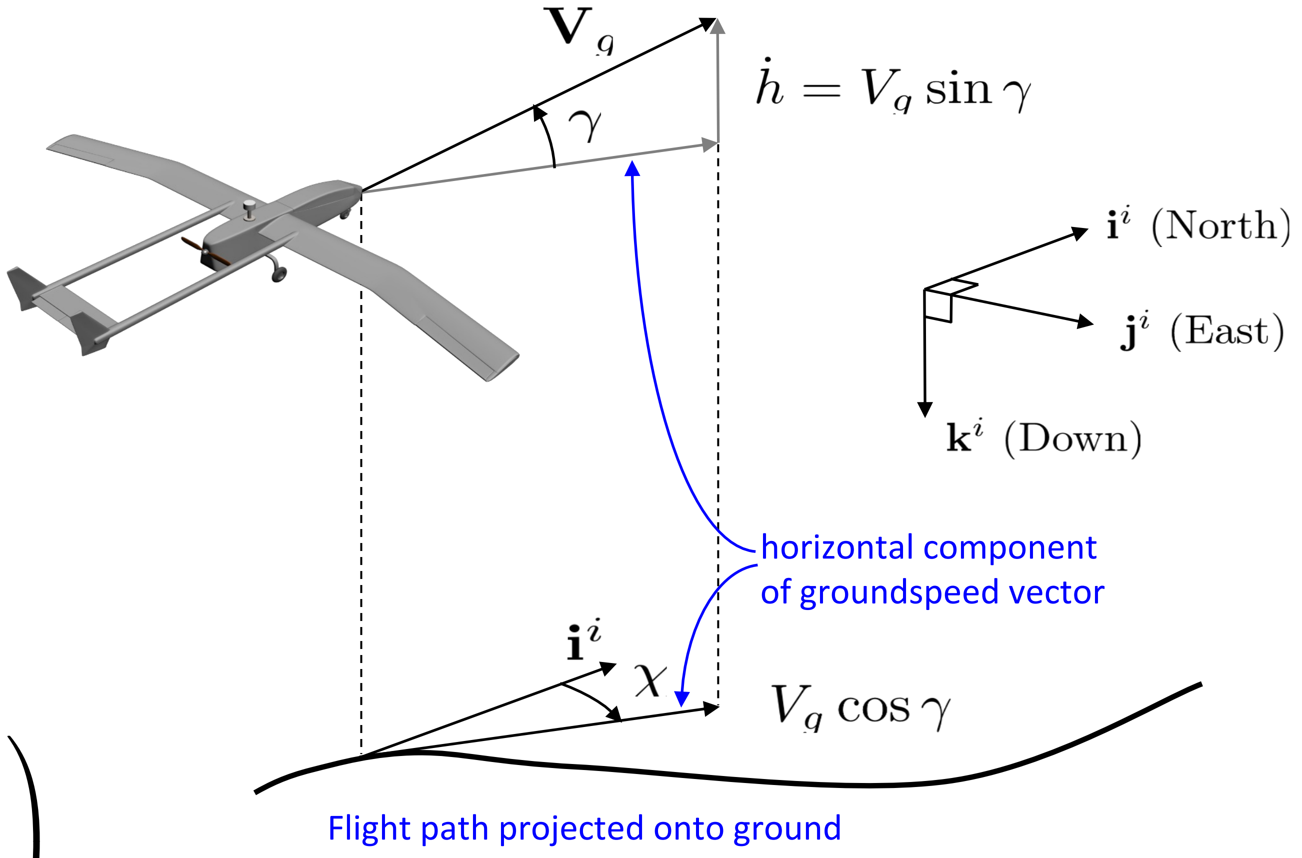
$$\mathbf{V}_g^i = V_g \begin{pmatrix} \cos\chi\cos\gamma \\ \sin\chi\cos\gamma \\ -\sin\gamma \end{pmatrix}$$

which gives:

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{h} \end{pmatrix} = V_g \begin{pmatrix} \cos\chi\cos\gamma \\ \sin\chi\cos\gamma \\ \sin\gamma \end{pmatrix}$$

Alternatively, using the Wind Triangle:

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{h} \end{pmatrix} = V_a \begin{pmatrix} \cos\psi\cos\gamma_a \\ \sin\psi\cos\gamma_a \\ \sin\gamma_a \end{pmatrix} + \begin{pmatrix} w_n \\ w_e \\ -w_d \end{pmatrix}$$



Kinematic Model of Controlled Flight

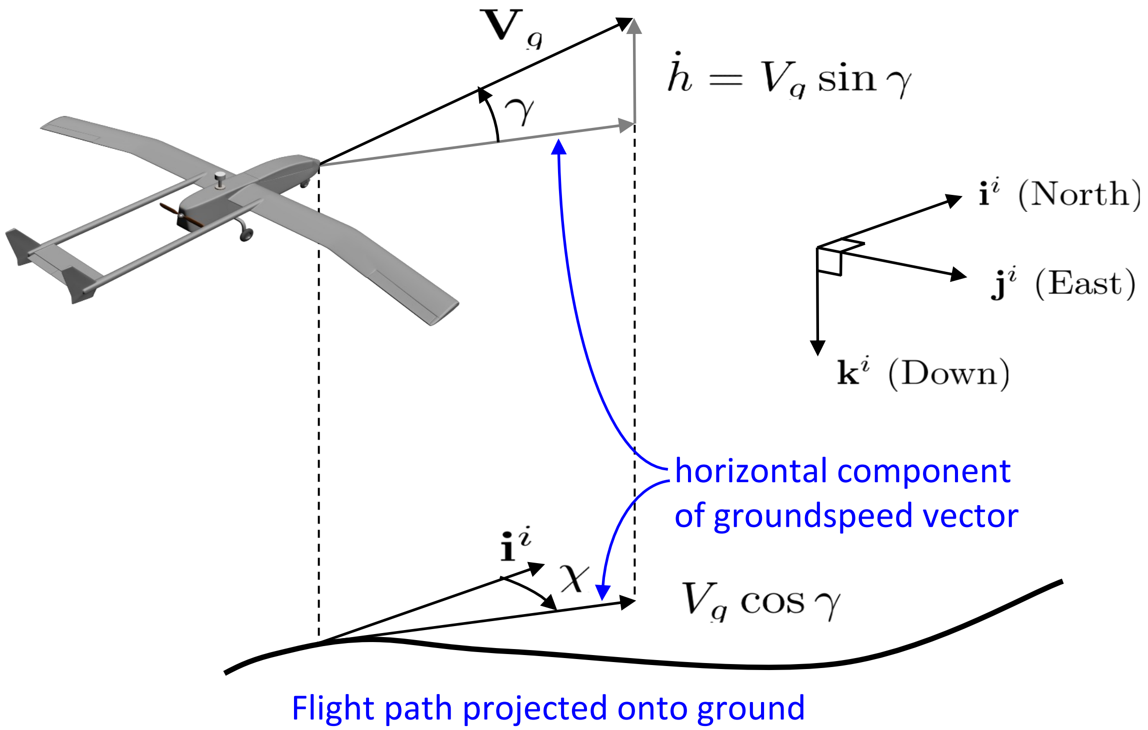
Beginning with

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{h} \end{pmatrix} = V_a \begin{pmatrix} \cos\psi\cos\gamma_a \\ \sin\psi\cos\gamma_a \\ \sin\gamma_a \end{pmatrix} + \begin{pmatrix} w_n \\ w_e \\ -w_d \end{pmatrix}$$

and assuming level flight, i.e., $\gamma_a = 0$,
and no down component of wind,
i.e., $w_d = 0$:

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{h} \end{pmatrix} = V_a \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix} + \begin{pmatrix} w_n \\ w_e \\ 0 \end{pmatrix}$$

This equation is typically called the Dubin's car model.



Coordinated Turn

From Chapter 2 we have

$$\dot{\chi} = \frac{g}{V_g} \tan\phi \cos(\chi - \psi)$$

For constant-altitude flight with no down component of wind:

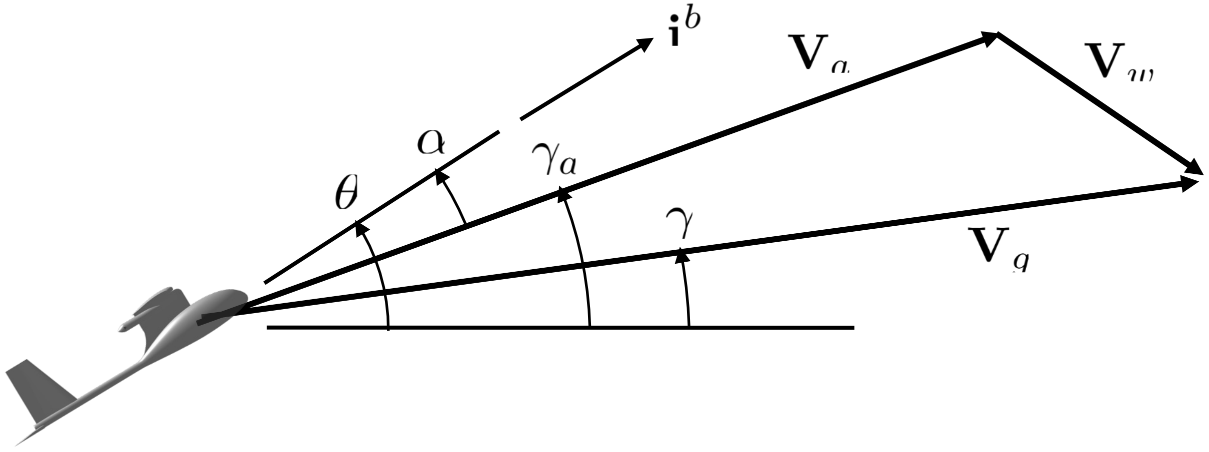
$$\dot{V}_g = \frac{\dot{V}_a}{\cos(\chi - \psi)} + V_g \dot{\chi} \tan(\chi - \psi)$$

$$\dot{\psi} = \frac{\dot{V}_a}{V_a} \tan(\chi - \psi) + \frac{V_g \dot{\chi}}{V_a \cos(\chi - \psi)}$$

If airspeed is constant, then we get the standard coordinated turn condition

$$\dot{\psi} = \frac{g}{V_a} \tan\phi$$

which is true even if when $w_n, w_e \neq 0$.



Differentiate both sides of vector wind-triangle equation (2.9). Solve resulting messy matrix equation.

Accelerating Climb

Summing forces gives

$$F_{\text{lift}} \cos\phi = \mathsf{m} V_g \dot{\gamma} + \mathsf{m} g \cos\gamma$$

Solving for $\dot{\gamma}$:

$$\dot{\gamma} = \frac{g}{V_g}\left(\frac{F_{\text{lift}}}{\mathsf{m}g}\cos\phi - \cos\gamma\right)$$

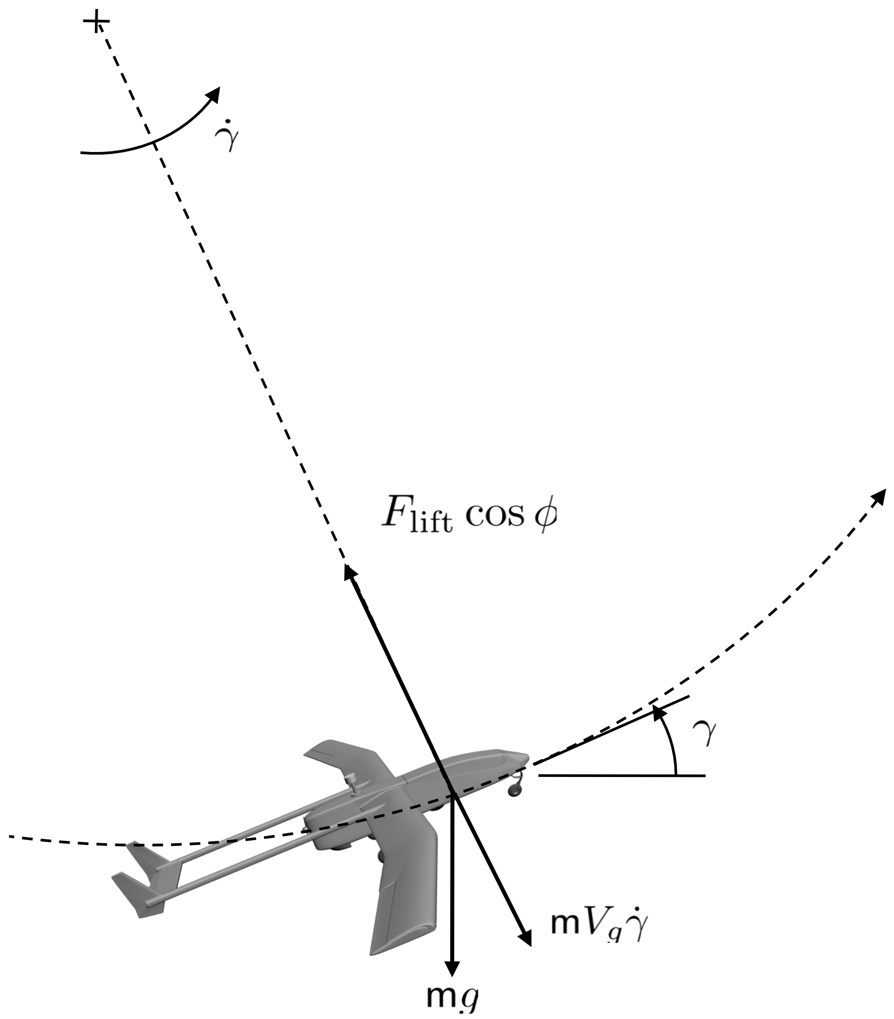
The load factor (often expressed in g's) is defined as

$$n_{lf} \stackrel{\triangle}{=} F_{\text{lift}}/\mathsf{m}g.$$

Note that for wings level, horizontal flight $n_{lf} = 1$. Therefore

$$\dot{\gamma} = \frac{g}{V_g}\left(n_{lf}\cos\phi - \cos\gamma\right)$$

In a constant climb where $\dot{\gamma} = 0$, we have $n_{lf} = \frac{\cos\gamma}{\cos\phi}$.



Kinematic Guidance Models

- Several guidance models can be derived, with varying levels of fidelity
- Choice of model depends on application

Kinematic Guidance Model - #1a

The simplest model (and the one used in Chapters 10, 11, 12) is given by

$$
\begin{aligned}
\dot{p}_n &= V_a \cos\psi + w_n \\
\dot{p}_e &= V_a \sin\psi + w_e \\
\ddot{\chi} &= b_{\dot{\chi}}(\dot{\chi}^c - \dot{\chi}) + b_{\chi}(\chi^c - \chi) \\
\ddot{h} &= b_{\dot{h}}(\dot{h}^c - \dot{h}) + b_h(h^c - h) \\
\dot{V}_a &= b_{V_a}(V_a^c - V_a)
\end{aligned}
$$

where ψ is given by (equation (2.12))

$$
\psi = \chi - \sin^{-1}\left(\frac{1}{V_a}\begin{pmatrix} w_n \\ w_e \end{pmatrix}^\top \begin{pmatrix} -\sin\chi \\ \cos\chi \end{pmatrix}\right)
$$

Kinematic Guidance Model - #1b

If the autopilot is designed to command heading instead of course, then the model becomes

$$
\begin{aligned}
\dot{p}_n &= V_a \cos\psi + w_n \\
\dot{p}_e &= V_a \sin\psi + w_e \\
\ddot{\psi} &= b_{\dot{\psi}}(\dot{\psi}^c - \dot{\psi}) + b_\psi(\psi^c - \psi) \\
\ddot{h} &= b_{\dot{h}}(\dot{h}^c - \dot{h}) + b_h(h^c - h) \\
\dot{V}_a &= b_{V_a}(V_a^c - V_a)
\end{aligned}
$$

Kinematic Guidance Models - #2a

A more accurate model is to command the roll angle and use the coordinated turn model for χ:

$$\begin{aligned}
\dot{p}_n &= V_a \cos\psi + w_n \\
\dot{p}_e &= V_a \sin\psi + w_e \\
\dot{\chi} &= \frac{g}{V_g} \tan\phi \cos(\chi - \psi) \\
\ddot{h} &= b_{\dot{h}}(\dot{h}^c - \dot{h}) + b_h(h^c - h) \\
\dot{V}_a &= b_{V_a}(V_a^c - V_a) \\
\dot{\phi} &= b_\phi(\phi^c - \phi)
\end{aligned}$$

where ψ and V_g are given by (equations (2.10) and (2.12))

$$\psi = \chi - \sin^{-1}\left(\frac{1}{V_a}\begin{pmatrix} w_n \\ w_e \end{pmatrix}^\top \begin{pmatrix} -\sin\chi \\ \cos\chi \end{pmatrix}\right)$$

$$V_g = w_n \cos\chi + w_e \sin\chi + \sqrt{(w_n\cos\chi + w_e \sin\chi)^2 + V_a^2 - w_n^2 - w_e^2}$$

Kinematic Guidance Models - #2b

Or, in terms of heading we have

$$
\begin{aligned}
\dot{p}_n &= V_a \cos\psi + w_n \\
\dot{p}_e &= V_a \sin\psi + w_e \\
\dot{\psi} &= \frac{g}{V_a}\tan\phi \\
\ddot{h} &= b_{\dot{h}}(\dot{h}^c - \dot{h}) + b_h(h^c - h) \\
\dot{V}_a &= b_{V_a}(V_a^c - V_a) \\
\dot{\phi} &= b_\phi(\phi^c - \phi)
\end{aligned}
$$

where χ and V_g are computed from the wind triangle if needed by the autopilot

Kinematic Guidance Models - #3

More accurate still is to command the flight path angle

$$
\begin{aligned}
\dot{p}_n &= V_a \cos\psi \cos\gamma_a + w_n \\
\dot{p}_e &= V_a \sin\psi \cos\gamma_a + w_e \\
\dot{h} &= V_a \sin\gamma_a - w_d \\
\dot{\psi} &= \frac{g}{V_a} \tan\phi \\
\dot{\gamma} &= b_\gamma(\gamma^c - \gamma) \\
\dot{V}_a &= b_{V_a}(V_a^c - V_a) \\
\dot{\phi} &= b_\phi(\phi^c - \phi)
\end{aligned}
$$

where γ_a is given by equation (2.11)

$$
\gamma_a = \sin^{-1}\left(\frac{V_g \sin\gamma + w_d}{V_a}\right)
$$

and χ and V_g are computed from the wind triangle if needed by the autopilot

Kinematic Guidance Models - #4

More accurate still is to command the load factor

$$
\begin{aligned}
\dot{p}_n &= V_a \cos\psi \cos\gamma_a + w_n \\
\dot{p}_e &= V_a \sin\psi \cos\gamma_a + w_e \\
\dot{h} &= V_a \sin\gamma_a - w_d \\
\dot{\psi} &= \frac{g}{V_a} \tan\phi \\
\dot{\gamma} &= \frac{g}{V_g} (n_{lf} \cos\phi - \cos\gamma) \\
\dot{V}_a &= b_{V_a} (V_a^c - V_a) \\
\dot{\phi} &= b_\phi (\phi^c - \phi) \\
\dot{n}_{lf} &= b_n (n_{lf}^c - n_{lf})
\end{aligned}
$$

where χ, V_g, and γ_a are computed from the wind triangle

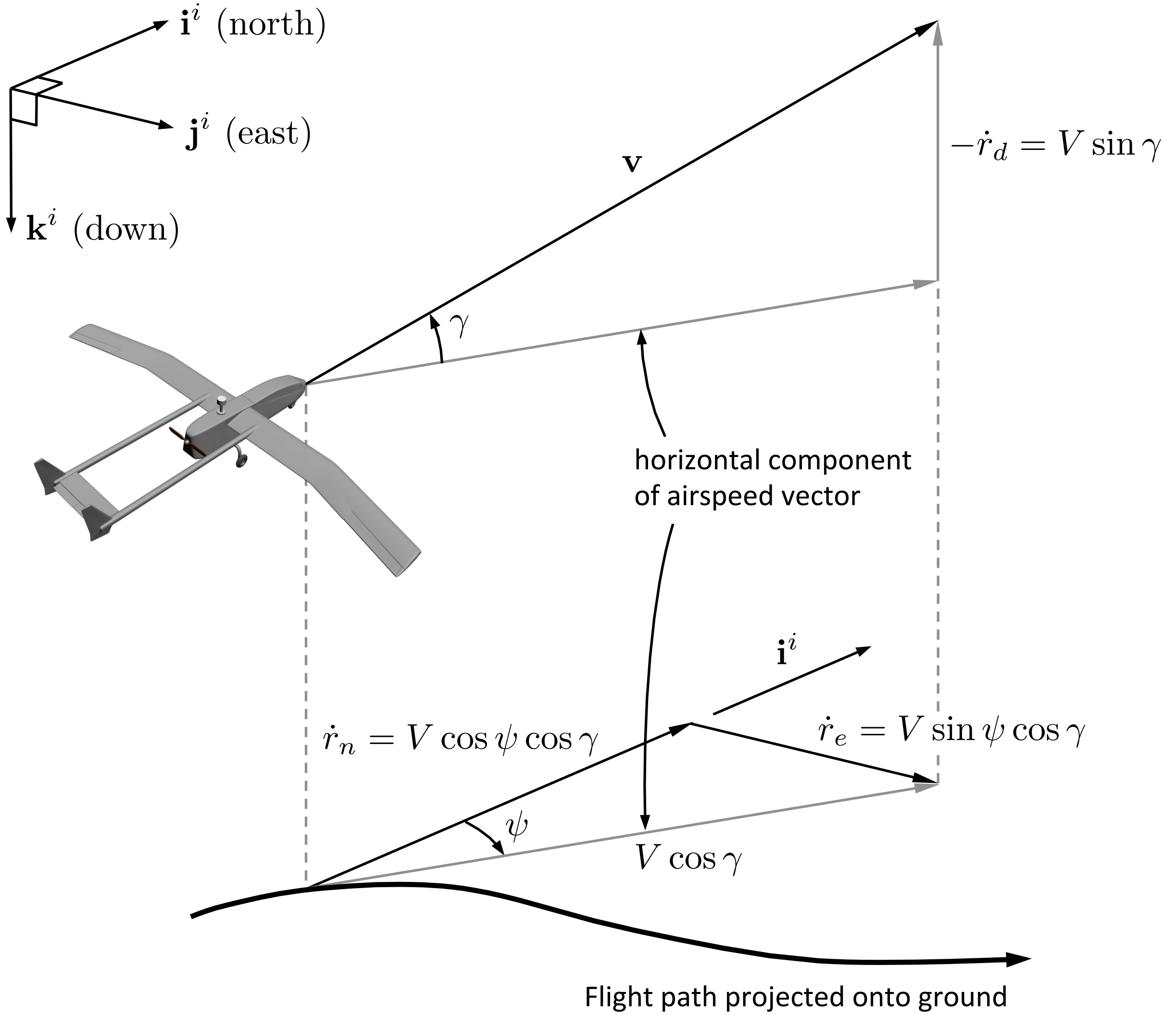
Dubins Airplane Model

The Dubin's airplane model is a particularly simple model often used in path planning:

$$\dot{r}_n = V \cos\psi \cos\gamma^c$$
$$\dot{r}_e = V \sin\psi \cos\gamma^c$$
$$\dot{r}_d = -V \sin\gamma^c$$
$$\dot{\psi} = \frac{g}{V} \tan\phi^c.$$

Assumes constant velocity V, and subject the constraints:

$$|\phi^c| \leq \bar{\phi}$$
$$|\gamma^c| \leq \bar{\gamma}.$$



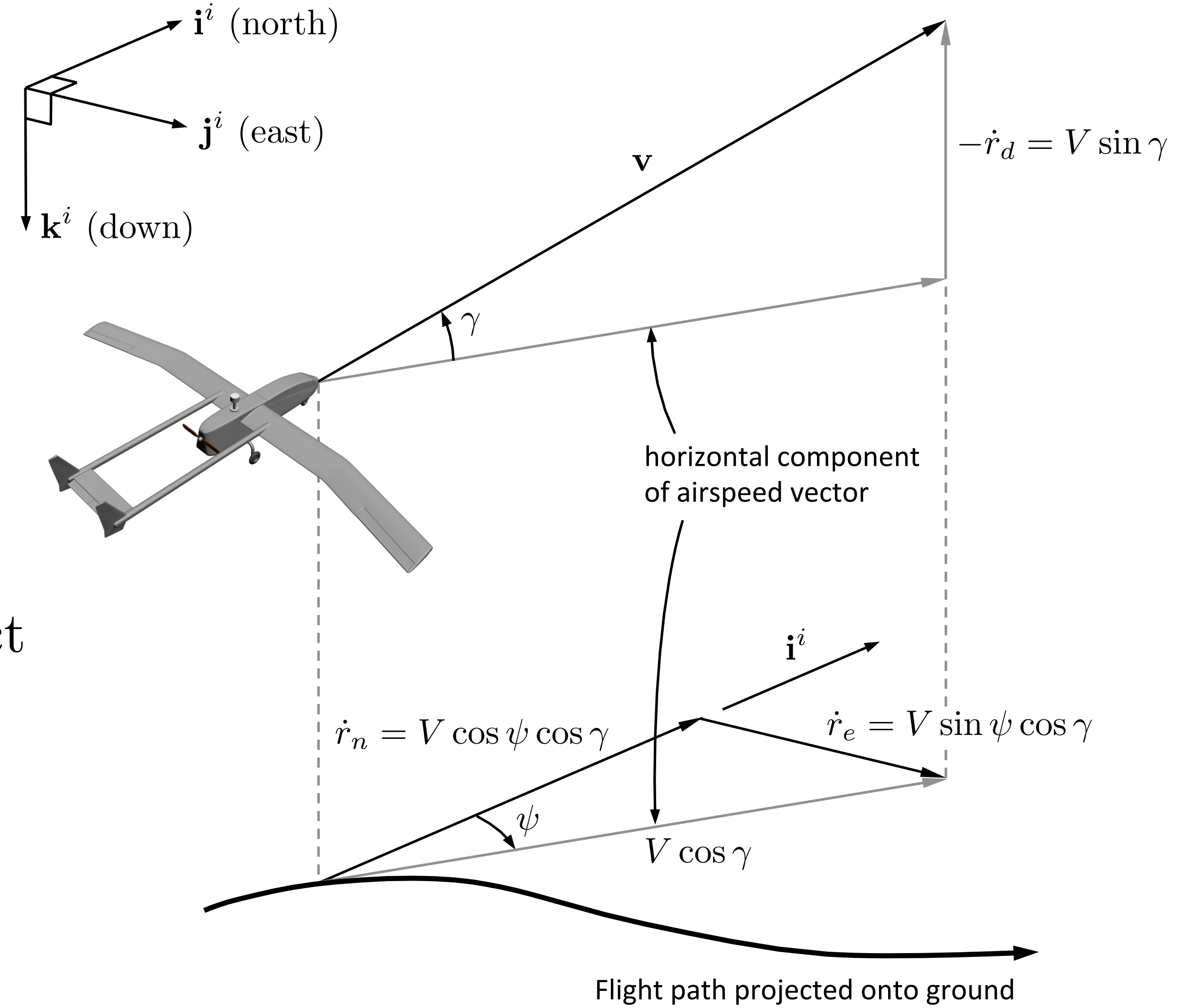
Dubins Airplane Model (simplified)

Often simplified as:

$$\dot{r}_n = V\cos\psi$$
$$\dot{r}_e = V\sin\psi$$
$$\dot{r}_d = -V\sin\gamma^c$$
$$\dot{\psi} = \frac{g}{V}\tan\phi^c.$$

Assumes constant velocity V, and subject the constraints:

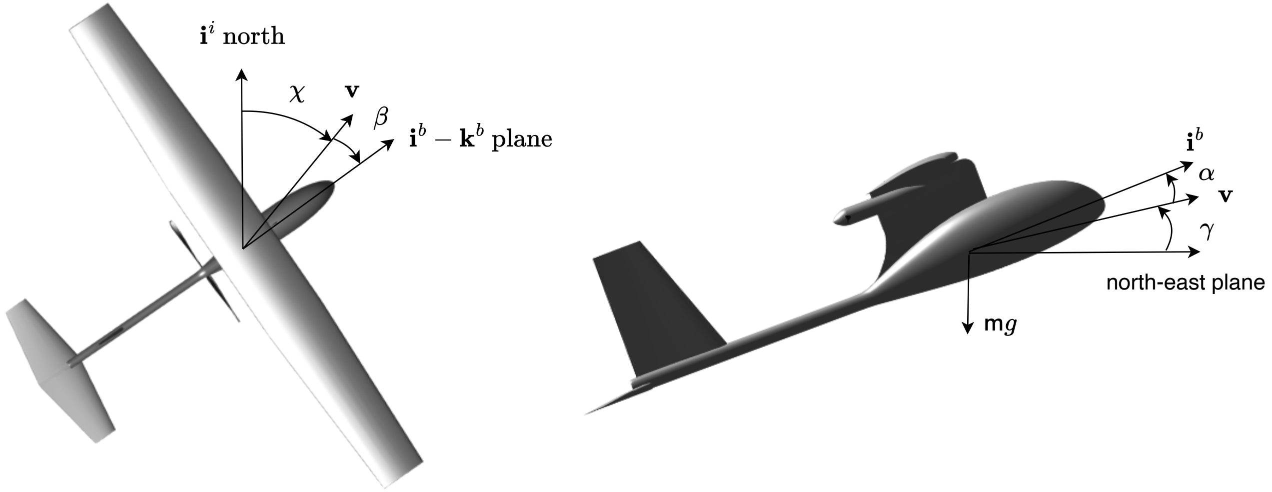
$$|\phi^c| \leq \bar{\phi}$$
$$|\gamma^c| \leq \bar{\gamma}.$$



Dynamic Guidance Models

Standard kinematic expression for evolution of position:

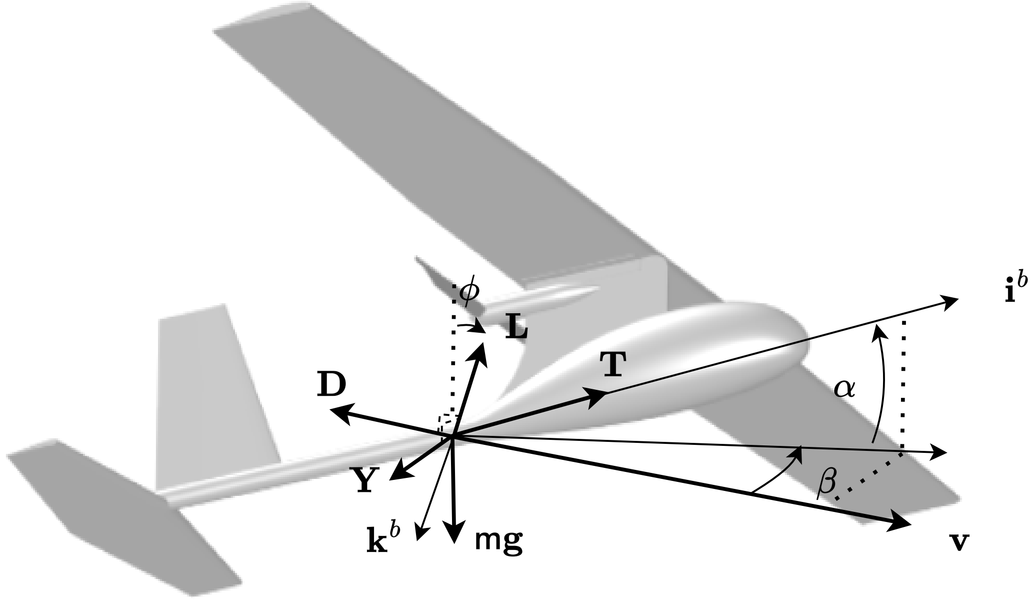
$$\dot{p}_n = V_g \cos\chi \cos\gamma$$
$$\dot{p}_e = V_g \sin\chi \cos\gamma$$
$$\dot{p}_d = -V_g \sin\gamma.$$



Dynamic Guidance Models

Thrust in the wind frame:

$$\mathbf{T}^w = R_b^w \mathbf{T}^b$$
$$= \begin{pmatrix} \cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} T\cos\alpha\cos\beta \\ -T\cos\alpha\sin\beta \\ -T\sin\alpha \end{pmatrix}.$$



Aerodynamic forces in the wind frame:

$$\mathbf{F}^w_{aero} = \begin{pmatrix} -D \\ -Y \\ -L \end{pmatrix}$$

Dynamic Guidance Models

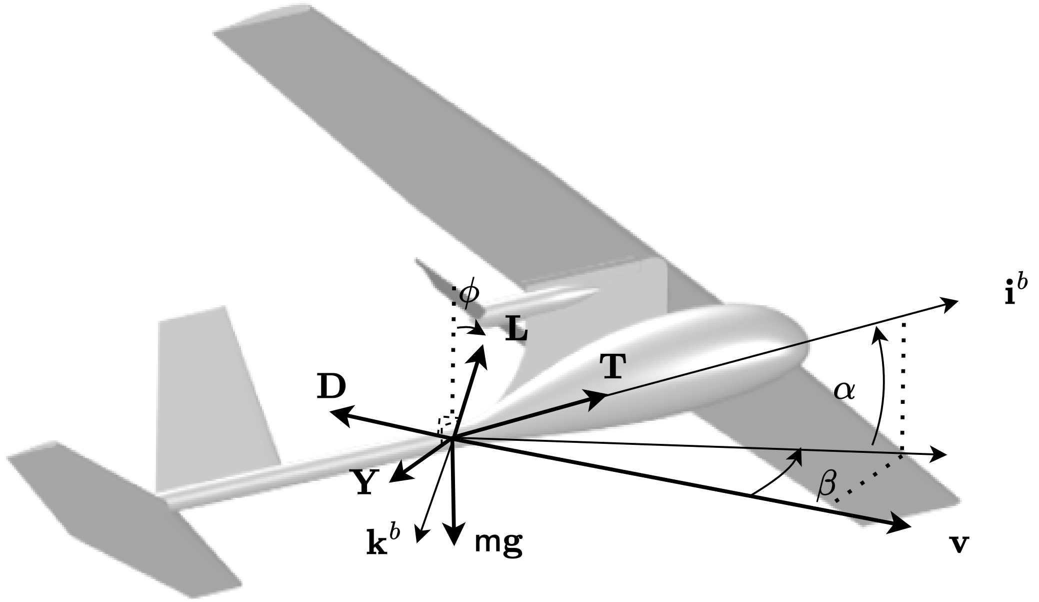
Total forces in the wind frame:

$$\mathbf{F}^w = \begin{pmatrix} -D + T\cos\alpha\cos\beta \\ -Y - T\cos\alpha\sin\beta \\ -L - T\sin\alpha \end{pmatrix}.$$

Total forces in the vehicle-2 (unrolled) frame:

$$\mathbf{F}^{v2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} -D + T\cos\alpha\cos\beta \\ -Y - T\cos\alpha\sin\beta \\ -L - T\sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} -D + T\cos\alpha\cos\beta \\ -(Y + T\cos\alpha\sin\beta)\cos\phi - (L + T\sin\alpha)\sin\phi \\ (Y + T\cos\alpha\sin\beta)\sin\phi - (L + T\sin\alpha)\cos\phi \end{pmatrix}.$$

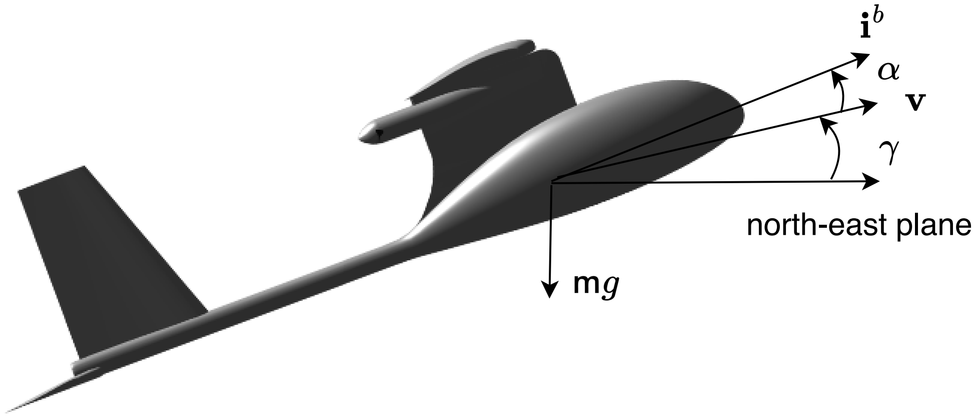


Dynamic Guidance Models

Along the $\mathbf{i}^{v2}$ axis, the magnitude of the velocity vector evolves as:

$$\mathsf{m}\dot{V}_g = T\cos\alpha\cos\beta - D - \mathsf{m}g\sin\gamma$$

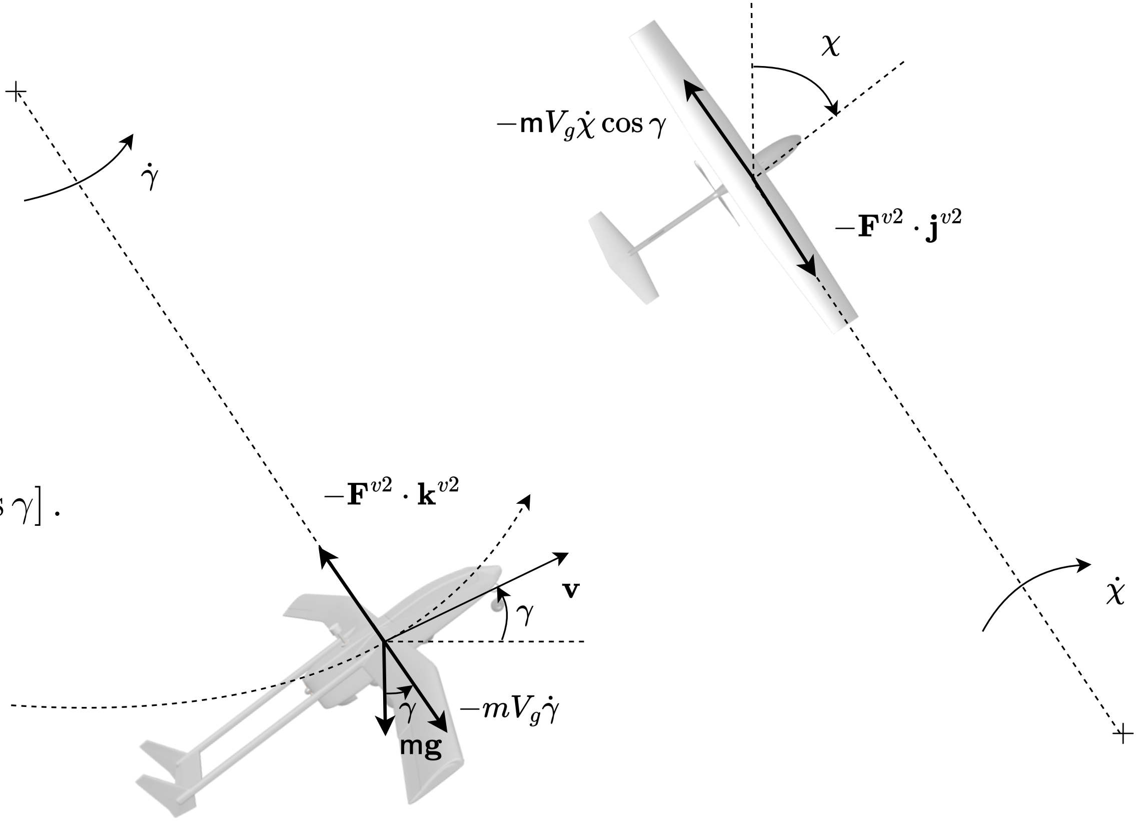
$$\implies \dot{V}_g = \frac{T}{\mathsf{m}}\cos\alpha\cos\beta - \frac{D}{\mathsf{m}} - g\sin\gamma.$$



Dynamic Guidance Models

Given:

$$\mathbf{F}^{v2} = \begin{pmatrix} -D + T\cos\alpha\cos\beta \\ -(Y + T\cos\alpha\sin\beta)\cos\phi - (L + T\sin\alpha)\sin\phi \\ (Y + T\cos\alpha\sin\beta)\sin\phi - (L + T\sin\alpha)\cos\phi \end{pmatrix}.$$



Longitudinal direction:

$$\mathsf{m}V_g\dot{\gamma} = -(Y + T\cos\alpha\sin\beta)\sin\phi + (L + T\sin\alpha)\cos\phi - \mathsf{m}g\cos\gamma$$

$$\implies \dot{\gamma} = \frac{1}{\mathsf{m}V}\left[(L + T\sin\alpha)\cos\phi - (Y + T\cos\alpha\sin\beta)\sin\phi - \mathsf{m}g\cos\gamma\right].$$

Lateral direction:

$$\mathsf{m}V_g\dot{\chi}\cos\gamma = (Y + T\cos\alpha\sin\beta)\cos\phi + (L + T\sin\alpha)\sin\phi$$

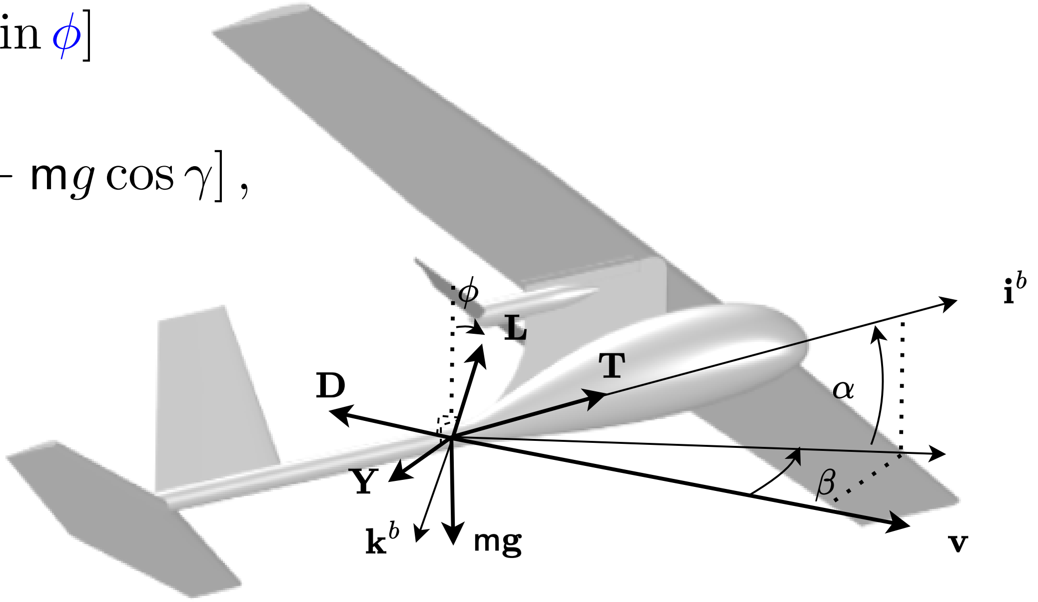
$$\implies \dot{\chi} = \frac{1}{\mathsf{m}V_g\cos\gamma}\left[(Y + T\cos\alpha\sin\beta)\cos\phi + (L + T\sin\alpha)\sin\phi\right].$$

Dynamic Guidance Models: Summary

$$\dot{p}_n = V_g \cos\chi \cos\gamma$$
$$\dot{p}_e = V_g \sin\chi \cos\gamma$$
$$\dot{h} = V_g \sin\gamma$$
$$\dot{V}_g = \frac{T}{\mathrm{m}} \cos\alpha \cos\beta - \frac{D}{\mathrm{m}} - g \sin\gamma$$
$$\dot{\chi} = \frac{1}{\mathrm{m} V_g \cos\gamma} \left[(Y + T\cos\alpha \sin\beta)\cos\phi + (L + T\sin\alpha)\sin\phi \right]$$
$$\dot{\gamma} = \frac{1}{\mathrm{m} V_g} \left[(L + T\sin\alpha)\cos\phi - (Y + T\cos\alpha\sin\beta)\sin\phi - \mathrm{m} g \cos\gamma \right],$$

where

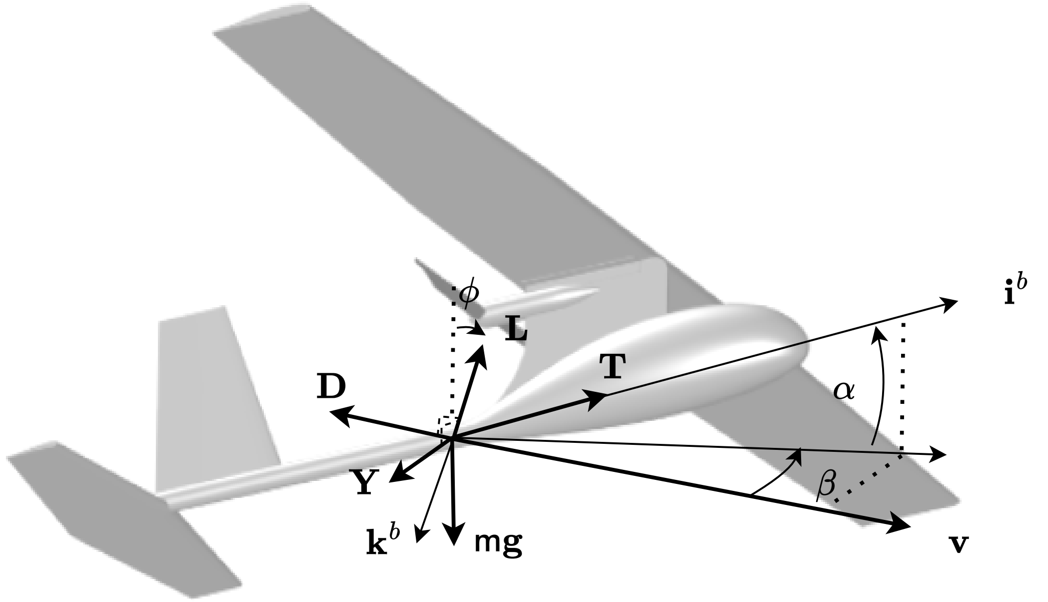
$$L(\alpha) = \frac{1}{2}\rho V_a^2 S \left(C_{L_0} + C_{L_\alpha}\alpha \right)$$
$$D(\alpha) = \frac{1}{2}\rho V_a^2 S \left(C_{D_0} + C_{D_\alpha}\alpha + C_{D_{\alpha^2}}\alpha^2 \right)$$
$$Y(\beta) = \frac{1}{2}\rho V_a^2 S C_{Y_\beta}\beta,$$



Dynamic Guidance Models: Summary (simplified)

When sideslip is regulated to zeros:

$$\dot{p}_n = V_g \cos\chi \cos\gamma$$
$$\dot{p}_e = V_g \sin\chi \cos\gamma$$
$$\dot{h} = V_g \sin\gamma$$
$$\dot{V}_g = \frac{T}{\mathrm{m}} \cos\alpha - \frac{D}{\mathrm{m}} - g\sin\gamma$$
$$\dot{\chi} = \frac{1}{\mathrm{m}V_g\cos\gamma}(L + T\sin\alpha)\sin\phi$$
$$\dot{\gamma} = \frac{1}{\mathrm{m}V_g}\left[(L + T\sin\alpha)\cos\phi - \mathrm{m}g\cos\gamma\right],$$



where

$$L(\alpha) = \frac{1}{2}\rho V_a^2 S\left(C_{L_0} + C_{L_\alpha}\alpha\right)$$
$$D(\alpha) = \frac{1}{2}\rho V_a^2 S\left(C_{D_0} + C_{D_\alpha}\alpha + C_{D_{\alpha^2}}\alpha^2\right)$$