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Q1. Asymptotic Notation

(a) $\log(50) \rightarrow \Theta(1)$ $(2^{10}+1)(2^{10}+n) \rightarrow \Theta(n)$

$\dfrac{n^2+\log n}{\sqrt{n}+\log n} \rightarrow \dfrac{n^2}{n^{\frac{1}{2}}} \rightarrow \Theta(n^{\frac{3}{2}})$ $4^{2n+1} \rightarrow 2^{4n+2} \rightarrow \Theta(2^{4n})$

$\sqrt{\log n+1} \rightarrow \Theta(\sqrt{\log n})$ $4^{\frac{n}{2}+1} \rightarrow 2^n \cdot 4 \rightarrow \Theta(2^n)$

$(1^n+2^n)(3^n+4^n) = 1^n 3^n + 1^n \cdot 4^n + 2^n \cdot 3^n + 2^n \cdot 4^n$

$= 3^n + 4^n + 2^n(\cancel{3^n} + 4^n) \rightarrow \Theta(2^{3n})$

$= 2^n \cdot 2^{2n}$

$\log 2^n \rightarrow n \cdot \log 2 \rightarrow \Theta(n)$

$2^{\log n} \rightarrow n \rightarrow \Theta(n)$

$(\log(\frac{n}{2}))^2 \rightarrow (\log n - \log 2)^2 \rightarrow (\log n)^2 \rightarrow \Theta((\log n)^2)$

☆ Rank:

$\therefore \log(50) \rightarrow \Theta(1)$, ~~$(\log(\frac{n}{2}))^2 \rightarrow \Theta((\log n)^2)$~~

$\sqrt{\log n+1} \rightarrow \Theta(\sqrt{\log n})$

$(\log(\frac{n}{2}))^2 \rightarrow \Theta((\log n)^2)$

$(2^{10}+1)(2^{10}+n) \rightarrow \Theta(n)$

$2^{\log n} \rightarrow \Theta(n)$

$\log 2^n \rightarrow \Theta(n)$

$\dfrac{n^2+\log n}{\sqrt{n}+\log n} \rightarrow \Theta(n^{\frac{3}{2}})$

$4^{\frac{n}{2}+1} \rightarrow \Theta(2^n)$

$(1^n+2^n)(3^n+4^n) \rightarrow \Theta(2^{3n})$

$4^{2n+1} \rightarrow \Theta(2^{4n})$

non-decreasing

(b) Q1. counter: $f(n) = n\log n$ $\because n\log n$ not $\geq cn^2$ ~~when n=10~~
if $f(n) = \Omega(n^2) \geq cn^2$ $\therefore$ False $n \log n \leq n^2$ ~~n=100~~

1. If $f(n)$ is $O(n\log n)$, $f(n)$ is also $\Omega(n^2)$? False

2. If $f(n)$ is $O(n^2)$, $f(n)$ is also $O(2^n)$? True

$\because f(n)$ is $O(n^2)$ for $d>0$, when $n \geq 10$ when $n>4$

$\therefore f(n) \leq d \cdot n^2$ $n^2 \leq 2^n$ $2^n > n^2$ $\therefore O(n^2) \subseteq O(2^n)$ $\therefore$ True.

$\therefore f(n) \leq dn^2 \leq d \cdot 2^n$

(c) $f(n) = \sqrt{n}\log n + n^2 - n$ is $\Theta(n^2)$ $n < n^2$

$O(n^2)$: $f(n) \leq \_\ g(n)$ $g(n) = n^2$ $\sqrt{n}\log n < n^2$

$\therefore f(n) \leq c \cdot n^2$ $\therefore f(n)$ is $O(n^2)$

$= n^2 + n^2 + n^2 = 3n^2$ $c=3$

$\Omega(n^2)$:

$f(n) \geq \_ \cdot n^2$ $f(n) = (\sqrt{n}\log n - n) + n^2$ ~~$n^2 \geq \sqrt{n}\log n - n$~~

$\because \sqrt{n}\log n - n > 0$ $c=1$ $\therefore c_1 n^2 \leq \sqrt{n}\log + n^2 - n$

$\therefore f(n) = n^2 + w \geq 1 \cdot n^2$ $\therefore \Omega(n^2)$ $\leq c_2 n^2$

$\therefore k=1$

$\therefore \Theta(n^2)$ ~~as long~~ when $k=1$ all $n \geq 1$.

$f(n) = \dfrac{3n^2+100}{n-2\log n} \Rightarrow \dfrac{n^2}{n} = n$ is $\Theta(n)$

$O(n)$: $f(n) \leq c \cdot n$ $f(n) \Rightarrow \dfrac{3n^2}{n} = 3n$ $c=3$

$\therefore O(n)$ $n - 2\log n > 0$

$\Omega(n)$: $f(n) \geq c \cdot n$ $f(n) \rightarrow \dfrac{3n^2}{n} = 3n$ $n > 2\log n$

when $c=1$ ~~f(n)~~. $3n > n$ $k=3$

$\therefore \Omega(n)$ $\therefore \Theta(n)$ when $k=3$ $n \geq 3$

$f(n) = n + n^2 + n^3 + 2^n$ is $\Theta(2^n)$

$O(2^n)$: $f(n) \rightarrow 2^n + 2^n + 2^n + 2^n = 4 \cdot 2^n$ $\therefore c=4$

$f(n) \leq 4 \cdot 2^n$ $\therefore O(2^n)$

$\Omega(2^n)$: $f(n) = 2^n + (n+n^2+n^3) \leq 1 \cdot 2^n$ $c=1$ $\therefore \Omega(2^n)$

$\therefore \Theta(2^n)$ when $k=1$ all $n \geq 1$ $\because$ when ~~k=~~ $n \geq 1$ $2^n > n^3$

$\therefore k=1$

Q $f(n) = \sqrt{n^4+n} - n$ is $\Theta(n^2)$

$O(n^2)$ $f(n) \to n^2 + n^{\frac{1}{2}} - n \le n^2 + n^2 + n^2 = 3 \cdot n^2$ $c=3$

$f(n) \le c \cdot n^2$

$\therefore O(n^2)$

$\Omega(n^2)$

$f(n) \ge c \cdot n^2$

$f(n) \to n^2 + n^{\frac{1}{2}} - n \ge n^2$ $c=1$

$\therefore \Theta(n^2)$ ~~k=10 n≥10~~ $k=1$ $n \ge 1$ ~~n≥1 n≥10~~

~~Q2. Sorting Algo.~~

$\because \sqrt{n^4+n} - n = n^2 + \frac{1}{2n} - n$

$\therefore c_1 n^2 \le f(n) \le c_2 n$

$\because n \ge 1 \Rightarrow \frac{1}{2n} - n$ is a small adjust

$\therefore k=1$

Q2. Sorting Algorithms

(a) $L = (a, b, c, d)$ 4 numbers ?

Insert Sort :

worse case : reverse order

$(d, c, b, a) \xrightarrow[1]{c \Rightarrow d} (c, d, b, a) \xrightarrow[2]{b \Rightarrow c, d} (b, c, d, a)$

$\xrightarrow[3]{a \Rightarrow b, c, d} (a, b, c, d)$

compare : $\frac{n(n-1)}{2} = \frac{4 \times 3}{2} = 6$

swap = compare $= 6$

Merge Sort :

worse case ~~(reverse order)~~ (a, b, c, d)

Divide : (a, b) (c, d)

a b c d

Merge : (a, b) compare+=1 (c, d) compare +=1 $\Rightarrow ((a, b), (c, d))$ compare +=2

$\therefore$ compare $= 1 + 1 + 2 = 4$

(b) 3 5 6 8 7 9 2 1 4 0 ~~(original)~~ (initial)

The Question changed

1. 3568792140 same with initialize Array

$\because$ ② is Selection , ④ is Merge , ③ is Insert or Bubble

$\therefore$ ① is Insert or Bubble

2. 0568792143

$\because$ The smallest element 0 is placed first and swap with 3.

$\therefore$ is Selection Sort.

3. 356789 2140

~~$\because$ 8 and 7 are swapped, 8 and 7 are adjacent element~~

$\because$ Insert 7 to a correct position

~~$\therefore$ is Bubble Sort~~

$\therefore$ is Insert sort ~~or Bubble~~. And 356789 is the sorted part.

4. 356 879 1204

$\because$ 879 is unsorted so is not Insert or Bubble and 0 is not placed first

$\therefore$ ~~is~~ (2,1) (4,0) are sort $\therefore$ Merge Sort

(c) Q2.

1. Yes. The new updates avoid the usual merge step when it's unnecessary

2. ~~In~~ the best case is the new check succeed ~~at~~ every merge call, and the merge skip. Which means A[q] < A[q+1]
For n element, there are n-1 subpromblems.
So the number of comparisons in the best case is n-1

(b) ~~3~~ 3 5 6 8 7 9 2 1 4 0

1. ~~3~~ 3 5 6 8 7 9 2 1 4 0 is original order
excluded the answer below, it's Merge Sort,

2. 0 5 6 8 7 9 2 1 4 3
∵ The smallest 0 is placed first and swap 3
∴ Selection Sort / insert: no insert element into correct posi
Bubble: No swap neighbor or push the large one
Merge: Only one element change, into end.

3. 3 5 6 7 8 9 2 1 4 0
∵ Insert 7 to correct position
∴ Insertion Sort
It can be bubble, but ④ is definitely Bubble, so Insertion

4. 3 5 6 7 8 2 1 4 0 9
∵ 9 (biggest one) move to end and 8 move to correct position
∴ Bubble Sort

Q3 : Recurrences

$s \quad q_1 \quad q_2 \quad f$

(a) $q_1 : \frac{1}{3} \qquad q_2 : \frac{2}{3} \qquad n = f - s + 1$

$n_1 = f - q_1 + 1 \approx f - \frac{2s+f}{3} \approx \frac{2(f-s)}{3} \approx \frac{2}{3}n$

$n_2 = q_2 - s + 1 \approx \frac{s+2f}{3} - s \approx \frac{2f-2s}{3} \approx \frac{2}{3}n$

① $T(n) = 3 \cdot T(\frac{2}{3}n) + \cancel{O(1)}\ c \qquad \because n<4$ InsertSort is $O(16)$

$a = 3 \qquad b = \frac{3}{2} \qquad f(n) = c$ is $O(1)$

$k = \log_{\frac{3}{2}} 3 \approx 2.71 \qquad n^k = n^{2.71} > f(n)$

② $\therefore T(n)$ is $\theta(n^{2.71})$ is $O(n^{2.71})$

③ input $[\overset{}{\underset{0}{4}}, \overset{q_1}{\underset{1}{3}}, \overset{}{\underset{2}{5}}, \overset{q_2}{\underset{3}{1}}, \underset{4}{6}, \underset{5}{2}]$ $n = 6 \qquad f - s = 5$

(1,5) → 3 5 1 6 2 → 1 2 3 5 6

(0,3) 4 3 5 1 → 1 3 4 5

(1,5) 3 5 1 6 2 → 1 2 3 5 6

~~It'll directly use InsertionSort to sort the Array~~.

⇓

It'll correctly sort

1 2 3 4 5 6

(b) ① Input $[\underset{0}{1}, \cdots, \underset{11}{12}] \to q = 5 \quad A[5] = 6 \to [\underset{6}{7}, \underset{11}{12}] \to q = 8 \quad A[8] = 9$

$\to [\underset{9}{10}, \cdots, \underset{11}{12}] \to q = 10 \quad A[10] = 11 \to [\underset{11}{12}] \to A[11] = 12 \to$ back Return

print : 6 9 11 12 11 9 6

② $T(n) = T(n/2) + c \qquad a = 1 \quad b = 2 \quad f(n) = c$

$k = \log_2 1 = 0 \qquad n^k = 1 \qquad f(n)$ is $O(1) = O(n^k)$ $\therefore T(n)$ is $\theta(\log n)$

③ Assume: $T(n)$ is $O(\log n)$

goal: $T(n) \le d \cdot \log n \qquad d > 0$

$T(n/2) \le d\log(\frac{n}{2}) = d\log n - d\log 2$

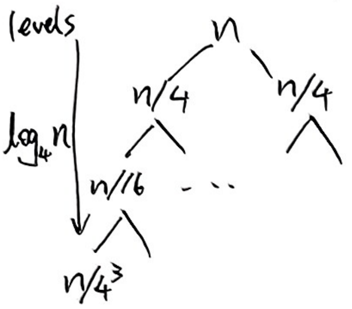
$T(n) = T(n/2) + c_1 \le d\log n + c_1 - d\log 2 = d\log n + c_2 \le d\log n$

as long as $\quad c_1 - d\log 2 \le 0 \quad c_1 \le d\log 2 \quad c_1 \le d$

$\therefore T(n)$ is $O(\log n)$ as long as $c_1 \le d$

(c) Recursion Tree

① $T(n) = 2T(n/4) + n$ $\qquad T(n/4) = 2T(n/16) + n/4$



$T(n)$

$T(n/4)$ $\frac{n}{2}$

$T(n/16)$ $\frac{n}{4}$

$T(n/4^3)$ $\frac{n}{8}$

Total levels:

$$\sum_{k=0}^{L} \frac{n}{2^k} = \sum_{k=0}^{L} \cdot \left(\frac{1}{2}\right)^k \cdot n = c \cdot n$$

$\therefore T(n)$ is $\Theta(n)$

② $T(n) = 2T(n/4) + 1$

1

2

4

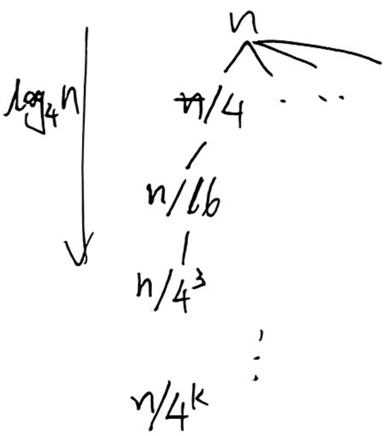
2^k

$$\sum_{k=0}^{\log_4 n} 2^k = \frac{2^{\log_4 n + 1} - 1}{2-1} = 2 \cdot n^{\frac{1}{2}} - 1$$

$\therefore T(n)$ is $O(n^{\frac{1}{2}})$

$$2^{\log_4 n} = 2^{\log_2 n^{\frac{1}{2}}} = n^{\frac{1}{2}}$$

③ $T(n) = 16T(n/4) + n$



n

$4n$

$16n$

$4^3 n$

$4^k n$

$$\sum_{k=0}^{\log_4 n} 4^k \cdot n = n \cdot \frac{4^{\log_4 n + 1} - 1}{4-1} = \frac{n(4n-1)}{3} = \frac{4n^2 - n}{3}$$

$\therefore T(n)$ is $\Theta(n^2)$

(d) ① $T(n) = 6T(n/2) + n^2 \log n$ $\qquad a = 6 \quad b = 2 \quad f(n) = n^2 \log n$

$k = \log_2 6 \quad n^k = n^{2.585} > n^2 \log n \qquad \therefore T(n)$ is $\Theta(n^{\log_2 6})$

② $T(n) = 16T(n/4) + \log n \qquad a = 16 \quad b = 4 \quad f(n) = \log n$

$k = \log_4 16 = 2 \quad n^2 > \log n \qquad \therefore T(n)$ is $\Theta(n^2)$

③ $T(n) = 6T(n/4) + n^2 \qquad a = 6 \quad b = 4 \quad f(n) = n^2$

$k = \log_4 6 \quad n^k = n^{1.292} < n^2 \qquad \therefore T(n)$ is $\Theta(n^2)$