Bar construction and group (co)homology

Bar construction

Here is what we call a 'bar construction', which is deceptively simple but will turn out to be highly useful.

Let X be a set. Consider the collection of sets

$$K_\bullet = \{K_0, K_1, K_2, \cdots\}$$

given by

$$K_r := X^{r+1}.$$

One can view K_r as the collection of maps from $\{0, 1, \cdots, r\}$ to X.

For each $i = 0, 1, \cdots, r$, there is a map

$$s_i \colon X^{r+1} \to X^r$$

which removes the i-th component.

Bar construction, continued

One can form a chain complex out of a bar construction, too.

Here we explain the construction of the desired chain complex. Let

$$f\colon \{0, \cdots, r\} \longrightarrow X$$

be a function. Then, for each $i = 0, 1, \cdots, r$, there is a map

$$f_i\colon \{0, \cdots, r-1\} \longrightarrow X$$

defined by the operation "omit i". Namely,

$$f_i(j) = \begin{cases} f(j) & \text{if } j < i \\ f(j+1) & \text{if } j \geq i. \end{cases}$$

Bar construction, continued

As before, X is a set to which we are applying the 'bar construction' to get $K_\bullet$, $K_r = X^{r+1}$. Take

$$C_r(K_\bullet, A)$$
$$= \{\text{finite formal } A\text{-linear combinations of } f\colon \{0, \cdots, r\} \to X\}.$$

and define

$$df := \sum_{i=0}^{r} (-1)^i f_i$$

for $f \in C_r(K_\bullet, A)$.

Group cohomology via bar construction.

Now take the set $X = G$ to be a group. Let L be an A-module on which G acts on the left. Consider

$$C^r(G, L) := \{\phi\colon G^{r+1} \to L\colon \text{any } G\text{-equivariant function}\}.$$

Being G-equivariant means $g \cdot \phi(g_0, \cdots, g_r) = \phi(gg_0, \cdots, gg_r)$.

Proposition

$(C^r(G, L), d)$ *forms a cochain complex.*

Group cohomology via bar construction

Definition

The group cohomology of G with coefficients in L is defined to be

$$H^r\left(C^\bullet\left(G, L\right)\right)$$

for $r = 0, 1, \cdots$.

If L is an A-module on which G acts, then each $H^r(G, L)$ is an A-module.

Interpretation of H^0

Define

$$L^G := \{x \in L \colon gx = x, \forall g \in G\}.$$

Proposition

We have $H^0(G, L) = L^G$*.*

Proof.

Unfold the definition of being G-equivariant, and use the condition $d\phi(g, h) = 0$ for all $g, h \in G$. $\square$

Interpretation of H^1 when $L = \mathbb{Z}$ with trivial G-action

For a group G, let $[G, G]$ be its commutator subgroup. It is a normal subgroup, and $G/[G, G]$ is the largest abelian quotient of G.

As an abelian group $G/[G, G]$ is presented by generators $[g]$ for each $g \in G$ and relations $[gh] = [hg]$ for each $g, h \in G$.

Proposition

There is a natural bijection $H^1(G, \mathbb{Z}) \simeq \mathrm{Hom}\,(G/[G, G], \mathbb{Z})$.

Proof.

Let $\phi \in Z^1(G, \mathbb{Z})$ be a cocycle; $\phi\colon G \times G \to \mathbb{Z}$ such that $d\phi = 0$. Define

$$f_\phi(a) := \phi(a, 1)$$

for $a \in G$. We wish to show that $f_\phi(a)$ induces a homomorphism $G/[G, G] \to \mathbb{Z}$ and that the association $\phi \mapsto f_\phi$ induces the desired bijection.

Proof continued

Claim: f_ϕ is well-defined.

Let $a = g^{-1}h^{-1}gh \in [G, G]$. Want to show $f_\phi(a) = 0$.

$$\begin{aligned} f_\phi(g^{-1}h^{-1}gh) &= \phi(g^{-1}h^{-1}gh, 1) \\ &= \phi(gh, gh) \\ &= \phi(h, h) \\ &= \phi(1, 1) \\ &= \phi(1, 1) - \phi(1, 1) + \phi(1, 1) \\ &= d\phi(1, 1, 1) \\ &= 0 \end{aligned}$$

In particular, $\phi(1, 1) = 0$.

Proof continued

Claim: The map

$$f_\phi \colon G/[G,G] \to \mathbb{Z}$$

satisfies $f_\phi(g^{-1}) = -f_\phi(g)$.

$$\begin{aligned}
0 =& d\phi(g,1,g) \\
=& \phi(1,g) - \phi(g,g) + \phi(g,1) \\
=& \phi(1,g) - \phi(1,1) + \phi(g,1) \\
=& \phi(1,g) + \phi(g,1) \\
=& \phi(g^{-1},1) + \phi(g,1).
\end{aligned}$$

$\Rightarrow \phi(g^{-1},1) = -\phi(g,1)$
$\Rightarrow f_\phi(g^{-1}) = -f_\phi(g)$

Proof continued

Claim: f_ϕ is a group homomorphism.

$$\begin{aligned}
0 &= d\phi(g^{-1}, h, 1) \\
&= \phi(h, 1) - \phi(g^{-1}, 1) + \phi(g^{-1}, h) \\
&= \phi(h, 1) + \phi(g, 1) + \phi(h^{-1}g^{-1}, 1) \\
&= \phi(h, 1) + \phi(g, 1) - \phi(gh, 1) \\
&= f_\phi(h) + f_\phi(g) - f_\phi(gh)
\end{aligned}$$

$\Rightarrow f_\phi(gh) = f_\phi(g) + f_\phi(h)$ for all $g, h \in G$.

Proof continued

So far, we have constructed a map

$$Z^1(G, \mathbb{Z}) \to \mathrm{Hom}(G/[G, G], \mathbb{Z})$$

by sending $\phi \mapsto f_\phi$.

We will show that it factors through $H^1(G, \mathbb{Z})$. Suppose that ϕ is of the form

$$\phi = d\eta$$

where η is an arbitrary G-equivariant map $\eta \colon G \to \mathbb{Z}$.

Want to show $f_\phi(a) = 0$ for all $a \in G$.

$$\begin{aligned} f_\phi(a) &= d\eta(a, 1) \\ &= \eta(1) - \eta(a) \\ &= \eta(1) - \eta(1) \\ &= 0 \end{aligned}$$

Proof continued

We have constructed a map

$$H^1(G, \mathbb{Z}) \to \mathrm{Hom}(G/[G, G], \mathbb{Z})$$

sending $\phi \mapsto f_\phi$. We need to show that it is a bijection. We will construct an inverse.

Let $f \colon G/[G, G] \to \mathbb{Z}$ be a homomorphism. Lift it to an inverse $\tilde{f} \colon G \to \mathbb{Z}$. Define

$$\phi_f(g, h) = \tilde{f}(h^{-1}g).$$

Claim: $d\phi_f = 0$.

$$\begin{aligned}
d\phi_f(g, h, k) &= \phi_f(h, k) - \phi_f(g, k) + \phi_f(g, h) \\
&= \tilde{f}(k^{-1}h) - \tilde{f}(k^{-1}g) + \tilde{f}(h^{-1}g) \\
&= f(k^{-1}h) - f(k^{-1}g) + f(h^{-1}g) \\
&= 0
\end{aligned}$$

Proof continued

The desired bijection

$$H^1(G, \mathbb{Z}) \simeq \mathrm{Hom}(G/[G, G], \mathbb{Z}) \qquad (1)$$

is induced by $\phi \mapsto f_\phi$ with inverse $f \mapsto \phi_f$. $\square$

The bijection (1) also preserves the additive structures on both sides.
$\Rightarrow$ It is a group isomorphism.