Simplicial complex

Simplicial complexes

A simplicial complex models a space combinatorially.

Definition

A simplicial complex $K_\bullet$ consists of sets $K_0, K_1, K_2, \cdots$ which satisfy the following conditions.

1. If $\sigma \in K_r$, then $\#\sigma = r + 1$.
2. If $\sigma \in K_r$ and $\tau \subset \sigma$, then $\tau \in K_r$ for some r.

Given a simplicial complex $K_\bullet$, an element $\sigma \in K_r$ is considered an r-dimensional simplex whose vertices are chosen from K_0.

A chain complex associated to $K_\bullet$

Let A be a ring. We want to construct a chain complex from a given simplicial complex $K_\bullet$.

We need to build a graded A-module, say $C_\bullet(K_\bullet, A)$, and a differential d on it such that $d^2 = 0$.

$C_\bullet(K_\bullet, A)$

The formula for $C_\bullet(K_\bullet, A)$ is easy.

Define $C_r(K_\bullet, A)$ to be the free A-module generated by K_r. In other words, $C_r(K_\bullet, A)$ consists of formal A-linear combinations of r-simplices in $K_\bullet$.

$d\colon C_r(K_\bullet, A) \to C_{r-1}(K_\bullet, A)$

To define d, we need an auxiliary choice; a total order on K_0, say $<$.

Once such an order is fixed, a simplex $\sigma \in K_r$ can be unambiguously written as

$$\sigma = \{v_0, v_1, \cdots, v_r\}$$

with $v_0 < v_1 < \cdots < v_r$.

For $i = 0, \cdots, r$, let $\sigma_i = \{v_0, v_1, \cdots, v_r\} - \{v_i\}$, an $r-1$-simplex. The formula

$$d\{v_0, v_1, \cdots, v_r\} = \sum_{i=0}^{r} (-1)^i \sigma_i$$

defines a map of degree -1 on $C_\bullet(K_\bullet, A)$.

$d^2 = 0$

Proposition

Suppose that $K_\bullet$ is a simplicial complex. The map $d\colon C_\bullet(K_\bullet, A) \to C_\bullet(K_\bullet, A)$ satisfies $d^2 = 0$.

Proof.

Let σ be an r-simplex. If $r = 0, 1$, there is nothing to prove. Let $r \geq 2$ and $\tau \subset \sigma$ be a codimension two subsimplex. Then, τ appears in $d^2\sigma$ twice with opposite signs. $\square$

Dependence on the choice of ordering

The ordering on K_0 affects d, but not $C_\bullet(K_\bullet, A)$.

Let d' be another differential on $C_\bullet(K_\bullet, A)$ arising from a different choice of ordering. Comparing two orderings provide a permutation of σ for every $\sigma \in K_\bullet$. The map

$$f\colon \sigma \mapsto \pm\sigma$$

where the sign is chosen to be the sign of permutation on σ, induces an A-module isomorphism

$$f\colon C_\bullet(K_\bullet, A) \xrightarrow{\simeq} C_\bullet(K_\bullet, A)$$

such that $fd = d'f$.

Homology of a chain complex

If $M_\bullet$ is a chain complex with differential d, define

$$H_r(M_\bullet) = \frac{\ker(M_r \to M_{r-1})}{\operatorname{im}(M_{r+1} \to M_r)}$$

to be the r-th homology group of $M_\bullet$.

Proposition

The homology group $H_\bullet(C_\bullet(K_\bullet, A))$ of a simplicial complex $K_\bullet$ is well-defined.

Proof.

The A-module isomorphism f introduced earlier commutes with differentials. It implies that f induces a map, say $H(f)$ on homology groups. It is straightforward to show that $H(f)$ is an isomorphism. $\square$