Simplicial complexes and simplicial sets

Simplicial complex and associated chain complex

Let $K_\bullet$ be a simplicial complex. We have seen how to construct $C_\bullet(K_\bullet, A)$, the associated chain complex.

In practice, it can be quite annoying to keep track of ordering on K_0 we had to choose.

A way to overcome this problem is to consider all the possible orderings at once. To do it, view an ordering on $\sigma \in K_r$ as a bijection

$$f \colon \{0, 1, \cdots, r\} \to \sigma$$

and consider the cochain complex generated, in degree r, by all such bijections as σ runs over K_r.

Differential

Let $\widetilde{C}_\bullet(K_\bullet, A)$ be the resulting graded A-module.

Differential in this setting can be defined similarly;

$$df = \sum_{i=0}^{r} (-1)^i f_i$$

where f_i is obtained from the ordered simplex f from removing the i-th vertex.

Comparing two chain complexes

Let $K_\bullet$ be a simplicial complex. Choose an ordering on K_0. We have a map

$$\phi\colon C_\bullet(K_\bullet, A) \to \widetilde{C}_\bullet(K_\bullet, A).$$

Proposition

The map ϕ satisfies $d\phi = \phi d$.

Comparing two chain complexes, continued

The map

$$\phi \colon C_\bullet(K_\bullet, A) \to \widetilde{C}_\bullet(K_\bullet, A).$$

has a natural section, say ψ, given by the following formula. Let $f' \colon \{0, 1, \cdots, r\} \to \sigma$ be a bijection. Then there is a chosen ordering $f \colon \{0, 1, \cdots, r\} \to \sigma$ which we used for $C_\bullet$. We get a sign $\epsilon(f, f') = \pm 1$.

$$\psi(f') = \epsilon(f, f') f.$$

Proposition

Two maps ϕ and ψ induce isomorphisms at the level of homology groups. Furthermore, they are inverses to each other.

Even bigger complex

We can relax the condition that

$$f : \{0, 1, \cdots, r\} \to \sigma$$

is a bijection. Instead, one can consider arbitary maps.

It becomes more cumbersome to make geometric interpretations, but algebraically more convenient.