Chain complexes

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Modular forms and homological algebra

Goal

We want to study modular forms from an algebraic point of view.

Here, 'algebraic' means that we will define modular forms using 'homological algebra'.

In a way, homological algebra enriches the theory of rings and modules.

modules

Let A be a commutative ring with unity. A (left) module over A is consists of

- an abelian group M
- an action of A on M.

Tha latter means a map

$$\begin{aligned} A \times M &\to M \\ (a, m) &\mapsto a \cdot m \end{aligned}$$

such that the associated map

$$A \to \mathrm{End}_{\mathbb{Z}}(M)$$

is a ring homomorphism.

Such an M is often called an A-module.
We often omit "$\cdot$" as we do with ring multiplication

maps betweein modules

Let A be a commutative ring with unity. Let M and N be two modules over A. A homomorphism from M to N is a map

$$\phi\colon M \longrightarrow N$$

such that

$$\phi(a \cdot m) = a \cdot \phi(m)$$

for all $a \in A$ and all $m \in M$

examples of modules

Example ($A = \mathbb{Z}$)

A $\mathbb{Z}$-module is the same as an abelian group. A $\mathbb{Z}$-homomorphism is the same as a group homomorphism.

Example (A is a field)

If A is a field, then an A-module is the same as a vector space over A. A A-homomorphism is the same as an A-linear map.

graded modules

Let A be a commutative ring with unity. A $\mathbb{Z}$-graded module $M_\bullet$ over A consists of a family of A-modules

$$M_r$$

for each $r \in \mathbb{Z}$.

Sometimes, we view $M_\bullet$ as a direct sum

$$M_\bullet = \bigoplus_{r \in \mathbb{Z}} M_r$$

of A-modules.

One can consider positively, negatively, non-positively, non-negatively modules as well. One can talk about supports of $M_\bullet$ as well.

elements in a graded module

Let $M_\bullet = \bigoplus_{d \in \mathbb{Z}} M_r$ be a graded A-module.

An element $m \in M_\bullet$ is called homogeneous of degree r if

$$m = m_r$$

with $m_r \in M_r$.

A general element $m \in M_\bullet$ is a finite sum of homogeneous elements.

When we pick an element $m \in M_\bullet$, we typically mean a homogeneous element.

maps between a graded modules

If $M_\bullet$ and $N_\bullet$ are A-modules, finitely supported, then the abelian group

$$\mathrm{Hom}_A(M, N)$$

consisting of A-homomorphisms is naturally $\mathbb{Z}$-graded.

An A-homomorphism ϕ of degree s consists of a family

$$M_r \to N_{r+s}$$

of A-homomorphisms for every $r \in \mathbb{Z}$.

elements in a graded module

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chain complex

A chain complex over A consists of

- a graded A-module $M_\bullet$
- a map $d\colon M_\bullet \to M_\bullet$ of degree -1

such that $d^2 = 0$.

Much of gomological algebra is about chain complexes.