Bar construction and group (co)homology

Bar construction

Here is what we call a ‘bar construction’, which is deceptively simple but will turn out to be highly useful.

Let X be a set. Consider the collection of sets

$$K_\bullet = \{K_0, K_1, K_2, \cdots\}$$

given by

$$K_r := X^{r+1}.$$

One can view K_r as the collection of maps from $\{0, 1, \cdots, r\}$ to X.

Bar construction, continued

One can form a chain complex out of a bar construction, too.

Here we explain the construction of the desired chain complex. Let

$$f\colon \{0, \cdots, r\} \longrightarrow X$$

be a function. Then, for each $i = 0, 1, \cdots, r$, there is a map

$$f_i\colon \{0, \cdots, r-1\} \longrightarrow X$$

defined by the operation "omit i". Namely,

$$f_i(j) = \begin{cases} f(j) & \text{if } j < i \\ f(j+1) & \text{if } j \geq i. \end{cases}$$

Bar construction, continued

As before, X is a set to which we are applying the 'bar construction' to get $K_\bullet$, $K_r = X^{r+1}$. Take

$$
\begin{aligned}
&C_r(K_\bullet, A) \\
&= \{\text{finite formal } A\text{-linear combinations of } f\colon \{0, \cdots, r\} \to X\}.
\end{aligned}
$$

and define

$$df := \sum_{i=0}^{r} f_i$$

for $f \in C_r(K_\bullet, A)$.

Group (co)homology via bar construction.

Now take the set $X = G$ to be a group. Let L be an A-module on which G acts on the left. Consider

$$C^r(G, L) := \{\phi \colon G^{r+1} \to L \colon \text{any } G\text{-equivariant function}\}.$$

Being G-equivariant means $g \cdot \phi(g_0, \cdots, g_r) = \phi(gg_0, \cdots, gg_r)$.

Proposition

$(C^r(G, L), d)$ *forms a cochain complex.*

Group cohomology via bar construction

Definition

The group cohomology of G with coefficients in L is defined to be

$$H^r\left(C^\bullet\left(G, L\right)\right)$$

for $r = 0, 1, \cdots$.