Homology versus Cohomology

The slogan

Cohomology is dual to homology.

Chain complex and its homology groups

Recall: if

$$(M_\bullet, d)$$

is a chain complex, then its r-th homology group is defined by

$$H_r(M_\bullet) = \frac{\ker(M_r \to M_{r-1})}{\operatorname{im}(M_{r+1} \to M_r)}.$$

We view $(M_\bullet, d)$ as a chain complex that computes (or represents) the homology groups.

The slogan, continued

Taking dual of homology

1. means dualizing $(M_\bullet, d)$ and
2. is not necessarily the same as dualizing the homology groups.

Simplicial cohomology

We have seen simplicial homology. So let us dualize it.

Let $K_\bullet$ be a simplicial complex. We considered

$C_r(K_\bullet, A) := \{\text{finite formal } A\text{-linear combinations of elements in } K_r\}.$

Dualizing $C_r(K_\bullet, A)$ means considering

$$C^r(K_\bullet, A) := \mathrm{Hom}\left(C_r(M_\bullet, A), A\right).$$

Note that

$$C^r(K_\bullet, A) = \{A\text{-valued functions on } K_r\}$$

and we have a map

$$C^r(K_\bullet, A) \to C^{r+1}(K_\bullet, A)$$

induced by d. Now it is a map of degree $+1$, which also sqaures to zero.

Cochain complex

Definition

A cochain complex over A is a pair $(M^\bullet, d)$ where $M^\bullet$ is a graded A-module and d is a map of degree $+1$ such that $d^2 = 0$.

Cohomology groups are defined to be

$$H^r(M^\bullet) := \frac{\ker\left(M^r \to M^{r+1}\right)}{\operatorname{im}\left(M^{r-1} \to M^r\right)}.$$

It is customary to write

$$Z^r := \ker(M^r \to M^{r+1})$$
$$B_r := \operatorname{im}(M^{r-1} \to M^r)$$

so that we have

$$H^r(M^\bullet) = Z^r / B^r.$$