inhomogeneous bar construction

Let G be a group and L be an A-module on which G acts. Recall that the bar complex consists of

$$C^r(G, L) = \{G\text{-equivariant functions } G^{r+1} \to L\}.$$

Its typical element is called a cochain.

If you choose a set of representatives for the orbits $G \backslash G^{r+1}$, then a cochain is determined by its values on the chosen representatives. For example,

$$\{1\} \times G^r \hookrightarrow G^{r+1}$$

is such a set of representatives.

Define

$$\bar{C}^r(G, L) = \{\text{all functions } G^r \to L\}.$$

There is a natural restriction map

$$\begin{aligned} C^r(G, L) &\to \bar{C}^r(G, L) \\ \phi &\mapsto \bar{\phi} \end{aligned}$$

such that

$$\bar{\phi}(g_1, \cdots, g_r) := \phi(1, g_1, \cdots, g_r).$$

The map

$$C^r(G, L) \to \bar{C}^r(G, L)$$

has a inverse; it is given by

$$\begin{aligned} \bar{C}^r(G, L) &\to C^r(G, L) \\ \phi &\mapsto \tilde{\phi} \end{aligned}$$

where

$$\tilde{\phi}(g_0, \cdots, g_r) = g_0 \cdot \phi(g_0^{-1} g_1, \cdots, g_0^{-1} g_r).$$

Let us try to transfer the differential on $C^\bullet(G, L)$ to $\bar{C}^\bullet(G, L)$. An obvious attempt is

$$d\phi := \overline{\left(d\tilde{\phi}\right)}.$$

That is to say, if $\phi \in \bar{C}^r(G, L)$, then

$$\begin{aligned}
&d\phi(g_1, \cdots, g_{r+1}) \\
=&\overline{\left(d\tilde{\phi}\right)}(g_1, \cdots, g_{r+1}) \\
=&\left(d\tilde{\phi}\right)(1, g_1, \cdots, g_{r+1}) \\
=&\tilde{\phi}(g_1, \cdots, g_{r+1}) - \tilde{\phi}(1, g_2, \cdots, g_{r+1}) + \cdots + (-1)^{r+2}\tilde{\phi}(1, g_1, \cdots, g_r) \\
=&g_1 \cdot \phi(g_1^{-1}g_2, \cdots, g_1^{-1}g_{r+1}) \\
&- \phi(g_2, \cdots, g_{r+1}) + \cdots + (-1)^{r+2}\tilde{\phi}(g_1, \cdots, g_r).
\end{aligned}$$

Clearly, $d^2 = 0$ on $\bar{C}^\bullet(G, L)$. The pair

$$(\bar{C}^\bullet(G, L), d)$$

is called the (inhomogeneous) bar complex.

H^0 by inhomogeneous bar complex

An element

$$\phi \in \bar{C}^0(G, L)$$

is a function $\phi \colon G^0 \to L$. Let $G^0 = \{*\}$. The condition $d\phi = 0$ is equivalent to

$$(d\phi)(g) = g \cdot \phi(*) - \phi(*) = 0$$

for all $g \in G$. Thus, we recover

$$H^0(G, L) = L^G.$$

H^1 by inhomogeneous bar complex

Take $\phi \in \bar{C}^1(G, L)$. It is a function $\phi \colon G \to L$. The condition $d\phi = 0$ is equivalent to

$$d\phi(g_1, g_2) = g_1 \cdot \phi(g_1^{-1} g_2) - \phi(g_2) + \phi(g_1) = 0$$

for all $g_1, g_2 \in G$. It is equivalent, by the substitution $g_2 \to g_1 g_2$, to

$$\phi(g_1 g_2) = g_1 \cdot \phi(g_2) + \phi(g_1)$$

for all $g_1, g_2 \in G$.

H^1 by inhomogeneous bar complex, continued

Let us describe

$$d\colon \bar{C}^0(G,L) \to \bar{C}^1(G,L).$$

Recall that a 0-cochain $\eta \in \bar{C}^0(G,L)$ is determined by its unique value, say $x \in L$. Then,

$$(d\eta)(g) = g \cdot x - x.$$

H^1 by inhomogeneous bar complex, continued

We conclude that $H^1(G, L)$ is generated by functions $\phi\colon G \to L$ satisfying the 'cocycle condition'

$$\phi(g_1 g_2) = g_1 \cdot \phi(g_2) + \phi(g_1)$$

modulo the relations $\phi = \phi'$ if and only if there exists $x \in L$ such that

$$\left(\phi - \phi'\right)(g) = g \cdot x - x$$

for all $g \in G$.