

Algorithm Foundations of Data Science and Engineering

Lecture 8: Community Detection

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May 15, 2020

Outline

Course Project

Motivation

Modularity

- Graph Model

- Definition

- Variants

Modularity Matrix

- Two communities

- Multiple Community Partitioning

Louvain Method

- Introduction

- Algorithm

- Analysis

Image Compression via PCA

Image compression is used to minimize the amount of memory needed to represent an image.

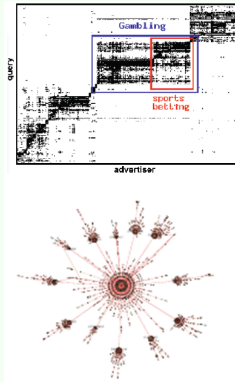
1. Downloading dataset (NWPU VHR-10 dataset is an aerial photography dataset, which can be downloaded from URL ^{a)});
2. Image compression
 - standardizing the sizes
 - PCA-based image compression
 - Evaluating the performance of image compression
3. Writing a report in Chinese or English (You will be given a report template)

Your project report and source code are due by Jun. 28, 2020.

^a<http://www.escience.cn/people/gongcheng/NWPU-VHR-10.html>

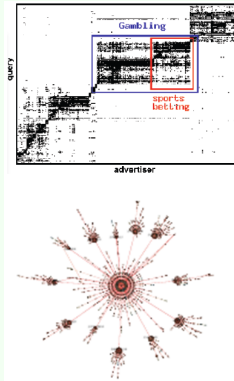
Network and communities

We often think of networks being organized into modules, clusters, communities.



Network and communities

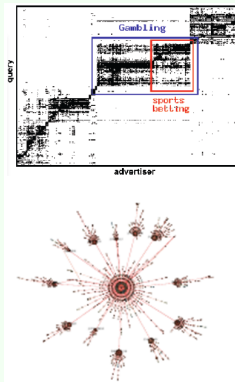
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- Network visualization.

Network and communities

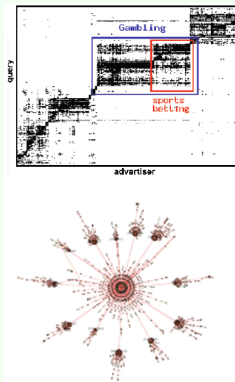
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- Network visualization.
- Find densely linked clusters.

Network and communities

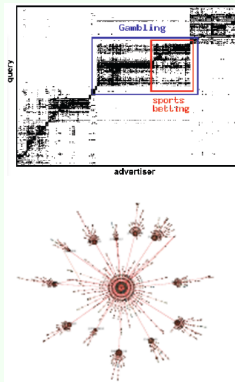
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- Find densely linked clusters.
- Find micro-markets by partitioning the query VS. advertiser graph.

Network and communities

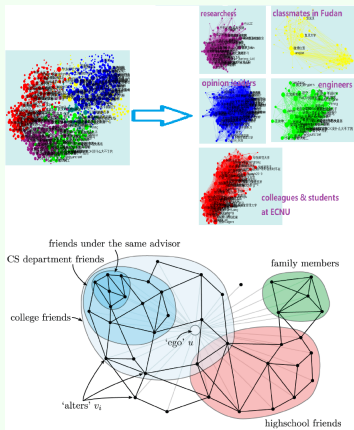
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- Network visualization.
- Find densely linked clusters.
- Find micro-markets by partitioning the query VS. advertiser graph.
- Spammer detection (water army).

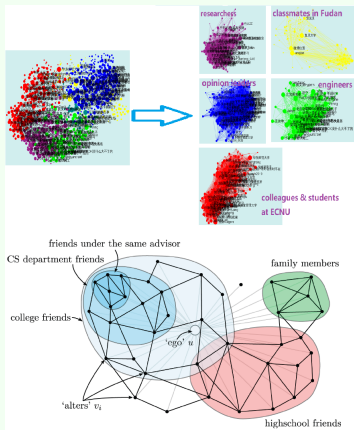
Network and communities Cont'd

Discovering social circles, circles of trust:



Network and communities Cont'd

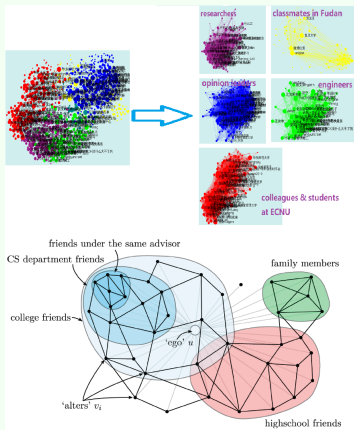
Discovering social circles, circles of trust:



■ Trust network.

Network and communities Cont'd

Discovering social circles, circles of trust:



- Trust network.
- Social circles.

Problem setting

Community structure indicates that the network divides naturally into groups of nodes with dense connections internally and sparser connections between groups.

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 - Global structure.

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- Community structures:
 - Global structure.
 - Local structure.

Global community structures

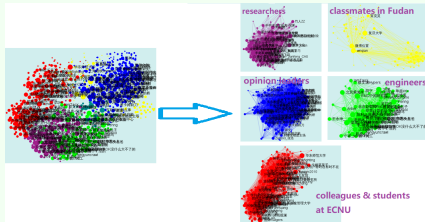
Goal

Partition nodes of a network into disjoint sets.

Global community structures

Goal

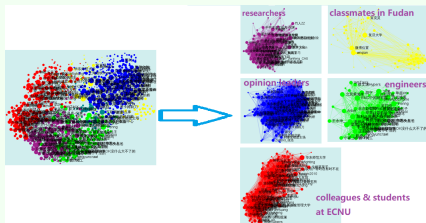
Partition nodes of a network into disjoint sets.



Global community structures

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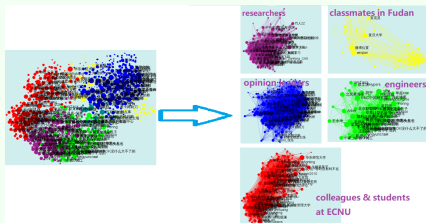


- Clustering based on vertex similarity

Global community structures

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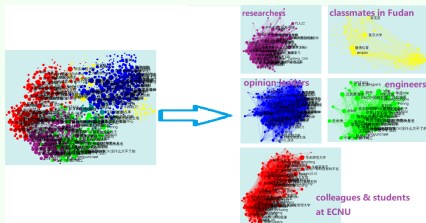


- Clustering based on vertex similarity
- Latent space models

Global community structures

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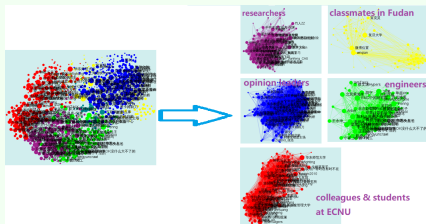


- Clustering based on vertex similarity
- Latent space models
- Spectral clustering

Global community structures

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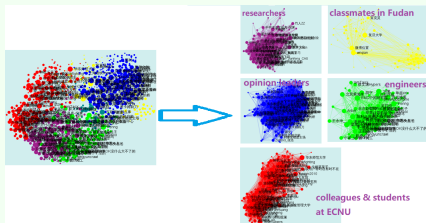


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- Modularity maximization

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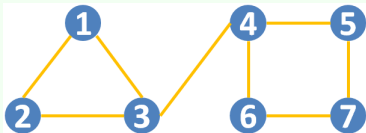


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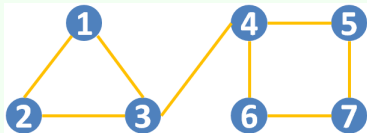
In the study of complex networks, a network is said to have community structure if the nodes of the network can be easily grouped into (potentially overlapping) sets of nodes such that each set of nodes is densely connected internally.

What makes a good community?

Input an undirected graph
 $G = (V, E)$:



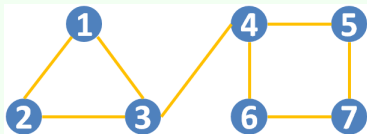
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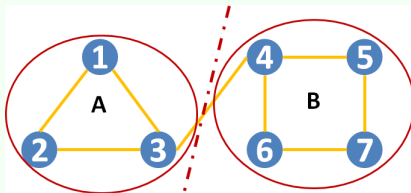
- Partitioning task: Divide vertices into 2 disjoint groups A and $B = V \setminus A$.

What makes a good community?



Input an undirected graph
 $G = (V, E)$:

- Partitioning task: Divide vertices into 2 disjoint groups A and $B = V \setminus A$.
- How can we define a “good” community in G ?



What makes a good community? Cont'd

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Express community quality as a function of the “edge cut” of the community, where cut is the set of edges (edge weights) with only one node in the community, and can be defined as

$$cut(A) = \sum_{i \in A, j \notin A} w_{ij}.$$

What makes a good community? Cont'd

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What makes a good community? Cont'd

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A good community makes a minimal cut. A cut is minimum if the size or weight of the cut is not larger than the size of any other cut. There are polynomial-time methods to solve the min-cut problem, notably the EdmondsKarp algorithm, which complexity is $O(|V||E|^2)$.

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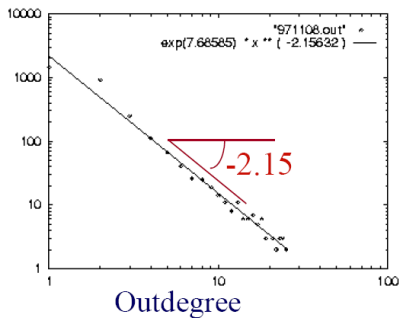
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Power-law I

Frequency

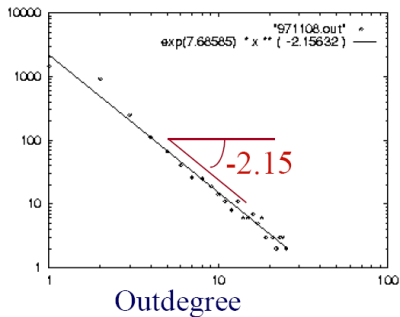


Internet topology [SIGCOMM 99]

- Out-degree distribution is plotted in log-log scale.

Power-law I

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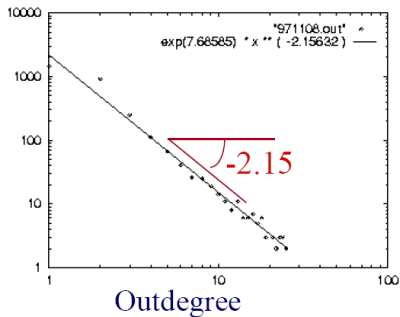


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 - 80% of a company's profits come from 20% of the time its staff spent

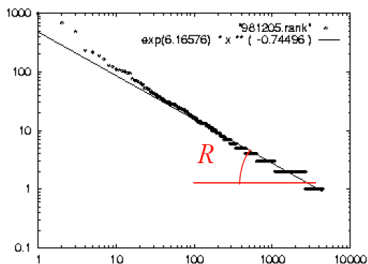
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 - 80% of a company's profits come from 20% of the time its staff spent
 - 80% of a company's sales are made by 20% of its sales staff

Power-law II

outdegree

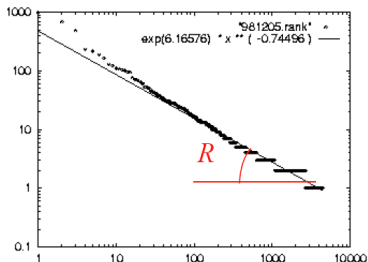


Rank: nodes in decreasing outdegree order

Rank of out-degrees [ICDE 09]

Power-law II

outdegree

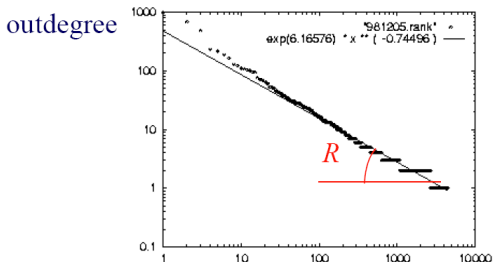


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- Vertices are ranked in decreasing out-degree order, and plotted in log-log scale.

Power-law II

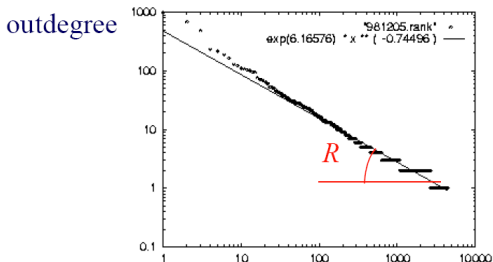


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- It forms a line with a slope ~ -0.74

Power-law II

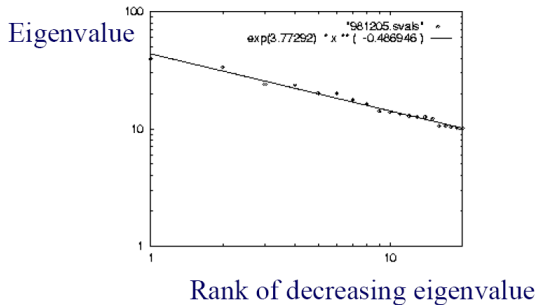


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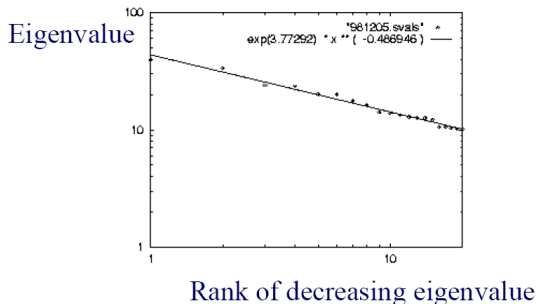
- Vertices are ranked in decreasing out-degree order, and plotted in log-log scale.
- It forms a line with a slope ~ -0.74
- $deg. = rank^{-0.74}$

Power-law III



Rank of eigenvalues [ICDE 09]

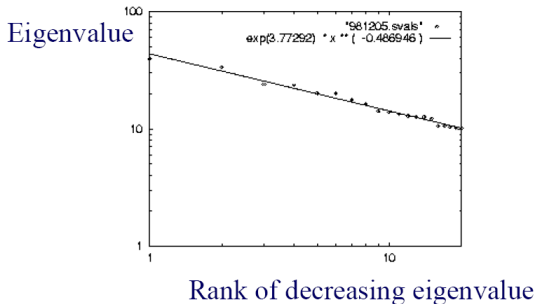
Power-law III



Rank of eigenvalues [ICDE 09]

- Eigenvalues of adjacency matrix (top 20) are ranked in decreasing order, and plotted in log-log scale.

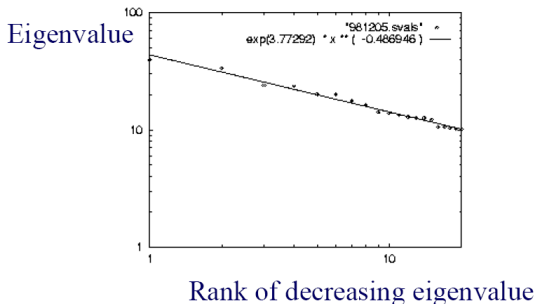
Power-law III



Rank of eigenvalues [ICDE 09]

- Eigenvalues of adjacency matrix (top 20) are ranked in decreasing order, and plotted in log-log scale.
- It forms a line with a slope ~ -0.48

Power-law III

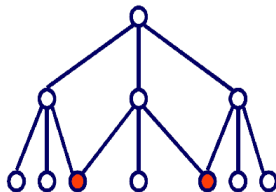
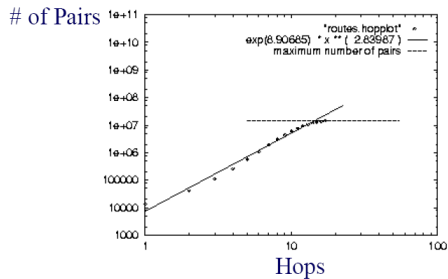


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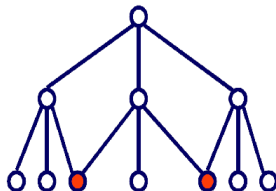
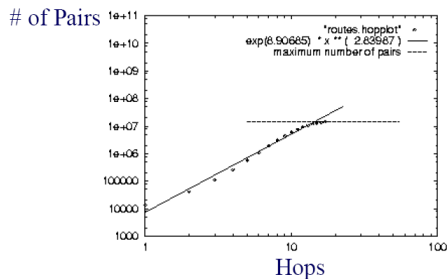
■ $eigen. = rank^{-0.48}$

Power-law IV



Hop plot [ICDE 09]

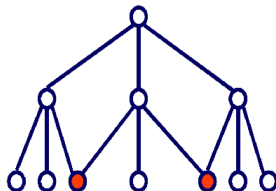
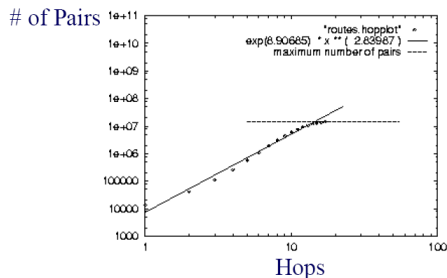
Power-law IV



Hop plot [ICDE 09]

- How many neighbors within 1, 2, \dots , h hops? ($\sum_{i=1}^h avg.i$)

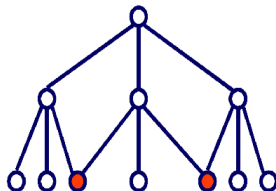
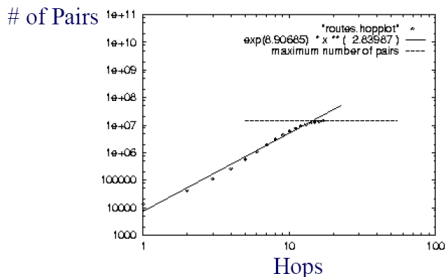
Power-law IV



Hop plot [ICDE 09]

- How many neighbors within $1, 2, \dots, h$ hops? ($\sum_{i=1}^h avg.i$)
- Pairs of vertices are plotted in log-log scale. It forms a line with a slope ~ 2.83

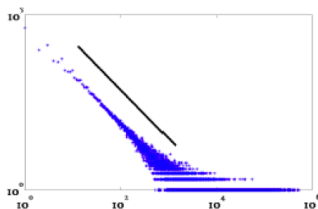
Power-law IV



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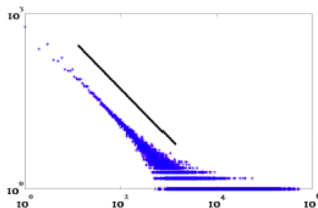
- How many neighbors within $1, 2, \dots, h$ hops? ($\sum_{i=1}^h avg.^i$)
- Pairs of vertices are plotted in log-log scale. It forms a line with a slope ~ 2.83
- $pairs. = hop^{2.83}$

Power-law V



Counting of triangle [ICDM 08]

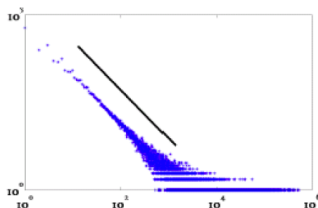
Power-law V



Counting of triangle [ICDM 08]

- X-axis: # of triangles a vertex participates in

Power-law V



Counting of triangle [ICDM 08]

- X-axis: # of triangles a vertex participates in
- Y-axis: count of such vertices
- In log-log scale, the plot is almost linear.

Erdos-Renyi model

Erdős-Renyi model is known as the random graph model, which generates undirected random graphs.

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Erdos-Renyi model

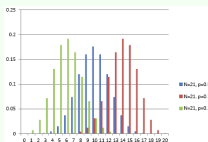
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- Degree distribution:
 - $P(\text{node has degree } k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$
 - Follows binomial distribution with mean $(N-1)p$ and variance $(N-1)p(1-p)$ (not power-law distribution).

Erdos-Renyi model

Erdős-Renyi model is known as the random graph model, which generates undirected random graphs.

- Parameters: N (# vertices) and p (probability of forming an edge)
- For each possible node pair, the approach generates an edge with probability p . Thus, # edges = $\frac{pN(N-1)}{2}$.
- Degree distribution:
 - $P(\text{node has degree } k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$
 - Follows binomial distribution with mean $(N-1)p$ and variance $(N-1)p(1-p)$ (not power-law distribution).



Scale-free network

Preferential attachment model

The more connected a node is, the more likely it is to receive new links (namely, Rich gets Richer, Matthew Effect or Paretos Law, etc.).

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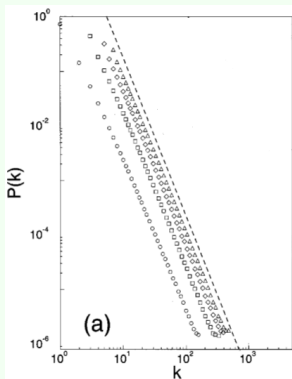
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- Power law with exponent $\alpha = 2 + \frac{1}{m}$ [Science 1965]

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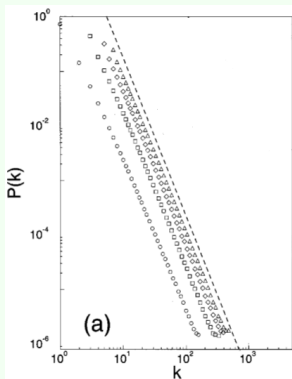
Model



Barabasi Albert model

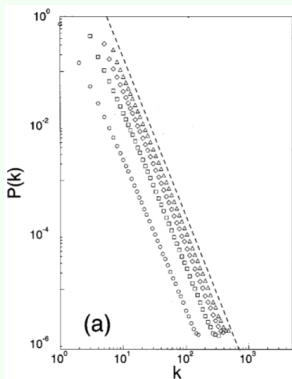
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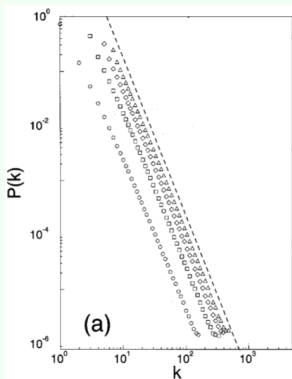
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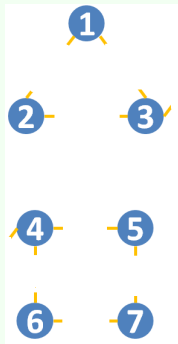
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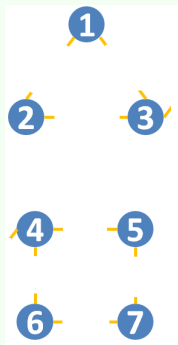
- Start with an initial network of m_0 (≥ 2) nodes, and the degree of each node ≥ 1 , otherwise it will always remain isolated.
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- Results in a single connected component with power-law degree distribution with $\alpha = 3$ [Reviews of Modern Physics 2003].

Null model

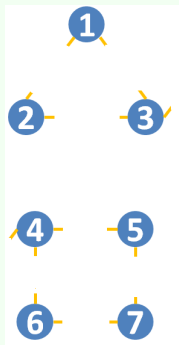


Null model

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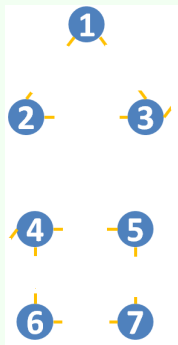
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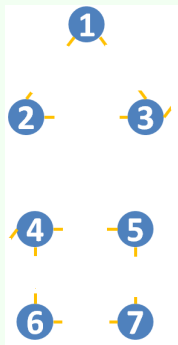
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We have:

$$\sum_{i,j} E(X_{ij}) = \sum_{i,j} P_{ij} = 2m$$

$$\sum_j E(X_{ij}) = \sum_j P_{ij} = k_i$$

Null model Cont'd

Since P_{ij} is related to k_i and k_j , let $P_{ij} = f(k_i)f(k_j)$.

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- Therefore,

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Binomial explanation for Null model

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$$Z_{ij} = \begin{cases} 1, & e_{ij} \in E \\ 0, & \text{otherwise.} \end{cases}$$

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- Thus, the expectation number of edge between vertices v_i and v_j is

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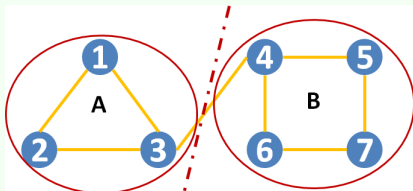
Modularity [Newman 2006]:

$$Q = \frac{1}{2m} \sum_{i,j} (A_{ij} - \frac{k_i k_j}{2m}) \delta(C_i, C_j),$$

where m and C_i denote # edges and the i -th community in the graph, k_i is the degree of vertex v_i , and

$$\delta(C_i, C_j) = \begin{cases} 1, & \text{if } C_i = C_j; \\ 0, & \text{otherwise.} \end{cases}$$

Example of modularity

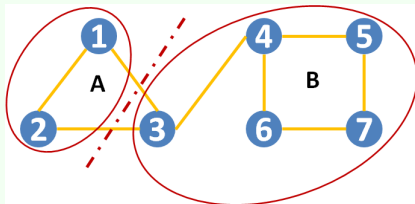


$$m = 8, k_1 = 2, k_2 = 2, k_3 = 3, \\ k_4 = 3, k_5 = 2, k_6 = 2, k_7 = 2$$

Thus, the modularity of this partition is

$$Q = \frac{1}{2m} \sum_{i,j} (A_{ij} - \frac{k_i k_j}{16}) \delta(C_i, C_j) = \frac{1}{16} [(0 - \frac{k_1 k_1}{16}) + 2(1 - \frac{k_1 k_2}{16}) + 2(1 - \frac{k_1 k_3}{16}) \\ + (0 - \frac{k_2 k_2}{16}) + 2(1 - \frac{k_2 k_3}{16}) + (0 - \frac{k_3 k_3}{16}) + (0 - \frac{k_4 k_4}{16}) + 2(1 - \frac{k_4 k_5}{16}) \\ + 2(1 - \frac{k_4 k_6}{16}) + 2(0 - \frac{k_4 k_7}{16}) + (0 - \frac{k_5 k_5}{16}) + 2(0 - \frac{k_5 k_6}{16}) + 2(1 - \frac{k_5 k_7}{16}) \\ + (0 - \frac{k_6 k_6}{16}) + 2(1 - \frac{k_6 k_7}{16}) + (0 - \frac{k_7 k_7}{16})] = \frac{47}{128}$$

Example of modularity Cont'd



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Intuition of modularity

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 - Consider G' as a multigraph.

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29 / 58 ■ It can be used as a measure to evaluate the communities quality.

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where m and C_i denote total number of edges and the i -th community in the graph, k_i^{out} and k_i^{in} are the out-degree and in-degree of vertex v_i , and

$$\delta(C_i, C_j) = \begin{cases} 1, & \text{if } C_i = C_j; \\ 0, & \text{otherwise.} \end{cases}$$

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- The modularity can be rewritten as

$$\begin{aligned} Q &= \frac{1}{4m} \sum_{i,j} (A_{ij} - \frac{k_i k_j}{2m}) (s_i s_j + 1) \\ &= \frac{1}{4m} \sum_{i,j} (A_{ij} - \frac{k_i k_j}{2m}) s_i s_j = \frac{1}{4m} \mathbf{s}^T B \mathbf{s}, \end{aligned}$$

where $B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$ is called modularity matrix.

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- This observation is reminiscent of the best-known methods of graph partitioning, spectral partitioning.
- We proceed by writing \mathbf{s} as a linear combination of the normalized eigenvectors \mathbf{u}_i of B s.t., $\mathbf{s} = \sum_{i=1}^n a_i \mathbf{u}_i$ with $a_i = \mathbf{u}_i^T \cdot \mathbf{s}$. Then

$$Q = \frac{1}{4m} \sum_i a_i \mathbf{u}_i^T B \sum_j a_j \mathbf{u}_j = \frac{1}{4m} \sum_i (\mathbf{u}_i^T \cdot \mathbf{s})^2 \beta_i,$$

where β_i is the eigenvalue of B corresponding to eigenvector

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- If there were no other constraints on \mathbf{s} , we would simply choose \mathbf{s} proportional to the eigenvector \mathbf{u}_1 since the eigenvectors are orthogonal.
- Unfortunately, there is another constraint on the problem imposed by the restriction of the elements of \mathbf{s} to the values ± 1 , which means \mathbf{s} cannot normally be chosen parallel to \mathbf{u}_1 .

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- As a result, we compute the leading eigenvector of the modularity matrix and divide the vertices into two groups according to the signs of the elements in this vector.

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- In doing this it is crucial to note that it is not correct.
 - The modularity will change if edges are deleted;
 - Any subsequent maximization of modularity would thus maximize the wrong quantity.

Extension Cont'd

Instead, the correct approach is to write the additional contribution $\triangle Q$ to the modularity upon further dividing a group g of size n_g in two as

$$\begin{aligned}\triangle Q &= \frac{1}{2m} \left[\frac{1}{2} \sum_{i,j \in g} B_{ij} (s_i s_j + 1) - \sum_{i,j \in g} B_{ij} \right] \\ &= \frac{1}{4m} \left[\sum_{i,j \in g} B_{ij} s_i s_j - \sum_{i,j \in g} B_{ij} \right] \\ &= \frac{1}{4m} \sum_{i,j \in g} \left[B_{ij} - \delta_{ij} \sum_{k \in g} B_{ik} \right] s_i s_j \\ &= \frac{1}{4m} \sum_{i,j \in g} \left[B_{ij} - \delta_{ij} \sum_{k \in g} B_{ik} \right] s_i s_j = \frac{1}{4m} \mathbf{s}^T B^{(g)} \mathbf{s}\end{aligned}$$

where $B_{ij}^{(g)} = B_{ij} - \delta_{ij} \sum_{k \in g} B_{ik}$ is the new modularity matrix.

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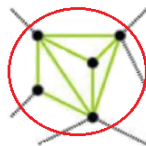
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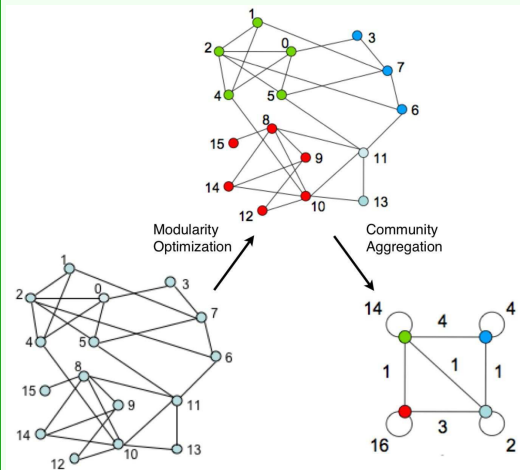
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Louvain method Cont'd

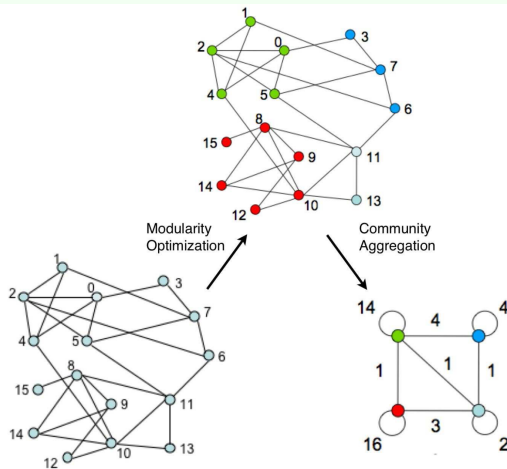
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Louvain method Cont'd

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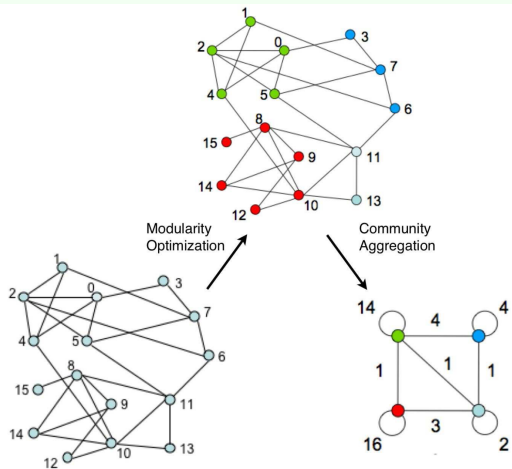
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Louvain method Cont'd

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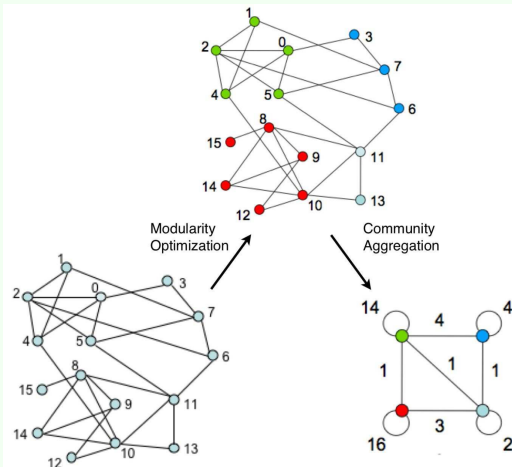
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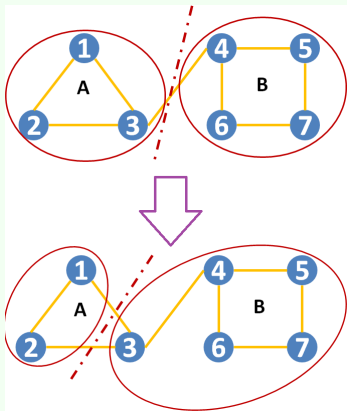
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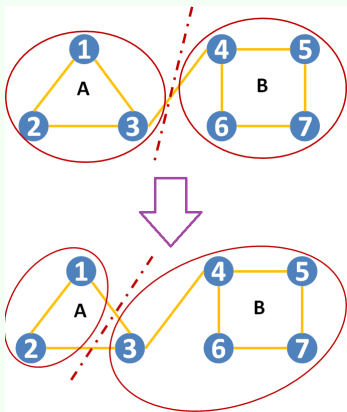
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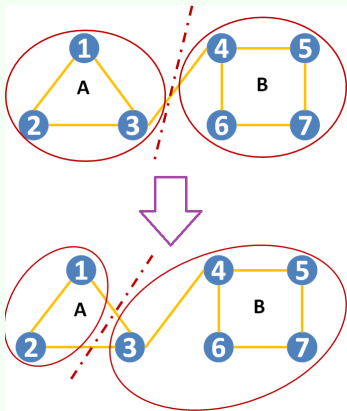
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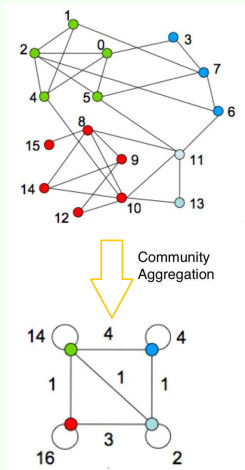
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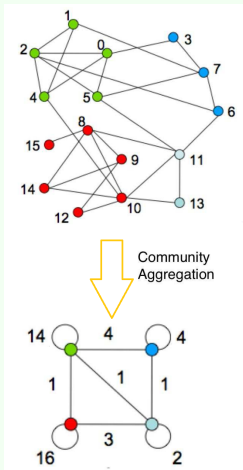
Finally, $\Delta Q = \Delta Q(v_i \rightarrow C) + \Delta Q(D \rightarrow v_i)$.

Louvain: Phase 2 (Reconstructing)



The partitions obtained in the first phase are contracted into super-nodes, and the weighted network is created as follows

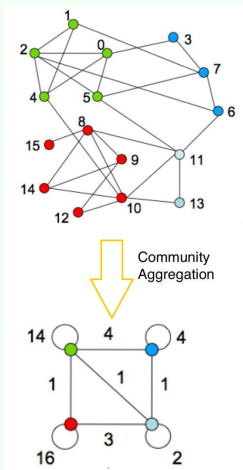
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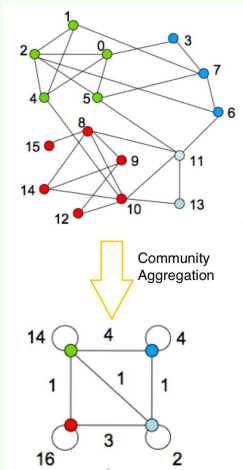
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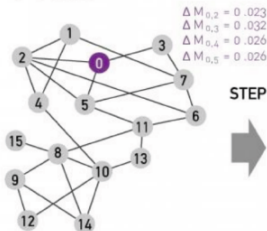


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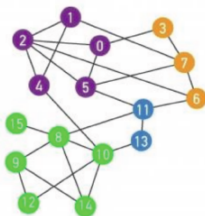
- Super-nodes are connected if there is at least one edge between vertices of the corresponding communities;
- The weight of the edge between the two supernodes is the sum of the weights from all edges between their corresponding partitions;
- After aggregation, the graph becomes a weighted graph.

Louvain method

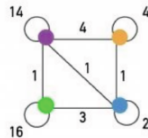
1ST PASS



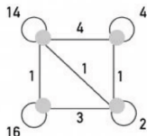
STEP I



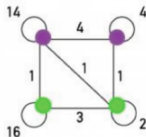
STEP II



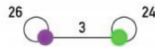
2ND PASS



STEP I



STEP II



Outline

Course Project

Motivation

Modularity

- Graph Model

- Definition

- Variants

Modularity Matrix

- Two communities

- Multiple Community Partitioning

Louvain Method

- Introduction

- Algorithm

- Analysis

Prons of Louvain method

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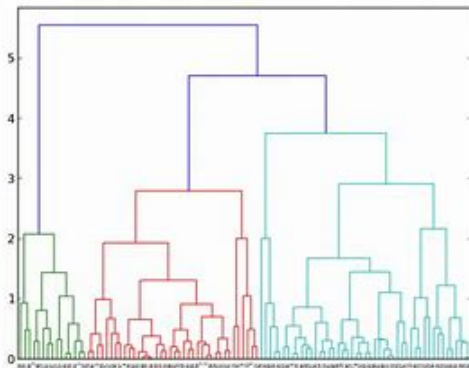
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- The time complexity of this algorithm is $O(|E|)$, e.g., the time is less than 1 minutes for finding the community from a graph of 1 million vertices.
- The number of communities is not a hyper-parameter.
- It can be used to evaluate the quality of community structure;

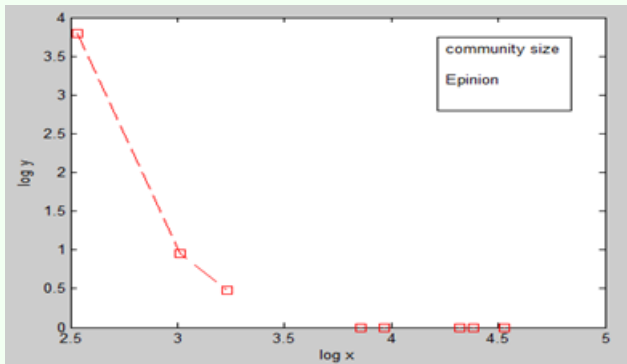
Prons of Louvain method Cont'd

Hierarchical partitions



Cons of Louvain method

The sizes of communities follow the rule of Power-law.



One of the major drawbacks of the Louvain algorithm is resolution limit.

Experimental result

	Karate	Arxiv	Internet	Web nd.edu
Nodes/links	34/77	9k/24k	70k/351k	325k/1M
CNM	.38/0s	.772/3.6s	.692/799s	.927/5034s
PL	.42/0s	.757/3.3s	.729/575s	.895/6666s
WT	.42/0s	.761/0.7s	.667/62s	.898/248s
Our algorithm	.42/0s	.813/0s	.781/1s	.935/3s

	Phone	Web uk-2005	Web WebBase 2001
Nodes/links	2.6M/6.3M	39M/783M	118M/1B
CNM	-/-	-/-	-/-
PL	-/-	-/-	-/-
WT	.56/464s	-/-	-/-
Our algorithm	.769/134s	.979/738s	.984/152mn

Take-home messages

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