Algorithm Foundations of Data Science and Engineering

Lecture 9: Integer Programming

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Outline

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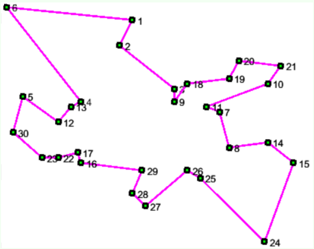

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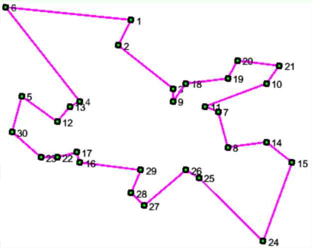

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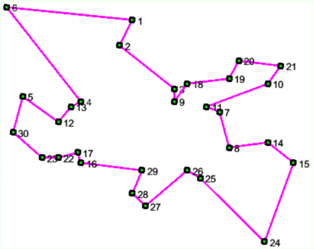

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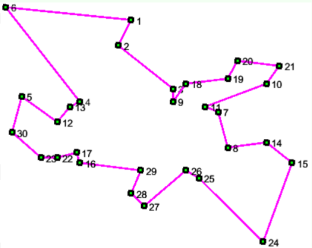

- **Input:** a set of n points;
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- **Objective function:** minimize the length of the tour.

It can be applied into scheduling problem, vehicle routing, aircraft routing, etc.

TSP formulation

The TSP can be defined on an undirected graph $G = (V, E)$ if it is symmetric (directed VS. asymmetric), $V = \{1, \cdots, n\}$ is the vertex set, $E \subset V \times V$ is an edge set, and a cost matrix C_{ij} is defined on E.

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Set covering problem: SCP

Input: Universe set $U = \{u_1, u_2, \cdots, u_n\}$
Subsets $S = \{s_i | s_i \subset U, 1 \leq i \leq m\}$
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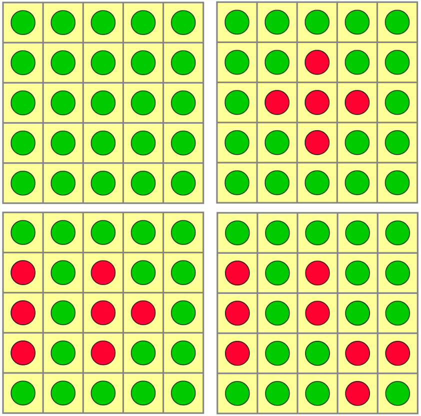
A game of fiver



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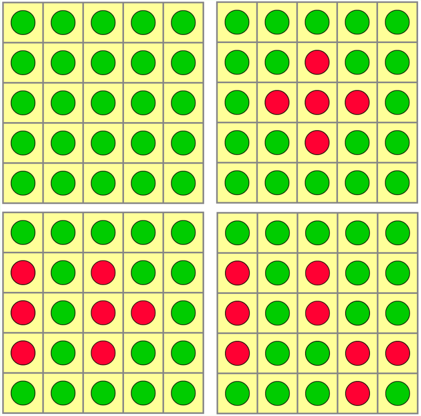

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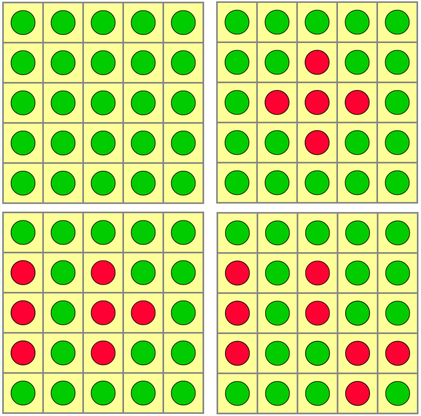

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- Can you make all of the circles red?

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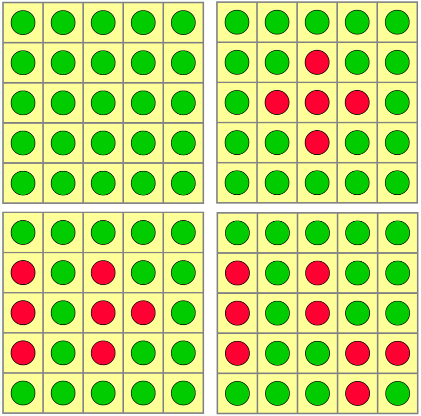
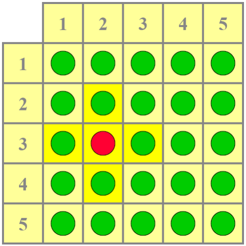


- Click on a circle, and flip its color and that of adjacent colors;
- Can you make all of the circles red?
- Click on $(3,3),(3,1)$ and $(4,4)$, sequentially.

Next: an optimization problem whose solution solves the problem in the fewest moves.

Fiver formulation



Let

$$x_{ij} \begin{cases} 1, & \text{if row i and column j} \\ & \text{in the square is clicked.} \\ 0, & \text{otherwise.} \end{cases}$$

Minimize: $\sum_i^5 \sum_j^5 x_{ij}$

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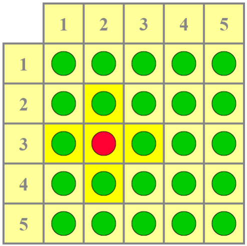

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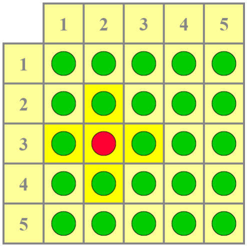

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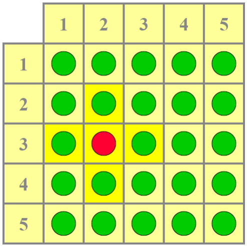

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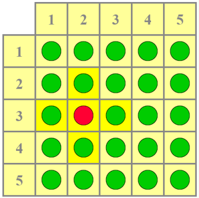

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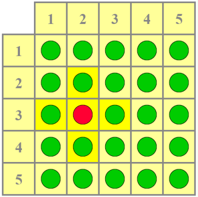

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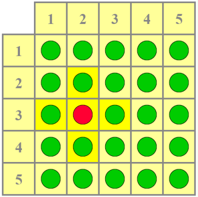

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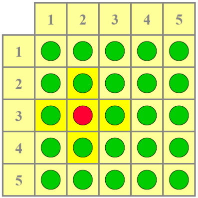

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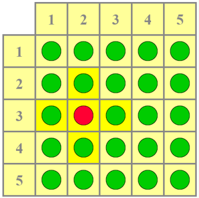

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$x_{ij} = 0$ otherwise
$0 \leq y_{ij} \leq 2$, and $y_{ij} \in Z^+$ for all $1 \leq i, j \leq 5$

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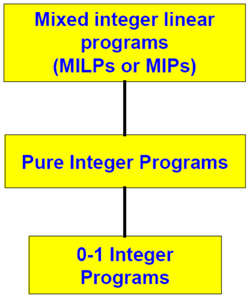
& 20x_1 + 10x_2 \geq 110 \\
& 0 \leq x_1, x_2 \in Z
\end{aligned}
$$

20 TVs + 20 laundries
Cost: 360

40 TVs + 10 laundries
Cost: 400

Task: ship 180 TVs and 110 laundries.

Type of Integer programming



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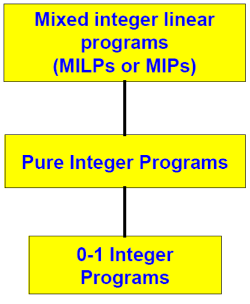

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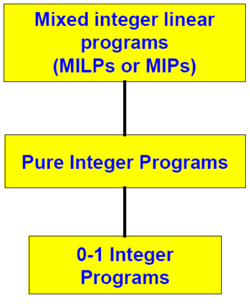

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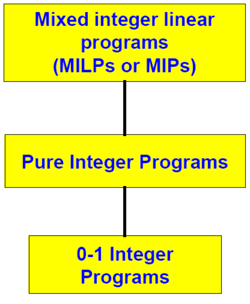

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Note, pure integer programming instances that are unbounded can have an infinite number of solutions. But they have a finite number of solutions if the variables are bounded.

Outline

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Integer programs: a linear program plus the additional constraints that some or all of the variables must be integer valued.

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  - You must select x_1 or x_2 or both, then $x_1 + x_2 \ge 1$;
- Modeling logical constraints that involve non-binary variables is much harder. But we will try to make it as simple as possible.

Logical constraint

If constraint $x \leq 2$ or $x \geq 6$, choose a binary variable w s.t.,

$$w = \begin{cases} 1, & x \leq 2; \\ 0, & x \geq 6. \end{cases}$$

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When M is a larger number, then it become IP constraints

$$\begin{aligned} & x \leq 2 + M(1 - w) \\ & x \geq 6 - Mw \\ & w \in \{0, 1\} \end{aligned}$$

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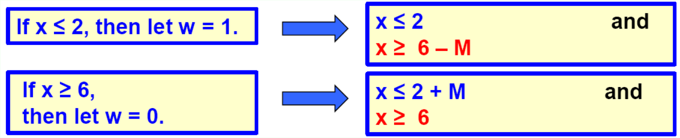

In both cases, the IP constraints are satisfied.

Modeling "or" constraint

$x_1 + 2x_2 \geq 12$ or
$4x_2 - 10x_3 \leq 1$

Logical constraints

If w = 1, then $x_1 + 2x_2 \geq 12$

If w = 0, then $4x_2 - 10x_3 \leq 1$

$x_1 + 2x_2 \geq 12 - M(1 - w)$ or
$4x_2 - 10x_3 \leq 1 + Mw$
$w \in \{0, 1\}$

IP constraints

Suppose that (x, w) is feasible, for the IP.

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Suppose that (x, w) is feasible, for the IP. Therefore, the logical constraints are satisfied.

If $x_1 + 2x_2 \geq 12$, then let w = 1	$\Rightarrow$	$x_1 + 2x_2 \geq 12$ AND $4x_2 - 10x_3 \leq 1 + M$.
Else $4x_2 - 10x_3 \leq 1$ then let w = 0	$\Rightarrow$	$x_1 + 2x_2 \geq 12 - M$ AND $4x_2 - 10x_3 \leq 1$.

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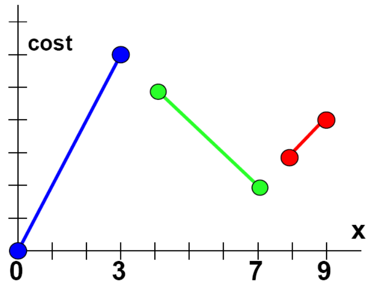
Else $4x_2 - 10x_3 \leq 1$ then let w = 0	$\Rightarrow$	$x_1 + 2x_2 \geq 12 - M$ AND $4x_2 - 10x_3 \leq 1$.

The logical constraints are equivalent to the IP constraints.

Outline

Modeling piecewise linear functions



$$y = \begin{cases} 2x, & \text{if } 0 \leq x \leq 3 \\ 9 - x, & \text{if } 4 \leq x \leq 7 \\ -5 + x, & \text{if } 8 \leq x \leq 9 \end{cases}$$

Assume that x is integer valued. We will create an IP formulation so that the variable y is correctly modeled.

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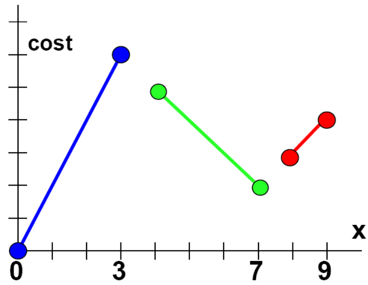

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Create new binary and integer variables

$w_1 = \begin{cases} 1 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$	$x_1 = \begin{cases} x & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$
$w_2 = \begin{cases} 1 & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$	$x_2 = \begin{cases} x & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$
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$w_1 + w_2 + w_3 = 1$
$x = x_1 + x_2 + x_3$ x_i integer $\forall\ i$

IP constraints

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$$y = 2x_1 + (9w_2 - x_2) + (-5w_3 + x_3).$$

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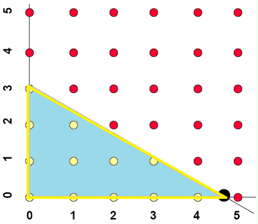

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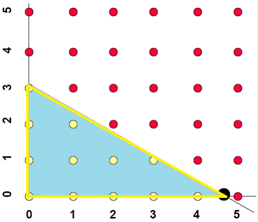

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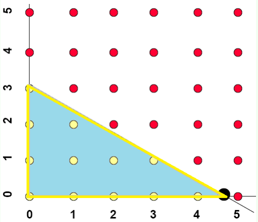

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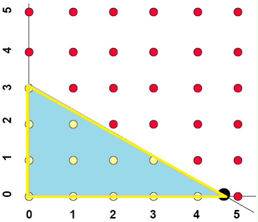

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Solve LP (ignore integrality) get $x = \frac{24}{5}, y = 0$ and $z = 14.4$.

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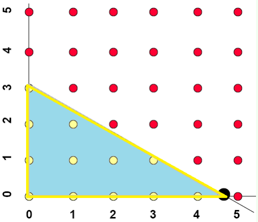

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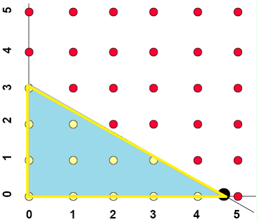

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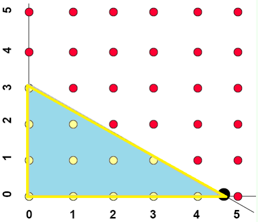

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Truncate, get $x = 4, y = 0$, and $z = 12$. Same solution value at $x = 0, y = 3$.
Optimal is $x = 3, y = 1$, and $z = 13$.

Feasible region for two constraints

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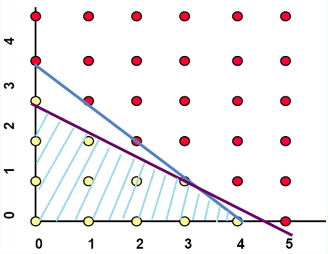

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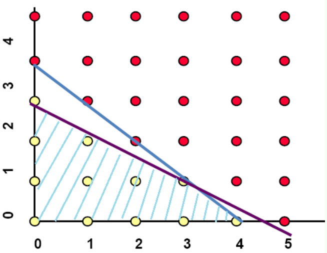

- More constraints will result in a smaller feasible region;

Feasible region for two constraints

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Subject to: $x + y \leq 4$

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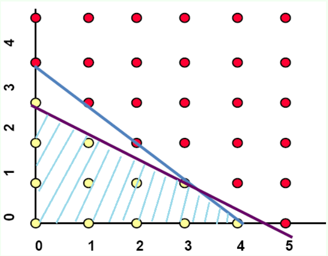

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- That is, the search space will be reduced;

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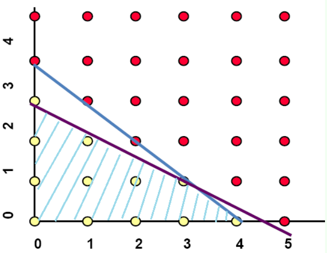

- More constraints will result in a smaller feasible region;
- That is, the search space will be reduced;
- Much, much harder than solving linear programs.

Outline

Complete enumeration

Prize	iPad	server	Brass Rat	Au Bon Pain	6.041 tutoring	15.053 dinner
Points	**5**	**7**	**4**	**3**	**4**	**6**
Utility	**16**	**22**	**12**	**8**	**11**	**19**

Maximize: $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

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Maximize: $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

Subject to: $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

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Maximize: $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

Subject to: $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$x_i \in \{0, 1\}$ for $1 \leq i \leq 6$

Complete enumeration

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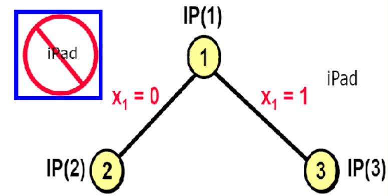
- Each node of the tree represents the original problem plus additional constraints.

An enumeration tree



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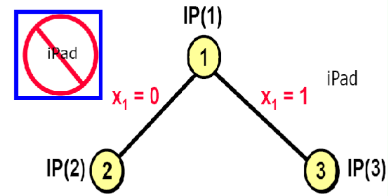

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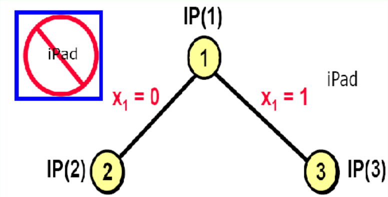

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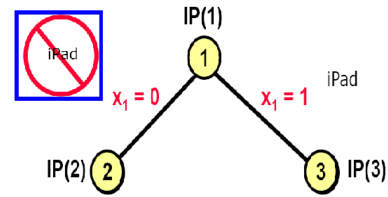

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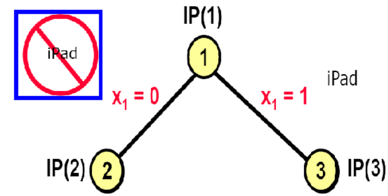

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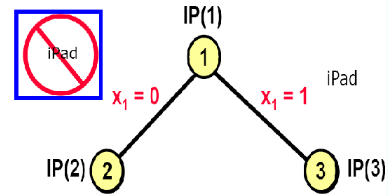

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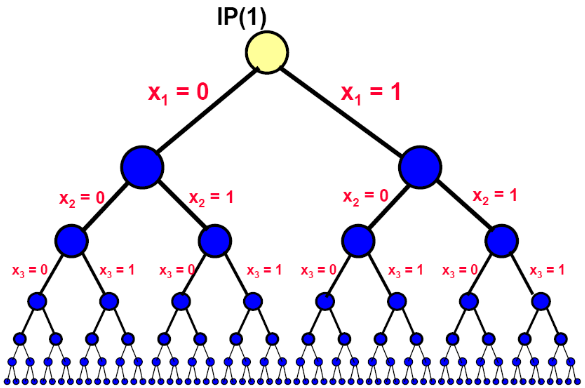

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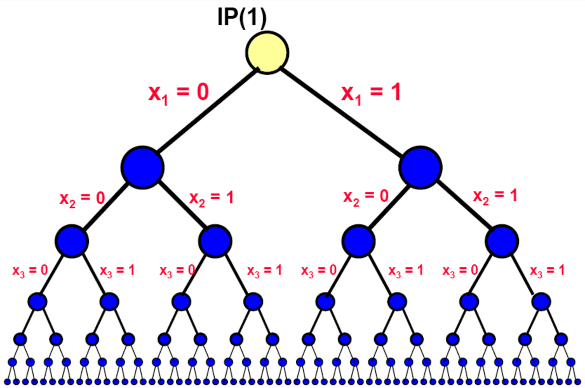

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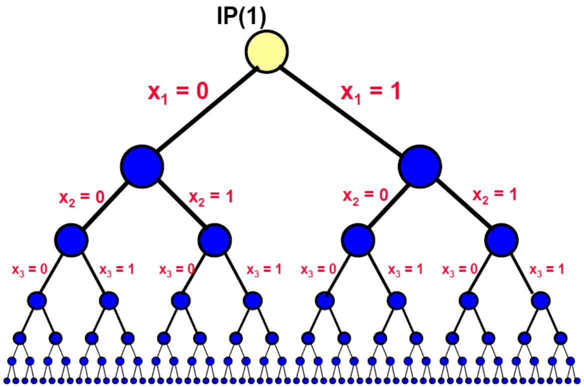

- Number of leaves of the tree: 64;
- If there are n variables, the number of leaves is 2^n.

A complete enumeration

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- Let n = number of binary variables;

n=30	1 sec.	n=60	31 years
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- Suppose that we could evaluate 1 trillion solutions per second, and instantaneously eliminate 99.9999999% of all solutions as not worth considering

n=70	1 sec.	n=100	31 years
n=80	17 minutes	n=110	31,000 years
n=90	11.6 days		

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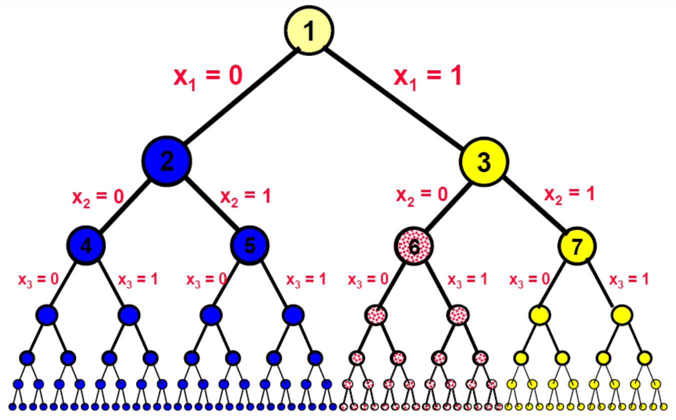

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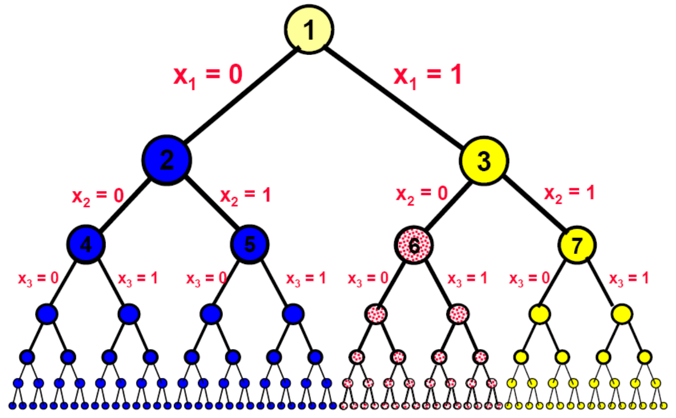

If we can eliminate an entire subtree in one step, we can eliminate a fraction of all complete solutions at in a single step.

A simpler problem to work with

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

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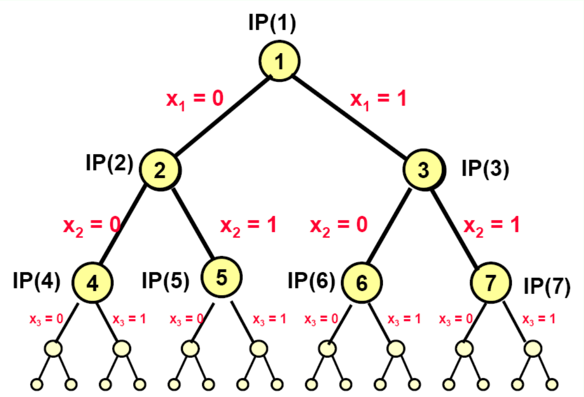

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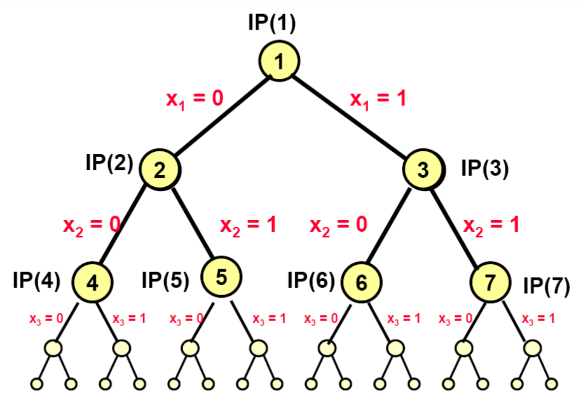

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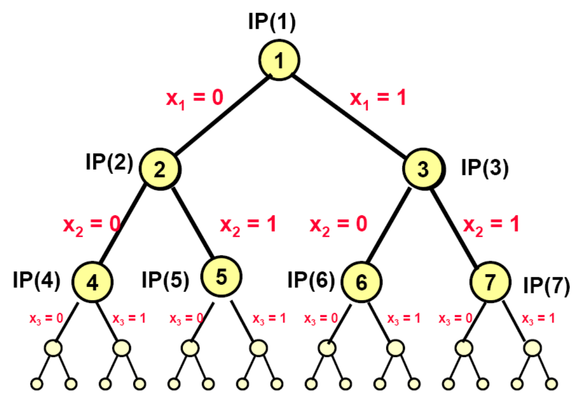

- In a branch and bound tree, the nodes represent IPs;
- What is the optimal objective value for IP(4)?

Eliminating subtrees

We eliminate a subtree if

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- For example, after we solved IP(4), you don't need to look at its children.

Outline

The LP relaxation of the IP

If we drop the requirements that variables be integer, we call it the **LP relaxation of the IP**.

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Optimal solution for LP(4): $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, z = 24.$

If the optimal solution for LP(k) is feasible for IP(k), then it is also optimal for IP(k).

The LP relaxation solves the IP Cont'd

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- If LP(k) is infeasible, then IP(k) is infeasible.
- In this example, the LHS of the constraint is at least 13. There is no way that the constraint can be satisfied by fractional values or integer values of x_3 and x_4.

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Incumbent solution

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Recall that we don't solve IP(k) directly. Instead, we solve its LP relaxation. We can use this to obtain bounds.

Pruning branches

Based on the observation

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Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0$

$0 \leq x_i \leq 1$ for $i = 2$ to 4

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- Suppose that the incumbent is $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, z_I = 26$;

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Recall that we don't solve IP(k) directly. Instead, we solve its LP relaxation. We can use this to obtain bounds.

The branch and bound algorithm

while there is some **active** nodes **do**

select an **active** node *j*

mark j as **inactive**

Solve LP(j): denote solution as x(j);

Case 1 -- **if** $z_{LP}(j) \leq z_I$ **then** ***prune*** node j;

Case 2 -- **if** $z_{LP}(j) > z_I$ and
if x(j) is feasible for IP(j)

then Incumbent := x(j), and $z_I := z_{LP}(j)$;

then ***prune*** node j;

Case 3 -- If **if** $z_{LP}(j) > z_I$ and
if x(j) is not feasible for IP(j) **then**

mark the children of node *j* as **active**

endwhile

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Case 3 -- If **if** $z_{LP}(j) > z_I$ and
if x(j) is not feasible for IP(j) **then**

mark the children of node *j* as **active**

endwhile

Under which condition can we not prune active node j from the B&B Tree for a maximization problem?

Example of B&B algorithm

LP(1)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$0 \leq x_i \leq 1$ for i = 1 to 4

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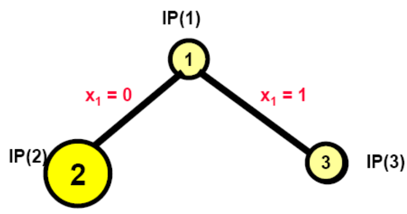

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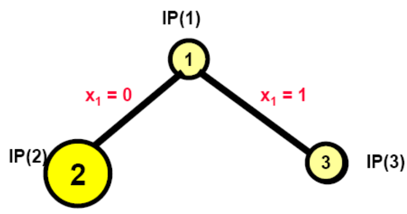

Optimal solution for LP(2) is: $x_1 = 0, x_2 = 1, x_3 = 1, x_4 = \frac{3}{4}, z_{LP}(2) = 25;$

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1$

$0 \leq x_i \leq 1$ for i = 2 to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1$

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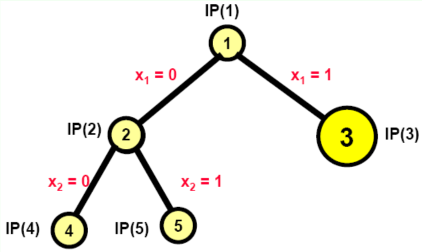

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

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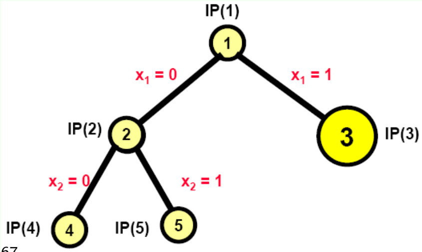

Optimal solution for LP(3) is:
$x_1 = 1, x_2 = 0, x_3 = \frac{1}{4}, x_4 = 0, z_{LP}(3) = 28;$

Example: Node 4

LP(4)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0$, $x_2 = 0$

$0 \leq x_i \leq 1$ for i = 3 to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.

Example: Node 4

LP(4)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 0$
$0 \leq x_i \leq 1$ for i = 3 to 4

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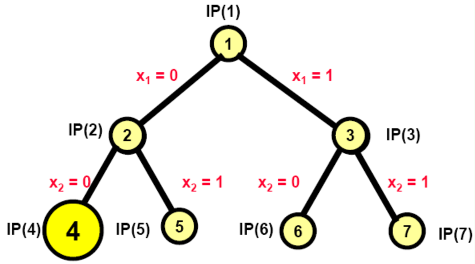


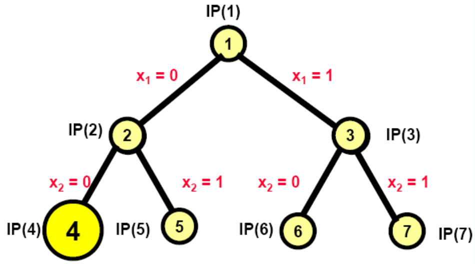
Example: Node 4

LP(4)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 0$
$0 \leq x_i \leq 1$ for $\text{i} = 3$ to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.



Optimal solution for LP(4) is:
$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, z_{LP}(4) = 24;$

Example: Node 4

LP(4)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
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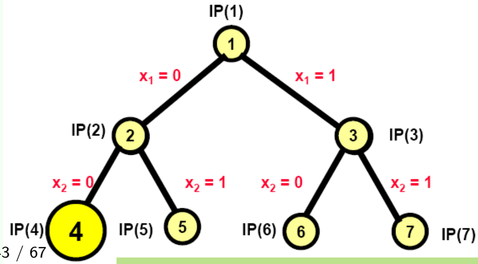
No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.



Optimal solution for LP(4) is:
$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, z_{LP}(4) = 24$; Pruned.

Example: Node 5

LP(5)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0$, $x_2 = 1$
$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution
$z_I = 24$.

Example: Node 5

LP(5)

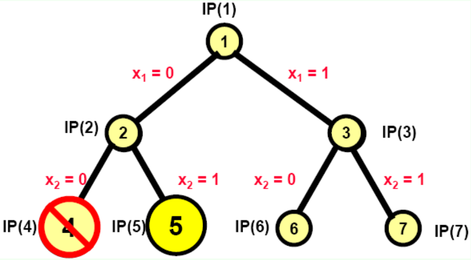
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1$

$0 \leq x_i \leq 1$ for i $= 3$ to 4

Incumbent solution $z_I = 24$.



Example: Node 5

LP(5)

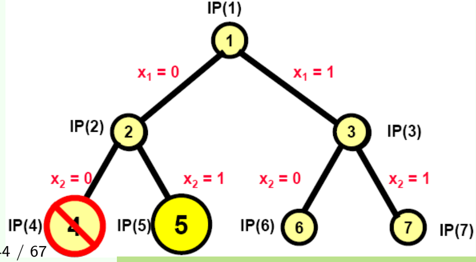
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1$

$0 \leq x_i \leq 1$ for $\ \mathrm{i} = 3\ $ to 4

Incumbent solution $z_I = 24$.



Optimal solution for LP(5) is: $x_1 = 0, x_2 = 1, x_3 = 1, x_4 = \frac{3}{4}, z_{LP}(5) = 25$;

Example: Node 6

LP(6)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 1,\ x_2 = 0$
$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution
$z_I = 24$.

Example: Node 6

LP(6)

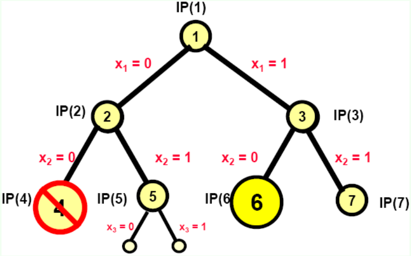
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0$

$0 \leq x_i \leq 1$ for i $= 3$ to 4

Incumbent solution $z_I = 24$.



Example: Node 6

LP(6)

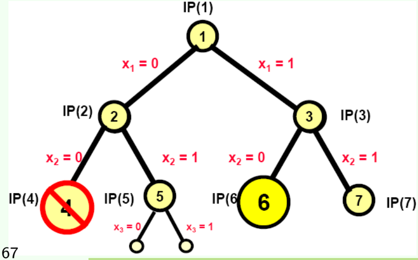
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution $z_I = 24$.



Optimal solution for LP(6) is: $x_1 = 1, x_2 = 0, x_3 = \frac{1}{5}, x_4 = 0, z_{LP}(6) = 28$;

Example: Node 7

LP(7)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1$, $x_2 = 1$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution $z_I = 24$.

Example: Node 7

LP(7)

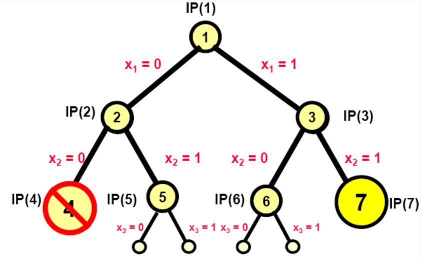
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1, \ x_2 = 1$

$0 \leq x_i \leq 1$ for i $= 3$ to 4

Incumbent solution $z_I = 24$.



Example: Node 7

LP(7)

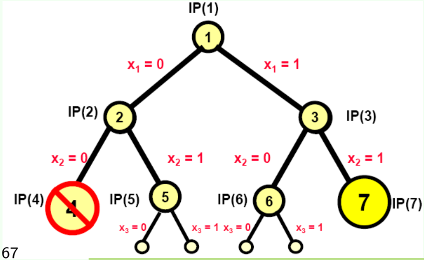
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 1$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution $z_I = 24$.



Optimal solution for LP(7) is:
$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, z_{LP}(7) = 26$;

Example: Node 8

LP(8)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 1,\ x_3 = 0$
$0 \leq x_4 \leq 1$

Incumbent solution
$z_I = 26$.

Example: Node 8

LP(8)

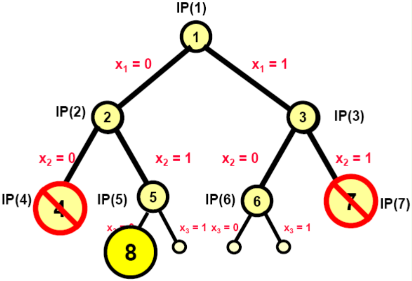
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1,\ x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Example: Node 8

LP(8)

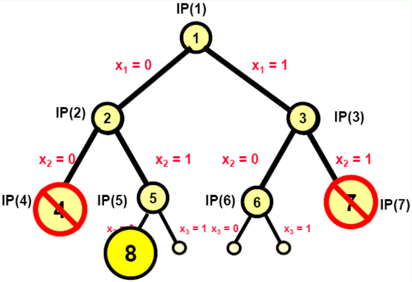
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0, \; x_2 = 1, \; x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(8) is: $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1, z_{LP}(8) = 6$;

Example: Node 8

LP(8)

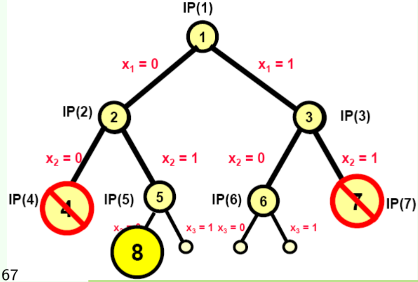
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1,\ x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(8) is: $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1, z_{LP}(8) = 6$; Pruned.

Example: Node 9

LP(9)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 1,\ x_3 = 1$
$0 \leq x_4 \leq 1$

Incumbent solution
$z_I = 26$.

Example: Node 9

LP(9)

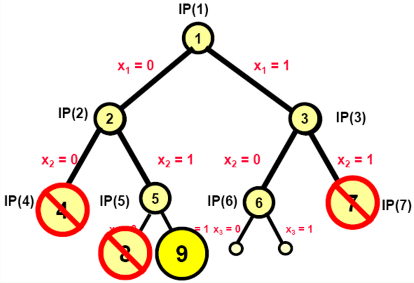
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



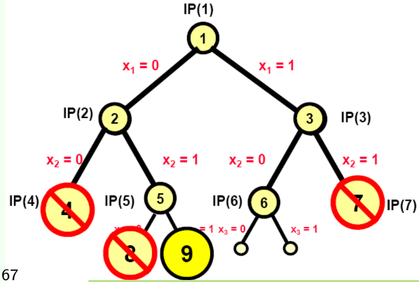
Example: Node 9

LP(9)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 1,\ x_3 = 1$
$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(9) is:
$x_1 = 0, x_2 = 1, x_3 = 1, x_4 = \frac{3}{4}, z_{LP}(9) = 25;$

Example: Node 10

LP(10)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.

Example: Node 10

LP(10)

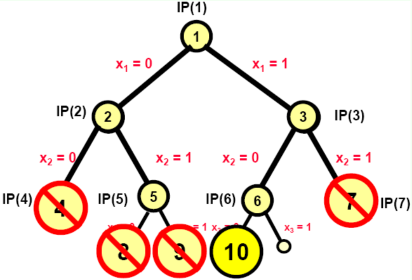
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



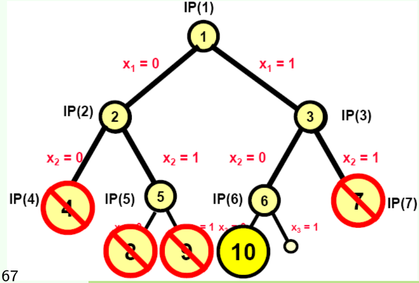
Example: Node 10

LP(10)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \le 9$
$x_1 = 1,\ x_2 = 0,\ x_3 = 0$
$0 \le x_4 \le 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(10) is:
$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = \frac{1}{4}, z_{LP}(10) = 25$;

Example: Node 11

LP(11)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.

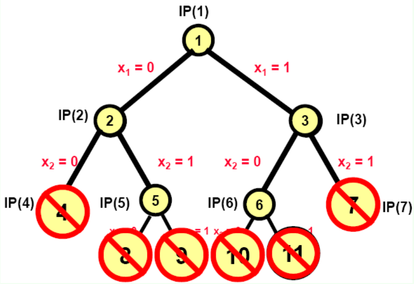
Example: Node 11

LP(11)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 1,\ x_2 = 0,\ x_3 = 1$
$0 \leq x_4 \leq 1$

Incumbent solution
$z_I = 26.$



Example: Node 11

LP(11)

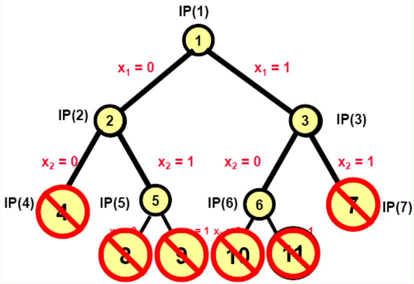
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(11): there is no feasible solution for LP(11).

Example: Node 11

LP(11)

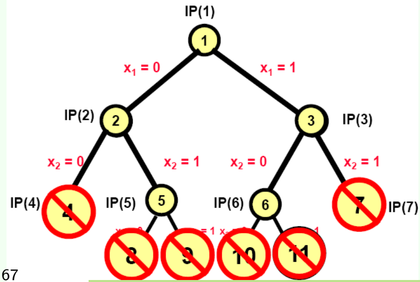
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(11): there is no feasible solution for LP(11). Pruned.

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  - Another technique is to use "rounding", e.g., $x_1 + x_2 \leq 1.5 \rightarrow x_1 + x_2 \leq 1$, or $z_{IP} \leq Z_{LP} = 5.5 \rightarrow z_{IP} \leq 5$.

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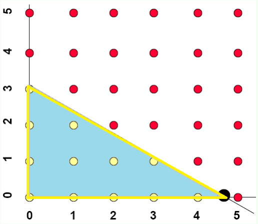

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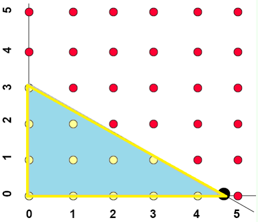

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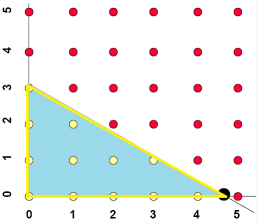

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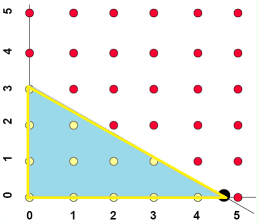

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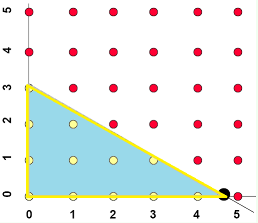

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  Note that LHS is integral, so RHS can be truncated, while it does not necessarily dominate original constraint.

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Valid inequality (subtract (2) from (1)):

$$0.2x_1 + 0.7x_2 + 0.6x_3 + 0.8x_4 \geq 0.9$$

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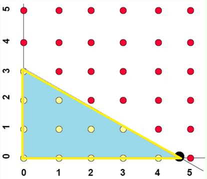

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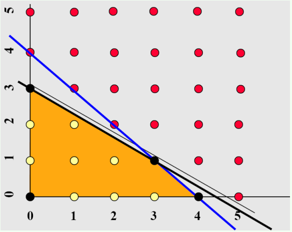

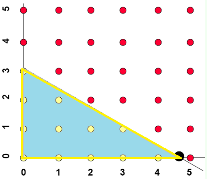

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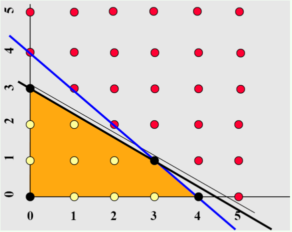

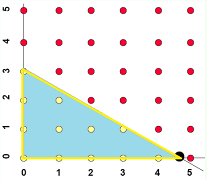

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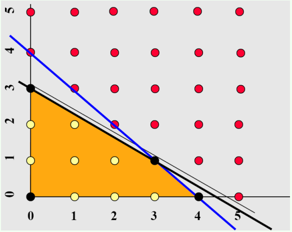

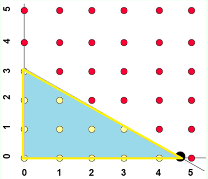

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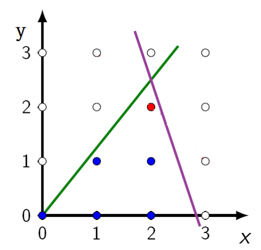

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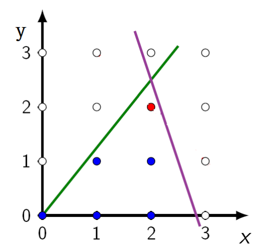

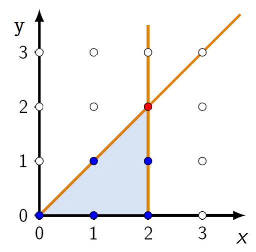

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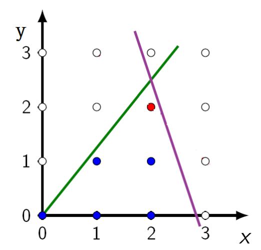

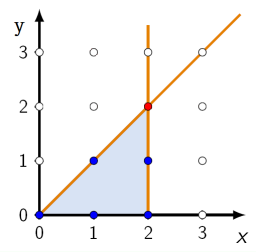

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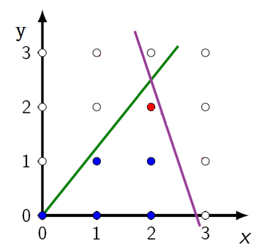

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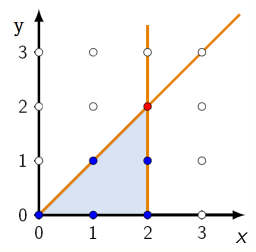

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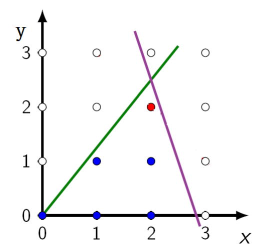

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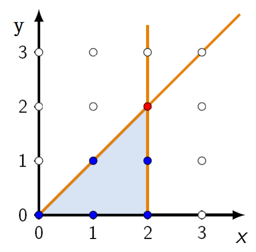

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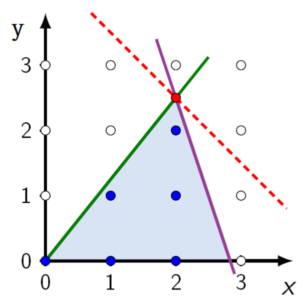

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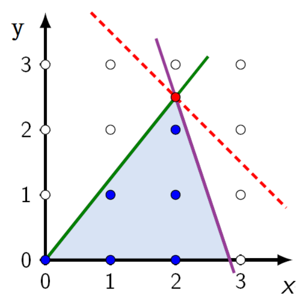

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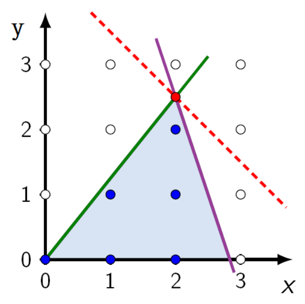

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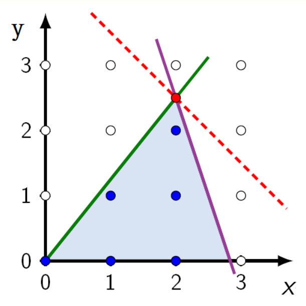

Optimal solution $= 4.5$.

- Remove integer constraint to obtain the LP relaxation;
- Optimal solution is an upper bound on the optimal cost;
- If solution is integral, it is optimal for the original problem.

Cutting plane method

Cutting plane algorithm:

Cutting plane method

Cutting plane algorithm:

Cutting plane method

Cutting plane algorithm:

Step 1: Solve LP relaxation

Cutting plane method

Cutting plane algorithm:

Step 1: Solve LP relaxation

Step 2: If LP solution is integral, it is optimal for the original problem. We are done!

Cutting plane method

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- Step 1: Solve LP relaxation
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- Step 3: If LP solution is not integral, find a linear constraint that excludes the LP solution but does not exclude any integer points (always possible);

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- Step 1: Solve LP relaxation
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- Step 4: Add the cut constraint to the problem. Return to step 1.

Cutting plane method

Cutting plane algorithm:

Step 1: Solve LP relaxation

Step 2: If LP solution is integral, it is optimal for the original problem. We are done!

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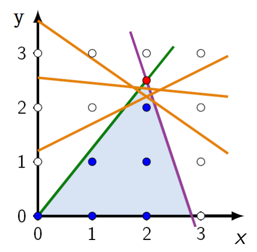

Cutting plane method

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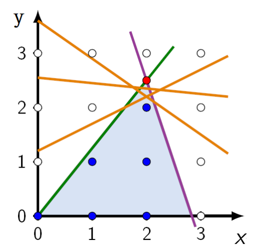

Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

$6x + 2y \leq 17$

Cutting plane method

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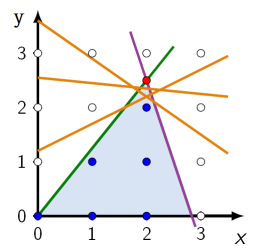

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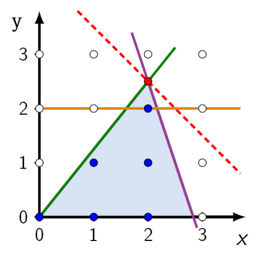
valid inequality

$0 \leq x, y \in Z$

Example of cutting plane



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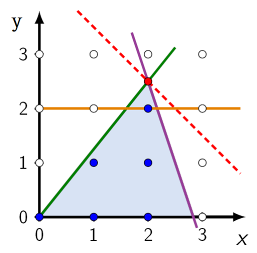

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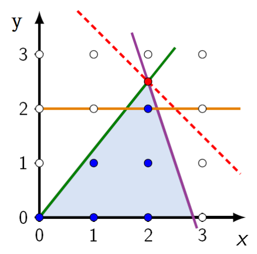

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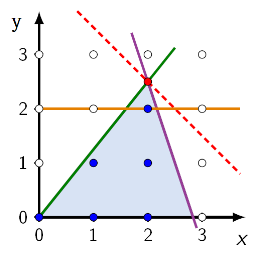
$y \leq 2$

$0 \leq x, y \in Z$

Example of cutting plane



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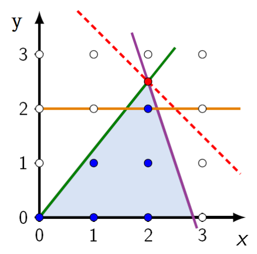
$0 \leq x, y \in Z$

- The constraint $y \leq 2$ is a valid cut because it excludes the optimal LP solution but does not exclude any integer points;

Example of cutting plane



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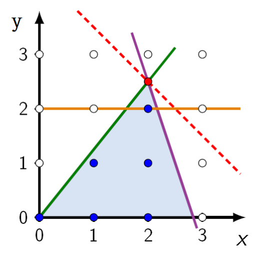
- Now solve the LP relaxation for this new problem.

Example of cutting plane



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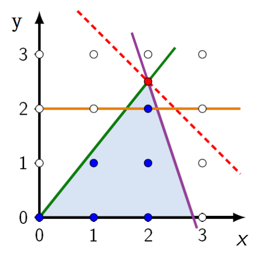
- The constraint $y \leq 2$ is a valid cut because it excludes the optimal LP solution but does not exclude any integer points;
- Now solve the LP relaxation for this new problem.

A cut must simultaneously exclude the LP solution while keeping all the feasible integer points.

Example of cutting plane



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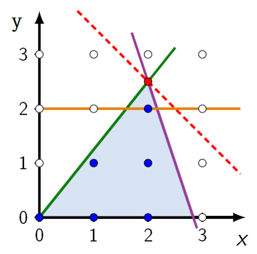
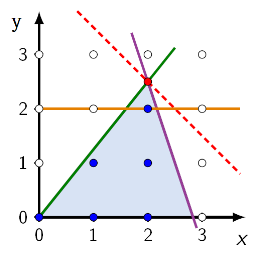
Example of cutting plane Cont'd



Example of cutting plane Cont'd



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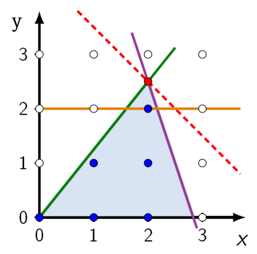

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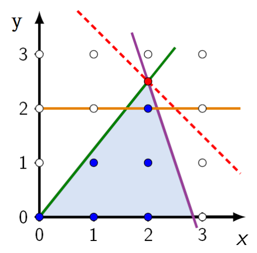
Example of cutting plane Cont'd



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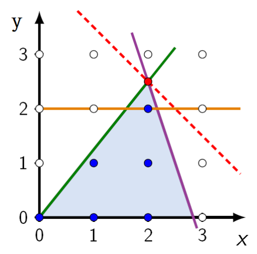

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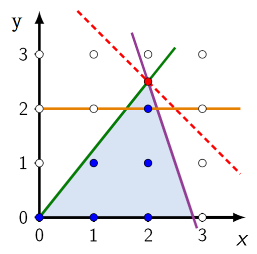

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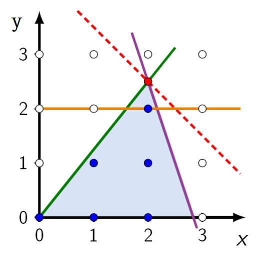
Example of cutting plane Cont'd



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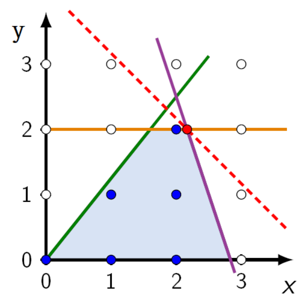

Maximize: $z = x + y$

Subject to:
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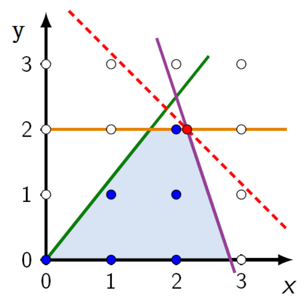
Optimal solution $= 4.1667$.

Example of cutting plane Cont'd



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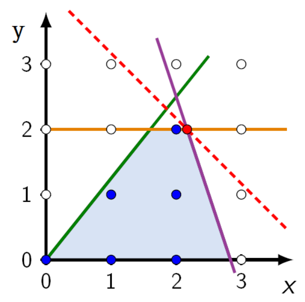
Optimal solution $= 4.1667$.

- Adding a cut reduces our upper bound because we are shrinking the feasible set (we added another constraint).

Example of cutting plane Cont'd



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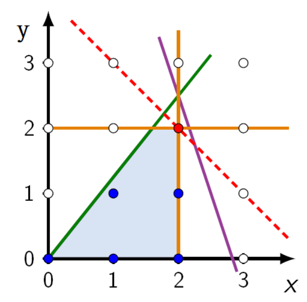
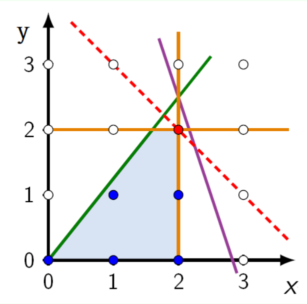
Optimal solution $= 4.1667$.

- Adding a cut reduces our upper bound because we are shrinking the feasible set (we added another constraint).
- Solution is still not an integer. Add another cut!

Example of cutting plane Cont'd



Example of cutting plane Cont'd



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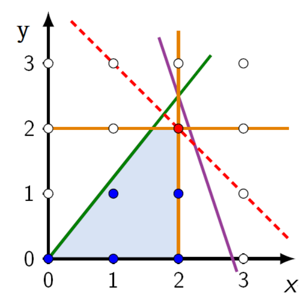

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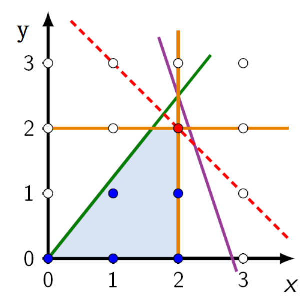

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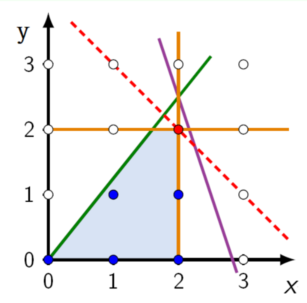
- Optimal solution $x = 2, y = 2, z = 4$;

Example of cutting plane Cont'd



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- Optimal solution $x = 2, y = 2, z = 4$;
- LP solution is integral, so it must also be optimal for the original integer problem.

Take-home messages

- Combinatorial Optimization
  - Motivated Examples
  - Constraint
  - Piecewise Objective Function
  - Feasible Region
- Branch and Bound
  - Enumeration Tree
  - LP Relation
  - Branch and Bound
- Cutting Planes
  - Valid Inequalities
  - Cutting Planes