Question 1

1 Introduction

The goal of this project is to implement three different numerical methods for solving a general ODE and study their accuracy. These numerical methods are as follows:

- The Euler method: $y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$
- The explicit trapezoidal method (RK2): $y_{n+1} = y_n + \Delta t \cdot \frac{1}{2} \cdot (f(t_n, y_n)) + f(t_{n+1}, y_{n+1}))$
- The classical Runge-Kutta method (RK4): $y_{n+1} = y_n + \Delta t(\frac{k_1+2k_2+2k_3+k_4}{6})$

We will use these to study a falling object under the force of gravity and drag force. This will be seen by comparing the numerical solution and the exact solution graphically.

2 Equations and values

All three methods will use the ODE bellow:

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{1}$$

where $y = y(t)$ is the unknown solution we seek to approximate, $f = f(t, y)$ is a given function, t_0 is the given initial time and y_0 the given initial condition. In order to check the accuracy, we consider the following system describing a falling object under the force of gravity and drag force:

$$\begin{aligned} \frac{dv}{dt} &= g - \frac{c_d}{m}v^2 \\ v(0) &= 0 \end{aligned} \tag{2}$$

where $g \approx 9.81\frac{m}{s^2}$ is the free-fall acceleration, $m = 75kg$ is the mass of the object and $c_d = 0.25\frac{kg}{m}$ is the drag coefficient. In this case $f(t, v) = g\frac{c_d}{m}v^2$ and the exact solution is:

$$v(t) = \sqrt{\frac{gm}{c_d}}tanh(t\sqrt{\frac{gc_d}{m}})$$

and we will take $t_f = 15s$ and a time step of $\Delta t = 0.1s, 0.05s, 0.025s, 0.0125s, 3.3s$.

3 Numerical and exact solutions

3.1 Graph

The Euler (blue circle), Trapezoidal (red circle), RK4 (yellow/orange circle) and Exact (purple line) solutions are represented below for the time-step dt = 3.3s:

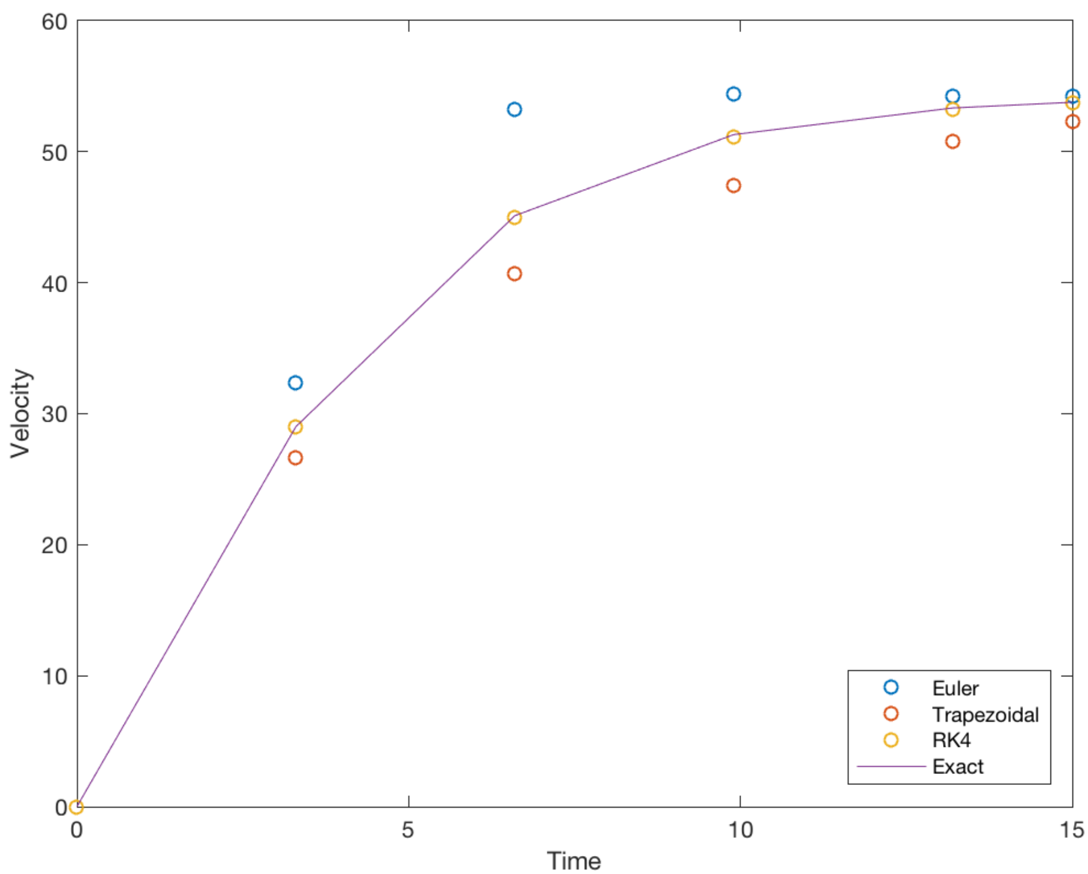

Figure 1: Euler/RK2/RK4/Exact

As we begin using higher order methods, we see less differnce between the exact solution and the numerical soluion. The Euler method has a large error, where as RK4 has virtually no error.

3.2 Tables

The following tables show the max error and order for Euler, Trapezoidal, and RK4 for the time steps $\Delta t = 0.1s, 0.05s, 0.025s, 0.0125s$

Time-Step	Max Error	Order
0.1	0.181522	1.004313
0.05	0.0904899	1.002141
0.025	0.0451779	1.001066
0.0125	0.0225722	1.000532

Table 1: Euler

Time-Step	Max Error	Order
0.1	0.00217233	2.013392
0.05	0.000538064	2.006675
0.025	0.000133895	2.003349
0.0125	$3.33961 \cdot 10^{-5}$	2.001673

Table 2: Trapezoidal

Method	Max Error	Order
0.1	$8.63907 \cdot 10^{-8}$	4.013019
0.05	$5.35091 \cdot 10^{-9}$	4.006024
0.025	$3.33038 \cdot 10^{-10}$	3.988442
0.0125	$2.09823 \cdot 10^{-11}$	4.197049

Table 3: RK4

Question 2

4 Order of accuracy via Taylors analysis

Considering the ODE bellow:

$$\begin{aligned} \frac{dy}{dt} &= f(t,y) \\ y(t_0) &= y_0 \end{aligned} \tag{3}$$

Where $y = y(t)$ is the unknown solution we seek to approximate and $f = f(t, y)$ is a given function we used a Taylors analysis to find the order of accuracy of the following scheme:

$$y_{n+1} = y_{n-1} + 2\Delta t f(t_n, y_n)$$

The local truncation error is defined as

$$E_{tr} = \frac{y(t_{n+1}) - y(t_{n-1})}{2\Delta t} - f(t_n, y_n) \tag{4}$$

The Taylor expansion of $y(t_{n+1})$ around point t_n can be approximated by

$$y(t_{n+1}) = y(t_n) + y'(t_n)\Delta t + \frac{1}{2}y''(t_n)\Delta t^2 + \frac{1}{6}y'''(t_n)\Delta t^3 + O((\Delta t)^4)$$

And the Taylor expansion of $y(t_{n-1})$ around point t_n can be approximated by

$$y(t_{n-1}) = y(t_n) - y'(t_n)\Delta t + \frac{1}{2}y''(t_n)\Delta t^2 - \frac{1}{6}y'''(t_n)\Delta t^3 + O((\Delta t)^4$$

We now take $y(t_{n+1})$ and $y(t_{n-1})$ and plug it back into equation (1) to get

$$E_{tr} = \frac{\begin{array}{c} y(t_n) + y'(t_n)\Delta t + \frac{1}{2}y''(t_n)\Delta t^2 + \frac{1}{6}y'''(t_n)\Delta t^3 + O((\Delta t)^4) \\ -(y(t_n) - y'(t_n)\Delta t + \frac{1}{2}y''(t_n)\Delta t^2 - \frac{1}{6}y'''(t_n)\Delta t^3 + O((\Delta t)^4)) \end{array}}{2\Delta t} - f(t_n, y_n)$$

Canceling out Δt, like terms, and 2 from combind terms we are left with

$$E_{tr} = \frac{1}{6}y'''(t_n)\Delta t^2 + y(t_n) - f(t_n, y_n)$$

The $y_n - f(t_n, y_n) = 0$ (ODE) and are now left with

$$E_{tr} = \frac{1}{6}y'''(t_n)\Delta t^2 \tag{5}$$

The leading term (and only term), which has the lowest power of Δt, is $\frac{1}{6}y'''(t_n)\Delta t^2$, therefore

$$E_{tr} = O((\Delta t)^2) \tag{6}$$

which shows that the equation has 2nd order of accuracy.

5 References

1 Daniil Bochkov, CS 111 - Introduction to Computational Science Homework 1 Fall 2017

2 Daniil Bochkov, CS 111 - Introduction to Computational Science Lecture 3 Accuracy Fall 2017