Homework 3

Introduction

The goal of this project is to simulate the flow around a rotating cylinder. The two spatial directions domain is $\Omega = (x_L, x_R)X(t_B, Y_T)$ with a time interval of $[t_{start}, t_{final}]$, velocity vector $\overrightarrow{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ and a function $c = c(t, x, y)$ satisfying the advection equation below:

$$\frac{\partial c}{\partial t} + \overrightarrow{v} \cdot \bigtriangledown c = f(t, x, y), \text{ for all } (x, y) \in \Omega \text{ and } t \in [t_{start}, t_{final}]$$

with boundary conditions:

$$c(t, x, y) = g(t, x, y), \text{ for all } (x, y) \in \Omega \text{ and } t \in [t_{start}, t_{final}]$$

where $\partial\Omega$ is the boundary of domain Ω, and initial data:

$$c(t_{start}, x, y) = c_{start}(x, y), \text{ for all } (x, y) \in \Omega$$

where $f(t, x, y), g(t, x, y)$ and $c_{start}(x, y)$ are given functions.

1 Problem 1

1.1 Part a

Implementing the upwind scheme for this problem with the following example:

$$\begin{array}{lrcl} \text{Domain:} & \Omega & = & [-1;1]X[-1;1] \\ \text{Velocity field:} & v_x(t,x,y) & = & -y \\ & v_y(t,x,y) & = & x \\ \text{Source term:} & f(t,x,y) & = & 0 \\ \text{Boundary conditions:} & g(t,x,y) & = & 0 \\ \text{Initial conditions:} & c_{start}(x,y) & = & \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{(x-R)^2+y^2}-r}{\epsilon}\right)\right) \end{array} \tag{1}$$

where $r = 0.2$, $R = 0.5$, and $\epsilon = 0.1$. The exact solution in this case is given by:

$$c_{exact}(t,x,y) = \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{(x-R\cos(t))^2+(y\sin(t))^2}-r}{\epsilon}\right)\right)$$

The advection equation from $t_{start} = 0$ to $t_{final} = 0.2$ using grid resolutions $N_x = N_y = 50, 100, 200, 400$ and the Courant number $C = 0.75$ is solved by discretizing the interval $[x_L, x_R]$ into a grid of N_x points:

$$x_1 = x_L, \quad x_2 = x_1 + \Delta x, \quad x_3 = x_2 + \Delta x, \quad ..., \quad x_{N_x} = x_R$$

with spatial step

$$\Delta x = \frac{x_R - x_L}{N_x - 1}$$

and also discretizing the interval $[y_B, y_T]$ into a grid of N_y points:

$$y_1 = y_B, \quad y_2 = y_1 + \Delta y, \quad y_3 = y_2 + \Delta y, \quad ..., \quad y_{N_y} = y_T$$

with spatial step

$$\Delta y = \frac{y_T - y_B}{N_y - 1}$$

The approximate $c(t,x,y)$ as a set of N_x and N_y values where $c_{i,j}(t), i = 1,...,N_x, j = 1,...,N_j$ denotes approximate values of function $c(t,x,y)$ at grid points $x_i, i = 1,...,N_x$ and $x_j = 1,...,N_y$. The values of function $c(t,x,y)$ at time $t = t_{start}$ at all grid points is given by the initial conditions:

$$c_1(t_{start}) = c_{start}(x_1, y_1), \quad c_2(t_{start}) = c_{start}(x_2, y_2), \quad ..., \quad c_{N_x}(t_{start}) = c_{start}(x_1 N_x, y_{N_y})$$

The spatial derivative $\frac{\partial c}{\partial x}(t, x_i, y_j)$ and $\frac{\partial c}{\partial y}(t, x_i, y_j)$ at grid point x_i, y_j can be approximated using values of function $c(t,x,y)$ at the same point x_i, y_j and its neighboring points. Forward difference and backward difference will be used for this task:

$$\begin{aligned}
\text{Forward difference: } \frac{\partial c}{\partial x}(t, x_i, y_j) &= \frac{c(t, x_{i+1}, y_j - c(t, x_i, y_j)}{\Delta x} + O(\Delta x) \\
\frac{\partial c}{\partial y}(t, x_i, y_j) &= \frac{c(t, x_i, y_{j+1} - c(t, x_i, y_j)}{\Delta y} + O(\Delta y) \\
\text{Backward difference: } \frac{\partial c}{\partial x}(t, x_i, y_j) &= \frac{c(t, x_i, y_j - c(t, x_{i-1}, y_j)}{\Delta x} + O(\Delta x) \\
\frac{\partial c}{\partial y}(t, x_i, y_j) &= \frac{c(t, x_i, y_j - c(t, x_i, y_{j-1})}{\Delta y} + O(\Delta y)
\end{aligned} \qquad (2)$$

From these we can get an approximation called the upwind discretization below:

$$v_x \frac{\partial c}{\partial x}(t, x_i, y_j) \approx \begin{cases} v\frac{c(t,x_i,y_j) - c(t,x_{i-1},y_j)}{\Delta x}, \text{if } v(t, x_i, y_j) \geq 0 \\ v\frac{c(t,x_{i+1},y_j) - c(t,x_i,y_j)}{\Delta x}, \text{if } v(t, x_i, y_j) < 0 \end{cases}$$

$$v_y \frac{\partial c}{\partial y}(t, x_i, y_j) \approx \begin{cases} v\frac{c(t,x_i,y_j)-c(t,x_i,y_{j-1})}{\Delta t}, \text{if } v(t, x_i, y_j) \geq 0 \\ v\frac{c(t,x_i,y_{j+1})-c(t,x_i,y_j)}{\Delta y}, \text{if } v(t, x_i, y_j) < 0 \end{cases}$$

The CFL stability condition for the 2D upwind scheme is given by:

$$\frac{v_x \Delta t}{\Delta x} + \frac{v_y \Delta t}{\Delta y} \leq 1$$

Given spatial steps Δx and Δy the time-step chosen can be determined by solving the equation below for Δt

$$C = \frac{v_x \Delta t}{\Delta x} + \frac{v_y \Delta t}{\Delta y}$$

and Δt becomes

$$\Delta t = \frac{c}{\frac{v_x}{\Delta x} + \frac{v_y}{\Delta y}}$$

The maximum values of v_x and v_y will yield us the smallest Δt and thus the highest accuracy. So in this case the maximum values are $v_x = 1$ and $v_y = 1$. And thus the equation becomes

$$\Delta t = \frac{c}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}$$

The results are shown in the table below:

Resolution	Error	Order
50	0.0694886	0
100	0.0377187	0.881498
200	0.0198717	0.924565
400	0.0102666	0.95276

Table 1: Errors and Order of accuracy

1.2 Part b

Simulating the flow around a rotating cylinder with radius $r_c = 0.2$ at $(x_c, y_c) = (0, 0)$. The velocity field outside the cylinder is given by:

$$v_x(t,x,y) = U(1 + r_c^2 \frac{y^2-x^2}{(x^2+y^2)^2}) + 2\alpha U r_c \frac{y}{(x^2+y^2)^{3/2}}$$

$$v_y(t,x,y) = -2Ur_c^2 \frac{xy}{(x^2+y^2)^2} - 2\alpha U r_c \frac{x}{(x^2+y^2)^{3/2}}$$

while inside the cylinder its:

$$v_x(t,x,y) = 0$$

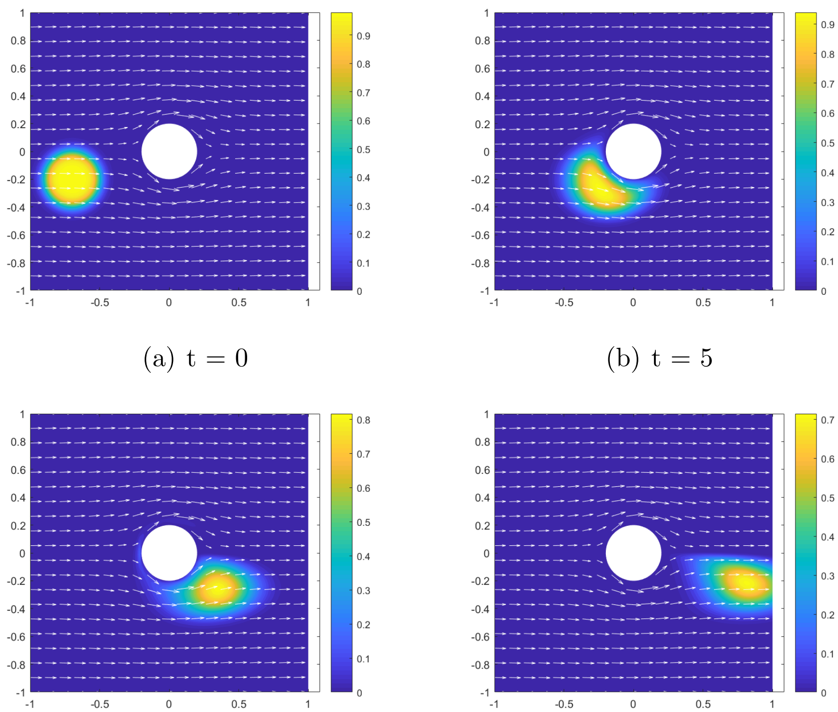
$$v_y(t,x,y) = 0$$

Parameter $U = 0.1$ is the free-stream velocity and α describes the rotational speed of the cylider. The rest of the parameters are:

$$\begin{array}{lrcl} \text{Domain:} & \Omega & = & [-1;1]X[-1;1] \\ \text{Source term:} & f(t,x,y) & = & 0 \\ \text{Boundary conditions:} & g(t,x,y) & = & 0 \\ \text{Initial conditions:} & c_{start}(x,y) & = & \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}-r}{\epsilon}\right)\right) \end{array} \tag{3}$$

where $r = 0.2, x_0 = -0.7, y_0 = -0.2, \epsilon = 0.05, t_{start} = 0, t_{final} = 15, N_x = N_y = 100, C = 0.75, \alpha = 0, 0.1, 0.2$. Snapshots are taken at time $t = 0, 5, 10, 15$

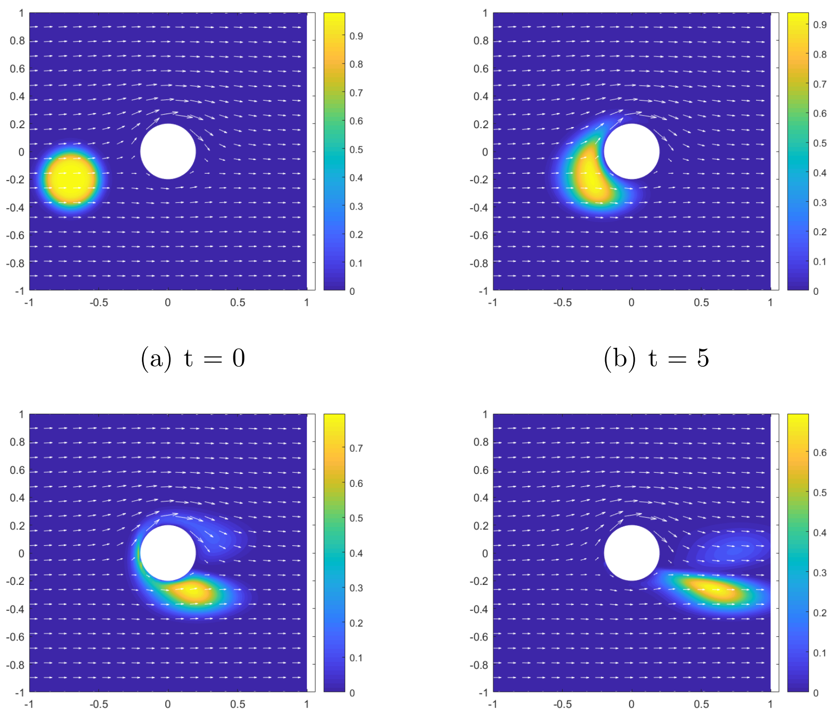
Please see the following pages for the results



(a) t = 0 (b) t = 5

(c) t = 10 (d) t = 15

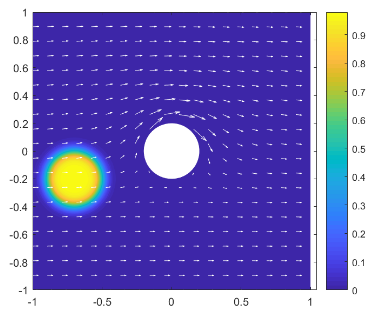
Figure 1: Flow around a rotating cylinder at $\alpha = 0$



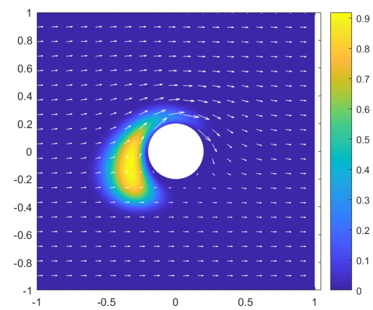
(a) t = 0 (b) t = 5

(c) t = 10 (d) t = 15

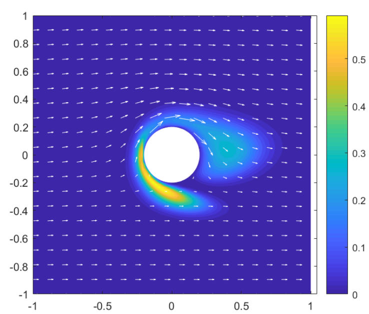
Figure 2: Flow around a rotating cylinder at $\alpha = 0.1$



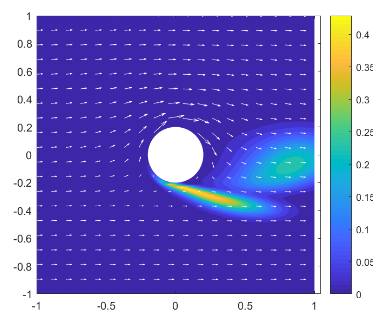
(a) t = 0



(b) t = 5



(c) t = 10



(d) t = 15

Figure 3: Flow around a rotating cylinder at $\alpha = 0.2$

2 References

1 Daniil Bochkov, CS 111 - Introduction to Computational Science Homework 3 Fall 2017

2 Daniil Bochkov, CS 111 - Introduction to Computational Science Lecture 10. Solving Advection Equation 2017

3 Daniil Bochkov, CS 111 - Introduction to Computational Science Lecture 9. Introduction to Partial Differential Equations. 2017