Interest Rate Modelling and Derivative Pricing

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Part V

Bermudan Swaption Pricing

Outline

Outline

Let's have another look at the cancellation option

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

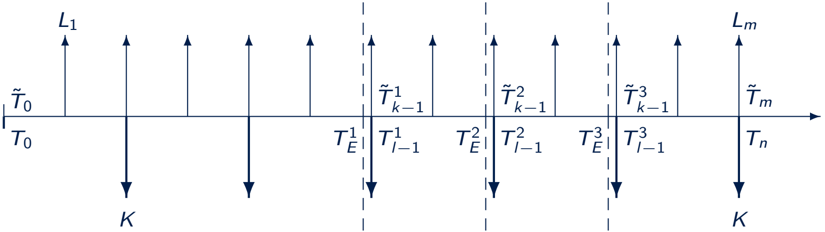
Start date: Oct 31, 2019

End date: Oct 31, 2039

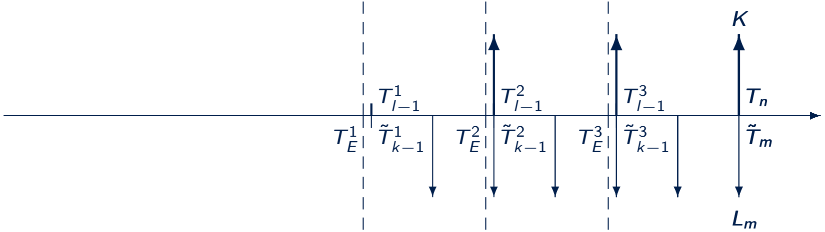
(semi-annually, act/360 day count, modified following, Target calendar)

Bank A may decide to early terminate deal in 10, 11, 12,..years.

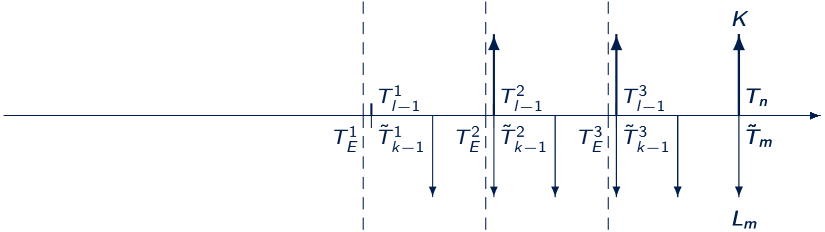
What does such a *Bermudan call right* mean?



[Bermudan cancellable swap] = [full swap] + [Bermudan option on opposite swap]



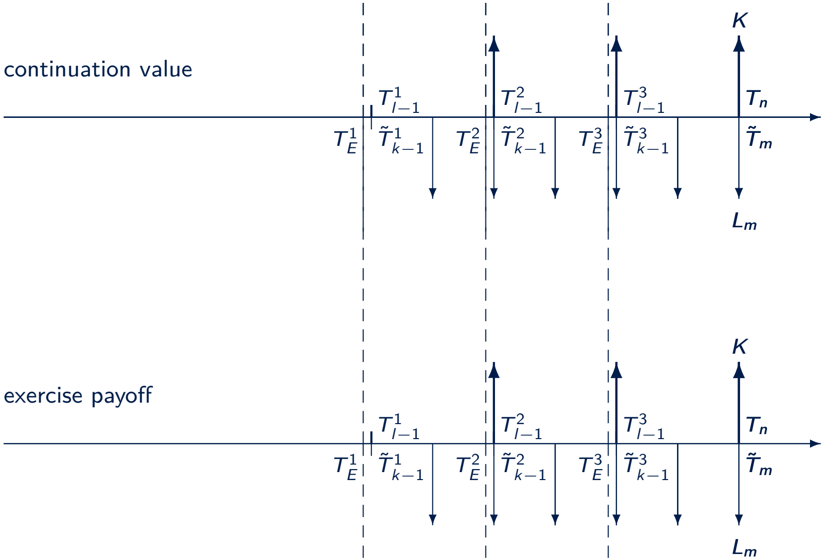
What is a Bermudan swaption?



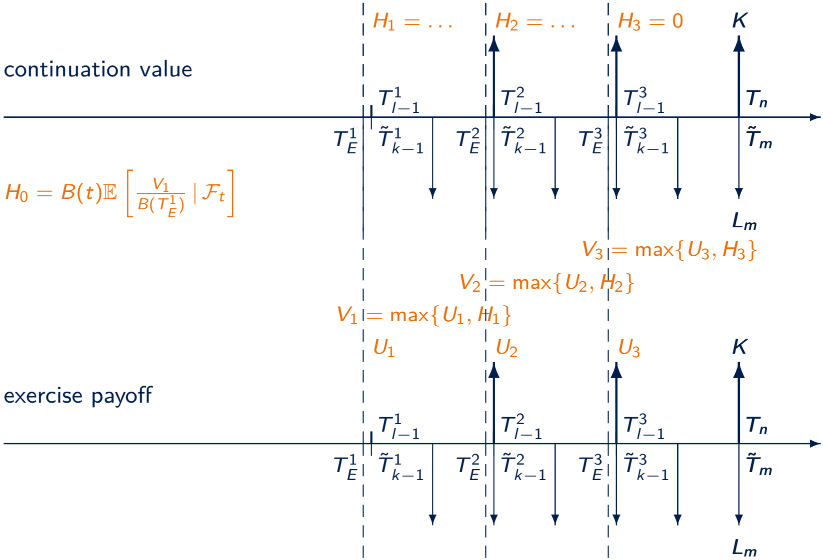
Bermudan swaption

A Bermudan swaption is an option to enter into a Vanilla swap with fixed rate K and final maturity T_n at various exercise dates $T_E^1, T_E^2, \ldots, T_E^{\bar{k}}$. If there is only one exercise date (i.e. $\bar{k} = 1$) then the Bermudan swaption equals a European swaption.

A Bermudan swaption can be priced via *backward induction*



A Bermudan swaption can be priced via *backward induction* - let's add some notation



First we specify the future payoff cash flows

- Choose a numeraire $B(t)$ and corresponding cond. expectations $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_t]$.
- Underlying payoff U_k if option is exercised

$$
\begin{aligned}
U_k &= B(T_E^k) \sum_{T_i \geq T_E^k} \mathbb{E}_{T_E^k}\left[\frac{X_i(T_i)}{B(T_i)}\right] \\
&= B(T_E^k) \underbrace{\left[\sum_{T_i \geq T_E^k} K \cdot \tau_i \cdot P(T_E^k, T_i) - \sum_{\tilde{T}_j \geq T_E^k} L^\delta(T_E^k, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(T_E^k, \tilde{T}_j)\right]}_{\text{future fixed leg minus future float leg}} \\
&= B(T_E^k) \left[\sum_{T_i \geq T_E^k} K \cdot \tau_i \cdot P(T_E^k, T_i)\right. \\
&\qquad \left. - P(T_E^k, \tilde{T}_{j_k}) - \sum_{\tilde{T}_j \geq T_E^k} P(T_E^k, \tilde{T}_{j-1}) \cdot \left[D(\tilde{T}_{j-1}, \tilde{T}_j) - 1\right] + P(T_E^k, \tilde{T}_m)\right].
\end{aligned}
$$

Then we specify the continuation value and optimal exercise

- Continuation value $H_k(t)$ ($T_E^k \leq t \leq T_E^{k+1}$) represents the time-$t$ value of the remaining option if not exercised.
- Option becomes worthless if not exercises at last exercise date $T_E^{\bar{k}}$. Thus last continuation value $H_{\bar{k}}(T_E^{\bar{k}}) = 0$.
- Recall that Bermudan option gives the right but not the obligation to enter into underlying at exercise.
- Rational agent will choose the maximum of payoff and continuation at exercise, i.e.

$$V_k = \max\left\{U_k, H_k(T_E^k)\right\}.$$

- V_k represents the Bermudan option value at exercise T_E^k. Thus we also must have for the continuation value

$$H_{k-1}(T_E^k) = V_k.$$

- Derivative pricing formula yields

$$H_{k-1}(T_E^{k-1}) = B(T_E^{k-1}) \cdot \mathbb{E}_{T_E^{k-1}}\left[\frac{H_{k-1}(T_E^k)}{B(T_E^k)}\right] = B(T_E^{k-1}) \cdot \mathbb{E}_{T_E^{k-1}}\left[\frac{V_k}{B(T_E^k)}\right].$$

We summarize the Bermudan pricing algorithm

Backward induction for Bermudan options

Consider a Bermudan option with $\bar{k}$ exercise dates T_E^k ($k = 1, \ldots \bar{k}$) and underlying future payoffs with (time-T_E^k) prices U_k.

Denote $H_k(t)$ the option's continuation value for $T_E^k \leq t \leq T_E^{k+1}$ and set $H_{\bar{k}}\left(T_E^{\bar{k}}\right) = 0$. Also set $T_E^0 = t$ (i.e. pricing time today).

The option price can be derived via the recursion

$$H_k\left(T_E^k\right) = B(T_E^k) \cdot \mathbb{E}_{T_E^k}\left[\frac{H_k(T_E^{k+1})}{B(T_E^{k+1})}\right] = B(T_E^k) \cdot \mathbb{E}_{T_E^k}\left[\frac{\max\left\{U_{k+1}, H_{k+1}(T_E^{k+1})\right\}}{B(T_E^{k+1})}\right]$$

for $k = \bar{k} - 1, \ldots, 0$. The Bermudan option price is given by

$$V^{\text{Berm}}(t) = H_0(t) = H_0(T_E^0).$$

Some more comments regarding Bermudan pricing ...

- Recursion for Bermudan pricing can be formally derived via theory of optimal stopping and Hamilton-Jacobi-Bellman (HJB) equation.
- For more details, see Sec. 18.2.2 in Andersen/Piterbarg (2010).
- For a single exercise date $\bar{k} = 1$ we get

$$H_0(t) = B(t) \cdot \mathbb{E}_t \left[\frac{\max\{U_1, 0\}}{B(T_E^1)} \right].$$

  This is the general pricing formula for a European swaption (if U_1 represents a Vanilla swap).

- In principle, recursion $H_k\left(T_E^k\right) = B(T_E^k) \cdot \mathbb{E}_{T_E^k}\left[\frac{\max\left\{U_{k+1}, H_{k+1}(T_E^{k+1})\right\}}{B(T_E^{k+1})}\right]$ holds for any payoffs U_k. However, computation

$$U_k = B(T_E^k) \sum_{T_i \geq T_E^k} \mathbb{E}_{T_E^k}\left[\frac{X_i(T_i)}{B(T_i)}\right]$$

  might pose additional challenges if cash flows $X_i(T_i)$ are more complex.

How do we price a Bermudan in practice?

- In principle, recursion algorithm for $H_k()$ is straight forward.
- Computational challenge is calculating conditional expectations

$$H_k\left(T_E^k\right) = B(T_E^k) \cdot \mathbb{E}_{T_E^k}\left[\frac{\max\left\{U_{k+1}, H_{k+1}(T_E^{k+1})\right\}}{B(T_E^{k+1})}\right].$$

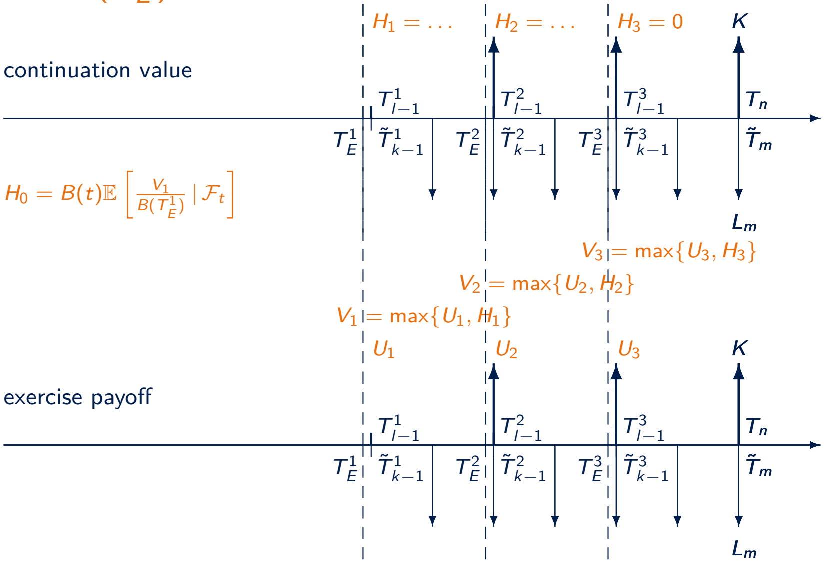
- Note, that this problem is an instance of the general option pricing problem

$$V(T_0) = B(T_0) \cdot \mathbb{E}\left[\frac{V(T_1)}{B(T_1)} \mid \mathcal{F}_{T_0}\right].$$

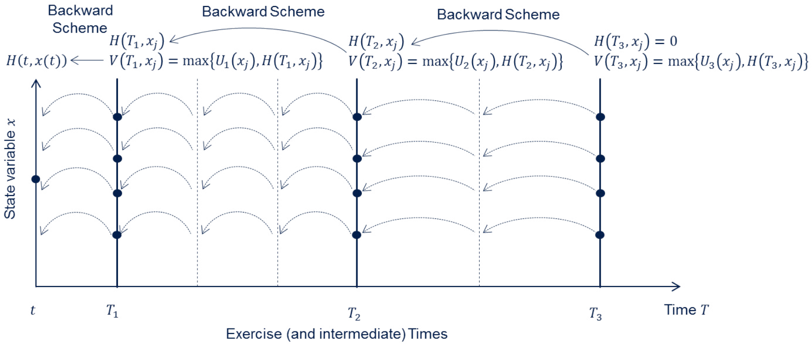
We can apply general option pricing methods to *roll-back* the Bermudan payoff.

Outline

Note that U_k, V_k and H_k depend on underlying state variable $x(T_E^k)$



Typically we need to discretise variables U_k, V_k and H_k on a grid of underlying state variables



Forthcomming, we discuss several methods to roll-back the payoffs.

Outline

Outline

Key idea using the conditional density function in the Hull-White model

Recall that

$$V(T_0) = B(T_0) \cdot \mathbb{E}\left[\frac{V(T_1)}{B(T_1)} \,|\, \mathcal{F}_{T_0}\right].$$

We choose the T_1-maturing zero coupon bond $P(t, T_1)$ as numeraire. Then

$$\begin{aligned} V(T_0) &= P(T_0, T_1) \cdot \mathbb{E}^{T_1}\left[V(T_1) \,|\, \mathcal{F}_{T_0}\right] \\ &= P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu,\sigma^2}(x) \cdot dx. \end{aligned}$$

State variable $x = x(T_1)$ is normally distributed with known mean and variance.

Hull-White model results yield density parameters of the state variable $x(T_1)$

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu,\sigma^2}(x) \cdot dx.$$

State variable $x = x(T_1)$ is normally distributed with mean

$$\mu = \mathbb{E}^{T_1}\left[x(T_1) \,|\, \mathcal{F}_{T_0}\right] = G'(T_0, T_1)\left[x(T_0) + G(T_0, T_1)y(T_0)\right]$$

and variance

$$\sigma^2 = \mathrm{Var}\left[x(T_1) \,|\, \mathcal{F}_{T_0}\right] = y(T_1) - G'(T_0, T_1)^2 y(T_0).$$

Thus

$$p_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

and

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx.$$

Integral against normal density needs to be computed numerically by quadrature methods

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx.$$

- We can apply general purpose quadrature rules to function

$$f(x) = \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

  - Select a grid $[x_0, \ldots, x_N]$ and approximate e.g. via
  - Trapezoidal rule

$$\int_{-\infty}^{+\infty} f(x) \cdot dx \approx \sum_{i=1}^{N} \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right](x_i - x_{i-1})$$

  - or Simpson's rule with equidistant grid $(h = x_i - x_{i-1})$ and even sub-intervalls, then

$$\int_{-\infty}^{+\infty} f(x) \cdot dx \approx \frac{h}{3} \cdot \left[f(x_0) + 2\sum_{j=1}^{N/2-1} f(x_{2j}) + 4\sum_{j=1}^{N/2} f(x_{2j-1}) + f(x_N)\right].$$

There are some details that need to be considered - Select your integration domain carefully

- Infinite integral is approximated by definite integral

$$\int_{-\infty}^{+\infty} f(x) \cdot dx \approx \int_{x_0}^{x_N} f(x) \cdot dx \approx \cdots .$$

- Finite integration boundaries need to be chosen carefully by taking into account variance of $x(t)$.
- One approach is calculating variance to option expiry T_1, $\hat{\sigma}^2 = \mathrm{Var}\left[x(T)\right] = y(T_1)$ and set

$$x_0 = -\lambda \cdot \hat{\sigma} \quad \text{and} \quad x_N = \lambda \cdot \hat{\sigma}.$$

- Note, that $\mathbb{E}^{T_1}\left[x(T_1)\right] = 0$, thus we don't apply a shift to the x-grid.

There are some details that need to be considered - Take care of the break-even point

- Note that convergence of quadrature rules depends on smoothness of integrand $f(x)$.
- Recall that

$$f(x) = V(x) \cdot p_{\mu,\sigma^2}(x) = \max\left\{U_{k+1}(x), H_{k+1}(x;\, T_E^{k+1})\right\} \cdot p_{\mu,\sigma^2}(x).$$

- Max-function is not smooth at $U_{k+1}(x) = H_{k+1}(x;\, T_E^{k+1})$.

Determine $x^\star$ (via interpolation of $H_{k+1}(\cdot)$ and numerical root search) such that

$$U_{k+1}(x^\star) = H_{k+1}(x^\star;\, T_E^{k+1})$$

and split integration

$$\int_{-\infty}^{+\infty} f(x) \cdot dx = \int_{-\infty}^{x^\star} f(x) \cdot dx + \int_{x^\star}^{+\infty} f(x) \cdot dx.$$

Can we exploit the structure of the integrand?

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx.$$

- Integral against normal distribution density can be solved more efficiently:

1. Use Gauss–Hermite quadrature.
2. Interpolate only $V(x; T_1)$ by cubic spline and integrate exact.

Outline

Gauss–Hermite quadrature is an efficient integration method for smooth integrands

- Gauss–Hermite quadrature is a particular form of Gaussian quadrature.
- Choose a degree parameter d, and approximate

$$\int_{-\infty}^{+\infty} f(x) \cdot e^{-x^2} dx \approx \sum_{k=1}^{d} w_k \cdot f(x_k)$$

  with x_k $(i = 1, 2, ..., d)$ being the roots of the physicists' version of the Hermite polynomial $H_d(x)$ and w_k are weights with

$$w_k = \frac{2^{d-1} d! \sqrt{\pi}}{d^2 \left[H_{d-1}(x_k)\right]^2}.$$

- Roots and weights can be obtained, e.g. via stored values and look-up tables.

Variable transformation allows application of Gauss–Hermite quadrature to Hull-White model integration

We get

$$\int_{-\infty}^{+\infty} \frac{V(x;T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} V(\sqrt{2}\sigma x + \mu; T_1) \cdot e^{-x^2} dx$$
$$\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{d} w_k \cdot V(\sqrt{2}\sigma x_k + \mu; T_1).$$

Some constraints need to be considered:

- Payoff $V(\cdot, T_1)$ is only available on the x-grid at T_1, thus $V(\cdot, T_1)$ needs to be interpolated.
- Gauss-Hermite quadrature does not take care of non-smooth payoff at break-even state $x^\star$, thus d needs to be sufficiently large to mitigate impact.

Outline

If we apply cubic spline interpolation anyway then we can also integrate exactly

Approximate $V(\cdot, T_1)$ via cubic spline on the grid $[x_0, \dots x_N]$ as

$$V(x, T_1) \approx C(x) = \sum_{i=0}^{N-1} \mathbb{1}_{\{x_i \le x < x_{i+1}\}} \sum_{k=0}^{d} c_k \cdot (x - x_i)^k \,.$$

Then

$$\begin{aligned}\int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu,\sigma^2}(x) \cdot dx &\approx \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \sum_{k=0}^{d} c_k \cdot (x - x_i)^k \cdot p_{\mu,\sigma^2}(x) \cdot dx \\ &= \sum_{i=0}^{N-1} \sum_{k=0}^{d} c_k \cdot \int_{x_i}^{x_{i+1}} (x - x_i)^k \cdot p_{\mu,\sigma^2}(x) \cdot dx.\end{aligned}$$

Thus, all we need is

$$I_{i,k} = \int_{x_i}^{x_{i+1}} (x - x_i)^k \cdot p_{\mu,\sigma^2}(x) \cdot dx.$$

We transform variables to make integration easier

First we apply the variable transformation $\bar{x} = (x - \mu)/\sigma$. This yields $p_{\mu,\sigma^2}(x) = p_{0,1}(\bar{x})/\sigma$ and

$$I_{i,k} = \int_{\bar{x}_i}^{\bar{x}_{i+1}} (\sigma\bar{x} + \mu - x_i)^k \cdot p_{0,1}(\bar{x}) \cdot \frac{dx}{\sigma}$$
$$= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^k (\bar{x} - \bar{x}_i)^k \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\bar{x}^2}{2}\right\}}_{\text{standard normal density}} d\bar{x}$$

with the shifted grid points $\bar{x}_i = (x_i - \mu)/\sigma$.
Denote $\Phi(\cdot)$ the cumulated standard normal distribution function. Then

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\bar{x}^2}{2}\right\} \quad \text{and} \quad \Phi''(x) = -x\Phi'(x).$$

As a sub-step we aim at solving the integral

$$\int_{\bar{x}_i}^{\bar{x}_{i+1}} \bar{x}^k \cdot \Phi'(\bar{x}) \cdot d\bar{x}.$$

We use cubic splines ($d = 3$) to keep formulas reasonably simple I

It turnes out that

$$F_0(\bar{x}) = \int \Phi'(\bar{x}) d\bar{x} = \Phi(\bar{x}),$$

$$F_1(\bar{x}) = \int \bar{x}\Phi'(\bar{x}) d\bar{x} = -\Phi'(\bar{x}),$$

$$F_2(\bar{x}) = \int \bar{x}^2\Phi'(\bar{x}) d\bar{x} = \Phi(\bar{x}) - x \cdot \Phi'(\bar{x}),$$

$$F_3(\bar{x}) = \int \bar{x}^3\Phi'(\bar{x}) d\bar{x} = -\left(\bar{x}^2 + 2\right) \cdot \Phi'(\bar{x}).$$

This yields for $I_{i,0}$

$$I_{i,0} = \int_{\bar{x}_i}^{\bar{x}_{i+1}} \Phi'(\bar{x}) \cdot dx = F_0(\bar{x}_{i+1}) - F_0(\bar{x}_i)$$

We use cubic splines ($d = 3$) to keep formulas reasonably simple II

and for $I_{i,1}$

$$\begin{aligned}
I_{i,1} &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma\left(\bar{x} - \bar{x}_i\right) \cdot \Phi'(\bar{x}) \cdot dx \\
&= \sigma \cdot \int_{\bar{x}_i}^{\bar{x}_{i+1}} \bar{x} \cdot \Phi'(\bar{x}) \cdot dx - \sigma \cdot \bar{x}_i \cdot I_{i,0} \\
&= \sigma \cdot \left[F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)\right] - \sigma \cdot \bar{x}_i \cdot I_{i,0}.
\end{aligned}$$

We may proceed similarly for $I_{i,2}$

$$\begin{aligned}
I_{i,2} &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^2\left(\bar{x} - \bar{x}_i\right)^2 \cdot \Phi'(\bar{x}) \cdot dx \\
&= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^2\left[\bar{x}^2 - 2\bar{x}_i\bar{x} + \bar{x}_i^2\right] \cdot \Phi'(\bar{x}) \cdot dx \\
&= \sigma^2\left[F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)\right] - 2\sigma^2\bar{x}_i\left[F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)\right] + \sigma^2\bar{x}_i^2 I_{i,0} \\
&= \sigma^2\left[F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)\right] - 2\sigma\bar{x}_i\left[I_{i,1} + \sigma \cdot \bar{x}_i \cdot I_{i,0}\right] + \sigma^2\bar{x}_i^2 I_{i,0} \\
&= \sigma^2\left[F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)\right] - 2\sigma\bar{x}_i I_{i,1} - \sigma^2\bar{x}_i^2 I_{i,0}
\end{aligned}$$

We use cubic splines ($d = 3$) to keep formulas reasonably simple III

and $I_{i,3}$

$$\begin{aligned}
I_{i,3} &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^3 \left(\bar{x} - \bar{x}_i\right)^3 \cdot \Phi'(\bar{x}) \cdot dx \\
&= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^3 \left[\bar{x}^3 - 3\bar{x}_i\bar{x}^2 + 3\bar{x}_i^2\bar{x} - \bar{x}_i^3\right] \cdot \Phi'(\bar{x}) \cdot dx \\
&= \sigma^3 \left[F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)\right] - 3\sigma^3\bar{x}_i \left[F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)\right] \\
&\quad + 3\sigma^3\bar{x}_i^2 \left[F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)\right] - \sigma^3\bar{x}_i^3 I_{i,0}.
\end{aligned}$$

Substituting terms as before yields

$$\begin{aligned}
I_{i,3} &= \sigma^3 \left[F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)\right] - 3\sigma\bar{x}_i \left[I_{i,2} + 2\sigma\bar{x}_i I_{i,1} + \sigma^2\bar{x}_i^2 I_{i,0}\right] \\
&\quad + 3\sigma^2\bar{x}_i^2 \left[I_{i,1} + \sigma \cdot \bar{x}_i \cdot I_{i,0}\right] - \sigma^3\bar{x}_i^3 I_{i,0} \\
&= \sigma^3 \left[F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)\right] - 3\sigma\bar{x}_i I_{i,2} - 3\sigma^2\bar{x}_i^2 I_{i,1} - \sigma^3\bar{x}_i^3 I_{i,0}.
\end{aligned}$$

Let's summarise the formulas...

We get

$$
\begin{aligned}
V(T_0) &= P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu,\sigma^2}(x) \cdot dx \\
&\approx P(x(T_0); T_0, T_1) \cdot \sum_{i=0}^{N-1} \sum_{k=0}^{3} c_k \cdot I_{i,k}
\end{aligned}
$$

with

$$
\begin{aligned}
I_{i,0} &= F_0(\bar{x}_{i+1}) - F_0(\bar{x}_i) \\
I_{i,1} &= \sigma \cdot [F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)] - \sigma \cdot \bar{x}_i \cdot I_{i,0} \\
I_{i,2} &= \sigma^2 [F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)] - 2\sigma\bar{x}_i I_{i,1} - \sigma^2 \bar{x}_i^2 I_{i,0} \\
I_{i,3} &= \sigma^3 [F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)] - 3\sigma\bar{x}_i I_{i,2} - 3\sigma^2 \bar{x}_i^2 I_{i,1} - \sigma^3 \bar{x}_i^3 I_{i,0}
\end{aligned}
$$

and anti-derivative functions $F_k(x)$ as before.

Integrating a cubic spline versus a normal density function occurs in various contexts of pricing methods

- Method already yields good accuracy for smaller number of grid points.
- For larger number of grid points accuracy benefit compared to e.g. Simpson integration seems not too much.
- Either way, use special treatment of break-even point $x^{\star}$.
- Computational effort can become significant for larger number of grid points.
  - Bermudan pricing requires N^2 evaluations of $\Phi(\cdot)$ and $\Phi'(\cdot)$ per exercise.

Outline

PDE methods for finance and pricing are extensively studied in the literature

- We present the basic principles and some aspects relevant for Bermudan bond option pricing.
- Further reading:
  - L. Andersen and V. Piterbarg. *Interest rate modelling, volume I to III.* Atlantic Financial Press, 2010, Sec. 2.
  - D. Duffy. *Finite Difference Methods in Financial Engineering.* Wiley Finance, 2006

Outline

We can adapt the Black-Scholes equation to our Hull-White model setting

- Recall that state variable $x(t)$ follows the risk-neutral dynamics

$$dx(t) = \underbrace{[y(t) - a \cdot x(t)]}_{\mu(t,x(t))} dt + \sigma(t) \cdot dW(t).$$

- Consider an option with price $V = V(t, x(t))$, option expiry time T and payoff $V(T, x(T)) = g(x(T))$.

- Derivative pricing formula yields that discounted option price is a martingale, i.e.

$$d\left(\frac{V(t, x(t))}{B(t)}\right) = \sigma_V(t, x(t)) \cdot dW(t).$$

How can we use this to derive a PDE?

Apply Ito's Lemma to the discounted option price

We get

$$d\left(\frac{V(t,x(t))}{B(t)}\right) = \frac{dV(t,x(t))}{B(t)} + V(t)d\left(\frac{1}{B(t)}\right).$$

With $d\left(B(t)^{-1}\right) = -r(t)\cdot B(t)^{-1}\cdot dt$ follows

$$d\left(\frac{V(t,x(t))}{B(t)}\right) = \frac{1}{B(t)}\left[dV(t,x(t)) - r(t)\cdot V(t)\cdot dt\right].$$

Applying Ito's Lemma to option price $V(t,x(t))$ gives

$$\begin{aligned} dV(t,x(t)) &= V_t\cdot dt + V_x\cdot dx(t) + \frac{1}{2}V_{xx}\cdot[dx(t)]^2 \\ &= \left[V_t + V_x\cdot\mu(t,x(t)) + \frac{1}{2}V_{xx}\cdot\sigma(t)^2\right]dt + V_x\cdot\sigma(t)\cdot dW(t) \end{aligned}$$

with partial derivatives $V_t = \partial V(t,x(t))/\partial t$, $V_x = \partial V(t,x(t))/\partial x$ and $V_{xx} = \partial^2 V(t,x(t))/\partial x^2$.

Combining results yields dynamics of discounted option price

$$d\left(\frac{V\left(t,x(t)\right)}{B(t)}\right) = \frac{1}{B(t)}\underbrace{\left[V_t + V_x \cdot \mu\left(t,x(t)\right) + \frac{1}{2}V_{xx}\cdot\sigma(t)^2 - r(t)\cdot V\right]}_{\mu_V(t,x(t))}dt$$

$$+ \underbrace{\frac{V_x\cdot\sigma(t)}{B(t)}}_{\sigma_V(t,x(t))}\cdot dW(t).$$

Martingale property of $\frac{V(t,x(t))}{B(t)}$ requires that drift vanishes. That is

$$\mu_V(t,x(t)) = V_t + V_x \cdot \mu\left(t,x(t)\right) + \frac{1}{2}V_{xx}\cdot\sigma(t)^2 - r(t)\cdot V = 0.$$

Substituting $\mu\left(t,x(t)\right) = y(t) - a\cdot x(t)$ and $r(t) = f(0,t) + x(t)$ yields pricing PDE.

We get the parabolic pricing PDE with terminal condition

Theorem (Derivative pricing PDE in Hull-White model)

Consider our Hull-White model setup and a derivative security with price process $V(t, x(t))$ that pays at time T the payoff $V(T, x(T)) = g(x(T))$. Further assume $V(T, x(T))$ has finite variance and is attainable. Then for $t < T$ the option price

$$V(t, x(t)) = B(t) \cdot \mathbb{E}^{\mathbb{Q}} \left[\frac{V(T, x(T))}{B(T)} \,|\, \mathcal{F}_t \right]$$

follows the PDE

$$V_t(t, x) + [y(t) - a \cdot x] \cdot V_x(t, x) + \frac{\sigma(t)^2}{2} \cdot V_{xx}(t, x) = [x + f(0, t)] \cdot V(t, x)$$

with terminal condition

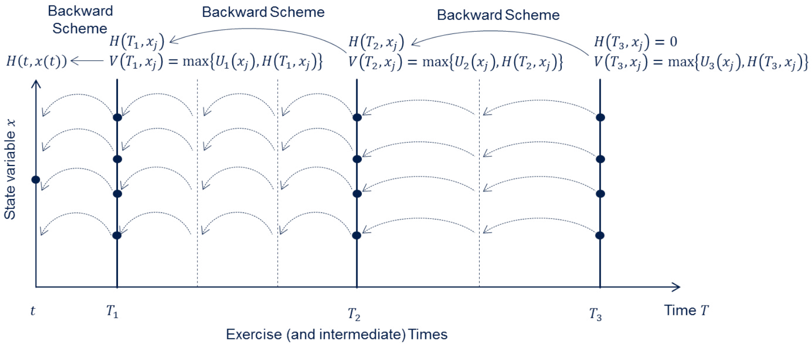
$$V(T, x) = g(x).$$

Proof.

Follows from derivation above.

□

How does this help for our Bermudan option pricing problem?



- We need option prices on a grid of state variables $[x_0, \ldots x_N]$

Solve Hull-White option pricing PDE backwards from exercise to exercise.

Pricing PDE is typically solved via finite difference scheme and time integration

- Use *method of lines (MOL)* to solve parabolic PDE:
  - First discretise state space.
  - Then integrate resulting system of ODEs with terminal condition in time-direction.
- We discuss basic (or general purpose) approach to solve PDE; for a detailed treatment see Andersen/Piterbarg (2010) or Duffy (2006).
- Some aspects may require special attention in the context of Hull-White model:
  - more problem-specific boundary discretisation,
  - non-equidistant grids with finer grid around break-even state $x^\star$,
  - upwinding schemes to deal with potentially dominant impact of convection term $[y(t) - a \cdot x] \cdot V_x(t, x)$ at the grid boundaries of x.

Outline

How do we discretise state space?

- PDE for $V(t, x(t))$ is defined on infinite domain $(-\infty, +\infty)$.
  - We don't get explicit boundary conditions from PDE derivation.
  - Thus, we require payoff-specific approximation.
  - Finally, we are interested in $V(0, 0)$.
- We use equidistant x-grid $x_0, \ldots, x_N$ with grid size h_x centered around zero and scaled via standard deviation of $x(T)$ at final maturity T ,

$$x_0 = -\lambda \cdot \hat{\sigma} \quad \text{and} \quad x_N = \lambda \cdot \hat{\sigma}$$

  with $\hat{\sigma}^2 = \text{Var}[x(T)] = y(T)$ and $\lambda \approx 5$.
- Why not shift the grid by expectation $\mathbb{E}[x(T)]$ (as suggested in the literature)?
  - Pricing PDE is independent of pricing measure (used for derivation).
  - There is no *natural* measure under which $\mathbb{E}[x(T)]$ should be calculated.
  - In T-forward measure $\mathbb{E}^T[x(T)] = 0$ anyway.

Differential operators in state-dimension are discretised via central finite differences

For now leave time t continuous. We use notation $V(\cdot, x)$.

For inner grid points x_i with $i = 1, \ldots, N-1$

$$V_x(\cdot, x_i) = \frac{V(\cdot, x_{i+1}) - V(\cdot, x_{i-1})}{2h_x} + \mathcal{O}(h_x^2) \quad \text{and}$$

$$V_{xx}(\cdot, x_i) = \frac{V(\cdot, x_{i+1}) - 2V(\cdot, x_i) + V(\cdot, x_{i-1})}{h_x^2} + \mathcal{O}(h_x^2).$$

At the boundaries we impose condition

$$V_{xx}(\cdot, x_0) = \lambda_0 \cdot V_x(\cdot, x_0) \quad \text{and} \quad V_{xx}(\cdot, x_N) = \lambda_N \cdot V_x(\cdot, x_N).$$

This yields one-sided first order partial derivative approximations

$$V_x(\cdot, x_0) \approx \frac{2\left[V(\cdot, x_1) - V(\cdot, x_0)\right]}{(2 + \lambda_0 h_x)\, h_x} \quad \text{and} \quad V_x(\cdot, x_N) \approx \frac{2\left[V(\cdot, x_N) - V(\cdot, x_{N-1})\right]}{(2 - \lambda_N h_x)\, h_x}.$$

Some initial comments regarding choice of $\lambda_{0,N}$

- Often, $\lambda_{0,N} = 0$ (also suggested in the literature).

- With $\lambda_{0,N} = 0$ we have $V_{xx}(\cdot, x_0) = V_{xx}(\cdot, x_N) = 0$ and

$$V_x(\cdot, x_0) = \frac{V(\cdot, x_1) - V(\cdot, x_0)}{h_x} + \mathcal{O}(h_x^2) \quad \text{and}$$

$$V_x(\cdot, x_N) = \frac{V(\cdot, x_N) - V(\cdot, x_{N-1})}{h_x} + \mathcal{O}(h_x^2).$$

- However, for bond options the choice $V_{xx}(\cdot, x_0) = V_{xx}(\cdot, x_N) = 0$ might be a poor approximation.

- We will discuss an alternative choice for $\lambda_{0,N}$ later.

Now consider PDE for each grid point individually

Define the vector-valued function $v(t)$ via

$$v(t) = [v_0(t), \dots, v_N(t)]^\top = [V(t, x_0), \dots, V(t, x_0)]^\top \in \mathbb{R}^{N+1}.$$

Then state discretisation yields for inner points x_i with $i = 1, \dots, N-1$,

$$v_i'(t) + [y(t) - a x_i] \frac{v_{i+1}(t) - v_{i-1}(t)}{2h_x} + \frac{\sigma(t)^2}{2} \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_x^2} = [x_i + f(0,t)]\, v_i(t)$$

and for the boundaries

$$v_0'(t) + \left[y(t) - a x_0 + \lambda_0 \frac{\sigma(t)^2}{2} \right] \frac{2\,[v_1(t) - v_0(t)]}{(2 + \lambda_0 h_x)\, h_x} = [x_0 + f(0,t)]\, v_0(t),$$

$$v_N'(t) + \left[y(t) - a x_N + \lambda_N \frac{\sigma(t)^2}{2} \right] \frac{2\,[v_N(t) - v_{N-1}(t)]}{(2 - \lambda_N h_x)\, h_x} = [x_N + f(0,t)]\, v_N(t).$$

As before, we have the terminal condition

$$v_i(T) = g(x_i).$$

Parabolic PDE is transformed into linear system of ODEs with terminal condition.

It is more convenient to write system of ODEs in matrix-vector notation

We get

$$v'(t) = M(t) \cdot v(t) = \begin{bmatrix} c_0 & u_0 & & \\ l_1 & \ddots & \ddots & \\ & \ddots & \ddots & u_{N-1} \\ & & l_N & c_N \end{bmatrix} \cdot v(t)$$

with time-dependent components $c_i,\ l_i,\ u_i\ (i = 1, \dots N-1)$,

$$c_i = \frac{\sigma(t)^2}{h_x^2} + x_i + f(0,t),\ l_i = -\frac{\sigma(t)^2}{2h_x^2} + \frac{y(t) - ax_i}{2h_x},\ u_i = -\frac{\sigma(t)^2}{2h_x^2} - \frac{y(t) - ax_i}{2h_x}$$

and

$$c_0 = \frac{2\left[y(t) - ax_0 + \lambda_0 \frac{\sigma(t)^2}{2}\right]}{(2 + \lambda_0 h_x)\, h_x} + x_0 + f(0,t), \quad c_N = -\frac{2\left[y(t) - ax_N + \lambda_N \frac{\sigma(t)^2}{2}\right]}{(2 - \lambda_N h_x)\, h_x} + x_0 + f(0,t),$$

$$u_0 = -\frac{2\left[y(t) - ax_0 + \lambda_0 \frac{\sigma(t)^2}{2}\right]}{(2 + \lambda_0 h_x)\, h_x}, \quad l_N = \frac{2\left[y(t) - ax_N + \lambda_N \frac{\sigma(t)^2}{2}\right]}{(2 - \lambda_N h_x)\, h_x}.$$

Outline

Linear system of ODEs can be solved by standard methods

We have

$$v'(t) = f\left(t, v(t)\right) = M(t) \cdot v(t).$$

We demonstrate time discretisation based on θ-*method*. Consider equidistant time grid $t = t_0, \ldots, t_M = T$ with step size h_t and approximation

$$\frac{v(t_{j+1}) - v(t_j)}{h_t} \approx f\left(t_{j+1} - \theta h_t, (1-\theta)v(t_{j+1}) + \theta v(t_j)\right)$$

for $\theta \in [0, 1]$.

- In general approximation yields method of order $\mathcal{O}(h_t)$.
- For $\theta = \frac{1}{2}$ approximation yields method of order $\mathcal{O}(h_t^2)$.

For our linear ODE we set $v^j = v(t_j)$, $M_\theta = M(t_{j+1} - \theta h_t)$ and get the scheme

$$\frac{v^{j+1} - v^j}{h_t} = M_\theta \left[(1-\theta)v^{j+1} + \theta v^j\right].$$

We get a recursion for the θ-method

Rearranging terms yields

$$[I + h_t \theta M_\theta]\, v^j = [I - h_t\,(1-\theta)\, M_\theta]\, v^{j+1}.$$

If $[I + h_t \theta M_\theta]$ is regular then we can solve for v^j via

$$v^j = [I + h_t \theta M_\theta]^{-1}\, [I - h_t\,(1-\theta)\, M_\theta]\, v^{j+1}.$$

Terminal condition is

$$v^M = [g(x_0), \dots, g(x_N)]^\top .$$

- Unless $\theta = 0$ (Explicit Euler scheme) we need to solve a linear equation system.
- Fortunately, matrix $[I + h_t \theta M_\theta]$ is tri-diagonal; solution requires $\mathcal{O}(M)$ operations.
- θ-method is A-stable for $\theta \geq \frac{1}{2}$.
- However, oscillations in solution may occur unless $\theta = 1$ (Implicit Euler scheme, L-stable).

Outline

Let's have another look at the boundary condition ...

We look at an example of a zero coupon bond option with payoff

$$V(x,T) = \left[P(x,T,T') - K\right]^+ .$$

For $x \ll 0$ option is far in-the-money and $V(x,t)$ can be approximated by intrinsic value

$$V(x,t) \approx \tilde{V}(x,t) = \left[P(x,t,T') - K\right]^+ = \left[\frac{P(0,T')}{P(0,t)} e^{-G(t,T)x - \frac{1}{2}G(t,T)^2 y(t)} - K\right]^+ .$$

This yields

$$\frac{\partial}{\partial x}\tilde{V}(x,t) = -G(t,T)\left[\tilde{V}(x,t) + K\right]$$

and

$$\frac{\partial^2}{\partial x^2}\tilde{V}(x,t) = \underbrace{-G(t,T)}_{\lambda}\frac{\partial}{\partial x}\tilde{V}(x,t).$$

Alternatively, for $x \gg 0$ option is far out-of-the-money and

$$\frac{\partial^2}{\partial x^2}\tilde{V}(x,t) = \frac{\partial}{\partial x}\tilde{V}(x,t) = 0.$$

We adapt that approximation to our general option pricing problem

- In principle, for a coupon bond underlying we could estimate $\lambda = \lambda(t)$ via option intrinsic value $\tilde{V}(x,t)$ and

$$\lambda(t) = \left[\frac{\partial^2}{\partial x^2}\tilde{V}(x,t)\right] / \frac{\partial}{\partial x}\tilde{V}(x,t) \quad \text{for} \quad \frac{\partial}{\partial x}\tilde{V}(x,t) \neq 0,$$

otherwise $\lambda(t) = 0$.

- We take a more rough approach by approximating λ based only on previous solution

$$\lambda_{0,N} = \left[\frac{\partial^2}{\partial x^2}V(x,t)\right] / \frac{\partial}{\partial x}V(x,t) \approx \left[\frac{\partial^2}{\partial x^2}V(x_{1,N-1}, t+h_t)\right] / \frac{\partial}{\partial x}V(x_{1,N-1}, t+h_t)$$

$$\approx \frac{v_{0,N-2}^{j+1} - 2v_{1,N-1}^{j+1} + v_{2,N}^{j+1}}{h_x^2} / \frac{v_{2,N}^{j+1} - v_{0,N-2}^{j+1}}{2h_x}$$

for $v_{2,N}^{j+1} - v_{0,N-2}^{j+1}/(2h_x) \neq 0$, otherwise $\lambda_{0,N} = 0$.

It turns out that accuracy of one-sided first order derivative approximation is of order $\mathcal{O}(h_x^2)$ I

Lemma

Assume $V = V(x)$ is twice continuously differentiable. Moreover, consider grid points x_{-1}, x_0, x_1 with equal spacing $h_x = x_1 - x_0 = x_0 - x_{-1}$. If there is a $\lambda_0 \in \mathbb{R}$ such that

$$V''(x_0) = \lambda_0 \cdot V'(x_0)$$

then

$$V'(x_0) = \frac{2\,[V(x_1) - V(x_0)]}{(2 + \lambda_0 h_x)\, h_x} + \mathcal{O}(h_x^2).$$

Proof:

Denote $v_i = V(x_i)$. We have from standard Taylor approximation

$$V''(x_0) = \frac{v_{-1} - 2v_0 + v_1}{h_x^2} + \mathcal{O}(h_x^2) \quad \text{and} \quad V'(x_0) = \frac{v_1 - v_{-1}}{2h_x} + \mathcal{O}(h_x^2).$$

From $V''(x_0) = \lambda \cdot V'(x_0)$ follows

$$\frac{v_{-1} - 2v_0 + v_1}{h_x^2} + \mathcal{O}(h_x^2) = \lambda_0 \left[\frac{v_1 - v_{-1}}{2h_x} + \mathcal{O}(h_x^2) \right].$$

It turns out that accuracy of one-sided first order derivative approximation is of order $\mathcal{O}(h_x^2)$ II

Multiplying with $2h_x^2$ gives the relation

$$2\,(v_{-1} - 2v_0 + v_1) + \mathcal{O}(h_x^4) = \lambda_0 h_x\,(v_1 - v_{-1}) + \mathcal{O}(h_x^4).$$

Reordering terms yields

$$(2 + \lambda_0 h_x)\,v_{-1} = 4v_0 + (\lambda_0 h_x - 2)\,v_1 + \mathcal{O}(h_x^4).$$

And solving for v_{-1} gives $v_{-1} = [4v_0 + (\lambda_0 h_x - 2)\,v_1]\,/\,(2 + \lambda_0 h_x) + \mathcal{O}(h_x^4)$.
Now, we substitute v_{-1} in the approximation for $V'(x)$. This gives

$$\begin{aligned}
V'(x_0) &= \frac{v_1 - \Big[[4v_0 + (\lambda_0 h_x - 2)\,v_1]\,/\,(2 + \lambda_0 h_x) + \mathcal{O}(h_x^4)\Big]}{2h_x} + \mathcal{O}(h_x^2) \\
&= \frac{(2 + \lambda_0 h_x)\,v_1 - [4v_0 + (\lambda_0 h_x - 2)\,v_1]}{2\,(2 + \lambda_0 h_x)\,h_x} + \mathcal{O}(h_x^2) + \mathcal{O}(h_x^3) \\
&= \frac{2v_1 - 4v_0 + 2v_1}{2\,(2 + \lambda_0 h_x)\,h_x} + \mathcal{O}(h_x^2) \\
&= \frac{2\,(v_1 - v_0)}{(2 + \lambda_0 h_x)\,h_x} + \mathcal{O}(h_x^2).
\end{aligned}$$

It turns out that accuracy of one-sided first order derivative approximation is of order $\mathcal{O}(h_x^2)$ III

- With constraint $V''(x_0) = \lambda \cdot V'(x_0)$ we can eliminate explicit dependence on second derivative $V''(x_0)$ and outer grid point $v_{-1} = V(x_{-1})$.
- Analogous result can be derived for upper boundery and down-ward approximation of first derivative.
- Resulting scheme is still second order accurate in state space direction.

Outline

We summarise the PDE pricing method

1. Discretise state space x on a grid $[x_0, \ldots, x_N]$ and specify time step size h_t and $\theta \in [0, 1]$.

2. Determine the terminal condition $v^{j+1} = \max\{U_{j+1}, H_{j+1}\}$ for the current valuation step.

3. Set up discretised linear operator M_θ of the resulting ODE system $\frac{d}{dt} v = M_\theta \cdot v$.

4. Incorporate appropriate product-specific boundary conditons.

5. Set up linear system $[I + h_t \theta M_\theta] v^j = [I - h_t (1 - \theta) M_\theta] v^{j+1}$.

6. Solve linear system for v^j by tri-diagonal matrix solver.

7. Repeat with step 3. until next exercise date or $t_j = 0$.

Outline

Monte Carlo methods are widely applied in various finance applications

- We demonstrate the basic principles for
  - path integration of Ito processes
  - exact simulation of Hull-White model paths
- There are many aspects that should also be considered, see e.g.
  - L. Andersen and V. Piterbarg. *Interest rate modelling, volume I to III.*
    Atlantic Financial Press, 2010, Sec. 3.
  - P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer, 2003

Outline

Monte Carlo (MC) pricing is based on the Strong Law of Large Numbers

Theorem (Strong Law of Large Numbers)

Let $Y_1, Y_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables with finite expectation $\mu < \infty$. Then the sample mean $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ converges to μ a.s. That is

$$\lim_{n\to\infty} \bar{Y}_n = \mu \quad a.s.$$

- We aim at calculating $V(t) = N(t) \cdot \mathbb{E}^N \left[V(T)/N(T) \,|\, \mathcal{F}_t \right]$.
- For MC pricing simulate future discounted payoffs $\left\{ \frac{V(T;\omega_i)}{N(T;\omega_i)} \right\}_{i=1,2,\ldots n}$.
- And estimate

$$V(t) = N(t) \cdot \frac{1}{n} \sum_{i=1}^{n} \frac{V(T;\omega_i)}{N(T;\omega_i)}.$$

Keep in mind that sample mean is still a random variable governed by central limit theorem

Theorem (Central Limit Theorem)

Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with finite expectation $\mu < \infty$ and standard deviation $\sigma < \infty$. Denote the sample mean $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Then

$$\frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1).$$

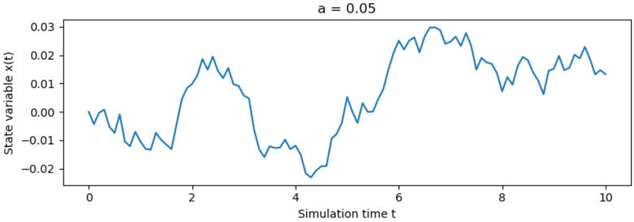
Moreover, for the variance estimator $s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}_n\right)^2$ we also have

$$\frac{\bar{Y}_n - \mu}{s_n/\sqrt{n}} \xrightarrow{d} N(0,1).$$

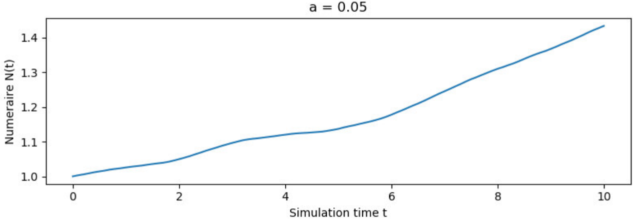
- Here, $N(0,1)$ is the standard normal distribution.
- $\xrightarrow{d}$ denotes convergence in distribution, i.e. $\lim_{n\to\infty} F_n(x) = F(x)$ for the corresponding cumulative distribution functions and all $x \in \mathbb{R}$ at which $F(x)$ is continuous.
- $s_n/\sqrt{n}$ is the standard error of the sample mean $\bar{Y}_n$.

How do we get our samples $V(T;\omega_i)/N(T;\omega_i)$?

1. Simulate state variables $x(t)$ on relevant dates t:



2. Simulate numeraire $N(t)$ on relevant dates t:



3. Calculate payoff $V(T, x(T))$ at observation/pay date T.

We need to simulate our state variables on the relevant observation dates

Consider the general dynamics for a process given as SDE

$$dX(t) = \mu(t, X(t)) \cdot dt + \sigma(t, X(t)) \cdot dW(t).$$

- Typically, we know initial value $X(t)$ ($t = 0$).
- We need $X(T)$ for some future time $T > t$.
- In Hull-White model and risk-neutral measure formulation we have

$$\mu(t, X(t)) = y(t) - a \cdot X(t), \quad \text{and}, \quad \sigma(t, X(t)) = \sigma(t).$$

There are several standard methods to solve above SDE. We will briefly discuss Euler method and Milstein method.

Euler method for SDEs is similar to Explicit Euler method for ODEs

- Specify a grid of simulation times $t = t_0, t_1, \ldots, t_M = T$.
- Calculate sequence of state variables

$$X_{k+1} = X_k + \mu(t_k, X_k)\,(t_{k+1} - t_k) + \sigma(t_k, X_k)\,[W(t_{k+1}) - W(t_k)]\,.$$

- Drift $\mu(t_k, X_k)$ and volatility $\sigma(t_k, X_k)$ are evaluated at current time t_k and state X_k.
- Increment of Brownian motion $W(t_{k+1}) - W(t_k)$ is normally distributed, i.e.

$$W(t_{k+1}) - W(t_k) = Z_k \cdot \sqrt{t_{k+1} - t_k} \quad \text{with} \quad Z_k \sim N(0,1).$$

Milstein method refines the simulation of the diffusion term

- Again, specify a grid of simulation times $t = t_0, t_1, \ldots, t_M = T$.
- Calculate sequence of state variables

$$
\begin{aligned}
X_{k+1} = X_k + \mu(t_k, X_k)\,(t_{k+1} - t_k) + \sigma(t_k, X_k)\,[W(t_{k+1}) - W(t_k)] \\
+ \frac{1}{2} \cdot \frac{\partial}{\partial x}\sigma(t_k, X_k) \cdot \sigma(t_k, X_k) \cdot \left[(W(t_{k+1}) - W(t_k))^2 - (t_{k+1} - t_k)\right].
\end{aligned}
$$

- Drift $\mu(t_k, X_k)$ and volatility $\sigma(t_k, X_k)$ are evaluated at current time t_k and state X_k.
- Requires calculation of derivative of volatility $\frac{\partial}{\partial x}\sigma(t_k, X_k)$ w.r.t. state variable.
- Increment of Brownian motion $W(t_{k+1}) - W(t_k)$ is normally distributed, i.e.

$$
W(t_{k+1}) - W(t_k) = Z_k \cdot \sqrt{t_{k+1} - t_k} \quad \text{with} \quad Z_k \sim N(0, 1).
$$

- With $\Delta_k = t_{k+1} - t_k$ iteration becomes

$$
X_{k+1} = X_k + \mu(t_k, X_k)\Delta_k + \sigma(t_k, X_k) Z_k \sqrt{\Delta_k} + \frac{1}{2}\frac{\partial\sigma(t_k, X_k)}{\partial x}\sigma(t_k, X_k)\left(Z_k^2 - 1\right)\Delta_k.
$$

How can we measure convergence of the methods?

- We distinguish strong order of convergence and weak order of convergence.
- Consider a discrete SDE solution $\left\{X_k^h\right\}_{k=0}^M$ with $X_k^h \approx X(t+kh)$, $h = \frac{T-t}{M}$.

Definition (Strong order of convergence)

The discrete solution X_M^h at final maturity $T = t + hM$ converges to the exact solution $X(T)$ with strong orderβ if there exists a constant C such that

$$\mathbb{E}\left[\left|X_M^h - X(T)\right|\right] \leq C \cdot h^\beta .$$

- Strong order of convergence focuses on convergence on the individual paths.
- Euler method has strong order of convergence of $\frac{1}{2}$ (given sufficient conditions on $\mu(\cdot)$ and $\sigma(\cdot)$).
- Milstein method has strong order of convergence of 1 (given sufficient conditions on $\mu(\cdot)$ and $\sigma(\cdot)$).

For derivative pricing we are typically interested in weak order of convergence

We need some context for weak order of convergence

- A function $f : \mathbb{R} \to \mathbb{R}$ is polynomially bounded if $|f(x)| \leq k(1+|x|)^q$ for constants k and q and all x.
- The set $\mathcal{C}_{\mathcal{P}}^{n}$ represents all functions that are n-times continuously differentiable and with 1st to nth derivative polynomially bounded.

Definition (Weak order of convergence)

The discrete solution X_M^h at final maturity $T = t + hM$ converges to the exact solution $X(T)$ with weak orderβ if there exists a constant C such that

$$\left|\mathbb{E}\left[f\left(X_M^h\right)\right] - \mathbb{E}\left[f\left(X(T)\right)\right]\right| \leq C \cdot h^{\beta} \quad \forall f \in \mathcal{C}_{\mathcal{P}}^{2\beta+2}$$

for sufficiently small h.

- Think of f as a payoff function, then weak order of convergence is related to convergence in price.
- Euler method and Milstein method can be shown to have weak order 1 convergence (given sufficient conditions on μ and σ).

Some comments regarding weak order of convergence

Error estimate

$$\left|\mathbb{E}\left[f\left(X_M^h\right)\right] - \mathbb{E}\left[f\left(X(T)\right)\right]\right| \leq C \cdot h^{\beta}$$

requires considerable assumptions regarding smoothness of $\mu(\cdot)$, $\sigma(\cdot)$ and test functions $f(\cdot)$.

- In practice payoffs are typically non-smooth at the strike.
- This limits applicability of more advanced schemes with theoretical higher order of convergence.
- A fairly simple approach of a higher order scheme is based on Richardson extrapolation:
  - this method is also applied to ODEs,
  - see Glassermann (2000), Sec. 6.2.4 for details.
- Typically, numerical testing is required to assess convergence in practice.

The choice of pricing measure is crucial for numeraire simulation

Consider risk-neutral measure, then

$$N(T) = B(T) = \exp\left\{\int_0^T r(s)ds\right\} = \exp\left\{\int_0^T [f(0,s) + x(s)]\, ds\right\}$$
$$= P(0,T)^{-1} \exp\left\{\int_0^T x(s)ds\right\}.$$

Requires simulation or approximation of $\int_0^T x(s)ds$.

Suppose $x(t_k)$ is simulated on a time grid $\{t_k\}_{k=0}^M$ then we approximate integral via Trapezoidal rule

$$\int_0^T x(s)ds \approx \sum_{i=1}^M \frac{x(t_{k-1}) + x(t_k)}{2} (t_k - t_{k-1}).$$

Numeraire simulation is done in parallel to state simulation

$$N(t_k) = \frac{P(0,t_{k-1})}{P(0,t_k)} \cdot N(t_{k-1}) \cdot \exp\left\{\frac{x(t_{k-1}) + x(t_k)}{2} (t_k - t_{k-1})\right\}.$$

Alternatively, we can simulate in T-forward measure for a fixed future time T

Select a future time $\bar{T}$ sufficiently large. Then $N(0) = P(0, \bar{T})$.
At any pay time $T \leq \bar{T}$ numeraire is directly available via zero coupon bond formula

$$N(T) = P(x(T), T, \bar{T}) = \frac{P(0, \bar{T})}{P(0, T)} e^{-G(T, T')x(T) - \frac{1}{2} G(T, T')^2 y(T)}.$$

However, $\bar{T}$-forward measure simulation needs consistent model formulation or change of measure.
In particular

$$\underbrace{dW^{\bar{T}}(t)}_{\text{B.M. in } \bar{T}\text{-forward measure}} = \underbrace{\sigma_P(t, \bar{T})}_{\text{ZCB volatility}} \cdot dt + \underbrace{dW(t)}_{\text{B.M. in risk-neutral measure}}.$$

Another commonly used numeraire for simulation is the discretely compounded bank account

- Consider a grid of simulation times $t = t_0, t_1, \ldots, t_M = T$.
- Assume we start with 1 EUR at $t = 0$, i.e. $N(0) = 1$.
- At each t_k we take numeraire $N(t_k)$ and buy zero coupon bond maturing at t_{k+1}. That is

$$N(t) = P(t, t_{k+1}) \cdot \frac{N(t_k)}{P(t_k, t_{k+1})} \quad \text{for} \quad t \in [t_k, t_{k+1}] .$$

Explicitly, define discretely compounded bank account as $\bar{B}(0) = 1$ and

$$\bar{B}(t) = \prod_{t_k < t} \frac{P(t, t_{k+1})}{P(t_k, t_{k+1})}.$$

We get

$$d\left(\frac{\bar{B}(t)}{P(t, t_{k+1})}\right) = \prod_{t_k < t} \frac{1}{P(t_k, t_{k+1})} \cdot d\left(\frac{P(t, t_{k+1})}{P(t, t_{k+1})}\right) = 0 \quad \text{for} \quad t \in [t_k, t_{k+1}] .$$

Simulating in $\bar{B}$-measure is equivalent to simulating in rolling t_{k+1}-forward measure.

Outline

Do we really need to solve the Hull-White SDE numerically?

Recall dynamics in T-forward measure

$$dx(t) = \left[y(t) - \sigma(t)^2 G(t, T) - a \cdot x(t)\right] \cdot dt + \sigma(t) \cdot dW^T(t).$$

That gives

$$x(T) = e^{-a(T-t)} \left[x(t) + \int_t^T e^{a(u-t)} \left(\left[y(u) - \sigma(u)^2 G(u, T)\right] du + \sigma(u) dW^T(u)\right)\right].$$

As a result $x(T) \sim N(\mu, \sigma^2)$ (conditional on t) with

$$\mu = \mathbb{E}^T \left[x(T) \,|\, \mathcal{F}_t\right] = G'(t, T) \left[x(t) + G(t, T) y(t)\right] \quad \text{and}$$

$$\sigma^2 = \mathrm{Var}\left[x(T) \,|\, \mathcal{F}_t\right] = y(T) - G'(t, T)^2 y(t).$$

We can simulate exactly

$$x(T) = \mu + \sigma \cdot Z \quad \text{with} \quad Z \sim N(0, 1).$$

Expectation calculation via $\mu = \mathbb{E}^{T}\left[x(T) \,|\, \mathcal{F}_t\right]$ requires carefull choice of numeraire

Consider grid of simulation times $t = t_0, t_1, \ldots, t_M = T$.
We simulate

$$x(t_{k+1}) = \mu_k + \sigma_k \cdot Z_k$$

with

$$\begin{aligned}
\mu_k &= G'(t_k, t_{k+1}) \left[x(t_k) + G(t_k, t_{k+1}) y(t_k) \right], \\
\sigma_k^2 &= y(t_{k+1}) - G'(t_k, t_{k+1})^2 y(t_k), \quad \text{and} \\
Z_k &\sim N(0, 1).
\end{aligned}$$

Grid point t_{k+1} must coincide with forward measure for $\mathbb{E}^{t_{k+1}}[\cdot]$ for each individual step $k \to k+1$.
Numeraire must be discretely compounded bank account $\bar{B}(t)$ and

$$\bar{B}(t_{k+1}) = \frac{\bar{B}(t_k)}{P(x(t_k), t_k, t_{k+1})}.$$

Recursion for $x(t_{k+1})$ and $\bar{B}(t_{k+1})$ fully specifies path simulation for pricing.

Some comments regarding Hull-White MC simulation ...

- We could also simulate in risk-neutral measure or $\bar{T}$-forward measure.
  - This might be advantegous if also FX or equities are modelled/simulated.
  - Requires adjustment of conditional expectation μ_k and numeraire $N(t_k)$ calculation.
  - Variance σ_k^2 is invariant to change of meassure in Hull-White model.

- Repeat path generation for as many paths $1, \ldots, n$ as desired (or computationally feasible).

- For Bermudan pricing we need to simulate x and N (at least) at exercise dates $T_E^1, \ldots, T_E^{\bar{k}}$.

- For calculation of Z_k use
  - pseudo-random numbers or
  - Quasi-Monte Carlo sequences.

  as proxies for independent $N(0, 1)$ random variables accross time steps and paths.

We illustrate MC pricing by means of a coupon bond option example

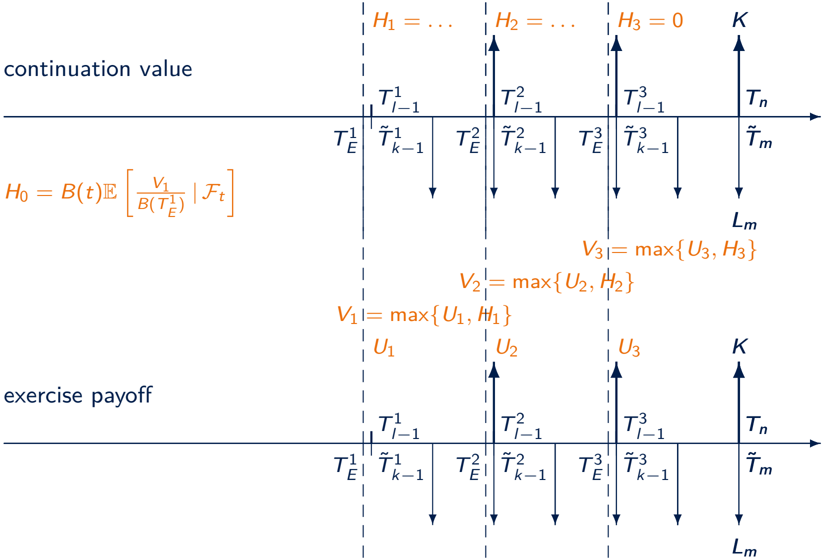
Consider coupon bond option expiring at T_E with coupons C_i paid at T_i ($i = 1, \ldots, u$, incl. strike and notional).

- Set $t_0 = 0$, $t_1 = T_E/2$ and $t_2 = T_E$ (two steps for illustrative purpose).
- Compute $2n$ independent $N(0,1)$ pseudo random numbers $Z^1, \ldots, Z^{2n}$.
- For all paths $j = 1, \ldots, n$ calculate:
  - μ_0^j, σ_0 and $\bar{B}^j(t_1)$; note μ_0^j and $\bar{B}^j(t_1)$ are equal for all paths j since $x(t_0) = 0$,
  - $x_1^j = \mu_0^j + \sigma_0 \cdot Z^j$,
  - μ_1^j, σ_1 and $\bar{B}^j(t_2)$; note now μ_1^j and $\bar{B}^j(t_2)$ depend on x_1^j,
  - $x_2^j = \mu_1^j + \sigma_1 \cdot Z^{n+j}$,
  - payoff $V^j(t_2) = \left[\sum_{i=1}^u C_i \cdot P(x_2^j, t_2, T_i)\right]^+$ at $t_2 = T_E$.
- Calculate option price (note $\bar{B}(0) = 1$)

$$V(0) = \bar{B}(0) \cdot \frac{1}{n} \sum_{j=1}^{n} \frac{V^j(t_2)}{\bar{B}^j(t_2)}.$$

Outline

Let's return to our Bermudan option pricing problem

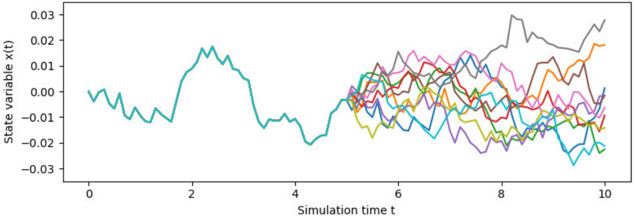


In this setting we need to calculate future conditional expectations

- Assume we already simulated paths for state variables x_k, underlyings U_k and numeraire B_k for all relevant dates t_k.
- We need continuation values H_k defined recursively via $H_{\bar{k}} = 0$ and

$$H_k = B_k \mathbb{E}_k \left[\frac{\max\left\{U_{k+1}, H_{k+1}\right\}}{B_{k+1}} \right].$$

- In principle, we could use nested Monte Carlo:



- In practice, nested Monte Carlo is typically computationally not feasible.

A key idea of American Monte Carlo is approximating conditional expectation via regression

Conditional expectation

$$H_k = \mathbb{E}_k \left[\frac{B_k}{B_{k+1}} \max \left\{ U_{k+1}, H_{k+1} \right\} \right]$$

is a function of the path $x(t)$ for $t \leq t_k$.

For non-path-dependent underlyings U_k, H_k can be written as function of $x_k = x(t_k)$, i.e.

$$H_k = H_k(x_k).$$

We aim at finding a regression operator

$$\mathcal{R}_k = \mathcal{R}_k \left[Y \right]$$

which we can use as proxy for H_k.

What do we mean by regression operator?

Denote $\zeta(\omega) = [\zeta_1(\omega), \ldots, \zeta_q(\omega)]^\top$ a set of basis functions (vector of random variables).

Let $Y = Y(\omega)$ be a target random variable.

Assume we have outcomes $\omega_1, \ldots, \omega_{\bar{n}}$ with control variables $\zeta(\omega_1), \ldots, \zeta(\omega_{\bar{n}})$ and observations $Y(\omega_1), \ldots, Y(\omega_{\bar{n}})$.

A regression operator $\mathcal{R}[Y]$is defined via

$$\mathcal{R}[Y](\omega) = \zeta(\omega)^\top \beta$$

where the regression coefficients β solve linear least squares problem

$$\left\| \begin{bmatrix} \zeta(\omega_1)^\top \beta - Y(\omega_1) \\ \vdots \\ \zeta(\omega_{\bar{n}})^\top \beta - Y(\omega_{\bar{n}}) \end{bmatrix} \right\|^2 \to \min .$$

Linear least squares system can be solved e.g. via QR factorisation or SVD.

A basic pricing scheme is obtained by replacing conditional expectation of future payoff by regression operator

Approximate $\tilde{H}_k \approx H_k$ via $\tilde{H}_{\bar{k}} = H_{\bar{k}} = 0$ and

$$\tilde{H}_k = \mathcal{R}_k \left[\frac{B_k}{B_{k+1}} \max \left\{ U_{k+1}, \tilde{H}_{k+1} \right\} \right] \quad \text{for} \quad k = \bar{k} - 1, \ldots, 1.$$

- Critical piece of this methodology is (for each step k)
  - choice of regression variables $\zeta_1, \ldots, \zeta_q$ and
  - calibration of regression operator $\mathcal{R}_k$ with coefficients β.
- Regression variables $\zeta_1, \ldots, \zeta_q$ must be calculated based on information up to t_k.
  - They must not look into the future to avoid upward bias.
- Control variables $\zeta(\omega_1), \ldots, \zeta(\omega_{\bar{n}})$ and observations $Y(\omega_1), \ldots, Y(\omega_{\bar{n}})$ for calibration should be simulated on paths independent from pricing.
  - Using same paths for calibration and payoff simulation also incorporates information on the future.

What are typical basis functions?

State variable approach

Set $\zeta_i = x(t_k)^{i-1}$ for $i = 1, \ldots, q$. Typical choice is $q \approx 4$ (i.e. polynomials of order 3). For multi-dimensional models we would set $\zeta_i = \prod_{j=1}^{d} x_j(t_k)^{p_{i,j}}$ with $\sum_{j=1}^{d} p_{i,j} \leq r$.

- Very generic and easy to incorporate.

Explanatory variable approach

Identify variables $y_1, \ldots y_{\bar{d}}$ relevant for the underlying option. Set basis functions as monomials

$$\zeta_i = \prod_{j=1}^{\bar{d}} y_j(t_k)^{p_{i,j}} \quad \text{with} \quad \sum_{j=1}^{\bar{d}} p_{i,j} \leq r.$$

- Can be chosen option-specific and incorporate information prior to t_k.
- Typical choices are co-terminal swap rates or Libor rates (observed at t_k).

Regression of the full underlying can be a bit rough - we may restrict regression to exercise decision only

For a given path consider

$$
\begin{aligned}
H_k &= \frac{B_k}{B_{k+1}} \max\left\{U_{k+1}, H_{k+1}\right\} \\
&= \frac{B_k}{B_{k+1}} \left[\mathbb{1}_{\left\{U_{k+1} > H_{k+1}\right\}} U_{k+1} + \left(1 - \mathbb{1}_{\left\{U_{k+1} > H_{k+1}\right\}}\right) H_{k+1}\right].
\end{aligned}
$$

Use regression to calculate $\mathbb{1}_{\left\{U_{k+1} > H_{k+1}\right\}}$.

Calculate $\mathcal{R}_k = \mathcal{R}_k\left[U_{k+1} - H_{k+1}\right]$, set $H_{\bar{k}} = 0$ and

$$
H_k = \frac{B_k}{B_{k+1}} \left[\mathbb{1}_{\{\mathcal{R}_k > 0\}} U_{k+1} + \left(1 - \mathbb{1}_{\{\mathcal{R}_k > 0\}}\right) H_{k+1}\right] \quad \text{for} \quad k = \bar{k} - 1, \ldots, 1.
$$

- Think of $\mathbb{1}_{\{\mathcal{R}_k > 0\}}$ as an exercise strategy (which might be sub-optimal).
- This approach is sometimes considered more accurate than regression on regression.
- For further reference, see also Longstaff/Schwartz (2001).

We summarise the American Monte Carlo method

1. Simulate n paths of state variables x_k^j, underlyings U_k^j and numeraires B_k^j ($j = 1, \ldots, n$) for all relevant times t_k ($k = 1, \ldots \bar{k}$).
2. Set $H_{\bar{k}}^j = 0$.
3. For $k = \bar{k} - 1, \ldots 1$ iterate:
   3.1 Calculate control variables $\left\{\zeta_i^j = \zeta_i(\omega_j)\right\}_{i=1,\ldots,q}^{j=1,\ldots,\hat{n}}$ and regression variables $Y^j = U_k^j - H_k^j$ for the first $\hat{n}$ paths ($\hat{n} \approx \frac{1}{4} n$).
   3.2 Calibrate regression operator $\mathcal{R}_k = \mathcal{R}_k[Y]$ which gives coefficients β.
   3.3 Calculate control variables $\left\{\zeta_i^j = \zeta_i(\omega_j)\right\}_{i=1,\ldots,q}^{j=\hat{n}+1,\ldots n}$ for remaining paths and (for all paths)

$$H_k^j = \frac{B_k^j}{B_{k+1}^j} \left[\mathbb{1}_{\{\mathcal{R}_k(\omega_j) > 0\}} U_{k+1}^j + \left(1 - \mathbb{1}_{\{\mathcal{R}_k(\omega_j) > 0\}}\right) H_{k+1}^j \right].$$

4. Calculate discounted payoffs for the paths $j = \hat{n} + 1, \ldots n$ not used for regression

$$H_0^j = \frac{B_k^j}{B_{k+1}^j} \max\left\{U_1^j, H_1^j\right\}.$$

5. Derive average $V(0) = \frac{1}{n-\hat{n}} \sum_{j=\hat{n}+1}^{n} H_0^j$.

Some comments regarding AMC for Bermudans in Hull-White model

- AMC implementations can be very bespoke and problem specific.
  - See literature for more details.
- More explanatory variables or too high polynomial degree for regression may deteriorate numerical solution.
  - This is particularly relevant for 1-factor models like Hull-White.
  - Single state variable or co-terminal swap rate should suffice.
- AMC with Hull-White for Bermudans is *not* the method of choice.
  - PDE and integration methods are directly applicable.
  - AMC is much slower and less accurate compared to PDE and integration.

AMC is the method of choice for high-dimensional models and/or path-dependent products.

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