Interest Rate Modelling and Derivative Pricing

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Part I

Introduction and Preliminaries

Outline

Outline

What is this lecture about?

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(semi-annually, act/360 day count, modified following, Target calendar)

Suppose, Bank A may decide to early terminate deal in 10, 11, 12,.. years

How does early termination option affect the present value and risk of the deal?

Organisational details first

- Lecture: Fri, 11:15 - 12:45 s.t., room RUD 25; 1.115
- Exercises: Fri, 09:15 - 10:45, room RUD 25; 1.115 (every second week)
- Office times: Fridays on request after the lecture

Exercises:

- Discuss and analyse practical examples and theory details
- Main tool: QuantLib (open source financial library)
- Python, some Excel, (bring your laptop)

Requirements:

- Present at least once during exercises
- exam planned on February 14, 2020

Literature and resources you will need

- Literature
  - L. Andersen and V. Piterbarg. *Interest rate modelling, volume I to III.*
    Atlantic Financial Press, 2010
  - D. Brigo and F. Mercurio. *Interest Rate Models - Theory and Practice.*
    Springer-Verlag, 2007
  - S. Shreve. *Stochastic Calculus for Finance II - Continuous-Time Models.*
    Springer-Verlag, 2004
- QuantLib web site `www.quantlib.org`
- Official source repository `www.github.com/lballabio`
- Some extensions which we might use `www.github.com/sschlenkrich`
- `https://www.applied-financial-mathematics.de/interest-rate-modelling-and-derivative-pricing`

Let's revisit the introductory example

Interbank swap deal example

Fixed interest rate

Pays 3% on 100mm EUR

Notional

Start date: Oct 31, 2019

Dates

End date: Oct 31, 2039

Market conventions

(annually, 30/360 day count, modified following, Target calendar)

Stochastic interest rates

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(semi-annually, act/360 day count, modified following, Target calendar)

Optionalities

Bank A may decide to early terminate deal in 10, 11, 12,.. years

Agenda covers static yield curve modelling, Vanilla rates models and term structure models

Interest Rate Modelling

- Stochastic calculus basics
- Static yield curve modelling and linear products
- Vanilla interest rate models
- HJM term structure modelling framework
- Classical Hull-White interest rate model
- Pricing methods for Bermudan swaptions

Model Calibration

- Multi-curve yield curve calibration
- Hull-White model calibration
- Numerical methods for model calibration

Sensitivity Calculation

- Delta and Vega specification
- Numerical methods for sensitivity calculation

Outline

We will work along three streams

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
Radon-Nikodym derivative & change of measure	Ito integral	Equivalent martingale measure & FTAP
Martingale	Martingale representation	Change of equiv. martingale meas.
Density process	Ito's lemma	Permissible trading strategy

Risk-neutral derivative pricing formula

Outline

Measure theory is independent of financial application

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
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Risk-neutral derivative pricing formula

We start with stochastic processes and probability space

Stochastic process (for assets or interest rate components)

$$X(t) = [X_1(t), \ldots, X_p(t)]^\top .$$

Probability space that drives stochastic process $(\Omega, \mathcal{F}, \mathbb{P})$

- Ω sample space with outcomes ω (typically increments of Brownian motions),
- $\mathcal{F}$ σ-algebra on Ω,
- $\mathbb{P}$ probability measure on $\mathcal{F}$.

Information flow is realised via filtration $\{\mathcal{F}_t,\ t \in [0, T]\}$

- $\mathcal{F}_t$ sub-algebra of $\mathcal{F}$ with $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t \leq s$,
- Assume $X(t)$ is adapted to filtration $\mathcal{F}_t$, i.e. $X(t)$ is fully observable at time t.

Measures can be linked by Radon–Nikodym derivative

Theorem (Radon–Nikodym derivative)

Let $\mathbb{P}$ and $\hat{\mathbb{P}}$ be equivalent probability measures on $(\Omega, \mathcal{F})$. Then there exists a unique (a.s.) non-negative random variable $R(\omega)$ with $\mathbb{E}^{\mathbb{P}}[R] = 1$, such that for all $A \in \mathcal{F}$

$$\hat{\mathbb{P}}(A) = \mathbb{E}^{\mathbb{P}}\left[R\,\mathbb{1}_{\{A\}}\right].$$

R is denoted Radon–Nikodym derivative.

It follows

$$\hat{\mathbb{P}}(A) = \int_A d\hat{\mathbb{P}} = \int_A R\,d\mathbb{P} = \mathbb{E}^{\mathbb{P}}\left[R\,\mathbb{1}_{\{A\}}\right].$$

and also for all measurable functions X (via algebraic induction)

$$\mathbb{E}^{\hat{\mathbb{P}}}[X] = \mathbb{E}^{\mathbb{P}}[R\,X].$$

Thus we may write

$$R = d\hat{\mathbb{P}}/d\mathbb{P}.$$

We will frequently need the change of measure for conditional expectations

Definition (Conditional expectation)

Let X be a random variable. The conditional expectation $\mathbb{E}^{\mathbb{P}}\left[X \mid \mathcal{F}_t\right]$ is defined as the stochastic variable that satisfies:

- $\mathbb{E}^{\mathbb{P}}\left[X \mid \mathcal{F}_t\right]$ is $\mathcal{F}_t$-measurable and
- for all $A \in \mathcal{F}_t$ we have

$$\int_A \mathbb{E}^{\mathbb{P}}\left[X \mid \mathcal{F}_t\right] d\mathbb{P} = \int_A X d\mathbb{P}.$$

Theorem (Baye's rule for conditional expectation)

Let $R = d\hat{\mathbb{P}}/d\mathbb{P}$ be the Radon–Nikodym derivative associated with $(\Omega, \mathcal{F}, \mathbb{P})$ and $\left(\Omega, \mathcal{F}, \hat{\mathbb{P}}\right)$ and X a random variable. Then

$$\mathbb{E}^{\hat{\mathbb{P}}}\left[X \mid \mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[R\,X \mid \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[R \mid \mathcal{F}_t\right]}.$$

We sketch the proof for change of measure

Proof.

We use the definition of conditional expectation and show that (for all $A \in \mathcal{F}_t$)

$$\int_A \mathbb{E}^{\mathbb{P}}[R\,X \mid \mathcal{F}_t]\,d\mathbb{P} = \int_A \mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]\,\mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\,d\mathbb{P}.$$

We have for the left side using conditional expectation and Radon–Nikodym derivative

$$\int_A \mathbb{E}^{\mathbb{P}}[R\,X \mid \mathcal{F}_t]\,d\mathbb{P} = \int_A X\,R\,d\mathbb{P} = \int_A X d\hat{\mathbb{P}}.$$

For the right side we get using conditional expectation

$$\int_A \mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]\,\mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\,d\mathbb{P} = \int_A \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\,R \mid \mathcal{F}_t\right]d\mathbb{P} = \int_A \mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\,R\,d\mathbb{P}.$$

Applying Radon–Nikodym derivative and again conditional expectation yields

$$\int_A \mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\,R\,d\mathbb{P} = \int_A \mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\,d\hat{\mathbb{P}} = \int_A X d\hat{\mathbb{P}}.$$

$\square$

Martingales allow derivation of expected future values

Sum of squares notation (Frobenius norm, L^2 norm for vectors)

For a matrix or vector $A \in \mathbb{R}^{m \times n}$ with elements $\{a_{i,j}\}_{i,j}$ we denote

$$|A| = \sqrt{\mathrm{tr}\left(AA^\top\right)} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}^2}.$$

Definition (Martingale)

Let $X(t)$ be an adapted vector-valued process with finite absolute expectation $\mathbb{E}^{\mathbb{P}}\left[|X(t)|\right] < \infty$ (under the probability measure $\mathbb{P}$) for all $t \in [0, T]$.
$X(t)$ is a martingale under $\mathbb{P}$ if for all $t, s \in [0, T]$ with $t \leq s$

$$X(t) = \mathbb{E}^{\mathbb{P}}\left[X(s) \mid \mathcal{F}_t\right] \quad a.s.$$

- Typically, martingale property is derived (by other results) for a process.
- Then we can use martingale property to obtain expectation of future values $X(T)$.

Density process describes change of measure for processes

Definition (Density process)

Denote $\zeta(t) = \mathbb{E}^{\mathbb{P}}\left[d\hat{\mathbb{P}}/d\mathbb{P} \mid \mathcal{F}_t\right]$ the density process of $\hat{\mathbb{P}}$ (relative to $\mathbb{P}$).

- Then $\zeta(t)$ is a $\mathbb{P}$-martingale with $\zeta(0) = \mathbb{E}^{\mathbb{P}}[\zeta(t)] = 1$.

Lemma (Change of measure for processes)

Let $X(t)$ be a $\mathcal{F}_t$ measurable random variable. Then

$$\mathbb{E}^{\hat{\mathbb{P}}}[X(T) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}\left[\frac{\zeta(T)}{\zeta(t)} X(T) \mid \mathcal{F}_t\right].$$

Proof.

Recall that $R = d\hat{\mathbb{P}}/d\mathbb{P}$. We have $\mathbb{E}^{\hat{\mathbb{P}}}[X(T) \mid \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{P}}[R\,X(T) \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]}$. Then

$$\mathbb{E}^{\mathbb{P}}[R\,X(T) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[R\,X(T) \mid \mathcal{F}_T] \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_T]\,X(T) \mid \mathcal{F}_t\right].$$

The result follows from the definition of $\zeta(t)$ via $\zeta(t) = \mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]$.

□

Density process may be used to define a new measure

Let $\zeta(t)$ be a $\mathbb{P}$-martingale with $\zeta(0) = 1$. We choose a final horizon time T and define the Radon–Nikodym derivative as $R(\omega) = \zeta(T, \omega)$.

The corresponding measure $\hat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ is

$$\hat{\mathbb{P}}(A) = \mathbb{E}^{\mathbb{P}}\left[R\, 1_{\{A\}}\right] = \mathbb{E}^{\mathbb{P}}\left[\zeta(T, \omega)\, \mathbb{1}_{\{A\}}\right].$$

We show that the density of $\hat{\mathbb{P}}$ indeed equals $\zeta(t)$.

Denote $\bar{\zeta}(t) = \mathbb{E}^{\mathbb{P}}\left[R \,|\, \mathcal{F}_t\right]$ the density of $\hat{\mathbb{P}}$. Then we get with the martingale property of $\zeta(t)$

$$\bar{\zeta}(t) = \mathbb{E}^{\mathbb{P}}\left[\zeta(T, \omega) \,|\, \mathcal{F}_t\right] = \zeta(t).$$

Outline

Diffusion processes are the basis for our models

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
Radon-Nikodym derivative & change of measure	Ito integral	Equivalent martingale measure & FTAP
Martingale	Martingale representation	Change of equiv. martingale meas.
Density process	Ito's lemma	Permissible trading strategy

Risk-neutral derivative pricing formula

Stochastic process is driven by Brownian motion

Information is generated by Brownian motion

- $W(t) = [W_1(t), \ldots, W_d(t)]^\top$ d-dimensional Brownian motion.
- $W_i(\cdot)$ independent of $W_j(\cdot)$ for $i \neq j$.
- Independent Gaussian increments $W_i(s) - W_i(t) \sim \mathcal{N}(0, s-t)$ for $s \geq t$.
- Typically, filtration $\mathcal{F}_t$ is generated by Brownian motion $W(\cdot)$, i.e. $\mathcal{F}_t = \sigma\left\{W(u),\, 0 \leq u \leq t\right\}$.

Definition (H^2 for volatility processes σ)

Let $\sigma : \mathbb{R} \times \Omega \to \mathbb{R}^{p \times d}$ be a volatility process adapted to the filtration generated by $\mathcal{F}_t$. We say that σ is in H^2 if for all $t \in [0, T]$ we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^t |\sigma(s,\omega)|^2 \, ds\right] < \infty.$$

Stochastic process is described as Ito process with Ito integral

$$X(t) = X(0) + \int_0^t \mu(s,\omega)\, ds + \int_0^t \sigma(s,\omega)\, dW(s)$$

or in differential notation

$$dX(t) = \mu(t,\omega)\, dt + \sigma(t,\omega)\, dW(t),$$

- vector-valued drift $\mu : \mathbb{R} \times \Omega \to \mathbb{R}^p$,
- matrix of volatilities $\sigma : \mathbb{R} \times \Omega \to \mathbb{R}^{p \times d}$,
- assume drift μ and volatility σ are adapted to $\mathcal{F}_t$ and σ is in H^2.

We consider the Ito integral as

$$\int_0^t \sigma(s,\omega)\, dW(s) = \lim_{n\to\infty} \sum_{i=1}^{n} \sigma(s_{i-1},\omega)\left[W(s_i) - W(s_{i-1})\right], \quad s_i = \frac{i}{n} t.$$

Ito integrals are important martingales for modelling

Theorem (Ito Integral properties)

Define the Ito integral $X(t) = \int_0^t \sigma(u,\omega)\, dW(u)$ *with* σ *is in* H^2. *Then*

1. $X(t)$ *is* $\mathcal{F}_t$*-measurable (i.e. we can calculate the distribution of* $X(t)$ *using* $(\Omega, \mathcal{F}, \mathbb{P})$*)*
2. $X(t)$ *is a continuous martingale*
3. $\mathbb{E}^{\mathbb{P}}\left[\left|X(t)^2\right|\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^t |\sigma(u,\omega)|^2\, du\right] < \infty$ *(Ito isometry)*
4. $\mathbb{E}^{\mathbb{P}}\left[X(t)X(s)^\top\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^{\min\{t,s\}} \sigma(u,\omega)\,\sigma(u,\omega)^\top\, dt\right]$ *(auto-covariance)*

Stochastic processes can be represented as Ito integrals

Theorem (Martingale representation theorem)

If $X(\cdot)$ is a (local) martingale adapted to the filtration $\mathcal{F}_t$ which is generated by Brownian motion $W(\cdot)$ then there exists a volatility process $\sigma(t,\omega)$ such that

$$dX(t) = \sigma(t,\omega)\, dW(t).$$

Moreover, if $X(\cdot)$ is a square-integrable martingale then σ is in H^2.

Ito's Lemma is one of the most relevant tools

Theorem (Ito's Lemma)

Let $X(t)$ be an Ito process and $f(\cdot)$ a twice continuous differentiable scalar function. Then

$$df\,(X(t)) = \nabla_X f(X)^\top dX(t) + \frac{1}{2} dX(t)^\top H_X f(x) dX(t)$$

with $\nabla_X f$ being the gradient of f and $H_X f(x)$ being the Hessian of f.

Here we use *calculus* $dW_i(t)dW_i(t) = dt$ and $dW_i(t)dW_j(t) = 0$ for $i \neq j$.

Corollary (Ito product rule)

Let $X_1(t)$ and $X_2(t)$ be scalar Ito processes. Then

$$d\left[X_1(t)X_2(t)\right] = X_1(t)dX_2(t) + X_2(t)dX_1(t) + dX_1(t)dX_2(t).$$

Outline

Pricing builds on measure theory and stochastic processes

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
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Risk-neutral derivative pricing formula

We specify our market based on assets and trading strategies

Financial Market

We assume p (dividend-free[1]) assets $X(t) = [X_1(t), \ldots, X_p(t)]^\top$ which are driven by Ito processes

$$dX(t) = \mu\,(t,\omega)\,dt + \sigma\,(t,\omega)\,dW(t).$$

Trading Strategy

A trading strategy represents a predictable adapted process (of asset weights)

$$\phi(t,\omega) = [\phi_1(t,\omega), \ldots, \phi_p(t,\omega)]^\top.$$

The value of the trading strategy (or corresponding portfolio) is

$$\pi(t) = \phi(t)^\top X(t).$$

[1] I.e. no intermediate payments

Self-financing strategies and arbitrage

Trading Gains and Self-financing Strategy

Trading gains (over a short period of time) are $\phi(t)^\top [X(t+dt) - X(t)]$.
This leads to the general specification $\int_t^T \phi(t)^\top dX(t)$.
A trading strategy is self-financing if portfolio changes are only induced by asset returns (no money inflow or outflow). That is

$$\pi(T) - \pi(t) = \int_t^T \phi(s)^\top dX(s).$$

Definition (Arbitrage)

An arbitrage opportunity is a self-financing strategy $\phi(\cdot)$ with $\pi(0) = 0$ and, for some $t \in [0, T]$,

$$\pi(t) \geq 0 \text{ a.s., and } \mathbb{P}(\pi(t) > 0) > 0.$$

Arbitrage needs to be precluded in a financial model.

Absence of arbitrage is closely related to equivalent martingale measures

Definition (Numeraire and equivalent martingale measure)

A numeraire is a positive asset $N(t)$ of our market. An equivalent martingale measure (corresponding to the numeraire $N(t)$) is a measure $\mathbb{Q}$ such that the normalised asset prices $[X_1(t)/N(t), \ldots, X_p(t)/N(t)]^\top$ are $\mathbb{Q}$-martingales.

Fundamental theorem of asset pricing

Assuming some restrictions on permissible trading strategies one can show that absence of arbitrage is "nearly equivalent" to the existence of an equivalent martingale measure.

Our models are all based on the assumption of no-arbitrage and the existence of an equivalent martingale measure.

Equivalent martingale measures exists for any numeraire

Suppose we have a numeraire $N(t)$ and an equivalent martingale measure $\mathbb{Q}^N$. Suppose we also have another numeraire $M(t)$. Define

$$\zeta(t) = \frac{M(t)}{N(t)} \frac{N(0)}{M(0)}.$$

Then

- $\mathbb{E}^N\left[\zeta(T) \,|\, \mathcal{F}_t\right] = \mathbb{E}^N\left[\frac{M(T)}{N(T)} \,|\, \mathcal{F}_t\right] \frac{N(0)}{M(0)} = \frac{M(t)}{N(t)} \frac{N(0)}{M(0)} = \zeta(t)$, thus $\zeta(t)$ is a $\mathbb{Q}^N$-martingale
- $\zeta(0) = \frac{M(0)}{N(0)} \frac{N(0)}{M(0)} = 1$

Define the new measure $\mathbb{Q}^M$ via the density $\zeta(t)$. Then for an asset $X_i(t)$

$$\mathbb{E}^M\left[\frac{X_i(T)}{M(T)} \,|\, \mathcal{F}_t\right] = \mathbb{E}^N\left[\frac{\zeta(T)}{\zeta(t)} \frac{X_i(T)}{M(T)} \,|\, \mathcal{F}_t\right] = \mathbb{E}^N\left[\frac{M(T)}{N(T)} \frac{N(\mathrm{t})}{M(t)} \frac{X_i(T)}{M(T)} \,|\, \mathcal{F}_t\right].$$

Taking out what is known and using the martingale property of measure $\mathbb{Q}^N$ yields

$$\mathbb{E}^M\left[\frac{X_i(T)}{M(T)} \,|\, \mathcal{F}_t\right] = \frac{N(\mathrm{t})}{M(t)} \mathbb{E}^N\left[\frac{X_i(T)}{N(T)} \,|\, \mathcal{F}_t\right] = \frac{N(\mathrm{t})}{M(t)} \frac{X_i(t)}{N(t)} = \frac{X_i(t)}{M(t)}.$$

$X_i(t)/M(t)$ is a $\mathbb{Q}^M$-martingale. Thus $\mathbb{Q}^M$ is an equivalent martingale measure for $M(t)$.

Trading strategies need to be permissible

Definition (Permissible trading strategy)

Let $X(t)$ be an Ito process and $\mathbb{Q}$ an equivalent martingale measure with numeraire $N(t)$. A self-financing trading strategy $\phi(t)$ is called permissible if

$$\int_0^t \phi(s)^\top d\left(\frac{X(s)}{N(s)}\right)$$

is a $\mathbb{Q}$-martingale.

Recall that $X(t)/N(t)$ is a $\mathbb{Q}$-martingale by construction. If $\phi(t)$ is sufficiently bounded then it is also permissible.

Theorem (Martingale property for trading strategies)

For any self-financing and permissible trading strategy $\phi(t)$ and an equivalent martingale measure $\mathbb{Q}$ with numeraire $N(t)$ the discounted portfolio price process $\pi(t)/N(t)$ is a martingale.

On average you can not beat the market when trading in the assets.

We proof the martingale property for trading strategies

Proof.

Recall that $\pi(t) = \phi(t)^\top X(t)$. The self-financing condition may be written as $d\pi(t) = \phi(t)^\top dX(t)$. Applying Ito's product rule yields

$$\begin{aligned} d\left[\frac{\pi(t)}{N(t)}\right] &= d\left[\pi(t)\frac{1}{N(t)}\right] = \frac{d\pi(t)}{N(t)} + \pi(t)d\left[\frac{1}{N(t)}\right] + d\pi(t)d\left[\frac{1}{N(t)}\right] \\ &= \frac{\phi(t)^\top dX(t)}{N(t)} + \phi(t)^\top X(t)d\left[\frac{1}{N(t)}\right] + \phi(t)^\top dX(t)d\left[\frac{1}{N(t)}\right] \\ &= \phi(t)^\top\left[\frac{dX(t)}{N(t)} + X(t)d\left[\frac{1}{N(t)}\right] + dX(t)d\left[\frac{1}{N(t)}\right]\right] \\ &= \phi(t)^\top d\left[\frac{X(t)}{N(t)}\right]. \end{aligned}$$

Now the assertion follows directly from the condition that $\phi(t)$ is permissible.

$\square$

Derivative pricing is closely related to trading strategies

Definition (Contingent claim)

A derivative security (or contingent claim) pays at time T the random variable $V(T)$ (no intermediate payments). We assume $V(T)$ has finite variance and is attainable. That is there exists a permissible trading strategy $\phi(\cdot)$ such that

$$V(T) = \phi(T)^\top X(T)\, a.s.$$

Then absence of arbitrage yields that the fair price $V(t)$ of the derivative security becomes

$$V(t) = \phi(t)^\top X(t) \text{ for all } t \in [0, T].$$

Consequently,

$$\frac{V(t)}{N(t)} = \frac{\phi(t)^\top X(t)}{N(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\phi(T)^\top X(T)}{N(T)} \,|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{V(T)}{N(T)} \,|\, \mathcal{F}_t\right].$$

Above arbitrage pricing formula is the foundation of derivative pricing.

Outline

We summarize the key results

$$\boxed{(\Omega, \mathcal{F}, \mathbb{P}),\ \mathcal{F}_t,\ t \in [0, T]}$$

$$\boxed{W(t) = [W_1(t), \ldots, W_d(t)]^\top}$$

$$\boxed{d\pi(T) = \phi(t)^\top dX(t)}$$

$$\boxed{\mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{P}}[R\,X \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]}}$$

$$\boxed{X(t) = \int_0^t \sigma(u, \omega)\, dW(u)}$$

$$\boxed{\frac{X(t)}{N(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(T)}{N(T)} \mid \mathcal{F}_t\right]}$$

$$\boxed{X(t) = \mathbb{E}^{\mathbb{P}}[X(s) \mid \mathcal{F}_t]}$$

$$\boxed{dX(t) = \sigma(u, \omega)\, dW(u)}$$

$$\boxed{\mathbb{E}^M\left[\frac{X_i(T)}{M(T)} \mid \mathcal{F}_t\right] = \mathbb{E}^N\left[\frac{N(t)}{M(t)}\,\frac{X_i(T)}{N(T)} \mid \mathcal{F}_t\right]}$$

$$\boxed{\zeta(t) = \mathbb{E}^{\mathbb{P}}\left[d\hat{\mathbb{P}}/d\mathbb{P} \mid \mathcal{F}_t\right]}$$

$$\boxed{df = f'\,dX + \frac{f''}{2}\,dX^2}$$

$$\boxed{\phi(t)^\top d\left[\frac{X(t)}{N(t)}\right] = \bar{\sigma}\, dW(t)}$$

$$\boxed{V(t)/N(t) = \mathbb{E}^{\mathbb{Q}}[V(T)/N(T) \mid \mathcal{F}_t]}$$

Outline

Outline

Basic Fixed Income Modelling

Market Setting

Discounted Cash Flow pricing

First we need to specify the assets in the market

Example (Overnight bank account)

- Suppose bank A deposits 1 EUR at ECB at time $T_0 = 0$ (today) with the right to withdraw money at T_1, say the next day.
- Bank A may leave deposit with ECB as long as they want
- Time T_i is measured in years (or year fraction) for simplicity
- ECB pays annualized interest rate r_i from T_i to T_{i+1}

Example also holds for deposits between two banks, e.g. bank A and bank B
What is the value of the deposit at a future time T_N?

Denote B_i the value of the deposit at time T_i. Then

$$B_0 = 1$$

and

$$B_i = B_{i-1} + r_{i-1} \cdot (T_i - T_{i-1}) \cdot B_{i-1} = [1 + r_{i-1} (T_i - T_{i-1})] \cdot B_{i-1}.$$

The most basic asset is the money market bank account

Definition (Short rate and (abstract) bank account)

Assume a process $r(t)$ (adapted to the filtration $\mathcal{F}_t$) for the instantaneous interest rate. The rate $r(t)$ is denoted the short rate. The continuous compounded bank account (or money market account) is an asset with price $B(t)$ given by $B(0) = 1$ and

$$dB(t) = r(t) \cdot B(t) \cdot dt.$$

It follows that the future price of the bank account becomes

$$B(t) = \exp\left\{ \int_0^t r(s)ds \right\}.$$

The most relevant assets are zero coupon bonds (ZCBs)

ZCBs are fixed future cash flows of unit notional, e.g. 1 EUR in 10y.

Definition (Zero Coupon Bond)

A zero coupon bond for maturity T is an asset with time-t asset price $P(t,T)$ for $t \leq T$ and $P(T,T) = 1$.

What is the time-t asset price of a zero coupon bond?

Use risk-neutral pricing formula!

Select money market account $B(t)$ as numeraire and denote $\mathbb{Q}$ the equivalent martingale measure.
Then

$$\frac{P(t,T)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{P(T,T)}{B(T)}\right] = \mathbb{E}^{\mathbb{Q}}\left[B(T)^{-1}\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_0^T r(s)ds\right\}\right].$$

Multiplying with $B(t) = \exp\left\{\int_0^t r(s)ds\right\}$ yields

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r(s)ds\right\}\right].$$

And what is the ZCB price in terms of money ...?

- Formula $P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]$ is a model-independent result
- To calculate it more concrete we need to specify a model/dynamics for short rate $r(t)$
- Suppose short rate is known deterministic function, then

$$P(t,T) = \exp\left\{-\int_t^T r(s)ds\right\}.$$

- Suppose short rate is fixed, i.e. $r(t) = r_0$, then (even simpler)

$$P(t,T) = e^{-r_0(T-t)}.$$

For our market we assume that today's prices $P(0,T)$ of all ZCBs (with maturity $T \geq 0$) are known.

Interest rate market consists of money market bank account and zero coupon bonds

Interest rate market

We consider a market consisting of the money market account $B(t)$ and zero coupon bonds $P(t, T)$ for $t \leq T$ as financial assets.

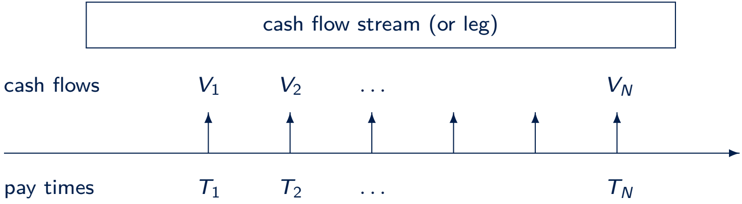
Interest rate derivatives

Interest rate derivatives are contingent claims (or baskets of contingent claims) depending on realisations of future zero coupon bonds.

- We may restrict modelling to discrete set of ZCBs $\{P(t, T_i)\}_i$ (vanilla models).
- Full continuum of ZCBs $\{P(t, T) \,|\, t \leq T\}$ is modelled via term structure models.

Outline

Discounted cash flow (DCF) pricing methodology ...



$$\frac{V(t)}{B(t)} = \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}} \left[\frac{V_i}{B(T_i)} \,|\, \mathcal{F}_t \right]$$

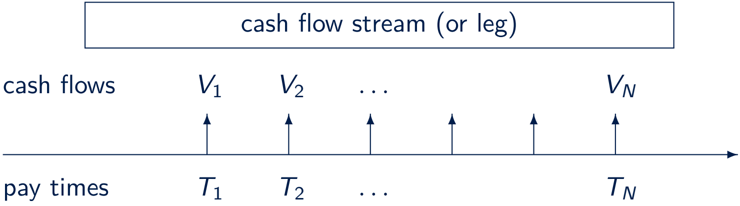
Denote $\mathbb{E}^{T_i}[\cdot]$ expectation(s) in T_i-forward measure(s) with zero coupon bond numeraire $P(t, T_i)$.
Then (change of measure)

$$\frac{V(t)}{B(t)} = \sum_{i=1}^{N} \mathbb{E}^{T_i} \left[\frac{P(t, T_i)}{B(t)} \cdot \frac{V_i}{P(T_i, T_i)} \,|\, \mathcal{F}_t \right].$$

Thus with $P(T_i, T_i) = 1$ follows

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i} \left[V_i \,|\, \mathcal{F}_t \right].$$

(DCF) ... is a model-independent concept



$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i}\left[V_i \,|\, \mathcal{F}_t\right]$$

- Present value is sum of discounted expected future cash flows
- If future cash flows are known (i.e. deterministic), then

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot V_i$$

- In general, challenge lies in calculating $\mathbb{E}^{T_i}\left[V_i \,|\, \mathcal{F}_t\right]$ using a model

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