DCS440 最优化理论 第二次作业

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Prob. 1

记

$$f_i(x) = a_i^\top x + b_i$$

有

$$\begin{aligned} f_i(x) = a_i^\top x + b_i, &\qquad f_i(y) = a_i^\top y + b_i \\ f_i(tx + (1-t)y) &= a_i^\top (tx + (1-t)y) + b_i \\ &= ta_i^\top x + tb_i + (1-t)a_i^\top y + (1-t)b_i \\ &= tf_i(x) + (1-t)f_i(y) \end{aligned}$$

故对于任意 i，$f_i(x)$ 为凸函数

又有

$$\begin{aligned} f(x) &= \max f_i(x) \\ f(tx + (1-t)y) &= \max f_i(tx + (1-t)y) \\ &\le \max(tf_i(x) + (1-t)f_i(y)) \\ &\le t \max f_i(t) + (1-t) \max f_i(y) \\ &= tf(x) + (1-t)f(y) \end{aligned}$$

即

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

$f(x)$ 为凸函数

Prob. 2

对于原问题

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Gx \le h \\ & Ax = b \end{aligned}$$

其拉格朗日函数为

$$\begin{aligned} L(x, \lambda, \nu) &= c^\top x + \lambda^\top (Gx - h) + \nu^\top (Ax - b) \\ &= (c^\top + \lambda^\top G + \nu^\top A)x - \lambda^\top h - \nu^\top b \end{aligned}$$

对偶函数为

$$\begin{aligned} g(\lambda, \nu) &= \inf\{(c^\top + \lambda^\top G + \nu^\top A)x\} - \lambda^\top h - \nu^\top b \\ &= \begin{cases} -\lambda^\top h - \nu^\top b, & c^\top + \lambda^\top G + \nu^\top A = 0 \\ -\infty, & \text{其它} \end{cases} \end{aligned}$$

故可化为对偶问题

$$
\begin{aligned}
\max_{\lambda,\nu} \quad & g(\lambda,\nu) = -\lambda^\top h - \nu^\top b \\
\text{s.t.} \quad & c^\top + \lambda^\top G + \nu^\top A = 0
\end{aligned}
$$

有 KTT 条件

$$
\begin{aligned}
Gx^* - h &\le 0 \\
Ax^* - b &= 0 \\
\lambda^* &\ge 0 \\
\lambda(Gx^* - h) &= 0
\end{aligned}
$$

$$
\nabla f_0(x^*) + \lambda \nabla (Gx^* - h) + \nu \nabla (Ax^* - b) = 0
$$

Prob. 3

令 $A_i x + b_i = y_i$

有

$$
\begin{aligned}
\min_{x} \quad & \frac{1}{2}||x - x_0||_2^2 + \sum_{i=1}^{N} ||y_i||_2 \\
\text{s.t.} \quad & y_i = A_i x + b_i
\end{aligned}
$$

有拉格朗日函数

$$
L(x, z_1, \cdots, z_n) = \sum_{i=1}^{N} ||y_i||_2 + \frac{1}{2}||x - x_0||_2^2 - \sum_{i=1}^{N} z_i^\top (y_i - A_i x - b_i)
$$

对 y_i 最小化有

$$
\inf_{y_i}(||y_i||_2 + z_i^\top y_i) = \begin{cases} 0, & ||z_i||_2 \le 1 \\ -\infty, & \text{其它} \end{cases}
$$

将对 x 的梯度设为 0，有 $x = x_0 + \sum_{i=1}^{N} A_i^\top z$

故

$$
g(z_1, \cdots, z_n) = \begin{cases} \sum_{i=1}^{N} (A_i x_0 + b_i)^\top z_i - \frac{1}{2}||\sum_{i=1}^{N} A_i^\top z_i||_2^2, & ||z_i||_2 \le 1, i = 1, 2, \cdots, N \\ -\infty, & \text{其它} \end{cases}
$$

故对偶问题为

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} (A_i x_0 + b_i)^\top z_i - \frac{1}{2}||\sum_{i=1}^{N} A_i^\top z_i||^2 \\
\text{s.t.} \quad & ||z_i||_2 \le 1, i = 1, 2, \cdots, N
\end{aligned}
$$