RIEMANNIAN GEOMETRY

Solution

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1 Chapter 1

2 Chapter 2: Riemannian Metrics

Problem 2-10

Proof. If $X = \text{grad} f$. Take local orthonormal frame $\{E_1, \cdots, E_n\}$, then $X = \text{grad} f = (E_i f) E_i$, so $Xf = (E_i f)^2 = |X|_g^2$. X is orthogonal to the level sets of f by proposition 2.37.

If $X \in \mathfrak{X}(M)$ satisfies $Xf \equiv |X|_g^2$ and X is orthogonal to the level sets of f at all regular points of f. Take $x \in M$ a regular point of f, df_p is surjective, then there exists an open neighborhood U of x s.t. $f|_U$ is a smooth submersion. Level sets (in fact regular ones) of f in U is a hypersurface, the normal vector space is 1 dimensional. Since X is orthogonal to level sets and also $\text{grad} f$, we have $X = g \cdot \text{grad} f$ for some smooth function g. Now $g|\text{grad} f|_g^2 = Xf = |X|_g^2 = g^2 |\text{grad} f|_g^2$, we have $g \equiv g^2$, so $g \equiv 1$. Hence $X = \text{grad} f$ at regular points of f. If x is not a regular point of f, then $0 = df_x(w) = \langle \text{grad} f|_x, w \rangle$ for $\forall w \in T_x M$, then $\text{grad} f|_x$ must be zero. Since X is non-vanishing, $|X|_g^2 = Xf = \langle X, \text{grad} f \rangle$ implies that f can not have critical points, so $X = \text{grad} f$ for all $x \in M$.

□

Problem 2-14

Proof. Since $M, \widetilde{M}$ are compact, and $\pi : \widetilde{M} \to M$ is k-sheeted Riemannian covering, we can choose finite covering $\{U_\alpha\}_{\alpha \in A}$ of M and $\{U_{\alpha\beta}\}_{(\alpha,\beta) \in A \times B}$ such that $\pi|_{U_{\alpha\beta}} : U_{\alpha\beta} \to U_\alpha$ is isometry. Now choose POU $\{\varphi_\alpha\}$ of $\{U_\alpha\}_{\alpha \in A}$, we have

$$\text{Vol}(\widetilde{M}) = \int_{\widetilde{M}} dV_{\widetilde{g}} = \sum_{\alpha\beta} \int_{U_{\alpha\beta}} \varphi_\alpha(\pi(x)) dV_{\widetilde{g}} = k \sum_{\alpha} \int_{U_\alpha} \varphi_\alpha(x) dV_g = k \text{Vol}(M).$$

□

Problem 2-15

Proof. Take local coordinate $\left\{x^1, \cdots, x^{k_1+k_2}\right\}$ of $M_1 \times_f M_2$, the first k_1 coordinate and last k_2 coordinate are coordinate for M_1 and M_2 respectively. We have

$$g_{ij}(x) = \begin{cases} g_{1,ij}(x) \qquad 1 \leq i, j \leq k_1 \\ f^2(x) g_{2,ij}(x) \qquad k_1 < i, j \leq k_1 + k_2 \\ 0 \qquad \text{otherwise} \end{cases}$$

Now $\det(g_{ij}) = f^{2k_2}\det(g_{1,ij})\det(g_{2,ij})$, so

$$\begin{aligned} dV_g &= \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^{k_1+k_2} \\ &= f^{k_2}\sqrt{\det(g_{1,ij})}dx^1 \wedge \cdots \wedge dx^{k_1} \wedge \sqrt{\det(g_{2,ij})}dx^{k_1+1} \wedge \cdots \wedge dx^{k_1+k_2} \\ &= f^{k_2} dV_{g_1} \wedge dV_{g_2} \end{aligned}$$

□

Problem 2-22

Proof. (a)

(b) □

3 Chapter 3: Model Riemannian Manifolds

Exercise 3.1. (**Problem 3-3**)

Let (M, g) be a Riemannian manifold.

(a) If M is isotropic at one point and it is homogeneous, then it is isotropic.

(b) If M is frame-homogeneous, then it is homogeneous and isotropic.

Proof. (a) Suppose M is isotropic at $p \in M$ and M is homogeneous. Let φ be the isometry s.t. $\varphi(q) = p$. For $\forall q \in M$ and unit vectors $v, u \in T_qM$, we have $d\varphi_q(v), d\varphi_q(u) \in T_pM$ are unit vectors. Since M is isotropic at p, there exists an isometry $\psi \in Iso(M,p)$ s.t. $d\psi_p(d\varphi_q(v)) = d\varphi_q(u)$. So $\varphi^{-1} \circ \psi \circ \varphi \in Iso(M,q)$ takes v to u. Hence M is isotropic.

(b) Obvious. □

Exercise 3.2. (**Problem 3-6**)

Show that two Riemannian metrics g_1 and g_2 are conformal if and only if they define the same angles but not necessarily the same lengths, and that a diffeomorphism is a conformal equivalence if and only if it preserves angles. [Hint: Let (E_i) be a local orthonormal frame for g_1, and consider the g_2 angle between E_i and $(\cos\theta)E_i + (\sin\theta)E_j$.]

Proof. □

4 Chapter 4: Connections

Problem 4-3

Proof. Prove by definition:

$$\begin{aligned}\widetilde{\Gamma}^k_{ij}\widetilde{E}_k &= \nabla_{\widetilde{E}_i}\widetilde{E}_j = \nabla_{\widetilde{E}_i}A^r_jE_r = \widetilde{E}_i(A^r_j)E_r + A^r_j\nabla_{\widetilde{E}_i}E_r \\ &= (A^q_iE_q)(A^r_j)E_r + A^r_jA^q_i\nabla_{E_q}E_r = A^q_iE_q(A^r_j)E_r + A^r_jA^q_i\Gamma^\ell_{qr}E_\ell\end{aligned}$$

Since $\widetilde{\Gamma}^k_{ij}\widetilde{E}_k = \widetilde{\Gamma}^k_{ij}A^p_kE_p$, by comparison we have

$$\widetilde{\Gamma}^k_{ij} = (A^{-1})^k_pA^q_iE_q(A^p_j) + (A^{-1})^k_pA^r_jA^q_i\Gamma^p_{qr}$$

□

Problem 4-4

Proof. It's clear that

□

5 Chapter 5

Problem 5-1

Proof. By definition we have $D^\flat(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$. Then

$$\begin{aligned} D^\flat(X,Y,Z) = -D^\flat(X,Z,Y) &\iff \langle \widetilde{\nabla}_X Y, Z\rangle - \langle \nabla_X Y, Z\rangle = -\langle \widetilde{\nabla}_X Z, Y\rangle + \langle \nabla_X Z, Y\rangle \\ &\iff \langle \widetilde{\nabla}_X Y, Z\rangle + \langle \widetilde{\nabla}_X Z, Y\rangle = \nabla_X \langle Y, Z\rangle = X\langle Y, Z\rangle \\ &\iff \widetilde{\nabla} \text{ is compatible with } g. \end{aligned}$$

□

Problem 5-2

Proof. We have

$$\begin{aligned} g_{jk}\omega_i^k + g_{ik}\omega_j^k = dg_{ij} &\iff \forall X \in \mathfrak{X}(M),\ g_{jk}\omega_i^k(X) + g_{ik}\omega_j^k(X) = dg_{ij}(X) \\ &\iff \forall X \in \mathfrak{X}(M),\ \langle \omega_i^k(X)E_k, E_j\rangle + \langle \omega_j^k(X)E_k, E_i\rangle = X\langle E_i, E_j\rangle \\ &\iff \forall X \in \mathfrak{X}(M),\ \langle \nabla_X E_i, E_j\rangle + \langle \nabla_X E_j, E_i\rangle = X\langle E_i, E_j\rangle \\ &\iff \nabla \text{ is compatible with } g. \end{aligned}$$

□

Problem 5-3

Proof. By proposition 5.5 it's easy to prove after calculation. □

Problem 5-6

Proof. (a) Since $\widetilde{X}, \widetilde{Y}$ are π-related to X, Y and π is a Riemannian submersion, we have $\langle \widetilde{X}_p, \widetilde{Y}_p\rangle = \langle X_{\pi(p)}, Y_{\pi(p)}\rangle = \langle X, Y\rangle \circ \pi$; Since $d\pi_p\left([\widetilde{X}, \widetilde{Y}]_p^H\right) = d\pi_p\left([\widetilde{X}, \widetilde{Y}]_p\right) = [X, Y]_{\pi(p)}$, so $[\widetilde{X}, \widetilde{Y}]^H = \widetilde{[X, Y]}$; Since $d\pi_p\left([\widetilde{X}, W]_p\right) = [X, 0]_{\pi(p)} = 0$, so $[\widetilde{X}, W]$ is vertical if W is vertical.

(b) For Levi-Civita connection we have

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = \frac{1}{2}\big(&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &-\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\big). \end{aligned}$$

So we have

$$\begin{aligned}
\langle \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \rangle &= \frac{1}{2} \Big(\widetilde{X} \langle \widetilde{Y}, \widetilde{Z} \rangle + \widetilde{Y} \langle \widetilde{Z}, \widetilde{X} \rangle - \widetilde{Z} \langle \widetilde{X}, \widetilde{Y} \rangle \\
&\quad - \langle \widetilde{Y}, [\widetilde{X}, \widetilde{Z}] \rangle - \langle \widetilde{Z}, [\widetilde{Y}, \widetilde{X}] \rangle + \langle \widetilde{X}, [\widetilde{Z}, \widetilde{Y}] \rangle \Big) \\
&= \frac{1}{2} \Big(\widetilde{X} \left(\langle Y, Z \rangle \circ \pi \right) + \widetilde{Y} \left(\langle Z, X \rangle \circ \pi \right) - \widetilde{Z} \left(\langle X, Y \rangle \circ \pi \right) \\
&\quad - \langle \widetilde{Y}, [\widetilde{X}, \widetilde{Z}]^H \rangle - \langle \widetilde{Z}, [\widetilde{Y}, \widetilde{X}]^H \rangle + \langle \widetilde{X}, [\widetilde{Z}, \widetilde{Y}]^H \rangle \Big) \\
&= \frac{1}{2} \left((X \langle Y, Z \rangle) + (Y \langle Z, X \rangle) - (Z \langle X, Y \rangle) \right. \\
&\quad \left. - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \right) \circ \pi \\
&= \langle \nabla_X Y, Z \rangle \circ \pi = \langle \widetilde{\nabla_X Y}, \widetilde{Z} \rangle + \frac{1}{2} \langle [\widetilde{X}, \widetilde{Y}]^V, \widetilde{Z} \rangle.
\end{aligned}$$

and

$$\begin{aligned}
\langle \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, W \rangle &= \frac{1}{2} \Big(\widetilde{X} \langle \widetilde{Y}, W \rangle + \widetilde{Y} \langle W, \widetilde{X} \rangle - W \langle \widetilde{X}, \widetilde{Y} \rangle \\
&\quad - \langle \widetilde{Y}, [\widetilde{X}, W] \rangle - \langle W, [\widetilde{Y}, \widetilde{X}] \rangle + \langle \widetilde{X}, [W, \widetilde{Y}] \rangle \Big) \\
&= \frac{1}{2} \Big(- \langle W, [\widetilde{Y}, \widetilde{X}]^V \rangle \Big) = \frac{1}{2} \langle [\widetilde{X}, \widetilde{Y}]^V, W \rangle + \langle \widetilde{\nabla_X Y}, W \rangle.
\end{aligned}$$

So for local frame of $\mathfrak{X}(\widetilde{M})$, $\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}$ satisfies the two formula above, then we have $\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2} [\widetilde{X}, \widetilde{Y}]^V$.

□

Problem 5-7

Problem 5-9

Problem 5-10

Proof. Since $\varphi(p) = \psi(p)$, $d\varphi_p = d\psi_p$, we have the following commute diagram:

$$\begin{tikzcd}
M \arrow[r, "\psi"] & \widetilde{M} \\
\mathcal{E}_p \arrow[u, "\exp_p"] \arrow[d, "\exp_p"'] \arrow[r, "d\psi_p", "d\varphi_p"'] & \widetilde{\mathcal{E}}_{\varphi(p)} \arrow[u, "\exp_{\psi(p)}"'] \arrow[d, "\exp_{\varphi(p)}"] \\
M \arrow[r, "\varphi"] & \widetilde{M}
\end{tikzcd}$$

By the commute diagram and $\exp_p$ local diffeomorphism, there exists a neighborhood U of p such that $\varphi \equiv \psi$ on U. Since M is connected, $\varphi \equiv \psi$ on M.

□

6 Chapter 6: Geodesics and Distance

Problem 6-1

Proof. (a) For $\forall p \in \gamma(I)$, there exists a uniformly normal neighborhood $p \in W$, then $W \subseteq \gamma(I)$, so $\gamma(I)$ is open; Let (p_i) be a sequence of points in $\gamma(I)$ and $(p_i) \to p$. Since p has a uniformly normal neighborhood W, $\exists N \in \mathbb{N}$ s.t. $\forall i > N$, $p_i \in W$. Then $p \in \gamma(I)$, so $\gamma(I)$ is closed. Since M is connected, γ is surjective.

(b) If γ is injective, then it's a bijective smooth map from I to M, we only need to show γ is a local isometry. The Riemannian metric has local representation $g_{11}(dt)^2$, since γ is unit-speed, we have $g(\gamma', \gamma') = g_{11} = 1$, then the local representation is $(dt)^2$, which shows that γ is a isometry.

(c) Let $\alpha(t) = \gamma(t + t_1)$, $\beta(t) = \gamma(t + t_2)$, then we have α, β are unit-speed geodesic, $\alpha(0) = \beta(0)$, $\alpha'(0) = \beta'(0)$. By the uniqueness of geodesic we have $\alpha \equiv \beta$, hence $\gamma(t + t_1) = \gamma(t + t_2)$, γ is periodic.

(d) WLOG, we assume that $t_2 > t_1$, then $\alpha(\frac{t_2 - t_1}{2}) = \beta(\frac{t_2 - t_1}{2})$, contradiction. □

Problem 6-4

Proof. (a) Since the equation is independent of coordinate, we take the normal coordinate centered at $\gamma(0)$. In this chart we have $\gamma(t) = \left(\gamma^1(t), \cdots, \gamma^n(t)\right)$ and $d_g\left(\gamma(0), \gamma(t)\right) = \sqrt{\sum \left(\gamma^i(t)\right)^2}$, then

$$\lim_{t \searrow 0} \frac{d_g\left(\gamma(0), \gamma(t)\right)}{t} = \lim_{t \searrow 0} \sqrt{\sum \left(\frac{\gamma^i(t)}{t}\right)^2} = \sqrt{\sum \left(\dot{\gamma}^i(0)\right)^2} = |\gamma'(0)|_g$$

(b) We only need to verify that for every $p \in M$, $g_p = \widetilde{g}_p$. From (a) we have $g_p(v, v) = \widetilde{g}_p(v, v)$ for all $v \in T_pM$, by polarization identity we have $g_p = \widetilde{g}_p$. □

Problem 6-5

Proof. (a) In the normal coordinates centered at p, let $u(q, v, t) = \exp_p^{-1}\left(\exp_q(tv)\right)$, it's clearly a smooth, then $f = |u|^2$ is smooth.

(b) $\frac{\partial f}{\partial t} = 2\langle u, \frac{\partial u}{\partial t}\rangle$, $\frac{\partial^2 f}{\partial t^2} = 2\langle \frac{\partial^2 u}{\partial t^2}, u\rangle + 2\left|\frac{\partial u}{\partial t}\right|^2$. Then for $q = p$, $u(q, v, t) = tv$, $\frac{\partial^2 f}{\partial t^2} = 2\left|\frac{\partial u}{\partial t}\right|^2$ is positive. Use continuity we have that if ϵ is small enough then $\frac{\partial^2 f}{\partial t^2} > 0$.

(c) Since q_1, q_2 are in the uniformly normal neighborhood of p, $\gamma(t)$ is of the form $\exp_{q_1}(tv)$ for some $v \in T_{q_1}M$, then $d_g(p, \gamma(t))^2 = f(q_1, v, t)$. By (b) we have $\frac{\partial^2 f}{\partial t^2} > 0$, so f is convex on where it's defined. Then $d_g(p, \gamma(t))$ is also convex on its domain, so it attains its maximum at one of the endpoints of γ.

(d) By (c) it's clear that $d_g(p, \gamma(t)) < \epsilon \implies \gamma(t) \in B_\epsilon(p)$, so the image of the unique geodesic segment lies in $B_\epsilon(p)$, hence geodesically convex. □

Problem 6-6

Proof. For $x, x' \in M$, if $d_g(x,x') < \operatorname{conv}(x)$, then the geodesic ball $B_\delta(x')$, where $\delta = \operatorname{conv}(x) - d_g(x,x')$, is contained in $B_{\operatorname{conv}(x)}(x)$: for $\forall x'' \in B_\delta(x')$, $d_g(x,x'') \le d_g(x,x') + d_g(x',x'') < \operatorname{conv}(x)$. It's clear that $B_\delta(x')$ is also geodesically convex since it's contained in a geodesically convex geodesic ball. Then $\operatorname{conv}(x') \ge \delta = \operatorname{conv}(x) - d_g(x,x')$, which means $\operatorname{conv}(x) - \operatorname{conv}(x') \le d_g(x,x')$. If $d_g(x,x') \ge \operatorname{conv}(x)$ this inequality naturally holds. By reversing the role of x, x' we then get: $|\operatorname{conv}(x) - \operatorname{conv}(x')| \le d_g(x,x')$. So $\operatorname{conv}(x)$ is continuous. □

Problem 6-7

Proof. (a) Suppose that $\lim\limits_{t\to 0} \frac{d_g\left(\exp_p tv, \exp_p tw\right)}{t} > c|v-w|_g$ for $c > 1$. In Riemannian normal coordinates on a convex geodesic ball centered at p, we have $g = \delta_{ij} + O(|x|^2)$, then we have $\frac{1}{c}|w|_{\overline{g}} < |w|_g < c|w|_{\overline{g}}$, which shows that

$$\frac{1}{c}\frac{d_{\overline{g}}\left(\exp_p tv, \exp_p tw\right)}{t} < \frac{d_g\left(\exp_p tv, \exp_p tw\right)}{t} < c\frac{d_{\overline{g}}\left(\exp_p tv, \exp_p tw\right)}{t}$$

Since $d_g\left(\exp_p tv, \exp_p tw\right) = |tv - tw|_{\overline{g}} = t|v-w|_{\overline{g}}$ and $|v-w|_{\overline{g}} = |v-w|_g$, we have $\lim\limits_{t\to 0} \frac{d_g\left(\exp_p tv, \exp_p tw\right)}{t} < c|v-w|_g$. Similarly we can argue the $'' < ''$ case. So we have

$$\lim_{t\to 0} \frac{d_g\left(\exp_p tv, \exp_p tw\right)}{t} = c|v-w|_g$$

(b) Since geodesics are locally minimizing curves and vise visa, we just need to show that for a minimizing segment $\gamma : [a,b] \to M$, $\varphi \circ I$ is minimizing. This is obvious since φ is a metric isometry.

(c) Define □

Exercise 6.1. (**Problem 6-10**)

A curve $\gamma : [0,b) \to M$ (with $0 < b \le \infty$) is said to **diverge to infinity** if for every compact set $K \subseteq M$, there is a time $T \in [0,b)$ such that $\gamma(T) \notin K$ for $t > T$. Prove that a connected Riemannian manifold is complete if and only if every regular curve that diverges to infinity has infinite length.

Proof. • If M is a Riemannian manifold and γ is a regular curve that diverges to infinity. Then $\exists T_R \in [0,b)$ such that $\gamma(T_R) \notin \exp_{\gamma(0)}\left(B_R(0)\right)$, so $L_g(\gamma) \ge R$ for all $R \ge 0$. Hence γ has infinite length.

• If every regular curve that diverges to infinity has infinite length. Suppose M is not complete, then there exists $p \in M$ and a unit-speed maximal geodesic γ starting at p such that γ is defined on $[0,b)$. (We only consider the positive part.) If there exists a compact set $K \subseteq M$ such that $\gamma\left([0,b)\right) \subseteq K$, then there exists $\epsilon > 0$ such that geodesics starting at points in K are defined at least on $(-\epsilon, \epsilon)$. Take $b - \epsilon < b_1 < b$, then we can paste two geodesics to form a larger one, which contradicts to γ is maximal. So γ diverges to infinity. By our assumption then γ has infinite length, so the domain of γ must be $\mathbb{R}$. Hence M is complete. $\square$

Exercise 6.2. (**Problem 6-11**)

Suppose (M, g) is a connected Riemannian manifold, $P \subseteq M$ is a connected embedded submanifold, and $\widehat{g}$ is the induced Riemannina metric on P.

(1) Show that $d_{\widehat{g}}(p, q) \geq d_g(p, q)$ for $p, q \in P$.

(2) Prove that if (M, g) is complete and P is closed in M, then $(P, \widehat{g})$ is complete.

(3) Give an example of a complete Riemannian manifold (M, g) and a connected embedded submanifold $P \subseteq M$ that is complete but not closed in M.

Proof. (1) Since admissible curves in P is also admissible curves in M, then after taking inf we have $d_{\widehat{g}}(p, q) \geq d_g(p, q)$ for $p, q \in P$.

(2) Let (p_i) be Cauchy sequence in $(P, \widehat{g})$. By (1) we see (p_i) is also Cauchy sequence in (M, g). Since (M, g) is complete, $p_i \to p \in M$. Since P is closed in M, we have $p \in P$. Hence $(P, \widehat{g})$ is complete.

(3) $\left\{x, \sin \frac{1}{x} : x > 0\right\} \subseteq \mathbb{R}^2$. $\square$

Exercise 6.3. (**Problem 6-12**)

Let (M, g) be a connected Riemannian manifold.

(1) Suppose there exists $\delta > 0$ such that for each $p \in M$, every maximal unit-speed geodesic starting at p is defined at least on an interval of the form $(-\delta, \delta)$. Prove that M is complete.

(2) Prove that if M has positive or infinite injectivity radius, then it is complete.

(3) Prove that if M is homogeneous, then it is complete.

(4) Given an example of a complete, connected Riemannian manifold that has zero injectivity radius.

Proof. $\square$

Exercise 6.4. (**Problem 6-14**)

Let (M, g) be a connected Riemannian manifold.

(1) Show that M is complete if and only if the compact subsets of M are exactly the closed and bounded ones.

(2) Show that M is compact if and only if it is complete and bounded.

Proof. (1) Suppose M is complete, then clearly compact subsets of M is closed and bounded. (Closed because M is Hausdorff, bounded because you can cover compact subset with finite many bounded balls.) If $A \subseteq M$ is closed and bounded, then for

$p \in A$, there exists $R > 0$ s.t. $A \subseteq \exp_p\left(\overline{B_R(0)}\right)$. Since $\exp_p(\overline{B_R(0)})$ is compact and A is closed, A is compact. Hence compacts subsets are exactly closed and bounded ones.

Suppose compact subsets of M are exactly closed and bounded ones. Let (p_i) be Cauchy sequence in M, then $\overline{\{p_i\}_i}$ is closed and bounded, which is compact by assumption. Then there exists convergent subsequence $p_{i_k} \to p \in \overline{\{p_i\}_i} \subseteq M$. Hence M is complete.

(2) Suppose M is compact, then clearly M is complete and also bounded by (1).

Suppose M is complete and bounded. Since M is closed itself, by (1) we have M is compact. $\square$

7 Chapter 7: Curvature

Exercise 7.1. (**Problem 7-7**)

Suppose (M, g) is a Riemannian manifold and $u \in C^\infty(M)$. Prove **Bochner's formula**:

$$\Delta\left(\frac{1}{2}|\mathrm{grad} u|^2\right) = |\nabla^2 u|^2 + \langle \mathrm{grad}(\Delta u), \mathrm{grad} u\rangle + Rc(\mathrm{grad} u, \mathrm{grad} u).$$

Proof. For $p \in M$, take a geodesic frame (E_i) near p. Since $\nabla^2 u = \nabla\left((\mathrm{grad} u)^\flat\right) = (\nabla \mathrm{grad} u)^\flat$, we have $\nabla^2(X, Y) = \langle \nabla_X \mathrm{grad} u, Y\rangle$. Then

$$\begin{aligned}
\Delta\left(\frac{1}{2}|\mathrm{grad} u|^2\right) &= \frac{1}{2}\sum_i E_i E_i \langle \mathrm{grad} u, \mathrm{grad} u\rangle \\
&= \sum_i E_i \langle \nabla_{E_i} \mathrm{grad} u, \mathrm{grad} u\rangle \\
&= \sum_i E_i \nabla^2 u(E_i, \mathrm{grad} u) \\
&= \sum_i E_i \left\langle \nabla_{\mathrm{grad} u} \mathrm{grad} u, E_i\right\rangle \\
&= \sum_i \left(\left\langle \nabla_{E_i} \nabla_{\mathrm{grad} u} \mathrm{grad} u, E_i\right\rangle + \left\langle \nabla_{\mathrm{grad} u} \mathrm{grad} u, \nabla_{E_i} E_i\right\rangle\right) \\
&= \sum_i \left(\langle R(E_i, \mathrm{grad} u)\mathrm{grad} u, E_i\rangle + \left\langle \nabla_{\mathrm{grad} u} \nabla_{E_i} \mathrm{grad} u, E_i\right\rangle \right.\\
&\quad \left. + \left\langle \nabla_{[E_i, \mathrm{grad} u]} \mathrm{grad} u, E_i\right\rangle\right) \\
&= Rc(\mathrm{grad} u, \mathrm{grad} u) + \sum_i \left(\left\langle \nabla_{\mathrm{grad} u} \nabla_{E_i} \mathrm{grad} u, E_i\right\rangle + \left\langle \nabla_{[E_i, \mathrm{grad} u]} \mathrm{grad} u, E_i\right\rangle\right).
\end{aligned}$$

And we have

$$\begin{aligned}
\sum_i \left\langle \nabla_{\mathrm{grad} u} \nabla_{E_i} \mathrm{grad} u, E_i\right\rangle &= \sum_i \left(\mathrm{grad} u \langle \nabla_{E_i} \mathrm{grad} u, E_i\rangle - \langle \nabla_{E_i} \mathrm{grad} u, \nabla_{\mathrm{grad} u} E_i\rangle\right) \\
&= \mathrm{grad} u(\Delta u) = \langle \mathrm{grad}(\Delta u), \mathrm{grad} u\rangle.
\end{aligned}$$

$$\begin{aligned}
\sum_i \left\langle \nabla_{[E_i, \mathrm{grad} u]} \mathrm{grad} u, E_i\right\rangle &= \sum_i \langle \nabla_{E_i} \mathrm{grad} u, [E_i, \mathrm{grad} u]\rangle \\
&= \sum_i \left\langle \nabla_{E_i} \mathrm{grad} u, \nabla_{E_i} \mathrm{grad} u - \nabla_{\mathrm{grad} u} E_i\right\rangle \\
&= \sum_i \langle \nabla_{E_i} \mathrm{grad} u, \nabla_{E_i} \mathrm{grad} u\rangle \\
&= |\nabla^2 u|^2.
\end{aligned}$$

Hence we have

$$\Delta\left(\frac{1}{2}|\mathrm{grad} u|^2\right) = |\nabla^2 u|^2 + \langle \mathrm{grad}(\Delta u), \mathrm{grad} u\rangle + Rc(\mathrm{grad} u, \mathrm{grad} u).$$

$\square$

Exercise 7.2. (**Problem 7-13**)

Let G be a Lie group with a bi-invariant metric g. Show that the following formula holds whenever X, Y, Z are left-invariant vector fields on G:

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$$

Proof. Just routine computation:

$$\begin{aligned}
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= \nabla_X \left(\frac{1}{2}[Y,Z]\right) - \nabla_Y \left(\frac{1}{2}[X,Z]\right) - \frac{1}{2}[[X,Y],Z] \\
&= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] + \frac{1}{2}[Z,[X,Y]] \\
&= \frac{1}{4}\left([X,[Y,Z]] + [Y,[Z,X]]\right) + \frac{1}{2}[Z,[X,Y]] \\
&= -\frac{1}{4}[Z,[X,Y]] + \frac{1}{2}[Z,[X,Y]] \\
&= \frac{1}{4}[Z,[X,Y]].
\end{aligned}$$

□

Exercise 7.3. (**Problem 7-14**)

Suppose $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian submersion. Using the notation and results of Problem 5-6, prove **O'Neill's formula**:

$$\begin{aligned}
Rm(W,X,Y,Z)\circ\pi = \widetilde{Rm}(\widetilde{W},\widetilde{X},\widetilde{Y},\widetilde{Z}) - \frac{1}{2}\left\langle [\widetilde{W},\widetilde{X}]^V, [\widetilde{Y},\widetilde{Z}]^V \right\rangle \\
- \frac{1}{4}\left\langle [\widetilde{W},\widetilde{Y}]^V, [\widetilde{X},\widetilde{Z}]^V \right\rangle + \frac{1}{4}\left\langle [\widetilde{W},\widetilde{Z}]^V, [\widetilde{X},\widetilde{Y}]^V \right\rangle.
\end{aligned}$$

Proof. By problem 5-6, we have $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\widetilde{X},\widetilde{Y}]^V$. Then

$$\widetilde{Rm}(\widetilde{W},\widetilde{X},\widetilde{Y},\widetilde{Z}) = \left\langle \widetilde{\nabla}_{\widetilde{W}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{W}}\widetilde{Y} - \widetilde{\nabla}_{[\widetilde{W},\widetilde{X}]}\widetilde{Y}, \widetilde{Z} \right\rangle.$$

Compute

$$\begin{aligned}
\widetilde{\nabla}_{\widetilde{W}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} &= \widetilde{\nabla}_{\widetilde{W}}\left(\widetilde{\nabla_X Y} + \frac{1}{2}[\widetilde{X},\widetilde{Y}]^V\right) \\
&= \widetilde{\nabla_W \nabla_X Y} + \frac{1}{2}[\widetilde{W}, \widetilde{\nabla_X Y}]^V + \frac{1}{2}\widetilde{\nabla}_{\widetilde{W}}[\widetilde{X},\widetilde{Y}]^V.
\end{aligned}$$

$$\begin{aligned}
0 = \widetilde{\nabla}_{\widetilde{W}}\left\langle [\widetilde{X},\widetilde{Y}]^V, \widetilde{Z} \right\rangle &= \left\langle \widetilde{\nabla}_{\widetilde{W}}[\widetilde{X},\widetilde{Y}]^V, \widetilde{Z} \right\rangle + \left\langle [\widetilde{X},\widetilde{Y}]^V, \widetilde{\nabla_W Z} + \frac{1}{2}[\widetilde{W},\widetilde{Z}]^V \right\rangle \\
&= \left\langle \widetilde{\nabla}_{\widetilde{W}}[\widetilde{X},\widetilde{Y}]^V, \widetilde{Z} \right\rangle + \left\langle [\widetilde{X},\widetilde{Y}]^V, \frac{1}{2}[\widetilde{W},\widetilde{Z}]^V \right\rangle.
\end{aligned}$$

$$0 = \widetilde{\nabla}_{\widetilde{Y}} \left\langle [\widetilde{W}, \widetilde{X}]^V, \widetilde{Z} \right\rangle = \left\langle \widetilde{\nabla}_{\widetilde{Y}} [\widetilde{W}, \widetilde{X}]^V, \widetilde{Z} \right\rangle + \left\langle [\widetilde{W}, \widetilde{X}]^V, \widetilde{\nabla}_{\widetilde{Y}} \widetilde{Z} \right\rangle$$
$$= \left\langle \widetilde{\nabla}_{\widetilde{Y}} [\widetilde{W}, \widetilde{X}]^V, \widetilde{Z} \right\rangle + \left\langle [\widetilde{W}, \widetilde{X}]^V, \frac{1}{2}[\widetilde{Y}, \widetilde{Z}]^V \right\rangle.$$

Then we have

$$\left\langle \widetilde{\nabla}_{\widetilde{W}} \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \right\rangle = \left\langle \widetilde{\nabla_W \nabla_X Y}, \widetilde{Z} \right\rangle - \frac{1}{4} \left\langle [\widetilde{X}, \widetilde{Y}]^V, [\widetilde{W}, \widetilde{Z}]^V \right\rangle$$
$$= \langle \nabla_W \nabla_X Y, Z \rangle \circ \pi - \frac{1}{4} \left\langle [\widetilde{X}, \widetilde{Y}]^V, [\widetilde{W}, \widetilde{Z}]^V \right\rangle.$$

$$\left\langle \widetilde{\nabla}_{\widetilde{X}} \widetilde{\nabla}_{\widetilde{W}} \widetilde{Y}, \widetilde{Z} \right\rangle = \left\langle \widetilde{\nabla_X \nabla_W Y}, \widetilde{Z} \right\rangle - \frac{1}{4} \left\langle [\widetilde{W}, \widetilde{Y}]^V, [\widetilde{X}, \widetilde{Z}]^V \right\rangle$$
$$= \langle \nabla_X \nabla_W Y, Z \rangle \circ \pi - \frac{1}{4} \left\langle [\widetilde{W}, \widetilde{Y}]^V, [\widetilde{X}, \widetilde{Z}]^V \right\rangle.$$

$$\left\langle \widetilde{\nabla}_{[\widetilde{W}, \widetilde{X}]} \widetilde{Y}, \widetilde{Z} \right\rangle = \left\langle \widetilde{\nabla}_{[\widetilde{W}, \widetilde{X}]^H + [\widetilde{W}, \widetilde{X}]^V} \widetilde{Y}, \widetilde{Z} \right\rangle$$
$$= \left\langle \widetilde{\nabla_{[W,X]} Y}, \widetilde{Z} \right\rangle + \left\langle \widetilde{\nabla}_{\widetilde{Y}} [\widetilde{W}, \widetilde{X}]^V + [[\widetilde{W}, \widetilde{X}]^V, \widetilde{Y}], \widetilde{Z} \right\rangle$$
$$= \left\langle \nabla_{[W,X]} Y, Z \right\rangle \circ \pi - \frac{1}{2} \left\langle [\widetilde{W}, \widetilde{X}]^V, [\widetilde{Y}, \widetilde{Z}]^V \right\rangle.$$

Hence we have

$$\widetilde{Rm}(\widetilde{W}, \widetilde{X}, \widetilde{Y}, \widetilde{Z}) = \langle \nabla_W \nabla_X Y, Z \rangle \circ \pi - \frac{1}{4} \left\langle [\widetilde{X}, \widetilde{Y}]^V, [\widetilde{W}, \widetilde{Z}]^V \right\rangle - \langle \nabla_X \nabla_W Y, Z \rangle \circ \pi$$
$$+ \frac{1}{4} \left\langle [\widetilde{W}, \widetilde{Y}]^V, [\widetilde{X}, \widetilde{Z}]^V \right\rangle - \left\langle \nabla_{[W,X]} Y, Z \right\rangle \circ \pi + \frac{1}{2} \left\langle [\widetilde{W}, \widetilde{X}]^V, [\widetilde{Y}, \widetilde{Z}]^V \right\rangle$$
$$= Rm(W, X, Y, Z) \circ \pi - \frac{1}{4} \left\langle [\widetilde{X}, \widetilde{Y}]^V, [\widetilde{W}, \widetilde{Z}]^V \right\rangle$$
$$+ \frac{1}{4} \left\langle [\widetilde{W}, \widetilde{Y}]^V, [\widetilde{X}, \widetilde{Z}]^V \right\rangle + \frac{1}{2} \left\langle [\widetilde{W}, \widetilde{X}]^V, [\widetilde{Y}, \widetilde{Z}]^V \right\rangle.$$

So we get the O'Neill's formula:

$$Rm(W, X, Y, Z) \circ \pi = \widetilde{Rm}(\widetilde{W}, \widetilde{X}, \widetilde{Y}, \widetilde{Z}) - \frac{1}{2} \left\langle [\widetilde{W}, \widetilde{X}]^V, [\widetilde{Y}, \widetilde{Z}]^V \right\rangle$$
$$- \frac{1}{4} \left\langle [\widetilde{W}, \widetilde{Y}]^V, [\widetilde{X}, \widetilde{Z}]^V \right\rangle + \frac{1}{4} \left\langle [\widetilde{W}, \widetilde{Z}]^V, [\widetilde{X}, \widetilde{Y}]^V \right\rangle.$$

$\square$

8 Chapter 8

Exercise 8.1. (**Problem 8-2**)

Let (M, g) be an embedded Riemannian hypersurface in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, let F be a local defining function for M, and let $N = \mathrm{grad} F / |\mathrm{grad} F|$.

(1) Show that the scalar fundamental form of M with respect to the unit normal N is given by

$$h(X, Y) = -\frac{\widetilde{\nabla}^2 F(X, Y)}{|\mathrm{grad} F|}$$

for all $X, Y \in \mathfrak{X}(M)$.

(2) Show that the mean curvature of M is given by

$$H = -\frac{1}{n} \mathrm{div}_{\widetilde{g}} \left(\frac{\mathrm{grad} F}{|\mathrm{grad} F|} \right),$$

where $n = \dim M$ and $\mathrm{div}_{\widetilde{g}}$ is the divergence operator of $\widetilde{g}$.

Proof. ... □

Exercise 8.2. (**Problem 8-9**)

Let $M \subseteq \mathbb{R}^{n+1}$ be a Riemannian hypersurface, and let N be a smooth unit normal vector field along M. At each point $p \in M$, $N_p \in T_p\mathbb{R}^{n+1}$ can be thought of as a unit vector in $\mathbb{R}^{n+1}$ and therefore as a point in $\mathbb{S}^n$. Thus each choice of unit normal vector field defines a smooth map $\nu : M \to \mathbb{S}^n$, called the **Gauss map of M**. Show that $\nu^* dV_{\mathring{g}} = (-1)^n K dV_g$, where K is the Gaussian curvature of M.

Proof. By Weingarten equation for hypersurface, we have: for every $X \in \mathfrak{X}(M)$, $\widetilde{\nabla}_X N = -sX$, so we have $d\nu = \nabla \nu = -s$. Then $K = \det s = (-1)^n \det d\nu$.

For $p \in M$, let $\{e_i\}$ be oriented orthonormal basis of T_pM. Then

$$\nu^* dV_{\mathring{g}}(e_1, \cdots, e_n) = \det(v_p, e_1, \cdots, e_n) = \det(d_p\nu) = (-1)^n \det s = (-1)^n K$$

Hence $\nu^* dV_{\mathring{g}} = (-1)^n K dV_g$. □

9 Chapter 9

10 Chapter 10: Jacobi Fields

Exercise 10.1. (**Problem 10-1**)

Suppose (M, g) is a Riemannian manifold and $p \in M$. Show that the second-order Taylor series of g in normal coordinates centered at p is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{iklj}(p)x^k x^l + O(|x|^3).$$

Proof. Let $\gamma(t) = (tv^1, \cdots, tv^n)$ be radical geodesic in normal coordinates centered at p, $J(t) = tw^i\partial_i\mid_{\gamma(t)}$ be Jacobi field along γ, let $v = (v^1, \cdots, v^n), w = (w^1, \cdots, w^n)$.

On the one hand, $|J(t)|^2 = t^2 g_{ij}(\gamma(t))w^i w^j$ near p.

On the other hand, let $f = |J(t)|^2 = \langle J(t), J(t)\rangle$, $RJ = R(J, \gamma')\gamma'$, we have

$$\begin{aligned}
&f = \langle J, J\rangle, \quad f' = 2\langle D_t J, J\rangle, \\
&f'' = -2\langle RJ, J\rangle + 2\langle D_t J, D_t J\rangle \\
&f''' = -2\langle D_t(RJ), J\rangle - 6\langle RD_t J, J\rangle = -2\langle (D_t R)J, J\rangle - 8\langle RD_t J, J\rangle \\
&f'''' = -2\langle D_t((D_t R)J), J\rangle - 2\langle (D_t R)J, D_t J\rangle - 8\langle D_t(RD_t J), J\rangle - 8\langle RD_t J, D_t J\rangle \\
&\quad = -2\langle (D_t^2 R)J, J\rangle - 2\langle (D_t R)(D_t J), J\rangle - 2\langle (D_t R)(D_t J), J\rangle - 8\langle (D_t R)(D_t J), J\rangle \\
&\qquad - 8\langle R(D_t^2 J), J\rangle - 8\langle RD_t J, D_t J\rangle \\
&\quad = -2\langle (D_t^2 R)J, J\rangle - 12\langle (D_t R)(D_t J), J\rangle + 8\langle RJ, RJ\rangle - 8\langle RD_t J, D_t J\rangle
\end{aligned}$$

So we have

$$\begin{aligned}
&f(0) = 0, \quad f'(0) = 0 \\
&f''(0) = 2\langle w, w\rangle, \quad f'''(0) = 0, \quad f''''(0) = -8\langle R(w, v)v, w\rangle
\end{aligned}$$

Hence we have

$$\begin{aligned}
t^2 g_{ij}(\gamma(t))w^i w^j = f(t) &= \sum_{i=0}^{4} \frac{1}{i!} f^{(i)}(0)t^i + O(|t|^5) \\
&= \delta_{ij}w^i w^j t^2 - \frac{1}{3}\sum_{k,l} R_{iklj}(\gamma(t))w^i v^k v^l w^j t^4 + O(|t|^5) \\
&= \delta_{ij}w^i w^j t^2 - \frac{1}{3}\sum_{k,l} R_{iklj}(\gamma(t))x^k x^l w^i w^j t^2 + O(|t|^5)
\end{aligned}$$

By comparison, we have $g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{iklj}(p)x^k x^l + O(|x|^3)$.

Remark. For higher order you can check this website. □