李代数作业

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Contents

1 第一次作业

Exercise 1.1. Let F be a field, prove that:

- when $\mathrm{char} F \neq 2$, $\mathfrak{sl}(2, F)$ is a simple Lie algebra.
- when $\mathrm{char} F = 2$, $\mathfrak{sl}(2, F)$ is not a simple Lie algebra.

Solution. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be basis of $\mathfrak{sl}(2, F)$, then $[x, y] = z$, $[z, x] = 2x$, $[z, y] = -2y$.

- When $\mathrm{char} F \neq 2$: Suppose there exists a non-trivial ideal I of $\mathfrak{sl}(2, F)$, then it contains a nonzero element $e = ax + by + cz$. Then $[x, [x, e]] = -2bx$, $[y, [y, e]] = -2ay$. If a or b is nonzero, then I contains x or y, which implies that $I = \mathfrak{sl}(2, F)$; If $a = b = 0$, then $e = cz \in I$, which also implies $I = \mathfrak{sl}(2, F)$. Hence $\mathfrak{sl}(2, F)$ is simple Lie algebra.
- When $\mathrm{char} F = 2$: Consider subspace Fz, by simple computation we can see that Fz is non-trivial ideal of $\mathfrak{sl}(2, F)$. Hence $\mathfrak{sl}(2, F)$ is not a simple Lie algebra.

□

Exercise 1.2. Let $\dim \mathfrak{g} < \infty$. Show that $\dim C(\mathfrak{g}) \neq \dim \mathfrak{g} - 1$.

Solution. Suppose $\dim C(\mathfrak{g}) = \dim \mathfrak{g} - 1 = n - 1$. Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathfrak{g}$ s.t. $\{e_1, \cdots, e_{n-1}\}$ is basis of $C(\mathfrak{g})$. Since $e_n \in \mathfrak{g} - C(\mathfrak{g})$, there exists e_i, $1 \leq i \leq n - 1$ s.t. $[e_i, e_n] \neq 0$, which contradicts to $e_i \in C(\mathfrak{g})$. Hence $\dim C(\mathfrak{g}) \neq \dim \mathfrak{g} - 1$. □

Exercise 1.3. We define $Heis_{2n+1}$ to be the Lie algebra with basis p_i, q_i, c where $[p_i, q_i] = c = -[q_i, p_i]$, $1 \leq i \leq n$, and all other bracketed pairs are 0. Classify all finite dimensional Lie algebra for which $\dim C(\mathfrak{g}) = \dim \mathfrak{g} - 2$. Let $\dim \mathfrak{g} = n$ and show either $\mathfrak{g} \simeq Ab_{n-3} \oplus Heis_3$ or $\mathfrak{g} \simeq Ab_{n-2} \oplus \mathfrak{h}$ where $\mathfrak{h}$ is the two-dimensional non-abelian Lie algebra.

Solution. Let $\{e_1, \cdots, e_{n-2}, y_1, y_2\}$ be a basis of $\mathfrak{g}$ s.t. $\{e_1, \cdots, e_{n-2}\}$ is basis of $C(\mathfrak{g})$.

- If $[y_1, y_2] \in C(\mathfrak{g})$: Let $c = [y_1, y_2]$, we may assume WLOG that $c = e_{n-2}$. Now $\{y_1, y_2, e_{n-2}\}$ forms the $Heis_3$ and $\{e_1, \cdots, e_{n-3}\}$ forms the Ab_{n-3}. Hence $\mathfrak{g} \simeq Ab_{n-3} \oplus Heis_3$.
- If $[y_1, y_2] \in L(y_1, y_2)$: Denote the Lie algebra generated by $\{y_1, y_2\}$ by $\mathfrak{h}$. By the classification of two-dimensional Lie algebra, we may assume WLOG that $[y_1, y_2] = y_1$. Hence $\mathfrak{g} \simeq Ab_{n-2} \oplus \mathfrak{h}$.

□

$\boxed{\text{Remark.}}$ We could also consider quotient Lie algebra $\mathfrak{g}/C(\mathfrak{g})$, the process is similar.