

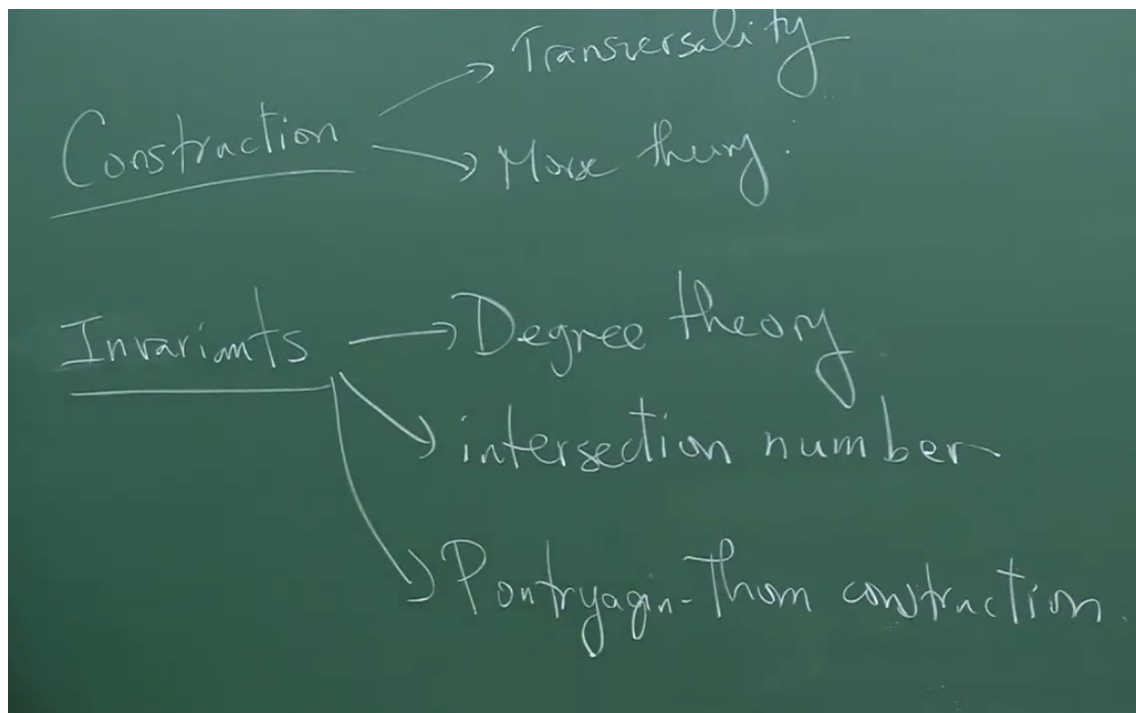
Differential Topology

Misuzu

January 20, 2025

Contents

| | | |
|----------|---|-----------|
| 1 | Review of Differentiable Manifold | 4 |
| 1.1 | Jet bundles | 5 |
| 1.2 | Whitney C^∞ -Topology | 6 |
| 1.3 | Transversality Theorem | 11 |
| 1.4 | Whitney Immersions and Embeddings | 15 |
| 1.5 | Morse Functions | 17 |
| 2 | Intersection Theory | 19 |
| 2.1 | Manifolds with boundary and orientation | 19 |



Bibliography.

- (1) Guillemin - Pollack, *Differential Topology*.
- (2) Hirsch, *Differential Topology*.
- (3) Golubitsky-Guillemin, *Stable Mapping and Their Singularities*.
- (4) Bredon, *Geometry and Topology*.

1 Review of Differentiable Manifold

Definition 1.1: Topological Space

A topological space is a pair (X, T) , where $T \subseteq \mathcal{P}(X)$ such that

- $\emptyset, X \in T$
- $\{U_\alpha\}_{\alpha \in I} \subseteq T \implies \bigcup_{\alpha \in I} U_\alpha \in T$
- $U_1, \dots, U_n \subseteq T \implies U_1 \cap \dots \cap U_n \in T$

Fixing (X, T) , the elements of T are called open sets.

1.1 Jet bundles

Definition 1.2:

Let X, Y be smooth manifolds, $f, g : X \rightarrow Y$ smooth.

(1) We write $f \sim_k g$ at $p \in X$ if $f(p) = g(p)$ and given charts $\varphi : U \rightarrow \mathbb{R}^n$ around p , $\psi : V \rightarrow \mathbb{R}^m$ around $f(p)$

$$\frac{\partial^{|\alpha|} (\psi \circ f \circ \varphi^{-1})_j}{\partial x^\alpha} (\varphi(p)) = \frac{\partial^{|\alpha|} (\psi \circ g \circ \varphi^{-1})_j}{\partial x^\alpha} (\varphi(p)), \quad \forall |\alpha| \leq k, 1 \leq j \leq m$$

It follows from the chain rule that \sim_k is an equivalence relation.

(2) $J^k(X, Y)_{p,q} = \{f : X \rightarrow Y \text{ smooth} \mid f(p) = q\} / \sim_k$, called the space of k -jets at p with value q .

(3) $J^k(X, Y) = \bigsqcup_{\substack{p \in X \\ q \in Y}} J^k(X, Y)_{p,q}$.

Example 1.1:

(1) $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$,

$$J^k(U, V)_{x,y} \xrightarrow{\Phi} B_{n,m}^k, [f] \mapsto (p_1(x), \dots, p_m(x))$$

where $p_j(x)$ is the Taylor polynomial of $f_j(x)$ without the constant term, $B_{n,m}^k = \{\text{polynomial functions } \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ of degree } \leq k \text{ with no constant term}\}$. Φ is a bijection. $J^k(U, V) \cong U \times V \times B_{n,m}^k$.

(2) $J^1(M, \mathbb{R}) \xrightarrow{\text{bijection}} \mathbb{R} \times T^*M, [f]_x \mapsto (f(x), df_x)$.

(3) $J^1(\mathbb{R}, M) \cong \mathbb{R} \times TM$.

Definition 1.3:

(1) $\varphi : Y \rightarrow Z$ smooth. Then $\varphi_* : J^k(X, Y) \rightarrow J^k(X, Z), [f]_x \mapsto [\varphi \circ f]_x$.

(2) $\varphi : Z \rightarrow X$ diffeo. Then $\varphi^* : J^k(X, Y) \rightarrow J^k(Z, Y), [f]_x \mapsto [f \circ \varphi]_{\varphi^{-1}(x)}$.

Remark. These operations are well-defined and natural (functionality). In particular, if $\varphi : Y \rightarrow Z$ diffeo, then φ_* is bijection; $\varphi : Z \rightarrow X$ diffeo, then φ^* is bijection.

Suppose $\sigma \in J^k(X, Y), \sigma = [f]_x$.

Define $\alpha(\sigma) = x, \beta(\sigma) = f(x)$, called the source of σ and target of σ respectively, then $\alpha : J^k(X, Y) \rightarrow X, \beta : J^k(X, Y) \rightarrow Y$. We will define the local topology around σ and a smooth structure near σ .

Fix charts $\varphi : U \rightarrow \mathbb{R}^n, \psi : V \rightarrow \mathbb{R}^m$ around x and $f(x)$ respectively, $f(U) \subseteq V$. Let

$$\tau_{U,V} : J^k(U, V) \longrightarrow J^k(\varphi(U), \psi(V)) \cong \varphi(U) \times \psi(V) \times B_{n,m}^k, \sigma \mapsto (\varphi^{-1})^* \psi_* \sigma$$

Since $\varphi(U) \times \psi(V) \times B_{n,m}^k \subseteq \mathbb{R}^N$, use $\tau_{U,V}$ to topologize $J^k(U, V)$ and hence $J^k(X, Y)$. It's easy to see that this topology doesn't depend on the choice of charts.

Exercise. Let $\tilde{\varphi} : U \rightarrow \mathbb{R}^n, \tilde{\psi} : V \rightarrow \mathbb{R}^m$ be other charts, then $\tau_{\tilde{\varphi}, \tilde{\psi}} \circ \tau_{\varphi, \psi}^{-1}$ is smooth. So $J^k(X, Y)$ has an induced smooth structure.

Lemma 1.1:

- (1) $J^k(X, Y)$ is a manifold of dimension $n + m \binom{n+k}{k}$.
- (2) $\alpha : J^k(X, Y) \rightarrow X, \beta : J^k(X, Y) \rightarrow Y, \alpha \times \beta : J^k(X, Y) \rightarrow X \times Y$ are smooth surjective submersions.
- (3) $\varphi : Y \rightarrow Z$ smooth, then φ_* is smooth; $\varphi : Z \rightarrow X$ diffeomorphism, then φ^* is diffeomorphism.

Definition 1.4:

Let $f \in C^\infty(X, Y)$. Its k -jet $j^k f$ is the function

$$j^k f : X \rightarrow J^k(X, Y), x \mapsto [f]_x$$

Remark. $J^k(X, Y)$ is usually not a vector bundle over X, Y or $X \times Y$. If $Y = \mathbb{R}^m$, then $J^k(X, Y)$ is a vector bundle over X .

Definition 1.5:

Let E, B, F be manifolds, and $\pi : E \rightarrow B$ is a surjective submersion. We say that π is a fiber bundle with fiber F if $\forall b \in B, \exists U \subseteq B$ neighborhood of b and a diffeomorphism $\Phi : \pi^{-1}(U) \rightarrow U \times F$ such that the diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ \pi \searrow & & \swarrow pr_1 \\ & U & \end{array}$$

Exercise. $J^k(X, Y)$ is a fiber bundle over $X, Y, X \times Y$, $f \in C^\infty(X, Y)$ gives rise to a section $j^k f : X \rightarrow J^k(X, Y)$.

1.2 Whitney C^∞ -Topology

Let X, Y be smooth manifolds. For $U \subseteq J^k(X, Y)$ open, let

$$M(U) = \{f \in C^\infty(X, Y) \mid j^k f(X) \subseteq U\}$$

Note that $M(U) \cap M(V) = M(U \cap V)$, so $\{M(U) \mid U \subseteq J^k(X, Y) \text{ open}\}$ is a basis for a topology on $C^\infty(X, Y)$, which is called the C^k -topology. Let W_k be the C^k -topology.

Lemma 1.2:

$$k \leq l \implies W_k \subseteq W_l.$$

Proof. Suppose $k \leq l$. There exists a surjective continuous map:

$$\pi_{k,l} : J^l(X, Y) \rightarrow J^k(X, Y), [f]_x \mapsto [f]_x$$

$\pi_{k,l} \circ j^l f = j^k f$. If $U \subseteq J^k(X, Y)$ is open, then $\pi_{k,l}^{-1}(U) \subseteq J^l(X, Y)$ is open. So $M(U) = M(\pi_{k,l}^{-1}(U))$. Therefore $W_k \subseteq W_l$. \square

Definition 1.6:

The (Whitney) C^∞ -topology is the topology on $C^\infty(X, Y)$ generated by $\bigcup_{k \in \mathbb{N}} W_k$.

Recall that every manifold M admits a Riemannian metric, which induced a metric space structure on the manifold (M, d) . Moreover, we may assume that d is complete.

Why? (1) \exists smooth proper function $f : M \rightarrow \mathbb{R}$; (2) For any metric d on M , we can define $\tilde{d}(x, y) = d(x, y) + |f(x) - f(y)|$, \tilde{d} is complete.

Let d be a (complete) metric on $J^k(X, Y)$.

Definition 1.7:

Let $\delta : X \rightarrow \mathbb{R}_{>0}$ continuous and $f \in C^\infty(X, Y)$, let

$$B_\delta^k(f) = \{g \in C^\infty(X, Y) \mid d(j^k f(x), j^k g(x)) < \delta(x)\}$$

Proposition 1.1:

$\{B_\delta(f) \mid \delta : X \rightarrow \mathbb{R}_{>0}\}$ is a basis for C^k -topology at f . (neighborhood basis)

Proof. $f \in B_\delta(f)$.

Step 1. $B_\delta(f)$ is open. We claim that

$$B_\delta(f) = M(U), \quad U = \{\sigma \in J^k(X, Y) \mid d(j^k f(\alpha(\sigma)), \sigma) < \delta(\alpha(\sigma))\}$$

Define $\Delta : J^k(X, Y) \rightarrow \mathbb{R}$, $\Delta = \delta \circ \alpha - d(j^k f \circ \alpha(\cdot), \cdot)$, so $U = \Delta^{-1}(0, \infty)$ is open.

Step 2. Let $\mathcal{U} \subseteq C^\infty(X, Y)$ be an open neighborhood of f (in C^k -topology), then there exists $U \subseteq J^k(X, Y)$ open set such that $f \in M(U) \subseteq \mathcal{U}$. We claim that $\exists \delta \in C(X, \mathbb{R}_{>0})$ such that $f \in B_\delta(f) \subseteq M(U)$.

For each $x \in X$, let

$$m(x) = \inf \{d(\sigma, j^k f(x)) \mid \sigma \in \alpha^{-1}(x) \cap (J^k(X, Y) \setminus U)\}$$

It's strictly bigger than 0 for every $x \in X$ because U is open, $m(x)$ could be ∞ for some x . We can choose $\delta : X \rightarrow \mathbb{R}_{>0}$ continuous such that $0 < \delta(x) < m(x)$. Then

$$g \in B_\delta(f) \implies d(j^k f(x), j^k g(x)) < \delta(x) < m(x), \quad \forall x \in X$$

which implies $j^k g(x) \in U, \forall x \in X$. So $B_\delta(f) \subseteq M(U)$. \square

Obs. $B_\delta(f)$ is roughly the set of functions whose partial derivatives up to order k are close enough to f 's.

To make this more precise, let $\Phi = \{\varphi_i : U_i \rightarrow \mathbb{R}^n\}_{i \in I}$ locally finite atlas of X , $\mathcal{K} = \{K_i\}_{i \in I}$ family of compact sets of X , $K_i \subseteq U_i$, $\Psi = \{\psi_i : V_i \rightarrow \mathbb{R}^m\}_{i \in I}$ atlas for Y , $\mathcal{E} = \{\epsilon_i\}_{i \in I}$, $\epsilon_i > 0$. Define

$$\mathcal{N}^k(f; \Phi, \Psi, \mathcal{K}, \mathcal{E}) = \{g \in C^\infty(X, Y) \mid g(K_i) \subseteq V_i \text{ and } \|D^r(\psi_i \circ f \circ \varphi_i^{-1})(x) - D^r(\psi_i \circ g \circ \varphi_i^{-1})(x)\| < \epsilon_i, \forall i, x \in X, r \leq k\}$$

Exercise. Prove that $\{\mathcal{N}^k(f; \Phi, \Psi, \mathcal{K}, \mathcal{E})\}$ is a basis for the C^k -topology.

Remark. If X is compact, then we can find a countable basis of f given by $\{B_{\delta_n}(f)\}$, where $\delta_n = \frac{1}{n}$. So C^k -topology is first countable. Moreover,

$$f_n \xrightarrow{C^k} f \Leftrightarrow \frac{\partial^{|\alpha|} f_n}{\partial x^\alpha} \xrightarrow{\text{uniformly}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}, \quad \forall |\alpha| \leq k$$

Proposition 1.2:

Suppose $\{f_n\}_{n \in \mathbb{N}} \subseteq C^\infty(X, Y)$ such that $f_n \xrightarrow{C^k} f$. Then $\exists K \subseteq X$ compact such that $f_n \equiv f$ in $X \setminus K$ for $n \gg 0$ and $j^k f_n|_K \xrightarrow{\text{uniformly}} j^k f$.

Proof. Suppose $f_n \xrightarrow{C^k} f$ and let $\{K_i\}_{i \in \mathbb{N}}$ exhaustion by compact sets such that $K_i \subseteq \text{int}(K_{i+1})$. Assume, by contradiction, that $\nexists K \subseteq X$ compact set, such that $f_n \equiv f$ on $X \setminus K$. So for each i , $\exists x_i \in K_i, n_i$ such that $f_{n_i}(x_i) \neq f(x_i)$. WLOG, $n_1 < n_2 < \dots$, $a_i = d(j^k f_{n_i}(x_i), j^k f(x_i)) > 0$. Let $\delta : X \rightarrow \mathbb{R}_+$ such that $\delta(x_i) = a_i/2$. Then $f_{n_i} \notin B_\delta(f)$, so $f_{n_i} \not\rightarrow f$. \square

Definition 1.8:

A topological space is Baire if the countable intersection of open and dense subsets is dense.

Theorem 1.1:

Let X, Y be smooth manifolds. Then $C^\infty(X, Y)$ is Baire in the C^∞ -topology.

Proof. Fix complete metric d_k on $J^k(X, Y)$. Let $\{U_n\}_{n \in \mathbb{N}}$ dense open subsets of $C^\infty(X, Y)$ in the C^∞ -topology. Let $V \subseteq C^\infty(X, Y)$ non-empty open set. We want to show that $\bigcap_{n \in \mathbb{N}} U_n \cap V \neq \emptyset$.

Since V is open, $\exists Z \subseteq J^{k_0}(X, Y)$ open such that $M(\overline{Z}) \subseteq V$. It's enough to show that $M(\overline{Z}) \cap \bigcap_{n \in \mathbb{N}} U_n \neq \emptyset$.

We can construct f_i inductively, $\{k_i\} \subseteq \mathbb{N}$, $Z_i \subseteq J^{k_i}(X, Y)$ open sets such that

- (1) $f_i \in M(Z) \cap \bigcap_{s=1}^i M(Z_s)$
- (2) $M(\overline{Z_i}) \subseteq U_i$
- (3) $d_s(j^s f_i(x), j^s f_{i-1}(x)) < 1/2^i, \forall x \in X, 0 \leq s \leq i$

Since $M(Z) \cap U_1$ is open and non-empty, we can find $Z_1 \subseteq J^{k_1}(X, Y)$ non-empty such that $M(\overline{Z_1}) \subseteq M(Z) \cap U_1$. Take $f_1 \in M(Z_1)$ and it satisfies (1) and (2). Say we've chosen (f_s, k_s, Z_s) for $s \leq i-1$. Let $D_i = B_{\frac{1}{2^i}}^0(f_{i-1}) \cap B_{\frac{1}{2^i}}^1(f_{i-1}) \cap \cdots \cap B_{\frac{1}{2^i}}^i(f_{i-1})$ open in C^∞ -topology, $f_{i-1} \in M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i$. Since U_i is open and dense, $M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i \cap U_i \neq \emptyset$ and open, so we can find $\emptyset \neq Z_i \subseteq J^{k_i}(X, Y)$ such that $M(\overline{Z_i}) \subseteq M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i \cap U_i$. Choose $f_i \in M(Z_i)$, it satisfies the three conditions.

For a fixed s , the condition (3) tells that $\{j^s f_i(x)\}$ is a Cauchy sequence in $J^k(X, Y)$, it converges to $g^s(x)$, $g^0(x) \in J^0(X, Y) = X \times Y$, $g^0(x) = (x, g(x))$.

Exercise. $g \in C^\infty(X, Y)$ and $j^s g = g^s$. (Look in a compact set and in charts)

Then $g = \lim_{i \rightarrow \infty} f_i$ in the C^∞ -topology. $f_i \in M(Z) \implies g \in M(\overline{Z})$, $f_i \in M(Z_s)$ for $i \geq s$, so $g \in M(\overline{Z_s})$ for $\forall s$, hence $g \in M(\overline{Z}) \cap \bigcap_{n \in \mathbb{N}} M(\overline{Z_s}) \subseteq M(\overline{Z}) \cap \bigcap_{n \in \mathbb{N}} U_n$. \square

Proposition 1.3:

Let X, Y be smooth manifolds. Then $j^k : C^\infty(X, Y) \rightarrow C^\infty(X, J^k(X, Y))$ is continuous in the C^∞ -topology.

Proof. $U \subseteq J^\ell(X, J^k(X, Y))$ open, so $M(U)$ is open set in the C^ℓ -topology of $C^\infty(X, J^k(X, Y))$. We need to show that $(j^k)^{-1}(M(U))$ is open in $C^\infty(X, Y)$. Consider

$$\alpha^{k, \ell} : J^{k+\ell}(X, Y) \rightarrow J^\ell(X, J^k(X, Y)), \alpha^{k, \ell}(j^{k+\ell} f(x)) = j^\ell(j^k f)(x)$$

This is a smooth embedding. So $(j^k)^{-1}(M(U)) = M((\alpha^{k, \ell})^{-1}(U))$ is open in $C^{k+\ell}$ -topology. \square

Proposition 1.4:

$\phi : Y \rightarrow Z$ smooth. Then $\tilde{\phi}_* : C^\infty(X, Y) \rightarrow C^\infty(X, Z)$, $f \mapsto \phi \circ f$ is continuous in the C^∞ -topology.

Proposition 1.5:

Let X, Y, Z be smooth manifolds. Then $C^\infty(X, Y) \times C^\infty(X, Z) \rightarrow C^\infty(X, Y \times Z)$, $(f, g) \mapsto f \times g$ is a homeomorphism in the C^∞ -topology.

Appendix. About existence of proper function on manifolds.

Next, we use partitions of unity to construct a special kind of smooth function. If M is a topological space, an **exhaustion function for M** is a continuous function $f: M \rightarrow \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ (called a **sublevel set of f**) is compact for each $c \in \mathbb{R}$. The name comes from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}((-\infty, n])$ form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and $g: \mathbb{B}^n \rightarrow \mathbb{R}$ given by

$$f(x) = |x|^2, \quad g(x) = \frac{1}{1 - |x|^2}$$

are smooth exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.28 (Existence of Smooth Exhaustion Functions). *Every smooth manifold with or without boundary admits a smooth positive exhaustion function.*

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^\infty$ be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j \psi_j(p).$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \geq \sum_j \psi_j(p) = 1$.

To see that f is an exhaustion function, let $c \in \mathbb{R}$ be arbitrary, and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^N \bar{V}_j$, then $\psi_j(p) = 0$ for $1 \leq j \leq N$, so

$$f(p) = \sum_{j=N+1}^{\infty} j \psi_j(p) \geq \sum_{j=N+1}^{\infty} N \psi_j(p) = N \sum_{j=1}^{\infty} \psi_j(p) = N > c.$$

Equivalently, if $f(p) \leq c$, then $p \in \bigcup_{j=1}^N \bar{V}_j$. Thus $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^N \bar{V}_j$ and is therefore compact. \square

1.3 Transversality Theorem

Definition 1.9:

Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold. We say that f is transverse to W (write $f \pitchfork W$) if

$$df_x(T_x X) + T_{f(x)} W = T_{f(x)} Y, \quad \forall x \in f^{-1}(W)$$

[Note.] $X_1, X_2 \subseteq Y$ submanifolds, $X_1 \pitchfork X_2 \Leftrightarrow T_x X_1 + T_x X_2 = T_x Y$ for $\forall x \in X_1 \cap X_2$. It's just inclusion of one submanifold transverse to another submanifold.

Proposition 1.6:

Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold such that $\dim X + \dim W < \dim Y$. Then $f \pitchfork W \Leftrightarrow f(X) \cap W = \emptyset$.

Proof. The proof is easy. □

Theorem 1.2:

Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold such that $f \pitchfork W$. If $f^{-1}(W) \neq \emptyset$, then $f^{-1}(W)$ is a submanifold of X of codim $f^{-1}(W) = \text{codim } W$. In particular, if $\dim X = \text{codim } W$, then $f^{-1}(W)$ consists of isolated points.

Proof. Let $p \in f^{-1}(W)$, $n = \dim X, m = \dim Y, k = \dim W$. Let $\varphi : U \rightarrow \mathbb{R}^m$ be a chart around $f(p)$ such that $\varphi(U \cap W) \subseteq \mathbb{R}^k \times \{0\}$. Let $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^{m-k}$ be the orthogonal projection along $\mathbb{R}^k \times \{0\}$, $\phi = \pi \circ \varphi$. Then $\phi : U \rightarrow \mathbb{R}^{m-k}$ is a submersion and $\phi^{-1}(0) = U \cap W$.

[Claim.] $f \pitchfork W$ at $p \Leftrightarrow p$ is a regular point of $\phi \circ f$.

Since $\phi^{-1}(0) = U \cap W$, $\ker d\phi_{f(p)} = T_{f(p)} W$. Transversality assumption gives that $df_p(T_p X) + T_{f(p)} W = T_{f(p)} Y = T_{f(p)} U$, which implies that $d(\phi \circ f)_p(T_p X) = d\phi_{f(p)} T_{f(p)} U$. And the converse is easy to proof.

Now $f \pitchfork W$ on $U \Leftrightarrow 0$ is a regular value of $\phi \circ f : f^{-1}(U) \rightarrow \mathbb{R}^{m-k}$. By the implicit function theorem, $(\phi \circ f)^{-1}(0) = f^{-1}(U \cap W)$ is a submanifold of $f^{-1}(U) \subseteq X$ open set of codimension $m - k$. So $f^{-1}(W)$ is a submanifold of X of codimension $m - k$. □

Proposition 1.7:

Let X, Y be manifolds, $W \subseteq Y$ submanifold which is a closed subset. Then $T_W := \{f \in C^\infty(X, Y) \mid f \pitchfork W\}$ is open in the C^∞ -topology.

Proof. We show that T_W is open in the C^1 -topology. Let

$$U = \{\sigma = j^1 f(x) \in J^1(X, Y) \mid f(x) \notin W \text{ or } df_x(T_x X) + T_{f(x)} W = T_{f(x)} Y\}$$

It's easy to see that $T_W = M(U) = \{f \in C^\infty(X, Y) \mid j^1 f(X) \subseteq U\}$

[Claim.] U is open.

We will show that $V = J^1(X, Y) \setminus U$ is closed. To prove that, take $\{\sigma_n\} \subseteq V$ such that $\sigma_n \rightarrow \sigma \in J^1(X, Y)$, we need to show that $\sigma \in V$. Consider continuous map $\beta : J^1(X, Y) \rightarrow Y$, then $\beta(\sigma_n) \rightarrow \beta(\sigma)$. Since $\beta(\sigma_n) \in W$ and W is closed, we have $\beta(\sigma) \in W$, which mean that $\sigma = j^1 f(x), f(x) \in W$.

Now choose charts around x and $f(x)$, $\varphi : \tilde{U} \rightarrow \mathbb{R}^n$, $\psi : \tilde{V} \rightarrow \mathbb{R}^m$, $\psi(\tilde{V} \cap W) = \mathbb{R}^k \times \{0\}$, $\varphi(x) = 0, \psi(f(x)) = 0$. $f \pitchfork W$ at $x \Leftrightarrow \psi \circ f \circ \varphi^{-1} \pitchfork \mathbb{R}^k \times \{0\}$ at $0 \Leftrightarrow 0$ is a regular value of $\pi \circ \psi \circ f \circ \varphi^{-1}$ where $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^{m-k}$ orthogonal projection $\Leftrightarrow \pi \circ d(\psi \circ f \circ \varphi^{-1})_0$ has rank $m - k$.

Let $F = \{A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}) \mid \text{rank } A < m - k\}$. In a neighborhood \mathcal{N} of σ , fixing φ, ψ we obtain a map

$$\eta : \mathcal{N} \subseteq J^1(X, Y) \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}), \quad j^1 g(x) \mapsto \pi \circ d(\psi \circ f \circ \varphi^{-1})_{\varphi(x)}$$

$V \cap \mathcal{N} = \eta^{-1}(F)$, η is continuous.

Exercise. F is closed. □

Remark. The condition that W is closed is necessary.

Lemma 1.3:

Let X, Y, B manifolds, $W \subseteq Y$ submanifold, let $j : B \rightarrow C^\infty(X, Y)$ map (not necessary continuous).

$$\Phi : X \times B \rightarrow Y, \quad \Phi(x, b) = j(b)(x)$$

Suppose Φ is smooth and $\Phi \pitchfork W$. Then $\{b \in B \mid j(b) \pitchfork W\}$ is dense in B .

Proof. Let $W_\Phi = \Phi^{-1}(W) \subseteq X \times B$ be the submanifold, $pr : X \times B \rightarrow B$ the projection and $\pi = pr|_{W_\Phi}$.

Claim. b is a regular value of $\pi \implies j(b) \pitchfork W$.

Suppose b is a regular value of π .

(1) $b \notin \text{im } \pi$, then $\nexists x \in X$ such that $\Phi(x, b) \in W$, so $j(b)(X) \cap W = \emptyset$, which means $j(b) \pitchfork W$.

(2) If $\dim W_\Phi < \dim B$, then b is a regular value of π , so $b \notin \text{im } \pi$, therefore by (1) we have $j(b) \pitchfork W$.

(3) If $\dim W_\Phi \geq \dim B$. Let b be a regular value of π and $x \in X$. If $(x, b) \notin W_\Phi$, then $j(b)(x) \notin W$, so $j(b) \pitchfork W$ at x . If $(x, b) \in W_\Phi$, then $\pi(T_{(x,b)}W_\Phi) = T_b B$, which gives $T_{(x,b)}(X \times B) = T_{(x,b)}W_\Phi + T_{(x,b)}(X \times \{b\})$, so $T_{j(b)(x)}Y = T_{j(b)(x)}W + (dj(b))_x T_x X$, so $j(b) \pitchfork W$ at x . □

Corollary 1.1:

Let $G : X \times B \rightarrow Y$ smooth, $\Phi(x, b) = j^k G_b(x)$. If $\Phi \pitchfork W$, where $W \subseteq J^k(X, Y)$ submanifold. Then $\{b \in B \mid j^k G_b \pitchfork W\}$ is dense in B .

Theorem 1.3: Thom Transversality Theorem

Let X, Y manifolds, $W \subseteq J^k(X, Y)$ submanifold, let

$$T_W = \{f \in C^\infty(X, Y) \mid j^k f \pitchfork W\}$$

Then T_W is a residual subset of $C^\infty(X, Y)$ (residual subset means countable intersection of open and dense sets). Moreover if W is closed, then T_W is open.

Proof. For each $\sigma \in W$, let $W_\sigma \subseteq W$, $U_\sigma \subseteq X$, $V_\sigma \subseteq Y$ open neighborhood of σ , $\alpha(\sigma)$, $\beta(\sigma)$ respectively and charts $\varphi_\sigma : U_\sigma \rightarrow \mathbb{R}^n$, $\psi_\sigma : V_\sigma \rightarrow \mathbb{R}^m$ such that:

- (a) $\overline{W_\sigma} \subseteq W$ and is compact.
- (b) $\overline{U_\sigma}$ is compact.
- (c) $\alpha(\overline{W_\sigma}) \subseteq U_\sigma$ and $\beta(\overline{W_\sigma}) \subseteq V_\sigma$.
- (d) $\psi_\sigma(V_\sigma) = \mathbb{R}^m$.

We say that $g \pitchfork W$ on A if $g \pitchfork W$ for $\forall x \in g^{-1}(A)$. Let

$$T_\sigma = \{f \in C^\infty(X, Y) \mid j^k f \pitchfork W \text{ on } \overline{W_\sigma}\}$$

$T_W = \bigcap_{\sigma \in W} T_\sigma$. Since W is 2-countable, there exists a countable covering $\{W_{\sigma_i}\}_{i=1}^\infty$ of W . So $T_W = \bigcap_{i=1}^\infty T_{\sigma_i}$.

Claim. T_σ is open and dense.

(1) T_σ is open. Let $\tilde{T}_\sigma = \{g \in C^\infty(X, J^k(X, Y)) \mid g \pitchfork W \text{ on } \overline{W_j}\}$. By previous proposition we have \tilde{T}_σ is open, then $T_\sigma = (j^k)^{-1}(\tilde{T}_\sigma)$ is open.

(2) T_σ is dense. Let $f \in C^\infty(X, Y)$, we will construct a sequence $\{g_n\} \subseteq C^\infty(X, Y)$ such that $g_n \in T_\sigma$ and $g_n \xrightarrow{C^\infty} f$. The idea is to define $\Phi : X \times B \rightarrow J^k(X, Y)$, $\Phi(x, b) = j^k g_b(x)$, where $g_b(x)$ is a polynomial perturbation of f , such that $\Phi \pitchfork W$.

Fix smooth functions $\rho_1 : \mathbb{R}^n \rightarrow [0, 1]$, $\rho_2 : \mathbb{R}^m \rightarrow [0, 1]$ such that $\rho_1 \equiv 1$ in a neighborhood of $\varphi(\alpha(\overline{W_\sigma}))$, $\text{supp } \rho_1 \subseteq \varphi(U_\sigma)$; $\rho_2 \equiv 1$ in a neighborhood of $\psi(\beta(\overline{W_\sigma}))$, $\text{supp } \rho_2$ is compact. Let $B = \{\text{polynomial maps } \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ of degree } \leq k\}$.

For $b \in B$, let

$$g_b(x) = \begin{cases} \psi^{-1}(\psi(f(x)) + b(\varphi(x))\rho_1(\varphi(x))\rho_2(\psi(f(x)))) & \text{if } x \in U_\sigma, f(x) \in V_\sigma \\ f(x) & \text{if } x \notin U_\sigma \text{ or } f(x) \notin V_\sigma \end{cases}$$

$$G : X \times B \rightarrow Y, G(x, b) = g_b(x).$$

Exercise. G is smooth.

Let $\Phi : X \times B \rightarrow J^k(X, Y)$, $\Phi(x, b) = j^k g_b(x)$, so Φ is smooth.

Claim. $\exists \tilde{B} \subseteq B$ open neighborhood of $0 \in B$ such that $\Phi|_{X \times \tilde{B}} \pitchfork W$ on $\overline{W_\sigma}$.

Assuming the claim, apply the previous lemma, $\exists \{b_n\} \subseteq \tilde{B}$ such that $b_n \rightarrow 0$ and $j^k g_{b_n} \pitchfork (W \cap \overline{W_\sigma})$, this also implies $g_{b_n} \xrightarrow{C^\infty} f$ and $j^k g_{b_n} \pitchfork W$ on $\overline{W_\sigma}$. So T_σ is dense.

Proof of the claim: Let $\epsilon = \frac{1}{2}d(\psi(\beta(\overline{W_j})), \rho_2^{-1}([0, 1])) > 0$, define

$$\tilde{B} = \{b \in B \mid \|b(x)\| < \epsilon, \forall x \in \text{supp } \rho_1\}$$

We fix $b \in \tilde{B}$ such that $\Phi(x, b) \in \overline{W_\sigma}$. We will show that Φ is a local diffeomorphism near (x, b) . Since $\Phi(x, b) \in \overline{W_\sigma}$, $x \in \alpha(\overline{W_\sigma})$, $g_b(x) \in \beta(\overline{W_\sigma})$. $\psi(g_b(x)) = \psi(f(x)) + b(\varphi(x))\rho_1(\varphi(x))\rho_2(\psi(f(x))) = \psi(f(x)) + b(\varphi(x))$. Because $\|b(\varphi(x))\| < \epsilon$, $\forall x \in \text{supp } \rho_1$, then $\rho_2(\psi(g_b(x))) = 1$. So $\psi \circ g_b(x) = \psi(f(x)) + b(\varphi(x))$ in a neighborhood of (x, b) . σ' is sufficiently close to σ , so we can find a unique polynomial b' so that $\sigma' = j^k(\psi^{-1}(f(\varphi(\alpha(\sigma'))))) + b'(\varphi(\alpha(\sigma')))$. So we have constructed a local inverse for every $(x, b) \in \Phi^{-1}(\overline{W_\sigma})$, then $\Phi \pitchfork W$ on $\overline{W_\sigma}$. \square

Corollary 1.2:

Let X, Y manifolds, $f \in C^\infty(X, Y)$, $W \subseteq J^k(X, Y)$ submanifold such that $\alpha(\overline{W}) \subseteq U$ open set. Then $\exists \{g_n\} \subseteq C^\infty(X, Y)$ such that $j^k g_n \pitchfork W$, $g_n \rightarrow f$ and $g_n = f$ outside U .

Proof. The same as the theorem above but we choose $U_\sigma \subseteq U$ for $\forall \sigma \in W$. \square

Corollary 1.3: Elementary Transversality Theorem

Let X, Y manifolds, $W \subseteq Y$ submanifold.

- (a) $T_W = \{f \in C^\infty(X, Y) \mid f \pitchfork W\}$ is dense in $C^\infty(X, Y)$. Moreover if W is closed, then T_W is open.
- (b) Let $U_1, U_2 \subseteq X$ open sets such that $\overline{U_1} \subseteq U_2$, let $f \in C^\infty(X, Y)$, $V \subseteq C^\infty(X, Y)$ near f and open. Then there is $\{g_n\} \in C^\infty(X, Y)$ such that $g_n \xrightarrow{C^\infty} f$, $g_n = f$ in U_1 and $g_n \pitchfork W$ outside U_2 .

Multijets

Definition 1.10:

Let X, Y manifolds. For $s \in \mathbb{N}$, define

$$X^{(s)} = \{(x_1, \dots, x_s) \in X^s \mid x_i \neq x_j, i \neq j\}$$

$\alpha^s = \alpha \times \dots \times \alpha : J^k(X, Y)^s \rightarrow X^s$, let $J_s^k(X, Y) = (\alpha^s)^{-1}(X^{(s)}) \subseteq J^k(X, Y)^s$ open, so $J_s^k(X, Y)$ is a manifold. $f \in C^\infty(X, Y)$ gives rise to

$$j_s^k f : X^{(s)} \rightarrow J_s^k(X, Y), j_s^k f(x_1, \dots, x_s) = (j^k f(x_1), \dots, j^k f(x_s))$$

Theorem 1.4: Thom Transversality for multijets

Let X, Y manifolds, $W \subseteq J_s^k(X, Y)$ submanifold. Let

$$T_W = \{f \in C^\infty(X, Y) \mid j_s^k f \pitchfork W\}$$

Then T_W is residual. Moreover, if W is compact, then T_W is open.

1.4 Whitney Immersions and Embeddings

Let X^n, Y^m manifolds, $\sigma = j^1 f(x) \in J^1(X, Y)$. Then $df_x : T_x X \rightarrow T_{f(x)} Y$ depends only on σ . Define $\text{rank}(\sigma) = \text{rank}(df_x)$ and $\text{corank}(\sigma) = \min(m, n) - \text{rank}(\sigma)$. Let $S_r = \{\sigma \in J^1(X, Y) \mid \text{corank}(\sigma) = r\}$.

Lemma 1.4:

f is an immersion ($n \leq m$) or submersion ($n \geq m$) $\Leftrightarrow j^1 f(X) \cap \bigcup_{r \geq 1} S_r = \emptyset$.

Proof. f is not an immersion/submersion $\Leftrightarrow \exists x \in X$ such that $\text{rank}(df_x) \leq \min(m, n) - 1$
 $\Leftrightarrow \exists x \in X$ such that $\text{corank}(j^1 f(x)) \geq 1 \Leftrightarrow j^1 f(X) \cap S_r \neq \emptyset$ for some $r \geq 1$. \square

Proposition 1.8:

S_r is a submanifold of codimension $(n - q + r)(m - q + r)$, where $q = \min(n, m)$.

Proof. S_r is a bundle over $X \times Y$ with fiber $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) = \{A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \mid \text{corank}(A) = r\}$.

Claim. $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of codimension $(n - q + r)(m - q + r)$.
 So $S_r \subseteq J^1(X, Y)$ is a subbundle over $X \times Y$.

Proof of the claim: Let $M \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m)$, let $k = q - r$. We can choose basis of \mathbb{R}^n and \mathbb{R}^m so that

$$[M] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A \text{ is an invertible } k \times k \text{ matrix}$$

So in a neighborhood U of M , every other M' will be represented as

$$[M'] = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \quad A' \text{ is an invertible } k \times k \text{ matrix}$$

$$\text{So rank } [M'] = \text{rank} \begin{bmatrix} I^k & 0 \\ -C'(A')^{-1} & I_{m-k} \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \text{rank} \begin{bmatrix} A' & B' \\ 0 & D' - C'(A')^{-1}B' \end{bmatrix}$$

Then $\text{rank } [M'] = k \Leftrightarrow D' - C'(A')^{-1}B' = 0$. $M' \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U \Leftrightarrow D' - C'(A')^{-1}B' = 0$. Let

$$\varphi : U \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \rightarrow \mathcal{L}(\mathbb{R}^{n-k}, \mathbb{R}^{m-k}), \quad \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \mapsto D' - C'(A')^{-1}B'$$

φ is a submersion, so $\varphi^{-1}(0) = \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U$ is a submanifold of codimension $(n - q + r)(m - q + r)$. \square

Obs. $\mathcal{L}^0(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is open. So $S_0 \subseteq J^1(X, Y)$ open submanifold, then $\bigcup_{r \geq 1} S_r$ is closed.

Lemma 1.5:

Suppose $n \leq m$. Then $\text{Imm}(X, Y) = \{f : X \rightarrow Y \text{ smooth immersion}\}$ is an open subset of $C^\infty(X, Y)$.

Proof. $\text{Imm}(X, Y) = M(S_0)$. □

Theorem 1.5: Whitney Immersion

Let X^n, Y^m be manifolds such that $m \geq 2n$. Then $\text{Imm}(X, Y)$ is open and dense subset of $C^\infty(X, Y)$.

Proof. $\min(n, m) = n$, so for $r \geq 1$, $\text{codim } S_r = (n - q + r)(m - q + r) = r(n + r) \geq n + 1$. So $j^1 f \pitchfork S_r \Leftrightarrow j^1 f(X) \cap S_r = \emptyset$ since $\dim X = n < n + 1 \leq \text{codim } S_r$.

$$\text{Imm}(X, Y) = \left\{ f \in C^\infty(X, Y) \mid j^1 f(X) \cap \bigcup_{r \geq 1} S_r = \emptyset \right\} = \left\{ f \in C^\infty(X, Y) \mid j^1 f \pitchfork \bigcup_{r \geq 1} S_r \right\}$$

By the Thom transversality theorem, $\text{Imm}(X, Y)$ is dense and open. □

Theorem 1.6: Whitney Injective Immersion Theorem

Let X^n, Y^m be manifolds such that $m \geq 2n + 1$. Then the set of injective immersions is residual.

Proof. $\text{Imm}(X, Y)$ is open and dense, we need to show

$$\text{Inj}(X, Y) = \{f \in C^\infty(X, Y) \mid f \text{ is injective}\} \text{ is residual}$$

Recall $J_2^0(X, Y) = X^{(2)} \times Y^2 = \{(x_1, x_2, y_1, y_2) \in X^2 \times Y^2 \mid x_1 \neq x_2\}$, let

$$W = X^{(2)} \times \Delta Y = \{(x_1, x_2, y, y) \mid x_1 \neq x_2\} \subseteq J_2^0(X, Y)$$

f is injective iff $j_2^0 f(X^{(2)}) \cap W = \emptyset$. Codimension of W is dimension of Y , so f is injective iff $j_2^0 f \pitchfork W$ from the proof of previous theorem. By the Thom transversality theorem for multijets, we have $\text{Inj}(X, Y)$ is residual. □

Lemma 1.6:

Let X manifold. Then $\text{Prop}(X, \mathbb{R}^m) = \{f \in C^\infty(X, \mathbb{R}^m) \mid f \text{ is proper}\}$ is non-empty and open.

Proof. Recall that there exists a proper map $X \rightarrow \mathbb{R}$, compose this map with a linear injection $\mathbb{R} \rightarrow \mathbb{R}^m$ to obtain a proper map.

Now let $f \in \text{Prop}(X, \mathbb{R}^m)$. For $x \in X$, define $V_x = \{y \in \mathbb{R}^m \mid d(y, f(x)) < 1\}$. So $V_x \subseteq \mathbb{R}^m$ open. Let $V = \bigcup_{x \in X} \{x\} \times V_x$, then $V \subseteq X \times \mathbb{R}^m = J^0(X, \mathbb{R}^m)$ is open. $f \in M(V)$ because $j^0 f(x) = (x, f(x))$, $d(f(x), f(x)) = 0$, so $f(x) \in V_x$.

Claim. $M(V) \subseteq \text{Prop}(X, \mathbb{R}^m)$.

If $g \in M(V)$, then $d(g(x), f(x)) < 1 \forall x \in X$, so $g^{-1}(\overline{B}_r(0)) \subseteq f^{-1}(\overline{B}_{r+1}(0))$. Since f is proper, $f^{-1}(\overline{B}_{r+1}(0))$ is compact, therefore $g^{-1}(\overline{B}_r(0))$ is compact, hence g is proper. \square

Corollary 1.4: Whitney Embedding Theorem

Let X^n manifold. Then there exists $X \hookrightarrow \mathbb{R}^{2n+1}$.

Proof. $\text{Inj}(X, \mathbb{R}^{2n+1}) \cap \text{Imm}(X, \mathbb{R}^{2n+1}) \cap \text{Prop}(X, \mathbb{R}^{2n+1}) \neq \emptyset$. \square

1.5 Morse Functions

Definition 1.11:

Let $f : X \rightarrow \mathbb{R}$ smooth and $p \in \text{Crit}(f)$ ($df_p = 0$). Define the Hessian of f to be the bilinear map:

$$D^2f_p : T_pX \times T_pX \rightarrow \mathbb{R}, \quad D^2f_p \left(\left. \frac{\partial}{\partial x_i} \right|_p, \left. \frac{\partial}{\partial x_j} \right|_p \right) = \left. \frac{\partial^2}{\partial x_i \partial x_j} \right|_{\varphi(p)} (f \circ \varphi^{-1})$$

where $\varphi = (x_1, \dots, x_n)$ is a chart around p . We say that p is non-degenerate if D^2f_p is non-degenerate.

Exercise. D^2f_p doesn't depend on the choice of a chart whenever $p \in \text{Crit}(f)$.

Let $f : X \rightarrow \mathbb{R}$ smooth, $df : TX \rightarrow \mathbb{R}$, $(p, v) \in TX$, we have $d_{(p,v)}df : T_{(p,v)}TX \rightarrow \mathbb{R}$, $T_{(p,v)}TX$ is isomorphic to $T_pX \oplus T_pX$ but it's not natural.

Proposition 1.9:

$p \in \text{Crit}(f)$ is non-degenerate $\Leftrightarrow j^1f \pitchfork S_1$ at p .

Proof. This is a local question, we may assume $X = U \subseteq \mathbb{R}^n$, $J^1(X, \mathbb{R}) = U \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $\pi : J^1(X, \mathbb{R}) \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ submersion, $\pi^{-1}(0) = S_1 = \{j^1f(x) \mid df_x = 0\}$.

Claim. $j^1f \pitchfork S_1$ at $p \Leftrightarrow \pi \circ j^1f$ is a submersion at p .

Now $\pi \circ j^1f : U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $x \mapsto \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$ is a submersion at p iff $\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_p$ is invertible iff p is non-degenerate. \square

Definition 1.12:

$f \in C^\infty(X, \mathbb{R})$ is Morse if every $p \in \text{Crit}(f)$ is non-degenerate.

Corollary 1.5:

$f \in C^\infty(X, \mathbb{R})$ is Morse $\Leftrightarrow j^1f \pitchfork S_1$.

Theorem 1.7:

Let X manifold. Then $\{f \in C^\infty(X, \mathbb{R}) \mid f \text{ is Morse}\}$ is open and dense in $C^\infty(X, \mathbb{R})$.

Proof. Since $S_1 = J^1(X, \mathbb{R}) \setminus S_0$ is closed, by the corollary and Thom transversality theorem we complete the proof. \square

2 Intersection Theory

2.1 Manifolds with boundary and orientation

Definition 2.1:

A topological manifold with boundary is a 2-countable Hausdorff topological space such that every point $p \in X$ has a neighborhood which is homeomorphic to an open set in $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$.

Lemma 2.1:

Let X be a topological manifold with boundary, $p \in X$, $\varphi, \psi : U \rightarrow \mathbb{H}^n$ charts around p . Suppose $pr_1 \circ \varphi(p) = 0$, then $pr_1 \circ \psi(p) = 0$, where pr_1 is the canonical projection of \mathbb{H}^n to the first coordinate.

Proof. $\psi \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(U)$ is homeomorphic, then $\psi \circ \varphi^{-1} : \varphi(U) \setminus \varphi(p) \rightarrow \psi(U) \setminus \psi(p)$ is also homeomorphic. Since $pr_1 \circ \varphi(p) = 0$, $\varphi(U) \setminus \varphi(p)$ is contractible. If $pr_1 \circ \psi(p) \neq 0$, then $\psi(U) \setminus \psi(p) \simeq S^{n-1}$, S^{n-1} and contractible space have different homology group, so they can't be homeomorphic. \square

Definition 2.2:

Let X be a topological manifold with boundary. Then

$$\partial X = \{p \in X \mid \exists \varphi : U \rightarrow \mathbb{H}^n \text{ chart around } p \text{ s.t. } pr_1 \circ \varphi(p) = 0\}, \text{ int}(X) = X \setminus \partial X$$

Obs. $\text{int}(X)$ and ∂X are topological manifold without boundary of dimension n and $n - 1$ respectively.

Definition 2.3:

A map $f : \mathbb{H}^n \supseteq U \rightarrow \mathbb{H}^n$ is smooth if it admits a smooth extension to $\tilde{f} : \tilde{U} \rightarrow \mathbb{R}^n$, where $U \subseteq \tilde{U}$ in an open set in \mathbb{R}^n .

Definition 2.4:

We say that two charts $\varphi : U \rightarrow \mathbb{H}^n$, $\psi : V \rightarrow \mathbb{H}^n$ are compatible if $\psi \circ \varphi^{-1}$ is smooth. An atlas is a collection of charts whose domain cover X .

Definition 2.5:

A (smooth) manifold with boundary is a topological manifold with boundary endowed with a maximal (smooth) atlas.

Smooth manifold with boundary X induces smooth structure (without boundary) on $\text{int}(X)$ and ∂X .

Proposition 2.1:

Let $f \in C^\infty(X, \mathbb{R})$, $a \in \mathbb{R}$ regular value of f . Then $f^{-1}([a, +\infty))$ and $f^{-1}((-\infty, a])$ are manifolds with boundary.

Proof. $(a, +\infty) \subseteq \mathbb{R}$ is open then $f^{-1}((a, +\infty))$ is a manifold without boundary. Let $p \in f^{-1}(a)$, by the implicit function theorem, there exists a chart $\varphi : U \rightarrow \mathbb{R}^n$ such that $\varphi(p) = 0$ and $f \circ \varphi^{-1}(x_1, \dots, x_n) = a + x_1$. So we obtain a chart $\varphi|_{f^{-1}([a, +\infty)) \cap U} : \tilde{U} \rightarrow \mathbb{H}^n$. So $f^{-1}([a, +\infty))$ is a manifold with boundary. \square

Definition 2.6:

Let X be a manifold with boundary, $p \in X$, a curve centered at p is a smooth map $\gamma : [0, \epsilon) \rightarrow X$ or $\gamma : (-\epsilon, 0] \rightarrow X$ such that $\gamma(0) = p$. $T_p X$ is the equivalent classes of curves centered at p .

If $x \in \text{int}(X)$, then $T_x(\text{int}(X)) = T_x X$; If $x \in \partial X$, then $T_x X$ is still a n -dimensional vector space. Moreover, we have a canonical inclusion $T_x(\partial X) \subseteq T_x X$.

Proposition 2.2:

Let X, Y be manifolds with boundary, $y \in \text{int}(Y)$ regular value of $f : X \rightarrow Y$ and $\partial f := f|_{\partial X} : \partial X \rightarrow \partial Y$. Then $f^{-1}(y)$ is a manifold with boundary and $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X = (\partial f)^{-1}(y)$.

Example 2.1:

$f : \mathbb{H}^2 \rightarrow \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$, then $f^{-1}(1) = S^1 \cap \mathbb{H}^2$.

Exercise. Prove the proposition.

Theorem 2.1:

Let X, Y manifolds with boundary, $W \subseteq Y$ submanifold, $\partial W = \partial Y = \emptyset$. Suppose $f \pitchfork W$ and $\partial f \pitchfork W$, then $f^{-1}(W)$ is a manifold with boundary, $\partial(f^{-1}(W)) = f^{-1}(W) \cap \partial X$.

Proof. $f|_{\text{int}(X)} \pitchfork W$ is a manifold without boundary. Let $x \in f^{-1}(W) \cap \partial X$, $\pi : V \subseteq Y \rightarrow \mathbb{R}^{m-k}$ be a submersion such that $\pi^{-1}(0) = W \cap V$. As in the case without boundary: $f \pitchfork W$ at x iff x is a regular point of $\pi \circ f$, $\partial f \pitchfork W$ at x iff x is a regular point of $\pi \circ \partial f$. The result follows from the proposition above. \square

Obs. It's easy to see that $\partial f \pitchfork W$ at $x \implies f \pitchfork W$ at x .

Theorem 2.2: Sard's Theorem

Let X manifold with boundary, Y manifold, $f : X \rightarrow Y$. Then

$$\{y \in Y \mid y \text{ is a critical value of } f \text{ or } \partial f\}$$

has measure zero.

Proof. $\text{Crit}(f) \cup \text{Crit}(\partial f) = \text{Crit}(f|_{\text{int}(X)}) \cup \text{Crit}(\partial f)$. □

Theorem 2.3: Thom Transversality Theorem

X manifold with boundary, Y manifold, $W \subseteq J^k(X, Y)$ submanifold, $\partial W \subseteq \alpha^{-1}(\partial X)$. Then

$$\{f \in C^\infty(X, Y) \mid j^k f \pitchfork W \text{ and } j^k(\partial f) \pitchfork W\}$$

is residual.

Corollary 2.1: Elementary Transversality Theorem

(1) X manifold with boundary, Y manifold and $W \subseteq Y$ submanifold $\partial W = \emptyset$. Then

$$\{f \in C^\infty(X, Y) \mid f \pitchfork W \text{ and } \partial f \pitchfork W\}$$

is residual.

(2) $f \in C^\infty(X, Y)$, $\partial f \pitchfork W$. There exists $\{g_n\} \subseteq C^\infty(X, Y)$ such that $g_n \xrightarrow{C^\infty} f$, $g_n \pitchfork W$ and $g_n \equiv f$ in a neighborhood of ∂X .

Definition 2.7:

Let V be a vector space. Define an equivalence relation on the set of bases of V as follows:

$$\{x_1, \dots, x_n\} \sim \{y_1, \dots, y_n\} \text{ if the linear map } T : V \rightarrow V, Tx_i = y_i \text{ has } \det T > 0$$

Obs. Given V , there are two equivalence classes.

Definition 2.8:

An orientation of V is a choice of such an equivalence class.

Definition 2.9:

Let X be a smooth manifold. An orientation on X is a choice of orientation on $T_p X$ for each $p \in X$ such that for each chart $\varphi : U \rightarrow \mathbb{R}^n$, $\varphi = (x_1, \dots, x_n)$, either

$$\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\} \text{ or } \left\{ -\frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\} \text{ is oriented for } \forall p \in U$$

Obs. Not all manifold admits an orientation.

Rmk. A connected orientable manifold has exactly two orientations. \mathbb{R}^n has a natural orientation.

Proposition 2.3:

Let X be an oriented manifold with boundary. Then ∂X has a natural orientation.

Proof. For $x \in \partial X$, $T_x(\partial X) \subseteq T_x X$. There exists a 1-dimensional vector bundle N over ∂X such that $N_x \oplus T_x(\partial X) = T_x X$ and a outward normal vector field $n \in \Gamma(N)$ which doesn't vanishes. If $\{v_1, \dots, v_{n-1}\}$ is a basis of $T_x(\partial X)$, then $\{n_x, v_1, \dots, v_{n-1}\}$ is a basis of $T_x X$. \square

Obs. X, Y oriented manifolds, $\partial Y = \emptyset$, then $X \times Y$ inherits a natural orientation.

Example 2.2:

\mathbb{R}

Proposition 2.4:

Let X^n, Y^m manifolds with boundary, $W^k \subseteq Y$ submanifold such that $\partial W = \partial Y = \emptyset$, let $f \in C^\infty(X, Y)$ such that $f \pitchfork W$ and $\partial f \pitchfork W$. Suppose X, Y, W oriented. Then $f^{-1}(W)$ has a natural orientation.

Proof. Let $Q = f^{-1}(W)$, NQ be the normal bundle of Q . \square