BASICS OF DIFFERENTIAL GEOMETRY

Principal Bundles and Characteristic Classes

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Contents

Introduction

Last semester:

- Geometry of vector bundles
- Basic Riemannian geometry
- Differential operators on manifolds

We will learn this semester:

- Theory of principal bundles
- Characteristic classes

1 Principal Bundles

In this section, we introduce the connections of principal bundles, which is closely related to the connections of vector bundles and simpler in some sense.

1.1 Lie Groups

Definition 1.1. Let G be a smooth manifold. G is a **Lie group** if G is a group and multiplication, inverse are smooth.

Let G be a Lie group, $g \in G$, we denote:

- $L_g : G \to G$, $h \mapsto gh$ (called left translation)
- $R_g : G \to G$, $h \mapsto hg$ (called right translation)
- $\mathfrak{X}^L(G) = \{X \in \mathfrak{X}(G) \mid \forall g \in G, (L_g)_* X = X\}$ (left invariant vector fields)

For $X \in \mathfrak{X}^L(G)$, $L_{g*}X = X$ means that X is L_g-related to X. Then for $\forall X, Y \in \mathfrak{X}^L(G)$, $L_{g*}([X,Y]) = [L_{g*}X, L_{g*}Y] = [X,Y]$, so $\mathfrak{X}^L(G)$ is closed under $[\cdot,\cdot]$

Definition 1.2. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a $\mathbb{K}$-vector space $\mathfrak{g}$ and a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we say $\mathfrak{g}$ is a **Lie algebra** if:
(1) $\forall X, Y \in \mathfrak{g}$, $[X,Y] = -[Y,X]$.
(2) $\forall X, Y, Z \in \mathfrak{g}$, $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi's identity).
And $[\cdot,\cdot]$ is called Lie bracket.

So by definition we have $\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)$ is a Lie algebra.

Definition 1.3. For Lie algebra $\mathfrak{g}, \mathfrak{h}$, a linear map $f : \mathfrak{g} \to \mathfrak{h}$ is called the **Lie algebra homomorphism** if: $\forall X, Y \in \mathfrak{g}$, $f([X,Y]) = [f(X), f(Y)]$
If f is in addition an isomorphism, then f is called a **Lie algebra isomorphism**.

Let $e \in G$ be the unit of G. Set $\iota : \mathfrak{X}^L(G) \to T_eG$, $X \mapsto X_e$. Then ι is a natural linear isomorphism. Let $\mathfrak{g} = T_eG$, so we can define the Lie bracket on $\mathfrak{g}$ s.t. ι is a Lie algebra isomorphism, i.e. setting $X^\sharp = \iota^{-1}(X)$, then $[X,Y] = \left[X^\sharp, Y^\sharp\right]_e$. Note that $X^\sharp_g = (L_g)_{*e}X$, $g \in G$.

Definition 1.4. Let G be Lie group, $\mathfrak{g} = T_eG$ with $[\cdot,\cdot]$ is called the **Lie algebra of** G. $\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)$ is also called the Lie algebra of G by the Lie algebra isomorphism ι.

Definition 1.5. Let G, H be Lie groups. A map $\rho : G \to H$ is a **Lie group homomorphism** if ρ is a smooth map and a group homomorphism. For the special case $(\mathbb{R}, +) \to G$, $t \mapsto g_t$, $\{g_t\}_{t\in\mathbb{R}}$ is called **one parameter subgroup of** G.

Proposition 1.1. Let G be Lie group and $\mathfrak{g}$ its Lie algebra. Then

(1) $\forall X \in \mathfrak{g}$, $X^\sharp = \iota^{-1}(X)$ is complete, i.e. $X^\sharp$ generates a flow $\{\varphi_t\}_{t\in\mathbb{R}}$.

(2) Set $\boxed{\exp_G(tX) = \varphi_t(e)} \in G$. Then $\varphi_t = R_{\exp_G(tX)}$.

(3) For $s, t \in \mathbb{R}$, $\exp_G(sX)\exp_G(tX) = \exp_G((s+t)X)$, i.e. $\{\exp_G(tX)\}_{t\in\mathbb{R}}$ is one parameter subgroup of G.

(4) $\mathfrak{g} \to \{\text{one parameter subgroup of } G\}$, $X \mapsto \{\exp_G(tX)\}_{t\in\mathbb{R}}$ is bijective.

Proof. (1) By ODE theory, $\exists \epsilon > 0$, $\gamma_e : (-\epsilon, \epsilon) \to G$ s.t. $\gamma_e(0) = e$, $\dfrac{d\gamma_e}{dt} = X^\sharp_{\gamma_e(t)}$.

$\boxed{\text{Claim 1.}}$ $\forall g \in G$, define $\gamma_g : (-\epsilon, \epsilon) \to G$, $t \mapsto g\gamma_e(t)$ is the integral curve of $X^\sharp$ with $\gamma_g(0) = g$.

Indeed, $\forall t \in (-\epsilon, \epsilon)$, $\dfrac{d\gamma_g}{dt}(t) = (L_g)_{*\gamma_e(t)}\dfrac{d\gamma_e}{dt}(t) = X^\sharp_{g\cdot\gamma_e(t)}$.

$\boxed{\text{Claim 2.}}$ $\gamma_e : (-\epsilon, \epsilon) \to G$ can be extended to integral curve $\gamma_e : \mathbb{R} \to G$ of $X^\sharp$ with $\gamma_e(0) = e$.

Set $\varphi_t = R_{\gamma_e(t)}$, then $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow generated by $X^\sharp$. So by uniqueness the following parts are easy. $\square$

By this proposition, we can define the exponential map $\exp_G : \mathfrak{g} \to G$.

Proposition 1.2. Let G, H be Lie groups with Lie algebra $\mathfrak{g}, \mathfrak{h}$. If $f : G \to H$ is Lie group homomorphism, then $f_{*e} : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. We only need to show that $X^\sharp$ and $(f_{*e}X)^\sharp$ are f-related. Since $X = \dfrac{d}{dt}\exp_G(tX)|_{t=0}$, we have $f_{*g}(X^\sharp_g) = \dfrac{d}{dt}f\left(g \cdot \exp_G(tX)\right)|_{t=0} = \dfrac{d}{dt}f(g)f\left(\exp_G(tX)\right)|_{t=0} = \left(L_{f(g)}\right)_{*e}(f_{*e}X) = (f_{*e}X)^\sharp_{f(g)}$. $\square$

Example 1.1. Let V be a $\mathbb{R}$-vector space, $G = GL(V)$, $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g} = End(V)$, the bracket is given as follows:

Proposition 1.3. $\forall X, Y \in End(V)$, $[X, Y] = XY - YX$.

Proof. For $X \in End(V)$, set matrix exponential $e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}$. Then $\{e_{tX}\}_{t\in\mathbb{R}}$ is a one parameter subgroup of G and $\frac{d}{dt}e^{tX}|_{t=0} = X$. So $\exp_G(tX) = e^{tX}$. Then

$[X, Y] = [X^\sharp, Y^\sharp]_e = \left(\mathcal{L}_{X^\sharp} Y^\sharp\right)_e = \frac{d}{dt}(\varphi_{-t})_{*e^{tX}} \left(Y^\sharp_{e^{tX}}\right)|_{t=0} = \frac{d}{dt}\frac{d}{ds}\varphi_{-t}\left(e^{tX}e^{sY}\right)|_{s,t=0} = XY - YX.$

□

Example 1.2. Set

- $O(n) = \{g \in GL(n; \mathbb{R}) \mid g^t g = E_n\}$ (orthogonal group)
- $SO(n) = \{g \in O(n) \mid \det g = 1\}$ (special orthogonal group)

we can check that $O(n), SO(n)$ are Lie subgroups of $GL(n; \mathbb{R})$.

$SO(n)$ is the unit component of $O(n)$, so $\mathfrak{o}(n) = \mathfrak{so}(n)$ (Lie algebra of $(O(n))$ and $SO(n)$). This is a Lie subalgebra of $End(\mathbb{R}^n)$ given by

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \left\{X \in End(\mathbb{R}^n) \mid X^t + X = O_n\right\}$$

where O_n is the zero matrix of size n.

Similarly, set

- $U(n) = \{g \in GL(n; \mathbb{C}) \mid g^* g = E_n\}$ (unitary group) where $g^* = \overline{g^t}$
- $SU(n) = \{g \in U(n) \mid \det g = 1\}$ (special unitary group)

We can check that

- $U(n), SU(n)$ are Lie subgroups of $GL(n; \mathbb{C})$
- $\mathfrak{u}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O\}$ (Lie algebra of $U(n)$)
- $\mathfrak{su}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O, \mathrm{tr}X = 0\}$ (Lie algebra of $SU(n)$)

$\boxed{\text{Note.}}$ A **Lie subgroup** H of G is a Lie group s.t.

- H is a subset of G
- inclusion map $H \hookrightarrow G$ is an embedding and group homomorphism

Fact: A closed subgroup of G is a Lie subgroup of G.

Definition 1.6. Let V be a $\mathbb{K}$-vector space, G be a Lie group. A Lie group homomorphism $\rho : G \to GL(V)$ is called a **representation of** V. The Lie algebra homomorphism $\rho_{*e} : \mathfrak{g} \to End(V)$ is called a **differential representation**.

Example 1.3. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. $\forall g \in G$, define a homomorphism

$$F_g : G \to G,\ h \mapsto ghg^{-1}$$

Note that $F_g \circ F_{g'} = F_{gg'}$. This induces a Lie algebra homomorphism $(F_g)_{*e} : \mathfrak{g} \to \mathfrak{g}$ which satisfies $(F_g)_{*e} \circ (F_{g'})_{*e} = (F_{gg'})_{*e}$. So we obtain a representation

$$Ad : G \to GL(\mathfrak{g}),\ g \mapsto (F_g)_{*e}$$

called **adjoint representation of** G. The differential representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ of Ad is given as follows.

Proposition 1.4. $\forall X, Y \in \mathfrak{g}$, $ad(X)(Y) = [X, Y]$.

Proof. Note that $F_g = R_{g^{-1}} \circ L_g$. Then

$$\begin{aligned} ad(X)(Y) &= \frac{d}{dt} Ad(\exp_G(tX))(Y)|_{t=0} \\ &= \frac{d}{dt} \left(R_{\exp_G(-tX)}\right)_{*\exp_G(tX)} \left(L_{\exp_G(tX)}\right)_{*e} (Y)|_{t=0} \\ &= [X^\sharp, Y^\sharp]_e = [X, Y] \end{aligned}$$

□

Recall that there is a exponential map in Riemannian geometry. The Riemannian exp and the Lie group exp are related as follows.

Definition 1.7. A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group G is said to be **bi-invariant** if $\forall g, h \in G$, $L_g^* R_h^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$.

Theorem 1.1. Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$. Then $\exp_e = \exp_G$.

To show this we describe the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$.

Lemma 1.1. $\forall X, Y \in \mathfrak{g}$, $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

Proof. By Koszul formula, we have

$$\begin{aligned} \langle \nabla_{X^\sharp} Y^\sharp, Z^\sharp \rangle = \frac{1}{2} \big(& X^\sharp \langle Y^\sharp, Z^\sharp \rangle + Y^\sharp \langle Z^\sharp, X^\sharp \rangle - Z^\sharp \langle X^\sharp, Y^\sharp \rangle \\ & - \langle Y^\sharp, [X^\sharp, Z^\sharp] \rangle - \langle Z^\sharp, [Y^\sharp, X^\sharp] \rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp] \rangle \big) \end{aligned}$$

Since for $\forall g \in G$, $X_g^\sharp = \dfrac{d}{dt} g \cdot \exp_G(tX) \,|_{t=0}$, we have

$$X^\sharp \langle Y^\sharp, Z^\sharp \rangle = \frac{d}{dt} \langle Y^\sharp_{g\cdot\exp_G(tX)}, Z^\sharp_{g\cdot\exp_G(tX)} \rangle_{g\cdot\exp_G(tX)} \,|_{t=0} = \frac{d}{dt} \langle Y, Z \rangle_e \,|_{t=0} = 0$$

Since $\langle \cdot, \cdot \rangle$ is bi-invariant,

$$L_g^* R_{g^{-1}}^* \langle \cdot, \cdot \rangle_e = \langle \cdot, \cdot \rangle_e \text{ for } \forall g \in G \iff \langle Ad(g)(\cdot), Ad(g)(\cdot) \rangle_e = \langle \cdot, \cdot \rangle_e$$

Setting $g = \exp_G(tZ)$ and $\frac{d}{dt} \,|_{t=0}$, we have $\langle ad(Z)(\cdot), \cdot \rangle_e + \langle \cdot, ad(Z)(\cdot) \rangle_e = 0$, which shows that $\langle Y^\sharp, [X^\sharp, Z^\sharp] \rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp] \rangle = 0$, so we have $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

□

The proof of the theorem completes once shown that $\exp_G(tX)$ is geodesic, which is left as an exercise.

Exercise 1.1. Prove the theorem.

Remark 1.1. Existence/uniqueness of bi-invariant metrics? Some facts from representation theory are needed, the argument here is not used after this remark.

$\boxed{\text{Existence}}$ When G is compact, $\exists$ bi-invariant metric using "averaging trick".

- We first define Ad-invariant inner product on $\mathfrak{g}$.
- Then extend it to the whole G by pulling back L_g.

$\boxed{\text{Note:}}$ $\exists$ bi-invariant on $G \iff \exists$ Ad-invariant inner product on $\mathfrak{g}$.

$$\begin{cases} (\Rightarrow) \text{ Trivial.} \\ (\Leftarrow) \text{ Given } Ad\text{-invariant inner product on } \mathfrak{g}, \text{ we can extend it to left-invariant metric} \\ \quad \text{on } G, \text{ this is also right-invariant by pullback of } R_h = R_h \circ L_{h^{-1}} \circ L_h = Ad(h^{-1}) \circ L_h \end{cases}$$

$\boxed{\text{Uniqueness}}$ When G is abelian, then $L_g = R_g$, so $\exists$ many bi-invariant metrics on G (Any inner product on $\mathfrak{g}$ induces left-invariant metric on $\mathfrak{g}$, by the note above it is bi-invariant). Suppose that $\exists$ Ad-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. By $\langle\cdot,\cdot\rangle$, we have an irreducible decomposition of $(\mathfrak{g}, Ad)$: $\mathfrak{g} = \mathfrak{g}_1^{\oplus n_1} \oplus \cdots \oplus \mathfrak{g}_r^{\oplus n_r}$, where $\mathfrak{g}_i$ is irreducible representation of G and $\mathfrak{g}_i \neq \mathfrak{g}_j$ for $i \neq j$. Then

$$\dim \{Ad\text{-invariant symmetric bilinear map } \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\} = \sum_{i=1}^{r} n_i^2$$

To see this, take $T \in \{Ad\text{-invariant symmetric bilinear map }\}$ and use Schur's lemma to

$$T_{ij} : \mathfrak{g}_i \hookrightarrow \mathfrak{g} \xrightarrow{x \mapsto T(x,\cdot)} \mathfrak{g}^* \overset{\langle\cdot,\cdot\rangle}{\cong} \mathfrak{g} \xrightarrow{proj.} \mathfrak{g}_j$$

Then $T_{ij} = \begin{cases} 0 & (i \neq j) \\ c \cdot id & (i = j) \text{ for } \exists c \in \mathbb{R} \end{cases}$, so uniqueness up to scalar multiplication holds only when $r = 1, n = 1$, i.e. $(\mathfrak{g}, Ad)$ is irreducible $\iff$ G is simple Lie group.

Definition 1.8. Let M be smooth manifold, G be Lie group with unit e. A smooth map

$$A : M \times G \to M,\ (x, g) \mapsto xg$$

is called the **right action of G on M** if

(1) $\forall x \in M,\ xe = x$

(2) $\forall x \in M,\ \forall g, g \in G,\ (xg)h = x(gh)$

We write the right action as $M \curvearrowleft G$.

Definition 1.9. Suppose $M \curvearrowleft G$.

(1) For $\forall g \in G$, set $R_g : M \to M$, $X \mapsto xg$ (called right translation).

(2) For $\forall X \in \mathfrak{g}$, define the **fundamental vector field** $X^\sharp \in \mathfrak{X}(M)$ by $X^\sharp_x = \dfrac{d}{dt} x \cdot \exp_G(tX) \mid_{t=0} = dA(x, \cdot)_e(X)$.

Here the notation $X^\sharp$ is the same as the left-invariant vector field on Lie group, we'll show that they have similar property:

Remark 1.2. (1) $\forall g \in G$, $\forall X \in \mathfrak{g}$, $(R_g)_* X^\sharp = (Ad(g^{-1})X)^\sharp$.

(2) $\forall X, Y \in \mathfrak{g}$, $[X^\sharp, Y^\sharp] = [X, Y]^\sharp$.

Proof. (1) $\forall x \in M$, $\left((R_g)_* X^\sharp\right)_x = (R_g)_* X^\sharp_{xg^{-1}} = \dfrac{d}{dt} xg^{-1} \exp_G(tX) g \mid_{t=0}$. Since we know $\{g^{-1} \exp_G(tX) g\}_{t \in \mathbb{R}}$ is a one parameter subgroup of G with $\dfrac{d}{dt} g^{-1} \exp_G(tX) g \mid_{t=0} = Ad(g^{-1})X$, then $g^{-1} \exp_G(tX) g = \exp_G(tAd(g^{-1})X)$, which gives (1).

(2) By definition, $\left\{\varphi_t = R_{\exp_G(tX)}\right\}_{t \in \mathbb{R}}$ is flow of $X^\sharp$. So

$$[X^\sharp, Y^\sharp] = \frac{d}{dt} (\varphi_{-t})_* Y^\sharp \mid_{t=0} = \frac{d}{dt} \left(Ad\left(\exp_G(tX)\right) Y\right)^\sharp \mid_{t=0} = (ad(X)(Y))^\sharp = [X, Y]^\sharp.$$

□

Remark 1.3. We can define the left action

$$A^L : G \times M \to M, \ (g, x) \mapsto gx$$

and also the fundamental vector field $X^\sharp_L \in \mathfrak{X}(M)$. The left and right actions are essentially the same, since the right action is given form the left action. Indeed, given A^L above, define A by $A(x, g) = A^L(g^{-1}, x) = g^{-1}x$, then $X^\sharp_L = -X^\sharp$ for $X \in \mathfrak{g}$. $[X^\sharp_L, Y^\sharp_L] = [X, Y]^\sharp = -[X, Y]^\sharp_L$.

Definition 1.10. Suppose $M \curvearrowleft G$.

(1) For $p \in M$, define $G_p = \{g \in G \mid pg = p\}$ (called **isotropy subgroup at** p).

(2) The G action is **free** of $G_p = \{e\}$ for $\forall p \in M$.

(3) The G action is **effective** if $\bigcap\limits_{p \in M} G_p = \{e\}$. In other words, $G \to \mathrm{Diff}(M)$ is injective.

1.2 Definition of Principal Bundles

Definition 1.11. Let P, M be smooth manifolds and G be Lie group. The map $\pi_P : P \to M$ is a **principal G-bundle** or **principal bundle with structure group G** if:

(1) $P \curvearrowleft G$.

(2) There exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of M and diffeomorphisms called local trivialization

$$\phi_\alpha : \pi_P^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$$

such that

(2.1) Denoting by $p_1 : U_\alpha \times G \to U_\alpha$ the projection, then $\pi_P = p_1 \circ \phi_\alpha$

(2.2) The G-action preserves each $\pi_P^{-1}(U_\alpha)$. Denoting the right G-action on $U_\alpha \times G$ by

$$(U_\alpha \times G) \times G \to U_\alpha \times G,\ ((x, h), g) \mapsto (x, h) \cdot g = (x, hg)$$

Then ϕ_α is G-equivalent, i.e. $\forall \xi \in \pi_P^{-1}(U_\alpha)$, $\forall g \in G$, $\phi_\alpha(\xi g) = \phi_\alpha(\xi) g$. Note that the G-action is free.

We often write $P|_U = \pi_P^{-1}(U)$ for open subset $U \subseteq M$ and $P_x = \pi_P^{-1}(x)$ for $x \in M$, P_x is called the **fiber of P at** x.

Recall that $e \in G$ is the unit, define a section $p_\alpha \in \Gamma(P|_{U_\alpha})$ on U_α: $\phi_\alpha(p_\alpha(x)) = (x, e)$, which is equivalent to $p_\alpha(x) = \phi_\alpha^{-1}(x, e)$. Define $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ by

$$\boxed{p_\alpha(x) g_{\alpha\beta}(x) = p_\beta(x)}$$

$\{g_{\alpha\beta}\}_{\alpha\beta}$ is called the **transition map** of $\pi_P : P \to M$. Note that $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$, we have $\boxed{g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)}$. Conversely, given open covering $\{U_\alpha\}_{\alpha \in A}$ of M and transition maps, we can recover principal G-bundle $\pi_P : P \to M$.

As before, for $g \in G$, we can define $R_g : P \to P$ the right translation and the fundamental vector field $X^\sharp$ generated by $X \in \mathfrak{g}$.

Definition 1.12. Let $\pi_P : P \to M$ be a principal G-bundle, $\rho : G \to GL(V)$ representation of G. Define the right G-action on $P \times V$ by

$$(P \times V) \times G \to P \times V,\ ((\xi, v), g) \mapsto \left(\xi g, \rho(g)^{-1} v\right)$$

$P \underset{\rho}{\times} V = (P \times V)/G$ is called the **associated vector bundle to** P.

Set $\xi \underset{\rho}{\times} v$ the equivalence class of $(\xi, v) \in P \times V$. Set $E = P \underset{\rho}{\times} V$, $\pi_E : E \to M$, $\xi \underset{\rho}{\times} v \mapsto \pi_P(\xi)$. Then $\pi_E : E \to M$ is a vector bundle.

The local trivialization of E are induced from those of P:

$$\phi_\alpha^E : E\mid_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V,\ p_\alpha(x) \underset{\rho}{\times} v \mapsto (x,v)$$

For $x \in U_\alpha \cap U_\beta$ and $v_\beta \in V$, $p_\beta(x) \underset{\rho}{\times} v_\beta = p_\alpha g_{\alpha\beta}(x) \underset{\rho}{\times} v_\beta = p_\alpha(x) \underset{\rho}{\times} \rho\left(g_{\alpha\beta}(x)\right) v_\beta$. The transition functions of E are given by $\{\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to GL(V)\}$.

We will explain some relations between P and E.

- First note that $\forall \xi \in P$, we have $\xi : V \xrightarrow{\cong} E_{\pi_P(\xi)}$, $v \mapsto \xi \underset{\rho}{\times} v$ is an isomorphism. For $\xi' \in P$ with $\xi' = \xi g$ for $g \in G$, we have $\xi^{-1}\left(\xi' \underset{\rho}{\times} v'\right) = \xi^{-1}\left(\xi \underset{\rho}{\times} \rho(g)v'\right) = \rho(g)v'$ for $v' \in V$.
- $\pi_P^* E$ is a trivial bundle. Indeed,
$$P \times V \underset{(\xi,\xi^{-1}(e)) \leftarrow (\xi,e)}{\xrightarrow{(\xi,v)\mapsto(\xi,\xi \underset{\rho}{\times} v)}} \pi_P^* E = \{(\xi,e) \in P \times E \mid \pi_P(\xi) = \pi_E(e)\} \text{ is isomorphism.}$$
- Next, for $s \in \Omega^q(E) = \Gamma\left(\Lambda^q T^*M \otimes E\right)$, define $\pi_P^* s \in \Omega^q\left(P;V\right)$ as follows (V-valued q-form on P)
  - For $q = 0$, $\left(\pi_P^* s\right)(\xi) = \xi^{-1}\left(s\left(\pi_P(\xi)\right)\right)$
  - For $q > 1$, $\forall \alpha \in \Omega^q(M)$, $\forall s \in \Omega^0(E) = \Gamma(E)$,
  $$\pi_P^*\left(\alpha \otimes s\right) = \pi_P^*\alpha \otimes \pi_P^* s$$
  The left one is pullback and the right one is define above. In other words, $\forall \xi \in P$, $\forall v_1, \cdots, v_q \in T_\xi P$,
  $$\left(\pi_P^* s\right)_\xi\left(v_1, \cdots, v_q\right) = \xi^{-1}\left(s_{\pi_P(\xi)}\left(\pi_{P*}(v_1), \cdots, \pi_{P*}(v_q)\right)\right)$$

  Notation: denote $\Omega_B^q(P;V)$ to be the elements $\widetilde{s}$ in $\Omega^q(P;V)$ satisfying:

  ($\star$) $\forall X \in \mathfrak{g}$, $i(X^\sharp)\widetilde{s} = 0$.

  ($\star\star$) $\forall g \in G$, $R_g^*\widetilde{s} = \rho(g)^{-1}\widetilde{s}$.

  called the **space of basic q-forms**. Note that $\Omega_B^q(P;V)$ depends on representation ρ.

Proposition 1.5. (**Important to study the relations between P and E**)

(1) $\pi_P^*\left(\Omega^q(E)\right) \subseteq \Omega_B^q(P;V)$ and $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega_B^q(P;V)$. E-valued q-forms on M are identified with basic q-forms on P.

(2) Recall the local trivialization $\phi_\alpha^E : E\mid_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$. For $s \in \Omega^q(E)$, suppose that $s\mid_{U_\alpha}$ corresponds to $s_\alpha \in \Omega^q(U_\alpha;V)$. Then $s_\alpha = p_\alpha^*\left(\pi_P^* s\right)$. So we regard $s \in \Omega^q(E)$ as a basic form, and then pullback by p_α is s_α.

Proof. (1) We show $\pi_P^*\left(\Omega^q(E)\right) \subseteq \Omega_B^q(P;V)$. Take $\forall s \in \Omega^q(E)$,

- $\boxed{\text{For } q=0}$ $(\star)$ is trivial; For $(\star\star)$: for $g \in G$, $\xi \in P$, we have
$$\left(R_g^* \pi_P^* s\right)(\xi) = \left(\pi_P^* s\right)\left(R_g \xi\right) = (\xi g)^{-1}\left(s(\pi_P(\xi g))\right) = (\xi g)^{-1}\left(s(\pi_P(\xi))\right)$$
By definition of ξ, we have: for $\forall v \in V$,
$$\xi(v) = \xi \underset{\rho}{\times} v = \xi g \underset{\rho}{\times} \rho(g)^{-1}(v) = (\xi g)\left(\rho(g)^{-1}(v)\right)$$
so $\xi = (\xi g) \circ \rho(g)^{-1}$, hence $(\xi g)^{-1} = \rho(g)^{-1} \circ \xi^{-1}$. Then
$$\left(R_g^* \pi_P^* s\right)(\xi) = \rho(g)^{-1}\left(\xi^{-1} s\left(\pi_P(\xi)\right)\right) = \left(\rho(g)^{-1}\left(\pi_P^* s\right)\right)(\xi).$$
- $\boxed{\text{For } q \geq 1}$ $(\star)$: Since $\pi_P(\xi g) = \pi_P(\xi)$, we have $\pi_{P*}(X^\sharp) = 0$, which implies $(\star)$; $(\star\star)$: For $\forall \alpha \in \Omega^q(M)$, $\forall s \in \Gamma(E)$, $\forall g \in G$, we have
$$R_g^*\left(\pi_P^*(\alpha \otimes s)\right) = R_g^* \pi_P^* \alpha \otimes R_g^* \pi_P^* s = \pi_P^* \alpha \otimes \rho(g)^{-1}\left(\pi_P^* s\right) = \rho(g)^{-1} \pi_P^*\left(\alpha \otimes s\right)$$
which finishes the proof of $(\star\star)$.

Next we show $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega_B^q(P;V)$:

- $\boxed{\text{Injectivity}}$ It is clear from the formula
$$\left(\pi_P^* s\right)_\xi\left(v_1, \cdots, v_q\right) = \xi^{-1}\left(s_{\pi_P(\xi)}\left(\pi_{P*}(v_1), \cdots, \pi_{P*}(v_q)\right)\right).$$
- $\boxed{\text{Surjectivity}}$ Take $\widetilde{s} \in \Omega_B^q(P;V)$,
  - When $q=0$, define $s \in \Omega^0(E) = \Gamma(E)$ by $s(x) = \xi \underset{\rho}{\times} \widetilde{s}(\xi)$ where $\xi \in \pi_P^{-1}(x)$. It is well-defined since $\xi g \underset{\rho}{\times} \widetilde{s}(\xi g) = \xi g \underset{\rho}{\times} (R_g^* \widetilde{s})(\xi) = \xi g \underset{\rho}{\times} \rho(g)^{-1} \widetilde{s}(\xi) = \xi \underset{\rho}{\times} \widetilde{s}(\xi)$. Then by definition we have $\pi_P^* s = \widetilde{s}$.
  - When $q \geq 1$, define $s \in \Omega^0(E) = \Gamma(E)$ by
$$s_x(w_1, \cdots, w_q) = \xi \underset{\rho}{\times} \widetilde{s}_\xi(\widetilde{w_1}, \cdots, \widetilde{w_q})$$
where $x \in M$, $w_i \in T_x M$, $\xi \in \pi_P^{-1}(x)$, $\pi_{P*}(\widetilde{w}_i) = w_i$. It's left as an exercise to check s is well-defined in this case.

(2) First we describe s_α clearly. Set $s|_{U_\alpha} = \sum \beta_i \otimes e_i$. Since
$$\phi_\alpha^E : E\,|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V,\ p_\alpha(x) \underset{\rho}{\times} v \mapsto (x, v),$$
we have $\phi_\alpha^E\left((e_i)_x\right) = (x, v_i(x))$ for a function $v_i : U_\alpha \to V$. Note that $(e_i)_x = p_\alpha(x) \underset{\rho}{\times} v_i(x)$. Then $s_\alpha = \sum \beta_i \otimes v_i$. Now we compute
$$p_\alpha^*\left(\pi_P^* s\right) = p_\alpha^*\left(\sum \pi_P^* \beta_i \otimes \pi_P^* e_i\right) = \sum \left(\pi_P \circ p_\alpha\right)^* \beta_i \otimes \left(\pi_P^* e_i\right) p_\alpha(x) = \sum \beta_i \otimes v_i(x).$$
So we have $p_\alpha^*\left(\pi_P^* s\right) = s_\alpha$. $\square$

Now we give a typical example of principal bundles.

Example 1.4. Let $\pi_E : E \to M$ be a vector bundle with rank r. For $x \in M$, set

- $P_x = \{\xi : \mathbb{K}^r \to E_x : \text{ linear isomorphism }\}$.
- $P = \bigsqcup_{x \in M} P_x$; $\pi_P : P \to M$, $\xi \mapsto x$ if $\xi \in P_x$.

We see that $\pi_P : P \to M$ is a principal $GL(r;\mathbb{K})$-bundle:

- The right action on P is given by:
$$P \times GL(r;\mathbb{K}) \to P, \ (\xi \times g) \mapsto \xi \circ g.$$
- To give a local trivialization, first note that
$$P_x \xrightarrow[\xi \mapsto \{\xi(\epsilon_1), \cdots, \xi(\epsilon_r)\}]{\cong} \{\text{basis of } E_x\},$$
where $\epsilon_i = (0, \cdots, \underset{i\text{-th}}{1}, \cdots, 0)^t$. If $\{e_1, \cdots, e_r\} \subseteq \Gamma(E|_{U_\alpha})$ is local frame of E over $U_\alpha \subseteq M$, define $p_\alpha \in \Gamma(P|_{U_\alpha})$ by
$$p_\alpha : U_\alpha \to P|_{U_\alpha}, \ x \mapsto (e_1(x), \cdots, e_r(x)),$$
which induces a local trivialization
$$\phi^P_\alpha : P|_{U_\alpha} \to U_\alpha \times GL(r;\mathbb{K}), \ \xi \mapsto \left(\pi_P(\xi), \left(p_\alpha\left(\pi_P(\xi)\right)\right)^{-1}\xi\right)$$
The inverse of this map is $(x, g) \mapsto p_\alpha(x) \cdot g$. We see that ϕ^P_α is $GL(r;\mathbb{K})$-equivalent.

So $\pi_P : P \to M$ is a principal $GL(r;\mathbb{K})$-bundle. This is called the **frame bundle** of $\pi_E : E \to M$. Also note that transition maps of E is the transition maps of P. Indeed, if $\{f_1, \cdots, f_r\} \subseteq \Gamma(E|_{U_\alpha})$ is another local frame, the transition map $g_{\alpha\beta}$ satisfies $(f_1, \cdots, f_r) = (e_1, \cdots, e_r)g_{\alpha\beta}$, and this is exactly $p_\beta = p_\alpha g_{\alpha\beta}$.

1.3 Connections on Principal Bundles

In this subsection we study properties of connection on principal bundle and its relation between connection on associated vector bundle.

Definition 1.13. Let $\pi_P : P \to M$ be principal G-bundle.

(1) A distribution $\{H_\xi \subseteq T_\xi P\}_{\xi \in P}$ is a **connection** on P if

(1-1) $\forall \xi \in P$, $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$.

(1-2) $\{H_\xi \subseteq T_\xi P\}_{\xi \in P}$ is G-invariant, i.e. $\forall \xi \in P$, $\forall g \in G$, $(R_g)_{*\xi} H_\xi = H_{\xi g}$.

H_ξ, $\ker(\pi_P)_{*\xi}$ are called **horizontal/vertical subspaces**.

(2) A $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P;\mathfrak{g})$ on P is a **connection form** if

(2-1) $\forall X \in \mathfrak{g}$, $\theta(X^\sharp) = X$.

(2-2) $\forall g \in G$, $R_g^*\theta = Ad(g^{-1})\theta$.

These 2 notions are the same in the following sense:

Theorem 1.2. Let $\pi_P : P \to M$ be principal G-bundle.

(1) If $\theta \in \Omega^1(P;\mathfrak{g})$ is a connection form, a distribution $\{\ker\theta_\xi\}_{\xi\in P} = \{v \in T_\xi P \mid \theta_\xi(v) = 0\}_{\xi\in P}$ is a connection on P.

(2) $\{\text{connection form}\} \leftrightarrow \{\text{connection on } P\}$, $\theta \leftrightarrow \{\ker\theta_\xi\}_{\xi\in P}$ is bijective.

Proof. (1) We check that $\{\ker\theta_\xi\}_{\xi\in P}$ satisfies (1-1), (1-2):

(1-1) Note that $\ker(\pi_P)_{*\xi} = \left\{X^\sharp_\xi \in T_\xi P \mid X \in \mathfrak{g}\right\}$, then for $\forall v \in T_\xi P$, we have $\theta(v) \in \mathfrak{g}$ and $v = \theta(v)^\sharp_\xi + \left(v - \theta(v)^\sharp_\xi\right)$, which implies that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus \ker\theta_\xi$ ($\ker(\pi_P)_{*\xi} \cap \ker\theta_\xi = \{0\}$ is obvious).

(1-2) Take $\forall v \in \ker\theta_\xi$. By (2-2), $\forall g \in G$, we have $\left(R_g^*\theta\right)_\xi = Ad(g^{-1})\theta_\xi$, the left hand side is $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right)$, so we have $(R_g)_{*\xi}(v) \in \ker\theta_{\xi g}$, hence $(R_g)_{*\xi}(\ker\theta_\xi) \subseteq \ker\theta_{\xi g}$. Replacing (g,ξ) with $(g^{-1},\xi g)$, we have $(R_{g^{-1}})_{*\xi g}(\ker\theta_{\xi g}) \subseteq \ker\theta_\xi$. So $(R_g)_{*\xi}(\ker\theta_\xi) = \ker\theta_{\xi g}$, $\{\ker\theta_\xi\}_{\xi\in P}$ is a connection on P.

(2) $\boxed{\text{Injectivity}}$ Let θ, θ' be connection forms with $\ker\theta_\xi = \ker\theta'_\xi$ $\forall\xi \in P$. We show that $\forall v \in T_\xi P$, $\theta_\xi(v) = \theta'_\xi(v)$. By (1), v is described as $v = X^\sharp_\xi + w$ for $X^\sharp_\xi \in \ker(\pi_P)_{*\xi}$ and $w \in \ker\theta_\xi = \ker\theta'_\xi$. So $\theta_\xi(v) = \theta_\xi(X^\sharp_\xi) = X = \theta'_\xi(v)$.

$\boxed{\text{Surjectivity}}$ Take $\forall\{H_\xi\}_{\xi\in P}$ a connection on P. By (1-1), we can define $\theta \in \Omega^1(P;\mathfrak{g})$ by

$$\theta_\xi(v) = \begin{cases} 0 & (v \in H_\xi) \\ X & (v = X^\sharp_\xi \text{ for } X \in \mathfrak{g}) \end{cases}$$

By definition, $\ker\theta_\xi = H_\xi$, we check (2-1), (2-2).

(2-1) Holds by definition of θ_ξ.

(2-2) $\forall\xi \in P$, $\forall g \in G$, we show that $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = Ad(g^{-1})\theta_\xi$ on $T_\xi P$. Recall that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$, if $v \in H_\xi$, the equality holds by definition and (1-2); for $\forall X \in \mathfrak{g}$,

$$(R_g)_{*\xi}\left(X^\sharp_\xi\right) = (R_g)_{*\xi}\frac{d}{dt}\xi\exp_G(tX)\mid_{t=0} = \frac{d}{dt}\xi g \cdot g^{-1}\exp_G(tX)g\mid_{t=0} = \left(Ad(g^{-1})X\right)^\sharp_{\xi g}$$

So $\theta_{\xi g}\left((R_g)_{*\xi}(X^\sharp_\xi)\right) = Ad(g^{-1})X = Ad(g^{-1})\theta_\xi(X^\sharp_\xi)$, hence the equality holds. So we have $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = Ad(g^{-1})\theta_\xi$ on $T_\xi P$. $\square$

The next proposition says that a connection form θ on P induces a connection ∇^E of the associated vector bundle E. The relation between θ and local connection form of ∇^E us also given.

Proposition 1.6. Let $\pi_P : P \to M$ be a principal bundle, $\rho : G \to GL(V)$ a representation of G with differential representation $\rho_* : \mathfrak{g} \to End(V)$. Denote by

$\theta \in \Omega^1(P;\mathfrak{g})$ a connection form. Set $E = P \underset{\rho}{\times} V$ its associated vector bundle. Then,

(1) $(d + \rho_*(\theta)\wedge)\,\Omega^q_B(P;V) \subseteq \Omega^{q+1}_B(P;V)$. Here

- d: standard exterior derivative.
- $\rho_*(\theta) \in \Omega^1(P;End(V))$ acts on $\Omega^q_B(P;V)$ by wedging on differential form parts and composing $End(V), V$-parts.

(2) Recall that $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega^q_B(P;V)$. Then we can define $\nabla^E : \Omega^0(E) \to \Omega^1(E)$ by $(\pi_P^*)^{-1} \circ (d + \rho_*(\theta)\wedge) \circ \pi_P^*$.

(3) Recall that a local section $p_\alpha \in \Gamma(P|_{U_\alpha})$ induces a local trivialization $\phi^E_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$. Then

$$\begin{array}{ccc} \Omega^0(E|_{U_\alpha}) & \xrightarrow{\nabla^E|_{U_\alpha}} & \Omega^1(E|_{U_\alpha}) \\ \downarrow \cong & \circlearrowleft & \cong \downarrow \\ \Omega^0(U_\alpha;V) & \xrightarrow{d+\rho_*(p_\alpha^*\theta)\wedge} & \Omega^1(U_\alpha;V) \end{array}$$

(4) Recall that a connection ∇^E induces the exterior derivative $d^{\nabla^E} : \Omega^q(E) \to \Omega^{q+1}(E)$. Then

$$\begin{array}{ccc} \Omega^q(E) & \xrightarrow{d^{\nabla^E}} & \Omega^{q+1}(E) \\ \pi_P^* \ \downarrow \cong & \circlearrowleft & \cong \downarrow \ \pi_P^* \\ \Omega^q_B(P;V) & \xrightarrow{d+\rho_*(\theta)\wedge} & \Omega^{q+1}_B(P;V) \end{array}$$

Remark 1.4. In [Kobayashi-Nomizu, *Foundation of differential geometry* Vol 1, chapter 2, section 5], for any principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$, $\forall V$ vector space, the *exterior covariant derivative* $D : \Omega^q(P;V) \to \Omega^{q+1}(P;V)$ is defined by $(D\widetilde{s})(v_0,\cdots,v_q) = (d\widetilde{s})(hv_0,\cdots,hv_q)$ for $v_i \in TP$, where $h : TP \to \ker\theta$ is the projection. If in addition, given a representation $\rho : G \to GL(V)$ and $\widetilde{s} \in \Omega^q_B(P;V)$, we have $D\widetilde{s} = (d + \rho_*(\theta)\wedge)(\widetilde{s})$.

Proof. (1) Take $\forall \widetilde{s} \in \Omega^q_B(P;V)$, recall that $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\widetilde{s} = 0. \\ \forall g \in G, R_g^*\widetilde{s} = \rho(g)^{-1}\widetilde{s}. \end{cases}$. We show that $(d + \rho_*(\theta)\wedge)\,\widetilde{s}$ also satisfies the same property.

- $\forall X \in \mathfrak{g}$, we have

$$\mathcal{L}_{X^\sharp}\widetilde{s} = \frac{d}{dt} R^*_{\exp_G(tX)}\widetilde{s}\ |_{t=0} = \frac{d}{dt}\rho\left(\exp_G(tX)\right)^{-1}\widetilde{s}\ |_{t=0} = -\rho_*(X)\widetilde{s}.$$

Since $\mathcal{L}_{X^\sharp}\widetilde{s} = i(X^\sharp)d\widetilde{s} + d\left(i(X^\sharp)\widetilde{s}\right)$ and $i(X^\sharp)\widetilde{s} = 0$, we have $i(X^\sharp)d\widetilde{s} = -\rho_*(X)\widetilde{s}$. Hence $i(X^\sharp)\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = i(X^\sharp)d\widetilde{s} + \rho_*\left(\theta(X^\sharp)\right)\widetilde{s} - \rho_*(\theta)\wedge i(X^\sharp)\widetilde{s} = 0$.

- For $\forall g \in G$, we have

$$R_g^*\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = dR_g^*\widetilde{s} + \rho_*\left(R_g^*\theta\right)\wedge R_g^*\widetilde{s} = d\left(\rho(g)^{-1}\widetilde{s}\right) + \rho_*\left(Ad(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s}.$$

Since $\rho(g)^{-1}$ acts only on V-part, $d\left(\rho(g)^{-1}\widetilde{s}\right) = \rho(g)^{-1}d\widetilde{s}$. Note that $\forall X \in \mathfrak{g}$,

$$\frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)g\right)\rho(g)^{-1}\mid_{t=0} = \frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)\right)\mid_{t=0}$$

and $g^{-1}\exp_G(tX)g = \exp_G(tAd(g^{-1})X)$, we have

$$\rho_*\left(Ad(g^{-1})X\right)\rho(g)^{-1} = \rho(g)^{-1}\rho_*(X).$$

This implies that

$$\rho_*\left(Ad(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s} = \rho(g)^{-1}\left(\rho_*(\theta)\wedge\widetilde{s}\right).$$

Then we obtain

$$R_g^*\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = \rho(g)^{-1}\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right),$$

so $(d+\rho_*(\theta)\wedge)(\widetilde{s}) \in \Omega_B^{q+1}(P;V)$.

(2) $\nabla^E = (\pi_P^*)^{-1}\circ(d+\rho_*(\theta)\wedge)\circ\pi_P^*$, we check the Leibniz rule, i.e. for $\forall f \in C^\infty(M)$, $\forall s \in \Gamma(E)$, we show $\nabla^E(fs) = df\otimes s + f\nabla^E s$. This is left as an exercise.

(3) Since for $s \in \Omega^q(E)$, $s|_{U_\alpha}$ corresponds to $p_\alpha^*(\pi_P^* s)$. We compute

$$p_\alpha^*\pi_P^*\left(\nabla^E s\right) = p_\alpha^*\left((d+\rho_*(\theta)\wedge)\pi_P^* s\right) = p_\alpha^* d\left(\pi_P^* s\right) + \rho_*\left(p_\alpha^*\theta\right)\wedge p_\alpha^*\pi_P^* s = \left(d+\rho_*(p_\alpha^*\theta)\wedge\right)\left(p_\alpha^*\pi_P^* s\right).$$

(4) Since d^{∇^E} is given by $d^{\nabla^E}(s\otimes\alpha) = \nabla^E s\wedge\alpha + s\otimes d\alpha$ for $s \in \Gamma(E)$, $\alpha \in \Omega^q(M)$, we have

$$\begin{aligned}\pi_P^*\left(d^{\nabla^E}(s\otimes\alpha)\right) &= \pi_P^*\left(\nabla^E s\wedge\alpha + s\otimes d\alpha\right) = (d+\rho_*(\theta)\wedge)\pi_P^* s\wedge\pi_P^*\alpha + \pi_P^* s\otimes\pi_P^* d\alpha \\ &= d\left(\pi_P^* s\otimes\pi_P^*\alpha\right) + \rho_*(\theta)\wedge\left(\pi_P^* s\otimes\pi_P^*\alpha\right) = (d+\rho_*(\theta)\wedge)\left(\pi_P^*(s\otimes\alpha)\right).\end{aligned}$$

□

Exercise 1.2. Prove that ∇^E defined above is a connection.

Example 1.5. Given a vector bundle $\pi_E : E \to M$, let $\pi_P : P \to M$ be the frame bundle. Consider the trivial representation $id : GL(r;\mathbb{K}) \to GL(r;\mathbb{K})$. Then

Definition 1.14. Let $\pi_P : P \to M$ be principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$.

(1) $\Omega = d\theta + \frac{1}{2}[\theta\wedge\theta] \in \Omega^2(P;\mathfrak{g})$ is called the **curvature** of θ. ($[\theta\wedge\theta]$ means taking the wedge product of differential form part and taking Lie bracket of $\mathfrak{g}$-part)

(2) For $\forall X \in \mathfrak{X}(M)$, $\exists! \widetilde{X} \in \mathfrak{X}(P)$ s.t. $\begin{cases} (\pi_P)_* \widetilde{X} = X \\ \theta(\widetilde{X}) = 0 \end{cases}$. Then $\widetilde{X}$ is called the **horizontal lift of** X.

We see existence and uniqueness of $\widetilde{X}$ in (2) as follows: recall that $\forall \xi \in P$, $T_\xi P = \ker(\pi_P)_* \oplus \ker \theta_\xi$, so $(\pi_P)_* : \ker \theta_\xi \xrightarrow{\cong} T_{\pi_P(\xi)} M$. So we may set $\widetilde{X}_\xi = (\pi_P)_*^{-1} \left(X_{\pi_P(\xi)} \right)$. Since $(\pi_P)_*|_{\ker \theta_\xi}$ is isomorphism, uniqueness follows.

Remark 1.5. Recall exterior covariant derivative of Kobayashi-Nomizu, i.e. $D : \Omega^q(P;V) \to \Omega^{q+1}(P;V)$ is defined by $(D\widetilde{s})(v_0, \cdots, v_q) = (d\widetilde{s})(hv_0, \cdots, hv_q)$ for $v_i \in TP$, where $h : TP \to \ker \theta$ is the projection. Then $\boxed{\Omega = D\theta}$. Actually, Kobayashi-Nomizu defined curvature by $D\theta$, and shows the equality in (1). The equality is called the **structure equation**.

To show this, note the following:

Remark 1.6. Let $\{\xi_1, \cdots, \xi_\ell\}$ be a basis of $\mathfrak{g}$. Then $\theta = \sum \xi_i \otimes \theta_i = \sum \xi_i \theta_i$ where $\theta_i \in \Omega^1(P)$ and we omit the $\otimes$. Then by definition we have

$$\Omega = \sum \xi_i d\theta_i + \frac{1}{2} \sum [\xi_i, \xi_j] \theta_i \wedge \theta_j.$$

Note that

$$\theta_i \wedge \theta_j(u, v) = \theta_i(u)\theta_j(v) - \theta_j(u)\theta_i(v),$$

so we have $[\xi_j, \xi_i]\theta_j \wedge \theta_i = [\xi_i, \xi_j]\theta_i \wedge \theta_j$, then

$$[\theta \wedge \theta](u, v) = [\theta(u), \theta(v)] - [\theta(v), \theta(u)] = 2[\theta(u), \theta(v)],$$

so for $u, v \in TP$, we have $\boxed{\Omega(u, v) = d\theta(u, v) + [\theta(u), \theta(v)]}$. Now we show $\Omega = D\theta$. Since $TP = \ker(\pi_P)_* \oplus \ker \theta$, we have to show in the following cases:

- $\underline{u, v \in \ker \theta}$: $\Omega(u, v) = d\theta(u, v) = (D\theta)(u, v)$.
- $\underline{u, v \in \ker(\pi_P)_*}$: we may set $u = X^\sharp, v = Y^\sharp$ for $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned} \Omega(X^\sharp, Y^\sharp) &= d\theta(X^\sharp, Y^\sharp) + [X, Y] \\ &= X^\sharp \left(\theta(Y^\sharp) \right) - Y^\sharp \left(\theta(X^\sharp) \right) - \theta([X^\sharp, Y^\sharp]) + [X, Y] = 0. \end{aligned}$$

  Also $(D\theta)(X^\sharp, Y^\sharp) = 0$.
- $\underline{u \in \ker \theta, v = X^\sharp \text{ for } X \in \mathfrak{g}}$: extend u to a local horizontal vector field on P, which is still denoted as u. For example, extend $\pi_{P*}(u)$ to a local vector field on M, consider its horizontal lift. Then

$$\Omega(u, X^\sharp) = d\theta(u, X^\sharp) = u(\theta(X^\sharp)) - X^\sharp(\theta(u)) - \theta([u, X^\sharp]) = -\theta([u, X^\sharp])$$

Now we show that $[u, X^\sharp] \in \Gamma(\ker\theta)$, then $\theta([u, X^\sharp]) = 0$. Recall that $\{R_{\exp_G(tX)}\}_{t\in\mathbb{R}}$ is the flow of $X^\sharp$, so $[X^\sharp, u] = \frac{d}{dt}\left(R_{\exp_G(-tX)}\right)_* u \mid_{t=0}$. Since for $\forall g \in G$, $\theta\left((R_g)_* u\right) = \left(R_g^*\theta\right)(u) = Ad(g^{-1})\theta(u) = 0$, we have $\theta([X^\sharp, u]) = 0$, hence $\Omega(u, X^\sharp) = (D\theta)(u, X^\sharp)$.

So we have $\Omega = D\theta$.

Theorem 1.3. Let $\pi_P : P \to M$ be principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. Denote by $\Omega \in \Omega^2(P;\mathfrak{g})$ the curvature of θ. For $\forall X, Y \in \mathfrak{X}(M)$, let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(P)$ be the horizontal lifts respectively. Then $\Omega(\widetilde{X}, \widetilde{Y}) = -\theta([\widetilde{X}, \widetilde{Y}])$.

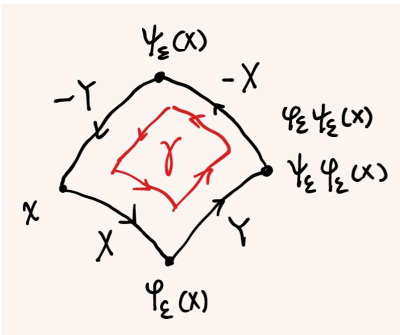
Proof. Since $\widetilde{X}, \widetilde{Y} \in \Gamma(\ker\theta)$, we have

$$\Omega(\widetilde{X}, \widetilde{Y}) = d\theta(\widetilde{X}, \widetilde{Y}) = \widetilde{X}\left(\theta(\widetilde{Y})\right) - \widetilde{Y}\left(\theta(\widetilde{X})\right) - \theta([\widetilde{X}, \widetilde{Y}]) = -\theta([\widetilde{X}, \widetilde{Y}]).$$

□

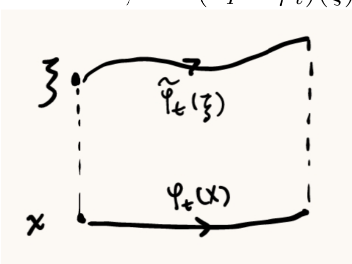
This theorem will imply that the curvature measures how "curved" the connection is.

Take a local vector field X, Y on M s.t. $[X, Y] = 0$. Let $\{\varphi_t\}$, $\{\psi_t\}$ be local flow of X, Y respectively. We know that $[X, Y] = 0 \Leftrightarrow \varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ $(\star)$. Now fix $x \in M$, consider

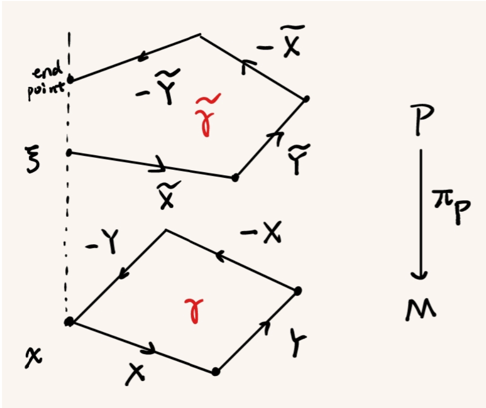


By $(\star)$, γ is a closed curve. We want to know what happens if we "lift" γ. Let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(P)$ be horizontal lifts of X, Y respectively. Let $\{\widetilde{\varphi}_t\}$, $\{\widetilde{\psi}_t\}$ be a local flow of $\widetilde{X}, \widetilde{Y}$ respectively.

Note that since $\frac{d}{dt}(\pi_P \circ \widetilde{\varphi}_t) = (\pi_P)_*(\widetilde{X} \circ \widetilde{\varphi}_t) = X \circ (\pi_P \circ \widetilde{\varphi}_t)$, then for $\forall \xi \in \pi_P^{-1}(x)$, $\{(\pi_P \circ \widetilde{\varphi}_t)(\xi)\}$ is the integral curve of X, i.e. $(\pi_P \circ \widetilde{\varphi}_t)(\xi) = \varphi_t(x)$ $(\star\star)$.



Consider a similar path $\widetilde{\gamma}$ in P form $\widetilde{X}, \widetilde{Y}$. In general $[\widetilde{X}, \widetilde{Y}] \neq 0$, so $\widetilde{\gamma}$ is not always a closed curve. By $(\star\star)$, so (end point of $\widetilde{\gamma}$) $\in \pi_P^{-1}(x)$.



Now recall the flow of $[\widetilde{X}, \widetilde{Y}]$ is given by $\widetilde{\alpha}_t = \widetilde{\psi}_{-\sqrt{t}} \circ \widetilde{\varphi}_{-\sqrt{t}} \circ \widetilde{\psi}_{\sqrt{t}} \circ \widetilde{\varphi}_{\sqrt{t}}$, hence (end point of $\widetilde{\gamma}$) $= \widetilde{\alpha}_{\epsilon^2}(\xi)$. Thus the "distance" between initial point ξ and end point $\widetilde{\alpha}_{\epsilon^2}(\xi)$ is given by "$\widetilde{\alpha}_{\epsilon^2}(\xi) - \xi$". On the other hand, "$\lim\limits_{t \to 0} \frac{\widetilde{\alpha}_t(\xi) - \xi}{dt}$" $= \frac{d\widetilde{\alpha}_t(\xi)}{dt}\ |_{t=0} = [\widetilde{X}, \widetilde{Y}]_\xi$, so $[\widetilde{X}, \widetilde{Y}]_\xi$ measures the "infinitesimal distance" between the initial point and end point of $\widetilde{\gamma}$. In addition, since $(\pi_P)_*([\widetilde{X}, \widetilde{Y}]) = [X, Y] = 0$, we have $[\widetilde{X}, \widetilde{Y}] \in \Gamma(\ker(\pi_{P*}))$. Since $\ker(\pi_{P*})_\xi \xrightarrow{\cong} \mathfrak{g}$, we have $[\widetilde{X}, \widetilde{Y}]_\xi \cong \theta_\xi\left([\widetilde{X}, \widetilde{Y}]_\xi\right) = -\Omega_\xi(\widetilde{X}, \widetilde{Y})$. So we see the curvature measures how "curved" the connection is.

Zero curvature means that a connection (horizontal subspace) is not "curved". This is made clear by the following.

Corollary 1.1. Suppose $\Omega = 0 \in \Omega^2(P; \mathfrak{g})$. Then

(1) Distribution $D = \{\ker \theta_\xi\}_{\xi \in P}$ is (completely) integrable.

(2) For a representation $\rho : G \to GL(V)$, set $E = P \underset{\rho}{\times} V$. Let ∇^E be the induced connection on E form θ. Then for $\forall x \in M$, $\exists\ U$ an open neighborhood of x, $\exists \phi : E|_U \xrightarrow{\cong} U \times V$ local trivialization s.t. $\nabla^E|_U$ is identified with $d : \Omega^0(U; V) \to \Omega^1(U; V)$.

Proof. (1) For $\forall u_1, u_2 \in \Gamma(D)$, we show $[u_1, u_2] \in \Gamma(D)$.

Let $\{X_i\}$ be a frame of $TM|_U$ on open subset U, and let $\{\widetilde{X}_i\} \subseteq \mathfrak{X}(P|_U)$ be the horizontal lift. Note that for $\forall \xi \in P|_U$, $\{(\widetilde{X}_i)_\xi\} \subseteq D_\xi$ is a basis. So locally

$$u_i = \sum f_{ij} \widetilde{X}_j \text{ for } f_{ij} \in C^\infty(P|_U).$$

By theorem 1.3, we have $\theta([\widetilde{X}_i, \widetilde{X}_j]) = -\Omega(\widetilde{X}_i, \widetilde{X}_j) = 0$. Hence $[\widetilde{X}_i, \widetilde{X}_i] \in \Gamma(D|_{P_U})$. So $[u_1, u_2] = \sum [f_{1j} \widetilde{X}_j, f_{2k} \widetilde{X}_k] \in \Gamma(D|_{P|_U})$ and D is integrable.

(2) Fix $\forall x \in M$ and $\forall \xi \in P_x$. By (1), there exists a submanifold $\widetilde{U} \subseteq P$ s.t. $\forall q \in \widetilde{U}$, $T_q \widetilde{U} = D_q \subseteq T_q P$. Shrinking $\widetilde{U}$ if necessary, we have $\pi_P|_{\widetilde{U}} : \widetilde{U} \to \pi_P(\widetilde{U})$ is a diffeomorphism (by inverse function theorem). Now define $p \in \Gamma(P|_U)$ by $p = (\pi_P|_{\widetilde{U}})^{-1} : U \to P$. Then $p^* \theta = 0$.

Recall that a local section p of P induces a local trivialization

$$E|_U = P \underset{\rho}{\times} V|_U \xrightarrow{\cong} U \times V,\ p(x) \underset{\rho}{\times} v \mapsto (x, v)$$

By proposition 1.6, via this identification, $\nabla^E|_U$ corresponds to $d+\rho_*(p^*\theta)=d$. $\square$

Proposition 1.7. Let $\pi_P; P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. Denote by $\Omega \in \Omega^2(P;\mathfrak{g})$ the curvature of θ. Then

(1) $\Omega \in \Omega^2_B(P;\mathfrak{g})$ w.r.t. representation $(\mathfrak{g}, Ad)$.

(2) (Bianchi identity) $(d+ad(\theta)\wedge)\Omega = 0 \in \Omega^3_B(P;\mathfrak{g})$.

Remark 1.7. Using the exterior covariant derivative D of Kobayashi-Nomizu, we have $D\Omega = (d+ad(\theta)\wedge)\Omega$. So (2) says that $D\Omega = 0$. It is because for any representation $\rho : G \to GL(V)$, $\forall \widetilde{s} \in \Omega^q_B(P;V)$, we already know $D\widetilde{s} = (d+\rho_*(\theta)\wedge)(\widetilde{s})$. Then set $(V,\rho)=(\mathfrak{g},Ad)$, $\widetilde{s}=\Omega$.

Proof. (1) We show that $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\Omega = 0. \\ \forall g \in G, R_g^*\Omega = Ad(g^{-1})\Omega. \end{cases}$.

- $\mathcal{L}_{X^\sharp}\theta = \dfrac{d}{dt}R^*_{\exp_G(tX)}\theta|_{t=0} = \dfrac{d}{dt}Ad\left(\exp_G(-tX)\right)\theta|_{t=0} = -ad(X)\theta.$
  Since $\mathcal{L}_{X^\sharp}\theta = i(X^\sharp)d\theta + d\left(i(X^\sharp)\theta\right) = i(X^\sharp)d\theta$, we have $i(X^\sharp)d\theta = -ad(X)\theta$. So
  $$i(X^\sharp)\Omega = i(X^\sharp)d\theta + \frac{1}{2}i(X^\sharp)[\theta\wedge\theta] = -ad(X)\theta + \frac{1}{2}\left([X,\theta]-[\theta,X]\right) = 0$$
- For $\forall g \in G$,
  $$\begin{aligned} R_g^*\Omega &= R_g^*d\theta + \frac{1}{2}R_g^*[\theta\wedge\theta] = dR_g^*\theta + \frac{1}{2}[R_g^*\theta\wedge R_g^*\theta] \\ &= \mathrm{d}Ad(g^{-1})\theta + \frac{1}{2}[Ad(g^{-1})\theta\wedge Ad(g^{-1})\theta] = Ad(g^{-1})\left(d\theta + \frac{1}{2}[\theta\wedge\theta]\right) \\ &= Ad(g^{-1})\Omega \end{aligned}$$

So we see that $\Omega \in \Omega^2_B(P;\mathfrak{g})$.
(2) We have
$$\begin{aligned} (d+ad(\theta)\wedge)\Omega &= (d+ad(\theta)\wedge)\left(d\theta + \frac{1}{2}[\theta\wedge\theta]\right) \\ &= \frac{1}{2}d[\theta\wedge\theta] + [\theta\wedge d\theta] + \frac{1}{2}[\theta\wedge[\theta\wedge\theta]]. \end{aligned}$$

Since $d[\theta\wedge\theta] = [d\theta\wedge\theta] - [\theta\wedge d\theta] = -2[\theta\wedge d\theta]$, we have $\frac{1}{2}d[\theta\wedge\theta]+[\theta\wedge d\theta]=0$. For a basis $\{\xi_i\}_1^\ell$ of $\mathfrak{g}$. Set $\theta = \sum \xi_i\theta_i$ for $\theta_i \in \Omega^1(P)$. Then
$$\begin{aligned} [\theta[\theta\wedge\theta]] &= \sum [\xi_i,[\xi_j,\xi_k]]\theta_i\wedge\theta_j\wedge\theta_k \\ &= \frac{1}{3}\sum\left\{[\xi_i,[\xi_j,\xi_k]]+[\xi_j,[\xi_k,\xi_i]]+[\xi_k,[\xi_i,\xi_j]]\right\}\theta_i\wedge\theta_j\wedge\theta_k \\ &= 0 \text{ (by Jacobi identity).} \end{aligned}$$

$\square$

Proposition 1.8. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. Denote by $\Omega \in \Omega^2(P;\mathfrak{g})$ the curvature of θ. For a representation $\rho : G \to GL(V)$, set $E = P \underset{\rho}{\times} V$. Recall $\rho_* : \mathfrak{g} \to End(V)$. Note that ρ induces a map

$$\widetilde{\rho} : G \to GL(End(V)),\ g \mapsto \left(T \mapsto \rho(g) \circ T \circ \rho(g)^{-1}\right)$$

(hence $End(E) = P \underset{\widetilde{\rho}}{\times} End(V)$) Then

(1) $\rho_*(\Omega) = d\rho_*(\theta) + \rho_*(\theta) \wedge \rho_*(\theta) \in \Omega^2_B(P; End(V))$ w.r.t. $(End(V), \widetilde{\rho})$.

(2) $(d + \rho_*(\theta)\wedge) \circ (d + \rho_*(\theta)\wedge) = \rho_*(\Omega)\wedge : \Omega^q_B(P;V) \to \Omega^{q+2}_B(P;V)$.

Proof. (1) For a basis $\{\xi_i\}_1^\ell$ of $\mathfrak{g}$. Set $\theta = \sum \xi_i\theta_i$ for $\theta_i \in \Omega^1(P)$. Then

$$\begin{aligned}
\rho_*(\Omega) &= \rho_*\left(\sum_i \xi_i d\theta_i + \frac{1}{2}\sum_{i,j}[\xi_i,\xi_j]\theta_i \wedge \theta_j\right) \\
&= \sum_i \rho_*(\xi_i)d\theta_i + \frac{1}{2}\sum_{i,j}[\rho_*(\xi_i),\rho_*(\xi_j)]\theta_i \wedge \theta_j \\
&= d\rho_*(\sum_i \xi_i\theta_i) + \sum_{i,j}\rho_*(\xi_i)\rho_*(\xi_j)\theta_i \wedge \theta_j \\
&= d\rho_*(\theta) + \rho_*(\theta) \wedge \rho_*(\theta)
\end{aligned}$$

Next we show $\rho_*(\Omega)$ is basic, i.e. $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\Omega = 0. \\ \forall g \in G, R_g^*\Omega = Ad(g^{-1})\Omega. \end{cases}$. Recall that $\Omega \in \Omega^2_B(P;\mathfrak{g})$ w.r.t. $(\mathfrak{g}, Ad)$,

- Since ρ_* acts only on $\mathfrak{g}$-part, we have $i(X^\sharp)\rho_*(\Omega) = \rho_*\left(i(X^\sharp)\Omega\right) = 0$.
- $R_g^*\rho_*(\Omega) = \rho_*(R_g^*\Omega) = \rho_*\left(Ad(g^{-1})\Omega\right)$. Since for $\forall X \in \mathfrak{g}$,

$$\rho_*\left(Ad(g^{-1})X\right) = \frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)g\right)|_{t=0} = \rho(g^{-1})\rho_*(X)\rho(g),$$

  we have $R_g^*\rho_*(\Omega) = \rho(g^{-1})\rho_*(\Omega)\rho(g) = \widetilde{\rho}(g^{-1})\rho_*(\Omega)$.

(2) Note that for $\forall \widetilde{s} \in \Omega^q_B(P;V)$,

$$(d \circ \rho_*(\theta)\wedge)(\widetilde{s}) = d\left(\rho_*(\theta) \wedge \widetilde{s}\right) = (d\rho_*(\theta)) \wedge \widetilde{s} - \rho_*(\theta) \wedge d\widetilde{s},$$

so we have $d \circ \rho_*(\theta)\wedge = (d\rho_*(\theta)) \wedge -(\rho_*(\theta)\wedge) \circ d$. Then

$$(d + \rho_*(\theta)\wedge) \circ (d + \rho_*(\theta)\wedge) = d \circ \rho_*(\theta) \wedge + \rho_*(\theta) \wedge \circ d + \rho_*(\theta) \wedge \rho_*(\theta)\wedge = \rho_*(\Omega)\wedge$$

□

Remark 1.8. This proposition is interpreted as follows.

Recall the isomorphism

1.4 Holonomy Groups

In this section, we introduce the holonomy group of a connection and study the properties. Roughly speaking, the curvature measures how a connection is curved locally, but the holonomy group measures how a connection is curved globally.

First, we formulate pullbacks of principal bundles and connections.

- Let $\pi_P : P \to M$ be principal G-bundle and $f : N \to M$ a smooth map. The **pullback** $\pi_{f^*P} : f^*P \to N$ of $\pi_P; P \to M$ is defined by
$$f^*P = \{(x, \xi) \in N \times P | f(x) = \pi_P(\xi)\}$$
For $x \in N$, we have $(f^*P)_x = \pi_{f^*P}^{-1}(x) = P_{f(x)}$. Setting $\widetilde{f} : f^*P \to P$, $(x, \xi) \mapsto \xi$. Then $\pi_P \circ \widetilde{f} = f \circ \pi_{f^*P}$.
The right G-action on f^*P is given by
$$f^*P \times G \to f^*P,\ ((x, \xi), g) \mapsto (x, \xi g).$$
Let $\left\{\phi_\alpha^P : P|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times G\right\}_{\alpha \in A}$ be a family of local trivialization of P, $pr_2 : U_\alpha \times G \to G$ projection. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open covering of N. Define local trivialization of f^*P by
$$\phi_\alpha^{f^*P} : \pi_{f^*P}^{-1}\left(f^{-1}(U_\alpha)\right) \xrightarrow{\cong} f^{-1}(U_\alpha) \times G,\ (x, \xi) \mapsto \left(x, (pr_2 \circ \phi_\alpha^P)(\xi)\right).$$
In terms of local sections, if $p_\alpha \in \Gamma(P|_{U_\alpha})$ is induced form ϕ_α^P, then $\phi_\alpha^{f^*P}$ induces f^*p_α given by $(f^*p_\alpha)(x) = (x, (p_\alpha \circ f)(x))$.
If $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ are transition maps of P, then
$$\left\{f^*g_{\alpha\beta} : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to G\right\}$$
are transition maps of f^*P.
- Let $\theta \in \Omega^1(P; \mathfrak{g})$ be a connection form. Then $\widetilde{f}^*\theta \in \Omega^1(f^*P; \mathfrak{g})$ is a connection form on f^*P. For $\forall x \in N$, $\widetilde{f} : (f^*P)_x \cong P_{f(x)} \xrightarrow{id} P_{f(x)}$ is identity map on each fiber, so θ, $\widetilde{f}^*\theta$ have the same fiberwise property. Definition of connections require some properties for G-action, which is fiberwise.

Next we consider the associated vector bundle. For a representation $\rho : G \to GL(V)$, set
$$\begin{cases} E = P \underset{\rho}{\times} V \\ \nabla^E : \text{ induced connection form on } E \text{ from } \theta \end{cases}$$
Then $f^*E = f^*P \underset{\rho}{\times} V$.

For $x \in N$, $(f^*E)_x \cong E_{f(x)}$, $(f^*P)_x \underset{\rho}{\times} V \cong P_{f(x)} \underset{\rho}{\times} V = E_{f(x)}$.

Proposition 1.9. The pullback ∇^{f^*E} of ∇^E by f is induced connection on f^*E from $\widetilde{f}^*\theta$.

Proof. We give 2 proofs depending on how ∇^{f^*E} is defined.

(1) $\boxed{\text{Local proof}}$ Let $\{A_\alpha\}$ be connection forms of ∇^E, i.e. $\nabla^E|_{U_\alpha} \cong d + A_\alpha$. Then ∇^{f^*E} is a connection with connection forms $\{f^*A_\alpha\}$. Since $A_\alpha = \rho_*(p_\alpha^*\theta)$, so $f^*A_\alpha = \rho_*(f^*p_\alpha^*\theta)$ since f^* acts only on differential form part. Next we check what is induced local connection on f^*E from $\widetilde{f}^*\theta$. By proposition before, we have the local connection form $\rho_*\Big((f^*p_\alpha)^*\widetilde{f}^*\theta\Big)$. Since $\widetilde{f} \circ (f^*p_\alpha) = p_\alpha \circ f$, we have $\rho_*\Big((f^*p_\alpha)^*\widetilde{f}^*\theta\Big) = \rho_*(f^*p_\alpha^*\theta)$, which completes the proof.

(2) $\boxed{\text{Global proof}}$ We use the fact that ∇^{f^*E} is the unique connection satisfying $\nabla^{f^*E}(f^*s) = f^*(\nabla^E s)$ for $\forall s \in \Gamma(E)$. Let ∇' be induced connection on f^*E from $\widetilde{f}^*\theta$, we show ∇' satisfies the same property then completes the proof.

We can define $\pi_{f^*P}^* : \Omega^q(f^*E) \xrightarrow{\cong} \Omega_B^q(f^*P;V)$ in the same way, and we have the commutative diagram

$$\begin{array}{ccc} \Omega^q(E) & \xrightarrow{f^*} & \Omega^q(f^*E) \\ \pi_P^* \downarrow\cong & \circlearrowleft & \cong\downarrow \pi_{f^*P}^* \\ \Omega_B^q(P;V) & \xrightarrow{\widetilde{f}^*} & \Omega_B^q(f^*P;V) \end{array}$$

Then we can compute

$$\begin{aligned} \nabla'(f^*s) &= \left((\pi_{f^*P}^*)^{-1} \circ \left(d + \rho_*(\widetilde{f}^*\theta)\right) \circ \pi_{f^*P}^* \circ f^*\right)(s) \\ &= (\pi_{f^*P}^*)^{-1}\widetilde{f}^*\left(((d + \rho_*(\theta)) \circ \pi_P^*)(s)\right) \\ &= f^*(\pi_P^*)^{-1}\left(((d + \rho_*(\theta)) \circ \pi_P^*)(s)\right) \\ &= f^*\nabla^E s). \end{aligned}$$

□

Next we describe parallel transport of a vector bundle in terms of principal bundles. First, we introduce horizontal lift.

Proposition 1.10. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. Let $c : [0,1] \to M$ be a C^∞ curve, then for $\forall \xi \in P_{c(0)}$, there exists a unique C^∞ curve $\widetilde{c} : [0,1] \to P$ s.t.

$$\begin{cases} (1)\ \pi_P \circ \widetilde{c} = c \\ (2)\ \widetilde{c}(0) = \xi \\ (3)\ \text{For } \forall t \in [0,1],\ \frac{d\widetilde{c}}{dt}(t) \in \ker \theta_{\widetilde{c}(t)}\ \text{(horizontal)} \end{cases}$$

This $\widetilde{c}$ is called the **horizontal lift of** c.

Proof. Since P is locally trivial, there exists $\widetilde{\gamma} : [0,1] \to P$ s.t.

$$\begin{cases} \pi_P \circ \widetilde{\gamma} = c \\ \widetilde{\gamma}(0) = \xi \end{cases}$$

We will find $g : [0,1] \to G$ s.t.

$$\begin{cases} g(0) = e \\ \widetilde{\eta}(t) = \widetilde{\gamma}(t) \cdot g(t) \text{ is horizontal} \end{cases}$$

so $\widetilde{\eta}$ is the horizontal lift as required.

We determine the equation that g should satisfy, to do this, we compute $\frac{d\widetilde{\eta}}{dt}$. Let $A : P \times G \to P$ be the right action, then

$$\begin{aligned} \frac{d\widetilde{\eta}}{dt}(t) &= \frac{d}{dt} A\left(\widetilde{\gamma}(t), g(t)\right) = (dA)_{(\widetilde{\gamma}(t), g(t))} \left(\frac{d}{dt} \left(\widetilde{\gamma}(t), g(t)\right) \right) \\ &= (dA)_{(\widetilde{\gamma}(t), g(t))} \left(\left(\frac{d}{ds} \widetilde{\gamma}(t+s)|_{s=0}, 0 \right) + \left(0, \frac{d}{ds} g(t+s)|_{s=0} \right) \right) \\ &= \frac{d}{ds} A\left(\widetilde{\gamma}(t+s), g(t)\right)|_{s=0} + \frac{d}{ds} A\left(\widetilde{\gamma}(t), g(t+s)\right)|_{s=0} \\ &= \frac{d}{ds} A\left(\widetilde{\gamma}(t+s), g(t)\right)|_{s=0} + \frac{d}{ds} \left(\widetilde{\gamma}(t) \cdot g(t) \cdot g(t)^{-1} \cdot g(t+s)\right)|_{s=0} \\ &= (R_{g(t)})_* \left(\frac{d\widetilde{\gamma}}{dt}(t) \right) + \left(\frac{d}{ds} g(t)^{-1} g(t+s)|_{s=0} \right)^\sharp_{\widetilde{\gamma}(t) g(t)} \quad \text{(def. of fund. v.f.).} \end{aligned}$$

By the computation we have $\widetilde{\eta}$ is horizontal $\iff$

$$\begin{aligned} 0 &= \theta_{\widetilde{\eta}(t)} \left(\frac{d\widetilde{\eta}}{dt}(t) \right) \\ &= \theta_{\widetilde{\gamma}(t) \cdot g(t)} \left((R_{g(t)})_* \left(\frac{d\widetilde{\gamma}}{dt}(t) \right) \right) + \left(L_{g(t)^{-1}}\right)_* \frac{dg}{dt}(t) \\ &= Ad\left(g(t)^{-1}\right) \theta_{\widetilde{\gamma}(t)} \left(\frac{d\widetilde{\gamma}}{dt}(t) \right) + \left(L_{g(t)^{-1}}\right)_* \frac{dg}{dt}(t) \\ &= \left(L_{g(t)^{-1}}\right)_* \left(R_{g(t)}\right)_* \theta_{\widetilde{\gamma}(t)} \left(\frac{d\widetilde{\gamma}}{dt}(t) \right) + \left(L_{g(t)^{-1}}\right)_* \frac{dg}{dt}(t) \\ \iff \left(R_{g(t)^{-1}}\right)_* \left(\frac{dg}{dt}(t) \right) &= -\theta_{\widetilde{\gamma}(t)} \left(\frac{d\widetilde{\gamma}}{dt}(t) \right) \qquad (\star) \end{aligned}$$

So we study $(\star)$.

First we show uniqueness of horizontal lift. If $\widetilde{\gamma}$, $\widetilde{\eta}$ are horizontal lifts of c, then $\pi_P \circ \widetilde{\gamma} = c = \pi_P \circ \widetilde{\eta} \implies \widetilde{\eta}(t) = \widetilde{\gamma}(t) \cdot g(t)$ for $g : [0,1] \to G$. Since $\widetilde{\gamma}$ is horizontal, $(\star)$ implies $\frac{dg}{dt} = 0$, hence $g(t) \equiv e$, $\widetilde{\eta} = \widetilde{\gamma}$.

$\boxed{\text{Claim. (Existence)}}$ For any $X : [0,1] \to \mathfrak{g}$, $\exists!$ $g : [0,1] \to G$ s.t.

$$\begin{cases} \left(R_{g(t)^{-1}}\right)_* \left(\frac{dg}{dt}(t)\right) = X(t) \\ g(0) = e \end{cases} \qquad (\star\star)$$

$\boxed{\text{Note}}$ When $X(t)$ is t-independent, $(\star\star)$ is flow equation of right invariant vector field $X(0)^{\sharp_R}$ given by $\left(X(0)^{\sharp_R}\right)_g = (R_g)_* X(0)$, $g \in G$. So $(\star\star)$ is "t-dependent" version.

Now we prove the claim. The proof of uniqueness is similar to the proof above. The proof of existence is similar to the proof for left invariant vector field. $\square$

Remark 1.9. When $G \subseteq GL(r; \mathbb{K})$, we can take $g(t) = \exp(\int_0^t X(s)ds)$.

Now we describe parallel transport of vector bundles in terms of principal bundles.

Proposition 1.11. Let $\pi_P : P \to M$ be principal G-bundle with a connection form $\theta \in \Omega^1(P; \mathfrak{g})$. For a representation $\rho : G \to GL(V)$, $E := P \underset{\rho}{\times} V$, ∇^E be the induced connection on E from θ. Take $\forall c : [0,1] \to M$ and any horizontal lift $\widetilde{c}$ of c, define $\gamma \in \Gamma(c^*E)$ by $\gamma(t) = (t, \widetilde{c}(t)) \underset{\rho}{\times} v(t)$ for $v : [0,1] \to V$. Then

(1) $\left(\nabla^{c^*E}_{\frac{d}{dt}} \gamma\right)_t = (t, \widetilde{c}(t)) \underset{\rho}{\times} \frac{dv}{dt}(t)$, where ∇^{c*E} is induced connection on c^*E from ∇^E.

(2) Parallel transport along c w.r.t ∇^E is given by

$$P_c : E_{c(0) \to E_{c(1)}}, \ \widetilde{c}(0) \underset{\rho}{\times} v \mapsto \widetilde{c}(1) \underset{\rho}{\times} v.$$

Proof. (1) First note: $\forall X \in \mathfrak{X}(M)$, let $\widetilde{X} \in \mathfrak{X}(P)$ s.t. $(\pi_P)_*(\widetilde{X}) = X$, then

$$\begin{array}{ccc} \Omega^{q+1}(E) & \xrightarrow{i(X)} & \Omega^q(E) \\ \pi_P^* \downarrow \cong & \circlearrowleft & \cong \downarrow \pi_P^* \\ \Omega_B^{q+1}(P; V) & \xrightarrow{i(\widetilde{X})} & \Omega_B^q(P; V) \end{array}$$

Indeed, for $s = \alpha \otimes e$ where $\alpha \in \Omega^{q+1}(M), e \in \Gamma(E)$,

$$\pi_P^* \left(i(X)s\right) = \pi_P^* \left(i(X)\alpha\right) \otimes \pi_P^* e = i(\widetilde{X}) \pi_P^* s.$$

Use the following notation:

$$\begin{array}{ccc} c^*P & \xrightarrow{c_P} & P \\ \pi_{c^*P} \downarrow & \circlearrowleft & \downarrow \pi_P \\ [0,1] & \xrightarrow{c} & M \end{array}$$

By proposition before, we have

$$\nabla^{c^*E}\gamma = \left((\pi_{c^*P}^*)^{-1} \circ (d + \rho_*(c_P^*\theta)) \circ \pi_{c^*P}\right)(\gamma)$$

Denote by $\widetilde{\frac{d}{dt}} \in \mathfrak{X}(c^*P)$ the horizontal lift of $\frac{d}{dt}$, then

$$\begin{aligned} \nabla^{c^*E}_{\frac{d}{dt}}\gamma &= i\left(\frac{d}{dt}\right)\left(\nabla^{c^*E}\gamma\right) \\ &= (\pi_{c^*P}^*)^{-1}\left(i\left(\widetilde{\frac{d}{dt}}\right)(d + \rho_*(c_P^*\theta))\,(\pi_{c^*P}\gamma)\right). \end{aligned}$$

Since $\forall \widetilde{s} \in \Omega_B^0(c^*P; V),\ \forall t \in [0,1]$,

$$\left((\pi_{c^*P}^*)^{-1}(\widetilde{s})\right)_t = \xi \underset{\rho}{\times} \widetilde{s}(\xi), \quad \text{for } \xi \in \pi_{c^*P}^{-1}(t).$$

For $(t, \widetilde{c}(t)) \in \pi_{c^*P}^{-1}(t)$, we have

$$\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t)\cdot g)} = \left(\frac{d}{dt}, \frac{d}{dt}\widetilde{c}(t)\cdot g\right), \quad t \in [0,1], g \in G.$$

Then we have

$$\nabla^{c^*E}_{\frac{d}{dt}}\gamma = (t, \widetilde{c}(t)) \underset{\rho}{\times} i\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t))}(d + \rho_*(c_P^*\theta))\,(\pi_{c^*P}\gamma).$$

We compute

$$i\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t))} d(\pi_{c^*P}^*\gamma) = \frac{d}{dt}(\pi_{c^*P}^*\gamma)(t, \widetilde{c}(t)) = \frac{d}{dt}(t, \widetilde{c}(t))^{-1}(\gamma(t)) = \frac{d}{dt}v(t)$$

$$i\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t))}(c_P^*\theta) = \theta\left(\frac{d}{dt}c_P(t, \widetilde{c}(t))\right) = 0.$$

Hence $\left(\nabla^{c^*E}_{\frac{d}{dt}}\gamma\right)_t = (t, \widetilde{c}(t)) \underset{\rho}{\times} \frac{dv}{dt}(t)$.

(2) By (1),

$$\nabla^{c^*E}_{\frac{d}{dt}}\gamma = 0 \iff \frac{dv}{dt} = 0 \iff v \text{ is constant.}$$

So it implies (2) by definition of parallel transport.

$\square$

Next we formulate the holonomy group. Let $\pi_P : P \to M$ be a principal G-bundle.

For $x \in M$, set

$$\Omega_x \coloneqq \{c : [0,1] \to M \text{ piecewise } C^\infty \text{ curve} \mid c(0) = c(1) = x\}$$

$$\Omega_x^0 \coloneqq \{c : [0,1] \to M \text{ piecewise } C^\infty \text{ curve} \mid c(0) = c(1) = x,\ [c] = 1 \in \pi_1(M,x)\}$$

Proposition 1.12. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. For $x \in M$, $\xi \in \pi_P^{-1}(x)$, define

$$\Phi_\xi : \Omega_x \to G, \quad \widetilde{c}(1) = \xi \cdot \Phi_\xi(c)$$

where $\widetilde{c}$ is the horizontal lift of c with $\widetilde{c}(0) = \xi$. Then

(1) For $h \in G$, $\Phi_{\xi h}(c) = h^{-1}\Phi_\xi(c)h$.

(2) For $c_1, c_2 \in \Omega_x$, set $c_3 \coloneqq c_2 \cdot c_1$ (c_1 concatenates c_2). Then

$$\Phi_\xi(c_3) = \Phi_\xi(c_2)\Phi_\xi(c_1).$$

(3) Let $\gamma : [0,1] \to M$ be a piecewise C^∞ curve. Set $\gamma(0) = y, \gamma(1) = x$, $\eta \in P_y, \xi \in P_x$. Let $\widetilde{\gamma}$ be the horizontal lift of γ with $\widetilde{\gamma}(0) = \eta$. Set $\widetilde{\gamma}(1) = \xi \cdot h$ for $h \in G$. For $c \in \Omega_x$, define $c' \in \Omega_y$ by $c' = \gamma^{-1} \cdot c \cdot \gamma$. Then

$$\Phi_\eta(c') = h^{-1}\Phi_\xi(c)h.$$

Proof. (1) First note that $\widetilde{c}(t) \cdot h$ is the horizontal lift of c with initial point ξh by the G-invariant property of $\ker\theta$. Now by definition we have

$$\xi\Phi_\xi(c)h = \widetilde{c}(1)h = (\widetilde{c}h)(1) = \xi h \cdot \Phi_{\xi h}(c).$$

Since G-action is free, the proof is finished.

(2) Let $\widetilde{c}_i$ be horizontal lift of c_i with $\widetilde{c}_i(0) = \xi$, then $\widetilde{c}_3 = (\widetilde{c}_2 \cdot \Phi_\xi(c_1)) \cdot \widetilde{c}_1$ is horizontal lift of c_3 with $\widetilde{c}_3(0) = \xi$ (draw a picture to help you understand). Then by definition we have

$$\widetilde{c}_3(1) = \xi\Phi_\xi(c_3) = \widetilde{c}_2(1) \cdot \Phi_\xi(c_1) = \xi\Phi_\xi(c_2)\Phi_\xi(c_1).$$

(3) We want to determine the horizontal lift $\widetilde{c'}$ of c' with $\widetilde{c'}(0) = \eta$. Indeed, we have $\widetilde{c'} = (\widetilde{\gamma}^{-1} \cdot (h^{-1}\Phi_\xi(c)h)) \cdot (\widetilde{c} \cdot h) \cdot \widetilde{\gamma}$ (Draw a picture to help you understand). So we have

$$\widetilde{c'}(1) = \eta \cdot \Phi_\eta(c') = \widetilde{\gamma}^{-1}(1) \cdot h^{-1}\Phi_\xi(c)h.$$

□

Definition 1.15. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. For $\xi \in P$, define

- $Hol_\xi(P,\theta) \coloneqq \Phi_\xi(\Omega_x) = \{\Phi_\xi(c) \mid c \in \Omega_x\} \subseteq G$ called **holonomy group**.
- $Hol_\xi^0(P,\theta) \coloneqq \Phi_\xi(\Omega_x^0) \subseteq G$ called **restricted holonomy group**.

Remark 1.10. $Hol_\xi(P,\theta)$ is a Lie subgroup of G (not necessarily closed). $Hol^0_\xi(P,\theta)$ is unit component of $Hol_\xi(P,\theta)$. For reference, see Kobayashi-Nomizu book.

$Hol_\xi(P,\theta)$, $Hol^0_\xi(P,\theta)$ depends on $\xi \in P$, but the conjugacy class is independent of choice of $\xi \in P$ (just the proposition above).

Recall that holonomy group is also defined for a connection on a vector bundle. For a vector bundle $E \to M$, ∇^E connection on E, take $x \in M$, we have

$$Hol_x(\nabla^E) = \{P_c : T_xM \to T_xM \mid c \in \Omega_x\}$$

where P_c is parallel transport along c and similar definition for $Hol^0_x(\nabla^E)$.

Lemma 1.2. Let $\pi_P : P \to M$ be a principal G-bundle with $\theta \in \Omega^1(P;\mathfrak{g})$ be a connection form. Let ∇^E be the induced connection on associated vector bundle E for a representation $\rho : G \to GL(V)$. Then for $x \in M$, $\xi \in P_x$,

- $Hol_x(\nabla^E) = \xi \circ \rho(Hol_\xi(P,\theta)) \circ \xi^{-1}$.
- $Hol^0_x(\nabla^E) = \xi \circ \rho(Hol^0_\xi(P,\theta)) \circ \xi^{-1}$.

where $\xi : V \to E_x$, $v \mapsto \xi \underset{\rho}{\times} v$.

Proof. The proof is straightforward from proposition 1.11 and 1.12. □

The smaller the holonomy group, the smaller the connection is "globally curved". In the following we show that if holonomy group is small, we can make structure group smaller and make principal bundle itself smaller.

Definition 1.16. Let $\pi_P : P \to M$ be a principal G-bundle, $\pi_Q : Q \to M$ be a principal H-bundle.

(1) Q is a **subbundle** of P if

- $\exists$ an embedding $\iota_Q : Q \hookrightarrow P$ s.t. $\pi_P \circ \iota_Q = \pi_Q$.
- $\exists$ an injective homomorphism $\iota_H : H \hookrightarrow G$ s.t. $\forall \eta \in Q, \forall h \in H$,

$$\iota_Q(\eta h) = \iota_Q(\eta)\iota_H(h).$$

In this case, we can say that the structure group G of P is **reduced** to H.

(2) In the setting of (1), a connection form $\theta \in \Omega^1(P;\mathfrak{g})$ is said to be **reducible** to Q if for $\forall \eta \in Q$, $(\ker\theta)_{\iota_Q(\eta)} \subseteq (\iota_Q)_{*\eta}(T_\eta Q)$.

Suppose Q is a subbundle of P. For sumplicity, suppose that H is a Lie subgroup of G and ι_H the inclusion. We identify H with $\iota_H(H)$ and the corresponding Lie algebra. In this setting, we have

Lemma 1.3. Let $\theta \in \Omega^1(P;\mathfrak{g})$ be connection form of P. Then θ is reducible to Q iff $\iota_Q^*\theta$ is $\mathfrak{h}$-valued. In this case, $\iota_Q^*\theta$ is a connection form on Q.

Proof. We show

($\Rightarrow$) $\forall \eta \in Q, \forall v \in T_\eta Q$, we have $(\iota_Q)_*(v) = w_1 + w_2 \in \ker \theta_{\iota_Q(\eta)} \oplus \ker(\pi_P)_{*\iota_Q(\eta)}$. Since θ is reducible, $\exists v_1 \in T_\eta Q$ s.t. $w_1 = (\iota_Q)_*(v_1)$ and then $w_2 = (\iota_Q)_*(v_2)$ for $v_2 = v - v_1$. Then

$$0 = (\pi_P)_*(w_2) = (\pi_Q)_*(v_2) \iff v_2 = X_\eta^\sharp \text{ for } X \in \mathfrak{h}.$$

Thus

$$\begin{aligned}
\theta_{\iota_Q(\eta)}\left((\iota_Q)_*(v)\right) &= \theta_{\iota_Q(\eta)}\left((\iota_Q)_*(X_\eta^\sharp)\right) \\
&= \theta_{\iota_Q(\eta)}\left(\frac{d}{dt}\iota_Q(\eta \cdot \exp_H(tX))\right)|_{t=0} \\
&= \theta_{\iota_Q(\eta)}\left(\frac{d}{dt}\iota_Q(\eta) \cdot \exp_H(tX)\right)|_{t=0} \\
&= \theta_{\iota_Q(\eta)}\left(\frac{d}{dt}\iota_Q(\eta) \cdot \exp_G(tX)\right)|_{t=0} \\
&= \theta_{\iota_Q(\eta)}(X^\sharp_{\iota_Q(\eta)}) \\
&= X \in \mathfrak{h}.
\end{aligned}$$

($\Leftarrow$) Under the assumption it's easily shown that $\iota_Q^*\theta$ is a connection form on Q. We show that θ is reducible. Since $\iota_Q^*\theta$ is a connection form on Q, we have

$$(\star)\begin{cases} \dim\ker(\iota_Q^*\theta)_\eta = \dim M, & \forall \eta \in Q \\ \dim\ker\theta_\xi = \dim M, & \forall \xi \in P \end{cases}$$

Then $\ker\theta_{\iota_Q(\eta)} = (\iota_Q)_*\left(\ker(\iota_Q^*\theta)_\eta\right) \subseteq (\iota_Q)_*(T_\eta Q)$ for $\forall \eta \in Q$ by dimension condition.

□

Exercise 1.3. Show that $\iota_Q^*\theta$ is a connection form on Q.

More generally we have the following

Lemma 1.4. Let $\pi_P : P \to M$ be a principal G-bundle, $\pi_Q : Q \to M$ be a principal H-bundle. Suppose $H \subseteq G$ and Q is a subbundle of P with embedding ι_Q. Assume that $\exists$ a subspace $\mathfrak{m} \subseteq \mathfrak{g}$ s.t.

$$\begin{cases} \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \\ Ad(H)\mathfrak{m} = \mathfrak{m} \end{cases}$$

Denote by $P_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$, $P_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ the projections. Then for $\forall$ connection form $\theta \in \Omega^1(P; \mathfrak{g})$,

- $P_{\mathfrak{h}}(\iota_Q^* \theta) \in \Omega^1(Q; \mathfrak{h})$ is a connection form on Q.
- $P_{\mathfrak{m}}(\iota_Q^* \theta) \in \Omega^1(Q; \mathfrak{m})$ is basic w.r.t. H-representation $(\mathfrak{m}, Ad)$.

Exercise 1.4. Prove this lemma. For reference you can check [Kobayashi-Nomizu, Vol I, p.83, Prop 6.4].

Example 1.6. Let $\pi_E : E \to M$ be a vector bundle of rank r, $\pi_P : P \to M$ be the frame bundle. Recall that the frame bundle $P = \bigsqcup_{x \in M} P_x$ where $P_x = \{\xi : \mathbb{K}^r \to E_x \mid \text{linear isomorphism}\}$. In addition, we consider a fiber metric h on E. Denoted by $(\cdot, \cdot)_{std}$ the standard inner product on $\mathbb{K}^r$. For $x \in M$, set

$$\begin{cases} Q_x := \{\xi \in P_x \mid h_x(\xi(\cdot), \xi(\cdot)) = (\cdot, \cdot)_{std}\} \\ Q := \bigsqcup_{x \in M} Q_x \\ \pi_Q : Q \to M, \quad \xi \mapsto x \text{ if } \xi \in Q_x \end{cases}$$

Setting $H = \begin{cases} U(r) & (if\ \mathbb{K} = \mathbb{C}) \\ O(r) & (if\ \mathbb{K} = \mathbb{R}) \end{cases}$, we can show that Q is a principal H-bundle and a subbundle of the frame bundle P. We call Q the **frame bundle of** (E, h).

Now suppose that a connection form $\theta \in \Omega^1(P; \mathfrak{g})$ is given ($\mathfrak{g} = End(\mathbb{K}^r)$). Denote by ∇^θ the induced connection on E (or $E^* \otimes E^*$) from θ. Then we have the following lemma:

Lemma 1.5. $\nabla^\theta h = 0 \iff \theta$ is reducible to Q.

Proof. Note that $h \in \Omega^0(E^* \otimes E^*)$.

- We first describe ∇^θ on $E^* \otimes E^*$ explicitly. Set $V := \mathbb{K}^r$, then

  $$P \underset{\rho'}{\times} (V^* \otimes V^*) \overset{\cong}{\to} E^* \otimes E^*, \quad \xi \underset{\rho'}{\times} h_0 \mapsto h_0(\xi^{-1}(\cdot), \xi^{-1}(\cdot)),\ h_0 \in V^* \otimes V^*$$

  where

  $$\rho' : G = GL(V) \to GL(V^* \otimes V^*), \quad \rho'(g)(h_0) = h_0(g^{-1}(\cdot), g^{-1}(\cdot)),\ h_0 \in V^* \otimes V^*$$

Via $\pi_P^* : \Omega^q(E^* \otimes E^*) \xrightarrow{\cong} \Omega^Q(P; V^* \otimes V^*)$, from proposition 1.6 we have:

$$\begin{array}{ccc} \Omega^0(E^* \otimes E^*) & \xrightarrow{\nabla^\theta} & \Omega^1(E^* \otimes E^*) \\ \pi_P^* \ \downarrow \ \cong & \circlearrowleft & \pi_P^* \ \downarrow \ \cong \\ \Omega^0_B(P; V^* \otimes V^*) & \xrightarrow{d+\rho'_*(\theta)} & \Omega^1_B(P; V^* \otimes V^*) \end{array}$$

- Denote by $\iota_Q : Q \hookrightarrow P$ the embedding, then

$$Q \underset{\rho'|_H}{\times} (V^* \otimes V^*) \xrightarrow{\iota_Q \times id} P \underset{\rho'}{\times} (V^* \otimes V^*) \cong E^* \otimes E^*$$

is an isomorphism, $\dim(V^* \otimes V^*) = \operatorname{rank}(E^* \otimes E^*)$. So we can define $\Omega^q_B(Q; V^* \otimes V^*)$ for H-representation $(V^* \otimes V^*, \rho'|_H)$, $\Omega^q_B(Q; V^* \otimes V^*) \cong \Omega^q(E^* \otimes E^*)$. Recall the pullback ι_Q^* of $\Omega^q(P; V^* \otimes V^*)$ to $\Omega^q(Q; V^* \otimes V^*)$ via ι_Q.

Claim.

(a) $\iota_Q^*(\Omega^q_B(P; V^* \otimes V^*) \subseteq \Omega^q_B(Q; V^* \otimes V^*)$.

(b)

$$\begin{array}{ccc} \Omega^q_B(P; V^* \otimes V^*) & \xrightarrow{\iota_Q^*} & \Omega^q_B(Q; V^* \otimes V^*) \\ \pi_P^* \ \nwarrow \ \cong & \circlearrowleft & \cong \ \nearrow \ \pi_Q^* \\ & \Omega^q(E^* \otimes E^*) & \end{array}$$

In particular, ι_Q^* is an isomorphism.

Proof of the claim.

(a) Take $\forall \widetilde{s} \in \Omega^q_B(P; V^* \otimes V^*)$. for $\forall X \in \mathfrak{h}$, $\forall \eta \in Q$,

$$\begin{aligned} (\iota_Q)_*(X^\sharp_\eta) &= \frac{d}{dt}\iota\left(\eta \cdot \exp_H(tX)\right)|_{t=0} = \frac{d}{dt}\iota_Q(\eta) \cdot \exp_H(tX)|_{t=0} \\ &= \frac{d}{dt}\iota_Q(\eta) \cdot \exp_G(tX)|_{t=0} = X^\sharp_{\iota_Q(\eta)}. \end{aligned}$$

So $i(X^\sharp_\eta)(\iota_Q^* \widetilde{s})_\eta = \iota_Q^*\left(i(X^\sharp)\widetilde{s}\right)_\eta = 0$.

For $\forall h \in H$, $\forall \eta \in Q$, we have $\iota_Q(\eta \cdot h) = \iota_Q(\eta) \cdot h \iff \iota_Q \circ R_h = R_h \circ \iota_Q$. Then

$$R_h^* \iota_Q^* \widetilde{s} = \iota_Q^* R_h^* \widetilde{s} = \rho'(h^{-1}) \iota_Q^* \widetilde{s}.$$

Hence $\iota_Q^* \widetilde{s}$ is basic.

(b) This will follow from $\pi_Q = \pi_P \circ \iota_Q$. We only have to show the commutativity when $q = 0$ since the differential form part is the original pullback.

When $q = 0$, for $\forall s \in \Omega^0(E^* \otimes E^*)$, $\forall \eta \in Q$, we have

$$(\iota_Q^* \pi_P^* s)(\eta) = (\pi_P^* s)(\iota_Q(\eta)) = \iota_Q(\eta)^{-1}\left(s_{(\pi_P \circ \iota_Q)(\eta)}\right) = (\pi_Q^* s)(\eta).$$

hence finishing the claim.

Now we show the lemma.

$$
\begin{aligned}
\nabla^\theta h = 0 &\iff (d + \rho'_*(\theta))\pi_P^* h = 0 \\
&\underset{\iota_Q^*\ iso.}{\iff} \iota_Q^* \left(d + \rho'_*(\theta)\right) \pi_P^* h = 0 \\
&\iff \left(d + \rho'_*(\iota_Q^*\theta)\right) \underbrace{\iota_Q^* \pi_P^* h}_{=\pi_Q^* h} = 0
\end{aligned}
$$

For $\forall \eta \in Q_y$, $y \in M$, we have $(\pi_Q^* h)(\eta) = \eta^{-1}(h_y)$, where $\eta : V^* \otimes V^* \to (E^* \otimes E^*)_y$, $h_0 \mapsto h_0(\eta^{-1}(\cdot), \eta^{-1}(\cdot))$. Since $\eta \in Q_y \iff h_y(\eta(\cdot), \eta(\cdot)) = (\cdot, \cdot)_{std}$, we have

$$
\eta^{-1}(h_y) = h_y(\eta(\cdot), \eta(\cdot)) = (\cdot, \cdot)_{std}.
$$

Hence $\pi_Q^* h \equiv (\cdot, \cdot)_{std}$, and then

$$
\begin{aligned}
\nabla^\theta h = 0 &\iff \rho'_*(\iota_Q^*\theta)\left((\cdot, \cdot)_{std}\right) = 0 \\
&\iff \left((\iota_Q^*\theta)(\cdot), \cdot\right)_{std} + \left(\cdot, (\iota_Q^*\theta)(\cdot)\right)_{std} = 0 \\
&\iff \iota_Q^*\theta \text{ is } \mathfrak{h}\text{-valued} \\
&\iff \theta \text{ is reducible to } Q.
\end{aligned}
$$

□

Remark 1.11. The same argument works if h is replaced with other geometric object, i.e. suppose a tensor $\varphi \in \Gamma(\bigotimes^* E \otimes \bigotimes^* E^*)$, $\varphi_0 \in \Gamma(\bigotimes^* \mathbb{K}^r \otimes \bigotimes^* (\mathbb{K}^r)^*)$ are given (such as fiber metric and standard inner product). Consider

$$
\begin{cases}
Q_x^\varphi := \left\{ \xi : \mathbb{K}^r \xrightarrow{\cong} E_x \mid \xi \cdot \varphi_x = \varphi_0 \right\}, & x \in M \\
Q^\varphi := \bigsqcup_{x \in M} Q_x^\varphi
\end{cases}
$$

The fiber of $Q^\varphi \cong \{g \in GL(r; \mathbb{K}) \mid g \cdot \varphi_0 = \varphi_0\}$. Then for a connection form θ on P,

$$
\nabla^\theta \varphi = 0 \iff \theta \text{ is reducible to } Q^\varphi.
$$

Such a situation can be found when we consider a manifold with special holonomy, which will be explained shortly later.

Let $\pi_P : P \to M$ be a principal G-bundle with a connection $\theta \in \Omega^1(P; \mathfrak{g})$. The horizontal lift of a piecewise C^∞ curve in M is called a piecewise C^∞ **horizontal curve**.

For $\xi, \eta \in P$, define

$$
\xi \sim \eta \iff \exists \text{ piecewise } C^\infty \text{ horizontal curve connecting } \xi \text{ and } \eta.
$$

Then we can check that $\sim$ is equivalence relation.

Theorem 1.4. (**Reduction Theorem**) Let $\pi_P : P \to M$ be a principal G-bundle with a connection $\theta \in \Omega^1(P; \mathfrak{g})$. Fix $\xi \in P$, set $Q := Q(\xi) = \{\eta \in P \mid \xi \sim \eta\}$, $H := Hol_\xi(P, \theta) \subseteq G$. Then

- $\pi_P|_Q : Q \to M$ is a principal H-bundle and Q is a subbundle of P.
- θ is reducible to Q.

Proof. (1) [Kobayashi-Nomizu, Vol I, p.83, Thm 7.1]

(2) For $\forall \eta \in Q$, we show $(\ker\theta)_\eta \subseteq T_\eta Q$. Take $\forall v \in (\ker\theta)_\eta$, then $\exists\ \widetilde{c} : [0,1] \to P$ horizontal s.t. $\widetilde{c}(0) = \eta$, $\frac{d\widetilde{c}}{dt}(0) = v$. Thus for $\forall t$, $\widetilde{c}(t) \sim \eta \sim \xi$, i.e. $\widetilde{c}(t) \in Q$, so $v \in T_\eta Q$. $\square$

Remark 1.12. $Q = Q(\xi)$ is called **holonomy subbundle** for a connection $\theta \in \Omega^1(P; \mathfrak{g})$.

- For $\xi, \eta \in P$,
$$Q(\xi) = Q(\eta) \iff \xi, \eta \text{ are joined by horizontal curve}$$
So under $\sim$, either $Q(\xi) = Q(\eta)$ or $Q(\xi) \cap Q(\eta) = \emptyset$, i.e. $P = \bigsqcup Q(\xi)$.
- $Q(\xi)$, $\xi \in P$ are mutually isomorphic.

Proposition 1.13. For a given connection θ, holonomy subbundle is smallest subbundle to which θ is reducible, i.e. for $\forall$ subbundle $\iota_Q : Q \to P$ s.t. θ is reducible to Q, we have $Q(\iota_Q(\eta)) \subseteq \iota_Q(Q)$ for $\forall \eta \in Q$.

Proof. Fix $\eta_0 \in Q$, $\xi_0 := \iota_Q(\eta_0)$. Take $\forall \xi \in Q(\xi_0)$, we show $\xi \in \iota_Q(Q)$. By definition, $\exists$ piecewise C^∞ horizontal curve $\widetilde{c} : [0,1] \to P$ s.t. $\widetilde{c}(0) = \xi_0$, $\widetilde{c}(1) = \xi$. Set $c := \pi_P \circ \widetilde{c}$, take $\widetilde{c}^Q$ be the horizontal lift of c for $(Q, \iota_Q^*\theta)$. We can show $\widetilde{c} = \iota_Q \circ \widetilde{c}^Q$ (use uniqueness of horizontal lift), hence finishing the proof. $\square$

1.5 H-Structures and Intrinsic Torsion

Let M be a n-dim smooth manifold. Consider $E = TM \to M$. Recall frame bundle $\pi_P : P \to M$ of E: $P = \bigsqcup_{x \in M} P_x$,
$$P_x := \left\{\xi : \mathbb{R}^n \xrightarrow{\cong} T_x M \mid \text{linear isomorphism}\right\}, \quad x \in M.$$
P is a principal $GL(n; \mathbb{R})$-bundle.

Definition 1.17. For a subgroup $H \subseteq GL(n; \mathbb{R})$, an **H-structure** on M is a subbundle Q of P with fiber H.

We can define a special differential form on H-structure: Let $\pi_Q : Q \to M$ be H-structure on M. Define the **tautological 1-form (solder form)** $\omega \in \Omega^1(Q;\mathbb{R}^n)$ by $\omega_\eta(v) := \eta^{-1}\left((\pi_Q)_*(v)\right)$ for $\eta \in Q$, $v \in T_\eta Q$. We can check properties of ω:

- Let $\{e_1,\dots,e_n\}$ be a frame on $U \underset{\text{open}}{\subseteq} M$. This define a local section p on P:

$$p(x) : \mathbb{R}^n \xrightarrow{\cong} T_xM, \quad (x^1,\dots,x^n)^t \mapsto (e_1,\dots,e_n)(x^1,\dots,x^n)^t.$$

  Suppose that $p(x) \in Q$ for $\forall x \in U$, then $p^*\omega \in \Omega^1(U;\mathbb{R}^n)$ satisfies $(p^*\omega)(e_i) = \varepsilon_i$, where $\varepsilon_i = (0,\dots,\underset{\text{i-th}}{1},\dots,0)^t$. So $p^*\omega = \sum e^i \otimes \varepsilon_i$ where $\{e^i\}$ is the dual of $\{e_i\}$.

- Let $i : H \hookrightarrow GL(n;\mathbb{R})$ be inclusion, $V = \mathbb{R}^n$. We can consider i as an H-representation on V and $Q \underset{i}{\times} V \xrightarrow{\cong} TM$, $\eta \underset{i}{\times} v \mapsto \eta(v)$. So we have an isomorphism

$$\pi_Q^* : \Omega^q(TM) \xrightarrow{\cong} \Omega_B^q(Q;V).$$

  We can check $\omega \in \Omega_B^1(Q;V)$:

  - $\forall X \in \mathfrak{h}$, $i(X^\sharp)\omega = 0$.
  - $\forall h \in H$, $\forall \eta \in Q$, $(R_h^*\omega)_\eta = h^{-1}\eta^{-1}(\underbrace{(\pi_Q)_*(R_h)_*}_{(\pi_Q)_*}(\cdot)) = h^{-1}\omega_\eta$.

  Since we know $\Omega^1(TM) = \Gamma(T^*M \otimes TM)$, take $id_{TM} \in \Gamma(T^*M \otimes TM)$, we will show $\boxed{\pi_Q^*(id_{TM}) = \omega}$. Indeed, $\forall \eta \in Q$, $\forall v \in T_\eta Q$, we have

$$\left(\pi_Q^*(id_{TM})\right)_\eta(v) = \eta^{-1}\left((id_{TM})_{\pi_Q(\eta)}(\pi_Q)_*(v)\right) = \omega_\eta(v).$$

Now take a connection form $\theta \in \Omega^1(Q;\mathfrak{h})$, we will define the torsion of θ next. Recall that θ induces a connection $\nabla = \nabla^\theta$ on $TM \cong Q \underset{i}{\times} V$. By proposition 1.6 we have:

$$\begin{array}{ccc}
\Omega^1(TM) & \xrightarrow{d^\nabla} & \Omega^2(TM) \\
\pi_Q^* \downarrow\cong & \circlearrowleft & \cong\downarrow \pi_Q^* \\
\Omega_B^1(Q;V) & \underset{=\ (d+\theta\wedge)}{\xrightarrow{d+\rho_*(\theta)\wedge}} & \Omega_B^2(Q;V)
\end{array}$$

Recall the torsion $T^\nabla \in \Omega^2(TM)$ of ∇: $T^\nabla = d^\nabla(id_{TM})$. More explicitly, $\forall X,Y \in \mathfrak{X}(M)$,

$$\left(d^\nabla(id_{TM})\right)(X,Y) = \nabla_X\left(id_{TM}(Y)\right) - \nabla_Y\left(id_{TM}(X)\right) - id([X,Y]) = \nabla_XY - \nabla_YX - [X,Y].$$

Then

$$\pi_Q^*\left(d^\nabla(id_{TM})\right) = (d+\theta\wedge)\pi_Q^*(id_{TM}) \iff d\omega + \theta\wedge\omega = \pi_Q^*(T^\nabla) \quad (\star).$$

We will rewrite RHS.

Proposition 1.14. $\forall s \in \Omega^q(TM)$, $\exists\, T_s : Q \to V \otimes \bigwedge^q V^*$ an H-equivariant map s.t.

$$\pi_Q^* s = \frac{1}{q!} T_s(\underbrace{\omega \wedge \cdots \wedge \omega}_{q})$$

Proof. By definition, $\forall \eta \in Q$, $\forall v_1, \ldots, v_q \in T_\eta Q$,

$$(\pi_Q^* s)_\eta(v_1, \ldots, v_q) = \eta^{-1}\left(s_{\pi_Q(\eta)}\left((\pi_Q)_*(v_1), \ldots, (\pi_Q)_*(v_q)\right)\right).$$

Since $(\pi_Q)_*(v_i) = \eta\eta^{-1}(\pi_Q)_*(v_i) = \eta\omega_\eta(v_i)$, define $T_s(\eta) := \eta^{-1}\left(s_{\pi_Q(\eta)}\left(\eta(\cdot), \ldots, \eta(\cdot)\right)\right)$, then set $\omega = \sum \omega^i \varepsilon_i$ for $\omega^i \in \Omega^1(Q)$, we have

$$\begin{aligned}
\left(\frac{1}{q!} T_s(\omega \wedge \cdots \wedge \omega)\right)(v_1, \ldots, v_q) &= \frac{1}{q!} \sum_{i_1, \ldots, i_q} T_s(\varepsilon_{i_1}, \ldots, \varepsilon_{i_q})(\omega^{i_1} \wedge \cdots \wedge \omega^{i_q})(v_1, \ldots, v_q) \\
&= \frac{1}{q!} \sum_{i_1, \ldots, i_q} T_s(\varepsilon_{i_1}, \ldots, \varepsilon_{i_q}) \det\left(\omega^{i_k}(v_l)\right) \\
&= \frac{1}{q!} \sum_{i_1, \ldots, i_q} T_s(\varepsilon_{i_1}, \ldots, \varepsilon_{i_q}) \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) \omega^{i_1}(v_{\sigma(1)}) \cdots \omega^{i_q}(v_{\sigma(q)}) \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) T_s\left(\omega(v_{\sigma(1)}), \ldots, \omega(v_{\sigma(q)})\right) \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} T_s\left(\omega(v_1), \ldots, \omega(v_q)\right) \\
&= T_s\left(\omega(v_1), \ldots, \omega(v_q)\right) \\
&= (\pi_Q^* s)_\eta(v_1, \ldots, v_q).
\end{aligned}$$

□

Remark 1.13. T_s is understand as follows. Consider $s \in \Omega^q(TM) = \Omega^0(TM \otimes \bigwedge^q T^*M)$, then we have

$$\widetilde{\pi_Q}^* : \Omega^0(TM \otimes \bigwedge^q T^*M) \to \Omega^0_B(Q; V \otimes \bigwedge^q V^*).$$

For $\eta \in Q$, $(\widetilde{\pi_Q}^* s)(\eta) = \widetilde{\eta}^{-1}(s_{\pi_Q(\eta)})$, $T_s = \widetilde{\pi_Q}^* s$ where

$$\widetilde{\eta} : V \otimes \bigwedge^q V^* \to T_x M \otimes \bigwedge^q T_x^* M, \quad v \otimes \alpha \mapsto \eta(v) \otimes \alpha(\eta^{-1}(\cdot), \ldots, \eta^{-1}(\cdot))$$

Definition 1.18. By this proposition, $T_\theta := \widetilde{\pi_Q}^* T^\nabla \in \Omega^0_B(Q; V \otimes \bigwedge^2 V^*)$ satisfies

$$\pi_Q^*(T^\nabla) = \frac{1}{2} T_\theta(\omega \wedge \omega).$$

T_θ is called the **torsion of** θ.

Then $(\star)$ becomes:

$$d\omega + \theta \wedge \omega = \frac{1}{2}T_\theta(\omega \wedge \omega) \quad \textbf{Cartan's first structure equation}$$

Next we define intrinsic torsion.

Lemma 1.6. Take connections $\theta_1, \theta_2 \in \Omega^1(Q; \mathfrak{h})$ on Q with torsion $T_{\theta_1}, T_{\theta_2}$ respectively.

(1) $\exists\, p : Q \to \mathfrak{h} \otimes V^*$ an H-equivariant map s.t. $\theta_1 - \theta_2 = p(\omega)$. Conversely, for $\forall \widetilde{p} : Q \to \mathfrak{h} \otimes V^*$ an H-equivariant map, $\forall \theta$ connection, we have $\theta + \widetilde{p}\omega$ is a connection form on Q.

2 Complex Manifolds

Roughly speaking, complex manifold is smooth manifold on which holomorphic functions are defined. It's fundamental objects in many fields such as differential geometry, function theory of several complex variables, algebraic geometry and mathematical physics.

2.1 Complex Manifolds and Complex Differential Forms

We start from introducing complex differential forms and holomorphic functions on an open subset of $\mathbb{C}^m$.

Set: (1) $V \subseteq \mathbb{C}^m$ open subset; (2) $(z^1, \ldots, z^m)$ standard coordinates on V; (3) $z^i = x^i + \sqrt{-1}y^i$, $x^i, y^i \in \mathbb{R}$.

For $p \in V$, set

$$\left(\frac{\partial}{\partial z^i}\right)_p := \frac{1}{2}\left(\left(\frac{\partial}{\partial x^i}\right)_p - \sqrt{-1}\left(\frac{\partial}{\partial y^i}\right)_p\right), \left(\frac{\partial}{\partial \bar{z}^i}\right)_p := \frac{1}{2}\left(\left(\frac{\partial}{\partial x^i}\right)_p + \sqrt{-1}\left(\frac{\partial}{\partial y^i}\right)_p\right),$$

$$(dz^i)_p := (dx^i)_p + \sqrt{-1}(dy^i)_p, \quad (d\bar{z}^i)_p := (dx^i)_p - \sqrt{-1}(dy^i)_p.$$

$$(T_p\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C} : \text{ the complexification of } T_p\mathbb{C}^m$$

Then it's clear that $\left(\frac{\partial}{\partial z^i}\right)_p, \left(\frac{\partial}{\partial \bar{z}^i}\right)_p \in (T_p\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C}$, $(dz^i)_p, (d\bar{z}^i)_p \in (T_p^*\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C}$. We see that $\left\{\left(\frac{\partial}{\partial z^i}\right)_p, \left(\frac{\partial}{\partial \bar{z}^i}\right)_p\right\}_{i=1}^m$ is a basis of $(T_p\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C}$, same for the cotangent case.

Proposition 2.1. For a C^1-function $f = g + \sqrt{-1}h : V \to \mathbb{C}$, set

$$(df)_p := (dg)_p + \sqrt{-1}(dh)_p \in T_p^*\mathbb{C}^m \otimes_{\mathbb{R}} \mathbb{C}.$$

Then

$$(df)_p = \left(\left(\frac{\partial}{\partial z^i}\right)_p f\right)(dz^i)_p + \left(\left(\frac{\partial}{\partial \bar{z}^i}\right)_p f\right)(d\bar{z}^i)_p.$$

Proof. Routine computation. □

Definition 2.1. A C^1-function $f : V \to \mathbb{C}$ is a **holomorphic function** if $\frac{\partial f}{\partial \bar{z}^i} = 0$ on V. For an open subset $W \subseteq \mathbb{C}^n$, a map $F = (f^1, \ldots, f^n) : V \to W$ is a **holomorphic map** if each f^i is a holomorphic function.

Remark 2.1. It is known that a holomorphic function is analytic. In particular, it's a C^∞ function.

Lemma 2.1. The composition of holomorphic maps is a holomorphic map.

Definition 2.2. A topological space M is an m-dimensional **complex manifold** if

(1) M is a Hausdorff space.

(2) $\exists \{U_\alpha\}_{\alpha \in A}$ open cover of M, $\forall \alpha \in A$, $\exists \varphi_\alpha : U_\alpha \xrightarrow{\text{homeo.}} V_\alpha \underset{\text{open}}{\subseteq} \mathbb{C}^m$ s.t. if $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \text{ is a holomorphic map}$$

$\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called a **holomorphic coordinate system**, φ_α is called local coordinates. m-dimensional complex manifold is a $2m$-dimensional smooth manifold.

A C^1-map