BASICS OF DIFFERENTIAL GEOMETRY

Principle Bundles and Characteristic Classes

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Contents

Introduction

Last semester:

- Geometry of vector bundles
- Basic Riemannian geometry
- Differential operators on manifolds

We will learn this semester:

- Theory of principle bundles
- Characteristic classes

Before entering formal study, I want to summarize the history of principle bundle and characteristic class, hoping it can provide some motivation for studying these stuff. The following are given by DeepSeek.

Principle Bundles

Principal bundles emerged from the interplay of differential geometry, topology, and theoretical physics. This document traces their evolution, emphasizing how mathematical innovations and physical intuitions reinforced one another.

Early Foundations (1920s–1930s)

Élie Cartan and Moving Frames

- In the 1920s, Élie Cartan revolutionized differential geometry using **moving frames** (repère mobile). By attaching a local orthonormal frame $\{e_i\}$ to each point on a manifold M, he encoded geometric data (curvature, torsion) via **connection forms** ω_i^j, satisfying the Cartan structure equation:

  $$d\omega^j = \omega^i \wedge \omega_i^j + \Omega^j,$$

  where Ω^j is the curvature. This implicitly defined the **frame bundle** $F(M)$, a principal $\mathrm{GL}(n, \mathbb{R})$-bundle over M.

Hermann Weyl's Gauge Theory

- In 1918, Hermann Weyl proposed a failed unified theory of gravity and electromagnetism by introducing a **gauge symmetry** (scale invariance). By the 1920s, he reinterpreted this as a phase symmetry $\psi \mapsto e^{i\theta}\psi$, linking it to the group $\mathrm{U}(1)$. Though not yet framed in bundle terms, this presaged the idea of a principal G-bundle with G as the symmetry group.
- **Connection to Cartan**: Weyl's gauge transformations generalized Cartan's local frame adjustments, but with a focus on physics. Cartan's connection forms would

later formalize Weyl's intuition.

Formalization (1940s–1950s)

Ehresmann Connections and Fiber Bundles

- Charles Ehresmann, a student of Cartan, axiomatized connections in the 1940s. An **Ehresmann connection** on a principal G-bundle $P \xrightarrow{\pi} M$ is a splitting $TP = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \ker(d\pi)$. The connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfies:

$$\omega(A^\sharp) = A, \quad R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega,$$

  for $A \in \mathfrak{g}$, $A^\sharp$ the fundamental vector field, and R_g the right G-action.
- **Bridge to Physics**: Ehresmann's work provided the geometric language later used by physicists to describe gauge fields.

Topology of Fiber Bundles

- Hassler Whitney (1935) and Norman Steenrod (*The Topology of Fibre Bundles*, 1951) formalized fiber bundles. Steenrod showed that equivalence classes of principal G-bundles over X correspond to homotopy classes of maps $X \to BG$, where BG is the **classifying space** of G.
- **Cross-pollination**: Chern-Weil theory (1940s) linked curvature to characteristic classes (e.g., Chern classes $c_k \in H^{2k}(M, \mathbb{Z})$), connecting differential geometry (Cartan, Ehresmann) to algebraic topology (Steenrod).

Physics and Gauge Theory (1950s–1970s)

Yang-Mills Theory

- In 1954, Yang and Mills generalized Maxwell's theory by replacing U(1) with SU(2). A Yang-Mills field is a connection ∇ on a principal SU(2)-bundle, with curvature F_∇ governing particle interactions.
- **Mathematical Impact**: Yang-Mills equations $d_\nabla F_\nabla = 0$, $d_\nabla \star F_\nabla = J$ drove advances in PDEs and 4-manifold topology.

Geometric Unification

- By the 1970s, Kobayashi (Kobayashi-Nomizu, *Foundations of Differential Geometry*) and physicists like Trautman formalized gauge theories using principal bundles. The **adjoint bundle** $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$ became key for symmetry-breaking mechanisms.

- **Legacy of Cartan**: Cartan's structure equations reappeared as the Maurer-Cartan equation in gauge theory:

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = \Omega.$$

Modern Developments (1980s–Present)

Topological Quantum Field Theory (TQFT)

- Principal bundles underpin **Donaldson invariants** (1983) and **Seiberg-Witten theory** (1994), which study connections on SU(2)-bundles over 4-manifolds. These revealed exotic smooth structures, linking analysis (Yang-Mills) to topology.

Algebraic Geometry and Arithmetic

- Grothendieck recast principal bundles as **torsors** in étale topology, enabling applications to the Langlands program. The moduli stack $\mathrm{Bun}_G(X)$ of G-bundles over a scheme X is central to geometric Langlands.

Conclusion

The history of principal bundles illustrates a dialogue between abstraction and application: Cartan's frames motivated Weyl's gauge theory; Ehresmann's connections enabled Yang-Mills; and Grothendieck's algebraic reformulations bridged number theory and physics. Each advance recontextualized earlier work, showing mathematics as an evolving tapestry of ideas.

1 Principle Bundles

In this section, we introduce the connections of principle bundles, it's closely related to the connections of vector bundles.

1.1 Lie Groups

Definition 1.1. Let G be a smooth manifold. G is a *Lie group* if G is a group s.t. multiplication and inverse are smooth.

Let G be a Lie group, $g \in G$, we denote:

- $L_g : G \to G$, $h \mapsto gh$ (left translation)
- $R_g : G \to G$, $h \mapsto hg$ (right translation)
- $\mathfrak{X}^L(G) = \{X \in \mathfrak{X}(G) \mid \forall g \in G, (L_g)_* X = X\}$ (left invariant vector fields)

For $X \in \mathfrak{X}^L(G)$, $L_{g*}X = X$ means that X is L_g-related to X. Then for $\forall X, Y \in \mathfrak{X}^L(G)$, $L_{g*}([X, Y]) = [L_{g*}X, L_{g*}Y] = [X, Y]$, so $\mathfrak{X}^L(G)$ is closed under $[\cdot, \cdot]$

Definition 1.2. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a $\mathbb{K}$-vector space $\mathfrak{g}$ and a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we say $\mathfrak{g}$ is a *Lie algebra* if:

(1) $\forall X, Y \in \mathfrak{g}$, $[X, Y] = -[Y, X]$

(2) $\forall X, Y, Z \in \mathfrak{g}$, $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

$[\cdot, \cdot]$ is called Lie bracket.

So by definition we have $\left(\mathfrak{X}^L(G), [\cdot, \cdot]\right)$ is a Lie algebra.

Definition 1.3. For Lie algebra $\mathfrak{g}, \mathfrak{h}$, a linear map $f : \mathfrak{g} \to \mathfrak{h}$ is called the *Lie algebra homomorphism* if: $\forall X, Y \in \mathfrak{g}$, $f([X, Y]) = [f(X), f(Y)]$

If f is in addition an isomorphism, then f is called a *Lie algebra isomorphism.*

Let $e \in G$ be the unit of G. Set $\iota : \mathfrak{X}^L(G) \to T_eG$, $X \mapsto X_e$. Then ι is a linear isomorphism. Let $\mathfrak{g} = T_eG$, so we can define the Lie bracket on $\mathfrak{g}$ s.t. ι is a Lie algebra isomorphism, i.e. setting $X^\sharp = \iota^{-1}(X)$, $[X, Y] = \left[X^\sharp, Y^\sharp\right]_e$. Note that $X^\sharp_g = (L_g)_{*e}X$, $g \in G$.

Definition 1.4. Let G be Lie group, $\mathfrak{g} = T_eG$ with $[\cdot, \cdot]$ is called the *Lie algebra of* G. $\left(\left(\mathfrak{X}^L(G), [\cdot, \cdot]\right)\right.$ is also called the Lie algebra of $\left.G\right)$

Definition 1.5. Let G, H be Lie groups. A map $\rho : G \to H$ is a *Lie group homomorphism* if ρ is a smooth map and a group homomorphism. For the special case $(\mathbb{R},+) \to G$, $t \mapsto g_t$, $\{g_t\}_{t\in\mathbb{R}}$ is called *one parameter subgroup of* G.

Proposition 1.1. Let G be Lie group and $\mathfrak{g}$ its Lie algebra. Then

(1) $\forall X \in \mathfrak{g}$, $X^\sharp = \iota^{-1}(X)$ is complete, i.e. $X^\sharp$ generates a flow $\{\varphi_t\}_{t\in\mathbb{R}}$.

(2) Set $\boxed{\exp_G(tX) = \varphi_t(e)} \in G$. Then $\varphi_t = R_{\exp_G(tX)}$.

(3) For $s,t \in \mathbb{R}$, $\exp_G(sX)\exp_G(tX) = \exp_G((s+t)X)$, i.e. $\{\exp_G(tX)\}_{t\in\mathbb{R}}$ is one parameter subgroup of G.

(4) $\mathfrak{g} \to \{\text{one parameter subgroup of } G\}$, $X \mapsto \{\exp_G(tX)\}_{t\in\mathbb{R}}$ is bijective.

Proof. (1) By ODE theory, $\exists\epsilon > 0$, $\gamma_e : (-\epsilon,\epsilon) \to G$ s.t. $\gamma_e(0) = e$, $\dfrac{d\gamma_e}{dt} = X^\sharp_{\gamma_e(t)}$.

$\boxed{\text{Claim 1.}}$ $\forall g \in G$, define $\gamma_g : (-\epsilon,\epsilon) \to G$, $t \mapsto g\gamma_e(t)$ is the integral curve of $X^\sharp$ with $\gamma_g(0) = g$.

Indeed, $\forall t \in (-\epsilon,\epsilon)$, $\dfrac{d\gamma_g}{dt}(t) = (L_g)_{*\gamma_e(t)}\dfrac{d\gamma_e}{dt}(t) = X^\sharp_{g\cdot\gamma_e(t)}$.

$\boxed{\text{Claim 2.}}$ $\gamma_e : (-\epsilon,\epsilon) \to G$ can be extended to integral curve $\gamma_e : \mathbb{R} \to G$ of $X^\sharp$ with $\gamma_e(0) = e$.

Set $\varphi_t = R_{\gamma_e(t)}$, then $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow generated by $X^\sharp$. So the following are easy. $\square$

By this proposition, we can define the exponential map $\exp_G : \mathfrak{g} \to G$.

Proposition 1.2. Let G, H be Lie groups with Lie algebra $\mathfrak{g}, \mathfrak{h}$. If $f : G \to H$ is Lie group homomorphism, then $f_{*e} : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. We only need to show that $X^\sharp$ and $(f_{*e}X)^\sharp$ are f-related. Since $X = \dfrac{d}{dt}\exp_G(tX)|_{t=0}$, we have $f_{*g}(X^\sharp_g) = \dfrac{d}{dt}f\left(g\cdot\exp_G(tX)\right)|_{t=0} = \dfrac{d}{dt}f(g)f\left(\exp_G(tX)\right)|_{t=0} = \left(L_{f(g)}\right)_{*e}(f_{*e}X) = (f_{*e}X)^\sharp_{f(g)}$. $\square$

Example 1.1. Let V be a $\mathbb{R}$-vector space, $G = GL(V)$, $\mathfrak{g}$ Lie algebra of G. Then $\mathfrak{g} = End(V)$, the bracket is given as follows:

Proposition 1.3. $\forall X, Y \in End(V)$, $[X,Y] = XY - YX$.

Proof. For $X \in End(V)$, set matrix exponential $e^{tX} = \sum_{k=0}^{\infty}\frac{(tX)^k}{k!}$. Then $\{e_{tX}\}_{t\in\mathbb{R}}$ is a one parameter subgroup of G and $\frac{d}{dt}e^{tX}|_{t=0} = X$. So $\exp_G(tX) = e^{tX}$. Then

$[X,Y] = [X^\sharp, Y^\sharp]_e = \left(\mathcal{L}_{X^\sharp}Y^\sharp\right)_e = \dfrac{d}{dt}(\varphi_{-t})_{*e^{tX}}\left(Y^\sharp_{e^{tX}}\right)|_{t=0} = \dfrac{d}{dt}\dfrac{d}{ds}\varphi_{-t}\left(e^{tX}e^{sY}\right)|_{s=t=0} = XY - YX.$

$\square$

Example 1.2. Set

- $O(n) = \{g \in GL(n;\mathbb{R}) \mid g^t g = E_n\}$ (orthogonal group)
- $SO(n) = \{g \in O(n) \mid \det g = 1\}$ (special orthogonal group)

we can check that $O(n), SO(n)$ are Lie subgroups of $GL(n;\mathbb{R})$.

$SO(n)$ is the unit component of $O(n)$, so $\mathfrak{o}(n) = \mathfrak{so}(n)$ (Lie algebra of $(O(n))$ and $SO(n)$). This is a Lie subalgebra of $End(\mathbb{R}^n)$ given by

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in End(\mathbb{R}^n) \mid X^t + X = O_n\}$$

where O_n is the zero matrix of size n.

Similarly, set

- $U(n) = \{g \in GL(n;\mathbb{C}) \mid g^* g = E_n\}$ (unitary group) where $g^* = \overline{g^t}$
- $SU(n) = \{g \in U(n) \mid \det g = 1\}$ (special unitary group)

We can check that

- $U(n), SU(n)$ are Lie subgroups of $GL(n;\mathbb{C})$
- $\mathfrak{u}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O\}$ (Lie algebra of $U(n)$)
- $\mathfrak{su}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O, \mathrm{tr} X = 0\}$ (Lie algebra of $SU(n)$)

$\boxed{\text{Note.}}$ A Lie subgroup H of G is a Lie group s.t.

- H is a subset of G
- inclusion map $H \hookrightarrow G$ is an embedding and group homomorphism

$\boxed{\text{Fact.}}$ A closed subgroup of G is a Lie subgroup of G.

Definition 1.6. Let V be a $\mathbb{K}$-vector space, G a Lie group. A Lie group homomorphism $\rho : G \to GL(V)$ is called a *representation of V*. The Lie algebra homomorphism $\rho_{*e} : \mathfrak{g} \to End(V)$ is called a *differential representation*.

Example 1.3. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. $\forall g \in G$, define a homomorphism

$$F_g : G \to G,\ h \mapsto ghg^{-1}$$

Note that $F_g \circ F_{g'} = F_{gg'}$. This induces a Lie algebra homomorphism $(F_g)_{*e} : \mathfrak{g} \to \mathfrak{g}$ which satisfies $(F_g)_{*e} \circ (F_{g'})_{*e} = (F_{gg'})_{*e}$. So we obtain a representation

$$Ad : G \to GL(\mathfrak{g}),\ g \mapsto (F_g)_{*e}$$

called *adjoint representation of G*. The differential representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ of Ad is given as follows.

Proposition 1.4. $\forall X, Y \in \mathfrak{g}$, $ad(X)(Y) = [X, Y]$.

Proof. Note that $F_g = R_{g^{-1}} \circ L_g$. Then

$ad(X)(Y) = \frac{d}{dt}Ad(\exp_G(tX))(Y)|_{t=0} = \frac{d}{dt}\left(R_{\exp_G(-tX)}\right)_{*\exp_G(tX)}\left(L_{\exp_G(tX)}\right)_{*e}(Y)|_{t=0} = [X^\sharp, Y^\sharp]_e = [X, Y]$. □

Recall that there is a exponential map in Riemannian geometry. The Riemannian exp and the Lie group exp are related as follows.

Definition 1.7. A Riemannian metric $\langle\cdot,\cdot\rangle$ on a Lie group G is said to be *bi-invariant* if $\forall g, h \in G$, $L_g^* R_h^* \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$.

Theorem 1.1. Let G be a Lie group with a bi-invariant metric $\langle\cdot,\cdot\rangle$. Then $\exp_e = \exp_G$.

To show this we describe the Levi-Civita connection ∇ of $\langle\cdot,\cdot\rangle$.

Lemma 1.1. $\forall X, Y \in \mathfrak{g}$, $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

Proof. By Koszul formula, we have

$$\begin{aligned}\langle \nabla_{X^\sharp} Y^\sharp, Z^\sharp\rangle = \frac{1}{2}\Big(&X^\sharp\langle Y^\sharp, Z^\sharp\rangle + Y^\sharp\langle Z^\sharp, X^\sharp\rangle - Z^\sharp\langle X^\sharp, Y^\sharp\rangle \\ &-\langle Y^\sharp, [X^\sharp, Z^\sharp]\rangle - \langle Z^\sharp, [Y^\sharp, X^\sharp]\rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp]\rangle\Big)\end{aligned}$$

Since for $\forall g \in G$, $X_g^\sharp = \frac{d}{dt} g \cdot \exp_G(tX) \,|_{t=0}$, we have

$$X^\sharp\langle Y^\sharp, Z^\sharp\rangle = \frac{d}{dt}\langle Y^\sharp_{g\cdot\exp_G(tX)}, Z^\sharp_{g\cdot\exp_G(tX)}\rangle_{g\cdot\exp_G(tX)} \,|_{t=0} = \frac{d}{dt}\langle Y, Z\rangle_e \,|_{t=0} = 0$$

Since $\langle\cdot,\cdot\rangle$ is bi-invariant,

$$L_g^* R_{g^{-1}}^* \langle\cdot,\cdot\rangle_e = \langle\cdot,\cdot\rangle_e \text{ for } \forall g \in G \iff \langle Ad(g)(\cdot), Ad(g)(\cdot)\rangle_e = \langle\cdot,\cdot\rangle_e$$

Setting $g = \exp_G(tZ)$ and $\frac{d}{dt}\,|_{t=0}$, we have $\langle ad(Z)(\cdot), \cdot\rangle_e + \langle\cdot, ad(Z)(\cdot)\rangle_e = 0$, which shows that $\langle Y^\sharp, [X^\sharp, Z^\sharp]\rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp]\rangle = 0$, so we have $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

□

The proof of the theorem completes once shown that $\exp_G(tX)$ is geodesic, which is left as an exercise.

Exercise 1.1. Prove the theorem.

Remark 1.1. Existence/uniqueness of bi-invariant metrics? Some facts from representation theory are needed, the argument here is not used after this remark.

$\boxed{\text{Existence}}$ When G is compact, $\exists$ bi-invariant metric using "averaging trick".

- We first define Ad-invariant inner product on $\mathfrak{g}$.
- Then extend it to the whole G by pulling back L_g.

$\boxed{\text{Note:}}$ $\exists$ bi-invariant on $G \iff \exists$ Ad-invariant inner product on $\mathfrak{g}$.

$$\begin{cases} (\Rightarrow) \text{ Trivial.} \\ (\Leftarrow) \text{ Given } Ad\text{-invariant inner product on } \mathfrak{g}, \text{ we can extend it to left-invariant metric} \\ \quad \text{on } G, \text{ this is also right-invariant by pullback of } R_h = R_h \circ L_{h^{-1}} \circ L_h = Ad(h^{-1}) \circ L_h \end{cases}$$

$\boxed{\text{Uniqueness}}$ When G is abelian, then $L_g = R_g$, so $\exists$ many bi-invariant metrics on G (Any inner product on $\mathfrak{g}$ induces left-invariant metric on $\mathfrak{g}$, by the note above it is bi-invariant). Suppose that $\exists$ Ad-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. By $\langle \cdot, \cdot \rangle$, we have an irreducible decomposition of $(\mathfrak{g}, Ad)$: $\mathfrak{g} = \mathfrak{g}_1^{\oplus n_1} \oplus \cdots \oplus \mathfrak{g}_r^{\oplus n_r}$, where $\mathfrak{g}_i$ is irreducible representation of G and $\mathfrak{g}_i \neq \mathfrak{g}_j$ for $i \neq j$. Then

$$\dim \{Ad\text{-invariant symmetric bilinear map } \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\} = \sum_{i=1}^{r} n_i^2$$

To see this, take $T \in \{Ad\text{-invariant symmetric bilinear map }\}$ and use Schur's lemma to

$$T_{ij} : \mathfrak{g}_i \hookrightarrow \mathfrak{g} \xrightarrow{x \mapsto T(x, \cdot)} \mathfrak{g}^* \overset{\langle \cdot, \cdot \rangle}{\cong} \mathfrak{g} \xrightarrow{proj.} \mathfrak{g}_j$$

Then $T_{ij} = \begin{cases} 0 & (i \neq j) \\ c \cdot id & (i = j) \text{ for } \exists c \in \mathbb{R} \end{cases}$, so uniqueness up to scalar multiplication holds only when $r = 1, n = 1$, i.e. $(\mathfrak{g}, Ad)$ is irreducible $\iff$ G is simple Lie group.

Definition 1.8. Let M be smooth manifold, G be Lie group with unit e. A smooth map

$$A : M \times G \to M, \ (x, g) \mapsto xg$$

is called the *right action of G on M* if

(1) $\forall x \in M$, $xe = x$

(2) $\forall x \in M$, $\forall g, g \in G$, $(xg)h = x(gh)$

We write the right action as $M \curvearrowleft G$.

Definition 1.9. Suppose $M \curvearrowleft G$.

(1) For $\forall g \in G$, set $R_g : M \to M$, $X \mapsto xg$ (*right translation*).

(2) For $\forall X \in \mathfrak{g}$, define the *fundamental vector field* $X^\sharp \in \mathfrak{X}(M)$ by $X^\sharp_x = \frac{d}{dt} x \cdot \exp_G(tX) \mid_{t=0} = dA(x, \cdot)_e(X)$.

Here the notation $X^\sharp$ is the same as the left-invariant vector field on Lie group, we'll show that they have the same property:

Remark 1.2. (1) $\forall g \in G$, $\forall X \in \mathfrak{g}$, $(R_g)_* X^\sharp = \left(Ad(g^{-1})X\right)^\sharp$.

(2) $\forall X, Y \in \mathfrak{g}$, $[X^\sharp, Y^\sharp] = [X, Y]^\sharp$.

Proof. (1) $\forall x \in M$, $\left((R_g)_* X^\sharp\right)_x = (R_g)_* X^\sharp_{xg^{-1}} = \dfrac{d}{dt} xg^{-1} \exp_G(tX)g \;|_{t=0}$. Since $\left\{g^{-1}\exp_G(tX)g\right\}_{t\in\mathbb{R}}$ is a one parameter subgroup of G with $\dfrac{d}{dt} g^{-1}\exp_G(tX)g \;|_{t=0}= Ad(g^{-1})X$, then $g^{-1}\exp_G(tX)g = \exp_G(tAd(g^{-1})X)$, which gives (1).

(2) By definition, $\left\{\varphi_t = R_{\exp_G(tX)}\right\}_{t\in\mathbb{R}}$ is flow of $X^\sharp$. So

$$[X^\sharp, Y^\sharp] = \frac{d}{dt}(\varphi_{-t})_* Y^\sharp \;|_{t=0}= \frac{d}{dt}\left(Ad\left(\exp_G(tX)\right)Y\right)^\sharp \;|_{t=0}= \left(ad(X)(Y)\right)^\sharp = [X, Y]^\sharp.$$

□

Remark 1.3. We can define the left action

$$A^L : G \times M \to M,\ (g, x) \mapsto gx$$

and also the fundamental vector field $X_L^\sharp \in \mathfrak{X}(M)$. The left and right actions are essentially the same, since the right action is given form the left action. Indeed, given A^L above, define A by $A(x, g) = A^L(g^{-1}, x) = g^{-1}x$, then $X_L^\sharp = -X^\sharp$ for $X \in \mathfrak{g}$. $[X_L^\sharp, Y_L^\sharp] = [X, Y]^\sharp = -[X, Y]_L^\sharp$.

Definition 1.10. Suppose $M \curvearrowleft G$.

(1) For $p \in M$, define $G_p = \{g \in G \mid pg = p\}$ (*isotropy subgroup at* p).

(2) The G action is *free* of $G_p = \{e\}$ for $\forall p \in M$.

(3) The G action is *effective* if $\bigcap_{p\in M} G_p = \{e\}$. In other words, $G \to \mathrm{Diff}(M)$ is injective.

1.2 Definition of Principle Bundles

Definition 1.11. Let P, M be smooth manifolds and G be Lie group. The map $\pi_P : P \to M$ is a *principle G-bundle* or *principle bundle with structure group G* if:

(1) $P \curvearrowleft G$.

(2) There exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of M and diffeomorphisms called local trivialization

$$\phi_\alpha : \pi_P^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$$

such that

(2.1) Denoting by $p_1 : U_\alpha \times G \to U_\alpha$ the projection, then $\pi_P = p_1 \circ \phi_\alpha$

(2.2) The G-action preserves each $\pi_P^{-1}(U_\alpha)$. Denoting the right G-action on $U_\alpha \times G$ by

$$(U_\alpha \times G) \times G \to U_\alpha \times G,\ ((x,h),g) \mapsto (x,h)\cdot g = (x,hg)$$

Then ϕ_α is G-equivalent, i.e. $\forall \xi \in \pi_P^{-1}(U_\alpha)$, $\forall g \in G$, $\phi_\alpha(\xi g) = \phi_\alpha(\xi) g$. Note that the G-action is free.

We often write $P|_U = \pi_P^{-1}(U)$ for open subset $U \subseteq M$ and $P_x = \pi_P^{-1}(x)$ for $x \in M$, P_x is called the fiber of P at x.

Recall that $e \in G$ is the unit, define a section $p_\alpha \in \Gamma(P|_{U_\alpha})$ on U_α: $\phi_\alpha(p_\alpha(x)) = (x,e)$, which is equivalent to $p_\alpha(x) = \phi_\alpha^{-1}(x,e)$. Define $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ by $\boxed{p_\alpha(x) g_{\alpha\beta}(x) = p_\beta(x)}$, $\{g_{\alpha\beta}\}_{\alpha\beta}$ is called the transition map of $\pi_P : P \to M$. Note that $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$, we have $\boxed{g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)}$. Conversely, given open covering $\{U_\alpha\}_{\alpha\in A}$ of M and transition maps, we can recover principle G-bundle $\pi_P : P \to M$.

As before, for $g \in G$, we can define $R_g : P \to P$ the right translation and the fundamental vector field $X^\sharp$ generated by $X \in \mathfrak{g}$.

Definition 1.12. Let $\pi_P : P \to M$ be a principle G-bundle, $\rho : G \to GL(V)$ representation of G. Define the right G-action on $P \times V$ by

$$(P \times V) \times G \to P \times V,\ ((\xi, v), g) \mapsto \left(\xi g, \rho(g)^{-1} v\right)$$

$P \underset{\rho}{\times} V = (P \times V)/G$ is called the *associated vector bundle to P*.

Set $\xi \underset{\rho}{\times} v$ the equivalence class of $(\xi, v) \in P \times V$. Set $E = P \underset{\rho}{\times} V$, $\pi_E : E \to M$, $\xi \underset{\rho}{\times} v \mapsto \pi_P(\xi)$. Then $\pi_E : E \to M$ is a vector bundle.

The local trivialization of E are induced from those of P:

$$\phi_\alpha^E : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V,\ p_\alpha(x) \underset{\rho}{\times} v \mapsto (x,v)$$

For $x \in U_\alpha \cap U_\beta$ and $v_\beta \in V$, $p_\beta(x) \underset{\rho}{\times} v_\beta = p_\alpha g_{\alpha\beta}(x) \underset{\rho}{\times} v_\beta = p_\alpha(x) \underset{\rho}{\times} \rho(g_{\alpha\beta}(x)) v_\beta$. The transition functions of E are given by $\{\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to GL(V)\}$.

We will explain some relations between P and E.

- First note that $\forall \xi \in P$, we have $\xi : V \xrightarrow{\cong} E_{\pi_P(\xi)}$, $v \mapsto \xi \underset{\rho}{\times} v$ is an isomorphism. For $\xi' \in P$ with $\xi' = \xi g$ for $g \in G$, we have $\xi^{-1}\left(\xi' \underset{\rho}{\times} v'\right) = \xi^{-1}\left(\xi \underset{\rho}{\times} \rho(g) v'\right) = \rho(g) v'$ for $v' \in V$.
- $\pi_P^* E$ is a trivial bundle. Indeed,

$$P \times V \underset{(\xi, \xi^{-1}(e)) \leftarrow (\xi, e)}{\xrightarrow{(\xi,v) \mapsto (\xi, \xi \underset{\rho}{\times} v)}} \pi_P^* E = \{(\xi, e) \in P \times E \mid \pi_P(\xi) = \pi_E(e)\} \text{ is isomorphism.}$$

- Next, for $s \in \Omega^q(E) = \Gamma(\Lambda^q T^* M \otimes E)$, define $\pi_P^* s \in \Omega^q(P; V)$ as follows (V-

valued q-form on P)

- For $q=0$, $(\pi_P^* s)(\xi) = \xi^{-1}(s(\pi_P(\xi)))$
- For $q>1$, $\forall \alpha \in \Omega^q(M)$, $\forall s \in \Omega^0(E) = \Gamma(E)$,

$$\pi_P^*(\alpha \otimes s) = \pi_P^* \alpha \otimes \pi_P^* s$$

The left one is pullback and the right one is define above. In other words, $\forall \xi \in P$, $\forall v_1, \cdots, v_q \in T_\xi P$,

$$(\pi_P^* s)_\xi (v_1, \cdots, v_q) = \xi^{-1}\left(s_{\pi_P(\xi)}(\pi_{P*}(v_1), \cdots, \pi_{P*}(v_q))\right)$$

Notation: denote $\Omega_B^q(P;V)$ to be the elements $\widetilde{s}$ in $\Omega^q(P;V)$ satisfying:

(1) $\forall X \in \mathfrak{g}$, $i(X^\sharp)\widetilde{s} = 0$.

(2) $\forall g \in G$, $R_g^* \widetilde{s} = \rho(g)^{-1}\widetilde{s}$.

called the *space of basic q-forms*. Note that $\Omega_B^q(P;V)$ depends on representation ρ.

Proposition 1.5. (**Important to study the relations between P and E**)

(1) $\pi_P^*(\Omega^q(E)) \subseteq \Omega_B^q(P;V)$ and $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega_B^q(P;V)$. E-valued q-forms on M are identified with basic q-forms on P.

(2) Recall the local trivialization $\phi_\alpha^E : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$. For $s \in \Omega^q(E)$, suppose that $s|_{U_\alpha}$ corresponds to $s_\alpha \in \Omega^q(U_\alpha;V)$. Then $s_\alpha = p_\alpha^*(\pi_P^* s)$. So we regard $s \in \Omega^q(E)$ as a basic form, and then pullback by p_α is s_α.

Proof. (1) We show $\pi_P^*(\Omega^q(E)) \subseteq \Omega_B^q(P;V)$. Take $\forall s \in \Omega^q(E)$,

- $\boxed{\text{For } q=0}$ (1) is trivial; For (2): for $g \in G$, $\xi \in P$, we have

$$\left(R_g^* \pi_P^* s\right)(\xi) = (\pi_P^* s)(R_g \xi) = (\xi g)^{-1}(s(\pi_P(\xi g))) = (\xi g)^{-1}(s(\pi_P(\xi)))$$

By definition of ξ, we have: for $\forall v \in V$,

$$\xi(v) = \xi \underset{\rho}{\times} v = \xi g \underset{\rho}{\times} \rho(g)^{-1}(v) = (\xi g)\left(\rho(g)^{-1}(v)\right)$$

so $\xi = (\xi g) \circ \rho(g)^{-1}$, hence $(\xi g)^{-1} = \rho(g)^{-1} \circ \xi^{-1}$. Then

$$\left(R_g^* \pi_P^* s\right)(\xi) = \rho(g)^{-1}\left(\xi^{-1} s(\pi_P(\xi))\right) = \left(\rho(g)^{-1}(\pi_P^* s)\right)(\xi).$$

- $\boxed{\text{For } q \geq 1}$ (1): Since $\pi_P(\xi g) = \pi_P(\xi)$, we have $\pi_{P*}(X^\sharp) = 0$, which implies (1); (2): For $\forall \alpha \in \Omega^q(M)$, $\forall s \in \Gamma(E)$, $\forall g \in G$, we have

$$R_g^*(\pi_P^*(\alpha \otimes s)) = R_g^* \pi_P^* \alpha \otimes R_g^* \pi_P^* s = \pi_P^* \alpha \otimes \rho(g)^{-1}(\pi_P^* s) = \rho(g)^{-1} \pi_P^*(\alpha \otimes s)$$

which finishes the proof of (2).

Next we show $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega_B^q(P;V)$:

- $\boxed{\text{Injectivity}}$ It is clear from the formula

$$(\pi_P^* s)_\xi (v_1, \cdots, v_q) = \xi^{-1} \left(s_{\pi_P(\xi)} (\pi_{P*}(v_1), \cdots, \pi_{P*}(v_q)) \right).$$

- $\boxed{\text{Surjectivity}}$ Take $\widetilde{s} \in \Omega_B^q(P; V)$,
  - When $q = 0$, define $s \in \Omega^0(E) = \Gamma(E)$ by $s(x) = \xi \underset{\rho}{\times} \widetilde{s}(\xi)$ where $\xi \in \pi_P^{-1}(x)$. It is well-defined since $\xi g \underset{\rho}{\times} \widetilde{s}(\xi g) = \xi g \underset{\rho}{\times} (R_g^* \widetilde{s})(\xi) = \xi g \underset{\rho}{\times} \rho(g)^{-1} \widetilde{s}(\xi) = \xi \underset{\rho}{\times} \widetilde{s}(\xi)$. Then by definition we have $\pi_P^* s = \widetilde{s}$.
  - When $q \geq 1$, define $s \in \Omega^0(E) = \Gamma(E)$ by

$$s_x(w_1, \cdots, w_q) = \xi \underset{\rho}{\times} \widetilde{s}_\xi(\widetilde{w_1}, \cdots, \widetilde{w_q})$$

where $x \in M$, $w_i \in T_x M$, $\xi \in \pi_P^{-1}(x)$, $\pi_{P*}(\widetilde{w_i}) = w_i$. It's left as an exercise to check s is well-defined in this case.

(2) First we describe s_α clearly. Set $s|_{U_\alpha} = \sum \beta_i \otimes e_i$. Since

$$\phi_\alpha^E : E\,|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V,\ p_\alpha(x) \underset{\rho}{\times} v \mapsto (x, v),$$

we have $\phi_\alpha^E\left((e_i)_x\right) = (x, v_i(x))$ for a function $v_i : U_\alpha \to V$. Note that $(e_i)_x = p_\alpha(x) \underset{\rho}{\times} v_i(x)$. Then $s_\alpha = \sum \beta_i \otimes v_i$. Now we compute

$$p_\alpha^* (\pi_P^* s) = p_\alpha^* \left(\sum \pi_P^* \beta_i \otimes \pi_P^* e_i \right) = \sum (\pi_P \circ p_\alpha)^* \beta_i \otimes (\pi_P^* e_i)\, p_\alpha(x) = \sum \beta_i \otimes v_i(x).$$

So we have $p_\alpha^* (\pi_P^* s) = s_\alpha$. □

Now we give a typical example of principle bundles.

Example 1.4. Let $\pi_E : E \to M$ be a vector bundle with rank r. For $x \in M$, set

- $P_x = \{\xi : \mathbb{K}^r \to E_x :$ linear isomorphism $\}$.
- $P = \bigsqcup_{x \in M} P_x$; $\pi_P : P \to M$, $\xi \mapsto x$ if $\xi \in P_x$.

We see that $\pi_P : P \to M$ is a principle $GL(r; \mathbb{K})$-bundle:

- The right action on P is given by:

$$P \times GL(r; \mathbb{K}) \to P,\ (\xi \times g) \mapsto \xi \circ g.$$

- To give a local trivialization, first note that

$$P_x \xrightarrow[\xi \mapsto \{\xi(\epsilon_1), \cdots, \xi(\epsilon_r)\}]{\cong} \{\text{basis of } E_x\},$$

where $\epsilon_i = (0, \cdots, \underset{i\text{-th}}{1}, \cdots, 0)^t$. If $\{e_1, \cdots, e_r\} \subseteq \Gamma(E|_{U_\alpha})$ is local frame of E over $U_\alpha \subseteq M$, define $p_\alpha \in \Gamma(P|_{U_\alpha})$ by

$$p_\alpha : U_\alpha \to P|_{U_\alpha},\ x \mapsto (e_1(x), \cdots, e_r(x)),$$

which induces a local trivialization

$$\phi_\alpha^P : P|_{U_\alpha} \to U_\alpha \times GL(r;\mathbb{K}),\ \xi \mapsto \Big(\pi_P(\xi), (p_\alpha(\pi_P(\xi)))^{-1}\xi\Big)$$

The inverse of this map is $(x,g) \mapsto p_\alpha(x)\cdot g$. We see that ϕ_α^P is $GL(r;\mathbb{K})$-equivalent.

So $\pi_P : P \to M$ is a principle $GL(r;\mathbb{K})$-bundle. This is called the *frame bundle* of $\pi_E : E \to M$. Also note that transition maps of E is the transition maps of P. Indeed, if $\{f_1,\cdots,f_r\} \subseteq \Gamma(E|_{U_\alpha})$ is another local frame, the transition map $g_{\alpha\beta}$ satisfies $(f_1,\cdots,f_r) = (e_1,\cdots,e_r)g_{\alpha\beta}$, and this is exactly $p_\beta = p_\alpha g_{\alpha\beta}$.

1.3 Connections on Principle Bundles

In this subsection we study properties of connection on principle bundle and its relation between connection on associated vector bundle.

Definition 1.13. Let $\pi_P : P \to M$ be principle G-bundle.

(1) A distribution $\{H_\xi \subseteq T_\xi P\}_{\xi\in P}$ is a *connection* on P if

(1-1) $\forall \xi \in P$, $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$.

(1-2) $\{H_\xi \subseteq T_\xi P\}_{\xi\in P}$ is G-invariant, i.e. $\forall\xi\in P$, $\forall g \in G$, $(R_g)_{*\xi}H_\xi = H_{\xi g}$.

H_ξ, $\ker(\pi_P)_{*\xi}$ are called *horizontal/vertical subspaces.*

(2) A $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P;\mathfrak{g})$ on P is a *connection form* if

(2-1) $\forall X \in \mathfrak{g}$, $\theta(X^\sharp) = X$.

(2-2) $\forall g \in G$, $R_g^*\theta = Ad(g^{-1})\theta$.

These 2 notions are the same in the following sense:

Theorem 1.2. Let $\pi_P : P \to M$ be principle G-bundle.

(1) If $\theta \in \Omega^1(P;\mathfrak{g})$ is a connection form, a distribution $\{\ker\theta_\xi\}_{\xi\in P} = \{v \in T_\xi P \mid \theta_\xi(v) = 0\}_{\xi\in P}$ is a connection on P.

(2) $\{\text{connection form}\} \to \{\text{connection on } P\}$, $\theta \mapsto \{\ker\theta_\xi\}_{\xi\in P}$ is bijective.

Proof. (1) We check that $\{\ker\theta_\xi\}_{\xi\in P}$ satisfies (1-1), (1-2):

(1-1) Note that $\ker(\pi_P)_{*\xi} = \Big\{X_\xi^\sharp \in T_\xi P \mid X \in \mathfrak{g}\Big\}$, then for $\forall v \in T_\xi P$, we have $\theta(v) \in \mathfrak{g}$ and $v = \theta(v)_\xi^\sharp + \Big(v - \theta(v)_\xi^\sharp\Big)$, which implies that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus \ker\theta_\xi$ ($\ker(\pi_P)_{*\xi} \cap \ker\theta_\xi = \{0\}$ is obvious).

(1-2) Take $\forall v \in \ker\theta_\xi$. By (2-2), $\forall g \in G$, we have $\big(R_g^*\theta\big)_\xi = Ad(g^{-1})\theta_\xi$, the left hand side is $\theta_{\xi g}((R_g)_{*\xi}(\cdot))$, so we have $(R_g)_{*\xi}(v) \in \ker\theta_{\xi g}$, hence $(R_g)_{*\xi}(\ker\theta_\xi) \subseteq \ker\theta_{\xi g}$. Replacing (g,ξ) with $(g^{-1},\xi g)$, we have $(R_{g^{-1}})_{*\xi g}(\ker\theta_{\xi g}) \subseteq \ker\theta_\xi$. So $(R_g)_{*\xi}(\ker\theta_\xi) = \ker\theta_{\xi g}$, $\{\ker\theta_\xi\}_{\xi\in P}$ is a connection on P.

(2) $\boxed{\text{Injectivity}}$ Let θ, θ' be connection forms with $\ker \theta_\xi = \ker \theta'_\xi$ $\forall \xi \in P$. We show that $\forall v \in T_\xi P$, $\theta_\xi(v) = \theta'_\xi(v)$. By (1), v is described as $v = X^\sharp_\xi + w$ for $X^\sharp_\xi \in \ker(\pi_P)_{*\xi}$ and $w \in \ker \theta_\xi = \ker \theta'_\xi$. So $\theta_\xi(v) = \theta_\xi(X^\sharp_\xi) = X = \theta'_\xi(v)$.

$\boxed{\text{Surjectivity}}$ Take $\forall \{H_\xi\}_{\xi \in P}$ a connection on P. By (1-1), we can define $\theta \in \Omega^1(P; \mathfrak{g})$ by

$$\theta_\xi(v) = \begin{cases} 0 & (v \in H_\xi) \\ X & (v = X^\sharp_\xi \text{ for } X \in \mathfrak{g}) \end{cases}$$

By definition, $\ker \theta_\xi = H_\xi$, we check (2-1), (2-2).

(2-1) Holds by definition of θ_ξ.

(2-2) $\forall \xi \in P$, $\forall g \in G$, we show that $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = Ad(g^{-1})\theta_\xi$ on $T_\xi P$. Recall that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$, if $v \in H_\xi$, the equality holds by definition and (1-2); for $\forall X \in \mathfrak{g}$,

$$(R_g)_{*\xi}\left(X^\sharp_\xi\right) = (R_g)_{*\xi} \frac{d}{dt} \xi \exp_G(tX) \mid_{t=0} = \frac{d}{dt} \xi g \cdot g^{-1} \exp_G(tX) g \mid_{t=0} = \left(Ad(g^{-1})X\right)^\sharp_{\xi g}$$

So $\theta_{\xi g}\left((R_g)_{*\xi}(X^\sharp_\xi)\right) = Ad(g^{-1})X = Ad(g^{-1})\theta_\xi(X^\sharp_\xi)$, hence the equality holds. So we have $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = Ad(g^{-1})\theta_\xi$ on $T_\xi P$. $\square$

The next proposition says that a connection form θ on P induces a connection ∇^E of the associated vector bundle E. The relation between θ and local connection form of ∇^E us also given.

Proposition 1.6. Let $\pi_P : P \to M$ be a principle bundle, $\rho : G \to GL(V)$ a representation of G with differential representation $\rho_* : \mathfrak{g} \to End(V)$. Denote by $\theta \in \Omega^1(P; \mathfrak{g})$ a connection form. Set $E = P \underset{\rho}{\times} V$ its associated vector bundle. Then,

(1) $(d + \rho_*(\theta)\wedge)\, \Omega^q_B(P; V) \subseteq \Omega^{q+1}_B(P; V)$. Here

- d: standard exterior derivative.
- $\rho_*(\theta) \in \Omega^1(P; End(V))$ acts on $\Omega^q_B(P; V)$ by wedging on differential form parts and composing $End(V), V$-parts.

(2) Recall that $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega^q_B(P; V)$. Then we can define $\nabla^E : \Omega^0(E) \to \Omega^1(E)$ by $(\pi_P^*)^{-1} \circ (d + \rho_*(\theta)\wedge) \circ \pi_P^*$.

(3) Recall that a local section $p_\alpha \in \Gamma(P|_{U_\alpha})$ induces a local trivialization $\phi^E_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$. Then

$$\begin{tikzcd} \Omega^0(E|_{U_\alpha}) \arrow[r, "\nabla^E|_{U_\alpha}"] \arrow[d, phantom, "\wr\|"] \arrow[dr, phantom, "\circlearrowright"] & \Omega^1(E|_{U_\alpha}) \arrow[d, phantom, "\wr\|"] \\ \Omega^0(U_\alpha; V) \arrow[r, "d+\rho_*(p_\alpha^*\theta)\wedge"'] & \Omega^1(U_\alpha; V) \end{tikzcd}$$

(4) Recall that a connection ∇^E induces the exterior derivative d^{∇^E} : $\Omega^q(E) \to \Omega^{q+1}(E)$. Then

$$\begin{tikzcd}
\Omega^q(E) \arrow[r, "d^{\nabla^E}"] \arrow[d, "\pi_P^*"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \Omega^{q+1}(E) \arrow[d, "\pi_P^*"', "\cong"] \\
\Omega^q_B(P;V) \arrow[r, "d+\rho_*(\theta)\wedge"'] & \Omega^{q+1}_B(P;V)
\end{tikzcd}$$

Remark 1.4. In [Kobayashi-Nomizu, *Foundation of differential geometry* Vol 1, chapter 2, section 5], for any principle G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$, $\forall V$ vector space, the *exterior covariant derivative* $D : \Omega^q(P;V) \to \Omega^{q+1}(P;V)$ is defined by $(D\widetilde{s})(v_0, \cdots, v_q) = (d\widetilde{s})(hv_0, \cdots, hv_q)$ for $v_i \in TP$, where $h : TP \to \ker\theta$ is the projection. If in addition, given a representation $\rho : G \to GL(V)$ and $\widetilde{s} \in \Omega^q_B(P;V)$, we have $D\widetilde{s} = (d + \rho_*(\theta)\wedge)(\widetilde{s})$.

Proof. (1) Take $\forall \widetilde{s} \in \Omega^q_B(P;V)$, recall that $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\widetilde{s} = 0. \\ \forall g \in G, R_g^*\widetilde{s} = \rho(g)^{-1}\widetilde{s}. \end{cases}$. We show that $(d + \rho_*(\theta)\wedge)\,\widetilde{s}$ also satisfies the same property.

- $\forall X \in \mathfrak{g}$, we have
$$\mathcal{L}_{X^\sharp}\widetilde{s} = \frac{d}{dt}R^*_{\exp_G(tX)}\widetilde{s}\,|_{t=0} = \frac{d}{dt}\rho\left(\exp_G(tX)\right)^{-1}\widetilde{s}\,|_{t=0} = -\rho_*(X)\widetilde{s}.$$
Since $\mathcal{L}_{X^\sharp}\widetilde{s} = i(X^\sharp)d\widetilde{s} + d\left(i(X^\sharp)\widetilde{s}\right)$ and $i(X^\sharp)\widetilde{s} = 0$, we have $i(X^\sharp)d\widetilde{s} = -\rho_*(X)\widetilde{s}$. Hence $i(X^\sharp)\left((d + \rho_*(\theta)\wedge)(\widetilde{s})\right) = i(X^\sharp)d\widetilde{s} + \rho_*\left(\theta(X^\sharp)\right)\widetilde{s} - \rho_*(\theta) \wedge i(X^\sharp)\widetilde{s} = 0$.
- For $\forall g \in G$, we have
$$R_g^*\left((d + \rho_*(\theta)\wedge)(\widetilde{s})\right) = dR_g^*\widetilde{s} + \rho_*\left(R_g^*\theta\right)\wedge R_g^*\widetilde{s} = d\left(\rho(g)^{-1}\widetilde{s}\right) + \rho_*\left(Ad(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s}.$$
Since $\rho(g)^{-1}$ acts only on V-part, $d\left(\rho(g)^{-1}\widetilde{s}\right) = \rho(g)^{-1}d\widetilde{s}$. Note that $\forall X \in \mathfrak{g}$,
$$\frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)g\right)\rho(g)^{-1}\,|_{t=0} = \frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)\right)|_{t=0}$$
and $g^{-1}\exp_G(tX)g = \exp_G(tAd(g^{-1})X)$, we have
$$\rho_*\left(Ad(g^{-1})X\right)\rho(g)^{-1} = \rho(g)^{-1}\rho_*(X).$$
This implies that
$$\rho_*\left(Ad(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s} = \rho(g)^{-1}\left(\rho_*(\theta)\wedge\widetilde{s}\right).$$
Then we obtain
$$R_g^*\left((d + \rho_*(\theta)\wedge)(\widetilde{s})\right) = \rho(g)^{-1}\left((d + \rho_*(\theta)\wedge)(\widetilde{s})\right),$$

so $(d+\rho_*(\theta)\wedge)(\widetilde{s}) \in \Omega^{q+1}_B(P;V)$.

(2) $\nabla^E = (\pi_P^*)^{-1}\circ(d+\rho_*(\theta)\wedge)\circ\pi_P^*$, we check the Leibniz rule, i.e. for $\forall f\in C^\infty(M)$, $\forall s\in\Gamma(E)$, we show $\nabla^E(fs) = df\otimes s + f\nabla^E s$. This is left as an exercise.

(3) Since for $s\in\Omega^q(E)$, $s|_{U_\alpha}$ corresponds to $p_\alpha^*(\pi_P^* s)$. We compute

$$p_\alpha^*\pi_P^*\left(\nabla^E s\right) = p_\alpha^*\left((d+\rho_*(\theta)\wedge)\,\pi_P^* s\right) = p_\alpha^* d\left(\pi_P^* s\right) + \rho_*\left(p_\alpha^*\theta\right)\wedge p_\alpha^*\pi_P^* s = \left(d+\rho_*(p_\alpha^*\theta)\wedge\right)\left(p_\alpha^*\pi_P^* s\right).$$

(4) Since d^{∇^E} is given by $d^{\nabla^E}(s\otimes\alpha) = \nabla^E s\wedge\alpha + s\otimes d\alpha$ for $s\in\Gamma(E)$, $\alpha\in\Omega^q(M)$, we have

$$\begin{aligned}\pi_P^*\left(d^{\nabla^E}(s\otimes\alpha)\right) &= \pi_P^*\left(\nabla^E s\wedge\alpha + s\otimes d\alpha\right) = (d+\rho_*(\theta)\wedge)\,\pi_P^* s\wedge\pi_P^*\alpha + \pi_P^* s\otimes\pi_P^* d\alpha \\ &= d\left(\pi_P^* s\otimes\pi_P^*\alpha\right) + \rho_*(\theta)\wedge\left(\pi_P^* s\otimes\pi_P^*\alpha\right) = (d+\rho_*(\theta)\wedge)\left(\pi_P^*(s\otimes\alpha)\right).\end{aligned}$$

□

Exercise 1.2. Prove that ∇^E defined above is a connection.

Example 1.5. Given a vector bundle $\pi_E : E\to M$, let $\pi_P : P\to M$ be the frame bundle. Consider the trivial representation $id : GL(r;\mathbb{K})\to GL(r;\mathbb{K})$. Then

Definition 1.14. Let $\pi_P : P\to M$ be principle G-bundle with a connection form $\theta\in\Omega^1(P;\mathfrak{g})$.

(1) $\Omega = d\theta + \frac{1}{2}[\theta\wedge\theta]\in\Omega^2(P;\mathfrak{g})$ is called the *curvature* of θ. ($[\theta\wedge\theta]$ means taking the wedge product of differential form part and taking Lie bracket of $\mathfrak{g}$-part)

(2) For $\forall X\in\mathfrak{X}(M)$, $\exists!\widetilde{X}\in\mathfrak{X}(P)$ s.t. $\begin{cases}(\pi_P)_*\widetilde{X} = X\\ \theta(\widetilde{X}) = 0\end{cases}$. Then $\widetilde{X}$ is called the *horizontal lift of* X.

We see existence and uniqueness of $\widetilde{X}$ in (2) as follows: recall that $\forall\xi\in P$, $T_\xi P = \ker(\pi_P)_*\oplus\ker\theta_\xi$, so $(\pi_P)_* : \ker\theta_\xi \xrightarrow{\cong} T_{\pi_P(\xi)}M$. So we may set $\widetilde{X}_\xi = (\pi_P)_*^{-1}\left(X_{\pi_P(\xi)}\right)$. Since $(\pi_P)_*|_{\ker\theta_\xi}$ is isomorphism, uniqueness follows.

Remark 1.5. Recall exterior covariant derivative of Kobayashi-Nomizu, i.e. $D : \Omega^q(P;V)\to\Omega^{q+1}(P;V)$ is defined by $(D\widetilde{s})(v_0,\cdots,v_q) = (d\widetilde{s})(hv_0,\cdots,hv_q)$ for $v_i\in TP$, where $h : TP\to\ker\theta$ is the projection. Then $\boxed{\Omega = D\theta}$. Actually, Kobayashi-Nomizu defined curvature by $D\theta$, and shows the equality in (1). The equality is called the *structure equation*.

To show this, note the following:

Remark 1.6. Let $\{\xi_1, \cdots, \xi_\ell\}$ be a basis of $\mathfrak{g}$. Then $\theta = \sum \xi_i \otimes \theta_i = \sum \xi_i \theta_i$ where $\theta_i \in \Omega^1(P)$ and we omit the $\otimes$. Then by definition we have

$$\Omega = \sum \xi_i d\theta_i + \frac{1}{2} \sum [\xi_i, \xi_j] \theta_i \wedge \theta_j.$$

Note that

$$\theta_i \wedge \theta_j(u, v) = \theta_i(u)\theta_j(v) - \theta_j(u)\theta_i(v),$$

so we have

$$[\theta \wedge \theta](u, v) = [\theta(u), \theta(v)] - [\theta(v), \theta(u)] = 2[\theta(u), \theta(v)],$$

then for $u, v \in TP$, we have $\boxed{\Omega(u, v) = d\theta(u, v) + [\theta(u), \theta(v)]}$. Now we show $\Omega = D\theta$. Since $TP = \ker(\pi_P)_* \oplus \ker \theta$, we have to show in the following cases:

- $\underline{u, v \in \ker \theta}$: $\Omega(u, v) = d\theta(u, v) = (D\theta)(u, v)$.
- $\underline{u, v \in \ker(\pi_P)_*}$: we may set $u = X^\sharp, v = Y^\sharp$ for $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned} \Omega(X^\sharp, Y^\sharp) &= d\theta(X^\sharp, Y^\sharp) + [X, Y] \\ &= X^\sharp \left(\theta(Y^\sharp)\right) - Y^\sharp \left(\theta(X^\sharp)\right) - \theta([X^\sharp, Y^\sharp]) + [X, Y] = 0. \end{aligned}$$

  Also $(D\theta)(X^\sharp, Y^\sharp) = 0$.
- $\underline{u \in \ker \theta, v = X^\sharp \text{ for } X \in \mathfrak{g}}$: extend u to a local horizontal vector field on P, which is still denoted as u. For example, extend $\pi_{P*}(u)$ to a local vector field on M, consider its horizontal lift. Then

$$\Omega(u, X^\sharp) = d\theta(u, X^\sharp) = u(\theta(X^\sharp)) - X^\sharp(\theta(u)) - \theta([u, X^\sharp]) = -\theta([u, X^\sharp])$$

  Now we show that $[u, X^\sharp] \in \Gamma(\ker \theta)$, then $\theta([u, X^\sharp]) = 0$. Recall that $\left\{R_{\exp_G(tX)}\right\}_{t \in \mathbb{R}}$ is the flow of $X^\sharp$, so $[X^\sharp, u] = \frac{d}{dt}\left(R_{\exp_G(-tX)}\right)_* u \,|_{t=0}$. Since for $\forall g \in G$, $\theta\left((R_g)_* u\right) = \left(R_g^* \theta\right)(u) = Ad(g^{-1})\theta(u) = 0$, we have $\theta([X^\sharp, u]) = 0$, hence $\Omega(u, X^\sharp) = (D\theta)(u, X^\sharp)$.

So we have $\Omega = D\theta$.

Theorem 1.3. Let $\pi_P : P \to M$ be principle G-bundle with a connection form $\theta \in \Omega^1(P; \mathfrak{g})$. Denote by $\Omega \in \Omega^2(P; \mathfrak{g})$ the curvature of θ. For $\forall X, Y \in \mathfrak{X}(M)$, let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(P)$ be the horizontal lifts respectively. Then $\Omega(\widetilde{X}, \widetilde{Y}) = -\theta([\widetilde{X}, \widetilde{Y}])$.

Proof. Since $\widetilde{X}, \widetilde{Y} \in \Gamma(\ker \theta)$, we have

$$\Omega(\widetilde{X}, \widetilde{Y}) = d\theta(\widetilde{X}, \widetilde{Y}) = \widetilde{X}\left(\theta(\widetilde{Y})\right) - \widetilde{Y}\left(\theta(\widetilde{X})\right) - \theta([\widetilde{X}, \widetilde{Y}]) = -\theta([\widetilde{X}, \widetilde{Y}]).$$

□

This theorem will imply that the curvature measures how "curved" the connection is.

Take a local vector field X, Y on M s.t. $[X, Y] = 0$. Let $\{\varphi_t\}$, $\{\psi_t\}$ be local flow of X, Y respectively. We know that $[X, Y] = 0 \Leftrightarrow \varphi_t \circ \psi_s = \psi_s \circ \varphi_t$. Now fix $x \in M$, consider