BASICS OF DIFFERENTIAL GEOMETRY

Principal Bundles and Characteristic Classes

Ethereal Kasumi

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Contents

Introduction

Last semester:

- Geometry of vector bundles
- Basic Riemannian geometry
- Differential operators on manifolds

We will learn this semester:

- Theory of principal bundles
- Characteristic classes

1 Principal Bundles

In this section, we introduce the connections of principal bundles, which is closely related to the connections of vector bundles and simpler in some sense. Unless otherwise stated, manifolds are assumed to be without boundary and connected.

1.1 Lie Groups

Definition 1.1. Let G be a smooth manifold. G is a **Lie group** if G is a group and multiplication, inverse are smooth.

Let G be a Lie group, $g \in G$, we denote:

- $L_g : G \to G$, $h \mapsto gh$ (called left translation)
- $R_g : G \to G$, $h \mapsto hg$ (called right translation)
- $\mathfrak{X}^L(G) = \{X \in \mathfrak{X}(G) \mid \forall g \in G, (L_g)_* X = X\}$ (left invariant vector fields)

For $X \in \mathfrak{X}^L(G)$, $L_{g*}X = X$ means that X is L_g-related to X. Then for $\forall X, Y \in \mathfrak{X}^L(G)$, $L_{g*}([X,Y]) = [L_{g*}X, L_{g*}Y] = [X,Y]$, so $\mathfrak{X}^L(G)$ is closed under $[\cdot,\cdot]$

Definition 1.2. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a $\mathbb{K}$-vector space $\mathfrak{g}$ and a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we say $\mathfrak{g}$ is a **Lie algebra** if:

(1) $\forall X, Y \in \mathfrak{g}$, $[X,Y] = -[Y,X]$.

(2) $\forall X, Y, Z \in \mathfrak{g}$, $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi's identity).

And $[\cdot,\cdot]$ is called Lie bracket.

So by definition we have $\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)$ is a Lie algebra.

Definition 1.3. For Lie algebra $\mathfrak{g}, \mathfrak{h}$, a linear map $f : \mathfrak{g} \to \mathfrak{h}$ is called the **Lie algebra homomorphism** if: $\forall X, Y \in \mathfrak{g}$, $f([X,Y]) = [f(X), f(Y)]$

If f is in addition an isomorphism, then f is called a **Lie algebra isomorphism**.

Let $e \in G$ be the unit of G. Set $\iota : \mathfrak{X}^L(G) \to T_eG$, $X \mapsto X_e$. Then ι is a natural linear isomorphism. Let $\mathfrak{g} = T_eG$, so we can define the Lie bracket on $\mathfrak{g}$ s.t. ι is a Lie algebra isomorphism, i.e. setting $X^\sharp = \iota^{-1}(X)$, then $[X,Y] = \left[X^\sharp, Y^\sharp\right]_e$. Note that $X^\sharp_g = (L_g)_{*e}X$, $g \in G$.

Definition 1.4. Let G be Lie group, $\mathfrak{g} = T_eG$ with $[\cdot,\cdot]$ defined above is called the **Lie algebra of** G. $\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)$ is also called the Lie algebra of G by the Lie algebra isomorphism ι.

Definition 1.5. Let G, H be Lie groups. A map $\rho : G \to H$ is a **Lie group homomorphism** if ρ is a smooth map and a group homomorphism. For the special case $(\mathbb{R}, +) \to G$, $t \mapsto g_t$, $\{g_t\}_{t\in\mathbb{R}}$ is called **one parameter subgroup of** G.

Proposition 1.1. Let G be Lie group and $\mathfrak{g}$ its Lie algebra. Then

(1) $\forall X \in \mathfrak{g}$, $X^\sharp = \iota^{-1}(X)$ is complete, i.e. $X^\sharp$ generates a flow $\{\varphi_t\}_{t\in\mathbb{R}}$.

(2) Set $\boxed{\exp_G(tX) = \varphi_t(e)} \in G$. Then $\varphi_t = R_{\exp_G(tX)}$.

(3) For $s, t \in \mathbb{R}$, $\exp_G(sX)\exp_G(tX) = \exp_G((s+t)X)$, i.e. $\{\exp_G(tX)\}_{t\in\mathbb{R}}$ is one parameter subgroup of G.

(4) $\mathfrak{g} \to \{\text{one parameter subgroup of } G\}$, $X \mapsto \{\exp_G(tX)\}_{t\in\mathbb{R}}$ is bijective.

Proof. (1) By ODE knowledge, $\exists\ \epsilon > 0$, $\gamma_e : (-\epsilon, \epsilon) \to G$ s.t. $\gamma_e(0) = e$, $\dfrac{d\gamma_e}{dt} = X^\sharp_{\gamma_e(t)}$.

$\boxed{\text{Claim 1.}}$ $\forall g \in G$, define $\gamma_g : (-\epsilon, \epsilon) \to G$, $t \mapsto g\gamma_e(t)$ is the integral curve of $X^\sharp$ with $\gamma_g(0) = g$.

Indeed, $\forall t \in (-\epsilon, \epsilon)$, $\dfrac{d\gamma_g}{dt}(t) = (L_g)_{*\gamma_e(t)} \dfrac{d\gamma_e}{dt}(t) = X^\sharp_{g\cdot\gamma_e(t)}$.

$\boxed{\text{Claim 2.}}$ $\gamma_e : (-\epsilon, \epsilon) \to G$ can be extended to integral curve $\gamma_e : \mathbb{R} \to G$ of $X^\sharp$ with $\gamma_e(0) = e$.

Set $\varphi_t = R_{\gamma_e(t)}$, then $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow generated by $X^\sharp$. So by uniqueness the following parts are easy. $\square$

By this proposition, we can define the exponential map $\exp_G : \mathfrak{g} \to G$.

Proposition 1.2. Let G, H be Lie groups with Lie algebra $\mathfrak{g}, \mathfrak{h}$. If $f : G \to H$ is Lie group homomorphism, then $f_{*e} : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. We only need to show that $X^\sharp$ and $(f_{*e}X)^\sharp$ are f-related. Since $X = \dfrac{d}{dt}\exp_G(tX)|_{t=0}$, we have

$$f_{*g}(X^\sharp_g) = \frac{d}{dt} f\left(g \cdot \exp_G(tX)\right)|_{t=0} = \frac{d}{dt} f(g) f\left(\exp_G(tX)\right)|_{t=0} = \left(L_{f(g)}\right)_{*e}(f_{*e}X) = (f_{*e}X)^\sharp_{f(g)}.$$

$\square$

Example 1.1. Let V be a $\mathbb{R}$-vector space, $G = \mathrm{GL}(V)$, $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g} = \mathrm{End}(V)$, the bracket is given as follows:

Proposition 1.3. $\forall X, Y \in \mathrm{End}(V)$, $[X, Y] = XY - YX$.

Proof. For $X \in \mathrm{End}(V)$, set matrix exponential $e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}$. Then $\left\{e^{tX}\right\}_{t\in\mathbb{R}}$ is a one parameter subgroup of G and $\frac{d}{dt}e^{tX}|_{t=0} = X$. So $\exp_G(tX) = e^{tX}$. Then
$[X,Y] = [X^\sharp, Y^\sharp]_e = \left(\mathcal{L}_{X^\sharp}Y^\sharp\right)_e = \dfrac{d}{dt}\left(\varphi_{-t}\right)_{*e^{tX}}\left(Y^\sharp_{e^{tX}}\right)|_{t=0} = \dfrac{d}{dt}\dfrac{d}{ds}\varphi_{-t}\left(e^{tX}e^{sY}\right)|_{s,t=0} = XY - YX$.

□

Example 1.2. Set

- $\mathrm{O}(n) = \{g \in \mathrm{GL}(n;\mathbb{R}) \mid g^t g = E_n\}$ (orthogonal group)
- $\mathrm{SO}(n) = \{g \in \mathrm{O}(n) \mid \det g = 1\}$ (special orthogonal group)

we can check that $\mathrm{O}(n), \mathrm{SO}(n)$ are Lie subgroups of $\mathrm{GL}(n;\mathbb{R})$.

$\mathrm{SO}(n)$ is the unit component of $\mathrm{O}(n)$, so $\mathfrak{o}(n) = \mathfrak{so}(n)$ (Lie algebra of $(\mathrm{O}(n))$ and $\mathrm{SO}(n)$). This is a Lie subalgebra of $\mathrm{End}(\mathbb{R}^n)$ given by

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \left\{X \in \mathrm{End}(\mathbb{R}^n) \mid X^t + X = O_n\right\}$$

where O_n is the zero matrix of size n.

Similarly, set

- $\mathrm{U}(n) = \{g \in \mathrm{GL}(n;\mathbb{C}) \mid g^* g = E_n\}$ (unitary group) where $g^* = \overline{g^t}$
- $\mathrm{SU}(n) = \{g \in \mathrm{U}(n) \mid \det g = 1\}$ (special unitary group)

We can check that

- $\mathrm{U}(n), \mathrm{SU}(n)$ are Lie subgroups of $\mathrm{GL}(n;\mathbb{C})$
- $\mathfrak{u}(n) = \{X \in \mathrm{End}(\mathbb{C}^n) \mid X^* + X = O\}$ (Lie algebra of $\mathrm{U}(n)$)
- $\mathfrak{su}(n) = \{X \in \mathrm{End}(\mathbb{C}^n) \mid X^* + X = O, \mathrm{tr}X = 0\}$ (Lie algebra of $\mathrm{SU}(n)$)

A **Lie subgroup** H of G is a Lie group s.t.

- H is a subset of G
- inclusion map $H \hookrightarrow G$ is an injective immersion and group homomorphism

Fact: A closed subgroup of G is a Lie subgroup of G.

Definition 1.6. Let V be a $\mathbb{K}$-vector space, G be a Lie group. A Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$ is called a **representation of** V. The Lie algebra homomorphism $\rho_{*e} : \mathfrak{g} \to \mathrm{End}(V)$ is called a **differential representation**.

Example 1.3. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. $\forall g \in G$, define a homomorphism

$$F_g : G \to G,\ h \mapsto ghg^{-1}$$

Note that $F_g \circ F_{g'} = F_{gg'}$. This induces a Lie algebra homomorphism $(F_g)_{*e}$:

$\mathfrak{g} \to \mathfrak{g}$ which satisfies $(F_g)_{*e} \circ (F_{g'})_{*e} = (F_{gg'})_{*e}$. So we obtain a representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \ g \mapsto (F_g)_{*e}$$

called **adjoint representation of** G. The differential representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ of Ad is given as follows.

Proposition 1.4. $\forall X, Y \in \mathfrak{g}$, $\mathrm{ad}(X)(Y) = [X, Y]$.

Proof. Note that $F_g = R_{g^{-1}} \circ L_g$. Then

$$\begin{aligned}
\mathrm{ad}(X)(Y) &= \frac{d}{dt} \mathrm{Ad}(\exp_G(tX))(Y)|_{t=0} \\
&= \frac{d}{dt} \left(R_{\exp_G(-tX)}\right)_{*\exp_G(tX)} \left(L_{\exp_G(tX)}\right)_{*e} (Y)|_{t=0} \\
&= [X^\sharp, Y^\sharp]_e = [X, Y]
\end{aligned}$$

□

Recall that there is a exponential map in Riemannian geometry. The Riemannian exp and the Lie group exp are related as follows.

Definition 1.7. A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group G is said to be **bi-invariant** if $\forall g, h \in G$, $L_g^* R_h^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$.

Theorem 1.1. Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$. Then $\exp_e = \exp_G$.

To show this we describe the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$.

Lemma 1.1. $\forall X, Y \in \mathfrak{g}$, $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

Proof. By Koszul formula, we have

$$\begin{aligned}
\langle \nabla_{X^\sharp} Y^\sharp, Z^\sharp \rangle = \frac{1}{2} \big(& X^\sharp \langle Y^\sharp, Z^\sharp \rangle + Y^\sharp \langle Z^\sharp, X^\sharp \rangle - Z^\sharp \langle X^\sharp, Y^\sharp \rangle \\
& - \langle Y^\sharp, [X^\sharp, Z^\sharp] \rangle - \langle Z^\sharp, [Y^\sharp, X^\sharp] \rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp] \rangle \big)
\end{aligned}$$

Since for $\forall g \in G$, $X_g^\sharp = \dfrac{d}{dt} g \cdot \exp_G(tX) \mid_{t=0}$, we have

$$X^\sharp \langle Y^\sharp, Z^\sharp \rangle = \frac{d}{dt} \langle Y^\sharp_{g \cdot \exp_G(tX)}, Z^\sharp_{g \cdot \exp_G(tX)} \rangle_{g \cdot \exp_G(tX)} \mid_{t=0} = \frac{d}{dt} \langle Y, Z \rangle_e \mid_{t=0} = 0$$

Since $\langle \cdot, \cdot \rangle$ is bi-invariant,

$$L_g^* R_{g^{-1}}^* \langle \cdot, \cdot \rangle_e = \langle \cdot, \cdot \rangle_e \text{ for } \forall g \in G \iff \langle \mathrm{Ad}(g)(\cdot), \mathrm{Ad}(g)(\cdot) \rangle_e = \langle \cdot, \cdot \rangle_e$$

Setting $g = \exp_G(tZ)$ and $\frac{d}{dt}\mid_{t=0}$, we have $\langle \mathrm{ad}(Z)(\cdot), \cdot\rangle_e + \langle \cdot, \mathrm{ad}(Z)(\cdot)\rangle_e = 0$, which shows that $\langle Y^\sharp, [X^\sharp, Z^\sharp]\rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp]\rangle = 0$, so we have $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

□

The proof of the theorem completes once shown that $\exp_G(tX)$ is geodesic, which is left as an exercise.

Exercise 1.1. Prove the theorem.

Remark 1.1. Existence/uniqueness of bi-invariant metrics? Some facts from representation theory are needed, the argument here is not used after this remark.

$\boxed{\text{Existence}}$ When G is compact, $\exists$ bi-invariant metric using "averaging trick".

- We first define Ad-invariant inner product on $\mathfrak{g}$.
- Then extend it to the whole G by pulling back L_g.

$\boxed{\text{Note:}}$ $\exists$ bi-invariant on $G \iff \exists$ Ad-invariant inner product on $\mathfrak{g}$.

$$\begin{cases} (\Rightarrow) \text{ Trivial.} \\ (\Leftarrow) \text{ Given } \mathrm{Ad}\text{-invariant inner product on } \mathfrak{g}, \text{ we can extend it to left-invariant metric} \\ \quad \text{on } G, \text{ this is also right-invariant by pullback of } R_h = R_h \circ L_{h^{-1}} \circ L_h = \mathrm{Ad}(h^{-1}) \circ L_h \end{cases}$$

$\boxed{\text{Uniqueness}}$ When G is abelian, then $L_g = R_g$, so $\exists$ many bi-invariant metrics on G (Any inner product on $\mathfrak{g}$ induces left-invariant metric on $\mathfrak{g}$, by the note above it is bi-invariant). Suppose that $\exists$ Ad-invariant inner product $\langle \cdot, \cdot\rangle$ on $\mathfrak{g}$. By $\langle \cdot, \cdot\rangle$, we have an irreducible decomposition of $(\mathfrak{g}, \mathrm{Ad})$: $\mathfrak{g} = \mathfrak{g}_1^{\oplus n_1} \oplus \cdots \oplus \mathfrak{g}_r^{\oplus n_r}$, where $\mathfrak{g}_i$ is irreducible representation of G and $\mathfrak{g}_i \neq \mathfrak{g}_j$ for $i \neq j$. Then

$$\dim\{\mathrm{Ad}\text{-invariant symmetric bilinear map } \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\} = \sum_{i=1}^{r} n_i^2$$

To see this, take $T \in \{\mathrm{Ad}\text{-invariant symmetric bilinear map }\}$ and use Schur's lemma to

$$T_{ij} : \mathfrak{g}_i \hookrightarrow \mathfrak{g} \xrightarrow{x \mapsto T(x,\cdot)} \mathfrak{g}^* \overset{\langle\cdot,\cdot\rangle}{\cong} \mathfrak{g} \xrightarrow{proj.} \mathfrak{g}_j$$

Then $T_{ij} = \begin{cases} 0 & (i \neq j) \\ c \cdot \mathrm{id} & (i = j) \text{ for } \exists c \in \mathbb{R} \end{cases}$, so uniqueness up to scalar multiplication holds only when $r = 1, n = 1$, i.e. $(\mathfrak{g}, \mathrm{Ad})$ is irreducible $\iff$ G is simple Lie group.

Definition 1.8. Let M be smooth manifold, G be Lie group with unit e. A smooth map

$$A : M \times G \to M, \ (x, g) \mapsto xg$$

is called the **right action of G on M** if

(1) $\forall x \in M$, $xe = x$

(2) $\forall x \in M$, $\forall g, g \in G$, $(xg)h = x(gh)$

We write the right action as $M \curvearrowleft G$.

Definition 1.9. Suppose $M \curvearrowleft G$.

(1) For $\forall g \in G$, set $R_g : M \to M$, $X \mapsto xg$ (called right translation).

(2) For $\forall X \in \mathfrak{g}$, define the **fundamental vector field** $X^\sharp \in \mathfrak{X}(M)$ by $X^\sharp_x = \dfrac{d}{dt} x \cdot \exp_G(tX) \mid_{t=0} = dA(x, \cdot)_e(X)$.

Here the notation $X^\sharp$ is the same as the left-invariant vector field on Lie group, we'll show that they have similar property:

Remark 1.2. (1) $\forall g \in G$, $\forall X \in \mathfrak{g}$, $(R_g)_* X^\sharp = (\mathrm{Ad}(g^{-1})X)^\sharp$.

(2) $\forall X, Y \in \mathfrak{g}$, $[X^\sharp, Y^\sharp] = [X, Y]^\sharp$.

Proof. (1) $\forall x \in M$, $\left((R_g)_* X^\sharp\right)_x = (R_g)_* X^\sharp_{xg^{-1}} = \dfrac{d}{dt} xg^{-1} \exp_G(tX) g \mid_{t=0}$. Since we know $\{g^{-1} \exp_G(tX) g\}_{t \in \mathbb{R}}$ is a one parameter subgroup of G with $\dfrac{d}{dt} g^{-1} \exp_G(tX) g \mid_{t=0} = \mathrm{Ad}(g^{-1})X$, then $g^{-1} \exp_G(tX) g = \exp_G(t \,\mathrm{Ad}(g^{-1})X)$, which gives (1).

(2) By definition, $\left\{\varphi_t = R_{\exp_G(tX)}\right\}_{t \in \mathbb{R}}$ is flow of $X^\sharp$. So

$$[X^\sharp, Y^\sharp] = \frac{d}{dt} (\varphi_{-t})_* Y^\sharp \mid_{t=0} = \frac{d}{dt} \left(\mathrm{Ad}\left(\exp_G(tX)\right) Y\right)^\sharp \mid_{t=0} = (\mathrm{ad}(X)(Y))^\sharp = [X, Y]^\sharp.$$

□

Remark 1.3. We can define the left action

$$A^L : G \times M \to M, \ (g, x) \mapsto gx$$

and also the fundamental vector field $X^\sharp_L \in \mathfrak{X}(M)$. The left and right actions are essentially the same, since the right action is given form the left action. Indeed, given A^L above, define A by $A(x, g) = A^L(g^{-1}, x) = g^{-1}x$, then $X^\sharp_L = -X^\sharp$ for $X \in \mathfrak{g}$. $[X^\sharp_L, Y^\sharp_L] = [X, Y]^\sharp = -[X, Y]^\sharp_L$.

Definition 1.10. Suppose $M \curvearrowleft G$.

(1) For $p \in M$, define $G_p = \{g \in G \mid pg = p\}$ (called **isotropy subgroup at** p).

(2) The G action is **free** of $G_p = \{e\}$ for $\forall p \in M$.

(3) The G action is **effective** if $\bigcap\limits_{p \in M} G_p = \{e\}$. In other words, $G \to \mathrm{Diff}(M)$ is injective.

1.2 Definition of Principal Bundles

Definition 1.11. Let P, M be smooth manifolds and G be Lie group. The map $\pi_P : P \to M$ is a **principal G-bundle** or **principal bundle with structure group** G if:

(1) $P \curvearrowleft G$.

(2) There exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of M and diffeomorphisms called local trivialization

$$\phi_\alpha : \pi_P^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$$

such that

(2.1) Denote $p_1 : U_\alpha \times G \to U_\alpha$ the projection, then $\pi_P = p_1 \circ \phi_\alpha$

(2.2) The G-action preserves each $\pi_P^{-1}(U_\alpha)$. Denoting the right G-action on $U_\alpha \times G$ by

$$(U_\alpha \times G) \times G \to U_\alpha \times G,\ ((x, h), g) \mapsto (x, h) \cdot g = (x, hg)$$

Then ϕ_α is G-equivalent, i.e. $\forall \xi \in \pi_P^{-1}(U_\alpha)$, $\forall g \in G$, $\phi_\alpha(\xi g) = \phi_\alpha(\xi) g$. Note that the G-action is free.

We often write $P\mid_U = \pi_P^{-1}(U)$ for open subset $U \subseteq M$ and $P_x = \pi_P^{-1}(x)$ for $x \in M$, P_x is called the **fiber of** P **at** x.

Recall that $e \in G$ is the unit, define a **section** $p_\alpha \in \Gamma(P\mid_{U_\alpha})$ on U_α: $\phi_\alpha(p_\alpha(x)) = (x, e)$, which is equivalent to $p_\alpha(x) = \phi_\alpha^{-1}(x, e)$. Define $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ by

$$\boxed{p_\alpha(x) g_{\alpha\beta}(x) = p_\beta(x)}$$

$\{g_{\alpha\beta}\}_{\alpha\beta}$ is called the **transition map** of $\pi_P : P \to M$. Note that $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$, we have $\boxed{g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)}$. Conversely, given open covering $\{U_\alpha\}_{\alpha \in A}$ of M and transition maps, we can recover principal G-bundle $\pi_P : P \to M$.

As before, for $g \in G$, we can define $R_g : P \to P$ the right translation and the fundamental vector field $X^\sharp$ generated by $X \in \mathfrak{g}$.

Definition 1.12. Let $\pi_P : P \to M$ be a principal G-bundle, $\rho : G \to \mathrm{GL}(V)$ representation of G. Define the right G-action on $P \times V$ by

$$(P \times V) \times G \to P \times V,\ ((\xi, v), g) \mapsto \left(\xi g, \rho(g)^{-1} v\right)$$

$P \times_\rho V = (P \times V) / G$ is called the **associated vector bundle to** P.

Set $\xi \times_\rho v$ the equivalence class of $(\xi, v) \in P \times V$. Set $E = P \times_\rho V$, $\pi_E : E \to M$, $\xi \times_\rho v \mapsto \pi_P(\xi)$. Then $\pi_E : E \to M$ is a vector bundle.

The local trivialization of E are induced from those of P:

$$\phi_\alpha^E : E\mid_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V,\ p_\alpha(x)\times_\rho v \mapsto (x,v)$$

For $x\in U_\alpha\cap U_\beta$ and $v_\beta\in V$, $p_\beta(x)\times_\rho v_\beta = p_\alpha g_{\alpha\beta}(x)\times_\rho v_\beta = p_\alpha(x)\times_\rho \rho\left(g_{\alpha\beta}(x)\right)v_\beta$. The transition functions of E are given by $\{\rho(g_{\alpha\beta}) : U_\alpha\cap U_\beta \to \mathrm{GL}(V)\}$.

We will explain some relations between P and E.

- First note that $\forall\ \xi\in P$, we have $\boxed{\xi : V \xrightarrow{\cong} E_{\pi_P(\xi)},\ v\mapsto \xi\times_\rho v \text{ is an isomorphism.}}$ For $\xi'\in P$ with $\xi'=\xi g$ for $g\in G$, we have

$$\xi^{-1}\left(\xi'\times_\rho v'\right) = \xi^{-1}\left(\xi\times_\rho \rho(g)v'\right) = \rho(g)v'$$

  for $v'\in V$.

- π_P^*E is a trivial bundle. Indeed,

$$P\times V \underset{(\xi,\xi^{-1}(e))\leftarrow(\xi,e)}{\xrightarrow{(\xi,v)\mapsto(\xi,\xi\times_\rho v)}} \pi_P^*E = \{(\xi,e)\in P\times E \mid \pi_P(\xi)=\pi_E(e)\} \text{ is isomorphism.}$$

- Next, for $s\in\Omega^q(E)=\Gamma\left(\Lambda^q T^*M\otimes E\right)$, define $\pi_P^*s\in\Omega^q\left(P;V\right)$ as follows (V-valued q-form on P)

  - For $q=0$, $\underline{\left(\pi_P^*s\right)(\xi) = \xi^{-1}\left(s\left(\pi_P(\xi)\right)\right)}$
  - For $q>1$, $\forall\alpha\in\Omega^q(M)$, $\forall s\in\Omega^0(E)=\Gamma(E)$,

$$\underline{\pi_P^*\left(\alpha\otimes s\right) = \pi_P^*\alpha\otimes\pi_P^*s}$$

    The left one is pullback and the right one is define above. In other words, $\forall\xi\in P$, $\forall v_1,\cdots,v_q\in T_\xi P$,

$$\underline{\left(\pi_P^*s\right)_\xi\left(v_1,\cdots,v_q\right) = \xi^{-1}\left(s_{\pi_P(\xi)}\left(\pi_{P*}(v_1),\cdots,\pi_{P*}(v_q)\right)\right)}$$

  Notation: denote $\Omega_B^q(P;V)$ to be the elements $\widetilde{s}$ in $\Omega^q(P;V)$ satisfying:

  ($\star$) $\forall X\in\mathfrak{g}$, $i(X^\sharp)\widetilde{s}=0$.

  ($\star\star$) $\forall g\in G$, $R_g^*\widetilde{s}=\rho(g)^{-1}\widetilde{s}$.

  called the **space of basic q-forms**. Note that $\Omega_B^q(P;V)$ depends on representation ρ.

Proposition 1.5. (**Important to study the relations between P and E**)

(1) $\pi_P^*\left(\Omega^q(E)\right)\subseteq\Omega_B^q(P;V)$ and $\pi_P^* : \Omega^q(E)\xrightarrow{\cong}\Omega_B^q(P;V)$ is isomorphism. E-valued q-forms on M are identified with basic q-forms on P.

(2) Recall the local trivialization $\phi_\alpha^E : E\mid_{U_\alpha}\xrightarrow{\cong} U_\alpha\times V$. For $s\in\Omega^q(E)$, suppose that $s\mid_{U_\alpha}$ corresponds to $s_\alpha\in\Omega^q(U_\alpha;V)$. Then $s_\alpha = p_\alpha^*\left(\pi_P^*s\right)$. So we regard $s\in\Omega^q(E)$ as a basic form, and then pullback by p_α is s_α.

Remark 1.4. The identification $\Omega^q(E) \cong \Omega^q_B(P;V)$ in proposition 1.5 is fundamental to modern geometry and gauge theory. The shift from E-valued forms on M to V-valued forms on P is motivated by three main advantages:

- An E-valued form $s \in \Omega^q(E)$ takes values in fibers E_x that vary with $x \in M$. By pulling back to P, we use the isomorphism $\xi : V \to E_x$ (where ξ is a point in the fiber of P) to "anchor" the values in a single, fixed vector space V. This simplifies the algebraic treatment of sections.
- On the base manifold M, a global section of E must be described via local data $\{s_\alpha\}$ satisfying transition rules $s_\alpha = g_{\alpha\beta}s_\beta$. On the principal bundle P, the "twisting" of the bundle is replaced by the global equivariance condition:

  $$R_g^*\tilde{s} = \rho(g)^{-1}\tilde{s}$$

  This turns a topological problem (glueing charts) into a representation-theoretic one (symmetry under the group action).
- The exterior derivative d is not intrinsically defined for E-valued forms on M because there is no canonical way to compare vectors in different fibers E_x and E_y. However, for V-valued forms on P, the standard exterior derivative d is perfectly well-defined because the target V is a fixed vector space. This allows us to perform differential calculus on P before a connection is even introduced.

Proof. (1) We show $\pi_P^*(\Omega^q(E)) \subseteq \Omega^q_B(P;V)$. Take $\forall s \in \Omega^q(E)$,

- $\boxed{\text{For } q = 0}$ $(\star)$ is trivial; For $(\star\star)$: for $g \in G$, $\xi \in P$, we have

  $$\left(R_g^*\pi_P^*s\right)(\xi) = \left(\pi_P^*s\right)(R_g\xi) = (\xi g)^{-1}\left(s(\pi_P(\xi g))\right) = (\xi g)^{-1}\left(s(\pi_P(\xi))\right)$$

  By definition of ξ, we have: for $\forall v \in V$,

  $$\xi(v) = \xi \times_\rho v = \xi g \times_\rho \rho(g)^{-1}(v) = (\xi g)\left(\rho(g)^{-1}(v)\right)$$

  so $\xi = (\xi g) \circ \rho(g)^{-1}$, hence $(\xi g)^{-1} = \rho(g)^{-1} \circ \xi^{-1}$. Then

  $$\left(R_g^*\pi_P^*s\right)(\xi) = \rho(g)^{-1}\left(\xi^{-1}s\left(\pi_P(\xi)\right)\right) = \left(\rho(g)^{-1}\left(\pi_P^*s\right)\right)(\xi).$$

- $\boxed{\text{For } q \geq 1}$ $(\star)$: Since $\pi_P(\xi g) = \pi_P(\xi)$, we have $\pi_{P*}(X^\sharp) = 0$, which implies $(\star)$; $(\star\star)$: For $\forall \alpha \in \Omega^q(M)$, $\forall s \in \Gamma(E)$, $\forall g \in G$, we have

  $$R_g^*\left(\pi_P^*(\alpha \otimes s)\right) = R_g^*\pi_P^*\alpha \otimes R_g^*\pi_P^*s = \pi_P^*\alpha \otimes \rho(g)^{-1}\left(\pi_P^*s\right) = \rho(g)^{-1}\pi_P^*\left(\alpha \otimes s\right)$$

  which finishes the proof of $(\star\star)$.

Next we show $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega^q_B(P;V)$:

- $\boxed{\text{Injectivity}}$ It is clear from the formula

$$(\pi_P^* s)_\xi (v_1, \cdots, v_q) = \xi^{-1} \left(s_{\pi_P(\xi)} (\pi_{P*}(v_1), \cdots, \pi_{P*}(v_q)) \right).$$

- $\boxed{\text{Surjectivity}}$ Take $\widetilde{s} \in \Omega^q_B(P; V)$,
  - When $q = 0$, define $s \in \Omega^0(E) = \Gamma(E)$ by $s(x) = \xi \times_\rho \widetilde{s}(\xi)$ where $\xi \in \pi_P^{-1}(x)$. It is well-defined since $\xi g \times_\rho \widetilde{s}(\xi g) = \xi g \times_\rho (R_g^* \widetilde{s})(\xi) = \xi g \times_\rho \rho(g)^{-1} \widetilde{s}(\xi) = \xi \times_\rho \widetilde{s}(\xi)$. Then by definition we have $\pi_P^* s = \widetilde{s}$.
  - When $q \geq 1$, define $s \in \Omega^q(E)$ by

$$s_x(w_1, \cdots, w_q) = \xi \times_\rho \widetilde{s}_\xi(\widetilde{w_1}, \cdots, \widetilde{w_q})$$

    where $x \in M$, $w_i \in T_x M$, $\xi \in \pi_P^{-1}(x)$, $\pi_{P*}(\widetilde{w_i}) = w_i$. Now we show this definition is independent of the choice of $\widetilde{w}_i$ and ξ.
    * Fix $\xi \in P$, since $\ker(\pi_{P*})_\xi = \left\{ X_\xi^\sharp \mid X \in \mathfrak{g} \right\}$, the $\widetilde{w}_i$ we choose only differs by a $X_\xi^\sharp$ for $X \in \mathfrak{g}$. Since $i(X^\sharp)\widetilde{s} = 0$, the difference$X_\xi^\sharp$ is irrelevant.
    * For $g \in G$,

$$\begin{aligned} &\xi g \times_\rho \widetilde{s}_{\xi g}((R_g)_* \widetilde{w}_1, \cdots, (R_g)_* \widetilde{w}_q) \quad \text{(we use the independence of } \widetilde{w}_i \text{ here)} \\ = &\xi g \times_\rho (R_g^* \widetilde{s})_\xi (\widetilde{w}_1, \cdots, \widetilde{w}_q) \\ = &\xi g \times_\rho \rho(g)^{-1} (\widetilde{s}_\xi(\widetilde{w}_1, \cdots, \widetilde{w}_q)) \\ = &\xi \times_\rho \widetilde{s}_\xi(\widetilde{w}_1, \cdots, \widetilde{w}_q). \end{aligned}$$

    Hence it is well-defined.

(2) First we describe s_α clearly. Set $s|_{U_\alpha} = \sum \beta_i \otimes e_i$. Since

$$\phi_\alpha^E : E\mid_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V, \; p_\alpha(x) \times_\rho v \mapsto (x, v),$$

we have $\phi_\alpha^E((e_i)_x) = (x, v_i(x))$ for a function $v_i : U_\alpha \to V$. Note that $(e_i)_x = p_\alpha(x) \times_\rho v_i(x)$. Then $s_\alpha = \sum \beta_i \otimes v_i$. Now we compute

$$p_\alpha^* (\pi_P^* s) = p_\alpha^* \left(\sum \pi_P^* \beta_i \otimes \pi_P^* e_i \right) = \sum (\pi_P \circ p_\alpha)^* \beta_i \otimes (\pi_P^* e_i) \, p_\alpha(x) = \sum \beta_i \otimes v_i(x).$$

So we have $p_\alpha^* (\pi_P^* s) = s_\alpha$. $\square$

Now we give a typical example of principal bundles.

Example 1.4. Let $\pi_E : E \to M$ be a vector bundle with rank r. For $x \in M$, set

- $P_x = \{\xi : \mathbb{K}^r \to E_x : \text{ linear isomorphism } \}$.
- $P = \bigsqcup_{x \in M} P_x$; $\pi_P : P \to M$, $\xi \mapsto x$ if $\xi \in P_x$.

We see that $\pi_P : P \to M$ is a principal $\mathrm{GL}(r; \mathbb{K})$-bundle:

- The right action on P is given by:

$$P \times \mathrm{GL}(r;\mathbb{K}) \to P,\ (\xi \times g) \mapsto \xi \circ g.$$

- To give a local trivialization, first note that

$$P_x \xrightarrow[\xi \mapsto \{\xi(\epsilon_1),\cdots,\xi(\epsilon_r)\}]{\cong} \{\text{basis of } E_x\},$$

where $\epsilon_i = (0, \cdots, \underset{i\text{-th}}{1}, \cdots, 0)^t$. If $\{e_1, \cdots, e_r\} \subseteq \Gamma(E|_{U_\alpha})$ is local frame of E over $U_\alpha \subseteq M$, define $p_\alpha \in \Gamma(P|_{U_\alpha})$ by

$$p_\alpha : U_\alpha \to P|_{U_\alpha},\ x \mapsto (e_1(x), \cdots, e_r(x)),$$

which induces a local trivialization

$$\phi_\alpha^P : P|_{U_\alpha} \to U_\alpha \times \mathrm{GL}(r;\mathbb{K}),\ \xi \mapsto \left(\pi_P(\xi), (p_\alpha(\pi_P(\xi)))^{-1}\xi\right)$$

The inverse of this map is $(x, g) \mapsto p_\alpha(x) \cdot g$. We see that ϕ_α^P is $\mathrm{GL}(r;\mathbb{K})$-equivalent.

So $\pi_P : P \to M$ is a principal $\mathrm{GL}(r;\mathbb{K})$-bundle. This is called the **frame bundle** of $\pi_E : E \to M$. Also note that transition maps of E is the transition maps of P. Indeed, if $\{f_1, \cdots, f_r\} \subseteq \Gamma(E|_{U_\alpha})$ is another local frame, the transition map $g_{\alpha\beta}$ satisfies $(f_1, \cdots, f_r) = (e_1, \cdots, e_r)g_{\alpha\beta}$, and this is exactly $p_\beta = p_\alpha g_{\alpha\beta}$.

Example 1.5. Let $\pi_E : (E, h) \to M$ be a Riemannian vector bundle with rank r and bundle metric h. Let $(\mathbb{K}^r, \langle\cdot,\cdot\rangle)$ be the standard inner product space. Set

- $P_x = \{\xi : \mathbb{K}^r \to E_x : \text{linear isometry}\}$.
- $P = \bigsqcup_{x \in M} P_x$; $\pi_P : P \to M$, $\xi \mapsto x$ if $\xi \in P_x$.

We see that $\pi_P : P \to M$ is a principal $\mathrm{O}(r;\mathbb{K})$-bundle (where O is $\mathrm{O}(r)$ for $\mathbb{R}$ and $\mathrm{U}(r)$ for $\mathbb{C}$):

- The right action on P is given by:

$$P \times \mathrm{O}(r;\mathbb{K}) \to P, \quad (\xi \times g) \mapsto \xi \circ g.$$

Since g is an isometry of $\mathbb{K}^r$ and ξ is an isometry into E_x, the composition $\xi \circ g$ is an isometry.

- To give a local trivialization, let $\{e_1, \ldots, e_r\} \subseteq \Gamma(E|_{U_\alpha})$ be a **local orthonormal frame** of E over $U_\alpha \subseteq M$. Define $p_\alpha \in \Gamma(P|_{U_\alpha})$ by

$$p_\alpha : U_\alpha \to P|_{U_\alpha}, \quad x \mapsto (e_1(x), \ldots, e_r(x)),$$

which induces a local trivialization

$$\phi_\alpha^P : P|_{U_\alpha} \to U_\alpha \times \mathrm{O}(r;\mathbb{K}), \quad \xi \mapsto (\pi_P(\xi), (p_\alpha(\pi_P(\xi)))^{-1}\xi).$$

This is called the **orthonormal frame bundle**. Its transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{O}(r; \mathbb{K})$ are exactly the transition maps of E relative to orthonormal frames.

Example 1.6. Consider the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. Let $M = \mathbb{CP}^n$ be the complex projective space. Set

- $P = S^{2n+1}$ and $G = S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.
- $\pi : P \to M$ given by $\pi(z_0, \dots, z_n) = [z_0 : \dots : z_n]$.

$\pi : S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1-bundle, known as the **Hopf bundle**:

- The right action of S^1 on S^{2n+1} is given by scalar multiplication:
$$S^{2n+1} \times S^1 \to S^{2n+1}, \quad (\mathbf{z}, \lambda) \mapsto \mathbf{z} \cdot \lambda = (z_0\lambda, \dots, z_n\lambda).$$
This is well-defined as $|\mathbf{z} \cdot \lambda| = |\mathbf{z}| \cdot |\lambda| = 1$.
- On the chart $U_j = \{[z] : z_j \neq 0\}$, define the local section $p_j : U_j \to S^{2n+1}$ by
$$p_j([z]) = \frac{1}{\|\mathbf{z}\|}\left(z_0 \frac{\bar{z}_j}{|z_j|}, \dots, z_n \frac{\bar{z}_j}{|z_j|}\right).$$
- The transition maps $g_{ij} : U_i \cap U_j \to S^1$ satisfy $p_j = p_i \cdot g_{ij}$. Computing this gives $g_{ij}([z]) = \frac{z_j}{|z_j|}\frac{|z_i|}{z_i}$, which are smooth maps into S^1.

The Hopf bundle is topologically non-trivial (it has no global section).

1.3 Connections on Principal Bundles

In this subsection we study properties of connection on principal bundle and its relation between connection on associated vector bundle.

Definition 1.13. Let $\pi_P : P \to M$ be principal G-bundle.

(1) A distribution $\{H_\xi \subseteq T_\xi P\}_{\xi \in P}$ is a **connection** on P if

(1-1) $\forall \xi \in P$, $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$.

(1-2) $\{H_\xi \subseteq T_\xi P\}_{\xi \in P}$ is G-invariant, i.e. $\forall \xi \in P$, $\forall g \in G$, $(R_g)_{*\xi} H_\xi = H_{\xi g}$.

H_ξ, $\ker(\pi_P)_{*\xi}$ are called **horizontal/vertical subspaces** separately.

(2) A $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P; \mathfrak{g})$ on P is a **connection form** if

(2-1) $\forall X \in \mathfrak{g}$, $\theta(X^\sharp) = X$.

(2-2) $\forall g \in G$, $R_g^*\theta = \mathrm{Ad}(g^{-1})\theta$.

These 2 notions of connection are identical in the following sense:

Theorem 1.2. Let $\pi_P : P \to M$ be principal G-bundle.

(1) If $\theta \in \Omega^1(P;\mathfrak{g})$ is a connection form, a distribution $\{\ker\theta_\xi\}_{\xi\in P} = \{v \in T_\xi P \mid \theta_\xi(v) = 0\}_{\xi\in P}$ is a connection on P.

(2) $\{\text{connection form}\} \leftrightarrow \{\text{connection on } P\}$, $\theta \leftrightarrow \{\ker\theta_\xi\}_{\xi\in P}$ is bijective.

Proof. (1) We check that $\{\ker\theta_\xi\}_{\xi\in P}$ satisfies (1-1), (1-2):

(1-1) Note that $\ker(\pi_P)_{*\xi} = \left\{X^\sharp_\xi \in T_\xi P \mid X \in \mathfrak{g}\right\}$, then for $\forall v \in T_\xi P$, we have $\theta(v) \in \mathfrak{g}$ and $v = \theta(v)^\sharp_\xi + \left(v - \theta(v)^\sharp_\xi\right)$, which implies that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus \ker\theta_\xi$ ($\ker(\pi_P)_{*\xi} \cap \ker\theta_\xi = \{0\}$ is obvious).

(1-2) Take $\forall v \in \ker\theta_\xi$. By (2-2), $\forall g \in G$, we have $\left(R_g^*\theta\right)_\xi = \mathrm{Ad}(g^{-1})\theta_\xi$, the left hand side is $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right)$, so we have $(R_g)_{*\xi}(v) \in \ker\theta_{\xi g}$, hence $(R_g)_{*\xi}(\ker\theta_\xi) \subseteq \ker\theta_{\xi g}$. Replacing (g,ξ) with $(g^{-1},\xi g)$, we have $(R_{g^{-1}})_{*\xi g}(\ker\theta_{\xi g}) \subseteq \ker\theta_\xi$. So $(R_g)_{*\xi}(\ker\theta_\xi) = \ker\theta_{\xi g}$, $\{\ker\theta_\xi\}_{\xi\in P}$ is a connection on P.

(2) $\boxed{\text{Injectivity}}$ Let θ, θ' be connection forms with $\ker\theta_\xi = \ker\theta'_\xi$ $\forall\xi \in P$. We show that $\forall v \in T_\xi P$, $\theta_\xi(v) = \theta'_\xi(v)$. By (1), v is described as $v = X^\sharp_\xi + w$ for $X^\sharp_\xi \in \ker(\pi_P)_{*\xi}$ and $w \in \ker\theta_\xi = \ker\theta'_\xi$. So $\theta_\xi(v) = \theta_\xi(X^\sharp_\xi) = X = \theta'_\xi(v)$.

$\boxed{\text{Surjectivity}}$ Take $\forall\,\{H_\xi\}_{\xi\in P}$ a connection on P. By (1-1), we can define $\theta \in \Omega^1(P;\mathfrak{g})$ by

$$\theta_\xi(v) = \begin{cases} 0 & (v \in H_\xi) \\ X & (v = X^\sharp_\xi \text{ for } X \in \mathfrak{g}) \end{cases}$$

By definition, $\ker\theta_\xi = H_\xi$, we check (2-1), (2-2).

(2-1) Holds by definition of θ_ξ.

(2-2) $\forall\xi \in P$, $\forall g \in G$, we show that $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = \mathrm{Ad}(g^{-1})\theta_\xi$ on $T_\xi P$. Recall that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$, if $v \in H_\xi$, the equality holds by definition and (1-2); for $\forall X \in \mathfrak{g}$,

$$(R_g)_{*\xi}\left(X^\sharp_\xi\right) = (R_g)_{*\xi}\frac{d}{dt}\xi\exp_G(tX)\mid_{t=0} = \frac{d}{dt}\xi g\cdot g^{-1}\exp_G(tX)g\mid_{t=0} = \left(\mathrm{Ad}(g^{-1})X\right)^\sharp_{\xi g}$$

So $\theta_{\xi g}\left((R_g)_{*\xi}(X^\sharp_\xi)\right) = \mathrm{Ad}(g^{-1})X = \mathrm{Ad}(g^{-1})\theta_\xi(X^\sharp_\xi)$, hence the equality holds. So we have $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = \mathrm{Ad}(g^{-1})\theta_\xi$ on $T_\xi P$. □

The next proposition says that a connection form θ on P induces a connection ∇^E of the associated vector bundle E. The relation between θ and local connection form of ∇^E us also given.

Proposition 1.6. Let $\pi_P : P \to M$ be a principal bundle, $\rho : G \to \mathrm{GL}(V)$ a representation of G with differential representation $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$. Denote by $\theta \in \Omega^1(P;\mathfrak{g})$ a connection form. Set $E = P \times_\rho V$ its associated vector bundle. Then,

(1) $(d+\rho_*(\theta)\wedge)\,\Omega^q_B(P;V) \subseteq \Omega^{q+1}_B(P;V)$. Here

- d: standard exterior derivative.
- $\rho_*(\theta) \in \Omega^1(P;\mathrm{End}(V))$ acts on $\Omega^q_B(P;V)$ by wedging on differential form parts and composing $\mathrm{End}(V), V$-parts.

(2) Recall that $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega^q_B(P;V)$. Then we can define $\nabla^E : \Omega^0(E) \to \Omega^1(E)$ by $(\pi_P^*)^{-1} \circ (d+\rho_*(\theta)\wedge) \circ \pi_P^*$.

(3) Recall that a local section $p_\alpha \in \Gamma(P|_{U_\alpha})$ induces a local trivialization $\phi^E_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$. Then

$$\begin{tikzcd}
\Omega^0(E|_{U_\alpha}) \arrow[r, "\nabla^E|_{U_\alpha}"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^1(E|_{U_\alpha}) \arrow[d, "\cong"] \\
\Omega^0(U_\alpha;V) \arrow[r, "d+\rho_*(p_\alpha^*\theta)\wedge"'] & \Omega^1(U_\alpha;V)
\end{tikzcd}$$

(4) Recall that a connection ∇^E induces the exterior derivative $d^{\nabla^E} : \Omega^q(E) \to \Omega^{q+1}(E)$. Then

$$\begin{tikzcd}
\Omega^q(E) \arrow[r, "d^{\nabla^E}"] \arrow[d, "\pi_P^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{q+1}(E) \arrow[d, "\cong"', "\pi_P^*"] \\
\Omega^q_B(P;V) \arrow[r, "d+\rho_*(\theta)\wedge"'] & \Omega^{q+1}_B(P;V)
\end{tikzcd}$$

Remark 1.5. In [Kobayashi-Nomizu, *Foundation of differential geometry* Vol 1, chapter 2, section 5], for any principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$, $\forall\, V$ vector space, the *exterior covariant derivative* $D : \Omega^q(P;V) \to \Omega^{q+1}(P;V)$ is defined by $(D\widetilde{s})(v_0,\cdots,v_q) = (d\widetilde{s})(hv_0,\cdots,hv_q)$ for $v_i \in TP$, where $h : TP \to \ker\theta$ is the projection. If in addition, given a representation $\rho : G \to \mathrm{GL}(V)$ and $\widetilde{s} \in \Omega^q_B(P;V)$, we have $D\widetilde{s} = (d+\rho_*(\theta)\wedge)(\widetilde{s})$.

Proof. (1) Take $\forall \widetilde{s} \in \Omega^q_B(P;V)$, recall that $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\widetilde{s} = 0. \\ \forall g \in G, R_g^*\widetilde{s} = \rho(g)^{-1}\widetilde{s}. \end{cases}$. We show that $(d+\rho_*(\theta)\wedge)\,\widetilde{s}$ also satisfies the same property.

- $\forall X \in \mathfrak{g}$, we have
$$\mathcal{L}_{X^\sharp}\widetilde{s} = \frac{d}{dt} R^*_{\exp_G(tX)}\widetilde{s}\,|_{t=0} = \frac{d}{dt}\rho\left(\exp_G(tX)\right)^{-1}\widetilde{s}\,|_{t=0} = -\rho_*(X)\widetilde{s}.$$
Since $\mathcal{L}_{X^\sharp}\widetilde{s} = i(X^\sharp)d\widetilde{s} + d\left(i(X^\sharp)\widetilde{s}\right)$ and $i(X^\sharp)\widetilde{s} = 0$, we have $i(X^\sharp)d\widetilde{s} = -\rho_*(X)\widetilde{s}$. Hence $i(X^\sharp)\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = i(X^\sharp)d\widetilde{s} + \rho_*\left(\theta(X^\sharp)\right)\widetilde{s} - \rho_*(\theta)\wedge i(X^\sharp)\widetilde{s} = 0.$

- For $\forall g \in G$, we have

  $$R_g^*\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = dR_g^*\widetilde{s}+\rho_*\left(R_g^*\theta\right)\wedge R_g^*\widetilde{s} = d\left(\rho(g)^{-1}\widetilde{s}\right)+\rho_*\left(\mathrm{Ad}(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s}.$$

  Since $\rho(g)^{-1}$ acts only on V-part, $d\left(\rho(g)^{-1}\widetilde{s}\right) = \rho(g)^{-1}d\widetilde{s}$. Note that $\forall X \in \mathfrak{g}$,

  $$\frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)g\right)\rho(g)^{-1}\,|_{t=0} = \frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)\right)|_{t=0}$$

  and $g^{-1}\exp_G(tX)g = \exp_G(t\,\mathrm{Ad}(g^{-1})X)$, we have

  $$\rho_*\left(\mathrm{Ad}(g^{-1})X\right)\rho(g)^{-1} = \rho(g)^{-1}\rho_*(X).$$

  This implies that

  $$\rho_*\left(\mathrm{Ad}(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s} = \rho(g)^{-1}\left(\rho_*(\theta)\wedge\widetilde{s}\right).$$

  Then we obtain

  $$R_g^*\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = \rho(g)^{-1}\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right),$$

  so $(d+\rho_*(\theta)\wedge)(\widetilde{s}) \in \Omega_B^{q+1}(P;V)$.

(2) $\nabla^E = (\pi_P^*)^{-1}\circ(d+\rho_*(\theta)\wedge)\circ\pi_P^*$, we check the Leibniz rule, i.e. for $\forall f \in C^\infty(M)$, $\forall s \in \Gamma(E)$, we show $\nabla^E(fs) = df\otimes s + f\nabla^E s$. Let $D = d+\rho_*(\theta)\wedge$ be the operator acting on $\Omega_B^q(P;V)$, we define their pullbacks to P as $\tilde{s} = \pi_P^* s$ and $\tilde{f} = \pi_P^* f = f\circ\pi_P$. Note that $\pi_P^*(fs) = \tilde{f}\tilde{s}$. Applying D to the section $\tilde{f}\tilde{s}$:

$$\begin{aligned}
D(\tilde{f}\tilde{s}) &= d(\tilde{f}\tilde{s}) + \rho_*(\theta)\wedge(\tilde{f}\tilde{s})\\
&= (d\tilde{f}\otimes\tilde{s} + \tilde{f}d\tilde{s}) + \tilde{f}(\rho_*(\theta)\wedge\tilde{s})\\
&= d\tilde{f}\otimes\tilde{s} + \tilde{f}(d\tilde{s} + \rho_*(\theta)\wedge\tilde{s})\\
&= d\tilde{f}\otimes\tilde{s} + \tilde{f}D\tilde{s}.
\end{aligned}$$

Now, applying the inverse pullback $(\pi_P^*)^{-1}$ to both sides:

$$\begin{aligned}
\nabla^E(fs) &= (\pi_P^*)^{-1}D\pi_P^*(fs)\\
&= (\pi_P^*)^{-1}(d\tilde{f}\otimes\tilde{s} + \tilde{f}D\tilde{s})\\
&= (\pi_P^*)^{-1}(d\tilde{f}\otimes\tilde{s}) + (\pi_P^*)^{-1}(\tilde{f}D\tilde{s}).
\end{aligned}$$

Since $\pi_P^*(df) = d(\pi_P^* f) = d\tilde{f}$ and π_P^* commutes with scalar multiplication by f, we have:

$$\begin{aligned}
\nabla^E(fs) &= df\otimes s + f\left((\pi_P^*)^{-1}D\pi_P^* s\right)\\
&= df\otimes s + f\nabla^E s.
\end{aligned}$$

Since ∇^E satisfies both linearity and the Leibniz rule, it is a well-defined connection on E.

(3) Since for $s \in \Omega^q(E)$, $s|_{U_\alpha}$ corresponds to $p_\alpha^*\left(\pi_P^* s\right)$. We compute

$$\begin{aligned}
p_\alpha^*\pi_P^*\left(\nabla^E s\right) &= p_\alpha^*\left((d+\rho_*(\theta)\wedge)\,\pi_P^* s\right)\\
&= p_\alpha^* d\left(\pi_P^* s\right) + \rho_*\left(p_\alpha^*\theta\right)\wedge p_\alpha^*\pi_P^* s
\end{aligned}$$

$$
\begin{aligned}
&= (d + \rho_*(p_\alpha^*\theta)\wedge)\,(p_\alpha^*\pi_P^* s) \\
&= (d + \rho_*(p_\alpha^*\theta)\wedge)\, s|_{U_\alpha}.
\end{aligned}
$$

(4) Since d^{∇^E} is given by $d^{\nabla^E}(s \otimes \alpha) = \nabla^E s \wedge \alpha + s \otimes d\alpha$ for $s \in \Gamma(E)$, $\alpha \in \Omega^q(M)$, we have

$$
\begin{aligned}
\pi_P^*\left(d^{\nabla^E}(s \otimes \alpha)\right) &= \pi_P^*\left(\nabla^E s \wedge \alpha + s \otimes d\alpha\right) \\
&= (d + \rho_*(\theta)\wedge)\,\pi_P^* s \wedge \pi_P^*\alpha + \pi_P^* s \otimes \pi_P^* d\alpha \\
&= d\left(\pi_P^* s \otimes \pi_P^*\alpha\right) + \rho_*(\theta) \wedge \left(\pi_P^* s \otimes \pi_P^*\alpha\right) \\
&= (d + \rho_*(\theta)\wedge)\left(\pi_P^*(s \otimes \alpha)\right).
\end{aligned}
$$

□

Example 1.7. Given a vector bundle $\pi_E : E \to M$, let $\pi_P : P \to M$ be the frame bundle. Consider the trivial representation $\mathrm{id} : \mathrm{GL}(r;\mathbb{K}) \to \mathrm{GL}(r;\mathbb{K})$. Then $P \times_{\mathrm{id}} \mathbb{K}^r \cong E$ via the map:

$$
(e_1, \ldots, e_r) \times_{\mathrm{id}} \begin{pmatrix} x^1 \\ \vdots \\ x^r \end{pmatrix} \longmapsto \sum_{i=1}^{r} x^i e_i
$$

where $(e_1, \ldots, e_r)$ is a basis of E_x. We show that:

$$
\{\text{connection form on } P\} \longrightarrow \{\text{connection on } E\}
$$

$$
\theta \longmapsto (\pi_P^*)^{-1} \circ (d + \theta) \circ \pi_P^*
$$

is bijective.

We construct the inverse as follows. Given a connection ∇^E on E, take $e_\alpha = (e_{\alpha_1}, \ldots, e_{\alpha_r})$ as a frame of $E|_{U_\alpha}$. Let $A_\alpha \in \Omega^1(U_\alpha; \mathrm{End}(\mathbb{K}^r))$ be a local connection form of ∇^E, i.e., $\nabla^E e_\alpha = e_\alpha A_\alpha$ or $\nabla^E|_{U_\alpha} \cong d + A_\alpha$, meaning that:

$$
\begin{tikzcd}
\Omega^0(E|_{U_\alpha}) \arrow[r, "\nabla^E|_{U_\alpha}"] \arrow[d, "\cong"] & \Omega^1(E|_{U_\alpha}) \arrow[d, "\cong"] \\
\Omega^0(U_\alpha; \mathbb{K}^r) \arrow[r, "d+A_\alpha"] & \Omega^1(U_\alpha; \mathbb{K}^r)
\end{tikzcd}
$$

e_α defines $p_\alpha \in \Gamma(P|_{U_\alpha})$. We construct a connection form $\theta \in \Omega^1(P; \mathfrak{g})$ such that $p_\alpha^*\theta = A_\alpha$ $(\star)$. If this is proved, we obtain the statement.

Indeed, let ∇^θ be the induced connection on E from θ and θ^{∇^E} be the induced connection on P from ∇^E by $p_\alpha^*\theta = A_\alpha$. Then we have $\theta^{\nabla^\theta} = \theta$ and $\nabla^{\theta^{\nabla^E}} = \nabla^E$. Indeed, by proposition 1.6 (3), $\nabla^\theta|_{U_\alpha} \cong d + p_\alpha^*\theta$. So by $(\star)$, for $\forall\ \alpha$

$$
p_\alpha^*\theta^{\nabla^\theta} = p_\alpha^*\theta \xrightarrow{G\text{-equiv}} \theta^{\nabla^\theta} = \theta \text{ on } \pi_P^{-1}(U_\alpha) \implies \theta^{\nabla^\theta} = \theta.
$$

Similarly,

$$\nabla^{\theta^{\nabla^E}}|_{U_\alpha} \cong d + p_\alpha^* \theta^{\nabla^E} = d + A_\alpha \cong \nabla^E|_{U_\alpha} \implies \nabla^{\theta^{\nabla^E}} = \nabla^E.$$

We only have to show ($\star$). We locally define θ first, and then show that they patch together. Recall that if $\{g_{\alpha\beta}\}$ are transition maps, we have:

$$d + A_\beta = g_{\alpha\beta}^{-1} \circ (d + A_\alpha) \circ g_{\alpha\beta} \iff A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

Recall the Maurer-Cartan form $\theta_{MC} \in \Omega^1(G; \mathfrak{g})$ on a Lie group G given by $(\theta_{MC})_g(X) := (L_{g^{-1}})_*(X)$. For $\forall g, h \in G$,

$$\begin{aligned} (R_g^* \theta_{MC})_h &= (\theta_{MC})_{hg}((R_g)_*(\cdot)) \\ &= (L_{g^{-1}})_*(L_{h^{-1}})_*(R_g)_*(\cdot) \\ &= \mathrm{Ad}(g^{-1})(\theta_{MC})_h. \end{aligned}$$

Note that if G is embedded in $\mathrm{GL}(r; \mathbb{K})$, $(\theta_{MC})_g = g^{-1} dg$. Set $G = \mathrm{GL}(r; \mathbb{K})$. Regard $A_\alpha \in \Omega^1(U_\alpha; \mathrm{End}(\mathbb{K}^r))$ and $\theta_{MC} \in \Omega^1(G; \mathfrak{g})$ as differential forms on $U_\alpha \times G$. Define $\theta_\alpha \in \Omega^1(U_\alpha \times G; \mathfrak{g})$ by:

$$(\theta_\alpha)_{(x,g)} = g^{-1}(A_\alpha)_x g + (\theta_{MC})_g$$

Check: $\forall X \in \mathfrak{g}$, $\theta_\alpha(X^\#) = X$ and $R_g^* \theta_\alpha = \mathrm{Ad}(g^{-1})\theta_\alpha$. Recall the local trivialization $\phi_\alpha : P|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times G$. We show:

$$\phi_\alpha^* \theta_\alpha = \phi_\beta^* \theta_\beta \quad \text{on } P|_{U_\alpha \cap U_\beta} \quad (\star\star)$$

If this is proved, we define θ by $\theta|_{P|_{U_\alpha}} = \phi_\alpha^* \theta_\alpha$. Then $p_\alpha^* \theta = A_\alpha$.

Since $p_\alpha g_{\alpha\beta} = p_\beta$ on $U_\alpha \cap U_\beta$, the transition map is

$$g_{\alpha\beta}^P : \underbrace{(U_\alpha \cap U_\beta) \times G}_{\subseteq U_\beta \times G} \xrightarrow{\phi_\beta^{-1}} P|_{U_\alpha \cap U_\beta} \xrightarrow{\phi_\alpha} \underbrace{(U_\alpha \cap U_\beta) \times G}_{\subseteq U_\alpha \times G},$$

$$(x, g) \mapsto p_\beta(x) g = p_\alpha(x) g_{\alpha\beta}(x) g \mapsto (x, g_{\alpha\beta}(x) g).$$

We compute:

$$\begin{aligned} (g_{\alpha\beta}^P)^* \theta_\alpha &= (g_{\alpha\beta} g)^{-1} A_\alpha (g_{\alpha\beta} g) + (g_{\alpha\beta} g)^{-1} d(g_{\alpha\beta} g) \\ &= g^{-1}(g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}) g + g^{-1} dg \\ &= g^{-1} A_\beta g + \theta_{MC} = \theta_\beta. \end{aligned}$$

We now have the bijective correspondence:

$$\{\text{connection form on } P\} \longleftrightarrow \{\text{connection on } E\}$$

Recall that given $\pi_E : E \to M$ vector bundle and a representation $\rho_W : \mathrm{GL}(r; \mathbb{K}) \to \mathrm{GL}(W)$, an associated vector bundle $E_W = P \times_{\rho_W} W$ is defined. Given a connection ∇^E on E, a connection ∇^{E_W} on E_W is induced. If $\nabla^E|_{U_\alpha} \cong d + A_\alpha$, then ∇^{E_W} is defined by:

$$\nabla^{E_W}|_{U_\alpha} \cong d + \rho_{W*}(A_\alpha)$$

In terms of the frame bundle P, we have $E_W \cong P \times_{\rho_W} W$. Given a connection form θ on P such that $p_\alpha^*\theta = A_\alpha$, the induced connection on E_W from θ is $d+\rho_{W*}(p_\alpha^*\theta) = d + \rho_{W*}(A_\alpha)$, which is exactly ∇^{E_W}.

Definition 1.14. Let $\pi_P : P \to M$ be principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$.

(1) $\Omega = d\theta + \frac{1}{2}[\theta \wedge \theta] \in \Omega^2(P;\mathfrak{g})$ is called the **curvature** of θ. ($[\theta \wedge \theta]$ means taking the wedge product of differential form part and taking Lie bracket of $\mathfrak{g}$-part)

(2) For $\forall X \in \mathfrak{X}(M)$, $\exists!\ \widetilde{X} \in \mathfrak{X}(P)$ s.t. $\begin{cases} (\pi_P)_* \widetilde{X} = X \\ \theta(\widetilde{X}) = 0 \end{cases}$. Then $\widetilde{X}$ is called the **horizontal lift of** X.

We see existence and uniqueness of $\widetilde{X}$ in (2) as follows: recall that $\forall \xi \in P$, $T_\xi P = \ker(\pi_P)_* \oplus \ker\theta_\xi$, so $(\pi_P)_* : \ker\theta_\xi \xrightarrow{\cong} T_{\pi_P(\xi)}M$. So we may set $\widetilde{X}_\xi = (\pi_P)_*^{-1}\left(X_{\pi_P(\xi)}\right)$. Since $(\pi_P)_*|_{\ker\theta_\xi}$ is isomorphism, uniqueness follows.

Remark 1.6. Recall exterior covariant derivative of Kobayashi-Nomizu, i.e. $D : \Omega^q(P;V) \to \Omega^{q+1}(P;V)$ is defined by $(D\widetilde{s})(v_0,\cdots,v_q) = (d\widetilde{s})(hv_0,\cdots,hv_q)$ for $v_i \in TP$, where $h : TP \to \ker\theta$ is the projection. Then $\boxed{\Omega = D\theta}$. Actually, Kobayashi-Nomizu defined curvature by $D\theta$, and shows the equality in (1). The equality is called the **structure equation**.

To show this, note the following:

Remark 1.7. Let $\{\xi_1,\cdots,\xi_\ell\}$ be a basis of $\mathfrak{g}$. Then $\theta = \sum \xi_i \otimes \theta_i = \sum \xi_i\theta_i$ where $\theta_i \in \Omega^1(P)$ and we omit the $\otimes$. Then by definition we have

$$\Omega = \sum \xi_i d\theta_i + \frac{1}{2}\sum_{i,j}[\xi_i,\xi_j]\theta_i \wedge \theta_j.$$

Note that

$$\theta_i \wedge \theta_j(u,v) = \theta_i(u)\theta_j(v) - \theta_j(u)\theta_i(v),$$

so we have $[\xi_j,\xi_i]\theta_j \wedge \theta_i = [\xi_i,\xi_j]\theta_i \wedge \theta_j$, then

$$[\theta \wedge \theta](u,v) = [\theta(u),\theta(v)] - [\theta(v),\theta(u)] = 2[\theta(u),\theta(v)],$$

so for $u,v \in TP$, we have $\boxed{\Omega(u,v) = d\theta(u,v) + [\theta(u),\theta(v)]}$. Now we show $\Omega = D\theta$. Since $TP = \ker(\pi_P)_* \oplus \ker\theta$, we have to show in the following cases:

- $u, v \in \ker\theta$: $\Omega(u,v) = d\theta(u,v) = (D\theta)(u,v)$.
- $u, v \in \ker(\pi_P)_*$: we may set $u = X^\sharp, v = Y^\sharp$ for $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned}\Omega(X^\sharp, Y^\sharp) &= d\theta(X^\sharp, Y^\sharp) + [X, Y] \\ &= X^\sharp\left(\theta(Y^\sharp)\right) - Y^\sharp\left(\theta(X^\sharp)\right) - \theta([X^\sharp, Y^\sharp]) + [X, Y] = 0.\end{aligned}$$

  Also $(D\theta)(X^\sharp, Y^\sharp) = 0$.
- $u \in \ker\theta, v = X^\sharp$ for $X \in \mathfrak{g}$: extend u to a local horizontal vector field on P, which is still denoted as u. For example, extend $\pi_{P*}(u)$ to a local vector field on M, consider its horizontal lift. Then

$$\Omega(u, X^\sharp) = d\theta(u, X^\sharp) = u(\theta(X^\sharp)) - X^\sharp\left(\theta(u)\right) - \theta([u, X^\sharp]) = -\theta([u, X^\sharp])$$

  Now we show that $[u, X^\sharp] \in \Gamma(\ker\theta)$, then $\theta([u, X^\sharp]) = 0$. Recall that $\left\{R_{\exp_G(tX)}\right\}_{t\in\mathbb{R}}$ is the flow of $X^\sharp$, so $[X^\sharp, u] = \frac{d}{dt}\left(R_{\exp_G(-tX)}\right)_* u \mid_{t=0}$. Since for $\forall g \in G$, $\theta\left((R_g)_* u\right) = \left(R_g^*\theta\right)(u) = \mathrm{Ad}(g^{-1})\theta(u) = 0$, we have $\theta([X^\sharp, u]) = 0$, hence $\Omega(u, X^\sharp) = (D\theta)(u, X^\sharp)$.

So we have $\Omega = D\theta$.

Theorem 1.3. Let $\pi_P : P \to M$ be principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. Denote by $\Omega \in \Omega^2(P;\mathfrak{g})$ the curvature of θ. For $\forall X, Y \in \mathfrak{X}(M)$, let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(P)$ be the horizontal lifts respectively. Then $\Omega(\widetilde{X}, \widetilde{Y}) = -\theta([\widetilde{X}, \widetilde{Y}])$.

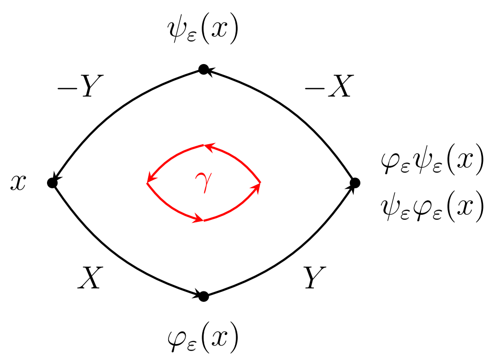
Proof. Since $\widetilde{X}, \widetilde{Y} \in \Gamma(\ker\theta)$, we have

$$\Omega(\widetilde{X}, \widetilde{Y}) = d\theta(\widetilde{X}, \widetilde{Y}) = \widetilde{X}\left(\theta(\widetilde{Y})\right) - \widetilde{Y}\left(\theta(\widetilde{X})\right) - \theta([\widetilde{X}, \widetilde{Y}]) = -\theta([\widetilde{X}, \widetilde{Y}]).$$

$\square$

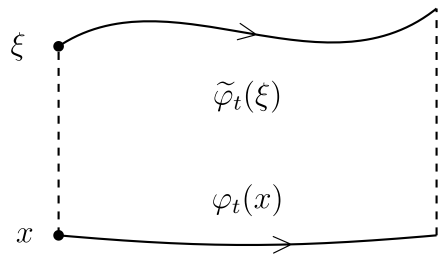
This theorem will imply that the curvature measures how "curved" the connection is.

Take local vector fields X, Y on M s.t. $[X, Y] = 0$. Let $\{\varphi_t\}$, $\{\psi_t\}$ be local flow of X, Y respectively. We know that $[X, Y] = 0 \Leftrightarrow \varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ $(\star)$. Now fix $x \in M$, consider

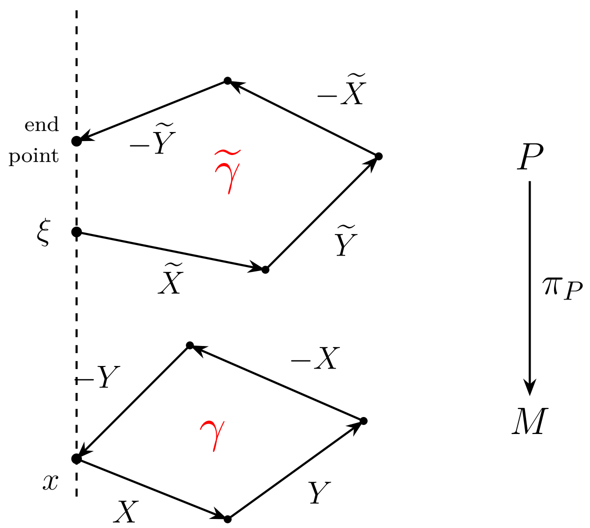


By (⋆), γ is a closed curve. We want to know what happens if we "lift" γ. Let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(P)$ be horizontal lifts of X, Y respectively. Let $\{\widetilde{\varphi}_t\}$, $\{\widetilde{\psi}_t\}$ be a local flow of $\widetilde{X}, \widetilde{Y}$ respectively.

Note that since $\frac{d}{dt}(\pi_P \circ \widetilde{\varphi}_t) = (\pi_P)_*(\widetilde{X} \circ \widetilde{\varphi}_t) = X \circ (\pi_P \circ \widetilde{\varphi}_t)$, then for $\forall \xi \in \pi_P^{-1}(x)$, $\{(\pi_P \circ \widetilde{\varphi}_t)(\xi)\}$ is the integral curve of X, i.e. $(\pi_P \circ \widetilde{\varphi}_t)(\xi) = \varphi_t(x)$ (⋆⋆).



Consider a similar path $\widetilde{\gamma}$ in P form $\widetilde{X}, \widetilde{Y}$. In general $[\widetilde{X}, \widetilde{Y}] \neq 0$, so $\widetilde{\gamma}$ is not always a closed curve. By (⋆⋆), so (end point of $\widetilde{\gamma}$) $\in \pi_P^{-1}(x)$.



Now recall the flow of $[\widetilde{X}, \widetilde{Y}]$ is given by $\widetilde{\alpha}_t = \widetilde{\psi}_{-\sqrt{t}} \circ \widetilde{\varphi}_{-\sqrt{t}} \circ \widetilde{\psi}_{\sqrt{t}} \circ \widetilde{\varphi}_{\sqrt{t}}$, hence (end point of $\widetilde{\gamma}$) $= \widetilde{\alpha}_{\epsilon^2}(\xi)$. Thus the "distance" between initial point ξ and end point $\widetilde{\alpha}_{\epsilon^2}(\xi)$ is given by "$\widetilde{\alpha}_{\epsilon^2}(\xi) - \xi$". On the other hand, "$\lim\limits_{t \to 0} \frac{\widetilde{\alpha}_t(\xi) - \xi}{dt}$" $= \frac{d\widetilde{\alpha}_t(\xi)}{dt} \big|_{t=0} = [\widetilde{X}, \widetilde{Y}]_\xi$, so $[\widetilde{X}, \widetilde{Y}]_\xi$ measures the "infinitesimal distance" between the initial point and end point of $\widetilde{\gamma}$. In addition, since $(\pi_P)_*([\widetilde{X}, \widetilde{Y}]) = [X, Y] = 0$, we have $[\widetilde{X}, \widetilde{Y}] \in \Gamma(\ker(\pi_{P*}))$. Since $\ker(\pi_{P*})_\xi \xrightarrow{\cong} \mathfrak{g}$, we have $[\widetilde{X}, \widetilde{Y}]_\xi \cong \theta_\xi\left([\widetilde{X}, \widetilde{Y}]_\xi\right) = -\Omega_\xi(\widetilde{X}, \widetilde{Y})$. So we see the curvature measures how "curved" the connection is.

Exercise 1.2. (From Frank W. Warner *Foundations of Differentiable Manifolds and Lie Groups*)

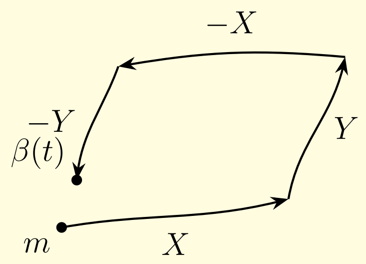
Let X and Y be C^∞ vector fields on M with corresponding local 1-parameter groups X_t and Y_t. Let $m \in M$, and let

$$\beta(t) = Y_{-\sqrt{t}} X_{-\sqrt{t}} Y_{\sqrt{t}} X_{\sqrt{t}}(m)$$

for t in $(-\varepsilon, \varepsilon)$, for a sufficiently small ε.

Prove that

$$(1) \quad [X,Y]|_m f = \lim_{t\to 0} \frac{f(\beta(t)) - f(\beta(0))}{t}.$$



If $\beta(t)$ were a smooth curve at $t = 0$, then the right-hand side of (1) would simply be the effect of the tangent vector to β at $t = 0$ applied to the function f. However, because of the $\sqrt{t}$, β is not generally smooth at $t = 0$. Thus the existence of the limit in (1) is part of the problem, and the problem asserts that this curve β, even though it is not smooth at $t = 0$, defines a tangent vector in M_m in the usual way, and this vector is precisely $[X,Y]|_m$.

Zero curvature means that a connection (horizontal subspace) is not "curved". This is made clear by the following.

Corollary 1.1. Suppose $\Omega = 0 \in \Omega^2(P;\mathfrak{g})$. Then

(1) Distribution $D = \{\ker\theta_\xi\}_{\xi\in P}$ is (completely) integrable.

(2) For a representation $\rho : G \to \mathrm{GL}(V)$, set $E = P \times_\rho V$. Let ∇^E be the induced connection on E form θ. Then for $\forall x \in M$, $\exists\, U$ an open neighborhood of x, $\exists \phi : E|_U \xrightarrow{\cong} U \times V$ local trivialization s.t. $\nabla^E|_U$ is identified with $d : \Omega^0(U;V) \to \Omega^1(U;V)$.

Proof. (1) For $\forall u_1, u_2 \in \Gamma(D)$, we show $[u_1,u_2] \in \Gamma(D)$.

Let $\{X_i\}$ be a frame of $TM|_U$ on open subset U, and let $\{\widetilde{X}_i\} \subseteq \mathfrak{X}(P|_U)$ be the horizontal lift. Note that for $\forall \xi \in P|_U$, $\{(\widetilde{X}_i)_\xi\} \subseteq D_\xi$ is a basis. So locally

$$u_i = \sum f_{ij}\widetilde{X}_j \text{ for } f_{ij} \in C^\infty(P|_U).$$

By theorem 1.3, we have $\theta([\widetilde{X}_i, \widetilde{X}_j]) = -\Omega(\widetilde{X}_i, \widetilde{X}_j) = 0$. Hence $[\widetilde{X}_i, \widetilde{X}_i] \in \Gamma(D|_{P_U})$. So $[u_1,u_2] = \sum [f_{1j}\widetilde{X}_j, f_{2k}\widetilde{X}_k] \in \Gamma(D|_{P|_U})$ and D is integrable.

(2) Fix $\forall x \in M$ and $\forall \xi \in P_x$. By (1), there exists a submanifold $\widetilde{U} \subseteq P$ s.t. $\forall q \in \widetilde{U}$, $T_q\widetilde{U} = D_q \subseteq T_qP$. Shrinking $\widetilde{U}$ if necessary, we have $\pi_P|_{\widetilde{U}} : \widetilde{U} \to \pi_P(\widetilde{U})$ is a

diffeomorphism (by inverse function theorem). Now define $p \in \Gamma(P|_U)$ by $p = (\pi_P|_{\widetilde{U}})^{-1} : U \to P$. Then $p^*\theta = 0$.

Recall that a local section p of P induces a local trivialization

$$E|_U = P \times_\rho V|_U \xrightarrow{\cong} U \times V,\ p(x) \times_\rho v \mapsto (x, v)$$

By proposition 1.6, via this identification, $\nabla^E|_U$ corresponds to $d + \rho_*(p^*\theta) = d$. □

Proposition 1.7. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. Denote by $\Omega \in \Omega^2(P;\mathfrak{g})$ the curvature of θ. Then

(1) $\Omega \in \Omega^2_B(P;\mathfrak{g})$ w.r.t. representation $(\mathfrak{g}, \mathrm{Ad})$.

(2) (Bianchi identity) $(d + \mathrm{ad}(\theta)\wedge)\Omega = 0 \in \Omega^3_B(P;\mathfrak{g})$.

Remark 1.8. Using the exterior covariant derivative D of Kobayashi-Nomizu, we have $D\Omega = (d + \mathrm{ad}(\theta)\wedge)\Omega$. So (2) says that $D\Omega = 0$. It is because for any representation $\rho : G \to \mathrm{GL}(V)$, $\forall \widetilde{s} \in \Omega^q_B(P;V)$, we already know $D\widetilde{s} = (d + \rho_*(\theta)\wedge)(\widetilde{s})$. Then set $(V,\rho) = (\mathfrak{g}, \mathrm{Ad})$, $\widetilde{s} = \Omega$.

Proof. (1) We show that $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\Omega = 0. \\ \forall g \in G, R_g^*\Omega = \mathrm{Ad}(g^{-1})\Omega. \end{cases}$.

- $\mathcal{L}_{X^\sharp}\theta = \dfrac{d}{dt} R^*_{\exp_G(tX)}\theta|_{t=0} = \dfrac{d}{dt}\mathrm{Ad}\left(\exp_G(-tX)\right)\theta|_{t=0} = -\,\mathrm{ad}(X)\theta$.
  Since $\mathcal{L}_{X^\sharp}\theta = i(X^\sharp)d\theta + d\left(i(X^\sharp)\theta\right) = i(X^\sharp)d\theta$, we have $i(X^\sharp)d\theta = -\,\mathrm{ad}(X)\theta$. So
  $$i(X^\sharp)\Omega = i(X^\sharp)d\theta + \frac{1}{2}i(X^\sharp)[\theta\wedge\theta] = -\,\mathrm{ad}(X)\theta + \frac{1}{2}\left([X,\theta] - [\theta, X]\right) = 0$$
- For $\forall g \in G$,
  $$\begin{aligned} R_g^*\Omega &= R_g^* d\theta + \frac{1}{2}R_g^*[\theta\wedge\theta] = dR_g^*\theta + \frac{1}{2}[R_g^*\theta \wedge R_g^*\theta] \\ &= \mathrm{d}\,\mathrm{Ad}(g^{-1})\theta + \frac{1}{2}[\mathrm{Ad}(g^{-1})\theta\wedge \mathrm{Ad}(g^{-1})\theta] = \mathrm{Ad}(g^{-1})\left(d\theta + \frac{1}{2}[\theta\wedge\theta]\right) \\ &= \mathrm{Ad}(g^{-1})\Omega \end{aligned}$$

So we see that $\Omega \in \Omega^2_B(P;\mathfrak{g})$.

(2) Recall from proposition 1.4 that $\mathrm{ad}(\theta)\wedge d\theta = [\theta\wedge d\theta]$ and similar for the $[\theta\wedge\theta]$ part. We have

$$\begin{aligned} (d + \mathrm{ad}(\theta)\wedge)\Omega &= (d + \mathrm{ad}(\theta)\wedge)\left(d\theta + \frac{1}{2}[\theta\wedge\theta]\right) \\ &= \frac{1}{2}d[\theta\wedge\theta] + [\theta\wedge d\theta] + \frac{1}{2}[\theta\wedge[\theta\wedge\theta]]. \end{aligned}$$

Since $d[\theta\wedge\theta] = [d\theta\wedge\theta] - [\theta\wedge d\theta] = -2[\theta\wedge d\theta]$, we have $\frac{1}{2}d[\theta\wedge\theta] + [\theta\wedge d\theta] = 0$. For a basis $\{\xi_i\}_1^\ell$ of $\mathfrak{g}$. Set $\theta = \sum \xi_i \theta_i$ for $\theta_i \in \Omega^1(P)$. Then

$$\begin{aligned}
[\theta[\theta \wedge \theta]] &= \sum [\xi_i, [\xi_j, \xi_k]] \theta_i \wedge \theta_j \wedge \theta_k \\
&= \frac{1}{3} \sum \{[\xi_i, [\xi_j, \xi_k]] + [\xi_j, [\xi_k, \xi_i]] + [\xi_k, [\xi_i, \xi_j]]\} \theta_i \wedge \theta_j \wedge \theta_k \\
&= 0 \text{ (by Jacobi identity).}
\end{aligned}$$

□

Proposition 1.8. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P; \mathfrak{g})$. Denote by $\Omega \in \Omega^2(P; \mathfrak{g})$ the curvature of θ. For a representation $\rho : G \to \mathrm{GL}(V)$, set $E = P \times_\rho V$. Recall $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$. Note that ρ induces a map

$$\widetilde{\rho} : G \to \mathrm{GL}(\mathrm{End}(V)), \ g \mapsto \left(T \mapsto \rho(g) \circ T \circ \rho(g)^{-1}\right)$$

(hence $\mathrm{End}(E) = P \times_{\widetilde{\rho}} \mathrm{End}(V)$) Then

(1) $\rho_*(\Omega) = d\rho_*(\theta) + \rho_*(\theta) \wedge \rho_*(\theta) \in \Omega^2_B(P; \mathrm{End}(V))$ w.r.t. $(\mathrm{End}(V), \widetilde{\rho})$.

(2) $(d + \rho_*(\theta)\wedge) \circ (d + \rho_*(\theta)\wedge) = \rho_*(\Omega)\wedge : \Omega^q_B(P; V) \to \Omega^{q+2}_B(P; V)$.

Proof. (1) For a basis $\{\xi_i\}_1^\ell$ of $\mathfrak{g}$. Set $\theta = \sum \xi_i \theta_i$ for $\theta_i \in \Omega^1(P)$. Then

$$\begin{aligned}
\rho_*(\Omega) &= \rho_* \left(\sum_i \xi_i d\theta_i + \frac{1}{2} \sum_{i,j} [\xi_i, \xi_j] \theta_i \wedge \theta_j \right) \\
&= \sum_i \rho_*(\xi_i) d\theta_i + \frac{1}{2} \sum_{i,j} [\rho_*(\xi_i), \rho_*(\xi_j)] \theta_i \wedge \theta_j \\
&= d\rho_*(\sum_i \xi_i \theta_i) + \sum_{i,j} \rho_*(\xi_i) \rho_*(\xi_j) \theta_i \wedge \theta_j \\
&= d\rho_*(\theta) + \rho_*(\theta) \wedge \rho_*(\theta)
\end{aligned}$$

Next we show $\rho_*(\Omega)$ is basic, i.e. $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\rho_*(\Omega) = 0. \\ \forall g \in G, R_g^*\Omega = \widetilde{\rho}(g^{-1})\rho_*(\Omega). \end{cases}$. Recall that $\Omega \in \Omega^2_B(P; \mathfrak{g})$ w.r.t. $(\mathfrak{g}, \mathrm{Ad})$,

- Since ρ_* acts only on $\mathfrak{g}$-part, we have $i(X^\sharp)\rho_*(\Omega) = \rho_* \left(i(X^\sharp)\Omega\right) = 0$.
- $R_g^* \rho_*(\Omega) = \rho_*(R_g^*\Omega) = \rho_* \left(\mathrm{Ad}(g^{-1})\Omega\right)$. Since for $\forall X \in \mathfrak{g}$,

$$\rho_* \left(\mathrm{Ad}(g^{-1})X\right) = \frac{d}{dt} \rho \left(g^{-1} \exp_G(tX) g\right) |_{t=0} = \rho(g^{-1})\rho_*(X)\rho(g),$$

we have $R_g^*\rho_*(\Omega) = \rho(g^{-1})\rho_*(\Omega)\rho(g) = \widetilde{\rho}(g^{-1})\rho_*(\Omega)$.

(2) Note that for $\forall \widetilde{s} \in \Omega^q_B(P; V)$,

$$(d \circ \rho_*(\theta)\wedge)(\widetilde{s}) = d\left(\rho_*(\theta) \wedge \widetilde{s}\right) = (d\rho_*(\theta)) \wedge \widetilde{s} - \rho_*(\theta) \wedge d\widetilde{s},$$

so we have $d \circ \rho_*(\theta)\wedge = (d\rho_*(\theta)) \wedge -(\rho_*(\theta)\wedge) \circ d$. Then

$$(d + \rho_*(\theta)\wedge) \circ (d + \rho_*(\theta)\wedge) = d \circ \rho_*(\theta) \wedge + \rho_*(\theta) \wedge \circ d + \rho_*(\theta) \wedge \rho_*(\theta)\wedge = \rho_*(\Omega)\wedge$$

□

Remark 1.9. This proposition is interpreted as follows.

Recall the isomorphism $\pi_P^* = \pi_{P,E}^* : \Omega^q(E) \xrightarrow{\cong} \Omega_B^q(P;V)$. For $s \in \Omega^0(E)$,

$$(\pi_{P,E}^* s)(\xi) = \xi_E^{-1}(s_{\pi_P(\xi)}) \quad \text{for } \xi \in P,$$

where $\xi_E : V \xrightarrow{\cong} E_{\pi_P(\xi)}$, $v \mapsto \xi \times_\rho v$.

By **??**,

$$\begin{tikzcd} \Omega^q(E) \arrow[r, "d^{\nabla^E}"] \arrow[d, "\pi_{P,E}^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{q+1}(E) \arrow[d, "\cong"', "\pi_{P,E}^*"] \\ \Omega_B^q(P;V) \arrow[r, "d+\rho_*(\theta)\wedge"'] & \Omega_B^{q+1}(P;V) \end{tikzcd}$$

Thus

$$(2) \iff \pi_{P,E}^* \circ \underbrace{(d^{\nabla^E} \circ d^{\nabla^E})}_{R^{\nabla^E}\wedge} \circ (\pi_{P,E}^*)^{-1} = \rho_*(\Omega)\wedge \ \leftarrow (\star)$$

((1) is a simplification of (2).)

($\star$) is also described as follows. Since $\mathrm{End}(E) = P \times_{\tilde{\rho}} \mathrm{End}(V)$, we have

$$\pi_{P,\mathrm{End}(E)}^* : \Omega^q(\mathrm{End}(E)) \xrightarrow{\cong} \Omega_B^q(P;\mathrm{End}(V))$$

For $\tau \in \Omega^0(\mathrm{End}(E))$, $(\pi_{P,\mathrm{End}(E)}^*\tau)(\xi) = \xi_{\mathrm{End}(E)}^{-1}(\tau_{\pi_P(\xi)})$, where

$$\xi_{\mathrm{End}(E)} : \mathrm{End}(V) \longrightarrow \mathrm{End}(E)_{\pi_P(\xi)} = (P \times_{\tilde{\rho}} \mathrm{End}(V))_{\pi_P(\xi)}$$

$$T \longmapsto \xi \times_{\tilde{\rho}} T$$

We show

$$\pi_{P,\mathrm{End}(E)}^*(R^{\nabla^E}) = \rho_*(\Omega)$$

To show this, we show

$$\forall \tau \in \Omega^q(\mathrm{End}(E)), \forall s \in \Omega^{q'}(E), \quad \pi_{P,E}^*(\tau(s)) = (\pi_{P,\mathrm{End}(E)}^*\tau)(\pi_{P,E}^* s), \text{ i.e.}$$

$$\begin{tikzcd} \Omega^q(\mathrm{End}(E)) \times \Omega^{q'}(E) \arrow[r] \arrow[d, "(\pi_{P,\mathrm{End}(E)}^*,\pi_{P,E}^*)"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{q+q'}(E) \arrow[d, "\pi_{P,E}^*"', "\cong"] \\ \Omega_B^q(P;\mathrm{End}(V)) \times \Omega_B^{q'}(P;V) \arrow[r] & \Omega_B^{q+q'}(P;V) \end{tikzcd}$$

Since we only take the wedge product and pullback for differential form part, we only have to show when $q = q' = 0$.

For $\forall \tau \in \Omega^0(\mathrm{End}(E)), \forall s \in \Omega^0(E)$, then $(\pi^*_{P,\mathrm{End}(E)}\tau)(\pi^*_{P,E}s) \in \Omega^0_B(P;V)$ is obvious, and for $\forall \xi \in P$,

$$\begin{aligned}\left((\pi^*_{P,\mathrm{End}(E)}\tau)(\pi^*_{P,E}s)\right)(\xi) &= \underbrace{(\pi^*_{P,\mathrm{End}(E)}\tau)(\xi)}_{\in \mathrm{End}(V)}\underbrace{(\pi^*_{P,E}s)(\xi)}_{\in V} \\ &= \left(\xi^{-1}_{\mathrm{End}(E)}(\tau_{\pi_P(\xi)})\right)\left(\xi_E^{-1}(s_{\pi_P(\xi)})\right)\end{aligned}$$

Set $x = \pi_P(\xi)$. The composition is given by

$$\underbrace{(\xi_{\mathrm{End}(E)}(T))}_{\in \mathrm{End}(E)_x}\underbrace{(\xi_E(v))}_{\in E_x} = \underbrace{\xi_E(T(v))}_{\in E_x} \quad \text{for } T \in \mathrm{End}(V), v \in V.$$

(i.e. $(\xi \times_{\tilde{\rho}} T)(\xi \times_\rho v) = \xi \times_\rho T(v)$, well-definedness is checked easily.)

Now setting $\begin{cases} T = \xi^{-1}_{\mathrm{End}(E)}(\tau_x) \\ v = \xi_E^{-1}(s_x) \end{cases}$, we have $\tau_x(s_x) = \xi_E(T(v))$.

So we obtain

$$\begin{aligned}\left((\pi^*_{P,\mathrm{End}(E)}\tau)(\pi^*_{P,E}s)\right)(\xi) &= \xi_E^{-1}(\tau_x(s_x)) \\ &= \left(\pi^*_{P,E}(\tau(s))\right)(\xi)\end{aligned}$$

Then $(\star)$ is described as

$$\pi^*_{P,\mathrm{End}(E)}(R^{\nabla^E}) = \rho_*(\Omega) \quad (\star\star)$$

Hence the curvature of ∇^E is identified with the curvature of θ (with ρ_*).

Finally, we note that Bianchi identity for principal bundles induces Bianchi identity for associated vector bundles.

Recall that Bianchi identity for vector bundles says that

$$d^{\nabla^{\mathrm{End}(E)}} R^{\nabla^E} = 0 \in \Omega^3(\mathrm{End}(E))$$

for any connection ∇^E.

By proposition 1.6, we have

$$\begin{tikzcd} \Omega^2(\mathrm{End}(E)) \arrow[r, "d^{\nabla^{\mathrm{End}(E)}}"] \arrow[d, "\pi^*_{P,\mathrm{End}(E)}"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^3(\mathrm{End}(E)) \arrow[d, "\pi^*_{P,\mathrm{End}(E)}"', "\cong"] \\ \Omega^2_B(P;\mathrm{End}(V)) \arrow[r, "d+\tilde{\rho}_*(\theta)\wedge"'] & \Omega^3_B(P;\mathrm{End}(V)) \end{tikzcd}$$

So by $(\star\star)$, Bianchi identity for E is equivalent to

$$(d + \widetilde{\rho}_*(\theta)\wedge)\rho_*(\Omega) = 0.$$

Since $\widetilde{\rho}_*(X)(T) = \rho_*(X) \circ T - T \circ \rho_*(X)$ for $X \in \mathfrak{g}, T \in \mathrm{End}(V)$, we have

$$\begin{aligned}(d + \widetilde{\rho}_*(\theta)\wedge)\rho_*(\Omega) &= d\rho_*(\Omega) + \rho_*(\theta) \wedge \rho_*(\Omega) - \rho_*(\Omega) \wedge \rho_*(\theta) \\ &= d\rho_*(\Omega) + [\rho_*(\theta) \wedge \rho_*(\Omega)] \\ &= \rho_* \underbrace{(d\Omega + [\theta \wedge \Omega])}_{=0} = 0\end{aligned}$$

1.4 Holonomy Groups

In this section, we introduce the holonomy group of a connection and study the properties. Roughly speaking, the curvature measures how a connection is curved locally, while curvature gives you an "infinitesimal" or "local" snapshot of how the bundle is twisting, holonomy tells you the cumulative global effect of that twisting.

The primary reason to study holonomy is that it is the "integral" of curvature. The Ambrose-Singer Theorem (Local-to-Global Bridge) states that the Lie algebra $\mathfrak{hol}_\xi$ of the holonomy group at $\xi \in P$ is exactly the subspace of the Lie algebra $\mathfrak{g}$ spanned by all curvature values:

$$\mathfrak{hol}_\xi = \mathrm{span}\{\Omega_\eta(X, Y) \mid \eta \in P(\xi),\ X, Y \in H_\eta P\}$$

where $P(\xi) \subseteq P$ is the set of points reachable from ξ via horizontal curves. Since a Lie group is generated by its Lie algebra via the exponential map, saying "the Lie algebra is spanned by curvature" is mathematically equivalent to saying "the group is the integral of curvature." This result is non-trivial because it allows us to recover a global group structure purely from local curvature forms.

In the case of a flat connection, the local geometry is trivial. However, the holonomy is generally non-trivial and is determined entirely by the fundamental group $\pi_1(M)$. In this context, holonomy is called the *monodromy representation* $\rho : \pi_1(M) \to G$. This identifies holonomy as the precise tool for studying how the topology of the base manifold M "twists" the bundle P, even in the absence of local curvature.

First, we formulate pullbacks of principal bundles and connections.

- Let $\pi_P : P \to M$ be principal G-bundle and $f : N \to M$ a smooth map. The **pullback** $\pi_{f^*P} : f^*P \to N$ of $\pi_P; P \to M$ is defined by

  $$f^*P = \{(x, \xi) \in N \times P \mid f(x) = \pi_P(\xi)\}$$

  For $x \in N$, we have $(f^*P)_x = \pi_{f^*P}^{-1}(x) = P_{f(x)}$. Setting $\widetilde{f} : f^*P \to P$, $(x, \xi) \mapsto \xi$. Then $\pi_P \circ \widetilde{f} = f \circ \pi_{f^*P}$.
  The right G-action on f^*P is given by

  $$f^*P \times G \to f^*P,\ ((x, \xi), g) \mapsto (x, \xi g).$$

  Let $\left\{\phi_\alpha^P : P|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times G\right\}_{\alpha \in A}$ be a family of local trivialization of P, $\mathrm{pr}_2 : U_\alpha \times G \to G$ projection. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open covering of N. Define local trivialization of f^*P by

  $$\phi_\alpha^{f^*P} : \pi_{f^*P}^{-1}\left(f^{-1}(U_\alpha)\right) \xrightarrow{\cong} f^{-1}(U_\alpha) \times G,\ (x, \xi) \mapsto \left(x, (\mathrm{pr}_2 \circ \phi_\alpha^P)(\xi)\right).$$

  In terms of local sections, if $p_\alpha \in \Gamma(P|_{U_\alpha})$ is induced form ϕ_α^P, then $\phi_\alpha^{f^*P}$ induces f^*p_α given by $(f^*p_\alpha)(x) = (x, (p_\alpha \circ f)(x))$.

If $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ are transition maps of P, then

$$\left\{f^* g_{\alpha\beta} : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to G\right\}$$

are transition maps of f^*P.

- Let $\theta \in \Omega^1(P; \mathfrak{g})$ be a connection form. Then $\widetilde{f}^*\theta \in \Omega^1(f^*P; \mathfrak{g})$ is a connection form on f^*P. For $\forall x \in N$, $\widetilde{f} : (f^*P)_x \cong P_{f(x)} \xrightarrow{\text{id}} P_{f(x)}$ is identity map on each fiber, so θ, $\widetilde{f}^*\theta$ have the same fiberwise property. Definition of connections require some properties for G-action, which is fiberwise.

Next we consider the associated vector bundle. For a representation $\rho : G \to \mathrm{GL}(V)$, set

$$\begin{cases} E = P \times_\rho V \\ \nabla^E : \text{ induced connection form on } E \text{ from } \theta \end{cases}$$

Then $f^*E = f^*P \times_\rho V$.

For $x \in N$, $(f^*E)_x \cong E_{f(x)}$, $(f^*P)_x \times_\rho V \cong P_{f(x)} \times_\rho V = E_{f(x)}$.

Proposition 1.9. The pullback ∇^{f^*E} of ∇^E by f is induced connection on f^*E from $\widetilde{f}^*\theta$.

Proof. We give 2 proofs depending on how ∇^{f^*E} is defined.

(1) $\boxed{\text{Local proof}}$ Let $\{A_\alpha\}$ be connection forms of ∇^E, i.e. $\nabla^E|_{U_\alpha} \cong d + A_\alpha$. Then ∇^{f^*E} is a connection with connection forms $\{f^*A_\alpha\}$. Since $A_\alpha = \rho_*(p_\alpha^*\theta)$, so $f^*A_\alpha = \rho_*(f^*p_\alpha^*\theta)$ since f^* acts only on differential form part. Next we check what is induced local connection on f^*E from $\widetilde{f}^*\theta$. By proposition before, we have the local connection form $\rho_*\left((f^*p_\alpha)^*\widetilde{f}^*\theta\right)$. Since $\widetilde{f} \circ (f^*p_\alpha) = p_\alpha \circ f$, we have $\rho_*\left((f^*p_\alpha)^*\widetilde{f}^*\theta\right) = \rho_*(f^*p_\alpha^*\theta)$, which completes the proof.

(2) $\boxed{\text{Global proof}}$ We use the fact that ∇^{f^*E} is the unique connection satisfying $\nabla^{f^*E}(f^*s) = f^*(\nabla^E s)$ for $\forall s \in \Gamma(E)$. Let ∇' be induced connection on f^*E from $\widetilde{f}^*\theta$, we show ∇' satisfies the same property then completes the proof.

We can define $\pi_{f^*P}^* : \Omega^q(f^*E) \xrightarrow{\cong} \Omega_B^q(f^*P; V)$ in the same way, and we have the commutative diagram

$$\begin{tikzcd} \Omega^q(E) \arrow[r, "f^*"] \arrow[d, "\pi_P^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^q(f^*E) \arrow[d, "\cong"', "\pi_{f^*P}^*"] \\ \Omega_B^q(P; V) \arrow[r, "\widetilde{f}^*"'] & \Omega_B^q(f^*P; V) \end{tikzcd}$$

Then we can compute

$$\begin{aligned} \nabla'(f^*s) &= \left((\pi_{f^*P}^*)^{-1} \circ \left(d + \rho_*(\widetilde{f}^*\theta)\right) \circ \pi_{f^*P}^* \circ f^*\right)(s) \\ &= (\pi_{f^*P}^*)^{-1}\widetilde{f}^*\left(((d + \rho_*(\theta)) \circ \pi_P^*)(s)\right) \end{aligned}$$

$$
\begin{aligned}
&= f^*(\pi_P^*)^{-1}\left(((d + \rho_*(\theta)) \circ \pi_P^*)\,(s)\right) \\
&= f^*\nabla^E s).
\end{aligned}
$$

□

Next we describe parallel transport of a vector bundle in terms of principal bundles. First, we introduce horizontal lift.

Proposition 1.10. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P; \mathfrak{g})$. Let $c : [0,1] \to M$ be a C^∞ curve, then for $\forall\ \xi \in P_{c(0)}$, there exists a unique C^∞ curve $\widetilde{c} : [0,1] \to P$ s.t.

$$
\begin{cases}
(1)\ \pi_P \circ \widetilde{c} = c \\
(2)\ \widetilde{c}(0) = \xi \\
(3)\ \text{For } \forall t \in [0,1],\ \frac{d\widetilde{c}}{dt}(t) \in \ker \theta_{\widetilde{c}(t)}\ \text{(horizontal)}
\end{cases}
$$

This $\widetilde{c}$ is called the **horizontal lift of** c.

Proof. Since P is locally trivial, there exists $\widetilde{\gamma} : [0,1] \to P$ s.t.

$$
\begin{cases}
\pi_P \circ \widetilde{\gamma} = c \\
\widetilde{\gamma}(0) = \xi
\end{cases}
$$

We will find $g : [0,1] \to G$ s.t.

$$
\begin{cases}
g(0) = e \\
\widetilde{\eta}(t) = \widetilde{\gamma}(t) \cdot g(t) \text{ is horizontal}
\end{cases}
$$

so $\widetilde{\eta}$ is the horizontal lift as required.

We determine the equation that g should satisfy, to do this, we compute $\frac{d\widetilde{\eta}}{dt}$. Let $A : P \times G \to P$ be the right action, then

$$
\begin{aligned}
\frac{d\widetilde{\eta}}{dt}(t) &= \frac{d}{dt} A\left(\widetilde{\gamma}(t), g(t)\right) = (dA)_{(\widetilde{\gamma}(t), g(t))}\left(\frac{d}{dt}\left(\widetilde{\gamma}(t), g(t)\right)\right) \\
&= (dA)_{(\widetilde{\gamma}(t), g(t))}\left(\left(\frac{d}{ds}\widetilde{\gamma}(t+s)|_{s=0}, 0\right) + \left(0, \frac{d}{ds} g(t+s)|_{s=0}\right)\right) \\
&= \frac{d}{ds} A\left(\widetilde{\gamma}(t+s), g(t)\right)|_{s=0} + \frac{d}{ds} A\left(\widetilde{\gamma}(t), g(t+s)\right)|_{s=0} \\
&= \frac{d}{ds} A\left(\widetilde{\gamma}(t+s), g(t)\right)|_{s=0} + \frac{d}{ds}\left(\widetilde{\gamma}(t) \cdot g(t) \cdot g(t)^{-1} \cdot g(t+s)\right)|_{s=0} \\
&= (R_{g(t)})_*\left(\frac{d\widetilde{\gamma}}{dt}(t)\right) + \left(\frac{d}{ds} g(t)^{-1} g(t+s)|_{s=0}\right)^\sharp_{\widetilde{\gamma}(t) g(t)} \quad \text{(def. of fund. v.f.)}.
\end{aligned}
$$

By the computation we have $\widetilde{\eta}$ is horizontal $\iff$

$$\begin{aligned}
0 &= \theta_{\widetilde{\eta}(t)}\left(\frac{d\widetilde{\eta}}{dt}(t)\right) \\
&= \theta_{\widetilde{\gamma}(t)\cdot g(t)}\left((R_{g(t)})_*\left(\frac{d\widetilde{\gamma}}{dt}(t)\right)\right) + \left(L_{g(t)^{-1}}\right)_*\frac{dg}{dt}(t) \\
&= \mathrm{Ad}\left(g(t)^{-1}\right)\theta_{\widetilde{\gamma}(t)}\left(\frac{d\widetilde{\gamma}}{dt}(t)\right) + \left(L_{g(t)^{-1}}\right)_*\frac{dg}{dt}(t) \\
&= \left(L_{g(t)^{-1}}\right)_*\left(R_{g(t)}\right)_*\theta_{\widetilde{\gamma}(t)}\left(\frac{d\widetilde{\gamma}}{dt}(t)\right) + \left(L_{g(t)^{-1}}\right)_*\frac{dg}{dt}(t) \\
\iff \left(R_{g(t)^{-1}}\right)_*\left(\frac{dg}{dt}(t)\right) &= -\theta_{\widetilde{\gamma}(t)}\left(\frac{d\widetilde{\gamma}}{dt}(t)\right) \qquad (\star)
\end{aligned}$$

So we study $(\star)$.

First we show uniqueness of horizontal lift. If $\widetilde{\gamma}$, $\widetilde{\eta}$ are horizontal lifts of c, then $\pi_P \circ \widetilde{\gamma} = c = \pi_P \circ \widetilde{\eta} \implies \widetilde{\eta}(t) = \widetilde{\gamma}(t)\cdot g(t)$ for $g : [0,1] \to G$. Since $\widetilde{\gamma}$ is horizontal, $(\star)$ implies $\frac{dg}{dt} = 0$, hence $g(t) \equiv e$, $\widetilde{\eta} = \widetilde{\gamma}$.

Claim. (Existence)

For any $X : [0,1] \to \mathfrak{g}$, $\exists!\ g : [0,1] \to G$ s.t.

$$\begin{cases} \left(R_{g(t)^{-1}}\right)_*\left(\frac{dg}{dt}(t)\right) = X(t) \\ g(0) = e \end{cases} \qquad (\star\star)$$

Note When $X(t)$ is t-independent, $(\star\star)$ is flow equation of right invariant vector field $X(0)^{\sharp_R}$ given by $\left(X(0)^{\sharp_R}\right)_g = (R_g)_* X(0)$, $g \in G$. So $(\star\star)$ is "t-dependent" version.

Now we prove the claim. The proof of uniqueness is similar to the proof above. The proof of existence is similar to the proof for left invariant vector field. $\square$

Remark 1.10. When $G \subseteq \mathrm{GL}(r;\mathbb{K})$, we can take $g(t) = \exp(\int_0^t X(s)ds)$.

Now we describe parallel transport of vector bundles in terms of principal bundles.

Proposition 1.11. Let $\pi_P : P \to M$ be principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. For a representation $\rho : G \to \mathrm{GL}(V)$, $E := P \times_\rho V$, ∇^E be the induced connection on E from θ. Take $\forall c : [0,1] \to M$ and any horizontal lift $\widetilde{c}$ of c, define $\gamma \in \Gamma(c^*E)$ by $\gamma(t) = (t, \widetilde{c}(t)) \times_\rho v(t)$ for $v : [0,1] \to V$. Then

(1) $\left(\nabla^{c^*E}_{\frac{d}{dt}}\gamma\right)_t = (t, \widetilde{c}(t)) \times_\rho \frac{dv}{dt}(t)$, where ∇^{c*E} is induced connection on c^*E from ∇^E.

(2) Parallel transport along c w.r.t ∇^E is given by

$$P_c : E_{c(0)} \to E_{c(1)},\ \widetilde{c}(0) \times_\rho v \mapsto \widetilde{c}(1) \times_\rho v.$$

Proof. (1) First note: $\forall X \in \mathfrak{X}(M)$, let $\widetilde{X} \in \mathfrak{X}(P)$ s.t. $(\pi_P)_*(\widetilde{X}) = X$, then

$$\begin{tikzcd}
\Omega^{q+1}(E) \arrow[r, "i(X)"] \arrow[d, "\pi_P^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^q(E) \arrow[d, "\cong"', "\pi_P^*"] \\
\Omega_B^{q+1}(P;V) \arrow[r, "i(\widetilde{X})"'] & \Omega_B^q(P;V)
\end{tikzcd}$$

Indeed, for $s = \alpha \otimes e$ where $\alpha \in \Omega^{q+1}(M), e \in \Gamma(E)$,

$$\pi_P^* \left(i(X)s\right) = \pi_P^* \left(i(X)\alpha\right) \otimes \pi_P^* e = i(\widetilde{X})\pi_P^* s.$$

Use the following notation:

$$\begin{tikzcd}
c^*P \arrow[r, "c_P"] \arrow[d, "\pi_{c^*P}"'] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[d, "\pi_P"] \\
{[0,1]} \arrow[r, "c"'] & M
\end{tikzcd}$$

By proposition before, we have

$$\nabla^{c^*E}\gamma = \left((\pi_{c^*P}^*)^{-1} \circ (d + \rho_*(c_P^*\theta)) \circ \pi_{c^*P}\right)(\gamma)$$

Denote by $\widetilde{\frac{d}{dt}} \in \mathfrak{X}(c^*P)$ the horizontal lift of $\frac{d}{dt}$, then

$$\begin{aligned}
\nabla^{c^*E}_{\frac{d}{dt}}\gamma &= i\left(\frac{d}{dt}\right)\left(\nabla^{c^*E}\gamma\right) \\
&= (\pi_{c^*P}^*)^{-1}\left(i\left(\widetilde{\frac{d}{dt}}\right)(d + \rho_*(c_P^*\theta))\left(\pi_{c^*P}\gamma\right)\right).
\end{aligned}$$

Since $\forall \widetilde{s} \in \Omega_B^0(c^*P;V),\ \forall t \in [0,1]$,

$$\left((\pi_{c^*P}^*)^{-1}(\widetilde{s})\right)_t = \xi \times_\rho \widetilde{s}(\xi), \quad \text{for } \xi \in \pi_{c^*P}^{-1}(t).$$

For $(t, \widetilde{c}(t)) \in \pi_{c^*P}^{-1}(t)$, we have

$$\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t)\cdot g)} = \left(\frac{d}{dt}, \frac{d}{dt}\widetilde{c}(t)\cdot g\right), \quad t \in [0,1], g \in G.$$

Then we have

$$\nabla^{c^*E}_{\frac{d}{dt}}\gamma = (t, \widetilde{c}(t)) \times_\rho i\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t))}(d + \rho_*(c_P^*\theta))\left(\pi_{c^*P}\gamma\right).$$

We compute

$$i\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t))} d(\pi_{c^*P}^*\gamma) = \frac{d}{dt}(\pi_{c^*P}^*\gamma)(t,\widetilde{c}(t)) = \frac{d}{dt}(t,\widetilde{c}(t))^{-1}(\gamma(t)) = \frac{d}{dt}v(t)$$

$$i\left(\widetilde{\frac{d}{dt}}\right)_{(t,\widetilde{c}(t))}(c_P^*\theta) = \theta\left(\frac{d}{dt}c_P(t,\widetilde{c}(t))\right) = 0.$$

Hence $\left(\nabla^{c^*E}_{\frac{d}{dt}}\gamma\right)_t = (t,\widetilde{c}(t)) \times_\rho \frac{dv}{dt}(t)$.

(2) By (1),

$$\nabla^{c^*E}_{\frac{d}{dt}}\gamma = 0 \iff \frac{dv}{dt} = 0 \iff v \text{ is constant.}$$

So it implies (2) by definition of parallel transport.

□

Next we formulate the holonomy group. Let $\pi_P : P \to M$ be a principal G-bundle. For $x \in M$, set

$$\Omega_x := \{c : [0,1] \to M \text{ piecewise } C^\infty \text{ curve} \mid c(0) = c(1) = x\}$$

$$\Omega_x^0 := \{c : [0,1] \to M \text{ piecewise } C^\infty \text{ curve} \mid c(0) = c(1) = x,\ [c] = 1 \in \pi_1(M,x)\}$$

Proposition 1.12. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$. For $x \in M$, $\xi \in \pi_P^{-1}(x)$, define

$$\Phi_\xi : \Omega_x \to G, \quad \widetilde{c}(1) = \xi \cdot \Phi_\xi(c)$$

where $\widetilde{c}$ is the horizontal lift of c with $\widetilde{c}(0) = \xi$. Then

(1) For $h \in G$, $\Phi_{\xi h}(c) = h^{-1}\Phi_\xi(c)h$.

(2) For $c_1, c_2 \in \Omega_x$, set $c_3 := c_2 \cdot c_1$ (c_1 concatenates c_2). Then

$$\Phi_\xi(c_3) = \Phi_\xi(c_2)\Phi_\xi(c_1).$$

(3) Let $\gamma : [0,1] \to M$ be a piecewise C^∞ curve. Set $\gamma(0) = y, \gamma(1) = x$, $\eta \in P_y, \xi \in P_x$. Let $\widetilde{\gamma}$ be the horizontal lift of γ with $\widetilde{\gamma}(0) = \eta$. Set $\widetilde{\gamma}(1) = \xi \cdot h$ for $h \in G$. For $c \in \Omega_x$, define $c' \in \Omega_y$ by $c' = \gamma^{-1} \cdot c \cdot \gamma$. Then

$$\Phi_\eta(c') = h^{-1}\Phi_\xi(c)h.$$

Proof. (1) First note that $\widetilde{c}(t) \cdot h$ is the horizontal lift of c with initial point ξh by the G-invariant property of $\ker\theta$. Now by definition we have

$$\xi\Phi_\xi(c)h = \widetilde{c}(1)h = (\widetilde{c}h)(1) = \xi h \cdot \Phi_{\xi h}(c).$$

Since G-action is free, the proof is finished.

(2) Let $\widetilde{c}_i$ be horizontal lift of c_i with $\widetilde{c}_i(0) = \xi$, then $\widetilde{c}_3 = (\widetilde{c}_2 \cdot \Phi_\xi(c_1)) \cdot \widetilde{c}_1$ is horizontal lift of c_3 with $\widetilde{c}_3(0) = \xi$ (draw a picture to help you understand). Then by definition we have

$$\widetilde{c}_3(1) = \xi\Phi_\xi(c_3) = \widetilde{c}_2(1) \cdot \Phi_\xi(c_1) = \xi\Phi_\xi(c_2)\Phi_\xi(c_1).$$

(3) We want to determine the horizontal lift $\widetilde{c}'$ of c' with $\widetilde{c}'(0) = \eta$. Indeed, we have

$\widetilde{c}' = (\widetilde{\gamma}^{-1} \cdot (h^{-1}\Phi_\xi(c)h)) \cdot (\widetilde{c} \cdot h) \cdot \widetilde{\gamma}$ (Draw a picture to help you understand). So we have

$$\widetilde{c}'(1) = \eta \cdot \Phi_\eta(c') = \widetilde{\gamma}^{-1}(1) \cdot h^{-1}\Phi_\xi(c)h.$$

□

Definition 1.15. Let $\pi_P : P \to M$ be a principal G-bundle with a connection form $\theta \in \Omega^1(P; \mathfrak{g})$. For $\xi \in P$, define

- $\mathrm{Hol}_\xi(P, \theta) \coloneqq \Phi_\xi(\Omega_x) = \{\Phi_\xi(c) \mid c \in \Omega_x\} \subseteq G$ called **holonomy group**.
- $\mathrm{Hol}^0_\xi(P, \theta) \coloneqq \Phi_\xi(\Omega^0_x) \subseteq G$ called **restricted holonomy group**.

Remark 1.11. $\mathrm{Hol}_\xi(P, \theta)$ is a Lie subgroup of G (not necessarily closed). $\mathrm{Hol}^0_\xi(P, \theta)$ is unit component of $\mathrm{Hol}_\xi(P, \theta)$. There is a theorem proved by Kuranishi and Yamabe says that every path-connected subgroup of a Lie group is a Lie subgroup. But a weaker theorem suffices for our purpose. Some related arguments are put in the end of this subsection. For more details, see [1].

$\mathrm{Hol}_\xi(P, \theta)$, $\mathrm{Hol}^0_\xi(P, \theta)$ depends on $\xi \in P$, but the conjugacy class is independent of choice of $\xi \in P$ (just apply proposition 1.12).

Recall that holonomy group is also defined for a connection on a vector bundle. For a vector bundle $E \to M$, ∇^E connection on E, take $x \in M$, we have

$$\mathrm{Hol}_x(\nabla^E) = \{P_c : T_xM \to T_xM \mid c \in \Omega_x\}$$

where P_c is parallel transport along c and similar definition for $\mathrm{Hol}^0_x(\nabla^E)$.

Lemma 1.2. Let $\pi_P : P \to M$ be a principal G-bundle with $\theta \in \Omega^1(P; \mathfrak{g})$ be a connection form. Let ∇^E be the induced connection on associated vector bundle E for a representation $\rho : G \to \mathrm{GL}(V)$. Then for $x \in M$, $\xi \in P_x$,

- $\mathrm{Hol}_x(\nabla^E) = \xi \circ \rho(\mathrm{Hol}_\xi(P, \theta)) \circ \xi^{-1}$.
- $\mathrm{Hol}^0_x(\nabla^E) = \xi \circ \rho(\mathrm{Hol}^0_\xi(P, \theta)) \circ \xi^{-1}$.

where $\xi : V \to E_x$, $v \mapsto \xi \times_\rho v$.

Proof. The proof is straightforward from proposition 1.11 and proposition 1.12. □

The smaller the holonomy group, the smaller the connection is “globally curved”. In the following we show that if holonomy group is small, we can make structure group smaller and make principal bundle itself smaller.

Definition 1.16. Let $\pi_P : P \to M$ be a principal G-bundle, $\pi_Q : Q \to M$ be a principal H-bundle.

(1) Q is a **subbundle** of P if

- $\exists$ an embedding $\iota_Q : Q \hookrightarrow P$ s.t. $\pi_P \circ \iota_Q = \pi_Q$.
- $\exists$ an injective homomorphism $\iota_H : H \hookrightarrow G$ s.t. $\forall \eta \in Q, \forall h \in H$,

$$\iota_Q(\eta h) = \iota_Q(\eta)\iota_H(h).$$

In this case, we can say that the structure group G of P is **reduced** to H.

(2) In the setting of (1), a connection form $\theta \in \Omega^1(P; \mathfrak{g})$ is said to be **reducible** to Q if for $\forall \eta \in Q$, $(\ker \theta)_{\iota_Q(\eta)} \subseteq (\iota_Q)_{*\eta}(T_\eta Q)$.

Suppose Q is a subbundle of P. For sumplicity, suppose that H is a Lie subgroup of G and ι_H the inclusion. We identify H with $\iota_H(H)$ and the corresponding Lie algebra. In this setting, we have

Lemma 1.3. Let $\theta \in \Omega^1(P; \mathfrak{g})$ be connection form of P. Then θ is reducible to Q iff $\iota_Q^*\theta$ is $\mathfrak{h}$-valued. In this case, $\iota_Q^*\theta$ is a connection form on Q.

Proof. We show

($\Rightarrow$) $\forall \eta \in Q, \forall v \in T_\eta Q$, we have $(\iota_Q)_*(v) = w_1 + w_2 \in \ker \theta_{\iota_Q(\eta)} \oplus \ker(\pi_P)_{*\iota_Q(\eta)}$. Since θ is reducible, $\exists v_1 \in T_\eta Q$ s.t. $w_1 = (\iota_Q)_*(v_1)$ and then $w_2 = (\iota_Q)_*(v_2)$ for $v_2 = v - v_1$. Then

$$0 = (\pi_P)_*(w_2) = (\pi_Q)_*(v_2) \iff v_2 = X^\sharp_\eta \text{ for } X \in \mathfrak{h}.$$

Thus

$$\begin{aligned}
\theta_{\iota_Q(\eta)}\left((\iota_Q)_*(v)\right) &= \theta_{\iota_Q(\eta)}\left((\iota_Q)_*(X^\sharp_\eta)\right) \\
&= \theta_{\iota_Q(\eta)}\left(\frac{d}{dt}\iota_Q(\eta \cdot \exp_H(tX))\right)|_{t=0} \\
&= \theta_{\iota_Q(\eta)}\left(\frac{d}{dt}\iota_Q(\eta) \cdot \exp_H(tX)\right)|_{t=0} \\
&= \theta_{\iota_Q(\eta)}\left(\frac{d}{dt}\iota_Q(\eta) \cdot \exp_G(tX)\right)|_{t=0} \\
&= \theta_{\iota_Q(\eta)}(X^\sharp_{\iota_Q(\eta)}) \\
&= X \in \mathfrak{h}.
\end{aligned}$$

($\Leftarrow$) Under the assumption it's easily shown that $\iota_Q^*\theta$ is a connection form on Q. We show that θ is reducible. Since $\iota_Q^*\theta$ is a connection form on Q, we have

$$(\star)\begin{cases}\dim\ker(\iota_Q^*\theta)_\eta = \dim M, & \forall\eta\in Q\\ \dim\ker\theta_\xi = \dim M, & \forall\xi\in P\end{cases}$$

Then $\ker\theta_{\iota_Q(\eta)} = (\iota_Q)_*\left(\ker(\iota_Q^*\theta)_\eta\right)\subseteq(\iota_Q)_*(T_\eta Q)$ for $\forall\eta\in Q$ by dimension condition.

□

Exercise 1.3. Show that $\iota_Q^*\theta$ is a connection form on Q.

More generally we have the following

Lemma 1.4. Let $\pi_P : P\to M$ be a principal G-bundle, $\pi_Q : Q\to M$ be a principal H-bundle. Suppose $H\subseteq G$ and Q is a subbundle of P with embedding ι_Q. Assume that $\exists$ a subspace $\mathfrak{m}\subseteq\mathfrak{g}$ s.t.

$$\begin{cases}\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}\\ \mathrm{Ad}(H)\mathfrak{m}=\mathfrak{m}\end{cases}$$

Denote by $P_\mathfrak{m} : \mathfrak{g}\to\mathfrak{m}$, $P_\mathfrak{h} : \mathfrak{g}\to\mathfrak{h}$ the projections. Then for $\forall$ connection form $\theta\in\Omega^1(P;\mathfrak{g})$,

- $P_\mathfrak{h}(\iota_Q^*\theta)\in\Omega^1(Q;\mathfrak{h})$ is a connection form on Q.
- $P_\mathfrak{m}(\iota_Q^*\theta)\in\Omega^1(Q;\mathfrak{m})$ is basic w.r.t. H-representation $(\mathfrak{m},\mathrm{Ad})$.

Proof. Let $\omega=\iota_Q^*\theta$ be the restriction of the connection form θ to the subbundle Q. We define the $\mathfrak{h}$-component and $\mathfrak{m}$-component of ω as:

$$\omega_\mathfrak{h}=P_\mathfrak{h}\circ\omega\quad\text{and}\quad\omega_\mathfrak{m}=P_\mathfrak{m}\circ\omega.$$

Thus, $\omega(X)=\omega_\mathfrak{h}(X)+\omega_\mathfrak{m}(X)$ for any $X\in TQ$.

1. $\omega_\mathfrak{h}$ is a connection form on Q.

- Let $A\in\mathfrak{h}\subseteq\mathfrak{g}$. Let $A^\sharp$ denote the fundamental vector field on Q generated by A. Since Q is a subbundle of P and the action of H on Q is the restriction of the action of G on P, $A^\sharp$ is also the restriction of the fundamental vector field on P generated by A. Since θ is a connection form on P, we have $\theta(A^\sharp)=A$. Restricting to Q, we have $\omega(A^\sharp)=A$. Applying the projections:

$$\omega_\mathfrak{h}(A^\sharp)=P_\mathfrak{h}(A)=A\quad\text{and}\quad\omega_\mathfrak{m}(A^\sharp)=P_\mathfrak{m}(A)=0.$$

- Let $a\in H$ and $X\in T_uQ$ for some $u\in Q$. Since θ is a connection form on P, it satisfies the equivariance property $\theta((R_a)_*X)=\mathrm{Ad}(a^{-1})\theta(X)$. Since R_a maps Q to

Q because $a \in H$, restricting to Q , we have:

$$\omega((R_a)_* X) = \mathrm{Ad}(a^{-1})\omega(X).$$

Substituting the decomposition $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}}$:

$$\omega_{\mathfrak{h}}((R_a)_* X) + \omega_{\mathfrak{m}}((R_a)_* X) = \mathrm{Ad}(a^{-1}) \left(\omega_{\mathfrak{h}}(X) + \omega_{\mathfrak{m}}(X)\right).$$

Distributing on the right side:

$$\mathrm{RHS} = \mathrm{Ad}(a^{-1})\omega_{\mathfrak{h}}(X) + \mathrm{Ad}(a^{-1})\omega_{\mathfrak{m}}(X).$$

Since $\mathfrak{h}$ is a Lie algebra, $\mathrm{Ad}(H)\mathfrak{h} \subseteq \mathfrak{h}$. By the hypothesis of the Lemma, $\mathrm{Ad}(H)\mathfrak{m} = \mathfrak{m}$. Therefore:

$$\mathrm{Ad}(a^{-1})\omega_{\mathfrak{h}}(X) \in \mathfrak{h} \quad \text{and} \quad \mathrm{Ad}(a^{-1})\omega_{\mathfrak{m}}(X) \in \mathfrak{m}.$$

Comparing the $\mathfrak{h}$-components of both sides, we obtain:

$$\omega_{\mathfrak{h}}((R_a)_* X) = \mathrm{Ad}(a^{-1})\omega_{\mathfrak{h}}(X).$$

This confirms that $\omega_{\mathfrak{h}}$ is a connection form on Q.

2. $\omega_{\mathfrak{m}}$ is basic.

- We already showed in step 1(a) that for any fundamental vector field $A^\sharp$, $\omega_{\mathfrak{m}}(A^\sharp) = 0$. Since vertical vectors are spanned by fundamental vector fields, $\omega_{\mathfrak{m}}$ vanishes on vertical vectors.
- Comparing the $\mathfrak{m}$-components of the equation derived in step 1(b), we obtain:

$$\omega_{\mathfrak{m}}((R_a)_* X) = \mathrm{Ad}(a^{-1})\omega_{\mathfrak{m}}(X).$$

Thus, $\omega_{\mathfrak{m}}$ satisfies the required equivariance condition.

□

Example 1.8. Let $\pi_E : E \to M$ be a vector bundle of rank r, $\pi_P : P \to M$ be the frame bundle. Recall that the frame bundle $P = \bigsqcup_{x \in M} P_x$ where $P_x = \{\xi : \mathbb{K}^r \to E_x \mid \text{linear isomorphism }\}$. In addition, we consider a fiber metric h on E. Denoted by $(\cdot,\cdot)_{std}$ the standard inner product on $\mathbb{K}^r$. For $x \in M$, set

$$\begin{cases} Q_x \coloneqq \{\xi \in P_x \mid h_x(\xi(\cdot),\xi(\cdot)) = (\cdot,\cdot)_{std}\} \\ Q \coloneqq \bigsqcup_{x \in M} Q_x \\ \pi_Q : Q \to M, \quad \xi \mapsto x \text{ if } \xi \in Q_x \end{cases}$$

Setting $H = \begin{cases} \mathrm{U}(r) & (if\ \mathbb{K} = \mathbb{C}) \\ \mathrm{O}(r) & (if\ \mathbb{K} = \mathbb{R}) \end{cases}$, we can show that Q is a principal H-bundle and a subbundle of the frame bundle P. We call Q the **frame bundle of** (E, h).

Now suppose that a connection form $\theta \in \Omega^1(P; \mathfrak{g})$ is given ($\mathfrak{g} = \mathrm{End}(\mathbb{K}^r)$). Denote by ∇^θ the induced connection on E (or $E^* \otimes E^*$) from θ. Then we have the following lemma:

Lemma 1.5. $\nabla^\theta h = 0 \iff \theta$ is reducible to Q.

Proof. Note that $h \in \Omega^0(E^* \otimes E^*)$.

- We first describe ∇^θ on $E^* \otimes E^*$ explicitly. Set $V := \mathbb{K}^r$, then

$$P \times_{\rho'} (V^* \otimes V^*) \xrightarrow{\cong} E^* \otimes E^*, \quad \xi \times_{\rho'} h_0 \mapsto h_0(\xi^{-1}(\cdot), \xi^{-1}(\cdot)),\ h_0 \in V^* \otimes V^*$$

where

$$\rho' : G = \mathrm{GL}(V) \to \mathrm{GL}(V^* \otimes V^*), \quad \rho'(g)(h_0) = h_0(g^{-1}(\cdot), g^{-1}(\cdot)),\ h_0 \in V^* \otimes V^*$$

Via $\pi_P^* : \Omega^q(E^* \otimes E^*) \xrightarrow{\cong} \Omega^Q(P; V^* \otimes V^*)$, from proposition 1.6 we have:

$$\begin{tikzcd} \Omega^0(E^* \otimes E^*) \arrow[r, "\nabla^\theta"] \arrow[d, "\pi_P^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^1(E^* \otimes E^*) \arrow[d, "\cong"', "\pi_P^*"] \\ \Omega^0_B(P; V^* \otimes V^*) \arrow[r, "d+\rho'_*(\theta)"'] & \Omega^1_B(P; V^* \otimes V^*) \end{tikzcd}$$

- Denote by $\iota_Q : Q \hookrightarrow P$ the embedding, then

$$Q \times_{\rho'|_H} (V^* \otimes V^*) \xrightarrow{\iota_Q \times \mathrm{id}} P \times_{\rho'} (V^* \otimes V^*) \cong E^* \otimes E^*$$

is an isomorphism, $\dim(V^* \otimes V^*) = \mathrm{rank}(E^* \otimes E^*)$. So we can define $\Omega^q_B(Q; V^* \otimes V^*)$ for H-representation $(V^* \otimes V^*, \rho'|_H)$, $\Omega^q_B(Q; V^* \otimes V^*) \cong \Omega^q(E^* \otimes E^*)$. Recall the pullback ι_Q^* of $\Omega^q(P; V^* \otimes V^*)$ to $\Omega^q(Q; V^* \otimes V^*)$ via ι_Q.

Claim:

(a) $\iota_Q^*(\Omega^q_B(P; V^* \otimes V^*) \subseteq \Omega^q_B(Q; V^* \otimes V^*)$.

(b)

$$\begin{tikzcd} \Omega^q_B(P; V^* \otimes V^*) \arrow[rr, "\iota_Q^*"] & & \Omega^q_B(Q; V^* \otimes V^*) \\ & \Omega^q(E^* \otimes E^*) \arrow[ul, "\pi_P^*", "\cong"'] \arrow[ur, "\pi_Q^*"', "\cong"] & \end{tikzcd}$$

In particular, ι_Q^* is an isomorphism.

Proof of the claim.

(a) Take $\forall \tilde{s} \in \Omega^q_B(P; V^* \otimes V^*)$. for $\forall X \in \mathfrak{h}$, $\forall \eta \in Q$,

$$\begin{aligned} (\iota_Q)_*(X^\sharp_\eta) &= \frac{d}{dt}\iota\left(\eta \cdot \exp_H(tX)\right)|_{t=0} = \frac{d}{dt}\iota_Q(\eta) \cdot \exp_H(tX)|_{t=0} \\ &= \frac{d}{dt}\iota_Q(\eta) \cdot \exp_G(tX)|_{t=0} = X^\sharp_{\iota_Q(\eta)}. \end{aligned}$$

So $i(X^\sharp_\eta)(\iota_Q^*\widetilde{s})_\eta = \iota_Q^*\left(i(X^\sharp)\widetilde{s}\right)_\eta = 0$.
For $\forall h \in H$, $\forall \eta \in Q$, we have $\iota_Q(\eta \cdot h) = \iota_Q(\eta)\cdot h \iff \iota_Q \circ R_h = R_h \circ \iota_Q$. Then

$$R_h^*\iota_Q^*\widetilde{s} = \iota_Q^* R_h^*\widetilde{s} = \rho'(h^{-1})\iota_Q^*\widetilde{s}.$$

Hence $\iota_Q^*\widetilde{s}$ is basic.
(b) This will follow from $\pi_Q = \pi_P \circ \iota_Q$. We only have to show the commutativity when $q = 0$ since the differential form part is the original pullback.
When $q = 0$, for $\forall s \in \Omega^0(E^* \otimes E^*)$, $\forall \eta \in Q$, we have

$$(\iota_Q^*\pi_P^* s)(\eta) = (\pi_P^* s)(\iota_Q(\eta)) = \iota_Q(\eta)^{-1}\left(s_{(\pi_P\circ\iota_Q)(\eta)}\right) = (\pi_Q^* s)(\eta).$$

hence finishing the claim.

Now we show the lemma.

$$\begin{aligned}
\nabla^\theta h = 0 &\iff (d + \rho'_*(\theta))\pi_P^* h = 0 \\
&\underset{\iota_Q^*\ iso.}{\iff} \iota_Q^*\left(d + \rho'_*(\theta)\right)\pi_P^* h = 0 \\
&\iff \left(d + \rho'_*(\iota_Q^*\theta)\right)\underbrace{\iota_Q^*\pi_P^* h}_{=\pi_Q^* h} = 0
\end{aligned}$$

For $\forall \eta \in Q_y$, $y \in M$, we have $(\pi_Q^* h)(\eta) = \eta^{-1}(h_y)$, where

$$\eta : V^* \otimes V^* \to (E^* \otimes E^*)_y, \quad h_0 \mapsto h_0(\eta^{-1}(\cdot), \eta^{-1}(\cdot)).$$

Since $\eta \in Q_y \iff h_y(\eta(\cdot), \eta(\cdot)) = (\cdot,\cdot)_{std}$, we have

$$\eta^{-1}(h_y) = h_y(\eta(\cdot), \eta(\cdot)) = (\cdot,\cdot)_{std}.$$

Hence $\pi_Q^* h \equiv (\cdot,\cdot)_{std}$, and then

$$\begin{aligned}
\nabla^\theta h = 0 &\iff \rho'_*(\iota_Q^*\theta)\left((\cdot,\cdot)_{std}\right) = 0 \\
&\iff \left((\iota_Q^*\theta)(\cdot), \cdot\right)_{std} + \left(\cdot, (\iota_Q^*\theta)(\cdot)\right)_{std} = 0 \\
&\iff \iota_Q^*\theta \text{ is } \mathfrak{h}\text{-valued} \\
&\iff \theta \text{ is reducible to } Q.
\end{aligned}$$

□

Remark 1.12. The same argument works if h is replaced with other geometric object, i.e. suppose a tensor $\varphi \in \Gamma(\bigotimes^* E \otimes \bigotimes^* E^*)$, $\varphi_0 \in \Gamma(\bigotimes^* \mathbb{K}^r \otimes \bigotimes^* (\mathbb{K}^r)^*)$ are given (such as fiber metric and standard inner product). Consider

$$\begin{cases} Q_x^\varphi := \left\{\xi : \mathbb{K}^r \xrightarrow{\cong} E_x \mid \xi \cdot \varphi_x = \varphi_0\right\}, & x \in M \\ Q^\varphi := \bigsqcup_{x\in M} Q_x^\varphi \end{cases}$$

The fiber of $Q^\varphi \cong \{g \in \mathrm{GL}(r;\mathbb{K}) \mid g \cdot \varphi_0 = \varphi_0\}$. Then for a connection form θ on

P,

$$\nabla^\theta \varphi = 0 \iff \theta \text{ is reducible to } Q^\varphi.$$

Such a situation can be found when we consider a manifold with special holonomy, which will be explained shortly later.

Let $\pi_P : P \to M$ be a principal G-bundle with a connection $\theta \in \Omega^1(P; \mathfrak{g})$. The horizontal lift of a piecewise C^∞ curve in M is called a piecewise C^∞ **horizontal curve**.

For $\xi, \eta \in P$, define

$$\xi \sim \eta \iff \exists \text{ piecewise } C^\infty \text{ horizontal curve connecting } \xi \text{ and } \eta.$$

Then we can check that $\sim$ is equivalence relation.

Theorem 1.4. (**Reduction Theorem**) Let $\pi_P : P \to M$ be a principal G-bundle with a connection $\theta \in \Omega^1(P; \mathfrak{g})$. Fix $\xi \in P$, set $Q \coloneqq Q(\xi) = \{\eta \in P \mid \xi \sim \eta\}$, $H \coloneqq \mathrm{Hol}_\xi(P, \theta) \subseteq G$. Then

- $\pi_P|_Q : Q \to M$ is a principal H-bundle and Q is a subbundle of P.
- θ is reducible to Q.

Proof. (1) We fix $\xi \in P$ and consider the holonomy bundle $Q = \{\eta \in P \mid \xi \sim \eta\}$ and the holonomy group $H = \mathrm{Hol}_\xi(P, \theta)$. We aim to show that Q is a principal H-bundle and a subbundle of P.

Since M is paracompact and connected, we know from previous results that H is a Lie subgroup of G. To prove that Q defines a reduced subbundle, we verify the following four properties regarding the subset Q and the subgroup H:

1. **Surjectivity:** Since M is connected, any point $x \in M$ can be joined to $\pi_P(\xi)$ by a piecewise differentiable curve. The horizontal lift of this curve starting at ξ exists and ends at some point u in the fiber over x. By definition, u is joined to ξ by a horizontal curve, so $u \in Q$. Thus, Q covers all of M.
2. **Right Invariance:** Let $\eta \in Q$ and $h \in H$. By definition, η can be joined to ξ by a horizontal curve. Since $h \in \mathrm{Hol}_\xi(P, \theta)$, h is realized by parallel displacement along a loop at ξ. Concatenating the path from η to ξ with the loop generating h shows that $\eta \cdot h$ can be joined to ξ by a horizontal curve. Thus $\eta \cdot h \in Q$.
3. **Fiber Transitivity:** Let $u, v \in Q$ be in the same fiber, i.e., $\pi_P(u) = \pi_P(v)$. Since both are in Q, there are horizontal curves joining ξ to u and ξ to v. Combining the path from u to ξ and ξ to v creates a piecewise horizontal loop starting at u and ending at v. This implies $v = u \cdot g$ for some g in the holonomy group of the connection at u. Since the holonomy groups at points on the same horizontal path are conjugate, there exists $h \in H$ such that $v = u \cdot h$.

4. **Existence of Local Sections in Q:** We must show that for every $x \in M$, there is a local section taking values in Q. Let $x^1, \ldots, x^n$ be a local coordinate system centered at x. Let U be a convex neighborhood contained in the coordinate patch. Fix $u_0 \in Q$ such that $\pi_P(u_0) = x$. For any $y \in U$, let τ_y be the radial line segment from x to y. Let $\sigma(y)$ be the point obtained by parallel displacement of u_0 along τ_y. Since τ_y is a curve, $\sigma(y)$ is connected to u_0 (and thus to ξ) by a horizontal curve. Therefore, $\sigma(y) \in Q$. The map $y \mapsto \sigma(y)$ defines a smooth local section $\sigma : U \to P$ such that $\sigma(U) \subset Q$.

Since Q and H satisfy conditions (1) through (4), by lemma 1.6, Q is a reduced subbundle of P with structure group H. $\square$

(2) For $\forall \eta \in Q$, we show $(\ker\theta)_\eta \subseteq T_\eta Q$. Take $\forall v \in (\ker\theta)_\eta$, then $\exists\, \widetilde{c} : [0,1] \to P$ horizontal s.t. $\widetilde{c}(0) = \eta$, $\frac{d\widetilde{c}}{dt}(0) = v$. Thus for $\forall t$, $\widetilde{c}(t) \sim \eta \sim \xi$, i.e. $\widetilde{c}(t) \in Q$, so $v \in T_\eta Q$. $\square$

Lemma 1.6. Let Q be a subset of P and H a Lie subgroup of G. Assume:

1. The projection π_P maps Q onto M;
2. Q is stable under H;
3. H acts transitively on the fibers of Q;
4. Every point $x \in M$ has a neighborhood U and a cross section $\sigma : U \to P$ such that $\sigma(U) \subset Q$.

Then $\pi_Q : Q \to M$ (principal H-bundle) is a reduced subbundle of P.

Proof. For a neighborhood U and section σ as given in condition (4), we define a local trivialization. For any $u \in \pi_P^{-1}(U)$, let $x = \pi_P(u)$. Since G acts transitively on the fibers of P, there is a unique $g \in G$ such that $u = \sigma(x) \cdot g$. Define a map $\psi : \pi_P^{-1}(U) \to U \times G$ by $\psi(u) = (x, g)$., this is a local trivialization of P.

We examine the restriction of ψ to $Q \cap \pi_P^{-1}(U)$. If $u \in Q$, then $u = \sigma(x) \cdot g$. Since both u and $\sigma(x)$ are in Q, and Q is invariant under H and transitive on fibers with respect to H, the element g must belong to H. Conversely, if $h \in H$, then $\sigma(x) \cdot h \in Q$. Thus, ψ maps $Q \cap \pi_P^{-1}(U)$ bijectively onto $U \times H$.

We introduce a differentiable structure on Q by requiring that these restricted maps $\psi|_Q : Q \cap \pi_P^{-1}(U) \to U \times H$ be diffeomorphisms. This makes Q a differentiable manifold and a principal fiber bundle over M with structure group H. Since the inclusion $Q \hookrightarrow P$ is smooth and an embedding on fibers, Q is a subbundle of P. $\square$

Remark 1.13. $Q = Q(\xi)$ is called **holonomy subbundle** for a connection $\theta \in \Omega^1(P; \mathfrak{g})$.

- For $\xi, \eta \in P$,
$$Q(\xi) = Q(\eta) \iff \xi, \eta \text{ are joined by horizontal curve}$$
So under $\sim$, either $Q(\xi) = Q(\eta)$ or $Q(\xi) \cap Q(\eta) = \emptyset$, i.e. $P = \bigsqcup Q(\xi)$.
- $Q(\xi)$, $\xi \in P$ are mutually isomorphic.

Proposition 1.13. For a given connection θ, holonomy subbundle is smallest subbundle to which θ is reducible, i.e. for $\forall$ subbundle $\iota_Q : Q \to P$ s.t. θ is reducible to Q, we have $Q(\iota_Q(\eta)) \subseteq \iota_Q(Q)$ for $\forall \eta \in Q$.

Proof. Fix $\eta_0 \in Q$, $\xi_0 \coloneqq \iota_Q(\eta_0)$. Take $\forall \xi \in Q(\xi_0)$, we show $\xi \in \iota_Q(Q)$. By definition, $\exists$ piecewise C^∞ horizontal curve $\widetilde{c} : [0, 1] \to P$ s.t. $\widetilde{c}(0) = \xi_0$, $\widetilde{c}(1) = \xi$. Set $c \coloneqq \pi_P \circ \widetilde{c}$, take $\widetilde{c}^Q$ be the horizontal lift of c for $(Q, \iota_Q^* \theta)$. We can show $\widetilde{c} = \iota_Q \circ \widetilde{c}^Q$ (use uniqueness of horizontal lift), hence finishing the proof. □

Appendix

Theorem 1.5. Let $\pi_P : P \to M$ be a principal G-bundle whose base manifold M is connected and paracompact. Let $\mathrm{Hol}_\xi(P, \theta)$ and $\mathrm{Hol}^0_\xi(P, \theta)$ be the holonomy group and the restricted holonomy group of a connection form θ with reference point $\xi \in P$. Then:

(a) $\mathrm{Hol}^0_\xi(P, \theta)$ is a connected Lie subgroup of G.

(b) $\mathrm{Hol}^0_\xi(P, \theta)$ is a normal subgroup of $\mathrm{Hol}_\xi(P, \theta)$ and the quotient
$$\mathrm{Hol}_\xi(P, \theta) / \mathrm{Hol}^0_\xi(P, \theta)$$
is countable.

Consequently, $\mathrm{Hol}_\xi(P, \theta)$ is a Lie subgroup of G whose identity component is $\mathrm{Hol}^0_\xi(P, \theta)$.

Proof. **Proof of (a):** We aim to show that every element of $\mathrm{Hol}^0_\xi(P, \theta)$ can be joined to the identity element by a piecewise differentiable curve of class C^k in G which lies entirely within $\mathrm{Hol}^0_\xi(P, \theta)$. If we establish this, then by theorem 1.6, it follows that $\mathrm{Hol}^0_\xi(P, \theta)$ is a connected Lie subgroup of G.

Let $a \in \mathrm{Hol}^0_\xi(P, \theta)$ be an element obtained by parallel displacement along a piecewise differentiable loop τ (homotopic to 0). By the Factorization Lemma, τ is equivalent to a product of small lassos of the form $\tau_1^{-1} \cdot \mu \cdot \tau_1$, where μ is a small loop at b contained in a coordinate neighborhood and τ_1 is a piecewise differentiable curve from a to b. It suffices to show that the group element defined by such a lasso can be connected to the identity.

Let μ be a loop at $y \in M$ inside a coordinate chart. We can contract μ to the point y using a homotopy $f(t,s)$ (where $s \in [0,1]$ is the deformation parameter), such that $f(t,0)$ is μ and $f(t,1)$ is the constant trivial loop. This homotopy generates a path of group elements $b(s) \in \mathrm{Hol}^0_v(P,\theta)$ connecting the original element $b(0) = b$ to the identity $b(1) = e$.

The crucial step is to ensure this path $b(s)$ is differentiable. This is guaranteed by lemma 1.7. Thus, the element a is connected to the identity by a differentiable path, satisfying the condition of Theorem 1.6.

Proof of (b): If τ is any loop and μ is a null-homotopic loop, the conjugate $\tau \cdot \mu \cdot \tau^{-1}$ is also null-homotopic. Thus, conjugation maps $\mathrm{Hol}^0_\xi(P,\theta)$ into itself, proving it is a normal subgroup.

Consider the map from the fundamental group $\pi_1(M) \to \mathrm{Hol}_\xi(P,\theta)/\mathrm{Hol}^0_\xi(P,\theta)$. Since M is connected and paracompact, $\pi_1(M)$ is countable. As this map is surjective, the quotient group is also countable. □

The following lemma was required in the proof of Theorem 1.5 to ensure that deforming a loop results in a smooth path in the Lie group.

Lemma 1.7. Let $f : I \times I \to M$ be a differentiable mapping of class C^k. Let $\xi_0(s)$ be a differentiable curve of class C^k in P s.t. $\pi_P(\xi_0(s)) = f(0,s)$.

For each fixed s, let $\xi_1(s)$ be the point in P obtained by parallel displacement of $\xi_0(s)$ along the curve $t \mapsto f(t,s)$. Then the curve $s \mapsto \xi_1(s)$ is differentiable of class C^k.

Proof. Let $F(t,s)$ be a lift of $f(t,s)$ into P. Let $v_t(s) = F(t,s)$. Parallel transport is defined by a horizontal lift. For each fixed s, we look for a curve $a_t(s)$ in G such that the curve $u_t(s) = v_t(s)a_t(s)$ is horizontal. The condition that $u_t(s)$ is horizontal translates to a differential equation on G:

$$\dot{a}_t(s)a_t(s)^{-1} = -\theta(\dot{v}_t(s))$$

where θ is the connection form. This is an ordinary linear differential equation with a parameter s. By the standard theory of ODEs, the solution to a differential equation depends differentiably on parameters as smoothly as the equation itself does. Since f (and thus $\dot{v}$) is C^k, the solution $a_1(s)$ is C^k in s. Consequently, $\xi_1(s) = u_1(s)$ is differentiable. □

The following theorem provides the theoretical basis for why the arc-wise connected restricted holonomy group must be a Lie subgroup.

Theorem 1.6. Let G be a Lie group and H a subgroup of G. If every element of H can be joined to the identity e by a piecewise differentiable curve of class C^1 contained entirely in H, then H is a Lie subgroup of G.

Proof. **Step 1: Constructing the Lie Algebra.** Let S be the set of vectors $X \in T_e(G)$ which are tangent to differentiable curves contained in H. Identifying $T_e(G)$ with the Lie algebra $\mathfrak{g}$, we show S is a Lie subalgebra:

- Closed under scaling: If $x(t) \in H$, then $x(rt) \in H$.
- Closed under addition: Using the product path $x(t)y(t)$.
- Closed under Lie bracket: Using the commutator path $x(t)y(t)x(t)^{-1}y(t)^{-1}$ (which relates to $[X, Y]$).

Step 2: Constructing the Integral Manifold. Since S is a subalgebra, it defines an involutive distribution on G via left translation: $x \mapsto L_x S$. By the Frobenius Theorem, there exists a unique maximal integral manifold K through the identity e corresponding to S. K is a connected Lie subgroup of G.

Step 3: Showing $H = K$. *Direction $K \supseteq H$:* Any curve in H starting at e has its tangent vectors in S (by definition). Thus, the curve lies in the integral manifold K. Since H is generated by such paths to elements, $H \subseteq K$.

Direction $H \supseteq K$: Consider a basis $X_1, \ldots, X_k$ of S. These are tangent to curves in H. By the Inverse Function Theorem applied to the map $(t_1, \ldots, t_k) \mapsto \exp(t_1 X_1) \cdots \exp(t_k X_k)$, the image of this map covers a neighborhood of the identity in K. Since these curves are in H, a neighborhood of e in K lies in H. Since K is connected and generated by any neighborhood of the identity, $K \subseteq H$.

Thus $H = K$, proving H is a Lie subgroup. $\square$

1.5 H-Structures and Intrinsic Torsion

Let M be a n-dim smooth manifold. Consider $E = TM \to M$. Recall frame bundle $\pi_P : P \to M$ of E: $P = \bigsqcup_{x\in M} P_x$,

$$P_x := \left\{\xi : \mathbb{R}^n \xrightarrow{\cong} T_xM \mid \text{linear isomorphism}\right\}, \quad x \in M.$$

P is a principal $\mathrm{GL}(n;\mathbb{R})$-bundle.

Definition 1.17. For a subgroup $H \subseteq \mathrm{GL}(n;\mathbb{R})$, an **H-structure** on M is a subbundle Q of P with fiber H.

We can define a special differential form on H-structure: Let $\pi_Q : Q \to M$ be H-structure on M. Define the **tautological 1-form (solder form)** $\omega \in \Omega^1(Q;\mathbb{R}^n)$ by $\omega_\eta(v) := \eta^{-1}\left((\pi_Q)_*(v)\right)$ for $\eta \in Q$, $v \in T_\eta Q$. We can check properties of ω:

- Let $\{e_1, \dots, e_n\}$ be a frame on $U \underset{\text{open}}{\subseteq} M$. This define a local section p on P:

  $$p(x) : \mathbb{R}^n \xrightarrow{\cong} T_xM, \quad (x^1, \dots, x^n)^t \mapsto (e_1, \dots, e_n)(x^1, \dots, x^n)^t.$$

  Suppose that $p(x) \in Q$ for $\forall x \in U$, then $p^*\omega \in \Omega^1(U;\mathbb{R}^n)$ satisfies $(p^*\omega)(e_i) = \varepsilon_i$, where $\varepsilon_i = (0, \dots, \underset{\text{i-th}}{1}, \dots, 0)^t$. So $p^*\omega = \sum e^i \otimes \varepsilon_i$ where $\{e^i\}$ is the dual of $\{e_i\}$.

- Let $i : H \hookrightarrow \mathrm{GL}(n;\mathbb{R})$ be inclusion, $V = \mathbb{R}^n$. We can consider i as an H-representation on V and $Q \times_i V \xrightarrow{\cong} TM$, $\eta \times_i v \mapsto \eta(v)$. So we have an isomorphism

  $$\pi_Q^* : \Omega^q(TM) \xrightarrow{\cong} \Omega^q_B(Q;V).$$

  We can check $\omega \in \Omega^1_B(Q;V)$:

  - $\forall X \in \mathfrak{h}$, $i(X^\sharp)\omega = 0$.
  - $\forall h \in H$, $\forall \eta \in Q$, $\left(R_h^*\omega\right)_\eta = h^{-1}\eta^{-1}(\underbrace{(\pi_Q)_*(R_h)_*}_{(\pi_Q)_*}(\cdot)) = h^{-1}\omega_\eta$.

  Since we know $\Omega^1(TM) = \Gamma(T^*M \otimes TM)$, take $\mathrm{id}_{TM} \in \Gamma(T^*M \otimes TM)$, we will show $\boxed{\pi_Q^*(\mathrm{id}_{TM}) = \omega}$. Indeed, $\forall \eta \in Q$, $\forall v \in T_\eta Q$, we have

  $$\left(\pi_Q^*(\mathrm{id}_{TM})\right)_\eta(v) = \eta^{-1}\left((\mathrm{id}_{TM})_{\pi_Q(\eta)}(\pi_Q)_*(v)\right) = \omega_\eta(v).$$

Now take a connection form $\theta \in \Omega^1(Q;\mathfrak{h})$, we will define the torsion of θ next. Recall that θ induces a connection $\nabla = \nabla^\theta$ on $TM \cong Q \times_i V$. By proposition 1.6 we have:

$$\begin{tikzcd}
\Omega^1(TM) \arrow[r, "d^\nabla"] \arrow[d, "\pi_Q^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^2(TM) \arrow[d, "\pi_Q^*", "\cong"'] \\
\Omega^1_B(Q;V) \arrow[r, "d+\rho_*(\theta)\wedge", "=d+\theta\wedge"'] & \Omega^2_B(Q;V)
\end{tikzcd}$$

Recall the torsion $T^\nabla \in \Omega^2(TM)$ of ∇: $T^\nabla = d^\nabla(\mathrm{id}_{TM})$. More explicitly, $\forall X, Y \in \mathfrak{X}(M)$, $\left(d^\nabla(\mathrm{id}_{TM})\right)(X,Y) = \nabla_X\left(\mathrm{id}_{TM}(Y)\right) - \nabla_Y\left(\mathrm{id}_{TM}(X)\right) - \mathrm{id}([X,Y]) = \nabla_X Y - \nabla_Y X - [X,Y]$.
Then

$$\pi_Q^*\left(d^\nabla(\mathrm{id}_{TM})\right) = (d+\theta\wedge)\pi_Q^*(\mathrm{id}_{TM}) \iff d\omega + \theta\wedge\omega = \pi_Q^*(T^\nabla) \quad (\star).$$

We will rewrite RHS.

Proposition 1.14. $\forall s \in \Omega^q(TM)$, $\exists\ T_s : Q \to V \otimes \Lambda^q V^*$ an H-equivariant map s.t.

$$\pi_Q^* s = \frac{1}{q!} T_s(\underbrace{\omega\wedge\cdots\wedge\omega}_{q})$$

Proof. By definition, $\forall \eta \in Q$, $\forall v_1, \dots, v_q \in T_\eta Q$,

$$(\pi_Q^* s)_\eta(v_1,\dots,v_q) = \eta^{-1}\left(s_{\pi_Q(\eta)}\left((\pi_Q)_*(v_1),\dots,(\pi_Q)_*(v_q)\right)\right).$$

Since $(\pi_Q)_*(v_i) = \eta\eta^{-1}(\pi_Q)_*(v_i) = \eta\omega_\eta(v_i)$, define $T_s(\eta) \coloneqq \eta^{-1}\left(s_{\pi_Q(\eta)}\left(\eta(\cdot),\dots,\eta(\cdot)\right)\right)$, then set $\omega = \sum \omega^i \varepsilon_i$ for $\omega^i \in \Omega^1(Q)$, we have

$$\begin{aligned}
\left(\frac{1}{q!}T_s(\omega\wedge\cdots\wedge\omega)\right)(v_1,\dots,v_q) &= \frac{1}{q!}\sum_{i_1,\dots,i_q} T_s(\varepsilon_{i_1},\dots,\varepsilon_{i_q})(\omega^{i_1}\wedge\cdots\wedge\omega^{i_q})(v_1,\dots,v_q) \\
&= \frac{1}{q!}\sum_{i_1,\dots,i_q} T_s(\varepsilon_{i_1},\dots,\varepsilon_{i_q}) \det\left(\omega^{i_k}(v_l)\right) \\
&= \frac{1}{q!}\sum_{i_1,\dots,i_q} T_s(\varepsilon_{i_1},\dots,\varepsilon_{i_q}) \sum_{\sigma\in S_q} \mathrm{sgn}(\sigma)\omega^{i_1}(v_{\sigma(1)})\cdots\omega^{i_q}(v_{\sigma(q)}) \\
&= \frac{1}{q!}\sum_{\sigma\in S_q} \mathrm{sgn}(\sigma) T_s\left(\omega(v_{\sigma(1)}),\dots,\omega(v_{\sigma(q)})\right) \\
&= \frac{1}{q!}\sum_{\sigma\in S_q} T_s\left(\omega(v_1),\dots,\omega(v_q)\right) \\
&= T_s\left(\omega(v_1),\dots,\omega(v_q)\right) \\
&= (\pi_Q^* s)_\eta(v_1,\dots,v_q).
\end{aligned}$$

□

Remark 1.14. T_s is understand as follows. Consider $s \in \Omega^q(TM) = \Omega^0(TM \otimes \Lambda^q T^*M)$, then we have

$$\widetilde{\pi_Q}^* : \Omega^0(TM\otimes\Lambda^q T^*M) \to \Omega^0_B(Q; V\otimes\Lambda^q V^*).$$

For $\eta \in Q$, $(\widetilde{\pi_Q}^* s)(\eta) = \widetilde{\eta}^{-1}(s_{\pi_Q(\eta)})$, $T_s = \widetilde{\pi_Q}^* s$ where

$$\widetilde{\eta} : V\otimes\Lambda^q V^* \to T_xM \otimes \Lambda^q T_x^*M, \quad v\otimes\alpha \mapsto \eta(v)\otimes\alpha(\eta^{-1}(\cdot),\dots,\eta^{-1}(\cdot))$$

Definition 1.18. By this proposition, $T_\theta := \widetilde{\pi_Q}^* T^\nabla \in \Omega^0_B(Q; V \otimes \Lambda^2 V^*)$ satisfies

$$\pi_Q^*(T^\nabla) = \frac{1}{2} T_\theta(\omega \wedge \omega).$$

T_θ is called the **torsion of** θ.

Then $(\star)$ becomes:

$$d\omega + \theta \wedge \omega = \frac{1}{2} T_\theta(\omega \wedge \omega) \quad \textbf{Cartan's first structure equation}$$

Next we define intrinsic torsion.

Lemma 1.8. Take connections $\theta_1, \theta_2 \in \Omega^1(Q; \mathfrak{h})$ on Q with torsion $T_{\theta_1}, T_{\theta_2}$ respectively.

(1) $\exists\, p : Q \to \mathfrak{h} \otimes V^*$ an H-equivariant map s.t. $\theta_1 - \theta_2 = p(\omega)$. Conversely, for $\forall \widetilde{p} : Q \to \mathfrak{h} \otimes V^*$ an H-equivariant map, $\forall \theta$ connection, we have $\theta + \widetilde{p}\omega$ is a connection form on Q.

- $T_{\theta_1} - T_{\theta_2} = -\delta(p)$, where

$$\delta : \mathfrak{h} \otimes V^* \hookrightarrow V \otimes V^* \otimes V^* \xrightarrow{\mathrm{id}_V \otimes (\cdot \wedge \cdot)} V \otimes \Lambda^2 V^* \quad (H\text{-equivalent})$$

  We can show that δ is isomorphism when $\mathfrak{h} = \mathfrak{so}(n)$, thus δ is injective when $\mathfrak{h} \subseteq \mathfrak{so}(n)$.

Proof. (1) By definition of connection form, we have

$$\begin{cases} \forall X \in \mathfrak{h},\ (\theta_1 - \theta_2)(X^\sharp) = 0 \\ \forall h \in H,\ R_h^*(\theta_1 - \theta_2) = \mathrm{Ad}(h^{-1})(\theta_1 - \theta_2) \end{cases} \iff \theta_1 - \theta_2 \in \Omega^1_B(Q; \mathfrak{h}).$$

Here we consider representation $(\mathfrak{h}, \mathrm{Ad})$. Then by proposition 1.14 we obtain (1). More precisely, since $\pi^*_{Q,\mathfrak{h}} : \Omega^1(\underbrace{Q \times_{\mathrm{Ad}} \mathfrak{h}}_{:=\mathrm{Ad}(Q)}) \xrightarrow{\cong} \Omega^1_B(Q; \mathfrak{h})$, $\exists\, \tau \in \Omega^1(\mathrm{Ad}(Q))$ s.t. $\theta_1 - \theta_2 = \pi^*_{Q,\mathfrak{h}}(\tau)$. For $\forall \eta \in Q$, $\forall v \in T_\eta Q$, we have

$$(\theta_1 - \theta_2)_\eta(v) = \eta_{\mathfrak{h}}^{-1}\left(\tau_{\pi_Q(\eta)}\left((\pi_Q)_*(v)\right)\right) = \eta_{\mathfrak{h}}^{-1}\left(\tau_{\pi_Q(\eta)}\left(\eta(\omega_\eta(v))\right)\right)$$

where

$$\eta_{\mathfrak{h}} : \mathfrak{h} \xrightarrow{\cong} \mathrm{Ad}(Q)_{\pi_Q(\eta)} = Q_{\pi_Q(\eta)} \times_{\mathrm{Ad}} \mathfrak{h}, \quad X \mapsto \eta \times_{\mathrm{Ad}} X.$$

So we can define $p : Q \to \mathfrak{h} \otimes V^*$ by $p(\eta) = \eta_{\mathfrak{h}}^{-1}\left(\tau_{\pi_Q(\eta)}(\eta(\cdot))\right)$ to obtain $\theta_1 - \theta_2 = p(\omega)$.

(2) By Cartan's first structure equation we have

$$(\theta_1 - \theta_2) \wedge \omega = \frac{1}{2}(T_{\theta_1} - T_{\theta_2})(\omega \wedge \omega).$$

Set $\omega = \sum \omega^i \varepsilon_i$ for $\omega^i \in \Omega^1(Q)$, then

$$p(\omega) \wedge \omega = \sum p(\varepsilon_i)(\varepsilon_j)\omega^i \wedge \omega^j \implies p = \sum p(\varepsilon_j) \otimes \varepsilon^j = \sum p(\varepsilon_j)(\varepsilon_i) \otimes \varepsilon^i \otimes \varepsilon^j.$$

So $\delta(p) = \sum_{i,j} p(\varepsilon_j)(\varepsilon_i) \otimes \varepsilon^i \wedge \varepsilon^j \implies \delta(p)(\varepsilon_i, \varepsilon_j) = p(\varepsilon_j)(\varepsilon_i) - p(\varepsilon_i)(\varepsilon_j)$, then

$$p(\omega) \wedge \omega = -\frac{1}{2}\delta(p)(\omega \wedge \omega).$$

So $(T_{\theta_1} - T_{\theta_2})(\omega \wedge \omega) = -\delta(p)(\omega \wedge \omega)$ $(\star)$. We want to say $(\star)$ implies the conclusion in (2). For $\forall \eta \in Q$, $v \in T_\eta Q$, we have $\omega_\eta(v) = \eta^{-1}\left((\pi_Q)_*(v)\right)$, so ω_η is suejective, hence completing the proof. When $\mathfrak{h} = \mathfrak{so}(n) \cong \Lambda^2 V^*$, δ is injective implies δ is isomorphic. This is just linear algebra. ∎

$\delta : \mathfrak{h} \otimes V^* \to V \otimes \Lambda^2 V^*$ H-equivalent induces $\delta : \Omega^0_B(Q; \mathfrak{h} \otimes V^*) \to \Omega^0_B(Q; V \otimes \Lambda^2 V^*)$, $\widetilde{s} \mapsto \delta(\widetilde{s})$. Recall that $T_\theta \in \Omega^0_B(Q; V \otimes \Lambda^2 V^*)$ for connection form θ on Q. So by the previous lemma we have $[T_\theta] \in \dfrac{\Omega^0_B(Q; V \otimes \Lambda^2 V^*)}{\delta\left(\Omega^0_B(Q; \mathfrak{h} \otimes V^*)\right)}$ is independent of choice of θ.

Definition 1.19. $[T_\theta]$ is called the **intrinsic torsion** of Q. We call the H-structure Q **torsion-free** if $[T_\theta] = 0$.

Remark 1.15. We can define intrinsic torsion in vector bundle setting. Under the notation in lemma 1.6, we have

$$\begin{tikzcd}
p \in & \Omega^0_B(Q; \mathfrak{h} \otimes V^*) \arrow[r, "\delta"] \arrow[dr, phantom, "\circlearrowright"] & \Omega^0_B(Q; V \otimes \Lambda^2 V^*) & \ni T_\theta \\
\tau \in \arrow[u, mapsto] & \Omega^0(\mathrm{Ad}(Q) \otimes T^*M) \arrow[r, dashed, "\delta_M"'] \arrow[u, "\widetilde{\pi}^*_{Q,\mathfrak{h}}", "\cong"'] & \Omega^0(TM \otimes \Lambda^2 T^*M) \arrow[u, "\cong", "\widetilde{\pi}^*_Q"'] & \ni T_\nabla \arrow[u, mapsto]
\end{tikzcd}$$

So intrinsic torsion of Q corresponds to $[T_\nabla] \in \dfrac{\Omega^2(TM)}{\delta_M(\Omega^1(\mathrm{Ad}(Q)))}$.

Lemma 1.9. *H-structure Q is torsion-free $\iff$ $\exists$ a connection $\theta \in \Omega^1(Q; \mathfrak{h})$ s.t. $T_\theta = 0$ (torsion-free connection).*

Proof. $(\Leftarrow)$ is trivial. For $(\Rightarrow)$, take $\forall$ connection θ_0 on Q, Q is torsion-free implies that $\exists\, \widetilde{p} \in \Omega^0_B(Q; \mathfrak{h} \otimes V^*)$ s.t. $T_{\theta_0} = \delta(\widetilde{p})$. Set $\theta := \theta_0 + \widetilde{p}(\omega)$, we have $T_\theta = 0$.

∎

Proposition 1.15. *M admits a torsion-free H-structure Q $\iff$ $\exists$ a connection form $\theta \in \Omega^1(P; \mathfrak{gl}(n, \mathbb{R}))$ s.t. $T_\theta = 0$ and $\exists\, \xi \in P$, $\mathrm{Hol}_\xi(P, \theta) \subseteq H$.*

Proof. $(\Rightarrow)$ $\exists$ a connection $\theta_Q \in \Omega^1(Q; \mathfrak{h})$ s.t. $T_{\theta_Q} = 0$. We can extend θ_Q to a unique connection form $\theta \in \Omega^1(P; \mathfrak{gl}(n, \mathbb{R}))$ s.t.

$$\begin{cases} \iota_Q^* \theta = \theta_Q \\ \mathrm{Hol}_\eta(Q, \theta_Q) = \mathrm{Hol}_{\iota(\eta)}(P, \theta) \end{cases}$$

Proof of the existence and uniqueness of θ can be found in [Kobayashi-Nomizu, Vol I, p.79, Prop 6.1]. The idea is outlined as follows: For $\xi \in P$, set $\xi = \iota_Q(\eta) \cdot g$ for $\eta \in Q$, $g \in \mathrm{GL}(n, \mathbb{R})$. Define $H_\xi := (R_g)_* \left((\iota_Q)_* (\ker \theta_\eta) \right)$. Then $\{H_\xi\}_{\xi \in P}$ is a well-defined connection on P. By construction this is the unique connection we want. Setting $\xi := \iota_Q(\eta)$ for $\forall \eta \in Q$, we see that $\mathrm{Hol}_\xi(P, \theta) = \mathrm{Hol}_\eta(Q, \theta_Q) \subseteq H$. We also have $T_\theta = 0$. Indeed, let ω_P, ω_Q be tautological 1-forms on P, Q. Then we have $\iota_Q^* \omega_P = \omega_Q$. Pullback Cartan's first structure equation on θ from ι_Q we have $T_\theta \circ \iota_Q = 0$ by our assumption. Since T_θ is $\mathrm{GL}(n; \mathbb{R})$-equivalent, we have $T_\theta = 0$.

($\Leftarrow$) Consider holonomy subbundle $Q(\xi)$. By reduction theorem, θ is reducible to $Q(\xi)$. So $\theta_{Q(\xi)} := \theta|_{Q(\xi)} \in \Omega^1(Q(\xi); \mathfrak{hol}_\xi(P, \theta))$ is a connection form on $Q(\xi)$. As above, we see that $T_{\theta_{Q(\xi)}} = 0$. Set $Q := Q(\xi) \times_i H$, where $i : \mathrm{Hol}_\xi(P, \theta) \hookrightarrow H$. We see that Q is an H-structure and $\theta_{Q(\xi)}$ extends to a connection θ_Q on Q s.t. $T_{\theta_Q} = 0$.

$\square$

Remark 1.16. In geometry, we often consider the case $\mathfrak{h} \subseteq \mathfrak{so}(n)$ ($H \subseteq \mathrm{O}(n)$). In this case, we can simplify as follows.

Recall that

$$\mathfrak{so}(n) \times \mathfrak{so}(n) \to \mathbb{R}, \quad (X, Y) \mapsto -tr(XY) = tr(X^t Y)$$

is $\mathrm{O}(n)$-invariant inner product on $\mathfrak{so}(n)$. Using this, we have the orthogonal decomposition $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ (Note that $\mathfrak{h}$, $\mathfrak{h}^\perp$ are H-invariant). Recall

$$\delta : \mathfrak{so}(n) \otimes V^* \xrightarrow{\cong} V \otimes \Lambda^2 V^* \quad (H\text{-equivariant iso.})$$

So it implies

$$V \otimes \Lambda^2 V^* = \delta(\mathfrak{h} \otimes V^*) \oplus \delta(\mathfrak{h}^\perp \otimes V^*) \implies \frac{V \otimes \Lambda^2 V^*}{\delta(\mathfrak{h} \otimes V^*)} \cong \mathfrak{h}^\perp \otimes V^* \quad (H\text{-equivariant iso.})$$

$\boxed{\text{Claim.}}$ $\dfrac{\Omega^0_B(Q; V \otimes \Lambda^2 V^*)}{\delta\left(\Omega^0_B(Q; \mathfrak{h} \otimes V^*)\right)} \cong \Omega^0_B\left(Q; \dfrac{V \otimes \Lambda^2 V^*}{\delta(\mathfrak{h} \otimes V^*)}\right) \cong \Omega^0_B\left(Q; \mathfrak{h}^\perp \otimes V^*\right)$, $[\widetilde{s}] \mapsto [\widetilde{s}]$.

So by this claim we consider intrinsic torsion of Q as an element of $\Omega^0_B\left(Q; \mathfrak{h}^\perp \otimes V^*\right)$. The torsion-free case is the most important H-structure, but we can consider some weaker cases of H-structures. We often consider the irreducible decomposition of H-representation $\mathfrak{h}^\perp \otimes V^* = V_1 \oplus V_2 \oplus \cdot$ and consider the H-structure whose intrinsic torsion is V_i-valued (e.g. nearly Kähler manifolds).

$\boxed{\text{Proof of the Claim.}}$ Surjectivity is clear. For injectivity, consider $\widetilde{s} \in \Omega^0_B(Q; V \otimes \Lambda^2 V^*)$. Suppose $\widetilde{s}$ is $\delta(\mathfrak{h} \otimes V^*)$-valued. Since $\delta' := \delta|_{\mathfrak{h} \otimes V^*}$ is H-equivalent isomorphism, we have $(\delta')^{-1}(\widetilde{s}) \in \Omega^0_B(Q;\mathfrak{h}\otimes V^*)$, so $\widetilde{s} = \delta\left((\delta')^{-1}(\widetilde{s})\right) \in \delta\left(\Omega^0_B(Q;\mathfrak{h}\otimes V^*)\right)$.

Let (M,g) be a Riemannian manifold, $Q :=$ frame bundle of (TM,g) (see example 1.6). Then Q is an $\mathrm{O}(n)$-structure on M.

Corollary 1.2. $\exists\,!$ torsion-free connection on Q.

Proof. Set $H = \mathrm{O}(n)$. Then by the above remark we see that intrinsic torsion of Q belongs to $\Omega^0_B(Q;\{0\}) = \{0\}$. So $\exists$ a torsion-free connection on Q. For uniqueness, let θ_1, θ_2 be torsion-free connection, then we have p in lemma 1.6. From $\ker\delta = \{0\}$ implies $p = 0$ and then $\theta_1 = \theta_2$. $\square$

This corollary actually shows existence and uniqueness of Levi-Civita conenction of g.

Recall the 1-1 correspondence

$$\{\text{connection on frame bundle } P\} \leftrightarrow \{\text{connection on } TM\}, \quad \theta \mapsto \nabla^\theta.$$

A connection ∇ on TM is Levi-Civita connection of g iff $\nabla g = 0$ and $T^\nabla = 0$. From previous proposition we know that

$$\nabla^\theta g = 0 \iff \theta \text{ is reducible to } Q, \quad T_\theta = 0 \iff T^{\nabla^\theta} = 0.$$

Hence we have

$$\begin{array}{ccc}
\theta \in & \left\{\begin{matrix}\theta: \text{ connection on } P \\ \theta \text{ is reducible to } Q \\ T_\theta = 0\end{matrix}\right\} & \longleftrightarrow \left\{\begin{matrix}\text{LC connection} \\ \text{on } TM\end{matrix}\right\} \\
\Big\downarrow & \Big\updownarrow & \\
\theta|_Q \in & \left\{\begin{matrix}\theta_Q: \text{ connection on } Q \\ T_{\theta_Q} = 0\end{matrix}\right\} &
\end{array}$$

From the corollary we show the existence and uniqueness of LC connection.

1.6 Holonomy Groups of Riemannian Manifolds

Let (M,g) be a Riemannian manifold. We assume M is simply connected for simplicity. Then M is orientable, so fix the orientation. Consider the oriented orthonormal frame bundle $\pi_Q : Q \to M$, $Q_x := \left\{\eta : \mathbb{R}^n \xrightarrow{\cong} T_xM \mid \eta^* g_x = (\cdot,\cdot)_{std},\ \eta^*(vol_g)_x = vol_{std}\right\}$. Then it's principal $\mathrm{SO}(n)$-bundle on M. The Levi-Civita connection ∇ of g defineds a connection form $\theta \in \Omega^1(P;\mathfrak{gl}(n,\mathbb{R}))$. Since $\nabla g = 0$, $\nabla vol_g = 0$, so θ is reducible to Q (see lemma 1.5 and remark 1.11). So we obtain a connection form $\theta_g = \theta|_Q \in \Omega^1(Q;\mathfrak{so}(n))$.

Definition 1.20. Fix $\eta \in Q$, M simply connected implies that $\mathrm{Hol}_\eta(Q, \theta_g) = \mathrm{Hol}^0_\eta(Q, \theta_g) \subseteq \mathrm{SO}(n)$. Since the conjugacy class is independent of the choice of $\eta \in Q$, we call this conjugacy class in $\mathrm{SO}(n)$ the **holonomy group** $Hol(g)$ ($Hol(M, g)$) of g.

Remark 1.17. We can describe $Hol(g)$ in terms of LC connection ∇ as follows. $TM = P \times_{\mathrm{id}} \mathbb{R}^n$, ∇ is induced connection from θ. By lemma 1.2 we know that $\mathrm{Hol}_\xi(P, \theta) = \xi^{-1} \circ \mathrm{Hol}_x(\nabla) \circ \xi$ for $x \in M$, $\xi \in P_x$. Since θ is reducible to θ, $\mathrm{Hol}_\eta(Q, \theta_g) = \mathrm{Hol}_\eta(P, \theta)$ for $\forall \eta \in Q$. Thus $\mathrm{Hol}_\eta(Q, \theta_g) = \eta^{-1} \circ \mathrm{Hol}_x(\nabla) \circ \eta$, then $Hol(g)$ is conjugacy class of $\mathrm{Hol}_x(\nabla)$ described in an oriented orthonormal basis of T_xM.

Theorem 1.7. (**Berger Theorem**)

Let (M, g) be a simply connected n-dimensional Riemannian manifold. Suppose M is **not** locally reducible (locally reducible means locally M is isometric to product of Riemannian manifolds) and **not** a symmetric space. Then $Hol(M, g)$ is one of the following:

$$\mathrm{SO}(n),\ \mathrm{U}(m)\ (n = 2m, m \geq 2),\ \mathrm{SU}(m)\ (n = 2m, m \geq 2)$$
$$Sp(m)\ (n = 4m, m \geq 2),\quad \underbrace{\boxed{Sp(m)Sp(1)}}_{= Sp(m) \times Sp(1)/\{\pm 1\}}\quad (n = 4m, m \geq 2)$$
$$G_2\ (n = 7),\ Spin(7)\ (n = 8)$$

We will give some remarks for each cases.

Definition 1.21. A Riemannian manifold (M, g) is a **Riemannian symmetric space** if for $\forall p \in M$, $\exists\ s_p : M \to M$ isometry s.t.

- $s_p \circ s_p = \mathrm{id}_M$.
- p is an isolated fixed point of s_p.

For a Riemannian symmetric space, set G be isometry group of M, K be isotropic group of p. It is known that $M \cong G/K$, so Riemannian symmetric spaces are classified by Lie group/algebra theory, and their holonomy groups are computed.

2 Complex Manifolds

Roughly speaking, complex manifold is smooth manifold on which holomorphic functions are defined. It's fundamental objects in many fields such as differential geometry, function theory of several complex variables, algebraic geometry and mathematical physics. We'll focus on the differential geometry part.

2.1 Complex Manifolds and Complex Differential Forms

Before we move onto complex geometry, we start from introducing complex differential forms and holomorphic functions on an open subset of $\mathbb{C}^m$.

Notations: (1) $V \subseteq \mathbb{C}^m$ open subset; (2) $(z^1, \dots, z^m)$ standard coordinates on V; (3) $z^i = x^i + \sqrt{-1}y^i$, $x^i, y^i \in \mathbb{R}$. For $p \in V$, set

$$\left(\frac{\partial}{\partial z^i}\right)_p \coloneqq \frac{1}{2}\left(\left(\frac{\partial}{\partial x^i}\right)_p - \sqrt{-1}\left(\frac{\partial}{\partial y^i}\right)_p\right), \left(\frac{\partial}{\partial \bar{z}^i}\right)_p \coloneqq \frac{1}{2}\left(\left(\frac{\partial}{\partial x^i}\right)_p + \sqrt{-1}\left(\frac{\partial}{\partial y^i}\right)_p\right),$$

$$(dz^i)_p \coloneqq (dx^i)_p + \sqrt{-1}(dy^i)_p, \quad (d\bar{z}^i)_p \coloneqq (dx^i)_p - \sqrt{-1}(dy^i)_p.$$

$$(T_p\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C} : \text{ the complexification of } T_p\mathbb{C}^m$$

Then it's clear that $\left(\frac{\partial}{\partial z^i}\right)_p, \left(\frac{\partial}{\partial \bar{z}^i}\right)_p \in (T_p\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C}$, $(dz^i)_p, (d\bar{z}^i)_p \in (T_p^*\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C}$. We see that $\left\{\left(\frac{\partial}{\partial z^i}\right)_p, \left(\frac{\partial}{\partial \bar{z}^i}\right)_p\right\}_{i=1}^m$ is a basis of $(T_p\mathbb{C}^m) \otimes_{\mathbb{R}} \mathbb{C}$ and similar for the cotangent case.

Proposition 2.1. For a C^1-function $f = g + \sqrt{-1}h : V \to \mathbb{C}$, set

$$(df)_p \coloneqq (dg)_p + \sqrt{-1}(dh)_p \in T_p^*\mathbb{C}^m \otimes_{\mathbb{R}} \mathbb{C}.$$

Then

$$(df)_p = \left(\left(\frac{\partial}{\partial z^i}\right)_p f\right)(dz^i)_p + \left(\left(\frac{\partial}{\partial \bar{z}^i}\right)_p f\right)(d\bar{z}^i)_p.$$

Proof.

$$\begin{aligned}
(RHS) &= \frac{1}{2}\left(\left(\frac{\partial}{\partial x^i}\right)_p - \sqrt{-1}\left(\frac{\partial}{\partial y^i}\right)_p\right) f \left((dx^i)_p + \sqrt{-1}(dy^i)_p\right) \\
&\quad + \frac{1}{2}\left(\left(\frac{\partial}{\partial x^i}\right)_p + \sqrt{-1}\left(\frac{\partial}{\partial y^i}\right)_p\right) f \left((dx^i)_p - \sqrt{-1}(dy^i)_p\right) \\
&= \frac{\partial f}{\partial x^i}(p)(dx^i)_p + \frac{\partial f}{\partial y^i}(p)(dy^i)_p \\
&= \frac{\partial g}{\partial x^i}(p)(dx^i)_p + \frac{\partial g}{\partial y^i}(p)(dy^i)_p + \sqrt{-1}\left(\frac{\partial h}{\partial x^i}(p)(dx^i)_p + \frac{\partial h}{\partial y^i}(p)(dy^i)_p\right) \\
&= (dg)_p + \sqrt{-1}(dh)_p = (df)_p
\end{aligned}$$

□

Definition 2.1. A C^1-function $f : V \to \mathbb{C}$ is a **holomorphic function** if $\frac{\partial f}{\partial \bar{z}^i} = 0$ on V. For an open subset $W \subseteq \mathbb{C}^n$, a map $F = (f^1, \ldots, f^n) : V \to W$ is a **holomorphic map** if each component f^i is a holomorphic function.

Remark 2.1. It is known that a holomorphic function is analytic. In particular, it's a C^∞ function.

Lemma 2.1. The composition of holomorphic maps is a holomorphic map.

Definition 2.2. A topological space M is an m-dimensional **complex manifold** if

(1) M is a Hausdorff space.

(2) $\exists \{U_\alpha\}_{\alpha \in A}$ open cover of M, $\forall \alpha \in A$, $\exists \varphi_\alpha : U_\alpha \xrightarrow{\text{homeo.}} V_\alpha \underset{\text{open}}{\subseteq} \mathbb{C}^m$ s.t. if $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \text{ is a holomorphic map}$$

$\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called a **holomorphic coordinate system**, φ_α is called local coordinates. m-dimensional complex manifold is a $2m$-dimensional smooth manifold.

A C^1-map

References

[1] Kobayashi, S. and Nomizu, K. *Foundations of differential geometry, volume 1*. Volume 1. John Wiley & Sons, 1963.