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# DIFFERENTIAL TOPOLOGY

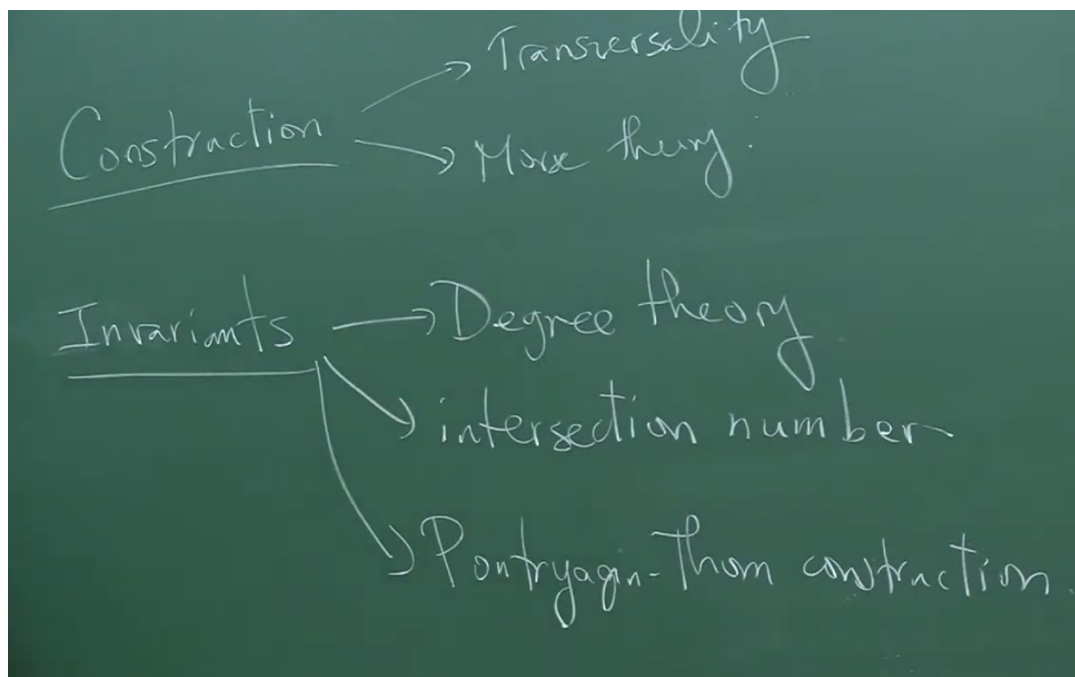
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## 1 Review of Differentiable Manifold

**Definition 1.1.** A topological space is a pair  $(X, T)$ , where  $T \subseteq \mathcal{P}(X)$  such that

- $\emptyset, X \in T$
- $\{U_\alpha\}_{\alpha \in I} \subseteq T \implies \bigcup_{\alpha \in I} U_\alpha \in T$
- $U_1, \dots, U_n \subseteq T \implies U_1 \cap \dots \cap U_n \in T$

Fixing  $(X, T)$ , the elements of  $T$  are called open sets.

## 1.1 Jet bundles

**Definition 1.2.** Let  $X, Y$  be smooth manifolds,  $f, g : X \rightarrow Y$  smooth.

(1) We write  $f \sim_k g$  at  $p \in X$  if  $f(p) = g(p)$  and given charts  $\varphi : U \rightarrow \mathbb{R}^n$  around  $p$ ,  $\psi : V \rightarrow \mathbb{R}^m$  around  $f(p)$

$$\frac{\partial^{|\alpha|} (\psi \circ f \circ \varphi^{-1})_j}{\partial x^\alpha} (\varphi(p)) = \frac{\partial^{|\alpha|} (\psi \circ g \circ \varphi^{-1})_j}{\partial x^\alpha} (\varphi(p)), \quad \forall |\alpha| \leq k, 1 \leq j \leq m$$

It follows from the chain rule that  $\sim_k$  is an equivalence relation.

(2)  $J^k(X, Y)_{p,q} = \{f : X \rightarrow Y \text{ smooth} \mid f(p) = q\} / \sim_k$ , called the space of  $k$ -jets at  $p$  with value  $q$ .

$$(3) J^k(X, Y) = \bigsqcup_{\substack{p \in X \\ q \in Y}} J^k(X, Y)_{p,q}.$$

**Example 1.1.** (1)  $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ ,

$$J^k(U, V)_{x,y} \xrightarrow{\Phi} B_{n,m}^k, [f] \mapsto (p_1(x), \dots, p_m(x))$$

where  $p_j(x)$  is the Taylor polynomial of  $f_j(x)$  without the constant term,  $B_{n,m}^k = \{\text{polynomial functions } \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ of degree } \leq k \text{ with no constant term}\}$ .  $\Phi$  is a bijection.  $J^k(U, V) \cong U \times V \times B_{n,m}^k$ .

$$(2) J^1(M, \mathbb{R}) \xrightarrow{\text{bijection}} \mathbb{R} \times T^*M, [f]_x \mapsto (f(x), df_x).$$

$$(3) J^1(\mathbb{R}, M) \cong \mathbb{R} \times TM.$$

**Definition 1.3.** (1)  $\varphi : Y \rightarrow Z$  smooth. Then  $\varphi_* : J^k(X, Y) \rightarrow J^k(X, Z)$ ,  $[f]_x \mapsto [\varphi \circ f]_x$ .

(2)  $\varphi : Z \rightarrow X$  diffeo. Then  $\varphi^* : J^k(X, Y) \rightarrow J^k(Z, Y)$ ,  $[f]_x \mapsto [f \circ \varphi]_{\varphi^{-1}(x)}$ .

**Remark.** These operations are well-defined and natural (functionality). In particular, if  $\varphi : Y \rightarrow Z$  diffeo, then  $\varphi_*$  is bijection;  $\varphi : Z \rightarrow X$  diffeo, then  $\varphi^*$  is bijection.

Suppose  $\sigma \in J^k(X, Y)$ ,  $\sigma = [f]_x$ .

Define  $\alpha(\sigma) = x, \beta(\sigma) = f(x)$ , called the source of  $\sigma$  and target of  $\sigma$  respectively, then  $\alpha : J^k(X, Y) \rightarrow X, \beta : J^k(X, Y) \rightarrow Y$ . We will define the local topology around  $\sigma$  and a smooth structure near  $\sigma$ .

Fix charts  $\varphi : U \rightarrow \mathbb{R}^n, \psi : V \rightarrow \mathbb{R}^m$  around  $x$  and  $f(x)$  respectively,  $f(U) \subseteq V$ . Let

$$\tau_{U,V} : J^k(U, V) \longrightarrow J^k(\varphi(U), \psi(V)) \cong \varphi(U) \times \psi(V) \times B_{n,m}^k, \sigma \mapsto (\varphi^{-1})^* \psi_* \sigma$$

Since  $\varphi(U) \times \psi(V) \times B_{n,m}^k \subseteq \mathbb{R}^N$ , use  $\tau_{U,V}$  to topologize  $J^k(U, V)$  and hence  $J^k(X, Y)$ . It's easy to see that this topology doesn't depend on the choice of charts.

**Exercise.** Let  $\tilde{\varphi} : U \rightarrow \mathbb{R}^n, \tilde{\psi} : V \rightarrow \mathbb{R}^m$  be other charts, then  $\tau_{\tilde{\varphi}, \tilde{\psi}} \circ \tau_{\varphi, \psi}^{-1}$  is smooth. So  $J^k(X, Y)$  has an induced smooth structure.

**Lemma 1.1.** (1)  $J^k(X, Y)$  is a manifold of dimension  $n + m \binom{n+k}{k}$ .

(2)  $\alpha : J^k(X, Y) \rightarrow X, \beta : J^k(X, Y) \rightarrow Y, \alpha \times \beta : J^k(X, Y) \rightarrow X \times Y$  are smooth surjective submersions.

(3)  $\varphi : Y \rightarrow Z$  smooth, then  $\varphi_*$  is smooth;  $\varphi : Z \rightarrow X$  diffeomorphism, then  $\varphi^*$  is diffeomorphism.

**Definition 1.4.** Let  $f \in C^\infty(X, Y)$ . Its  $k$ -jet  $j^k f$  is the function

$$j^k f : X \rightarrow J^k(X, Y), \quad x \mapsto [f]_x$$

**Remark.**  $J^k(X, Y)$  is usually not a vector bundle over  $X, Y$  or  $X \times Y$ . If  $Y = \mathbb{R}^m$ , then  $J^k(X, Y)$  is a vector bundle over  $X$ .

**Definition 1.5.** Let  $E, B, F$  be manifolds, and  $\pi : E \rightarrow B$  is a surjective submersion. We say that  $\pi$  is a fiber bundle with fiber  $F$  if  $\forall b \in B, \exists U \subseteq B$  neighborhood of  $b$  and a diffeomorphism  $\Phi : \pi^{-1}(U) \rightarrow U \times F$  such that the diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ \pi \searrow & & \swarrow pr_1 \\ & U & \end{array}$$

**Exercise.**  $J^k(X, Y)$  is a fiber bundle over  $X, Y, X \times Y$ ,  $f \in C^\infty(X, Y)$  gives rise to a section  $j^k f : X \rightarrow J^k(X, Y)$ .

## 1.2 Whitney $C^\infty$ -Topology

Let  $X, Y$  be smooth manifolds. For  $U \subseteq J^k(X, Y)$  open, let

$$M(U) = \left\{ f \in C^\infty(X, Y) \mid j^k f(X) \subseteq U \right\}$$

Note that  $M(U) \cap M(V) = M(U \cap V)$ , so  $\{M(U) \mid U \subseteq J^k(X, Y) \text{ open}\}$  is a basis for a topology on  $C^\infty(X, Y)$ , which is called the  $C^k$ -topology. Let  $W_k$  be the  $C^k$ -topology.

**Lemma 1.2.**  $k \leq l \implies W_k \subseteq W_l$ .

*Proof.* Suppose  $k \leq l$ . There exists a surjective continuous map:

$$\pi_{k,l} : J^l(X, Y) \rightarrow J^k(X, Y), \quad [f]_x \mapsto [f]_x$$

$\pi_{k,l} \circ j^l f = j^k f$ . If  $U \subseteq J^k(X, Y)$  is open, then  $\pi_{k,l}^{-1}(U) \subseteq J^l(X, Y)$  is open. So  $M(U) = M(\pi_{k,l}^{-1}(U))$ . Therefore  $W_k \subseteq W_l$ .  $\square$

**Definition 1.6.** The (Whitney)  $C^\infty$ -topology is the topology on  $C^\infty(X, Y)$  generated by  $\bigcup_{k \in \mathbb{N}} W_k$ .

Recall that every manifold  $M$  admits a Riemannian metric, which induced a metric space structure on the manifold  $(M, d)$ . Moreover, we may assume that  $d$  is complete.

Why? (1)  $\exists$  smooth proper function  $f : M \rightarrow \mathbb{R}$ ; (2) For any metric  $d$  on  $M$ , we can define  $\tilde{d}(x, y) = d(x, y) + |f(x) - f(y)|$ ,  $\tilde{d}$  is complete.

Let  $d$  be a (complete) metric on  $J^k(X, Y)$ .

**Definition 1.7.** Let  $\delta : X \rightarrow \mathbb{R}_{>0}$  continuous and  $f \in C^\infty(X, Y)$ , let

$$B_\delta^k(f) = \left\{ g \in C^\infty(X, Y) \mid d(j^k f(x), j^k g(x)) < \delta(x) \right\}$$

**Proposition 1.1.**  $\{B_\delta(f) \mid \delta : X \rightarrow \mathbb{R}_{>0}\}$  is a basis for  $C^k$ -topology at  $f$ . (neighborhood basis)

*Proof.*  $f \in B_\delta(f)$ .

**Step 1.**  $B_\delta(f)$  is open. We claim that

$$B_\delta(f) = M(U), \quad U = \left\{ \sigma \in J^k(X, Y) \mid d(j^k f(\alpha(\sigma)), \sigma) < \delta(\alpha(\sigma)) \right\}$$

Define  $\Delta : J^k(X, Y) \rightarrow \mathbb{R}$ ,  $\Delta = \delta \circ \alpha - d(j^k f \circ \alpha(\cdot), \cdot)$ , so  $U = \Delta^{-1}(0, \infty)$  is open.

**Step 2.** Let  $\mathcal{U} \subseteq C^\infty(X, Y)$  be an open neighborhood of  $f$  (in  $C^k$ -topology), then there exists  $U \subseteq J^k(X, Y)$  open set such that  $f \in M(U) \subseteq \mathcal{U}$ . We claim that  $\exists \delta \in C(X, \mathbb{R}_{>0})$  such that  $f \in B_\delta(f) \subseteq M(U)$ .

For each  $x \in X$ , let

$$m(x) = \inf \left\{ d(\sigma, j^k f(x)) \mid \sigma \in \alpha^{-1}(x) \cap (J^k(X, Y) \setminus U) \right\}$$

It's strictly bigger than 0 for every  $x \in X$  because  $U$  is open,  $m(x)$  could be  $\infty$  for some  $x$ . We can choose  $\delta : X \rightarrow \mathbb{R}_{>0}$  continuous such that  $0 < \delta(x) < m(x)$ . Then

$$g \in B_\delta(f) \implies d(j^k f(x), j^k g(x)) < \delta(x) < m(x), \quad \forall x \in X$$

which implies  $j^k g(x) \in U$ ,  $\forall x \in X$ . So  $B_\delta(f) \subseteq M(U)$ .  $\square$

**Obs.**  $B_\delta(f)$  is roughly the set of functions whose partial derivatives up to order  $k$  are close enough to  $f$ 's.

To make this more precise, let  $\Phi = \{\varphi_i : U_i \rightarrow \mathbb{R}^n\}_{i \in I}$  locally finite atlas of  $X$ ,  $\mathcal{K} = \{K_i\}_{i \in I}$  family of compact sets of  $X$ ,  $K_i \subseteq U_i$ ,  $\Psi = \{\psi_i : V_i \rightarrow \mathbb{R}^m\}_{i \in I}$  atlas for  $Y$ ,

$\mathcal{E} = \{\epsilon_i\}_{i \in I}$ ,  $\epsilon_i > 0$ . Define

$$\mathcal{N}^k(f; \Phi, \Psi, \mathcal{K}, \mathcal{E}) = \{g \in C^\infty(X, Y) \mid g(K_i) \subseteq V_i \text{ and} \\ \|D^r(\psi_i \circ f \circ \varphi_i^{-1})(x) - D^r(\psi_i \circ g \circ \varphi_i^{-1})(x)\| < \epsilon_i, \forall i, x \in X, r \leq k\}$$

**Exercise.** Prove that  $\{\mathcal{N}^k(f; \Phi, \Psi, \mathcal{K}, \mathcal{E})\}$  is a basis for the  $C^k$ -topology.

**Remark.** If  $X$  is compact, then we can find a countable basis of  $f$  given by  $\{B_{\delta_n}(f)\}$ , where  $\delta_n = \frac{1}{n}$ . So  $C^k$ -topology is first countable. Moreover,

$$f_n \xrightarrow{C^k} f \Leftrightarrow \frac{\partial^{|\alpha|} f_n}{\partial x^\alpha} \xrightarrow{\text{uniformly}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}, \quad \forall |\alpha| \leq k$$

**Proposition 1.2.** Suppose  $\{f_n\}_{n \in \mathbb{N}} \subseteq C^\infty(X, Y)$  such that  $f_n \xrightarrow{C^k} f$ . Then  $\exists K \subseteq X$  compact such that  $f_n \equiv f$  in  $X \setminus K$  for  $n \gg 0$  and  $j^k f_n|_K \xrightarrow{\text{uniformly}} j^k f$ .

*Proof.* Suppose  $f_n \xrightarrow{C^k} f$  and let  $\{K_i\}_{i \in \mathbb{N}}$  exhaustion by compact sets such that  $K_i \subseteq \text{int}(K_{i+1})$ . Assume, by contradiction, that  $\nexists K \subseteq X$  compact set, such that  $f_n \equiv f$  on  $X \setminus K$ . So for each  $i$ ,  $\exists x_i \in K_i, n_i$  such that  $f_{n_i}(x_i) \neq f(x_i)$ . WLOG,  $n_1 < n_2 < \dots$ ,  $a_i = d(j^k f_{n_i}(x_i), j^k f(x_i)) > 0$ . Let  $\delta : X \rightarrow \mathbb{R}_+$  such that  $\delta(x_i) = a_i/2$ . Then  $f_{n_i} \notin B_\delta(f)$ , so  $f_{n_i} \not\rightarrow f$ .  $\square$

**Definition 1.8.** A topological space is Baire if the countable intersection of open and dense subsets is dense.

**Theorem 1.1.** Let  $X, Y$  be smooth manifolds. Then  $C^\infty(X, Y)$  is Baire in the  $C^\infty$ -topology.

*Proof.* Fix complete metric  $d_k$  on  $J^k(X, Y)$ . Let  $\{U_n\}_{n \in \mathbb{N}}$  dense open subsets of  $C^\infty(X, Y)$  in the  $C^\infty$ -topology. Let  $V \subseteq C^\infty(X, Y)$  non-empty open set. We want to show that  $\bigcap_{n \in \mathbb{N}} U_n \cap V \neq \emptyset$ .

Since  $V$  is open,  $\exists Z \subseteq J^{k_0}(X, Y)$  open such that  $M(\overline{Z}) \subseteq V$ . It's enough to show that  $M(\overline{Z}) \cap \bigcap_{n \in \mathbb{N}} U_n \neq \emptyset$ .

We can construct  $f_i$  inductively,  $\{k_i\} \subseteq \mathbb{N}$ ,  $Z_i \subseteq J^{k_i}(X, Y)$  open sets such that

- (1)  $f_i \in M(Z) \cap \bigcap_{s=1}^i M(Z_s)$
- (2)  $M(\overline{Z_i}) \subseteq U_i$
- (3)  $d_s(j^s f_i(x), j^s f_{i-1}(x)) < 1/2^i, \forall x \in X, 0 \leq s \leq i$

Since  $M(Z) \cap U_1$  is open and non-empty, we can find  $Z_1 \subseteq J^{k_1}(X, Y)$  non-empty such that  $M(\overline{Z_1}) \subseteq M(Z) \cap U_1$ . Take  $f_1 \in M(Z_1)$  and it satisfies (1) and (2). Say we've chosen  $(f_s, k_s, Z_s)$  for  $s \leq i-1$ . Let  $D_i = B_{\frac{1}{2^i}}^0(f_{i-1}) \cap B_{\frac{1}{2^i}}^1(f_{i-1}) \cap \dots \cap B_{\frac{1}{2^i}}^{i-1}(f_{i-1})$  open in  $C^\infty$ -topology,  $f_{i-1} \in M(Z) \cap M(Z_1) \cap \dots \cap M(Z_{i-1}) \cap D_i$ . Since  $U_i$  is open and dense,  $M(Z) \cap M(Z_1) \cap \dots \cap M(Z_{i-1}) \cap D_i \cap U_i \neq \emptyset$  and open, so we can find  $\emptyset \neq Z_i \subseteq J^{k_i}(X, Y)$



such that  $M(\overline{Z_i}) \subseteq M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i \cap U_i$ . Choose  $f_i \in M(Z_i)$ , it satisfies the three conditions.

For a fixed  $s$ , the condition (3) tells that  $\{j^s f_i(x)\}$  is a Cauchy sequence in  $J^k(X, Y)$ , it converges to  $g^s(x)$ ,  $g^0(x) \in J^0(X, Y) = X \times Y$ ,  $g^0(x) = (x, g(x))$ .

**Exercise.**  $g \in C^\infty(X, Y)$  and  $j^s g = g^s$ . (Look in a compact set and in charts)

Then  $g = \lim_{i \rightarrow \infty} f_i$  in the  $C^\infty$ -topology.  $f_i \in M(Z) \implies g \in M(\overline{Z})$ ,  $f_i \in M(Z_s)$  for  $i \geq s$ , so  $g \in M(\overline{Z_s})$  for  $\forall s$ , hence  $g \in M(\overline{Z}) \cap \bigcap_{n \in \mathbb{N}} M(\overline{Z_s}) \subseteq M(\overline{Z}) \cap \bigcap_{n \in \mathbb{N}} U_n$ .  $\square$

**Proposition 1.3.** Let  $X, Y$  be smooth manifolds. Then  $j^k : C^\infty(X, Y) \rightarrow C^\infty(X, J^k(X, Y))$  is continuous in the  $C^\infty$ -topology.

*Proof.*  $U \subseteq J^\ell(X, J^k(X, Y))$  open, so  $M(U)$  is open set in the  $C^\ell$ -topology of  $C^\infty(X, J^k(X, Y))$ . We need to show that  $(j^k)^{-1}(M(U))$  is open in  $C^\infty(X, Y)$ . Consider

$$\alpha^{k, \ell} : J^{k+\ell}(X, Y) \rightarrow J^\ell(X, J^k(X, Y)), \quad \alpha^{k, \ell}(j^{k+\ell} f(x)) = j^\ell(j^k f)(x)$$

This is a smooth embedding. So  $(j^k)^{-1}(M(U)) = M((\alpha^{k, \ell})^{-1}(U))$  is open in  $C^{k+\ell}$ -topology.  $\square$

**Proposition 1.4.**  $\phi : Y \rightarrow Z$  smooth. Then  $\widetilde{\phi}_* : C^\infty(X, Y) \rightarrow C^\infty(X, Z)$ ,  $f \mapsto \phi \circ f$  is continuous in the  $C^\infty$ -topology.

**Proposition 1.5.** Let  $X, Y, Z$  be smooth manifolds. Then  $C^\infty(X, Y) \times C^\infty(X, Z) \rightarrow C^\infty(X, Y \times Z)$ ,  $(f, g) \mapsto f \times g$  is a homeomorphism in the  $C^\infty$ -topology.

**Appendix.** About existence of proper function on manifolds (from GTM218).

“ If  $M$  is a topological space, an exhaustion function for  $M$  is a continuous function  $f : M \rightarrow \mathbb{R}$  with the property that the set  $f^{-1}((-\infty, c])$  (called a sublevel set of  $f$ ) is compact for each  $c \in \mathbb{R}$ . The name comes from the fact that as  $n$  ranges over the positive integers, the sublevel sets  $f^{-1}((-\infty, n])$  form an exhaustion of  $M$  by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \mathbb{B}^n \rightarrow \mathbb{R}$  given by

$$f(x) = |x|^2, \quad g(x) = \frac{1}{1 - |x|^2}$$

are smooth exhaustion functions. Of course, if  $M$  is compact, any continuous real-valued function on  $M$  is an exhaustion function, so such functions are interesting only for noncompact manifolds.

**Proposition 2.28** (Existence of Smooth Exhaustion Functions). Every smooth manifold with or without boundary admits a smooth positive exhaustion function.

*Proof.* Let  $M$  be a smooth manifold with or without boundary, let  $\{V_j\}_{j=1}^\infty$  be any countable open cover of  $M$  by precompact open subsets, and let  $\{\psi_j\}$  be a smooth partition of unity subordinate to this cover. Define  $f \in C^\infty(M)$  by

$$f(p) = \sum_{j=1}^{\infty} j \psi_j(p).$$

Then  $f$  is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because  $f(p) \geq \sum_j j \psi_j(p) = 1$ . To see that  $f$  is an exhaustion function, let  $c \in \mathbb{R}$  be arbitrary, and choose a positive integer  $N > c$ . If  $p \notin \bigcup_{j=1}^N \bar{V}_j$ , then  $\psi_j(p) = 0$  for  $1 \leq j \leq N$ , so

$$f(p) = \sum_{j=N+1}^{\infty} j \psi_j(p) \geq \sum_{j=N+1}^{\infty} N \psi_j(p) = N \sum_{j=1}^{\infty} \psi_j(p) = N > c.$$

Equivalently, if  $f(p) \leq c$ , then  $p \in \bigcup_{j=1}^N \bar{V}_j$ . Thus  $f^{-1}((-\infty, c])$  is a closed subset of the compact set  $\bigcup_{j=1}^N \bar{V}_j$  and is therefore compact."

### 1.3 Transversality Theorem

**Definition 1.9.** Let  $X, Y$  be manifolds,  $f \in C^\infty(X, Y)$ ,  $W \subseteq Y$  submanifold. We say that  $f$  is transverse to  $W$  (write  $f \pitchfork W$ ) if

$$df_x(T_x X) + T_{f(x)} W = T_{f(x)} Y, \quad \forall x \in f^{-1}(W)$$

**Note.**  $X_1, X_2 \subseteq Y$  submanifolds,  $X_1 \pitchfork X_2 \Leftrightarrow T_x X_1 + T_x X_2 = T_x Y$  for  $\forall x \in X_1 \cap X_2$ .

It's just inclusion of one submanifold transverse to another submanifold.

**Proposition 1.6.** Let  $X, Y$  be manifolds,  $f \in C^\infty(X, Y)$ ,  $W \subseteq Y$  submanifold such that  $\dim X + \dim W < \dim Y$ . Then  $f \pitchfork W \Leftrightarrow f(X) \cap W = \emptyset$ .

*Proof.* The proof is easy. □

**Theorem 1.2.** Let  $X, Y$  be manifolds,  $f \in C^\infty(X, Y)$ ,  $W \subseteq Y$  submanifold such that  $f \pitchfork W$ . If  $f^{-1}(W) \neq \emptyset$ , then  $f^{-1}(W)$  is a submanifold of  $X$  of codim  $f^{-1}(W) = \text{codim } W$ . In particular, if  $\dim X = \text{codim } W$ , then  $f^{-1}(W)$  consists of isolated points.

*Proof.* Let  $p \in f^{-1}(W)$ ,  $n = \dim X, m = \dim Y, k = \dim W$ . Let  $\varphi : U \rightarrow \mathbb{R}^m$  be a chart around  $f(p)$  such that  $\varphi(U \cap W) \subseteq \mathbb{R}^k \times \{0\}$ . Let  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^{m-k}$  be the orthogonal projection along  $\mathbb{R}^k \times \{0\}$ ,  $\phi = \pi \circ \varphi$ . Then  $\phi : U \rightarrow \mathbb{R}^{m-k}$  is a submersion and  $\phi^{-1}(0) = U \cap W$ .

**Claim.**  $f \pitchfork W$  at  $p \Leftrightarrow p$  is a regular point of  $\phi \circ f$ .

Since  $\phi^{-1}(0) = U \cap W$ ,  $\ker d\phi_{f(p)} = T_{f(p)}W$ . Transversality assumption gives that  $df_p(T_pX) + T_{f(p)}W = T_{f(p)}Y = T_{f(p)}U$ , which implies that  $d(\phi \circ f)_p(T_pX) = d\phi_{f(p)}T_{f(p)}U$ . And the converse is easy to proof.

Now  $f \pitchfork W$  on  $U \Leftrightarrow 0$  is a regular value of  $\phi \circ f : f^{-1}(U) \rightarrow \mathbb{R}^{m-k}$ . By the implicit function theorem,  $(\phi \circ f)^{-1}(0) = f^{-1}(U \cap W)$  is a submanifold of  $f^{-1}(U) \subseteq X$  open set of codimension  $m - k$ . So  $f^{-1}(W)$  is a submanifold of  $X$  of codimension  $m - k$ .  $\square$

**Proposition 1.7.** Let  $X, Y$  be manifolds,  $W \subseteq Y$  submanifold which is a closed subset. Then  $T_W := \{f \in C^\infty(X, Y) \mid f \pitchfork W\}$  is open in the  $C^\infty$ -topology.

*Proof.* We show that  $T_W$  is open in the  $C^1$ -topology. Let

$$U = \{\sigma = j^1 f(x) \in J^1(X, Y) \mid f(x) \notin W \text{ or } df_x(T_x X) + T_{f(x)}W = T_{f(x)}Y\}$$

It's easy to see that  $T_W = M(U) = \{f \in C^\infty(X, Y) \mid j^1 f(X) \subseteq U\}$

**Claim.**  $U$  is open.

We will show that  $V = J^1(X, Y) \setminus U$  is closed. To prove that, take  $\{\sigma_n\} \subseteq V$  such that  $\sigma_n \rightarrow \sigma \in J^1(X, Y)$ , we need to show that  $\sigma \in V$ . Consider continuous map  $\beta : J^1(X, Y) \rightarrow Y$ , then  $\beta(\sigma_n) \rightarrow \beta(\sigma)$ . Since  $\beta(\sigma_n) \in W$  and  $W$  is closed, we have  $\beta(\sigma) \in W$ , which mean that  $\sigma = j^1 f(x), f(x) \in W$ .

Now choose charts around  $x$  and  $f(x)$ ,  $\varphi : \tilde{U} \rightarrow \mathbb{R}^n$ ,  $\psi : \tilde{V} \rightarrow \mathbb{R}^m$ ,  $\psi(\tilde{V} \cap W) = \mathbb{R}^k \times \{0\}$ ,  $\varphi(x) = 0, \psi(f(x)) = 0$ .  $f \pitchfork W$  at  $x \Leftrightarrow \psi \circ f \circ \varphi^{-1} \pitchfork \mathbb{R}^k \times \{0\}$  at  $0 \Leftrightarrow 0$  is a regular value of  $\pi \circ \psi \circ f \circ \varphi^{-1}$  where  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^{m-k}$  orthogonal projection  $\Leftrightarrow \pi \circ d(\psi \circ f \circ \varphi^{-1})_0$  has rank  $m - k$ .

Let  $F = \{A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}) \mid \text{rank } A < m - k\}$ . In a neighborhood  $\mathcal{N}$  of  $\sigma$ , fixing  $\varphi, \psi$  we obtain a map

$$\eta : \mathcal{N} \subseteq J^1(X, Y) \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}), \quad j^1 g(x) \mapsto \pi \circ d(\psi \circ f \circ \varphi^{-1})_{\varphi(x)}$$

$V \cap \mathcal{N} = \eta^{-1}(F)$ ,  $\eta$  is continuous.

**Exercise.**  $F$  is closed.  $\square$

**Remark.** The condition that  $W$  is closed is necessary.

**Lemma 1.3.** Let  $X, Y, B$  manifolds,  $W \subseteq Y$  submanifold, let  $j : B \rightarrow C^\infty(X, Y)$  map (not necessary continuous).

$$\Phi : X \times B \rightarrow Y, \quad \Phi(x, b) = j(b)(x)$$

Suppose  $\Phi$  is smooth and  $\Phi \pitchfork W$ . Then  $\{b \in B \mid j(b) \pitchfork W\}$  is dense in  $B$ .

*Proof.* Let  $W_\Phi = \Phi^{-1}(W) \subseteq X \times B$  be the submanifold,  $pr : X \times B \rightarrow B$  the projection and  $\pi = pr|_{W_\Phi}$ .

**Claim.**  $b$  is a regular value of  $\pi \implies j(b) \pitchfork W$ .

Suppose  $b$  is a regular value of  $\pi$ .

(1)  $b \notin \text{im } \pi$ , then  $\nexists x \in X$  such that  $\Phi(x, b) \in W$ , so  $j(b)(X) \cap W = \emptyset$ , which means  $j(b) \nmid W$ .

(2) If  $\dim W_\Phi < \dim B$ , then  $b$  is a regular value of  $\pi$ , so  $b \notin \text{im } \pi$ , therefore by (1) we have  $j(b) \nmid W$ .

(3) If  $\dim W_\Phi \geq \dim B$ . Let  $b$  be a regular value of  $\pi$  and  $x \in X$ . If  $(x, b) \notin W_\Phi$ , then  $j(b)(x) \notin W$ , so  $j(b) \nmid W$  at  $x$ . If  $(x, b) \in W_\Phi$ , then  $\pi(T_{(x,b)}W_\Phi) = T_bB$ , which gives  $T_{(x,b)}(X \times B) = T_{(x,b)}W_\Phi + T_{(x,b)}(X \times \{b\})$ , so  $T_{j(b)(x)}Y = T_{j(b)(x)}W + (dj(b))_x T_xX$ , so  $j(b) \mid W$  at  $x$ .  $\square$

**Corollary 1.1.** Let  $G : X \times B \rightarrow Y$  smooth,  $\Phi(x, b) = j^k G_b(x)$ . If  $\Phi \mid W$ , where  $W \subseteq J^k(X, Y)$  submanifold. Then  $\{b \in B \mid j^k G_b \mid W\}$  is dense in  $B$ .

**Theorem 1.3. (Thom Transversality Theorem.)**

Let  $X, Y$  manifolds,  $W \subseteq J^k(X, Y)$  submanifold, let

$$T_W = \left\{ f \in C^\infty(X, Y) \mid j^k f \mid W \right\}$$

Then  $T_W$  is a residual subset of  $C^\infty(X, Y)$  (residual subset means countable intersection of open and dense sets). Moreover if  $W$  is closed, then  $T_W$  is open.

*Proof.* For each  $\sigma \in W$ , let  $W_\sigma \subseteq W$ ,  $U_\sigma \subseteq X$ ,  $V_\sigma \subseteq Y$  open neighborhood of  $\sigma$ ,  $\alpha(\sigma)$ ,  $\beta(\sigma)$  respectively and charts  $\varphi_\sigma : U_\sigma \rightarrow \mathbb{R}^n$ ,  $\psi_\sigma : V_\sigma \rightarrow \mathbb{R}^m$  such that:

- (a)  $\overline{W_\sigma} \subseteq W$  and is compact.
- (b)  $\overline{U_\sigma}$  is compact.
- (c)  $\alpha(\overline{W_\sigma}) \subseteq U_\sigma$  and  $\beta(\overline{W_\sigma}) \subseteq V_\sigma$ .
- (d)  $\psi_\sigma(V_\sigma) = \mathbb{R}^m$ .

We say that  $g \mid W$  on  $A$  if  $g \mid W$  for  $\forall x \in g^{-1}(A)$ . Let

$$T_\sigma = \left\{ f \in C^\infty(X, Y) \mid j^k f \mid W \text{ on } \overline{W_\sigma} \right\}$$

$T_W = \bigcap_{\sigma \in W} T_\sigma$ . Since  $W$  is 2-countable, there exists a countable covering  $\{W_{\sigma_i}\}_{i=1}^\infty$  of  $W$ . So  $T_W = \bigcap_{i=1}^\infty T_{\sigma_i}$ .

**Claim.**  $T_\sigma$  is open and dense.

(1)  $T_\sigma$  is open. Let  $\tilde{T}_\sigma = \{g \in C^\infty(X, J^k(X, Y)) \mid g \mid W \text{ on } \overline{W_j}\}$ . By previous proposition we have  $\tilde{T}_\sigma$  is open, then  $T_\sigma = (j^k)^{-1}(\tilde{T}_\sigma)$  is open.

(2)  $T_\sigma$  is dense. Let  $f \in C^\infty(X, Y)$ , we will construct a sequence  $\{g_n\} \subseteq C^\infty(X, Y)$  such that  $g_n \in T_\sigma$  and  $g_n \xrightarrow{C^\infty} f$ . The idea is to define  $\Phi : X \times B \rightarrow J^k(X, Y)$ ,  $\Phi(x, b) = j^k g_b(x)$ , where  $g_b(x)$  is a polynomial perturbation of  $f$ , such that  $\Phi \mid W$ .

Fix smooth functions  $\rho_1 : \mathbb{R}^n \rightarrow [0, 1]$ ,  $\rho_2 : \mathbb{R}^m \rightarrow [0, 1]$  such that  $\rho_1 \equiv 1$  in a neighborhood of  $\varphi(\alpha(\overline{W_\sigma}))$ ,  $\text{supp } \rho_1 \subseteq \varphi(U_\sigma)$ ;  $\rho_2 \equiv 1$  in a neighborhood of  $\psi(\beta(\overline{W_\sigma}))$ ,  $\text{supp } \rho_2$  is compact. Let  $B = \{\text{polynomial maps } \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ of degree } \leq k\}$ .

For  $b \in B$ , let

$$g_b(x) = \begin{cases} \psi^{-1}(\psi(f(x)) + b(\varphi(x))\rho_1(\varphi(x))\rho_2(\psi(f(x)))) & \text{if } x \in U_\sigma, f(x) \in V_\sigma \\ f(x) & \text{if } x \notin U_\sigma \text{ or } f(x) \notin V_\sigma \end{cases}$$

$G : X \times B \rightarrow Y$ ,  $G(x, b) = g_b(x)$ .

**Exercise.**  $G$  is smooth.

Let  $\Phi : X \times B \rightarrow J^k(X, Y)$ ,  $\Phi(x, b) = j^k g_b(x)$ , so  $\Phi$  is smooth.

**Claim.**  $\exists \tilde{B} \subseteq B$  open neighborhood of  $0 \in B$  such that  $\Phi|_{X \times \tilde{B}} \pitchfork W$  on  $\overline{W}_\sigma$ .

Assuming the claim, apply the previous lemma,  $\exists \{b_n\} \subseteq \tilde{B}$  such that  $b_n \rightarrow 0$  and  $j^k g_{b_n} \pitchfork (W \cap \overline{W}_\sigma)$ , this also implies  $g_{b_n} \xrightarrow{C^\infty} f$  and  $j^k g_{b_n} \pitchfork W$  on  $\overline{W}_\sigma$ . So  $T_\sigma$  is dense.

Proof of the claim: Let  $\epsilon = \frac{1}{2}d(\psi(\beta(\overline{W}_j)), \rho_2^{-1}([0, 1])) > 0$ , define

$$\tilde{B} = \{b \in B \mid \|b(x)\| < \epsilon, \forall x \in \text{supp } \rho_1\}$$

We fix  $b \in \tilde{B}$  such that  $\Phi(x, b) \in \overline{W}_\sigma$ . We will show that  $\Phi$  is a local diffeomorphism near  $(x, b)$ . Since  $\Phi(x, b) \in \overline{W}_\sigma$ ,  $x \in \alpha(\overline{W}_\sigma)$ ,  $g_b(x) \in \beta(\overline{W}_\sigma)$ .  $\psi(g_b(x)) = \psi(f(x)) + b(\varphi(x))\rho_1(\varphi(x))\rho_2(\psi(f(x))) = \psi(f(x)) + b(\varphi(x))$ . Because  $\|b(\varphi(x))\| < \epsilon$ ,  $\forall x \in \text{supp } \rho_1$ , then  $\rho_2(\psi(g_b(x))) = 1$ . So  $\psi \circ g_b(x) = \psi(f(x)) + b(\varphi(x))$  in a neighborhood of  $(x, b)$ .  $\sigma'$  is sufficiently close to  $\sigma$ , so we can find a unique polynomial  $b'$  so that  $\sigma' = j^k(\psi^{-1}(f(\varphi(\alpha(\sigma'))))) + b'(\varphi(\alpha(\sigma')))$ . So we have constructed a local inverse for every  $(x, b) \in \Phi^{-1}(\overline{W}_\sigma)$ , then  $\Phi \pitchfork W$  on  $\overline{W}_\sigma$ .  $\square$

**Corollary 1.2.** Let  $X, Y$  manifolds,  $f \in C^\infty(X, Y)$ ,  $W \subseteq J^k(X, Y)$  submanifold such that  $\overline{\alpha(W)} \subseteq U$  open set. Then  $\exists \{g_n\} \subseteq C^\infty(X, Y)$  such that  $j^k g_n \pitchfork W$ ,  $g_n \rightarrow f$  and  $g_n = f$  outside  $U$ .

*Proof.* The same as the theorem above but we choose  $U_\sigma \subseteq U$  for  $\forall \sigma \in W$ .  $\square$

**Corollary 1.3. (Elementary Transversality Theorem.)**

Let  $X, Y$  manifolds,  $W \subseteq Y$  submanifold.

(a)  $T_W = \{f \in C^\infty(X, Y) \mid f \pitchfork W\}$  is dense in  $C^\infty(X, Y)$ . Moreover if  $W$  is closed, then  $T_W$  is open.

(b) Let  $U_1, U_2 \subseteq X$  open sets such that  $\overline{U_1} \subseteq U_2$ , let  $f \in C^\infty(X, Y)$ ,  $V \subseteq C^\infty(X, Y)$  near  $f$  and open. Then there is  $\{g_n\} \in C^\infty(X, Y)$  such that  $g_n \xrightarrow{C^\infty} f$ ,  $g_n = f$  in  $U_1$  and  $g_n \pitchfork W$  outside  $U_2$ .

**Definition 1.10. (Multijets.)** Let  $X, Y$  manifolds. For  $s \in \mathbb{N}$ , define

$$X^{(s)} = \{(x_1, \dots, x_s) \in X^s \mid x_i \neq x_j, i \neq j\}$$

$$\alpha^s = \alpha \times \dots \times \alpha : J^k(X, Y)^s \rightarrow X^s, \text{ let } J_s^k(X, Y) = (\alpha^s)^{-1}(X^{(s)}) \subseteq J^k(X, Y)^s$$

open, so  $J_s^k(X, Y)$  is a manifold.  $f \in C^\infty(X, Y)$  gives rise to

$$j_s^k f : X^{(s)} \rightarrow J_s^k(X, Y), \quad j_s^k f(x_1, \dots, x_s) = (j^k f(x_1), \dots, j^k f(x_s))$$

**Theorem 1.4. (Thom Transversality for multijets.)**

Let  $X, Y$  manifolds,  $W \subseteq J_s^k(X, Y)$  submanifold. Let

$$T_W = \left\{ f \in C^\infty(X, Y) \mid j_s^k f \pitchfork W \right\}$$

Then  $T_W$  is residual. Moreover, if  $W$  is compact, then  $T_W$  is open.

### 1.4 Whitney Immersions and Embeddings

Let  $X^n, Y^m$  manifolds,  $\sigma = j^1 f(x) \in J^1(X, Y)$ . Then  $df_x : T_x X \rightarrow T_{f(x)} Y$  depends only on  $\sigma$ . Define  $\text{rank}(\sigma) = \text{rank}(df_x)$  and  $\text{corank}(\sigma) = \min(m, n) - \text{rank}(\sigma)$ . Let  $S_r = \{ \sigma \in J^1(X, Y) \mid \text{corank}(\sigma) = r \}$ .

**Lemma 1.4.**  $f$  is an immersion ( $n \leq m$ ) or submersion ( $n \geq m$ )  $\Leftrightarrow j^1 f(X) \cap \bigcup_{r \geq 1} S_r = \emptyset$ .

*Proof.*  $f$  is not an immersion/submersion  $\Leftrightarrow \exists x \in X$  such that  $\text{rank}(df_x) \leq \min(m, n) - 1$   
 $\Leftrightarrow \exists x \in X$  such that  $\text{corank}(j^1 f(x)) \geq 1 \Leftrightarrow j^1 f(X) \cap S_r \neq \emptyset$  for some  $r \geq 1$ .  $\square$

**Proposition 1.8.**  $S_r$  is a submanifold of codimension  $(n - q + r)(m - q + r)$ , where  $q = \min(n, m)$ .

*Proof.*  $S_r$  is a bundle over  $X \times Y$  with fiber  $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) = \{ A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \mid \text{corank}(A) = r \}$ .

**Claim.**  $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$  is a submanifold of codimension  $(n - q + r)(m - q + r)$ . So  $S_r \subseteq J^1(X, Y)$  is a subbundle over  $X \times Y$ .

Proof of the claim: Let  $M \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m)$ , let  $k = q - r$ . We can choose basis of  $\mathbb{R}^n$  and  $\mathbb{R}^m$  so that

$$[M] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A \text{ is an invertible } k \times k \text{ matrix}$$

So in a neighborhood  $U$  of  $M$ , every other  $M'$  will be represented as

$$[M'] = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \quad A' \text{ is an invertible } k \times k \text{ matrix}$$

$$\text{So rank } [M'] = \text{rank} \begin{bmatrix} I^k & 0 \\ -C'(A')^{-1} & I_{m-k} \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \text{rank} \begin{bmatrix} A' & B' \\ 0 & D' - C'(A')^{-1}B' \end{bmatrix}$$

Then  $\text{rank}[M'] = k \Leftrightarrow D' - C'(A')^{-1}B' = 0$ .  $M' \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U \Leftrightarrow D' - C'(A')^{-1}B' = 0$ . Let

$$\varphi : U \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \rightarrow \mathcal{L}(\mathbb{R}^{n-k}, \mathbb{R}^{m-k}), \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \mapsto D' - C'(A')^{-1}B'$$

$\varphi$  is a submersion, so  $\varphi^{-1}(0) = \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U$  is a submanifold of codimension  $(n - q + r)(m - q + r)$ . □

**Obs.**  $\mathcal{L}^0(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$  is open. So  $S_0 \subseteq J^1(X, Y)$  open submanifold, then  $\bigcup_{r \geq 1} S_r$  is closed.

**Lemma 1.5.** Suppose  $n \leq m$ . Then

$$\text{Imm}(X, Y) = \{f : X \rightarrow Y \text{ smooth immersion}\}$$

is an open subset of  $C^\infty(X, Y)$ .

*Proof.*  $\text{Imm}(X, Y) = M(S_0)$ . □

**Theorem 1.5. (Whitney Immersion.)** Let  $X^n, Y^m$  be manifolds such that  $m \geq 2n$ . Then  $\text{Imm}(X, Y)$  is open and dense subset of  $C^\infty(X, Y)$ .

*Proof.*  $\min(n, m) = n$ , so for  $r \geq 1$ ,  $\text{codim } S_r = (n - q + r)(m - q + r) = r(n + r) \geq n + 1$ . So  $j^1 f \pitchfork S_r \Leftrightarrow j^1 f(X) \cap S_r = \emptyset$  since  $\dim X = n < n + 1 \leq \text{codim } S_r$ .

$$\text{Imm}(X, Y) = \left\{ f \in C^\infty(X, Y) \mid j^1 f(X) \cap \bigcup_{r \geq 1} S_r = \emptyset \right\} = \left\{ f \in C^\infty(X, Y) \mid j^1 f \pitchfork \bigcup_{r \geq 1} S_r \right\}$$

By the Thom transversality theorem,  $\text{Imm}(X, Y)$  is dense and open. □

**Theorem 1.6. (Whitney Injective Immersion Theorem.)** Let  $X^n, Y^m$  be manifolds such that  $m \geq 2n + 1$ . Then the set of injective immersions is residual.

*Proof.*  $\text{Imm}(X, Y)$  is open and dense, we need to show

$$\text{Inj}(X, Y) = \{f \in C^\infty(X, Y) \mid f \text{ is injective}\} \text{ is residual}$$

Recall  $J_2^0(X, Y) = X^{(2)} \times Y^2 = \{(x_1, x_2, y_1, y_2) \in X^2 \times Y^2 \mid x_1 \neq x_2\}$ , let

$$W = X^{(2)} \times \Delta Y = \{(x_1, x_2, y, y) \mid x_1 \neq x_2\} \subseteq J_2^0(X, Y)$$

$f$  is injective iff  $j_2^0 f(X^{(2)}) \cap W = \emptyset$ . Codimension of  $W$  is dimension of  $Y$ , so  $f$  is injective iff  $j_2^0 f \pitchfork W$  from the proof of previous theorem. By the Thom transversality theorem for multijets, we have  $\text{Inj}(X, Y)$  is residual. □

**Lemma 1.6.** Let  $X$  manifold. Then

$$\text{Prop}(X, \mathbb{R}^m) = \{f \in C^\infty(X, \mathbb{R}^m) \mid f \text{ is proper}\}$$

is non-empty and open.

*Proof.* Recall that there exists a proper map  $X \rightarrow \mathbb{R}$ , compose this map with a linear injection  $\mathbb{R} \rightarrow \mathbb{R}^m$  to obtain a proper map.

Now let  $f \in \text{Prop}(X, \mathbb{R}^m)$ . For  $x \in X$ , define  $V_x = \{y \in \mathbb{R}^m \mid d(y, f(x)) < 1\}$ . So  $V_x \subseteq \mathbb{R}^m$  open. Let  $V = \bigcup_{x \in X} \{x\} \times V_x$ , then  $V \subseteq X \times \mathbb{R}^m = J^0(X, \mathbb{R}^m)$  is open.  $f \in M(V)$  because  $j^0 f(x) = (x, f(x))$ ,  $d(f(x), f(x)) = 0$ , so  $f(x) \in V_x$ .

**Claim.**  $M(V) \subseteq \text{Prop}(X, \mathbb{R}^m)$ .

If  $g \in M(V)$ , then  $d(g(x), f(x)) < 1 \forall x \in X$ , so  $g^{-1}(\overline{B}_r(0)) \subseteq f^{-1}(\overline{B}_{r+1}(0))$ . Since  $f$  is proper,  $f^{-1}(\overline{B}_{r+1}(0))$  is compact, therefore  $g^{-1}(\overline{B}_r(0))$  is compact, hence  $g$  is proper.  $\square$

**Corollary 1.4. (Whitney Embedding Theorem.)** Let  $X^n$  manifold. Then there exists  $X \hookrightarrow \mathbb{R}^{2n+1}$ .

*Proof.*  $\text{Inj}(X, \mathbb{R}^{2n+1}) \cap \text{Imm}(X, \mathbb{R}^{2n+1}) \cap \text{Prop}(X, \mathbb{R}^{2n+1}) \neq \emptyset$ .  $\square$

## 1.5 Morse Functions

**Definition 1.11.** Let  $f : X \rightarrow \mathbb{R}$  smooth and  $p \in \text{Crit}(f)$  ( $df_p = 0$ ). Define the Hessian of  $f$  to be the bilinear map:

$$D^2 f_p : T_p X \times T_p X \rightarrow \mathbb{R}, \quad D^2 f_p \left( \left. \frac{\partial}{\partial x_i} \right|_p, \left. \frac{\partial}{\partial x_j} \right|_p \right) = \left. \frac{\partial^2}{\partial x_i \partial x_j} \right|_{\varphi(p)} (f \circ \varphi^{-1})$$

where  $\varphi = (x_1, \dots, x_n)$  is a chart around  $p$ . We say that  $p$  is non-degenerate if  $D^2 f_p$  is non-degenerate.

**Exercise.**  $D^2 f_p$  doesn't depend on the choice of a chart whenever  $p \in \text{Crit}(f)$ .

Let  $f : X \rightarrow \mathbb{R}$  smooth,  $df : TX \rightarrow \mathbb{R}$ ,  $(p, v) \in TX$ , we have  $d_{(p,v)} df : T_{(p,v)} TX \rightarrow \mathbb{R}$ ,  $T_{(p,v)} TX$  is isomorphic to  $T_p X \oplus T_p X$  but it's not natural.

**Proposition 1.9.**  $p \in \text{Crit}(f)$  is non-degenerate  $\Leftrightarrow j^1 f \pitchfork S_1$  at  $p$ .

*Proof.* This is a local question, we may assume  $X = U \subseteq \mathbb{R}^n$ ,  $J^1(X, \mathbb{R}) = U \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ ,  $\pi : J^1(X, \mathbb{R}) \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$  submersion,  $\pi^{-1}(0) = S_1 = \{j^1 f(x) \mid df_x = 0\}$ .

**Claim.**  $j^1 f \pitchfork S_1$  at  $p \Leftrightarrow \pi \circ j^1 f$  is a submersion at  $p$ .

Now  $\pi \circ j^1 f : U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ ,  $x \mapsto \left( \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$  is a submersion at  $p$  iff  $\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_p$  is invertible iff  $p$  is non-degenerate.  $\square$



**Definition 1.12.**  $f \in C^\infty(X, \mathbb{R})$  is Morse if every  $p \in \text{Crit}(f)$  is non-degenerate.

**Corollary 1.5.**  $f \in C^\infty(X, \mathbb{R})$  is Morse  $\Leftrightarrow j^1 f \pitchfork S_1$ .

**Theorem 1.7.** Let  $X$  manifold. Then  $\{f \in C^\infty(X, \mathbb{R}) \mid f \text{ is Morse}\}$  is open and dense in  $C^\infty(X, \mathbb{R})$ .

*Proof.* Since  $S_1 = J^1(X, \mathbb{R}) \setminus S_0$  is closed, by the corollary and Thom transversality theorem we complete the proof.  $\square$

## 2 Intersection Theory

### 2.1 Manifolds with boundary and orientation

**Definition 2.1.** A topological manifold with boundary is a 2-countable Hausdorff topological space such that every point  $p \in X$  has a neighborhood which is homeomorphic to an open set in  $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ .

**Lemma 2.1.** Let  $X$  be a topological manifold with boundary,  $p \in X$ ,  $\varphi, \psi : U \rightarrow \mathbb{H}^n$  charts around  $p$ . Suppose  $pr_1 \circ \varphi(p) = 0$ , then  $pr_1 \circ \psi(p) = 0$ , where  $pr_1$  is the canonical projection of  $\mathbb{H}^n$  to the first coordinate.

*Proof.*  $\psi \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(U)$  is homeomorphic, then  $\psi \circ \varphi^{-1} : \varphi(U) \setminus \varphi(p) \rightarrow \psi(U) \setminus \psi(p)$  is also homeomorphic. Since  $pr_1 \circ \varphi(p) = 0$ ,  $\varphi(U) \setminus \varphi(p)$  is contractible. If  $pr_1 \circ \psi(p) \neq 0$ , then  $\psi(U) \setminus \psi(p) \simeq S^{n-1}$ ,  $S^{n-1}$  and contractible space have different homology group, so they can't be homeomorphic.  $\square$

**Definition 2.2.** Let  $X$  be a topological manifold with boundary. Then

$$\partial X = \{p \in X \mid \exists \varphi : U \rightarrow \mathbb{H}^n \text{ chart around } p \text{ s.t. } pr_1 \circ \varphi(p) = 0\}, \text{ int}(X) = X \setminus \partial X$$

[Obs.]  $\text{int}(X)$  and  $\partial X$  are topological manifold without boundary of dimension  $n$  and  $n - 1$  respectively.

**Definition 2.3.** A map  $f : \mathbb{H}^n \supseteq U \rightarrow \mathbb{H}^n$  is smooth if it admits a smooth extension to  $\tilde{f} : \tilde{U} \rightarrow \mathbb{R}^n$ , where  $U \subseteq \tilde{U}$  in an open set in  $\mathbb{R}^n$ .

**Definition 2.4.** We say that two charts  $\varphi : U \rightarrow \mathbb{H}^n$ ,  $\psi : V \rightarrow \mathbb{H}^n$  are compatible if  $\psi \circ \varphi^{-1}$  is smooth. An atlas is a collection of charts whose domain cover  $X$ .

**Definition 2.5.** A (smooth) manifold with boundary is a topological manifold with boundary endowed with a maximal (smooth) atlas.

Smooth manifold with boundary  $X$  induces smooth structure (without boundary) on  $\text{int}(X)$  and  $\partial X$ .

**Proposition 2.1.** Let  $f \in C^\infty(X, \mathbb{R})$ ,  $a \in \mathbb{R}$  regular value of  $f$ . Then  $f^{-1}([a, +\infty))$  and  $f^{-1}((-\infty, a])$  are manifolds with boundary.

*Proof.*  $(a, +\infty) \subseteq \mathbb{R}$  is open then  $f^{-1}((a, +\infty))$  is a manifold without boundary. Let  $p \in f^{-1}(a)$ , by the implicit function theorem, there exists a chart  $\varphi : U \rightarrow \mathbb{R}^n$  such that

$\varphi(p) = 0$  and  $f \circ \varphi^{-1}(x_1, \dots, x_n) = a + x_1$ . So we obtain a chart  $\varphi|_{f^{-1}([a, +\infty)) \cap U} : \tilde{U} \rightarrow \mathbb{H}^n$ . So  $f^{-1}([a, +\infty))$  is a manifold with boundary.  $\square$

**Definition 2.6.** Let  $X$  be a manifold with boundary,  $p \in X$ , a curve centered at  $p$  is a smooth map  $\gamma : [0, \epsilon) \rightarrow X$  or  $\gamma : (-\epsilon, 0] \rightarrow X$  such that  $\gamma(0) = p$ .  $T_p X$  is the equivalent classes of curves centered at  $p$ .

If  $x \in \text{int}(X)$ , then  $T_x(\text{int}(X)) = T_x X$ ; If  $x \in \partial X$ , then  $T_x X$  is still a  $n$ -dimensional vector space. Moreover, we have a canonical inclusion  $T_x(\partial X) \subseteq T_x X$ .

**Proposition 2.2.** Let  $X, Y$  be manifolds with boundary,  $y \in \text{int}(Y)$  regular value of  $f : X \rightarrow Y$  and  $\partial f := f|_{\partial X} : \partial X \rightarrow Y$ . Then  $f^{-1}(y)$  is a manifold with boundary and  $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X = (\partial f)^{-1}(y)$ .

**Example 2.1.**  $f : \mathbb{H}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto x^2 + y^2$ , then  $f^{-1}(1) = S^1 \cap \mathbb{H}^2$ .

**Exercise 2.1.** Prove the proposition.

**Theorem 2.1.** Let  $X, Y$  manifolds with boundary,  $W \subseteq Y$  submanifold,  $\partial W = \partial Y = \emptyset$ . Suppose  $f \pitchfork W$  and  $\partial f \pitchfork W$ , then  $f^{-1}(W)$  is a manifold with boundary,  $\partial(f^{-1}(W)) = f^{-1}(W) \cap \partial X$ .

*Proof.*  $f|_{\text{int}(X)} \pitchfork W$  is a manifold without boundary. Let  $x \in f^{-1}(W) \cap \partial X$ ,  $\pi : V \subseteq Y \rightarrow \mathbb{R}^{m-k}$  be a submersion such that  $\pi^{-1}(0) = W \cap V$ . As in the case without boundary:  $f \pitchfork W$  at  $x$  iff  $x$  is a regular point of  $\pi \circ f$ ,  $\partial f \pitchfork W$  at  $x$  iff  $x$  is a regular point of  $\pi \circ \partial f$ . The result follows from the proposition above.  $\square$

**[Obs.]** It's easy to see that  $\partial f \pitchfork W$  at  $x \implies f \pitchfork W$  at  $x$ .

**Theorem 2.2. (Sard's Theorem.)** Let  $X$  manifold with boundary,  $Y$  manifold,  $f : X \rightarrow Y$ . Then

$$\{y \in Y \mid y \text{ is a critical value of } f \text{ or } \partial f\}$$

has measure zero.

*Proof.*  $\text{Crit}(f) \cup \text{Crit}(\partial f) = \text{Crit}(f|_{\text{int}(X)}) \cup \text{Crit}(\partial f)$ .  $\square$

**Theorem 2.3. (Thom Transversality Theorem.)**  $X$  manifold with boundary,

$Y$  manifold,  $W \subseteq J^k(X, Y)$  submanifold,  $\partial W \subseteq \alpha^{-1}(\partial X)$ . Then

$$\left\{ f \in C^\infty(X, Y) \mid j^k f \pitchfork W \text{ and } j^k(\partial f) \pitchfork W \right\}$$

is residual.

**Corollary 2.1. (Elementary Transversality Theorem.)**

(1)  $X$  manifold with boundary,  $Y$  manifold and  $W \subseteq Y$  submanifold  $\partial W = \emptyset$ .

Then

$$\{f \in C^\infty(X, Y) \mid f \pitchfork W \text{ and } \partial f \pitchfork W\}$$

is residual.

(2)  $f \in C^\infty(X, Y)$ ,  $\partial f \pitchfork W$ . There exists  $\{g_n\} \subseteq C^\infty(X, Y)$  such that  $g_n \xrightarrow{C^\infty} f$ ,  $g_n \pitchfork W$  and  $g_n \equiv f$  in a neighborhood of  $\partial X$ .

**Definition 2.7.** Let  $V$  be a vector space. Define an equivalence relation on the set of bases of  $V$  as follows:

$$\{x_1, \dots, x_n\} \sim \{y_1, \dots, y_n\} \text{ if the linear map } T : V \rightarrow V, Tx_i = y_i \text{ has } \det T > 0$$

**Obs.** Given  $V$ , there are two equivalence classes.

**Definition 2.8.** An orientation of  $V$  is a choice of such an equivalence class.

**Definition 2.9.** Let  $X$  be a smooth manifold. An orientation on  $X$  is a choice of orientation on  $T_p X$  for each  $p \in X$  such that for each chart  $\varphi : U \rightarrow \mathbb{R}^n$ ,  $\varphi = (x_1, \dots, x_n)$ , either

$$\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\} \text{ or } \left\{ -\frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\} \text{ is oriented for } \forall p \in U$$

**Obs.** Not all manifold admits an orientation.

**Rmk.** A connected orientable manifold has exactly two orientations.  $\mathbb{R}^n$  has a natural orientation.

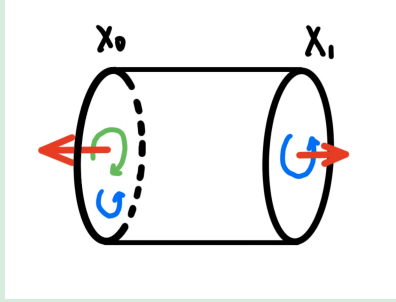
**Proposition 2.3.** Let  $X$  be an oriented manifold with boundary. Then  $\partial X$  has a natural orientation.

*Proof.* For  $x \in \partial X$ ,  $T_x(\partial X) \subseteq T_x X$ . There exists a 1-dimensional vector bundle  $N$  over  $\partial X$  such that  $N_x \oplus T_x(\partial X) = T_x X$  and a outward normal vector field  $n \in \Gamma(N)$  which doesn't vanishes. If  $\{v_1, \dots, v_{n-1}\}$  is a basis of  $T_x(\partial X)$ , then  $\{n_x, v_1, \dots, v_{n-1}\}$  is a basis of  $T_x X$ .  $\square$

**[Obs.]**  $X, Y$  oriented manifolds,  $\partial Y = \emptyset$ , then  $X \times Y$  inherits a natural orientation.

**Example 2.2.** Let  $X$  oriented manifold without boundary,  $I = [0, 1]$ , the  $I \times X$  is oriented manifold with boundary.

$\partial(I \times X) = (\partial I) \times X = \{0\} \times X \cup \{1\} \times X$ . Let  $X_1 = \{1\} \times X$ ,  $X_0 = \{0\} \times X$ , they have induced orientation since they are diffeomorphic to  $X$ , but this orientation may not compatible with the induced boundary orientation.



**Proposition 2.4.** Let  $X^n, Y^m$  manifolds with boundary,  $W^k \subseteq Y$  submanifold such that  $\partial W = \partial Y = \emptyset$ , let  $f \in C^\infty(X, Y)$  such that  $f \pitchfork W$  and  $\partial f \pitchfork W$ . Suppose  $X, Y, W$  oriented. Then  $f^{-1}(W)$  has natural orientation.

*Proof.* Let  $Q = f^{-1}(W)$ ,  $NQ$  be the normal bundle of  $Q$  (for every  $x \in Q$ ,  $N_x Q \oplus T_x Q = T_x X$ ).  $df_x(T_x Q) = T_{f(x)} W$ .

**[Claim.]**  $df_x|_{N_x Q}$  is injective.

$f \pitchfork W$ , so  $df_x(T_x X) + T_{f(x)} W = T_{f(x)} Y$ , then  $df_x(N_x Q) + T_{f(x)} W = T_{f(x)} Y$ ,  $\dim df_x(N_x Q) = \dim N_x Q$ , so  $df_x$  is injective.

Since  $T_{f(x)} W, T_{f(x)} Y$  are oriented, it induces an orientation on  $df_x(N_x Q)$  by  $df_x(N_x Q) \oplus T_{f(x)} W = T_{f(x)} Y$ , hence induces orientation on  $N_x Q$ . By  $N_x Q \oplus T_x Q = T_x X$  we have an orientation on  $T_x Q$ .  $\square$

**Corollary 2.2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $a \in \mathbb{R}^m$  regular value of  $f$ . Then  $f^{-1}(a)$  is orientable.

**Exercise 2.2.**  $f \pitchfork W$  and  $\partial f \pitchfork W$ .  $\partial f^{-1}(W) = (\partial f)^{-1}(W)$ . Let  $X, Y, W$  are oriented, then for natural orientation,  $[\partial f^{-1}(W)] = (-1)^{\text{codim} W} [(\partial f)^{-1} W]$ .

## 2.2 Intersection Number

**Theorem 2.4. (Classification of 1-Manifolds.)**

Let  $X$  compact and connected 1-manifold, then  $X$  is diffeomorphic to either  $[0, 1]$  or  $S^1$ .

Let  $X, Y, W$  be oriented manifolds without boundary such that  $X$  is compact and  $W \subseteq Y$  closed subset and  $\dim X + \dim W = \dim Y$ . Let  $f : X \rightarrow Y$ ,  $f \pitchfork W$ , then  $\dim f^{-1}(W) = 0$ , so  $f^{-1}(W)$  is a set of isolated points. By compactness and orientation assumption,  $f^{-1}(W)$  is a finite number of points with signs. Define intersection number 
$$I(f, W) = \sum_{p \in f^{-1}(W)} \text{sign}(p).$$

**Definition 2.10.** Let  $X, Y, W$  be oriented manifolds without boundary such that  $X$  is compact and  $W \subseteq Y$  closed subset and  $\dim X + \dim W = \dim Y$ . Let  $f : X \rightarrow Y$ ,  $f \pitchfork W$ . Define intersection number 
$$I(f, W) = \sum_{p \in f^{-1}(W)} \text{sign}(p).$$

**Recall.**  $df_p(T_p X) \oplus T_{f(p)} W = T_{f(p)} Y$ ,  $\text{sign}(p) = +1$  iff orientation match.

For now we always assume that  $X, Y, W$  be oriented manifolds without boundary such that  $X$  is compact and  $W \subseteq Y$  closed subset and  $\dim X + \dim W = \dim Y$ .

**Proposition 2.5.** Let  $f_0, f_1 \in C^\infty(X, Y)$  smoothly homotopic and transverse to  $W$ , then  $I(f_0, W) = I(f_1, W)$ .

*Proof.* Let  $Z = [0, 1] \times X$ ,  $F : [0, 1] \times X \rightarrow Y$  the smooth homotopy of  $f_0, f_1$ . Since  $\partial([0, 1] \times X) = \{1\} \times X \cup [-\{0\} \times X]$ , by the lemma below we have  $0 = I(\partial F, W) = I(f_1, W) - I(f_0, W)$ .  $\square$

**Remark 2.1.** Two smooth maps are smoothly homotopic iff they are homotopic.

**Lemma 2.2.** Suppose  $X = \partial Z$ , where  $Z$  compact oriented manifold with boundary,  $f : X \rightarrow Y$ ,  $f \pitchfork W$ . Suppose that  $f$  can be extend to  $F : Z \rightarrow Y$ . Then  $I(f, W) = 0$ .

*Proof.* Since  $f = F|_{\partial Z}$  and  $f \pitchfork W$ ,  $F \pitchfork W$  on  $\partial Z$ . We can perturb  $F$  so that  $F \pitchfork W$  in all  $Z$  and  $F|_{\partial Z} = f$ .  $F^{-1}(W)$  is an oriented manifold such that  $\partial F^{-1}(W) = \pm f^{-1}(W)$ . Since  $\dim F^{-1}(W) = 1$ ,  $F^{-1}(W)$  is a compact 1-manifold. So it's a disjoint union of copies of  $[0, 1]$  and  $S^1$ ,  $\partial F^{-1}(W)$  is an even number of points and number of positive sign is the same as the negative.  $\square$

If  $f \in C^\infty(X, Y)$  not necessarily transverse to  $W$ , we can take  $g \simeq f$  such that  $g \pitchfork W$  and define  $I(f, W) = I(g, W)$ , by the proposition above it's well-defined.

**Definition 2.11. (Degree.)** Let  $X, Y$  oriented manifolds without boundary and  $X$  compact,  $Y$  connected,  $\dim X = \dim Y$ . If  $f \in C^\infty(X, Y)$ , define  $\deg(f) = I(f, \{y\})$ ,  $y \in Y$ .

**Proposition 2.6.**  $I(f, \{y\})$  doesn't depend on  $y$ .

*Proof.* We may assume WLOG that  $f \pitchfork \{y\}$ , so  $y$  is a regular value of  $f$ .  $f^{-1}(y) = \{x_1, \dots, x_k\}$  is a finite set of points. Let  $U_1, \dots, U_k$  be small disjoint neighborhood of these points. Since  $x_i$  is a regular point and  $\dim X = \dim Y$ , we may assume  $f|_{U_i}$  is a diffeomorphism. Then it's easy to see that  $I(f, \{y\})$  is locally constant if  $y$  varies, so it's independent of  $y$ .  $\square$

**Proposition 2.7.** Let  $X, Y$  oriented manifolds without boundary and  $X$  compact,  $Y$  connected,  $\dim X = \dim Y$ . If  $f \simeq g : X \rightarrow Y$ , then  $\deg f = \deg g$ .

**Definition 2.12. (Winding Number.)** Let  $X$  compact, oriented manifold without boundary,  $\dim X = n$  (often  $X = S^n$ ). Let  $f : X \rightarrow \mathbb{R}^{n+1}$  and  $z \in \mathbb{R}^{n+1}$  such that  $z \notin f(X)$ . Let  $u : X \rightarrow S^n, x \mapsto \frac{f(x)-z}{\|f(x)-z\|}$ . Define  $\text{wind}(f; z) := \deg u = I(u, \{a\})$ .

**Proposition 2.8.** Let  $\{f_t : X \rightarrow \mathbb{R}^{n+1} \setminus \{z\}\}_{t \in [0,1]}$  homotopy of maps. Then  $\text{wind}(f_0; z) = \text{wind}(f_1; z)$ .

**Proposition 2.9.** Suppose  $X = \partial D$ ,  $D$  compact and oriented,  $f$  extends to a map  $F : D \rightarrow \mathbb{R}^{n+1}$ . Then  $\text{wind}(f; z) = I(F, \{z\})$ .

**Note.** Let  $X$  oriented manifolds with or without boundary,  $Y, W$  be oriented manifolds without boundary such that  $X$  is compact and  $W \subseteq Y$  closed subset and  $\dim X + \dim W = \dim Y$ . Let  $f : X \rightarrow Y$ ,  $f \pitchfork W$ ,  $f^{-1}(W) \cap \partial X = \emptyset$ . Then we can define the intersection number for  $X$  with boundary.

*Proof.* Since  $f(X) \cap \{z\} = \emptyset$ ,  $f \pitchfork \{z\}$ . We may assume WLOG, that  $F \pitchfork \{z\}$ . If  $F^{-1}(z) = \emptyset$ , then  $u : X \rightarrow S^n$  can be extended to a map  $U : D \rightarrow S^n$ , so  $\deg u = 0$ .

Suppose  $F^{-1}(z) = \{x_1, \dots, x_k\}$ , by implicit function theorem,  $F$  is a local diffeomorphism, near each  $x_i$ . We can find disjoint neighborhoods  $U_i$  near  $x_i$ ,  $V$  of  $z$  such that  $F|_{U_i} : U_i \rightarrow V$  is a diffeomorphism. We can choose orientation preserving charts  $\varphi_i : U_i \rightarrow \mathbb{R}^{n+1}$ ,  $\psi : V \rightarrow \mathbb{R}^{n+1}$  such that  $\psi \circ F \circ \varphi_i^{-1} = \text{id}$  or reflection in the first coordinate. Let  $\epsilon > 0$  such that  $B^{n+1}(\epsilon) \subseteq \varphi_i(U_i)$  for all  $i$ , let  $Z = D \setminus \bigcup_i \varphi_i^{-1}(B^{n+1}(\epsilon))$ , then  $[\partial Z] = [\partial D] \cup \bigcup_i [-\varphi_i^{-1}(\partial B^{n+1}(\epsilon))]$ ,

$$0 = \deg(U|_{\partial Z}) = I(U|_{\partial Z}, \{a\}) = I(U|_{\partial D}, \{a\}) - \sum_i I(U|_{\varphi_i^{-1}(\partial B^{n+1}(\epsilon))}, \{a\})$$

So  $\text{wind}(f; z) = \sum_i \text{wind}(f_i; z)$ , where  $f_i = F|_{\varphi_i^{-1}(\partial B^{n+1}(\epsilon))}$ . Now we need to compute  $\text{wind}(f_i; z)$ .

Since  $\varphi_i, \psi$  are orientation preserving,  $\text{wind}(f_i; z) = \text{wind}(\psi \circ f_i \circ \varphi_i^{-1}; 0) = \sigma(i)$ , where  $\sigma(i) = 1$  if  $\psi \circ f_i \circ \varphi_i^{-1} = \text{id}$ ;  $\sigma(i) = -1$  if  $\psi \circ f_i \circ \varphi_i^{-1}$  is a reflection. Since  $F^{-1}(z) = x_i$  in  $\varphi_i^{-1}(B^{n+1}(\epsilon))$ ,  $\text{sign}(x_i) = \sigma(i)$ , so  $\sum_i \text{wind}(f_i; z) = \sum_i \sigma(i) = I(F, \{z\})$ .  $\square$

**Theorem 2.5. (Borsuk-Ulam.)** Let  $f : S^n \rightarrow \mathbb{R}^n$  smooth. Then  $\exists x \in S^n$  such that  $f(x) = -f(-x)$ .

*Proof.* Suppose not, then  $g(x) = f(x) - f(-x)$  is an odd function that doesn't vanish, let  $\tilde{g} : S^n \rightarrow \mathbb{R}^{n+1}$ ,  $\tilde{g} = g \times 0$ , then  $\tilde{g}$  is odd and  $\tilde{g}(x) \neq 0$ .

Claim.  $\text{wind}(\tilde{g}, 0)$  is odd. Prove by induction.

Assuming the claim,  $\text{wind}(\tilde{g}, 0) = \deg\left(\frac{\tilde{g}}{\|\tilde{g}\|}\right)$  is odd, but  $(0, \dots, 0, 1) \notin \frac{\tilde{g}}{\|\tilde{g}\|}(S^n)$ , so  $\deg\left(\frac{\tilde{g}}{\|\tilde{g}\|}\right) = 0$ , contradiction.  $\square$

**Theorem 2.6. (Jordan-Brouwer.)** Let  $X \subseteq \mathbb{R}^n$  be a compact connected hypersurface. Then  $\mathbb{R}^n \setminus X = U_1 \sqcup U_2$ ,  $U_1, U_2$  open, connected,  $U_1 \cup X$  is a compact manifold with boundary  $\partial U_1 = X$  and  $U_2 \cup X$  is a manifold with boundary  $X$  and  $U_2$  is unbounded.

Set up mod 2 intersection theory.

$X, Y, W$  manifolds,  $X$  compact,  $W \subseteq Y$  closed,  $\partial W = \partial Y = \emptyset$ ,  $f : X \rightarrow Y$  such that  $f(\partial X) \cap W = \emptyset$ ,  $f \pitchfork W$ ,  $\dim X + \dim W = \dim Y$ , define  $I_2(f, W) = \#f^{-1}(W) \pmod 2$ . Similarly we can define  $\deg_2(f)$  and  $\text{wind}_2(f, z)$ .

*Proof.* Step 1.  $\mathbb{R}^n \setminus X$  has at most two connected components.

Fix  $x \in X$  and  $x \in U$  open connected set. Take  $z \in \mathbb{R}^n \setminus X$ , let  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$  be a curve connecting  $x$  and  $z$ . WLOG, we can assume  $\gamma \pitchfork X$ ,  $\gamma \cap X = \{x, p_1, \dots, p_k\}$ , where  $\gamma(t_i) = p_i$  and  $\gamma|_{[t_k+\epsilon, 1]}$  doesn't intersect  $X$ . We can find a curve  $\tilde{\gamma} \subseteq X$  connecting  $x$  and  $p_k$ , a neighborhood of this path is diffeomorphic to  $B^{n-1} \times (-1, 1)$ , we can connect  $\gamma(t_k + \epsilon)$  to  $U \setminus X$  without crossing  $X$  inside  $B^{n-1} \times (-1, 1)$ .

Step 2. If  $z_0$  and  $z_1$  can be connected in  $\mathbb{R}^n \setminus X$ , then  $\text{wind}_2(X, z_0) = \text{wind}_2(X, z_1)$ .

If  $\gamma(t)$  is a path connecting  $z_0$  and  $z_1$  in  $\mathbb{R}^n \setminus X$ , then  $u_t(x) = \frac{x - \gamma(t)}{\|x - \gamma(t)\|}$  is a homotopy between  $u_{z_0}$  and  $u_{z_1}$ .

Step 3. If  $\ell$  is a line segment connecting  $z_0$  and  $z_1$ ,  $z_0, z_1 \notin X$ ,  $\ell \pitchfork X$  and  $\ell \cap X = \{p\}$ , then  $\text{wind}_2(X, z_0) = \text{wind}_2(X, z_1) + 1$ .

Step 4. Consider  $U_1 = \{z \in \mathbb{R}^n \setminus X \mid \text{wind}_2(X, z) = -1\}$ ,  $U_2 = \{z \in \mathbb{R}^n \setminus X \mid \text{wind}_2(X, z) = 0\}$ .  $\square$



**Definition 2.13. (Index.)** Let  $X$  be oriented manifold without boundary and let  $V \in \mathfrak{X}(X)$ . Let  $p \in X$  such that  $V_p = 0$ . Suppose there exists an open set  $U$  near  $p$  such that  $V_x = 0$  for  $x \in U$  iff  $x = p$ . Choose an oriented chart  $\varphi : U \rightarrow \mathbb{R}^n$  such that  $\varphi(p) = 0$ ,  $\varphi(U) = B^n(\epsilon)$ . It induces a map  $\bar{\varphi} : TX|_U \rightarrow U \times \mathbb{R}^n$ ,  $\bar{V} = pr_2 \circ \bar{\varphi} \circ V \circ \varphi^{-1} : B^n(\epsilon) \rightarrow \mathbb{R}^n$ . Define  $\text{ind}_p(V) = \text{wind}(\bar{V}_{\partial B^n(\epsilon)}, 0)$ .

**Theorem 2.7. (Poincaré-Hopf Theorem.)** Suppose all zeros of  $V$  are isolated and  $X$  is compact. Then  $\sum_{V_p=0} \text{ind}_p(V) = \chi(X)$ .

**Note.** If  $X$  oriented compact manifold,  $\Delta = \{(x, x) \in X \times X\}$  is a submanifold of  $X \times X$ , then we can talk about  $I(\Delta, \Delta) \in \mathbb{Z}$  (after perturb  $\Delta$  it can intersect itself transversely). We define Euler characteristic  $\chi(X) = I(\Delta, \Delta)$ .

**Example 2.3.**

**Definition 2.14.** Let  $V \in \mathfrak{X}(X)$  and  $p \in X$  isolated zero.  $p$  is said to be simple if 0 is a regular value of  $\bar{V}$ , where  $\bar{V} : B^n(\epsilon) \rightarrow \mathbb{R}^n$  is a local representation of  $V$  (i.e.  $d\bar{V}_0$  is an isomorphism).

**Exercise 2.3.** If  $p$  is simple, then in a local representation,  $\bar{V} = id$  or  $\bar{V} = \text{reflection at the first coordinate}$ .

**Obs.** If  $p$  is an isolated zero of  $V$ , then there exists  $\tilde{V} \in \mathfrak{X}(X)$  which coincides with  $V$  outside a small neighborhood  $p \in U$  such that all the zeros of  $\tilde{V}$  in  $U$  are simple. Moreover, if  $\{p_1, \dots, p_n\} \subseteq U$  are zeros of  $\tilde{V}$ , then  $\text{ind}_p(V) = \sum_i \text{ind}_{p_i}(\tilde{V})$ .

*Proof.* We may assume WLOG that all zeros of  $V$  are simple. Let  $f_t : X \rightarrow X$  be the flow of  $V$ . Note that if  $\epsilon$  is sufficiently small, then  $\text{Fix}(f_t) = \text{Zero}(V)$  for  $\forall 0 < t < \epsilon$ .  $\Delta = j^0 id(X)$ ,  $Gr(f_t) = j^0 f_t(X) = \{(x, f_t(x)) \mid x \in X\}$ .

**Claim.**  $\Delta \pitchfork Gr(f_t)$  and for each  $x \in \text{Zero}(V)$ ,  $\text{ind}_x(V)$  is the sign of the intersection  $(x, x) \in \Delta \pitchfork Gr(f_t) \leftrightarrow \text{Fix}(f_t)$ .

If the claim is true, then  $\sum_{V_x=0} \text{ind}_x V = I(\Delta, Gr(f_t)) = I(\Delta, Gr(f_0)) = \chi(X)$ .

*Proof of the claim:* We can look in oriented charts:  $V : B^n(\epsilon) \rightarrow \mathbb{R}^n$  such that 0 is a regular value and  $V^{-1}(0) = \{0\}$ . We may assume WLOG that  $V = id$  or reflection, let  $q = \pm 1$ , then we write  $V(x_1, \dots, x_n) = (qx_1, \dots, x_n)$ . The flow of  $V$  is:  $f_t(x_1, \dots, x_n) = (e^{qt}x_1, \dots, e^t x_n)$ . Then locally we have  $\Delta \cap Gr(f_t) = \{(0, 0)\}$ , now we need to show  $T_{(0,0)}\Delta + T_{(0,0)}Gr(f_t) = T_{(0,0)}(\mathbb{R}^n \times \mathbb{R}^n)$ . Let  $\{e_1, \dots, e_n\}$  be canonical basis of  $\mathbb{R}^n$ , it induces orientation on  $\mathbb{R}^n \times \mathbb{R}^n$  given by  $\{e_1 \times 0, \dots, e_n \times 0, 0 \times e_1, \dots, 0 \times e_n\}$ . Also

we have induced orientation on  $\Delta = j^0 id(X)$  and  $Gr(f_t) = j^0 f_t(X)$ :

$$\Delta : \{e_1 \times e_1, \dots, e_n \times e_n\}; Gr(f_t) : \{e_1 \times e^{qt} e_1, \dots, e_n \times e^t e_n\}$$

Putting these linearly independent sets together, we obtain a matrix:

$$\begin{bmatrix} \text{Id} & \text{Id} \\ \hline \text{Id} & \begin{matrix} e^{qt} & & \\ & \ddots & \\ & & e^t \end{matrix} \end{bmatrix}$$

where  $\det = (e^{qt} - 1)(e^t - 1)^{n-1} \neq 0$ . Note that the sign of the determination is the same as  $q$ . This completes the proof of the claim.  $\square$

**Theorem 2.8. (Lefschetz Fixed Point Theorem.)**

Let  $X$  be an oriented compact manifold without boundary,  $f : X \rightarrow X$  smooth. Define lefschetez index  $\mathcal{L}(f) = I(\Delta, Gr(f))$ . If  $\mathcal{L}(f) \neq 0$ , then  $f$  has a fixed point.

*Proof.* If  $f$  doesn't have a fixed point, then  $\Delta \cap Gr(f) = \emptyset$ . So  $\mathcal{L}(f) = 0$ .  $\square$

If  $\Delta \cap Gr(f)$ , then for each  $(x, x) \in \Delta \cap Gr(f)$ , define  $\mathcal{L}_x(f) = \text{sign of the intersection } (x, x)$ .

**Exercise 2.4.**  $\mathcal{L}_x(f) = \text{sign of } \det(df_x - I)$ .

**Theorem 2.9. (Hopf.)**  $f, g : S^n \rightarrow S^n$ , then  $f \simeq g \Leftrightarrow \deg f = \deg g$ . Moreover, if  $X^n$  is compact oriented connected manifold without boundary,  $f, g : X \rightarrow S^n$ , then  $f \simeq g \Leftrightarrow \deg f = \deg g$ .

*Proof.* Let  $y \in S^n$  be a regular value of  $f$  and  $g$ .  $Q_0 = f^{-1}(y)$ ,  $Q_1 = g^{-1}(y)$  are 0-dim oriented submanifold.  $\deg f = \#Q_0$ ,  $\deg g = \#Q_1$ .

**Step 1.** Construct a homotopy between  $\tilde{f}, \tilde{g}$  s.t.  $\tilde{f}^{-1}(y) = Q_0$ ,  $\tilde{g}^{-1}(y) = Q_1$ .

Let  $C$  be an oriented 1-manifold in  $[0, 1] \times X$  s.t.  $\partial C = \{1\} \times Q_1 - \{0\} \times Q_0$ . Now we define an embedding  $\varphi : C \times \mathbb{R}^n \rightarrow [0, 1] \times X$  s.t.  $\varphi(x, 0) = x$  and  $\frac{\partial \varphi}{\partial v}(x, 0)$  specified.

**Step 2.**  $f \simeq \tilde{f}, g \simeq \tilde{g}$ .  $\square$

### 2.3 Pontryagin-Thom Construction

Let  $X^n$  oriented compact manifold.

**Goal.** Classify maps  $X \rightarrow S^m$ ,  $n \geq m$ , up to homotopy.

If  $n = m$  then the degree will do it, we now want to generalize it, which is called framed cobordism.

**Definition 2.15.** Let  $Q^k \subseteq X$  submanifold without boundary. A framing is a choice of trivialization of the normal bundle  $NQ \xrightarrow{\cong} Q \times \mathbb{R}^{n-k}$ .

An equivalent definition: Let  $V(NQ) = \coprod_{x \in Q} V(N_x Q)$ , where  $V(N_x Q) = \{(v_1, \dots, v_{n-k}) \mid \{v_1, \dots, v_{n-k}\} \text{ is a basis of } N_x Q\}$ . We can check that  $V(NQ) \rightarrow Q$  is a fiber bundle. A framing is a section  $Q \rightarrow V(NQ)$ . If  $Q$  is oriented, we usually assume that the framing is position w.r.t. the orientation.

**Example 2.4.** Every knot in  $S^3$  admits a framing.

Take  $f : X \rightarrow S^m$ ,  $a$  is a regular value. Then  $f^{-1}(a)$  admits a natural framing from  $f : df_x : N_x Q \rightarrow T_a S^m$  is an isomorphism, fix an oriented basis of  $T_a S^m$ , the preimage is a basis of  $N_x Q$ .

**Remark 2.2.** If  $n = m$ ,  $Q$  is a collection of points, a framing modulo isotopy is the same as an orientation.

**Definition 2.16.** Let  $Q_0, Q_1 \subseteq X$  be oriented compact submanifolds without boundary. A cobordism from  $Q_0$  to  $Q_1$  is an oriented compact manifold  $W \subseteq [0, 1] \times X$  s.t.  $W \cap ([0, \epsilon) \times X \cup (1 - \epsilon, 1] \times X) = [0, \epsilon) \times Q_0 \cup (1 - \epsilon, 1] \times Q_1$  and  $\partial W \subseteq \partial([0, 1] \times X)$ . In particular,  $\partial W = \{1\} \times Q_1 \cup (-\{0\} \times Q_0)$  and also  $W \cap \partial([0, 1] \times X)$ .

If  $g$  is a Riemannian metric on  $X$ , then we let  $\tilde{g} = dt \otimes dt + g$ . Clearly  $\frac{\partial}{\partial t}$  is orthogonal to  $\{t\} \times X$ . Let  $W$  be a cobordism from  $Q_0$  to  $Q_1$ , we have  $\varphi : \partial W \rightarrow Q_1 \sqcup (-Q_0)$  an orientation preserving diffeomorphism then  $NW|_{\{0\} \times Q_0} \cong \varphi^* NQ_0$ ,  $NW|_{\{1\} \times Q_1} \cong \varphi^* NQ_1$