Riemannian Geometry

Solution to Lee′s Book

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1 Chapter 4: Connections

Problem 4-3

Proof. Prove by definition:

$$\begin{aligned}\widetilde{\Gamma}_{ij}^k \widetilde{E}_k &= \nabla_{\widetilde{E}_i} \widetilde{E}_j = \nabla_{\widetilde{E}_i} A_j^r E_r = \widetilde{E}_i(A_j^r) E_r + A_j^r \nabla_{\widetilde{E}_i} E_r \\ &= (A_i^q E_q)(A_j^r) E_r + A_j^r A_i^q \nabla_{E_q} E_r = A_i^q E_q(A_j^r) E_r + A_j^r A_i^q \Gamma_{qr}^\ell E_\ell\end{aligned}$$

Since $\widetilde{\Gamma}_{ij}^k \widetilde{E}_k = \widetilde{\Gamma}_{ij}^k A_k^p E_p$, by comparison we have

$$\widetilde{\Gamma}_{ij}^k = (A^{-1})_p^k A_i^q E_q(A_j^p) + (A^{-1})_p^k A_j^r A_i^q \Gamma_{qr}^p$$

□

Problem 4-4

Proof. It's clear that

□

2 Chapter 5

Problem 5-1

Proof. By definition we have $D^\flat(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$. Then

$$\begin{aligned} D^\flat(X,Y,Z) = -D^\flat(X,Z,Y) &\iff \langle \widetilde{\nabla}_X Y, Z\rangle - \langle \nabla_X Y, Z\rangle = -\langle \widetilde{\nabla}_X Z, Y\rangle + \langle \nabla_X Z, Y\rangle \\ &\iff \langle \widetilde{\nabla}_X Y, Z\rangle + \langle \widetilde{\nabla}_X Z, Y\rangle = \nabla_X \langle Y, Z\rangle = X\langle Y, Z\rangle \\ &\iff \widetilde{\nabla} \text{ is compatible with } g. \end{aligned}$$

□

Problem 5-2

Proof. We have

$$\begin{aligned} g_{jk}\omega_i^k + g_{ik}\omega_j^k = dg_{ij} &\iff \forall X \in \mathfrak{X}(M),\ g_{jk}\omega_i^k(X) + g_{ik}\omega_j^k(X) = dg_{ij}(X) \\ &\iff \forall X \in \mathfrak{X}(M),\ \langle \omega_i^k(X)E_k, E_j\rangle + \langle \omega_j^k(X)E_k, E_i\rangle = X\langle E_i, E_j\rangle \\ &\iff \forall X \in \mathfrak{X}(M),\ \langle \nabla_X E_i, E_j\rangle + \langle \nabla_X E_j, E_i\rangle = X\langle E_i, E_j\rangle \\ &\iff \nabla \text{ is compatible with } g. \end{aligned}$$

□

Problem 5-3

Proof. By proposition 5.5 it's easy to prove after calculation. □

Problem 5-6

Proof. (a) Since $\widetilde{X}, \widetilde{Y}$ are π-related to X, Y and π is a Riemannian submersion, we have $\langle \widetilde{X}_p, \widetilde{Y}_p\rangle = \langle X_{\pi(p)}, Y_{\pi(p)}\rangle = \langle X, Y\rangle \circ \pi$; Since $d\pi_p\left([\widetilde{X}, \widetilde{Y}]_p^H\right) = d\pi_p\left([\widetilde{X}, \widetilde{Y}]_p\right) = [X,Y]_{\pi(p)}$, so $[\widetilde{X}, \widetilde{Y}]^H = \widetilde{[X,Y]}$; Since $d\pi_p\left([\widetilde{X}, W]_p\right) = [X, 0]_{\pi(p)} = 0$, so $[\widetilde{X}, W]$ is vertical if W is vertical.

(b) For Levi-Civita connection we have

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = \frac{1}{2}\big(& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & -\langle Y, [X,Z]\rangle - \langle Z, [Y,X]\rangle + \langle X, [Z,Y]\rangle\big). \end{aligned}$$

So we have

$$\begin{aligned}
\langle \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \rangle &= \frac{1}{2} \Big(\widetilde{X} \langle \widetilde{Y}, \widetilde{Z} \rangle + \widetilde{Y} \langle \widetilde{Z}, \widetilde{X} \rangle - \widetilde{Z} \langle \widetilde{X}, \widetilde{Y} \rangle \\
&\quad - \langle \widetilde{Y}, [\widetilde{X}, \widetilde{Z}] \rangle - \langle \widetilde{Z}, [\widetilde{Y}, \widetilde{X}] \rangle + \langle \widetilde{X}, [\widetilde{Z}, \widetilde{Y}] \rangle \Big) \\
&= \frac{1}{2} \Big(\widetilde{X} \left(\langle Y, Z \rangle \circ \pi \right) + \widetilde{Y} \left(\langle Z, X \rangle \circ \pi \right) - \widetilde{Z} \left(\langle X, Y \rangle \circ \pi \right) \\
&\quad - \langle \widetilde{Y}, [\widetilde{X}, \widetilde{Z}]^H \rangle - \langle \widetilde{Z}, [\widetilde{Y}, \widetilde{X}]^H \rangle + \langle \widetilde{X}, [\widetilde{Z}, \widetilde{Y}]^H \rangle \Big) \\
&= \frac{1}{2} \left((X \langle Y, Z \rangle) + (Y \langle Z, X \rangle) - (Z \langle X, Y \rangle) \right. \\
&\quad \left. - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \right) \circ \pi \\
&= \langle \nabla_X Y, Z \rangle \circ \pi = \langle \widetilde{\nabla_X Y}, \widetilde{Z} \rangle + \frac{1}{2} \langle [\widetilde{X}, \widetilde{Y}]^V, \widetilde{Z} \rangle.
\end{aligned}$$

and

$$\begin{aligned}
\langle \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, W \rangle &= \frac{1}{2} \Big(\widetilde{X} \langle \widetilde{Y}, W \rangle + \widetilde{Y} \langle W, \widetilde{X} \rangle - W \langle \widetilde{X}, \widetilde{Y} \rangle \\
&\quad - \langle \widetilde{Y}, [\widetilde{X}, W] \rangle - \langle W, [\widetilde{Y}, \widetilde{X}] \rangle + \langle \widetilde{X}, [W, \widetilde{Y}] \rangle \Big) \\
&= \frac{1}{2} \Big(- \langle W, [\widetilde{Y}, \widetilde{X}]^V \rangle \Big) = \frac{1}{2} \langle [\widetilde{X}, \widetilde{Y}]^V, W \rangle + \langle \widetilde{\nabla_X Y}, W \rangle.
\end{aligned}$$

So for local frame of $\mathfrak{X}(\widetilde{M})$, $\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}$ satisfies the two formula above, then we have $\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla_X Y} + \dfrac{1}{2} [\widetilde{X}, \widetilde{Y}]^V$.

□

Problem 5-7

Problem 5-9

Problem 5-10

Proof. Since $\varphi(p) = \psi(p)$, $d\varphi_p = d\psi_p$, we have the following commute diagram:

$$\begin{tikzcd}
M \arrow[r, "\psi"] & \widetilde{M} \\
\mathcal{E}_p \arrow[u, "\exp_p"] \arrow[d, "\exp_p"'] \arrow[r, "d\psi_p", "d\varphi_p"'] & \widetilde{\mathcal{E}}_{\varphi(p)} \arrow[u, "\exp_{\psi(p)}"'] \arrow[d, "\exp_{\varphi(p)}"] \\
M \arrow[r, "\varphi"] & \widetilde{M}
\end{tikzcd}$$

By the commute diagram and $\exp_p$ local diffeomorphism, there exists a neighborhood U of p such that $\varphi \equiv \psi$ on U. Since M is connected, $\varphi \equiv \psi$ on M.

$\square$

3 Chapter 6

Problem 6-1

Proof. (a) For $\forall p \in \gamma(I)$, there exists a uniformly normal neighborhood $p \in W$, then $W \subseteq \gamma(I)$, so $\gamma(I)$ is open; Let (p_i) be a sequence of points in $\gamma(I)$ and $(p_i) \to p$. Since p has a uniformly normal neighborhood W, $\exists N \in \mathbb{N}$ s.t. $\forall i > N$, $p_i \in W$. Then $p \in \gamma(I)$, so $\gamma(I)$ is closed. Since M is connected, γ is surjective.

(b) If γ is injective, then it's a bijective smooth map from I to M, we only need to show γ is a local isometry. The Riemannian metric has local representation $g_{11}(dt)^2$, since γ is unit-speed, we have $g(\gamma', \gamma') = g_{11} = 1$, then the local representation is $(dt)^2$, which shows that γ is a isometry.

(c) Let $\alpha(t) = \gamma(t + t_1)$, $\beta(t) = \gamma(t + t_2)$, then we have α, β are unit-speed geodesic, $\alpha(0) = \beta(0)$, $\alpha'(0) = \beta'(0)$. By the uniqueness of geodesic we have $\alpha \equiv \beta$, hence $\gamma(t + t_1) = \gamma(t + t_2)$, γ is periodic.

(d) WLOG, we assume that $t_2 > t_1$, then $\alpha(\frac{t_2 - t_1}{2}) = \beta(\frac{t_2 - t_1}{2})$, contradiction. □

Problem 6-4

Proof. (a) Since the equation is independent of coordinate, we take the normal coordinate centered at $\gamma(0)$. In this chart we have $\gamma(t) = (\gamma^1(t), \cdots, \gamma^n(t))$ and $d_g(\gamma(0), \gamma(t)) = \sqrt{\sum (\gamma^i(t))^2}$, then

$$\lim_{t \searrow 0} \frac{d_g(\gamma(0), \gamma(t))}{t} = \lim_{t \searrow 0} \sqrt{\sum \left(\frac{\gamma^i(t)}{t}\right)^2} = \sqrt{\sum (\dot{\gamma}^i(0))^2} = |\gamma'(0)|_g$$

(b) We only need to verify that for every $p \in M$, $g_p = \widetilde{g}_p$. From (a) we have $g_p(v, v) = \widetilde{g}_p(v, v)$ for all $v \in T_pM$, by polarization identity we have $g_p = \widetilde{g}_p$. □

Problem 6-5

Proof. (a) In the normal coordinates centered at p, let $u(q, v, t) = \exp_p^{-1}\left(\exp_q(tv)\right)$, it's clearly a smooth, then $f = |u|^2$ is smooth.

(b) $\frac{\partial f}{\partial t} = 2\langle u, \frac{\partial u}{\partial t}\rangle$, $\frac{\partial^2 f}{\partial t^2} = 2\langle \frac{\partial^2 u}{\partial t^2}, u\rangle + 2\left|\frac{\partial u}{\partial t}\right|^2$. Then for $q = p$, $u(q, v, t) = tv$, $\frac{\partial^2 f}{\partial t^2} = 2\left|\frac{\partial u}{\partial t}\right|^2$ is positive. Use continuity we have that if ϵ is small enough then $\frac{\partial^2 f}{\partial t^2} > 0$.

(c) Since q_1, q_2 are in the uniformly normal neighborhood of p, $\gamma(t)$ is of the form $\exp_{q_1}(tv)$ for some $v \in T_{q_1}M$, then $d_g(p, \gamma(t))^2 = f(q_1, v, t)$. By (b) we have $\frac{\partial^2 f}{\partial t^2} > 0$, so f is convex on where it's defined. Then $d_g(p, \gamma(t))$ is also convex on its domain, so it attains its maximum at one of the endpoints of γ.

(d) By (c) it's clear that $d_g(p, \gamma(t)) < \epsilon \implies \gamma(t) \in B_\epsilon(p)$, so the image of the unique geodesic segment lies in $B_\epsilon(p)$, hence geodesically convex. □

Problem 6-6

Proof. For $x, x' \in M$, if $d_g(x, x') < \operatorname{conv}(x)$, then the geodesic ball $B_\delta(x')$, where $\delta = \operatorname{conv}(x) - d_g(x, x')$, is contained in $B_{\operatorname{conv}(x)}(x)$: for $\forall x'' \in B_\delta(x')$, $d_g(x, x'') \leq d_g(x, x') + d_g(x', x'') < \operatorname{conv}(x)$. It's clear that $B_\delta(x')$ is also geodesically convex since it's contained in a geodesically convex geodesic ball. Then $\operatorname{conv}(x') \geq \delta = \operatorname{conv}(x) - d_g(x, x')$, which means $\operatorname{conv}(x) - \operatorname{conv}(x') \leq d_g(x, x')$. If $d_g(x, x') \geq \operatorname{conv}(x)$ this inequality naturally holds. By reversing the role of x, x' we then get: $|\operatorname{conv}(x) - \operatorname{conv}(x')| \leq d_g(x, x')$. So $\operatorname{conv}(x)$ is continuous. □

Problem 6-7

Proof. (a) □