Notebook

Spectral Sequence

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NOTE
IN

SS 1

Recall For X space, $H^*(X)$ is a graded ring via cup product

$$H^*(X) = \bigoplus_n H^n(X). \qquad x \in H^n(X),\ y \in H^m(X) \Rightarrow xy \in H^{n+m}(X)$$

In fact, it's graded commutative: $xy = (-1)^{|x||y|} yx$

If $A \hookrightarrow X$ a CW pair, we have

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X,A) \longrightarrow 0$$

$\Rightarrow$ long exact sequence in homology:

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow \underset{\substack{\text{SII} \\ \tilde{H}_n(X/A)}}{H_n(X,A)} \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

also in cohomology:

$$\cdots \longleftarrow H^n(A) \longleftarrow H^n(X) \longleftarrow H^n(X,A) \xleftarrow{\delta} H^{n-1}(A) \longleftarrow \cdots$$

We often try to compute $H^*(X)$ from A and X/A.

$\boxed{\text{Goal}}$: Extend this technique.

$A \hookrightarrow X \longrightarrow X/A$ cofiber sequence. $\longrightarrow$ long exact sequence

$$\begin{tikzcd} A_0 \arrow[r, hook] & A_1 \arrow[r, hook] \arrow[d] & X \arrow[d] \\ & A_1/A_0 & X/A_1 \end{tikzcd}$$

$\Longrightarrow$ use two les to bootstrap up to $H^*(X)$

$$\begin{tikzcd} A_0 \arrow[r, hook] & A_1 \arrow[r, hook] \arrow[d] & A_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & A_m \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & A_n = X \arrow[d] \\ & A_1/A_0 & A_2/A_1 & & A_m/A_{m-1} & & A_n/A_{n-1} \end{tikzcd}$$

$\Longrightarrow$ use a spectral sequence (lots of les's sewn together) to bootstrap to $H^*(X)$.

SS2 - Graded rings and modules.

Let R be a ring.

Category of graded R-modules

objects: $A_* = \bigoplus_n A_n$ (indexed on any set, usually $\mathbb{Z}$)

elements of A_n are homogeneous of degree n

morphisms: $\mathrm{Hom}(A_*, B_*) = \prod_n \mathrm{Hom}(A_n, B_n)$

degree preserving maps

tensors: $(A_* \otimes B_*)_n = \bigoplus_{p+q=n} (A_p \otimes B_q)$

adjoint

hom: $\mathrm{Hom}^k(A_*, B_*) = \prod_n \mathrm{Hom}(A_n, B_{n+k})$

i.e. sequence of maps raising degree by k.

Def. A graded ring A_* is a graded abelian group with a degree zero map $A_* \otimes A_* \longrightarrow A_*$ (+ associative, unital)

$\Longleftrightarrow A_* = \bigoplus_n A_n$ for A_n abelian group and $A_m A_n \subseteq A_{m+n}$

(We can define graded algebra with abelian group in the above definition replaced by R-module)

Ex $H^*(X)$

Def. A differential graded algebra (dga) A_* is a graded ring with a degree $\begin{cases} +1 & \text{(cohomological)} \\ -1 & \text{(homological)} \end{cases}$ map $d: A_* \to A_*$ s.t.

1) $d \circ d = d^2 = 0$; 2) Leibniz rule: $d(xy) = d(x)y + (-1)^{|x|} x\, d(y)$

Ex $C^*(X)$ – singular cochain complex.

Def. A filtration of an R-module M is a sequence of submodules

$0 \subseteq \cdots \subseteq F_{-1}M \subseteq F_0 M \subseteq F_1 M \subseteq \cdots \subseteq M = \bigcup_n F_n M$

For us, usually bounded below ($F_n M = 0$ for some n)

Def. The <u>associated graded module</u> of a filtered module M is the graded module $Gr(M)$ with $Gr_n(M) = F_nM/F_{n-1}M$

$Gr(M)$ approximates M, but we're left with extension problems.

Ex. If $Gr_0(M) = \mathbb{Z}$, $Gr_1(M) = \mathbb{Z}/2$ and $Gr(M)$ has no negative part, then

$$0 \to F_0M \hookrightarrow F_1M \to F_1M/F_0M \to 0 \quad \text{ses, could be}$$

(where $F_0M = \mathbb{Z}$ and $F_1M/F_0M = \mathbb{Z}/2$)

1) $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$. trivial extension.

2) $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$, nontrivial extension.

Rmk. If M is a K-module for K a field, then all extensions are trivial. Same holds if $Gr_n(M)$ are free R-modules. $\forall n$.

SS3. Spectral sequence definition

Def. A cohomological spectral sequence is a sequence of bigraded R-modules $E_r^{p,q}$, $r \geq 1$, $p,q \in \mathbb{Z}$ together with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ s.t. $d_r^2 = 0$ (d_r has bidegree $(r,-r+1)$, total degree 1) and $E_{r+1} = H^{*,*}(E_r, d_r)$. i.e.

$$E_{r+1}^{p,q} = \ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) / \operatorname{Im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q})$$

Let $E_\infty = \underset{r}{\underrightarrow{\operatorname{colim}}}\, E_r$.

Def. We say the SS (spectral sequence) <u>converges</u> to the graded R-module M_* and we write $E_r^{p,q} \Rightarrow M_{p+q}$ if

$E_r^{p,q} \twoheadrightarrow E_{r+1}^{p,q}$

a) For each (p,q) there exists an r_0 s.t. $d_r^{p,q} = 0$ for $r \geq r_0$

b) There is a filtration of M_* so that for each n, $E_\infty^{p,n-p} = \underset{r}{\underrightarrow{\operatorname{colim}}}\, E_r^{p,n-p}$ is isomorphic to the associated graded module $Gr(M_*)_n$.

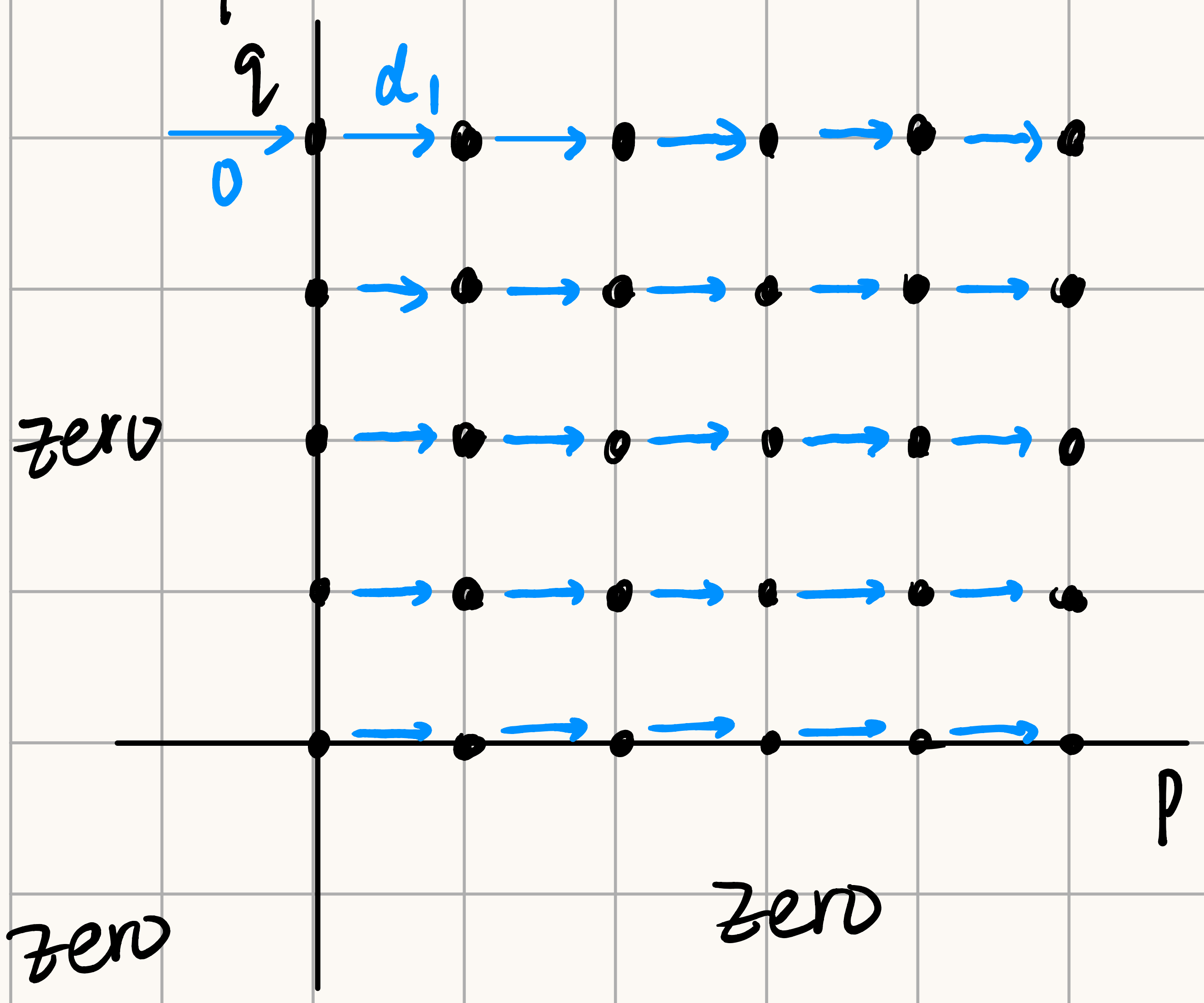
Everything here is almost the same for a homological SS.

Def. A first quadrant SS has $E_r^{p,q} = 0$ whenever $p<0$ or $q<0$

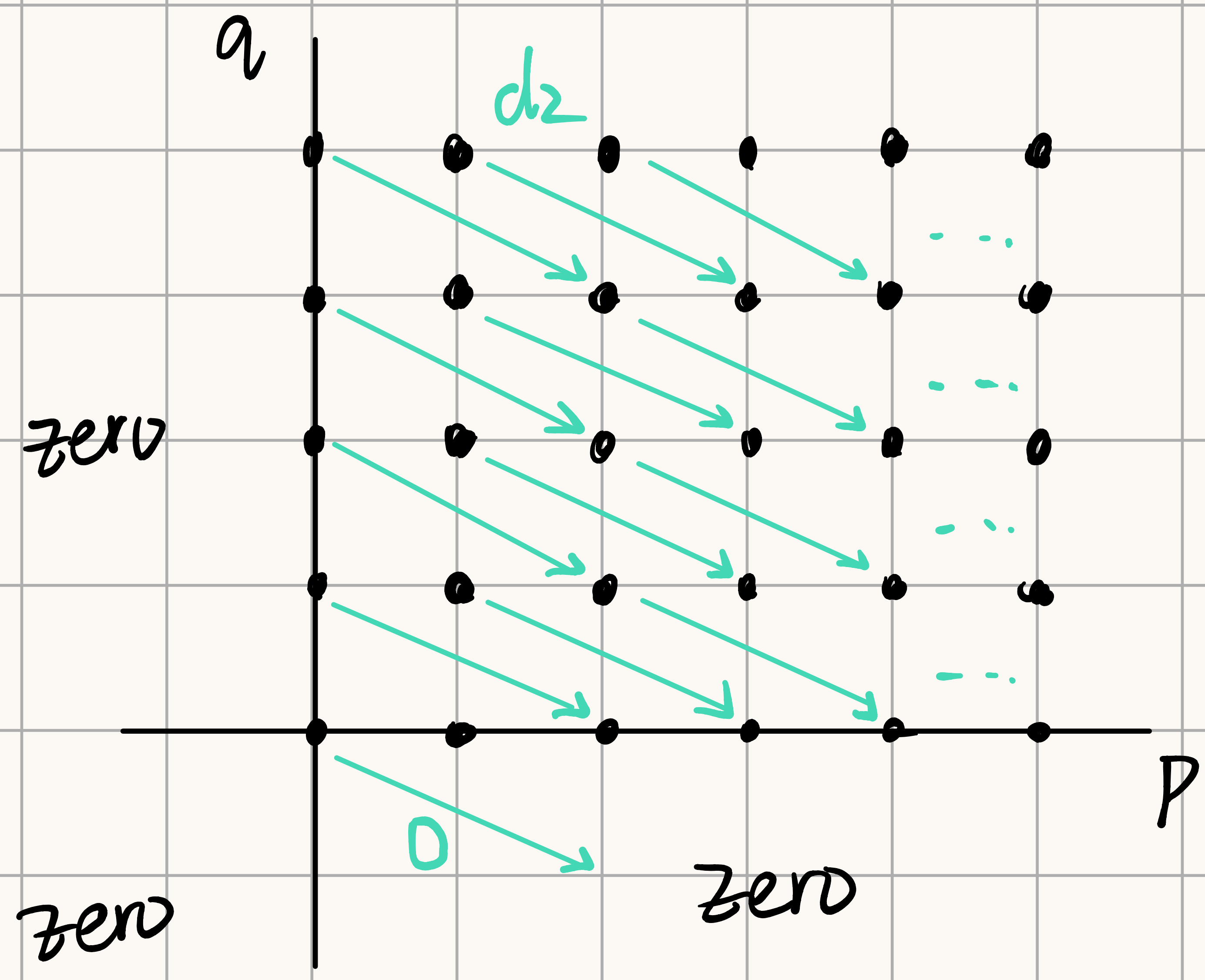
Rmk. A 1st quadrant SS converges in a stronger way. For each (p,q) there exists r_0 s.t. $E_r^{p,q} = E_\infty^{p,q}$ for all $r \geq r_0$.

Rmk. Sometimes convergence is even better. We say SS collapses at E_{r_0} if $\exists\, r_0$ s.t. $\forall (p,q)$ and $r \geq r_0$, $E_r^{p,q} = E_\infty^{p,q}$

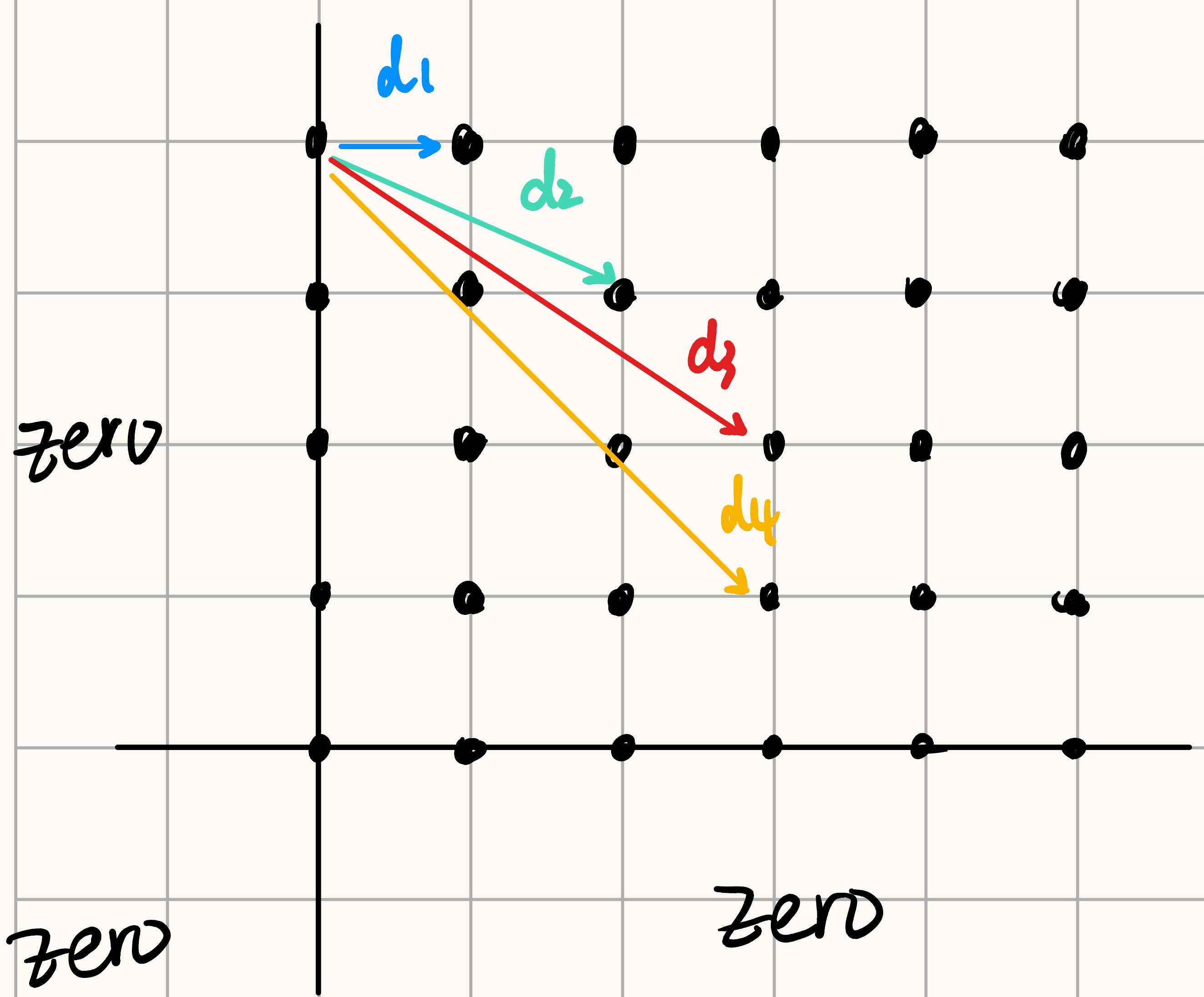
1st quadrant SS:



E_1



E_2 (the rots are different from E_1's since they're $H^{*,*}(E_1)$)



often draw all pages at once, but remember the rots change every time!

Def. $E_*^{*,*}$ is a SS of algebras if $(E_r^{*,*}, d_r)$ is a dga (use total degree to specify Leibniz rule) and $E_{r+1}^{*,*} = H^{*,*}(E_r)$ as algebras.

SS 5. Homology 1st quadrant SS

Refs: Hatcher ch. 5, Mosher + Tangora Ch. 7

Recall A 1st quadrant homological SS has $E^r_{p,q} = 0$ if $p<0$ or $q<0$ and $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ has bidegree $(-r, r-1)$

The E^∞-page will be an associated graded for a filtration of some module we're interested in $Gr(m)$

SS6 Unraveled exact couple

Let X a CW complex. $\phi = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ a filtration

Each inclusion $X_{p-1} \hookrightarrow X_p$ induces a les in homology

$$\cdots \longrightarrow H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \xrightarrow{j} H_n(X_p, X_{p-1}) \xrightarrow{\partial = k} H_{n-1}(X_{p-1}) \longrightarrow \cdots$$

$\hookrightarrow$ sew these all together.

Unraveled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_{p-2}) \arrow[d] & & & & \\
\cdots \arrow[r] & H_n(X_{p-1}) \arrow[r] \arrow[d, "i"] & H_n(X_{p-1}, X_{p-2}) \arrow[r] & H_n(X_{p-2}) \arrow[d, "i"] & & \\
\cdots \arrow[r] & H_n(X_p) \arrow[r, "j"] \arrow[d, "i"] & H_n(X_p, X_{p-1}) \arrow[r, "k"] & H_{n-1}(X_{p-1}) \arrow[r, "j"] \arrow[d, "i"] & H_{n-1}(X_{p-1}, X_{p-2}) \arrow[r] & \cdots \\
& H_n(X_{p+1}) \arrow[r] & H_n(X_{p+1}, X_p) \arrow[r, "k"] & H_{n-1}(X_p) \arrow[r, "j"] \arrow[d] & H_{n-1}(X_p, X_{p-1}) \arrow[r, "k"] & \cdots \\
& & & H_{n-1}(X_{p+1}) \arrow[r] & H_{n-1}(X_{p+1}, X_p) \arrow[r] & \cdots
\end{tikzcd}$$

$H_n(X)$ * relative interpolates $H_{n-1}(X)$

Claim. There's a SS of the form $(n = p+q)$

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \Longrightarrow H_{p+q}(X), \quad d^1 = jk$$

Let $E^2_{p,q} = \ker d^1 / \operatorname{Im} d^1$ $\quad d^2 = ?$

If $d^1(a) = 0$ then $jk(a) = 0$

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_{p-2}) \arrow[d] & & & \\
\cdots \arrow[r] & H_n(X_{p-1}) \arrow[r] \arrow[d, "i"] & H_n(X_{p-1}, X_{p-2}) \arrow[r] & H_n(X_{p-2}) \arrow[r] \arrow[d, "i"] & H_{n-1}(X_{p-2}, X_{p-3}) \\
\cdots \arrow[r] & H_n(X_p) \arrow[r, "j"] & H_n(X_p, X_{p-1}) \arrow[r, "k"] & H_{n-1}(X_{p-1}) \arrow[r, "j"] & H_{n-1}(X_{p-1}, X_{p-2}) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
& & b \arrow[r, mapsto] \arrow[d, mapsto] & j(b) \\
a \arrow[r, mapsto] \arrow[rrru, bend left, "d_2"] & k(a) \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

bule sequence is exact.

Should have $d^2 : E^2_{p,q} \longrightarrow E^2_{p-2,q+2-1}$. Why is d^2 well-defined?

Why is $d^2(a) \in E^2_{p-2,q+2-1} = \ker(d^1)/\operatorname{Im}(d^1)$? ...

If $a \in \operatorname{Im} d^1$, then $k(a) = 0$, so well-defined.

$d^2(a) = j(b)$. so $d^1(j(b)) = jkj(b) = 0$ and so $j(b) \in \ker d^1$

d^3 – same idea, lift through i again $\quad \cdots \quad d^{r+1}$ "=" $j(i^{-1})^r i$