BASICS OF DIFFERENTIAL GEOMETRY 2

Notes of BIMSA course

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Contents

Introduction

Last semester:

- Geometry of vector bundles
- Basic Riemannian geometry
- Differential operators on manifolds

We will learn this semester:

- Theory of principle bundles
- characteristic classes
- Basics of complex manifold, Kähler manifold, symplectic manifold.

1 Principle Bundles

In this section, we introduce the connections of principle bundles, it's closely related to the connections of vector bundles.

1.1 Lie Groups

Definition 1.1. Let G be a smooth manifold. G is a *Lie group* if G is a group s.t. multiplication and inverse are smooth.

Let G be a Lie group, $g \in G$, we denote:

- $L_g : G \to G$, $h \mapsto gh$ (left translation)
- $R_g : G \to G$, $h \mapsto hg$ (right translation)
- $\mathfrak{X}^L(G) = \{X \in \mathfrak{X}(G) \mid \forall g \in G, (L_g)_* X = X\}$ (left invariant vector fields)

For $X \in \mathfrak{X}^L(G)$, $L_{g*}X = X$ means that X is L_g-related to X. Then for $\forall X, Y \in \mathfrak{X}^L(G)$, $L_{g*}([X,Y]) = [L_{g*}X, L_{g*}Y] = [X,Y]$, so $\mathfrak{X}^L(G)$ is closed under $[\cdot,\cdot]$

Definition 1.2. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a $\mathbb{K}$-vector space $\mathfrak{g}$ and a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we say $\mathfrak{g}$ is a *Lie algebra* if:

(1) $\forall X, Y \in \mathfrak{g}$, $[X,Y] = -[Y,X]$

(2) $\forall X, Y, Z \in \mathfrak{g}$, $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

$[\cdot,\cdot]$ is called Lie bracket.

So by definition we have $\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)$ is a Lie algebra.

Definition 1.3. For Lie algebra $\mathfrak{g}, \mathfrak{h}$, a linear map $f : \mathfrak{g} \to \mathfrak{h}$ is called the *Lie algebra homomorphism* if: $\forall X, Y \in \mathfrak{g}$, $f([X,Y]) = [f(X), f(Y)]$

If f is in addition an isomorphism, then f is called a *Lie algebra isomorphism*.

Let $e \in G$ be the unit of G. Set $\iota : \mathfrak{X}^L(G) \to T_eG$, $X \mapsto X_e$. Then ι is a linear isomorphism. Let $\mathfrak{g} = T_eG$, so we can define the Lie bracket on $\mathfrak{g}$ s.t. ι is a Lie algebra isomorphism, i.e. setting $X^\sharp = \iota^{-1}(X)$, $[X,Y] = \left[X^\sharp, Y^\sharp\right]_e$. Note that $X^\sharp_g = (L_g)_{*e}X$, $g \in G$.

Definition 1.4. Let G be Lie group, $\mathfrak{g} = T_eG$ with $[\cdot,\cdot]$ is called the *Lie algebra of G*. $\left(\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)\right.$ is also called the Lie algebra of $\left.G\right)$

Definition 1.5. Let G, H be Lie groups. A map $\rho : G \to H$ is a *Lie group homomorphism* if ρ is a smooth map and a group homomorphism. For the special case $(\mathbb{R},+) \to G$, $t \mapsto g_t$, $\{g_t\}_{t\in\mathbb{R}}$ is called *one parameter subgroup of* G.

Proposition 1.1. Let G be Lie group and $\mathfrak{g}$ its Lie algebra. Then

(1) $\forall X \in \mathfrak{g}$, $X^\sharp = \iota^{-1}(X)$ is complete, i.e. $X^\sharp$ generates a flow $\{\varphi_t\}_{t\in\mathbb{R}}$.

(2) Set $\boxed{\exp_G(tX) = \varphi_t(e)} \in G$. Then $\varphi_t = R_{\exp_G(tX)}$.

(3) For $s,t \in \mathbb{R}$, $\exp_G(sX)\exp_G(tX) = \exp_G((s+t)X)$, i.e. $\{\exp_G(tX)\}_{t\in\mathbb{R}}$ is one parameter subgroup of G.

(4) $\mathfrak{g} \to \{\text{one parameter subgroup of } G\}$, $X \mapsto \{\exp_G(tX)\}_{t\in\mathbb{R}}$ is bijective.

Proof. (1) By ODE theory, $\exists\epsilon > 0$, $\gamma_e : (-\epsilon,\epsilon) \to G$ s.t. $\gamma_e(0) = e$, $\dfrac{d\gamma_e}{dt} = X^\sharp_{\gamma_e(t)}$.

$\boxed{\text{Claim 1.}}$ $\forall g \in G$, define $\gamma_g : (-\epsilon,\epsilon) \to G$, $t \mapsto g\gamma_e(t)$ is the integral curve of $X^\sharp$ with $\gamma_g(0) = g$.

Indeed, $\forall t \in (-\epsilon,\epsilon)$, $\dfrac{d\gamma_g}{dt}(t) = (L_g)_{*\gamma_e(t)}\dfrac{d\gamma_e}{dt}(t) = X^\sharp_{g\cdot\gamma_e(t)}$.

$\boxed{\text{Claim 2.}}$ $\gamma_e : (-\epsilon,\epsilon) \to G$ can be extended to integral curve $\gamma_e : \mathbb{R} \to G$ of $X^\sharp$ with $\gamma_e(0) = e$.

Set $\varphi_t = R_{\gamma_e(t)}$, then $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow generated by $X^\sharp$. So the following are easy. $\square$

By this proposition, we can define the exponential map $\exp_G : \mathfrak{g} \to G$.

Proposition 1.2. Let G, H be Lie groups with Lie algebra $\mathfrak{g}, \mathfrak{h}$. If $f : G \to H$ is Lie group homomorphism, then $f_{*e} : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. We only need to show that $X^\sharp$ and $(f_{*e}X)^\sharp$ are f-related. Since $X = \dfrac{d}{dt}\exp_G(tX)|_{t=0}$, we have $f_{*g}(X^\sharp_g) = \dfrac{d}{dt}f\left(g\cdot\exp_G(tX)\right)|_{t=0} = \dfrac{d}{dt}f(g)f\left(\exp_G(tX)\right)|_{t=0} = \left(L_{f(g)}\right)_{*e}(f_{*e}X) = (f_{*e}X)^\sharp_{f(g)}$. $\square$

Example 1.1. Let V be a $\mathbb{R}$-vector space, $G = GL(V)$, $\mathfrak{g}$ Lie algebra of G. Then $\mathfrak{g} = End(V)$, the bracket is given as follows:

Proposition 1.3. $\forall X, Y \in End(V)$, $[X,Y] = XY - YX$.

Proof. For $X \in End(V)$, set matrix exponential $e^{tX} = \sum\limits_{k=0}^{\infty}\frac{(tX)^k}{k!}$. Then $\{e_{tX}\}_{t\in\mathbb{R}}$ is a one parameter subgroup of G and $\frac{d}{dt}e^{tX}|_{t=0} = X$. So $\exp_G(tX) = e^{tX}$.

Then $[X,Y] = [X^\sharp, Y^\sharp]_e = \left(\mathcal{L}_{X^\sharp}Y^\sharp\right)_e = \dfrac{d}{dt}(\varphi_{-t})_{*e^{tX}}\left(Y^\sharp_{e^{tX}}\right)|_{t=0} = \dfrac{d}{dt}\dfrac{d}{ds}\varphi_{-t}\left(e^{tX}e^{sY}\right)|_{s=t=0} = XY - YX$. $\square$

Example 1.2. Set

- $O(n) = \{g \in GL(n;\mathbb{R}) \mid g^t g = E_n\}$ (orthogonal group)
- $SO(n) = \{g \in O(n) \mid \det g = 1\}$ (special orthogonal group)

we can check that $O(n), SO(n)$ are Lie subgroups of $GL(n;\mathbb{R})$.

$SO(n)$ is the unit component of $O(n)$, so $\mathfrak{o}(n) = \mathfrak{so}(n)$ (Lie algebra of $(O(n))$ and $SO(n)$). This is a Lie subalgebra of $End(\mathbb{R}^n)$ given by

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \left\{X \in End(\mathbb{R}^n) \mid X^t + X = O_n\right\}$$

where O_n is the zero matrix of size n.

Similarly, set

- $U(n) = \{g \in GL(n;\mathbb{C}) \mid g^* g = E_n\}$ (unitary group) where $g^* = \overline{g^t}$
- $SU(n) = \{g \in U(n) \mid \det g = 1\}$ (special unitary group)

We can check that

- $U(n), SU(n)$ are Lie subgroups of $GL(n;\mathbb{C})$
- $\mathfrak{u}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O\}$ (Lie algebra of $U(n)$)
- $\mathfrak{su}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O, \mathrm{tr}X = 0\}$ (Lie algebra of $SU(n)$)

$\boxed{\text{Note.}}$ A Lie subgroup H of G is a Lie group s.t.

- H is a subset of G
- inclusion map $H \hookrightarrow G$ is an embedding and group homomorphism

$\boxed{\text{Fact.}}$ A closed subgroup of G is a Lie subgroup of G.

Definition 1.6. Let V be a $\mathbb{K}$-vector space, G a Lie group. A Lie group homomorphism $\rho : G \to GL(V)$ is called a *representation of V*. The Lie algebra homomorphism $\rho_{*e} : \mathfrak{g} \to End(V)$ is called a *differential representation*.

Example 1.3. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. $\forall g \in G$, define a homomorphism

$$F_g : G \to G,\ h \mapsto ghg^{-1}$$

Note that $F_g \circ F_{g'} = F_{gg'}$. This induces a Lie algebra homomorphism $(F_g)_{*e} : \mathfrak{g} \to \mathfrak{g}$ which satisfies $(F_g)_{*e} \circ (F_{g'})_{*e} = (F_{gg'})_{*e}$. So we obtain a representation

$$Ad : G \to GL(\mathfrak{g}),\ g \mapsto (F_g)_{*e}$$

called *adjoint representation of G*. The differential representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ of Ad is given as follows.

Proposition 1.4. $\forall X, Y \in \mathfrak{g}$, $ad(X)(Y) = [X, Y]$.

Proof. Note that $F_g = R_{g^{-1}} \circ L_g$. Then

$ad(X)(Y) = \frac{d}{dt} Ad(\exp_G(tX))(Y)|_{t=0} = \frac{d}{dt} \left(R_{\exp_G(-tX)}\right)_{*\exp_G(tX)} \left(L_{\exp_G(tX)}\right)_{*e} (Y)|_{t=0} =$ $[X^\sharp, Y^\sharp]_e = [X, Y]$. □

Recall that there is a exponential map in Riemannian geometry. The Riemannian exp and the Lie group exp are related as follows.

Definition 1.7. A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group G is said to be *bi-invariant* if $\forall g, h \in G$, $L_g^* R_h^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$.

Theorem 1.1. Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$. Then $\exp_e = \exp_G$.

To show this we describe the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$.

Lemma 1.1. $\forall X, Y \in \mathfrak{g}$, $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

Proof. By Koszul formula, we have

$$\begin{aligned}\langle \nabla_{X^\sharp} Y^\sharp, Z^\sharp \rangle = \frac{1}{2} \Big(& X^\sharp \langle Y^\sharp, Z^\sharp \rangle + Y^\sharp \langle Z^\sharp, X^\sharp \rangle - Z^\sharp \langle X^\sharp, Y^\sharp \rangle \\ & - \langle Y^\sharp, [X^\sharp, Z^\sharp] \rangle - \langle Z^\sharp, [Y^\sharp, X^\sharp] \rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp] \rangle \Big)\end{aligned}$$

Since for $\forall g \in G$, $X_g^\sharp = \frac{d}{dt} g \cdot \exp_G(tX) \mid_{t=0}$, we have

$$X^\sharp \langle Y^\sharp, Z^\sharp \rangle = \frac{d}{dt} \langle Y^\sharp_{g \cdot \exp_G(tX)}, Z^\sharp_{g \cdot \exp_G(tX)} \rangle_{g \cdot \exp_G(tX)} \mid_{t=0} = \frac{d}{dt} \langle Y, Z \rangle_e \mid_{t=0} = 0$$

Since $\langle \cdot, \cdot \rangle$ is bi-invariant,

$$L_g^* R_{g^{-1}}^* \langle \cdot, \cdot \rangle_e = \langle \cdot, \cdot \rangle_e \text{ for } \forall g \in G \iff \langle Ad(g)(\cdot), Ad(g)(\cdot) \rangle_e = \langle \cdot, \cdot \rangle_e$$

Setting $g = \exp_G(tZ)$ and $\frac{d}{dt} \mid_{t=0}$, we have $\langle ad(Z)(\cdot), \cdot \rangle_e + \langle \cdot, ad(Z)(\cdot) \rangle_e = 0$, which shows that $\langle Y^\sharp, [X^\sharp, Z^\sharp] \rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp] \rangle = 0$, so we have $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

□

The proof of the theorem completes once shown that $\exp_G(tX)$ is geodesic, which is left as an exercise.

Exercise 1.1. Prove the theorem.

Remark 1.1. Existence/uniqueness of bi-invariant metrics? Some facts from representation theory are needed, the argument here is not used after this remark.

$\boxed{\text{Existence}}$ When G is compact, $\exists$ bi-invariant metric using "averaging trick".

- We first define Ad-invariant inner product on $\mathfrak{g}$.
- Then extend it to the whole G by pulling back L_g.

$\boxed{\text{Note:}}$ $\exists$ bi-invariant on $G \iff \exists$ Ad-invariant inner product on $\mathfrak{g}$.

$$\begin{cases} (\Rightarrow) \text{ Trivial.} \\ (\Leftarrow) \text{ Given } Ad\text{-invariant inner product on } \mathfrak{g}, \text{ we can extend it to left-invariant metric} \\ \quad \text{on } G, \text{ this is also right-invariant by pullback of } R_h = R_h \circ L_{h^{-1}} \circ L_h = Ad(h^{-1}) \circ L_h \end{cases}$$

$\boxed{\text{Uniqueness}}$ When G is abelian, then $L_g = R_g$, so $\exists$ many bi-invariant metrics on G (Any inner product on $\mathfrak{g}$ induces left-invariant metric on $\mathfrak{g}$, by the note above it is bi-invariant). Suppose that $\exists$ Ad-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. By $\langle \cdot, \cdot \rangle$, we have an irreducible decomposition of $(\mathfrak{g}, Ad)$: $\mathfrak{g} = \mathfrak{g}_1^{\oplus n_1} \oplus \cdots \oplus \mathfrak{g}_r^{\oplus n_r}$, where $\mathfrak{g}_i$ is irreducible representation of G and $\mathfrak{g}_i \neq \mathfrak{g}_j$ for $i \neq j$. Then

$$\dim \{Ad\text{-invariant symmetric bilinear map } \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\} = \sum_{i=1}^{r} n_i^2$$

To see this, take $T \in \{Ad\text{-invariant symmetric bilinear map }\}$ and use Schur's lemma to

$$T_{ij} : \mathfrak{g}_i \hookrightarrow \mathfrak{g} \xrightarrow{x \mapsto T(x, \cdot)} \mathfrak{g}^* \overset{\langle \cdot, \cdot \rangle}{\cong} \mathfrak{g} \xrightarrow{proj.} \mathfrak{g}_j$$

Then $T_{ij} = \begin{cases} 0 & (i \neq j) \\ c \cdot id & (i = j) \text{ for } \exists c \in \mathbb{R} \end{cases}$, so uniqueness up to scalar multiplication holds only when $r = 1, n = 1$, i.e. $(\mathfrak{g}, Ad)$ is irreducible $\iff$ G is simple Lie group.

Definition 1.8. Let M be smooth manifold, G be Lie group with unit e. A smooth map

$$A : M \times G \to M, \ (x, g) \mapsto xg$$

is called the *right action of G on M* if

(1) $\forall x \in M$, $xe = x$

(2) $\forall x \in M$, $\forall g, g \in G$, $(xg)h = x(gh)$

We write the right action as $M \curvearrowleft G$.

Definition 1.9. Suppose $M \curvearrowleft G$.

(1) For $\forall g \in G$, set $R_g : M \to M$, $X \mapsto xg$ (*right translation*).

(2) For $\forall X \in \mathfrak{g}$, define the *fundamental vector field* $X^\sharp \in \mathfrak{X}(M)$ by $X^\sharp_x = \dfrac{d}{dt} x \cdot \exp_G(tX) \mid_{t=0} = dA(x, \cdot)_e(X)$.

Here the notation $X^\sharp$ is the same as the left-invariant vector field on Lie group, we'll show that they have the same property:

Remark 1.2. (1) $\forall g \in G$, $\forall X \in \mathfrak{g}$, $(R_g)_* X^\sharp = \left(Ad(g^{-1})X\right)^\sharp$.

(2) $\forall X, Y \in \mathfrak{g}$, $[X^\sharp, Y^\sharp] = [X, Y]^\sharp$.

Proof. (1) $\forall x \in M$, $\left((R_g)_* X^\sharp\right)_x = (R_g)_* X^\sharp_{xg^{-1}} = \frac{d}{dt} xg^{-1} \exp_G(tX)g \mid_{t=0}$. Since $\left\{g^{-1}\exp_G(tX)g\right\}_{t\in\mathbb{R}}$ is a one parameter subgroup of G with $\frac{d}{dt} g^{-1}\exp_G(tX)g \mid_{t=0} = Ad(g^{-1})X$, then $g^{-1}\exp_G(tX)g = \exp_G(tAd(g^{-1})X)$, which gives (1).

(2) By definition, $\left\{\varphi_t = R_{\exp_G(tX)}\right\}_{t\in\mathbb{R}}$ is flow of $X^\sharp$. So

$$[X^\sharp, Y^\sharp] = \frac{d}{dt}(\varphi_{-t})_* Y^\sharp \mid_{t=0} = \frac{d}{dt}\left(Ad\left(\exp_G(tX)\right)Y\right)^\sharp \mid_{t=0} = \left(ad(X)(Y)\right)^\sharp = [X, Y]^\sharp.$$

□

Remark 1.3. We can define the left action

$$A^L : G \times M \to M,\ (g, x) \mapsto gx$$

and also the fundamental vector field $X_L^\sharp \in \mathfrak{X}(M)$. The left and right actions are essentially the same, since the right action is given form the left action. Indeed, given A^L above, define A by $A(x, g) = A^L(g^{-1}, x) = g^{-1}x$, then $X_L^\sharp = -X^\sharp$ for $X \in \mathfrak{g}$. $[X_L^\sharp, Y_L^\sharp] = [X, Y]^\sharp = -[X, Y]_L^\sharp$.

Definition 1.10. Suppose $M \curvearrowleft G$.

(1) For $p \in M$, define $G_p = \{g \in G \mid pg = p\}$ (*isotropy subgroup at* p).

(2) The G action is *free* of $G_p = \{e\}$ for $\forall p \in M$.

(3) The G action is *effective* if $\bigcap_{p\in M} G_p = \{e\}$. In other words, $G \to \mathrm{Diff}(M)$ is injective.

1.2 Definition of Principle Bundles

Definition 1.11. Let P, M be smooth manifolds and G be Lie group. The map $\pi_P : P \to M$ is a *principle G-bundle* or *principle bundle with structure group G* if:

(1) $P \curvearrowleft G$.

(2) There exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of M and diffeomorphisms called local trivialization

$$\phi_\alpha : \pi_P^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$$

such that

(2.1) Denoting by $p_1 : U_\alpha \times G \to U_\alpha$ the projection, then $\pi_P = p_1 \circ \phi_\alpha$

(2.2) The G-action preserves each $\pi_P^{-1}(U_\alpha)$. Denoting the right G-action on $U_\alpha \times G$ by

$$(U_\alpha \times G) \times G \to U_\alpha \times G,\ ((x,h),g) \mapsto (x,h)\cdot g = (x,hg)$$

Then ϕ_α is G-equivalent, i.e. $\forall \xi \in \pi_P^{-1}(U_\alpha)$, $\forall g \in G$, $\phi_\alpha(\xi g) = \phi_\alpha(\xi)g$. Note that the G-action is free.

We often write $P\mid_U = \pi_P^{-1}(U)$ for open subset $U \subseteq M$ and $P_x = \pi_P^{-1}(x)$ for $x \in M$, P_x is called the fiber of P at x.

Recall that $e \in G$ is the unit, define a section $p_\alpha \in \Gamma(P\mid_{U_\alpha})$ on U_α: $\phi_\alpha(p_\alpha(x)) = (x,e)$, which is equivalent to $p_\alpha(x) = \phi_\alpha^{-1}(x,e)$. Define $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ by $\boxed{p_\alpha(x)g_{\alpha\beta}(x) = p_\beta(x)}$, $\{g_{\alpha\beta}\}_{\alpha\beta}$ is called the transition map of $\pi_P : P \to M$. Note that $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$, we have $\boxed{g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)}$. Conversely, given open covering $\{U_\alpha\}_{\alpha\in A}$ of M and transition maps, we can recover principle G-bundle $\pi_P : P \to M$.

As before, for $g \in G$, we can define $R_g : P \to P$ the right translation and the fundamental vector field $X^\sharp$ generated by $X \in \mathfrak{g}$.

Definition 1.12. Let $\pi_P : P \to M$ be a principle G-bundle, $\rho : G \to GL(V)$ representation of G. Define the right G-action on $P \times V$ by

$$(P \times V) \times G \to P \times V,\ ((\xi,v),g) \mapsto \left(\xi g, \rho(g)^{-1}v\right)$$

$P \underset{\rho}{\times} V = (P \times V)/G$ is called the *associated vector bundle to P*.

Set $\xi \underset{\rho}{\times} v$ the equivalence class of $(\xi,v) \in P \times V$. Set $E = P \underset{\rho}{\times} V$, $\pi_E : E \to M$, $\xi \underset{\rho}{\times} v \mapsto \pi_P(\xi)$. Then $\pi_E : E \to M$ is a vector bundle.

The local trivialization of E are induced from those of P:

$$\phi_\alpha^E : E\mid_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V,\ p_\alpha(x) \underset{\rho}{\times} v \mapsto (x,v)$$

For $x \in U_\alpha \cap U_\beta$ and $v_\beta \in V$, $p_\beta(x) \underset{\rho}{\times} v_\beta = p_\alpha g_{\alpha\beta}(x) \underset{\rho}{\times} v_\beta = p_\alpha(x) \underset{\rho}{\times} \rho(g_{\alpha\beta}(x))v_\beta$. The transition functions of E are given by $\{\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to GL(V)\}$.

We will explain some relations between P and E.

- First note that $\forall \xi \in P$, we have $\xi : V \xrightarrow{\cong} E_{\pi_P(\xi)}$, $v \mapsto \xi \underset{\rho}{\times} v$ is an isomorphism. For $\xi' \in P$ with $\xi' = \xi g$ for $g \in G$, we have $\xi^{-1}\left(\xi' \underset{\rho}{\times} v'\right) = \xi^{-1}\left(\xi \underset{\rho}{\times} \rho(g)v'\right) = \rho(g)v'$ for $v' \in V$.
- $\pi_P^* E$ is a trivial bundle. Indeed,
$$P \times V \xrightarrow[(\xi,\xi^{-1}(e)) \leftarrow (\xi,e)]{(\xi,v)\mapsto(\xi,\xi \underset{\rho}{\times} v)} \pi_P^*E = \{(\xi,e) \in P \times E \mid \pi_P(\xi) = \pi_E(e)\} \text{ is isomorphism.}$$
- Next, for $s \in \Omega^q(E) = \Gamma(\Lambda^q T^*M \otimes E)$, define $\pi_P^* s \in \Omega^q(P;V)$ as follows (V-

valued q-form on P)

- For $q=0$, $(\pi_P^* s)(\xi) = \xi^{-1}(s(\pi_P(\xi)))$
- For $q > 1$, $\forall \alpha \in \Omega^q(M)$, $\forall s \in \Omega^0(E) = \Gamma(E)$,

$$\pi_P^*(\alpha \otimes s) = \pi_P^* \alpha \otimes \pi_P^* s$$

The left one is pullback and the right one is define above. In other words, $\forall \xi \in P$, $\forall v_1, \cdots, v_q \in T_\xi P$,

$$(\pi_P^* s)_\xi (v_1, \cdots, v_q) = \xi^{-1}\left(s_{\pi_P(\xi)}(\pi_{P*}(v_1), \cdots, \pi_{P*}(v_q))\right)$$

Notation: denote $\Omega_B^q(P;V)$ to be the elements $\widetilde{s}$ in $\Omega^q(P;V)$ satisfying:

(1) $\forall X \in \mathfrak{g}$, $i(X^\sharp)\widetilde{s} = 0$.

(2) $\forall g \in G$, $R_g^* \widetilde{s} = \rho(g)^{-1}\widetilde{s}$.

called the *space of basic q-forms*. Note that $\Omega_B^q(P;V)$ depends on representation ρ.

Proposition 1.5. (**Important to study the relations between P and E**)

(1) $\pi_P^*(\Omega^q(E)) \subseteq \Omega_B^q(P;V)$ and $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega_B^q(P;V)$. E-valued q-forms on M are identified with basic q-forms on P.

(2) Recall the local trivialization $\phi_\alpha^E : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$. For $s \in \Omega^q(E)$, suppose that $s|_{U_\alpha}$ corresponds to $s_\alpha \in \Omega^q(U_\alpha; V)$. Then $s_\alpha = p_\alpha^*(\pi_P^* s)$. So we regard $s \in \Omega^q(E)$ as a basic form, and then pullback by p_α is s_α.

Proof. (1) We show $\pi_P^*(\Omega^q(E)) \subseteq \Omega_B^q(P;V)$. Take $\forall s \in \Omega^q(E)$,

- $\boxed{\text{For } q=0}$ (1) is trivial; For (2): for $g \in G$, $\xi \in P$, we have

$$\left(R_g^* \pi_P^* s\right)(\xi) = (\pi_P^* s)(R_g \xi) = (\xi g)^{-1}(s(\pi_P(\xi g))) = (\xi g)^{-1}(s(\pi_P(\xi)))$$

By definition of ξ, we have: for $\forall v \in V$,

$$\xi(v) = \xi \underset{\rho}{\times} v = \xi g \underset{\rho}{\times} \rho(g)^{-1}(v) = (\xi g)(\rho(g)^{-1}(v))$$

so $\xi = (\xi g) \circ \rho(g)^{-1}$, hence $(\xi g)^{-1} = \rho(g)^{-1} \circ \xi^{-1}$. Then

$$\left(R_g^* \pi_P^* s\right)(\xi) = \rho(g)^{-1}\left(\xi^{-1} s(\pi_P(\xi))\right) = \left(\rho(g)^{-1}(\pi_P^* s)\right)(\xi).$$

- $\boxed{\text{For } q \geq 1}$ (1): Since $\pi_P(\xi g) = \pi_P(\xi)$, we have $\pi_{P*}(X^\sharp) = 0$, which implies (1); (2): For $\forall \alpha \in \Omega^q(M)$, $\forall s \in \Gamma(E)$, $\forall g \in G$, we have

$$R_g^*(\pi_P^*(\alpha \otimes s)) = R_g^* \pi_P^* \alpha \otimes R_g^* \pi_P^* s = \pi_P^* \alpha \otimes \rho(g)^{-1}(\pi_P^* s) = \rho(g)^{-1} \pi_P^*(\alpha \otimes s)$$

which finishes the proof of (2).

Next we show $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega_B^q(P;V)$:

- $\boxed{\text{Injectivity}}$ It is clear from the formula

$$(\pi_P^* s)_\xi (v_1, \cdots, v_q) = \xi^{-1} \left(s_{\pi_P(\xi)} (\pi_{P*}(v_1), \cdots, \pi_{P*}(v_q)) \right).$$

- $\boxed{\text{Surjectivity}}$ Take $\widetilde{s} \in \Omega_B^q(P; V)$,
  - When $q = 0$, define $s \in \Omega^0(E) = \Gamma(E)$ by $s(x) = \xi \underset{\rho}{\times} \widetilde{s}(\xi)$ where $\xi \in \pi_P^{-1}(x)$. It is well-defined since $\xi g \underset{\rho}{\times} \widetilde{s}(\xi g) = \xi g \underset{\rho}{\times} (R_g^* \widetilde{s})(\xi) = \xi g \underset{\rho}{\times} \rho(g)^{-1} \widetilde{s}(\xi) = \xi \underset{\rho}{\times} \widetilde{s}(\xi)$. Then by definition we have $\pi_P^* s = \widetilde{s}$.
  - When $q \geq 1$, define $s \in \Omega^0(E) = \Gamma(E)$ by

$$s_x(w_1, \cdots, w_q) = \xi \underset{\rho}{\times} \widetilde{s}_\xi(\widetilde{w_1}, \cdots, \widetilde{w_q})$$

    where $x \in M$, $w_i \in T_x M$, $\xi \in \pi_P^{-1}(x)$, $\pi_{P*}(\widetilde{w_i}) = w_i$. It's left as an exercise to check s is well-defined in this case.

(2) First we describe s_α clearly. Set $s|_{U_\alpha} = \sum \beta_i \otimes e_i$. Since

$$\phi_\alpha^E : E\,|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V, \; p_\alpha(x) \underset{\rho}{\times} v \mapsto (x, v),$$

we have $\phi_\alpha^E((e_i)_x) = (x, v_i(x))$ for a function $v_i : U_\alpha \to V$. Note that $(e_i)_x = p_\alpha(x) \underset{\rho}{\times} v_i(x)$. Then $s_\alpha = \sum \beta_i \otimes v_i$. Now we compute

$$p_\alpha^* (\pi_P^* s) = p_\alpha^* \left(\sum \pi_P^* \beta_i \otimes \pi_P^* e_i \right) = \sum (\pi_P \circ p_\alpha)^* \beta_i \otimes (\pi_P^* e_i)\, p_\alpha(x) = \sum \beta_i \otimes v_i(x).$$

So we have $p_\alpha^* (\pi_P^* s) = s_\alpha$. □

Now we give a typical example of principle bundles.

Example 1.4. Let $\pi_E : E \to M$ be a vector bundle with rank r. For $x \in M$, set

- $P_x = \{\xi : \mathbb{K}^r \to E_x : \text{ linear isomorphism } \}$.
- $P = \coprod_{x \in M} P_x$; $\pi_P : P \to M$, $\xi \mapsto x$ if $\xi \in P_x$.

We see that $\pi_P : P \to M$ is a principle $GL(r; \mathbb{K})$-bundle:

- The right action on P is given by:

$$P \times GL(r; \mathbb{K}) \to P, \; (\xi \times g) \mapsto \xi \circ g.$$

- To give a local trivialization, first note that

$$P_x \xrightarrow[\xi \mapsto \{\xi(\epsilon_1), \cdots, \xi(\epsilon_r)\}]{\cong} \{\text{basis of } E_x\},$$

  where $\epsilon_i = (0, \cdots, \underset{i\text{-th}}{1}, \cdots, 0)^t$. If $\{e_1, \cdots, e_r\} \subseteq \Gamma(E|_{U_\alpha})$ is local frame of E over $U_\alpha \subseteq M$, define $p_\alpha \in \Gamma(P|_{U_\alpha})$ by

$$p_\alpha : U_\alpha \to P|_{U_\alpha}, \; x \mapsto (e_1(x), \cdots, e_r(x)),$$

which induces a local trivialization

$$\phi_\alpha^P : P|_{U_\alpha} \to U_\alpha \times GL(r;\mathbb{K}),\ \xi \mapsto \left(\pi_P(\xi), (p_\alpha(\pi_P(\xi)))^{-1}\xi\right)$$

The inverse of this map is $(x, g) \mapsto p_\alpha(x) \cdot g$. We see that ϕ_α^P is $GL(r;\mathbb{K})$-equivalent.

So $\pi_P : P \to M$ is a principle $GL(r;\mathbb{K})$-bundle. This is called the *frame bundle* of $\pi_E : E \to M$. Also note that transition maps of E is the transition maps of P. Indeed, if $\{f_1, \cdots, f_r\} \subseteq \Gamma(E|_{U_\alpha})$ is another local frame, the transition map $g_{\alpha\beta}$ satisfies $(f_1, \cdots, f_r) = (e_1, \cdots, e_r)g_{\alpha\beta}$, and this is exactly $p_\beta = p_\alpha g_{\alpha\beta}$.

1.3 Connections on Principle Bundles

In this subsection we study properties of connection on principle bundle and its relation between connection on associated vector bundle.

Definition 1.13. Let $\pi_P : P \to M$ be principle G-bundle.

(1) A distribution $\{H_\xi \subseteq T_\xi P\}_{\xi\in P}$ is a *connection* on P if

(1-1) $\forall \xi \in P$, $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$.

(1-2) $\{H_\xi \subseteq T_\xi P\}_{\xi\in P}$ is G-invariant, i.e. $\forall\xi \in P$, $\forall g \in G$, $(R_g)_{*\xi}H_\xi = H_{\xi g}$.

H_ξ, $\ker(\pi_P)_{*\xi}$ are called *horizontal/vertical subspaces.*

(2) A $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P;\mathfrak{g})$ on P is a *connection form* if

(2-1) $\forall X \in \mathfrak{g}$, $\theta(X^\sharp) = X$.

(2-2) $\forall g \in G$, $R_g^*\theta = Ad(g^{-1})\theta$.

These 2 notions are the same in the following sense:

Theorem 1.2. Let $\pi_P : P \to M$ be principle G-bundle.

(1) If $\theta \in \Omega^1(P;\mathfrak{g})$ is a connection form, a distribution $\{\ker\theta_\xi\}_{\xi\in P} = \{v \in T_\xi P \mid \theta_\xi(v) = 0\}_{\xi\in P}$ is a connection on P.

(2) $\{\text{connection form}\} \to \{\text{connection on } P\}$, $\theta \mapsto \{\ker\theta_\xi\}_{\xi\in P}$ is bijective.

Proof. (1) We check that $\{\ker\theta_\xi\}_{\xi\in P}$ satisfies (1-1), (1-2):

(1-1) Note that $\ker(\pi_P)_{*\xi} = \left\{X_\xi^\sharp \in T_\xi P \mid X \in \mathfrak{g}\right\}$, then for $\forall v \in T_\xi P$, we have $\theta(v) \in \mathfrak{g}$ and $v = \theta(v)_\xi^\sharp + \left(v - \theta(v)_\xi^\sharp\right)$, which implies that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus \ker\theta_\xi$ ($\ker(\pi_P)_{*\xi} \cap \ker\theta_\xi = \{0\}$ is obvious).

(1-2) Take $\forall v \in \ker\theta_\xi$. By (2-2), $\forall g \in G$, we have $\left(R_g^*\theta\right)_\xi = Ad(g^{-1})\theta_\xi$, the left hand side is $\theta_{\xi g}((R_g)_{*\xi}(\cdot))$, so we have $(R_g)_{*\xi}(v) \in \ker\theta_{\xi g}$, hence $(R_g)_{*\xi}(\ker\theta_\xi) \subseteq \ker\theta_{\xi g}$. Replacing (g, ξ) with $(g^{-1}, \xi g)$, we have $(R_{g^{-1}})_{*\xi g}(\ker\theta_{\xi g}) \subseteq \ker\theta_\xi$. So $(R_g)_{*\xi}(\ker\theta_\xi) = \ker\theta_{\xi g}$, $\{\ker\theta_\xi\}_{\xi\in P}$ is a connection on P.

(2) $\boxed{\text{Injectivity}}$ Let θ, θ' be connection forms with $\ker \theta_\xi = \ker \theta'_\xi$ $\forall \xi \in P$. We show that $\forall v \in T_\xi P$, $\theta_\xi(v) = \theta'_\xi(v)$. By (1), v is described as $v = X^\sharp_\xi + w$ for $X^\sharp_\xi \in \ker(\pi_P)_{*\xi}$ and $w \in \ker \theta_\xi = \ker \theta'_\xi$. So $\theta_\xi(v) = \theta_\xi(X^\sharp_\xi) = X = \theta'_\xi(v)$.

$\boxed{\text{Surjectivity}}$ Take $\forall \{H_\xi\}_{\xi \in P}$ a connection on P. By (1-1), we can define $\theta \in \Omega^1(P; \mathfrak{g})$ by

$$\theta_\xi(v) = \begin{cases} 0 & (v \in H_\xi) \\ X & (v = X^\sharp_\xi \text{ for } X \in \mathfrak{g}) \end{cases}$$

By definition, $\ker \theta_\xi = H_\xi$, we check (2-1), (2-2).

(2-1) Holds by definition of θ_ξ.

(2-2) $\forall \xi \in P$, $\forall g \in G$, we show that $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = Ad(g^{-1})\theta_\xi$ on $T_\xi P$. Recall that $T_\xi P = \ker(\pi_P)_{*\xi} \oplus H_\xi$, if $v \in H_\xi$, the equality holds by definition and (1-2); for $\forall X \in \mathfrak{g}$,

$$(R_g)_{*\xi}\left(X^\sharp_\xi\right) = (R_g)_{*\xi} \frac{d}{dt} \xi \exp_G(tX)\,|_{t=0} = \frac{d}{dt} \xi g \cdot g^{-1} \exp_G(tX) g\,|_{t=0} = \left(Ad(g^{-1})X\right)^\sharp_{\xi g}$$

So $\theta_{\xi g}\left((R_g)_{*\xi}(X^\sharp_\xi)\right) = Ad(g^{-1})X = Ad(g^{-1})\theta_\xi(X^\sharp_\xi)$, hence the equality holds. So we have $\theta_{\xi g}\left((R_g)_{*\xi}(\cdot)\right) = Ad(g^{-1})\theta_\xi$ on $T_\xi P$. □

The next proposition says that a connection form θ on P induces a connection ∇^E of the associated vector bundle E. The relation between θ and local connection form of ∇^E us also given.

Proposition 1.6. Let $\pi_P : P \to M$ be a principle bundle, $\rho : G \to GL(V)$ a representation of G with differential representation $\rho_* : \mathfrak{g} \to End(V)$. Denote by $\theta \in \Omega^1(P; \mathfrak{g})$ a connection form. Set $E = P \underset{\rho}{\times} V$ its associated vector bundle. Then,

(1) $(d + \rho_*(\theta)\wedge)\, \Omega^q_B(P; V) \subseteq \Omega^{q+1}_B(P; V)$. Here

- d: standard exterior derivative.
- $\rho_*(\theta) \in \Omega^1(P; End(V))$ acts on $\Omega^q_B(P; V)$ by wedging on differential form parts and composing $End(V), V$-parts.

(2) Recall that $\pi_P^* : \Omega^q(E) \xrightarrow{\cong} \Omega^q_B(P; V)$. Then we can define $\nabla^E : \Omega^0(E) \to \Omega^1(E)$ by $(\pi_P^*)^{-1} \circ (d + \rho_*(\theta)\wedge) \circ \pi_P^*$.

(3) Recall that a local section $p_\alpha \in \Gamma(P|_{U_\alpha})$ induces a local trivialization $\phi^E_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$. Then

$$\begin{tikzcd} \Omega^0(E|_{U_\alpha}) \arrow[r, "\nabla^E|_{U_\alpha}"] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\circlearrowright"] & \Omega^1(E|_{U_\alpha}) \arrow[d, phantom, "\cong" rotate=90] \\ \Omega^0(U_\alpha; V) \arrow[r, "d+\rho_*(p_\alpha^*\theta)\wedge"'] & \Omega^1(U_\alpha; V) \end{tikzcd}$$

(4) Recall that a connection ∇^E induces the exterior derivative $d^{\nabla^E} : \Omega^q(E) \to \Omega^{q+1}(E)$. Then

$$\begin{tikzcd}
\Omega^q(E) \arrow[r, "d^{\nabla^E}"] \arrow[d, "\pi_P^*"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \Omega^{q+1}(E) \arrow[d, "\pi_P^*"', "\cong"] \\
\Omega_B^q(P;V) \arrow[r, "d+\rho_*(\theta)\wedge"'] & \Omega_B^{q+1}(P;V)
\end{tikzcd}$$

Remark 1.4. In [Kobayashi-Nomizu, *Foundation of differential geometry* Vol 1, chapter 2, section 5], for any principle G-bundle with a connection form $\theta \in \Omega^1(P;\mathfrak{g})$, $\forall V$ vector space, the *exterior covariant derivative* $D : \Omega^q(P;V) \to \Omega^{q+1}(P;V)$ is defined by $(D\widetilde{s})(v_0,\cdots,v_q) = (d\widetilde{s})(hv_0,\cdots,hv_q)$ for $v_i \in TP$, where $h : TP \to \ker\theta$ is the projection. If in addition, given a representation $\rho : G \to GL(V)$ and $\widetilde{s} \in \Omega_B^q(P;V)$, we have $D\widetilde{s} = (d+\rho_*(\theta)\wedge)(\widetilde{s})$.

Proof. (1) Take $\forall \widetilde{s} \in \Omega_B^q(P;V)$, recall that $\begin{cases} \forall X \in \mathfrak{g}, i(X^\sharp)\widetilde{s} = 0. \\ \forall g \in G, R_g^*\widetilde{s} = \rho(g)^{-1}\widetilde{s}. \end{cases}$. We show that $(d+\rho_*(\theta)\wedge)\,\widetilde{s}$ also satisfies the same property.

- $\forall X \in \mathfrak{g}$, we have
$$\mathcal{L}_{X^\sharp}\widetilde{s} = \frac{d}{dt}R^*_{\exp_G(tX)}\widetilde{s}\mid_{t=0} = \frac{d}{dt}\rho\left(\exp_G(tX)\right)^{-1}\widetilde{s}\mid_{t=0} = -\rho_*(X)\widetilde{s}.$$
Since $\mathcal{L}_{X^\sharp}\widetilde{s} = i(X^\sharp)d\widetilde{s} + d\left(i(X^\sharp)\widetilde{s}\right)$ and $i(X^\sharp)\widetilde{s} = 0$, we have $i(X^\sharp)d\widetilde{s} = -\rho_*(X)\widetilde{s}$. Hence $i(X^\sharp)\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = i(X^\sharp)d\widetilde{s} + \rho_*\left(\theta(X^\sharp)\right)\widetilde{s} - \rho_*(\theta)\wedge i(X^\sharp)\widetilde{s} = 0$.
- For $\forall g \in G$, we have
$$R_g^*\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = dR_g^*\widetilde{s} + \rho_*\left(R_g^*\theta\right)\wedge R_g^*\widetilde{s} = d\left(\rho(g)^{-1}\widetilde{s}\right) + \rho_*\left(Ad(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s}.$$
Since $\rho(g)^{-1}$ acts only on V-part, $d\left(\rho(g)^{-1}\widetilde{s}\right) = \rho(g)^{-1}d\widetilde{s}$. Note that $\forall X \in \mathfrak{g}$,
$$\frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)g\right)\rho(g)^{-1}\mid_{t=0} = \frac{d}{dt}\rho\left(g^{-1}\exp_G(tX)\right)\mid_{t=0}$$
and $g^{-1}\exp_G(tX)g = \exp_G(tAd(g^{-1})X)$, we have
$$\rho_*\left(Ad(g^{-1})X\right)\rho(g)^{-1} = \rho(g)^{-1}\rho_*(X).$$
This implies that
$$\rho_*\left(Ad(g^{-1})\theta\right)\wedge\rho(g)^{-1}\widetilde{s} = \rho(g)^{-1}\left(\rho_*(\theta)\wedge\widetilde{s}\right).$$
Then we obtain
$$R_g^*\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right) = \rho(g)^{-1}\left((d+\rho_*(\theta)\wedge)(\widetilde{s})\right),$$

so $(d+\rho_*(\theta)\wedge)(\widetilde{s}) \in \Omega_B^{q+1}(P;V)$.

(2) $\nabla^E = (\pi_P^*)^{-1} \circ (d+\rho_*(\theta)\wedge) \circ \pi_P^*$, we check the Leibniz rule, i.e. for $\forall f \in C^\infty(M)$, $\forall s \in \Gamma(E)$, we show $\nabla^E(fs) = df \otimes s + f\nabla^E s$. This is left as an exercise.

(3) Since for $s \in \Omega^q(E)$, $s|_{U_\alpha}$ corresponds to $p_\alpha^*(\pi_P^* s)$. We compute

$$p_\alpha^*\pi_P^*\left(\nabla^E s\right) = p_\alpha^*\left((d+\rho_*(\theta)\wedge)\,\pi_P^* s\right) = p_\alpha^* d\left(\pi_P^* s\right) + \rho_*\left(p_\alpha^*\theta\right) \wedge p_\alpha^*\pi_P^* s = \left(d+\rho_*(p_\alpha^*\theta)\wedge\right)\left(p_\alpha^*\pi_P^* s\right).$$

(4) Since d^{∇^E} is given by $d^{\nabla^E}(s\otimes\alpha) = \nabla^E s \wedge \alpha + s \otimes d\alpha$ for $s \in \Gamma(E)$, $\alpha \in \Omega^q(M)$, we have

$$\begin{aligned}\pi_P^*\left(d^{\nabla^E}(s\otimes\alpha)\right) &= \pi_P^*\left(\nabla^E s\wedge\alpha + s\otimes d\alpha\right) = (d+\rho_*(\theta)\wedge)\,\pi_P^* s \wedge \pi_P^*\alpha + \pi_P^* s \otimes \pi_P^* d\alpha \\ &= d\left(\pi_P^* s \otimes \pi_P^*\alpha\right) + \rho_*(\theta) \wedge \left(\pi_P^* s \otimes \pi_P^*\alpha\right) = (d+\rho_*(\theta)\wedge)\left(\pi_P^*(s\otimes\alpha)\right).\end{aligned}$$

□

Exercise 1.2. Prove that ∇^E defined above is a connection.

Example 1.5. Given a vector bundle $\pi_E : E \to M$, let $\pi_P : P \to M$ be the frame bundle. Consider the trivial representation $id : GL(r;\mathbb{K}) \to GL(r;\mathbb{K})$. Then