Differential Topology

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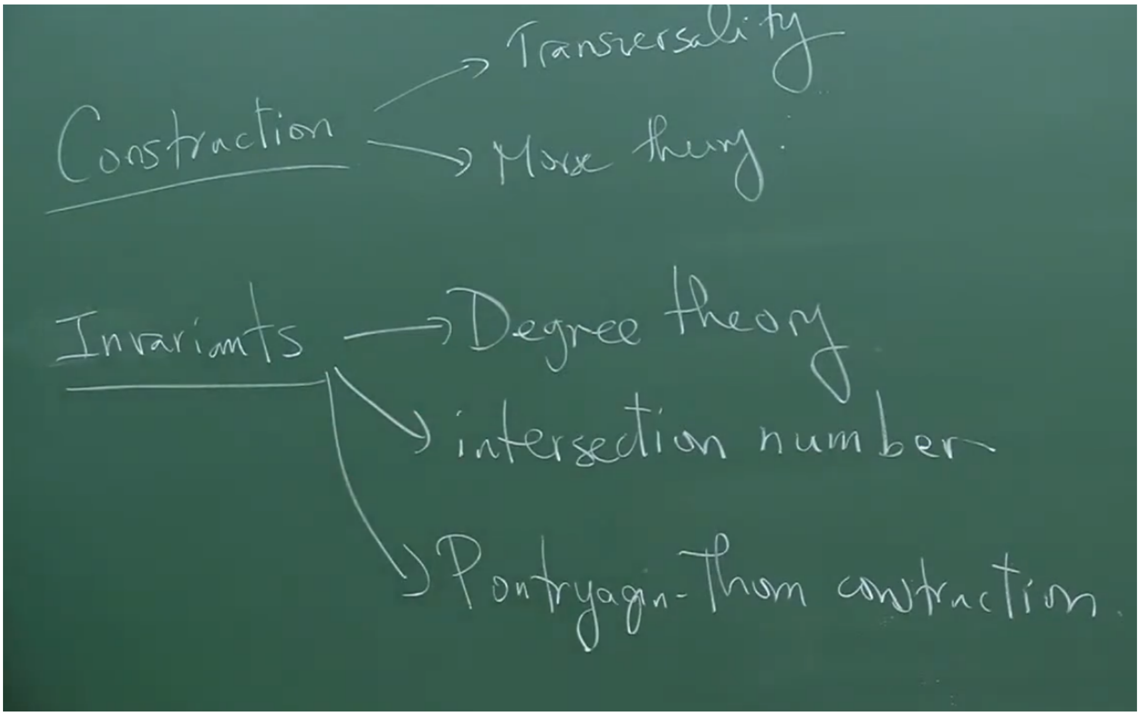

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1 Review of Differentiable Manifold

Definition 1.1: Topological Space

A topological space is a pair (X, T), where $T \subseteq \mathcal{P}(X)$ such that

- $\emptyset, X \in T$
- $\{U_\alpha\}_{\alpha \in I} \subseteq T \implies \bigcup_{\alpha \in I} U_\alpha \in T$
- $U_1, \ldots, U_n \subseteq T \implies U_1 \cap \cdots \cap U_n \in T$

Fixing (X, T), the elements of T are called open sets.

1.1 Jet bundles

Definition 1.2:

Let X, Y be smooth manifolds, $f, g : X \to Y$ smooth.
(1) We write $f \sim_k g$ at $p \in X$ if $f(p) = g(p)$ and given charts $\varphi : U \to \mathbb{R}^n$ around p, $\psi : V \to \mathbb{R}^m$ around $f(p)$

$$\frac{\partial^{|\alpha|}\left(\psi \circ f \circ \varphi^{-1}\right)_j}{\partial x^\alpha}(\varphi(p)) = \frac{\partial^{|\alpha|}\left(\psi \circ g \circ \varphi^{-1}\right)_j}{\partial x^\alpha}(\varphi(p)), \ \forall |\alpha| \leq k, 1 \leq j \leq m$$

It follows from the chain rule that $\sim_k$ is an equivalence relation.
(2) $J^k(X,Y)_{p,q} = \{f : X \to Y \text{ smooth} \mid f(p) = q\} / \sim_k$, called the space of k-jets at p with value q.
(3) $J^k(X,Y) = \bigsqcup_{\substack{p\in X\\ q\in Y}} J^k(X,Y)_{p,q}$.

Example 1.1:

(1) $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$,

$$J^k(U,V)_{x,y} \xrightarrow{\Phi} B^k_{n,m}, \ [f] \mapsto (p_1(x), \cdots, p_m(x))$$

where $p_j(x)$ is the Taylor polynomial of $f_j(x)$ without the constant term, $B^k_{n,m} = \{\text{polynomial functions } \mathbb{R}^n \to \mathbb{R}^m \text{ of degree} \leq k \text{ with no constant term}\}$. Φ is a bijection. $J^k(U,V) \cong U \times V \times B^k_{n,m}$.
(2) $J^1(M, \mathbb{R}) \xrightarrow{bijection} \mathbb{R} \times T^*M, \ [f]_x \mapsto (f(x), df_x)$.
(3) $J^1(\mathbb{R}, M) \cong \mathbb{R} \times TM$.

Definition 1.3:

(1) $\varphi : Y \to Z$ smooth. Then $\varphi_* : J^k(X,Y) \to J^k(X,Z), \ [f]_x \mapsto [\varphi \circ f]_x$.
(2) $\varphi : Z \to X$ diffeo. Then $\varphi^* : J^k(X,Y) \to J^k(Z,Y)$, $[f]_x \mapsto [f \circ \varphi]_{\varphi^{-1}(x)}$.

$\boxed{\text{Remark.}}$ These operations are well-defined and natural (functionality). In particular, if $\varphi : Y \to Z$ diffeo, then φ_* is bijection; $\varphi : Z \to X$ diffeo, then φ^* is bijection.

Suppose $\sigma \in J^k(X,Y), \ \sigma = [f]_x$.

Define $\alpha(\sigma) = x, \beta(\sigma) = f(x)$, called the source of σ and target of σ respectively, then $\alpha : J^k(X,Y) \to X, \beta : J^k(X,Y) \to Y$. We will define the local topology around σ and a smooth structure near σ.

Fix charts $\varphi : U \to \mathbb{R}^n, \psi : V \to \mathbb{R}^m$ around x and $f(x)$ respectively, $f(U) \subseteq V$. Let

$$\tau_{U,V} : J^k(U,V) \longrightarrow J^k\left(\varphi(U), \psi(V)\right) \cong \varphi(U) \times \psi(V) \times B^k_{n,m},\ \sigma \mapsto (\varphi^{-1})^* \psi_* \sigma$$

Since $\varphi(U) \times \psi(V) \times B^k_{n,m} \subseteq \mathbb{R}^N$, use $\tau_{U,V}$ to topologize $J^k(U,V)$ and hence $J^k(X,Y)$. It's easy to see that this topology doesn't depend on the choice of charts.

$\boxed{\text{Exercise.}}$ Let $\widetilde{\varphi} : U \to \mathbb{R}^n$, $\widetilde{\psi} : V \to \mathbb{R}^m$ be other charts, then $\tau_{\widetilde{\varphi},\widetilde{\psi}} \circ \tau^{-1}_{\varphi,\psi}$ is smooth. So $J^k(X,Y)$ has an induced smooth structure.

Lemma 1.1:

(1) $J^k(X,Y)$ is a manifold of dimension $n + m\binom{n+k}{k}$.
(2) $\alpha : J^k(X,Y) \to X, \beta : J^k(X,Y) \to Y, \alpha \times \beta : J^k(X,Y) \to X \times Y$ are smooth surjective submersions.
(3) $\varphi : Y \to Z$ smooth, then φ_* is smooth; $\varphi : Z \to X$ diffeomorphism, then φ^* is diffeomorphism.

Definition 1.4:

Let $f \in C^\infty(X,Y)$. Its k-jet $j^k f$ is the function

$$j^k f : X \to J^k(X,Y),\ x \mapsto [f]_x$$

$\boxed{\text{Remark.}}$ $J^k(X,Y)$ is usually not a vector bundle over X, Y or $X \times Y$. If $Y = \mathbb{R}^m$, then $J^k(X,Y)$ is a vector bundle over X.

Definition 1.5:

Let E, B, F be manifolds, and $\pi : E \to B$ is a surjective submersion. We say that π is a fiber bundle with fiber F if $\forall b \in B$, $\exists U \subseteq B$ neighborhood of b and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times F$ such that the diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}$$

$\boxed{\text{Exercise.}}$ $J^k(X,Y)$ is a fiber bundle over $X, Y, X \times Y$, $f \in C^\infty(X,Y)$ gives rise to a section $j^k f : X \to J^k(X,Y)$.

1.2 Whitney C^∞-Topology

Let X, Y be smooth manifolds. For $U \subseteq J^k(X,Y)$ open, let

$$M(U) = \left\{ f \in C^\infty(X,Y) \mid j^k f(X) \subseteq U \right\}$$

Note that $M(U) \cap M(V) = M(U \cap V)$, so $\{M(U) \mid U \subseteq J^k(X,Y) \text{ open}\}$ is a basis for a topology on $C^\infty(X,Y)$, which is called the C^k-*topology*. Let W_k be the C^k-topology.

Lemma 1.2:

$k \leq l \implies W_k \subseteq W_l$.

Proof. Suppose $k \leq l$. There exists a surjective continuous map:

$$\pi_{k,l} : J^l(X,Y) \to J^k(X,Y),\ [f]_x \mapsto [f]_x$$

$\pi_{k,l} \circ j^l f = j^k f$. If $U \subseteq J^k(X,Y)$ is open, then $\pi_{k,l}^{-1}(U) \subseteq J^l(X,Y)$ is open. So $M(U) = M(\pi_{k,l}^{-1}(U))$. Therefore $W_k \subseteq W_l$. $\square$

Definition 1.6:

The (Whitney) C^∞-topology is the topology on $C^\infty(X,Y)$ generated by $\bigcup\limits_{k \in \mathbb{N}} W_k$.

Recall that every manifold M admits a Riemannian metric, which induced a metric space structure on the manifold (M,d). Moreover, we may assume that d is complete.

Why? (1) $\exists$ smooth proper function $f : M \to \mathbb{R}$; (2) For any metric d on M, we can define $\widetilde{d}(x,y) = d(x,y) + |f(x) - f(y)|$, $\widetilde{d}$ is complete.

Let d be a (complete) metric on $J^k(X,Y)$.

Definition 1.7:

Let $\delta : X \to \mathbb{R}_{>0}$ continuous and $f \in C^\infty(X,Y)$, let

$$B_\delta^k(f) = \left\{g \in C^\infty(X,Y) \mid d\left(j^k f(x), j^k g(x)\right) < \delta(x)\right\}$$

Proposition 1.1:

$\{B_\delta(f) \mid \delta : X \to \mathbb{R}_{>0}\}$ is a basis for C^k-topology at f. (neighborhood basis)

Proof. $f \in B_\delta(f)$.

$\boxed{\text{Step 1.}}$ $B_\delta(f)$ is open. We claim that

$$B_\delta(f) = M(U),\ U = \left\{\sigma \in J^k(X,Y) \mid d\left(j^k f(\alpha(\sigma)), \sigma\right) < \delta\left(\alpha(\sigma)\right)\right\}$$

Define $\Delta : J^k(X,Y) \to \mathbb{R}$, $\Delta = \delta \circ \alpha - d\left(j^k f \circ \alpha(\cdot), \cdot\right)$, so $U = \Delta^{-1}(0,\infty)$ is open.

$\boxed{\text{Step 2.}}$ Let $\mathcal{U} \subseteq C^\infty(X,Y)$ be an open neighborhood of f (in C^k-topology), then there exists $U \subseteq J^k(X,Y)$ open set such that $f \in M(U) \subseteq \mathcal{U}$. We claim that $\exists \delta \in C(X, \mathbb{R}_{>0})$ such that $f \in B_\delta(f) \subseteq M(U)$.

For each $x \in X$, let

$$m(x) = \inf \left\{ d\left(\sigma, j^k f(x)\right) \mid \sigma \in \alpha^{-1}(x) \cap \left(J^k(X,Y) \backslash U\right) \right\}$$

It's strictly bigger than 0 for every $x \in X$ because U is open, $m(x)$ could be ∞ for some x. We can choose $\delta : X \to \mathbb{R}_{>0}$ continuous such that $0 < \delta(x) < m(x)$. Then

$$g \in B_\delta(f) \implies d\left(j^k f(x), j^k g(x)\right) < \delta(x) < m(x), \ \forall x \in X$$

which implies $j^k g(x) \in U$, $\forall x \in X$. So $B_\delta(f) \subseteq M(U)$. □

$\boxed{\text{Obs.}}$ $B_\delta(f)$ is roughly the set of functions whose partial derivatives up to order k are close enough to f's.

To make this more precise, let $\Phi = \{\varphi_i : U_i \to \mathbb{R}^n\}_{i \in I}$ locally finite atlas of X, $\mathcal{K} = \{K_i\}_{i \in I}$ family of compact sets of X, $K_i \subseteq U_i$, $\Psi = \{\psi_i : V_i \to \mathbb{R}^m\}_{i \in I}$ atlas for Y, $\mathcal{E} = \{\epsilon_i\}_{i \in I}$, $\epsilon_i > 0$. Define

$$\begin{aligned} \mathcal{N}^k\left(f; \Phi, \Psi, \mathcal{K}, \mathcal{E}\right) = \{ g \in C^\infty(X,Y) \mid g(K_i) \subseteq V_i \text{ and} \\ ||D^r(\psi_i \circ f \circ \varphi_i^{-1})(x) - D^r(\psi_i \circ g \circ \varphi_i^{-1})(x)|| < \epsilon_i, \forall i, x \in X, r \leq k \} \end{aligned}$$

$\boxed{\text{Exercise.}}$ Prove that $\left\{\mathcal{N}^k\left(f; \Phi, \Psi, \mathcal{K}, \mathcal{E}\right)\right\}$ is a basis for the C^k-topology.

$\boxed{\text{Remark.}}$ If X is compact, then we can find a countable basis of f given by $\{B_{\delta_n}(f)\}$, where $\delta_n = \frac{1}{n}$. So C^k-topology is first countable. Moreover,

$$f_n \xrightarrow{C^k} f \Leftrightarrow \frac{\partial^{|\alpha|} f_n}{\partial x^\alpha} \xrightarrow{\text{uniformly}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}, \ \forall |\alpha| \leq k$$

Proposition 1.2:

Suppose $\{f_n\}_{n \in \mathbb{N}} \subseteq C^\infty(X,Y)$ such that $f_n \xrightarrow{C^k} f$. Then $\exists K \subseteq X$ compact such that $f_n \equiv f$ in $X \backslash K$ for $n \gg 0$ and $j^k f_n|_K \xrightarrow{\text{uniformly}} j^k f$.

Proof. Suppose $f_n \xrightarrow{C^k} f$ and let $\{K_i\}_{i \in \mathbb{N}}$ exhaustion by compact sets such that $K_i \subseteq \text{int}(K_{i+1})$. Assume, by contradiction, that $\nexists K \subseteq X$ compact set, such that $f_n \equiv f$ on $X \backslash K$. So for each i, $\exists x_i \in K_i, n_i$ such that $f_{n_i}(x_i) \neq f(x_i)$. WLOG, $n_1 < n_2 < \cdots$, $a_i = d\left(j^k f_{n_i}(x_i), j^k f(x_i)\right) > 0$. Let $\delta : X \to \mathbb{R}_+$ such that $\delta(x_i) = a_i/2$. Then $f_{n_i} \notin B_\delta(f)$, so $f_{n_i} \nrightarrow f$. □

Definition 1.8:

A topological space is Baire if the countable intersection of open and dense subsets is dense.

Theorem 1.1:

Let X, Y be smooth manifolds. Then $C^\infty(X,Y)$ is Baire in the C^∞-topology.

Proof. Fix complete metric d_k on $J^k(X,Y)$. Let $\{U_n\}_{n\in\mathbb{N}}$ dense open subsets of $C^\infty(X,Y)$ in the C^∞-topology. Let $V \subseteq C^\infty(X,Y)$ non-empty open set. We want to show that $\bigcap\limits_{n\in\mathbb{N}} U_n \cap Y \neq \emptyset$.

Since V is open, $\exists Z \subseteq J^{k_0}(X,Y)$ open such that $M(\overline{Z}) \subseteq V$. It's enough to show that $M(\overline{Z}) \cap \bigcap\limits_{n\in\mathbb{N}} U_n \neq \emptyset$.

We can construct f_i inductively, $\{k_i\} \subseteq \mathbb{N}$, $Z_i \subseteq J^{k_i}(X,Y)$ open sets such that

(1) $f_i \in M(Z) \cap \bigcap\limits_{s=1}^{i} M(Z_s)$

(2) $M(\overline{Z_i}) \subseteq U_i$

(3) $d_s\left(j^s f_i(x), j^s f_{i-1}(x)\right) < 1/2^i,\ \forall x \in X, 0 \le s \le i$

Since $M(Z)\cap U_1$ is open and non-empty, we can find $Z_1 \subseteq J^{k_1}(X,Y)$ non-empty such that $M(\overline{Z_1}) \subseteq M(Z) \cap U_1$. Take $f_1 \in M(Z_1)$ and it satisfies (1) and (2). Say we've chosen (f_s, k_s, Z_s) for $s \le i-1$. Let $D_i = B^0_{\frac{1}{2^i}}(f_{i-1}) \cap B^1_{\frac{1}{2^i}}(f_{i-1}) \cap \cdots \cap B^i_{\frac{1}{2^i}}(f_{i-1})$ open in C^∞-topology, $f_{i-1} \in M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i$. Since U_i is open and dense, $M(Z)\cap M(Z_1)\cap\cdots\cap M(Z_{i-1})\cap D_i \cap U_i \neq \emptyset$ and open, so we can find $\emptyset \neq Z_i \subseteq J^{k_i}(X,Y)$ such that $M(\overline{Z_i}) \subseteq M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i \cap U_i$. Choose $f_i \in M(Z_i)$, it satisfies the three conditions.

For a fixed s, the condition (3) tells that $\{j^s f_i(x)\}$ is a Cauchy sequence in $J^k(X,Y)$, it converges to $g^s(x)$, $g^0(x) \in J^0(X,Y) = X \times Y$, $g^0(x) = (x, g(x))$.

$\boxed{\text{Exercise.}}$ $g \in C^\infty(X,Y)$ and $j^s g = g^s$. (Look in a compact set and in charts)

Then $g = \lim\limits_{i\to\infty} f_i$ in the C^∞-topology. $f_i \in M(Z) \implies g \in M(\overline{Z})$, $f_i \in M(Z_s)$ for $i \ge s$, so $g \in M(\overline{Z_s})$ for $\forall s$, hence $g \in M(\overline{Z}) \cap \bigcap\limits_{n\in\mathbb{N}} M(\overline{Z_s}) \subseteq M(\overline{Z}) \cap \bigcap\limits_{n\in\mathbb{N}} U_n$. $\square$

Proposition 1.3:

Let X, Y be smooth manifolds. Then $j^k : C^\infty(X,Y) \to C^\infty\left(X, J^k(X,Y)\right)$ is continuous in the C^∞-topology.

Proof. $U \subseteq J^\ell\left(X, J^k(X,Y)\right)$ open, so $M(U)$ is open set in the C^ℓ-topology of $C^\infty\left(X, J^k(X,Y)\right)$. We need to show that $(j^k)^{-1}(M(U))$ is open in $C^\infty(X,Y)$. Consider

$$\alpha^{k,\ell} : J^{k+\ell}(X,Y) \to J^\ell\left(X, J^k(X,Y)\right),\ \alpha^{k,\ell}\left(j^{k+\ell} f(x)\right) = j^\ell(j^k f)(x)$$

This is a smooth embedding. So $(j^k)^{-1}(M(U)) = M\left((\alpha^{k,\ell})^{-1}(U)\right)$ is open in $C^{k+\ell}$-topology. $\square$

Proposition 1.4:

$\phi : Y \to Z$ smooth. Then $\widetilde{\phi_*} : C^\infty(X,Y) \to C^\infty(X,Z)$, $f \mapsto \phi \circ f$ is continuous in the C^∞-topology.

Proposition 1.5:

Let X, Y, Z be smooth manifolds. Then $C^\infty(X,Y) \times C^\infty(X,Z) \to C^\infty(X, Y\times Z)$, $(f,g) \mapsto f\times g$ is a homeomorphism in the C^∞-topology.

Appendix. About existence of proper function on manifolds.

Next, we use partitions of unity to construct a special kind of smooth function. If M is a topological space, an ***exhaustion function for M*** is a continuous function $f\colon M \to \mathbb{R}$ with the property that the set $f^{-1}\big((-\infty, c]\big)$ (called a ***sublevel set of f***) is compact for each $c \in \mathbb{R}$. The name comes from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}\big((-\infty, n]\big)$ form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{B}^n \to \mathbb{R}$ given by

$$f(x) = |x|^2, \qquad g(x) = \frac{1}{1-|x|^2}$$

are smooth exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.28 (Existence of Smooth Exhaustion Functions). *Every smooth manifold with or without boundary admits a smooth positive exhaustion function.*

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^\infty$ be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^\infty j\psi_j(p).$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \geq \sum_j \psi_j(p) = 1$.

To see that f is an exhaustion function, let $c \in \mathbb{R}$ be arbitrary, and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^N \bar{V}_j$, then $\psi_j(p) = 0$ for $1 \leq j \leq N$, so

$$f(p) = \sum_{j=N+1}^\infty j\psi_j(p) \geq \sum_{j=N+1}^\infty N\psi_j(p) = N\sum_{j=1}^\infty \psi_j(p) = N > c.$$

Equivalently, if $f(p) \leq c$, then $p \in \bigcup_{j=1}^N \bar{V}_j$. Thus $f^{-1}\big((-\infty, c]\big)$ is a closed subset of the compact set $\bigcup_{j=1}^N \bar{V}_j$ and is therefore compact. □

1.3 Transversality Theorem

Definition 1.9:

Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold. We say that f is transverse to W (write $f \pitchfork W$) if

$$df_x(T_x X) + T_{f(x)}W = T_{f(x)}Y,\ \forall x \in f^{-1}(W)$$

$\boxed{\text{Note.}}$ $X_1, X_2 \subseteq Y$ submanifolds, $X_1 \pitchfork X_2 \Leftrightarrow T_x X_1 + T_x X_2 = T_x Y$ for $\forall x \in X_1 \cap X_2$. It's just inclusion of one submanifold transverse to another submanifold.

Proposition 1.6:

Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold such that $\dim X + \dim W < \dim Y$. Then $f \pitchfork W \Leftrightarrow f(X) \cap W = \emptyset$.

Proof. The proof is easy. $\square$

Theorem 1.2:

Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold such that $f \pitchfork W$. If $f^{-1}(W) \neq \emptyset$, then $f^{-1}(W)$ is a submanifold of X of codim $f^{-1}(W) =$ codim W. In particular, if $\dim X =$ codim W, then $f^{-1}(W)$ consists of isolated points.

Proof. Let $p \in f^{-1}(W)$, $n = \dim X, m = \dim Y, k = \dim W$. Let $\varphi : U \to \mathbb{R}^m$ be a chart around $f(p)$ such that $\varphi(U \cap W) \subseteq \mathbb{R}^k \times \{0\}$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ be the orthogonal projection along $\mathbb{R}^k \times \{0\}$, $\phi = \pi \circ \varphi$. Then $\phi : U \to \mathbb{R}^{m-k}$ is a submersion and $\phi^{-1}(0) = U \cap W$.

$\boxed{\text{Claim.}}$ $f \pitchfork W$ at $p \Leftrightarrow p$ is a regular point of $\phi \circ f$.

Since $\phi^{-1}(0) = U \cap W$, $\ker d\phi_{f(p)} = T_{f(p)}W$. Transversality assumption gives that $df_p(T_p X) + T_{f(p)}W = T_{f(p)}Y = T_{f(p)}U$, which implies that $d(\phi \circ f)_p(T_p X) = d\phi_{f(p)}T_{f(p)}U$. And the converse is easy to proof.

Now $f \pitchfork W$ on $U \Leftrightarrow 0$ is a regular value of $\phi \circ f : f^{-1}(U) \to \mathbb{R}^{m-k}$. By the implicit function theorem, $(\phi \circ f)^{-1}(0) = f^{-1}(U \cap W)$ is a submanifold of $f^{-1}(U) \subseteq X$ open set of codimension $m - k$. So $f^{-1}(W)$ is a submanifold of X of codimension $m - k$. $\square$

Proposition 1.7:

Let X, Y be manifolds, $W \subseteq Y$ submanifold which is a closed subset. Then $T_W := \{f \in C^\infty(X, Y) \mid f \pitchfork W\}$ is open in the C^∞-topology.

Proof. We show that T_W is open in the C^1-topology. Let

$$U = \left\{\sigma = j^1 f(x) \in J^1(X, Y) \mid f(x) \notin W \text{ or } df_x(T_x X) + T_{f(x)}W = T_{f(x)}Y\right\}$$

It's easy to see that $T_W = M(U) = \{f \in C^\infty(X,Y) \mid j^1 f(X) \subseteq U\}$

$\boxed{\text{Claim.}}$ U is open.

We will show that $V = J^1(X,Y) \backslash U$ is closed. To prove that, take $\{\sigma_n\} \subseteq V$ such that $\sigma_n \to \sigma \in J^1(X,Y)$, we need to show that $\sigma \in V$. Consider continuous map $\beta : J^1(X,Y) \to Y$, then $\beta(\sigma_n) \to \beta(\sigma)$. Since $\beta(\sigma_n) \in W$ and W is closed, we have $\beta(\sigma) \in W$, which mean that $\sigma = j^1 f(x), f(x) \in W$.

Now choose charts around x and $f(x)$, $\varphi : \widetilde{U} \to \mathbb{R}^n$, $\psi : \widetilde{V} \to \mathbb{R}^m$, $\psi(\widetilde{V} \cap W) = \mathbb{R}^k \times \{0\}$, $\varphi(x) = 0, \psi(f(x)) = 0$. $f \pitchfork W$ at $x \Leftrightarrow \psi \circ f \circ \varphi^{-1} \pitchfork \mathbb{R}^k \times \{0\}$ at $0 \Leftrightarrow 0$ is a regular value of $\pi \circ \psi \circ f \circ \varphi^{-1}$ where $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ orthogonal projection $\Leftrightarrow$ $\pi \circ d(\psi \circ f \circ \varphi^{-1})_0$ has rank $m-k$.

Let $F = \left\{A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}) \mid \mathrm{rank}\ A < m-k\right\}$. In a neighborhood $\mathcal{N}$ of σ, fixing φ, ψ we obtain a map

$$\eta : \mathcal{N} \subseteq J^1(X,Y) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}),\ j^1 g(x) \mapsto \pi \circ d(\psi \circ f \circ \varphi^{-1})_{\varphi(x)}$$

$V \cap \mathcal{N} = \eta^{-1}(F)$, η is continuous.

$\boxed{\text{Exercise.}}$ F is closed. □

$\boxed{\text{Remark.}}$ The condition that W is closed is necessary.

Lemma 1.3:

Let X, Y, B manifolds, $W \subseteq Y$ submanifold, let $j : B \to C^\infty(X,Y)$ map (not necessary continuous).

$$\Phi : X \times B \to Y,\ \Phi(x,b) = j(b)(x)$$

Suppose Φ is smooth and $\Phi \pitchfork W$. Then $\{b \in B \mid j(b) \pitchfork W\}$ is dense in B.

Proof. Let $W_\Phi = \Phi^{-1}(W) \subseteq X \times B$ be the submanifold, $pr : X \times B \to B$ the projection and $\pi = pr|_{W_\Phi}$.

$\boxed{\text{Claim.}}$ b is a regular value of $\pi \implies j(b) \pitchfork W$.

Suppose b is a regular value of π.

(1) $b \notin \mathrm{im}\ \pi$, then $\nexists x \in X$ such that $\Phi(x,b) \in W$, so $j(b)(X) \cap W = \emptyset$, which means $j(b) \pitchfork W$.

(2) If $\dim W_\Phi < \dim B$, then b is a regular value of π, so $b \notin \mathrm{im}\ \pi$, therefore by (1) we have $j(b) \pitchfork W$.

(3) If $\dim W_\Phi \geq \dim B$. Let b be a regular value of π and $x \in X$. If $(x,b) \notin W_\Phi$, then $j(b)(x) \notin W$, so $j(b) \pitchfork W$ at x. If $(x,b) \in W_\Phi$, then $\pi\left(T_{(x,b)} W_\Phi\right) = T_b B$, which gives $T_{(x,b)}(X \times B) = T_{(x,b)} W_\Phi + T_{(x,b)}(X \times \{b\})$, so $T_{j(b)(x)} Y = T_{j(b)(x)} W + (dj(b))_x T_x X$, so $j(b) \pitchfork W$ at x. □

Corollary 1.1:

Let $G : X \times B \to Y$ smooth, $\Phi(x,b) = j^k G_b(x)$. If $\Phi \pitchfork W$, where $W \subseteq J^k(X,Y)$ submanifold. Then $\{b \in B \mid j^k G_b \pitchfork W\}$ is dense in B.

Theorem 1.3: Thom Transversality Theorem

Let X, Y manifolds, $W \subseteq J^k(X,Y)$ submanifold, let

$$T_W = \{f \in C^\infty(X,Y) \mid j^k f \pitchfork W\}$$

Then T_W is a residual subset of $C^\infty(X,Y)$ (residual subset means countable intersection of open and dense sets). Moreover if W is closed, then T_W is open.

Proof. For each $\sigma \in W$, let $W_\sigma \subseteq W$, $U_\sigma \subseteq X$, $V_\sigma \subseteq Y$ open neighborhood of σ, $\alpha(\sigma)$, $\beta(\sigma)$ respectively and charts $\varphi_\sigma : U_\sigma \to \mathbb{R}^n$, $\psi_\sigma : V_\sigma \to \mathbb{R}^m$ such that:

(a) $\overline{W_\sigma} \subseteq W$ and is compact.

(b) $\overline{U_\sigma}$ is compact.

(c) $\alpha(\overline{W_\sigma}) \subseteq U_\sigma$ and $\beta(\overline{W_\sigma}) \subseteq V_\sigma$.

(d) $\psi_\sigma(V_\sigma) = \mathbb{R}^m$.

We say that $g \pitchfork W$ on A if $g \pitchfork W$ for $\forall x \in g^{-1}(A)$. Let

$$T_\sigma = \{f \in C^\infty(X,Y) \mid j^k f \pitchfork W \text{ on } \overline{W_\sigma}\}$$

$T_W = \bigcap\limits_{\sigma \in W} T_\sigma$. Since W is 2-countable, there exists a countable covering $\{W_{\sigma_i}\}_{i=1}^\infty$ of W. So $T_W = \bigcap\limits_{i=1}^\infty T_{\sigma_i}$.

$\boxed{\text{Claim.}}$ T_σ is open and dense.

(1) T_σ is open. Let $\widetilde{T_\sigma} = \{g \in C^\infty(X, J^k(X,Y)) \mid g \pitchfork W \text{ on } \overline{W_j}\}$. By previous proposition we have $\widetilde{T_\sigma}$ is open, then $T_\sigma = (j^k)^{-1}(\widetilde{T_\sigma})$ is open.

(2) T_σ is dense. Let $f \in C^\infty(X,Y)$, we will construct a sequence $\{g_n\} \subseteq C^\infty(X,Y)$ such that $g_n \in T_\sigma$ and $g_n \xrightarrow{C^\infty} f$. The idea is to define $\Phi : X \times B \to J^k(X,Y)$, $\Phi(x,b) = j^k g_b(x)$, where $g_b(x)$ is a polynomial perturbation of f, such that $\Phi \pitchfork W$.

Fix smooth functions $\rho_1 : \mathbb{R}^n \to [0,1]$, $\rho_2 : \mathbb{R}^m \to [0,1]$ such that $\rho_1 \equiv 1$ in a neighborhood of $\varphi\left(\alpha(\overline{W_\sigma})\right)$, supp $\rho_1 \subseteq \varphi(U_\sigma)$; $\rho_2 \equiv 1$ in a neighborhood of $\psi\left(\beta(\overline{W_\sigma})\right)$, supp ρ_2 is compact. Let $B = \{\text{polynomial maps } \mathbb{R}^n \to \mathbb{R}^m \text{ of degree } \le k\}$.

For $b \in B$, let

$$g_b(x) = \begin{cases} \psi^{-1}\left(\psi\left(f(x)\right) + b\left(\varphi(x)\right)\rho_1\left(\varphi(x)\right)\rho_2\left(\psi\left(f(x)\right)\right)\right) & \text{if } x \in U_\sigma, f(x) \in V_\sigma \\ f(x) & \text{if } x \notin U_\sigma \text{ or } f(x) \notin V_\sigma \end{cases}$$

$G : X \times B \to Y$, $G(x,b) = g_b(x)$.

$\boxed{\text{Exercise.}}$ G is smooth.

Let $\Phi : X \times B \to J^k(X,Y)$, $\Phi(x,b) = j^k g_b(x)$, so Φ is smooth.

$\boxed{\text{Claim.}}$ $\exists \widetilde{B} \subseteq B$ open neighborhood of $0 \in B$ such that $\Phi|_{X\times\widetilde{B}} \pitchfork W$ on $\overline{W_\sigma}$.

Assuming the claim, apply the previous lemma, $\exists \{b_n\} \subseteq \widetilde{B}$ such that $b_n \to 0$ and $j^k g_{b_n} \pitchfork \left(W \cap \overline{W_\sigma}\right)$, this also implies $g_{b_n} \xrightarrow{C^\infty} f$ and $j^k g_{b_n} \pitchfork W$ on $\overline{W_\sigma}$. So T_σ is dense.

Proof of the claim: Let $\epsilon = \frac{1}{2} d\left(\psi\left(\beta(\overline{W_j})\right), \rho_2^{-1}([0,1))\right) > 0$, define

$$\widetilde{B} = \{b \in B \mid ||b(x)|| < \epsilon,\ \forall x \in \operatorname{supp} \rho_1\}$$

We fix $b \in \widetilde{B}$ such that $\Phi(x,b) \in \overline{W_\sigma}$. We will show that Φ is a local diffeomorphism near (x,b). Since $\Phi(x,b) \in \overline{W_\sigma}$, $x \in \alpha(\overline{W_\sigma})$, $g_b(x) \in \beta(\overline{W_\sigma})$. $\psi\left(g_b(x)\right) = \psi\left(f(x)\right) + b\left(\varphi(x)\right)\rho_1\left(\varphi(x)\right)\rho_2\left(\psi\left(f(x)\right)\right) = \psi\left(f(x)\right) + b\left(\varphi(x)\right)$. Because $||b\left(\varphi(x)\right)|| < \epsilon$, $\forall x \in \operatorname{supp} \rho_1$, then $\rho_2\left(\psi\left(g_b(x)\right)\right) = 1$. So $\psi \circ g_b(x) = \psi\left(f(x)\right) + b\left(\varphi(x)\right)$ in a neighborhood of (x,b). σ' is sufficiently close to σ, so we can find a unique polynomial b' so that $\sigma' = j^k\left(\psi^{-1}\left(f\left(\varphi\left(\alpha(\sigma')\right)\right)\right)\right) + b'\left(\varphi\left(\alpha(\sigma')\right)\right)$. So we have constructed a local inverse for every $(x,b) \in \Phi^{-1}(\overline{W_\sigma})$, then $\Phi \pitchfork W$ on $\overline{W_\sigma}$. $\square$

Corollary 1.2:

Let X,Y manifolds, $f \in C^\infty(X,Y)$, $W \subseteq J^k(X,Y)$ submanifold such that $\alpha(\overline{W}) \subseteq U$ open set. Then $\exists \{g_n\} \subseteq C^\infty(X,Y)$ such that $j^k g_n \pitchfork W$, $g_n \to f$ and $g_n = f$ outside U.

Proof. The same as the theorem above but we choose $U_\sigma \subseteq U$ for $\forall \sigma \in W$. $\square$

Corollary 1.3: Elementary Transversality Theorem

Let X,Y manifolds, $W \subseteq Y$ submanifold.

(a) $T_W = \{f \in C^\infty(X,Y) \mid f \pitchfork W\}$ is dense in $C^\infty(X,Y)$. Moreover if W is closed, then T_W is open.

(b) Let $U_1, U_2 \subseteq X$ open sets such that $\overline{U_1} \subseteq U_2$, let $f \in C^\infty(X,Y)$, $V \subseteq C^\infty(X,Y)$ near f and open. Then there is $\{g_n\} \in C^\infty(X,Y)$ such that $g_n \xrightarrow{C^\infty} f$, $g_n = f$ in U_1 and $g_n \pitchfork W$ outside U_2.

Definition 1.10: Multijets

Let X,Y manifolds. For $s \in \mathbb{N}$, define

$$X^{(s)} = \{(x_1, \dots, x_s) \in X^s \mid x_i \neq x_j,\ i \neq j\}$$

$\alpha^s = \alpha \times \cdots \times \alpha : J^k(X,Y)^s \to X^s$, let $J^k_s(X,Y) = (\alpha^s)^{-1}(X^{(s)}) \subseteq J^k(X,Y)^s$ open, so $J^k_s(X,Y)$ is a manifold. $f \in C^\infty(X,Y)$ gives rise to

$$j^k_s f : X^{(s)} \to J^k_s(X,Y),\ j^k_s f(x_1, \dots, x_s) = \left(j^k f(x_1), \dots, j^k f(x_s)\right)$$

Theorem 1.4: Thom Transversality for multijets

Let X, Y manifolds, $W \subseteq J^k_s(X,Y)$ submanifold. Let

$$T_W = \left\{ f \in C^\infty(X,Y) \mid j^k_s f \pitchfork W \right\}$$

Then T_W is residual. Moreover, if W is compact, then T_W is open.

1.4 Whitney Immersions and Embeddings

Let X^n, Y^m manifolds, $\sigma = j^1 f(x) \in J^1(X,Y)$. Then $df_x : T_x X \to T_{f(x)} Y$ depends only on σ. Define $\operatorname{rank}(\sigma) = \operatorname{rank}(df_x)$ and $\operatorname{corank}(\sigma) = \min(m,n) - \operatorname{rank}(\sigma)$. Let $S_r = \{\sigma \in J^1(X,Y) \mid \operatorname{corank}(\sigma) = r\}$.

Lemma 1.4:

f is an immersion ($n \leq m$) or submersion ($n \geq m$) $\Leftrightarrow j^1 f(X) \cap \bigcup_{r \geq 1} S_r = \emptyset$.

Proof. f is not an immersion/submersion $\Leftrightarrow \exists x \in X$ such that $\operatorname{rank}(df_x) \leq \min(m,n) - 1$ $\Leftrightarrow \exists x \in X$ such that $\operatorname{corank}(j^1 f(x)) \geq 1 \Leftrightarrow j^1 f(X) \cap S_r \neq \emptyset$ for some $r \geq 1$. $\square$

Proposition 1.8:

S_r is a submanifold of codimension $(n-q+r)(m-q+r)$, where $q = \min(n,m)$.

Proof. S_r is a bundle over $X \times Y$ with fiber $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) = \{A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \mid \operatorname{corank}(A) = r\}$.

$\boxed{\text{Claim.}}$ $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of codimension $(n-q+r)(m-q+r)$. So $S_r \subseteq J^1(X,Y)$ is a subbundle over $X \times Y$.

Proof of the claim: Let $M \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m)$, let $k = q - r$. We can choose basis of $\mathbb{R}^n$ and $\mathbb{R}^{m]}$ so that

$$[M] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \ A \text{ is an invertible } k \times k \text{ matrix}$$

So in a neighborhood U of M, every other M' will be represented as

$$[M'] = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \ A' \text{ is an invertible } k \times k \text{ matrix}$$

$$\text{So } \operatorname{rank}[M'] = \operatorname{rank} \begin{bmatrix} I^k & 0 \\ -C'(A')^{-1} & I_{m-k} \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A' & B' \\ 0 & D' - C'(A')^{-1}B' \end{bmatrix}$$

Then $\operatorname{rank}[M'] = k \Leftrightarrow D' - C'(A')^{-1}B' = 0$. $M' \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U \Leftrightarrow D' - C'(A')^{-1}B' = 0$. Let

$$\varphi : U \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathcal{L}(\mathbb{R}^{n-k}, \mathbb{R}^{m-k}), \ \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \mapsto D' - C'(A')^{-1}B'$$

φ is a submersion, so $\varphi^{-1}(0) = \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U$ is a submanifold of codimension $(n-q+r)(m-q+r)$.

□

$\boxed{\text{Obs.}}$ $\mathcal{L}^0(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is open. So $S_0 \subseteq J^1(X,Y)$ open submanifold, then $\bigcup_{r\geq 1} S_r$ is closed.

Lemma 1.5:

Suppose $n \leq m$. Then $\mathrm{Imm}(X,Y) = \{f : X \to Y \text{ smooth immersion}\}$ is an open subset of $C^\infty(X,Y)$.

Proof. $\mathrm{Imm}(X,Y) = M(S_0)$. □

Theorem 1.5: Whitney Immersion

Let X^n, Y^m be manifolds such that $m \geq 2n$. Then $\mathrm{Imm}(X,Y)$ is open and dense subset of $C^\infty(X,Y)$.

Proof. $\min(n,m) = n$, so for $r \geq 1$, codim $S_r = (n-q+r)(m-q+r) = r(n+r) \geq n+1$. So $j^1 f \pitchfork S_r \Leftrightarrow j^1 f(X) \cap S_r = \emptyset$ since $\dim X = n < n+1 \leq \text{codim } S_r$.

$$\mathrm{Imm}(X,Y) = \left\{ f \in C^\infty(X,Y) \mid j^1 f(X) \cap \bigcup_{r\geq 1} S_r = \emptyset \right\} = \left\{ f \in C^\infty(X,Y) \mid j^1 f \pitchfork \bigcup_{r\geq 1} S_r \right\}$$

By the Thom transversality theorem, $\mathrm{Imm}(X,Y)$ is dense and open. □

Theorem 1.6: Whitney Injective Immersion Theorem

Let X^n, Y^m be manifolds such that $m \geq 2n+1$. Then the set of injective immersions is residual.

Proof. $\mathrm{Imm}(X,Y)$ is open and dense, we need to show

$$\mathrm{Inj}(X,Y) = \{f \in C^\infty(X,Y) \mid f \text{ is injective}\} \text{ is residual}$$

Recall $J_2^0(X,Y) = X^{(2)} \times Y^2 = \{(x_1,x_2,y_1,y_2) \in X^2 \times Y^2 \mid x_1 \neq x_2\}$, let

$$W = X^{(2)} \times \Delta Y = \{(x_1,x_2,y,y) \mid x_1 \neq x_2\} \subseteq J_2^0(X,Y)$$

f is injective iff $j_2^0 f(X^{(2)}) \cap W = \emptyset$. Codimension of W is dimension of Y, so f is injective iff $j_2^0 f \pitchfork W$ from the proof of previous theorem. By the Thom transversality theorem for multijets, we have $\mathrm{Inj}(X,Y)$ is residual. □

Lemma 1.6:

Let X manifold. Then $\mathrm{Prop}(X, \mathbb{R}^m) = \{f \in C^\infty(X, \mathbb{R}^m) \mid f \text{ is proper}\}$ is non-empty and open.

Proof. Recall that there exists a proper map $X \to \mathbb{R}$, compose this map with a linear injection $\mathbb{R} \to \mathbb{R}^m$ to obtain a proper map.

Now let $f \in \mathrm{Prop}(X, \mathbb{R}^m)$. For $x \in X$, define $V_x = \{y \in \mathbb{R}^m \mid d(y, f(x)) < 1\}$. So $V_x \subseteq \mathbb{R}^m$ open. Let $V = \bigcup_{x \in X} \{x\} \times V_x$, then $V \subseteq X \times \mathbb{R}_m = J^0(X, \mathbb{R}^m)$ is open. $f \in M(V)$ because $j^0 f(x) = (x, f(x))$, $d(f(x), f(x)) = 0$, so $f(x) \in V_x$.

$\boxed{\text{Claim.}}$ $M(V) \subseteq \mathrm{Prop}(X, \mathbb{R}^m)$.

If $g \in M(V)$, then $d(g(x), f(x)) < 1 \ \forall x \in X$, so $g^{-1}\left(\overline{B}_r(0)\right) \subseteq f^{-1}\left(\overline{B}_{r+1}(0)\right)$. Since f is proper, $f^{-1}\left(\overline{B}_{r+1}(0)\right)$ is compact, therefore $g^{-1}\left(\overline{B}_r(0)\right)$ is compact, hence g is proper. ☐

Corollary 1.4: Whitney Embedding Theorem

Let X^n manifold. Then there exists $X \hookrightarrow \mathbb{R}^{2n+1}$.

Proof. $\mathrm{Inj}(X, \mathbb{R}^{2n+1}) \cap \mathrm{Imm}(X, \mathbb{R}^{2n+1}) \cap \mathrm{Prop}(X, \mathbb{R}^{2n+1}) \neq \emptyset$. ☐

1.5 Morse Functions

Definition 1.11:

Let $f : X \to \mathbb{R}$ smooth and $p \in \mathrm{Crit}(f)$ ($df_p = 0$). Define the Hessian of f to be the bilinear map:

$$D^2 f_p : T_pX \times T_pX \to \mathbb{R}, \ D^2 f_p \left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right) = \left.\frac{\partial^2}{\partial x_i \partial x_j}\right|_{\varphi(p)} (f \circ \varphi^{-1})$$

where $\varphi = (x_1, \dots, x_n)$ is a chart around p. We say that p is non-degenerate if $D^2 f_p$ is non-degenerate.

$\boxed{\text{Exercise.}}$ $D^2 f_p$ doesn't depend on the choice of a chart whenever $p \in \mathrm{Crit}(f)$.

Let $f : X \to \mathbb{R}$ smooth, $df : TX \to \mathbb{R}$, $(p, v) \in TX$, we have $d_{(p,v)} df : T_{(p,v)}TX \to \mathbb{R}$, $T_{(p,v)}TX$ is isomorphic to $T_pX \oplus T_pX$ but it's not natural.

Proposition 1.9:

$p \in \mathrm{Crit}(f)$ is non-degenerate $\Leftrightarrow j^1 f \pitchfork S_1$ at p.

Proof. This is a local question, we may assume $X = U \subseteq \mathbb{R}^n$, $J^1(X, \mathbb{R}) = U \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $\pi : J^1(X, \mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ submersion, $\pi^{-1}(0) = S_1 = \{j^1 f(x) \mid df_x = 0\}$.

$\boxed{\text{Claim.}}$ $j^1 f \pitchfork S_1$ at $p \Leftrightarrow \pi \circ j^1 f$ is a submersion at p.

Now $\pi \circ j^1 f : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $x \mapsto \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x) \right)$ is a submersion at p iff $\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_p$ is invertible iff p is non-degenerate. ☐

Definition 1.12:

$f \in C^\infty(X, \mathbb{R})$ is Morse if every $p \in \mathrm{Crit}(f)$ is non-degenerate.

Corollary 1.5:

$f \in C^\infty(X, \mathbb{R})$ is Morse $\Leftrightarrow j^1 f \pitchfork S_1$.

Theorem 1.7:

Let X manifold. Then $\{f \in C^\infty(X, \mathbb{R}) \mid f \text{ is Morse}\}$ is open and dense in $C^\infty(X, \mathbb{R})$.

Proof. Since $S_1 = J^1(X, \mathbb{R}) \backslash S_0$ is closed, by the corollary and Thom transversality theorem we complete the proof. □

2 Intersection Theory

2.1 Manifolds with boundary and orientation

Definition 2.1:

A topological manifold with boundary is a 2-countable Hausdorff topological space such that every point $p \in X$ has a neighborhood which is homeomorphic to an open set in $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$.

Lemma 2.1:

Let X be a topological manifold with boundary, $p \in X$, $\varphi, \psi : U \to \mathbb{H}^n$ charts around p. Suppose $pr_1 \circ \varphi(p) = 0$, then $pr_1 \circ \psi(p) = 0$, where pr_1 is the canonical projection of $\mathbb{H}^n$ to the first coordinate.

Proof. $\psi \circ \varphi^{-1} : \varphi(U) \to \psi(U)$ is homeomorphic, then $\psi \circ \varphi^{-1} : \varphi(U) \backslash \varphi(p) \to \psi(U) \backslash \psi(p)$ is also homeomorphic. Since $pr_1 \circ \varphi(p) = 0$, $\varphi(U) \backslash \varphi(p)$ is contractible. If $pr_1 \circ \psi(p) \neq 0$, then $\psi(U) \backslash \psi(p) \simeq S^{n-1}$, S^{n-1} and contractible space have different homology group, so they can't be homeomorphic. ☐

Definition 2.2:

Let X be a topological manifold with boundary. Then

$$\partial X = \{p \in X \mid \exists \varphi : U \to \mathbb{H}^n \text{ chart around } p \text{ s.t. } pr \circ \varphi(p) = 0\}, \ \operatorname{int}(X) = X \backslash \partial X$$

$\boxed{\text{Obs.}}$ $\operatorname{int}(X)$ and ∂X are topological manifold without boundary of dimension n and $n-1$ respectively.

Definition 2.3:

A map $f : \mathbb{H}^n \supseteq U \to \mathbb{H}^n$ is smooth if it admits a smooth extension to $\widetilde{f} : \widetilde{U} \to \mathbb{R}^n$, where $U \subseteq \widetilde{U}$ in an open set in $\mathbb{R}^n$.

Definition 2.4:

We say that two charts $\varphi : U \to \mathbb{H}^n$, $\psi : V \to \mathbb{H}^n$ are compatible if $\psi \circ \varphi^{-1}$ is smooth. An atlas is a collection of charts whose domain cover X.

Definition 2.5:

A (smooth) manifold with boundary is a topological manifold with boundary endowed with a maximal (smooth) atlas.

Smooth manifold with boundary X induces smooth structure (without boundary) on $\mathrm{int}(X)$ and ∂X.

Proposition 2.1:

Let $f \in C^\infty(X, \mathbb{R})$, $a \in \mathbb{R}$ regular value of f. Then $f^{-1}([a, +\infty))$ and $f^{-1}((-\infty, a])$ are manifolds with boundary.

Proof. $(a, +\infty) \subseteq \mathbb{R}$ is open then $f^{-1}((a, +\infty))$ is a manifold without boundary. Let $p \in f^{-1}(a)$, by the implicit function theorem, there exists a chart $\varphi : U \to \mathbb{R}^n$ such that $\varphi(p) = 0$ and $f \circ \varphi^{-1}(x_1, \ldots, x_n) = a + x_1$. So we obtain a chart $\varphi|_{f^{-1}([a,+\infty)) \cap U} : \widetilde{U} \to \mathbb{H}^n$. So $f^{-1}([a, +\infty))$ is a manifold with boundary. □

Definition 2.6:

Let X be a manifold with boundary, $p \in X$, a curve centered at p is a smooth map $\gamma : [0, \epsilon) \to X$ or $\gamma : (-\epsilon, 0] \to X$ such that $\gamma(0) = p$. T_pX is the equivalent classes of curves centered at p.

If $x \in \mathrm{int}(X)$, then $T_x(\mathrm{int}(X)) = T_xX$; If $x \in \partial X$, then T_xX is still a n-dimensional vector space. Moreover, we have a canonical inclusion $T_x(\partial X) \subseteq T_xX$.

Proposition 2.2:

Let X, Y be manifolds with boundary, $y \in \mathrm{int}(Y)$ regular value of $f : X \to Y$ and $\partial f := f|_{\partial X} : \partial X \to \partial Y$. Then $f^{-1}(y)$ is a manifold with boundary and $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X = (\partial f)^{-1}(y)$.

Example 2.1:

$f : \mathbb{H}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 + y^2$, then $f^{-1}(1) = S^1 \cap \mathbb{H}^2$.

Exercise. Prove the proposition.

Theorem 2.1:

Let X, Y manifolds with boundary, $W \subseteq Y$ submanifold, $\partial W = \partial Y = \emptyset$. Suppose $f \pitchfork W$ and $\partial f \pitchfork W$, then $f^{-1}(W)$ is a manifold with boundary, $\partial(f^{-1}(W)) = f^{-1}(W) \cap \partial X$.

Proof. $f|_{\text{int}(X)} \pitchfork W$ is a manifold without boundary. Let $x \in f^{-1}(W) \cap \partial X$, $\pi : V \subseteq Y \to \mathbb{R}^{m-k}$ be a submersion such that $\pi^{-1}(0) = W \cap V$. As in the case without boundary: $f \pitchfork W$ at x iff x is a regular point of $\pi \circ f$, $\partial f \pitchfork W$ at x iff x is a regular point of $\pi \circ \partial f$. The result follows from the proposition above. □

$\boxed{\text{Obs.}}$ It's easy to see that $\partial f \pitchfork W$ at $x \implies f \pitchfork W$ at x.

Theorem 2.2: Sard's Theorem

Let X manifold with boundary, Y manifold, $f : X \to Y$. Then

$$\{y \in Y \mid y \text{ is a critical value of } f \text{ or } \partial f\}$$

has measure zero.

Proof. $\text{Crit}(f) \cup \text{Crit}(\partial f) = \text{Crit}\left(f|_{\text{int}(X)}\right) \cup \text{Crit}(\partial f)$. □

Theorem 2.3: Thom Transversality Theorem

X manifold with boundary, Y manifold, $W \subseteq J^k(X,Y)$ submanifold, $\partial W \subseteq \alpha^{-1}(\partial X)$. Then

$$\left\{f \in C^\infty(X,Y) \mid j^k f \pitchfork W \text{ and } j^k(\partial f) \pitchfork W\right\}$$

is residual.

Corollary 2.1: Elementary Transversality Theorem

(1) X manifold with boundary, Y manifold and $W \subseteq Y$ submanifold $\partial W = \emptyset$. Then

$$\{f \in C^\infty(X,Y) \mid f \pitchfork W \text{ and } \partial f \pitchfork W\}$$

is residual.
(2) $f \in C^\infty(X,Y)$, $\partial f \pitchfork W$. There exists $\{g_n\} \subseteq C^\infty(X,Y)$ such that $g_n \xrightarrow{C^\infty} f$, $g_n \pitchfork W$ and $g_n \equiv f$ in a neighborhood of ∂X.

Definition 2.7:

Let V be a vector space. Define an equivalence relation on the set of bases of V as follows:

$$\{x_1, \dots, x_n\} \sim \{y_1, \dots, y_n\} \text{ if the linear map } T : V \to V, Tx_i = y_i \text{ has } \det T > 0$$

$\boxed{\text{Obs.}}$ Given V, there are two equivalence classes.

Definition 2.8:

An orientation of V is a choice of such an equivalence class.

Definition 2.9:

Let X be a smooth manifold. An orientation on X is a choice of orientation on T_pX for each $p \in X$ such that for each chart $\varphi : U \to \mathbb{R}^n, \varphi = (x_1, \ldots, x_n)$, either

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p \right\} \text{ or } \left\{ -\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p \right\} \text{ is oriented for } \forall p \in U$$

$\boxed{\text{Obs.}}$ Not all manifold admits an orientation.

$\boxed{\text{Rmk.}}$ A connected orientable manifold has exactly two orientations. $\mathbb{R}^n$ has a natural orientation.

Proposition 2.3:

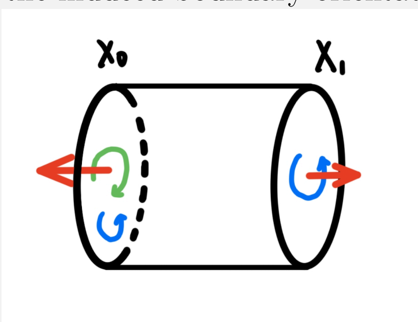
Let X be an oriented manifold with boundary. Then ∂X has a natural orientation.

Proof. For $x \in \partial X$, $T_x(\partial X) \subseteq T_xX$. There exists a 1-dimensional vector bundle N over ∂X such that $N_x \oplus T_x(\partial X) = T_xX$ and a outward normal vector field $n \in \Gamma(N)$ which doesn't vanishes. If $\{v_1, \ldots, v_{n-1}\}$ is a basis of $T_x(\partial X)$, then $\{n_x, v_1, \ldots, v_{n-1}\}$ is a basis of T_xX. □

$\boxed{\text{Obs.}}$ X, Y oriented manifolds, $\partial Y = \emptyset$, then $X \times Y$ inherits a natural orientation.

Example 2.2:

Let X oriented manifold without boundary, $I = [0, 1]$, the $I \times X$ is oriented manifold with boundary.
$\partial(I \times X) = (\partial I) \times X = \{0\} \times X \cup \{1\} \times X$. Let $X_1 = \{1\} \times X$, $X_0 = \{0\} \times X$, they have induced orientation since they are diffeomorphic to X, but this orientation may not compatible with the induced boundary orientation.



Proposition 2.4:

Let X^n, Y^m manifolds with boundary, $W^k \subseteq Y$ submanifold such that $\partial W = \partial Y = \emptyset$, let $f \in C^\infty(X, Y)$ such that $f \pitchfork W$ and $\partial f \pitchfork W$. Suppose X, Y, W oriented. Then $f^{-1}(W)$ has natural orientation.

Proof. Let $Q = f^{-1}(W)$, NQ be the normal bundle of Q (for evert $x \in Q$, $N_xQ \oplus T_xQ = T_xX$). $df_x(T_xQ) = T_{f(x)}W$.

$\boxed{\text{Claim.}}$ $df_x|_{N_xQ}$ is injective.

$f \pitchfork W$, so $df_x(T_xX) + T_{f(x)}W = T_{f(x)}Y$, then $df_x(N_xQ) + T_{f(x)}W = T_{f(x)}Y$, $\dim df_x(N_xQ) = \dim N_xQ$, so df_x is injective.

Since $T_{f(x)}W, T_{f(x)}Y$ are oriented, it induces an orientation on $df_x(N_xQ)$ by $df_x(N_xQ) \oplus T_{f(x)}W = T_{f(x)}Y$, hence induces orientation on N_xQ. By $N_xQ \oplus T_xQ = T_xX$ we have an orientation on T_xQ. □

Corollary 2.2:

Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^m$ regular value of f. Then $f^{-1}(a)$ is orientable.

$\boxed{\text{Exercise.}}$ $f \pitchfork W$ and $\partial f \pitchfork W$. $\partial f^{-1}(W) = (\partial f)^{-1}(W)$. Let X, Y, W are oriented, then for natural orientation, $[\partial f^{-1}(W)] = (-1)^{\mathrm{codim} W}[(\partial f)^{-1}W]$.

2.2 Intersection Number

Theorem 2.4: Classification of 1-Manifolds

Let X compact and connected 1-manifold, then X is diffeomorphic to either $[0, 1]$ or S^1.

Let X, Y, W be oriented manifolds without boundary such that X is compact and $W \subseteq Y$ closed subset and $\dim X + \dim W = \dim Y$. Let $f : X \to Y$, $f \pitchfork W$, then $\dim f^{-1}(W) = 0$, so $f^{-1}(W)$ is a set of isolated points. By compactness and orientation assumption, $f^{-1}(W)$ is a finite number of points with signs. Define $I(f, W) = \sum_{p \in f^{-1}(W)} \mathrm{sign}(p)$.

$\boxed{\text{Recall.}}$ $df_p(T_pX) \oplus T_{f(p)}W = T_{f(p)}Y$, $\mathrm{sign}(p) = +1$ iff orientation match.

For now we always assume that X, Y, W be oriented manifolds without boundary such that X is compact and $W \subseteq Y$ closed subset and $\dim X + \dim W = \dim Y$.

Proposition 2.5:

Let $f_0, f_1 \in C^\infty(X, Y)$ smoothly homotopic and transverse to W, then $I(f_0, W) = I(f_1, W)$.

Proof. Let $Z = [0,1] \times X$, $F : [0,1] \times X \to Y$ the smooth homotopy of f_0, f_1. Since $\partial([0,1] \times X) = \{1\} \times X \cup (-\{0\} \times X)$, by the lemma below we have $0 = I(\partial F, W) = I(f_1, W) - I(f_0, W)$. □

Lemma 2.2:

Suppose $X = \partial Z$, where Z compact oriented manifold with boundary, $f : X \to Y$, $f \pitchfork W$. Suppose that f can be extend to $F : Z \to Y$. Then $I(f, W) = 0$.

Proof. Since $f = F|_{\partial Z}$ and $f \pitchfork W$, $F \pitchfork W$ on ∂Z. We can perturb F so that $F \pitchfork W$ in all Z and $F|_{\partial Z} = f$. $F^{-1}(W)$ is an oriented manifold such that $\partial F^{-1}(W) = \pm f^{-1}(W)$. Since $\dim F^{-1}(W) = 1$, $F^{-1}(W)$ is a compact 1-manifold. So it's a disjoint union of copies of $[0,1]$ and S^1, $\partial F^{-1}(W)$ is an even number of points and number of positive sign is the same as the negative. □

Now if $f \in C^\infty(X, Y)$ not necessarily transverse to W, we can take $g \simeq f$ such that $g \pitchfork W$ and define $I(f, W) = I(g, W)$, by the proposition above it's well-defined.