DIFFERENTIAL TOPOLOGY

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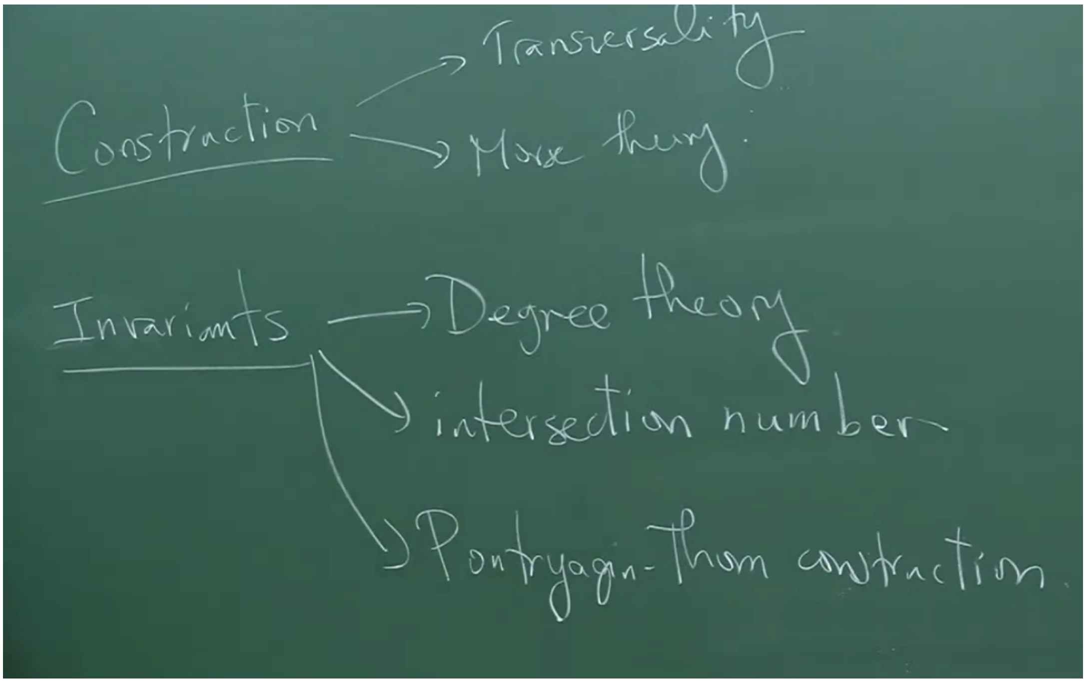
March 1, 2025

Contents



Bibliography.

(1) Guillemin - Pollack, *Differential Topology.*
(2) Hirsch, *Differential Topology.*
(3) Golubitsky-Guillemin, *Stable Mapping and Their Singularities.*
(4) Bredon, *Geometry and Topology.*

1 Review of Differentiable Manifold

Definition 1.1. A topological space is a pair (X, T), where $T \subseteq \mathcal{P}(X)$ such that

- $\emptyset, X \in T$
- $\{U_\alpha\}_{\alpha \in I} \subseteq T \implies \bigcup_{\alpha \in I} U_\alpha \in T$
- $U_1, \ldots, U_n \subseteq T \implies U_1 \cap \cdots \cap U_n \in T$

Fixing (X, T), the elements of T are called open sets.

1.1 Jet bundles

Definition 1.2. Let X, Y be smooth manifolds, $f, g : X \to Y$ smooth.

(1) We write $f \sim_k g$ at $p \in X$ if $f(p) = g(p)$ and given charts $\varphi : U \to \mathbb{R}^n$ around p, $\psi : V \to \mathbb{R}^m$ around $f(p)$

$$\frac{\partial^{|\alpha|}\left(\psi \circ f \circ \varphi^{-1}\right)_j}{\partial x^\alpha}(\varphi(p)) = \frac{\partial^{|\alpha|}\left(\psi \circ g \circ \varphi^{-1}\right)_j}{\partial x^\alpha}(\varphi(p)), \ \forall |\alpha| \leq k, 1 \leq j \leq m$$

It follows from the chain rule that $\sim_k$ is an equivalence relation.

(2) $J^k(X, Y)_{p,q} = \{f : X \to Y \text{ smooth} \mid f(p) = q\} / \sim_k$, called the space of k-jets at p with value q.

(3) $J^k(X, Y) = \bigsqcup\limits_{\substack{p \in X \\ q \in Y}} J^k(X, Y)_{p,q}$.

Example 1.1. (1) $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$,

$$J^k(U, V)_{x,y} \xrightarrow{\Phi} B^k_{n,m}, \ [f] \mapsto (p_1(x), \cdots, p_m(x))$$

where $p_j(x)$ is the Taylor polynomial of $f_j(x)$ without the constant term, $B^k_{n,m} = \{\text{polynomial functions } \mathbb{R}^n \to \mathbb{R}^m \text{ of degree} \leq k \text{ with no constant term}\}$. Φ is a bijection. $J^k(U, V) \cong U \times V \times B^k_{n,m}$.

(2) $J^1(M, \mathbb{R}) \xrightarrow{bijection} \mathbb{R} \times T^*M, \ [f]_x \mapsto (f(x), df_x)$.

(3) $J^1(\mathbb{R}, M) \cong \mathbb{R} \times TM$.

Definition 1.3. (1) $\varphi : Y \to Z$ smooth. Then $\varphi_* : J^k(X, Y) \to J^k(X, Z), \ [f]_x \mapsto [\varphi \circ f]_x$.

(2) $\varphi : Z \to X$ diffeo. Then $\varphi^* : J^k(X, Y) \to J^k(Z, Y)$, $[f]_x \mapsto [f \circ \varphi]_{\varphi^{-1}(x)}$.

Remark 1.1. These operations are well-defined and natural (functionality). In particular, if $\varphi : Y \to Z$ diffeo, then φ_* is bijection; $\varphi : Z \to X$ diffeo, then φ^* is bijection.

Suppose $\sigma \in J^k(X, Y), \ \sigma = [f]_x$.

Define $\alpha(\sigma) = x, \beta(\sigma) = f(x)$, called the source of σ and target of σ respectively, then $\alpha : J^k(X, Y) \to X, \beta : J^k(X, Y) \to Y$. We will define the local topology around σ and a smooth structure near σ.

Fix charts $\varphi : U \to \mathbb{R}^n, \psi : V \to \mathbb{R}^m$ around x and $f(x)$ respectively, $f(U) \subseteq V$. Let

$$\tau_{\varphi,\psi} : J^k(U, V) \longrightarrow J^k\left(\varphi(U), \psi(V)\right) \cong \varphi(U) \times \psi(V) \times B^k_{n,m}, \ \sigma \mapsto (\varphi^{-1})^* \psi_* \sigma$$

Since $\varphi(U) \times \psi(V) \times B^k_{n,m} \subseteq \mathbb{R}^N$, use $\tau_{\varphi,\psi}$ to topologize $J^k(U, V)$ and hence $J^k(X, Y)$. It's easy to see that this topology doesn't depend on the choice of charts.

Exercise 1.1. Let $\widetilde{\varphi} : U \to \mathbb{R}^n$, $\widetilde{\psi} : V \to \mathbb{R}^m$ be other charts, then $\tau_{\widetilde{\varphi},\widetilde{\psi}} \circ \tau_{\varphi,\psi}^{-1}$ is smooth. So $J^k(X,Y)$ has an induced smooth structure.

Lemma 1.1. (1) $J^k(X,Y)$ is a manifold of dimension $n + m\binom{n+k}{k}$.

(2) $\alpha : J^k(X,Y) \to X, \beta : J^k(X,Y) \to Y, \alpha \times \beta : J^k(X,Y) \to X \times Y$ are smooth surjective submersions.

(3) $\varphi : Y \to Z$ smooth, then φ_* is smooth; $\varphi : Z \to X$ diffeomorphism, then φ^* is diffeomorphism.

Definition 1.4. Let $f \in C^\infty(X,Y)$. Its k-jet $j^k f$ is the function

$$j^k f : X \to J^k(X,Y),\ x \mapsto [f]_x$$

Remark 1.2. $J^k(X,Y)$ is usually not a vector bundle over X, Y or $X \times Y$. If $Y = \mathbb{R}^m$, then $J^k(X,Y)$ is a vector bundle over X.

Definition 1.5. Let E, B, F be manifolds, and $\pi : E \to B$ is a surjective submersion. We say that π is a fiber bundle with fiber F if $\forall b \in B$, $\exists U \subseteq B$ neighborhood of b and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times F$ such that the diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

Exercise 1.2. $J^k(X,Y)$ is a fiber bundle over $X, Y, X \times Y$, $f \in C^\infty(X,Y)$ gives rise to a smooth section $j^k f : X \to J^k(X,Y)$.

1.2 Whitney C^∞-Topology

Let X, Y be smooth manifolds. For $U \subseteq J^k(X,Y)$ open, let

$$M(U) = \left\{ f \in C^\infty(X,Y) \mid j^k f(X) \subseteq U \right\}$$

Note that $M(U) \cap M(V) = M(U \cap V)$, so $\{M(U) \mid U \subseteq J^k(X,Y) \text{ open}\}$ is a basis for a topology on $C^\infty(X,Y)$, which is called the C^k-*topology*. Let W_k be the C^k-topology.

Lemma 1.2. $k \le l \implies W_k \subseteq W_l$.

Proof. Suppose $k \leq l$. There exists a surjective continuous map:

$$\pi_{k,l} : J^l(X,Y) \to J^k(X,Y),\ [f]_x \mapsto [f]_x$$

$\pi_{k,l} \circ j^l f = j^k f$. If $U \subseteq J^k(X,Y)$ is open, then $\pi_{k,l}^{-1}(U) \subseteq J^l(X,Y)$ is open. So $M(U) = M(\pi_{k,l}^{-1}(U))$. Therefore $W_k \subseteq W_l$. □

Definition 1.6. The (Whitney) C^∞-topology is the topology on $C^\infty(X,Y)$ generated by $\bigcup\limits_{k\in\mathbb{N}} W_k$.

Recall that every manifold M admits a Riemannian metric, which induced a metric space structure on the manifold (M,d). Moreover, we may assume that d is complete.

Why? (1) $\exists$ smooth proper function $f : M \to \mathbb{R}$; (2) For any metric d on M, we can define $\widetilde{d}(x,y) = d(x,y) + |f(x) - f(y)|$, $\widetilde{d}$ is complete.

Let d be a (complete) metric on $J^k(X,Y)$.

Definition 1.7. Let $\delta : X \to \mathbb{R}_{>0}$ continuous and $f \in C^\infty(X,Y)$, let

$$B^k_\delta(f) = \left\{g \in C^\infty(X,Y) \mid d\left(j^k f(x), j^k g(x)\right) < \delta(x)\right\}$$

Proposition 1.1. $\{B_\delta(f) \mid \delta : X \to \mathbb{R}_{>0}\}$ is a basis for C^k-topology at f. (neighborhood basis)

Proof. $f \in B_\delta(f)$.

$\boxed{\text{Step 1.}}$ $B_\delta(f)$ is open. We claim that

$$B_\delta(f) = M(U),\ U = \left\{\sigma \in J^k(X,Y) \mid d\left(j^k f(\alpha(\sigma)), \sigma\right) < \delta\left(\alpha(\sigma)\right)\right\}$$

Define $\Delta : J^k(X,Y) \to \mathbb{R},\ \Delta = \delta \circ \alpha - d\left(j^k f \circ \alpha(\cdot), \cdot\right)$, so $U = \Delta^{-1}(0,\infty)$ is open.

$\boxed{\text{Step 2.}}$ Let $\mathcal{U} \subseteq C^\infty(X,Y)$ be an open neighborhood of f (in C^k-topology), then there exists $U \subseteq J^k(X,Y)$ open set such that $f \in M(U) \subseteq \mathcal{U}$. We claim that $\exists \delta \in C(X, \mathbb{R}_{>0})$ such that $f \in B_\delta(f) \subseteq M(U)$.

For each $x \in X$, let

$$m(x) = \inf\left\{d\left(\sigma, j^k f(x)\right) \mid \sigma \in \alpha^{-1}(x) \cap \left(J^k(X,Y) \backslash U\right)\right\}$$

It's strictly bigger than 0 for every $x \in X$ because U is open, $m(x)$ could be ∞ for some x. We can choose $\delta : X \to \mathbb{R}_{>0}$ continuous such that $0 < \delta(x) < m(x)$. Then

$$g \in B_\delta(f) \implies d\left(j^k f(x), j^k g(x)\right) < \delta(x) < m(x),\ \forall x \in X$$

which implies $j^k g(x) \in U$, $\forall x \in X$. So $B_\delta(f) \subseteq M(U)$. □

$\boxed{\text{Obs.}}$ $B_\delta(f)$ is roughly the set of functions whose partial derivatives up to order k are close enough to f's.

To make this more precise, let $\Phi = \{\varphi_i : U_i \to \mathbb{R}^n\}_{i\in I}$ locally finite atlas of X, $\mathcal{K} = \{K_i\}_{i\in I}$ family of compact sets of X, $K_i \subseteq U_i$, $\Psi = \{\psi_i : V_i \to \mathbb{R}^m\}_{i\in I}$ atlas for Y, $\mathcal{E} = \{\epsilon_i\}_{i\in I}$, $\epsilon_i > 0$. Define

$$\begin{aligned}\mathcal{N}^k(f;\Phi,\Psi,\mathcal{K},\mathcal{E}) = \{g \in C^\infty(X,Y) \mid g(K_i) \subseteq V_i \text{ and}\\ \|D^r(\psi_i \circ f \circ \varphi_i^{-1})(x) - D^r(\psi_i \circ g \circ \varphi_i^{-1})(x)\| < \epsilon_i, \forall i, x \in X, r \le k\}\end{aligned}$$

$\boxed{\text{Exercise.}}$ Prove that $\{\mathcal{N}^k(f;\Phi,\Psi,\mathcal{K},\mathcal{E})\}$ is a basis for the C^k-topology.

$\boxed{\text{Remark.}}$ If X is compact, then we can find a countable basis of f given by $\{B_{\delta_n}(f)\}$, where $\delta_n = \frac{1}{n}$. So C^k-topology is first countable. Moreover,

$$f_n \xrightarrow{C^k} f \Leftrightarrow \frac{\partial^{|\alpha|} f_n}{\partial x^\alpha} \xrightarrow{\text{uniformly}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha},\ \forall |\alpha| \le k$$

Proposition 1.2. Suppose $\{f_n\}_{n\in\mathbb{N}} \subseteq C^\infty(X,Y)$ such that $f_n \xrightarrow{C^k} f$. Then $\exists K \subseteq X$ compact such that $f_n \equiv f$ in $X\backslash K$ for $n \gg 0$ and $j^k f_n|_K \xrightarrow{\text{uniformly}} j^k f$.

Proof. Suppose $f_n \xrightarrow{C^k} f$ and let $\{K_i\}_{i\in\mathbb{N}}$ exhaustion by compact sets such that $K_i \subseteq \operatorname{int}(K_{i+1})$. Assume, by contradiction, that $\nexists K \subseteq X$ compact set, such that $f_n \equiv f$ on $X\backslash K$. So for each i, $\exists x_i \in K_i, n_i$ such that $f_{n_i}(x_i) \ne f(x_i)$. WLOG, $n_1 < n_2 < \cdots$, $a_i = d\left(j^k f_{n_i}(x_i), j^k f(x_i)\right) > 0$. Let $\delta : X \to \mathbb{R}_+$ such that $\delta(x_i) = a_i/2$. Then $f_{n_i} \notin B_\delta(f)$, so $f_{n_i} \nrightarrow f$. $\square$

Definition 1.8. A topological space is Baire if the countable intersection of open and dense subsets is dense.

Theorem 1.1. Let X, Y be smooth manifolds. Then $C^\infty(X,Y)$ is Baire in the C^∞-topology.

Proof. Fix complete metric d_k on $J^k(X,Y)$. Let $\{U_n\}_{n\in\mathbb{N}}$ dense open subsets of $C^\infty(X,Y)$ in the C^∞-topology. Let $V \subseteq C^\infty(X,Y)$ non-empty open set. We want to show that $\bigcap\limits_{n\in\mathbb{N}} U_n \cap Y \ne \emptyset$.

Since V is open, $\exists Z \subseteq J^{k_0}(X,Y)$ open such that $M(\overline{Z}) \subseteq V$. It's enough to show that $M(\overline{Z}) \cap \bigcap\limits_{n\in\mathbb{N}} U_n \ne \emptyset$.

We can construct f_i inductively, $\{k_i\} \subseteq \mathbb{N}$, $Z_i \subseteq J^{k_i}(X,Y)$ open sets such that

(1) $f_i \in M(Z) \cap \bigcap\limits_{s=1}^{i} M(Z_s)$

(2) $M(\overline{Z_i}) \subseteq U_i$

(3) $d_s\left(j^s f_i(x), j^s f_{i-1}(x)\right) < 1/2^i,\ \forall x \in X, 0 \le s \le i$

Since $M(Z) \cap U_1$ is open and non-empty, we can find $Z_1 \subseteq J^{k_1}(X,Y)$ non-empty such that $M(\overline{Z_1}) \subseteq M(Z) \cap U_1$. Take $f_1 \in M(Z_1)$ and it satisfies (1) and (2). Say we've chosen (f_s, k_s, Z_s) for $s \leq i-1$. Let $D_i = B^0_{\frac{1}{2^i}}(f_{i-1}) \cap B^1_{\frac{1}{2^i}}(f_{i-1}) \cap \cdots \cap B^i_{\frac{1}{2^i}}(f_{i-1})$ open in C^∞-topology, $f_{i-1} \in M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i$. Since U_i is open and dense, $M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i \cap U_i \neq \emptyset$ and open, so we can find $\emptyset \neq Z_i \subseteq J^{k_i}(X,Y)$ such that $M(\overline{Z_i}) \subseteq M(Z) \cap M(Z_1) \cap \cdots \cap M(Z_{i-1}) \cap D_i \cap U_i$. Choose $f_i \in M(Z_i)$, it satisfies the three conditions.

For a fixed s, the condition (3) tells that $\{j^s f_i(x)\}$ is a Cauchy sequence in $J^k(X,Y)$, it converges to $g^s(x)$, $g^0(x) \in J^0(X,Y) = X \times Y$, $g^0(x) = (x, g(x))$.

Exercise. $g \in C^\infty(X,Y)$ and $j^s g = g^s$. (Look in a compact set and in charts)

Then $g = \lim\limits_{i \to \infty} f_i$ in the C^∞-topology. $f_i \in M(Z) \implies g \in M(\overline{Z})$, $f_i \in M(Z_s)$ for $i \geq s$, so $g \in M(\overline{Z_s})$ for $\forall s$, hence $g \in M(\overline{Z}) \cap \bigcap\limits_{n\in\mathbb{N}} M(\overline{Z_s}) \subseteq M(\overline{Z}) \cap \bigcap\limits_{n\in\mathbb{N}} U_n$. □

Proposition 1.3. Let X, Y be smooth manifolds. Then $j^k : C^\infty(X,Y) \to C^\infty\left(X, J^k(X,Y)\right)$ is continuous in the C^∞-topology.

Proof. $U \subseteq J^\ell\left(X, J^k(X,Y)\right)$ open, so $M(U)$ is open set in the C^ℓ-topology of $C^\infty\left(X, J^k(X,Y)\right)$. We need to show that $(j^k)^{-1}(M(U))$ is open in $C^\infty(X,Y)$. Consider

$$\alpha^{k,\ell} : J^{k+\ell}(X,Y) \to J^\ell\Big(X, J^k(X,Y)\Big),\ \alpha^{k,\ell}\Big(j^{k+\ell} f(x)\Big) = j^\ell(j^k f)(x)$$

This is a smooth embedding. So $(j^k)^{-1}(M(U)) = M\left((\alpha^{k,\ell})^{-1}(U)\right)$ is open in $C^{k+\ell}$-topology. □

Proposition 1.4. $\phi : Y \to Z$ smooth. Then $\widetilde{\phi_*} : C^\infty(X,Y) \to C^\infty(X,Z)$, $f \mapsto \phi \circ f$ is continuous in the C^∞-topology.

Proposition 1.5. Let X, Y, Z be smooth manifolds. Then $C^\infty(X,Y) \times C^\infty(X,Z) \to C^\infty(X, Y \times Z)$, $(f,g) \mapsto f \times g$ is a homeomorphism in the C^∞-topology.

Appendix. About existence of proper function on manifolds (from GTM218).

" If M is a topological space, an exhaustion function for M is a continuous function $f \colon M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$(called a sublevel set of f) is compact for each $c \in \mathbb{R}$. The name comes from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}((-\infty, n])$ form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{B}^n \to \mathbb{R}$ given by

$$f(x) = |x|^2, \quad g(x) = \frac{1}{1 - |x|^2}$$

are smooth exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.28 (Existence of Smooth Exhaustion Functions).Every smooth manifold with or without boundary admits a smooth positive exhaustion function.

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^{\infty}$be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j\ \psi_j(p).$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \geq \sum_j \psi_j(p) = 1$. To see that f is an exhaustion function, let $c \in \mathbb{R}$ be arbitrary, and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^{N} \bar{V}_j$,then $\psi_j(p) = 0$ for $1 \leq j \leq N$,so

$$f(p) = \sum_{j=N+1}^{\infty} j\ \psi_j(p) \geq \sum_{j=N+1}^{\infty} N\ \psi_j(p) = N \sum_{j=1}^{\infty} \psi_j(p) = N > c.$$

Equivalently, if $f(p) \leq c$, then $p \in \bigcup_{j=1}^{N} \bar{V}_j$. Thus $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^{N} \bar{V}_j$ and is therefore compact."

1.3 Transversality Theorem

Definition 1.9. Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold. We say that f is transverse to W (write $f \pitchfork W$) if

$$df_x(T_x X) + T_{f(x)} W = T_{f(x)} Y,\ \forall x \in f^{-1}(W)$$

$\boxed{\text{Note.}}$ $X_1, X_2 \subseteq Y$ submanifolds, $X_1 \pitchfork X_2 \Leftrightarrow T_x X_1 + T_x X_2 = T_x Y$ for $\forall x \in X_1 \cap X_2$. It's just inclusion of one submanifold transverse to another submanifold.

Proposition 1.6. Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold such that $\dim X + \dim W < \dim Y$. Then $f \pitchfork W \Leftrightarrow f(X) \cap W = \emptyset$.

Proof. The proof is easy. □

Theorem 1.2. Let X, Y be manifolds, $f \in C^\infty(X, Y)$, $W \subseteq Y$ submanifold such that $f \pitchfork W$. If $f^{-1}(W) \neq \emptyset$, then $f^{-1}(W)$ is a submanifold of X of codim $f^{-1}(W) =$ codim W. In particular, if $\dim X =$ codim W, then $f^{-1}(W)$ consists of isolated points.

Proof. Let $p \in f^{-1}(W)$, $n = \dim X, m = \dim Y, k = \dim W$. Let $\varphi : U \to \mathbb{R}^m$ be a chart around $f(p)$ such that $\varphi(U \cap W) \subseteq \mathbb{R}^k \times \{0\}$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ be the orthogonal projection along $\mathbb{R}^k \times \{0\}$, $\phi = \pi \circ \varphi$. Then $\phi : U \to \mathbb{R}^{m-k}$ is a submersion and $\phi^{-1}(0) = U \cap W$.

Claim. $f \pitchfork W$ at $p \Leftrightarrow p$ is a regular point of $\phi \circ f$.

Since $\phi^{-1}(0) = U \cap W$, $\ker d\phi_{f(p)} = T_{f(p)}W$. Transversality assumption gives that $df_p(T_pX) + T_{f(p)}W = T_{f(p)}Y = T_{f(p)}U$, which implies that $d(\phi \circ f)_p(T_pX) = d\phi_{f(p)}T_{f(p)}U$. And the converse is easy to proof.

Now $f \pitchfork W$ on $U \Leftrightarrow 0$ is a regular value of $\phi \circ f : f^{-1}(U) \to \mathbb{R}^{m-k}$. By the implicit function theorem, $(\phi \circ f)^{-1}(0) = f^{-1}(U \cap W)$ is a submanifold of $f^{-1}(U) \subseteq X$ open set of codimension $m-k$. So $f^{-1}(W)$ is a submanifold of X of codimension $m-k$. □

Proposition 1.7. Let X, Y be manifolds, $W \subseteq Y$ submanifold which is a closed subset. Then $T_W \coloneqq \{f \in C^\infty(X,Y) \mid f \pitchfork W\}$ is open in the C^∞-topology.

Proof. We show that T_W is open in the C^1-topology. Let

$$U = \left\{\sigma = j^1 f(x) \in J^1(X,Y) \mid f(x) \notin W \text{ or } df_x(T_xX) + T_{f(x)}W = T_{f(x)}Y\right\}$$

It's easy to see that $T_W = M(U) = \left\{f \in C^\infty(X,Y) \mid j^1 f(X) \subseteq U\right\}$

Claim. U is open.

We will show that $V = J^1(X,Y)\backslash U$ is closed. To prove that, take $\{\sigma_n\} \subseteq V$ such that $\sigma_n \to \sigma \in J^1(X,Y)$, we need to show that $\sigma \in V$. Consider continuous map $\beta : J^1(X,Y) \to Y$, then $\beta(\sigma_n) \to \beta(\sigma)$. Since $\beta(\sigma_n) \in W$ and W is closed, we have $\beta(\sigma) \in W$, which mean that $\sigma = j^1 f(x), f(x) \in W$.

Now choose charts around x and $f(x)$, $\varphi : \widetilde{U} \to \mathbb{R}^n$, $\psi : \widetilde{V} \to \mathbb{R}^m$, $\psi(\widetilde{V} \cap W) = \mathbb{R}^k \times \{0\}$, $\varphi(x) = 0, \psi(f(x)) = 0$. $f \pitchfork W$ at $x \Leftrightarrow \psi \circ f \circ \varphi^{-1} \pitchfork \mathbb{R}^k \times \{0\}$ at $0 \Leftrightarrow 0$ is a regular value of $\pi \circ \psi \circ f \circ \varphi^{-1}$ where $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ orthogonal projection $\Leftrightarrow$ $\pi \circ d(\psi \circ f \circ \varphi^{-1})_0$ has rank $m-k$.

Let $F = \left\{A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}) \mid \mathrm{rank}\ A < m-k\right\}$. In a neighborhood $\mathcal{N}$ of σ, fixing φ, ψ we obtain a map

$$\eta : \mathcal{N} \subseteq J^1(X,Y) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}),\ j^1 g(x) \mapsto \pi \circ d(\psi \circ f \circ \varphi^{-1})_{\varphi(x)}$$

$V \cap \mathcal{N} = \eta^{-1}(F)$, η is continuous. □

Exercise 1.3. F is closed.

Remark 1.3. The condition that W is closed is necessary.

Lemma 1.3. Let X, Y, B manifolds, $W \subseteq Y$ submanifold, let $j : B \to C^\infty(X,Y)$ map (not necessary continuous).

$$\Phi : X \times B \to Y,\ \Phi(x,b) = j(b)(x)$$

Suppose Φ is smooth and $\Phi \pitchfork W$. Then $\{b \in B \mid j(b) \pitchfork W\}$ is dense in B.

Proof. Let $W_\Phi = \Phi^{-1}(W) \subseteq X \times B$ be the submanifold, $pr : X \times B \to B$ the projection and $\pi = pr|_{W_\Phi}$.

$\boxed{\text{Claim.}}$ b is a regular value of $\pi \implies j(b) \pitchfork W$.

Suppose b is a regular value of π.

(1) $b \notin \operatorname{im} \pi$, then $\nexists x \in X$ such that $\Phi(x, b) \in W$, so $j(b)(X) \cap W = \emptyset$, which means $j(b) \pitchfork W$.

(2) If $\dim W_\Phi < \dim B$, then b is a regular value of π, so $b \notin \operatorname{im} \pi$, therefore by (1) we have $j(b) \pitchfork W$.

(3) If $\dim W_\Phi \geq \dim B$. Let b be a regular value of π and $x \in X$. If $(x, b) \notin W_\Phi$, then $j(b)(x) \notin W$, so $j(b) \pitchfork W$ at x. If $(x, b) \in W_\Phi$, then $\pi\left(T_{(x,b)} W_\Phi\right) = T_b B$, which gives $T_{(x,b)}(X \times B) = T_{(x,b)} W_\Phi + T_{(x,b)}(X \times \{b\})$, so $T_{j(b)(x)} Y = T_{j(b)(x)} W + (dj(b))_x T_x X$, so $j(b) \pitchfork W$ at x. ☐

Corollary 1.1. Let $G : X \times B \to Y$ smooth, $\Phi(x, b) = j^k G_b(x)$. If $\Phi \pitchfork W$, where $W \subseteq J^k(X, Y)$ submanifold. Then $\left\{b \in B \mid j^k G_b \pitchfork W\right\}$ is dense in B.

Proof. Let $j : B \to C^\infty(X, J^k(X, Y)), b \mapsto j^k G_b$. ☐

Theorem 1.3. (Thom Transversality Theorem.)

Let X, Y manifolds, $W \subseteq J^k(X, Y)$ submanifold, let

$$T_W = \left\{f \in C^\infty(X, Y) \mid j^k f \pitchfork W\right\}$$

Then T_W is a residual subset of $C^\infty(X, Y)$ (residual subset means countable intersection of open and dense sets). Moreover if W is closed, then T_W is open.

Proof. For each $\sigma \in W$, let $W_\sigma \subseteq W$, $U_\sigma \subseteq X$, $V_\sigma \subseteq Y$ open neighborhood of σ, $\alpha(\sigma)$, $\beta(\sigma)$ respectively and charts $\varphi_\sigma : U_\sigma \to \mathbb{R}^n$, $\psi_\sigma : V_\sigma \to \mathbb{R}^m$ such that:

(a) $\overline{W}_\sigma \subseteq W$ and is compact.

(b) $\overline{U}_\sigma$ is compact.

(c) $\alpha(\overline{W}_\sigma) \subseteq U_\sigma$ and $\beta(\overline{W}_\sigma) \subseteq V_\sigma$.

(d) $\psi_\sigma(V_\sigma) = \mathbb{R}^m$.

Let A be subset of W, we say that $g \pitchfork W$ on A if $g \pitchfork W$ for $\forall x \in g^{-1}(A)$. Let

$$T_\sigma = \left\{f \in C^\infty(X, Y) \mid j^k f \pitchfork W \text{ on } \overline{W}_\sigma\right\}$$

$T_W = \bigcap\limits_{\sigma \in W} T_\sigma$. Since W is 2-countable, there exists a countable covering $\{W_{\sigma_i}\}_{i=1}^\infty$ of W. So $T_W = \bigcap\limits_{i=1}^\infty T_{\sigma_i}$.

$\boxed{\text{Claim.}}$ T_σ is open and dense.

(1) T_σ is open. Let $\widetilde{T}_\sigma = \left\{g \in C^\infty\left(X, J^k(X, Y)\right) \mid g \pitchfork W \text{ on } \overline{W}_j\right\}$. By previous proposition we have $\widetilde{T}_\sigma$ is open, then $T_\sigma = (j^k)^{-1}(\widetilde{T}_\sigma)$ is open.

(2) T_σ is dense. Let $f \in C^\infty(X,Y)$, we will construct a sequence $\{g_n\} \subseteq C^\infty(X,Y)$ such that $g_n \in T_\sigma$ and $g_n \xrightarrow{C^\infty} f$. The idea is to define $\Phi : X \times B \to J^k(X,Y)$, $\Phi(x,b) = j^k g_b(x)$, where $g_b(x)$ is a polynomial perturbation of f, such that $\Phi \pitchfork W$.

Fix smooth functions $\rho_1 : \mathbb{R}^n \to [0,1]$, $\rho_2 : \mathbb{R}^m \to [0,1]$ such that $\rho_1 \equiv 1$ in a neighborhood of $\varphi\left(\alpha(\overline{W}_\sigma)\right)$, supp $\rho_1 \subseteq \varphi(U_\sigma)$; $\rho_2 \equiv 1$ in a neighborhood of $\psi\left(\beta(\overline{W}_\sigma)\right)$, supp ρ_2 is compact. Let $B = \{\text{polynomial maps } \mathbb{R}^n \to \mathbb{R}^m \text{ of degree } \leq k\}$.

For $b \in B$, let

$$g_b(x) = \begin{cases} \psi^{-1}\left(\psi\left(f(x)\right) + b\left(\varphi(x)\right)\rho_1\left(\varphi(x)\right)\rho_2\left(\psi\left(f(x)\right)\right)\right) & \text{if } x \in U_\sigma, f(x) \in V_\sigma \\ f(x) & \text{if } x \notin U_\sigma \text{ or } f(x) \notin V_\sigma \end{cases}$$

$G : X \times B \to Y$, $G(x,b) = g_b(x)$.

$\boxed{\text{Exercise.}}$ G is smooth.

Let $\Phi : X \times B \to J^k(X,Y)$, $\Phi(x,b) = j^k g_b(x)$, so Φ is smooth.

$\boxed{\text{Claim.}}$ $\exists \widetilde{B} \subseteq B$ open neighborhood of $0 \in B$ such that $\Phi|_{X \times \widetilde{B}} \pitchfork W$ on $\overline{W}_\sigma$.

Assuming the claim, apply the previous lemma, $\exists \{b_n\} \subseteq \widetilde{B}$ such that $b_n \to 0$ and $j^k g_{b_n} \pitchfork \left(W \cap \overline{W}_\sigma\right)$, this also implies $g_{b_n} \xrightarrow{C^\infty} f$ and $j^k g_{b_n} \pitchfork W$ on $\overline{W}_\sigma$. So T_σ is dense.

Proof of the claim: Let $\epsilon = \frac{1}{2} d\left(\psi\left(\beta(\overline{W}_j)\right), \rho_2^{-1}\left([0,1)\right)\right) > 0$, define

$$\widetilde{B} = \{b \in B \mid ||b(x)|| < \epsilon,\ \forall x \in \text{supp } \rho_1\}$$

We fix $b \in \widetilde{B}$ such that $\Phi(x,b) \in \overline{W}_\sigma$. We will show that Φ is a local diffeomorphism near (x,b). Since $\Phi(x,b) \in \overline{W}_\sigma$, $x \in \alpha(\overline{W}_\sigma)$, $g_b(x) \in \beta(\overline{W}_\sigma)$. $\psi\left(g_b(x)\right) = \psi\left(f(x)\right) + b\left(\varphi(x)\right)\rho_1\left(\varphi(x)\right)\rho_2\left(\psi\left(f(x)\right)\right) = \psi\left(f(x)\right) + b\left(\varphi(x)\right)$. Because $||b\left(\varphi(x)\right)|| < \epsilon$, $\forall x \in$ supp ρ_1, then $\rho_2\left(\psi\left(g_b(x)\right)\right) = 1$. So $\psi \circ g_b(x) = \psi\left(f(x)\right) + b\left(\varphi(x)\right)$ in a neighborhood of (x,b). σ' is sufficiently close to σ, so we can find a unique polynomial b' so that $\sigma' = j^k\left(\psi^{-1}\left(f\left(\varphi\left(\alpha(\sigma')\right)\right)\right)\right) + b'\left(\varphi\left(\alpha(\sigma')\right)\right)$. So we have constructed a local inverse for every $(x,b) \in \Phi^{-1}(\overline{W}_\sigma)$, then $\Phi \pitchfork W$ on $\overline{W}_\sigma$. $\square$

Corollary 1.2. Let X, Y manifolds, $f \in C^\infty(X,Y)$, $W \subseteq J^k(X,Y)$ submanifold such that $\overline{\alpha(W)} \subseteq U$ open set. Let V an open neighborhood of f in $C^\infty(X,Y)$. Then there exists a smooth mapping g in V such that $j^k g \pitchfork W$ and $g = f$ off U.

Proof. The same as the theorem above but we choose $U_\sigma \subseteq U$ for $\forall \sigma \in W$. $\square$

Corollary 1.3. (Elementary Transversality Theorem.)

Let X, Y manifolds, $W \subseteq Y$ submanifold.

(a) $T_W = \{f \in C^\infty(X,Y) \mid f \pitchfork W\}$ is dense in $C^\infty(X,Y)$. Moreover if W is closed, then T_W is open.

(b) Let $U_1, U_2 \subseteq X$ open sets such that $\overline{U}_1 \subseteq U_2$, let $f \in C^\infty(X,Y)$, $V \subseteq C^\infty(X,Y)$ near f and open. Then there is $g \in C^\infty(X,Y)$ such that $g = f$ on U_1 and $g \pitchfork W$ off U_2.

Proof. (a) Note that $J^0(X,Y) = X \times Y$ and $j^0 f(x) = (x, f(x))$. The projection $\beta : X \times Y \to Y$ is a submersion, so $\beta^{-1}(W)$ is a submanifold of $X \times Y$. If $j^0 f \pitchfork \beta^{-1}(W)$ at x then $f \pitchfork W$ at x, then we are done.

(b) Note that $W' = \beta^{-1}(W) \cap \big(X \times Y - \alpha^{-1}(\overline{U}_2)\big)$ is a submanifold of $X \times Y$ since $X \times Y - \alpha^{-1}(\overline{U}_2)$ is an open subset of $X \times Y$. Also $\alpha(W') \subseteq X - \overline{U}_1$, so there exists $g \in C^\infty(X,Y)$ s.t. $g = f$ off $X - \overline{U}_1$ and $j^0 g \pitchfork W'$, the latter condition is the same as $j^0 g \pitchfork \beta^{-1}(W)$ off U_2. So we have $g = f$ on U_1 and $g \pitchfork W$ off U_2. □

Exercise 1.4. Prove that "$j^0 f \pitchfork \beta^{-1}(W)$ at x then $f \pitchfork W$ at x" in (a).

Definition 1.10. (Multijets.) Let X, Y manifolds. For $s \in \mathbb{N}$, define

$$X^{(s)} = \{(x_1, \ldots, x_s) \in X^s \mid x_i \neq x_j,\ i \neq j\}$$

$\alpha^s = \alpha \times \cdots \times \alpha : J^k(X,Y)^s \to X^s$, let $J^k_s(X,Y) = (\alpha^s)^{-1}(X^{(s)}) \subseteq J^k(X,Y)^s$ open, so $J^k_s(X,Y)$ is a manifold. $f \in C^\infty(X,Y)$ gives rise to

$$j^k_s f : X^{(s)} \to J^k_s(X,Y),\ j^k_s f(x_1, \ldots, x_s) = \Big(j^k f(x_1), \ldots, j^k f(x_s)\Big)$$

Theorem 1.4. (Thom Transversality for multijets.)
Let X, Y manifolds, $W \subseteq J^k_s(X,Y)$ submanifold. Let

$$T_W = \Big\{ f \in C^\infty(X,Y) \mid j^k_s f \pitchfork W \Big\}$$

Then T_W is residual. Moreover, if W is compact, then T_W is open.

1.4 Whitney Immersions and Embeddings

Let X^n, Y^m manifolds, $\sigma = j^1 f(x) \in J^1(X,Y)$. Then $df_x : T_x X \to T_{f(x)} Y$ depends only on σ. Define $\operatorname{rank}(\sigma) = \operatorname{rank}(df_x)$ and $\operatorname{corank}(\sigma) = \min(m,n) - \operatorname{rank}(\sigma)$. Let $S_r = \big\{\sigma \in J^1(X,Y) \mid \operatorname{corank}(\sigma) = r\big\}$.

Lemma 1.4. f is an immersion ($n \leq m$) or submersion ($n \geq m$) $\Leftrightarrow j^1 f(X) \cap \bigcup\limits_{r \geq 1} S_r = \emptyset$.

Proof. f is not an immersion/submersion $\Leftrightarrow \exists x \in X$ such that $\operatorname{rank}(df_x) \leq \min(m,n) - 1$ $\Leftrightarrow \exists x \in X$ such that $\operatorname{corank}(j^1 f(x)) \geq 1 \Leftrightarrow j^1 f(X) \cap S_r \neq \emptyset$ for some $r \geq 1$. □

Proposition 1.8. S_r is a submanifold of codimension $(n-q+r)(m-q+r)$, where $q = \min(n,m)$.

Proof. S_r is a bundle over $X \times Y$ with fiber $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) = \{A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \mid \operatorname{corank}(A) = r\}$.

$\boxed{\text{Claim.}}$ $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of codimension $(n-q+r)(m-q+r)$. So $S_r \subseteq J^1(X, Y)$ is a subbundle over $X \times Y$.

Proof of the claim: Let $M \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m)$, let $k = q - r$. We can choose basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ so that

$$[M] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \ A \text{ is an invertible } k \times k \text{ matrix}$$

So in a neighborhood U of M, every other M' will be represented as

$$[M'] = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \ A' \text{ is an invertible } k \times k \text{ matrix}$$

$$\text{So rank}\,[M'] = \text{rank} \begin{bmatrix} I^k & 0 \\ -C'(A')^{-1} & I_{m-k} \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \text{rank} \begin{bmatrix} A' & B' \\ 0 & D' - C'(A')^{-1}B' \end{bmatrix}$$

Then $\text{rank}\,[M'] = k \Leftrightarrow D' - C'(A')^{-1}B' = 0$. $M' \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U \Leftrightarrow D' - C'(A')^{-1}B' = 0$. Let

$$\varphi : U \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathcal{L}(\mathbb{R}^{n-k}, \mathbb{R}^{m-k}), \ \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \mapsto D' - C'(A')^{-1}B'$$

φ is a submersion, so $\varphi^{-1}(0) = \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m) \cap U$ is a submanifold of codimension $(n-q+r)(m-q+r)$.

□

$\boxed{\text{Obs.}}$ $\mathcal{L}^0(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is open. So $S_0 \subseteq J^1(X, Y)$ open submanifold, then $\bigcup_{r \geq 1} S_r$ is closed.

Lemma 1.5. Suppose $n \leq m$. Then

$$\text{Imm}(X, Y) = \{f : X \to Y \text{ smooth immersion}\}$$

is an open subset of $C^\infty(X, Y)$.

Proof. $\text{Imm}(X, Y) = M(S_0)$. □

Theorem 1.5. (Whitney Immersion.) Let X^n, Y^m be manifolds such that $m \geq 2n$. Then $\text{Imm}(X, Y)$ is open and dense subset of $C^\infty(X, Y)$.

Proof. $\min(n, m) = n$, so for $r \geq 1$, $\text{codim}\ S_r = (n-q+r)(m-q+r) = r(n+r) \geq n+1$. So $j^1 f \pitchfork S_r \Leftrightarrow j^1 f(X) \cap S_r = \emptyset$ since $\dim X = n < n+1 \leq \text{codim}\ S_r$.

$$\text{Imm}(X, Y) = \left\{ f \in C^\infty(X, Y) \mid j^1 f(X) \cap \bigcup_{r \geq 1} S_r = \emptyset \right\} = \left\{ f \in C^\infty(X, Y) \mid j^1 f \pitchfork \bigcup_{r \geq 1} S_r \right\}$$

By the Thom transversality theorem, $\text{Imm}(X, Y)$ is dense and open. □

Theorem 1.6. (Whitney Injective Immersion Theorem.) Let X^n, Y^m be manifolds such that $m \geq 2n+1$. Then the set of injective immersions is residual.

Proof. $\mathrm{Imm}(X,Y)$ is open and dense, we need to show

$$\mathrm{Inj}(X,Y) = \{f \in C^\infty(X,Y) \mid f \text{ is injective}\} \text{ is residual}$$

Recall $J_2^0(X,Y) = X^{(2)} \times Y^2 = \left\{(x_1,x_2,y_1,y_2) \in X^2 \times Y^2 \mid x_1 \neq x_2\right\}$, let

$$W = X^{(2)} \times \Delta Y = \{(x_1,x_2,y,y) \mid x_1 \neq x_2\} \subseteq J_2^0(X,Y)$$

f is injective iff $j_2^0 f(X^{(2)}) \cap W = \emptyset$. Codimension of W is dimension of Y, so f is injective iff $j_2^0 f \pitchfork W$ from the proof of previous theorem. By the Thom transversality theorem for multijets, we have $\mathrm{Inj}(X,Y)$ is residual. $\square$

Lemma 1.6. Let X manifold. Then

$$\mathrm{Prop}(X,\mathbb{R}^m) = \{f \in C^\infty(X,\mathbb{R}^m) \mid f \text{ is proper}\}$$

is non-empty and open.

Proof. Recall that there exists a proper map $X \to \mathbb{R}$, compose this map with a linear injection $\mathbb{R} \to \mathbb{R}^m$ to obtain a proper map.

Now let $f \in \mathrm{Prop}(X,\mathbb{R}^m)$. For $x \in X$, define $V_x = \{y \in \mathbb{R}^m \mid d(y, f(x)) < 1\}$. So $V_x \subseteq \mathbb{R}^m$ open. Let $V = \bigcup_{x \in X} \{x\} \times V_x$, then $V \subseteq X \times \mathbb{R}_m = J^0(X,\mathbb{R}^m)$ is open. $f \in M(V)$ because $j^0 f(x) = (x, f(x))$, $d(f(x), f(x)) = 0$, so $f(x) \in V_x$.

$\boxed{\text{Claim.}}$ $M(V) \subseteq \mathrm{Prop}(X,\mathbb{R}^m)$.

If $g \in M(V)$, then $d(g(x), f(x)) < 1$ $\forall x \in X$, so $g^{-1}\left(\overline{B}_r(0)\right) \subseteq f^{-1}\left(\overline{B}_{r+1}(0)\right)$. Since f is proper, $f^{-1}\left(\overline{B}_{r+1}(0)\right)$ is compact, therefore $g^{-1}\left(\overline{B}_r(0)\right)$ is compact, hence g is proper. $\square$

Corollary 1.4. (Whitney Embedding Theorem.) Let X^n manifold. Then there exists $X \hookrightarrow \mathbb{R}^{2n+1}$.

Proof. $\mathrm{Inj}(X,\mathbb{R}^{2n+1}) \cap \mathrm{Imm}(X,\mathbb{R}^{2n+1}) \cap \mathrm{Prop}(X,\mathbb{R}^{2n+1}) \neq \emptyset$. $\square$

1.5 Morse Functions

Definition 1.11. Let $f : X \to \mathbb{R}$ smooth and $p \in \mathrm{Crit}(f)$ ($df_p = 0$). Define the Hessian of f to be the bilinear map:

$$D^2 f_p : T_pX \times T_pX \to \mathbb{R},\ D^2 f_p\left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p\right) = \left.\frac{\partial^2}{\partial x_i \partial x_j}\right|_{\varphi(p)} (f \circ \varphi^{-1})$$

where $\varphi = (x_1, \ldots, x_n)$ is a chart around p. We say that p is non-degenerate if $D^2 f_p$ is non-degenerate.

$\boxed{\text{Exercise.}}$ $D^2 f_p$ doesn't depend on the choice of a chart whenever $p \in \mathrm{Crit}(f)$.

Let $f : X \to \mathbb{R}$ smooth, $df : TX \to \mathbb{R}$, $(p, v) \in TX$, we have $d_{(p,v)} df : T_{(p,v)} TX \to \mathbb{R}$, $T_{(p,v)} TX$ is isomorphic to $T_p X \oplus T_p X$ but it's not natural.

Proposition 1.9. $p \in \mathrm{Crit}(f)$ is non-degenerate $\Leftrightarrow j^1 f \pitchfork S_1$ at p.

Proof. This is a local question, we may assume $X = U \subseteq \mathbb{R}^n$, $J^1(X, \mathbb{R}) = U \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $\pi : J^1(X, \mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ submersion, $\pi^{-1}(0) = S_1 = \left\{ j^1 f(x) \mid df_x = 0 \right\}$.

$\boxed{\text{Claim.}}$ $j^1 f \pitchfork S_1$ at $p \Leftrightarrow \pi \circ j^1 f$ is a submersion at p.

Now $\pi \circ j^1 f : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $x \mapsto \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x) \right)$ is a submersion at p iff $\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_p$ is invertible iff p is non-degenerate. □

Definition 1.12. $f \in C^\infty(X, \mathbb{R})$ is Morse if every $p \in \mathrm{Crit}(f)$ is non-degenerate.

Corollary 1.5. $f \in C^\infty(X, \mathbb{R})$ is Morse $\Leftrightarrow j^1 f \pitchfork S_1$.

Theorem 1.7. Let X manifold. Then $\{ f \in C^\infty(X, \mathbb{R}) \mid f \text{ is Morse} \}$ is open and dense in $C^\infty(X, \mathbb{R})$.

Proof. Since $S_1 = J^1(X, \mathbb{R}) \backslash S_0$ is closed, by the corollary and Thom transversality theorem we complete the proof. □

2 Intersection Theory

2.1 Manifolds with boundary and orientation

Definition 2.1. A topological manifold with boundary is a 2-countable Hausdorff topological space such that every point $p \in X$ has a neighborhood which is homeomorphic to an open set in $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$.

Lemma 2.1. Let X be a topological manifold with boundary, $p \in X$, $\varphi, \psi : U \to \mathbb{H}^n$ charts around p. Suppose $pr_1 \circ \varphi(p) = 0$, then $pr_1 \circ \psi(p) = 0$, where pr_1 is the canonical projection of $\mathbb{H}^n$ to the first coordinate.

Proof. $\psi \circ \varphi^{-1} : \varphi(U) \to \psi(U)$ is homeomorphic, then $\psi \circ \varphi^{-1} : \varphi(U) \backslash \varphi(p) \to \psi(U) \backslash \psi(p)$ is also homeomorphic. Since $pr_1 \circ \varphi(p) = 0$, $\varphi(U) \backslash \varphi(p)$ is contractible. If $pr_1 \circ \psi(p) \neq 0$, then $\psi(U) \backslash \psi(p) \simeq S^{n-1}$, S^{n-1} and contractible space have different homology group, so they can't be homeomorphic. □

Definition 2.2. Let X be a topological manifold with boundary. Then

$$\partial X = \{p \in X \mid \exists \varphi : U \to \mathbb{H}^n \text{ chart around } p \text{ s.t. } pr \circ \varphi(p) = 0\}, \ \mathrm{int}(X) = X \backslash \partial X$$

$\boxed{\text{Obs.}}$ $\mathrm{int}(X)$ and ∂X are topological manifold without boundary of dimension n and $n-1$ respectively.

Definition 2.3. A map $f : \mathbb{H}^n \supseteq U \to \mathbb{H}^n$ is smooth if it admits a smooth extension to $\widetilde{f} : \widetilde{U} \to \mathbb{R}^n$, where $U \subseteq \widetilde{U}$ in an open set in $\mathbb{R}^n$.

Definition 2.4. We say that two charts $\varphi : U \to \mathbb{H}^n$, $\psi : V \to \mathbb{H}^n$ are compatible if $\psi \circ \varphi^{-1}$ is smooth. An atlas is a collection of charts whose domain cover X.

Definition 2.5. A (smooth) manifold with boundary is a topological manifold with boundary endowed with a maximal (smooth) atlas.

Smooth manifold with boundary X induces smooth structure (without boundary) on $\mathrm{int}(X)$ and ∂X.

Proposition 2.1. Let $f \in C^\infty(X, \mathbb{R})$, $a \in \mathbb{R}$ regular value of f. Then $f^{-1}([a, +\infty))$ and $f^{-1}((-\infty, a])$ are manifolds with boundary.

Proof. $(a, +\infty) \subseteq \mathbb{R}$ is open then $f^{-1}((a, +\infty))$ is a manifold without boundary. Let $p \in f^{-1}(a)$, by the implicit function theorem, there exists a chart $\varphi : U \to \mathbb{R}^n$ such that

$\varphi(p)=0$ and $f\circ\varphi^{-1}(x_1,\dots,x_n)=a+x_1$. So we obtain a chart $\varphi|_{f^{-1}([a,+\infty))\cap U}:\widetilde{U}\to\mathbb{H}^n$. So $f^{-1}([a,+\infty))$ is a manifold with boundary. $\square$

Definition 2.6. Let X be a manifold with boundary, $p\in X$, a curve centered at p is a smooth map $\gamma:[0,\epsilon)\to X$ or $\gamma:(-\epsilon,0]\to X$ such that $\gamma(0)=p$. T_pX is the equivalent classes of curves centered at p.

If $x\in\operatorname{int}(X)$, then $T_x(\operatorname{int}(X))=T_xX$; If $x\in\partial X$, then T_xX is still a n-dimensional vector space. Moreover, we have a canonical inclusion $T_x(\partial X)\subseteq T_xX$.

Proposition 2.2. Let X,Y be manifolds with boundary, $y\in\operatorname{int}(Y)$ regular value of $f:X\to Y$ and $\partial f:=f|_{\partial X}:\partial X\to Y$. Then $f^{-1}(y)$ is a submanifold with boundary and $\partial\left(f^{-1}(y)\right)=f^{-1}(y)\cap\partial X=(\partial f)^{-1}(y)$.

Example 2.1. $f:\mathbb{H}^2\to\mathbb{R},\ (x,y)\mapsto x^2+y^2$, then $f^{-1}(1)=S^1\cap\mathbb{H}^2$.

Exercise 2.1. Prove the proposition.

$\boxed{\text{Hint.}}$ We just look locally at $x\in f^{-1}(y)$. If x happens to be in $\operatorname{int}(X)$ then we are done; if $x\in\partial X$, then locally f is a smooth map from open subset U of $\mathbb{H}^n$ to $\mathbb{R}^n$, so we can extend U a little bit to an open subset U' in $\mathbb{R}^n$, then we can use regular level set theorem to produce a submanifold W in U', note that $f^{-1}(y)=W\cap\mathbb{H}^n$. Now let $\pi:\mathbb{H}^n\to\mathbb{R}$ be the projection of the first coordinate, then we have $W\cap\mathbb{H}^n=\pi^{-1}([0,\infty))$. Now $x\in W$ is a regular point of π iff x is a regular point of ∂f, so we can use the proposition we have proven before.

Theorem 2.1. Let X,Y manifolds with boundary, $W\subseteq Y$ submanifold, $\partial W=\partial Y=\emptyset$. Suppose $f\pitchfork W$ and $\partial f\pitchfork W$, then $f^{-1}(W)$ is a manifold with boundary, $\partial\left(f^{-1}(W)\right)=f^{-1}(W)\cap\partial X$.

Proof. $f|_{\operatorname{int}(X)}\pitchfork W$ is a manifold without boundary. Let $x\in f^{-1}(W)\cap\partial X$, $\pi:V\subseteq Y\to\mathbb{R}^{m-k}$ be a submersion such that $\pi^{-1}(0)=W\cap V$. As in the case without boundary: $f\pitchfork W$ at x iff x is a regular point of $\pi\circ f$, $\partial f\pitchfork W$ at x iff x is a regular point of $\pi\circ\partial f$. The result follows from the proposition above. $\square$

$\boxed{\text{Obs.}}$ It's easy to see that $\partial f\pitchfork W$ at $x\implies f\pitchfork W$ at x.

Theorem 2.2. (Sard's Theorem.) Let X manifold with boundary, Y manifold, $f:X\to Y$. Then

$$\{y\in Y\mid y\text{ is a critical value of }f\text{ or }\partial f\}$$

has measure zero.

Proof. $\mathrm{Crit}(f) \cup \mathrm{Crit}(\partial f) = \mathrm{Crit}\left(f|_{\mathrm{int}(X)}\right) \cup \mathrm{Crit}(\partial f)$. □

Theorem 2.3. (Thom Transversality Theorem.) X manifold with boundary, Y manifold, $W \subseteq J^k(X,Y)$ submanifold, $\partial W \subseteq \alpha^{-1}(\partial X)$. Then

$$\left\{f \in C^\infty(X,Y) \mid j^k f \pitchfork W \text{ and } j^k(\partial f) \pitchfork W\right\}$$

is residual.

Corollary 2.1. (Elementary Transversality Theorem.)

(1) X manifold with boundary, Y manifold and $W \subseteq Y$ submanifold $\partial W = \emptyset$. Then

$$\{f \in C^\infty(X,Y) \mid f \pitchfork W \text{ and } \partial f \pitchfork W\}$$

is residual.

(2) $f \in C^\infty(X,Y)$, $\partial f \pitchfork W$. There exists $\{g_n\} \subseteq C^\infty(X,Y)$ such that $g_n \xrightarrow{C^\infty} f$, $g_n \pitchfork W$ and $g_n \equiv f$ in a neighborhood of ∂X.

Definition 2.7. Let V be a vector space. Define an equivalence relation on the set of bases of V as follows:

$$\{x_1, \ldots, x_n\} \sim \{y_1, \ldots, y_n\} \text{ if the linear map } T : V \to V, Tx_i = y_i \text{ has } \det T > 0$$

$\boxed{\text{Obs.}}$ Given V, there are two equivalence classes.

Definition 2.8. An orientation of V is a choice of such an equivalence class.

Definition 2.9. Let X be a smooth manifold. An orientation on X is a choice of orientation on T_pX for each $p \in X$ such that for each chart $\varphi : U \to \mathbb{R}^n, \varphi = (x_1, \ldots, x_n)$, either

$$\left\{\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p\right\} \text{ or } \left\{-\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p\right\} \text{ is oriented for } \forall p \in U$$

$\boxed{\text{Obs.}}$ Not all manifold admits an orientation.

$\boxed{\text{Rmk.}}$ A connected orientable manifold has exactly two orientations. $\mathbb{R}^n$ has a natural orientation.

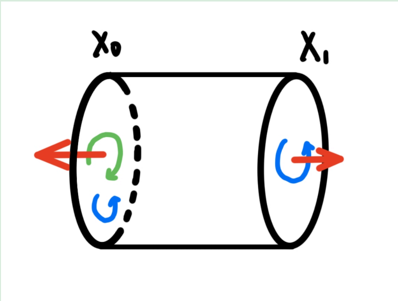
Proposition 2.3. Let X be an oriented manifold with boundary. Then ∂X has a natural orientation.

Proof. For $x \in \partial X$, $T_x(\partial X) \subseteq T_x X$. There exists a 1-dimensional vector bundle N over ∂X such that $N_x \oplus T_x(\partial X) = T_x X$ and a outward normal vector field $n \in \Gamma(N)$ which doesn't vanishes. If $\{v_1, \ldots, v_{n-1}\}$ is a basis of $T_x(\partial X)$, then $\{n_x, v_1, \ldots, v_{n-1}\}$ is a basis of $T_x X$. $\square$

Remark 2.1. X, Y oriented manifolds, $\partial Y = \emptyset$, then $X \times Y$ inherits a natural orientation induced from oriented basis of X matches with oriented basis of Y. Let $X = \{p\}$, then $X \times Y \cong Y$, $(p, y) \mapsto y$. If $\operatorname{sign}(p) = +$, then $\{p\} \times Y \to Y$ preserves orientation; If $\operatorname{sign}(p) = -$, then $\{p\} \times Y \to Y$ reverses orientation.

Example 2.2. Let X oriented manifold without boundary, $I = [0, 1]$, the $I \times X$ is oriented manifold with boundary.

$\partial(I \times X) = (\partial I) \times X = \{0\} \times X \cup \{1\} \times X$. Let $X_1 = \{1\} \times X$, $X_0 = \{0\} \times X$, they have induced orientation since they are diffeomorphic to X, but this orientation may not compatible with the induced boundary orientation, as the remark above says. So $\partial(I \times X) = X_1 \cup (-X_0)$.



Proposition 2.4. Let X^n, Y^m manifolds with boundary, $W^k \subseteq Y$ submanifold such that $\partial W = \partial Y = \emptyset$, let $f \in C^\infty(X, Y)$ such that $f \pitchfork W$ and $\partial f \pitchfork W$. Suppose X, Y, W oriented. Then $f^{-1}(W)$ has natural orientation.

Proof. Let $Q = f^{-1}(W)$, NQ be the normal bundle of Q (for every $x \in Q$, $N_x Q \oplus T_x Q = T_x X$). $df_x(T_x Q) = T_{f(x)} W$.

$\boxed{\text{Claim.}}$ $df_x|_{N_x Q}$ is injective.

$f \pitchfork W$, so $df_x(T_x X) + T_{f(x)} W = T_{f(x)} Y$, then $df_x(N_x Q) + T_{f(x)} W = T_{f(x)} Y$, then $\dim df_x(N_x Q) = m - k = \dim N_x Q$, so df_x is injective.

Since $T_{f(x)} W, T_{f(x)} Y$ are oriented, it induces an orientation on $df_x(N_x Q)$ by $df_x(N_x Q) \oplus T_{f(x)} W = T_{f(x)} Y$, hence induces orientation on $N_x Q$. By $N_x Q \oplus T_x Q = T_x X$ we have an orientation on $T_x Q$.

In other words, $\{v_1, \cdots, v_{m-k}\}$ of $N_x Q$ is positive if $\{df_x v_1, \cdots, df_x v_{m-k}, w_1, \cdots, w_k\}$ is oriented basis of $T_{f(x)} Y$, where $\{w_1, \cdots, w_k\}$ is oriented basis of $T_{f(x)} W$. Then we can use $N_x Q \oplus T_x Q = T_x X$ to induce an orientation on $T_x Q$. $\square$

Corollary 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^m$ regular value of f. Then $f^{-1}(a)$ is orientable.

Exercise 2.2. $f \pitchfork W$ and $\partial f \pitchfork W$. $\partial f^{-1}(W) = (\partial f)^{-1}(W)$. Let X, Y, W are oriented, then for natural orientation, $[\partial f^{-1}(W)] = (-1)^{\text{codim} W}[(\partial f)^{-1}W]$. (The orientation not necessarily matches)

2.2 Intersection Number

Theorem 2.4. (Classification of 1-Manifolds.)

Let X compact and connected 1-manifold, then X is diffeomorphic to either $[0, 1]$ or S^1.

Let X, Y, W be oriented manifolds without boundary such that X is compact and $W \subseteq Y$ closed subset and $\dim X + \dim W = \dim Y$. Let $f : X \to Y$, $f \pitchfork W$, then $\dim f^{-1}(W) = 0$, so $f^{-1}(W)$ is a set of isolated points. By compactness and orientation assumption, $f^{-1}(W)$ is a finite number of points with signs. Define intersection number $I(f, W) = \sum\limits_{p \in f^{-1}(W)} \text{sign}(p)$.

Definition 2.10. Let X, Y, W be oriented manifolds without boundary such that X is compact and $W \subseteq Y$ closed subset and $\dim X + \dim W = \dim Y$. Let $f : X \to Y$, $f \pitchfork W$. Define intersection number $I(f, W) = \sum\limits_{p \in f^{-1}(W)} \text{sign}(p)$.

$\boxed{\text{Recall.}}$ $df_p(T_p X) \oplus T_{f(p)}W = T_{f(p)}Y$, $\text{sign}(p) = +1$ iff orientation matches.

Example 2.3.

For now we always assume that X, Y, W be oriented manifolds without boundary such that X is compact and $W \subseteq Y$ closed subset and $\dim X + \dim W = \dim Y$.

Proposition 2.5. Let $f_0, f_1 \in C^\infty(X, Y)$ smoothly homotopic and transverse to W, then $I(f_0, W) = I(f_1, W)$.

Proof. Let $Z = [0, 1] \times X$, $F : [0, 1] \times X \to Y$ the smooth homotopy of f_0, f_1. Since $[\partial ([0, 1] \times X)] = [\{1\} \times X] \cup [-\{0\} \times X]$, by the lemma below we have $0 = I(\partial F, W) = I(f_1, W) - I(f_0, W)$. $\square$

Remark 2.2. Two smooth maps are smoothly homotopic iff they are homotopic.

Lemma 2.2. Suppose $X = \partial Z$, where Z compact oriented manifold with boundary, $f : X \to Y$, $f \pitchfork W$. Suppose that f can be extend to $F : Z \to Y$. Then $I(f, W) = 0$.

Proof. Since $f = F|_{\partial Z}$ and $f \pitchfork W$, $F \pitchfork W$ on ∂Z. We can perturb F so that $F \pitchfork W$ in all Z and $F|_{\partial Z} = f$. $F^{-1}(W)$ is an oriented manifold such that $\partial F^{-1}(W) = \pm f^{-1}(W)$. Since $\dim F^{-1}(W) = 1$, $F^{-1}(W)$ is a compact 1-manifold. So it's a disjoint union of copies of $[0, 1]$ and S^1, $\partial F^{-1}(W)$ is an even number of points and number of positive sign is the same as the negative. □

If $f \in C^\infty(X, Y)$ not necessarily transverse to W, we can take $g \simeq f$ such that $g \pitchfork W$ and define $I(f, W) = I(g, W)$, by the proposition above it's well-defined.

Definition 2.11. (Degree.) Let X, Y oriented manifolds without boundary and X compact, Y connected, $\dim X = \dim Y$. If $f \in C^\infty(X, Y)$, define $\deg(f) = I(f, \{y\})$, $y \in Y$.

Proposition 2.6. $I(f, \{y\})$ doesn't depend on y.

Proof. We may assume WLOG that $f \pitchfork \{y\}$, so y is a regular value of f. $f^{-1}(y) = \{x_1, \ldots, x_k\}$ is a finite set of points. Let $U_1, \ldots, U_k$ be small disjoint neighborhood of these points. Since x_i is a regular point and $\dim X = \dim Y$, we may assume $f|_{U_i}$ is a diffeomorphism. Then it's easy to see that $I(f, \{y\})$ is locally constant if y varies, so it's independent of y. □

Proposition 2.7. Let X, Y oriented manifolds without boundary and X compact, Y connected, $\dim X = \dim Y$. If $f \simeq g : X \to Y$, then $\deg f = \deg g$.

Definition 2.12. (Winding Number.) Let X compact, oriented manifold without boundary, $\dim X = n$ (often $X = S^n$). Let $f : X \to \mathbb{R}^{n+1}$ and $z \in \mathbb{R}^{n+1}$ such that $z \notin f(X)$. Let $u : X \to S^n, x \mapsto \dfrac{f(x) - z}{||f(x) - z||}$. Define $\operatorname{wind}(f; z) \coloneqq \deg u = I(u, \{a\})$.

Proposition 2.8. Let $\{f_t : X \to \mathbb{R}^{n+1} \setminus \{z\}\}_{t \in [o,1]}$ homotopy of maps. Then $\operatorname{wind}(f_0; z) = \operatorname{wind}(f_1; z)$.

Proof. For f_t, we have u_t as the definition shows, so $u_0 \sim u_1$. Then $\operatorname{wind}(f_0; z) = \deg(u_0) = \deg(u_1) = \operatorname{wind}(f_1; z)$. □

Example 2.4. If $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree n, then $\operatorname{wind}(p_{\partial B(R)}; 0) = n$ for $R \gg 0$ since $\frac{p}{|p|} \sim z^n$ on $\partial B(R)$.

Proposition 2.9. Suppose $X = \partial D$, D compact and oriented, f extends to a map $F : D \to \mathbb{R}^{n+1}$. Then $\operatorname{wind}(f; z) = I(F, \{z\})$.

Remark 2.3. Let X compact oriented manifolds with or without boundary, Y, W be oriented manifolds without boundary such that $W \subseteq Y$ closed subset and $\dim X + \dim W = \dim Y$. Let $f : X \to Y$, $f \pitchfork W$, $f^{-1}(W) \cap \partial X = \emptyset$. Then we can define the intersection number for X with boundary. If $\{f_t\}$ is homotopy of maps s.t, $f_t^{-1}(W) \cap \partial X = \emptyset$, then $I(f_0, W) = I(f_1, W)$. In particular, if $f(\partial X) \cap W = \emptyset$, then we can perturb f to find g s.t. $g \pitchfork W$, $f \sim g$ by maps not intersecting W, then define $I(f, W) = I(g, W)$.

Proof. Since $f(X) \cap \{z\} = \emptyset$, then in particular $f \pitchfork \{z\}$. We may assume WLOG, that $F \pitchfork \{z\}$. Let u be the corresponding map of f in the definition of winding number. If $F^{-1}(z) = \emptyset$, then $u : X \to S^n$ can be extended to a map $U : D \to S^n$, so $\deg u = 0$.

Suppose $F^{-1}(z) = \{x_1, \ldots, x_k\}$, by implicit function theorem, F is a local diffeomorphism near each x_i. We can find disjoint neighborhoods U_i near x_i, V of z such that $F|_{U_i} : U_i \to V$ is a diffeomorphism. We can choose orientation preserving charts $\varphi_i : U_i \to \mathbb{R}^{n+1}$, $\psi : V \to \mathbb{R}^{n+1}$ such that $\psi \circ F \circ \varphi_i^{-1} = id$ or reflection in the first coordinate. Let $\epsilon > 0$ small enough such that $B^{n+1}(\epsilon) \subseteq \varphi_i(U_i)$ for all i, let $Z = D \backslash \bigcup_i \varphi_i^{-1}\left(B^{n+1}(\epsilon)\right)$, then $z \notin F(Z)$, we have the map U. Also $[\partial Z] = [\partial D] \cup \bigcup_i \left[-\varphi_i^{-1}\left(\partial B^{n+1}(\epsilon)\right)\right]$ (draw a picture to see this), since $U|_{\partial Z}$ can be extended, we have

$$0 = \deg\left(U|_{\partial Z}\right) = I\left(U|_{\partial Z}, \{a\}\right) = I\left(U|_{\partial D}, \{a\}\right) - \sum_i I\left(U|_{\varphi_i^{-1}(\partial B^{n+1}(\epsilon))}, \{a\}\right)$$

So $\operatorname{wind}(f; z) = I\left(U|_{\partial D}, \{a\}\right) = \sum_i \operatorname{wind}(f_i; z)$, where $f_i = F|_{\varphi_i^{-1}(\partial B^{n+1}(\epsilon))}$. Now we need to compute $\operatorname{wind}(f_i; z)$.

Since φ_i, ψ are orientation preserving, $\operatorname{wind}(f_i; z) = \operatorname{wind}(\psi \circ f_i \circ \varphi_i^{-1}; 0) = \sigma(i)$, where $\sigma(i) = 1$ if $\psi \circ f_i \circ \varphi_i^{-1} = id$; $\sigma(i) = -1$ if $\psi \circ f_i \circ \varphi_i^{-1}$ is a reflection. Since $F^{-1}(z) = x_i$ in $\varphi_i^{-1}\left(B^{n+1}(\epsilon)\right)$, $\operatorname{sign}(x_i) = \sigma(i)$, so $\sum_i \operatorname{wind}(f_i; z) = \sum_i \sigma(i) = I(F, \{z\})$. □

Theorem 2.5. (Borsuk-Ulam.) Let $f : S^n \to \mathbb{R}^n$ smooth. Then $\exists x \in S^n$ such that $f(x) = f(-x)$.

Proof. Suppose not, then $g(x) = f(x) - f(-x)$ is an odd function that doesn't vanish, let $\widetilde{g} : S^n \to \mathbb{R}^{n+1}, \widetilde{g} = g \times 0$, then $\widetilde{g}$ is odd and $\widetilde{g}(x) \neq 0$.

$\boxed{\text{Claim.}}$ wind$(\widetilde{g}, 0)$ is odd (odd map has odd degree).

Assuming the claim, wind$(\widetilde{g}, 0) = \deg\left(\frac{\widetilde{g}}{||\widetilde{g}||}\right)$ is odd, but $(0, \ldots, 0, 1) \notin \frac{\widetilde{g}}{||\widetilde{g}||}(S^n)$, so $\deg\left(\frac{\widetilde{g}}{||\widetilde{g}||}\right) = 0$, contradiction.

Now we prove the claim by induction.

- $n = 1 : \widetilde{g} : S^1 \to \mathbb{R}^2 \setminus \{0\}$, $\widetilde{g}$ is odd, then $h = \frac{\widetilde{g}}{||\widetilde{g}||} : S^1 \to S^1$ is odd. Then h can be lifted to a map $\overline{h} : \mathbb{R} \to \mathbb{R}$ s.t. $\overline{h}(t + 2\pi) = \overline{h}(t) + 2\pi k$, for some $k \in \mathbb{Z}$. Since h is odd, then $\overline{h}(t + \pi) = \overline{h}(t) + \pi k$, where k is odd. Moreover $\deg h = k$, so wind$(\widetilde{g}, 0)$ is odd.
- Now suppose that every odd map $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ has odd winding number w.r.t. 0. Let $g : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ odd, let $S^{n-1} \subseteq S^n$ be the equator and let $h = g|_{S^{n-1}}$. Fix $a \in S^n$ be a regular value of $\frac{g}{||g||}$ and $\frac{h}{||h||}$, then $-a$ is also a regular value of both of them. Hence g and h intersect the line $\ell = \mathbb{R} \cdot a$ transversely. Moreover, $h(S^{n-1}) \cap \ell = \emptyset$. Now wind$(g, 0) = \deg\left(\frac{g}{||g||}\right) = \#\left(\frac{g}{||g||}\right)^{-1}(a) = \frac{1}{2}\#g^{-1}(\ell) = I(g_+, \ell)$ where $g_+ = g|_D$, D is the upper semisphere. Let $pr : \mathbb{R}^{n+1} \to \ell^\perp$, 0 is a regular value of $pr \circ g_+$ because $g_+ \pitchfork \ell$. And $(pr \circ g_+)^{-1}(0) = g_+^{-1}(\ell)$. Moreover, $pr \circ g_+|_{\partial D}$ is an odd function. Therefore

$$\text{wind}(g, 0) = I(g_+, \ell) = I(pr \circ g_+, 0) = \text{wind}(pr \circ g_+|_{\partial D}, 0)$$

  is odd by induction.

□

Theorem 2.6. (Jordan-Brouwer Separation Theorem.)

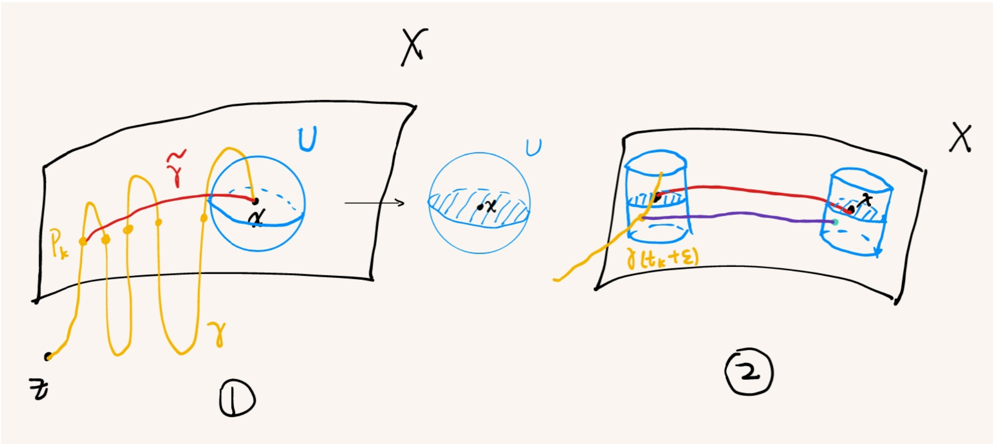
Let $X \subseteq \mathbb{R}^n$ be a compact connected hypersurface. Then $\mathbb{R}^n \backslash X = U_1 \sqcup U_2$, U_1, U_2 open, connected, $U_1 \cup X$ is a compact manifold with boundary X and $U_2 \cup X$ is a manifold with boundary X and U_2 is unbounded.

Remark 2.4. Set up mod 2 intersection theory.

X, Y, W manifolds, X compact, $W \subseteq Y$ closed, $\partial W = \partial Y = \emptyset$, $f : X \to Y$ such that $f(\partial X) \cap W = \emptyset$, $f \pitchfork W$, $\dim X + \dim W = \dim Y$, define $I_2(f, W) = \#f^{-1}(W) \mod 2$. Similarly we can define $\deg_2(f)$ and wind$_2(f, z)$.

Proof. $\boxed{\text{Step 1.}}$ $\mathbb{R}^n \backslash X$ has at most two connected components.

Fix $x \in X$ and $x \in U$ open connected set in $\mathbb{R}^n$. Take $z \in \mathbb{R}^n \backslash X$, let $\gamma : [0, 1] \to \mathbb{R}^n$ be a curve connecting x and z. WLOG, we can assume $\gamma \pitchfork X$, , $\gamma \cap X = \{x, p_1, \ldots, p_k\}$, where $\gamma(t_i) = p_i$ and $\gamma|_{[t_k+\epsilon,1]}$ doesn't intersect X. We can find a curve $\widetilde{\gamma} \subseteq X$ connecting x and p_k, a neighborhood of this path is diffeomorphic to $B^{n-1} \times (-1, 1)$, we can connect $\gamma(t_k + \epsilon)$ to $U \backslash X$ without crossing X inside $B^{n-1} \times (-1, 1)$.



$\boxed{\text{Step 2.}}$ If z_0 and z_1 can be connected in $\mathbb{R}^n \backslash X$, then $\text{wind}_2(X, z_0) = \text{wind}_2(X, z_1)$.

If $\gamma(t)$ is a path connecting z_0 and z_1 in $\mathbb{R}^n \backslash X$, then $u_t(x) = \frac{x-\gamma(t)}{||x-\gamma(t)||}$ is a homotopy between u_{z_0} and u_{z_1}.

$\boxed{\text{Step 3.}}$ If ℓ is a line segment connecting z_0 and z_1, where $z_0, z_1 \notin X$, $\ell \pitchfork X$ and $\ell \cap X = \{p\}$, then $\text{wind}_2(X, z_0) = \text{wind}_2(X, z_1) + 1$.

Let $v = z_1 - z_0 \in \mathbb{R}^n$. $\ell \pitchfork X$ implies that $u_{z_0} \pitchfork \left\{\frac{v}{||v||}\right\}$, $u_{z_1} \pitchfork \left\{\frac{v}{||v||}\right\}$. $u_{z_0}^{-1}\left(\frac{v}{||v||}\right) = u_{z_1}^{-1}\left(\frac{v}{||v||}\right) \bigsqcup \{p\}$, so $\deg_2(u_{z_0}) = \deg_2(u_{z_1}) + 1$, hence $\text{wind}_2(X, z_0) = \text{wind}_2(X, z_1) + 1$.

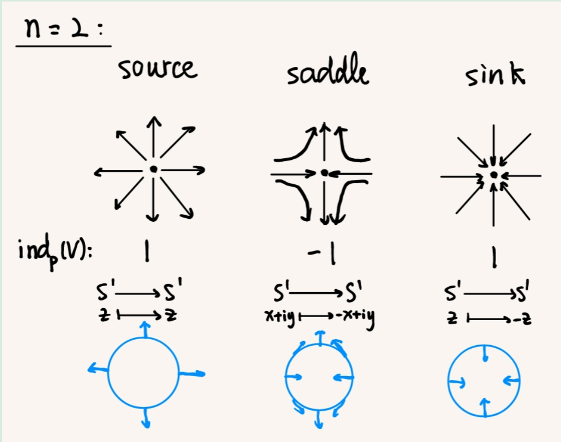
$\boxed{\text{Step 4.}}$ Let $U_1 = \{z \in \mathbb{R}^n \backslash X | \text{wind}_2(X, z) = 1\}$, $U_2 = \{z \in \mathbb{R}^n \backslash X | \text{wind}_2(X, z) = 0\}$. By previous steps, U_1 and U_2 are the connected components of $\mathbb{R}^n \backslash X$. It is simple to see that $\overline{U}_1 = U_1 \cup X$ and $\overline{U}_2 = U_2 \cup X$ are manifolds. Since X is compact, $X \subseteq B^n(R)$ for some R. If $z \notin B^n(R)$, $||z|| > R$, then for $x \in X$, $t \in [0, 1]$, $||tx - z|| \geq ||z|| - t||x|| > 0$, so u_z is homotopic to a constant map $\frac{z}{||z||}$, so $z \in U_2$, hence U_2 is unbounded. □

Definition 2.13. (Index.) Let X be oriented manifold without boundary and let $V \in \mathfrak{X}(X)$. Let $p \in X$ such that $V_p = 0$. Suppose there exists an open set U near p such that $V_x = 0$ for $x \in U$ iff $x = p$. Choose an oriented chart $\varphi : U \to \mathbb{R}^n$ such that $\varphi(p) = 0$, $\varphi(U) = B^n(\epsilon)$. It induces a map $\overline{\varphi} : TX|_U \to U \times \mathbb{R}^n$, $\overline{V} = pr_2 \circ \overline{\varphi} \circ V \circ \varphi^{-1} : B^n(\epsilon) \to \mathbb{R}^n$. Define $\text{ind}_p(V) = \text{wind}\left(\overline{V}_{\partial B^n(\epsilon)}, 0\right)$.

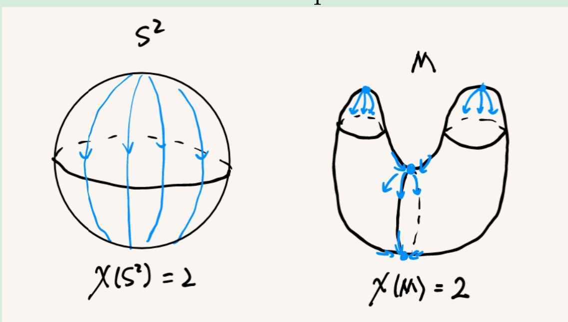
Theorem 2.7. (Poincaré-Hopf Theorem.) Suppose all zeros of V are isolated and X is compact. Then $\sum\limits_{V_p=0} \text{ind}_p(V) = \chi(X)$.

Remark 2.5. If X oriented compact manifold, $\Delta = \{(x, x) \in X \times X\}$ is a submanifold of $X \times X$, then we can talk about $I(\Delta, \Delta) \in \mathbb{Z}$ (after perturb Δ it can intersect itself transversely). We define Euler characteristic $\chi(X) = I(\Delta, \Delta)$.

Example 2.5. We look at some examples in $n = 2$ cases:



So by Poincaré-Hopf Theorem and the $n = 2$ cases we can directly "see" the Euler characteristic of a surface. For example:



Exercise 2.3. Find $V \in \mathfrak{X}(S^2)$ with a unique zero. (Then $\mathrm{ind}_p(V)$ must be 2)

Hint. What do think about when talking about a map $S^1 \to S^1$ with degree 2? Of course $z \mapsto z^2$! So near the zero the flow of this vector field should look like $\cdots$

Definition 2.14. Let $V \in \mathfrak{X}(X)$ and $p \in X$ isolated zero. p is said to be simple if 0 is a regular value of $\overline{V}$, where $\overline{V} : B^n(\epsilon) \to \mathbb{R}^n$ is a local representation of V (i.e. $d\overline{V}_0$ is an isomorphism).

Exercise 2.4. If p is simple, then in a local representation, $\overline{V} = id$ or $\overline{V}$ =reflection at the first coordinate.

Remark 2.6. If p is an isolated zero of V, then we can perturb V, there exists $\widetilde{V} \in \mathfrak{X}(X)$ which coincides with V outside a small neighborhood $p \in U$ such that all the zeros of $\widetilde{V}$ in U are simple. Moreover, if $\{p_1, \cdots, p_n\} \subseteq U$ are zeros of $\widetilde{V}$, then $\mathrm{ind}_p(V) = \sum_i \mathrm{ind}_{p_i}(\widetilde{V})$. The proof is similar to proposition 2.9.

Now we are going to prove the Poincaré-Hopf Theorem.

Proof. We may assume WLOG that all zeros of V are simple. Let $f_t : X \to X$ be the flow of V. Note that if ϵ is sufficiently small, then $\text{Fix}(f_t) = \text{Zero}(V)$ for $\forall 0 < t < \epsilon$. Let $\Delta = j^0 id(X) \subseteq X \times X$, $Gr(f_t) = j^0 f_t(X) = \{(x, f_t(x)) \mid x \in X\}$.

$\boxed{\text{Claim.}}$ $\Delta \pitchfork Gr(f_t)$ and for each $x \in \text{Zero}(V)$, $\text{ind}_x(V)$ is the sign of the intersection $(x,x) \in \Delta \cap Gr(f_t) \leftrightarrow \text{Fix}(f_t)$.

If the claim is true, then $\sum\limits_{V_x=0} \text{ind}_x V = I(\Delta, Gr(f_t)) = I(\Delta, Gr(f_0)) = \chi(X)$.

Proof of the claim: We can look in oriented charts: $V : B^n(\epsilon) \to \mathbb{R}^n$ such that 0 is a regular value and $V^{-1}(0) = \{0\}$. We may assume WLOG that $V = id$ or reflection, let $q = \pm 1$, then we write $V(x_1, \ldots, x_n) = (qx_1, \ldots, x_n)$. The flow of V is: $f_t(x_1, \ldots, x_n) = \left(e^{qt}x_1, \ldots, e^t x_n\right)$. Then locally we have $\Delta \cap Gr(f_t) = \{(0,0)\}$, now we need to show $T_{(0,0)}\Delta + T_{(0,0)}Gr(f_t) = T_{(0,0)}\left(\mathbb{R}^n \times \mathbb{R}^n\right)$. Let $\{e_1, \cdots, e_n\}$ be canonical basis of $\mathbb{R}^n$, it induces orientation on $\mathbb{R}^n \times \mathbb{R}^n$ given by $\{e_1 \times 0, \cdots, e_n \times 0, 0 \times e_1, \cdots, 0 \times e_n\}$. Also we have induced orientation on $\Delta = j^0 id(X)$ and $Gr(f_t) = j^0 f_t(X)$:

$$\Delta : \{e_1 \times e_1, \cdots, e_n \times e_n\}\,;\; Gr(f_t) : \left\{e_1 \times e^{qt}e_1, \cdots, e_n \times e^t e_n\right\}$$

Putting these linearly independent sets together, we obtain a matrix:

$$\left[\begin{array}{c|ccc} \text{Id} & & \text{Id} & \\ \hline & e^{qt} & & \\ \text{Id} & & \ddots & \\ & & & e^t \end{array}\right]$$

where $\det = (e^{qt} - 1)(e^t - 1)^{n-1} \neq 0$. Note that the sign of the determination is the same as q. This completes the proof of the claim.

□

Theorem 2.8. (Lefschetz Fixed Point Theorem.)
Let X be an oriented compact manifold without boundary, $f : X \to X$ smooth. Define lefschetez index $\mathcal{L}(f) = I(\Delta, Gr(f))$. If $\mathcal{L}(f) \neq 0$, then f has a fixed point.

Proof. If f doesn't have a fixed point, then $\Delta \cap Gr(f) = \emptyset$. So $\mathcal{L}(f) = 0$. □

If $\Delta \pitchfork Gr(f)$, then for each $(x,x) \in \Delta \cap Gr(f)$, define $\mathcal{L}_x(f) =$ sign of the intersection (x,x).

Exercise 2.5. $\mathcal{L}_x(f) =$ sign of $\det\left(df_x - I\right)$.

Theorem 2.9. (Hopf.) $f, g : S^n \to S^n$, then $f \simeq g \Leftrightarrow \deg f = \deg g$. Moreover, if X^n is compact oriented connected manifold without boundary, $f, g : X \to S^n$, then $f \simeq g \Leftrightarrow \deg f = \deg g$.

Proof. Let $y \in S^n$ be a regular value of f and g. $Q_0 = f^{-1}(y)$, $Q_1 = g^{-1}(y)$ are 0-dim oriented submanifold. $\deg f = \# Q_0$, $\deg g = \# Q_1$.

$\boxed{\text{Step 1.}}$ Construct a homotopy between $\widetilde{f}, \widetilde{g}$ s.t. $\widetilde{f}^{-1}(y) = Q_0$, $\widetilde{g}^{-1}(y) = Q_1$.

Let C be an oriented 1-manifold in $[0,1] \times X$ s.t. $\partial C = \{1\} \times Q_1 - \{0\} \times Q_0$. Now we define an embedding $\varphi : C \times \mathbb{R}^n \to [0,1] \times X$ s.t. $\varphi(x, 0) = x$ and $\frac{\partial \varphi}{\partial v}(x, 0)$ specified.

$\boxed{\text{Step 2.}}$ $f \simeq \widetilde{f}, g \simeq \widetilde{g}$. □

2.3 Pontryagin-Thom Construction

Let X^n oriented compact manifold.

$\boxed{\text{Goal.}}$ Classify maps $X \to S^m$, $n \geq m$, up to homotopy.

If $n = m$ then the degree will do it, we now want to generalize it, which is called framed cobordism.

Definition 2.15. Let $Q^k \subseteq X$ submanifold without boundary. A framing is a choice of trivialization of the normal bundle $NQ \xrightarrow{\cong} Q \times \mathbb{R}^{n-k}$.

An equivalent definition: Let $V(NQ) = \coprod_{x \in Q} V(N_x Q)$, where $V(N_x Q) = \{(v_1, \cdots, v_{n-k}) \mid \{v_1, \cdots, v_{n-k}\}$ is a basis of $N_x Q\}$. We can check that $V(NQ) \to Q$ is a fiber bundle. A framing is a section $Q \to V(NQ)$. If Q is oriented, we usually assume that the framing is position w.r.t. the orientation.

Example 2.6. Every knot in S^3 admits a framing.

Take $f : X \to S^m$, a is a regular value. Then $f^{-1}(a)$ admits a natural framing from f: $df_x : N_x Q \to T_a S^m$ is an isomorphism, fix an oriented basis of $T_a S^m$, the preimage is a basis of $N_x Q$.

Remark 2.7. If $n = m$, Q is a collection of points, a framing modulo isotopy is the same as an orientation.

Definition 2.16. Let $Q_0, Q_1 \subseteq X$ be oriented compact submanifolds without boundary. A cobordism from Q_0 to Q_1 is an oriented compact manifold $W \subseteq [0,1] \times X$ s.t. $W \cap ([0, \epsilon) \times X \cup (1-\epsilon, 1] \times X) = [0, \epsilon) \times Q_0 \cup (1-\epsilon, 1] \times Q_1$ and $\partial W \subseteq \partial([0,1] \times X)$. In particular, $\partial W = \{1\} \times Q_1 \cup (-\{0\} \times Q_0)$ and also $W \pitchfork \partial([0,1] \times X)$.

If g is a Riemannian metric on X, then we let $\widetilde{g} = dt \otimes dt + g$. Clearly $\frac{\partial}{\partial t}$ is orthogonal to $\{t\} \times X$. Let W be a cobordism from Q_0 to Q_1, we have $\varphi : \partial W \to Q_1 \sqcup (-Q_0)$ an orientation preserving diffeomorphism then $NW|_{\{0\} \times Q_0} \cong \varphi^* NQ_0$, $NW|_{\{1\} \times Q_1} \cong \varphi^* NQ_1$