BASICS OF DIFFERENTIAL GEOMETRY 2

Notes of BIMSA course

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Contents

Introduction.

Last semester:

- Geometry of vector bundles
- Basic Riemannian geometry
- Differential operators on manifolds

We will learn this semester:

- Theory of principle bundles
- characteristic classes
- Basics of complex manifold, Kähler manifold, symplectic manifold.

1 Principle Bundles

In this section, we introduce the connections of principle bundles, it's closely related to the connections of vector bundles.

1.1 Lie Groups

Definition 1.1. Let G be a smooth manifold. G is a Lie group if G is a group s.t. multiplication and inverse are smooth.

Let G be a Lie group, $g \in G$, we denote:

- $L_g : G \to G$, $h \mapsto gh$ (left translation)
- $R_g : G \to G$, $h \mapsto hg$ (right translation)
- $\mathfrak{X}^L(G) = \{X \in \mathfrak{X}(G) \mid \forall g \in G, (L_g)_* X = X\}$ (left invariant vector fields)

For $X \in \mathfrak{X}^L(G)$, $L_{g*}X = X$ means that X is L_g-related to X. Then for $\forall X, Y \in \mathfrak{X}^L(G)$, $L_{g*}([X,Y]) = [L_{g*}X, L_{g*}Y] = [X,Y]$, so $\mathfrak{X}^L(G)$ is closed under $[\cdot,\cdot]$

Definition 1.2. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a $\mathbb{K}$-vector space $\mathfrak{g}$ and a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we say $\mathfrak{g}$ is a Lie algebra if:

(1) $\forall X, Y \in \mathfrak{g}$, $[X,Y] = -[Y,X]$

(2) $\forall X, Y, Z \in \mathfrak{g}$, $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

$[\cdot,\cdot]$ is called Lie bracket.

So by definition we have $\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)$ is a Lie algebra.

Definition 1.3. For Lie algebra $\mathfrak{g}, \mathfrak{h}$, a linear map $f : \mathfrak{g} \to \mathfrak{h}$ is called the Lie algebra homomorphism if: $\forall X, Y \in \mathfrak{g}$, $f([X,Y]) = [f(X), f(Y)]$

If f is in addition an isomorphism, then f is called a Lie algebra isomorphism.

Let $e \in G$ be the unit of G. Set $\iota : \mathfrak{X}^L(G) \to T_eG$, $X \mapsto X_e$. Then ι is a linear isomorphism. Let $\mathfrak{g} = T_eG$, so we can define the Lie bracket on $\mathfrak{g}$ s.t. ι is a Lie algebra isomorphism, i.e. setting $X^\sharp = \iota^{-1}(X)$, $[X,Y] = \left[X^\sharp, Y^\sharp\right]_e$. Note that $X^\sharp_g = (L_g)_{*e}X$, $g \in G$.

Definition 1.4. Let G be Lie group, $\mathfrak{g} = T_eG$ with $[\cdot,\cdot]$ is called the Lie algebra of G. $\left(\left(\mathfrak{X}^L(G), [\cdot,\cdot]\right)\right.$ is also called the Lie algebra of $\left.G\right)$

Definition 1.5. Let G, H be Lie groups. A map $\rho : G \to H$ is a Lie group homomorphism if ρ is a smooth map and a group homomorphism. For the special case $(\mathbb{R},+) \to G$, $t \mapsto g_t$, $\{g_t\}_{t\in\mathbb{R}}$ is called one parameter subgroup of G.

Proposition 1.1. Let G be Lie group and $\mathfrak{g}$ its Lie algebra. Then

(1) $\forall X \in \mathfrak{g}$, $X^\sharp = \iota^{-1}(X)$ is complete, i.e. $X^\sharp$ generates a flow $\{\varphi_t\}_{t\in\mathbb{R}}$.

(2) Set $\boxed{\exp_G(tX) = \varphi_t(e)} \in G$. Then $\varphi_t = R_{\exp_G(tX)}$.

(3) For $s,t \in \mathbb{R}$, $\exp_G(sX)\exp_G(tX) = \exp_G((s+t)X)$, i.e. $\{\exp_G(tX)\}_{t\in\mathbb{R}}$ is one parameter subgroup of G.

(4) $\mathfrak{g} \to \{\text{one parameter subgroup of } G\}$, $X \mapsto \{\exp_G(tX)\}_{t\in\mathbb{R}}$ is bijective.

Proof. (1) By ODE theory, $\exists \epsilon > 0$, $\gamma_e : (-\epsilon,\epsilon) \to G$ s.t. $\gamma_e(0) = e$, $\dfrac{d\gamma_e}{dt} = X^\sharp_{\gamma_e(t)}$.

$\boxed{\text{Claim 1.}}$ $\forall g \in G$, define $\gamma_g : (-\epsilon,\epsilon) \to G$, $t \mapsto g\gamma_e(t)$ is the integral curve of $X^\sharp$ with $\gamma_g(0) = g$.

Indeed, $\forall t \in (-\epsilon,\epsilon)$, $\dfrac{d\gamma_g}{dt}(t) = (L_g)_{*\gamma_e(t)}\dfrac{d\gamma_e}{dt}(t) = X^\sharp_{g\cdot\gamma_e(t)}$.

$\boxed{\text{Claim 2.}}$ $\gamma_e : (-\epsilon,\epsilon) \to G$ can be extended to integral curve $\gamma_e : \mathbb{R} \to G$ of $X^\sharp$ with $\gamma_e(0) = e$.

Set $\varphi_t = R_{\gamma_e(t)}$, then $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow generated by $X^\sharp$. So the following are easy. $\square$

By this proposition, we can define the exponential map $\exp_G : \mathfrak{g} \to G$.

Proposition 1.2. Let G, H be Lie groups with Lie algebra $\mathfrak{g}, \mathfrak{h}$. If $f : G \to H$ is Lie group homomorphism, then $f_{*e} : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. We only need to show that $X^\sharp$ and $(f_{*e}X)^\sharp$ are f-related. Since $X = \dfrac{d}{dt}\exp_G(tX)|_{t=0}$, we have $f_{*g}(X^\sharp_g) = \dfrac{d}{dt} f\left(g \cdot \exp_G(tX)\right)|_{t=0} = \dfrac{d}{dt} f(g) f\left(\exp_G(tX)\right)|_{t=0} = \left(L_{f(g)}\right)_{*e}(f_{*e}X) = (f_{*e}X)^\sharp_{f(g)}$. $\square$

Example 1.1. Let V be a $\mathbb{R}$-vector space, $G = GL(V)$, $\mathfrak{g}$ Lie algebra of G. Then $\mathfrak{g} = End(V)$, the bracket is given as follows:

Proposition 1.3. $\forall X, Y \in End(V)$, $[X,Y] = XY - YX$.

Proof. For $X \in End(V)$, set matrix exponential $e^{tX} = \sum\limits_{k=0}^{\infty} \frac{(tX)^k}{k!}$. Then $\{e_{tX}\}_{t\in\mathbb{R}}$ is a one parameter subgroup of G and $\frac{d}{dt}e^{tX}|_{t=0} = X$. So $\exp_G(tX) = e^{tX}$.

Then $[X,Y] = [X^\sharp, Y^\sharp]_e = \left(\mathcal{L}_{X^\sharp}Y^\sharp\right)_e = \dfrac{d}{dt}(\varphi_{-t})_{*e^{tX}}\left(Y^\sharp_{e^{tX}}\right)|_{t=0} = \dfrac{d}{dt}\dfrac{d}{ds}\varphi_{-t}\left(e^{tX}e^{sY}\right)|_{s=t=0} = XY - YX$. $\square$

Example 1.2. Set

- $O(n) = \{g \in GL(n;\mathbb{R}) \mid g^t g = E_n\}$ (orthogonal group)
- $SO(n) = \{g \in O(n) \mid \det g = 1\}$ (special orthogonal group)

we can check that $O(n), SO(n)$ are Lie subgroups of $GL(n;\mathbb{R})$.

$SO(n)$ is the unit component of $O(n)$, so $\mathfrak{o}(n) = \mathfrak{so}(n)$ (Lie algebra of $(O(n))$ and $SO(n)$). This is a Lie subalgebra of $End(\mathbb{R}^n)$ given by

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \left\{X \in End(\mathbb{R}^n) \mid X^t + X = O_n\right\}$$

where O_n is the zero matrix of size n.

Similarly, set

- $U(n) = \{g \in GL(n;\mathbb{C}) \mid g^* g = E_n\}$ (unitary group) where $g^* = \overline{g^t}$
- $SU(n) = \{g \in U(n) \mid \det g = 1\}$ (special unitary group)

We can check that

- $U(n), SU(n)$ are Lie subgroups of $GL(n;\mathbb{C})$
- $\mathfrak{u}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O\}$ (Lie algebra of $U(n)$)
- $\mathfrak{su}(n) = \{X \in End(\mathbb{C}^n) \mid X^* + X = O, \mathrm{tr} X = 0\}$ (Lie algebra of $(SU(n))$)

$\boxed{\text{Note.}}$ A Lie subgroup H of G is a Lie group s.t.

- H is a subset of G
- inclusion map $H \hookrightarrow G$ is an embedding and group homomorphism

$\boxed{\text{Fact.}}$ A closed subgroup of G is a Lie subgroup of G.

Definition 1.6. Let V be a $\mathbb{K}$-vector space, G a Lie group. A Lie group homomorphism $\rho : G \to GL(V)$ is called a representation of V. The Lie algebra homomorphism $\rho_{*e} : \mathfrak{g} \to End(V)$ is called a differential representation.

Example 1.3. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. $\forall g \in G$, define a homomorphism

$$F_g : G \to G,\ h \mapsto ghg^{-1}$$

Note that $F_g \circ F_{g'} = F_{gg'}$. This induces a Lie algebra homomorphism $(F_g)_{*e} : \mathfrak{g} \to \mathfrak{g}$ which satisfies $(F_g)_{*e} \circ (F_{g'})_{*e} = (F_{gg'})_{*e}$. So we obtain a representation

$$Ad : G \to GL(\mathfrak{g}),\ g \mapsto (F_g)_{*e}$$

called adjoint representation of G. The differential representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ of Ad is given as follows.

Proposition 1.4. $\forall X, Y \in \mathfrak{g}$, $ad(X)(Y) = [X, Y]$.

Proof. Note that $F_g = R_{g^{-1}} \circ L_g$. Then

$ad(X)(Y) = \frac{d}{dt} Ad(\exp_G(tX))(Y)|_{t=0} = \frac{d}{dt} \left(R_{\exp_G(-tX)}\right)_{* \exp_G(tX)} \left(L_{\exp_G(tX)}\right)_{*e} (Y)|_{t=0} = [X^\sharp, Y^\sharp]_e = [X, Y]$. $\square$

Recall that there is a exponential map in Riemannian geometry. The Riemannian exp and the Lie group exp are related as follows.

Definition 1.7. A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group G is said to be bi-invariant if $\forall g, h \in G$, $L_g^* R_h^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$.

Theorem 1.1. Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$. Then $\exp_e = \exp_G$.

To show this we describe the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$.

Lemma 1.1. $\forall X, Y \in \mathfrak{g}$, $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$.

Proof. By Koszul formula, we have

$$\begin{aligned} \langle \nabla_{X^\sharp} Y^\sharp, Z^\sharp \rangle = \frac{1}{2} \Big(& X^\sharp \langle Y^\sharp, Z^\sharp \rangle + Y^\sharp \langle Z^\sharp, X^\sharp \rangle - Z^\sharp \langle X^\sharp, Y^\sharp \rangle \\ & - \langle Y^\sharp, [X^\sharp, Z^\sharp] \rangle - \langle Z^\sharp, [Y^\sharp, X^\sharp] \rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp] \rangle \Big) \end{aligned}$$

Since for $\forall g \in G$, $X_g^\sharp = \frac{d}{dt} g \cdot \exp_G(tX) \mid_{t=0}$, we have

$$X^\sharp \langle Y^\sharp, Z^\sharp \rangle = \frac{d}{dt} \langle Y^\sharp_{g \cdot \exp_G(tX)}, Z^\sharp_{g \cdot \exp_G(tX)} \rangle_{g \cdot \exp_G(tX)} \mid_{t=0} = \frac{d}{dt} \langle Y, Z \rangle_e \mid_{t=0} = 0$$

Since $\langle \cdot, \cdot \rangle$ is bi-invariant,

$$L_g^* R_{g^{-1}}^* \langle \cdot, \cdot \rangle_e = \langle \cdot, \cdot \rangle_e \text{ for } \forall g \in G \iff \langle Ad(g)(\cdot), Ad(g)(\cdot) \rangle_e = \langle \cdot, \cdot \rangle_e$$

Setting $g = \exp_G(tZ)$ and $\frac{d}{dt} \mid_{t=0}$, we have $\langle ad(Z)(\cdot), \cdot \rangle_e + \langle \cdot, ad(Z)(\cdot) \rangle_e = 0$, which shows that $\langle Y^\sharp, [X^\sharp, Z^\sharp] \rangle + \langle X^\sharp, [Z^\sharp, Y^\sharp] \rangle = 0$, so we have $\nabla_{X^\sharp} Y^\sharp = \frac{1}{2}[X, Y]^\sharp$. $\square$

The proof of the theorem completes once shown that $\exp_G(tX)$ is geodesic, which is left as an exercise.

Exercise 1.1. Prove the theorem.