Assignments

# Numerical Methods for Ordinary Differential Equations

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www.mop.uni-saarland.de/teaching/NODE19

— Summer Term 2019 —

**Submission Instructions:** Submit your solutions in the lecture hall before or directly after the lecture. *Clearly* write your *name* on the first sheet. Please use *A4 paper format* and *staple* all sheets together. Solutions that get separated and cannot be identified will not be evaluated. Solutions *should* be submitted in groups of 2 or 3 students. Instructions for practical exercises are given at the end of the assignment sheet.

## — Assignment 1 —

### Problem 1. [10 points]

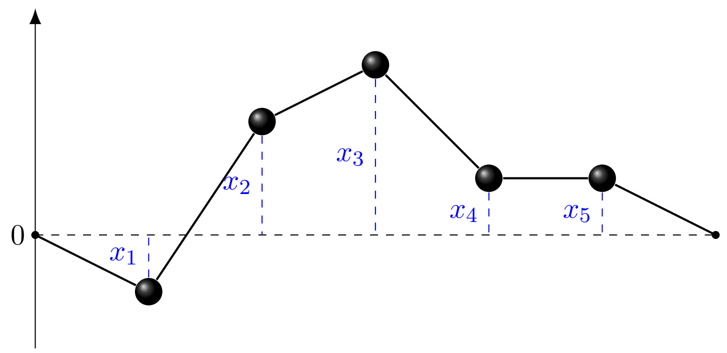
Consider the differential equation

$$x''(t) = -c\sin(x(t))\,,$$

where $c > 0$ and $x(t) \in \mathbb{R}$. Transform this ODE into a first-order autonomous ODE and show that the natural phase space is a cylinder, i.e., $\mathbb{R} \times S^1$ where $S^1$ is a circle.

### Problem 2. [10 points]

Let $\boldsymbol{x}(t) \in \mathbb{R}^5$ with $\boldsymbol{x} = (x_1, x_2, \dots, x_5)$ be the vertical displacements of points on a string with the same distance between neighboring points. The ends of the string are fixed at the same height 0. The following figure shows a certain configuration at time $t$:




We model the potential energy of each point as squared relative displacement between neighboring points

$$U = \frac{c}{2}x_1^2 + \frac{c}{2}\sum_{i=1}^{4}(x_{i+1} - x_i)^2 + \frac{c}{2}x_5^2$$

for some $c > 0$ that defines the elasticity of the string and the kinetic energy as

$$T = \frac{1}{2}\sum_{i=1}^{5}(x_i')^2\,.$$

Define the Lagrangian as

$$\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{x}') = \frac{1}{2}\sum_{i=1}^{5}(x_i'(t))^2 - \frac{c}{2}x_1^2 - \frac{c}{2}\sum_{i=1}^{4}(x_{i+1}(t) - x_i(t))^2 - \frac{c}{2}x_5^2 .$$

and formally derive the Euler–Lagrange Equations.

## Problem 3. [6 points]

The instructions of this practical exercise is given in the file `ex01_P1.py`. Replace

```
# >>> TODO <<<
```

with your code.

## Problem 4. [6 points]

The code in the file `ex01_P2.py` plots the following functions

$$g_x\colon [0, \frac{3\pi}{2}] \to \mathbb{R}, \qquad t \mapsto \cos(t)$$
$$g_y\colon [0, \frac{3\pi}{2}] \to \mathbb{R}, \qquad t \mapsto \sin(t)$$

in one coordinate system and (the image of) the curve

$$g\colon [0, \frac{3\pi}{2}] \to \mathbb{R}^2, \qquad t \mapsto (g_x(t), g_y(t))$$

in another coordinate system. The interval $[0, \frac{3\pi}{2}]$ is coarsely discretized (partitioned) by 5 grid points (knots).
Modify this code as follows:

- Change the functions $g_x$ and $g_y$ to

$$g_x\colon [0, \pi] \to \mathbb{R}, \qquad t \mapsto t\cos(t^2)$$
$$g_y\colon [0, \pi] \to \mathbb{R}, \qquad t \mapsto t\sin(t^2) .$$

- Discretize the domain with 150 grid points.
- Adapt the function $g$ according to the new functions $g_x$ and $g_y$.
- Visualize the new functions $g_x$, $g_y$ and $g$ analogously to the given code (the template).

## Problem 5. [8 points]

Consider the differential equation

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$$

with $\boldsymbol{x}\colon [0, 1] \to \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. The code in `ex01_P3.py` creates a grid of the phase space and initializes a figure. The goal is to visualize the vector field (one vector for each grid point) that is defined by the right hand side of the differential equation above. Further information is given in the file.
Visualize the vector field for the following three matrices

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and compare the results with 3 sentences (to be submitted with the theoretical assignments).

### Submission Instructions:

- Unpack the files using `tar -xzvf ex.tar.gz`.
- Fill-in the missing parts in the files that are provided as exercise. (Marked with `TODO`.)
- Make sure that the code can be executed using `python file.py`.
  - *Don't use exotic packages! (we check with python3)*
- Compress the files to `zip` or `tar.gz` format on a standard Linux machine.
  - *Submissions that cannot be unpacked on a standard Linux machine will receive no points.*
  - *Compress the files using* `tar -czvf Ex02_Surname1_Surname_2.tar.gz FOLDER`.
- If not done yet, rename the submission file to `Ex02_Surname1_Surname_2`.
- Send a *single* eMail *before the end of the lecture* on the submission date to

  Mahesh Chandra Mukkamala: `mukkamala@math.un-sb.de`.

  - *Only the first eMail will be considered!*
  - *You won't get points for late submissions!*