Announcements

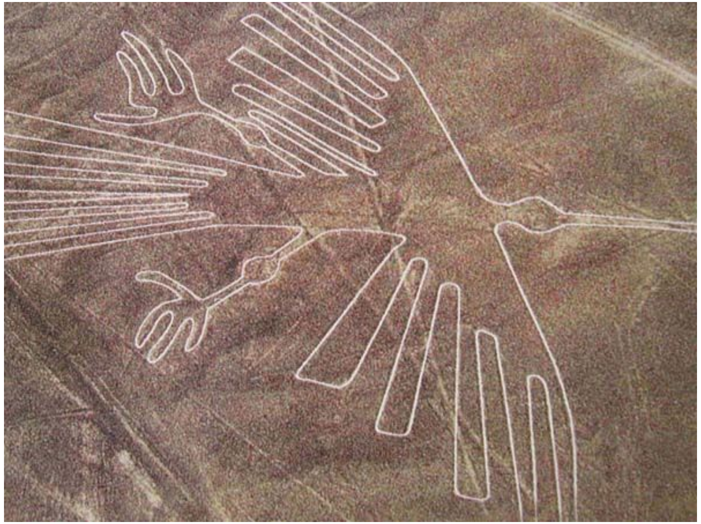
Project 2, Phase 2 due tonight (with 24 hour grace period).

Project 2 demos this are this week:

- See demo link in project 2 spec for the exact script we'll use for demos.
- You will let us know which commit to use from github.
    - Submissions from March 7th or later will incur a penalty on your 80 demo points. Will be done on a per day basis, e.g. submissions from March 7th will be 10% off, from March 8th will be 20% off, etc.
- Gold points are not part of the lab demo.

Gold points:

- No late penalty for submitting by March 9th.



CS61B

Lecture 20: Disjoint Sets

- Dynamic Connectivity and the Disjoint Sets Problem
- Quick Find
- Quick Union
- Weighted Quick Union
- Path Compression (CS170 Preview)

Meta-goals of the Coming Lectures: Data Structure Refinement

Project 2: A chance to see how a design evolves.

Next couple of weeks: Deriving classic solutions to interesting problems, with an emphasis on how set, map, and priority queue ADTs are implemented.

- Today: A chance to see how an implementation of an ADT can evolve and how different underlying data structures affect asymptotic runtime (using our formal notation).

Today's Goal: Dynamic Connectivity

Today: A case study in ADT implementation.

Goal: Given a series of pairwise connectedness declarations, determine if two items are connected transitively. Only care about yes vs. no.

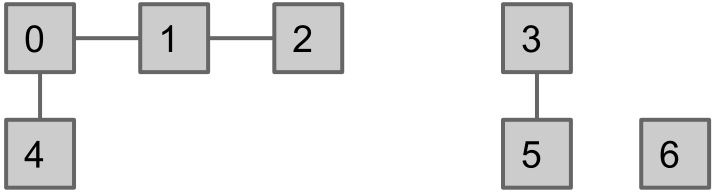
- Example: We have Mexico, USA, Canada, Ukraine, and Estonia.
    - USA is connected to Mexico.
    - USA is connected to Canada.
    - Is Mexico connected to Canada? **Yes**.
    - Ukraine is connected to Estonia.
    - Is Ukraine connected to USA? **No**.

Solvable using a simple ADT with cool implementation details.

The Dynamic Connectivity Problem

Goal: Given a series of pairwise integers connectedness declarations, determine if two integers (or items) are connected. Two operations:

- connect(p, q): Connect items p and q.
- isConnected(p, q): Are p and q connected?



```
connect(0, 1)
connect(1, 2)
connect(0, 4)
connect(3, 5)
isConnected(2, 4): true
isConnected(3, 0): false
```

The Dynamic Connectivity Problem

Goal: Given a series of pairwise integers connectedness declarations, determine if two integers (or items) are connected. Two operations:

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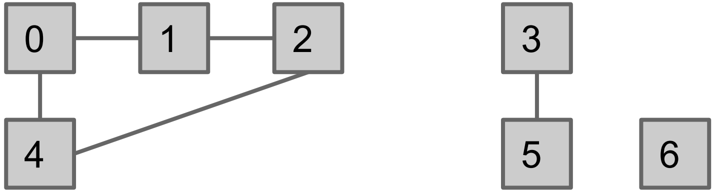

```
connect(0, 1)
connect(1, 2)
connect(0, 4)
connect(3, 5)
isConnected(2, 4): true
isConnected(3, 0): false
connect(4, 2)
```

The Dynamic Connectivity Problem

Goal: Given a series of pairwise integers connectedness declarations, determine if two integers (or items) are connected. Two operations:

- connect(p, q): Connect items p and q.
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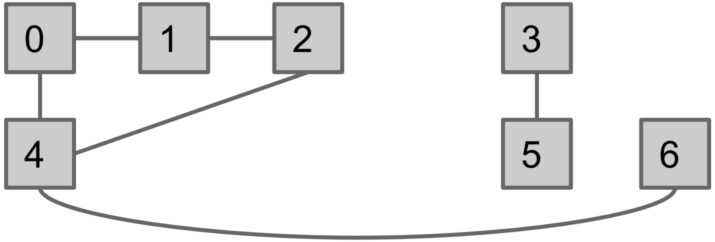
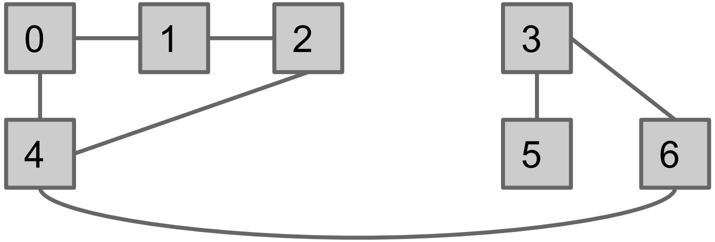


```
connect(0, 1)
connect(1, 2)
connect(0, 4)
connect(3, 5)
isConnected(2, 4): true
isConnected(3, 0): false
connect(4, 2)
connect(4, 6)
```

The Dynamic Connectivity Problem

Goal: Given a series of pairwise integers connectedness declarations, determine if two integers (or items) are connected. Two operations:

- connect(p, q): Connect items p and q.
- isConnected(p, q): Are p and q connected?



```
connect(0, 1)
connect(1, 2)
connect(0, 4)
connect(3, 5)
isConnected(2, 4): true
isConnected(3, 0): false
connect(4, 2)
connect(4, 6)
connect(3, 6)
```

The Dynamic Connectivity Problem

Goal: Given a series of pairwise integers connectedness declarations, determine if two integers (or items) are connected. Two operations:

- connect(p, q): Connect items p and q.
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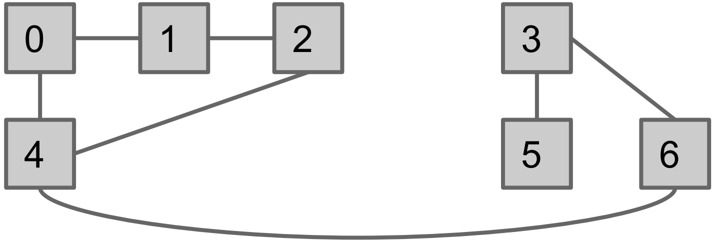


```
connect(0, 1)
connect(1, 2)
connect(0, 4)
connect(3, 5)
isConnected(2, 4): true
isConnected(3, 0): false
connect(4, 2)
connect(4, 6)
connect(3, 6)
isConnected(3, 0): true
```

The Disjoint Sets ADT

```
public interface DisjointSets {
    /** Connects two items P and Q. */
    void connect(int p, int q);

    /** Checks to see if two items are connected. */
    boolean isConnected(int p, int q);
}
```

```
connect(int p, int q)
isConnected(int p, int q)
```

Goal: Design an efficient DisjointSets implementation.

- Number of elements N can be huge.
- Number of method calls M can be huge.
- Calls to methods may be interspersed (e.g. can’t assume that we stop getting connect calls after some point).

The Naive Approach

Naive approach:

- Connecting two things: Record every single connecting line in some data structure.
- Checking connectedness: Do some sort of (??) iteration over the lines to see if one thing can be reached from the other.

A Better Approach: Connected Components

Rather than manually writing out every single connecting line, record the sets that something belongs to.

```
                           {0}, {1}, {2}, {3}, {4}, {5}, {6}, {7}
connect(0, 1)              {0, 1}, {2}, {3}, {4}, {5}, {6}, {7}
connect(1, 2)              {0, 1, 2}, {3}, {4}, {5}, {6}, {7}
connect(0, 4)              {0, 1, 2, 4}, {3}, {5}, {6}, {7}
connect(3, 5)              {0, 1, 2, 4}, {3, 5}, {6}, {7}
isConnected(2, 4): true
isConnected(3, 0): false
connect(4, 2)              {0, 1, 2, 4}, {3, 5}, {6}, {7}
connect(4, 6)              {0, 1, 2, 4, 6}, {3, 5}, {7}
connect(3, 6)              {0, 1, 2, 3, 4, 5, 6}, {7}
isConnected(3, 0): true
```

A Better Approach: Connected Components

A ***connected component*** is a maximal set of items that are mutually connected.

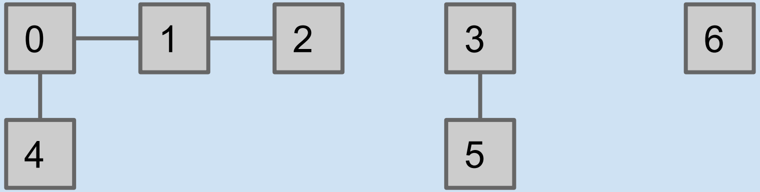
- Naive approach: Record every single connecting line somehow.
- Better approach: Model connectedness in terms of sets.
  - How things are connected isn't something we need to know.

`{ 0, 1, 2, 4 }, {3, 5}, {6}`

Quick Find

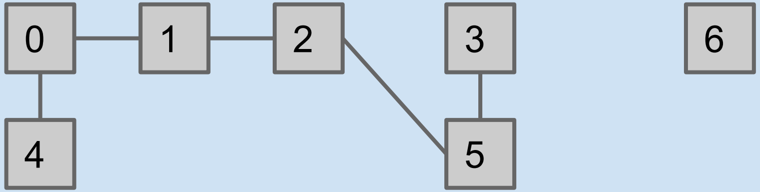
Challenge: Pick Data Structures to Support Tracking of Sets

Before connect(2, 5) operation:



`{ 0, 1, 2, 4 }, {3, 5}, {6}`

After connect(2, 5) operation:

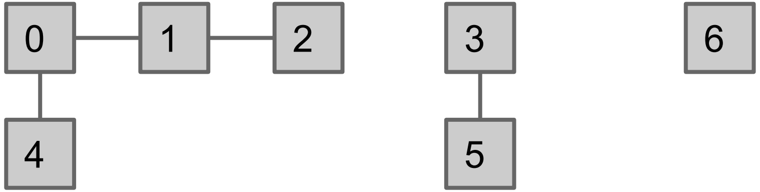


`{ 0, 1, 2, 4, 3, 5}, {6}`

Assume elements are numbered from 0 to N-1.

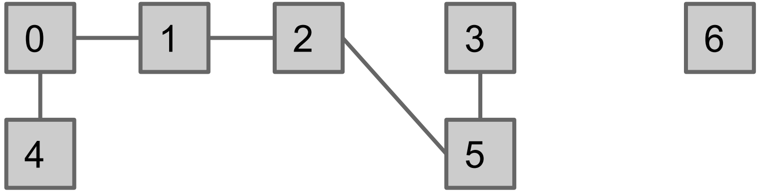
Challenge: Pick Data Structures to Support Tracking of Sets

Before connect(2, 5) operation:



`{ 0, 1, 2, 4 }, {3, 5}, {6}`

After connect(2, 5) operation:



`{ 0, 1, 2, 4, 3, 5}, {6}`

Idea #1:

A map from integers is also an array.

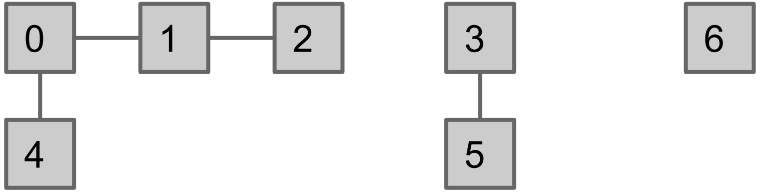
- Map<Integer, SetGuy>, where the Integer is the item in question.
  - map.get(1) ← return a reference to {0, 1, 2, 4}

Idea #2: List<HashSet>, where we move things around between sets

- Requires iterating through all the hashsets to find something.

Using an Array

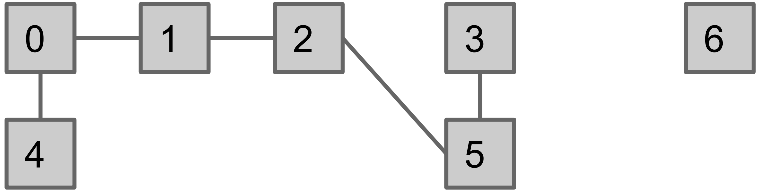
Before connect(2, 5) operation:



`{ 0, 1, 2, 4 }, {3, 5}, {6}`

`int[] id`	0	0	0	3	0	3	6
	0	1	2	3	4	5	6

After connect(2, 5) operation:



`{ 0, 1, 2, 4, 3, 5}, {6}`

`int[] id`	3	3	3	3	3	3	6
	0	1	2	3	4	5	6

One natural choice: int[] where ith entry gives set number of item i.

- `connect(p, q)`: Change entries that equal `id[p]` to `id[q]`

QuickFindDS

```java
public class QuickFindDS implements DisjointSets {
    private int[] id;

    public boolean isConnected(int p, int q) {
        return id[p] == id[q];
    }

    public void connect(int p, int q) {
        int pid = id[p];
        int qid = id[q];
        for (int i = 0; i < id.length; i++) {
            if (id[i] == pid) {
                id[i] = qid;
            }
        }...
```

Very fast: Two array accesses.

Relatively slow: N+2 to 2N+2 array accesses.

```java
public QuickFindDS(int N) {
    id = new int[N];
    for (int i = 0; i < N; i++)
        id[i] = i;
    }
}
```

Performance Summary

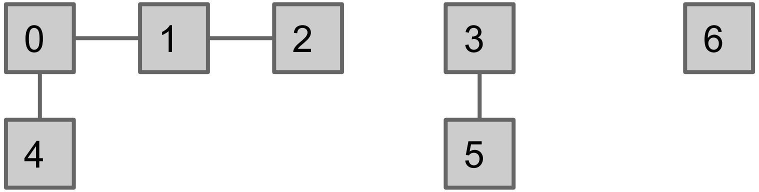
Implementation	constructor	connect	isConnected
QuickFindDS	Θ(N)	Θ(N)	Θ(1)

QuickFindDS is too slow: Connecting two items takes N time.

Quick Union

Improving the Connect Operation

Approach zero: Represent everything as boxes and lines. This was overkill.



→ ??? in Java.

Approach one: Represent everything as connected components. Represented connected components as an array.

`{ 0, 1, 2, 4 }, {3, 5}, {6}` → `int[] id`

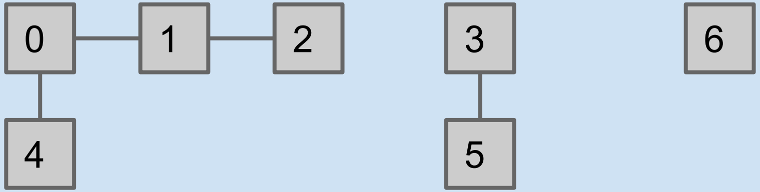
`int[] id`	0	0	0	3	0	3	6
	0	1	2	3	4	5	6

Approach two: We're still going to stick with connected components, but will represent connected components differently.

`{ 0, 1, 2, 4 }, {3, 5}, {6}` → ??? in Java.

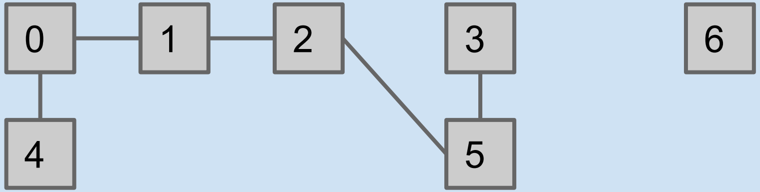
Improving the Connect Operation

A hard question: How could we change our set representation so that combining two sets into their union requires changing **one** value?



`{ 0, 1, 2, 4 }, {3, 5}, {6}`

`int[] id`	0	0	0	3	0	3	6
	0	1	2	3	4	5	6



`{ 0, 1, 2, 4, 3, 5}, {6}`

`int[] id`	3	3	3	3	3	3	6
	0	1	2	3	4	5	6

Improving the Connect Operation

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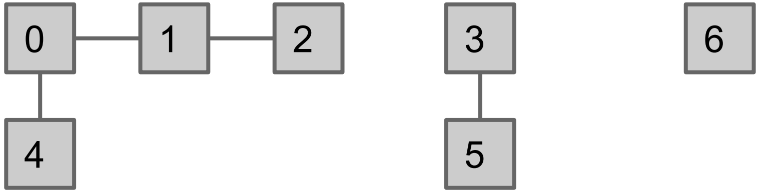
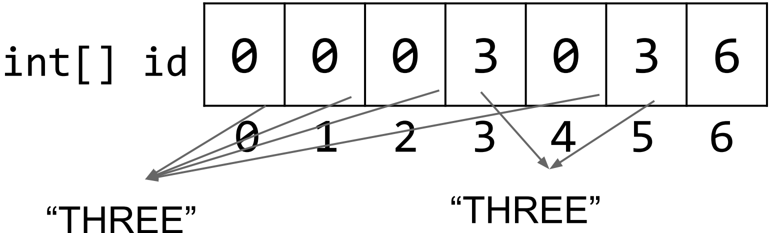
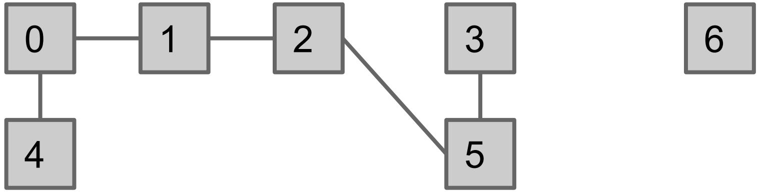


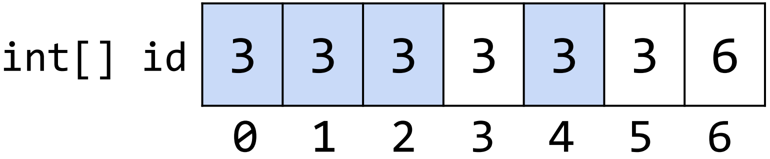
`{ 0, 1, 2, 4 }, {3, 5}, {6}`



`int[] id`	0	0	0	3	0	3	6
	0	1	2	3	4	5	6



`{ 0, 1, 2, 4, 3, 5}, {6}`



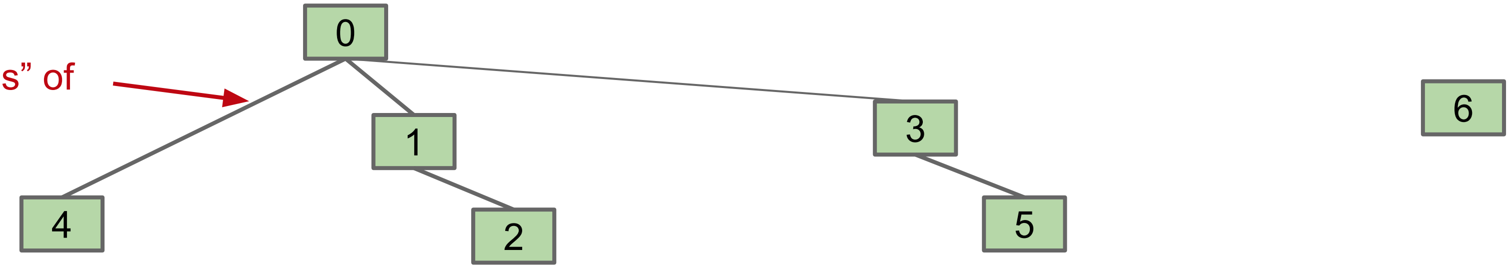
`int[] id`	3	3	3	3	3	3	6
	0	1	2	3	4	5	6

Improving the Connect Operation

Possibly harder question:

- Knowing that we only need to support the union/belongs operations, how can we represent a set such that the set union operation is very fast?
- Idea: Assign each node a parent (instead of an id).
  - An innocuous sounding, seemingly arbitrary solution.
  - Unlocks a pretty amazing universe of math that we won’t discuss.

`int[] parent`	0	0	1	0	0	3	6
	0	1	2	3	4	5	6



0 is the “boss” of this set.

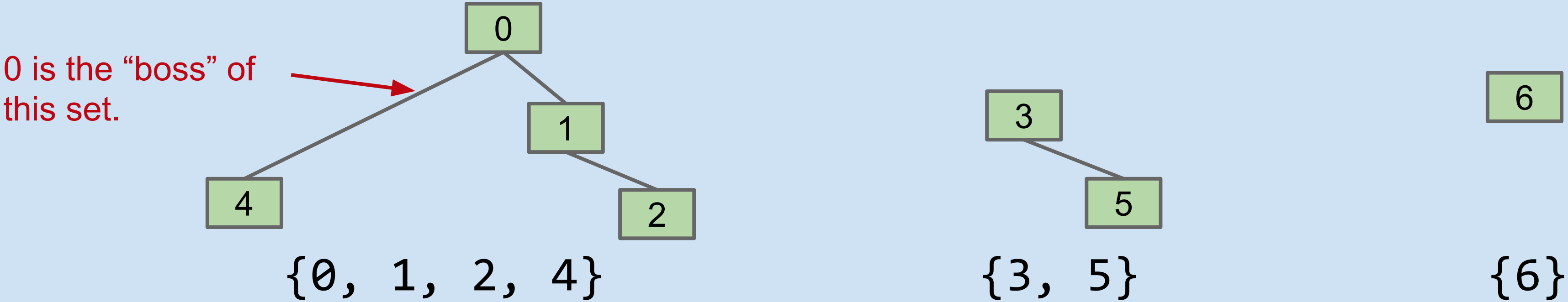
`{0, 1, 2, 4}` `{3, 5}` `{6}`

Improving the Connect Operation

connect(5, 2)

- How do we do this?
    - If you’re not sure where to start, consider: why can’t we just set id[5] to 2?

int[] parent	0	0	1	3	0	3	6
	0	1	2	3	4	5	6



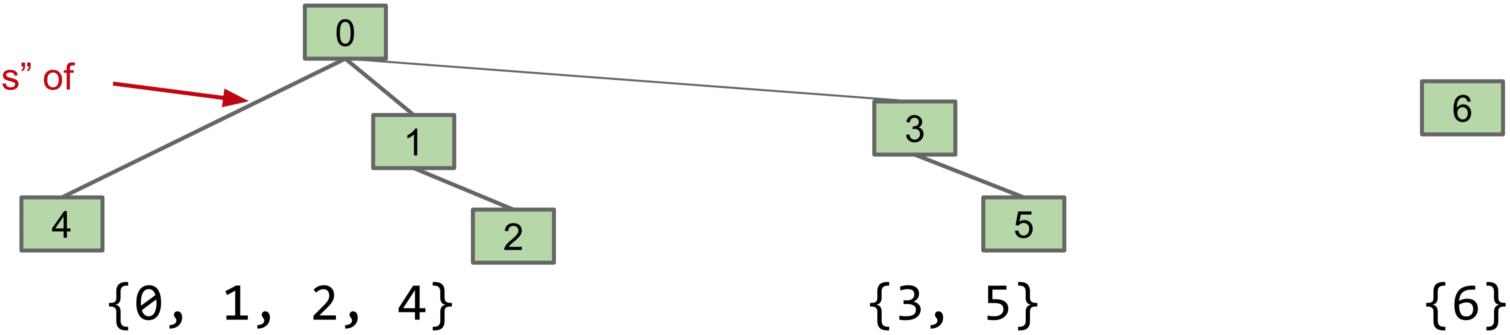
0 is the “boss” of this set.

Improving the Connect Operation

connect(5, 2)

- How do we do this?
    - Find the boss of 5. ← this isn’t free!
    - Find the boss of 2.
    - Change the value of the boss of 5 to boss of 2?

int[] parent	0	0	1	0	0	3	6
	0	1	2	**3**	4	5	6



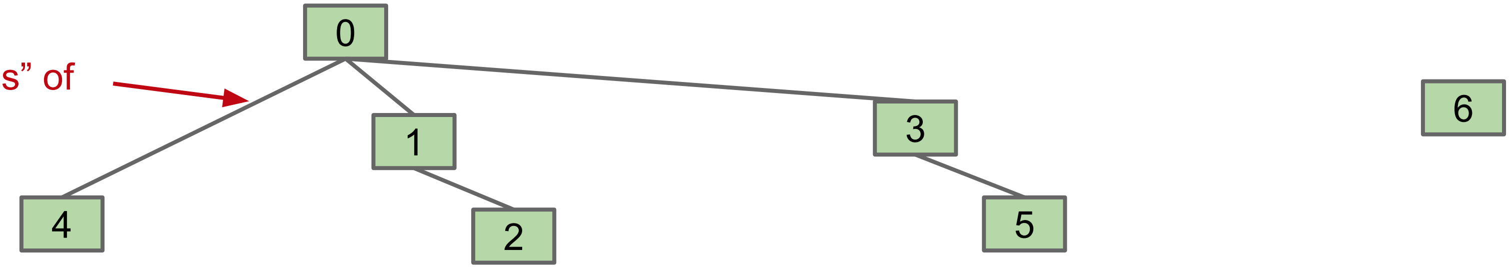
0 is the “boss” of this set.

Improving the Connect Operation

connect(5, 2)

- How do we do this?
  - If you're not sure where to start, consider: why can't we just set id[5] to 2?
  - make root(5) into a child of root(2).

int[] parent	0	0	1	0	0	3	6
	0	1	2	3	4	5	6



0 is the "boss" of this set.

{0, 1, 2, 4, 3, 5} {6}

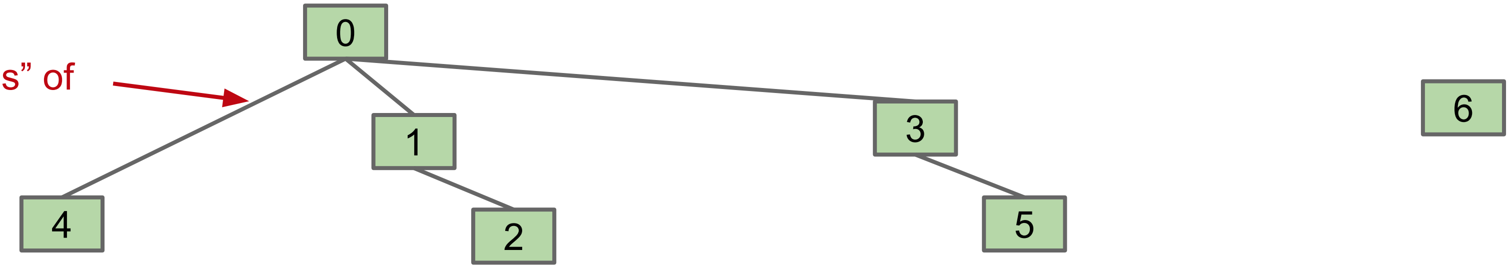
Set Union Using Rooted-Tree Representation

connect(5, 2)

- Make root(5) into a child of root(2).

`int[] parent`	0	0	1	0	0	3	6
	0	1	2	3	4	5	6

What are the potential performance issues with this approach?



0 is the “boss” of this set.

`{0, 1, 2, 4, 3, 5}` `{6}`

Set Union Using Rooted-Tree Representation

connect(5, 2)

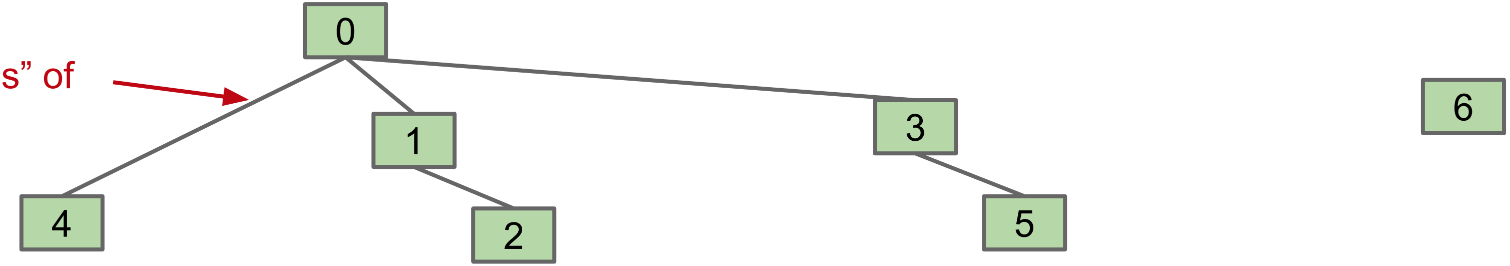
- Make root(5) into a child of root(2).

`int[] parent`	0	0	1	0	0	3	6
	0	1	2	3	4	5	6

What are the potential performance issues with this approach?

- Tree can get too tall!

finding boss is not free



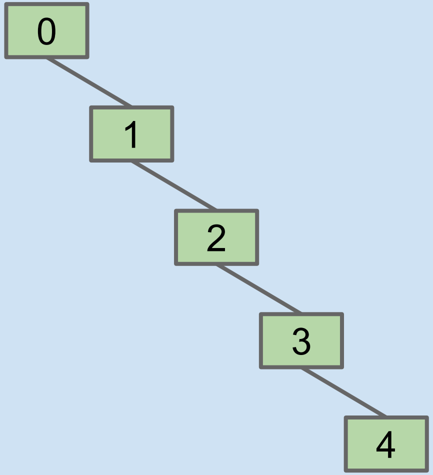
0 is the “boss” of this set.

`{0, 1, 2, 4, 3, 5}` `{6}`

The Worst Case

If we always connect the first items’ tree below the second item’s tree, we can end up with a tree of height M after M operations:

- connect(4, 3)
- connect(3, 2)
- connect(2, 1)
- connect(1, 0)



For N items, what’s the worst case runtime…

- For connect(p, q)?
- For isConnected(p, q)?

QuickUnionDS

```
public class QuickUnionDS implements DisjointSets {
    private int[] parent;
    public QuickUnionDS(int N) {
        parent = new int[N];
        for (int i = 0; i < N; i++)
            parent[i] = i;
        }

    private int find(int p) {
        while (p != parent[p])
            p = parent[p];
        return p;
    }
```

For N items, this means a worst case runtime of Θ(N).

```
public boolean isConnected(int p, int q) {
    return find(p) == find(q);
}

public void connect(int p, int q) {
    int i = find(p);
    int j = find(q);
    parent[i] = j;
}
```

Performance Summary

Implementation	Constructor	connect	isConnected
QuickFindDS	Θ(N)	Θ(N)	Θ(1)
QuickUnionDS	Θ(N)	O(N)	O(N)

QuickFindDS defect: QuickFindDS is too slow: Connecting two items takes N time in the worst case.

QuickUnion defect: Trees can get tall. Results in potentially even worse performance than QuickFind if tree is imbalanced.

Weighted Quick Union

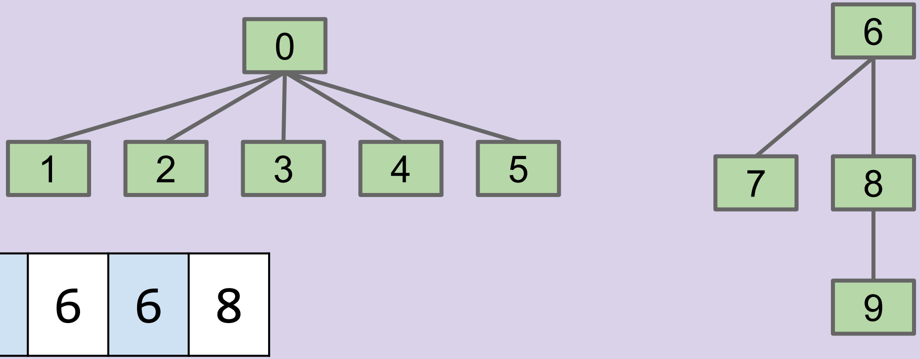
Weighted QuickUnion: http://yellkey.com/analysis

Modify quick-union to avoid tall trees.

- Track tree size (**number** of elements).
- New rule: Always link root of ***smaller*** tree ***to larger*** tree.

New rule: If we call connect(3, 8), which entry (or entries) of parent[] changes?

A. parent[3]
B. parent[0]
C. parent[8]
D. parent[6]



`int[] parent`	0	0	0	0	0	0	6	6	6	8
	0	1	2	3	4	5	6	7	8	9

Improvement #1: Weighted QuickUnion

Modify quick-union to avoid tall trees.

- Track tree size (**number** of elements).
- New rule: Always link root of ***smaller*** tree ***to larger*** tree.

New rule: If we call connect(3, 8), which entry (or entries) of parent[] changes?

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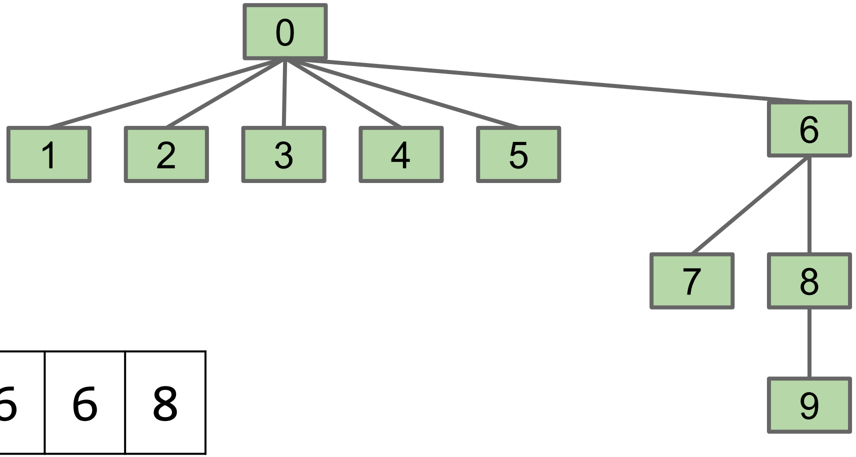


`int[] parent`	0	0	0	0	0	0	0	6	6	8
	0	1	2	3	4	5	6	7	8	9

Implementing WeightedQuickUnion

Minimal changes needed:

- Use `parent[]` array as before, but also add `size[]` array.
- `isConnected(int p, int q)` requires no changes.
- `connect(int p, int q)` needs two changes:
    - ○ Link root of smaller tree to larger tree.
    - ○ Update `size[]` array.

int[] parent	0	0	0	0	0	0	0	6	6	8
	0	1	2	3	4	5	6	7	8	9

Now the connect method looks like:

```
public void connect(int p, int q) {
    int i = find(p);
    int j = find(q);
    if (i == j) return;
    if (size[i] < size[j]) { parent[i] = j; size[j] += size[i]; }
    else { parent[j] = i; size[i] += size[j]; }
}
```

int[] size	10	1	1	1	1	1	4	1	2	1
	0	1	2	3	4	5	6	7	8	9

WeightedQuickUnion Performance

As before, connect and isConnected require time proportional to depth of the items involved.

Max depth of any item: log N

Very brief proof for the curious (not covered in lecture):

- Depth of an element x increases only when tree T1 that contains x is linked below some other tree T2.
    - The size of the tree at least doubles since weight(T2) ≥ weight(T1).
    - Tree containing x doubles at most log N times.

Performance Summary

Implementation	Constructor	connect	isConnected
QuickFindDS	Θ(N)	Θ(N)	Θ(1)
QuickUnionDS	Θ(N)	O(N)	O(N)
WeightedQuickUnionDS	Θ(N)	O(log N)	O(log N)

By tweaking QuickUnionDS we’ve achieved logarithmic time performance.

- Fast enough for most (all?) practical problems.

Performance Summary

Implementation	Constructor	connect	isConnected
QuickFindDS	Θ(N)	Θ(N)	Θ(1)
QuickUnionDS	Θ(N)	O(N)	O(N)
WeightedQuickUnionDS	Θ(N)	O(log N)	O(log N)

Performing M operations on a DisjointSet object with N elements:

- Runtime goes from O(MN) to O(N + M log N)
- For N = 10^9 and M = 10^9, time to run goes from 30 years to 6 seconds.
  - Key point: Good data structure unlocks solutions to problems that could otherwise not be solved!
- … pretty good for most problems, but could we do better?

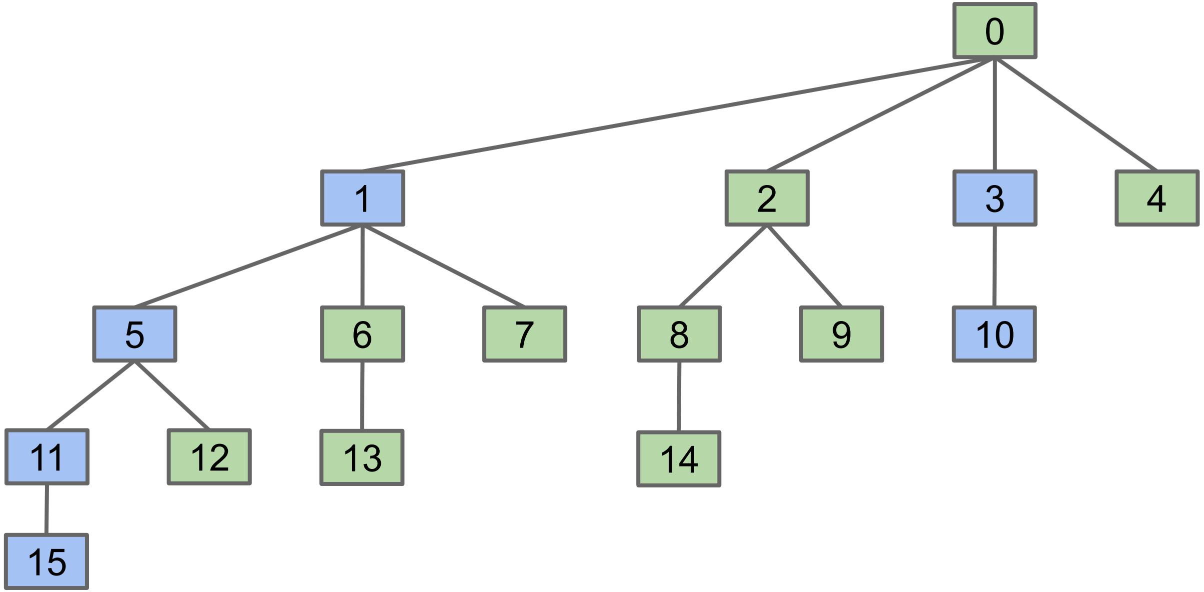
Path Compression

(CS170 Spoiler)

170 Spoiler: Path Compression: A Clever Idea

Below is the topology of the worst case if we use WeightedQuickUnion.

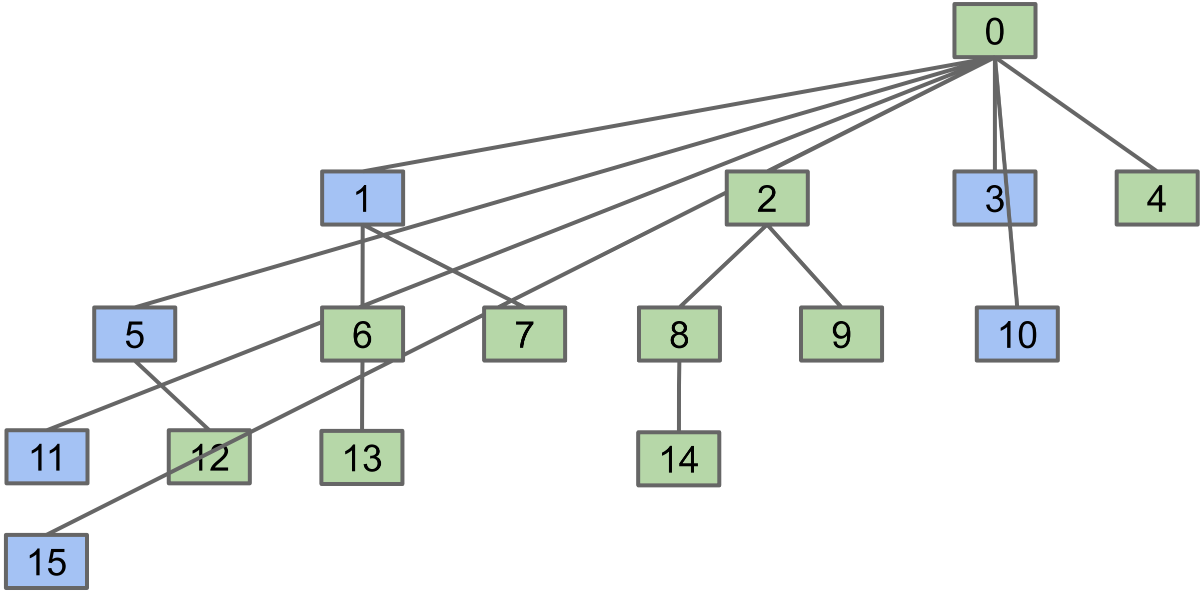
- Clever idea: When we do isConnected(15, 10), tie all nodes seen to the root.
  - Additional cost is insignificant (same order of growth).



170 Spoiler: Path Compression: A Clever Idea

Below is the topology of the worst case if we use WeightedQuickUnion

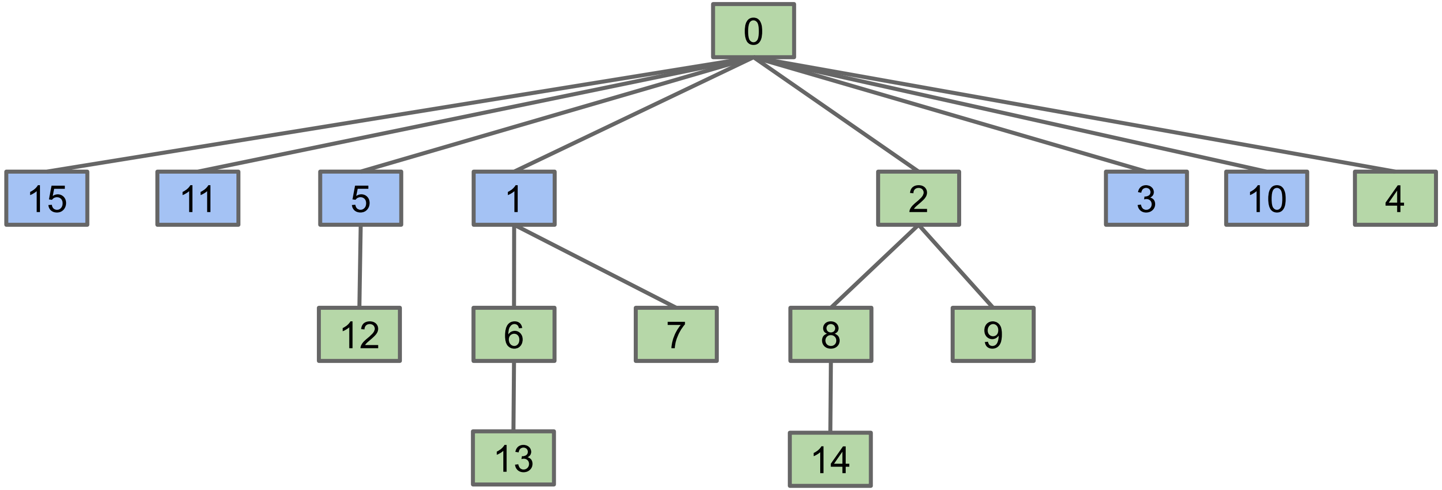
- Clever idea: When we do isConnected(15, 10), tie all nodes seen to the root.
    - Additional cost is insignificant (same order of growth).



170 Spoiler: Path Compression: A Clever Idea

Path compression results in a union/connected operations that are very very close to amortized constant time.

- M operations on N nodes is O(N + M lg* N) - you will see this in CS170.
- lg* is less than 5 for any realistic input.

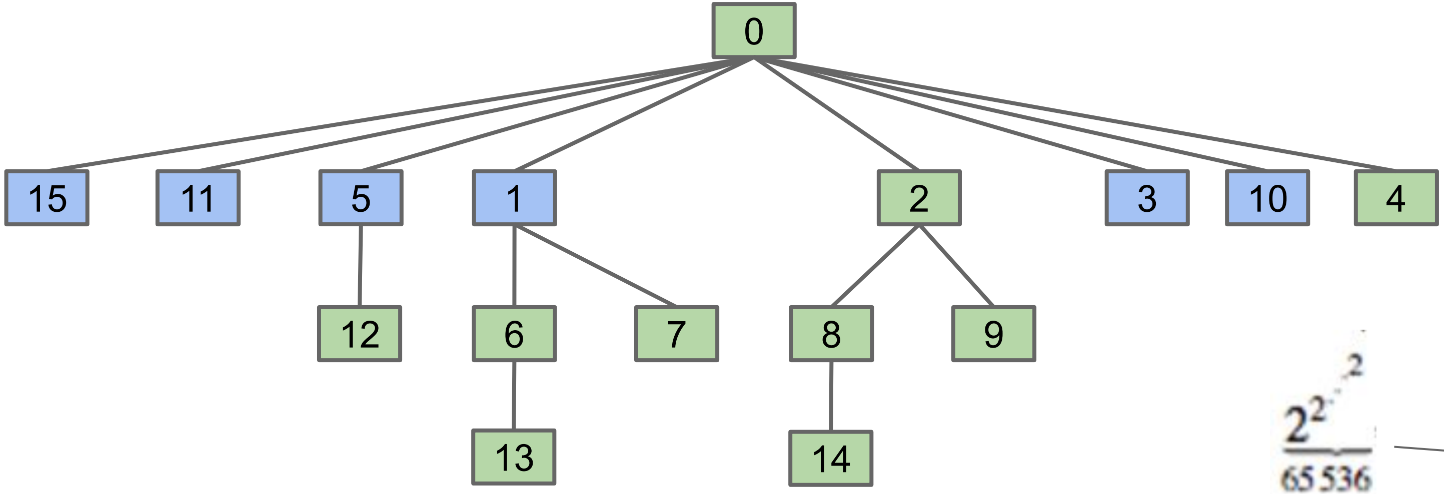


N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

170 Spoiler: Path Compression: A Clever Idea

Path compression results in a union/connected operations that are very very close to amortized constant time.

- M operations on N nodes is O(N + M lg* N) - you will see this in CS170.
- A tighter bound: O(N + M α(N)), where α is the inverse Ackermann function.
- The inverse ackermann function is less than 5 for all practical inputs!



N	α(N)
1	0
...	1
...	2
...	3
...	4
$\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{65536}$	5

170 Spoiler: Path Compression: A Clever Idea

- And we’re done! The end result of our iterative design process is the standard way disjoint sets are implemented today - quick union and path compression.
- The resulting code for find() is simple:

```
private int find(int p) {
    if (p == parent[p]) {
        return p;
    } else {
        parent[p] = find(parent[p]);
        return parent[p];
    }
}
```

Everything All Together...

```
public class WeightedQuickUnionDSWithPathCompression implements DisjointSets {
    private int[] parent; private int[] size;
    public WeightedQuickUnionDSWithPathCompression(int N) {
        parent = new int[N]; size = new int[N];
        for (int i = 0; i < N; i++) {
            parent[i] = i;
            size[i] = 1;
        }
    }

    private int find(int p) {
        if (p == parent[p]) {
           return p;
        } else {
            parent[p] = find(parent[p]);
            return parent[p];
        }
    }
```

```
public boolean isConnected(int p, int q) {
    return find(p) == find(q);
}
public void connect(int p, int q) {
    int i = find(p);
    int j = find(q);
    if (i == j) return;
    if (size[i] < size[j]) {
        parent[i] = j; size[j] += size[i];
    } else {
        parent[j] = i; size[i] += size[j];
    }
}
```

Performance Summary

Implementation	Runtime
QuickFindDS	Θ(NM)
QuickUnionDS	O(NM)
WeightedQuickUnionDS	O(N + M log N)
WeightedQuickUnionDSWithPathCompression	O(N + M α(N))

Runtimes are given assuming:

- We have a DisjointSets object of size N.
- We perform M operations, where an operation is defined as either a call to `connected` or `isConnected`.

Citations

Nazca Lines:
http://redicecreations.com/ul_img/24592nazca_bird.jpg

Implementation code adapted from Algorithms, 4th edition and Professor Jonathan Shewchuk's lecture notes on disjoint sets, where he presents a faster one-array solution. I would recommend taking a look. (http://www.cs.berkeley.edu/~jrs/61b/lec/33)

The proof of the inverse ackermann runtime for disjoint sets is given here:
http://www.uni-trier.de/fileadmin/fb4/prof/INF/DEA/Uebungen_LVA-Ankuendigungen/ws07/KAuD/effi.pdf
as originally proved by Tarjan here at UC Berkeley in 1975.

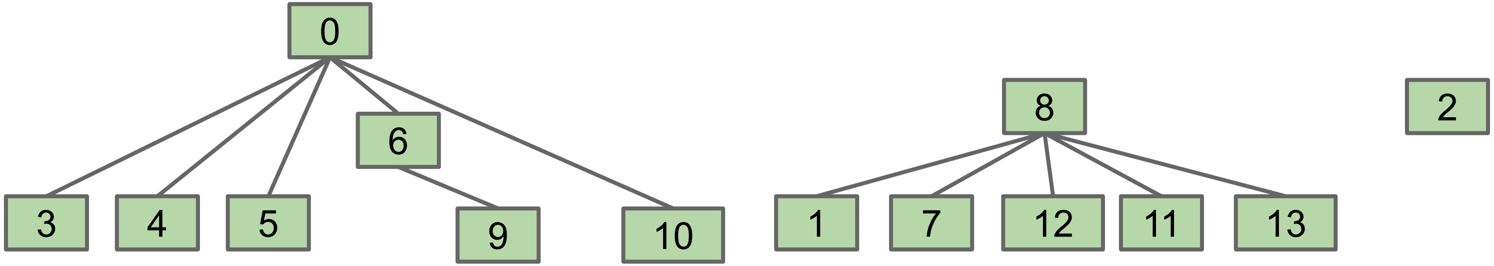
extra

Set Union Using Parent Representation

connect(11, 3)

- Any ideas how to do this quickly?

`int[] parent`	0	8	2	0	0	0	0	8	8	6	0	8	8	8
	0	1	2	3	4	5	6	7	8	9	10	11	12	13



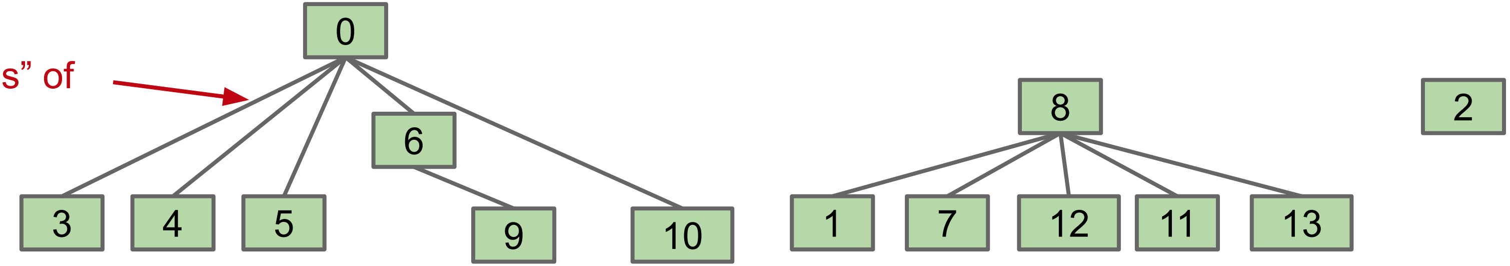
`{0, 3, 4, 5, 6, 9, 10}, {1, 7, 8, 11, 12, 13}, {2}`

Improving the Connect Operation

Possibly harder question:

- Knowing that we only need to support the union/belongs operations, how can we represent a set such that the set union operation is very fast?
- Idea: Assign each node a parent (instead of an id).
  - An innocuous sounding, seemingly arbitrary solution.
  - Unlocks a pretty amazing universe of math that we won’t discuss. D:

`int[] parent`	`0`	`8`	`2`	`0`	`0`	`0`	`0`	`8`	`8`	`6`	`0`	`8`	`8`	`8`
	`0`	`1`	`2`	`3`	`4`	`5`	`6`	`7`	`8`	`9`	`10`	`11`	`12`	`13`



0 is the “boss” of this set.

`{0, 3, 4, 5, 6, 9, 10}, {1, 7, 8, 11, 12, 13}, {2}`

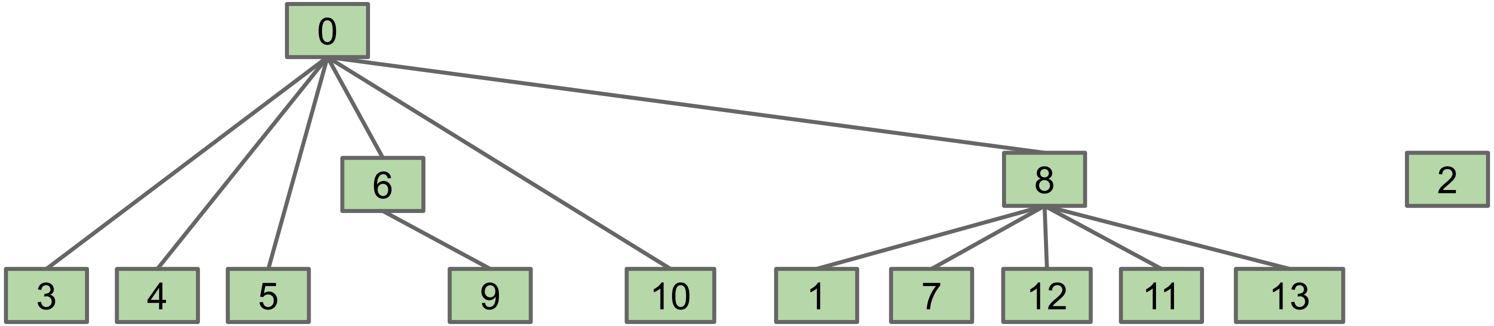
Set Union Using Rooted-Tree Representation

connect(11, 3)

- Make root(11) into a child of root(3).

`int[] id`	0	8	2	0	0	0	0	8	0	6	0	8	8	8
	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Anybody see any issues with this?



`{0, 3, 4, 5, 6, 9, 10, 1, 7, 8, 11, 12, 13}, {2}`

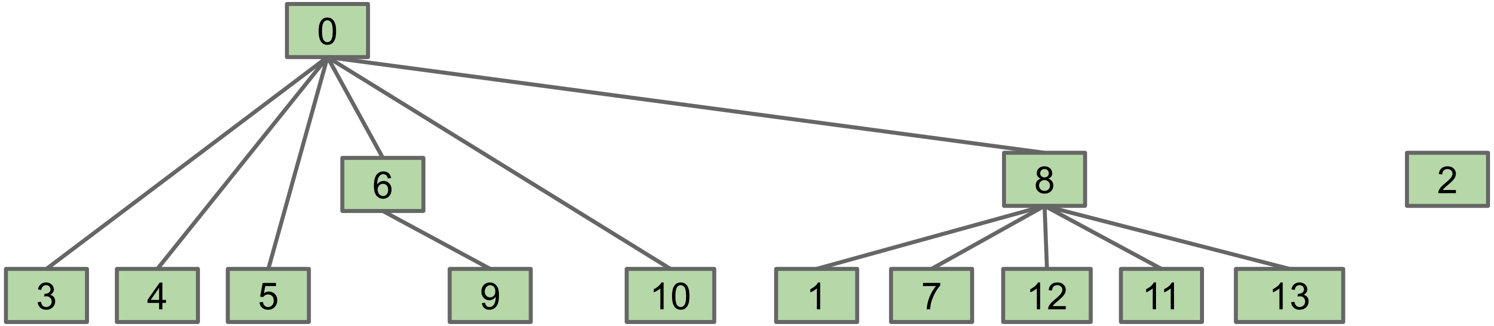
Set Union Using Rooted-Tree Representation

connect(11, 3)

- Make root(11) into a child of root(3).

int[] id	0	8	2	0	0	0	0	8	0	6	0	8	8	8
	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Anybody see any issues with this? Tree can get too tall!



{0, 3, 4, 5, 6, 9, 10, 1, 7, 8, 11, 12, 13}, {2}