1 Euler Angle Parametrization

You only need three angles (which we can achieve through rotations) to fully specify an object's orientation in 3D. We can denote these angles with respect to the 3 basis vectors in Euclidian space.

In general, we can apply these rotations either about the axes or on the axes. This is respectively called **extrinsic axes** and **intrinsic axes**. Each of the extrinsic and intrinsic axes approaches can be further split into the **Euler** (where you rotate one axis at a time) and **Tait-Bryan** (where you rotate all axes at once) Angles.

As a result, there are 24 ways that we can apply these rotations. This is because there are 3!=6 ways that you can apply the sequence of rotations. From the multiplication rule in combinatorics, we multiply by 2 (for the two Euler and Tait-Bryan angles) and multiply by 2 again (for the two extrinsic and intrinsic approaches) to get 24.

$$2 * 2 * 6 = 24$$

There is no practical difference to using extrinsic or intrinsic rotations. An extrinsic sequence of rotations is equivalent to an intrinsic set of rotations, just in reverse order.

Principal 3D Rotations

An implicit representation (in essence, overparametrizing the rotation) of 3D rotations is using rotation matrices. if we wanted to rotate about the X axis, then the rotation matrix of

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Where we rotate an angle ϕ about or on the x-axis. We can perform similar rotations about/on the y or z axis

$$Y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \quad Z = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These rotation angles (ϕ,θ,ψ), are called roll (X), pitch (Y) and yaw (Z). These terms and the order in which we specify them (X then Y then Z) is standard terminology in aerospace. We can combine these three rotations to achieve any orientation in 3D space.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Gimbal Lock

Parametrization in terms of Euler angles can lead to interesting and annoying results. A result that we want to avoid is called **gimbal lock**. This is a singularity that happens when we rotate by 90 degrees to a north or south pole (on a metaphorical sphere). The result of this singularity is losing a degree of freedom.

Let's say we wanted to specify a sequence of intrinsic rotations starting with a 90 degree Y rotation. The resulting rotation matrix would be:

$$Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

If we applied that to the X rotation matrix, we would get this:

$$\begin{bmatrix} 0 & 0 & 1 \\ \sin(\phi) & \cos(\phi) & 0 \\ -\cos(\phi) & \sin(\phi) & 0 \end{bmatrix}$$

This actually corresponds with a rotation about the Z axis. We can easily show this through plugging in 90 degrees into ϕ and applying it onto a unit vector in the x-direction:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This shows that we lose a degree of freedom if we pitch too much (by 90 degrees, to be exact) into the gimbal lock singularity. We are quite literally unable to rotate in the x direction (only y and z are allowed) unless we pitched out of this singularity.