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Recursive Explore and Check Abstraction Refinement

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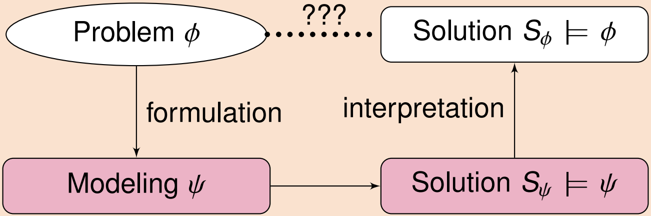
Abstraction: Idea & Motivation

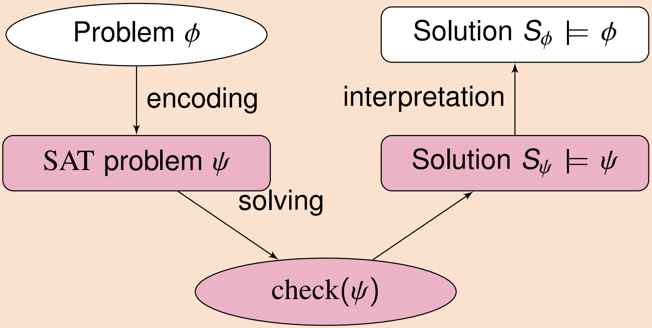
- Comes from: **Mathematical Modeling**



- Works for theoretical problems
- But what about practice?

Modeling: Propositional Formula

- For many **NP** problems: Encoding into SAT



- What is a SAT problem? a SAT solver?

The SATisfiability problem

The SAT problem

- Variables: w, x, y, z, . . . , a, b, c, . . .
- Literals: w, y, a, . . . , but also $\neg a$, $\neg c$, $\neg y$, . . .
- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses (CNF)
- Model: Mapping from variables to $\{0, 1\}$ that satisfies the SAT formula
- a Formula can be **SAT** or **UNSAT**.
- The SAT problem is **NP**-complete [Coo71]

The SATisfiability problem

The SAT problem

Example: $\psi = (a) \wedge (\neg a \vee b) \wedge (c \vee a) \wedge (\neg c \vee \neg b)$
One possible model M, s.t. $M \models \psi$

$$M = \{a = 1, b = 1, c = 0\}$$

The SAT problem

Example: $\psi = (a) \wedge (\neg a \vee b) \wedge (c \vee a) \wedge (\neg c \vee \neg b)$
One possible model M, s.t. $M \models \psi$

$$M = \{a = 1, b = 1, c = 0\}$$

- How to find quickly an M?
- How to prove that no such M exists?

With a SAT solver!

SAT solver

SAT solver

- Extremely efficient software
- Based on CDCL approach [SS99, MMZ+01]
- One of the current best is: Glucose [ES03a, AS09] ☺
- Able to solve efficiently problems with $\approx 10^8$ clauses

SAT solver: Disclaimer!

Disclaimer SAT is still **NP**-complete...

Name	sgen1-unsat-121-100.cnf
Category	CRAFTED
#Vars	121
#Clauses	252
Clause length	3

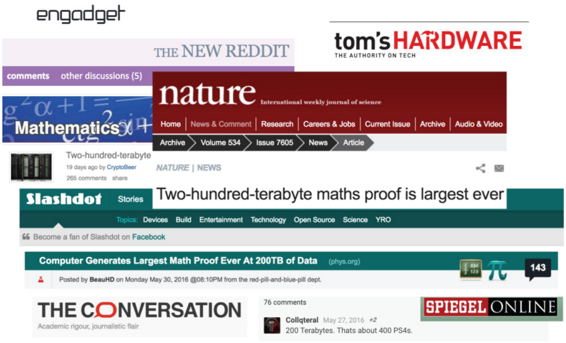
Solver Name	TraceID	Answer	CPU time	Wall clock time
SAT07 reference solver: SATzilla *CRAFTED* (complete)	1785603	? (exit code)	4998.65	5001.1
SATzilla2009_C *2009-03-22* (complete)	1825787	? (exit code)	4998.65	5000.32
VARSAT-industrial *2009-03-22* (complete)	1785604	? (TO)	5000.04	5001.91
glucose *1.0* (complete)	1784160	? (TO)	5000.04	5002.51
IUT_BMB_SAT *1.0* (complete)	1785601	? (TO)	5000.05	5002.21
MXC *2009-03-10* (complete)	1784161	? (TO)	5000.06	5001.81
SApperloT *base* (complete)	1785605	? (TO)	5000.06	5001.51
MiniSat *2.1 (Sat-race'08 Edition)* (complete)	1784159	? (TO)	5000.1	5002.21
clasp *1.2.0-SAT09-32* (complete)	1785600	? (TO)	5000.1	5013.71
precosat *236* (complete)	1784158	? (TO)	5000.1	5002.11
SAT07 reference solver: minisat *SAT 2007* (complete)	1785602	? (TO)	5000.11	5002.11
LySAT *c/2009-03-20* (complete)	1825454	? (TO)	5000.11	5002.61

http://www.cril.fr/SAT09/results/bench.php?idev=29&idbench=71111

SAT solver: Additional features

- Answer *SAT* and a model when the formula is satisfiable
- Answer *UNSAT*:
  - with a proof of unsatisfiability if asked [Gel02]
  - ...

SAT solver: Additional features



Information

- Published in SAT'16 [HKM16]
- Size of the proof of unsatisfiability: 200 Terabyte
- 16,000 CPU hours to check the proof

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SAT solver: Additional features

SAT solver: Additional features

- Answer *SAT* and a model when the formula is satisfiable
- Answer *UNSAT*:
  - with a proof of unsatisfiability if asked [Gel02]
  - A unsatisfiable core if asked [ES03a]
- Can work in an incremental way [ES03b, ES03a, ALS13]
- Can work under assumptions [ES03a]

Unsatisfiable core

Basically the “reason” why a formula is UNSAT (subset of clauses)

SAT solver

SAT solver: One limitation

- What happen when the encoding of the problem is too big ?
- Could be solved ‘easily’ but will not because of memory...

HCP via SAT: does not scale

- Ex. The Hamiltonian Cycle Problem (HCP)
- HCP: $\mathcal{O}(n^3)$ clauses [Pre03]
- Transitive relations for any three nodes
- HCP via SAT: hard to solve HCP of over 1000 nodes
- HCP solver ‘LKH’ scales up to 10,000 nodes

We need a SAT solver in a more complex procedure...

SAT solver : how to solve HCP efficiently?

V is a set of n nodes, A is a set of vertexes, and G = (V,A) is a digraph. $x_{ij} = 1 \leftrightarrow (i,j) \in A$ is used in a solution cycle.

$$\sum_{(i,j) \in A} x_{ij} = 1 \qquad \text{for each i = 1,...,n (out-degree)}$$

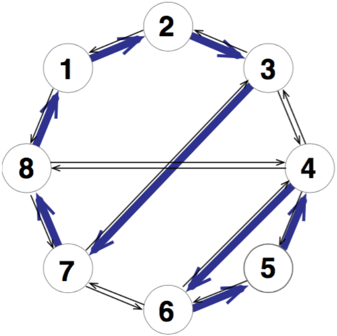
$$\sum_{(i,j) \in A} x_{ij} = 1 \qquad \text{for each j = 1,...,n (in-degree)}$$

$$\sum_{(i,j) \in S} x_{ij} \leq |S| - 1 \qquad S \subset V, 2 \leq |S| \leq n - 2 \text{ (connectivity)}$$

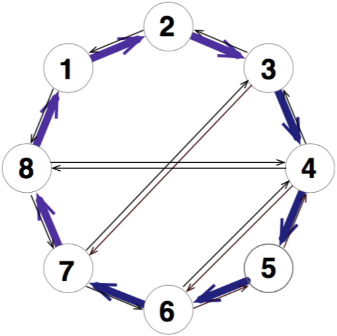
- in/out-degree constraints ensure that in/out-degrees are respectively exact one for each node in solution cycles
- connectivity constraint prohibits the formulation of sub-cycles

SAT solver : how to solve HCP efficiently?

- With only in/out-degree constraints, we have cycles but they may not be connected (Case A)
- With all constraints, we can find a Hamiltonian cycle (Case B)



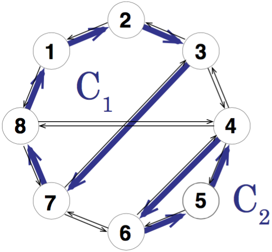
(Case A)
in/out-degree



(Case B)
in/out-degree + connectivity

HCP via SAT: no need to generate connectivity constraints

- Refine overall constraints by adding blocking clauses generated from counter examples [SLR+14].
- We can get lucky and find a Hamiltonian Cycle quickly



Blocking Clauses

C_1 $\neg x_{12} \vee \neg x_{23} \vee \neg x_{37} \vee \neg x_{78} \vee \neg x_{81}$

C_1' $\neg x_{87} \vee \neg x_{73} \vee \neg x_{32} \vee \neg x_{21} \vee \neg x_{18}$

C_2 $\neg x_{46} \vee \neg x_{65} \vee \neg x_{54}$

C_2' $\neg x_{45} \vee \neg x_{56} \vee \neg x_{64}$

HCP via SAT: no need to generate connectivity constraints

- Refine overall constraints by adding blocking clauses generated from counter examples [SLR+14].
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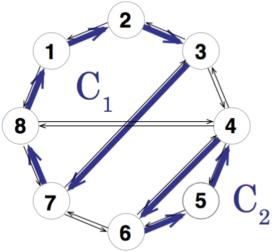

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This idea of going step by step and refining each step is called:

CEGAR: CounterExample Guided Abstraction Refinement

CounterExample Guided Abstraction Refinement

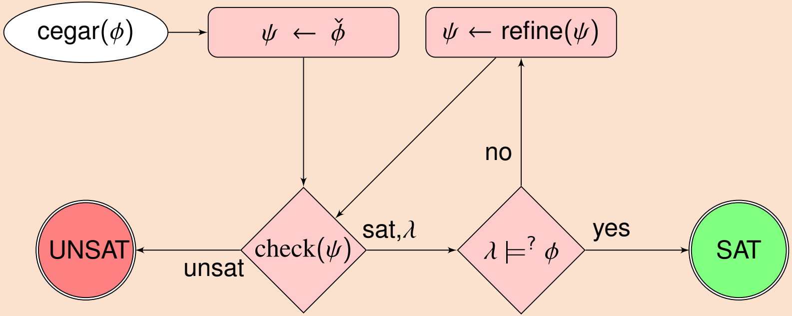
CEGAR: **C**ounter**E**xample **G**uided **A**bstraction **R**efinement

To solve a problem, we may need to consider only a small part of it [CGJ+03]

- To abstract problems: hoping it will be easier to solve
- Two variants of abstraction:
  - Under-abstraction: abstraction has **more** solutions
  - Over-abstraction: abstraction has **less** solutions
- CEGAR-over: CEGAR approach using over-abstractions
- CEGAR-under: CEGAR approach using under-abstractions

CEGAR using under-abstractions

CEGAR-under

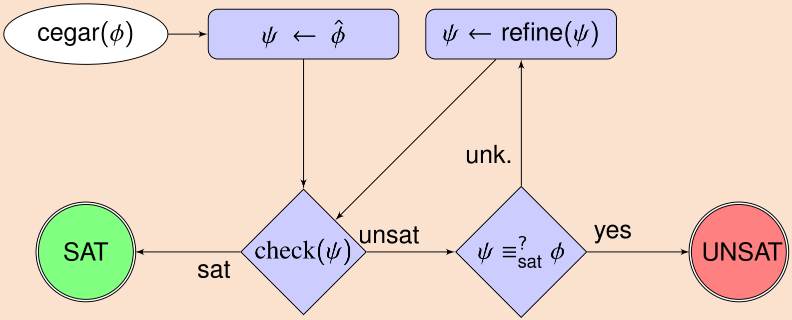


Example

SAT problem, by increasing step by step, the number of clauses

CEGAR using over-abstractions

CEGAR-over



Example

Planification problem, by increasing step by step, the horizon

CounterExample Guided Abstraction Refinement

Advantages

- If problem mainly satisfiable: CEGAR-over
- If problem mainly unsatisfiable: CEGAR-under
- Everytime check improves, CEGAR improves
- Many applications already use CEGAR

Drawbacks

- Not efficient when 50/50 chances of being SAT/UNSAT
- Not efficient when we need many refinement steps

Recursive Explore and Check Abstraction Refinement

Recursive **E**xplore and **C**heck **A**bstraction **R**efinement

- Called *RECAR* [LLdLM17]
- Inspired by CEGAR [CGJ+03]
- Rely on 5 very important assumptions

RECAR Assumptions

1. Function 'check' is sound, complete and terminates
2. $\mathit{isSAT}(\hat{\phi})$ implies $\mathit{isSAT}(\text{refine}(\hat{\phi}))$
3. $\exists. n \in \mathbb{N}$ s.t. $\text{refine}^n(\hat{\phi}) \equiv^?_{\text{sat}} \phi$.
4. $\text{isUNSAT}(\check{\phi})$ implies $\text{isUNSAT}(\phi)$
5. $\exists n \in \mathbb{N}$ s.t. $RC(\mathit{under}^n(\phi), \mathit{under}^{n+1}(\phi))$ is false.

Recursive Explore and Check Abstraction Refinement

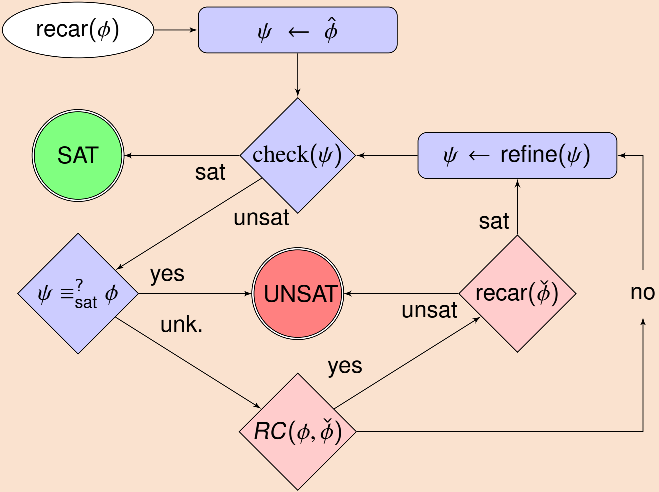
$\exists n \in \mathbb{N}$ s.t. $RC(under^n(\phi), under^{n+1}(\phi))$ is false.

RC function

- 'true' if we can do a recursive call, 'false' otherwise
- It compares $under^i(\phi)$ and $under^{i+1}(\phi)$
- It checks if $under^{i+1}(\phi)$ will be "easier to solve" than $under^i(\phi)$

Recursive Explore and Check Abstraction Refinement

RECAR



Recursive Explore and Check Abstraction Refinement

RECAR

- 2 levels of abstractions
  - One at the Oracle level $(\mathrm{check}(\psi))$
  - One at the Domain level (recursive call)
- Efficient even when 50/50 chance of being SAT/UNSAT
- Everytime check improves, RECAR improves
- The return of the recursive call can reduce the number of refinement
- Totally generic, can change SAT solver $\rightarrow$ FO solver?

RECAR for Modal Logic K

- Modal Logic K is **PSPACE**-complete [Lad77, Hal95]
- What is Modal Logic K?
- How we over-approximate a formula ϕ?
- How we under-approximate a formula ϕ?
- Is it competitive against a CEGAR approach?
- Is it competitive against the state-of-the-art approaches?

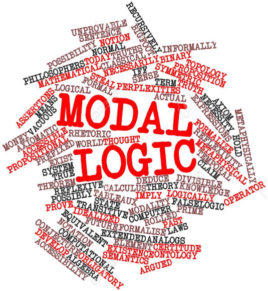
Modal Logic = Propositional Logic + $\Box$ and $\Diamond$

Modal Logic

- $\Box\phi$ means ϕ is necessarily true
- $\Diamond\phi$ means ϕ is possibly true

$$\Diamond\phi \leftrightarrow \neg\Box\neg\phi$$

$$\Box\phi \leftrightarrow \neg\Diamond\neg\phi$$



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Preliminaries: Kripke Structure

- $\mathbb{P}$ finite non-empty set of propositional variables

Kripke Structure [Kri59]

$M = \langle W, R, V \rangle$ with:

- W, a non-empty set of possible worlds
- R, a binary relation on W
- V, a function that associate to each $p \in \mathbb{P}$, the set of possible worlds where p is true

Pointed Kripke Structure: $\langle \mathcal{K}, w \rangle$

- $\mathcal{K}$: Kripke Structure
- w: a possible world in W

Preliminaries: Satisfaction Relation

Definition (Satisfaction Relation)

The relation $\models$ between Kripke Structures and formulae is recursively defined as follows:

$$
\begin{array}{lll}
\langle \mathcal{K}, w \rangle \models p & \text{iff} & w \in V(p) \\
\langle \mathcal{K}, w \rangle \models \neg\phi & \text{iff} & \langle \mathcal{K}, w \rangle \not\models \phi \\
\langle \mathcal{K}, w \rangle \models \phi_1 \wedge \phi_2 & \text{iff} & \langle \mathcal{K}, w \rangle \models \phi_1 \text{ and } \langle \mathcal{K}, w \rangle \models \phi_2 \\
\langle \mathcal{K}, w \rangle \models \phi_1 \vee \phi_2 & \text{iff} & \langle \mathcal{K}, w \rangle \models \phi_1 \text{ or } \langle \mathcal{K}, w \rangle \models \phi_2 \\
\langle \mathcal{K}, w \rangle \models \Box\phi & \text{iff} & (w, w') \in R \text{ implies } \langle \mathcal{K}, w' \rangle \models \phi \\
\langle \mathcal{K}, w \rangle \models \Diamond\phi & \text{iff} & (w, w') \in R \text{ and } \langle \mathcal{K}, w' \rangle \models \phi
\end{array}
$$

$\mathcal{K}$ that satisfied a formula ϕ will be called “Kripke model of ϕ”

Preliminaries: Example of a Kripke Structure

- ✓ $\phi_1 = \Box(\textcolor{cyan}{\bullet})$
- ✗ $\phi_2 = \Box\Diamond(\textcolor{orange}{\bullet})$
- ✓ $\phi_3 = \Diamond(\textcolor{red}{\bullet} \wedge \Diamond\neg\textcolor{green}{\bullet})$
- ✓ $\phi_4 = (\textcolor{cyan}{\bullet} \vee \textcolor{red}{\bullet} \vee \textcolor{orange}{\bullet})$
- ✗ $\phi_5 = \Diamond\Diamond(\textcolor{orange}{\bullet} \wedge \Box\neg\textcolor{orange}{\bullet})$

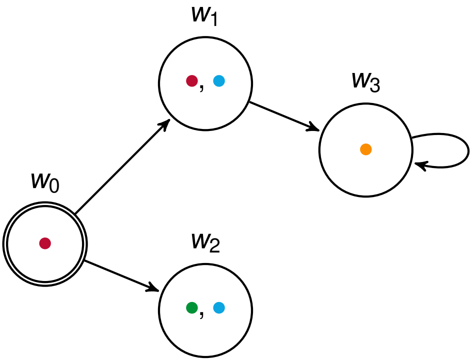

Figure: Example $\mathcal{K}$

MoSaiC

- Open-Source Modal Logic K solver
- Uses Glucose as internal SAT solver
- Uses a RECAR approach

MoSaiC

- Open-Source Modal Logic K solver
- Uses Glucose as internal SAT solver
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RECAR Assumptions: Reminder

✓1 Function ‘check’ is sound, complete and terminates

?2 *isSAT*$(\hat{\phi})$ implies *isSAT*(refine$(\hat{\phi})$)

?3 $\exists.n \in \mathbb{N}$ s.t. refine$^n(\hat{\phi}) \equiv^?_{\text{sat}} \phi$

4 isUNSAT$(\check{\phi})$ implies isUNSAT(ϕ)

5 $\exists n \in \mathbb{N}$ s.t. $RC(under^n(\phi), under^{n+1}(\phi))$ is false

MoSaiC: Over-Approximation

ϕ always in NNF and over(ϕ, i) in CNF thanks to Tseitin

$$
\begin{aligned}
\text{over}(\phi, n) &= \text{over}'(\phi, 0, n)\\
\text{over}'(p_k, i, n) &= p_{k,i}\\
\text{over}'(\neg p_k, i, n) &= \neg p_{k,i}\\
\text{over}'(\Box\phi, i, n) &= \bigwedge_{j=0}^{n} (r_{i,j} \rightarrow \text{over}'(\phi, j, n))\\
\text{over}'(\Diamond\phi, i, n) &= \bigvee_{j=0}^{n} (r_{i,j} \wedge \text{over}'(\phi, j, n))
\end{aligned}
$$

- $p_{k,i}$ means p_k is true in the world w_i
- $r_{i,j}$ means that there is a relation between worlds w_i and w_j

RECAR Assumptions: Reminder

✓1 Function 'check' is sound, complete and terminates

✓2 $isSAT(\hat{\phi})$ implies $isSAT(\text{refine}(\hat{\phi}))$

✓3 $\exists . n \in \mathbb{N}$ s.t. $\text{refine}^n(\hat{\phi}) \equiv^?_{\text{sat}} \phi$

?4 $\text{isUNSAT}(\check{\phi})$ implies $\text{isUNSAT}(\phi)$

?5 $\exists n \in \mathbb{N}$ s.t. $RC(under^n(\phi), under^{n+1}(\phi))$ is false

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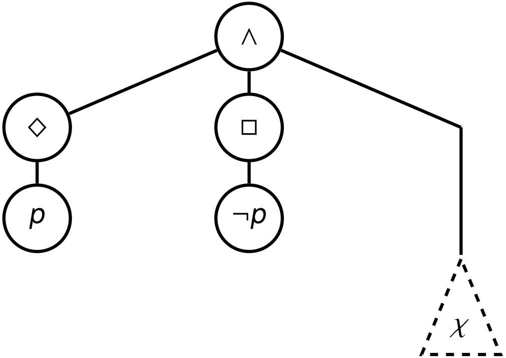
MoSaiC: Under-Approximation

Let's take an example, with χ huge but satisfiable...



Worst case for CEGAR using our 'over' function

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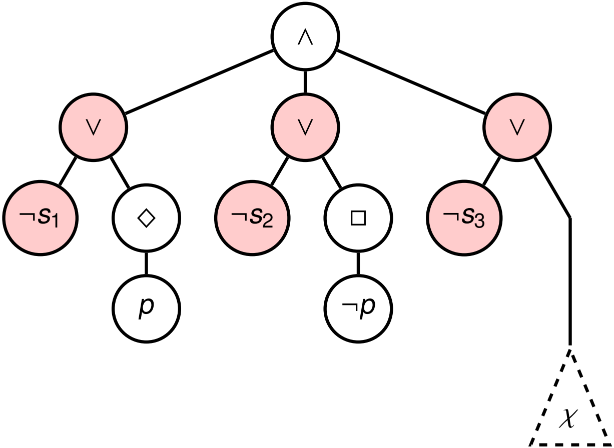
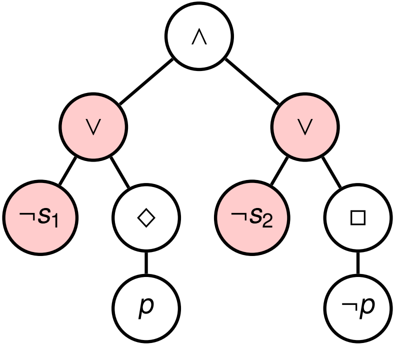
MoSaiC: Under-Approximation



Modern SAT solvers returns ‘the reason’ why a formula with n worlds is unsatisfiable ($core = \{s_1, s_2\}$)

MoSaiC: Under-Approximation

We want to cut what is not part of the 'unsatisfiability' ($s_i \notin core$)



We just create $\check{\phi}$ smaller than ϕ and easier to solve.
The function *RC* from RECAR just says here: did we cut something ?

MoSaiC: Under-Approximation

$$\text{under}(p, \mathit{core}) = p$$

$$\text{under}(\neg p, \mathit{core}) = \neg p$$

$$\text{under}(\Box\phi, \mathit{core}) = \Box(\text{under}(\phi, \mathit{core}))$$

$$\text{under}(\Diamond\phi, \mathit{core}) = \Diamond(\text{under}(\phi, \mathit{core}))$$

$$\text{under}((\phi \wedge \psi), \mathit{core}) = \text{under}(\phi, \mathit{core}) \wedge \text{under}(\psi, \mathit{core})$$

$$\text{under}((\psi \vee \chi), \mathit{core}) = \begin{cases} \text{under}(\chi, \mathit{core}) & \text{if } \psi = \neg s_i, s_i \in \mathit{core} \\ \top & \text{if } \psi = \neg s_i, s_i \notin \mathit{core} \\ (\text{under}(\psi, \mathit{core}) & \\ \vee\, \text{under}(\chi, \mathit{core})) & \text{otherwise} \end{cases}$$

- Unsatisfiable-cores: To create our under-approximations

RECAR Assumptions: Reminder

✓1 Function 'check' is sound, complete and terminates

✓2 $isSAT(\hat{\phi})$ implies $isSAT(\text{refine}(\hat{\phi}))$

✓3 $\exists.n \in \mathbb{N}$ s.t. $\text{refine}^n(\hat{\phi}) \equiv^?_{\text{sat}} \phi$

✓4 $\text{isUNSAT}(\check{\phi})$ implies $\text{isUNSAT}(\phi)$

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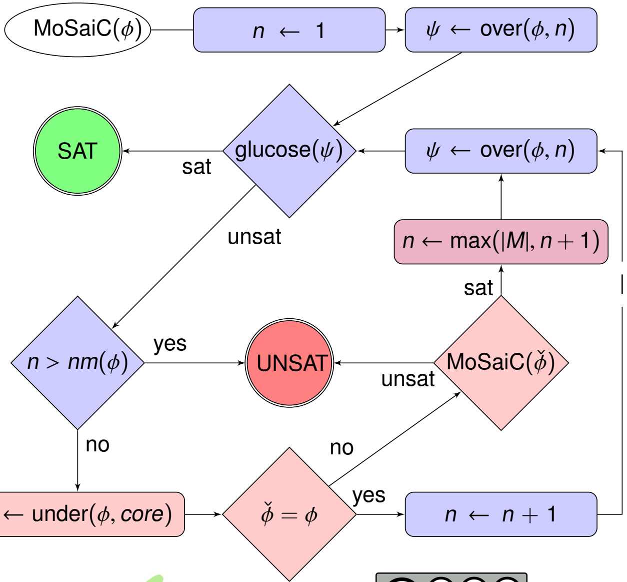
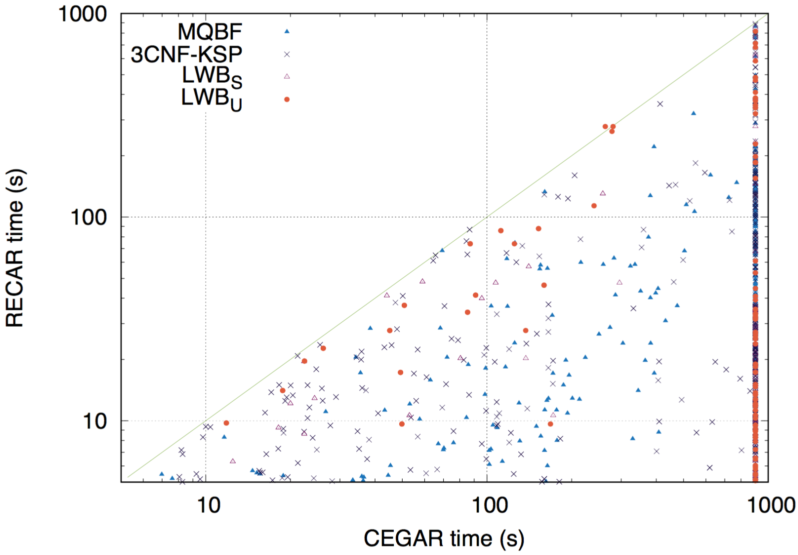
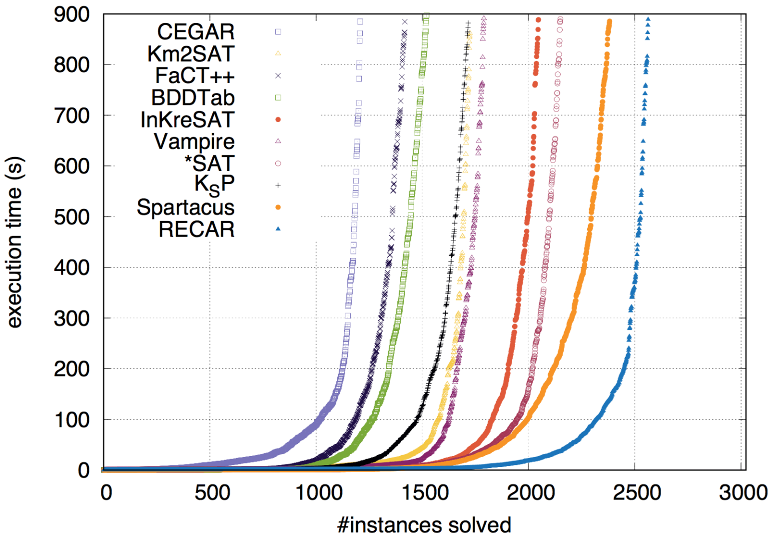
MoSaiC: RECAR for Modal Logic K



MoSaiC: RECAR for Modal Logic K



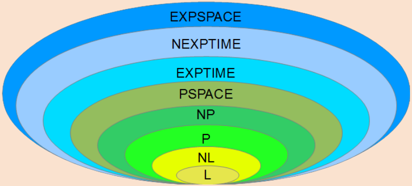
MoSaiC: RECAR for Modal Logic K



Conclusion

Abstractions according to complexity

- **PSPACE**: RECAR
- **NP**: CEGAR (over/under)



What is next ?

- RECAR for QBF (**PSPACE**)?
- RECAR for other modal logic?

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Perspective: Other modal logics

Sum-up of complexities in modal logics

NP
K5
K45
KB45
KD5
KD45
KT5

PSPACE
K
KT
KT4
KB
KD4
KD
K4
KDB
KBT

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