Notes 13.6: Probability and Binomial Expansions

I. Binomial Experiment

A. Definition, Requirements, Formula

- Some probability problems can be solved using binomial expansions. This is called a ‘binomial experiment”.
- A binomial experiment is an experiment which satisfies these four conditions
  - 1. A fixed number of trials
  - 2. Each trial is independent of the others
  - 3. There are only two outcomes
  - 4. The probability of each outcome remains constant from trial to trial.

Examples of binomial experiments

- Tossing a coin 20 times to see how many tails occur.
- Asking 200 people if they watch ABC news.
- Rolling a die to see if a 5 appears.

Examples which are NOT binomial experiments

- Rolling a die until a 6 appears (not a fixed number of trials)
- Asking 20 people how old they are (not two outcomes)
- Drawing 5 cards from a deck for a poker hand (done without replacement, so not independent)

I. Example of Probability and Binomial Expansion

- About 70% of the questions created by a random computer test generator for a science program are multiple choice. What is the probability that at least 3 of 5 questions generated for a short quiz are multiple choice?

We can represent the probability of each number of successes using a binomial expansion. Let S = probability of a question being multiple choice. Let F = probability of a question not being multiple choice. We can use Pascal's triangle and the patterns we learned last unit to expand the binomial.

$(S+F)^5 = S^5 + 5S^4 F + 10\ S^3F^2 + 10S^2F^3 + 5SF^4 + F^5$

Term	Rep the Prob that:
S^5	All 5 are MC
$5S^4F$	4 are MC and 1 is not
$10\ S^3F^2$	3 are MC and 2 are not
$10S^2F^3$	2 are MC and 3 are not
$+5SF^4$	1 is MC and 4 are not
F^5	zero are MC and 5 are not

What is the probability that at least 3 of 5 questions generated for a short quiz are multiple choice?

We need to substitute in the probabilities and add together all the terms that represent 3, 4, or 5 questions being multiple choice.

$(S+F) = S^5 + 5S^4 F + 10 S^3F^2 + 10S^2F^3 + 5SF^4 + F^5$

In this case, S=.7, and F = .3

$P(\text{at least 3 are MC}) = S^5 + 5S^4 F + 10 S^3F^2$

$P(\text{at least 3 are MC}) = (.7)^5 + 5(.7)^4(.3) + 10 (.7)^3(.3)^2$

$P(\text{at least 3 are MC}) = .16807 + .36015 + .3087$

$P(\text{at least 3 are MC}) = .83692$ or $\approx 84\%$

II. Using the Binomial Theorem:

- Remember that we can also use the Binomial Theorem to expand binomials.

$$(x+y)^n = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} x^{n-r} y^r$$

- Part of this should look like something we just learned:

$\frac{n!}{r!(n-r)!}$ is the formula for combinations ${}_nC_r$

- To make this simpler to use, we can rewrite this as

$$(x+y)^n = \sum_{r=0}^{n} {}_nC_r \, x^{n-r} y^r$$

III. Using Only One Term

- For binomial experiments, we are often looking for one term. Remember that when we are only looking for one term, we do not need the summation portion of the formula. We only need:

$$ {}_nC_r \, x^{n-r} \, y^r $$

Where,

x is the probability of success P(s)

y is the probability of failure P(f)

n is the number of trials

r is the number of failures (or think of it as the number subtracted from n to get the correct number of failures).

n-r is the number of successes

Examples

- Ex 1: Eight out of every 10 people who contract a certain viral infection can recover. If a group of 9 people become infected, what is the probability that exactly 5 people will recover?
- There are 9 people involved and there are only two outcomes: recover or don't recover. The events are independent of each other. So this is a binomial experiment. We are looking for the term where 5 people recover (which means 4 people do NOT recover), so x^5y^4. $x = \frac{8}{10}$ or $\frac{4}{5}$ $(probability\ of\ recover), y = \frac{2}{10}$ or $\frac{1}{5}$ (probability of not recovering), n = 9 (9 people, or trials), r = 4 (the number of failures).

$$_{nCr}\, x^{n-r}\, y^{r}$$

$$_9C_4 \left(\frac{4}{5}\right)^5 \left(\frac{1}{5}\right)^4$$

= .066 or 6.6%

- Ex 2: Find each probability if a number cube is tossed five times:

a. P(only one 4)

$${}_5C_4 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4$$

$$\frac{3125}{7776}$$

b. P(at least three 4s)

P(three 4s) + P(four 4s) + P(five 4s)

$${}_5C_2 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + {}_5C_1 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1 + {}_5C_0 \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^0$$

$$\frac{23}{648}$$

c. P(exactly five 4s)

$${}_5C_0 \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^0$$

$$\frac{1}{7776}$$

b. P(no more than two 4s)

P(no 4s) + P(one 4) + P(two 4s)

$${}_5C_5 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5 + {}_5C_4 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4 + {}_5C_3 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$$

$$\frac{625}{648}$$