The University of Georgia
Mathematics Education Program
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The Link Between Binomial Expansion and Probability

Binomials or any other two term quantities with integer exponents happen frequently in mathematics.
The first few powers are as follows:

$(a+b)^0 = 1$
$(a+b)^1 = a+b$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

There are some patterns observed as we increase the integer exponent of $(a+b)^n$.

--First, for the expansion of $(a+b)^n$, there are always n+1 terms.

--Second, the exponent of a in the first term is n and then decreases by 1 in each succeeding term all the way down to 0 in the last term.

--Finally, the exponent of b in the first term is 0 and then increases by 1 in each succeeding term all the way up to n in the last term.

The expansion of $(a+b)^n$ is expressed by the Binomial Theorem:

$$\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

Where $\binom{n}{r}$ is the symbolic representation of the coefficients of each term in the binomial expansion.
The coefficients can be calculated as follows:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The coefficients of the binomial expansion can be written in rows for each integer from 0 to n as follows

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10
1 11 55 165 330 462 462 330 165 55 11
1 12 66 220 495 792 924 792 495 220 66

The coefficients written in this manner form a triangle shape of values and is referred to as Pascal's Triangle.

There are certain patterns observed in Pascal's Triangle:

1.) **The first coefficient and the last coefficient in each row are equal to 1**

Symbolically, this is written as

$$\binom{n}{0} = 1 \text{ and } \binom{n}{n} = 1$$

Proof of $\binom{n}{0} = \binom{n}{n} = 1$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = \frac{n!}{n!} = 1$$

Similarly,

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{n!}{n! \cdot 1} = \frac{n!}{n!} = 1$$

2.) **The coefficients are symmetric about the median of each row**

Symbolically, this is written as

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof of $\binom{n}{r} = \binom{n}{n-r}$

$$\binom{n}{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)! \cdot r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

3.) **Finally, for each row in Pascal's Triangle where n≥2, each coefficient, except for the first and the last, can be obtained by adding the coefficients above it and to the left and right.**

Symbolically, this is written as

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Proof of $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!\cdot(n-r+1)!} + \frac{n!}{r!\cdot(n-r)!}$$

Since Least Common Denominator for the two fractions is ***r!(n-r+1)!*** we can rewrite the fractions as follows:

$$\frac{n!}{(r-1)!\cdot(n-r+1)!} + \frac{n!}{r!\cdot(n-r)!} = \frac{r\cdot n!}{r(r-1)!\cdot(n-r+1)!} + \frac{(n-r+1)\cdot n!}{r!\cdot(n-r)!\cdot(n-r+1)}$$

$$= \frac{r\cdot n!}{r!\cdot(n-r+1)!} + \frac{(n-r+1)\cdot n!}{r!\cdot(n-r+1)!}$$

$$= \frac{(r\cdot n!) + ((n-r+1)\cdot n!)}{r!\cdot(n-r+1)!} = \frac{(r+n-r+1)\cdot n!}{r!\cdot(n-r+1)!} = \frac{(n+1)\cdot n!}{r!\cdot(n-r+1)!}$$

$$= \frac{(n+1)!}{r!\cdot(n+1-r)!} = \binom{n+1}{r}$$

Now earlier we stated that the coefficients in each term can be calculated as follows:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This equation should be familiar as the formula to determine the number of combinations of n items taken r at a time.

So it follows that the coefficients in each term can be calculated using the equation for combinations:

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

But is there a way we can use the binomial theorem to find probability?

Example problem:

A two-sided coin is tossed 6 times with the possibility of obtaining a heads or tails.
What is the probability of getting 4 heads?

The two possibilities can be written as a binomial
$(H+T)^n$

where n is the number of times the coin is tossed.

So if the coin is tossed 6 times, we have the following binomial
$(H+T)^6$
along with its binomial expansion.

$1\,H^6T^0 \quad + \quad 6\,H^5T^1 \quad + \quad 15\,H^4T^2 \quad + \quad 20\,H^3T^3 \quad + \quad 15\,H^2T^4 \quad + \quad 6\,H^1T^5 \quad + \quad 1\,H^0T^6$

The coefficients of the binomial expansion represent the number of possibilities (or combinations) of the number of heads and tails (represented by the exponents on H and T) that follow the coefficient.

Number of ways a coin can be tossed and obtain

6 Heads (and 0 Tails) = 1
5 Heads (and 1 Tail) = 6
4 Heads (and 2 Tails) = 15
3 Heads (and 3 Tails) = 20
2 Heads (and 4 Tails) = 15
1 Head (and 5 Tails) = 6
0 Heads (and 6 Tails) = 1

In general, the sum of the coefficients of a binomial expansion $(a+b)^n = 2^n$
This sum represents the total number of possibilities when tossing a coin 6 times.
In this case the sum of the coefficients = $(1 + 6 + 15 + 20 + 15 + 6 + 1) = 2^6 = 64$

So, the probability of getting 4 heads is

$$\frac{\textit{Favorable outcomes}}{\textit{Possible outcomes}} = \frac{15}{64}$$

Summary

Binomial expansion using the Binomial Theorem and Pascal's Triangle as well as finding probability using combinations and permutations are topics taught in 9th grade GPS Math I. This connection between the two concepts should be taught to reinforce understanding of both. At minimum, it would serve as an excellent enrichment or differentiation for gifted and higher performing students in GPS Math I.