Worksheet 7 – Solution

Part 1: Time Response of First-Order Systems

1) For each of the following models, obtain the free response (zero-input response) and the time constant, if any.

a) $16\dot{x} + 14x = 0, \qquad x(0) = 6$

Take Laplace transform and solve the equation for $x(t)$

$$16\big(sX(s) - x(0)\big) + 14X(s) = 0 \quad \rightarrow \quad X(s) = \frac{16}{16s+14}x(0) = \frac{96}{16s+14} = \frac{\frac{96}{14}}{\frac{16}{14}s+1} = \frac{\frac{48}{7}}{\frac{8}{7}s+1}$$

$$Free\ Response \ \rightarrow \ x(t) = \frac{48}{7}e^{-\frac{7t}{8}} \quad \rightarrow \quad Time-constant = \frac{8}{7}$$

b) $13\dot{x} - 6x = 18u_s, \quad x(0) = -2$

Take Laplace transform and solve the equation for $x(t)$

$$13\big(sX(s) - x(0)\big) - 6X(s) = 18U_s(s) \quad \rightarrow \quad X(s) = \frac{18}{13s-6}U_s(s) + \frac{13}{13s-6}x(0)$$

$$Free\ Response \ \rightarrow X(s) = \frac{13}{13s-6}\,x(0) \quad \rightarrow \quad x(t) = \frac{-26}{13s-6} = \frac{-26/6}{\frac{13}{6}s-1} \quad \rightarrow \quad x(t) = -\frac{13}{3}e^{\frac{6t}{13}}$$

The system is unstable. No time-constant is defined.

c) $7\dot{x} - 5x = 0, \qquad x(0) = 9$

Take Laplace transform and solve the equation for $x(t)$

$$7\big(sX(s) - x(0)\big) - 5X(s) = 0 \quad \rightarrow \quad X(s) = \frac{7}{7s-5}x(0) = \frac{63}{7s-5} = \frac{\frac{63}{5}}{\frac{7}{5}s-1}$$

$$Free\ Response \ \rightarrow \ x(t) = \frac{63}{5}e^{\frac{5t}{7}}$$

The system is unstable. No time-constant is defined.

2) For the model $2\dot{x} + x = 10f(t)$, If $x(0) = 0$ and $f(t)$ is a unit-step, what is the steady-state response x_{ss}? How long does it take before 98% of the difference between $x(0)$ and x_{ss} is eliminated?

Take Laplace transform and solve the equation for $X(s)$

$$2\big(sX(s) - x(0)\big) + X(s) = 10F(s) \quad \rightarrow \quad X(s) = \frac{10}{2s+1}F(s) = \left(\frac{10}{2s+1}\right)\left(\frac{1}{s}\right) = \frac{10}{s(2s+1)}$$

The steady-state value is obtained from the final-value theorem:

$$x_{ss} = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\,\frac{10}{s(2s+1)} = 10$$

We have to find the settling time with 2% criteria:

$$t_s = 4\tau = 4(2) = 8\ sec$$

3) Obtain the response of the model $2\dot{v} + v = f(t)$, where $f(t) = 5t$ and $v(0) = 0$. Identify the transient and steady-state responses.

Take Laplace transform having $v(0) = 0, F(s) = \frac{5}{s^2}$, and then solve the equation for $v(t)$

$$2\big(sV(s) - v(0)\big) + V(s) = F(s) \quad \rightarrow \quad V(s) = \frac{1}{2s+1}F(s) = \left(\frac{1}{2s+1}\right)\left(\frac{5}{s^2}\right) = \frac{5}{s^2(2s+1)}$$

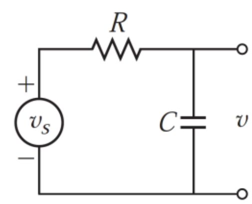
Applying partial fraction expansion and inverse Laplace transform:

$$V(s) = \frac{5}{2s^2(s+0.5)} = \frac{-10}{s} + \frac{5}{s^2} + \frac{10}{s+0.5} \quad \rightarrow \quad v(t) = -10 + 5t + 10e^{-0.5t}, \qquad t \geq 0$$

Transient response $\rightarrow$ $v_t(t) = 10e^{-0.5t}$

Steady-state response $\rightarrow$ $v_{ss}(t) = 5t - 10$

4) The RC circuit shown below has the parameter values $R = 3 \times 10^6\ \Omega$ and $C = 1\ \mu F$. If the initial capacitor voltage is 6V and the applied voltage is $v_s(t) = 12u_s(t)$, obtain the expression for the capacitor voltage response $v(t)$.



Apply KVL for the circuit:

$$v_s(t) = v_R(t) + v_c(t) \quad \rightarrow \quad v_s(t) = Ri(t) + v_c(t) \quad \rightarrow \quad v_s(t) = RC\frac{dv_c(t)}{dt} + v_c(t)$$

Take Laplace transform having $v_c(0) = 6$ and $V_s(s) = \frac{12}{s}$:

$$V_s(s) = RC\big(sV_c(s) - v_c(0)\big) + V_c(s) \quad \rightarrow \quad V_c(s) = \frac{1}{RCs+1}V_s(s) + \frac{RC}{RCs+1}v_c(0)$$

$$V_c(s) = \left(\frac{1}{3s+1}\right)\left(\frac{12}{s}\right) + \left(\frac{3}{3s+1}\right)(6) = \frac{12}{s(3s+1)} + \frac{18}{3s+1}$$

Take inverse Laplace transform:

$$v_c(t) = 12\left(1 - e^{-\frac{t}{3}}\right) + 18e^{-\frac{t}{3}}, \qquad t \geq 0$$

5) Consider the model: $6\dot{v} + 3v = \dot{g}(t) + g(t)$

where $v(0) = 0$. Obtain the response $v(t)$ if $g(t) = u_s(t)$.

Take Laplace transform having $v(0) = 0, g(0) = 0$ and $G(s) = \frac{1}{s}$:

$$6\big(sV(s) - v(0)\big) + 3V(s) = sG(s) - g(0) + G(s) \rightarrow \quad 6sV(s) + 3V(s) = s\left(\frac{1}{s}\right) + \frac{1}{s}$$

$$V(s) = \frac{s+1}{s(6s+3)} = \frac{s+1}{6s(s+0.5)} = \frac{1/3}{s} + \frac{-1/6}{s+0.5}$$

Take inverse Laplace transform:

$$v(t) = \frac{1}{3} - \frac{1}{6}e^{-0.5t}, \qquad t \geq 0$$

6) Obtain the response of the model $9\dot{v} + 3v = f(t)$, where $f(t) = 7t$ and $v(0) = 0$. Is the steady-state response parallel to $f(t)$?

Take Laplace transform having $v(0) = 0, g(0) = 0$ and $F(s) = \frac{7}{s^2}$:

$$9\big(sV(s) - v(0)\big) + 3V(s) = F(s) \quad \rightarrow \quad 9sV(s) + 3V(s) = \frac{7}{s^2}$$

$$V(s) = \frac{7}{s^2(9s+3)} = \frac{7}{9s^2(s+1/3)} = \frac{7}{s} + \frac{7/3}{s^2} + \frac{7}{s+1/3}$$

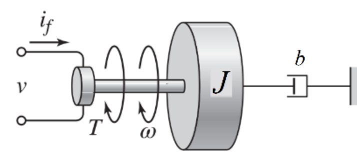
Take inverse Laplace transform:

$$v(t) = 7 + \frac{7}{3}t + 7e^{-t/3}, \qquad t \geq 0$$

Steady-state response $\rightarrow$ $v_{ss}(t) = 7 + \frac{7}{3}t$ which is not parallel to the input $7t$ because the slope is not same.

7) A certain rotational system has inertia $J = 20\ kg.m^2$ and a viscous damping constant $b = 10\ N.m.s/rad$. The torque $T(t)$ is applied by an electric motor. The equation of motion is:

$$50\frac{d\omega}{dt} + 10\omega = \tau(t)$$



The model of the motor's field current i_f in amperes is:

$$0.001\frac{di_f}{dt} + 5i_f = v(t)$$

where $v(t)$ is the voltage applied to the motor. The motor torque constant is $K_T = 25\ N.m/A$. Suppose the applied voltage is 10V. Determine the steady-state speed of the inertia and estimate the time required to reach that speed.

Transfer function of the mechanical subsystem is:

$$50\frac{d\omega}{dt} + 10\omega = \tau(t) \quad \rightarrow \quad 50s\Omega(s) + 10\Omega(s) = T(s) \quad \rightarrow \quad \frac{\Omega(s)}{T(s)} = \frac{1}{50s+10}$$

Transfer function of the electrical subsystem is:

$$0.001\frac{di_f}{dt} + 5i_f = v(t) \quad \rightarrow \quad 0.001sI_f(s) + 5I_f(s) = V(s) \quad \rightarrow \quad \frac{I_f(s)}{V(s)} = \frac{1}{0.001s+5}$$

The overall block diagram model and the overall transfer function model of the system is,

$$\frac{\Omega(s)}{V(s)} = \frac{25}{(0.001s+5)(50s+10)}$$

The steady-state speed is obtained using the final-value theorem:

$$\lim_{t\to\infty} \omega(t) = \lim_{s\to 0} s\Omega(s) \quad \rightarrow \quad \omega_{ss} = \omega(\infty) = \lim_{s\to 0} s\left(\frac{25}{(0.001s+5)(50s+10)}\right)\left(\frac{10}{s}\right) = 5\ rad/s$$

Time constant of the electrical subsystem is 0.001 sec.

Time constant of the mechanical subsystem is 5 sec.

Since the electrical subsystem is much **faster** than the mechanical subsystem, the overall time constant can be estimated as the time constant of the mechanical susystem.

Therefore, the estimated time to reach the final speed is about $4\tau = 4(5) = 20\ sec$

Part 2: Time Response of Second-Order Systems

1) If applicable, compute ζ, τ, ω_n and ω_d for the following roots, and find the corresponding characteristic pollynmial.

Having the general form of the second-order system characteristic equation and the pole locations:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad s = -\sigma \pm j\omega_d = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$$

a) $s = -2 \pm j6$

The characteristic polynomial is: $(s + 2 + j6)(s + 2 - j6) = s^2 + 4s + 40$

The system is stable with complex poles.

$\omega_n = \sqrt{40}, \qquad \zeta = \dfrac{2}{\sqrt{40}} = 0.316, \quad \omega_d = 6$

b) $s = -10,\ -10,$

The characteristic polynomial is: $(s + 10)(s + 10) = s^2 + 20s + 100$

The system is stable with repetitive real poles.

$\zeta = 1$

ω_n and ω_d are not defined because the free response is not oscillatory.

c) $s = -10,$

The characteristic polynomial is: $s + 10$

The system is first order, so ζ, ω_n and ω_d are not defined because the free response is not oscillatory.

Time constant is $\tau = 1/10$

d) $s = 1 \pm j5$

The characteristic polynomial is: $(s - 1 + j5)(s - 1 - j5) = s^2 - 2s + 26$

The system is unstable, so damping ratio ζ and time-constant τ are not defined.

$\omega_d = 5, \quad \omega_n = \sqrt{26}$

2) Find the response for the following models. The initial conditions are zero.

a) $3\ddot{x} + 21\dot{x} + 30x = 4t$

Take Laplace transform having all initial conditions are zero:

$$3s^2X(s) + 21sX(s) + 30X(s) = \frac{4}{s^2} \quad \rightarrow \quad X(s) = \frac{4}{3s^2(s^2 + 7s + 10)} = \frac{4}{3}\left(\frac{0.1}{s^2} - \frac{0.07}{s} + \frac{0.0833}{s+2} - \frac{0.0133}{s+5}\right)$$

Take inverse Laplace transform:

$$x(t) = \frac{4}{3}(0.1t - 0.07 + 0.0833e^{-2t} - 0.0133e^{-5t}), \qquad t \geq 0$$

b) $5\ddot{x} + 20\dot{x} + 20x = 7t$

Take Laplace transform having all initial conditions are zero:

$$5s^2X(s) + 20sX(s) + 20X(s) = \frac{7}{s^2} \quad \rightarrow \quad X(s) = \frac{7}{5s^2(s^2 + 4s + 4)} = \frac{7}{5}\left(\frac{0.25}{s^2} - \frac{0.25}{2} + \frac{0.25}{(s+2)^2} + \frac{0.25}{s+2}\right)$$

Take inverse Laplace transform:

$$x(t) = \frac{7}{5}(0.25t - 0.25 + 0.25te^{-2t} + 0.25e^{-2t}), \qquad t \geq 0$$

c) $2\ddot{x} + 8\dot{x} + 58x = 5t$

Take Laplace transform having all initial conditions are zero:

$$2s^2X(s) + 8sX(s) + 58X(s) = \frac{5}{s^2} \quad \rightarrow \quad X(s) = \frac{5}{2s^2(s^2 + 4s + 29)} = \frac{5}{2s^2((s+2)^2 + 25)}$$

$$X(s) = \frac{5}{2}\left(\frac{\frac{1}{29}}{s^2} - \frac{\frac{4}{29^2}}{s} - \frac{\frac{21}{29^2}}{(s+2)^2 + 25} + \frac{\frac{4}{29^2}(s+2)}{(s+2)^2 + 25}\right)$$

Take inverse Laplace transform:

$$x(t) = \frac{5}{2(29)^2}\left(29t - 4 + \frac{21}{5}e^{-2t}\sin 5t + 4e^{-2t}\cos 5t\right), \qquad t \geq 0$$

3) The characteristic equation of a certain system is $4s^2 + 6ds + 25d^2 = 0$, where d is a constant.

a) For what values of d is the system stable?

Having the general form of the second-order system characteristic equation and the pole locations:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad s = -\sigma \pm j\omega_d = -\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}$$

The system is stable if $\zeta\omega_n > 0$ and $\omega_n^2 > 0$.

The characteristic equation in standard form is: $s^2 + \frac{6}{4}ds + \frac{25}{4}d^2 = 0$

So, to have a stable system we must have $\frac{3}{4}d > 0$ and $\frac{25}{4}d^2 > 0$ or simply $d > 0$.

b) Is there a value of d for which the free response will consist of decaying oscillations?

The damping ratio is obtained as:

$$2\zeta\omega_n = \frac{6}{4}d \quad \rightarrow \quad \zeta = \frac{\frac{6}{4}d}{2\omega_n} = \frac{\frac{6}{4}d}{2\left(\frac{5}{2}d\right)} = \frac{6}{20} < 1$$

Thus there will be damped oscillations for any $d > 0$.

4) The characteristic equation of a certain system is $s^2 + 6bs + 5b - 10 = 0$, where b is a constant.

a) For what values of b is the system stable?

Having the general form of the second-order system characteristic equation and the pole locations:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad s = -\sigma \pm j\omega_d = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$$

The system is stable if $\zeta\omega_n > 0$ and $\omega_n^2 > 0$.

The characteristic equation in standard form is: $s^2 + 6bs + (5b - 10) = 0$

So, to have a stable system we must have $3b > 0$ and $5b - 10 > 0$ or simply $b > 2$.

b) Is there a value of b for which the free response will consist of decaying oscillations?

The damping ratio is obtained as:

$$2\zeta\omega_n = 6b \quad \rightarrow \quad \zeta = \frac{6b}{2(5b-10)} < 1 \quad \text{for all } \; b > 5$$

Thus there will be damped oscillations for $b > 5$.

5) Given the model: $\ddot{x} - (\mu + 2)\dot{x} + (2\mu + 5)x = 0$

a) Find the values of the parameter μ for which the system is: Stable, Marginally Stable and Unstable.

Having the general form of the second-order system characteristic equation and the pole locations:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad s = -\sigma \pm j\omega_d = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$$

The system is stable if $\zeta\omega_n > 0$ and $\omega_n^2 > 0$.

The characteristic equation in standard form is: $s^2 - (\mu + 2)s + (2\mu + 5) = 0$

To have a stable system we must have $\frac{-(\mu+2)}{2} > 0$ and $2\mu + 5 > 0$ or simply $-2.5 < \mu < -2$.

To have a marginally stable system we must have $-(\mu + 2) = 0$ or simply $\mu = -2$.

To have an unstable system we must have $\mu > -2$ and $\mu < -2.5$.

6) A certain armature-controlled DC motor has the characteristic equation

$$L_a J s^2 + (R_a J + bL_a)s + bR_a + K_b K_T = 0$$

Using the following parameter values:

$$K_b = K_T = 0.1\ N.m/A, \quad J = 6 \times 10^{-5}\ kg.m^2, \quad R_a = 0.6\ \Omega, \quad L_a = 4 \times 10^{-3}\ H$$

a) Obtain the expression for the damping ratio ζ and undamped natural frequency ω_n in terms of the damping b.

The characteristic equation becomes:

$$(24 \times 10^{-8})s^2 + (3.6 \times 10^{-5} + 4b \times 10^{-3})s + 0.6b + 0.01 = 0$$

Comparing with the standard form of $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$,

$$s^2 + \frac{(3.6 \times 10^{-5} + 4b \times 10^{-3})}{24 \times 10^{-8}} s + \frac{0.6b + 0.01}{24 \times 10^{-8}} = 0$$

The undamped natural frequency is:

$$\omega_n = \sqrt{\frac{0.6b + 0.01}{24 \times 10^{-8}}} = 5000\sqrt{\frac{0.6b + 0.01}{6}}$$

The damping ratio is:

$$2\zeta\omega_n = \frac{(3.6 \times 10^{-5} + 4b \times 10^{-3})}{24 \times 10^{-8}} \quad \rightarrow \quad \zeta = \frac{3.6 \times 10^{-5} + 4b \times 10^{-3}}{2(24 \times 10^{-8})5000\sqrt{\frac{0.6b + 0.01}{6}}} = \frac{0.09 + 10b}{\sqrt{6(0.6b + 0.01)}}$$

b) Find the required damping b to have a critically-damped response.

$$\zeta = 1 \quad \rightarrow \quad 1 = \frac{0.09 + 10b}{\sqrt{6(0.6b + 0.01)}} \quad \rightarrow \quad \sqrt{6(0.6b + 0.01)} = 0.09 + 10b$$

$$6(0.6b + 0.01) = 0.0081 + 100b^2 + 1.8b \qquad \rightarrow \qquad 100b^2 - 1.8b - 0.0519 = 0$$

Solving the quadratic equation we have: $b = 0.0335$ and $b = -0.0155$

Since the damping value should be a positive number, the required damping is: $b = 3.35 \times 10^{-2}\ N.s/m$

Part 3: Description and Specification of Step Response

1) Compute the maximum percent overshoot, maximum overshoot, peak time, rise time and settling time (2%) for the following model. The initial conditions are zero. Time is measured in second.

$$\ddot{y} + 4\dot{y} + 8y = 2u_s(t)$$

The system transfer function is obtained as:

$$s^2Y(s) + 4sY(s) + 8Y(s) = 2U(s) \quad \rightarrow \quad \frac{Y(s)}{U(s)} = \frac{2}{s^2 + 4s + 8}$$

The characteristic equation is: $s^2 + 4s + 8 = 0$

The standard form of the second-order system characteristic equation:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The undamped natural frequency is:

$\omega_n^2 = 8 \quad \rightarrow \quad \omega_n = \sqrt{8}$

The damping ratio is:

$2\zeta\omega_n = 4 \quad \rightarrow \quad \zeta = \dfrac{4}{2\sqrt{8}} = \dfrac{1}{\sqrt{2}} = 0.707$

The maximum overshoot and its percent are:

$O.S.\% = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\% = e^{-0.707\pi/\sqrt{1-0.707^2}} \times 100\% = 0.043 \times 100\% \quad \rightarrow \quad O.S.\% = 4.3\%$

The peak-time is:

$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{(\sqrt{8})\sqrt{1-0.707^2}} = 1.57\ sec$

The rise-time is:

$t_r \cong \dfrac{0.8 + 2.5\zeta}{\omega_n} = \dfrac{0.8 + 2.5(0.707)}{\sqrt{8}} = 0.91\ sec$

The 2% settling-time is:

$t_s \approx \dfrac{4}{\zeta\omega_n} = \dfrac{4}{0.707\sqrt{8}} = 2.00\ sec$

2) A certain system is described by the model

$$\ddot{x} + b\dot{x} + 4x = u_s(t)$$

Set the value of the damping constant b so that both of the following specifications are satisfied. Give priority to the overshoot specifications. If both cannot be satisfied, state the reason. Time is measured in seconds.

a) Maximum percent overshoot should be as small as possible and no greater than 20%.

b) Rise time should be as small as possible and no greater than 3s.

The system transfer function is obtained as:

$$s^2X(s) + bsX(s) + 4X(s) = U(s) \quad \rightarrow \quad \frac{X(s)}{U(s)} = \frac{1}{s^2 + bs + 4}$$

The characteristic equation is: $s^2 + bs + 4 = 0$

The standard form of the second-order system characteristic equation is:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The undamped natural frequency is:

$\omega_n^2 = 4 \quad \rightarrow \quad \omega_n = 2$

The damping ratio is:

$2\zeta\omega_n = b \quad \rightarrow \quad \zeta = \dfrac{b}{4}$

To have a maximum overshoot less than 20%, the damping ratio must be greater than 0.46:

$$\zeta = \frac{-\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}} = \frac{-\ln(0.2)}{\sqrt{\pi^2 + \ln^2(0.2)}} \quad \rightarrow \quad \zeta = 0.46$$

To have a rise-time less than 3sec, the damping ratio must be less than 2.08:

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n} = 3 \quad \rightarrow \quad \zeta = \frac{3\omega_n - 0.8}{2.5} = \frac{3(2) - 0.8}{2.5} = 2.08$$

Therefore, the damping ratio must be $0.46 \le \zeta \le 2.08$.

The required damping constant range is obtained as:

$0.46 \le \dfrac{b}{4} \le 2.08 \quad \rightarrow \quad 1.84 \le b \le 8.32$

3) A certain system is described by the model

$$9\ddot{x} + b\dot{x} + 4x = u_s(t)$$

Set the value of the damping constant b so that both of the following specifications are satisfied. Give priority to the overshoot specifications. If both cannot be satisfied, state the reason. Time is measured in seconds.

a) Maximum percent overshoot should be as small as possible and no greater than 20%.

b) Rise time should be as small as possible and no greater than 3s.

The system transfer function is obtained as:

$$9s^2X(s) + bsX(s) + 4X(s) = U(s) \quad \rightarrow \quad \frac{X(s)}{U(s)} = \frac{1}{9s^2 + bs + 4} = \frac{\frac{1}{9}}{s^2 + \frac{b}{9}s + \frac{4}{9}}$$

The characteristic equation is: $s^2 + \frac{b}{9}s + \frac{4}{9} = 0$

The standard form of the second-order system characteristic equation:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The undamped natural frequency is:

$\omega_n^2 = \dfrac{4}{9} \quad \rightarrow \quad \omega_n = \dfrac{2}{3}$

The damping ratio is:

$$2\zeta\omega_n = \frac{b}{9} \quad \rightarrow \quad \zeta = \frac{b}{18\omega_n} = \frac{b}{18\left(\frac{2}{3}\right)} = \frac{b}{12}$$

To have a maximum overshoot less than 20%, the damping ratio must be greater than 0.46:

$$\zeta = \frac{-\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}} = \frac{-\ln(0.2)}{\sqrt{\pi^2 + \ln^2(0.2)}} \quad \rightarrow \quad \zeta = 0.46$$

To have a rise-time less than 3sec, the damping ratio must be less than 0.48:

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n} = 3 \quad \rightarrow \quad \zeta = \frac{3\omega_n - 0.8}{2.5} = \frac{3\left(\frac{2}{3}\right) - 0.8}{2.5} = 0.48$$

Therefore, the damping ratio must be $0.46 \leq \zeta \leq 0.48$.

The required damping constant range is obtained as:

$$0.46 \leq \frac{b}{12} \leq 0.48 \quad \rightarrow \quad 5.52 \leq b \leq 5.76$$

4) Derive the fact that the peak time is the same for all characteristic roots having the same imaginary part.

The peak time is calculated by the following formula:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

The general form of the characteristic roots for a second-order under-damped system is:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{\zeta^2 - 1}$$

Comparing the peak time formula and the roots we can see that the peak time depends only on the imaginary part of the characteristic roots.

5) Derive the fact that the settling time is the same for all characteristic roots having the same real part.

The settling time (2%) is calculated by the following formula:

$$t_s = \frac{4}{\zeta\omega_n}$$

The general form of the characteristic roots for a second-order under-damped system is:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{\zeta^2 - 1}$$

Comparing the settling time formula and the roots we can see that the settling time depends only on the real part of the characteristic roots.