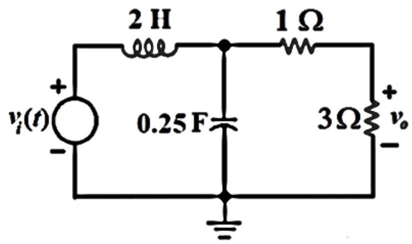
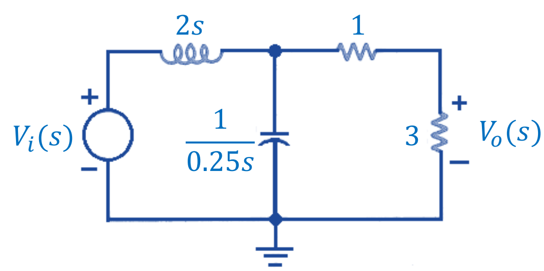
MENG 3020 – Midterm Exam Solution – Fall 2024

Question 1. Consider the following RLC network. The input is the applied voltage $v_i(t)$, and the output is the voltage across the 3Ω resistor v_o.



a) Redraw the RLC network in Laplace domain by applying the **complex impedance** method. Justify your answer.



Replace the passive element values with their impedances.

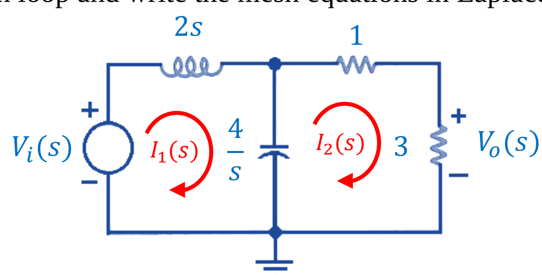
Replace all sources and time variables with their Laplace transform.

Impedance of inductance $\rightarrow \quad Ls = 2s$

Impedance of capacitor $\rightarrow \frac{1}{Cs} = \frac{1}{0.25s} = \frac{4}{s}$

Imdedance of resistances same of their resistance values.

b) Apply the **mesh analysis** for each loop and write the mesh equations in Laplace domain. Show your steps.



The mesh equations are obtained as follows:

Mesh 1 $\rightarrow \quad 2sI_1(s) + \frac{4}{s}\big(I_1(s) - I_2(s)\big) - V_i(s) = 0 \quad \rightarrow \quad \left(2s + \frac{4}{s}\right)I_1(s) - \frac{4}{s}I_2(s) = V_i(s) \qquad Eqn. (1)$

Mesh 2 $\rightarrow \quad 1I_2(s) + 3I_2(s) + \frac{4}{s}\big(I_2(s) - I_1(s)\big) = 0 \quad \rightarrow \quad -\frac{4}{s}I_1(s) + \left(4 + \frac{4}{s}\right)I_2(s) = 0 \qquad Eqn. (2)$

c) Simplify the obtained mesh equations in Step (b) to find the transfer function $V_o(s)/V_i(s)$. Show your steps.

Find $I_1(s)$ from Eqn. (2) and substitute in Eqn. (1):

From Eqn. (2) $\rightarrow \quad \frac{4}{s}I_1(s) = \left(4 + \frac{4}{s}\right)I_2(s) \quad \rightarrow \quad I_1(s) = (s+1)I_2(s)$

Replace $I_1(s)$ in Eqn. (1) $\rightarrow \quad \left(2s + \frac{4}{s}\right)(s+1)I_2(s) - \frac{4}{s}I_2(s) = V_i(s) \quad \rightarrow \quad \big((2s^2+4)(s+1) - 4\big)I_2(s) = sV_i(s)$

$\rightarrow \quad (2s^3 + 2s^2 + 4s)I_2(s) = sV_i(s)$

Replacing the $I_2(s) = \frac{V_o(s)}{3}$ in the above equation $\rightarrow \quad (2s^3 + 2s^2 + 4s)\frac{V_o(s)}{3} = sV_i(s)$

The transfer function is: $\rightarrow \quad \frac{V_o(s)}{V_i(s)} = \frac{3}{2s^2+2s+4}$

Question 2. Consider the following transfer function model of a dynamic system,

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \frac{\mathbf{2s+3}}{\boldsymbol{s^2+5s+6}}$$

a) Determine the <u>characteristic equation</u>, the <u>system order</u>, and the <u>poles</u> and <u>zeros</u> of the transfer function. Show your work.

Characteristic Equation $\rightarrow \quad s^2 + 5s + 6 = 0$

System Order $\rightarrow \quad 2nd - order$

Poles $\rightarrow \quad s^2 + 5s + 6 = (s+3)(s+2) = 0 \quad \rightarrow \quad s_1 = -3, \quad s_2 = -2$

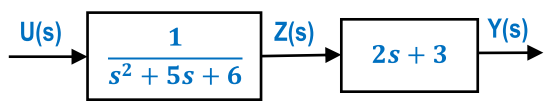
Zeroes $\rightarrow \quad 2s + 3 = 0 \quad \rightarrow \quad s = -\frac{3}{2} = -1.5$

b) Determine if the transfer function is <u>proper</u> or <u>strictly proper</u>. Justify your answer.

Since order of the numerator is less than the order of the denominator, the system is **strictly proper**.

c) Obtain a <u>state-space representation</u> of the system by selecting the state variables as the <u>phase variables</u>. Find the state equations and the output equation. Show the results in matrix-vector form. Show all your steps.

$$\frac{Y(s)}{U(s)} = \frac{Z(s)}{U(s)} \times \frac{Y(s)}{Z(s)} = \frac{1}{s^2+5s+6} \times (2s+3)$$



First, find the state equations from the part with denominator.

$$s^2Z(s) + 5sZ(s) + 6Z(s) = U(s) \quad \rightarrow \quad \ddot{z}(t) + 5\dot{z}(t) + 6z(t) = u(t)$$

Define the state variables as the phase variables:

$$q_1(t) = z(t) \quad \rightarrow \quad \dot{q}_1(t) = \dot{z}(t) \quad \rightarrow \quad \dot{q}_1(t) = q_2(t)$$

$$q_2(t) = \dot{z}(t) \quad \rightarrow \quad \dot{q}_2(t) = \ddot{z}(t) \quad \rightarrow \quad \dot{q}_2(t) = -5\dot{z}(t) - 6z(t) + u(t) \quad \rightarrow \quad \dot{q}_2(t) = -5q_2(t) - 6q_1(t) + u(t)$$

Find the output equation by considering the effect of the numerator term:

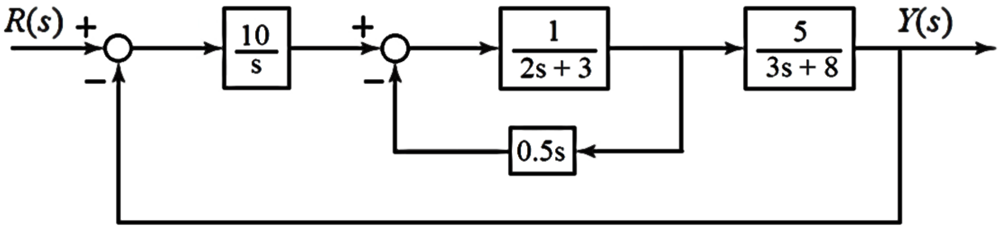
$$Y(s) = (2s+3)Z(s) \quad \rightarrow \quad Y(s) = 2sZ(s) + 3Z(s) \quad \rightarrow \quad y(t) = 2\dot{z}(t) + 3z(t) \quad \rightarrow \quad y(t) = 2q_2(t) + 3q_1(t)$$

Therefore, the state equations and the output equation are obtained as follows:

$$\dot{\mathbf{q}}(t) = \mathbf{Aq}(t) + \mathbf{Bu}(t) \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

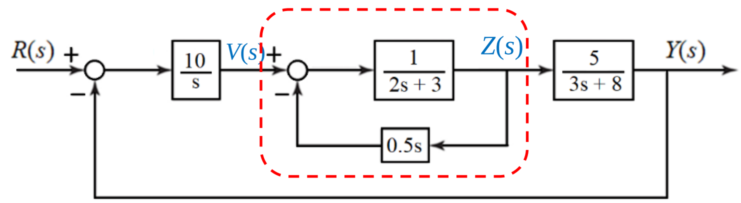
$$\boldsymbol{y}(t) = \mathbf{Cq}(t) + \mathbf{Du}(t) \quad \rightarrow \quad y(t) = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0]u(t)$$

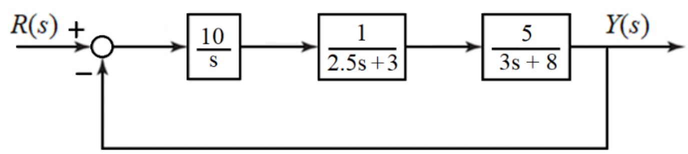
Question 3. Simplify the following block diagram and find the overall transfer function $\frac{Y(s)}{R(s)}$ in the simplest standard form. Show your work and the steps.



First simplify the internal feedback loop and replace it with Z(s)/V(s)

$$\frac{Z(s)}{V(s)} = \frac{\frac{1}{2s+3}}{1+\frac{0.5s}{2s+3}} = \frac{1}{2.5s+3}$$



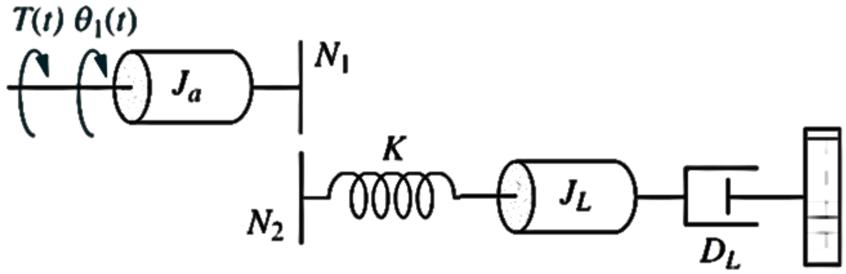


Then find the overall transfer function

$$\frac{Y(s)}{R(s)} = \frac{\frac{50}{s(2.5s+3)(3s+8)}}{1+\frac{50}{s(2.5s+3)(3s+8)}} = \frac{50}{s(2.5s+3)(3s+8)+50}$$

$$\frac{Y(s)}{R(s)} = \frac{50}{7.5s^3+29s^2+24s+50}$$

Question 4. Consider the following rotational system with the gear ratio of $\frac{N_2}{N_1} = \frac{5}{1}$.



a) Find the reflected value of the inertia J_a, torque T_1, and angular displacement θ_1, from side 1 to side 2 by considering the effect of the gear ratio. Show your work.

The reflected inertia from side 1 to side 2 is:

$$J_a \left(\frac{N_2}{N_1}\right)^2 = 25J_a$$

The reflected torque from side 1 to side 2 is:

$$T\left(\frac{N_2}{N_1}\right) = 5T$$

The reflected angular displacement from side 1 to side 2 is:

$$\theta_1\left(\frac{N_1}{N_2}\right) = \frac{\theta_1}{5}$$

b) Draw the equivalent system without gears based on the reflected values. Show the values in the system.