Module 3.4
L'Hopital's Rule and Indeterminate Forms

L'Hopital's Rule – simple, straightforward tool for evaluating special types of limits, the indeterminate forms.

$$\frac{0}{0}\ ;\ \frac{\infty}{\infty}\ \ ;\ \ 0\cdot\infty\ \ ;\ \ \infty-\infty;\ 0^{0}\ ;\ \infty^{0};\ 1^{\infty}$$

[Textbook CPL - 1 Link]

Motivation:

Example 1.

$$\lim_{x \to 2} \frac{x^3 - 8}{x^4 + 2x - 20} =$$

Example 2.

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} =$$

Direct substitution results in indeterminate form $\frac{0}{0}$.

L'Hospital or L'Hopital's Rule and Indeterminate Forms Rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval l that contains a (except possibly at a)

Suppose that:

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

Or that

$$\lim_{x \to} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \infty$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

If the limit on the right side exists (or is ∞ or $-\infty$)

(back to) **Example 1**.

$$\lim_{x \to 2} \frac{x^3 - 8}{x^4 + 2x - 20} =$$

Example 3: Find the limit

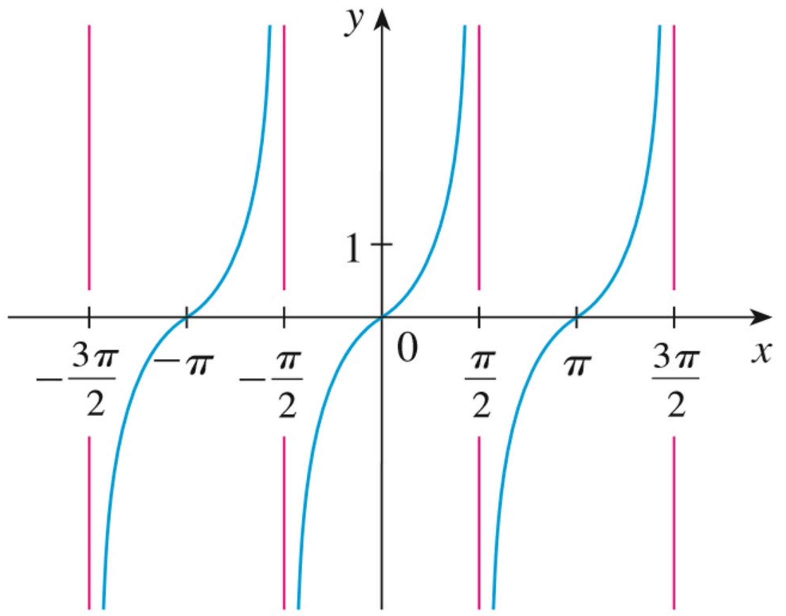
$$\lim_{t \to 1} \frac{t^8 - 1}{t^5 - 1}$$

(back to) **Example 2**.

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} =$$

Example 4: Find

$$\lim_{x \to \frac{\pi}{2}^+} \frac{\cos x}{1 - \sin x}$$



Example 5: Find

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

We cannot apply L'Hospital's Rule if we do not have an indetermination of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 6: Find

$$\lim_{x \to \pi^+} \frac{\sin x}{1 - \cos x}$$

Correct solution (direct substitution)

$$\lim_{x \to \pi^+} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{2} = 0$$

If we (incorrectly: without having an indetermination) try to apply L'Hospital's Rule, we will get

$$\lim_{x \to \pi^+} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^+} \frac{\cos x}{\sin x} = \lim_{x \to \pi^+} \cot x = \infty$$

a completely different result than the correct one

Other Types of Indetermination

Indetermination of the form $0 \cdot \infty$

Example 7: Find

$$\lim_{x \to 0^+} x \ln x$$

Solution:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$$

The last expression is an indetermination of the form $\frac{\infty}{\infty}$ so we can apply L'Hospital Rule

$$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$$

Other Types of Indetermination

Indetermination of the form $\infty - \infty$

Example 8: Find

$$\lim_{x\to 0}(\csc x - \cot x)$$

Solution:

$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$

Other Types of Indetermination

Solution (cont'd):

The last expression is an indetermination of the form $\frac{0}{0}$ so we can apply L'Hospital Rule

$$\lim_{x \to 0} \left(\frac{1 - \cos x}{\sin x} \right)$$

$$= \lim_{x \to 0} \frac{\sin x}{\cos x}$$

$$= \lim_{x \to 0} \tan x = 0$$

Other Types of Indetermination

Indetermination of the form $0^0, \infty^0, 1^\infty$

All these cases can be reduced to indeterminations of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking logarithms:

Example 9: Find $\lim_{x \to 0^+} x^x$

Other Types of Indetermination

Indetermination of the form $0^0, \infty^0, 1^\infty$

Recall: $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x =$

Example 9: Find $\lim\limits_{x \to 0^+} x^x$

Hint. Use the change of base formula $a^r = e^{r \ln x}$ and the continuity of an exponential function

$$\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x) \ln f(x)} = e^{\lim\limits_{x \to a}[g(x) \ln f(x)]}$$

Solution (cont'd):

$$\lim_{x \to 0^+} x^x =$$

Proceed with the Exercise 3.4