Worksheet 3 - Solution

PART 1: Laplace Transform Review

1) Obtain the Laplace transform of the following functions.

a) $x(t) = 15 + 3t^2$

$$X(s) = \frac{15}{s} + \frac{6}{s^3}$$

b) $x(t) = 8te^{-4t} + 2e^{-5t}$

$$X(s) = \frac{8}{(s+4)^2} + \frac{2}{s+5}$$

c) $x(t) = te^{-2t} sin4t$

If we define the $x(t) = ty(t)$, where $y(t) = e^{-2t} sin4t$. Then $X(s)$ is obtained as:

$$X(s) = -\frac{dY(s)}{ds}$$

Therefore, if $y(t) = e^{-2t} sin4t$, the $Y(s)$ is:

$$Y(s) = \frac{5}{(s+3)^2 + 5^2} = \frac{5}{s^2 + 6s + 34}$$

Then we can find the $X(s)$

$$X(s) = -\frac{dY(s)}{ds} = \frac{10s + 30}{(s^2 + 6s + 34)^2}$$

2) Use the initial-value and final-value theorems to determine $x(0)$ and $x(\infty)$ for the following transforms.

a) $X(s) = \frac{8}{2s+3}$

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\left(\frac{8}{2s+3}\right) = \frac{8}{2}$$

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left(\frac{8}{2s+3}\right) = 0$$

b) $X(s) = \frac{7}{2s^2+6s+3}$

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\left(\frac{7}{2s^2 + 6s + 3}\right) = 0$$

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left(\frac{7}{2s^2 + 6s + 3}\right) = 0$$

3) Obtain the inverse Laplace transform $x(t)$ for each of the following transforms.

a) $X(s) = \frac{3}{s(s+2)}$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+2}$$

$$C_1 = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{3}{s+2} = \frac{3}{2}$$

$$C_2 = \lim_{s \to -2} (s+2)X(s) = \lim_{s \to -2} \frac{3}{s} = -\frac{3}{2}$$

$$X(s) = \frac{3/2}{s} + \frac{-3/2}{s+2} \quad \rightarrow \quad x(t) = \frac{3}{2}(1 - e^{-2t})$$

b) $X(s) = \frac{10s+7}{s(s+3)}$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+3}$$

$$C_1 = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{10s+7}{s+3} = \frac{7}{3}$$

$$C_2 = \lim_{s \to -3} (s+3)X(s) = \lim_{s \to -3} \frac{10s+7}{s} = \frac{23}{3}$$

$$X(s) = \frac{7/3}{s} + \frac{23/3}{s+3} \quad \rightarrow \quad x(t) = \frac{7}{3} + \frac{23}{3}e^{-3t}$$

c) $X(s) = \frac{4s+7}{(s+2)(s+5)}$

$$X(s) = \frac{C_1}{s+2} + \frac{C_2}{s+5}$$

$$C_1 = \lim_{s \to -2} (s+2)X(s) = \lim_{s \to -2} \frac{4s+7}{s+5} = \frac{-1}{3}$$

$$C_2 = \lim_{s \to -5} (s+5)X(s) = \lim_{s \to -5} \frac{4s+7}{s+2} = \frac{13}{3}$$

$$X(s) = \frac{-1/3}{s+2} + \frac{13/3}{s+5} \quad \rightarrow \quad x(t) = -\frac{1}{3}e^{-2t} + \frac{13}{3}e^{-5t}$$

d) $X(s) = \frac{5}{s^2(s+4)}$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s+4}$$

$$C_1 = \lim_{s \to 0} \frac{d}{ds}[s^2 X(s)] = \lim_{s \to 0} \frac{d}{ds}\left[\frac{5}{s+4}\right] = \lim_{s \to 0} \frac{-5}{(s+4)^2} = \frac{-5}{16}$$

$$C_2 = \lim_{s \to 0} s^2 X(s) = \lim_{s \to 0} \frac{5}{s+4} = \frac{5}{4}$$

$$C_3 = \lim_{s \to -4} (s+4)X(s) = \lim_{s \to -4} \frac{5}{s^2} = \frac{5}{16}$$

$$X(s) = \frac{-5/16}{s} + \frac{5/4}{s^2} + \frac{5/16}{s+4} \quad \rightarrow \quad x(t) = -\frac{5}{16} + \frac{5}{4}t + \frac{5}{16}e^{-4t}$$

e) $X(s) = \frac{2}{s^2+16}$

$$X(s) = \frac{2}{s^2 + 4^2} = \frac{1}{2}\left(\frac{4}{s^2 + 4^2}\right) \quad \rightarrow \quad x(t) = \frac{1}{2}\sin 4t$$

f) $X(s) = \frac{7}{s^2+6s+13}$

$$X(s) = \frac{7}{(s^2 + 6s + 9) + 4} = \frac{7}{(s + 3)^2 + 2^2} = \frac{7}{2}\left(\frac{2}{(s + 3)^2 + 2^2}\right) \quad \rightarrow \quad x(t) = \frac{7}{2}e^{-3t}\sin 2t$$

g) $X(s) = \frac{2}{s(s^2+4s+13)}$

$$X(s) = \frac{C_1}{s} + \frac{C_2 s + C_3}{s^2 + 4s + 13}$$

$$C_1 = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\frac{2}{s^2 + 4s + 13} = \frac{2}{13}$$

$$2 = C_1(s^2 + 4s + 13) + s(C_2 s + C_3) \quad \rightarrow \quad 2 = (C_1 + C_2)s^2 + (4C_1 + C_3)s + 13C_1$$

$$C_1 = \frac{2}{13}, \quad C_2 = -\frac{2}{13}, \quad C_3 = -\frac{8}{13}$$

$$X(s) = \frac{\frac{2}{13}}{s} + \frac{-\frac{2}{13}s - \frac{8}{13}}{s^2 + 4s + 13} \quad \rightarrow \quad X(s) = \frac{\frac{2}{13}}{s} + \frac{-\frac{2}{13}s - \frac{8}{13}}{(s^2 + 4s + 4) + 9} = \frac{\frac{2}{13}}{s} - \frac{2}{13}\left(\frac{s + 4}{(s + 2)^2 + 9}\right)$$

$$X(s) = \frac{\frac{2}{13}}{s} - \frac{2}{13}\left(\frac{s + 2}{(s + 2)^2 + 9} + \frac{2}{(s + 2)^2 + 9}\right) = \frac{\frac{2}{13}}{s} - \frac{2}{13}\left(\frac{s + 2}{(s + 2)^2 + 9}\right) - \frac{4}{39}\left(\frac{3}{(s + 2)^2 + 9}\right)$$

$$\rightarrow \quad x(t) = \frac{2}{13} - \frac{2}{13}e^{-2t}\cos 3t - \frac{4}{39}e^{-2t}\sin 3t$$

h) $X(s) = \frac{16s^2+129s+200}{s(s+5)(s+8)}$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s + 5} + \frac{C_3}{s + 8}$$

$$C_1 = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\frac{16s^2 + 129s + 200}{(s + 5)(s + 8)} = \frac{200}{40} = 5$$

$$C_2 = \lim_{s\to -5}(s + 5)X(s) = \lim_{s\to -5}\frac{16s^2 + 129s + 200}{s(s + 8)} = \frac{-45}{-15} = 3$$

$$C_2 = \lim_{s\to -8}(s + 8)X(s) = \lim_{s\to -8}\frac{16s^2 + 129s + 200}{s(s + 5)} = \frac{192}{24} = 8$$

$$X(s) = \frac{5}{s} + \frac{3}{s + 5} + \frac{8}{s + 8} \quad \rightarrow \quad x(t) = 5 + 3e^{-5t} + 8e^{-8t}$$

i) $X(s) = \frac{12s^2+125s+1268}{(s+7)(s^2+8s+116)}$

$$X(s) = \frac{C_1}{s+7} + \frac{C_2 s + C_3}{s^2+8s+116}$$

$$C_1 = \lim_{s\to -7}(s+7)X(s) = \lim_{s\to -7}\frac{12s^2+125s+1268}{(s^2+8s+116)} = \frac{981}{109} = 9$$

$$12s^2+125s+1268 = C_1(s^2+8s+116) + (s+7)(C_2 s + C_3)$$

$$12s^2+125s+1268 = (C_1+C_2)s^2 + (8C_1+7C_2+C_3)s + 116C_1 + 7C_3$$

$$C_1 = 9\,,\quad C_2 = 3,\quad C_3 = 32$$

$$X(s) = \frac{9}{s+7} + \frac{3s+32}{s^2+8s+116} \quad\to\quad X(s) = \frac{9}{s+7} + \frac{3s+32}{(s^2+8s+16)+100} = \frac{9}{s+7} + \frac{3s+12+20}{(s+4)^2+100}$$

$$X(s) = \frac{9}{s+7} + \frac{3(s+4)}{(s+4)^2+100} + \frac{2(10)}{(s+4)^2+100}$$

$$\to\quad x(t) = 9e^{-7t} + 3e^{-4t}\cos 10t + 2e^{-4t}\sin 10t$$

4) Use the Laplace transform to solve the following ordinary differential equations.

a) $5\dot{x}(t) + 7x(t) = 0, \quad x(0) = 4$

$$5\big(sX(s) - x(0)\big) + 7X(s) = 0 \quad\to\quad 5(sX(s)-4) + 7X(s) = 0 \quad\to\quad (5s+7)X(s) - 20 = 0$$

$$\to\quad X(s) = \frac{20}{5s+7} \quad\to\quad x(t) = 4e^{-7t/5}$$

b) $5\dot{x}(t) + 7x(t) = 15, \quad x(0) = 0$

$$5\big(sX(s) - x(0)\big) + 7X(s) = \frac{15}{s} \quad\to\quad (5s+7)X(s) = \frac{15}{s}$$

$$\to\quad X(s) = \frac{15}{s(5s+7)} = \frac{3}{s(s+7/5)} = \frac{15/7}{s} + \frac{-15/7}{s+7/5} \quad\to\quad x(t) = \frac{15}{7} - \frac{15}{7}e^{-7t/5}$$

c) $\dot{x}(t) + 7x(t) = 4t, \quad x(0) = 5$

$$sX(s) - x(0) + 7X(s) = \frac{4}{s^2} \quad\to\quad sX(s) - 5 + 7X(s) = \frac{4}{s^2} \quad\to\quad (s+7)X(s) = \frac{4}{s^2} + 5$$

$$\to\quad X(s) = \frac{5s^2+4}{s^2(s+7)} \quad\to\quad X(s) = \frac{4/49}{s} + \frac{4/7}{s^2} + \frac{249/49}{s+7} \quad\to\quad x(t) = \frac{4}{49} + \frac{4}{7}t + \frac{249}{49}e^{-7t}$$

d) $\ddot{x}(t) + 7\dot{x}(t) + 10x(t) = 20, \quad x(0) = 5, \ \dot{x}(0) = 3$

$$s^2X(s) - sx(0) - \dot{x}(0) + 7(sX(s) - x(0)) + 10X(s) = \frac{20}{s}$$

$$\rightarrow \quad s^2X(s) - 5s - 3 + 7sX(s) - 35 + 10X(s) = \frac{20}{s} \quad \rightarrow \quad (s^2 + 7s + 10)X(s) = \frac{20}{s} + 5s + 38$$

$$\rightarrow \quad X(s) = \frac{5s^2 + 38s + 20}{s(s^2 + 7s + 10)} = \frac{5s^2 + 38s + 20}{s(s+2)(s+5)} = \frac{2}{s} + \frac{6}{s+2} + \frac{-3}{s+5}$$

$$\rightarrow \quad x(t) = 2 + 6e^{-2t} - 3e^{-5t}$$

e) $5\ddot{x}(t) + 20\dot{x}(t) + 20x(t) = 28, \quad x(0) = 5, \ \dot{x}(0) = 8$

$$5(s^2X(s) - sx(0) - \dot{x}(0)) + 20(sX(s) - x(0)) + 20X(s) = \frac{28}{s}$$

$$\rightarrow \quad 5s^2X(s) - 25s - 40 + 20sX(s) - 100 + 20X(s) = \frac{28}{s}$$

$$\rightarrow \quad (5s^2 + 20s + 20)X(s) = \frac{28}{s} + 25s + 140$$

$$\rightarrow \quad X(s) = \frac{25s^2 + 140s + 28}{s(5s^2 + 20s + 20)} = \frac{25s^2 + 140s + 28}{5s(s+2)^2} = \frac{7/5}{s} + \frac{18/5}{s+2} + \frac{76/5}{(s+2)^2}$$

$$\rightarrow \quad x(t) = \frac{7}{5} + \frac{18}{5}e^{-2t} + \frac{76}{5}te^{-2t}$$

f) $\ddot{x}(t) + 16x(t) = 144, \quad x(0) = 5, \ \dot{x}(0) = 12$

$$s^2X(s) - sx(0) - \dot{x}(0) + 16X(s) = \frac{144}{s} \quad \rightarrow \quad s^2X(s) - 5s - 12 + 16X(s) = \frac{144}{s}$$

$$\rightarrow \quad (s^2 + 16)X(s) = \frac{144}{s} + 5s + 12$$

$$\rightarrow \quad X(s) = \frac{5s^2 + 12s + 144}{s(s^2 + 16)} = \frac{9}{s} + \frac{-4s + 12}{s^2 + 16} = \frac{9}{s} + \frac{-4s}{s^2 + 16} + \frac{12}{s^2 + 16}$$

$$\rightarrow \quad x(t) = \frac{7}{5} - 4\cos 4t + 3\sin 4t$$

g) $\ddot{x}(t) + 14\dot{x}(t) + 49x(t) = 0, \quad x(0) = 1, \ \dot{x}(0) = 3$

$$s^2X(s) - sx(0) - \dot{x}(0) + 14(sX(s) - x(0)) + 49X(s) = 0$$

$$\rightarrow \quad s^2X(s) - s - 3 + 14sX(s) - 14 + 49X(s) = 0 \quad \rightarrow \quad (s^2 + 14s + 49)X(s) = s + 17$$

$$\rightarrow \quad X(s) = \frac{s + 17}{s^2 + 14s + 49} = \frac{s + 17}{(s+7)^2} = \frac{1}{s+7} + \frac{10}{(s+7)^2}$$

$$\rightarrow \quad x(t) = e^{-7t} + 10te^{-7t}$$

PART 2: Transfer Functions

1) For each of the following equations, determine the transfer function $X(s)/F(s)$ and compute the poles.

a) $10\dot{x}(t) + 14x(t) = 15f(t)$

$$10sX(s) + 14X(s) = 15F(s) \quad \rightarrow \quad \frac{X(s)}{F(s)} = \frac{15}{10s + 14}$$

Poles: $s = -7/5$

b) $6\ddot{x}(t) + 60\dot{x}(t) + 126x(t) = 7f(t)$

$$6s^2X(s) + 60sX(s) + 126X(s) = 7F(s) \quad \rightarrow \quad \frac{X(s)}{F(s)} = \frac{7}{6s^2 + 60s + 126}$$

Poles: $s_1 = -7,\ s_2 = -3$

c) $4\ddot{x}(t) + 56\dot{x}(t) + 232x(t) = 8\dot{f}(t) + 3f(t)$

$$4s^2X(s) + 56sX(s) + 232X(s) = 8sF(s) + 3F(s) \quad \rightarrow \quad \frac{X(s)}{F(s)} = \frac{8s + 3}{4s^2 + 56s + 232}$$

Poles: $s_{1,2} = -7 \pm 3j$

d) $10\dot{x}(t) + 14x(t) = 6\dot{f}(t) + 15f(t)$

$$10sX(s) + 14X(s) = 6sF(s) + 15F(s) \quad \rightarrow \quad \frac{X(s)}{F(s)} = \frac{6s + 15}{10s + 14}$$

Poles: $s = -7/5$

2) Obtain the transfer functions $X(s)/F(s)$ and $Y(s)/F(s)$ for each of the following models.

a) $4\dot{x}(t) = y(t), \qquad \dot{y}(t) = f(t) - 5y(t) - 17x(t)$

$$4sX(s) = Y(s) \qquad Eqn.(1)$$

$$sY(s) = F(s) - 5Y(s) - 17X(s) \qquad Eqn.(2)$$

Substitute Y(s) from Eqn. (1) into Eqn. (2):

$$4s^2X(s) = F(s) - 20sX(s) - 17X(s) \quad \rightarrow \quad \frac{X(s)}{F(s)} = \frac{1}{4s^2 + 20s + 17} \qquad Eqn.(4)$$

From Eqn. (1) and Eqn. (4):

$$X(s) = \frac{Y(s)}{4s} \quad \rightarrow \quad \frac{Y(s)}{F(s)} = \frac{4s}{4s^2 + 20s + 17}$$

b) $\dot{x}(t) = -3x(t) + 7y(t), \qquad \dot{y}(t) = f(t) - 8y(t) - 3x(t)$

$sX(s) = -3X(s) + 7Y(s) \qquad Eqn.(1)$

$sY(s) = F(s) - 8Y(s) - 3X(s) \qquad Eqn.(2)$

From Eqn. (1):

$$X(s) = \frac{7}{s+3}Y(s) \qquad Eqn.(3)$$

Substitute X(s) from Eqn. (3) into Eqn. (2):

$$sY(s) = F(s) - 8Y(s) - \frac{21Y(s)}{s+3} \quad \rightarrow \quad \frac{Y(s)}{F(s)} = \frac{s+3}{s^2+11s+45} \qquad Eqn.(4)$$

From Eqn. (3) and Eqn. (4):

$$\frac{X(s)}{F(s)} = \frac{7}{s^2+11s+45}$$

c) $4\dot{x}(t) = y(t), \qquad \dot{y}(t) = f(t) - 3y(t) - 12x(t)$

$4sX(s) = Y(s) \qquad Eqn.(1)$

$sY(s) = F(s) - 3Y(s) - 12X(s) \qquad Eqn.(2)$

Substitute Y(s) from Eqn. (1) into Eqn. (2):

$$4s^2X(s) = F(s) - 12sX(s) - 12X(s) \quad \rightarrow \quad \frac{X(s)}{F(s)} = \frac{1}{4s^2+12s+12} \qquad Eqn.(4)$$

From Eqn. (1) and Eqn. (4):

$$X(s) = \frac{Y(s)}{4s} \quad \rightarrow \quad \frac{Y(s)}{F(s)} = \frac{4s}{4s^2+12s+12}$$

$$\frac{X(s)}{F(s)} = \frac{1}{4s^2+12s+12}$$

PART 3: State-Space Model from Transfer Function Model

1) Obtain the state-space model for the following transfer function models. Select the state variables in phase-variable form. Draw the equivalent block diagram showing the state variables, input and output of the system.

a) $\frac{Y(s)}{F(s)} = \frac{4}{9s^2+2s+7}$

This is a strictly proper transfer function with a constant value in the numerator.

First, find the associate differential equation:

$$9s^2Y(s) + 2sY(s) + 7Y(s) = 4F(s) \quad \rightarrow \quad 9\ddot{y}(t) + 2\dot{y}(t) + 7y(t) = 4f(t)$$

Define the state variables:

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{x}_1(t) = \dot{y}(t) \quad \rightarrow \quad \dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{y}(t) \quad \rightarrow \quad \dot{x}_2(t) = \frac{1}{9}(4f(t) - 2\dot{y}(t) - 7y(t)) = \frac{4}{9}f(t) - \frac{2}{9}x_2(t) - \frac{7}{9}x_1(t)$$

Find the output in terms of the state variables and the input.

$$y(t) = x_1(t)$$

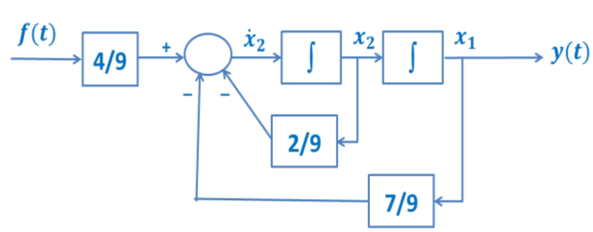
The system model has 2 state variables, 1 input, and 1 output.

Form the state equations and the output equations in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{7}{9} & -\frac{2}{9} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{4}{9} \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]f(t)$$

The block diagram to visualize the state variables, input, and output:

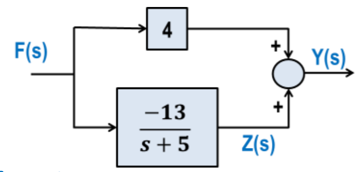


b) $\frac{Y(s)}{F(s)} = \frac{4s+7}{s+5}$

This is a proper transfer function.

First, we have to rewrite it as a summation of a constant term and a strictly proper function.

$$\frac{Y(s)}{F(s)} = \frac{4s+7}{s+5} = 4 + \frac{-13}{s+5}$$



The feed-forward matrix **D** is obtained as the constant term 4.

Then, determine the matrices **A, B**, and **C** from the strictly proper transfer function.

$$\frac{Z(s)}{F(s)} = \frac{-13}{s+5}$$

First, find the associate differential equation:

$$sZ(s) + 5Z(s) = -13F(s) \quad \rightarrow \quad \dot{z}(t) + 5z(t) = -13f(t)$$

Define the state variable:

$$x_1(t) = z(t)$$

Find the first derivative of the state variable and rewrite it in terms of the state variable and the input.

$$\dot{x}_1(t) = \dot{z}(t) \quad \rightarrow \quad \dot{x}_1(t) = -5x_1(t) - 13f(t)$$

Find the output in terms of the state variables and the input.

$$y(t) = 4f(t) + z(t) = 4f(t) + x_1(t)$$

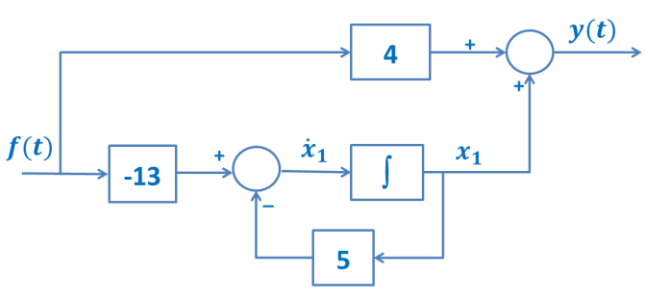
The system model has 1 state variable, 1 input, and 1 output.

Form the state equations and the output equations in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \dot{x}_1(t) = [-5]x_1(t) + [-13]f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = [1]x_1(t) + [4]f(t)$$

The block diagram to visualize the state variable, input, and output:

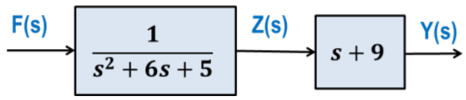


c) $\frac{Y(s)}{F(s)} = \frac{s+9}{s^2+6s+5}$

This is a strictly proper transfer function with a polynomial in the numerator.

Since, the numerator is a polynomial of s, we have to split it into two parts as below:

$$\frac{Y(s)}{F(s)} = \frac{s+9}{s^2+6s+5} = \left(\frac{1}{s^2+6s+5}\right)(s+9)$$



First, find the state equation from the part with the denominator:

$$\frac{Z(s)}{F(s)} = \frac{1}{s^2+6s+5}$$

$$s^2Z(s) + 6sZ(s) + 5Z(s) = F(s) \quad \rightarrow \quad \ddot{z}(t) + 6\dot{z}(t) + 5z(t) = f(t)$$

Define the state variables:

$$x_1(t) = z(t)$$

$$x_2(t) = \dot{z}(t)$$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{x}_1(t) = \dot{z}(t) \quad \rightarrow \quad \dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{z}(t) \quad \rightarrow \quad \dot{x}_2(t) = -6\dot{z}(t) - 5z(t) + f(t) = -6x_2(t) - 5x_1(t) + f(t)$$

Find the output equation by considering the effect of the block with the numerator.

$$Y(s) = (s+9)Z(s) \quad \rightarrow \quad Y(s) = sZ(s) + 9Z(s) \quad \rightarrow \quad y(t) = \dot{z}(t) + 9z(t) = x_2(t) + 9x_1(t)$$

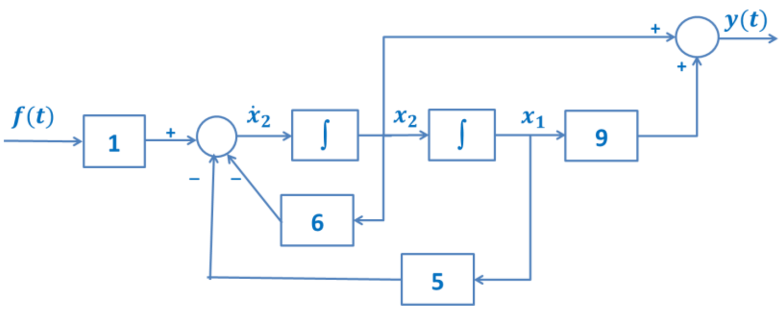
The system model has 2 state variables, 1 input, and 1 output.

Form the state equations and the output equations in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 9 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0] f(t)$$

The block diagram to visualize the state variables, input, and output:

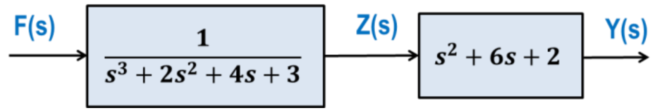


d) $\frac{Y(s)}{F(s)} = \frac{s^2+6s+2}{s^3+2s^2+4s+3}$

This is a strictly proper transfer function with a polynomial in the numerator.

Since, the numerator is a polynomial of s, we have to split it into two parts as below:

$$\frac{Y(s)}{F(s)} = \frac{s^2+6s+2}{s^3+2s^2+4s+3} = \left(\frac{1}{s^3+2s^2+4s+3}\right)(s^2+6s+2)$$



First, find the state equation from the part with the denominator:

$$\frac{Z(s)}{F(s)} = \frac{1}{s^3+2s^2+4s+3}$$

$$s^3Z(s) + 2s^2Z(s) + 4sZ(s) + 3Z(s) = F(s) \quad \rightarrow \quad \dddot{z}(t) + 2\ddot{z}(t) + 4\dot{z}(t) + 3z(t) = f(t)$$

Define the state variables:

$$x_1(t) = z(t)$$

$$x_2(t) = \dot{z}(t)$$

$$x_3(t) = \ddot{z}(t)$$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{x}_1(t) = \dot{z}(t) \quad \rightarrow \quad \dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{z}(t) \quad \rightarrow \quad \dot{x}_2(t) = x_3(t)$$

$$\dot{x}_3(t) = \dddot{z}(t) \quad \rightarrow \quad \dot{x}_3(t) = -2\ddot{z}(t) - 4\dot{z}(t) - 3z(t) + f(t) = -2x_3(t) - 4x_2(t) - 3x_1(t) + f(t)$$

Find the output equation by considering the effect of the block with the numerator.

$$Y(s) = (s^2+6s+2)Z(s) \quad \rightarrow \quad Y(s) = s^2Z(s) + 6sZ(s) + 2Z(s)$$

$$\rightarrow \quad y(t) = \ddot{z}(t) + 6\dot{z}(t) + 2z(t) = x_3(t) + 6x_2(t) + 2x_1(t)$$

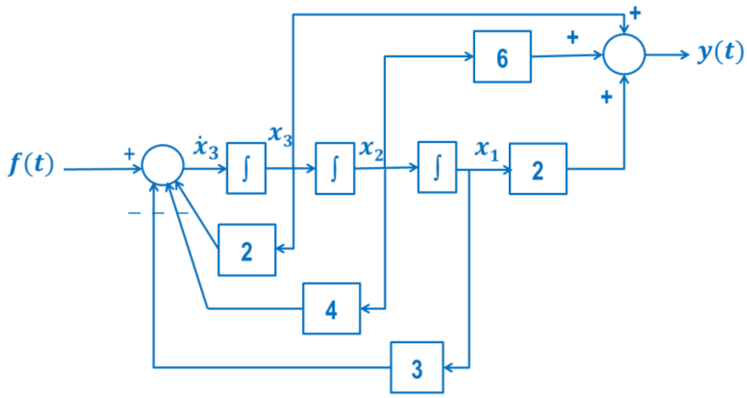
The system model has 3 state variables, 1 input, and 1 output.

Form the state equations and the output equations in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]f(t)$$

The block diagram to visualize the state variables, input, and output:

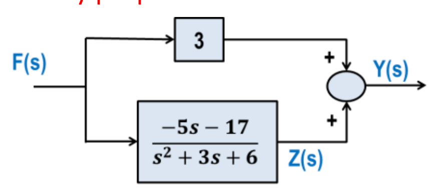


e) $\frac{Y(s)}{F(s)} = \frac{3s^2+4s+1}{s^2+3s+6}$

This is a proper transfer function.

First, we have to rewrite it as a summation of a constant term and a strictly proper function.

$$\frac{Y(s)}{F(s)} = \frac{3s^2+4s+1}{s^2+3s+6} = 3 + \frac{-5s-17}{s^2+3s+6}$$



The feed-forward matrix **D** is obtained as the constant term 3.

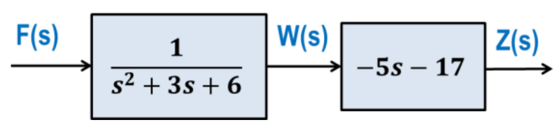
Then, determine the matrices **A, B**, and **C** from the strictly proper transfer function.

$$\frac{Z(s)}{F(s)} = \frac{-5s-17}{s^2+3s+6}$$

This is a strictly proper transfer function with a polynomial in the numerator.

Since, the numerator is a polynomial of s, we have to split it into two parts as below:

$$\frac{Z(s)}{F(s)} = \frac{-5s-17}{s^2+3s+6} = \left(\frac{1}{s^2+3s+6}\right)(-5s-17)$$



First, find the state equation from the part with the denominator:

$$\frac{W(s)}{F(s)} = \frac{1}{s^2+3s+6}$$

$$s^2W(s) + 3sW(s) + 6W(s) = F(s) \quad \rightarrow \quad \ddot{w}(t) + 3\dot{w}(t) + 6w(t) = f(t)$$

Define the state variables:

$x_1(t) = w(t)$

$x_2(t) = \dot{w}(t)$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{x}_1(t) = \dot{w}(t) \quad \rightarrow \quad \dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{w}(t) \quad \rightarrow \quad \dot{x}_2(t) = -3\dot{w}(t) - 6w(t) + f(t) = -3x_2(t) - 6x_1(t) + f(t)$$

Find the output equation by considering the effect of the block with the numerator.

$$Z(s) = (-5s - 17)W(s) \quad \rightarrow \quad Z(s) = -5sW(s) - 17W(s)$$

$$\rightarrow \quad z(t) = -5\dot{w}(t) - 17w(t) = -5x_2(t) - 17x_1(t)$$

Therefore, the output equation is:

$$y(t) = z(t) + 3f(t) = -5x_2(t) - 17x_1(t) + 3f(t)$$

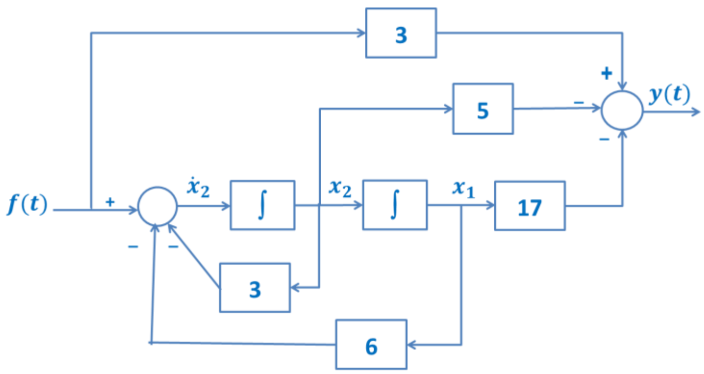
The system model has 2 state variables, 1 input, and 1 output.

Form the state equations and the output equations in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} -17 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [3] f(t)$$

The block diagram to visualize the state variables, input, and output:

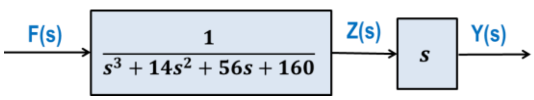


f) $\frac{Y(s)}{F(s)} = \frac{s}{s^3+14s^2+56s+160}$

This is a strictly proper transfer function with a polynomial in the numerator.

Since, the numerator is a polynomial of s, we have to split it into two parts as below:

$$\frac{Y(s)}{F(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160} = \left(\frac{1}{s^3 + 14s^2 + 56s + 160}\right)(s)$$



First, find the state equation from the part with the denominator:

$$\frac{Z(s)}{F(s)} = \frac{1}{s^3 + 14s^2 + 56s + 160}$$

$$s^3Z(s) + 14s^2Z(s) + 56sZ(s) + 160Z(s) = F(s) \quad \rightarrow \quad \dddot{z}(t) + 14\ddot{z}(t) + 56\dot{z}(t) + 160z(t) = f(t)$$

Define the state variables:

$x_1(t) = z(t)$

$x_2(t) = \dot{z}(t)$

$x_3(t) = \ddot{z}(t)$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$\dot{x}_1(t) = \dot{z}(t) \quad \rightarrow \quad \dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = \ddot{z}(t) \quad \rightarrow \quad \dot{x}_2(t) = x_3(t)$

$\dot{x}_3(t) = \dddot{z}(t) \quad \rightarrow \quad \dot{x}_3(t) = -14\ddot{z}(t) - 56\dot{z}(t) - 160z(t) + f(t) = -14x_3(t) - 56x_2(t) - 160x_1(t) + f(t)$

Find the <u>output equation</u> by considering the effect of the block with the <u>numerator</u>.

$Y(s) = sZ(s) \quad \rightarrow \quad y(t) = \dot{z}(t) = x_2(t)$

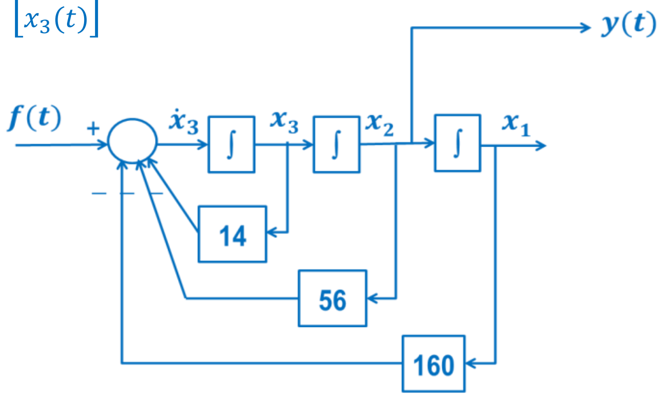
The system model has 3 state variables, 1 input, and 1 output.

Form the <u>state equations</u> and the <u>output equations</u> in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0] f(t)$$

The block diagram to visualize the state variables, input, and output:



g) $\frac{Y(s)}{F(s)} = \frac{24}{s^3+9s^2+26s+24}$

This is a strictly proper transfer function with a constant value in the numerator.

First, find the associate differential equation:

$$s^3Y(s) + 9s^2Y(s) + 26sY(s) + 24Y(s) = 24F(s) \quad \rightarrow \quad \dddot{y}(t) + 9\ddot{y}(t) + 26\dot{y}(t) + 24y(t) = 24f(t)$$

Define the state variables:

$x_1(t) = y(t)$

$x_2(t) = \dot{y}(t)$

$x_3(t) = \ddot{y}(t)$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$\dot{x}_1(t) = \dot{y}(t) \quad \rightarrow \quad \dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = \ddot{y}(t) \quad \rightarrow \quad \dot{x}_2(t) = x_3(t)$

$\dot{x}_3(t) = \dddot{y}(t) \quad \rightarrow \quad \dot{x}_3(t) = -9\ddot{y}(t) - 26\dot{y}(t) - 24y(t) + 24f(t) = -9x_3(t) - 26x_2(t) - 24x_1(t) + 24f(t)$

Find the output in terms of the state variables and the input.

$y(t) = x_1(t)$

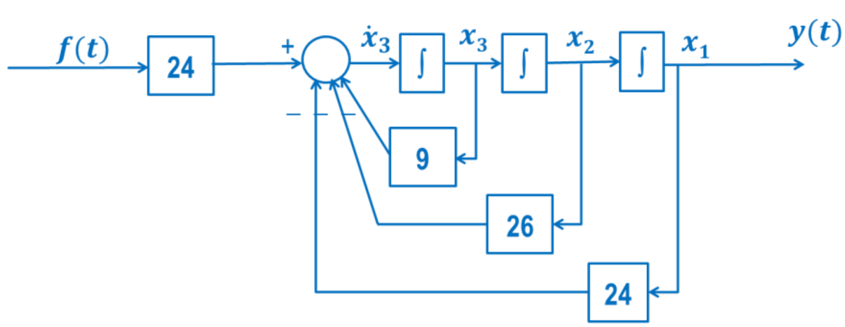
The system model has 3 state variables, 1 input, and 1 output.

Form the state equations and the output equations in the standard matrix-vector form.

$$\textit{State Equation} \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} f(t)$$

$$\textit{Output Equation} \quad \rightarrow \quad y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]f(t)$$

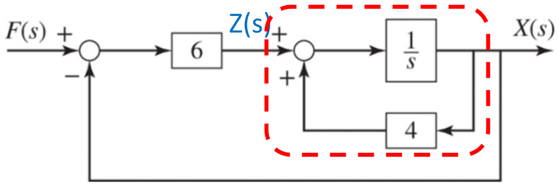
The block diagram to visualize the state variables, input, and output:



PART 3: Block Diagram Models

1) Obtain the transfer function $X(s)/F(s)$ for each of the following block diagrams.

a)



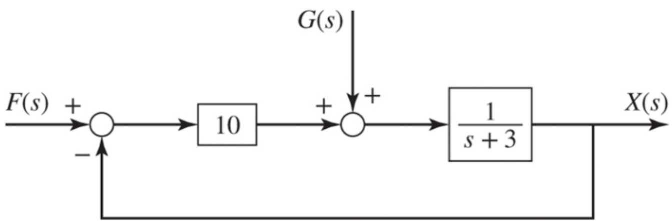
First find the transfer function of internal feedback. Note that it is a positive feedback loop,

$$\frac{X(s)}{Z(s)} = \frac{\frac{1}{s}}{1-\frac{4}{s}} = \frac{\frac{1}{s}}{\frac{s-4}{s}} = \frac{1}{s-4}$$

Then find the overall transfer function, which is a negative feedback loop,

$$\frac{X(s)}{F(s)} = \frac{6\left(\frac{1}{s-4}\right)}{1+6\left(\frac{1}{s-4}\right)} = \frac{\frac{6}{s-4}}{\frac{s-4+6}{s-4}} = \frac{6}{s+2}$$

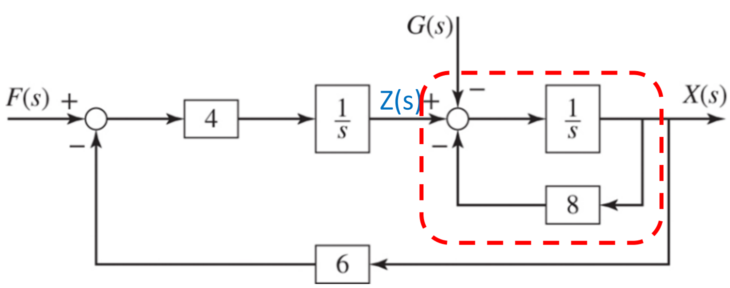
b)



This is a two-input one-output system. From the superposition principles, assume that G(s) = 0 and solve for X(s)/F(s), which is a negative feedback loop,

$$\frac{X(s)}{F(s)} = \frac{10\left(\frac{1}{s+3}\right)}{1+10\left(\frac{1}{s+3}\right)} = \frac{\frac{10}{s+3}}{\frac{s+3+10}{s+3}} = \frac{10}{s+13}$$

c)



This is a two-input one-output system. From the superposition principles, assume that G(s) = 0 and solve for X(s)/F(s):

First find the transfer function of internal feedback, which is a negative feedback loop:

$$\frac{X(s)}{Z(s)} = \frac{\frac{1}{s}}{1+\frac{8}{s}} = \frac{\frac{1}{s}}{\frac{s+8}{s}} = \frac{1}{s+8}$$

Then find the overall transfer function,

$$\frac{X(s)}{F(s)} = \frac{4\left(\frac{1}{s}\right)\left(\frac{1}{s+8}\right)}{1+4\left(\frac{1}{s}\right)\left(\frac{1}{s+8}\right)(6)} = \frac{\frac{4}{s(s+8)}}{1+\frac{24}{s(s+8)}} = \frac{\frac{4}{s(s+8)}}{\frac{s(s+8)+24}{s(s+8)}} = \frac{4}{s^2+8s+24}$$

2) Draw block diagram for the following equation. The output is $X(s)$, the inputs are $F(s)$ and $G(s)$.

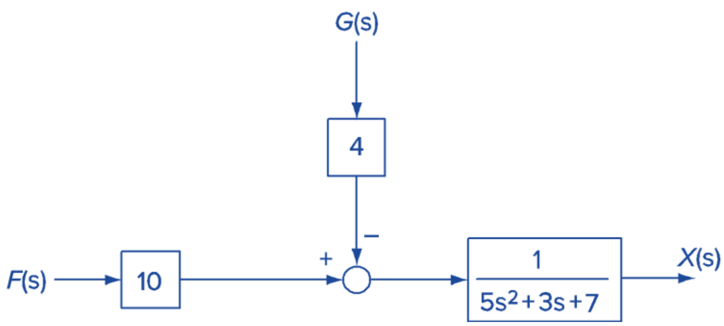
$$5\ddot{x}(t) + 3\dot{x}(t) + 7x(t) = 10f(t) - 4g(t)$$

Take Laplace transform:

$$5s^2X(s) + 3sX(s) + 7X(s) = 10F(s) - 4G(s)$$

Solve for X(s):

$$X(s) = \frac{10}{5s^2+3s+7}F(s) - \frac{4}{5s^2+3s+7}G(s)$$



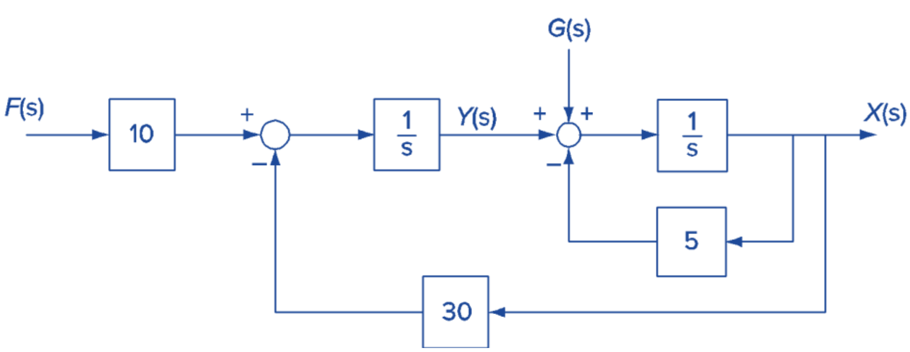
3) Draw the block diagram for the following model. The output is $X(s)$, the inputs are $F(s)$ and $G(s)$. Indicate the location of $Y(s)$ on the diagram.

$$\dot{x}(t) = y(t) - 5x(t) + g(t), \qquad \dot{y}(t) = 10f(t) - 30x(t)$$

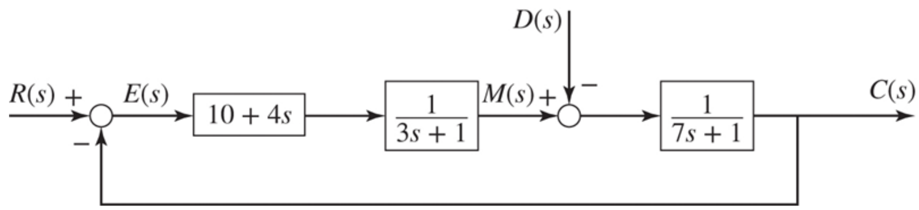
Take Laplace transform from each equation:

$$sX(s) = Y(s) - 5X(s) + G(s) \quad \rightarrow \quad X(s) = \frac{1}{s}[Y(s) - 5X(s) + G(s)]$$

$$sY(s) = 10F(s) - 30X(s) \quad \rightarrow \quad Y(s) = \frac{1}{s}[10F(s) - 30X(s)]$$



4) Given the following block diagram, derive the expression for the variables $C(s)$, $E(s)$, and $M(s)$ in terms of $R(s)$ and $D(s)$.



This is a two-input one-output system. Apply superposition principles to find the $C(s)$.

Assume G(s) = 0, solve for C(s)/R(s)

$$\frac{C(s)}{R(s)} = \frac{(10+4s)\left(\frac{1}{3s+1}\right)\left(\frac{1}{7s+1}\right)}{1+(10+4s)\left(\frac{1}{3s+1}\right)\left(\frac{1}{7s+1}\right)} = \frac{\frac{4s+10}{(3s+1)(7s+1)}}{\frac{(3s+1)(7s+1)+10+4s}{(3s+1)(7s+1)}} = \frac{4s+10}{21s^2+14s+11}$$

Assume R(s) = 0, solve for C(s)/D(s)

$$\frac{C(s)}{D(s)} = \frac{-\left(\frac{1}{7s+1}\right)}{1+(10+4s)\left(\frac{1}{3s+1}\right)\left(\frac{1}{7s+1}\right)} = \frac{\frac{-1}{7s+1}}{\frac{(3s+1)(7s+1)+10+4s}{(3s+1)(7s+1)}} = \frac{-(3s+1)}{21s^2+14s+11}$$

Therefore, the $C(s)$ is obtained in terms of $R(s)$ and $D(s)$

$$C(s) = \frac{4s+10}{21s^2+14s+11}R(s) - \frac{3s+1}{21s^2+14s+11}D(s)$$

From the block diagram:

$$E(s) = R(s) - C(s) = R(s) - \frac{4s+10}{21s^2+14s+11}R(s) + \frac{3s+1}{21s^2+14s+11}D(s)$$

$$E(s) = \frac{21s^2+10s+1}{21s^2+14s+11}R(s) + \frac{3s+1}{21s^2+14s+11}D(s)$$

From the block diagram:

$$M(s) = \frac{4s+10}{3s+1}E(s) = \frac{(4s+10)(7s+1)}{21s^2+14s+11}R(s) + \frac{4s+10}{21s^2+14s+11}D(s)$$