Signal Processing (MENG3520)

Module 3

Weijing Ma, Ph. D. P. Eng.

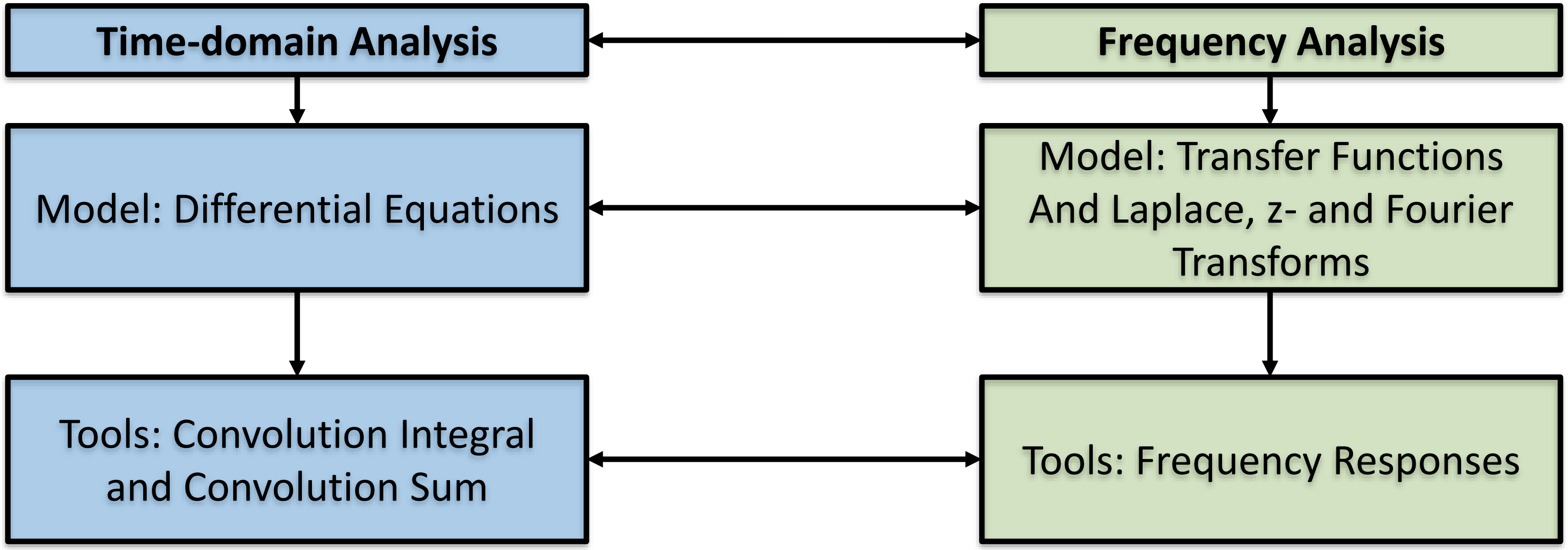
MODULE 3

TIME DOMAIN ANALYSIS OF LTI SYSTEMS

MODULE OUTLINE

- 3.1 System equations
- 3.2 Compute CT system response to initial conditions: the zero-input response
- 3.3 Compute CT system response to external input: the zero-state response
- 3.4 Unit step response and its relation to unit impulse response

Time-domain vs Frequency-domain Analysis



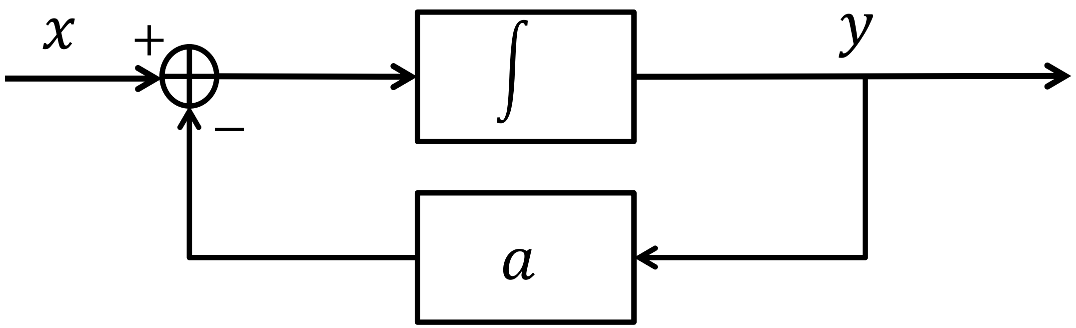
3.1

System Equations

System Equation

The System Equation relates the outputs of a system to its inputs.

Example: the system described in the block diagram below,



has a system equation,

$$\dot{y} + ay = x.$$

The solution to the system equation is the output y of the system, which can be uniquely defined by the system equation with input x and the initial conditions.

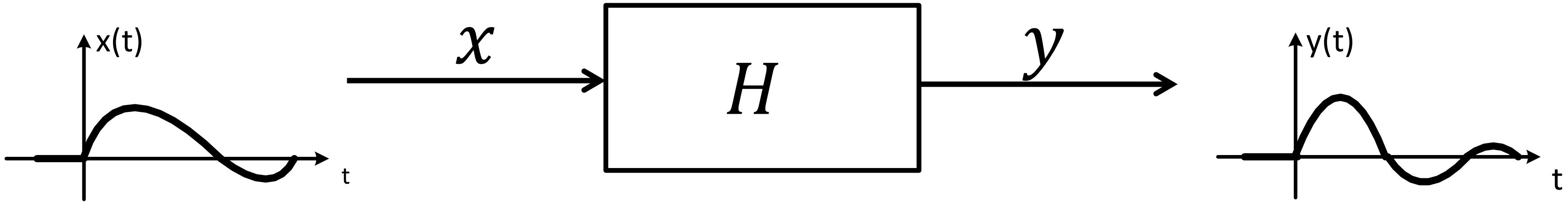
Solutions for the System Equation

- The solution to the system equation is the output y of the system.
- The solution can only be uniquely defined by the system equation with the following information known:
  - input signal x and,
  - initial conditions.
- It is obvious that the solution of the system equation will consequently consist of two parts.

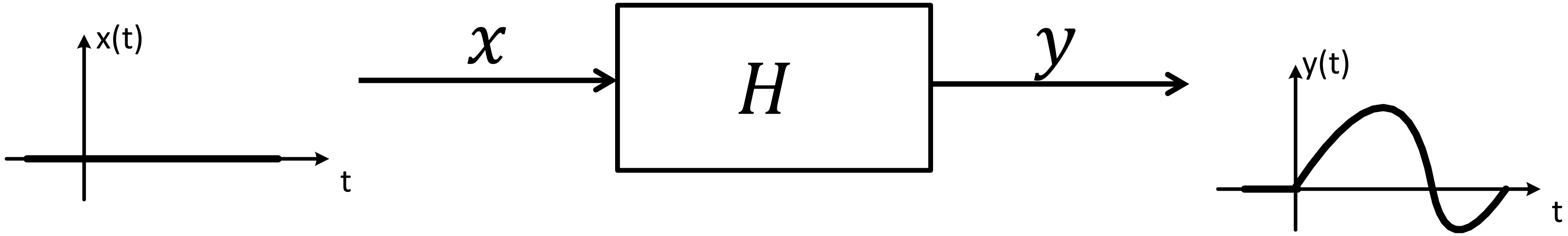
Solutions for the System Equation

The output y consists of two components: $y(t) = ZSR + ZIR$

Zero-state response (ZSR):



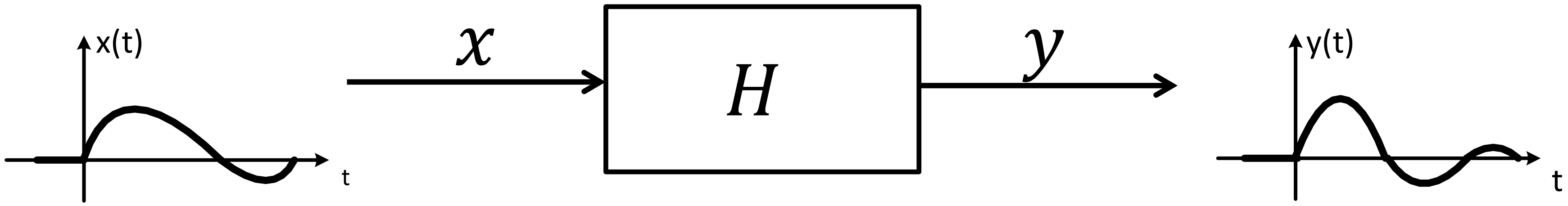
Zero-input response (ZIR):



Solutions for the System Equation

The output y consists of two components: $y(t) = ZSR + ZIR$

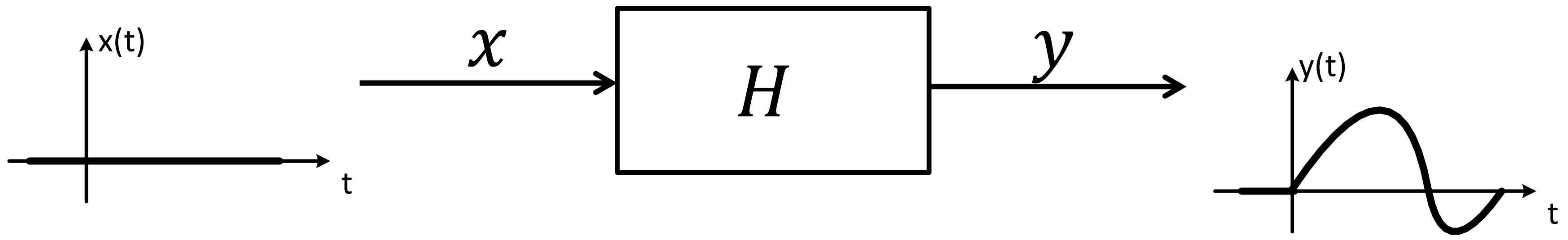
The zero-state response (ZSR), which is the output of the system with all initial conditions being zero. This part of the response is due to the input signal $x(t)$.



Solutions for the System Equation

The output y consists of two components:

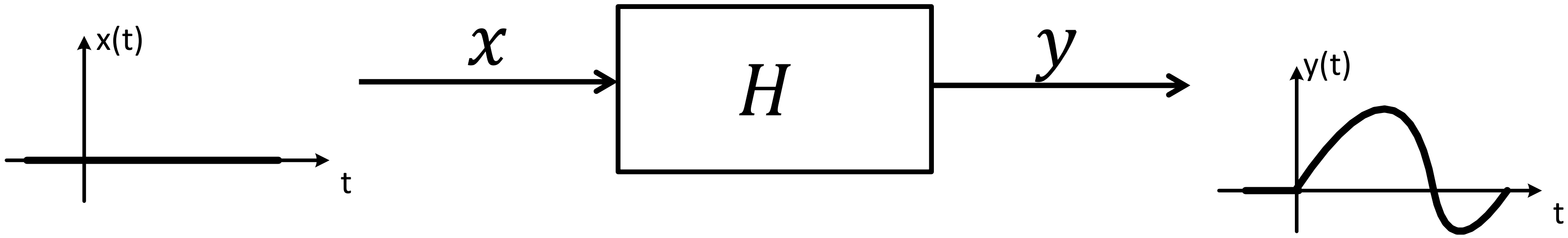
The zero-input response (ZIR), which is what the system does with no input at all. This part of the response is due to initial conditions. Examples of initial conditions: energy stored in capacitors and inductors if the system is an RLC circuit; the initial temperature distribution in a thermodynamics cooling problem; or the initial position and velocity of a moving object.



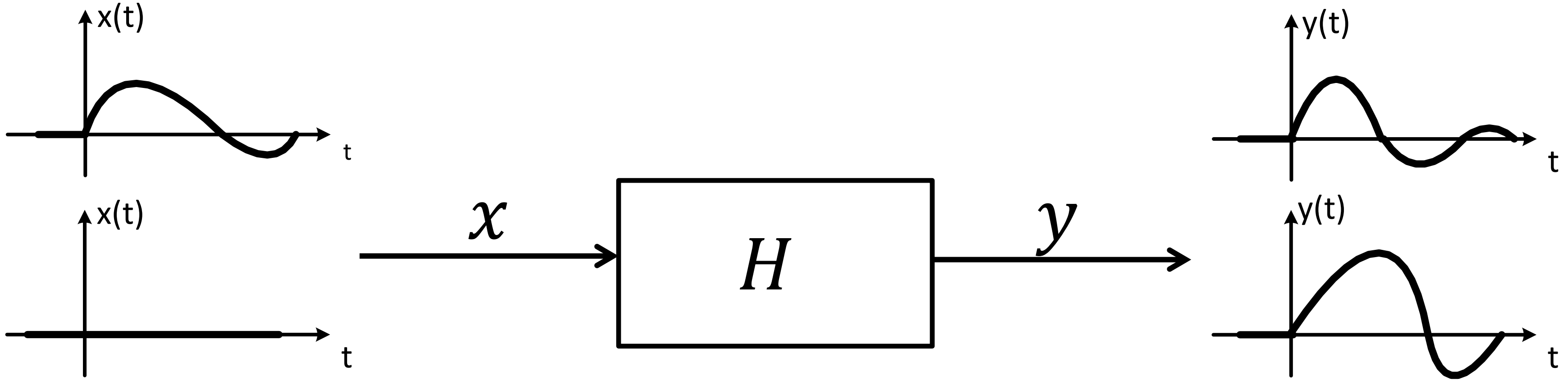
Solutions for the System Equation

The output y consists of two components: $y(t) = ZSR + ZIR$

Note: when determine whether a system H is linear, we ignores its zero-input response (ZIR), assuming it to be zero. Why?



Total Response for a Linear System – Using CT as Example



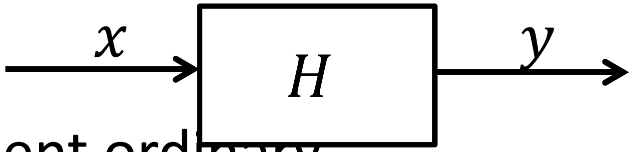
$$Total\ Response = \underbrace{zero - input\ response}_{when\ x(t)=0} + \underbrace{zero - state\ response}_{response\ to\ \boldsymbol{non-zero\ x(t)}}$$

$results\ from\ \boldsymbol{internal\ system\ conditions}\ only$ $\quad$ $all\ \boldsymbol{initial\ conditions\ are\ zero}$

$independent\ of\ x(t)$ $\quad$ $dependent\ on\ x(t)$

The two components are independent of each other. Each component can be computed independently.

General Form of CT System Equations



Many CT systems can be described by linear constant coefficient ordinary differential equations (LCCODE):

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t) \qquad (\mathrm{Eq.}\,1)$$

With given initial conditions:

$$y^{(N)}(0), \cdots, \dot{y}(0), y(0)$$

Where,

- N is the order of the system.
- For practical systems, $N \geq M$.
- $a_0, \cdots, a_N$ and $b_{N-M}, \cdots, b_N$ are the constant coefficients of the system → time invariant
- Output $y(t)$ can be fully determined with given input $x(t)$ and initial conditions.

Find the Total Response of the CT System Equation

Let's examine this system equation:

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t) \qquad (\text{Eq. }1)$$

Using operator notation D to represent d/dt, we can express the equation as:

$$(D^N + a_1 \; D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_{N-M}\, D^M + b_{N-M+1} \; D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

Or:

$$Q(D) y(t) = P(D) x(t) \qquad (\text{Eq. }2)$$

Where the polynomials $Q(D)$ and $P(D)$ are:

$$Q(D) = \; D^N + a_1 \; D^{N-1} + \cdots + a_{N-1} D + a_N$$
$$P(D) = \; b_{N-M}\, D^M + b_{N-M+1} \; D^{M-1} + \cdots + b_{N-1} D + b_N$$

3.2

COMPUTE ZERO-INPUT RESPONSE OF A CT SYSTEM

Compute Zero-Input Response of a CT System

The zero-input response $y_0(t)$ is the solution of $Eq. 2$ when input $x(t) = 0$ so that:

$$Q(D)y_0(t) = 0$$

With given initial conditions:

$$y^{(N)}(0), \cdots, \dot{y}(0), y(0)$$

Denote $D = \frac{d}{dt}$, we have:

$$(D^N + a_1 \ D^{N-1} + \cdots + a_{N-1}D + a_N)y(t) = 0, \qquad Eq. 3$$

Further examine $Eq. 3$ shows that a linear combination of $y_0(t)$ and its n successive derivatives is zero, not at some values of t, but **for all $\boldsymbol{t}$**.

Such a result is possible **if and only if** $y_0(t)$ and its n successive derivatives are of the same form. Otherwise, their sum can never add to zero for all values of t.

Compute Zero-Input Response of a CT System

Question: what function satisfy that itself and its n successive derivatives are of the same form?

Answer: We know that only an exponential function $y_0(t) = ce^{\lambda t}$ has this property (We have learned about this function already and will explore it further in future lectures).

So let us assume that $y_0(t) = ce^{\lambda t}$ is a solution to $Eq. 3$.

Since $y_0(t) = ce^{\lambda t}$, then,

$$Dy_0(t) = \frac{dy_0}{dt} = c\lambda e^{\lambda t}$$

$$D^2\, y_0(t) = \frac{d^2 y_0}{dt^2} = c\lambda^2 e^{\lambda t}$$

$$\vdots$$

$$D^N\, y_0(t) = \frac{d^N y_0}{dt^N} = c\lambda^N e^{\lambda t}$$

Substituting these results in $Eq.3$ we obtain

$$c(\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)e^{\lambda t} = 0$$

This results means that $\boldsymbol{ce^{\lambda t}}$ is indeed a **solution** of $Eq.3$, if λ satisfies $Eq.4$.

For a nontrivial solution of this equation,

$$\boxed{\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N \;=\; 0, \qquad Eq.4}$$

Note that the polynomial in $Eq. 4$ is identical to the polynomial $Q(D)$ in $Eq. 3$, with λ replacing D.

$$N + a_1 \ D^{N-1} + \cdots + a_{N-1}D + a_N)y(t) = 0, \qquad Eq. 3$$

$$D \ => \ \lambda$$

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N \ = \ 0, \qquad Eq. 4$$

Therefore, $Eq. 4$ can be expressed as

$$\boxed{Q(\lambda) = 0}$$

We can express $Q(\lambda)$ in factorized form:

$$\boxed{Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_N) = 0 \qquad Eq. 5}$$

And we can clearly see that $Q(\lambda)$ has n solution:

$\lambda_1, \lambda_2, \dots, \lambda_N$, if all $\lambda_{i,}$ are distinct.

Consequently, $Eq. 3$ has n possible solutions:

$c_1 e^{\lambda_1 t}$, $c_2 e^{\lambda_2 t}$, …, $c_n e^{\lambda_N t}$, with c_1, c_2, …, c_n as constants.

System Response to Internal Conditions: The Zero-Input Response

We can show that a general solution of zero-input response is given by the sum of these N solutions so that:

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t} \qquad Eq.6$$

Where $c_1, c_2, \ldots, c_N$ as constants determined by N constrains on the solution:
Initial condition constraints: $y^{(N)}(0), \cdots, \dot{y}(0), y(0)$ are given.

Characteristic Equation

$$\boxed{Q(\lambda) = 0}$$

Characteristic Roots

$$\boxed{\lambda_1, \lambda_2, \dots, \lambda_N}$$

Also known as **characteristic values**, **eigenvalues**, and **natural frequencies** of the system.

Characteristic Modes

$$\boxed{e^{\lambda_i t}\ (i = 1,2,\dots,N)}$$

Also known as ***eigen functions***, ***natural modes*** or simply ***modes*** of the system.

Special case: repeated roots

The solution of $Eq. 5$ assumes that the n characteristic roots $\lambda_1, \lambda_2, \dots, \lambda_n$ are **distinct**. If there are **repeated roots** (same root occurring more than once), the form of the solution is modified slightly. If root λ repeats r times, the differential equation is:

$$(D - \lambda)^r y_0(t) = 0 \qquad Eq. 7$$

The **characteristic modes** are

$$e^{\lambda_1 t}, te^{\lambda_1 t}, t^2 e^{\lambda_1 t}, \dots, , t^{r-1} e^{\lambda_1 t}$$

and the **solution** is:

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1}) e^{\lambda_1 t}$$

Special case: repeated roots

Consequently, for a system with the characteristic polynomial

r repeated modes and some distinct modes

$$Q(\lambda) = (\lambda - \lambda_1)^r (\lambda - \lambda_{r+1}) \dots (\lambda - \lambda_N)$$

The characteristic modes are:

$$e^{\lambda_1 t}, te^{\lambda_1 t}, t^2 e^{\lambda_1 t}, \dots, t^{r-1} e^{\lambda_1 t}, e^{\lambda_{r+1} t}, \dots, e^{\lambda_N t}$$

and the solution is:

$$y_0(t) = (c_1 + c_2 t + \dots + c_r t^{r-1}) e^{\lambda_1 t} + c_{r+1} e^{\lambda_{r+1} t} + \dots + c_N e^{\lambda_N t}$$

Special case: complex roots

- For a real system, complex roots must occur in pairs of conjugates if the coefficients of the characteristic polynomial $Q(\lambda)$ are to be real.
- Therefore, if $\alpha + j\beta$ is a characteristic root, $\alpha - j\beta$ must also be a characteristic root.
- The zero-input response corresponding to this pair of complex conjugate roots is:

$$\boxed{y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}}$$

- Since the response for a real system must also be real, $c_1\ and\ c_2$ have to be conjugates.

$$\boxed{c_1 = \frac{c}{2} e^{j\Theta} \qquad and \qquad c_2 = \frac{c}{2} e^{-j\Theta}}$$

Special case: complex roots

- If we substitute c_1 and c_2 in the following equation:

$$\boxed{y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}}$$

- This yields:

$$y_0(t) = \frac{c}{2} e^{j\theta} e^{(\alpha + j\beta)t} + \frac{c}{2} e^{-j\theta} e^{(\alpha - j\beta)t}$$

$$= \frac{c}{2} e^{\alpha t}\left[e^{j(\beta t + \theta)} + e^{-j(\beta t + \theta)}\right]$$

$$= c e^{\alpha t} \cos(\beta t + \theta)$$

Summary: Zero-input Response of a CT System

Distinct eigenvalues (roots)

$$\boxed{y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}}$$

Repeated eigenvalues (roots)

$$\boxed{y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1}) e^{\lambda_1 t}}$$

Summary: Zero-input Response of a CT System

Repeated eigenvalues (r times) and distinct eigenvalues ($N - r$ times)

$$\boxed{y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda_1 t} + c_{r+1}e^{\lambda_{r+1} t} + \cdots + c_N e^{\lambda_N t}}$$

Complex eigenvalues (roots)

$$\boxed{y_0(t) = \ c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}}$$

$$\boxed{c_1 = \frac{c}{2}e^{j\Theta} \qquad and \qquad c_2 = \frac{c}{2}e^{-j\Theta}}$$

$$\boxed{y_0(t) = e^{\alpha t}\cos(\beta t + \theta)}$$

Example: Find Zero-Input Response

1. $(D^2 + 3D + 2)y(t) = Dx(t)$ with initial conditions $y_0(0) = 0\ and\ \dot{y}_0(0) = -5.$

2. $(D^2 + 6D + 9)y(t) = (3D + 5)x(t)$ with initial conditions $y_0(0) = 3$ and $\dot{y}_0(0) = -7.$

3. $(D^2 + 4D + 40)y(t) = (D + 2)x(t)$ with initial conditions $y_0(0) = 2$ and $\dot{y}_0(0) = 16.78.$

Using MATLAB to Find the Polynomial Roots

We can use roots function to find the roots of any polynomial:

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0$$

$(\lambda^2 + 3\lambda + 2 = 0)$ $(\lambda^2 + 6\lambda + 9 = 0)$ $(\lambda^2 + 4\lambda + 40 = 0)$

```
>> r1 = roots([1   3   2])

r1 =

    -2
    -1
```

```
>> r2 = roots([1   6   9])

r2 =

  -3.0000 + 0.0000i
  -3.0000 - 0.0000i
```

```
>> r3 = roots([1   4   40])

r3 =

  -2.0000 + 6.0000i
  -2.0000 - 6.0000i
```

Using MATLAB to Find the constants $\boldsymbol{c_1, c_2, \ldots}$

When roots are distinct in a second order system:

$$\boxed{y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

$$y_0(0) = c_1 + c_2$$

$$\dot{y}_0(0) = c_1\lambda_1 + c_2\lambda_2$$

Using MATLAB to Find the constants $c_1, c_2, \ldots$

We want to solve this for c_1, c_2

$$y_0(0) = c_1 + c_2$$

$$\dot{y}_0(0) = c_1\lambda_1 + c_2\lambda_2$$

We can rewrite this with matrix representation:

$$\begin{bmatrix} y_0(0) \\ \dot{y}_0(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} y_0(0) \\ \dot{y}_0(0) \end{bmatrix}$$

We can use inv function in MATLAB to find c_1, c_2

Example

- Find zero-input response of system $(D^2 + 3D + 2)y(t) = Dx(t)$ with initial condition $y_0(0) = 0 \; and \; \dot{y}_0(0) = -5$.

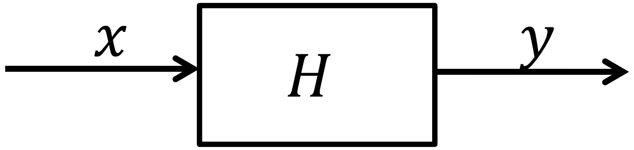
```
>> r1 = roots([1  3  2])

r1 =

    -2
    -1

>> c = inv([1 1; -2 -1])*[0;-5]

c =

     5
    -5
```

- Thus, zero-input response is $y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = 5e^{-2t} - 5e^{-t}$

3.3

Compute Zero-State Response of a CT System

General Form of CT System Equations



Let's review the linear constant coefficient ordinary differential equation (LCCODE) again:

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t) \qquad (\text{Eq. } 1)$$

With given all-zero initial conditions:

$$y^{(n)}(0) = 0, \cdots, \ddot{y}(0) = 0, \dot{y}(0) = 0, y(0) = 0$$

Discussion: is this system linear? Is this system time invariant?

What is the characteristic of LTI systems?

The Unit Impulse Response

- The impulse response $h(t)$ is the system response to an impulse input $\delta(t)$ applied at time t = 0 with all initial conditions zero at $t = 0^{-}$.
- An impulse is like a lightning, which strikes and then vanishes. But at that single moment, objects that have been struck are rearranged.
- Similarly, an impulse input $\delta(t)$ appears momentarily at t=0, and then it is gone forever. But in the moment, it generates energy; it creates nonzero initial conditions simultaneously within the system at $t = 0^{+}$.
- Therefore, system will have a response to this newly generated initial conditions.
- Therefore, the impulse response must consist of the system characteristic modes for $t \geq 0^{+}$.

The Unit Impulse Response

- To find the unit impulse response for a system given by the following differential equation:

$$(D^N + a_1 \ D^{N-1} + \cdots + a_{N-1}D + a_N)y(t) = (b_{N-M} \ D^M + b_{N-M+1} \ D^{M-1} + \cdots + b_{N-1}D + b_N)x(t)$$

- Firstly , we note that in a practical system, it is true that $m \leq n$.
- Secondly, at the single moment t = 0, at most the response can be an impulse $A_0\delta(t)$.
- Therefore, the form of complete impulse response is:

$$h(t) = A_0\delta(t) + \text{characteristic mode terms} \qquad \mathrm{t} \geq 0$$

- The next step is to find the coefficients for the characteristic mode terms and A_0, the method is called simplified impulse matching.

Compute Impulse Response From Simplified Impulse Matching

- Since the system equation is:

$$(D^N + a_1 \; D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_{N-M} \; D^M + b_{N-M+1} \; D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D)\, y(t) \; = \; P(D) x(t)$$

- And the unit impulse response is in the form of:

$$h(t) = \; b_0 \, \delta(t) + [P(D) \, y_n \, (t)] u(t)$$

- $b_0 = 0 \; if \; M < N.$
- $y_n(t)$ is a linear combination of the characteristic modes of the system subject to the following simplified initial conditions:

$$y_n(0) = \dot{y}_n(0) = \; \ddot{y}_n(0) = \; \ldots = \; y_n^{(N-2)}(0) = 0 \qquad and \quad y_n^{(N-1)}(0) = 1$$

- Where $y_n^{(k)}(0)$ is the value of kth derivative of $y_n(t)$ at $t = 0$.

Compute Impulse Response From Simplified Impulse Matching

We can express this set of conditions for various N (the system order) as follows:

- $N = 1$: $y_n(0) = 1$
- $N = 2$: $y_n(0) = 0, \text{and } \dot{y}_n(0) = 1$
- $N = 3$: $y_n(0) = \dot{y}_n(0) = 0 \text{ and } \ddot{y}_n(0) = 1$
- $N = 4$: $y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = 0, \text{and } y_n^{(3)}(0) = 1$

And so on.

With this set of constraints, it is easy to compute the linear coefficients for each characteristic terms in $y_n(t)$.

Example – Compute Impulse Response

Determine the impulse response h(t) for an LTIC system specified by the equation:

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

- Step 1. Identify the order of the system $N = 2$
- Step 2. Since $M = 1, M < N$, so $b_0 = 0$
- Step 3. To find $y_n(t)$ as the linear combination of the characteristic modes of equation $D^2 + 3D + 2 = 0$ with simplified initial condition: $y_n(0) = 0,$ and $\dot{y}_n(0) = 1$

  The characteristic modes of this equation is e^{-2t} and e^{-t}.

  $y_n(t) = K_1 e^{-2t} - K_2 e^{-t}$, with $y_n(0) = 0,$ and $\dot{y}_n(0) = 1$

  $y_n(0) = K_1 - K_2,$ and $\dot{y}_n(0) = -2K_1 + K_2 = 1,$ thus $K_1 = K_2 = -1$

$$y_n(t) = -1e^{-2t} + e^{-t}$$

Step 4. To find $h(t) = b_0\,\delta(t) + [P(D)y_n(t)]u(t)$.

$$\boxed{h(t) = b_0\,\delta(t) + [P(D)y_n(t)]u(t) = (2e^{-2t} - e^{-t})u(t)}$$

Example – Compute Impulse Response Using MATLAB

Determine the impulse response h(t) for an LTIC system specified by the equation:

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

- Step 1. Identify the order of the system $N = 2$
- Step 2. Since $M = 1, M < N$, so $b_0 = 0$
- Step 3. To find $y_n(t)$ as the linear combination of the characteristic modes of equation $D^2 + 3D + 2 = 0$ with simplified initial condition: $y_n(0) = 0$, and $\dot{y}_n(0) = 1$

```
>> syms y(t)
Dy = diff(y, t);          % First derivative of y with respect to t
D2y = diff(y, t, 2);      % Second derivative of y with respect to t
% Solve for y_n
y_n = dsolve(D2y + 3*Dy + 2*y == 0, y(0) == 0, Dy(0) == 1);
```

- Step 4. To find $h(t) = b_0\,\delta(t) + [P(D)y_n(t)]u(t)$.

```
h=diff(y_n)               % Compute impulse respone h
```

$$\boxed{h(t) = (2e^{-2t} - e^{-t})u(t)}$$

Total Response for a CT LTI System

$$Total\ Response = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{ZIR} + \underbrace{x(t) * h(t)}_{ZSR}$$

3.4

Unit Step Response and its Relation to Unit Impulse Response

The Unit Step Response

- To analyze an LTI system, it is critical to know the impulse response $h(t)$.
- However, $h(t)$ can be practically difficult to measure because in practice, input amplitude is often limited. A very short pulse then has very little energy and its response is hard to measure.
- A common alternative is to measure the step response $s(t)$, the response to a unit step input $u(t)$.

The Unit Step Response

- The relation between the unit step response and the unit impulse response

We know that:

$s(t) = u(t) * h(t)$ and $\delta(t) = \frac{d}{dt}u(t)$

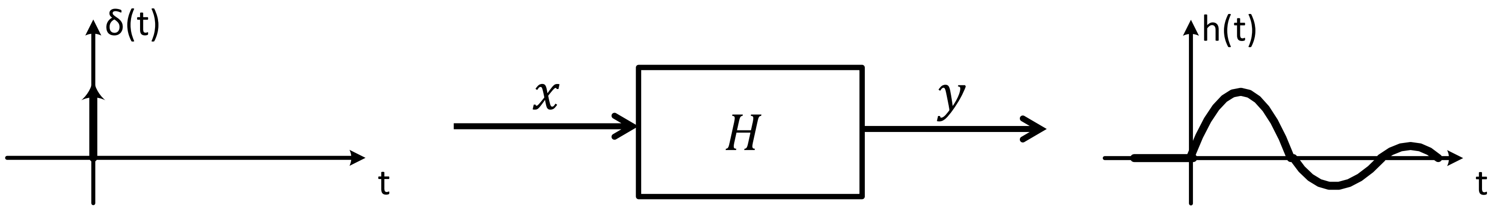
Then:

$$h(t) = \delta(t) * h(t) = \left(\frac{d}{dt}u(t)\right) * h(t) = \frac{d}{dt}\big(u(t) * h(t)\big) = \frac{d}{dt}s(t)$$

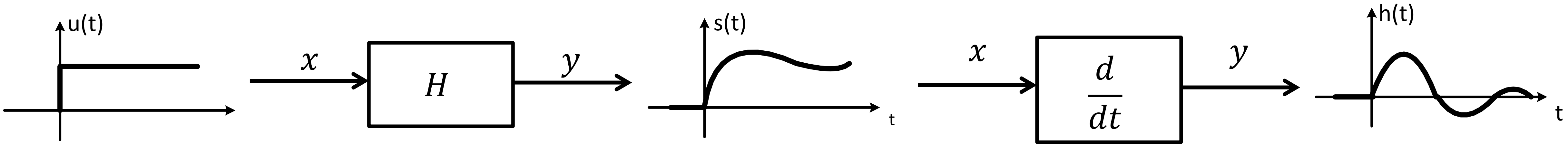
Conclusion: the unit impulse response $h(t)$ is the derivative of the unit step response $s(t)$.

The Unit Step Response

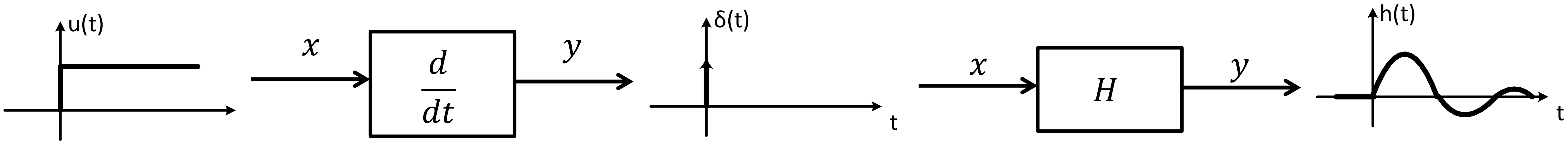
Instead of:



We use:



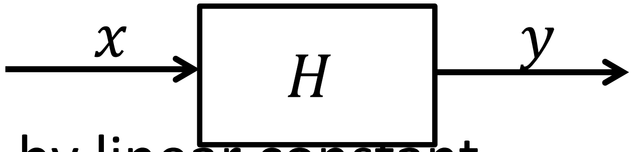
Which is the same as the direct measurement of the impulse response because:



3.5

TOTAL RESPONSE OF DT LTI SYSTEM

General Form of DT System Equations



Similar to CT systems, causal LTI DT systems can be described by linear constant coefficient N-th order difference equations:

$$y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_{N-M} x[n+M-N] + b_{N-M+1} x[n-(N-M+1)] + \cdots + b_N x[n-N]$$

With given initial conditions:

$$y[n-1], y[n-2], \cdots, y[n-N+1], y[n-N]$$

Where,

- N is the order of the system.
- For practical causal systems, $N \geq M$, without losing generality, let $N = M$.

$$y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N] \qquad (\text{Eq.} 8)$$

$$y[n] = -a_1 y[n-1] - \cdots - a_{N-1}\, y[n-N+1] - a_N y[n-N] + b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N] \qquad (\text{Eq.} 9)$$

Recursive Solution of DT System Equations

Based on this N-th order difference equations:

$$y[n] = -a_1 y[n-1] - \cdots - a_{N-1}\, y[n-N+1] - a_N y[n-N] + b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N] \qquad \text{(Eq. 9)}$$

The first method to solve this difference equation is the recursive approach.

Example 3.11. Solver iteratively $y[n] - 0.5y[n-1] = x[n]$ with initial condition $y[-1] = 16$ and causal input $x[n] = n^2 u[n]$.

Solution:

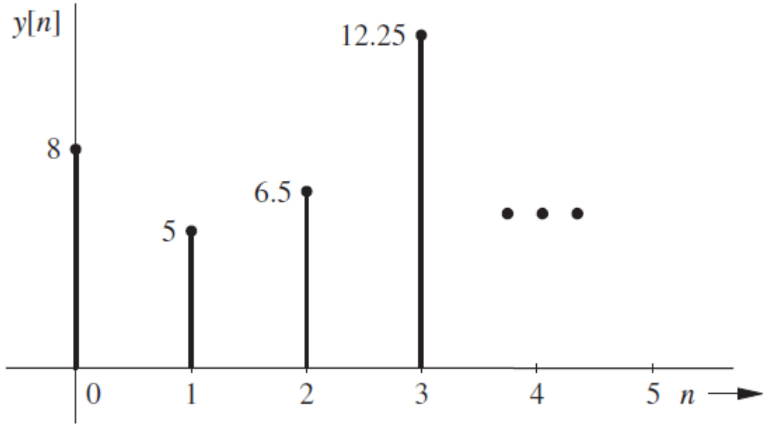
$y[n] = 0.5y[n-1] + x[n]$

$n = 0$: $y[0] = 0.5y[0-1] + x[0] = 0.5(16) + 0 = 8$

$n = 1$: $y[1] = 0.5y[1-1] + x[1] = 0.5(8) + 1 = 5$

$n = 2$: $y[2] = 0.5y[2-1] + x[2] = 0.5(5) + 4 = 6.5$

$n = 3$: $y[3] = 0.5y[3-1] + x[3] = 0.5(6.5) + 9 = 12.25$



General Form of DT System Equations

$$x \longrightarrow \boxed{H} \longrightarrow y$$

Similar to CT systems, causal LTI DT systems can be described by linear constant coefficient N-th order difference equations:

$$\boxed{y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_{N-M} x[n+M-N] + b_{N-M+1} x[n-(N-M+1)] + \cdots + b_N x[n-N]}$$

With given initial conditions:

$$y[n-1], y[n-2], \cdots, y[n-N+1], y[n-N]$$

Where,

- N is the order of the system.
- For practical causal systems, $N \geq M$, without losing generality, let $N = M$.

$$y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N] \qquad (\text{Eq. } 8)$$

Or

$$y[n] = -a_1 y[n-1] - \cdots - a_{N-1}\, y[n-N+1] - a_N y[n-N] + b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N] \qquad (\text{Eq. } 9)$$

Find the Total Response of the DT System Equation

Let’s examine this system equation:

$$y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N] \qquad \text{(Eq. 8)}$$

Using operator notation E to represent operation for advancing a sequency by one time unit,

$$Ex[n] \equiv x[n+1];\; E^2 x[n] \equiv x[n+2]; \cdots ; E^N x[n] \equiv x[n+N]$$

Substitute E and using time invariant nature of the system, Eq.8 becomes:

$$(E^N + a_1\, E^{N-1} + \cdots + a_{N-1}E + a_N) y[n-N] = (b_0\, E^N + b_1\, E^{N-1} + \cdots + b_{N-1}E + b_N) x[n-N]$$
$$(E^N + a_1\, E^{N-1} + \cdots + a_{N-1}E + a_N) y[n] = (b_0\, E^N + b_1\, E^{N-1} + \cdots + b_{N-1}E + b_N) x[n]$$

we can express the equation as:

$$Q(E) y[n] = P(E) x[n] \qquad \text{(Eq. 10)}$$

Where the polynomials $Q(E)$ and $P(E)$ are:

$$Q(E) = E^N + a_1\; E^{N-1} + \cdots + a_{N-1}E + a_N$$
$$P(E) = \; b_{N-M}\, E^M + b_{N-M+1}\; E^{M-1} + \cdots + b_{N-1}E + b_N$$

Find the Total Response of the DT System Equation

As you can see, the format of the DT system equation becomes very similar to its CT counter part. And the total response will be:

$$Total\ Response = \underbrace{zero - input\ response}_{\substack{when\ x[n]=0 \\ results\ from\ \boldsymbol{internal\ system\ conditions}\ only \\ independent\ of\ x[n]}} + \underbrace{zero - state\ response}_{\substack{response\ to\ \boldsymbol{non{-}zero\ x[n]} \\ all\ \boldsymbol{initial\ conditions\ are\ zero} \\ dependent\ on\ x[n]}}$$

Find the Total Response of the DT System Equation

The zero-input response $y_0[n]$ is the solution to: $Q(E)y[n] = 0$,

with $Q(E) = E^N + a_1 \; E^{N-1} + \cdots + a_{N-1}E + a_N$

and given initial conditions: $y_0[n-1], y_0[n-2], \cdots, y_0[n-N+1], y_0[n-N]$

Similar to the CT case, it can be shown that the zero-input response

$$y_0[n] = c_1\gamma_1^n + c_2\gamma_2^n + \cdots + c_N\gamma_N^n$$

- $\gamma_1, \gamma_2, \cdots, \gamma_N$ are the characteristic roots (eigenvalues) of the system.
- They can be computed as roots of the characteristic equation $Q(\gamma) = 0$.
- The corresponding characteristic modes (or characteristic polynomials) of the system are: $\gamma_1^n, \gamma_2^n, \cdots, \gamma_N^n$.
- The linear coefficients $c_1, c_2, \cdots c_N$ are computed using initial conditions.

Compute Impulse Response From Simplified Impulse Matching

- The impulse response $h[n]$ is the solution of the following equation for the input $\delta[n]$:

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)h[n] = (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N)\delta[n]$$

$$or, \qquad Q(E)h[n] \ = \ P(E)\delta[n]$$

- Subject to initial conditions: $h[0], h[1], \cdots, h[N-1]$ given.
- And the unit impulse response is in the form of:

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

- $y_c[n]$ is a linear combination of the characteristic modes of the system subject to the following the initial conditions for $h[n]$: $h[0], h[1], \cdots, h[N-1]$ given.

Total Response for a DT LTI System

$$Total\ Response = \underbrace{\sum_{k=1}^{N} c_k \gamma_k^n}_{ZIR} + \underbrace{x[n] * h[n]}_{ZSR}$$

Homework:

Review: example 2.1, 2.2, 3.18, 3.20.

Problems: 2.2-4, 2.2-10, 2.2-11, 3.4-9, 3.4-10, 3.4-12