HUMBER ENGINEERING

ENGI 1000 - Physics 1

WEEK 11 - MODULE 8

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Module 8

Rotational Motion of a Rigid Object About a Fixed Axis

- Rotational Kinematics
  - Angular Position, Velocity and Acceleration
  - Rigid Object Under Constant Angular Acceleration
- Rotational Dynamics
  - Torque
  - Rigid Object Under a Net Torque
  - Moment of Inertia

Introduction

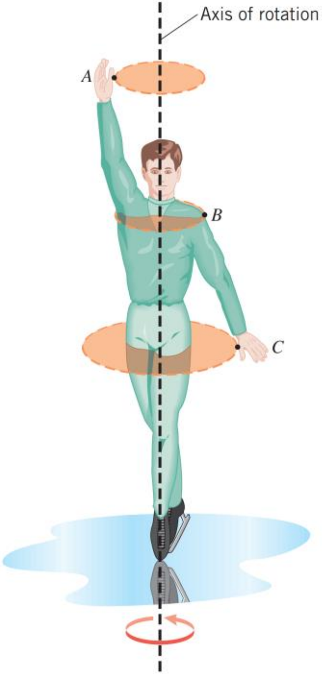
- Recall:
    - Applying a force to an object floating in space causes it to accelerate in the direction of that force.
    - Linear momentum of an object is equal to the product of its mass and its velocity.
    - Acceleration is the rate of change of velocity, and velocity is the rate of change of displacement (with respect to *time*).

- … But what if our applied force results in **rotational** motion instead of **translational** motion?

- We will see that the equations of motion are conceptually very similar, however they use the quantities of angular position, angular velocity, and angular acceleration.

Rotational Motion of a Rigid Object about a Fixed Axis

- When an extended object such as wheel rotates about its axis, at any time different parts of the object have different linear velocities and linear accelerations.
- Therefore, the motion cannot be analyzed by modeling the object as a particle.
- In analyzing the rotational motion, we assume that the object is rigid.
- A **rigid object** is one that is nondeformable during the motion; it means the relative locations of all particles of the object remain constant.
- In the simplest kind of rotational motion, points on a rigid object move on *circular paths*.
  - For example, the circular paths for points *A, B* and *C* on a spinning skater.
  - The centers of all such circular paths define a line, called the *axis of rotation.*



Angular Position

- **Angular Position (θ):** The angle between the position line of the point on the rigid object and the fixed reference line in space, which is often chosen as the x-axis.
- We use the polar coordinate system to show the location of objects.
- Consider the point P on the disc on the reference line at angle $\theta = 0$.
- As the disc rotates, the point traces out an arc of length s, which is measured along a circle of radius r.
- The arc length s is related to the angle θ as below

$$\boxed{s = r\theta}$$

The arc length

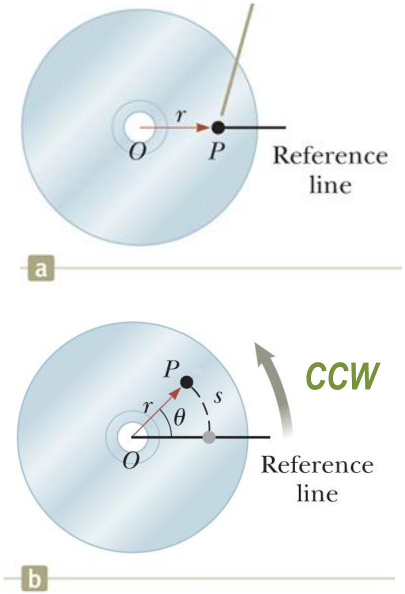
Distance from origin to *P*

The measured ***CCW angle*** from the fixed reference line (rad)

$$C = 2\pi r \Rightarrow 360° = 2\pi \text{ rad}$$

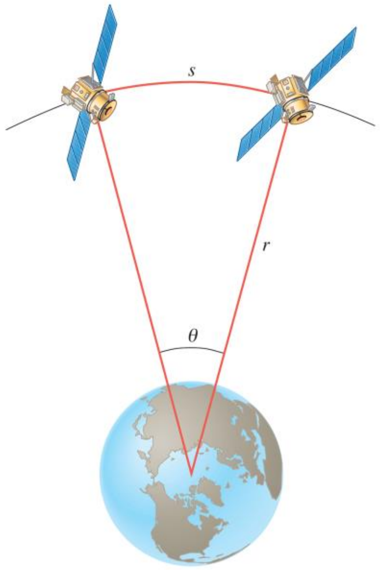
$$\theta(\text{rad}) = \frac{\pi}{180°}\theta(\text{deg})$$

$$1 \text{ rad} = \frac{360°}{2\pi} \approx 57.3°$$



Angular Position

Example 1 (Satellites): Synchronous communication satellites are put into an orbit whose radius is $r = 4.23 \times 10^7\ m$. The orbit is in the plane of the equator, and two adjacent satellites have an angular separation of $\theta = 2.00°$. Find the arc length s that separates the satellites.



To convert 2.00° into radians, we use the fact that 2π radians is equivalent to 360°

$$2.00° = (2.00\ deg)\left(\frac{2\pi\ rad}{360\ deg}\right) = 0.0349\ rad$$

The arc length between the satellites is

$$s = r\theta = (4.23 \times 10^7 m)(0.0349\ rad) = 1.48 \times 10^6\ m$$

Note that the radian, being a unitless quantity, is dropped from the final result, leaving the answer expressed in meters.

Average Angular Velocity

- **Angular Velocity (ω)** describes how fast a rigid object rotating about the fixed axis.
- The **average angular velocity** is defined as the ratio of the angular displacement of a rigid object to the time interval during that the angular displacement occurs.

$$\omega_{\text{avg}} \equiv \frac{\Delta\theta}{\Delta t}$$

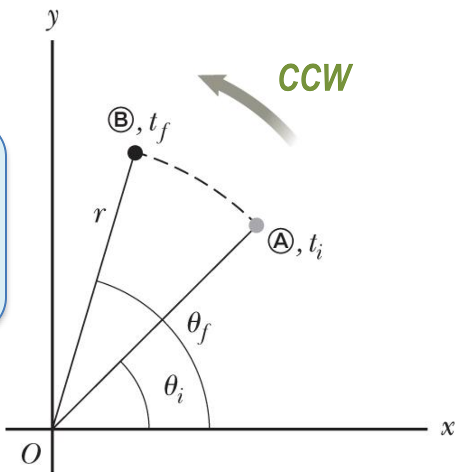
Average angular velocity (rad/s)

Elapsed time (s)

Angular displacement (rad)

$$\Delta\theta = \theta_f - \theta_i$$

$$\Delta t = t_f - t_i$$



- If the rotation is CCW $\rightarrow$ $\omega > 0$
- If the rotation is CW $\rightarrow$ $\omega < 0$

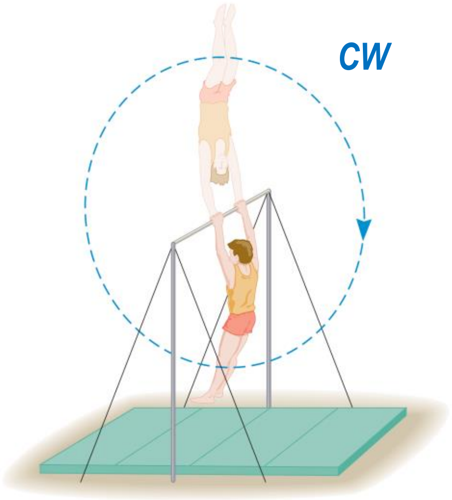
Average Angular Velocity

Example 2 (Gymnast on a High Bar): A gymnast on a high bar swings through two CW revolutions in a time of 1.90 s. Find the average angular velocity (in rad/s) of the gymnast.

Convert the angular displacement of the gymnast to radians

$$\Delta\theta = (-2.00\, rev)\left(\frac{2\pi\, rad}{1\, rev}\right) = -12.6\, rad$$

The minus sign shows a CW rotation



The average angular velocity is,

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{-12.6\, rad}{1.90\, s} = -6.63\, rad/s$$

Instantaneous Angular Velocity

- **Angular Velocity (ω)** describes how fast a rigid object rotating about the fixed axis.
- In analogy to translational motion, we can define the **instantaneous angular velocity** as the limit of the average angular velocity as Δt approaches zero.

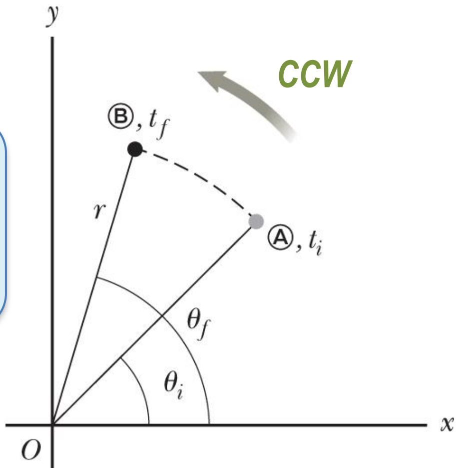
$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

Instantaneous angular velocity (rad/s)

Derivative of the angular position with respect to time.

$$\Delta \theta = \theta_f - \theta_i$$

$$\Delta t = t_f - t_i$$



Quick Quiz 1

- A rigid object rotates in a counterclockwise sense around a fixed axis. Each of the following pairs of quantities represents an initial angular position and a final angular position of the rigid object.

- Which of the sets can ***only*** occur if the rigid object rotates through more than 180°?
    a) 3 rad, 6 rad
    b) –1 rad, 1 rad
    c) 1 rad, 5 rad

Quick Quiz 2

- A rigid object rotates in a counterclockwise sense around a fixed axis. Each of the following pairs of quantities represents an initial angular position and a final angular position of the rigid object.
- Suppose the change in angular position for each of these pairs of values occurs in 1 s. Which choice represents the lowest average angular speed?
  a) 3 rad, 6 rad
  b) –1 rad, 1 rad
  c) 1 rad, 5 rad

Average Angular Acceleration

- **Angular Acceleration (α)** shows the rate of change in the angular velocity of a rotating rigid object.
- The **average angular acceleration** is defined as the ratio of the change in the angular velocity of a rotational rigid object to the time interval during the angular velocity change occurs.

$$\alpha_{\text{avg}} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

Average angular acceleration (rad/s^2)

Elapsed time (s)

Angular velocity change (rad/s)

- If the rotation is CCW and speeding up $\rightarrow$ $\alpha > 0$
- If the rotation is CW and slowing down $\rightarrow$ $\alpha > 0$
- If the rotation is CCW and slowing down $\rightarrow$ $\alpha < 0$
- If the rotation is CW and speeding up $\rightarrow$ $\alpha < 0$

Average Angular Acceleration

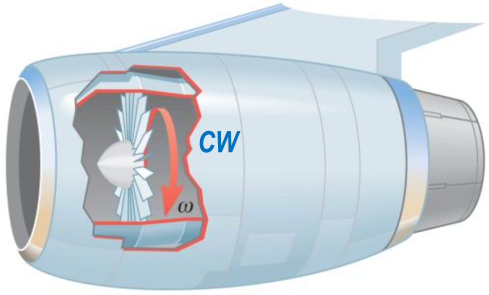
Example 3 (A Jet Revving its Engines): A jet awaiting clearance for takeoff is momentarily stopped on the runway. The fan blades of its front engine are rotating CW with an angular velocity of -110 rad/s, As the plane takes off, the angular velocity of the blades reaches -330 rad/s in the time of 14 s. Find the angular acceleration assuming it to be constant.

Apply the definition of average angular acceleration,

$$\alpha_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{(-330\, rad/s) - (-110\, rad/s)}{14\, s} = -16\, rad/s^2$$

The magnitude of the angular velocity increases by 16 rad/s

The negative sign indicates that the angular acceleration is in CW direction.



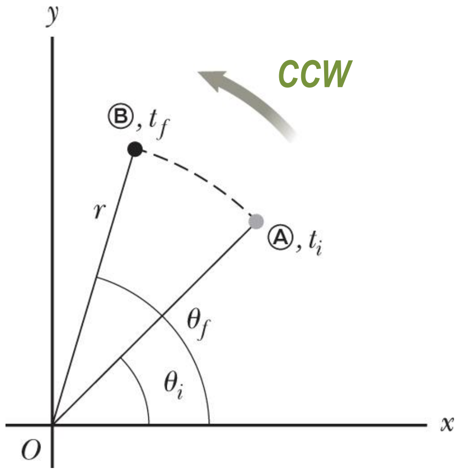
Instantaneous Angular Acceleration

- **Angular Acceleration (α)** shows the rate of change in the angular velocity of a rotating rigid object.
- In analogy to translational motion, we can define the **instantaneous angular acceleration** as the limit of the average angular acceleration as Δt approaches zero.

$$\boxed{\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}}$$

Instantaneous angular acceleration (rad/s^2)

Derivative of the angular velocity with respect to time.



$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}}$$

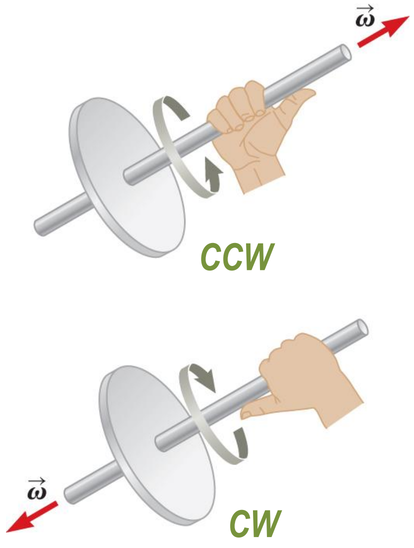
Direction of Angular Vectors

- Since the angular velocity and the angular acceleration are <u>vector quantities</u>, we need to determine the direction of these vectors.

- If the object rotates in xy plane, the direction of angular velocity is determined by using the **right-hand rule**.

  - When the four finger of the right hand are wrapped in the direction of rotation, the extended <u>right thumb</u> points in the direction of $\vec{\omega}$.
  - If the angular speed is increasing $\rightarrow$ $\vec{\alpha}$ is in the same direction as $\vec{\omega}$
  - If the angular speed is decreasing $\rightarrow$ $\vec{\alpha}$ is in the opposite direction of $\vec{\omega}$

- Since the rotation is about a fixed axis, we can use non-vector notations and indicate the vectors directions by assigning a positive and negative sign to ω and α.



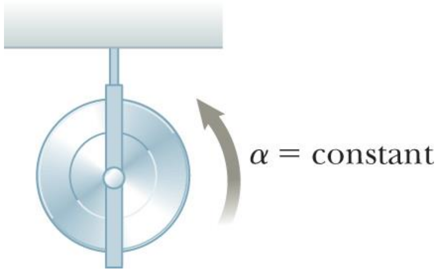
Rotational Kinematics

- In analogy to translational motion, we can derive the kinematic equations for rotational motion of a rigid object under constant angular acceleration motion.

Rigid Object Under Constant Angular Acceleration α	Particle Under Constant Acceleration a
$\theta\,, \quad \omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$	$x, \quad v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2v}{dt^2}$
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

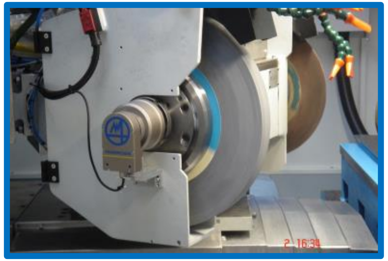
Rigid Object Under Constant Angular Acceleration Motion

- Imagine an object that undergoes a spinning motion such that its angular acceleration is constant.



Examples:

- During its spin cycle, the tub of a clothes washer begins from rest and accelerates up to its final spin speed
- A workshop grinding wheel is turned off and comes to rest under the action of a constant friction force in the bearings of the wheel
- The crankshaft of a diesel engine changes to a higher angular speed



Constant Angular Acceleration Motion

Example 4 (Rotating Wheel): A wheel rotates with a constant angular acceleration of 3.50 rad/s^2.

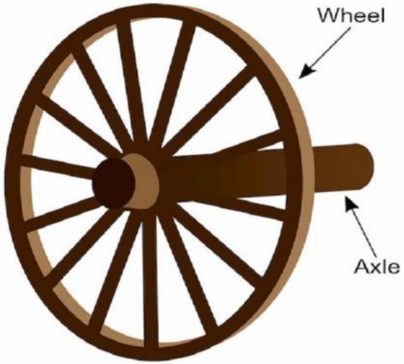
(a) If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, through what angular displacement does the wheel rotate in 2.00 s?

The rigid object under constant angular acceleration motion,

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \quad \rightarrow \quad \underbrace{\theta_f - \theta_i}_{\Delta\theta} = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = (2.00 \text{ rad/s})(2.00 \text{ s}) + \frac{1}{2}(3.50 \text{ rad/s}^2)(2.00 \text{ s})^2 = \boxed{11.0 \text{ rad}}$$

$$\Delta\theta = (11.0 \text{ rad})(180°/\pi \text{ rad}) = \boxed{630°}$$



Constant Angular Acceleration Motion

Example 4 (Rotating Wheel): A wheel rotates with a constant angular acceleration of 3.50 rad/s^2.

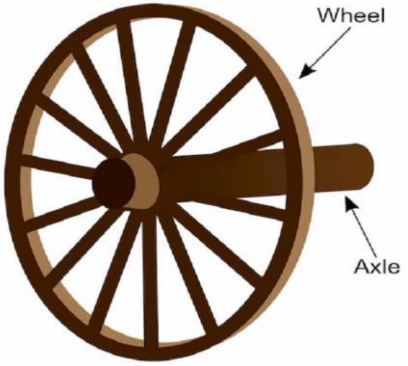
(b) Through how many revolutions has the wheel turned during this time interval?

$$\Delta\theta = 630^\circ \left(\frac{1\ \mathrm{rev}}{360^\circ}\right) = \boxed{1.75\ \mathrm{rev}}$$

(c) What is the angular speed of the wheel at t = 2.00 s?

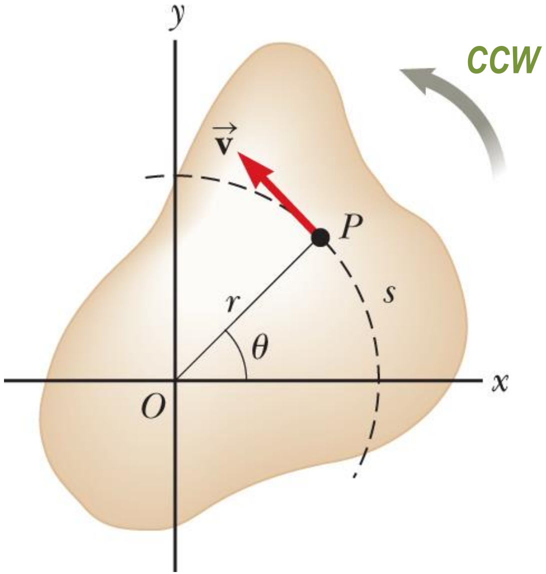
The rigid object under constant angular acceleration motion,

$$\begin{aligned}\omega_f &= \omega_i + \alpha t \\ &= (2.00\ \mathrm{rad/s}) + (3.50\ \mathrm{rad/s^2})(2.00\ \mathrm{s}) = 9.00\ \mathrm{rad/s}\end{aligned}$$



Angular and Translational Quantities

- Assume a rigid body rotates through an angle θ, any point on the body, like point P, moves on a circular arc of length s, and radius r.
- Recall the Circular Motion, point P has a translational velocity v that is always tangent to the circular path.
- Magnitude of the tangential velocity of point P is obtained as below



$$v = \frac{ds}{dt} \xrightarrow{s=r\theta} v = r\frac{d\theta}{dt}$$

$$\boxed{v = r\omega}$$

Translational velocity

Angular velocity

Angular and Translational Quantities

- Assume a rigid body rotates through an angle θ, any point on the body, like point P, moves on a circular arc of length s, and radius r.

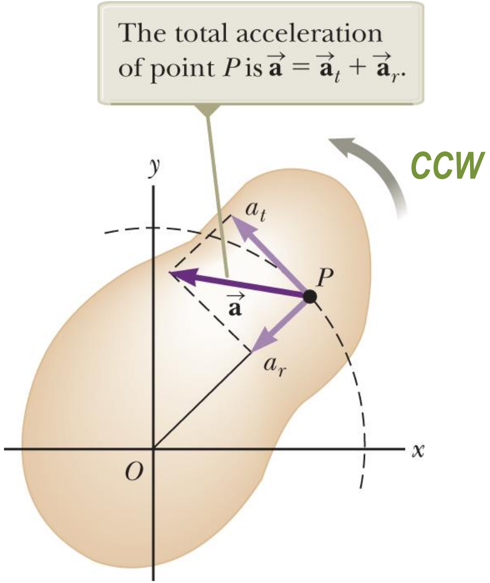
- <u>Recall the Circular Motion</u>, the point P is also experiences a total translational acceleration $\vec{a}$ that is the vector sum of the radial or centripetal acceleration $\vec{a}_r$ and the tangential acceleration $\vec{a}_t$:

$$\boxed{\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}_{\boldsymbol{t}} + \vec{\boldsymbol{a}}_{\boldsymbol{r}}}$$

$$a_t = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} \quad \Rightarrow \quad \boxed{a_t = r\alpha}$$

$$a_r = a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} \quad \Rightarrow \quad \boxed{a_r = r\omega^2}$$

$$a = \sqrt{{a_t}^2 + {a_r}^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} \quad \Rightarrow \quad \boxed{a = r\sqrt{\alpha^2 + \omega^4}}$$

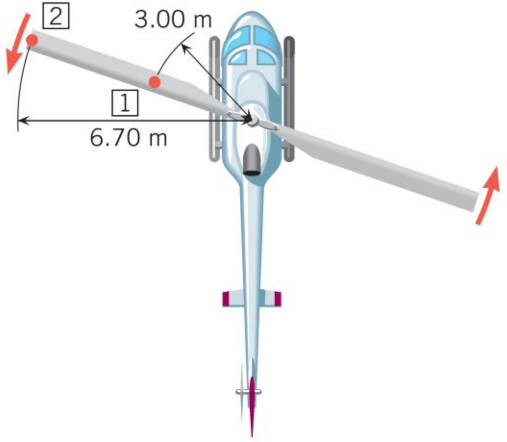


Angular and Translational Quantities

Example 5 (A Helicopter Blade): A helicopter blade has an angular speed of ω = 6.50 rev/s and has an angular acceleration of α = 1.30 rev/s^2. For points 1 and 2 on the blade,
(a) Find the magnitudes of the tangential speeds.

Converting the angular speed to rad/s from rev/s,

$$\omega = \left(6.50\frac{rev}{s}\right)\left(\frac{2\pi\, rad}{1\, rev}\right) = 40.8\, rad/s$$



The tangential speed for each point is,

$$\boxed{v = r\omega}$$

$$v_1 = r_1\omega = (3.00\, m)(40.8\, rad/s) = 122 \text{ m/s}$$

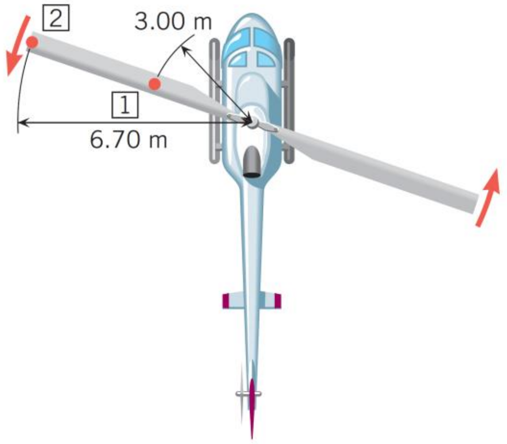
$$v_2 = r_2\omega = (6.70\, m)(40.8\, rad/s) = 273 \text{ m/s}$$

Angular and Translational Quantities

Example 5 (A Helicopter Blade): A helicopter blade has an angular speed of ω = 6.50 rev/s and has an angular acceleration of α = 1.30 rev/s^2. For points 1 and 2 on the blade,
(b) Find the magnitudes of the tangential accelerations.

Converting the angular acceleration to rad/s^2 from rev/s^2,

$$\alpha = \left(1.30\frac{rev}{s^2}\right)\left(\frac{2\pi\, rad}{1\, rev}\right) = 8.17\, rad/s^2$$



The tangential acceleration for each point is,

$$\boxed{a_t = r\alpha}$$

$$a_{t1} = r_1\alpha = (3.00\, m)(8.17\, rad/s^2) = 24.5\ \mathrm{m/s^2}$$

$$a_{t2} = r_2\alpha = (6.70\, m)(8.17\, rad/s^2) = 54.7\ \mathrm{m/s^2}$$

Quick Quiz 3

- Ethan and Rebecca are riding on a merry-go-round. Ethan rides on a horse at the outer rim of the circular platform, twice as far from the center of the circular platform as Rebecca, who rides on an inner horse.
- When the merry-go-round is rotating at a constant angular speed, what is Ethan's angular speed?
  a) twice Rebecca's
  b) the same as Rebecca's
  c) half of Rebecca's
  d) impossible to determine

Quick Quiz 4

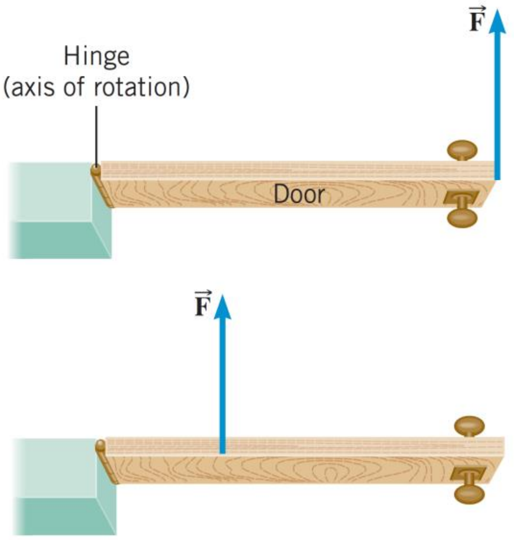
- Ethan and Rebecca are riding on a merry-go-round. Ethan rides on a horse at the outer rim of the circular platform, twice as far from the center of the circular platform as Rebecca, who rides on an inner horse.
- When the merry-go-round is rotating at a constant angular speed, describe Ethan’s tangential speed.
    a) twice Rebecca’s
    b) the same as Rebecca’s
    c) half of Rebecca’s
    d) impossible to determine

Rotational Dynamics

- In the study of translational motion, after investigating the kinematics of motion, we studied the dynamics or the cause of changes in motion: **Force**

- We follow the same plan here in rotational motion:

What is the cause of changes in rotational motion?



- Imagine trying to rotate a door by applying a force of magnitude F perpendicular to the door surface <u>near the hinges</u> and then at <u>various distances from the hinges.</u>

- You will achieve a <u>more rapid rate of rotation</u> for the door by applying the force near the doorknob than by applying it near the hinges.

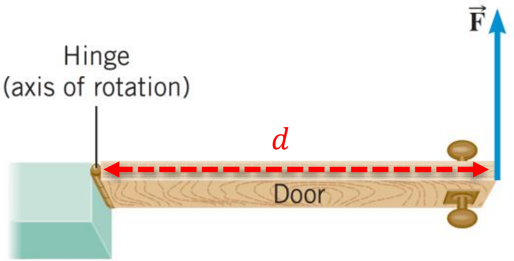
Why?

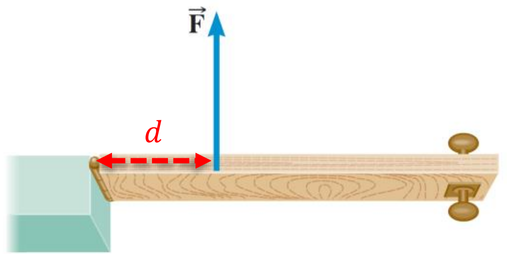
Torque

- **Torque (τ):** is a quantity to measure the tendency of a force to rotate an object about some axis.

$$\boxed{\tau \equiv Fd}$$

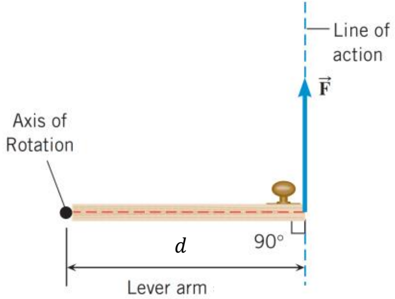
Torque (N.m) **Force (N)** **Moment arm Lever arm (m)**

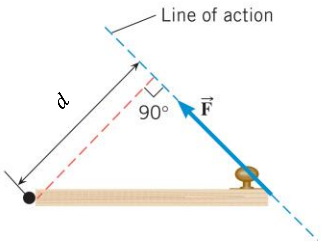


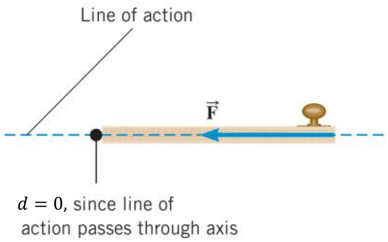


- Torque is a vector quantity, and its SI unit is (N.m)

- The ***lever arm*** is the distance between the line of action of the force and the axis of rotation, measured on a line that is perpendicular to both.







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Torque

- Consider the wrench that we wish to rotate around an axis that is perpendicular to the page and passes through the center of the bolt.
- The applied force $\vec{F}$ acts at an angle ϕ to the horizontal.
- We define the magnitude of the torque associated with the force $\vec{F}$ around the axis passing through O as below

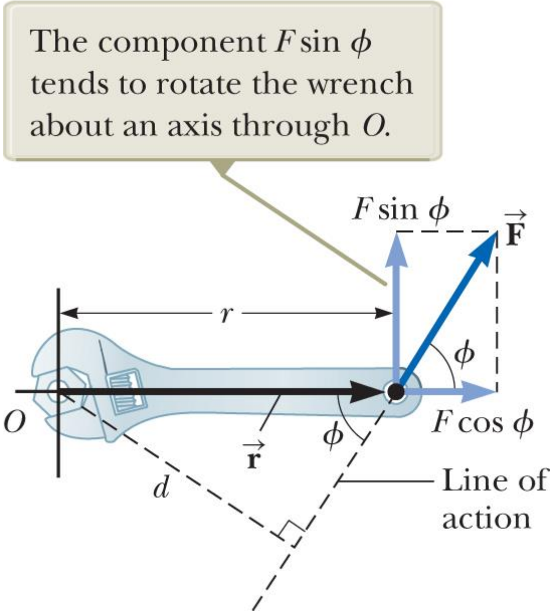
$$\tau \equiv Fd = Fr\sin\phi \ = r\underbrace{F\sin\phi}$$

Moment arm: The perpendicular distance from the rotation axis to the line of action of force

$$d = r\sin\phi$$

The distance between the rotation axis and the point of application of force

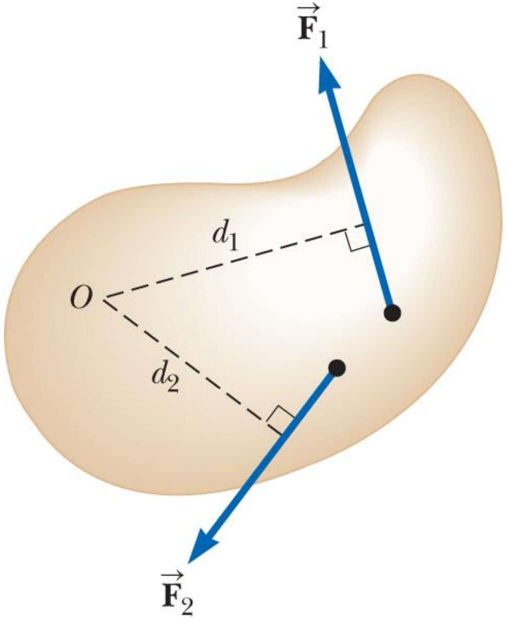
The component of the force tends to rotate the object about the axis



The Net Torque

- When two or more forces act on a rigid object, each tends to produce rotation about the axis.
- For example, $\vec{F}_2$ tends to rotate the object clockwise and $\vec{F}_1$ tends to rotate it counterclockwise.
- We use the convention that the sign of the torque resulting from a force is
  - Positive if the turning tendency of the force is CCW.
  - Negative if the turning tendency of the force is CW.
- The ***net torque*** about an axis through *O* is

$$\boxed{\sum \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2}$$

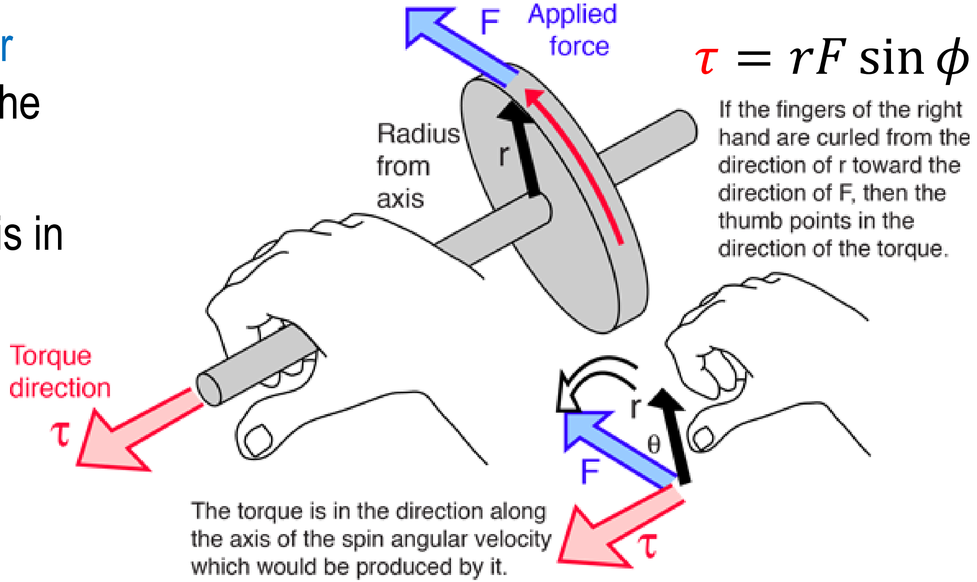


Direction of Torque

- The direction of torque is perpendicular to both the radius from the axis and to the force.
- It is conventional to choose it in the **right-hand rule** direction along the axis of rotation.

- The torque is in the direction of the angular velocity which would be produced by it in the absence of other influences.
- In general, the change in angular velocity is in the direction of the torque.

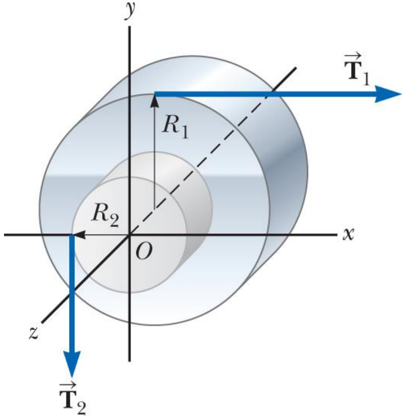
$$\tau = rF \sin \phi$$



The Net Torque

Example 6 (The Net Torque of a Cylinder): A one−piece cylinder is shaped with a core section protruding from the larger drum. The cylinder is free to rotate about the central z axis. A rope wrapped around the drum, which has radius R_1, exerts a force $\vec{T}_1$ to the right on the cylinder. A rope wrapped around the core, which has radius R_2, exerts a force $\vec{T}_2$ downward on the cylinder.

Imagine that the cylinder is a shaft in a machine. The force T_1 could be applied by a drive belt wrapped around the drum. The force T_2 could be applied by a friction brake at the surface of the core.



The Net Torque

Example 6 (The Net Torque of a Cylinder): A one−piece cylinder is shaped with a core section protruding from the larger drum. The cylinder is free to rotate about the central z axis. A rope wrapped around the drum, which has radius R_1, exerts a force $\vec{T}_1$ to the right on the cylinder. A rope wrapped around the core, which has radius R_2, exerts a force $\vec{T}_2$ downward on the cylinder.

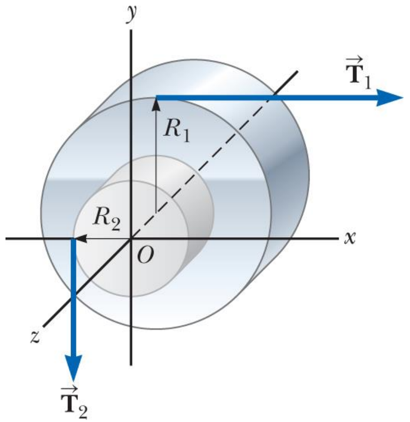
(a) What is the net torque acting on the cylinder about the rotation axis (which is the z axis in the figure)?

Evaluate the net torque about the rotation axis,

$$\sum \tau = \tau_1 + \tau_2 = R_2 T_2 - R_1 T_1$$

If the forces are equal $T_1 = T_2 \;\; \Rightarrow$ Since $R_1 > R_2$, the cylinder would rotate CW.

Because τ_1 would be more effective at turning it than would τ_2



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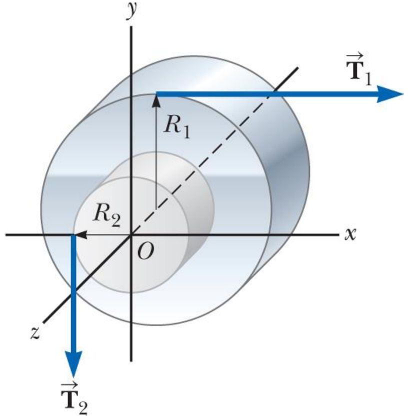
(b) Suppose T_1 = 5.0 N, R_1 = 1.0 m, T_2 = 15 N, and R_2 = 0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate starting from rest?



$$\sum \tau = \tau_1 + \tau_2 = R_2 T_2 - R_1 T_1$$

$$\sum \tau = (0.50\ \mathrm{m})(15\ \mathrm{N}) - (1.0\ \mathrm{m})(5.0\ \mathrm{N}) = \boxed{2.5\ \mathrm{N \cdot m}}$$

The net torque is positive, the cylinder begins to rotate in the counterclockwise (CCW) direction

Particle Under the Net Torque

- Recall the Circular Motion, if a particle of mass m rotates in a circle of radius r the net the force acting on the particle have a tangential and a radial component.

- The radial component $\sum \vec{F}_r$ is directed toward the center of the circle and is responsible for the radial or centripetal acceleration.

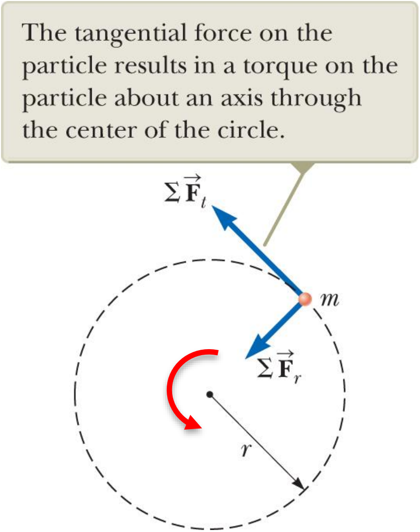
- The tangential component $\sum \vec{F}_t$ is tangent to the circle and is responsible for the tangential acceleration $\vec{a}_t$, which changes the particle speed.

$$\sum F_t = ma_t$$

- The magnitude of the net torque due to $\sum \vec{F}_t$ on the particle about the axis through the center of the circle is

$$\sum \tau = \sum F_t r = (ma_t)r = (mr\alpha)r = \underbrace{(mr^2)}_{\text{Moment of Inertia of particle } I}\alpha$$

$$\boxed{\sum \tau = I\alpha}$$



$$\boxed{\sum \vec{F} = \sum \vec{F}_r + \sum \vec{F}_t}$$

Rigid Object Under the Net Torque

- Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis.
- The object can be regarded as a collection of particles of mass m_i.
- Each particle rotates in a circle about the origin, and each has a tangential acceleration a_i produced by an external tangential force of magnitude F_i.
- For any given particle, we know from Newton's second law that

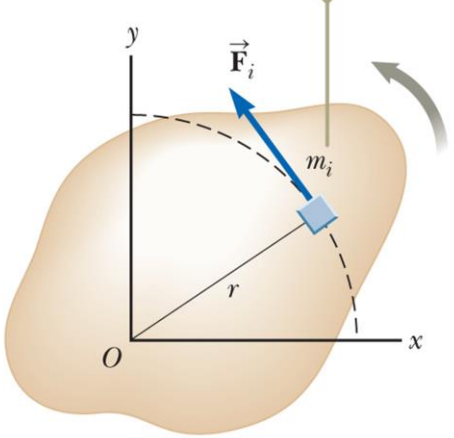
$$F_i = m_i a_i$$

- The external torque $\vec{\tau}_i$ associated with the force $\vec{F}_i$ acts about the origin and its magnitude is given by $$\tau_i = r_i F_i = r_i m_i a_i = m_i r_i{}^2 \alpha$$

- The net torque on the object about an axis due to all external forces:

$$\sum \tau_{ext} = \sum_i \tau_i = \sum_i m_i r_i{}^2 \alpha = \underbrace{\left(\sum_i m_i r_i{}^2 \right)}_{\text{Moment of Inertia of body } I} \alpha \Rightarrow \boxed{\sum \tau_{\text{ext}} = I\alpha}$$

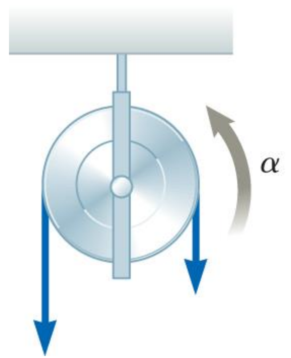
The particle of mass m_i of the rigid object experiences a torque in the same way that the particle in the previous figure does.



Rigid Object Under the Net Torque

- Imagine object free to rotate about a fixed axis.
- The cause of changes in rotational motion of this object is torque applied to the object:

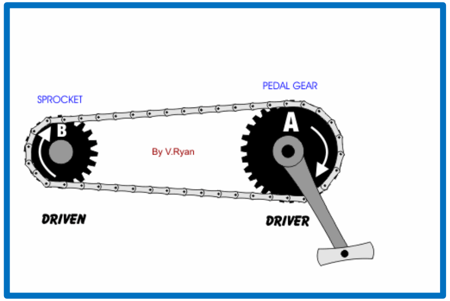
$$\sum \tau_{\text{ext}} = I\alpha$$

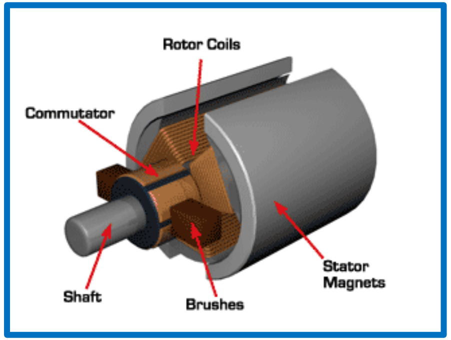


- Torque, moment of inertia, and angular acceleration must all be evaluated around same rotation axis.

Examples:

- A bicycle chain around the sprocket of a bicycle causes the rear wheel of the bicycle to rotate
- The armature of a motor rotates due to the torque exerted by a surrounding magnetic field





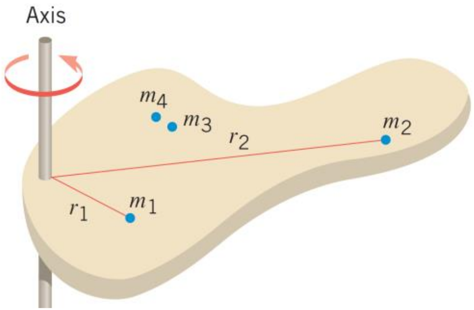
Moment of Inertia

- **Moment of Inertia (I)** of a rigid object is a quantity that depends on the masses of the particles making up the object and their distances from the rotation axis, which is calculated as below

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2 \Rightarrow \boxed{I = \sum_i m_i {r_i}^2}$$

- The moment of inertia plays the same role in rotational motion as the role that mass plays in translational motion

$$\boxed{\sum \tau_{\text{ext}} = I\alpha} \qquad \boxed{\sum F_{\text{ext}} = ma}$$

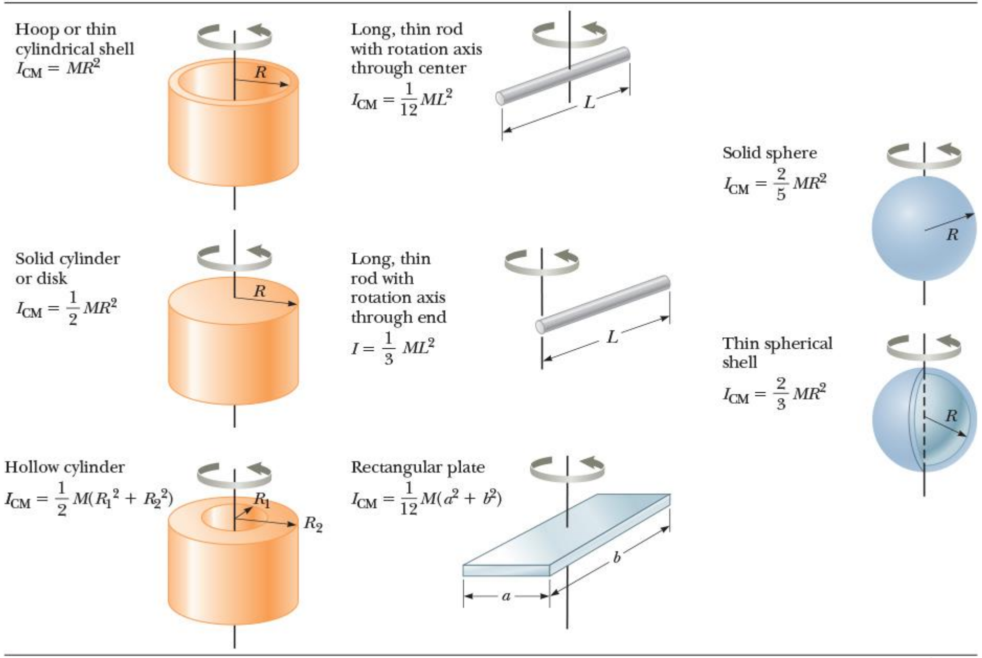


- The moment of inertia is the resistance of the object to changes in rotational motion. It depends on the
  - mass of the object
  - how the mass is distributed around the rotational axis

Moment of Inertia

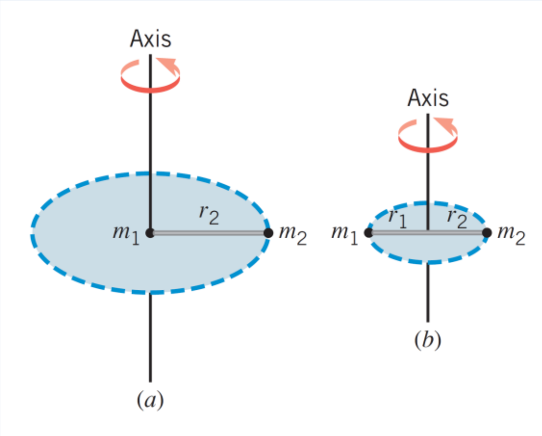
- The moment of inertia of rigid objects with high symmetry are easy to calculate. The rotation axis is the axis of symmetry.

TABLE 10.2 Moments of Inertia of Homogeneous Rigid Objects with Different Geometries



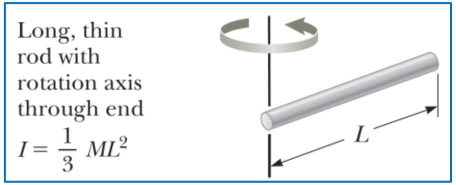
Quick Quiz 5

- Two particles each have a mass *M* and fixed to the ends of a thin rigid rod, whose mass can be ignored. The length of the rod is *L*. What is the moment of inertia when this object rotates relative to an axis that is perpendicular to the rod at (a) one end and (b) the center.
  a) $2ML^2$, ML^2
  b) $0.5ML^2$, $4ML^2$
  c) $0.25ML^2$, $2ML^2$
  d) ML^2 , $0.5ML^2$



Moment of Inertia

Example 7 (Rotating Rod): A uniform rod of length *L* and mass *M* is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane as in the figure. The rod is released from rest in the horizontal position.
What are the initial angular acceleration of the rod and the initial translational acceleration of its right end?

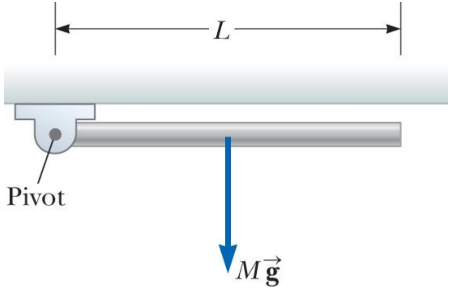


Evaluate the rigid object under net torque,

$$\boxed{\sum \tau_{ext} = I\alpha} \quad \rightarrow \quad \alpha = \frac{\sum \tau_{ext}}{I} = \frac{Mg(L/2)}{\frac{1}{3}ML^2} = \boxed{\frac{3g}{2L}}$$

$$\boxed{\sum \tau_{ext} = Fd} \quad \rightarrow \quad \sum \tau_{ext} = Mg\left(\frac{L}{2}\right)$$

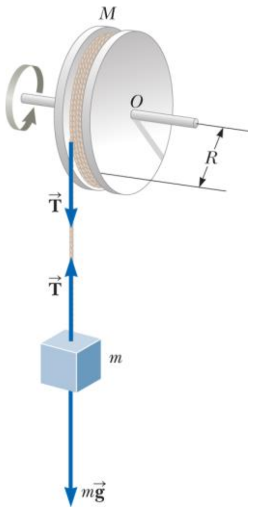
$$\boxed{a_t = r\alpha} \quad \rightarrow \quad a_t = L\alpha = \boxed{\frac{3}{2}g}$$



Moment of Inertia

Example 8 (Angular Velocity): A wheel of radius *R*, mass *M*, and moment of inertia *I* is mounted on a frictionless, horizontal axle as in the figure. A light cord wrapped around the wheel supports an object of mass *m*. When the wheel is released, the object accelerates downward, the cord unwraps off the wheel, and the wheel rotates CCW with an angular acceleration.

(a) Find expressions for the angular acceleration of the wheel, the translational acceleration of the object, and the tension in the cord.



Evaluate the rigid object under net torque:

$$\boxed{\sum \tau_{\text{ext}} = I\alpha} \rightarrow \alpha = \frac{\sum \tau_{\text{ext}}}{I} = \frac{TR}{I} \quad (1)$$

Apply Newton's second law for motion of the object:

$$\sum F_y = mg - T = ma \rightarrow a = \frac{mg - T}{m} \quad (2)$$

Moment of Inertia

Example 8 (Angular Velocity): A wheel of radius *R*, mass *M*, and moment of inertia *I* is mounted on a frictionless, horizontal axle as in the figure. A light cord wrapped around the wheel supports an object of mass *m*. When the wheel is released, the object accelerates downward, the cord unwraps off the wheel, and the wheel rotates CCW with an angular acceleration.

(a) Find expressions for the angular acceleration of the wheel, the translational acceleration of the object, and the tension in the cord.

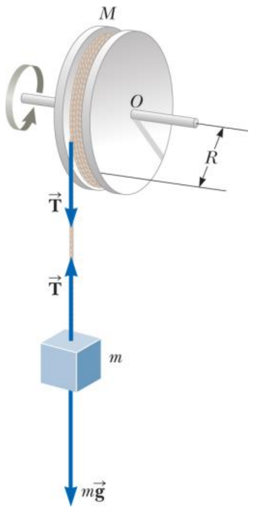
From Equations (1) and (2):

(1) $$\alpha = \frac{TR}{I}$$

(2) $$a = \frac{mg - T}{m}$$

Because the object and wheel are connected by a cord that does not slip, the translational acceleration of the suspended object is equal to the tangential acceleration of a point on the wheel's rim. Therefore, the angular acceleration a of the wheel and the translational acceleration of the object are related by

$$\boxed{a = R\alpha}$$



Moment of Inertia

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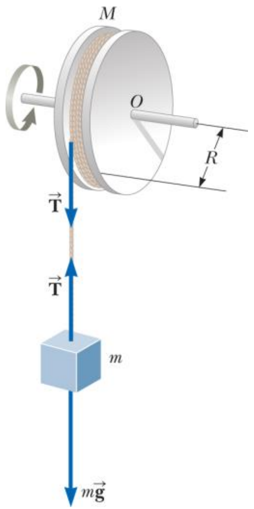

From Equations (1) and (2):

(1) $$\alpha = \frac{TR}{I}$$

(2) $$a = \frac{mg - T}{m}$$

$$\xrightarrow{a = R\alpha} \quad \frac{mg - T}{m} = \frac{TR^2}{I} \quad \Rightarrow \quad T = \boxed{\frac{mg}{1 + (mR^2/I)}}$$

$$a = \boxed{\frac{g}{1 + (I/mR^2)}}$$

$$\alpha = \frac{a}{R} \rightarrow \alpha = \boxed{\frac{g}{R + (I/mR)}}$$

Moment of Inertia

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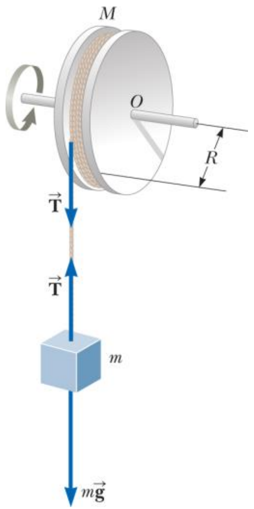
(b) What if the wheel were to become very massive so that *I* becomes very large?

What happens to the acceleration *a* of the object and the tension *T* ?



If the wheel becomes infinitely massive, we can imagine that the object of mass *m* will simply hang from the cord without causing the wheel to rotate.

We can show that mathematically by taking the limit $I \to \infty$

$$a = \lim_{I \to \infty} \frac{g}{1 + (I/mR^2)} = 0 \qquad T = \lim_{I \to \infty} \frac{mg}{1 + (mR^2/I)} = mg$$

The object simply hangs at rest in equilibrium between the gravitational force and the tension in the string.

THANK YOU

HUMBER

WE ARE HUMBER