HUMBER ENGINEERING

MENG 3510 – Control Systems
LECTURE 4

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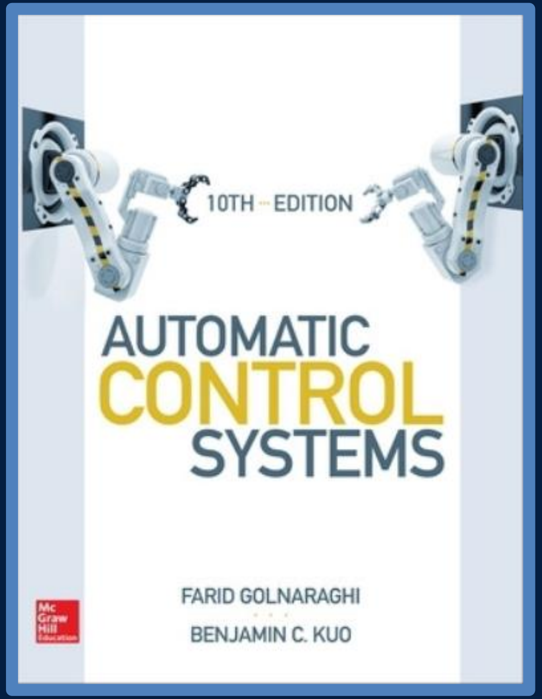
LECTURE 4
Stability of Linear Control Systems

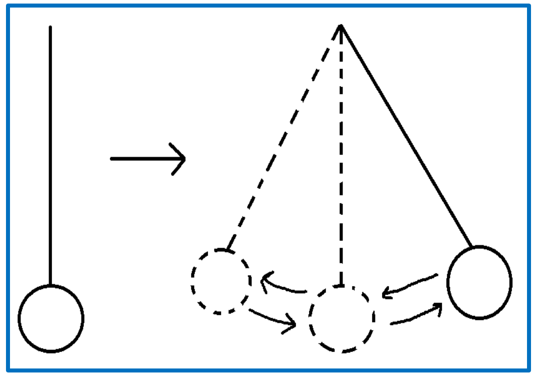
- Introduction to Stability
- Methods of Determining Stability
- Routh-Hurwitz Criterion
    - Special Cases
- Control Systems Stability Analysis
- Case Study: Antenna Control System
- Stability Analysis in LTI State-Space Models



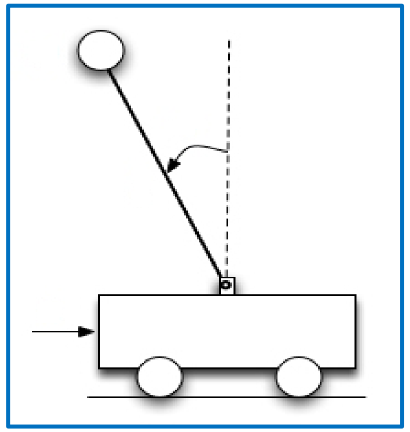
Chapter 5

Introduction to Stability

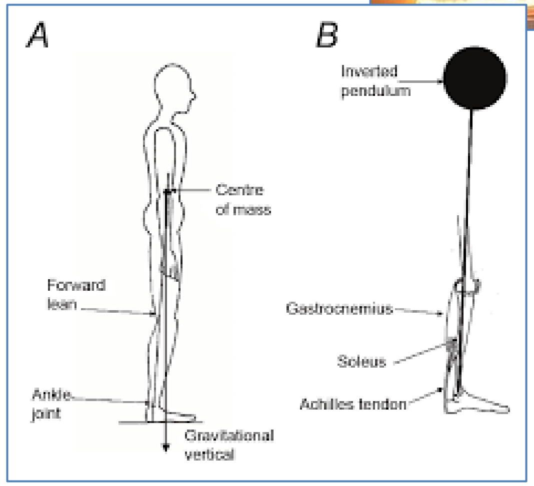
- In general, a stable system is the one that will
  - Remain at rest unless excited by an external source
  - Return to rest if all excitations are removed
- **Stability** is the most important specification of a control system.
- An unstable control system is generally useless.
- Simple pendulum hanging from the ceiling is a stable system, but an inverted pendulum is an unstable system.



Simple Pendulum



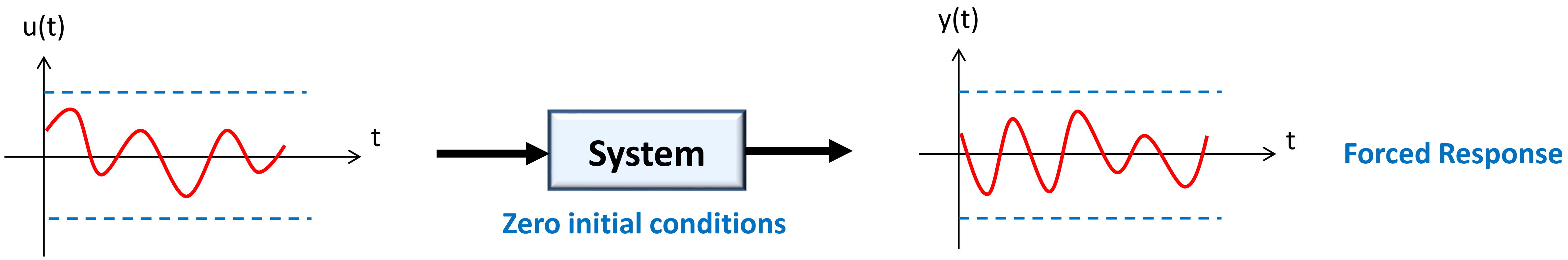
Inverted Pendulum



Introduction to Stability

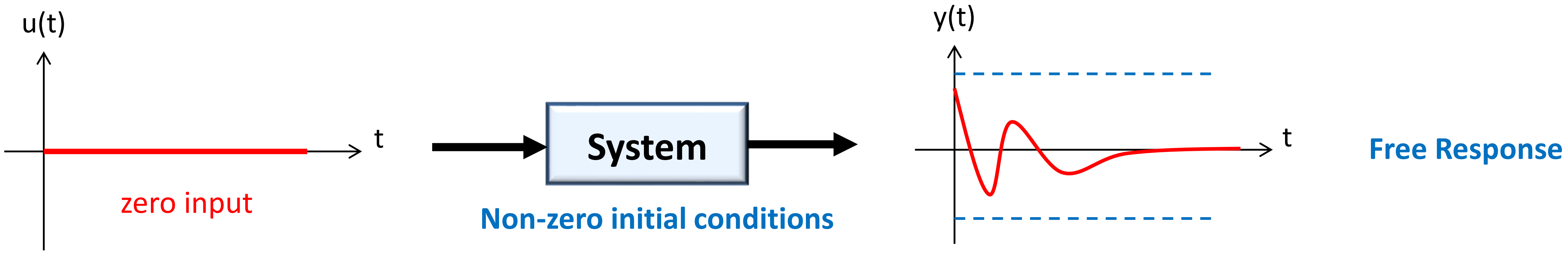
❑ Bounded-Input, Bounded-Output Stability (BIBO Stability)

- A system with zero initial conditions is **BIBO stable** if every bounded-input results in a bounded-output.



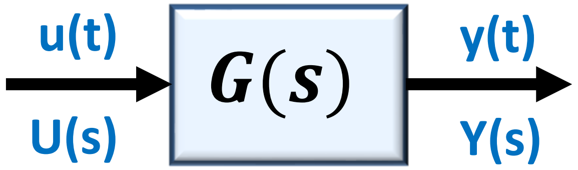
❑ Asymptotic Stability (Zero-Input Stability)

- A system with zero input is **asymptotically stable** if following conditions are satisfied:



- In **LTI systems** the asymptotic stability is equivalent to BIBO stability, but in general they are different.

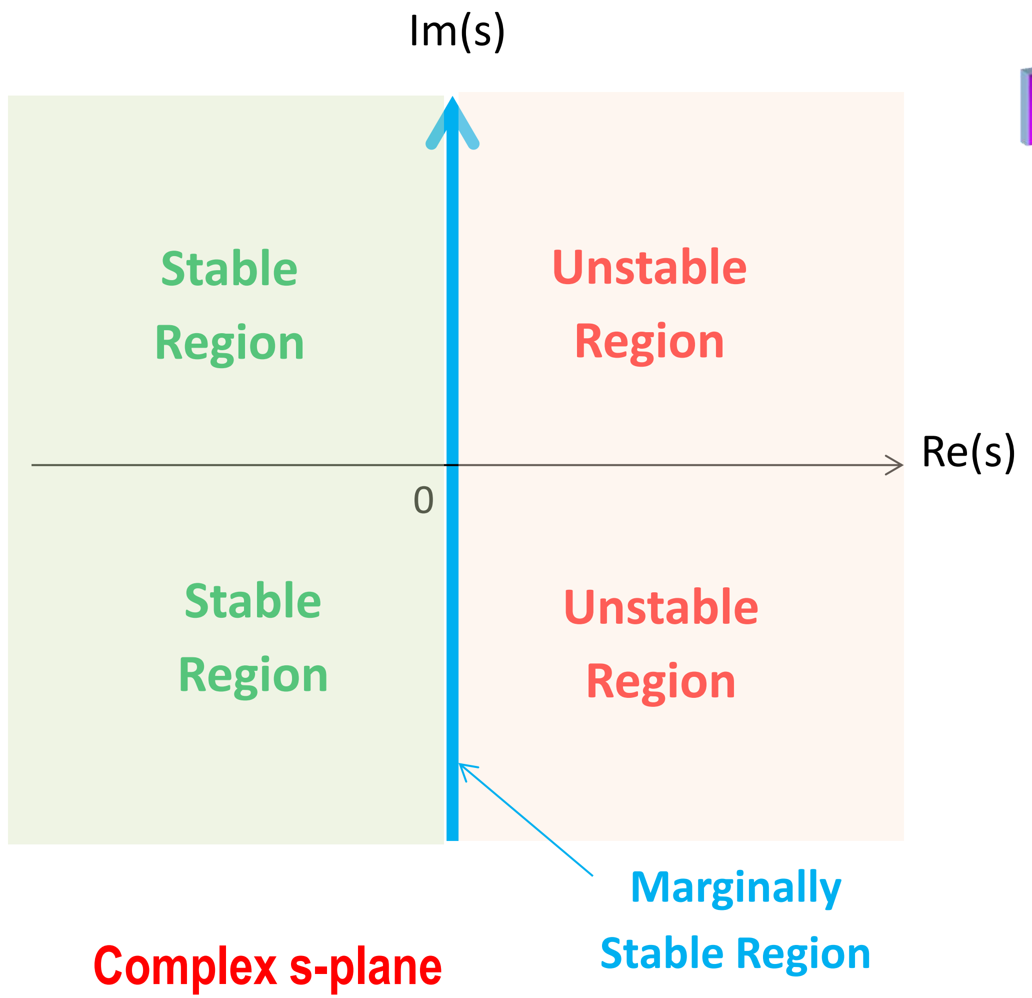
Introduction to Stability



❑ Stability of Linear Dynamic Systems

- Consider the a LTI dynamic system with input u(t), output y(t) and transfer function model of G(s)

A linear dynamic system is stable if and only if all poles of G(s) are in the left-half s-plane



Example 1

$$G_1(s) = \frac{10(s-1)}{(s+2)(s^2+5)} \qquad poles \rightarrow \begin{cases} s_1 = -2 \\ s_{2,3} = \pm j\sqrt{5} \end{cases}$$

Marginally Stable

$$G_2(s) = \frac{2(s+2)}{(s+10)(s+3)} \qquad poles \rightarrow \begin{cases} s_1 = -10 \\ s_2 = -3 \end{cases}$$

Stable

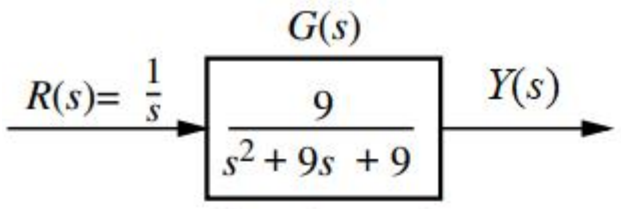
$$G_3(s) = \frac{10}{(s-10)(s^2+4)} \qquad poles \rightarrow \begin{cases} s_1 = 10 \\ s_2 = \pm j2 \end{cases}$$

Unstable

Introduction to Stability

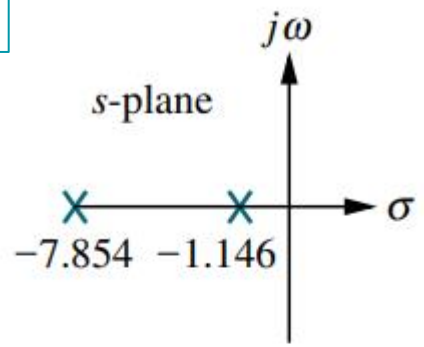
Example 2

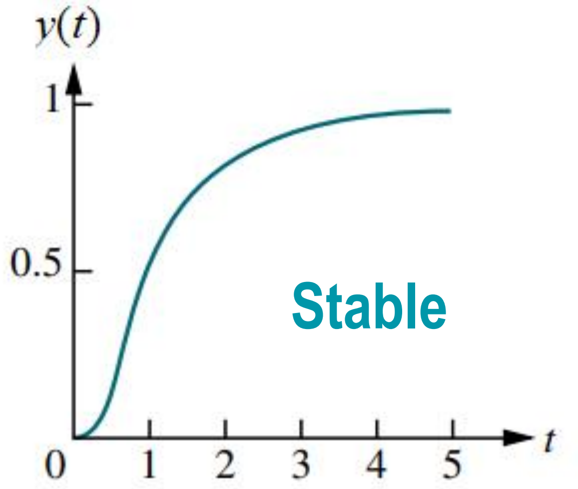
This example shows relationship of the step response, pole locations and stability in a second-order system.

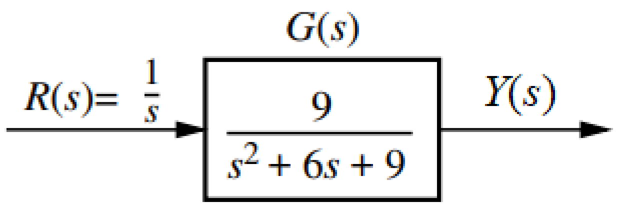


Overdamped

$\zeta = 1.5$
$s_1 = -7.854,$
$s_2 = -1.146$

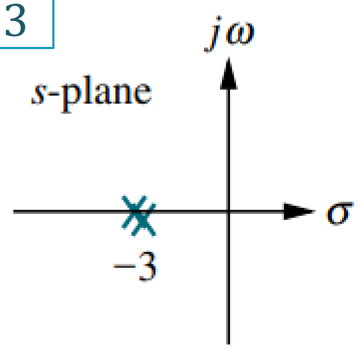


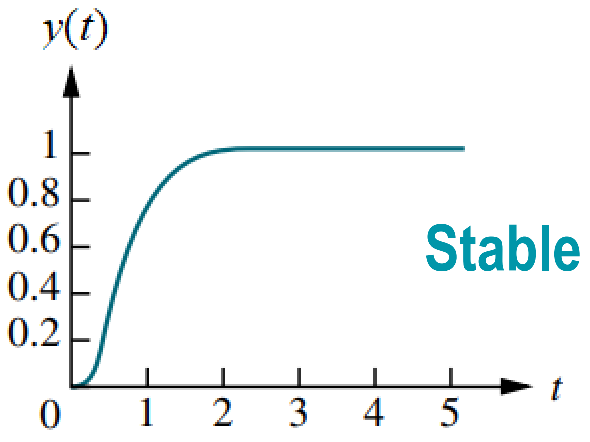


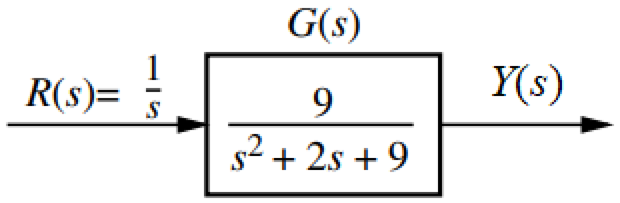


Critically damped

$\zeta = 1$
$s_1 = s_2 = -3$

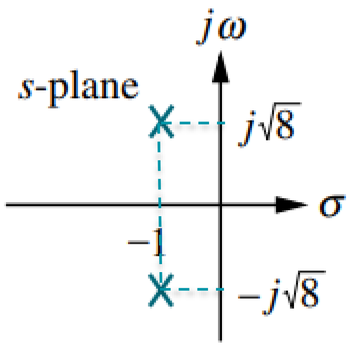


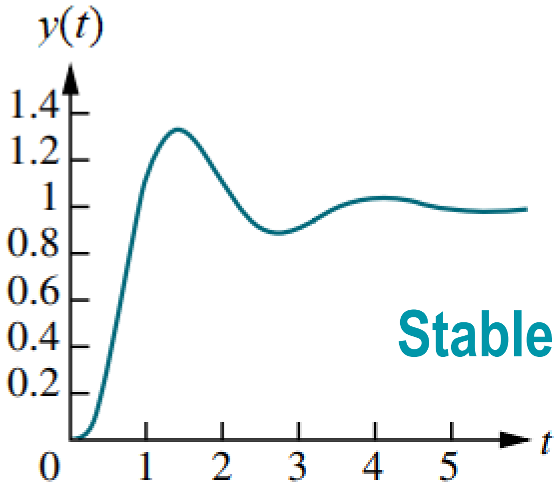




Underdamped

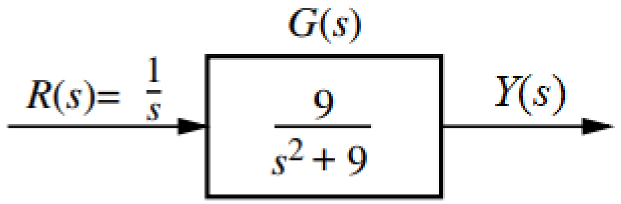
$\zeta = 0.33$
$s_{1,2} = -1 \pm j\sqrt{8}$





Introduction to Stability

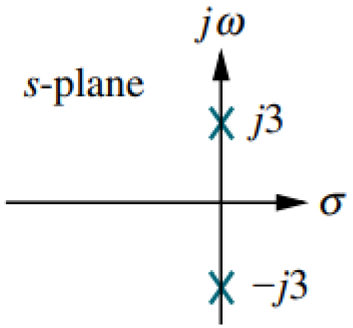
Example 2 This example shows relationship of the step response, pole locations and stability in a second-order system.

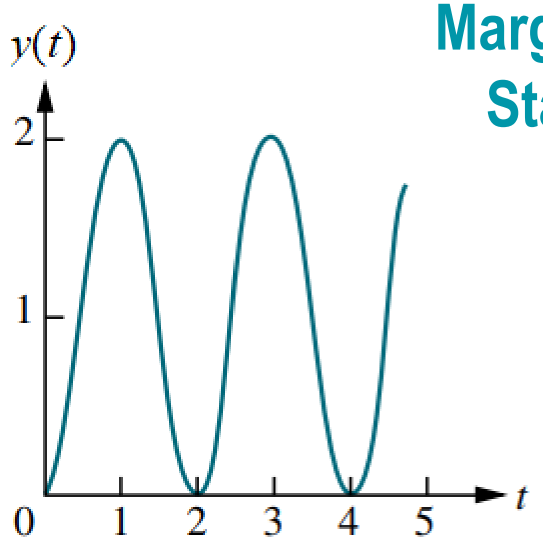


Undamped

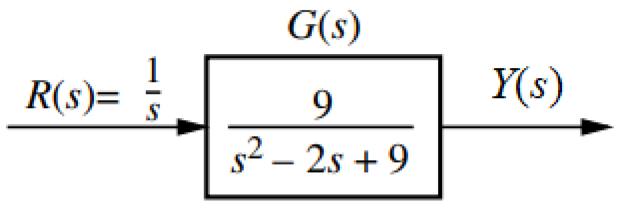
No damping

$\zeta = 0$
$s_{1,2} = \pm j3$



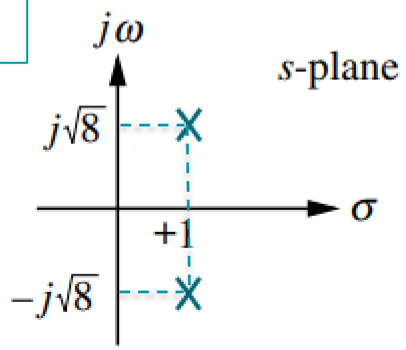


Marginally Stable

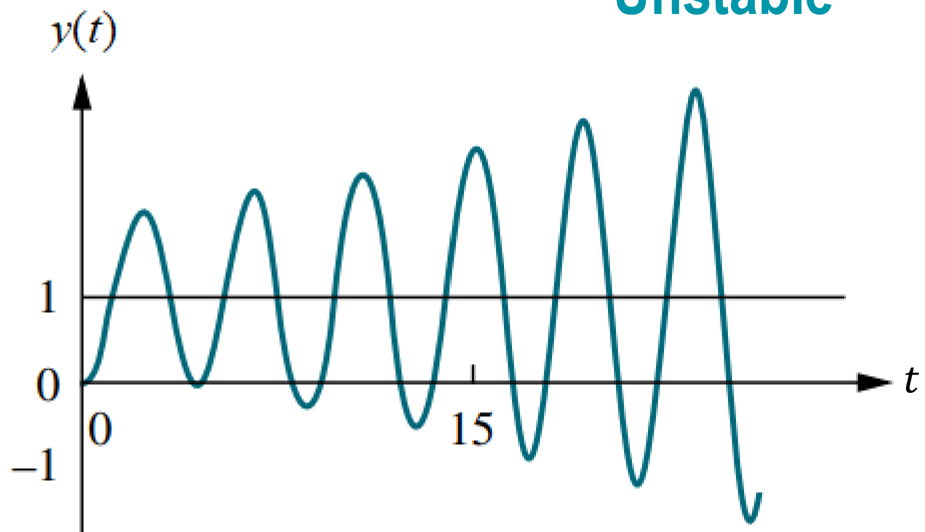


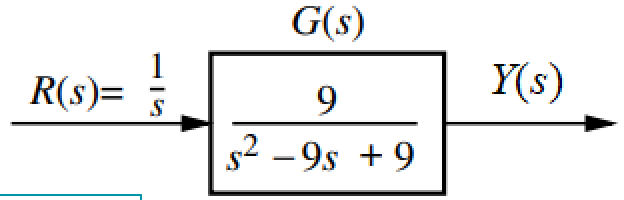
Unstable Oscillatory

Negative damping
$s_{1,2} = +1 \pm j\sqrt{8}$



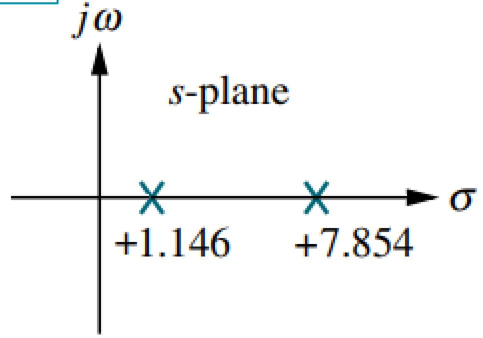
Unstable

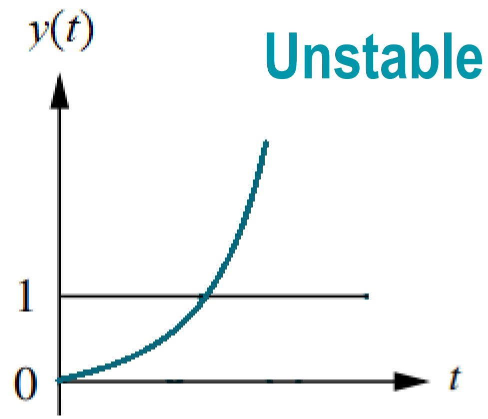




Negative damping
$s_1 = +7.854,$
$s_2 = +1.146$

Unstable Exponentially





Unstable

Methods of Determining Stability

- There are several methods to determine the stability of LTI systems **without** involving root solving.
  - Routh-Hurwitz Criterion
  - Root Locus Diagram
  - Bode Diagram
  - Nyquist Criterion

❑ Routh-Hurwitz Criterion

- Consider the following polynomial with real coefficients:

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

- **Routh-Hurwitz Criterion** is an algebraic method to find the number of roots of the polynomial in the right-half s-plane.

- **Routh-Hurwitz Criterion** gives the **necessary** and **sufficient** condition for stability.

Routh-Hurwitz Stability Criterion

1) Write the polynomial with real coefficients in the following form:

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

2) If any of the coefficients are **zero** or **negative**, the polynomial has roots on the **right-half s-plane**.

Example 3

Determine stability of the following LTI systems

$\boxed{G_1(s) = \dfrac{1}{3s^3 + s + 5}}$ ⟹ Characteristic equation $\rightarrow\ 3s^3 + 0s^2 + s + 5 = 0\ \rightarrow$ $G_1(s)$ is unstable

$\boxed{G_2(s) = \dfrac{1}{s^2 - 2s + 9}}$ ⟹ Characteristic equation $\rightarrow\ s^2 - 2s + 9 = 0\ \rightarrow$ $G_2(s)$ is unstable

$\boxed{G_3(s) = \dfrac{1}{3s^3 + s^2 + 2s + 1}}$ ⟹ **?** We have to create the Routh-Hurwitz table to check the stability.

Routh-Hurwitz Stability Criterion

3) If all coefficients are **positive** arrange them in a table according to the following pattern, which is called Routh-Hurwitz table. For example, for $n = 6$ we have:

$$a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

s^6	a_6	a_4	a_2	a_0
s^5	a_5	a_3	a_1	0
s^4	$b_1 = \dfrac{-\begin{vmatrix} a_6 & a_4 \\ a_5 & a_5 \end{vmatrix}}{a_5}$	$b_2 = \dfrac{-\begin{vmatrix} a_6 & a_2 \\ a_5 & a_1 \end{vmatrix}}{a_5}$	$b_3 = \dfrac{-\begin{vmatrix} a_6 & a_0 \\ a_5 & 0 \end{vmatrix}}{a_5}$	0
s^3	$c_1 = \dfrac{-\begin{vmatrix} a_5 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1}$	$c_2 = \dfrac{-\begin{vmatrix} a_5 & a_1 \\ b_1 & b_3 \end{vmatrix}}{b_1}$	$c_3 = \dfrac{-\begin{vmatrix} a_5 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1}$	0
s^2	$d_1 = \dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{c_1}$	$d_2 = \dfrac{-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}}{c_1}$	$d_3 = \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1}$	0
s^1	$e_1 = \dfrac{-\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix}}{d_1}$	$e_2 = \dfrac{-\begin{vmatrix} c_1 & c_3 \\ d_1 & d_3 \end{vmatrix}}{d_1}$	$e_3 = \dfrac{-\begin{vmatrix} c_1 & 0 \\ d_1 & 0 \end{vmatrix}}{d_1}$	0
s^0	$f_1 = \dfrac{-\begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix}}{e_1}$	$f_2 = \dfrac{-\begin{vmatrix} d_1 & d_3 \\ e_1 & e_3 \end{vmatrix}}{e_1}$	$f_3 = \dfrac{-\begin{vmatrix} d_1 & 0 \\ e_1 & 0 \end{vmatrix}}{e_1}$	0

4) Check number of sign changes in the first column.

a_6
a_5
b_1
c_1
d_1
e_1
f_1

Number of sign changes in the first column is equal to the number of roots in the right-half s-plane

Routh-Hurwitz Stability Criterion

Example 4

Determine stability of the following transfer function

$$G(s) = \frac{1}{3s^3 + s^2 + 2s + 1}$$

Characteristic eqn. $\rightarrow$ $3s^3 + s^2 + 2s + 1 = 0$ $\rightarrow$ The tabular is needed

The Routh-Hurwitz table for this polynomial is as below:

s^3	3	2
s^2	1	1
s^1	-1	0
s^0	1	0

$$b_1 = \frac{-\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}}{1} = \frac{-(3 \times 1 - 2 \times 1)}{1} = -1$$

$$c_1 = \frac{-\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}}{-1} = \frac{-(1 \times 0 - 1 \times (-1))}{-1} = 1$$

There are two sign changes in first column.

System is unstable and it has two poles in the right-half s-plane.

Routh-Hurwitz Stability Criterion

Example 5

Check whether roots of the following polynomial are in the right-half s-plane.

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

The Routh-Hurwitz table for this polynomial is as below:

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	0
s^1	$\frac{45}{7}$	0	0
s^0	10	0	0

$$b_1 = \frac{-\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}}{1} = \frac{-(2 \times 5 - 3 \times 1)}{1} = -7$$

$$b_2 = \frac{-\begin{vmatrix} 2 & 10 \\ 1 & 0 \end{vmatrix}}{1} = \frac{-(2 \times 0 - 10 \times 1)}{1} = 10$$

$$c_1 = \frac{-\begin{vmatrix} 1 & 5 \\ -7 & 10 \end{vmatrix}}{-7} = \frac{-(1 \times 10 - 5 \times (-7))}{-7} = \frac{45}{7}$$

$$d_1 = \frac{-\begin{vmatrix} -7 & 10 \\ \frac{45}{7} & 0 \end{vmatrix}}{\frac{45}{7}} = \frac{-(\,(-7) \times 0 - \frac{45}{7} \times 10)}{\frac{45}{7}} = 10$$

There are two sign changes in first column.

System is unstable and it has two poles in the right-half s-plane.

Routh-Hurwitz Stability Criterion

Special Cases

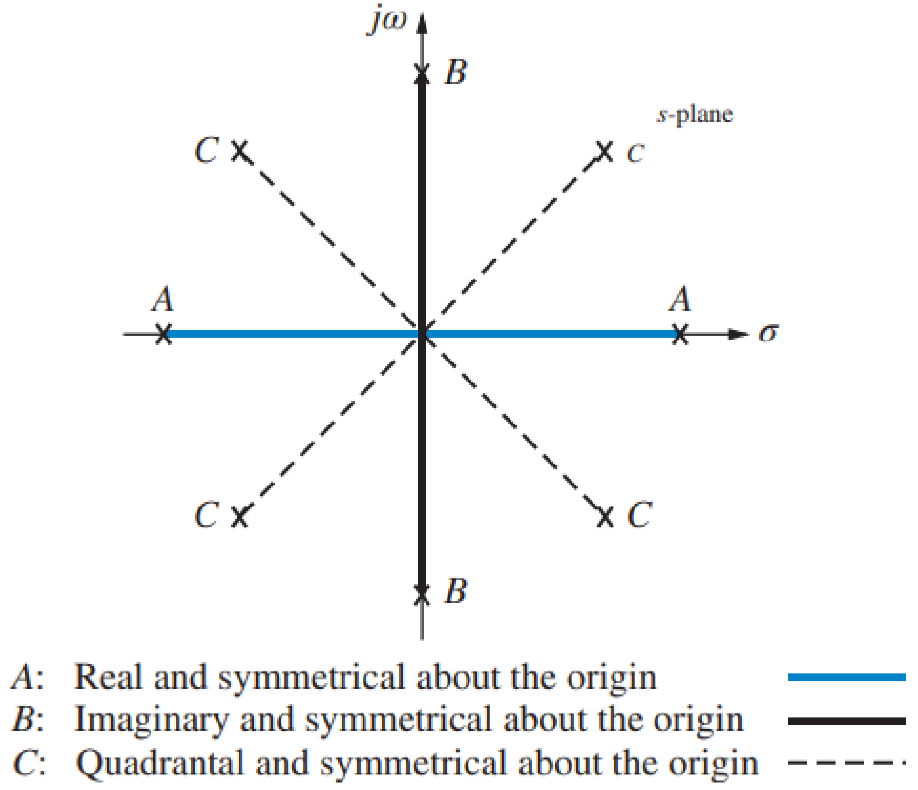
- Two special cases can arise when creating a Routh-Hurwitz table:

1) A zero in only the first column of a row

- Divide-by-zero problem when forming the next row.
- If a zero appears in the first column
  1. Replace the zero with a very small positive number $\varepsilon > 0$
  2. Complete the Routh-Hurwitz table as usual
  3. Take the limit as $\varepsilon \to 0$
  4. Evaluate the sign of the first-column entries

2) An entire row of zeros

- Indicates the presence of pairs of poles that are mirrored about the imaginary axis.
- If an entire row of zeros appears in a Routh-Hurwitz table
  1. Create an auxiliary polynomial from the row above the row of zeros, skipping every other power of s
  2. Differentiate the auxiliary polynomial with respect to s
  3. Replace the zero row with the coefficients of the resulting polynomial
  4. Complete the Routh-Hurwitz table as usual
  5. Evaluate the sign of the first-column entries



Routh-Hurwitz Stability Criterion

❑ Special Case1: Only First Term in a Row is Zero

Example 6 Check whether roots of the following polynomial are in the right-half s-plane.

$$\boldsymbol{s^4 + s^3 + 2s^2 + 2s + 3 = 0}$$

The Routh-Hurwitz table for this polynomial is as below:

s^4	1	2	3
s^3	1	2	0
s^2	**0**	3	0
s^1	$\infty \rightarrow$ Not acceptable		
s^0			

$$b_1 = \frac{-\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}{1} = \frac{-(1 \times 2 - 1 \times 2)}{1} = \mathbf{0}$$

$$b_2 = \frac{-\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}}{1} = \frac{-(1 \times 0 - 3 \times 1)}{1} = 3$$

$$c_1 = \frac{-\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}}{0} = \frac{-(1 \times 3 - 2 \times 0)}{0} = \infty$$

If only the first term in any row is zero, but the next term in the row is nonzero or there is no remaining term, then the zero term is replaced by a very small positive number ($\varepsilon > 0$) and the rest of the array evaluated as before.

Routh-Hurwitz Stability Criterion

❑ Special Case1: Only First Term in a Row is Zero

Example 6 Check whether roots of the following polynomial are in the right-half s-plane.

$$s^4 + s^3 + 2s^2 + 2s + 3 = 0$$

The Routh-Hurwitz table for this polynomial is as below:

s^4	1	2	3
s^3	1	2	0
s^2	ε	3	0
s^1	$\frac{-3}{\varepsilon}$		
s^0	3	0	0

Zero replaced by ε > 0

Zero replaced by ε > 0

$$b_1 = \frac{-\begin{vmatrix}1 & 2\\ 1 & 2\end{vmatrix}}{1} = \frac{-(1\times 2 - 1\times 2)}{1} = \mathbf{0}$$

$$b_2 = \frac{-\begin{vmatrix}1 & 3\\ 1 & 0\end{vmatrix}}{1} = \frac{-(1\times 0 - 3\times 1)}{1} = 3$$

$$c_1 = \frac{-\begin{vmatrix}1 & 2\\ \varepsilon & 3\end{vmatrix}}{\varepsilon} = \frac{-(1\times 3 - 2\times \varepsilon)}{\varepsilon} \approx \frac{-3}{\varepsilon}$$

$$d_1 = \frac{-\begin{vmatrix}\varepsilon & 3\\ \frac{-3}{\varepsilon} & 0\end{vmatrix}}{\frac{-3}{\varepsilon}} = \frac{-(\varepsilon\times 0 - 3\times(\frac{-3}{\varepsilon}))}{\frac{-3}{\varepsilon}} = 3$$

There are two sign changes in first column.

System is unstable and it has two poles in the right-half s-plane.

Routh-Hurwitz Stability Criterion

❑ Special Case2: All Terms in a Row are Zero

Example 7

Check number of roots of the following polynomial are in the right-half s-plane.

$$\boldsymbol{s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0}$$

The Routh-Hurwitz table for this polynomial is as below:

s^5	1	24	-25
s^4	2	48	-50
s^3	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
s^2			
s^1			
s^0			

All zero row

$$b_1 = \frac{-\begin{vmatrix} 1 & 24 \\ 2 & 48 \end{vmatrix}}{2} = \frac{-(1 \times 48 - 2 \times 24)}{2} = \mathbf{0}$$

$$b_2 = \frac{-\begin{vmatrix} 1 & -25 \\ 2 & -50 \end{vmatrix}}{2} = \frac{-(1 \times (-50) - 2 \times (-25))}{2} = \mathbf{0}$$

If all of the terms in a row are zero, first an auxiliary equation is formed from the previous non-zero row, then the all-zero row is replaced by coefficients of the derivative of the auxiliary equation and the rest of the array evaluated as before.

Routh-Hurwitz Stability Criterion

Special Case2: All Terms in a Row are Zero

Example 7

Check number of roots of the following polynomial are in the right-half s-plane.

$$\boldsymbol{s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0}$$

The Routh-Hurwitz table for this polynomial is as below:

s^5	1	24	-25
s^4	2	48	-50
s^3	**8**	**96**	**0**
s^2			
s^1			
s^0			

New row

$$b_1 = \frac{-\begin{vmatrix} 1 & 24 \\ 2 & 48 \end{vmatrix}}{2} = \frac{-(1 \times 48 - 2 \times 24)}{2} = \mathbf{0}$$

$$b_2 = \frac{-\begin{vmatrix} 1 & -25 \\ 2 & -50 \end{vmatrix}}{2} = \frac{-(1 \times (-50) - 2 \times (-25))}{2} = \mathbf{0}$$

The zeros are replaced by 8 and 96

The auxiliary equation is:

$$A(s) = 2s^4 + 48s^2 - 50$$

Take derivative of $A(s)$:

$$\frac{dA(s)}{ds} = 8s^3 + 96s$$

Routh-Hurwitz Stability Criterion

Special Case2: All Terms in a Row are Zero

Example 7

Check number of roots of the following polynomial are in the right-half s-plane.

$$\boldsymbol{s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0}$$

The Routh-Hurwitz table for this polynomial is as below:

s^5	1	24	-25
s^4	2	48	-50
s^3	$\mathbf{8}$	$\mathbf{96}$	0
s^2	24	-50	0
s^1	112.7	0	0
s^0	-50	0	0

$$c_1 = \frac{-\begin{vmatrix} 2 & 48 \\ 8 & 96 \end{vmatrix}}{8} = \frac{-(2 \times 96 - 8 \times 48)}{8} = 24$$

$$c_2 = \frac{-\begin{vmatrix} 2 & -50 \\ 8 & 0 \end{vmatrix}}{8} = \frac{-(2 \times 0 - (-50) \times 8)}{8} = -50$$

$$d_1 = \frac{-\begin{vmatrix} 8 & 96 \\ 24 & -50 \end{vmatrix}}{24} = \frac{-(8 \times (-50) - 24 \times 96)}{24} = 112.7$$

$$e_1 = \frac{-\begin{vmatrix} 24 & -50 \\ 112.7 & 0 \end{vmatrix}}{112.7} = \frac{-(24 \times 0 - (-50) \times 112.7)}{112.7} = -50$$

There are **one** sign change in first column.

System is unstable and it has one pole in the right-half s-plane.

Control Systems Stability Analysis via Routh-Hurwitz Criterion

Example 8

Consider the following transfer function of a system.

$$G_p(s) = \frac{1}{s^3 + s^2 + 2s - 4}$$

a) Determine stability of the system by Routh-Hurwitz criterion.

Characteristic equation $\rightarrow$ $s^3 + s^2 + 2s - 4 = 0$ $\rightarrow$ $G_p(s)$ is an unstable system

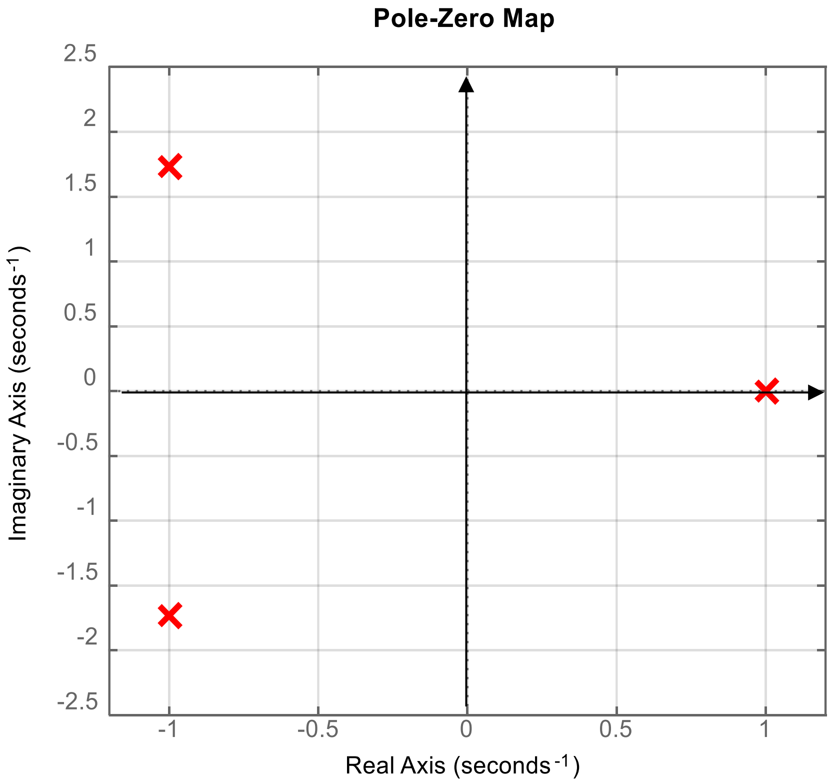
We can also check the pole locations and the step response of the system using MATLAB

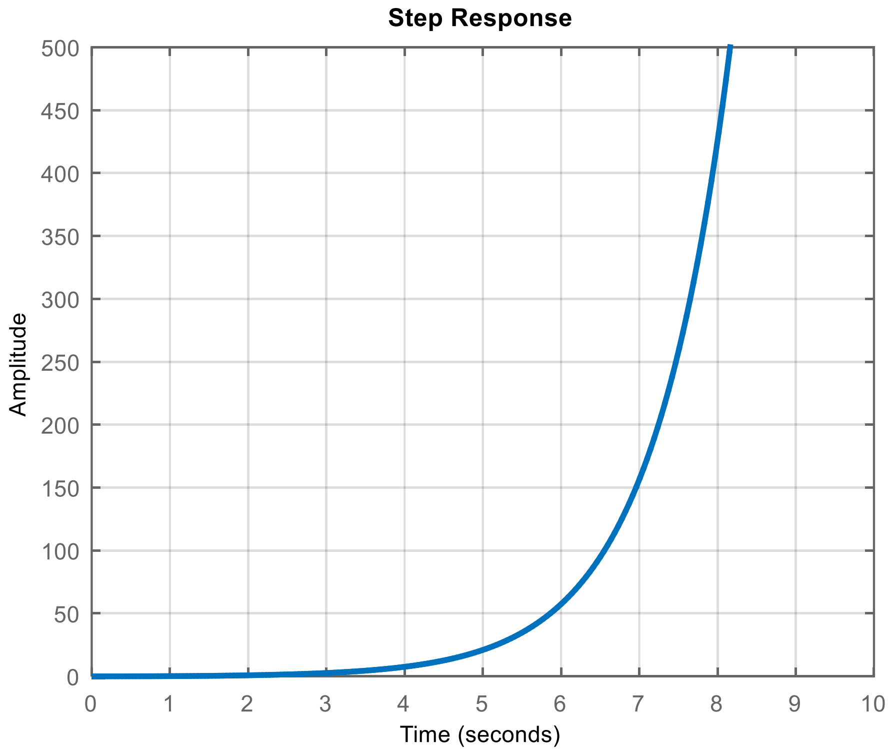
MATLAB

```
num = [1];
den = [1 1 2 -4];
G = tf(num,den);

pzmap(G)

step(G)
```





Unstable System

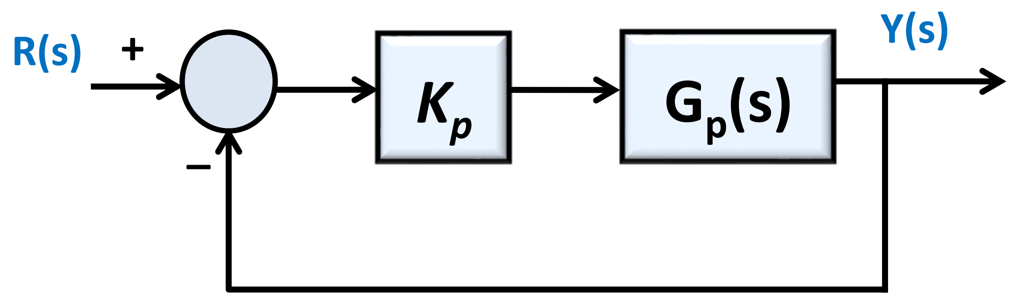
Control Systems Stability Analysis via Routh-Hurwitz Criterion

Example 8

Consider the following transfer function of a system.

$$G_p(s) = \frac{1}{s^3 + s^2 + 2s - 4}$$

b) Determine the range of the variable gain K_P for closed-loop stability.



First, find the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{K_p G_p(s)}{1 + K_P G_p(s)} = \frac{\dfrac{K_p}{s^3 + s^2 + 2s - 4}}{1 + \dfrac{K_p}{s^3 + s^2 + 2s - 4}}$$

$$\frac{Y(s)}{R(s)} = \frac{K_p}{\underbrace{s^3 + s^2 + 2s - 4 + K_p}}$$

Characteristic equation of the closed-loop system

$$\boxed{s^3 + s^2 + 2s - 4 + K_p = 0}$$

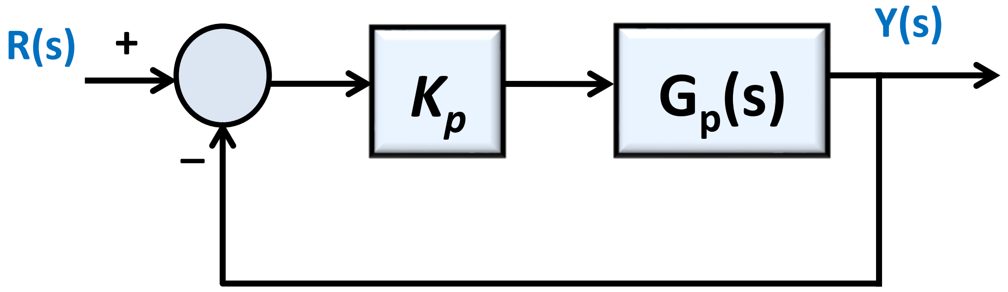
- The closed-loop characteristic equation and closed-loop poles are dependent on the gain K_p.
- Therefore, we can obtain a stable closed-loop system by selecting an appropriate range of gain K_p for stability.

Control Systems Stability Analysis via Routh-Hurwitz Criterion

Example 8 Consider the following transfer function of a system.

$$G_p(s) = \frac{1}{s^3 + s^2 + 2s - 4}$$

b) Determine the range of the variable gain K_P for closed-loop stability.



Next, apply the Routh-Hurwitz criterion to the closed-loop characteristic equation.

$$\boxed{s^3 + s^2 + 2s - 4 + K_p = 0}$$

s^3	1	2
s^2	1	$-4 + K_p$
s^1	$6 - K_p$	0
s^0	$-4 + K_p$	0

$$b_1 = \frac{-\begin{vmatrix} 1 & 2 \\ 1 & -4 + K_p \end{vmatrix}}{1} = \frac{-(1 \times (-4 + K_p) - 2 \times 1)}{1} = 6 - K_p$$

$$c_1 = \frac{-\begin{vmatrix} 1 & -4 + K_p \\ 6 - K_p & 0 \end{vmatrix}}{6 - K_p} = \frac{-\left(1 \times 0 - (-4 + K_p)(6 - K_p)\right)}{6 - K_p} = -4 + K_p$$

For stability, all terms in the first column must be positive:

$$6 - K_p > 0 \quad \rightarrow \quad K_p < 6$$

$$-4 + K_p > 0 \quad \rightarrow \quad K_p > 4$$

$$\boxed{4 < K_p < 6}$$

Necessary and sufficient condition for stability

- For $K_p = 4$ and $K_p = 6$ the closed-loop system is marginally stable.
- For $K_p < 4$ and $K_p > 6$ the closed-loop system is unstable.

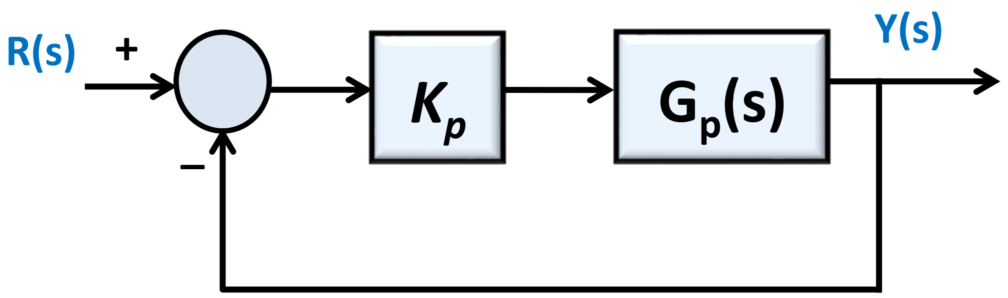
Control Systems Stability Analysis via Routh-Hurwitz Criterion

Example 8

Consider the following transfer function of a system.

$$G_p(s) = \frac{1}{s^3 + s^2 + 2s - 4}$$

b) Determine the range of the variable gain K_P for closed-loop stability.



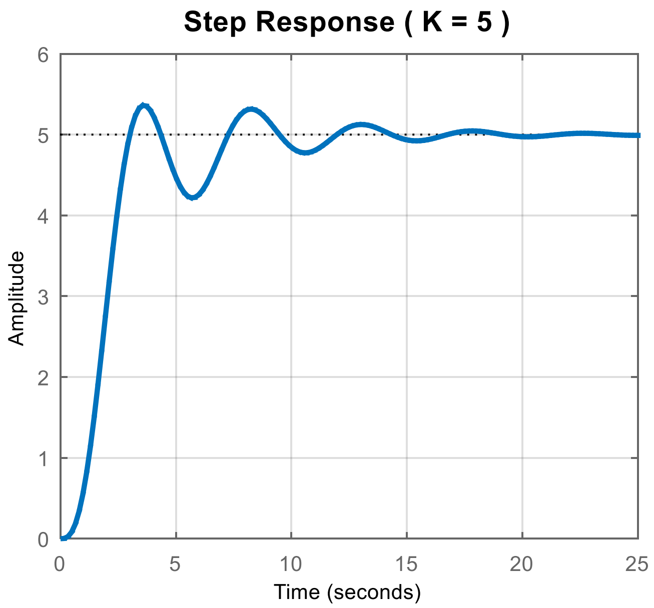
Closed-loop Transfer Function

$$\frac{Y(s)}{R(s)} = \frac{K_p}{s^3 + s^2 + 2s - 4 + K_p}$$

We can check the step response of the closed-loop system for the following range of gain values

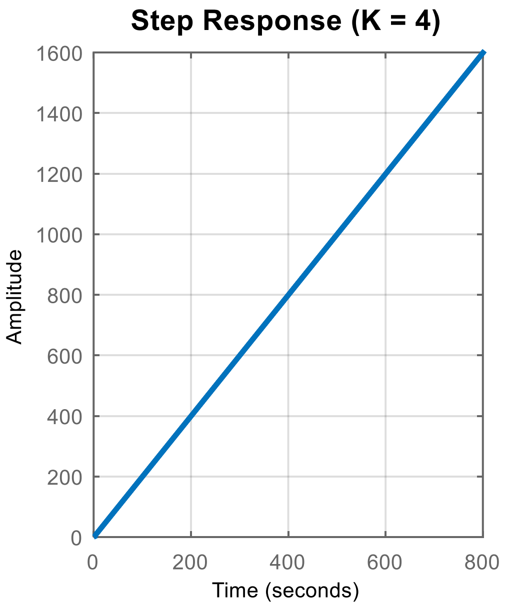
$4 < K_p < 6$

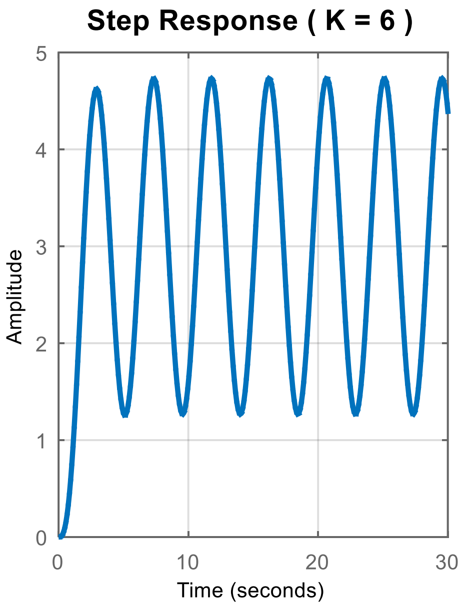
Stable System



$K_p = 4$ and $K_p = 6$

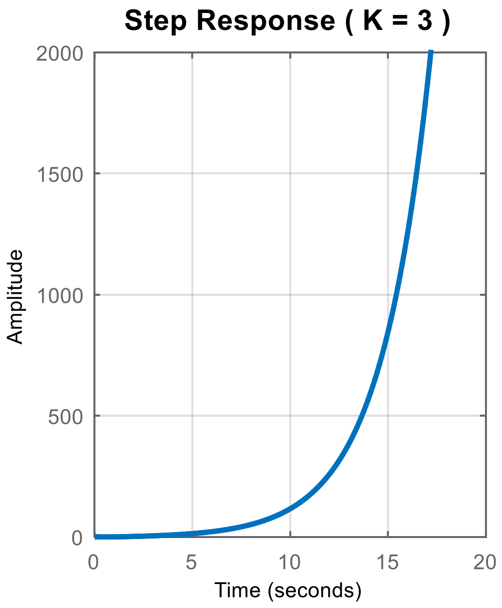
Marginally Stable System

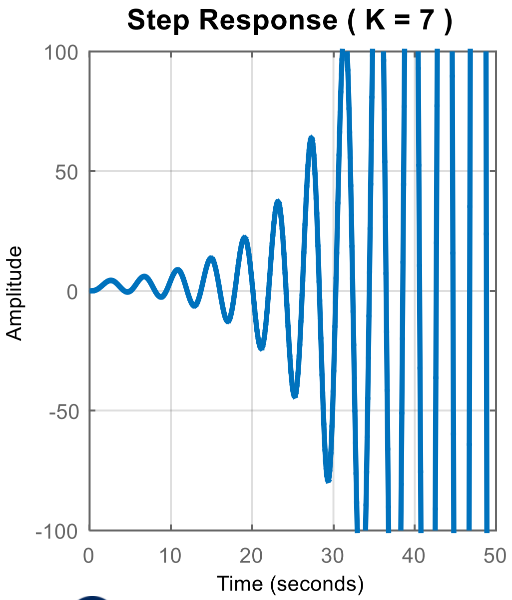




$K_p < 4$ and $K_p > 6$

Unstable System



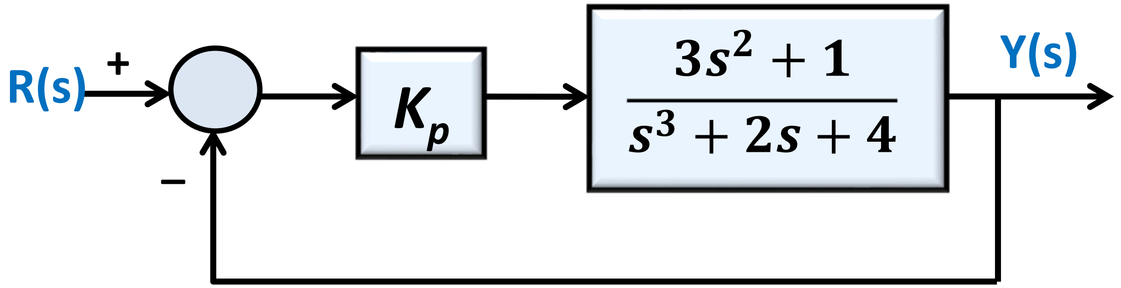


Control Systems Stability Analysis via Routh-Hurwitz Criterion

Example 9

Consider the following unity-feedback closed-loop system

$$G(s) = \frac{K_p(3s^2+1)}{s^3+2s+4} \quad , \qquad H(s) = 1$$



a) Determine order and type of the open-loop system.

G(*s*) is a third order, type 0 system with characteristic equation of

$$s^3+2s+4=0 \quad \rightarrow \quad \boldsymbol{n=3} \ , \qquad \boldsymbol{Type\ 0}$$

b) Determine stability of the open-loop system.

$$s^3+0s^2+2s+4=0 \quad \rightarrow$$ Based on the Routh-Hurwitz criterion, *G*(*s*) is unstable.

c) Determine the closed-loop transfer function.

Transfer function of the closed-loop system is

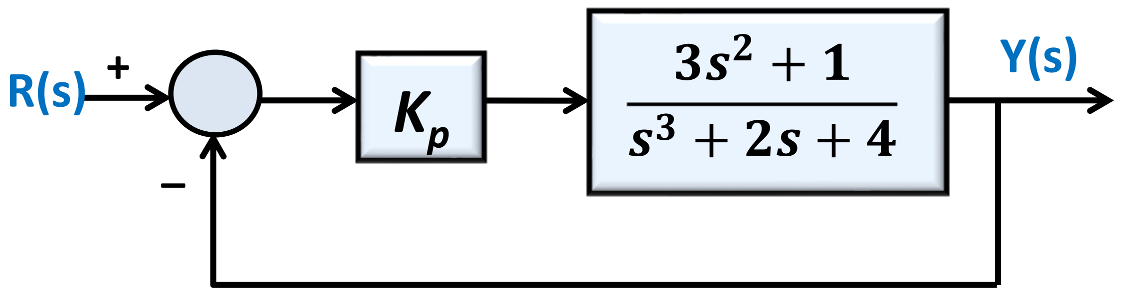
$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K_p(3s^2+1)}{s^3+2s+4}}{1+\dfrac{K_p(3s^2+1)}{s^3+2s+4}} = \frac{K_p(3s^2+1)}{s^3+3K_ps^2+2s+K_p+4}$$

Control Systems Stability Analysis via Routh-Hurwitz Criterion

Example 9

Consider the following unity-feedback closed-loop system

$$G(s) = \frac{K_p(3s^2+1)}{s^3+2s+4}\ , \qquad H(s) = 1$$



d) Determine range of the K_p for closed-loop stability.

Closed-loop system characteristic equation $\rightarrow$ $s^3 + 3K_ps^2 + 2s + K_p + 4 = 0$

The Routh-Hurwitz table:

s^3	1	2
s^2	$3K_p$	K_p+4
s^1	$\frac{5K_p-4}{3K_p}$	0
s^0	K_p+4	0

For stability, all terms in the first column must be positive:

$3K_p > 0 \quad \rightarrow \quad K_p > 0$

$5K_p - 4 > 0 \quad \rightarrow \quad K_p > 0.8$ $\Rightarrow$ $\boxed{K_p > 0.8}$

$K_p + 4 > 0 \quad \rightarrow \quad K_p > -4$

Stability Condition

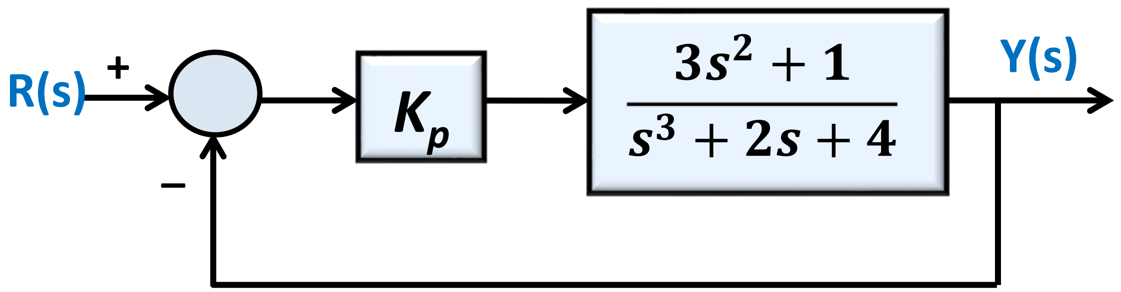
- For $K_p = 0.8$ the closed-loop system is marginally stable.
- For $K_p < 0.8$ the closed-loop system is unstable.

Control Systems Stability Analysis via Routh-Hurwitz Criterion

Example 9

Consider the following unity-feedback closed-loop system

$$G(s) = \frac{K_p(3s^2+1)}{s^3+2s+4}\ , \qquad H(s) = 1$$



e) Determine the gain K_p so that to achieve 5% steady-state error for unit-step input.

Step-error constant and steady-state error for unit-step input:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K_p(3s^2+1)}{s^3+2s+4} = \frac{K_p}{4}$$

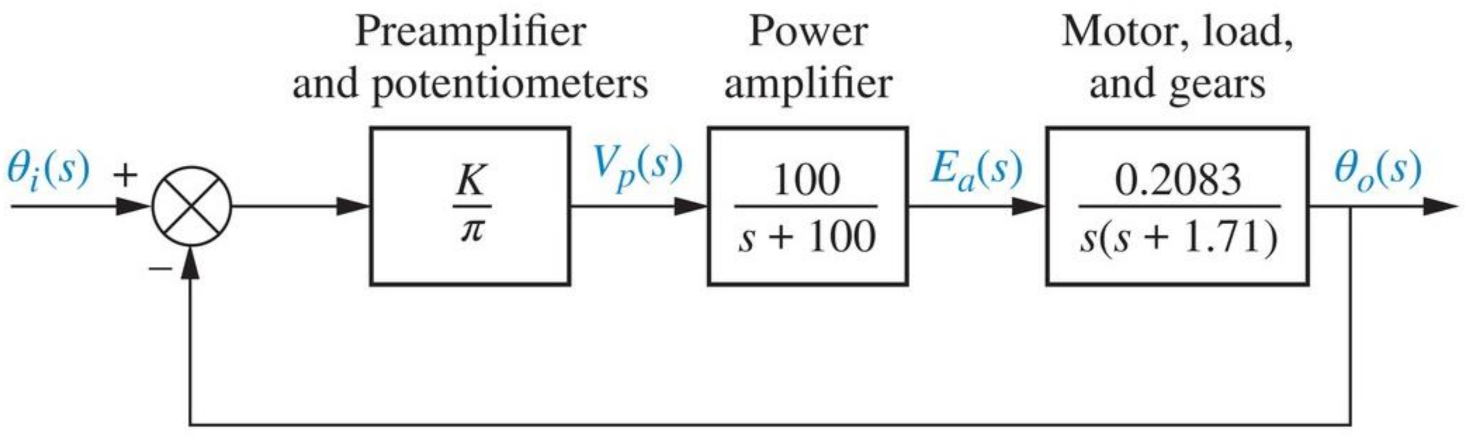
$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{1+\frac{K_p}{4}} = \frac{4}{4+K_p}$$

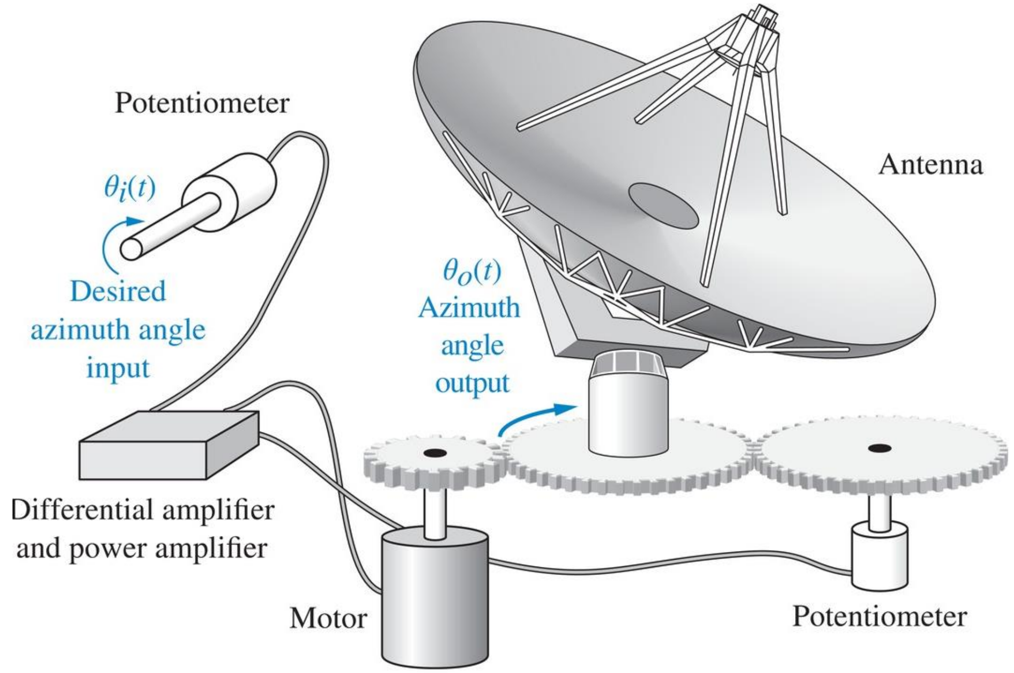
The required gain value to achieve the desired steady-state error of 5% is obtained as below

$$0.05 = \frac{4}{4+K_p} \quad \rightarrow \quad 0.2 + 0.05K_p = 4 \quad \rightarrow \quad K_p = \frac{3.8}{0.05} \quad \rightarrow \quad \boxed{K_p = 76}$$

Case Study: Antenna Control System

- Consider the ***motor-driven antenna azimuth position control system*** example from Lecture 1.
- We determined the block diagram of the control system as below:



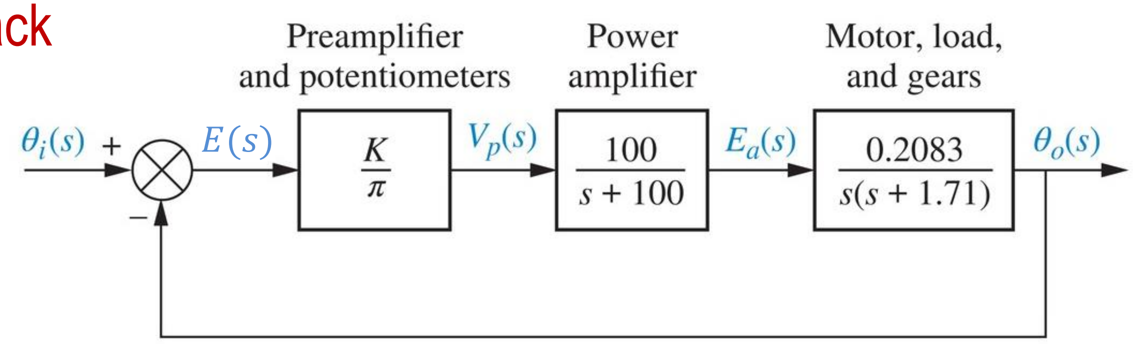


- In this part, we will find the <u>range of preamplifier gain *K*</u> required to keep the closed-loop system <u>stable</u>.

Case Study: Antenna Control System

For the antenna azimuth position control system with the given simplified unity-feedback block diagram model:

Find the range of preamplifier gain K required to keep the closed-loop system stable.



First, derive the closed-loop transfer function and the characteristic equation.

$$T(s) = \frac{6.63K}{s^3 + 101.71s^2 + 171s + 6.63K}$$

Create the Routh-Hurwitz table for the characteristic equation.

s^3	1	171
s^2	101.71	$6.63K$
s^1	$17392.41 - 6.63K$	0
s^0	$6.63K$	0

For stability, all terms in the first column must be positive:

$17392.41 - 6.63K > 0 \quad \rightarrow \quad K < 2623$

$6.63K > 0 \quad \rightarrow \quad K > 0$

$$\boxed{\mathbf{0 < K < 2623}}$$

Stability Condition

- For $K = 2623$ the closed-loop system is marginally stable.
- For $K > 2623$ and $K < 0$ the closed-loop system is unstable.

Quick Review

1) Answer the questions based on the given pole locations for each second order system.

   a) Which system has largest percent overshoot ?

   b) Which system has longest settling-time?

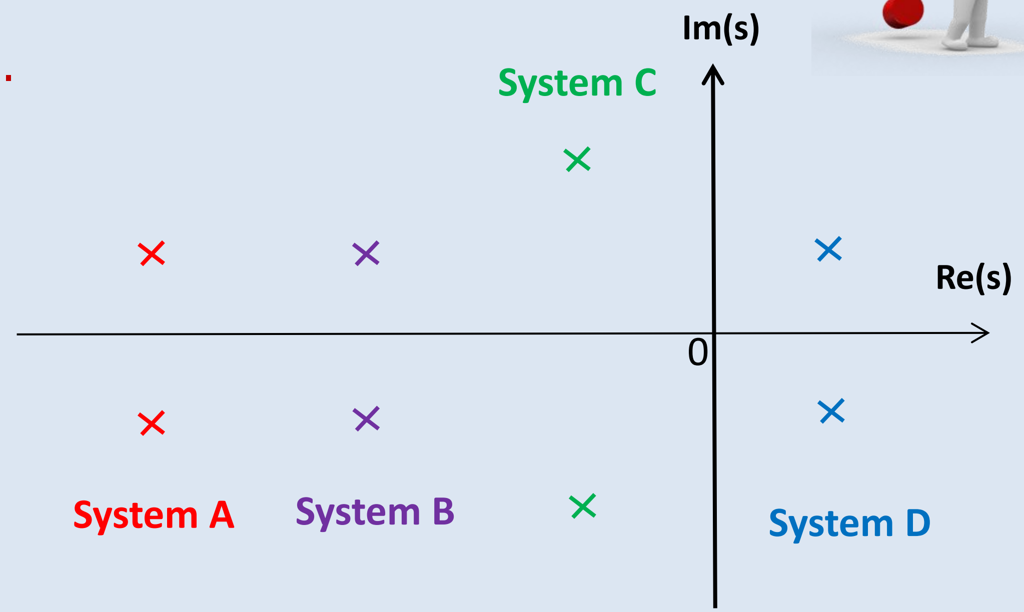
   c) Which system has shortest settling time?

   d) Which system is unstable?

   e) Which system has smallest percent overshoot?

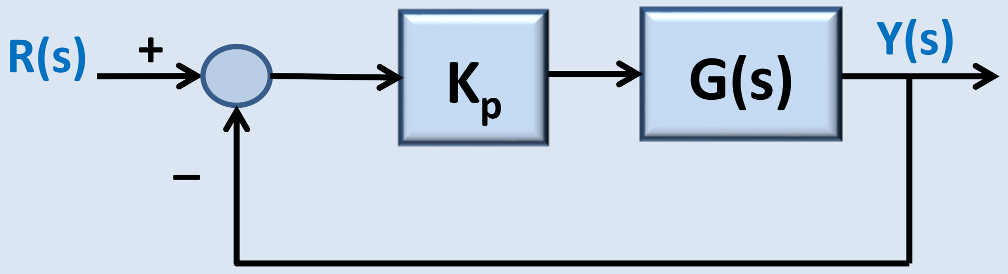
$$t_s \approx \frac{4}{\zeta\omega_n}$$

$$O.S. = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$



$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

2) Determine range of the gain K_p for closed-loop stability.



$$G(s) = \frac{1}{(s-1)(s+5)}$$

Stability Analysis of LTI State-Space Models

Transfer Function from State-space Model

- Determining the transfer function from the state space representation is called **reconstruction**.
- Consider a LTI system with the state space representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

- Take Laplace transform of the state space equations assuming the zero initial condition

$$\begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \longrightarrow (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \longrightarrow \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \longrightarrow \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathbf{Y}(\mathbf{s}) = \underbrace{[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]}_{\text{Transfer Function Matrix}}\,\mathbf{U}(s)$$

- For single-input-single-output (SISO) LTI systems the transfer function is obtained as below,

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

Stability Analysis of LTI State-Space Models

Reminder: Matrix Inverse

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$$

For 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \rightarrow \quad A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

For 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\mathrm{adj}(A) = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -(a_{12}a_{33} - a_{13}a_{32}) & a_{12}a_{23} - a_{13}a_{22} \\ -(a_{21}a_{33} - a_{23}a_{31}) & a_{11}a_{33} - a_{13}a_{31} & -(a_{11}a_{23} - a_{21}a_{13}) \\ a_{21}a_{32} - a_{22}a_{31} & -(a_{11}a_{32} - a_{12}a_{31}) & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

Stability Analysis of LTI State-Space Models

Example 10

Consider the following state space representation of a system

Determine transfer function of the system.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]\mathrm{u}(\mathrm{t})$$

The transfer function is determined by the following formula

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

First, find the $(s\mathbf{I} - \mathbf{A})^{-1}$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & -3 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} s+4 & 3 \\ -1 & s+5 \end{bmatrix} \longrightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 9s + 23} \begin{bmatrix} s+5 & -3 \\ 1 & s+4 \end{bmatrix}$$

Substitute the $(s\mathbf{I} - \mathbf{A})^{-1}$, $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{D}$ in the transfer function formula

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 9s + 23} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+5 & -3 \\ 1 & s+4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{1}{s^2 + 9s + 23} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3s - 3 \\ 6s + 27 \end{bmatrix} = \frac{1}{s^2 + 9s + 23} (3s - 3 + 12s + 54)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{15s + 51}{s^2 + 9s + 23}}$$

Transfer Function

Stability Analysis of LTI State-Space Models

Characteristic Polynomial and Eigenvalues

- Consider a LTI system with the following state space equations

$$\begin{cases} \dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) = \mathbf{C}\mathbf{x}(\mathbf{t}) + \mathbf{D}\mathbf{u}(\mathbf{t}) \end{cases}$$

- **Characteristic polynomial** of the system matrix is obtained as below

Characteristic Polynomial $\boxed{\det(\lambda\mathbf{I} - \mathbf{A}) = 0}$

- Given matrix $\mathbf{A}_{n\times n}$ with real arrays, the characteristic polynomial is a nth order monic polynomial with real coefficients

$$\boxed{\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0}$$

- The roots of the characteristic equation are called **eigenvalues** of the matrix **A**.
- The <u>eigenvalues</u> are always **real** or **complex conjugate** numbers.

Stability Analysis of LTI State-Space Models

- Consider a LTI system with the following state-space equations and characteristic polynomial

$$\begin{cases} \dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) = \mathbf{C}\mathbf{x}(\mathbf{t}) + \mathbf{D}\mathbf{u}(\mathbf{t}) \end{cases}$$

$$\boxed{\det(\lambda\mathbf{I} - \mathbf{A}) = 0}$$

Characteristic Polynomial

- Recall the reconstruction formula to obtain the transfer function:

$$\boxed{\frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}} \Rightarrow \frac{Y(s)}{U(s)} = \mathbf{C}\left(\frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right)\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A})}{\underbrace{\det(s\mathbf{I} - \mathbf{A})}_{\text{Characteristic Equation}}}$$

- Therefore, $\det(s\mathbf{I} - \mathbf{A}) = 0$ is the characteristic equation of the system, which is identical to the characteristic polynomial of the matrix **A**

- Therefore, the eigenvalues of the matrix **A** are identical to the system's poles with no pole-zero cancellation.

Stability of the system is analyzed by checking the eigenvalues of matrix A

Stability Analysis of LTI State-Space Models

Example 11

Determine stability of the following system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

First, find the characteristic polynomial of the matrix **A**

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{vmatrix} = \boxed{s^3 + 6s^2 + 11s + 6}$$

Characteristic Polynomial

Next, create the Routh-Hurwitz table for the characteristic equation.

s^3	1	11
s^2	6	6
s^1	10	0
s^0	6	0

Since there is no sign change in the first column all the eigenvalues are located on the left-half of the s-plane.
Therefore, the system is stable.

Stability Analysis of LTI State-Space Models

Example 11

Determine stability of the following system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

We can also determine the characteristic polynomial and the eigenvalues in MATLAB

```
A = [0 1 0;0 0 1;-6 -11 -6];
poly(A)
ans =
     1.0000     6.0000    11.0000     6.0000


eig(A)
ans =
   -1.0000
   -2.0000
   -3.0000
```

$$\boxed{\lambda^3 + 6\lambda^2 + 11\lambda + 6}$$

Characteristic Polynomial

$$\boxed{\begin{aligned} \lambda_1 &= -1, \\ \lambda_2 &= -2, \\ \lambda_3 &= -3 \end{aligned}}$$

Eigenvalues

Quick Review

1) A system is represented by state-space model of $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$.
   The values of K for a stable system are:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -K & -10 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

   a) $K < 1/2$

   b) $K > 1/2$

   c) $K = 1/2$

   d) The system is stable for all K.

THANK YOU

HUMBER

WE ARE HUMBER