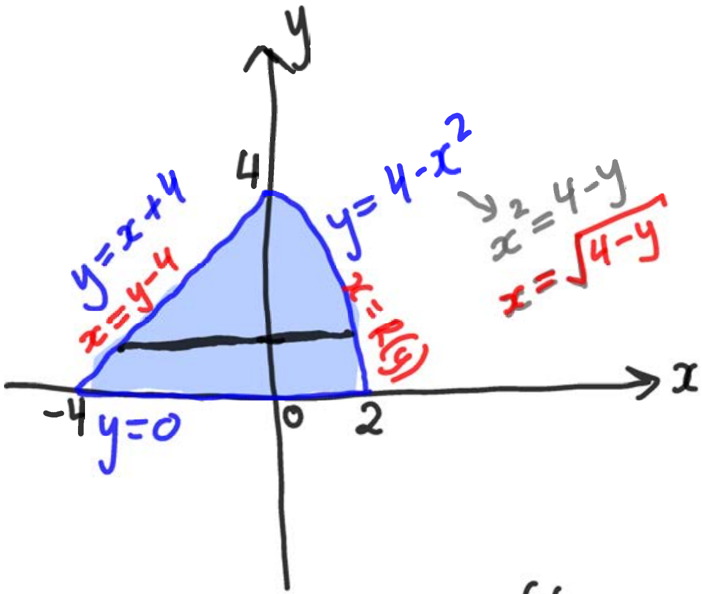


$x^2=4-y$

$x=\sqrt{4-y}$

$$\iint_R f(x,y)\,dA$$

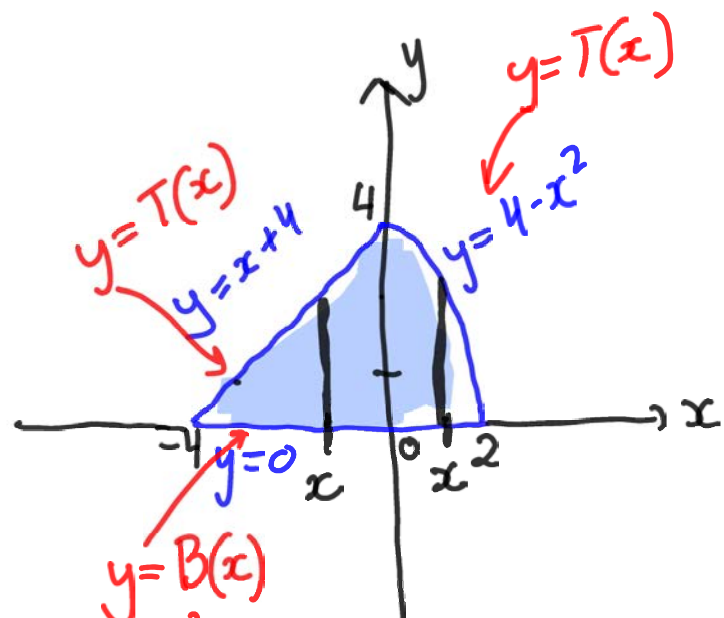
right boundary curve

<u>Order</u>: $dx\,dy$

$$\iint_R f(x,y)\,dA = \int_0^4 \int_{y-4}^{\sqrt{4-y}} f(x,y)\,dx\,dy$$

left boundary curve

<u>Order</u>: $dy\,dx$



$$\iint_R f(x,y)\,dA$$

$$= \int_{-4}^{0}\int_{0}^{x+4} f(x,y)\,dy\,dx + \int_{0}^{2}\int_{0}^{4-x^2} f(x,y)\,dy\,dx$$

The following result can help us compute double integrals quickly at times.

FRY Thm III.3.1.7

Theorem 9.6. Suppose

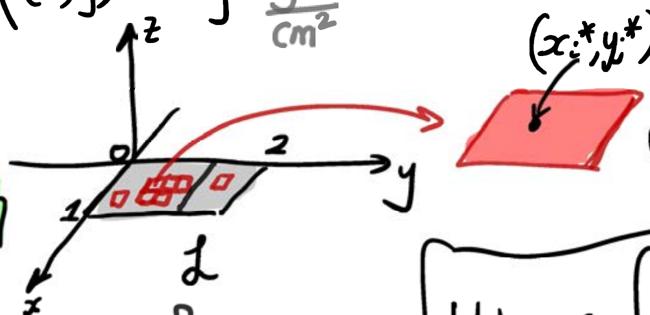
- $f(x,y)$ is continuous on the rectangle $\mathcal{R}=[a,b]\times[c,d]=\{(x,y):\ a\le x\le b,\ c\le y\le d\}$, and
- the function $f(x,y)$ can be written as the product $f(x,y)=g(x)h(y)$.

Then,

$$\iint_{\mathcal{R}} f(x,y)\ dA=\left(\int_a^b g(x)\ dx\right)\left(\int_c^d h(y)\ dy\right).$$

Example 9.7. Find the mass of a thin lamina described by the region $\mathcal{L}=\{(x,y):\ 0\le x\le 1,\ 0\le y\le 2\}$ whose density is given by the function $\rho(x,y)=3x^2y$.

rho $\rho(x,y)=3x^2y \quad \frac{gm}{cm^2}$



$\rho(x_i^*,y_i^*)\,\Delta x_i\Delta y_i$

$\mathcal{L}=[0,1]\times[0,2]$

$\mathcal{L}=\{(x,y):0\le x\le 1,\ 0\le y\le 2\}$

$$\text{total mass}=\iint \rho(x,y)\,dA$$

$$\frac{g}{\cancel{cm^2}}\ \cancel{cm^2}$$

$$\text{total mass}=\int_0^2\int_0^1 3x^2y\ dx\,dy$$

$\rho(x,y)=g(x)h(y)$

$$=\left(\int_0^2 y\,dy\right)\left(\int_0^1 3x^2\,dx\right)=\left[\tfrac{1}{2}y^2\right]_0^2\cdot\left[x^3\right]_0^1$$

$$=(2-0)\cdot(1-0)$$

$$=(2)(1)=2$$

* Double integrals help us calculate mass when working with 2-dimensional surfaces that have varying density.

* If the integrand $f(x,y)$ can be written as a product of two functions, one depending only on x and the other only on y:

$$f(x,y) = g(x)h(y)$$

AND

the bounds of integration are constants, then the double integral equals the product of two single integrals:

$$\int_a^b \int_c^d f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

FRY Thm III.3.1.3, Integrating over non-rectangular regions

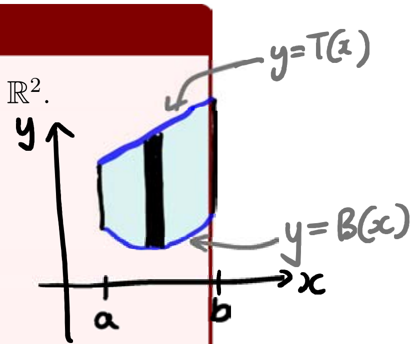
Theorem 9.8. Let f be a continuous function on a region $\mathcal{R} \subseteq \mathbb{R}^2$.

(a) If

$$\mathcal{R} = \{(x, y) : \ a \le x \le b, \ B(x) \le y \le T(x)\},$$

with $B(x)$ and $T(x)$ being continuous, then

$$\iint_{\mathcal{R}} f \ dA = \int_a^b \int_{B(x)}^{T(x)} f(x, y) \ dy \, dx.$$

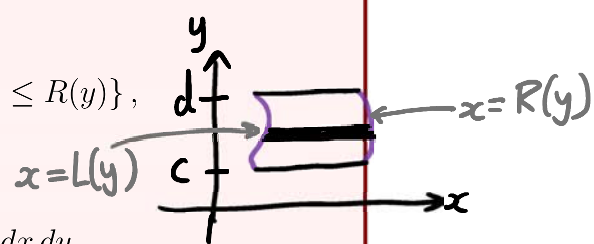


(b) If

$$\mathcal{R} = \{(x, y) : \ c \le y \le d, \ L(y) \le x \le R(y)\},$$

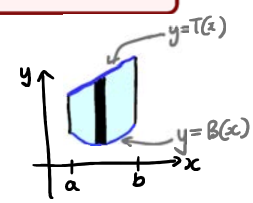
with $L(y)$ and $R(y)$ being continuous, then

$$\iint_{\mathcal{R}} f \ dA = \int_c^d \int_{L(y)}^{R(y)} f(x, y) \ dx \, dy.$$



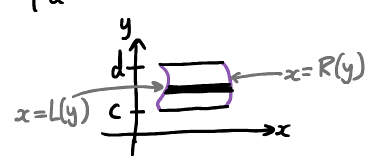
The FRY textbook sometimes uses the notation

$$\int_a^b dx \int_{B(x)}^{T(x)} dy \, f(x, y) \quad \text{for} \quad \int_a^b \int_{B(x)}^{T(x)} f(x, y) \ dy \, dx,$$



and

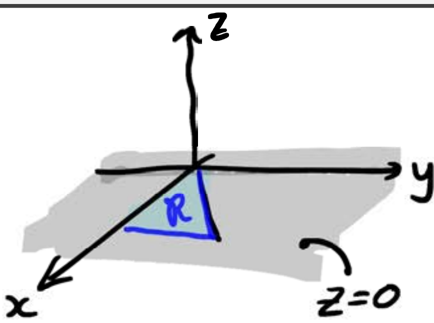
$$\int_c^d dy \int_{L(y)}^{R(y)} dx \, f(x, y) \quad \text{for} \quad \int_c^d \int_{L(y)}^{R(y)} f(x, y) \ dx \, dy.$$

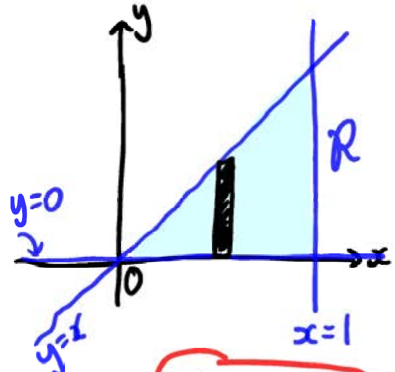


Example 9.9. (FRY Exercise III.3.1.7.12)
Find the volume V of the solid bounded above by the surface $z = f(x, y) = e^{-x^2}$ below by the plane $z = 0$ and over the triangle in the xy-plane formed by the lines $x = 1$, $y = 0$, and $y = x$.

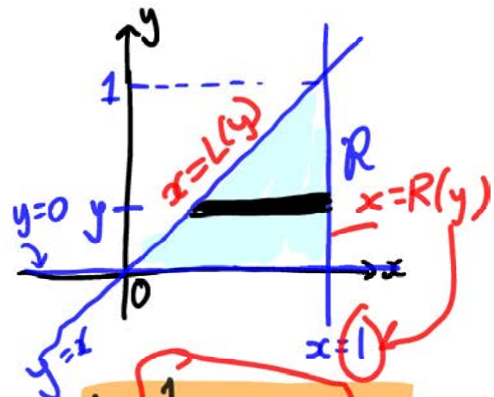
Goal: Volume $V = ?$

$z = f(x,y) = e^{-x^2}$





$$V = \iint_R f(x,y)\, dA$$



$$V = \int_0^1 \boxed{\int_0^x e^{-x^2}\, dy}\, dx$$

$$= \int_0^1 e^{-x^2} y \Big|_0^x \, dx$$

$$= \int_0^1 x e^{-x^2}\, dx$$

$$= -\frac{1}{2} e^{-x^2} \Big|_0^1 = -\frac{1}{2} e^{-1} - \left(-\frac{1}{2} e^{0}\right) = \frac{-1}{2e} + \frac{1}{2} = \frac{1}{2}\left(1 - \frac{1}{e}\right)$$

$$V = \int_0^1 \boxed{\int_y^1 e^{-x^2}\, dx}\, dy$$

Let $u = x^2$
Then $du = 2x\, dx$

☹ Substitution fails

* The order of integration ($dx\,dy$ vs $dy\,dx$) can be so consequential that with one order you <u>can</u> evaluate the integral while with the other it is very tough (or impossible) to make any progress.

Integration by Subtitution

$$\int_0^1 x e^{-x^2}\,dx = \int_0^{-1} \frac{-1}{2} e^u\,du = \frac{-1}{2}\int_0^{-1} e^u\,du = \frac{1}{2}\int_{-1}^{0} e^u\,du$$

Let $u = -x^2$

Then $\frac{du}{-2} = \frac{\cancel{-2}x\,dx}{\cancel{-2}}$

and $x\,dx = \frac{-1}{2}du$

When $x=0$,
$u=-0^2=0.$

When $x=1$,
$u=-1^2=-1$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

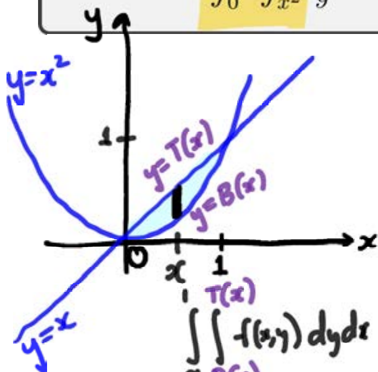
$$= \frac{1}{2}e^u\Big|_{-1}^{0}$$

$$= \frac{1}{2}\left(e^0 - e^{-1}\right)$$

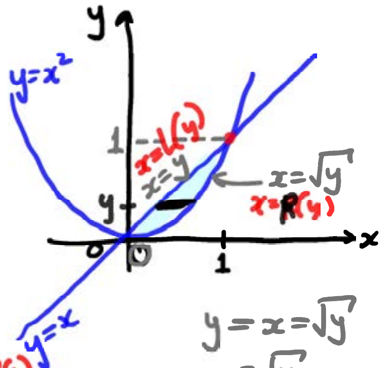
$$= \frac{1}{2}\left(1 - \frac{1}{e}\right)$$

Example 9.10. (FRY Exercise III.3.1.7.3e)

Evaluate $\displaystyle\int_0^1 \int_{x^2}^{x} \frac{x}{y} e^y \, dy dx.$



$$\int_0^1 \int_{B(x)}^{T(x)} f(x,y)\, dy dx$$



$$\int_0^1 \int_{L(y)}^{R(y)} f(x,y)\, dx dy$$

$y = x = \sqrt{y}$

$y = \sqrt{y}$

$y^2 = y$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0$ or $y = 1$

$$\int_0^1 \int_{x^2}^{x} \frac{x}{y} e^y \, dy\, dx$$

$$= \int_0^1 \int_y^{\sqrt{y}} \frac{x}{y} e^y \, dx\, dy$$

$$= \int_0^1 \frac{1}{2} x^2 \frac{e^y}{y} \Big|_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{1}{2} (\sqrt{y})^2 \frac{e^y}{y} - \frac{1}{2} y^2 \frac{e^y}{y} \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} e^y - \frac{1}{2} y e^y \right) dy$$

$$= \cdots$$

$$= \frac{1}{2} e - 1$$

* If you are finding it difficult to work out a double integral, try drawing the region of integration and see if you can switch the order of integration. With the other order, the integral may become easier to evaluate.

$$\int_0^1 \left(\tfrac{1}{2}e^y - \tfrac{1}{2}ye^y\right)dy$$

$$= \frac{1}{2}\int_0^1 (e^y - ye^y)\,dy$$

$$= \frac{1}{2}\int_0^1 (1-y)e^y\,dy$$

Integration by parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Let $u = 1-y$ and $dv = e^y dy$

Then $du = -dy$ and $v = e^y$

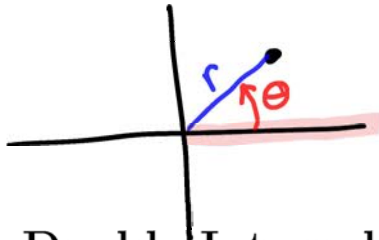
$$= \frac{1}{2}\left((1-y)e^y\Big|_0^1 - \int_0^1 e^y \cdot -dy \right)$$

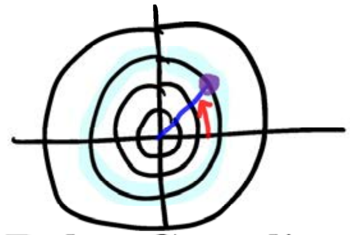
times

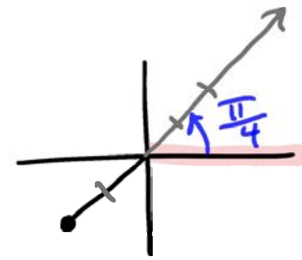
$$= \frac{1}{2}\left((1-y)e^y\Big|_0^1 + e^y\Big|_0^1 \right)$$

$$= \frac{1}{2}\left(-e^0 + e^1 - e^0\right)$$

$$= \frac{1}{2}(e-2) \quad = \frac{1}{2}e - 1$$







9.3 Double Integrals in Polar Coordinates

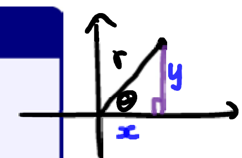
$r = -2, \theta = \frac{\pi}{4}$

FRY Defn III.3.2.1, Polar coordinates

Definition 9.11. The polar coordinates of any point (x, y) in the xy-plane are given by r and θ where r is the distance from the origin $(0, 0)$ to the point (x, y), and θ is the counter-clockwise angle from the positive x-axis to the line segment joining (x, y) to $(0, 0)$.

Polar angles are defined up to integer multiples of 2π. When the polar angle θ is negative, we rotate $|\theta|$ in clockwise direction from the positive x-axis. If r is negative, we rotate an angle θ and move a distance r along the ray opposite to the ray along which we usually move when r is positive.

FRY Eqn III.3.2.2, How are polar and Cartesian coordinates related?



Definition 9.12.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \arctan\left(\frac{y}{x}\right)$$

Pythagorean Theorem

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

$\tan\theta = \frac{y}{x}$

FRY Eqn III.3.2.5, The area element in polar coordinates

Definition 9.13. The area element in polar coordinates is

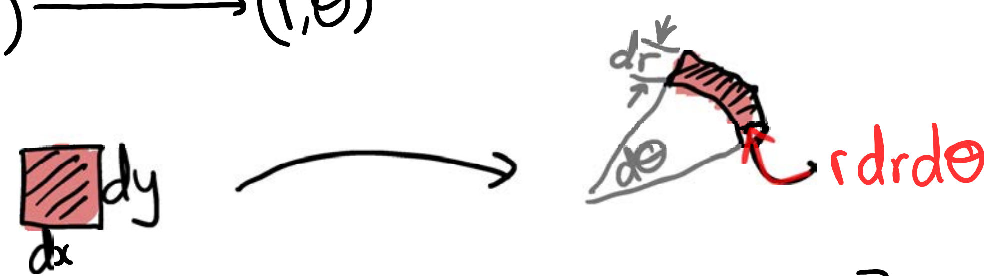
$$dA = r\,dr\,d\theta.$$

the absolute value of the determinant of the Jacobian matrix

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,\left|\det D\Phi\right|\,dr\,d\theta$$

absolute value

$$(x,y) \xrightarrow{\Phi} (r,\theta)$$



$$D\Phi = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\det D\Phi = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

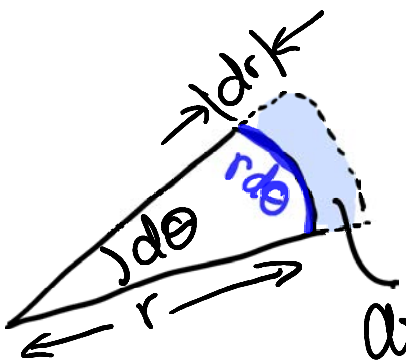
$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r \cdot 1$$

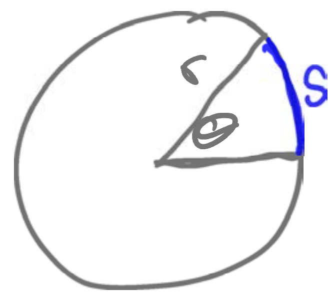
$$= r$$

$$|\det D\Phi| = |r| = r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,\overbrace{|\det D\Phi|}^{r}\,dr\,d\theta$$



$$\text{Area} \approx r\,d\theta\,dr$$



$$\theta = \frac{s}{r}$$

$$s = r\theta$$

Example 9.14. Let $\mathcal{R}$ be the region in the first quadrant bounded by

- $y = \sqrt{3}x$,
- $y = 0$, and
- $x^2 + y^2 = 9$.

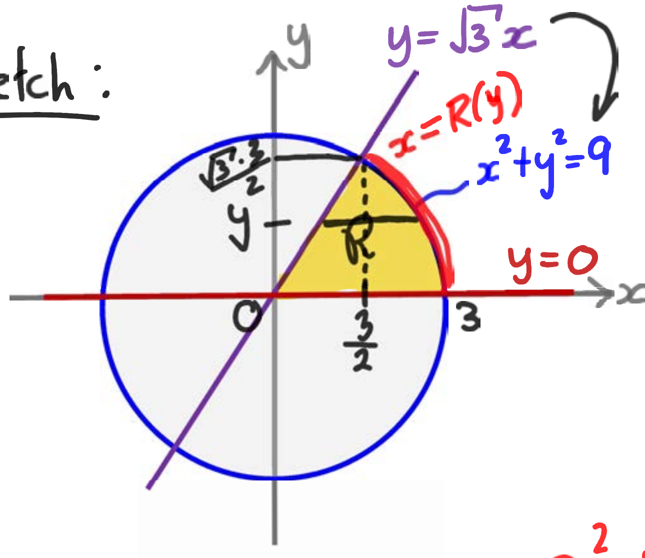
1. Sketch the region of integration $\mathcal{R}$.
2. Express $\displaystyle\iint_{\mathcal{R}} f(x, y)dA$ as an iterated integral in polar coordinates in two different ways.
3. If $f(x, y) = x + y$, evaluate the double integral.

region R
- first quadrant $(x \geq 0,\ y \geq 0)$
- $y = \sqrt{3}\,x$
- $y = 0$
- $x^2 + y^2 = 9$

Goal: Evaluate

$$\boxed{\iint_R (x+y)\,dA}$$

① Sketch:



$x^2 + 3x^2 = 9$

$4x^2 = 9$

$x^2 = \frac{9}{4}$

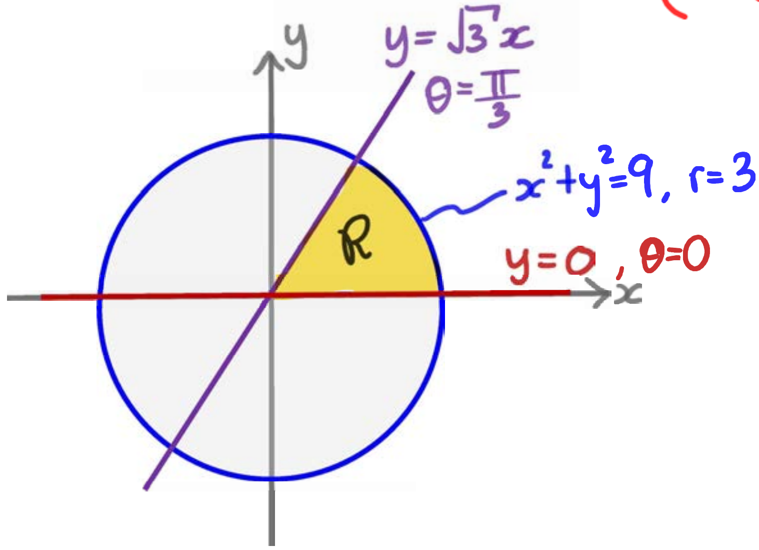
$x = \frac{3}{2} \quad y = \sqrt{3}\cdot\frac{3}{2}$

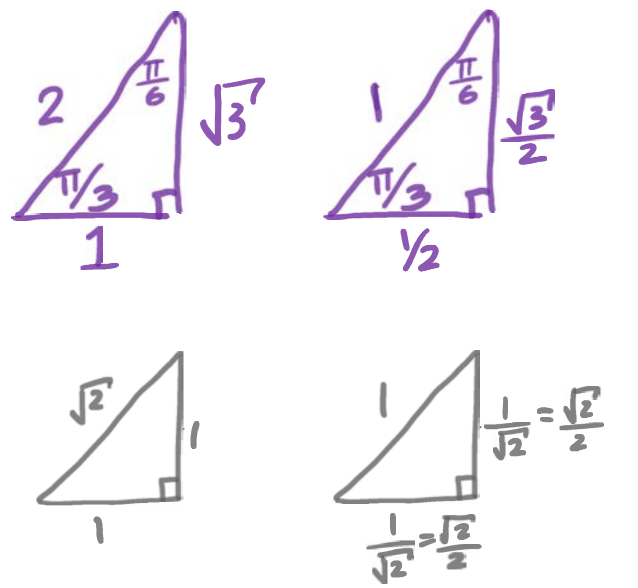
$$\int_0^{\frac{3\sqrt{3}}{2}} \int_{\frac{1}{\sqrt{3}}y}^{\sqrt{9-y^2}} (x+y)\,dx\,dy$$

$x^2 + y^2 = r^2$ circle of radius r centred at $(0,0)$

$(x-a) + (y-b)^2 = r^2$

② Double Integral:





$$\iint_R f(x,y)\,dA = \int_{\theta=0}^{\theta=\pi/3} \int_{r=0}^{r=3} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{r=0}^{r=3} \int_{\theta=0}^{\theta=\pi/3} f(r\cos\theta, r\sin\theta)\, r\,\overset{d\theta\,dr}{\cancel{dr}\,\cancel{d\theta}}$$

③ Let $f(x,y) = x + y = r\cos\theta + r\sin\theta$

$$\iint_R f(x,y)\,dA = \int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=3} (r\cos\theta + r\sin\theta)\underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{3}}\int_{r=0}^{r=3} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/3} \frac{1}{3}r^3(\cos\theta+\sin\theta)\Big|_{r=0}^{r=3}\,d\theta$$

$$= \int_0^{\pi/3} 9(\cos\theta+\sin\theta)\,d\theta$$

$$= 9\left[\sin\theta - \cos\theta\right]_0^{\pi/3}$$

$$= 9\left(\left[\sin(\pi/3) - \cos(\pi/3)\right] - \left[\sin(0) - \cos(0)\right]\right)$$

$$= 9\left(\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right] - [0-1]\right)$$

$$= 9\left(\frac{\sqrt{3}-1}{2} + 1\right)$$

$$= 9\left(\frac{\sqrt{3}-1}{2} + \frac{2}{2}\right) = \frac{9}{2}(\sqrt{3}+1) \approx 12.294$$

Let a thin lamina $\mathcal{L}$ lie in the first quadrant of the xy-plane with boundary curves

- $x^2 + y^2 = 5$,
- $x^2 + y^2 = 6$
- $y = x$, and
- $x = 0$.

1. Sketch $\mathcal{L}$, the region of integration.
2. Given that the lamina has a density $\delta(x, y)$, write an expression for the mass M of the lamina as a double integral in polar coordinates.
3. If $\delta(x, y) = \dfrac{2xy}{x^2 + y^2}$, find the mass M of the lamina.

Lamina $\mathcal{L}$

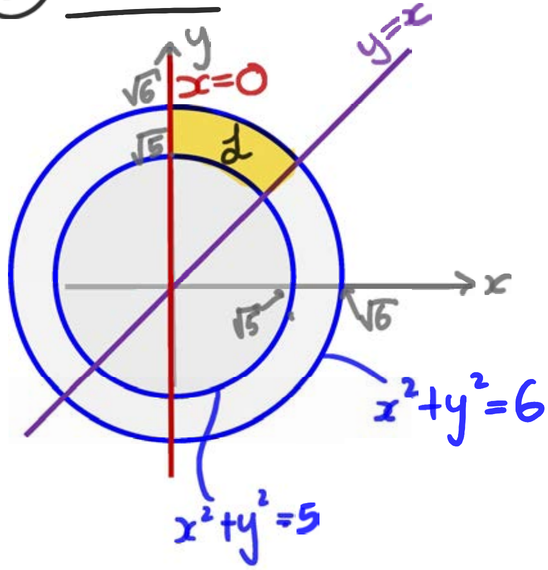
$x^2+y^2=5 \leftarrow r^2$

$x^2+y^2=6 \leftarrow r^2$

$y=x$

$x=0$

① Sketch

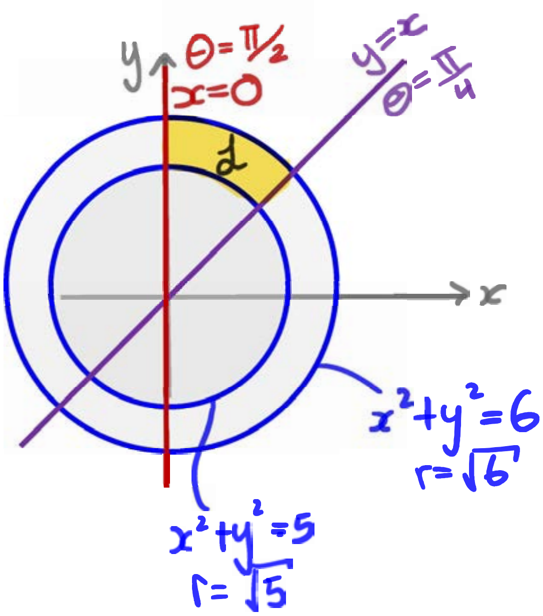


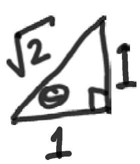
$$\iint_R \rho(x,y)\,dA$$

$$\iint \rho(r\cos\theta, r\sin\theta)\,\underline{r}\,dr\,d\theta$$

$$d^\circ = d^\circ \times \frac{\pi}{180^\circ}$$

② Set up integral for mass given density $\delta(x,y)$.



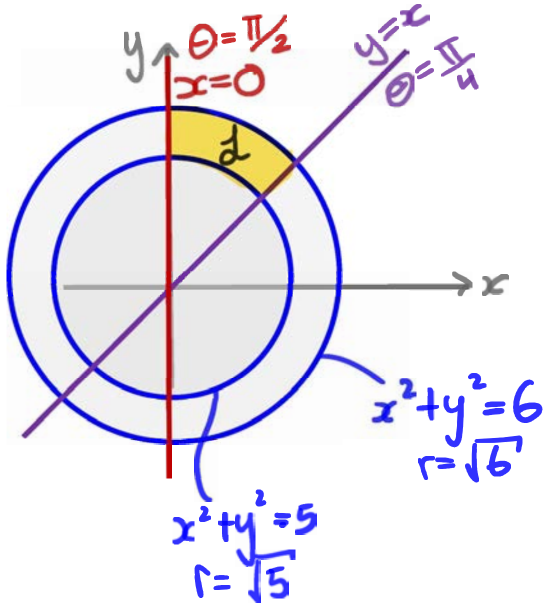


$\tan\theta = \frac{1}{1} \rightarrow \theta = \tan^{-1}(1)$
$= \pi/4$

$M = \text{Mass}(\mathcal{L})$

$$= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{5}}^{r=\sqrt{6}} \delta(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

③ Find mass M given density $\delta(x,y) = \dfrac{2xy}{x^2+y^2}$



$$M = \text{Mass}(\mathcal{L})$$

$$\delta(x,y) = \frac{2xy}{x^2+y^2} = \frac{2(r\cos\theta)(r\sin\theta)}{r^2} = 2\cos\theta\sin\theta = \sin(2\theta)$$

So $$M = \text{Mass}(\mathcal{L}) = \int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=\sqrt{5}}^{r=\sqrt{6}} \sin(2\theta)\, r\,dr\,d\theta$$

$$= \left(\int_{\theta=\pi/4}^{\theta=\pi/2} \sin(2\theta)\,d\theta\right)\left(\int_{r=\sqrt{5}}^{r=\sqrt{6}} r\,dr\right)$$

$$\int \sin(2\theta)\,d\theta = -\frac{1}{2}\cos(2\theta) + C$$

$$= \left[-\frac{1}{2}\cos(2\theta)\right]_{\pi/4}^{\pi/2} \cdot \left[\frac{1}{2}r^2\right]_{\sqrt{5}}^{\sqrt{6}}$$

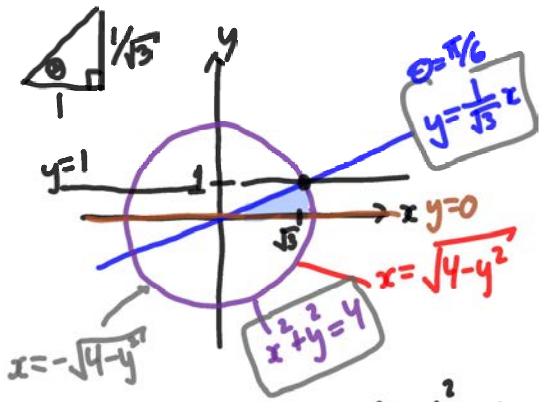
$$= \frac{1}{2}\cdot\left(\frac{6}{2}-\frac{5}{2}\right) = \frac{1}{4}$$

Example 9.15. Find the volume of the solid below $z = \ln(1 + x^2 + y^2)$ over the region $\mathcal{R} = \left\{(x, y) : \ 0 \le y \le 1, \ \sqrt{3}y \le x \le \sqrt{4 - y^2}\right\}$.

$\tan\theta = \dfrac{1/\sqrt{3}}{1}$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \pi/6$



$\sqrt{3}y = x$

$y = \frac{1}{\sqrt{3}}x$

$x = \sqrt{4-y^2}$

$x^2 = 4 - y^2$

$x^2 + y^2 = 4$

$x^2 + \left(\frac{1}{\sqrt{3}}x\right)^2 = 4$

$x^2 + \frac{1}{3}x^2 = 4$

$\frac{4}{3}x^2 = 4 \longrightarrow x^2 = 3 \longrightarrow x = \sqrt{3}$ or $\cancel{x = -\sqrt{3}}$

$y = \frac{1}{\sqrt{3}}x = \frac{1}{\sqrt{3}} \cdot \sqrt{3} = 1$

$(\sqrt{3}, 1)$

$\ln(1 + x^2 + y^2)$

$$\int_0^2 \int_0^{\pi/6} \ln(1 + r^2) \, \underbrace{r \, d\theta \, dr}_{dA}$$

$$= \left(\int_0^2 r \ln(1 + r^2) \, dr\right)\left(\int_0^{\pi/6} d\theta\right)$$

Let $u = 1 + r^2$

Then $du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$$= \left(\int_1^5 \frac{1}{2} \ln u \, du\right)\left(\frac{\pi}{6}\right)$$

by parts

$$= \frac{1}{2} \cdot \frac{\pi}{6}\Big[u \ln u - u\Big]_1^5 = \frac{\pi}{12}\Big([5\ln 5 - 5] - [\cancel{1 \cdot \ln 1}^{\,0} - 1]\Big) = \frac{\pi}{12}(5\ln 5 - 4)$$

Integration by parts

$$\int \ln x \, dx = \int \ln x \cdot 1 \, dx = \int \ln x \cdot (x)' \, dx$$

(Let $u = \ln x$, $dv = dx$) $= x \ln x - \int x (\ln x)' \, dx$

$$= x \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

9.4 References

References:

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