Signal Processing (MENG3520)

Module 2

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MODULE 2

LTI SYSTEMS

MODULE OUTLINE

2.1 System overview

2.2 System properties

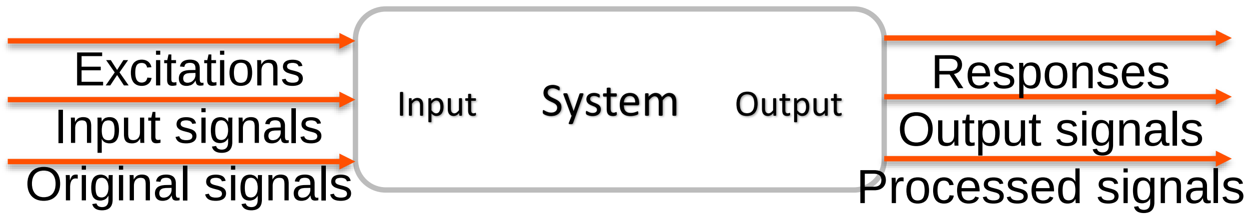
- Memory
- Causality
- Invertibility
- Stability
- Time invariance
- Linearity – additivity and homogeneity

2.3 Characteristics of LTI systems

2.1

SYSTEM OVERVIEW

- We have defined signals and systems in Module 1: systems process signals to produce a modified or transformed version of the original signal.
- A system is excited by excitations or input (original) signals applied at the input, response or output (processed) signals appear at the output.



A system with input x and output y can be described by the following:

$y = H\{x\}$, where H denotes an operator (i.e. transformation).

Alternatively, the same system can be described by the following:

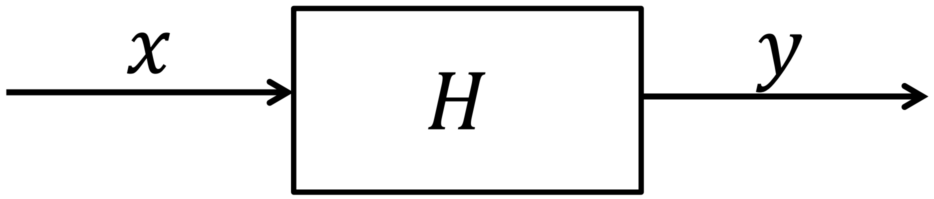
$$x \xrightarrow{H} y$$

Sometimes, the operator H is implied and it is written as

$$x \rightarrow y$$

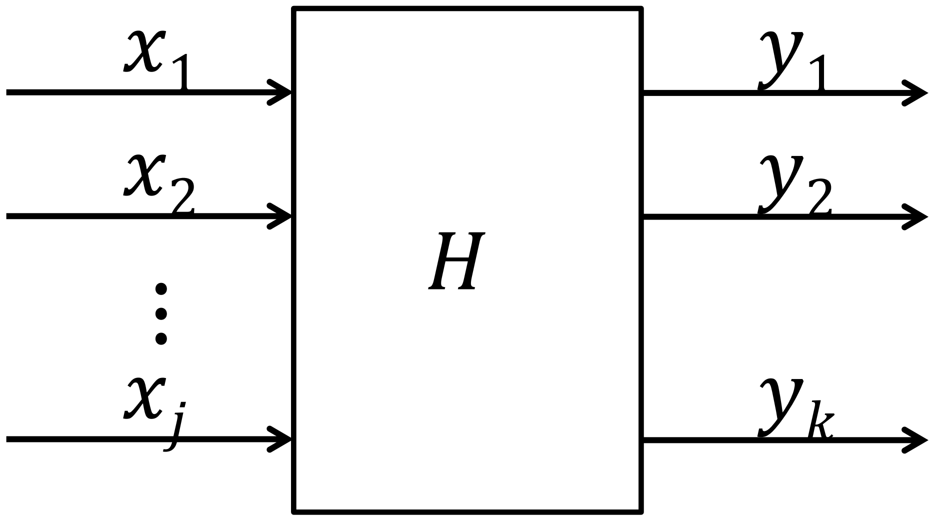
In systems analysis, it is common to represent systems using block diagrams.

A simple single input, single output system can be represented by the following block diagram:



In systems analysis, it is common to represent systems using block diagrams.

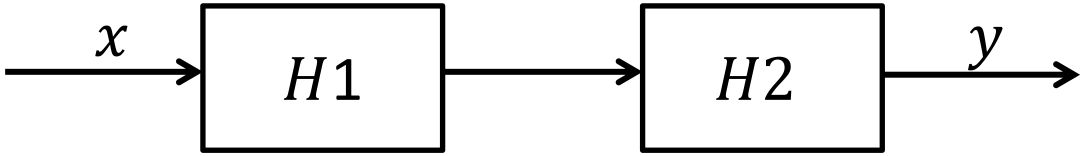
Sometimes, a system may have multiple inputs and multiple outputs. In this case, the system can be represented by the following block diagram:



- Interconnections of systems: many real systems are built as interconnections of several subsystems.
- These subsystems are often referred to as components.
- Components are smaller, simpler systems (operators) that is often standardized in the practice with its property known.
- The way these components are interconnected can be summarized into a few typical ways: series, parallel, series-parallel, feedback.

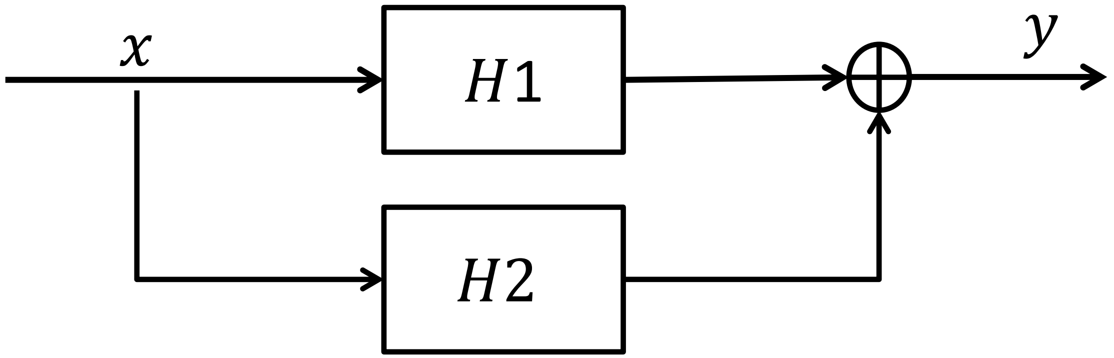
Basic ways in which systems can be inter-connected – series (cascade).

$$y = H2\{H1\{x\}\}$$



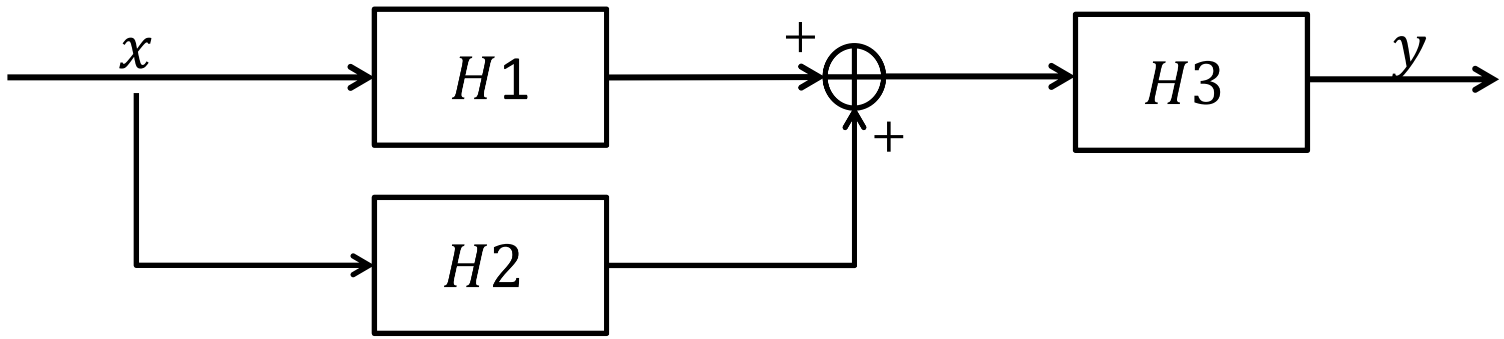
Basic ways in which systems can be inter-connected – parallel.

$$y = H1\{x\} + H2\{\mathrm{x}\}$$



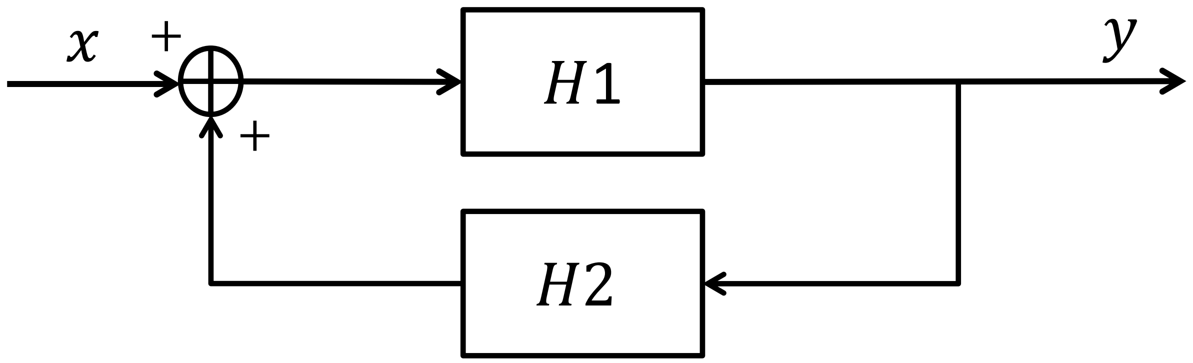
Basic ways in which systems can be inter-connected are: series-parallel.

$$y = H3\{H1\{x\} + H2\{x\}\}$$



Basic ways in which systems can be inter-connected – feedback.

$$y = H1\{x + H2\{y\}\}$$



Activity: decide what type of connections the following systems use:

(1) Several water tank meters reporting back individual water levels to the control centre.

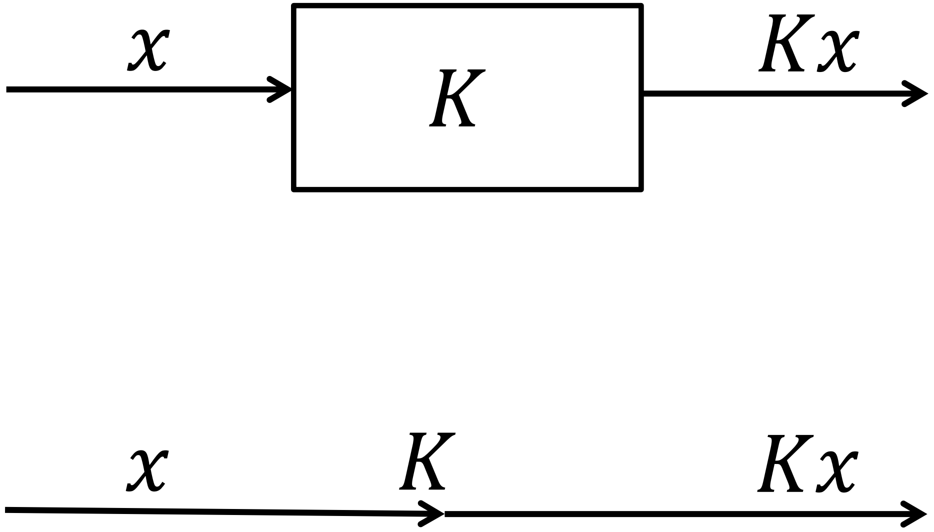
(2) A vehicle senses its velocity and adjust its fuel level to maintain speed.

(3) Two-tier security system requires a second password verification once the first one is confirmed.

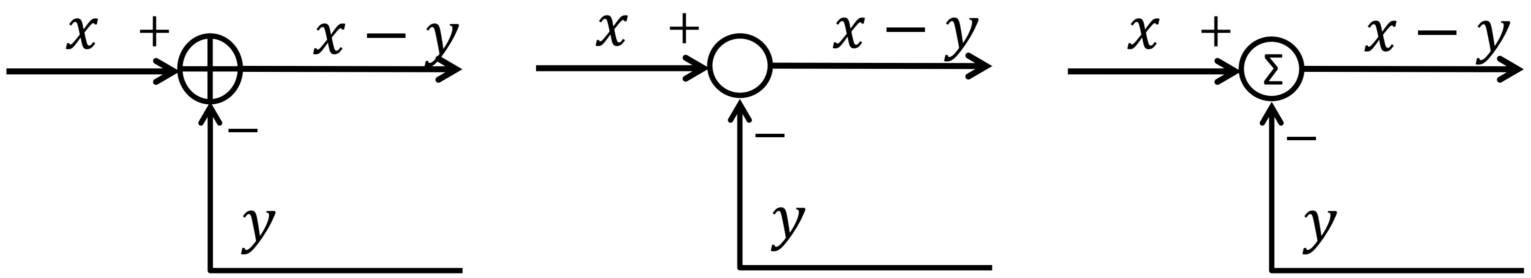
Graphical representations of some common components (operators).

- Amplifiers
- Summing junction
- Integrator

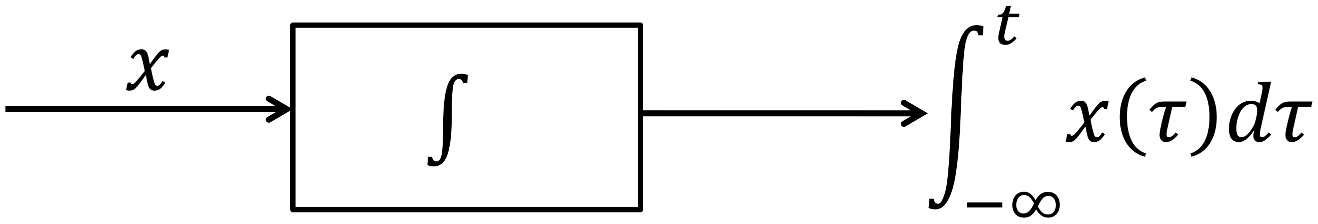
- Amplifiers



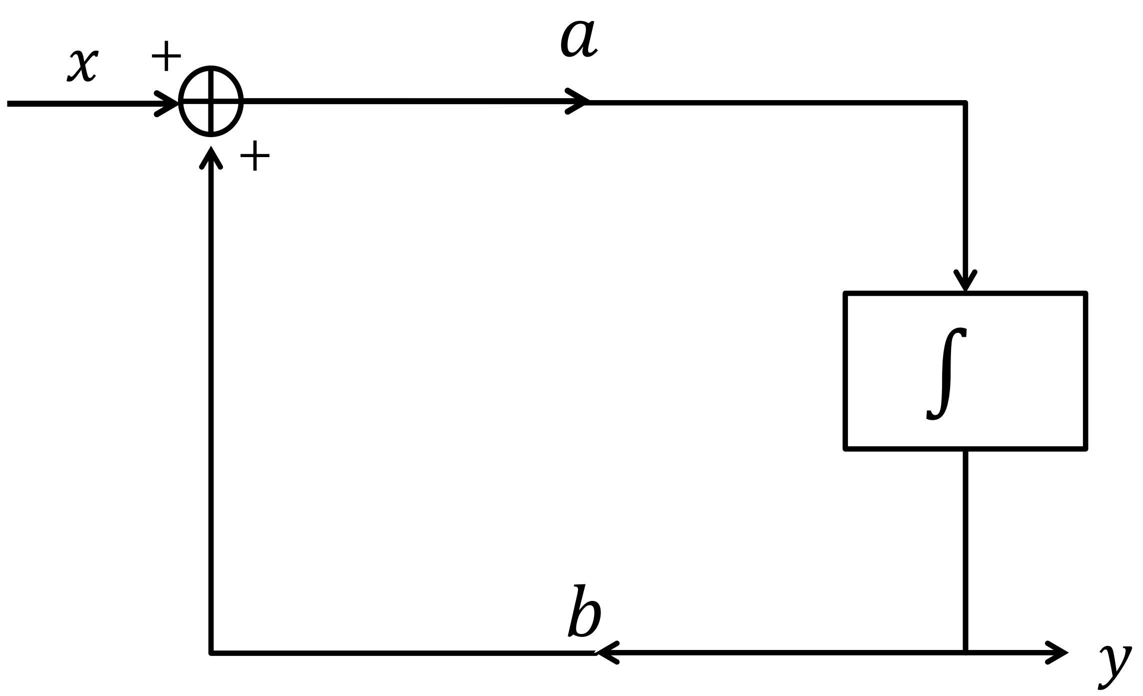
- Summing junction



- Integrator



- Activity: analyze the system shown on the block diagram below.



2.2

SYSTEM PROPERTIES

Important System Properties

- Memory
- Causality
- Invertibility
- Stability
- Time invariance
- Linearity – additivity and homogeneity

Memory means a system has a mechanism of accessing information that is not current.

Memory of a CT system: a CT system $y(t) = H\{x(t)\}$ has memory if for any real t_0, $y(t_0)$ depends on $x(t)$ where $t \neq t_0$.

Memory of a DT system: a DT system $y[n] = H\{x[n]\}$ has memory if for any integer n_0, $y[n_0]$ depends on $x[n]$ where $n \neq n_0$.

A system without memory is called memoryless.

- Memoryless system depends only on the current input and state of the system.
- Memoryless system is less flexible in modeling real-life systems.
- Memoryless system is simpler in implementing system design.

Activity: Determine whether the following systems are memoryless or with memory?

- $y(t) = \int_{-\infty}^{t} x(t)dt$
- $y[n] = n + \sin^2(x[n])$
- $y[n] = y[n-1] + x[n]$

Causality means the system does not anticipate future values of the input.

Causality of a CT system: a system $y(t) = H\{x(t)\}$ is causal if for any real t_0, $y(t_0)$ does not depend on $x(t)$ where $t > t_0$.

Causality of a DT system: a system $y[n] = H\{x[n]\}$ is causal if for any integer n_0, $y[n_0]$ does not depend on $x[n]$ where $n > n_0$.

Activity: Determine whether the following systems are causal:

- $y(t) = x(-t)$
- $y(t) = x(t+1)$
- $y(t) = \int_{-\infty}^{t} x(t)dt$
- $y[n+1] = x[n]$
- $y[n] = x[4n+1]$

Practicality of system causality:

- If independent variable t or n represents real time, then a system must be causal to be physically realizable in real-time.
- For noncausal systems with time as the independent variable t or n, they are only realizable with time delay.
- If independent variable t or n represents other physical quantities such as position, temperature, voltage, or recorded time, etc. then non-causal system can be useful and realizable in real-time.

Noncausal time-based systems are realizable if they are not real-time, meaning: delay is needed.

Noncausal systems are realizable with time delay!

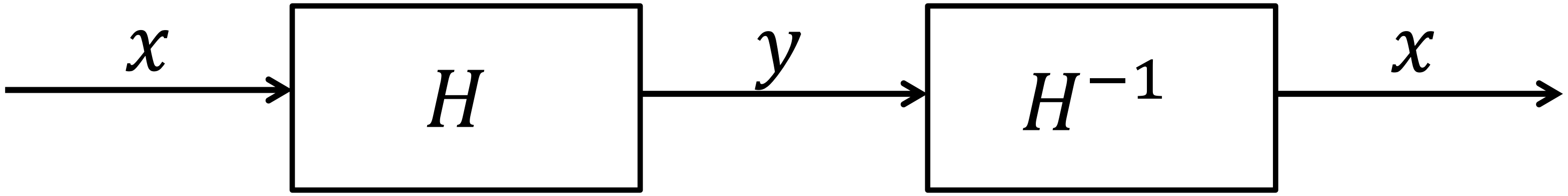
Invertibility of a system: a system $y = H\{x\}$ is invertible if there exist another system H^{-1} such that cascaded H and H^{-1} will generate the same output as the input:

$$x = H^{-1}\{y\} = H^{-1}\{H\{x\}\}$$

- An invertible system means that its input x can be uniquely determined by its output y through the inverse operator H^{-1}.
- An invertible system also means that a distinct input will produce a distinct output.

Benefit of system invertibility:

- An invertible system H must have a inverse system H^{-1} so that its effect can be completed undone / reversed.
- Invertible systems are commonly used in system design where lossless conversion is needed.



Stability of a system: a system $y = H\{x\}$ is stable if any bounded input produces a bounded output:

$$|x| < \infty \; implies \; |y| < \infty$$

- This bounded input implies bounded output is referred to as BIBO stable.
- If one bounded input leads to an unbounded output, this system is not BIBO stable.
- BIBO stable is highly desirable in practice because the system will face serious safety issues otherwise.

Time invariance of a system: a system $y = H\{x\}$ is time invariant (TI) if for any input x and real delay t_0 or n_0, the following is true:

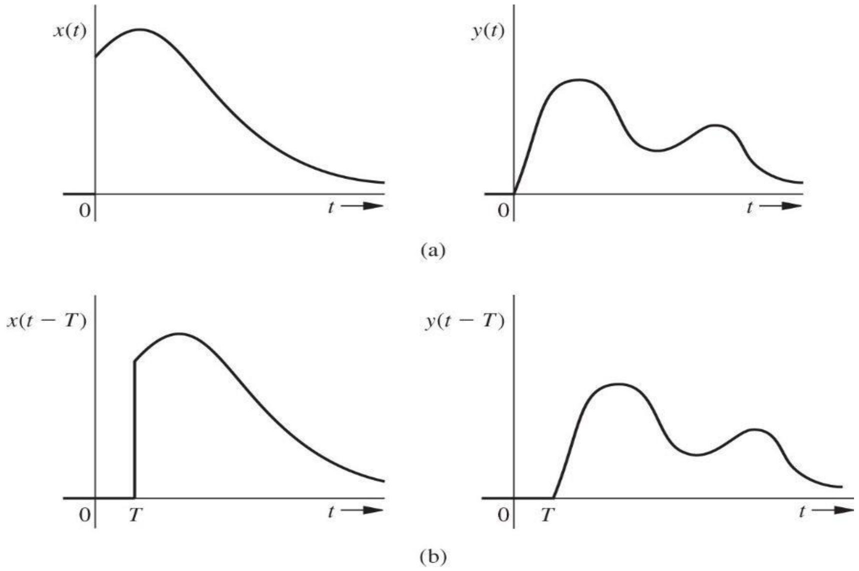
For CT system: $y(t - t_0) = H\{x(t - t_0)\}$

For DT system: $y[n - n_0] = H\{x[n - n_0]\}$

- Time invariance implies any time shift in the input signal will result in an identical time shift in the output signal.
- A time invariant system implies that this system does not change its behavior with respect to time.

- Time invariance is a highly desirable property for any system since its behavior does not change with respect to time, and thus easier to design and analyze.

- A system that is not time invariant is considered time varying.



- Activity: determine if the following systems are time invariant.
- $y(t) = 2x(t)$
- $y(t) = \sin\{x(t)\}$
- $y[n] = n \sin(x[n])$
- $y[n] \ = \ x[n] + x[n-1]$

Additivity of a system: a system $y = H\{x\}$ is additive if for any input signals $x1$ and $x2$, the following is true:

$$H\{x1 + x2\} = H\{x1\} + H\{x2\}$$

Homogeneity of a system: a system $y = H\{x\}$ is homogeneous if for any input signals x and every complex constant a, the following is true:

$$H\{ax\} = aH\{x\}$$

Linearity of a system: a system $y = H\{x\}$ is additive if for any input signals $x1$ and $x2$, and every complex constants $a1$ and $a2$, the following is true:

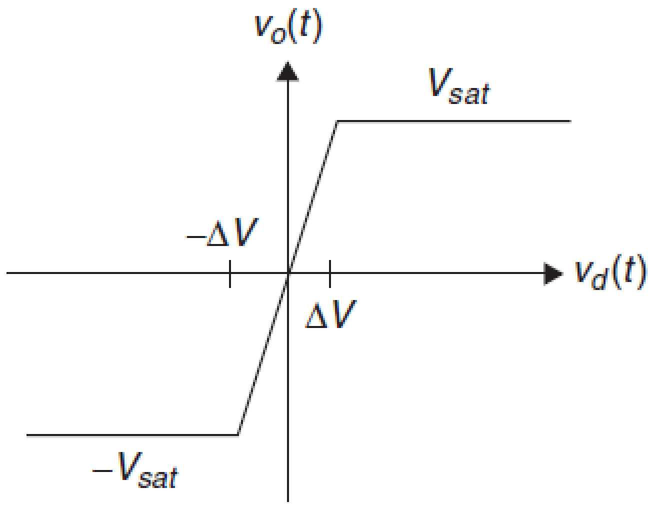
$$H\{a1x1 + a2x2\} = a1H\{x1\} + a2H\{x2\}$$

A linear system is both additive and homogenous.

Linearity is also referred as possess **superposition** property.

Activity: consider operational amplifier or op-amp. Is it a linear or nonlinear device?

- $v_o(t) = H\{v_+(t) - v_-(t)\} = H\{v_d(t)\}$
- $v_o(t) = av_d(t)$, for $-\Delta V \leq v_d(t) \leq \Delta V$,



Systems that are both linear and time invariant are called **Linear Time Invariant** (LTI) systems.

- LTI systems are the focus of this course and one of the most researched systems in engineering.
- Linearity and time invariance are independent of each other.
- Although many actual systems are nonlinear and time varying, linear models are used to approximate around an operating point the nonlinear behavior, and time-invariant models are used to approximate in short segments the system’s time-varying behavior.

2.3

Characteristics of LTI Systems

LTI characteristics:

- Two key properties of LTI system: time invariant and superposition.
- As a consequence, if the input of an LTI system can be represented as a linear combination of time-shifted basic signals, then the output of the system to such signals can be expressed the linear combination of time-shifted responses to these basic signals.

LTI systems analysis

Step 1. Represent any input signal as a linear combination of time-shifted basic signals;

Step 2. Find corresponding response to individual basic signals;

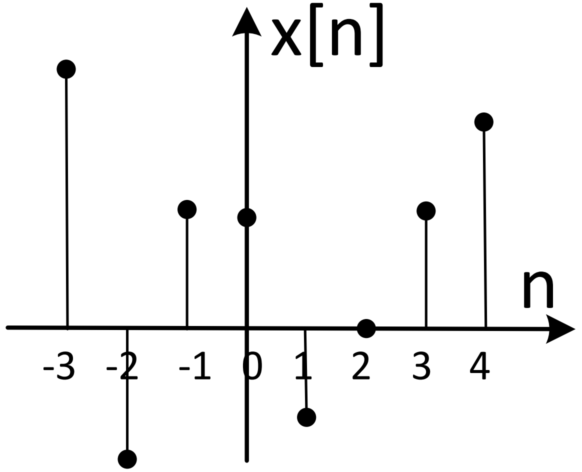
Step 3. Use superposition and time invariance to compute the resultant output signal.

Let’s start with a DT signal!

2.3.1

CHARACTERISTICS OF DT LTI SYSTEMS

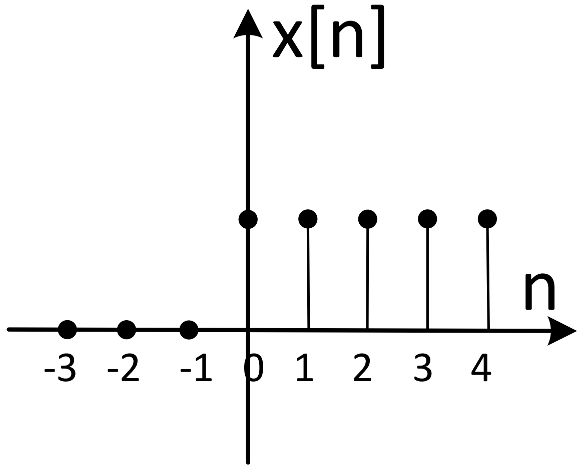
Decomposition of DT Signals



$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] = \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2]$$
$$+x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1]$$
$$+x[2]\delta[n-2] + x[3]\delta[n-3] + x[4]\delta[n-4] + \cdots$$

Example of DT Signal Decomposition

$$u[n] = \sum_{k=-\infty}^{+\infty} u[k]\delta[n-k] = \sum_{k=0}^{+\infty} \delta[n-k]$$



This is consistent with our previous understanding of the relation between the unit step sequence and the impulse sequence.

Representation of DT LTI System Response

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k\,[n]$$

Here $h_k\,[n] = H\{\delta[n-k]\}$ is response of this system to the time-shifted unit impulse $\delta[n-k]$. If the system is time invariant, then:

$$h_k\,[n] = H\{\delta[n-k]\} = h_0[n-k] \triangleq h[n-k]$$

Thus:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k\,[n] \; = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Representation of DT LTI System Response

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

How to interpret this equation?

LTI system response to any arbitrary signal $x[n]$ is the superposition of the time-shifted impulse responses weighted by the input signal.

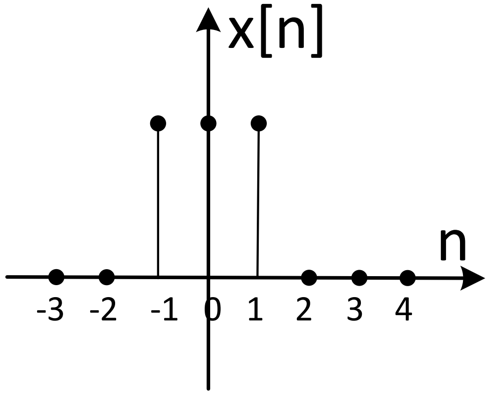
Representation of DT LTI System Response

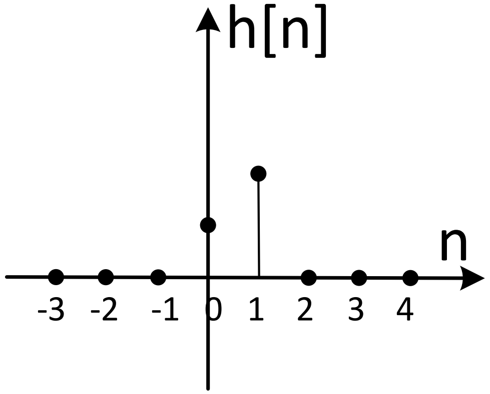
This operation is called **convolution sum** or **superposition sum**.
It is written as:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \triangleq x[n] * h[n]$$

Conclusion: for DT LTI systems, you only need to solve the difference equation for the system once to get the unit impulse response and then the response of any signal can be computed by convolution sum of the input signal and the unit impulse response.

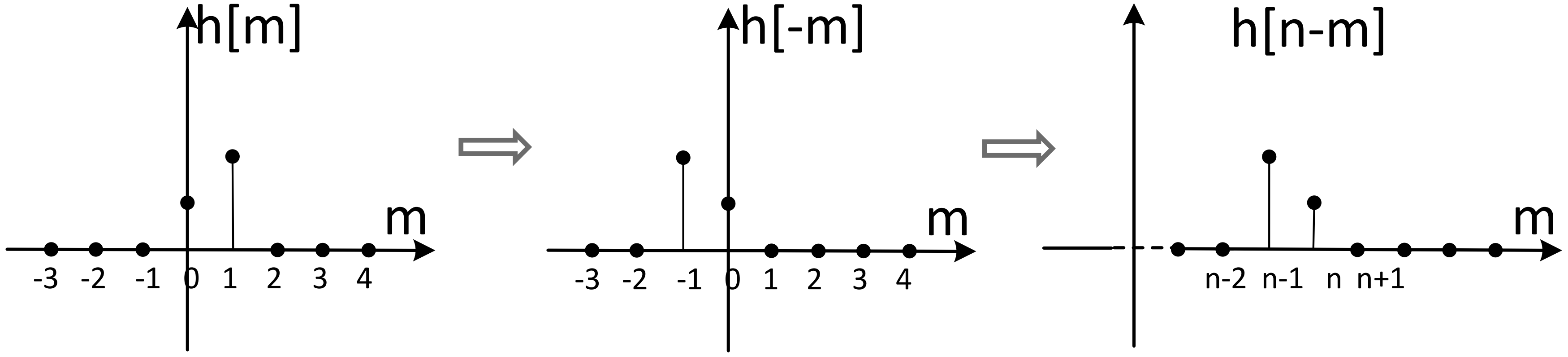
How Convolution sum is done





How Convolution sum is done

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]h[n-m]$$



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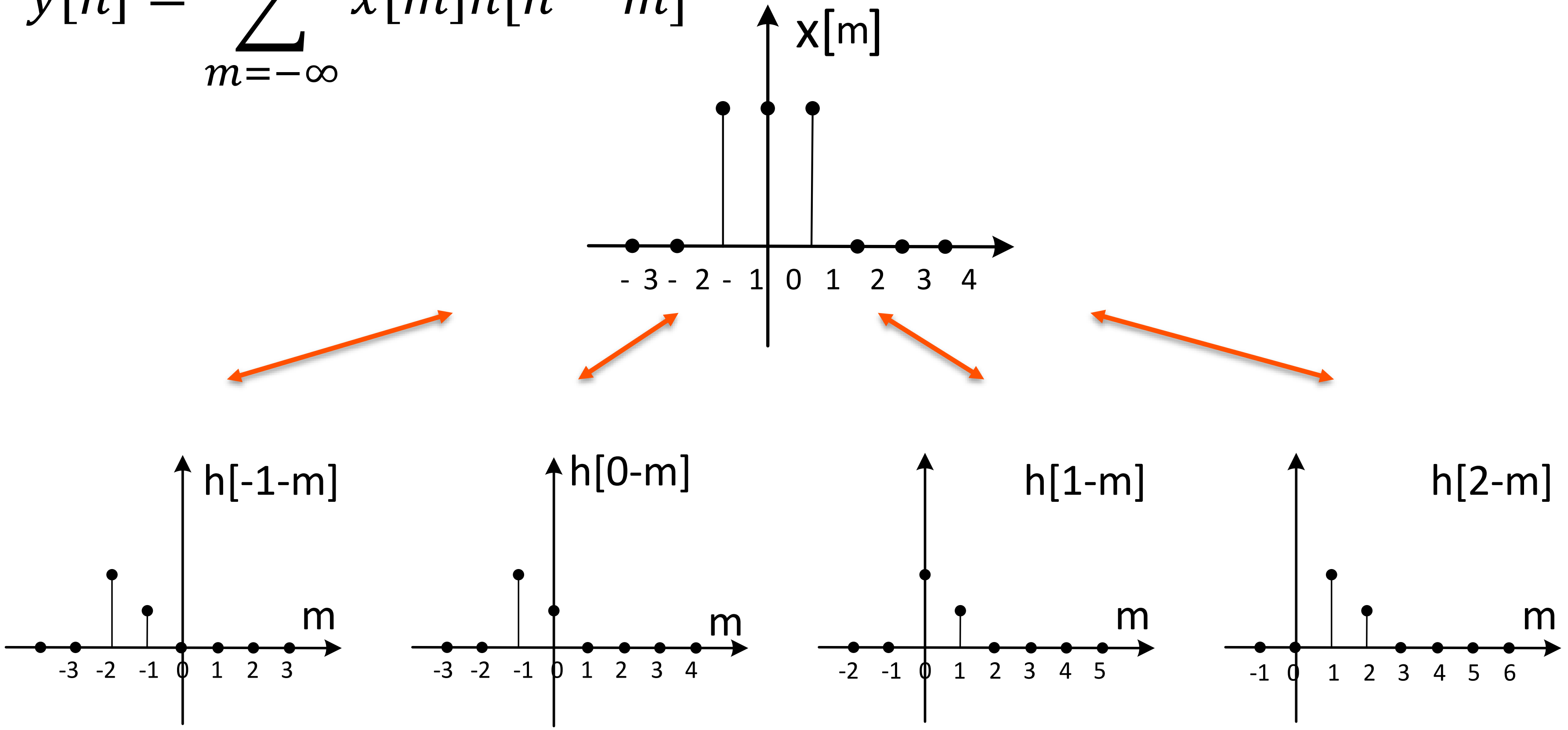
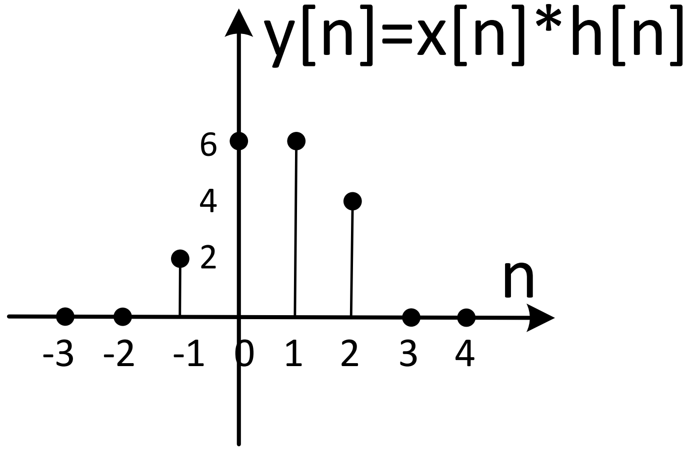


How Convolution sum is done



2.3.2

Characteristics of CT LTI Systems

Decomposition of CT Signal

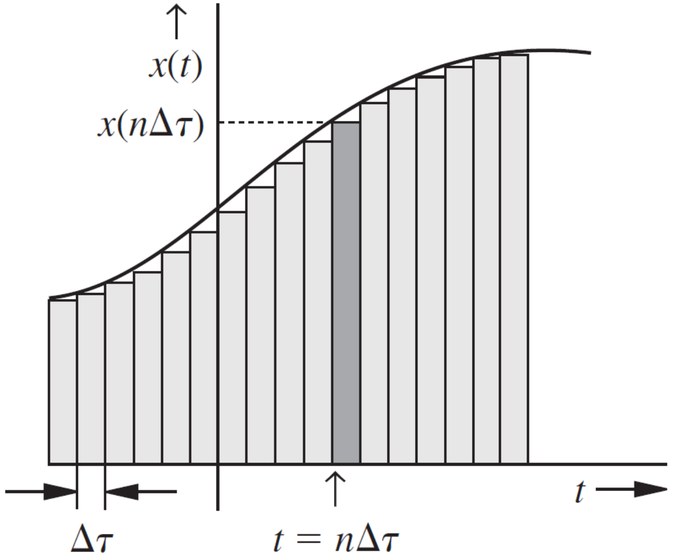
Approximate signal $x(t)$ using the sum of a sequence of rectangles.

$$\hat{x}(t) \triangleq \sum_{k=-\infty}^{+\infty} x(k\Delta\tau)\delta_{\Delta\tau}(t - k\Delta\tau)\Delta\tau$$

Here $\delta_{\Delta\tau}(t - k\Delta\tau)$ is the time shifted version of the function below:

$$\delta_{\Delta\tau}(t) = \begin{cases} \frac{1}{\Delta\tau}, 0 \leq t < \Delta\tau \\ 0, otherwise \end{cases}$$

Decomposition of CT Signals



When $\Delta\tau \to 0$:

$$x(t) \approx \hat{x}\,(t) = \lim_{\Delta\tau\,\to 0} \sum_{\tau} x(n\Delta\tau)\delta(t - n\Delta\tau)\Delta\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

Intuitively, $x(t)$ is the sum of a weighted shifted impulses.

Representation of CT LTI System Response

Similar to the DT case, the CT LTI system response

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta\tau)h(t - k\Delta\tau)\Delta\tau = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau$$

Here $h_\tau(t) = H\{\delta(t-\tau)\}$ is the response of this system to the unit impulse $\delta(t-\tau)$. If the system is time invariant:

$$h_\tau(t) = H\{\delta(t-\tau)\} = h_0(t-\tau) \triangleq h(t-\tau)$$

Thus:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

Representation of CT LTI System Response

Similar to the DT case, the CT LTI system response

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

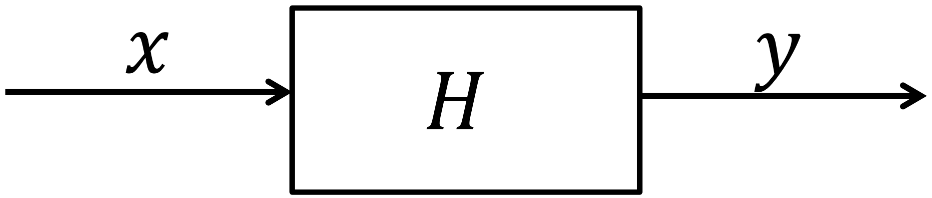
How to interpret this equation?

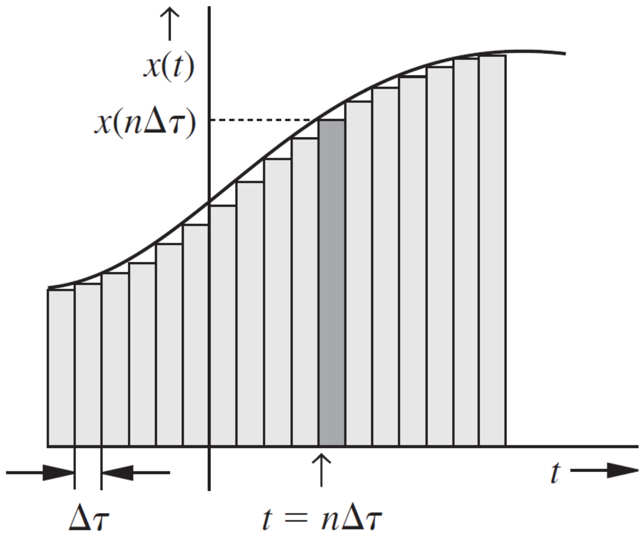
LTI system response to any arbitrary signal $x(t)$ is the intergral of the time shifted impulse responses $h(t-\tau)$ weighted by the input signal.

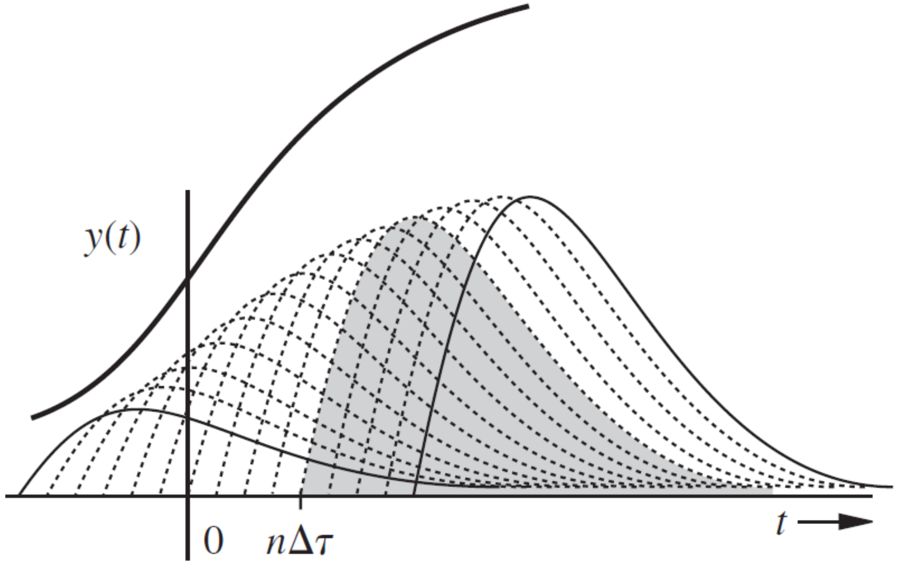
Representation of CT LTI System Response

How to interpret this equation?

LTI system response to any arbitrary signal $x(t)$ is the integral of the time shifted impulse responses $h(t-\tau)$ weighted by the input signal.







Representation of CT LTI System Response

This operation is called **convolution integral** or **superposition integral**. It is written as:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \triangleq x(t) * h(t)$$

Summary: for CT LTI system, only have to solve the differential equation for the system once for unit impulse response and then find response of any signal by using convolution integral.

Properties of Convolution

- The convolution operation is commutative.

$$x * h = h * x$$

- The convolution operation is associative.

$$x * (h_1 * h_2) = (x * h_1) * h_2$$

- The convolution operation is distributive with respect to addition.

$$x * (h_1 + h_2) = (x * h_1) + (x * h_2)$$

Summary: LTI Systems

An LTI system with its impulse response being h, for any input x, the corresponding zero-state output y is:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \triangleq x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \triangleq x(t) * h(t)$$

Homework:

Problems: 1.7-2, 1.7-3

Review examples 2.10,2.11,2.12 for Graphical Convolution.

Complete drills 2.10, 2.11, 2.12