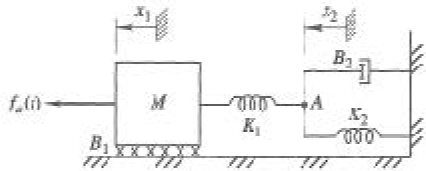
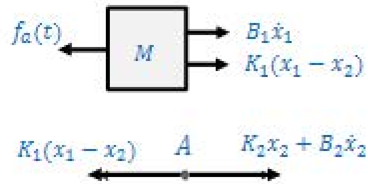
MENG 3020 – Quiz 1 Solution – Fall 2024

Question A translational mechanical system is shown below. The springs are undeflected when $x_1 = x_2 = 0$. The *system input* is the applied force $f_a(t)$ and the *system output* is the displacement x_2 of massless junction A.



Solve for the following questions:

a) Draw the free-body diagrams for mass M and junction A. Show all forces applied and write a set of ordinary differential equations of motion. Show your work.



From the free-body diagrams and applying the Newton's second law:

Mass M $\rightarrow$ $f_a(t) - K_1(x_1 - x_2) - B_1\dot{x}_1 = M\ddot{x}_1$

Junction A $\rightarrow$ $K_1(x_1 - x_2) = K_2x_2 + B_2\dot{x}_2$

The equation of motion is:

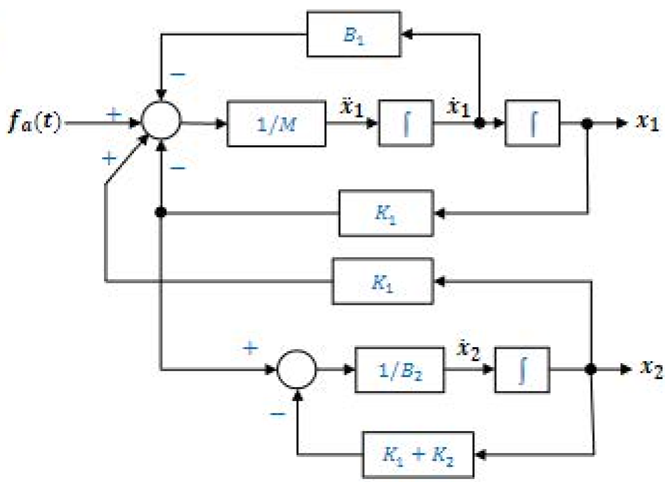
$$f_a(t) = M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2$$

$$B_2\dot{x}_2 + (K_1 + K_2)x_2 - K_1x_1 = 0$$

b) Complete the following block diagram model based on the equations of motion in Part (a).

$$\ddot{x}_1 = \frac{1}{M}(f_a(t) - B_1\dot{x}_1 - K_1x_1 + K_1x_2)$$

$$\dot{x}_2 = \frac{1}{B_2}(-(K_1 + K_2)x_2 + K_1x_1)$$



c) [8 marks] Assume $M = 1kg,\ B_1 = B_2 = 1\ N.s/m,\ K_1 = K_2 = 1\ N/m$. Select the appropriate state variables and develop a set of *state-variable equations* and *output equation*. Write the state-space equations in *matrix-vector* form. Show your work.

Having the equation of motion from Part (a):

$$f_a(t) = M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2$$

$$B_2\dot{x}_2 + (K_1 + K_2)x_2 - K_1x_1 = 0$$

Setting the mass, spring and damping values, the equations of motion will be:

$$f_a(t) = \ddot{x}_1 + \dot{x}_1 + x_1 - x_2$$

$$\dot{x}_2 + 2x_2 - x_1 = 0$$

The state variables q_1, q_2 and q_3 are selected as the displacement of the springs K_1, K_2 and velocity of the mass M.

$$q_1 = x_1 - x_2 \ \rightarrow\ \dot{q}_1 = \dot{x}_1 - \dot{x}_2 \ \rightarrow\ \dot{q}_1 = q_3 - q_1 + q_2 \qquad Eqn.\ (1)$$

$$q_2 = x_2 \ \rightarrow\ \dot{q}_2 = \dot{x}_2 \ \rightarrow\ \dot{q}_2 = x_1 - 2x_2 = (x_1 - x_2) - x_2 \ \rightarrow\ \dot{q}_2 = q_1 - q_2 \qquad Eqn.\ (2)$$

$$q_3 = \dot{x}_1 \ \rightarrow\ \dot{q}_3 = \ddot{x}_1 \ \rightarrow\ \dot{q}_3 = f_a(t) - \dot{x}_1 - x_1 + x_2 \ \rightarrow\ \dot{q}_3 = f_a(t) - q_3 - q_1 \qquad Eqn.\ (3)$$

The state-variable equations and output equation are obtained as:

$$\dot{q}_1 = q_3 - q_1 + q_2$$
$$\dot{q}_2 = q_1 - q_2$$
$$\dot{q}_3 = f_a(t) - q_3 - q_1$$

$$y = x_2 \ \rightarrow\ y = q_2$$

The state equation and output equation in the standard matrix-vector form are:

State Equation: $\dot{\mathbf{q}}(t) = \mathbf{A}\,\mathbf{q}(t) + \mathbf{B}\,\mathbf{u}(t)$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 0\\ -1 & 0 & -1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} f_a(t)$$

Output Equation: $\boldsymbol{y(t) = Cq(t) + Du(t)}$

$$y(t) = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + [0]u(t)$$

The state variables q_1, q_2 and q_3 can also be selected as the displacement of mass M, velocity of the mass M and the displacement of spring K_2 and. In this case the state-space model is obtained as below:

$$q_1 = x_1 \quad \rightarrow \quad \dot{q}_1 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_1 = q_2 \qquad Eqn.\ (1)$$

$$q_2 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_2 = \ddot{x}_1 \quad \rightarrow \quad \dot{q}_2 = f_a(t) - \dot{x}_1 - x_1 + x_2 \quad \rightarrow \quad \dot{q}_2 = f_a(t) - q_2 - q_1 + q_3 \qquad Eqn.\ (3)$$

$$q_3 = x_2 \quad \rightarrow \quad \dot{q}_3 = \dot{x}_2 \quad \rightarrow \quad \dot{q}_3 = x_1 - 2x_2 \quad \rightarrow \quad \dot{q}_3 = q_1 - 2q_3 \qquad Eqn.\ (2)$$

The state-variable equations and output equation are obtained as:

$\dot{q}_1 = q_2$
$\dot{q}_2 = f_a(t) - q_2 - q_1 + q_3$
$\dot{q}_3 = q_1 - 2q_3$

$y = x_2 \quad \rightarrow \quad y = q_3$

The state equation and output equation in the standard matrix-vector form are:

State Equation: $\dot{q}(t) = \mathbf{A}\,\mathbf{q}(t) + \mathbf{B}\,\mathbf{u}(t)$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} f_a(t)$$

Output Equation: $\boldsymbol{y(t) = Cq(t) + Du(t)}$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + [0]u(t)$$