Faculty of Applied Science and Technology

Laboratory 6 Fourier Series

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Note 1: This is an individual lab; please complete this on your own laptop.

Note 2: Lab report needs to be converted into .pdf format and submitted to SLATE -> Assignments folder before the deadline.

1. Learning outcome:

1.1 Implement continuous-time Fourier series as a different way to represent continuous-time periodic signals.

1.2 Investigate the Gibbs phenomenon.

1.3 Design a wave generating system that makes use of the Fourier components in a signal.

2. Background

Continuous-time Fourier series is a transform that converts continuous-time periodic functions into the frequency domain. Due to the periodic feature of the original function in the time domain, the transformed Fourier series has some interesting features.

2.1 Exponential Fourier Series

Let $x(t)$ to be a periodic CT function. It can be represented using the exponential Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Here, the series of sinusoids $e^{jk\omega_0 t}$ are harmonically-related with fundamental frequencies $k\omega_0$, the Fourier coefficient c_k can be computed using the integration below over any time interval of one period long:

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

2.2 Trigonometric Fourier Series

Exponential Fourier series is not the only Let $x(t)$ to be a periodic CT function. It can be represented using the trigonometric Fourier series, which means the periodic signal is written as an infinite sum of sines and cosines functions:

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

Here, the series of $\cos k\omega_0 t$ and $\sin k\omega_0 t$ are the harmonics, the Fourier coefficient a_0, a_k, b_k can be computed using the integration below over any time interval of one period long:

$$a_0 = \frac{1}{T}\int_T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_T x(t)\sin(k\omega_0 t)\,dt$$

2.3 Compact Trigonometric Fourier Series

Trigonometric Fourier series is very intuitive and easy to understand, however, this expression can be cumbersome since each harmonic needs two terms to represent. To make the Fourier series more compact, the compact trigonometric Fourier series is introduced.

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) = c_0 + \sum_{k=1}^{\infty} d_k(\cos(k\omega_0 t + \theta_k))$$

Here c_0, d_k and θ_k are related to a_k and b_k.

$$c_0 = a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$d_k = \sqrt{a_k^2 + b_k^2}$$

$$\theta_k = \begin{matrix} \tan^{-1}\left(-\frac{b_n}{a_n}\right) \\ \sin^{-1}\left(\frac{-b_k}{d_k}\right) \\ \cos^{-1}\left(\frac{a_k}{d_k}\right) \end{matrix}$$

3. Procedures

3.1 Continuous-time Fourier series.

A group of complex sinusoids is **harmonically related** if there exists a constant ω_0 such that the fundamental frequency for each of these sinusoids is an integer multiple of ω_0. A continuous signal $x(t)$ satisfies the following condition:

a. is a linear combination of a group of harmonically-related sinusoids: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$.

b. $\omega_0 = 4\pi$. c. $c_0 = 0, c_1 = 0.5, c_2 = -0.5, c_4 = 1$, others $= 0$.

Task 1 (10%) Please convert this into the trigonometric Fourier series. Record the process below.

$x(t) = 0.5e^{j4\pi t} + 0.5e^{-j4\pi t} - 0.5e^{j8\pi t} - 0.5e^{-j8\pi t} + 1e^{j16\pi t} + 1e^{-j16\pi t}$

$e^{j\theta} + e^{-j\theta} = 2cos\theta$

$x(t) = 2(0.5\cos(4\pi t) - 0.5\cos(8\pi t) + 1\cos(16\pi t)) \rightarrow \cos(4\pi t) - \cos(8\pi t) + 2\cos(16\pi t)$

Task 2 (10%) Consider the impact of the following changes to the original signal $x(t)$:

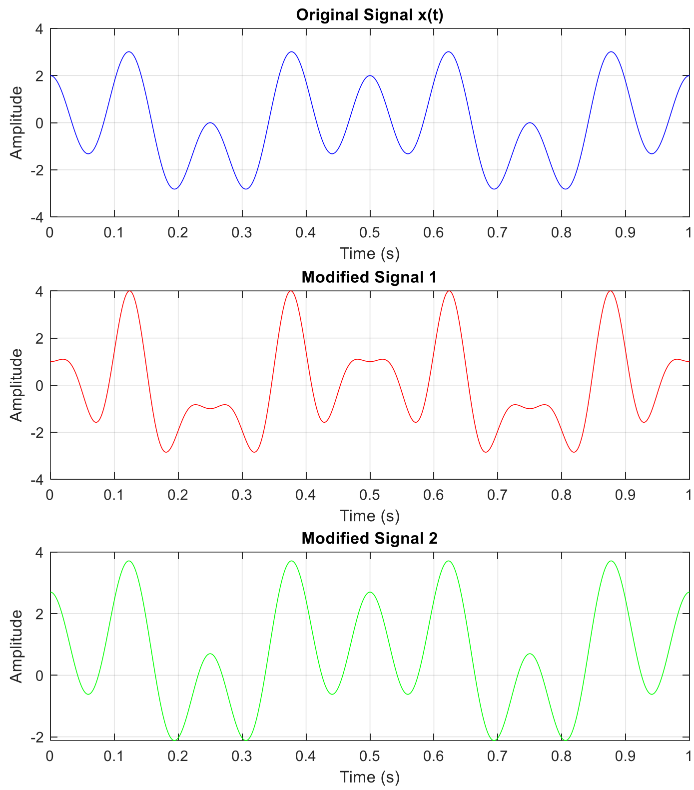
(a) If the function is changed to include one additional term $c_6 = -1$. All other conditions remain the same, what is the new fundamental frequency of this function?

It should still be 4π.

(b) If the function is changed to include one additional term $c_0 = 0.7$. What is the impact of these changes to $x(t)$?

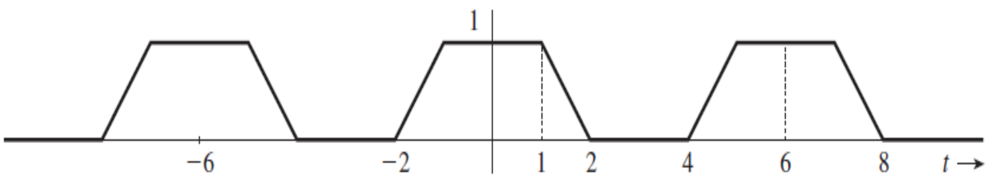
The entire frequency would go up by 0.7 but the rest should stay unchanged.

Task 3 (10%). Please plot the original signal $x(t)$ and the modified signals in Task 2 using MATLAB. Record your code and screenshots below.




```
t = linspace(0, 1, 1000);

omega = 4*pi;

x_og = cos(omega*t) - cos(2*omega*t) + 2*cos(4*omega*t);
x_m1 = x_og - cos(6*omega*t);
x_m2 = x_og +0.7;

figure;

subplot(3,1,1);
plot(t, x_og, 'b');
title('Original Signal x(t)');
xlabel('Time (s)'); ylabel('Amplitude'); grid on;

subplot(3,1,2);
plot(t,x_m1,'r');
title('Modified Signal 1');
xlabel('Time (s)'); ylabel('Amplitude'); grid on;

subplot(3,1,3);
plot(t,x_m2,'g');
title('Modified Signal 2');
xlabel('Time (s)'); ylabel('Amplitude'); grid on;
```

Task 4 (15%). Please compute the exponential and trigonometric Fourier series of the following signal, include the process and results below.



$$\begin{cases} t+2, & -2 \le t < -1 \\ 1, & -1 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & 2 \le t < 6 \end{cases}$$

Figure 1. Periodic signal $y(t)$

$$\text{Period } T = 6 \rightarrow \text{ Fundamental Hz} \rightarrow w_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \rightarrow c_k = \frac{1}{6}\int_{-2}^{6} x(t)e^{-jk\frac{\pi}{3}t}dt \rightarrow c_k = \frac{1}{6}\left[\int_{-2}^{-1}(t+2)\,e^{-jk\frac{\pi}{3}t}dt + \int_{-1}^{1} e^{-jk\frac{\pi}{3}t}dt + \int_{1}^{6}(2-t)\,e^{-jk\frac{\pi}{3}t}dt\right]$$

$$\left.\begin{array}{l} I_1 = (t+2)\dfrac{-e^{-jk\frac{\pi}{3}t}}{jk\frac{\pi}{3}}\Bigg|_{-2}^{-1} + \displaystyle\int_{-2}^{-1}\frac{e^{-jk\frac{\pi}{3}t}}{jk\frac{\pi}{3}}dt \rightarrow \left[\dfrac{-e^{jk\frac{\pi}{3}}}{jk\frac{\pi}{3}}\right] \rightarrow = \left[\dfrac{-e^{jk\frac{\pi}{3}}}{jk\frac{\pi}{3}}\right] + \dfrac{e^{jk\frac{\pi}{3}} - e^{2jk\frac{\pi}{3}}}{-jk\frac{\pi}{3}} \\ I_2 = \dfrac{e^{-jk\frac{\pi}{3}t}}{-jk\frac{\pi}{3}}\Bigg|_{-1}^{1} \dfrac{e^{-jk\frac{\pi}{3}} - e^{jk\frac{\pi}{3}}}{-jk\frac{\pi}{3}} \rightarrow \dfrac{-2j\sin\left(k\frac{\pi}{3}\right)}{-jk\frac{\pi}{3}} \rightarrow = \dfrac{2\sin\left(k\frac{\pi}{3}\right)}{k\frac{\pi}{3}} \\ I_3 = (2-t)\dfrac{-e^{-jk\frac{\pi}{3}t}}{jk\frac{\pi}{3}}\Bigg|_{1}^{6} + \displaystyle\int_{1}^{6}\frac{e^{-jk\frac{\pi}{3}t}}{jk\frac{\pi}{3}}dt \rightarrow = \left[(4)\dfrac{e^{-jk2\pi}}{jk\frac{\pi}{3}} + \dfrac{e^{-jk\frac{\pi}{3}}}{jk\frac{\pi}{3}}\right] + \dfrac{e^{-jk2\pi} + e^{-jk\frac{\pi}{3}}}{-jk\frac{\pi}{3}} \end{array}\right\} \boldsymbol{ck} = \frac{1}{6}\left(\frac{e^{2jk\frac{\pi}{3}} + 3e^{-jk2\pi}}{jk\frac{\pi}{3}} + \frac{2\sin\left(k\frac{\pi}{3}\right)}{k\frac{\pi}{3}}\right)$$

$$\boldsymbol{x(t) = \sum_{k=-\infty}^{\infty}\frac{1}{6}\left(\frac{e^{2jk\frac{\pi}{3}} + 3e^{-jk2\pi}}{jk\frac{\pi}{3}} + \frac{2\sin\left(k\frac{\pi}{3}\right)}{k\frac{\pi}{3}}\right)e^{jk\frac{\pi}{3}t}} \rightarrow x(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos k\omega_0 t + b_k \sin k\omega_0 t)$$

$$a_0(k=0) = \frac{1}{6}\left(\frac{e^0 + 3e^0}{0} + \frac{2\sin(0)}{0}\right) \rightarrow \frac{1}{6}(2) \rightarrow = \frac{1}{3}$$

$$a_k = \frac{2\sin\left(k\frac{\pi}{3}\right)}{k\pi} \rightarrow b_k = -\frac{\sin\left(2k\frac{\pi}{3}\right)}{k\pi}$$

$$\boldsymbol{x(t) = \frac{1}{3} + \sum_{k=1}^{\infty}\left(\frac{2\sin\left(k\frac{\pi}{3}\right)}{k\pi}\cos k\frac{\pi}{3}t - \frac{\sin\left(2k\frac{\pi}{3}\right)}{k\pi}\sin k\frac{\pi}{3}t\right)}$$

3.2 Square wave expression

A square wave is defined to defined as follows in the range $0 \le t < 1$:

$$x(t) = \begin{cases} 1, & 1/2 \le t < 1 \\ -1, & 0 \le t < 1/2 \end{cases}$$

In CTFS, this square wave can be expressed as a linear combination of a group of harmonically-related complex sinusoids. Using the example in the lecture, the corresponding Fourier series coefficients c_k can be expressed as:

$$c_k = \begin{cases} \dfrac{-j2}{(\pi k)}, & k \text{ is odd} \\ 0, & k \text{ is even} \end{cases}$$

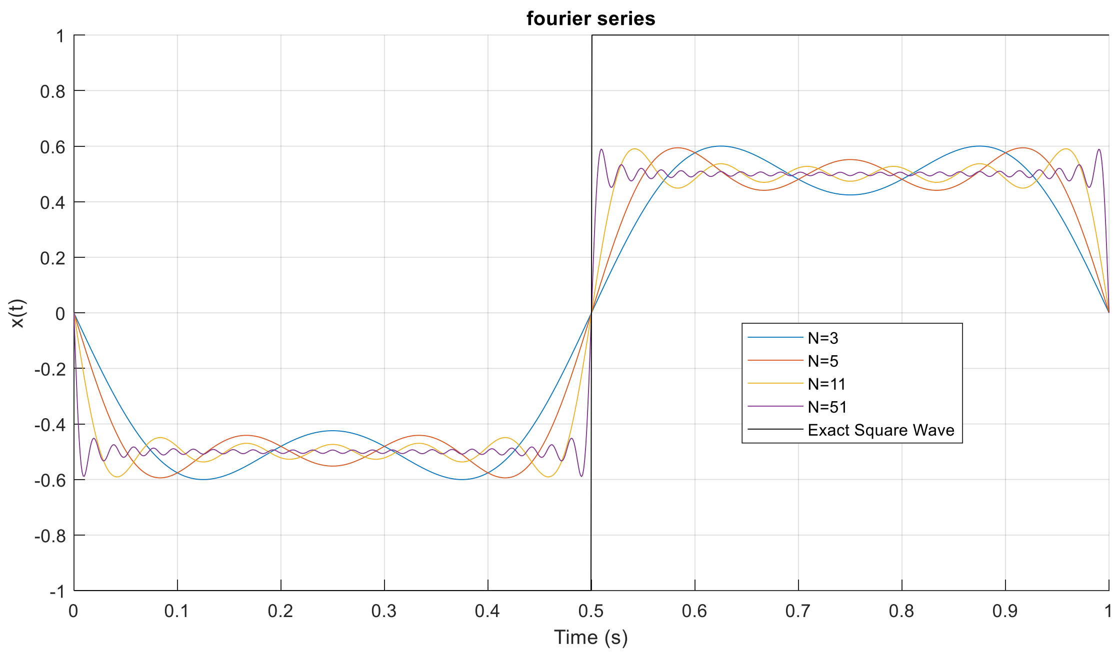
Task 5 (10%). Write the corresponding trigonometric Fourier series in the form of sines and/or cosines and record your calculation below.

$$c_k = \frac{a_k - jb_k}{2}, c_k = \frac{a_k + jb_k}{2} \rightarrow b_k = -\frac{2}{\pi k}, k\ is\ odd$$

$$x(t) = \sum_{k=odd}^{\infty} -\frac{2}{\pi k}\sin{(2\pi k t)}$$

Write a MATLAB program to approximate the square wave using the first N terms of the Fourier transform. Adjust the series length from short to long to see the impact of big or small N on the accuracy of the reconstructed square wave.

Task 6 (25%). Choose at least five different N values, and record the corresponding plots below. As the length of the truncated Fourier series increases, analyze the improvements in the approximation and identify any remaining issues. Record your code and analysis below.




```
t = linspace(0, 1, 1000);

Nv = [3, 5, 11, 51];

figure;
hold on;

for N = Nv
    aproxx = zeros(size(t));
    for k = 1:2:N
        aproxx = aproxx - (2 / (pi*k))*sin(2*pi*k*t);
    end
    plot(t,aproxx,'DisplayName', sprintf('N=%d',N));
end

truwave(t<0.5)=-1;
plot(t, truwave,'k','DisplayName','Exact Square Wave');
xlabel('Time (s)'); ylabel('x(t)');title('fourier series');legend;grid on;
hold off;
```

The smaller N is the less the approximation is, but the greater the N is the closer the approximation is.

3.3 Design a system that generates a sinusoidal wave from a square wave

From part 3.2, we know that a square wave contains many harmonically-related sinusoids. It is common that we want to extract a sinusoidal wave from a square wave. Based on what we have learned so far, design a system with a proper transfer function that will be able to complete the task.

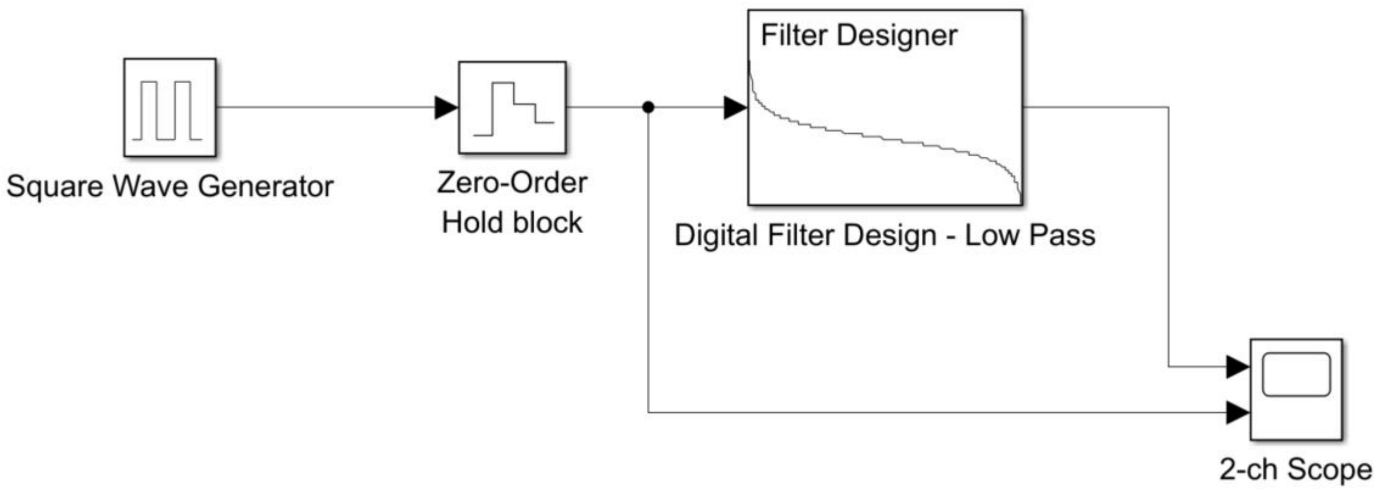
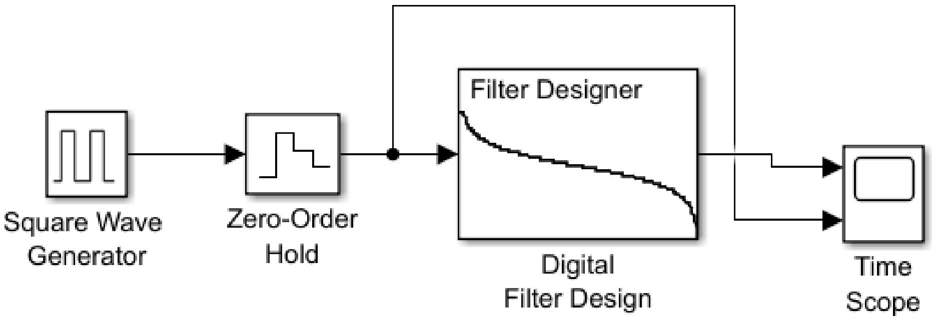
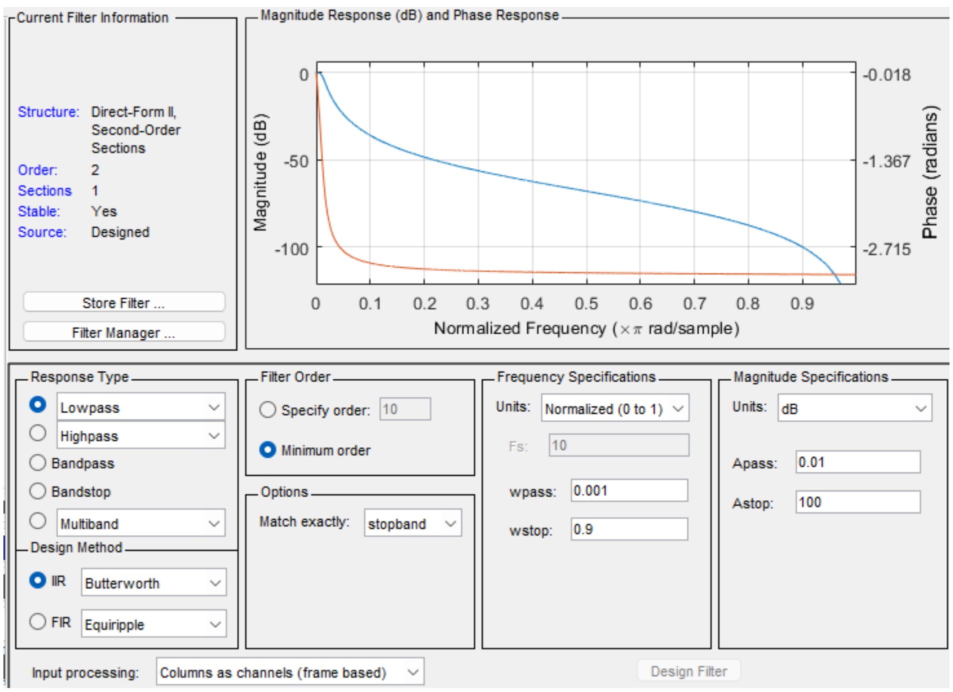


Figure 2. Block diagram of a Sinusoid Generator.

Task 7 (20%) Record your design and your filter parameters, and your scope screen in the space below. Explain the rationale for your design parameters and also explain the purpose of the zero-order hold block in this system.



Well the Zero-Order hold block holds the input for the sample period of the square wave generator, so the smaller you make the number the more "square" the wave would be.



Filter type: Lowpass allows frequencies below a certain frequency to cutoff
Design: butterworth provides a maximally flat frequency
Filter order: minimum order
Passband and stopband normalized 0.01 and 0.9
Magnitude specifications: Apass 0.01 db and Astop 100db effective noise reduction//removal