Chapter 6

Differentiability, Directional Derivatives

6.1 Differentiability

Shifrin Def 3.2.1: $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable

Definition 6.1. Let $U \subseteq \mathbb{R}^n$ be open, and let $\mathbf{a} \in U$. A function $\mathbf{f} : U \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - L(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

We call L the derivative of $\mathbf{f}$ at $\mathbf{a}$ and denote it by $D\mathbf{f}(\mathbf{a})$. If we let $\mathbf{x} = \mathbf{a} + \mathbf{h}$ and use the notation $D\mathbf{f}(\mathbf{a})$ instead of L, then, restating the above, $\mathbf{f}$ is differentiable at $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \Big(\mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})\Big)}{\|\mathbf{x}-\mathbf{a}\|} = \mathbf{0}.$$

The definition tells us that if $\mathbf{f}$ is differentiable at $\mathbf{a}$, then the "error" between $\mathbf{f}(\mathbf{x})$ and its (affine) linear approximation $\mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})$ goes to zero faster than $\|\mathbf{x}-\mathbf{a}\|$ goes to zero as $\mathbf{x} \to \mathbf{a}$. Colloquially speaking,

for $\mathbf{x}$ *close to* $\mathbf{a}$, the function $\mathbf{f}$ "looks flat,"

and the error between $\mathbf{f}(\mathbf{x})$ and its linear approximation $\mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})$ is "small" relative to the magnitude of the difference $\mathbf{x} - \mathbf{a}$.

This notion of differentiability, unlike the mere existence of partial (or directional) derivatives, allows us to say something about the continuity of the function at that point.

Shifrin Propn 3.2.2: Differentiability implies continuity

Proposition 6.2. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$.

The partial derivatives of the component functions of $\mathbf{f}$ give us the matrix (with respect to the standard basis) of the linear transformation that is the derivative.

Shifrin Propn 3.2.1: The derivative matrix is made up of partial derivatives

Proposition 6.3. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, where $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$ is differentiable at $\mathbf{a}$, then each of the partial derivatives, $\dfrac{\partial f_i}{\partial x_j}$ exist and

$$\big[D\mathbf{f}(\mathbf{a})\big] = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{a}) & \frac{\partial f_n}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

The matrix $\big[D\mathbf{f}(\mathbf{a})\big]$ is sometimes called the Jacobian matrix of $\mathbf{f}$ at $\mathbf{a}$.

The i^{th} row of the Jacobian matrix comes from taking the partial derivatives of the i^{th} component function f_i of $\mathbf{f}$ with respect to the n variables $x_1, x_2, \ldots, x_n$.

In the case that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$, then

$$[Df(\mathbf{a})] = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{a}) & \dfrac{\partial f}{\partial x_2}(\mathbf{a}) & \cdots & \dfrac{\partial f}{\partial x_n}(\mathbf{a}) \end{bmatrix} = \nabla f(\mathbf{a})^T.$$

The following result helps us bypass taking the limit and assessing the differentiability of a function using the original definition.

Shifrin Propn 3.2.4: $\mathbf{f} \in \mathcal{C}^1(U)$ implies $\mathbf{f}$ is differentiable

Proposition 6.4. If $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is of class $\mathcal{C}^1$ on U, that is, its first-order partial derivatives exist and are continuous, then $\mathbf{f}$ is differentiable.

6.2 Directional Derivatives and the Gradient

Shifrin Defn 3.1.2: Rate of change of $\mathbf{f}$ at $\mathbf{a}$ while moving with velocity $\mathbf{v}$

Definition 6.5. Let $U \subseteq \mathbb{R}^n$ be open and $\mathbf{a} \in U$. The rate of change of $\mathbf{f} : U \to \mathbb{R}^m$ at $\mathbf{a}$ while moving with velocity $\mathbf{v}$ is defined to be

$$D_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{v}) - \mathbf{f}(\mathbf{a})}{t},$$

provided this limit exists.

The rate of change of $\mathbf{f}$ at $\mathbf{a}$ while moving with velocity $\mathbf{v}$ depends on both the direction and magnitude (speed) of $\mathbf{v}$[1]. If we take a nonzero scalar multiple $c\mathbf{v}$ of $\mathbf{v}$, the resulting vector is in the same direction as $\mathbf{v}$ but has magnitude $|c|\|\mathbf{v}\|$. The resulting rate of change of $\mathbf{f}$ at $\mathbf{a}$ while moving with this new velocity $c\mathbf{v}$ is

$$\begin{aligned} D_{c\mathbf{v}}\mathbf{f}(\mathbf{a}) &= \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{a} + t(c\mathbf{v})) - \mathbf{f}(\mathbf{a})}{t} \\ &= \lim_{t \to 0} c \cdot \frac{\mathbf{f}(\mathbf{a} + ct\mathbf{v}) - \mathbf{f}(\mathbf{a})}{ct} \\ &= c \lim_{s \to 0} \frac{\mathbf{f}(\mathbf{a} + s\mathbf{v}) - \mathbf{f}(\mathbf{a})}{s} \\ &= cD_{\mathbf{v}}\mathbf{f}(\mathbf{a}), \end{aligned}$$

which is c times the rate of change of $\mathbf{f}$ at $\mathbf{a}$ while moving with a velocity of $\mathbf{v}$.

Consider the case where $f : \mathbb{R}^n \to \mathbb{R}$, that is, if f is a scalar-valued map. Here we may consider the composite map $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) = f(\mathbf{a} + t\mathbf{v}).$$

The function g composes the action of sending $t \in \mathbb{R}$ to a point on the line through $\mathbf{a}$ running in the direction of $\mathbf{v}$ (that is, $t \mapsto \mathbf{a} + t\mathbf{v}$) with f. In this scenario and with this perspective, if the rate of change of f at $\mathbf{a}$ while moving with velocity $\mathbf{v}$

[1]Given a nonzero vector $\mathbf{v}$, we may write it as $\mathbf{v} = \|\mathbf{v}\| \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, that is, as a product of a scalar (its magnitude or speed $\|\mathbf{v}\|$) and a unit vector (its "direction" $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$). Although we do not do so, some people use the word direction synonymously with unit vector.

exists, then

$$\begin{aligned} D_{\mathbf{v}} f(\mathbf{a}) &= \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t} \\ &= \lim_{t \to 0} \frac{g(t) - g(0)}{t} \\ &= \frac{dg}{dt}(0). \end{aligned}$$

That is, the rate of change of f at $\mathbf{a}$ while moving with velocity $\mathbf{v}$ is the derivative of g at 0.

If $\mathbf{f}$ is differentiable at the point $\mathbf{a}$, then the computation of the rate of change in the direction of any vector becomes easier.

Shifrin Propn 3.2.3, Rate of change when $\mathbf{f}$ is differentiable

Proposition 6.6. If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then for any $\mathbf{v} \in \mathbb{R}^n$,

$$D_{\mathbf{v}} \mathbf{f}(\mathbf{a}) = D\mathbf{f}(\mathbf{a})\mathbf{v}.$$

In other words, the rate of change of $\mathbf{f}$ at $\mathbf{a}$ in the direction of $\mathbf{v}$ equals the (linear map that is the) derivative of $\mathbf{f}$ at $\mathbf{a}$ acting on $\mathbf{v}$.

Rate of change when f is scalar-valued

Corollary 6.7. If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, that is, f is scalar-valued, and f is differentiable at $\mathbf{a} \in U$, then

$$D_{\mathbf{v}} \mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{a}) & \frac{\partial f}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f}{\partial x_n}(\mathbf{a}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \nabla f(\mathbf{a}) \cdot \mathbf{v},$$

where $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$.

Example 6.8. [FRY III.2.7 Exercise 11a] The air temperature $T(x, y, z)$ at a location (x, y, z) is given by

$$T(x, y, z) = 1 + x^2 + yz.$$

A bird passes through $(2, 1, 3)$ travelling towards $(4, 3, 4)$ with speed 2. At what rate does the air temperature it experiences change at this instant?

Given: $T(x,y,z) = 1 + x^2 + yz$

point $a = (2,1,3)$

moving towards point $(4,3,4)$ with speed 2.

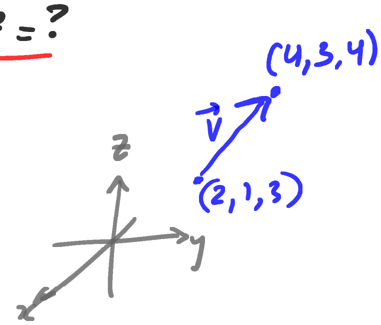
$P(2,1,3)$

$\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$\vec{a} = \langle 2,1,3 \rangle$

Goal: rate of change of temperature = ?

$D_{\vec{v}} T(a)$



$\vec{v} = \langle 4-2, 3-1, 4-3 \rangle$
$= \langle 2, 2, 1 \rangle$

Theorem: If $f(x,y,z)$ is differentiable, then

$D_{\vec{v}} f(a) = \nabla f(a) \cdot \vec{v}$

dot product

$T(x,y,z)$, being a polynomial, is differentiable. (Why?)

$$\boxed{D_{\vec{v}} T(a) = \nabla T(a) \cdot \vec{v}}$$

point $a = (2,1,3)$

$T(x,y,z) = 1 + x^2 + yz$

$$\underbrace{\nabla T(x,y,z)}_{\text{gradient of } T \text{ at } (x,y,z)} = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle = \langle 2x, z, y \rangle$$

$2x = 2(2)$ $\quad z = 3$ $\quad y = 1$

$$\nabla T(2,1,3) = \langle 4, 3, 1 \rangle$$

$$D_{\vec{v}} T(a) = \nabla T(a) \cdot \vec{v}$$
$$= \langle 4,3,1 \rangle \cdot \langle 2,2,1 \rangle$$
$$= (4)(2) + 3(2) + (1)(1)$$
$$= 15 \longleftarrow \text{WRONG}$$

For $\vec{v} = \vec{0}$.

$$\vec{v} = \underbrace{\|\vec{v}\|}_{\text{magnitude}} \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{direction}}$$

For us, we want to move in the direction $\frac{\langle 2,2,1 \rangle}{\|\langle 2,2,1\rangle\|}$

with speed 2

The "corrected" $\vec{v}$ is

$$\vec{v} = 2 \frac{\langle 2,2,1 \rangle}{3}$$

$3 \longleftarrow \|\langle 2,2,1 \rangle\| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$

$$= \frac{2}{3}\langle 2,2,1 \rangle$$
$$= \left\langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \right\rangle.$$

Correct rate of change:

$D_{\vec{v}} T(a) = \nabla T(a) \cdot \vec{v}$, since T is differentiable

rate of change of T at point a if moving in the direction $\frac{\vec{v}}{\|\vec{v}\|}$ with speed $\|\vec{v}\|$

$$= \langle 4,3,1 \rangle \cdot \left\langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \right\rangle$$
$$= \frac{16}{3} + \frac{12}{3} + \frac{2}{3}$$
$$= 10$$

$\frac{\partial T}{\partial x}(a)$ $\frac{\partial T}{\partial y}(a)$ $\frac{\partial T}{\partial z}(a)$

$\frac{{}^\circ C}{m}$

$\left\langle \frac{m}{s}, \frac{m}{s}, \frac{m}{s} \right\rangle$

$$\left\langle 4\frac{{}^\circ C}{m}, 3\frac{{}^\circ C}{m}, 1\frac{{}^\circ C}{m} \right\rangle \cdot \left\langle \frac{4}{3}\frac{m}{s}, \frac{4}{3}\frac{m}{s}, \frac{2}{3}\frac{m}{s} \right\rangle$$
$$= \frac{16}{3}\frac{{}^\circ C}{s} + \frac{12}{3}\frac{{}^\circ C}{s} + \frac{2}{3}\frac{{}^\circ C}{s}$$
$$= 10\frac{{}^\circ C}{s}$$

To determine the value of the rate of change of a function at a point solely based on the direction (and not the magnitude) of a vector, we introduce the notion of a directional derivative, which is the rate of change in the direction of a unit vector.

FRY Defn 2.7.5, The directional derivative

Definition 6.9. Let $U \subseteq \mathbb{R}^n$ be open and $\mathbf{a} \in U$. The directional derivative of $\mathbf{f} : U \to \mathbb{R}^m$ at $\mathbf{a}$ in the direction of a nonzero vector $\mathbf{v}$ is defined to be the rate of change of $\mathbf{f}$ at $\mathbf{a}$ in the direction of the corresponding unit vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$. That is,

$$\text{the directional derivative of } \mathbf{f} \text{ at } \mathbf{a} \text{ in the direction of } \mathbf{v} \ = D_{\mathbf{v}/\|\mathbf{v}\|}\mathbf{f}(\mathbf{a}).$$

If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, that is, f is scalar-valued, then the directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{v}$ equals

$$D_{\mathbf{v}/\|\mathbf{v}\|}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

We can relate the definition of the directional derivative to our earlier definition of the rate of change of a function $\mathbf{f}$ at a point $\mathbf{a}$ while moving with (nonzero) velocity $\mathbf{v}$. In the directional derivative, we are finding the rate of change of $\mathbf{f}$ at $\mathbf{a}$ while moving with a *unit* velocity, that is, the rate of change of the function at the given point while moving one unit of distance per unit time (in the direction of the velocity vector).

Keep the following notes in mind when working on problems involving directional derivatives.

(i) If the problem asks to find the "rate of change" of a function $\mathbf{f}$ at a point $\mathbf{a}$ in a direction $\mathbf{v}$ (where $\mathbf{v} \neq \mathbf{0}$) *without* specifying whether the magnitude (speed) of $\mathbf{v}$ is relevant, then work out the directional derivative $D_{\mathbf{v}/\|\mathbf{v}\|}\mathbf{f}(\mathbf{a})$.[2]

(ii) If the problem gives us or asks for the "slope" in the direction of a nonzero vector $\mathbf{v}$, then the value of the slope is the directional derivative of the involved function $\mathbf{f}$ at the given point $\mathbf{a}$ in the direction of $\mathbf{v}$. That is, the slope equals $D_{\mathbf{v}/\|\mathbf{v}\|}\mathbf{f}(\mathbf{a})$.

(iii) If the problem asks for "the direction" to move in for a function $\mathbf{f}$ starting from

[2] This can be frustrating at times because the phrase "rate of change" could also be interpreted as the rate of change of $\mathbf{f}$ at $\mathbf{a}$ while moving with a velocity $\mathbf{v}$.

the point **a**, it is asking for a unit vector **u**. Given a nonzero vector **v**, every positive scalar multiple of **v** points in the same direction as **v**. So we provide the single vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ as "the direction."

Example 6.10. The air temperature $T(x, y, z)$ at a location (x, y, z) is given by

$$T(x, y, z) = 1 + x^2 + yz.$$

A bird passes through $(2, 1, 3)$ travelling towards $(4, 3, 4)$, that is in the direction of $\langle 2, 2, 1\rangle$. What is the rate of change in air temperature that it experiences at this instant?

Given: $T(x,y,z) = 1 + x^2 + yz$

point $a = (2,1,3)$

moving in the direction of $\vec{v} = \langle 2,2,1\rangle$

Goal: rate of change in temperature at point a

= directional derivative

$= D_{\frac{\vec{v}}{\|\vec{v}\|}} T(2,1,3)$

$= \nabla T(2,1,3) \cdot \dfrac{\vec{v}}{\|\vec{v}\|}$

$= \langle 4,3,1\rangle \cdot \dfrac{\langle 2,2,1\rangle}{\|\langle 2,2,1\rangle\|}$

$= \langle 4,3,1\rangle \cdot \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$, because $\|\langle 2,2,1\rangle\| = \sqrt{2^2+2^2+1^2} = 3$

$= \frac{8}{3} + \frac{6}{3} + \frac{1}{3}$

$= 5$

$\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}$

$\underbrace{\nabla T(x,y,z)}_{\text{gradient of } T \text{ at } (x,y,z)} = \langle 2x, z, y\rangle$

Live Poll

Given: $f(x,y) = 3 - \frac{1}{4}x^2 - \frac{1}{9}y^2$

point $a=(2,3)$

$\vec{v} = \langle 1,1 \rangle$

Goal Find directional derivative of the function $f(x,y)$ at the point $a=(2,3)$ while moving in the direction of $\vec{v}=\langle 1,1\rangle$.

Solution: directional derivative $= D_{\frac{\vec{v}}{\|\vec{v}\|}} f(2,3)$

$= \nabla f(2,3) \cdot \frac{\vec{v}}{\|\vec{v}\|}$, since f, being a polynomial, is differentiable

$= \langle -1, \frac{-2}{3} \rangle \cdot \frac{\langle 1,1 \rangle}{\sqrt{2}}$ ← $\sqrt{1^2+1^2}$

$\nabla f(x,y) = \langle -\frac{1}{2}x, \frac{-2}{9}y \rangle$

$\nabla f(2,3) = \langle -\frac{1}{2}(2), -\frac{2}{9}(3) \rangle$

$= \langle -1, -\frac{2}{3} \rangle$

$= \langle -1, \frac{-2}{3} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= (-1)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-2}{3}\right)\left(\frac{1}{\sqrt{2}}\right)$

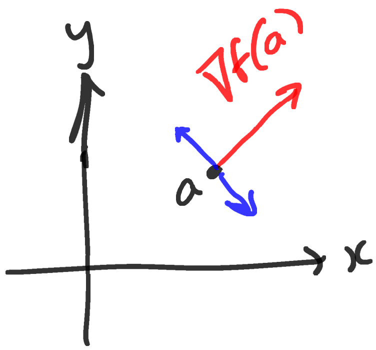
$= \frac{-5}{3\sqrt{2}} = \frac{-5\sqrt{2}}{6}$

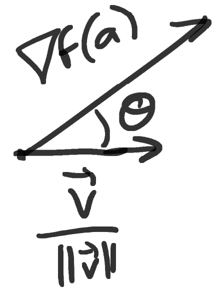
directional derivative

$$D_{\frac{\vec{v}}{\|\vec{v}\|}} f(a) = \nabla f(a) \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \|\nabla f(a)\| \overbrace{\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\|}^{=1} \cos\theta$$, where θ is the angle between $\nabla f(a)$ and $\frac{\vec{v}}{\|\vec{v}\|}$

$$= \|\nabla f(a)\| \cos\theta$$





Recall that we may express the dot product of two vectors $\mathbf{x}$ and $\mathbf{y}$ using the angle θ between them:

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta).$$

Since $\cos(\theta)$ is a number between -1 and 1, we get both a lower and an upper bound on the dot product:

$$-\|\mathbf{x}\| \|\mathbf{y}\| \leq \mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\| \|\mathbf{y}\|,$$

with the maximum occurring when the angle between $\mathbf{x}$ and $\mathbf{y}$ is zero (that is, $\mathbf{x}$ and $\mathbf{y}$ point in the same direction) and the minimum occurring when the angle between the two vectors is π (that is, $\mathbf{x}$ and $\mathbf{y}$ point in opposite directions).

Since, for a nonzero vector $\mathbf{v}$, the magnitude of the corresponding unit vector, $\left\| \dfrac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$, we get the following result.

Shifrin Propn 3.4.1, Directions of max increase/decrease

Theorem 6.11. Let $U \subseteq \mathbb{R}^n$ be open and $\mathbf{a} \in U$. Suppose $f : U \to \mathbb{R}$ is differentiable at $\mathbf{a}$. Then,

- The gradient $\nabla f(\mathbf{a})$ points in the direction in which the function f increases at its greatest rate.
- The greatest possible (maximal) rate of change of the function f at the point $\mathbf{a}$ is $\|\nabla f(\mathbf{a})\|$.
- The negative of the gradient, that is, $-\nabla f(\mathbf{a})$, points in the direction in which the function f decreases at its greatest rate.
- The most negative rate of change (the minimum rate of increase or the maximal rate of decrease) of the function f at the point $\mathbf{a}$ is $-\|\nabla f(\mathbf{a})\|$.
- If we move in a direction perpendicular to the gradient $\nabla f(\mathbf{a})$, then the function f remains constant. The curves along which the function's values (heights) remain constant are called level curves or contour curves. So, the gradient $\nabla f(\mathbf{a})$ is perpendicular to the level curve $f(\mathbf{x}) = f(\mathbf{a})$ at $\mathbf{a}$.

Example 6.12. (FRY Exercise III.2.7.2.8)
A hiker is walking on a mountain with height above the $z = 0$ plane given by $z = f(x, y) = 6 - xy^2$. The positive x-axis points east, and the positive y-axis points north. The hiker starts from the point $P(2, 1, 4)$.

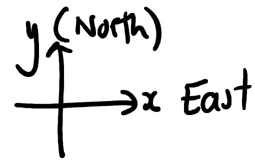
(a) In what direction should the hiker proceed from P to ascend along the steepest path? What is the slope along this path?

(b) In what direction should the hiker proceed from P to experience the fastest descent? What is the maximal rate of descent?

(c) Walking north from P, will the hiker start to ascend or descend? What is the slope (rate of ascent or descent)?

(d) In what direction should the hiker walk from P to remain at the same height?

$z = f(x,y) = 6 - xy^2$



point $P(2,1,4)$

a) To ascend along steepest path, the hiker should move in the direction $\dfrac{\nabla f(2,1)}{\|\nabla f(2,1)\|}$.

The slope of the path is $D_{\frac{\nabla f(2,1)}{\|\nabla f(2,1)\|}} f(2,1) = \|\nabla f(2,1)\|$

value of the directional derivative

$\nabla f(x,y) = \langle -y^2, -2xy \rangle$

$\nabla f(2,1) = \langle -1, -4 \rangle$

$\|\langle -1,-4 \rangle\| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{1+16} = \sqrt{17}$

The hiker should move in the direction $\dfrac{\langle -1,-4 \rangle}{\|\langle -1,-4 \rangle\|} = \left\langle \dfrac{-1}{\sqrt{17}}, \dfrac{-4}{\sqrt{17}} \right\rangle$.

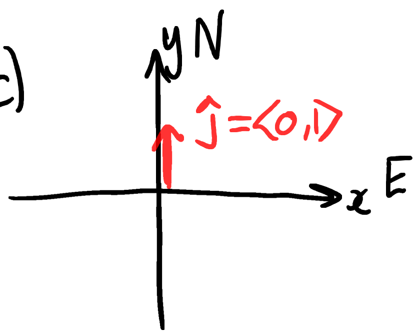
The slope along this path is $\|\nabla f(2,1)\| = \|\langle -1,-4 \rangle\| = \sqrt{17}$.

(b) To experience the fastest rate of descent, the hiker should move in the direction $-\dfrac{\nabla f(2,1)}{\|\nabla f(2,1)\|}$

$$= -\frac{\langle -1,-4\rangle}{\sqrt{17}} = \left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle$$

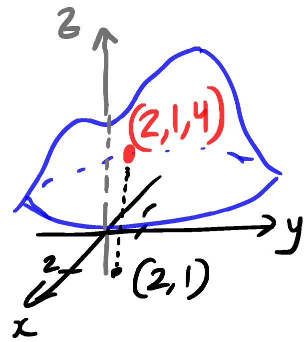
The maximal rate of decrease $= -\|\nabla f(2,1)\| = -\sqrt{17}$

(c)



$P(2,1,4)$

$z = f(x,y) = 6 - xy^2$



$$D_{\frac{\langle 0,1\rangle}{\|\langle 0,1\rangle\|}} f(2,1) = \nabla f(2,1) \cdot \frac{\langle 0,1\rangle}{\underbrace{\|\langle 0,1\rangle\|}_{=1}} = \langle -1,-4\rangle \cdot \langle 0,1\rangle = \underbrace{-4}_{\text{negative}}$$

If the hiker moves north, they'll descend. The slope (rate of descent) is -4.

(d) To stay at the same elevation, the hiker should move perpendicularly to the gradient $\nabla f(2,1) = \langle -1, -4 \rangle$.

The hiker could move in one of the following two directions:

$$\frac{\langle 4, -1 \rangle}{\|\langle 4, -1 \rangle\|} \quad \text{or} \quad \frac{\langle -4, 1 \rangle}{\|\langle -4, 1 \rangle\|}$$

$$= \left\langle \frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}} \right\rangle \quad \text{or} \quad \left\langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle$$

Check: $D_{\langle \frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}} \rangle} f(2,1) = \cdots = 0$

Live Poll Question:

$T(x,y,z) = 1 + x^2 + yz$

point $(2,1,3)$

a) direction of greatest increase in T at $(2,1,3)$

$$= \underbrace{\text{direction of}}_{\text{take unit vector}} \ \underbrace{\text{the gradient of } T \text{ at } (2,1,3)}_{\nabla T(2,1,3)}$$

take unit vector

$$\frac{\nabla T(2,1,3)}{\|\nabla T(2,1,3)\|}$$

$$\nabla T(x,y,z) = \langle 2x, z, y\rangle$$

$$\nabla T(2,1,3) = \langle 4,3,1\rangle$$

$$\|\nabla T(2,1,3)\| = \sqrt{4^2+3^2+1^2} = \sqrt{26}$$

$$= \frac{\langle 4,3,1\rangle}{\sqrt{26}}$$

$$= \left\langle \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}} \right\rangle$$

$$= \left\langle \frac{2\sqrt{26}}{13}, \frac{3\sqrt{26}}{26}, \frac{\sqrt{26}}{26} \right\rangle$$

b) maximal rate of increase in T at $(2,1,3)$

$$= \|\nabla T(2,1,3)\|$$

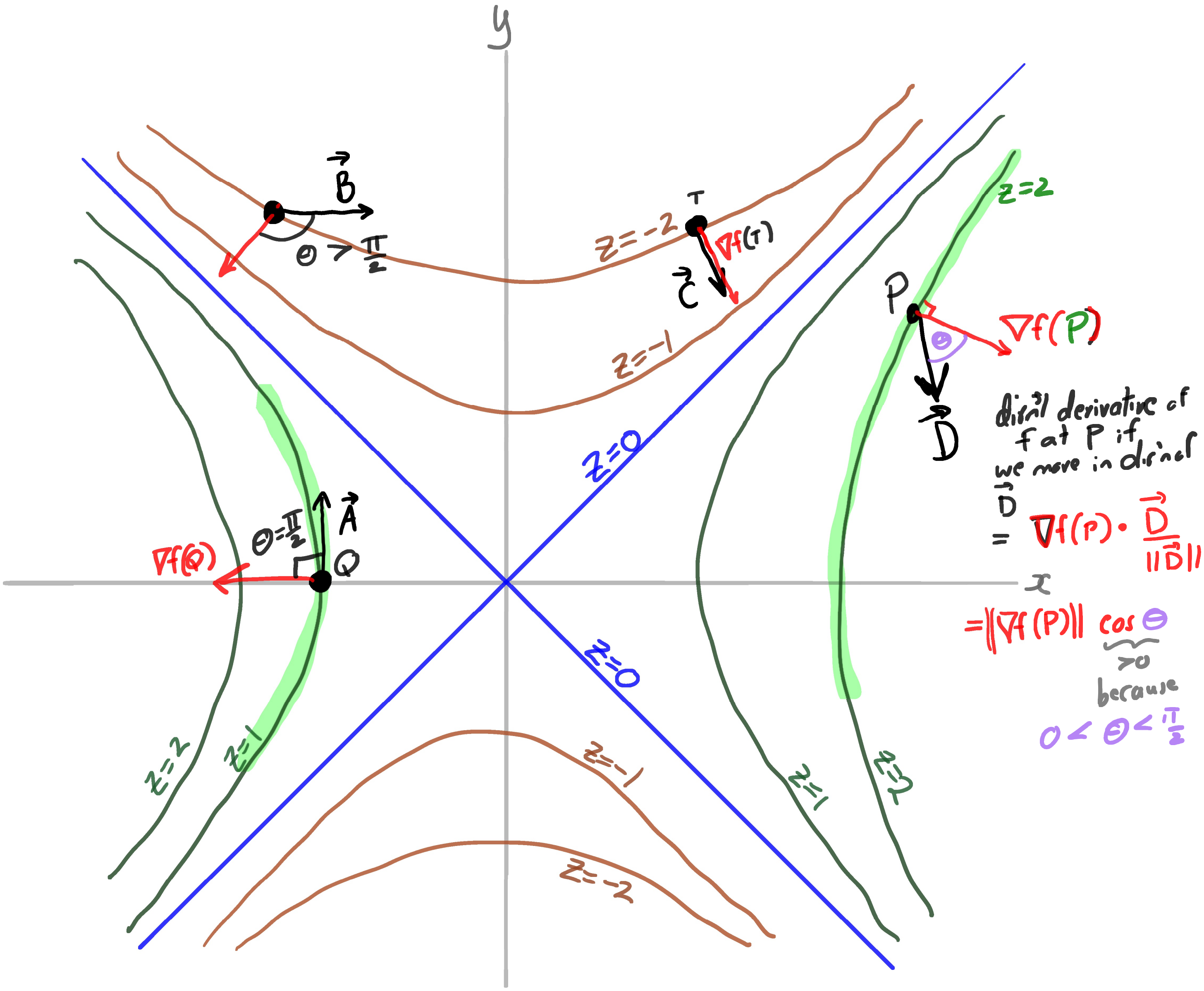
$$= \sqrt{26}$$

c) direction of greatest decrease in T at $(2,1,3)$

$$= -\frac{\nabla T(2,1,3)}{\|\nabla T(2,1,3)\|} = \left\langle \frac{-2\sqrt{26}}{13}, \frac{-3\sqrt{26}}{26}, \frac{-\sqrt{26}}{26} \right\rangle$$

d) maximal rate of decrease $= -\|\nabla T(2,1,3)\rangle = -\sqrt{26}$

Credit: Daniel An [SUNY Maritime]



For each of the vectors $\vec{A}$ through $\vec{D}$, state whether the directional derivative is positive, negative, or zero for the function at the indicated points on the surface z = f(x,y).

directional derivative in the direction of	will be
$\vec{A}$	0
$\vec{B}$	<0 (negative)
$\vec{C}$	>0 (positive)
$\vec{D}$	>0 (positive)

6.3 References

References:

1. Feldman J., Rechnitzer A., Yeager E., *CLP-3 Multivariable Calculus*, University of British Columbia, 2022.

2. Hubbard J.H., Hubbard B.B., *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach, 5th Edition*, Matrix Editions, 2009.

3. Shifrin T., *Multivariable Mathematics: Linear Algebra, Multivariable Calculus, and Manifolds*, John Wiley & Sons, 2005.

4. Stewart J., *Multivariable Calculus, Eighth Edition*, Cengage, 2016.