10.2 Triple Integrals in Spherical Coordinates

FRY Defn III.3.7.1, Spherical coordinates

Definition 10.5. The spherical coordinates of a point in three-dimensional space are denoted by ρ, θ, and ϕ, where

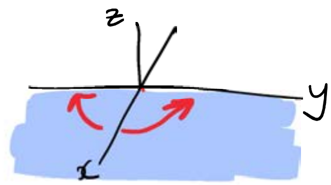
(i) ρ represent the distance from the origin $(0,0,0)$ to the point,

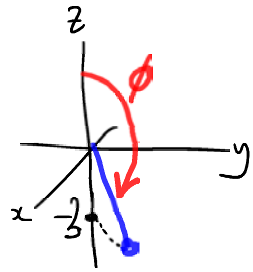
(ii) θ is the angle between the positive x-axis and the line segment from the origin to the projection of the point onto the xy-plane, and

(iii) ϕ is the angle between the z-axis and the line segment from the origin to the point.

The equations

- $\rho = \rho_0$, where ρ_0 is a constant, describes a sphere;
- $\theta = \theta_0$, where θ_0 is a constant, describes a plane; and
- $\phi = \phi_0$, where ϕ_0 is a constant, describes a cone.





Given (ρ, θ, ϕ),

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

Given (x, y, z),

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$

Notes:

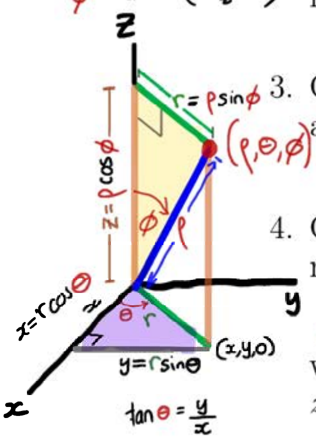
1. If the x- and y-coordinates are such that $(x, y, 0)$ lies in the second or third quadrant of the xy-plane, then we add π to $\arctan\left(\frac{y}{x}\right)$ to get the correct value for θ .

$r = \sqrt{x^2+y^2}$

$\tan\phi = \dfrac{\sqrt{x^2+y^2}}{z}$

$\phi = \arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)$

2. Similarly, when $z < 0$, in order to get ϕ to lie in the interval $[0, \pi]$, we add π to the above formula for ϕ. (Alternatively, if we don't want to worry about making such adjustments to get the correct value of ϕ, we could simply use the formula $\phi = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$).



3. Given the spherical coordinates (ρ, θ, ϕ), the cylindrical coordinates of the point are

$$\overset{r = \rho\sin\phi}{\cancel{r = \rho\sin\theta}}, \quad \theta = \theta, \quad \overset{z = \rho\cos\phi}{\cancel{z = \rho\cos\theta}}.$$

4. Given the cylindrical coordinates (r, θ, z), the corresponding spherical coordinates are

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \phi = \arctan\left(\frac{r}{z}\right),$$

$r = \sqrt{x^2+y^2}$

with the adjustment referred to above made to the arctan computation when $z < 0$.

If $\mathbf{g}$ denotes the change of variable transformation from (x, y, z)-coordinates into (ρ, θ, ϕ)-coordinates, then

$x = r\cos\theta$
$= \rho\sin\phi\cos\theta$

$y = r\sin\theta$
$= \rho\sin\phi\sin\theta$

$$D\mathbf{g}(\rho, \phi, \theta) = \begin{bmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}.$$

(x_1, y_1, z_1)
(x_2, y_2, z_2) distance $= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$(0,0,0)$
(x,y,z) distance $\rho = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$
$= \sqrt{x^2+y^2+z^2}$

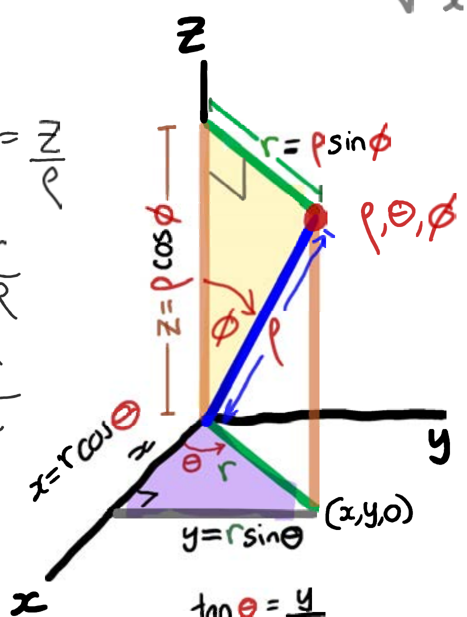
ρ rho
θ theta
φ, ϕ phi

$\cos\phi = \frac{z}{\rho}$

$\sin\phi = \frac{r}{\rho}$

$\tan\phi = \frac{r}{z}$

$\phi = \tan^{-1}\left(\frac{r}{z}\right)$



$\tan\theta = \frac{y}{x}$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

know x, y, z

$\rho = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}(y/x)$

$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

know ρ, θ, ϕ

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$x = r\cos\theta$
$= \rho\sin\phi\cos\theta$

$y = r\sin\theta$
$= \rho\sin\phi\sin\theta$

The determinant of the derivative matrix of the change of variables transformation $\mathbf{g}$, through cofactor expansion along the third row, is

$$\begin{aligned}
&\det D\mathbf{g}(\rho,\theta,\phi) \\
&= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta +\right) \\
&\qquad -\rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right) \\
&= \cos\phi\left(-\rho^2\sin\phi\cos\phi\right) - \rho\sin\phi\left(\rho\sin^2\phi\right) \\
&= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi \\
&= -\rho^2\sin\phi.
\end{aligned}$$

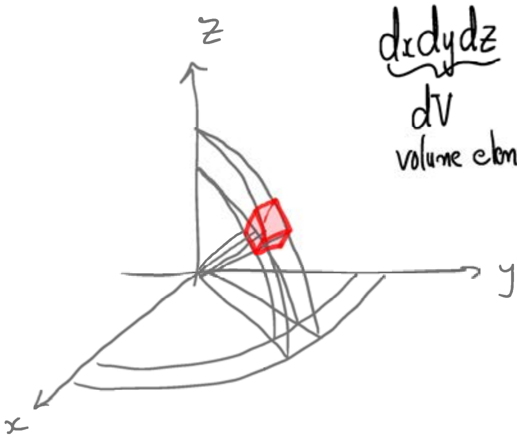
Thus,

$$|\det D\mathbf{g}(\rho,\theta,\phi)| = \overset{|-\rho^2\sin\phi|}{\cancel{|\rho^2\sin\phi|}} = \rho^2\sin\phi,$$

where we have dropped the absolute sign because both ρ^2 and $\sin\phi$ are nonnegative, the latter since $\phi\in[0,\pi]$. We use this information to adjust the volume element dV when changing from Cartesian to spherical coordinates:

$$\iiint_{\mathcal{R}} f(x,y,z)\ dV = \iiint_{\mathcal{R}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \underbrace{\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}_{dV},$$

though we may use a different order of integration depending on the domain of integration $\mathcal{R}$.



$\underbrace{dx\,dy\,dz}_{dV}$

volume element

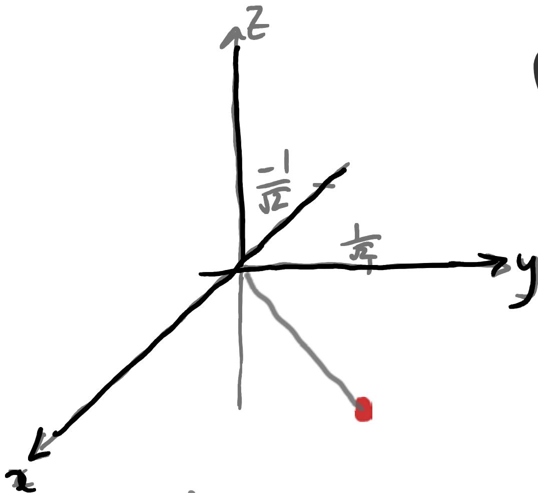
$(x,y,z) \xmapsto{\vec{g}} (\rho,\theta,\phi)$

$|\det D\vec{g}| = \rho^2\sin\phi$

$\rho = \sqrt{x^2+y^2+z^2}$ $\qquad \theta = \arctan\left(\frac{y}{x}\right)$ $\qquad \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

Example 10.6. Convert from the Cartesian coordinates $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{3}\right)$ to spherical coordinates.

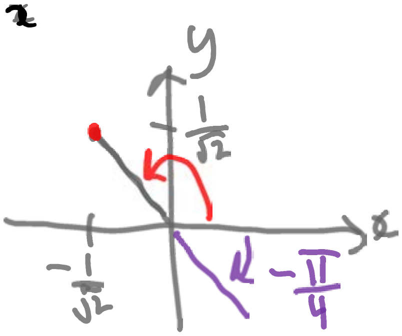
$x \quad y \quad z$



$\rho = \sqrt{\frac{1}{2}+\frac{1}{2}+3} = 2$

$\theta = \arctan\left(\frac{\frac{1}{\sqrt{2}}}{-1/\sqrt{2}}\right) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$\phi = \arctan\left(\frac{\sqrt{\frac{1}{2}+\frac{1}{2}}}{-\sqrt{3}}\right) = \frac{-\pi}{6} + \pi = \frac{5\pi}{6}$



Example 10.7. Rewrite the equation $3z^2 = x^2 + y^2$ in spherical coordinates.

know ρ, θ, ϕ (rho, theta, phi)

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$3(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$$

$$3\rho^2\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

$$3\rho^2\cos^2\phi = \rho^2\sin^2\phi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})$$

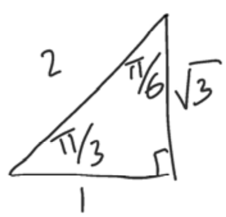
$$3\rho^2\cos^2\phi = \rho^2\sin^2\phi$$

$\boxed{\rho = 0 \text{ origin } (0,0,0)}$ or $\dfrac{3\cos^2\phi}{\cos^2\phi} = \dfrac{\sin^2\phi}{\cos^2\phi}$

$\tan\phi = \dfrac{\sin\phi}{\cos\phi}$

$$3 = \tan^2\phi$$

$\tan\phi = \sqrt{3}$ or $\tan\phi = -\sqrt{3}$



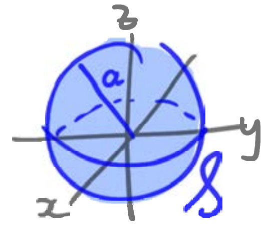
$\boxed{\phi = \frac{\pi}{3}}$ Cone

or $\phi = -\frac{\pi}{3} + \pi$

$\boxed{\phi = \frac{2\pi}{3}}$ Cone

Example 10.8. Show that the volume of the region $\mathcal{S}$ enclosed within a sphere of radius a centred at the origin is $\dfrac{4}{3}\pi a^3$.

Take a sphere S of $\boxed{\text{radius } a}$.



(ρ, θ, ϕ)

$0 \le \rho \le a$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

Volume of enclosed region $= \iiint_S dV = \frac{4}{3}\pi a^3$

$$= \int_0^a \int_0^{2\pi} \int_0^{\pi} 1 \; \underbrace{\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho}_{dV}$$

$$= \left(\int_0^a \rho^2 \, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right)$$

$$= \left[\frac{1}{3}\rho^3\right]_0^a (2\pi) \left[-\cos\phi\right]_0^{\pi}$$

$\cos\pi = -1$

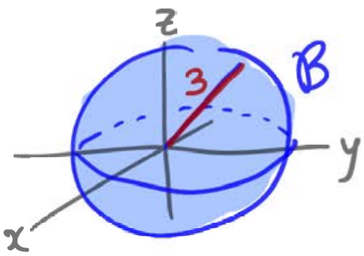
$-(-\cos 0)$
$= -(-1)$

$$= \left(\frac{1}{3}a^3\right)(2\pi)\underbrace{(1+1)}_{2}$$

$$= \frac{4\pi a^3}{3}$$

Example 10.9. Find the mass of a solid $\mathcal{B}$ enclosed within a sphere of radius 3 centred at the origin whose density is given by

$$\delta(x,y,z) = \frac{2x^2+2y^2+2z^2}{1+(x^2+y^2+z^2)^{5/2}}.$$



$$\text{Mass} = \iiint_{\mathcal{B}} \delta \, dV$$

$$\rho(x,y,z) = \frac{2x^2+2y^2+2z^2}{1+(x^2+y^2+z^2)^{5/2}} = \frac{2(x^2+y^2+z^2)}{1+(x^2+y^2+z^2)^{5/2}} = \frac{2\rho^2}{1+(\rho^2)^{5/2}} = \frac{2\rho^2}{1+\rho^5}$$

because ρ = distance from origin $= \sqrt{x^2+y^2+z^2}$

So $\rho^2 = x^2+y^2+z^2$

$0 \le \rho \le 3$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$$= \int_0^3 \int_0^{2\pi} \int_0^{\pi} \frac{2\rho^2}{1+\rho^5} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

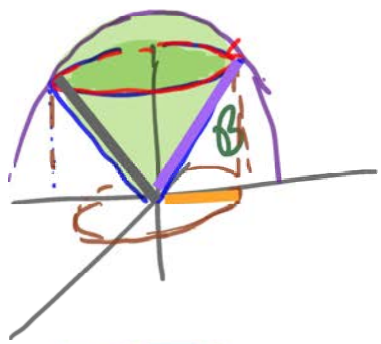
$$= \left(2\int_0^3 \frac{\rho^4}{1+\rho^5} d\rho \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\phi \, d\phi \right)$$

$$= \left[2 \cdot \frac{1}{5} \ln|1+\rho^5| \right]_0^3 (2\pi) \left[-\cos\phi \right]_0^{\pi}$$

$\vdots$

$$= \frac{8}{5}\pi \ln(244)$$

Example 10.10. (FRY Exercise III.3.7.5.16)
Let B denote the region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $x^2 + y^2 = z^2$. Compute $\iiint_B z^2 \, dV$.



sphere: $x^2+y^2+z^2 = \underset{\uparrow \atop 2^2}{4}$

$$\iiint_B z^2 \, dV = ?$$

Cone: $x^2+y^2=z^2$

$$\phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \arctan\left(\frac{\sqrt{z^2}}{z}\right) = \arctan\left(\frac{z}{z}\right) = \arctan(1) = \pi/4$$

rho

$0 \le \rho \le 2$

theta

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{4}$

phi

$$\iiint_B z^2 \, dV$$

Recall $z = \rho\cos\phi$

$$= \int_0^2 \int_0^{2\pi} \int_0^{\pi/4} \underbrace{(\rho\cos\phi)^2}_{\rho^2\cos^2\phi} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \left(\int_0^2 \rho^4 \, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/4} \cos^2\phi \sin\phi \, d\phi\right)$$

Let $u = \cos\phi$
Then $du = -\sin\phi \, d\phi$

$$= \left[\frac{1}{5}\rho^5\right]_0^2 (2\pi) \left[\frac{-1}{3}\cos^3\phi\right]_0^{\pi/4}$$

$$= \left(\frac{32}{5}\right)(2\pi)\left(-\frac{1}{3}\left[\left(\cos\frac{\pi}{4}\right)^3 - (\cos 0)^3\right]\right)$$

$$= \frac{64\pi}{5}\left(-\frac{1}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 1^3\right]\right)$$

$$\frac{\sqrt{2}}{\cancel{2}_1} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \quad \cancel{2}^{\,1}$$

$$= \frac{64\pi}{5}\left(-\frac{1}{3}\left[\frac{\sqrt{2}}{4} - 1\right]\right)$$

$$= \frac{-64\pi}{15}\left[\frac{\sqrt{2}}{4} - 1\right]$$

$$\approx 8.665$$

10.3 References

<u>References:</u>

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