Module 4 :

ANTIDERIVATIVES AND INTEGRALS

4.1 Introduction to antiderivatives; connection to differential equations and indefinite Integrals

4.2 Basic integration techniques (using Tables, rules, and U-substitution).

4.3 Definite integrals: definition and properties; The Evaluation Theorem (FTC, part 2)

4.4 The Fundamental Theorem of Calculus, part 1

Definite Integral and Area Under a Curve

Why area under a curve is useful to know?

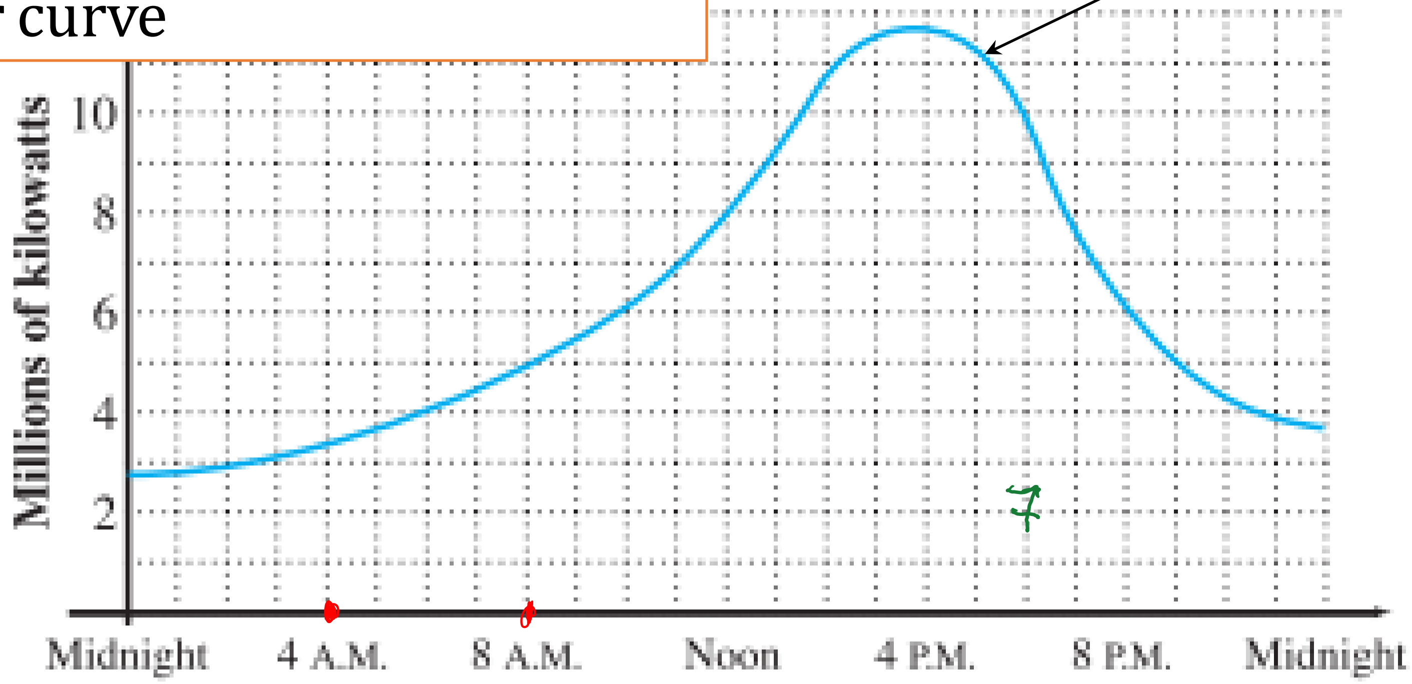
Electricity Consumption.

The graph shows the rate of use of electrical energy(in mill of kilowatts/hour) in a certain city on a very hot day.

Estimate the total usage of electricity on tha

Varying rate of use per unit of time

Total usage over period=Area under curve



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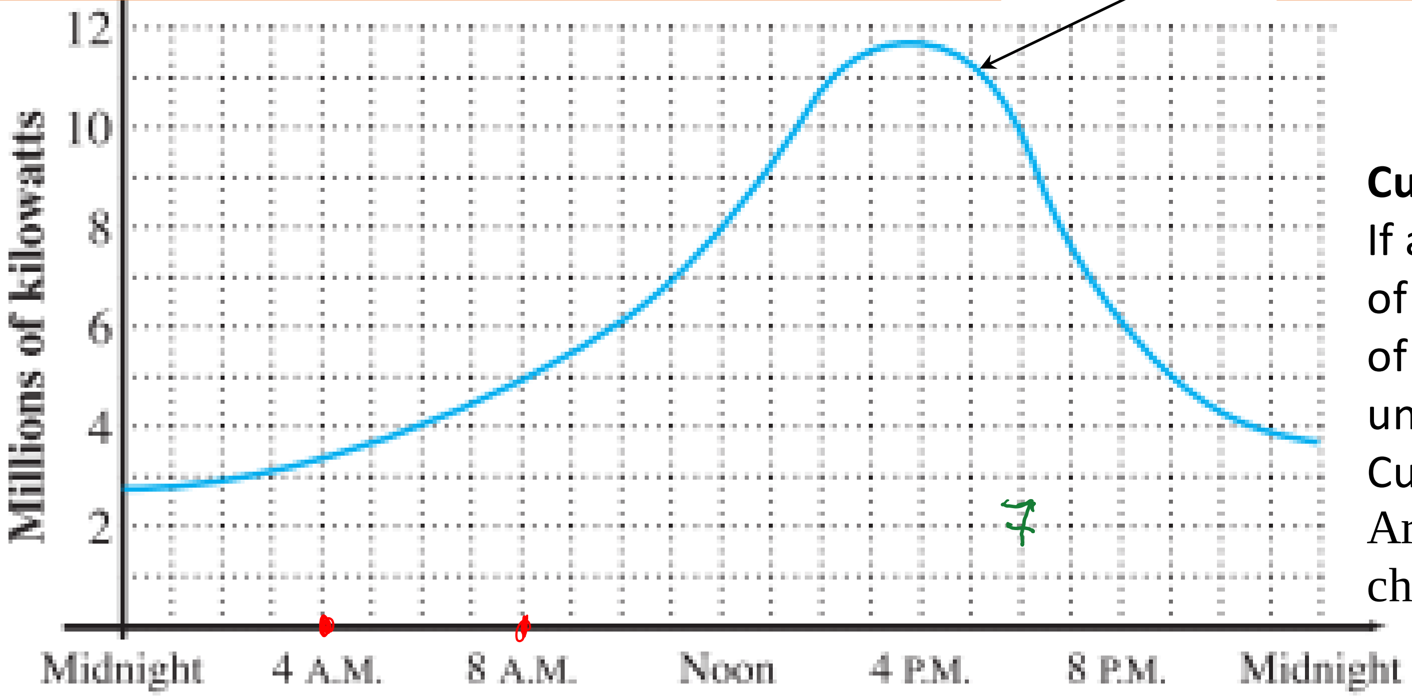

Curve and Area it bounds.
If a curve in 2D represents the rate of change of a quantity, then the total amount of change of the quantity is represented by the Area under the curve
Curve → rate of change
Area (under the curve) →total amount of change

Why area under a curve is useful to know?

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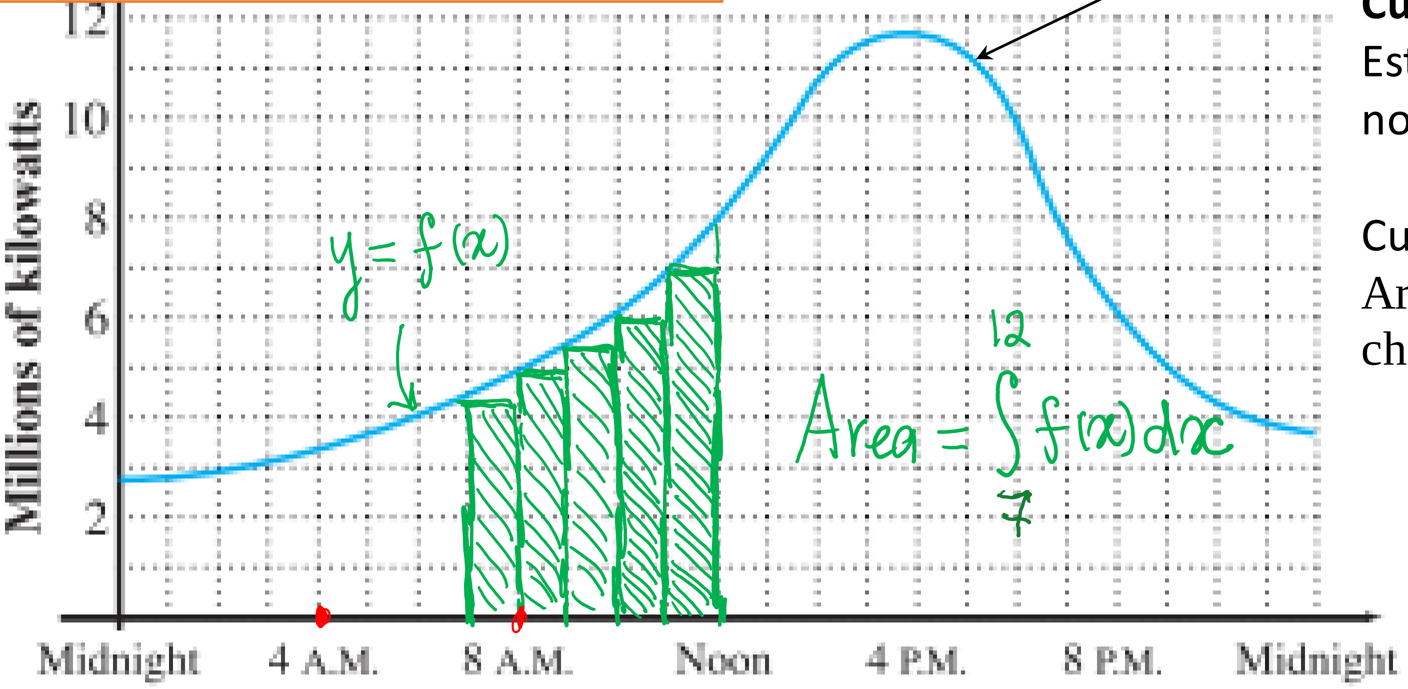


Curve and Area it bounds.

Estimate the amount used between 7 am and noon?

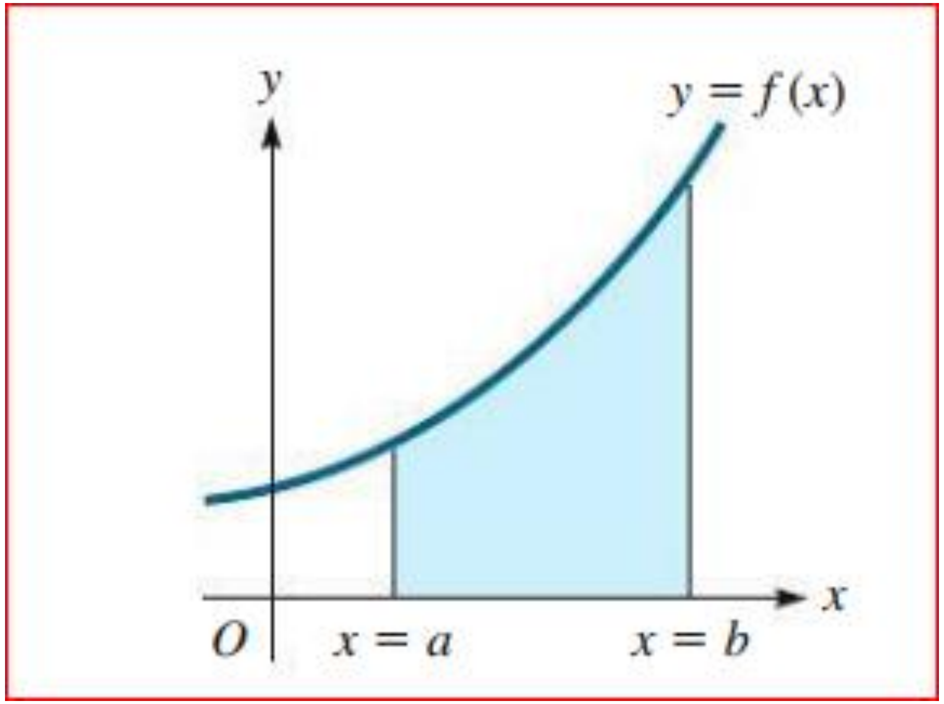
Curve → rate of change

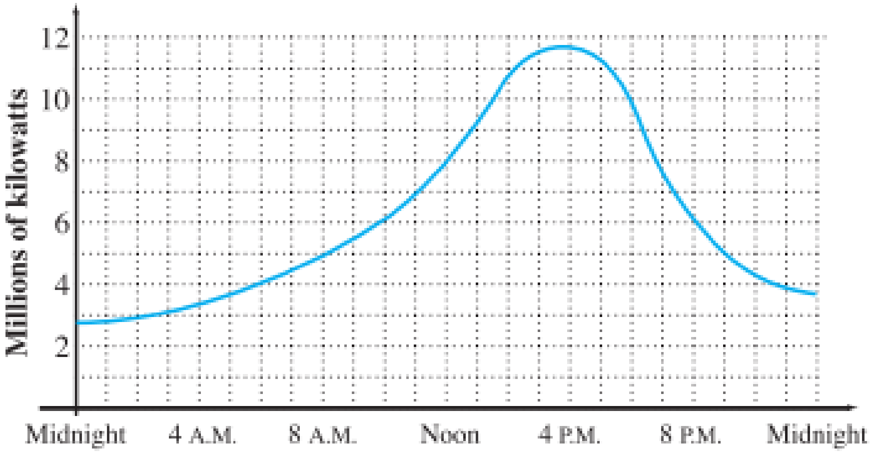
Area (under the curve) →total amount of change

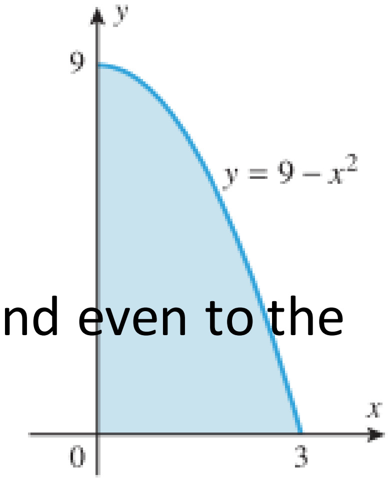
Task: find the area bounded by the curve, the x-axis, and the vertical lines $x = a$ and $x = b$.

Note. We limit our consideration to the *basic* method that works as long as the curve $y = f(x)$ bounding the area is:

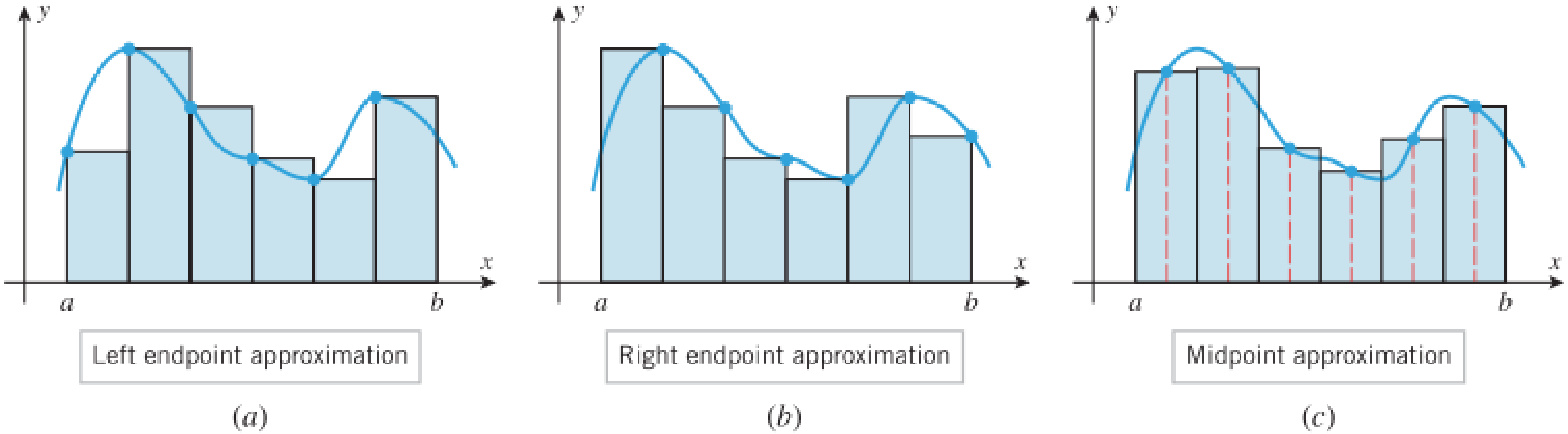
- Continuous over the interval $a \leq x \leq b$
- Positive $(f(x) > 0)$ on the interval $a \leq x \leq b$
  - "positive function" means geometrically that a curve is above the x-axis.
- Areas like that are shown on the right.
- this basic method extends to general areas and even to the discontinuous functions.







Numerical Approximations of Area



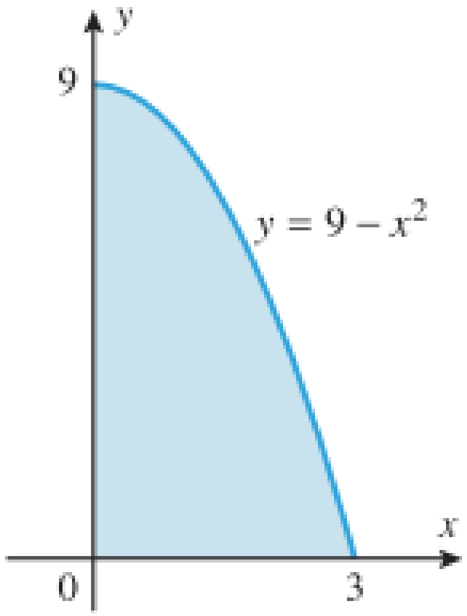
<u>Intuitive approach for finding area under the curve:</u>

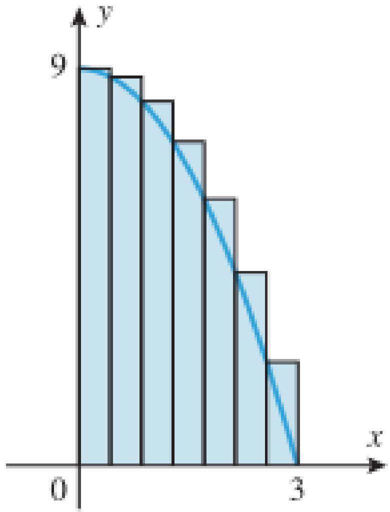
Partition the interval into subintervals, replace each portion of the area with the rectangle of approximately the same height as the curve over this subinterval, and then add up all such rectangles. It can be done in 3 different ways as presented in a), b) and c).

Find the approximations of the area under the curve $y = 9 - x^2$ from $x = 0$ to 3 with partitions n=10, n=20 and n=50

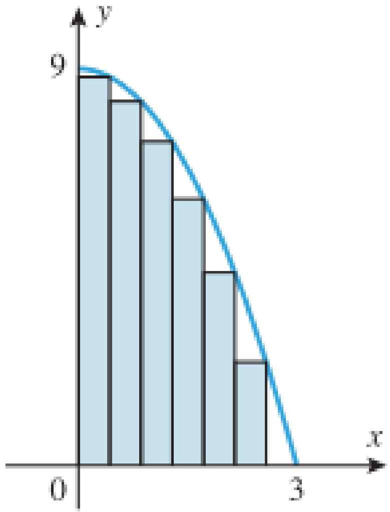
n	LEFT ENDPOINT APPROXIMATION	RIGHT ENDPOINT APPROXIMATION	MIDPOINT APPROXIMATION
10	19.305000	16.605000	18.022500
20	18.663750	17.313750	18.005625
50	18.268200	17.728200	18.000900

Exact Area = 18 units sq.

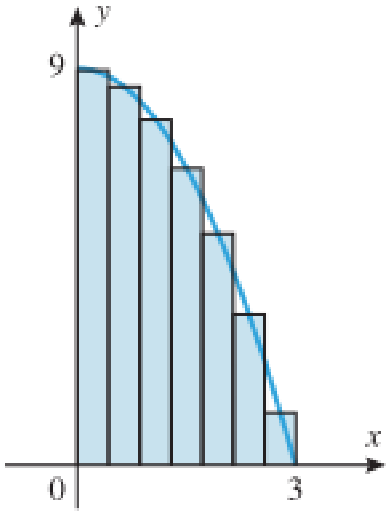




The left endpoint approximation overestimates the area.



The right endpoint approximation underestimates the area.



The midpoint approximation is better than the endpoint approximations.

Definite Integral Definition

Integration is Summation

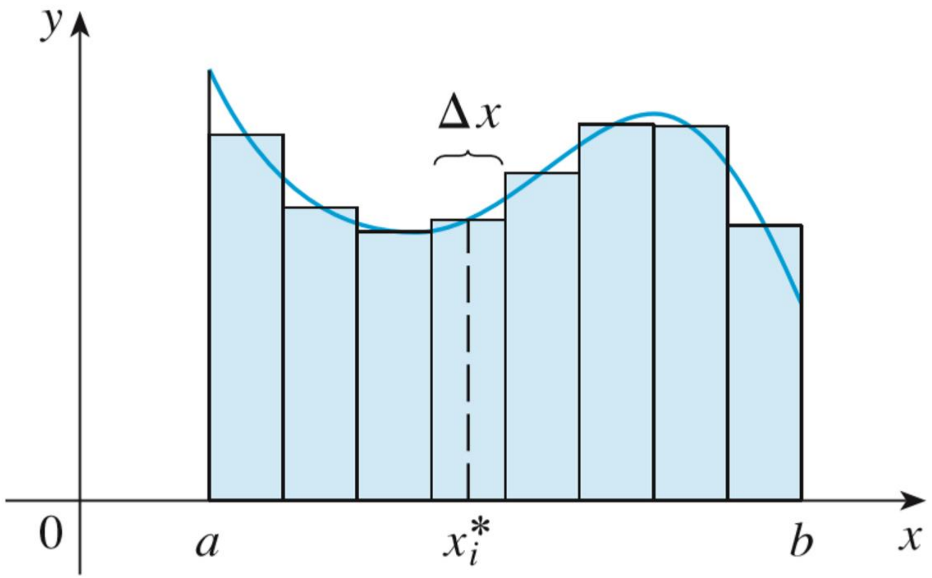
- https://youtu.be/ODwkTt0RMDg

- This is a less than 6 minutes video that
- Introduces the sigma notation, LHS, RHS and
- neatly connects the area under the curve to a **definite integral**

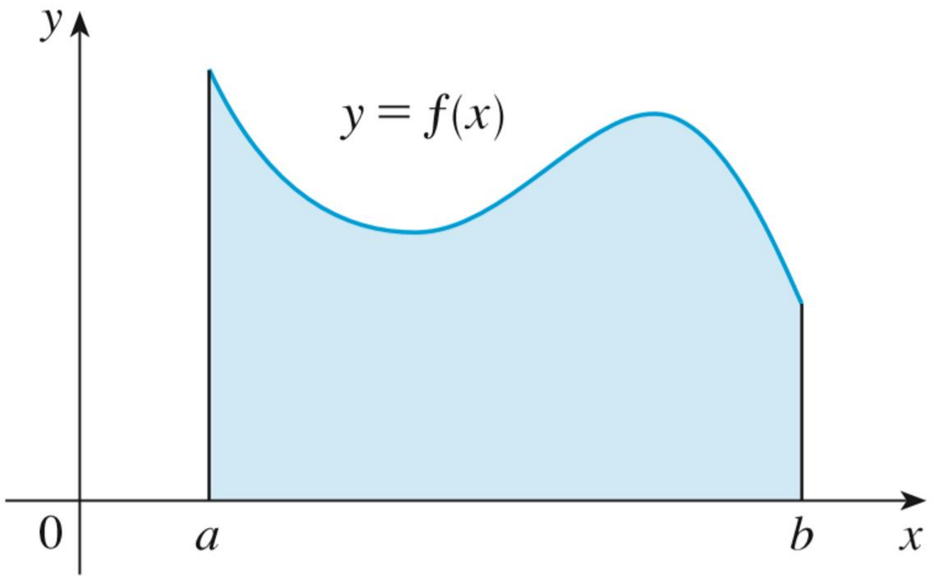
However, using process described in the video every time we need to find the area under the curve would be too time consuming. The short way of finding areas is formulated on the next slide as **The Fundamental Theorem of Calculus**(Part 2)
The FTC connects the DEFINITE(AREA) AND INDEFINITE INTEGRALS in an amazingly efficient way.

The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a *Riemann sum*

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$



If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.



If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

Definite Integral and The Fundamental Theorem of Calculus (Part 2)

- Powerful method for **evaluating** definite integrals using antiderivatives.

If f is continuous on an interval $[a, b]$, and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a) \text{ or } \int_a^b F'(x)dx = F(b) - F(a)$$

where

a – the lower limit of integration → LL

b – the upper limit of integration → UL

Verbally: the value of a definite integral is found by evaluating the function (antiderivative F found by integration) at the upper limit and subtracting the value of this function at the lower limit.
Two step process: 1. find indefinite integral; 2. sub in upper and lower limits, and subtract

EXAMPLE 1

<u>Verbally:</u> the value of a definite integral is found by evaluating the function (found by integration) at the upper limit and subtracting the value of this function at the lower limit.

Evaluate definite integrals:

upper limit

notation that indicates substitution of the upper and lower limits

$$\int_0^3 (9-x^2)dx = \underbrace{\left[9x - \frac{x^3}{3}\right]}_{\substack{F(x) \\ \text{antiderivative} \\ \text{or indefinite integral}}}\Bigg|_{x=0}^{x=3} = 9(3) - \frac{3^3}{3} - 0 = 18$$

Lower limit

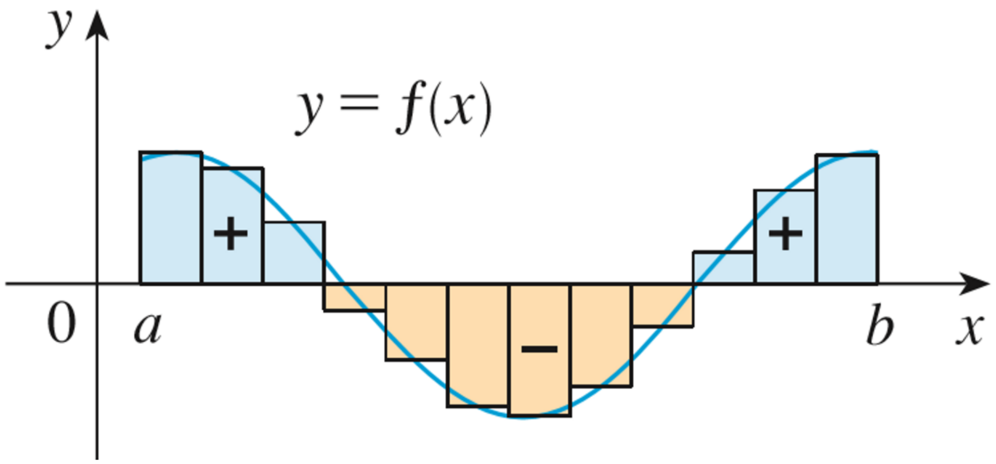
Note: the final result of integrating and evaluating is a **number**.

STEPS:

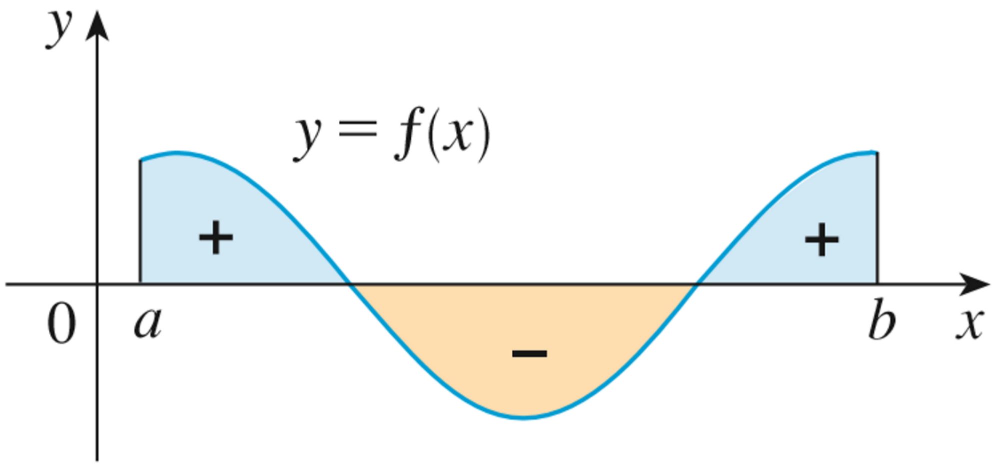
1. Integrate
2. Substitute the LI twice
3. Subtract

$$\int_0^2 x^4 dx = \left[\frac{x^5}{5}\right]\Bigg|_{x=0}^{x=2} = \frac{2^5}{5} - 0 = \frac{32}{5}$$

Net Signed Area between the Graph y=f9x) and the Interval [a, b]



$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.



$\int_a^b f(x)\,dx$ is the net area.

Discontinuities and Integrability

Theorem: Let f be a function that is defined on the finite closed interval $[a, b]$.

(a) If f has finitely many discontinuities in $[a, b]$ but is bounded on $[a, b]$, then f is integrable on $[a, b]$.

(b) If f is not bounded on $[a, b]$, then $[a, b]$, then f is not integrable on $[a, b]$

True or False?

If a function is integrable on [a,b], then it is continuous on [a,b]

Useful Properties of Definite Integral

1. $\int_a^a f(x)dx = 0\,;$ If UL=LL, then the interval $[a, a]$ does not exist.

2. Linearity (α, β are any constants, f&g are functions):
   - *Constant multiple can be carried out of integration*
   - *Expression with several terms can be integrated term by term.*

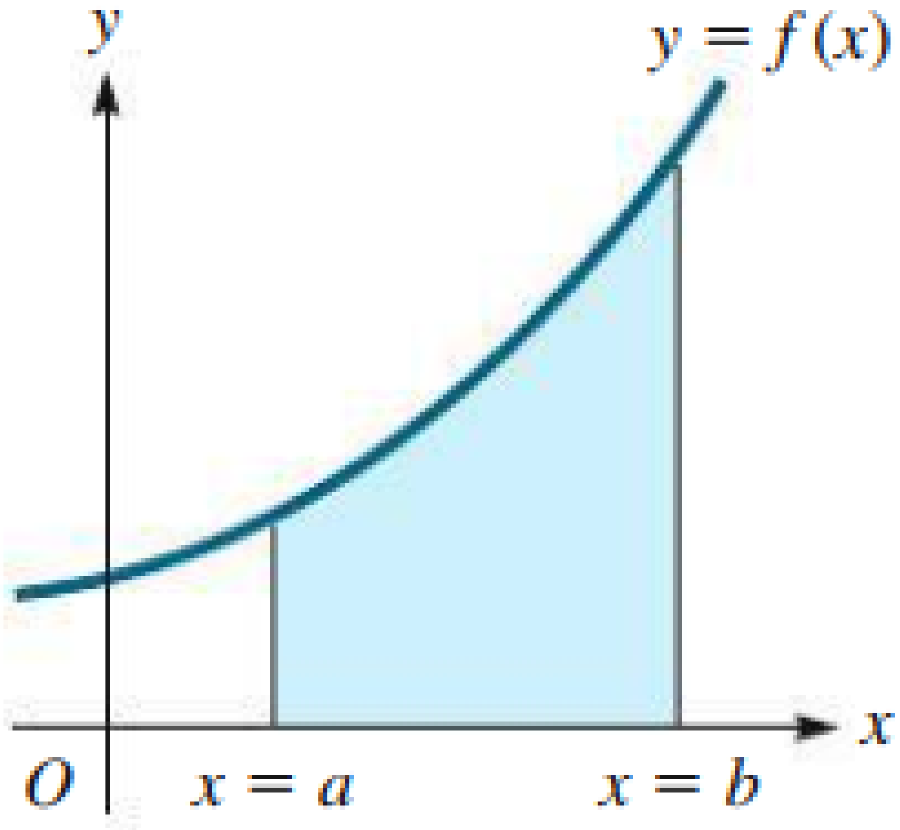
$$\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha \int_a^b f(x)dx \pm \beta \int_a^b g(x)dx$$

3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Useful Properties of Definite Integral

4. If $a \le c \le b$, then definite integral may be split in two:

$$\boxed{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx}$$



Computing the definite integral

EXAMPLE 2

$$\int_1^3 (2x^2 + 4)dx = \left[2\frac{x^3}{3} + 4x\right]\Big|_1^3 = \left[2\frac{(3)^3}{3} + 4(3)\right] - \left[2\frac{(1)^3}{3} + 4(1)\right]$$

$$= 2(9) + 12 - \left[\frac{2}{3} + 4\right] =$$

Note: the final result of integrating and evaluating is a number.

Answer

Computing the definite integral

EXAMPLE 3

$$\int_{2}^{5} (6x^2 - 3x + 5)dx =$$

Answer

Computing the definite integral

EXAMPLE 4

$$\int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_{1}^{2} = \ln 2 - \ln 1 = \ln 2$$

upper LI

Lower LI

Answer

Applications of the Definite Integral: Area under the Curve

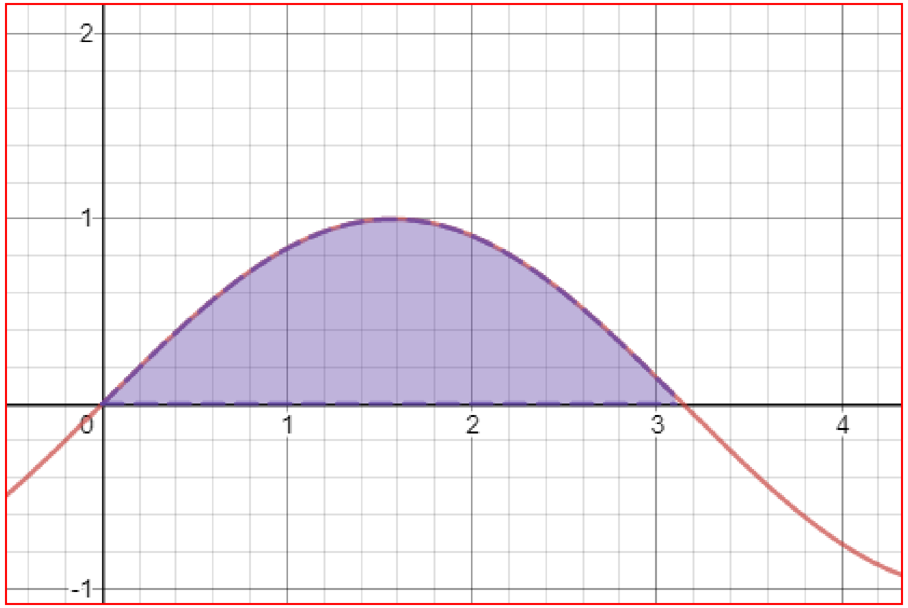
Find the area (in square units) bounded by each curve, the given lines, and the x-axis.

EXAMPLE 5. $y = 2x + \frac{1}{x^2}$ from $x = 1$ to $x = 4$

Answer

EXAMPLE 6. Find the area bounded by the $y = \sin x$ over the positive half-cycle.

$$\int_0^\pi \sin x \, dx =$$



Watch video:
https://humber.ca.panopto.com/Panopto/Pages/Viewer.aspx?id=483b07aa-02a2-435e-b426-ac4b0108d33c

Practice: U-sub with definite integration

a. $\int_0^5 \frac{10}{\sqrt{x+4}} dx$

b. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 8\sin(6x)\,dx$

c. $\int_0^4 \frac{x}{\sqrt{9+x^2}} dx$

ANSWERS: a. 20; b. -8/3; c. 2