HUMBER

Kinematics and Dynamics of Robots

Module 5

Rigid motions

We have now seen how to represent both positions and orientations. We combine these two concepts in this section to define a **rigid motion** and, in the next section, we derive an efficient matrix representation for rigid motions using the notion of homogeneous transformation.

Definition: *A rigid motion is an ordered pair* (d, R) *where* $d \in \mathbb{R}^3$ *and* $R \in SO(3)$*. The group of all rigid motions is known as the* **special Euclidean group** *and is denoted by* $SE(3)$*. We see then that* $SE(3) = \mathbb{R}^3 \times SO(3)$.

rigid motion is a pure translation together with a pure rotation. Let R_1^0 be the rotation matrix that specifies the orientation of frame $o_1x_1y_1z_1$ with respect to $o_0x_0y_0z_0$, and d be the vector from the origin of frame $o_0x_0y_0z_0$ to the origin of frame $o_1x_1y_1z_1$. Suppose the point p is rigidly attached to coordinate frame $o_1x_1y_1z_1$, with local coordinates p^1. We can express the coordinates of p with respect to frame $o_0x_0y_0z_0$ using

$$p^0 = R_1^0 p^1 + d^0$$

Now consider three coordinate frames $o_0x_0y_0z_0$, $o_1x_1y_1z_1$, and $o_2x_2y_2z_2$. Let d_1 be the vector from the origin of $o_0x_0y_0z_0$ to the origin of $o_1x_1y_1z_1$ and d_2 be the vector from the origin of $o_1x_1y_1z_1$ to the origin of $o_2x_2y_2z_2$. If the point p is attached to frame $o_2x_2y_2z_2$ with local coordinates p^2, we can compute its coordinates relative to frame $o_0x_0y_0z_0$ using

$$p^1 = R_2^1 p^2 + d_2^1$$

and

$$p^0 = R_1^0 p^1 + d_1^0$$

The composition of these two equations defines a third rigid motion, which we can describe by substituting the expression for p^1 from Equation 2.7b into Equation 2.7c

$$p^0 \quad = \quad R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$

Since the relationship between p^0 and p^2 is also a rigid motion, we can equally describe it as

$$p^0 \quad = \quad R_2^0 p^2 + d_2^0$$

Comparing Equations 2.7d and 2.7e we have the relationships

$$R_2^0 \quad = \quad R_1^0 R_2^1 \qquad (2.7f)$$

$$d_2^0 \quad = \quad d_1^0 + R_1^0 d_2^1 \qquad (2.7g)$$

Equation 2.7f shows that the orientation transformations can simply be multiplied together and Equation 2.7g shows that the vector from the origin o_0 to the origin o_2 has coordinates given by the sum of d_1^0 (the vector from o_0 to o_1 expressed with respect to $o_0x_0y_0z_0$) and $R_1^0 d_2^1$(the vector from o_1 to o_2, expressed in the orientation of the coordinate frame $o_0x_0y_0z_0$).

Homogeneous Transformations

One can easily see that the calculation leading to Equation 2.7d would quickly become intractable if a long sequence of rigid motions were considered. In this section we show how rigid motions can be represented in matrix form so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations.

In fact, a comparison of Equations 2.7f and 2.7g with the matrix identity

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

where 0 denotes the row vector (0, 0, 0), shows that the rigid motions can be represented by the set of matrices of the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \quad R \in SO(3), \quad d \in \mathbb{R}^3$$

Transformation matrices of the form given in Equation 2.7i are called **homogeneous transformations**. A homogeneous transformation is therefore nothing more than a matrix representation of a rigid motion and we will use $SE(3)$ interchangeably to represent both the set of rigid motions and the set of all 4×4 matrices H of the form given in Equation 2.7i.

Using the fact that R is orthogonal it is an easv exercise to show that the inverse transformation H^{-1} is given by

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

In order to represent the transformation given in Equation 2.7a by a matrix multiplication, we must augment the vectors p^0 and p^1 by the addition of a fourth component of 1 as follows,

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}$$

$$P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

The vectors P^0 and P^1 are known as homogeneous representations of the vectors p^0 and p^1, respectively. It can now be seen directly that the transformation given in Equation 2.7a is equivalent to the (homogeneous) matrix equation

$$P^0 = H_1^0 P^1$$

Homogeneous Transformations

A set of **basic homogeneous transformations** generating *SE*(3) is given by

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for translation and rotation about the *x*, *y*, *z*-axes, respectively.

The most general homogeneous transformation that we will consider may be written now as

$$H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the above equation $n = (n_x, n_y, n_z)$ is a vector representing the direction of x_1 in the $o_0x_0y_0z_0$ frame, $s = (s_x, s_y, s_z)$ represents the direction of y_1, and $a = (a_x, a_y, a_z)$ represents the direction of z_1. The vector $d = (d_x, d_y, d_z)$ represents the vector from the origin o_0 to the origin o_1 expressed in the frame $o_0x_0y_0z_0$. The rationale behind the choice of letters *n, s*, and *a* is explained later.

The same interpretation regarding composition and ordering of transformations holds for 4×4 homogeneous transformations as for 3×3 rotations. Given a homogeneous transformation H_1^0 relating two frames, if a second rigid motion, represented by $H \in SE(3)$ is performed relative to the current frame, then

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the fixed frame, then

$$H_2^0 = H H_1^0$$

Example

Find the homogeneous transformation matrix H that represents a rotation by angle a about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle θ about the current z-axis.

Questions?