Example 1.18. (Getting practice with the definition of convergence of a series)

Determine whether or not the series $\displaystyle\sum_{n=5}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges. Find the sum if it converges.

$$\sum_{n=5}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{5}-\frac{1}{6}+\frac{1}{6}-\frac{1}{7}+\frac{1}{7}-\frac{1}{8}+\cdots$$

Telescoping Series

$\{S_n\}_{n=5}^{\infty}$ sequence of partial sums

$$S_5 = \frac{1}{5}-\frac{1}{6}$$

$$S_6 = \frac{1}{5}\cancel{-\frac{1}{6}}\cancel{+\frac{1}{6}}-\frac{1}{7} = \frac{1}{5}-\frac{1}{7}$$

$$S_7 = \frac{1}{5}\cancel{-\frac{1}{6}}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{7}}+\cancel{\frac{1}{7}}-\frac{1}{8} = \frac{1}{5}-\frac{1}{8}$$

$\vdots$

$$S_N = \frac{1}{5}-\cancel{\frac{1}{6}}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{7}}+\cdots+\cancel{\frac{1}{N}}-\frac{1}{N+1} = \frac{1}{5}-\frac{1}{N+1}$$

$\vdots$

Since $\displaystyle\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{1}{5}-\frac{1}{N+1}\right) = \frac{1}{5}$, $\displaystyle\sum_{n=5}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{5}$.

As $N\to\infty$, $N+1\to\infty$. So $\frac{1}{N+1}\to 0$

The series in Example Eg. 1.18 is an example of a "telescoping series," which is a series in which nearly all the terms cancel each other out. Also note that, since $\frac{1}{n}-\frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n^2+n}$, the analysis above applies to the series $\displaystyle\sum_{n=5}^{\infty}\frac{1}{n^2+n}$ once we recognize that $\frac{1}{n^2+n}$ and $\frac{1}{n}-\frac{1}{n+1}$ are equivalent.

1.4.1 Geometric Series

FRY Example II.3.2.4, geometric series

Example 1.19. Let $a, r \in \mathbb{R}$ with $a \neq 0$. The series

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$$

is called the geometric series with first term a and ratio r. The sequence of partial sums of this geometric series consists of the terms S_N where

$$S_N = \sum_{n=0}^{N} ar^n = a + ar + ar^2 + \cdots + ar^N.$$

Note that

$$\sum_{n=0}^{N} ar^n = \begin{cases} \dfrac{a\left(1 - r^{N+1}\right)}{1 - r}, & \text{if } r \neq 1 \\ a(N+1), & \text{if } r = 1. \end{cases}$$

The series converges if and only if $|r| < 1$.

$$\text{When } |r| < 1, \ \sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}.$$

Example 1.20. (Like FRY Exercise II.3.2.2.11, a finite geometric series)
Evaluate $\displaystyle\sum_{n=0}^{4} 5 \cdot \left(\frac{4}{7}\right)^n$.

$N \to 4$

$$\sum_{n=0}^{4} 5\cdot\left(\frac{4}{7}\right)^n = \underbrace{5\cdot\left(\frac{4}{7}\right)^0 + 5\cdot\left(\frac{4}{7}\right)^1 + 5\cdot\left(\frac{4}{7}\right)^2 + 5\cdot\left(\frac{4}{7}\right)^3 + 5\cdot\left(\frac{4}{7}\right)^4}_{\text{a finite geometric series with } a=5,\ r=\frac{4}{7}}$$

$$= \frac{5\left[1 - \left(\frac{4}{7}\right)^{4+1}\right]}{1 - \frac{4}{7}} \qquad \frac{a\left(1 - r^{N+1}\right)}{1 - r}$$

$$= \frac{552\,405}{50\,421} \approx 10.956$$

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$$ Geometric Series

Case: $r=1$

When $r=1$, $S_N = a + a + a + \cdots + a = (N+1)a$

Since $\lim_{N\to\infty} S_N = \lim_{N\to\infty} (N+1)a = \begin{cases} \infty, & \text{if } a>0 \\ -\infty, & \text{if } a<0 \end{cases}$,

$\sum_{n=0}^{\infty} ar^n$ diverges.

Case: $r \neq 1$ $\quad S_N = \dfrac{(a + ar + ar^2 + \cdots + ar^N)(1-r)}{1-r}$, for $r \neq 1$

the N^{th} member of sequence of partial sums

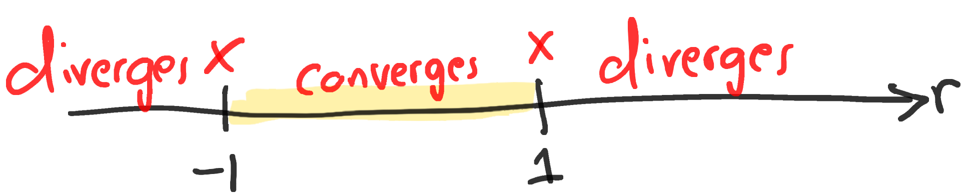
$$= \frac{a + ar + ar^2 + \cdots + ar^N - ar - ar^2 - ar^3 - \cdots - ar^{N+1}}{1-r}$$

$$= \frac{a - ar^{N+1}}{1-r}$$

$$= \frac{a(1-r^{N+1})}{1-r}$$

because $r^{N+1} \to 0$ when

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r} = \begin{cases} \dfrac{a}{1-r} & -1<r<1 \\ \text{DNE} & \text{when } |r| \geq 1 \end{cases}$$



So $$\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r} & -1<r<1 \\ \text{DNE (diverges)} & \text{when } |r| \geq 1 \end{cases}$$

Example 1.21. (Like FRY Exercise II.3.2.2.16, geometric series)
Determine whether the given series converges? If yes, to what value? If not, why not?

(i) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n$

(ii) $\displaystyle\sum_{n=5}^{\infty}\left(\frac{4}{7}\right)^n$

(iii) $\displaystyle\sum_{j=0}^{\infty}5\cdot\left(\frac{4}{7}\right)^j$

(iv) $\displaystyle\sum_{k=0}^{\infty}5\cdot\left(\frac{-4}{7}\right)^k$

(v) $\displaystyle\sum_{m=0}^{\infty}5\cdot\left(\frac{7}{4}\right)^m$

(i) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n = 1+\frac{4}{7}+\frac{16}{49}+\frac{64}{343}+\cdots$

Geometric Series: $a=1$, $r=\frac{4}{7}$

Since $|r|=\left|\frac{4}{7}\right|=\frac{4}{7}<1$, the series converges and

$$\sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n = \frac{1}{1-4/7}=\frac{7}{3}\approx 2.333$$

$\frac{a}{1-r}$

(ii) see below

(iii) $\displaystyle\sum_{j=0}^{\infty}5\cdot\left(\frac{4}{7}\right)^j = 5+5\cdot\frac{4}{7}+5\cdot\left(\frac{4}{7}\right)^2+\cdots$

$= 5+\frac{20}{7}+\frac{80}{49}+\frac{320}{343}+\cdots$

Geometric Series: $a=5$, $r=\frac{4}{7}$

Since $|r| = \left|\frac{4}{7}\right| = \frac{4}{7} < 1$, the series converges and

$$\sum_{j=0}^{\infty} 5\cdot\left(\frac{4}{7}\right)^j = \frac{5}{1-\frac{4}{7}} = \frac{35}{3} \approx 11.667$$

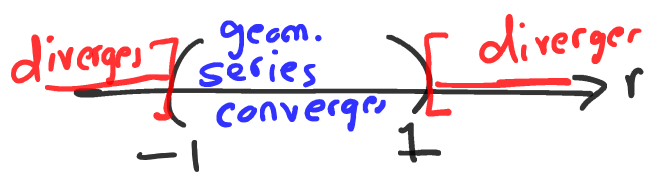
$$\frac{a}{1-r}$$

(iv) $\displaystyle\sum_{k=0}^{\infty} 5\left(\frac{-4}{7}\right)^k$ $\qquad \displaystyle\sum_{n=0}^{\infty} ar^n$

This is a geometric series with $a=5$ and $r = -\frac{4}{7}$.

Since $|r| = \left|-\frac{4}{7}\right| = \frac{4}{7} < 1$, the series converges.

Since $-\frac{4}{7}$ is strictly between -1 and 1



Moreover, $\displaystyle\sum_{k=0}^{\infty} 5\left(\frac{-4}{7}\right)^k = \frac{5}{1-(-4/7)} = \frac{35}{11} \approx 3.182$

(v) $\displaystyle\sum_{m=0}^{\infty} 5\cdot\left(\frac{7}{4}\right)^m$

Geometric series with $a=5$, $r=\frac{7}{4}$.

Since $|r| = \left|\frac{7}{4}\right| = \frac{7}{4}$, the series diverges.

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(ii) $\displaystyle\sum_{n=5}^{\infty}\left(\frac{4}{7}\right)^n = \left(\frac{4}{7}\right)^5 + \left(\frac{4}{7}\right)^6 + \left(\frac{4}{7}\right)^7 + \cdots$ geometric series

Solution 1

$$= \left(\frac{4}{7}\right)^5 \Big[\underbrace{1 + \frac{4}{7} + \left(\frac{4}{7}\right)^2 + \cdots}_{\text{geom. series with } a=1,\ r=\frac{4}{7}}\Big]$$

see part(i)

$$= \left(\frac{4}{7}\right)^5 \cdot \frac{7}{3}$$

$$= \frac{1024}{7203} \approx 0.142$$

Solution 2:

$$\sum_{n=5}^{\infty}\left(\frac{4}{7}\right)^n = \left(\frac{4}{7}\right)^5 + \left(\frac{4}{7}\right)^6 + \left(\frac{4}{7}\right)^7 + \cdots$$

$$= \underbrace{1 + \frac{4}{7} + \left(\frac{4}{7}\right)^2 + \left(\frac{4}{7}\right)^3 + \left(\frac{4}{7}\right)^4 + \left(\frac{4}{7}\right)^5 + \left(\frac{4}{7}\right)^6 + \left(\frac{4}{7}\right)^7 + \cdots}_{} - \underbrace{\left(1 + \frac{4}{7} + \left(\frac{4}{7}\right)^2 + \left(\frac{4}{7}\right)^3 + \left(\frac{4}{7}\right)^4\right)}_{}$$

$$= \sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n - \sum_{n=0}^{4}\left(\frac{4}{7}\right)^n$$

$$= \frac{7}{3} - \frac{5261}{2401}$$

$$= \frac{1024}{7203}$$

$$\approx 0.142$$

$$\sum_{n=5}^{\infty}\left(\frac{4}{7}\right)^n$$

$$= \underbrace{\sum_{n=0}^{4}\left(\frac{4}{7}\right)^n + \sum_{n=5}^{\infty}\left(\frac{4}{7}\right)^n}_{} - \sum_{n=0}^{4}\left(\frac{4}{7}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n - \sum_{n=0}^{4}\left(\frac{4}{7}\right)^n$$

$$= \frac{7}{3} - \frac{5261}{2401}$$

$$\approx 0.142$$

Live Poll

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

$$= \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{3}{10} \cdot \sum_{n=0}^{\infty} 1 \cdot \left(\frac{1}{10}\right)^n$$

- geom. series with $a = 1$, $r = \frac{1}{10}$
- converges since $|r| = \frac{1}{10} < 1$

$$= \frac{3}{10} \cdot \frac{1}{1 - 1/10}$$

$$= \frac{\cancel{3}^{1}}{\cancel{10}_{1}} \cdot \frac{\cancel{10}^{1}}{\cancel{9}_{3}}$$

$$= \frac{1}{3}$$

1.4.2 Telescoping Series

Telescoping series are another family of series in which we can write down a closed form expression for the partial sums. In such series, nearly all of the terms in the sum cancel each other out.

Example 1.22. (FRY Example II.3.2.8, a telescoping series)
Does the series $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$ converge? Why or why not? If yes, what number does the series converge to?

$1+\frac{1}{n} = \frac{n}{n}+\frac{1}{n} = \frac{n+1}{n}$

$$\log\left(1+\tfrac{1}{1}\right)+\log\left(1+\tfrac{1}{2}\right)+\log\left(1+\tfrac{1}{3}\right)+\cdots$$

$$\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty}\log\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty}\left(\log(n+1)-\log(n)\right)$$

$$= \log 2 - \underbrace{\log 1}_{0} + \log 3 - \log 2 + \log 4 - \log 3 + \log(5) - \log(4) + \cdots$$

Let's look at sequence of partial $\{S_N\}_{N=1}^{\infty}$: $\log 1 = 0$

$$S_1 = \sum_{n=1}^{1}\left(\log(n+1)-\log n\right) = \log(1+1)-\log(1) = \log 2 - \log 1 = \log 2 - 0 = \log 2$$

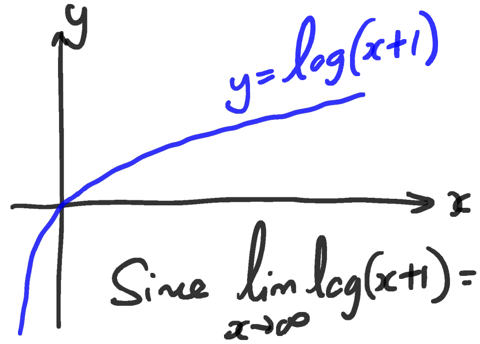
$$S_2 = \sum_{n=1}^{2}\left(\log(n+1)-\log n\right) = \log(1+1)-\log(1) + \log(2+1) - \log 2$$
$$= \cancel{\log 2} - \underbrace{\log 1}_{=0} + \log 3 \cancel{- \log 2}$$
$$= \log 3$$

$$S_3 = \sum_{n=1}^{3}\left(\log(n+1)-\log n\right) = \log(1+1)-\log(1) + \log(2+1) - \log 2 + \log(3+1) - \log 3$$
$$= \cancel{\log 2} - \underbrace{\log 1}_{=0} \cancel{+ \log 3} \cancel{- \log 2} + \log 4 \cancel{- \log 3}$$
$$= \log 4$$

$$\vdots$$

$$S_N = \cancel{\log 2} - \underbrace{\log 1}_{=0} + \cancel{\log 3} - \cancel{\log 2} + \cdots + \log(N+1) - \cancel{\log N}$$

$$= \log(N+1)$$



Since $\displaystyle\lim_{x\to\infty} \log(x+1) = \infty$,

We also have $\displaystyle\lim_{N\to\infty} \log(N+1) = \infty$

Since $\displaystyle\lim_{N\to\infty} S_N = \lim_{N\to\infty} \log(N+1) = \infty$

Since the sequence of partial sums, $\{S_N\}_{N=1}^{\infty}$, diverges, so too does the series $\displaystyle\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$.

1.4.3 The arithmetic of series

FRY Theorem II.3.2.9, arithmetic of series

Theorem 1.23. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with

$$\sum_{n=1}^{\infty} a_n = S \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = T.$$

Then

(i) The sum of convergent series is convergent: $\sum_{n=1}^{\infty} (a_n + b_n) = S + T$.

(ii) The difference of convergent series is convergent: $\sum_{n=1}^{\infty} (a_n - b_n) = S - T$.

(iii) The product of a convergent series with a "scalar" is convergent: $\sum_{n=1}^{\infty} c\, a_n = cS$, where c is a constant.

Example 1.24. (Like FRY Example II.3.2.10)
Does the series $\sum_{n=5}^{\infty} \left(\frac{3 \cdot 4^{n+1}}{8 \cdot 7^n} + \frac{2}{n} - \frac{2}{n+1} \right)$ converge? Why or why not? If yes, what does it converge to?

$$\sum_{n=5}^{\infty}\left(\frac{3\cdot 4^{n+1}}{8\cdot 7^{n}}+\frac{2}{n}-\frac{2}{n+1}\right) = \sum_{n=5}^{\infty}\left(\frac{3\cdot 4\cdot 4^{n}}{8\cdot 7^{n}}+\frac{2}{n}-\frac{2}{n+1}\right)$$

$$= \sum_{n=5}^{\infty}\left(\frac{12}{8}\cdot\left(\frac{4}{7}\right)^{n}+2\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)$$

$$\frac{12}{8} = \frac{\cancel{4}\cdot 3}{\cancel{4}\cdot 2} = \frac{3}{2}$$

$$= \frac{3}{2}\sum_{n=5}^{\infty}\left(\frac{4}{7}\right)^{n} + 2\sum_{n=5}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

see earlier work

$$= \frac{3}{2}\cdot\frac{1024}{7203} + 2\cdot\frac{1}{5} = \frac{7362}{12005} \approx 0.613$$

1.5 References

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