HUMBER ENGINEERING

MENG-3020
SYSTEMS MODELING & SIMULATION

LECTURE 2

HUMBER

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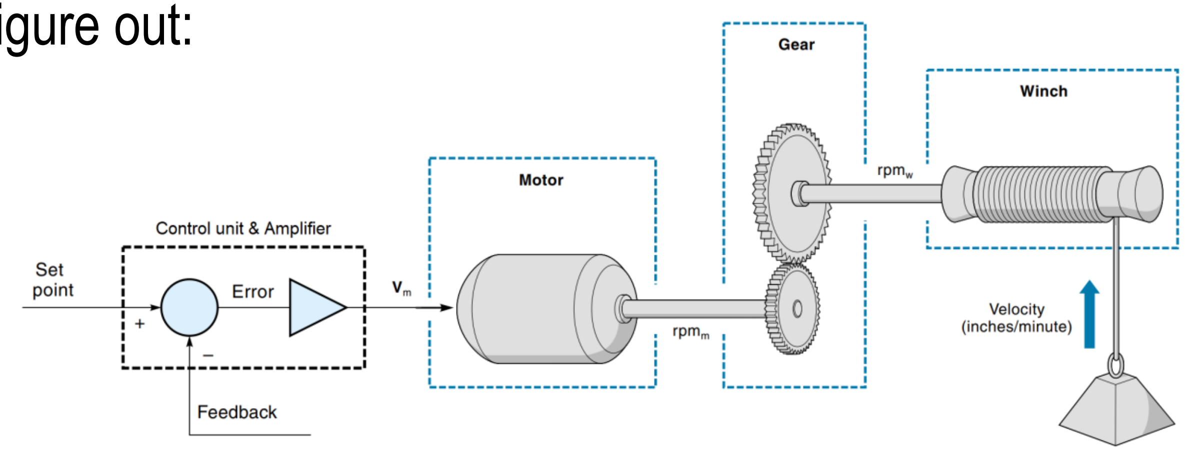
LECTURE 2
Translational Mechanical Systems

- Modeling of Translational Mechanical Systems
- Variables & Elements
- Element Laws
- Interconnection Laws
- Obtaining the System Model (Equation of Motion)
- Solving the Equation of Motion

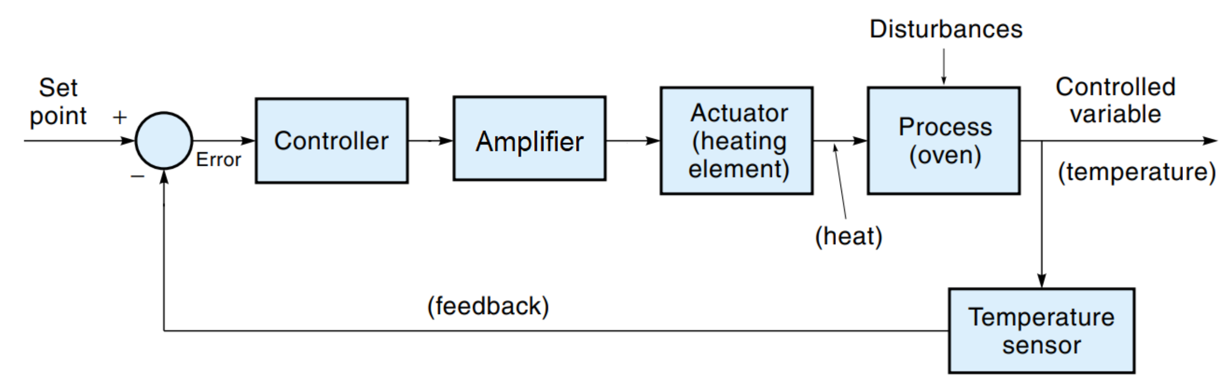
What We Already Know?

- A dynamic system is a collection of components and circuits connected together to perform a useful function.
- Each element in the system converts energy from one form to another.
  - For example, a temperature sensor as converting degrees to volts or a motor as converting volts to rpm.

- In order to design, analysis, and control of dynamic systems, we need to first understand the input-output relationship of the system elements and find a **model** for them. This helps us to figure out:
  - How components affect each other and the overall system.
  - How the output of the system will react to different inputs.
  - How to analyze, predict and control the system behavior.



- **System modeling** means finding a mathematical equation describing how the output of the system is related to its input.



How to find the system model?

What We Already Know?

❑ Lump-Parametric Modeling

- Find the input-output relationships for systems by considering them to be composed of just a few simple basic elements and applying the physical laws from first-principles.
- Mainly used to model electrical, mechanical, and electromechanical systems.
- The model is obtained as a differential equation, then shown in any standard form of, Transfer Function model, State-Space model or Block diagram model.
  - **Electrical systems** can be considered as compose of basic elements, which can be represented by resistors, capacitors, inductors and op-amps.
    - These are characterized by voltage –current relationships for components and the laws of interconnection Kirchhoff's Voltage Law (KVL) and Current Law (KCL)
  - **Mechanical systems** can be considered as compose of basic elements, which can be represented by springs, dampers and masses, and are characterized by Newton's Laws of motion.

❑ Empirical Modeling

- An experimental approach, which mainly used to model thermal and fluid (hydraulic and pneumatic) processes.
- Some experiments are performed on the system to collect input-output data, a model is then fitted to the collected data by assigning suitable numerical values to its parameters.

Modeling of Mechanical Systems

- The motion of elements of mechanical systems can be described as:
  - **Translational Motion**
  - **Rotational Motion**

- The equations governing the motion of mechanical systems are called the ***equation of motion*** that often directly or indirectly formulated by applying **Newton's law of motion** to the **free-body diagram** (FBD).

Translational Motion $\boxed{\sum F_{ext} = Ma}$ $\boxed{\sum T_{ext} = J\alpha}$ **Rotational Motion**

- The number of equations of motion required is equal to the number of ***linearly independent*** motions or the number of ***degrees of freedom***.
  - **Step 1:** Identify reference point and positive direction of motion.
  - **Step 2**: Draw a free-body diagram for each mass/point of motion.
  - **Step 3**: For each free-body diagram, find the forces acting on the body due only to its own motion and the forces create by the adjacent motion.
  - **Step 4**: Use Newton's law on each body to form the differential equation of motion.

Translational Mechanical Systems: Variables & Elements

- The translational motion is defined as a motion that takes place along a **straight** or **curved** path.
- The **variables** that are used to describe the translational motion are:
  - $f(t)$: Force (N)
  - $x(t)$: Displacement (m)
  - $v(t)$: Velocity (m/s)
  - $a(t)$: Acceleration (m/s^2)

- All these variables are function of time.
- Displacements are measured with respect to reference condition, which is the equilibrium position of the body.
- Velocities and accelerations are normally expressed as the derivatives of the corresponding displacement.

$$v(t) = \frac{dx(t)}{dt} = x'(t) = \dot{x}(t) \qquad a(t) = \frac{d^2x(t)}{dt^2} = x''(t) = \ddot{x}(t)$$

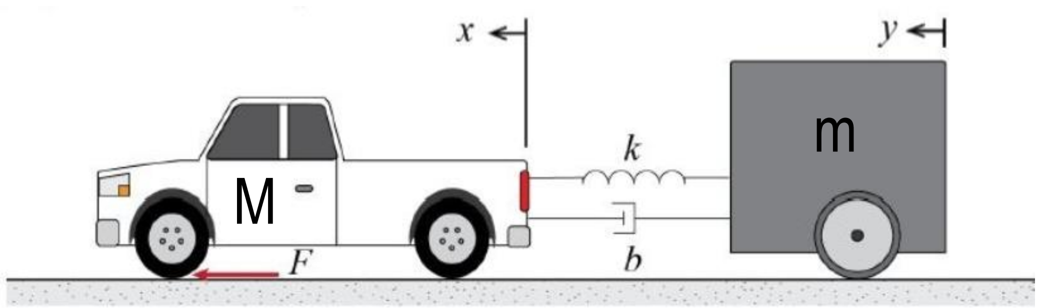
- The **elements** that we include in translational systems are:
  - **Inertia Element** $\rightarrow$ **Mass**
  - **Stiffness Element** $\rightarrow$ **Spring**
  - **Damping Element** $\rightarrow$ **Damper**

Translational Mechanical Systems: Example

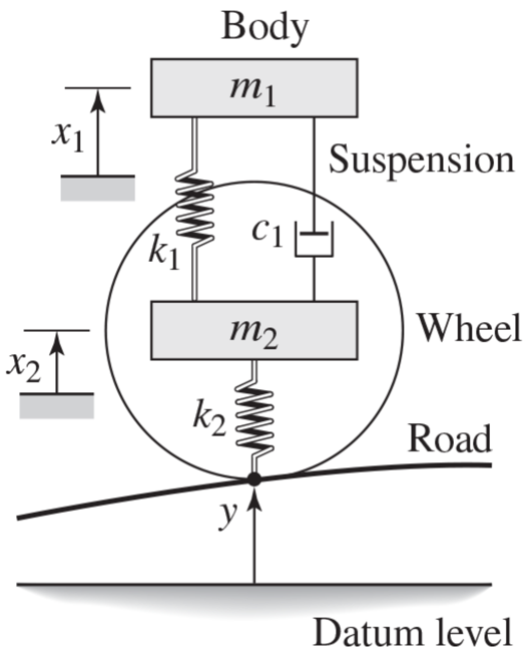
- Some real-word examples of translational motion systems.

○ Truck Pulling a Cart

Input: Applied engine force
Outputs: Displacement of truck and cart



○ Vehicle Suspension System

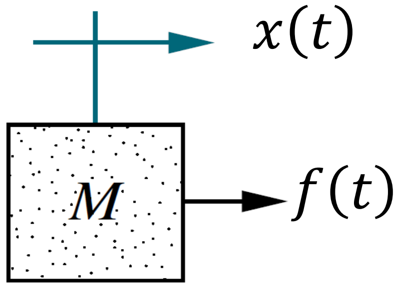


Input: Vertical displacement due to road bumps
Outputs: Vertical displacement of car

Translational Mechanical Systems: Element Laws

❑ Inertia Element: Mass

- If a force is applied on a body, then it is opposed by an opposing force due to inertia.
- The inertia of a body can be represented by a **mass.**
- From the Newton's second law, the applied force $f(t)$ is proportional to the acceleration $a(t)$ of the body.
- The M is the mass of the body. The unit is (kg).



$$\boxed{f(t) = Ma(t) = M\frac{dv(t)}{dt} = M\frac{d^2x(t)}{dt^2}}$$

- Energy in a mass is stored as kinetic energy if the mass is in motion, and as potential energy if the mass has a vertical displacement relative to the reference point.

$$\boxed{KE = \frac{1}{2}Mv^2} \qquad \boxed{PE = Mgh}$$

where $g = 9.8\ m/s^2$ is gravitational acceleration and h is the height of mass above its reference position.

Translational Mechanical Systems: Element Laws

Stiffness Element

- All physical objects deform somewhat under the action of externally applied forces.
- When the deformation is negligible for the purpose of the analysis, we can treat the object as a rigid body.
- Sometimes, an elastic element is <u>intentionally</u> included in the system, as with a spring in a vehicle suspension.
- Sometimes the element is <u>not intended to be elastic</u> but deforms anyway because it is subjected to large forces, such as cables, beams and rods.
- This can be the case with the boom or cables of a large crane that lifts a heavy load.
- In such cases, we must include the deformation and corresponding forces in our analysis.

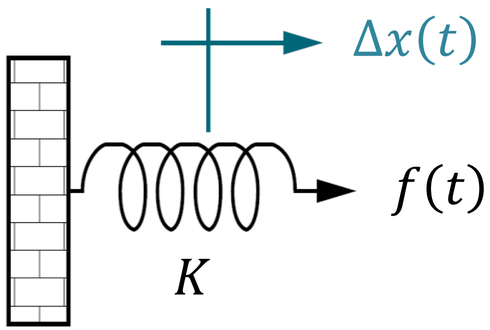
Translational Mechanical Systems: Element Laws

Stiffness Element: Spring

- Any mechanical element that undergoes a change in shape when subjected to a force can be characterized by a **stiffness element**, provided only that an algebraic relationship exists between the elongation and the force.
- The stiffness of an element can be represented by an **ideal spring.**

- If a force is applied to a spring, then it is opposed by an opposing force due to the **elasticity** of the spring.
- From the Hooke's law, the applied force $f(t)$ to a spring is proportional to the displacement $\Delta x(t)$ of the spring.
- The K is the spring constant. The unit is (N/m).

$$\boxed{f(t) = K\Delta x(t)}$$



- Potential energy stored in a spring that has been stretched or compressed, and for a linear spring that energy is given by:

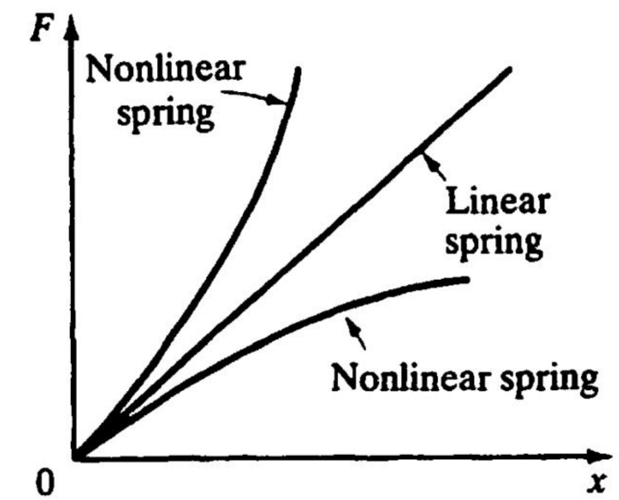
$$\boxed{PE = \frac{1}{2}K(\Delta x)^2}$$

- An ideal spring element is massless.
- A real spring element can be represented by an ideal element either by neglecting its mass or by including it in another mass in the system.

Translational Mechanical Systems: Element Laws

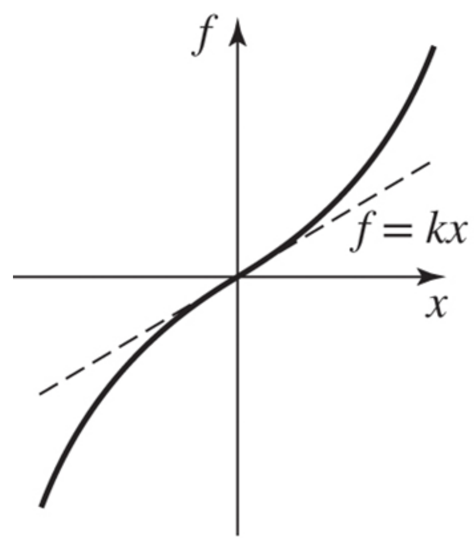
❑ Properties of Spring

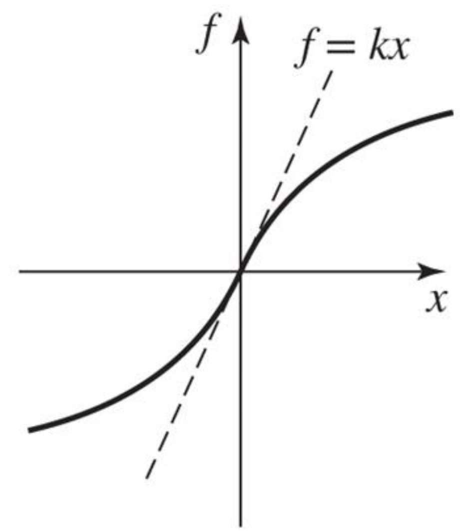
- For practical springs, the solution of linearity may be good only for relatively small net displacements.
- When a linear spring is stretched a point is reached in which the force per unit displacement begins to change and the spring becomes a nonlinear spring.



- Spring constants indicate stiffness of the spring:
  - Large value of spring constants correspond to a hard spring,
  - Small value of spring constants to a soft spring.
- The reciprocal of the spring constant is called compliance or mechanical capacitance.
- Mechanical capacitance indicates the softness of the spring.

Nonlinear characteristics of a hardening spring



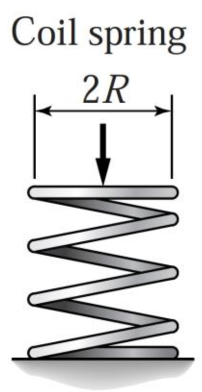


Nonlinear characteristics of a softening spring

Translational Mechanical Systems: Element Laws

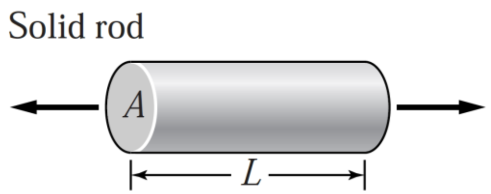
❑ Spring Constant

- The spring constant can be determined <u>experimentally</u> by a tension test or <u>analytically</u> from the geometry and material properties.



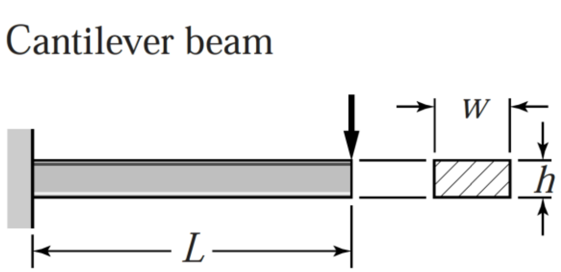
$$K = \frac{Gd^4}{64nR^3}$$

d = wire diameter
n = number of coils
G = Shear modulus of elasticity



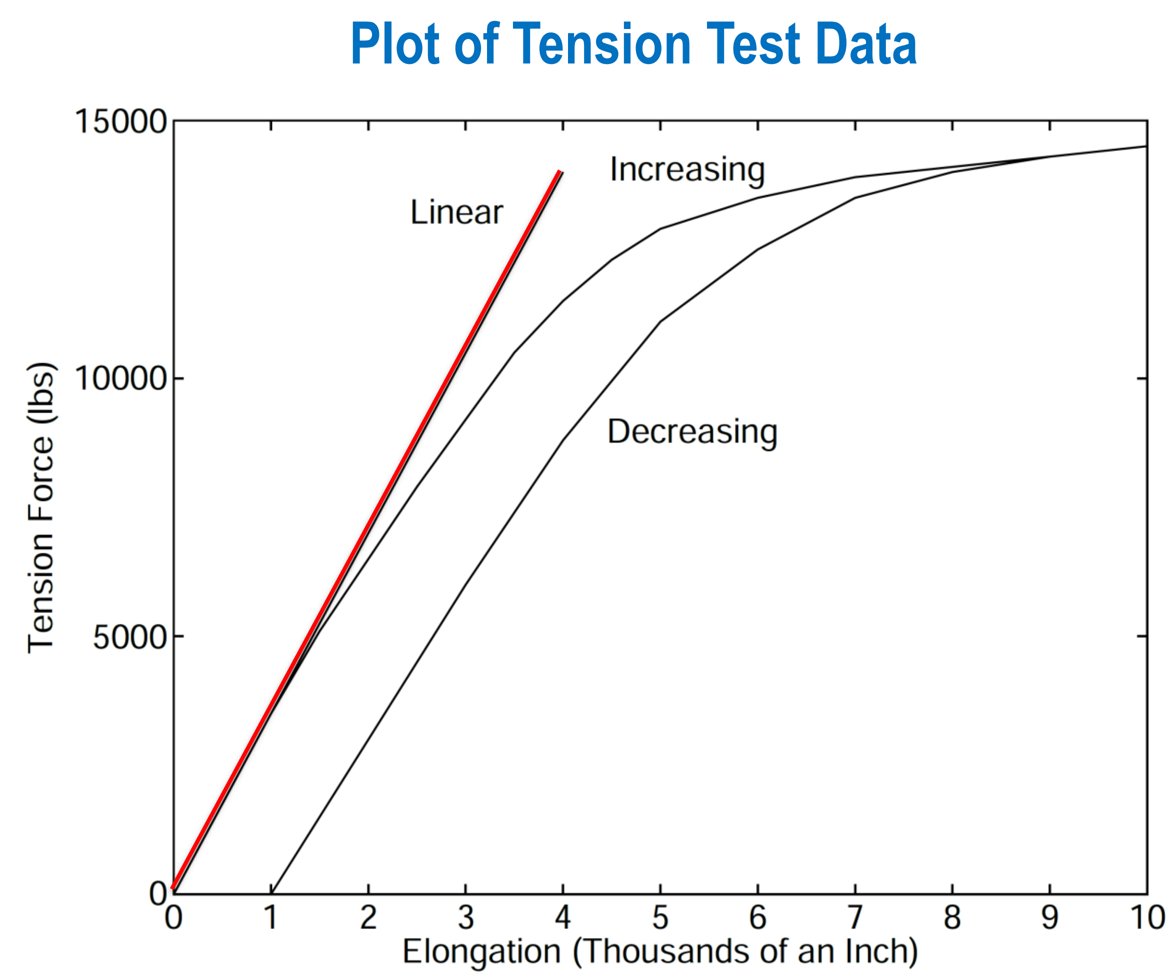
$$K = \frac{EA}{L}$$

A = rod area
L = rod length
E = Modulus of elasticity



$$K = \frac{Ewh^3}{4L^3}$$

w = beam width
h = beam thickness
L = beam length
E = Modulus of elasticity

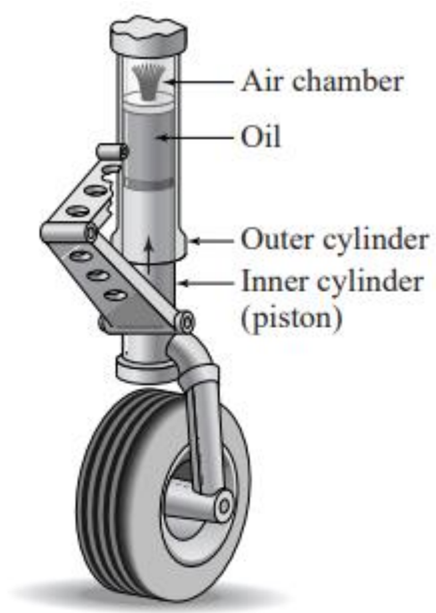


Translational Mechanical Systems: Element Laws

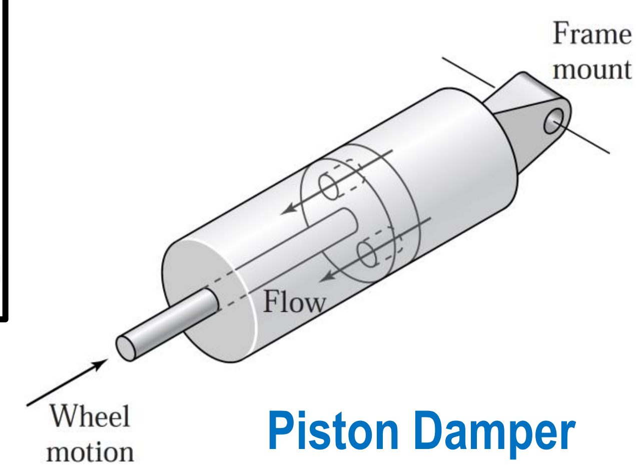
❑ Damping Element

- A damping element or damper is an element that resists relative velocity across it.
- A common example of a damping element is a dashpot (a shock absorber), which consists of a piston and an oil filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by oil because oil must flow around the piston from one side to the other.
- An example from everyday life of a device that contains a damping element as well as a spring element is the door closer
- An oleo strut is a pneumatic air–oil hydraulic shock absorber used on the landing gear of most large aircraft and many smaller ones. This design cushions the impacts of landing and damps out vertical oscillations.

Oleo Strut



Shock Absorber



Piston Damper

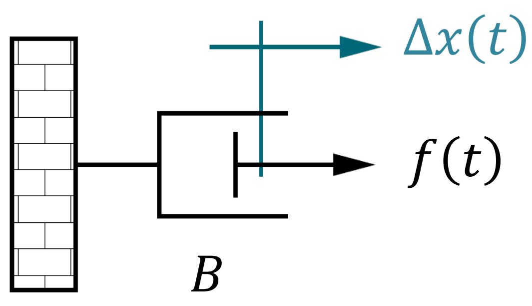
Pneumatic
Door Closer

Translational Mechanical Systems: Element Laws

❑ Damping Element

- Essentially, the damper absorbs energy, and the absorbed energy is dissipated as heat that flows away to the surroundings.
- Damping can exist whenever there is a fluid resistance force produced by a fluid layer moving relative to a solid surface.
- Engineering systems can exhibit damping in bearings and other surfaces lubricated to prevent wear.
- Damping elements can be deliberately included as part of the design.
- Drag friction or damping effect in a system can be represented by a linear damper.

- If a force is applied to a viscus damper, then it is opposed by an opposing force due to **viscus friction** of the damper.
- The applied force $f(t)$ is proportional to the velocity $v(t)$ of the motion.
- The B is the viscus friction coefficient. The unit is (N.s/m)

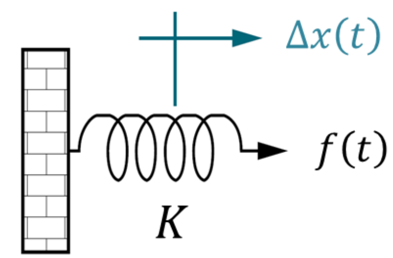
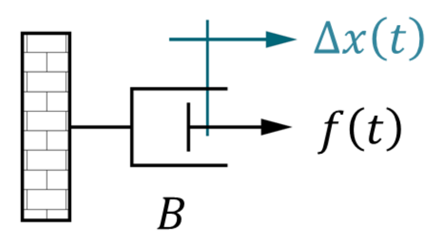
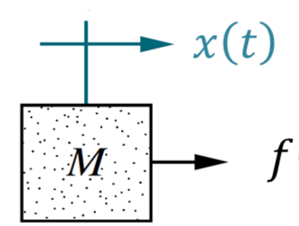


$$\boxed{f(t) = B\Delta v(t) = B\frac{d\Delta x(t)}{dt}}$$

Translational Mechanical Systems: Element Laws

❑ Summary

- Table shows summary of the Force-velocity, and Force-displacement translational relationships for Spring, Viscous dampers, and Mass.

Element	Force-velocity	Force-displacement
Spring 	$f(t) = K\int_0^t \Delta v(t)dt$	$f(t) = K\Delta x(t)$
Viscous damper 	$f(t) = B\Delta v(t)$	$f(t) = B\frac{d\Delta x(t)}{dt}$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$

Translational Mechanical Systems: Interconnection Laws

❑ Newton's Second Law

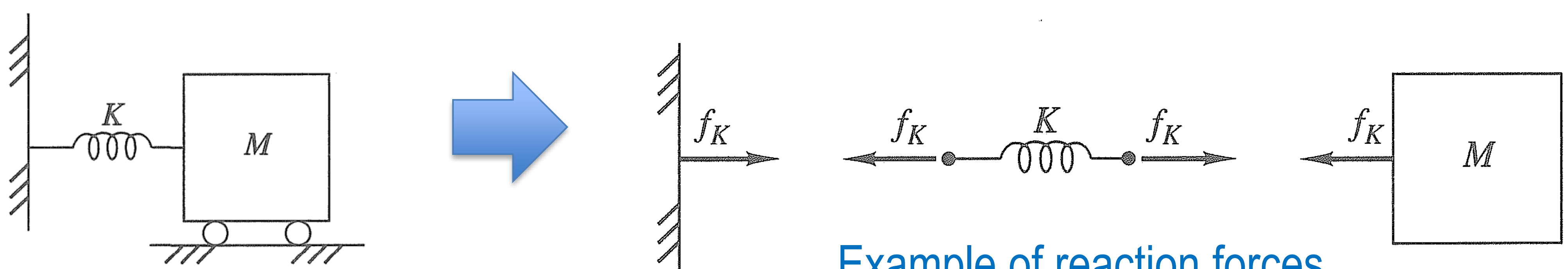
- Suppose that forces are acting on a body of mass m. If $\sum F$ is the sum of all forces acting on mass m through the center of mass in a given direction, then,

$$\boxed{\sum f_{ext} = ma}$$

where $a(t)$ is the resulting absolute acceleration in that direction.

❑ Newton's Third Law: The Law of Reaction Forces

- In order to relate the forces exerted by the elements of friction and stiffness to the forces acting on a mass or junction point, we need Newton's third law regarding reaction forces.
- Accompanying any force of one element on another, there is a reaction force on the first element of equal magnitude and opposite direction.



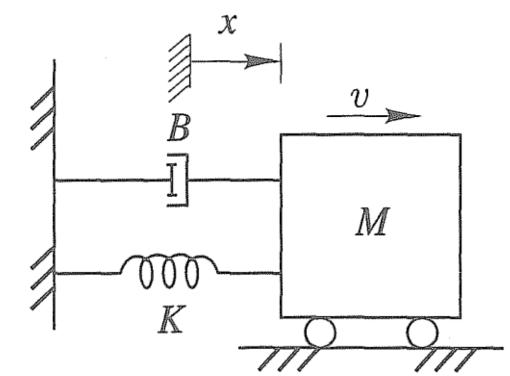
Example of reaction forces

Translational Mechanical Systems: Interconnection Laws

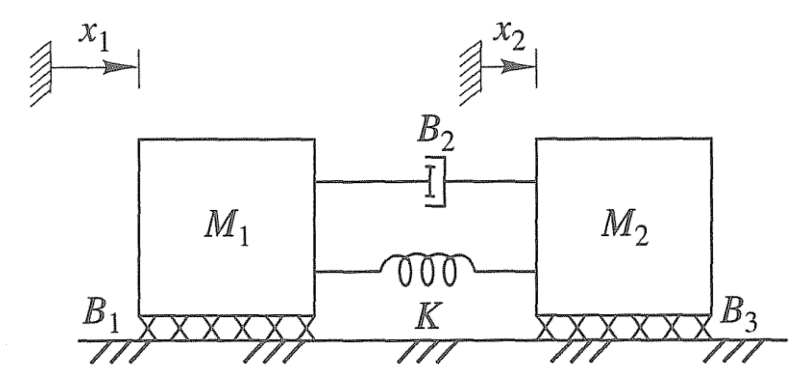
❑ The Law for Displacement

- If the ends of two elements are connected, those ends are forced to move with the same displacement and velocity.

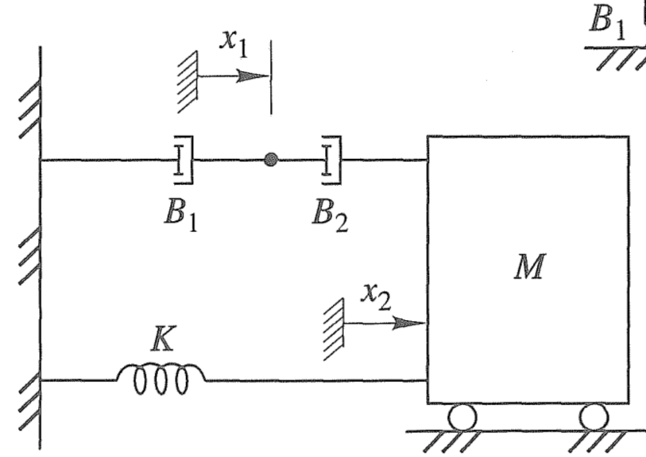
- For example, because the damper and spring are both connected between the wall and the mass, the right ends of both elements have the same displacement x and move with the same velocity v.



- In this system, where B_2 and K are connected between two moving masses, the elongation of both elements is $x_2 - x_1$.

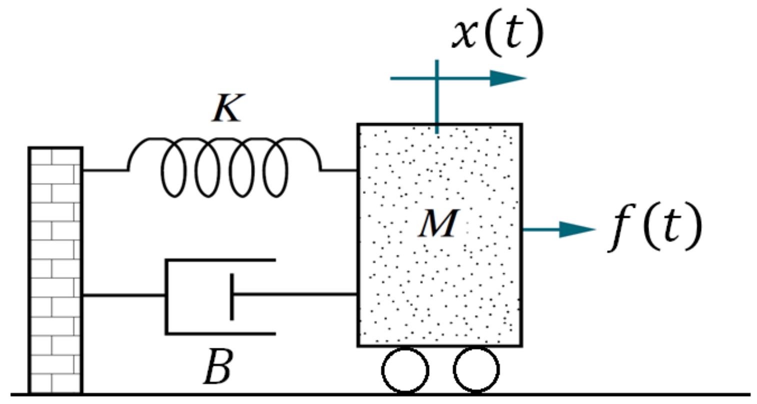


- Let x_1 and x_2 denote displacements measured with respect to reference position. Then the respective elongations of B_1, B_2 and K are x_1, $x_2 - x_1$ and x_2.



Modeling of Translational Mechanical Systems

Example 1 Find the equation of motion for the given mass-spring-damper system. Assume that the input is the applied force, and the output is the displacement of the mass.



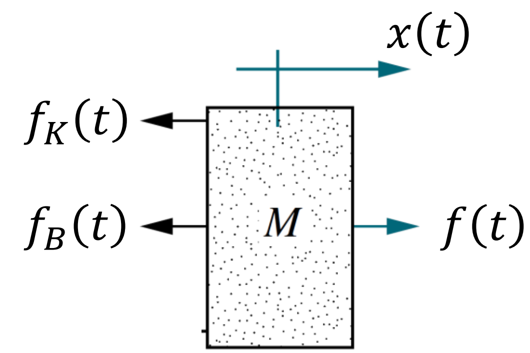
Assume that the positive direction for motion is to the right.

Draw the free-body diagram of the system for the mass M.

Apply Newton's second law to find the equation of motion.

$$\sum f_{ext} = ma \quad \rightarrow \quad f(t) - f_K(t) - f_B(t) = Ma(t)$$

$$f(t) - Kx(t) - B\frac{dx(t)}{dt} = M\frac{d^2x(t)}{dt^2}$$



Free-body Diagram

The equation of motion is obtained as a second-order differential equation,

$$\boxed{M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) = f(t)}$$

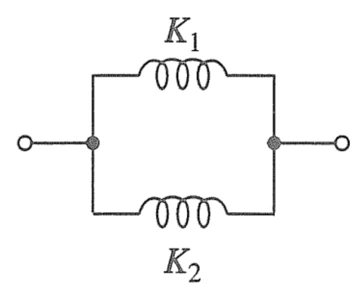
Second-order Differential Equation

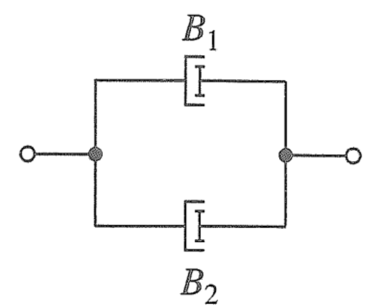
Modeling of Translational Mechanical Systems

❑ Parallel & Series Combination

- In some cases, two or more springs or dampers can be replaced by a single equivalent element.
  - Two springs or dampers are said to be **in parallel** if the first end of each is attached to the same body and if the remaining ends are also attached to a common body.
  - The key requirement for parallel elements is that respective ends move with the **same** displacement.

$$\boxed{K_{eq} = K_1 + K_2}$$

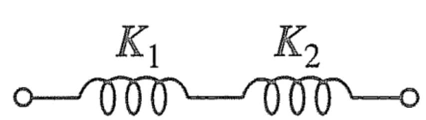


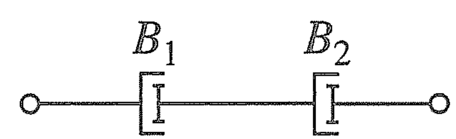


$$\boxed{B_{eq} = B_1 + B_2}$$

- Two springs or dampers are said to be **in series** if they are joined at only one end of each element and if there is **no other element connected to their common junction**.

$$\boxed{\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}}$$

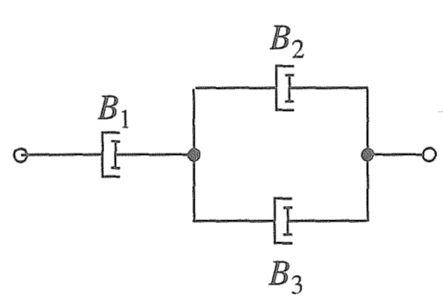


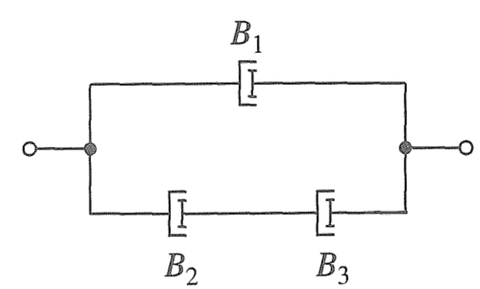


$$\boxed{\frac{1}{B_{eq}} = \frac{1}{B_1} + \frac{1}{B_2}}$$

- It is also possible to have combination of series and parallel connections. For example:

$$B_{eq} = \frac{B_1(B_2 + B_3)}{B_1 + B_2 + B_3}$$



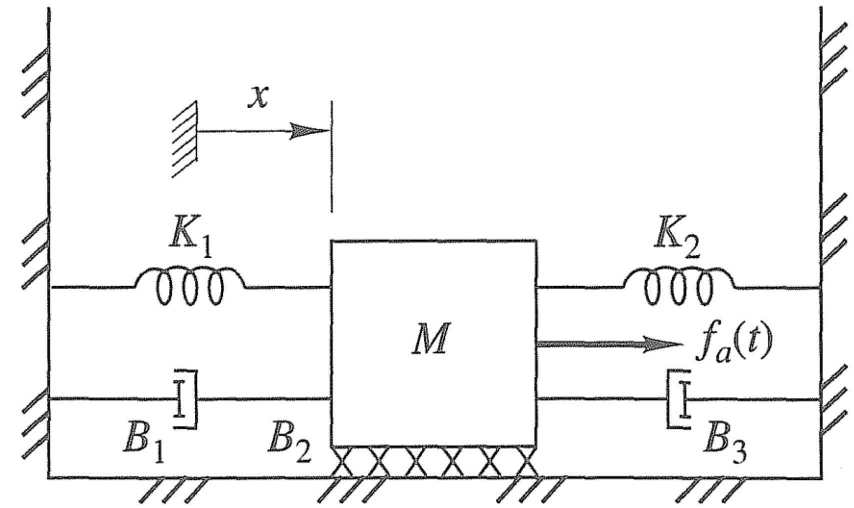


$$B_{eq} = B_1 + \frac{B_2 B_3}{B_2 + B_3}$$

Modeling of Translational Mechanical Systems

Example 2

Find the equation describing the motion of the mass in the translational system shown below. Show that the two springs can be replaced by a single equivalent spring, and the three friction elements by an equivalent element.



Assume that the positive direction for motion is to the right.

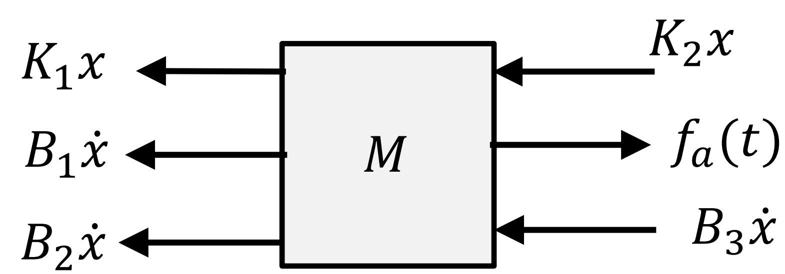
Draw the free-body diagram of the system for the mass M.

Apply Newton's second law to find the equation of motion.

$$\sum f_{ext} = ma$$

Mass M $\rightarrow$ $f_a(t) - K_1x - B_1\dot{x} - B_2\dot{x} - B_3\dot{x} - K_2x = M\ddot{x}$

$$f_a(t) = M\ddot{x} + (B_1 + B_2 + B_3)\dot{x} + (K_1 + K_2)x$$



Free-body Diagram

We can see that the two springs and the three damping elements can be replaced by an equivalent element.

$$f_a(t) = M\ddot{x} + B_{eq}\dot{x} + K_{eq}x$$

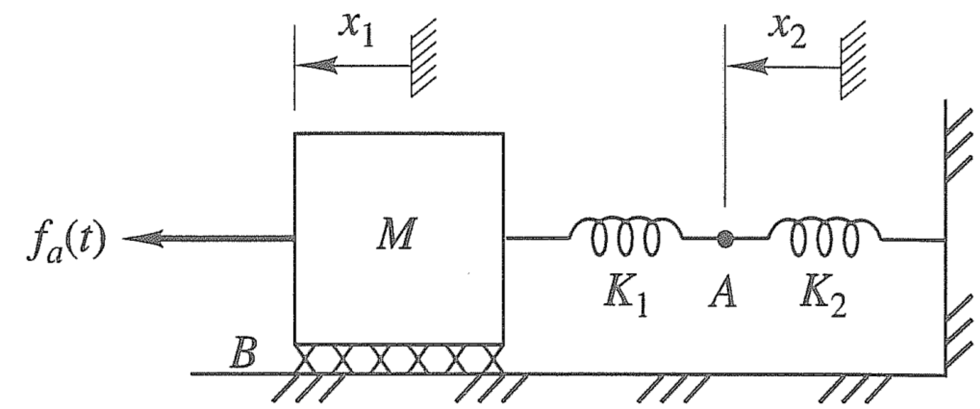
$$\boxed{B_{eq} = B_1 + B_2 + B_3}$$

$$\boxed{K_{eq} = K_1 + K_2}$$

Modeling of Translational Mechanical Systems

Example 3

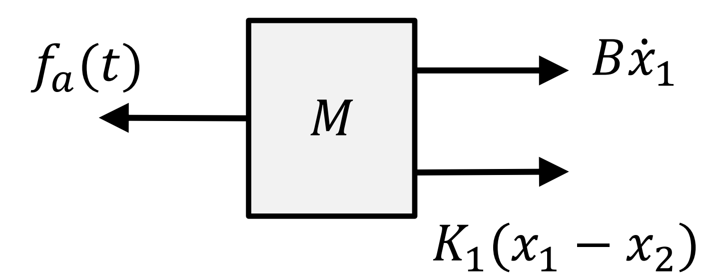
When $x_1 = x_2 = 0$, the two springs are neither stretched nor compressed. Draw free-body diagrams for the mass M and for the **massless junction A**, then write the equation of motion of the system. Find K_{eq} for a single spring that could replace the combination of K_1 and K_2.



Assume that the positive direction for motion is to the left.

Mass M $\rightarrow$ $f_a(t) - B\dot{x}_1 - K_1(x_1 - x_2) = M\ddot{x}_1$

Junction A $\rightarrow$ $K_1(x_1 - x_2) = K_2 x_2$

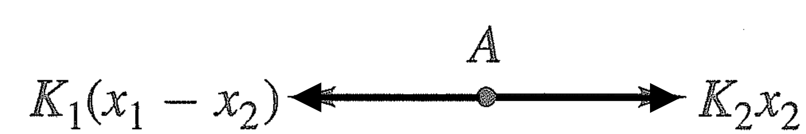


Solve the second equation for x_2 in terms of x_1 gives:

$$x_2 = \left(\frac{K_1}{K_1 + K_2}\right)x_1 \quad \rightarrow \text{ two displacement proportional to one another}$$

Substituting the x_2 into the first equation we have:

$$f_a(t) - B\dot{x}_1 - K_1\left(x_1 - \frac{K_1}{K_1 + K_2}x_1\right) = M\ddot{x}_1 \quad \rightarrow \quad f_a(t) = M\ddot{x}_1 + B\dot{x}_1 + \frac{K_1 K_2}{K_1 + K_2}x_1$$



This equation describes the system formed when the two springs are replaced by a single spring for:

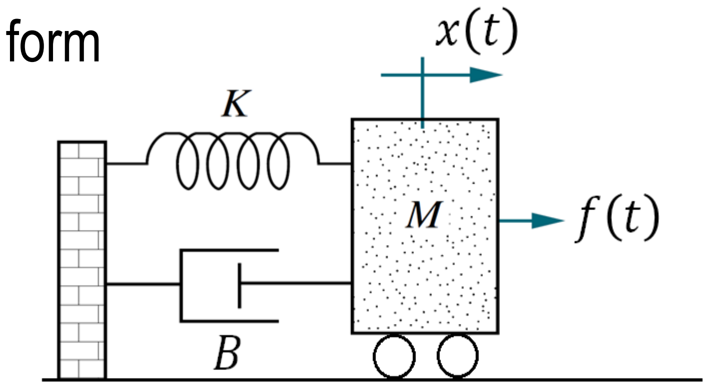
$$\boxed{K_{eq} = \frac{K_1 K_2}{K_1 + K_2}}$$

$$f_a(t) = M\ddot{x}_1 + B\dot{x}_1 + K_{eq}x_1$$

Modeling of Translational Mechanical Systems

❑ Solving the Equation of Motion

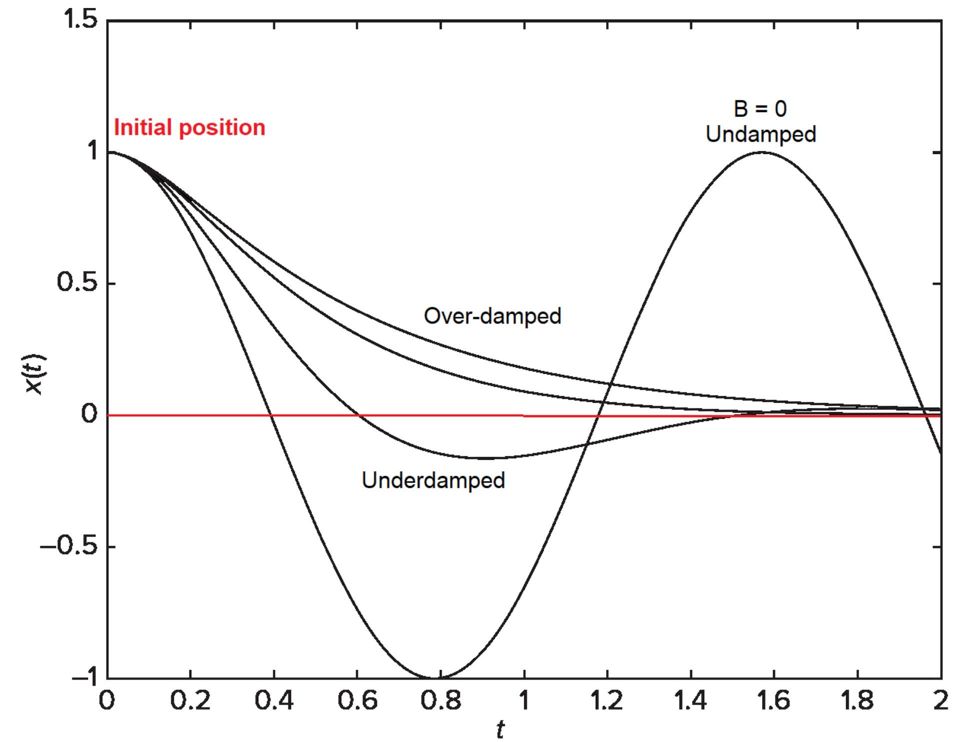
- We have seen that the equation of motion of mass-spring-damper systems has the general form of second-order ODE,



$$\boxed{M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f(t)}$$

where $f(t)$ is an applied external force other than gravity.

- Solution of the equation of motion mainly depends on the value of the damping element.



- Assume that the applied force $f(t)$ is zero, and we slightly moved and released the mass with the initial condition of $x(0) = 1, \dot{x}(0) = 0$
  - For no damping or friction $(\mathrm{B} = 0)$, the system is neutrally stable, and the mass oscillates with a constant amplitude and the natural frequency.
  - As damping is increased slightly, the system becomes stable, and mass still oscillates but with smaller frequency.
  - As the damping is increased further, the mass no longer oscillates because the damping force is large enough to limit the velocity and prevent the mass from overshooting the equilibrium position.

Modeling of Translational Mechanical Systems

Review of ODE Trial-Solution

- Table shows a summary of the trial-solution for the first-order and second-order ODE for constant input:

Equation	Solution Form
First order: $\dot{x}(t) + ax(t) = b, \quad a \neq 0$	$x(t) = \frac{b}{a} + Ce^{-at}$
Second order: $\ddot{x}(t) + a\dot{x}(t) + bx(t) = c, \quad b \neq 0$	
1. $(a^2 > 4b)$ distinct, real roots: $s_1,\ s_2$	$x(t) = \frac{c}{b} + C_1 e^{s_1 t} + C_2 e^{s_2 t}$
2. $(a^2 = 4b)$ repeated, real roots: $s_1,\ s_1$	$x(t) = \frac{c}{b} + (C_1 + C_2 t)e^{s_1 t}$
3. $(a = 0, b > 0)$ imaginary roots: $s = \pm j\omega_n, \quad \omega_n = \sqrt{b}$	$x(t) = \frac{c}{b} + C_1 \sin\omega_n t + C_2 \cos\omega_n t$
4. $(a \neq 0, a^2 < 4b)$ complex roots: $s = \sigma \pm j\omega_d$ $\sigma = -\frac{a}{2}, \quad \omega_d = \frac{1}{2}\sqrt{4b - a^2}$	$x(t) = \frac{c}{b} + e^{\sigma t}(C_1 \sin\omega_d t + C_2 \cos\omega_d t)$

- The coefficients C, C_1 and C_2 are determined from the given initial conditions $x(0)$ and $\dot{x}(0)$

Modeling of Translational Mechanical Systems

❑ Review of Types of Response

- The first-order ODE dynamic models has the following general form, where $f(t)$ is the input or forcing function, and $x(t)$ is the output variable.

$$\dot{x}(t) + ax(t) = f(t)$$

- If the **forcing input is a constant**, $f(t) = b$, and non-zero initial conditions $x(0)$, the general solution can be shown as

$$\dot{x}(t) + ax(t) = b \quad \rightarrow \quad x(t) = \frac{b}{a} + Ce^{at}$$

- The coefficient C is determined from the initial conditions at $t = 0$:

$$x(0) = \frac{b}{a} + C \quad \rightarrow \quad C = x(0) - \frac{b}{a}$$

- Therefore, the general solution can be rearranged and shown as two terms:

$$x(t) = \frac{b}{a} + \left(x(0) - \frac{b}{a}\right)e^{at} \qquad \rightarrow \qquad x(t) = \underbrace{x(0)e^{-at}}_{\substack{\text{Free response} \\ \text{Zero-input response}}} + \underbrace{\frac{b}{a}(1 - e^{-at})}_{\substack{\text{Forced response} \\ \text{Zero-initial condition response}}}$$

  - Free Response: The part of the response that depends on the initial conditions.
  - Forced Response: The part of the response due to the forcing function.

Modeling of Translational Mechanical Systems

Review of Types of Response

- Consider the general solution of first-order ODE:

$$x(t) = x(0)e^{-at} + \frac{b}{a}(1 - e^{-at})$$

- The general solution can also be rearranged as follows to distinguish the transient and the steady-state responses:

$$x(t) = \underbrace{\frac{b}{a}}_{\text{Steady-state response}} + \underbrace{\left(x(0) - \frac{b}{a}\right)e^{-at}}_{\text{Transient Response}}$$

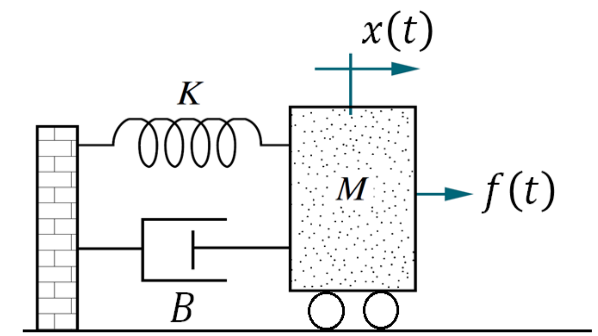
  - Transient Response: The part of the response that disappears with time.
  - Steady-state Response: The part of the response that remains with time.

- The same concepts are applicable for the response of the second-order ODE systems.

Modeling of Translational Mechanical Systems

Solving the Equation of Motion

Underdamped Response



In underdamped system, the damping force is not large enough to overcome the oscillation:

$$M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f(t)$$

Assume $M = 50\text{kg}$, $B = 30\ Ns/m$ and $K = 100N/m$ and the applied force is $f(t) = 100N$:

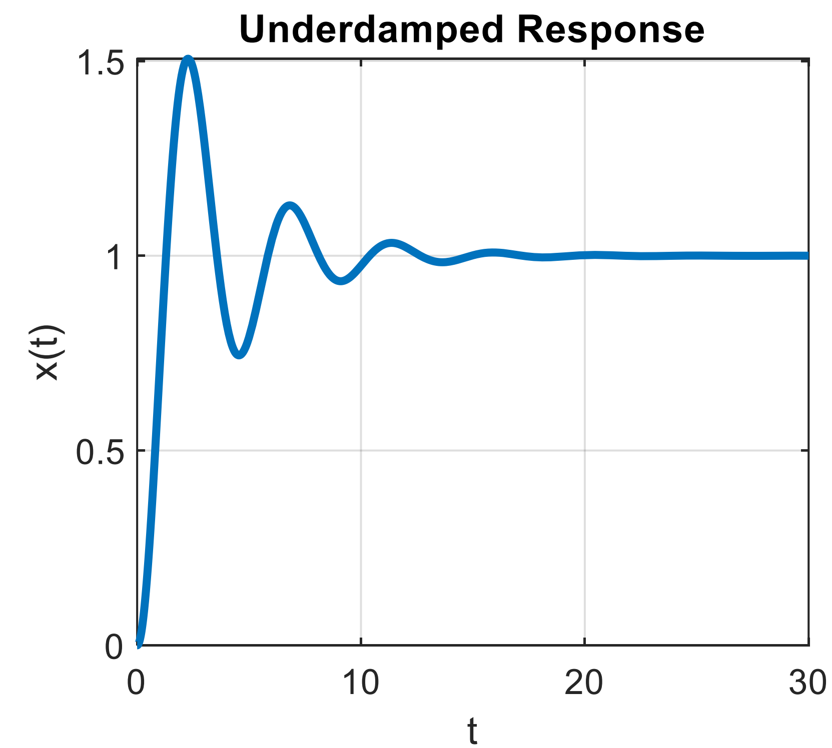
$$50\ddot{x}(t) + 30\dot{x}(t) + 100x(t) = 100 \quad \rightarrow \quad \ddot{x}(t) + 0.6\dot{x}(t) + 2x(t) = 2$$

- Solution of the equation of motion for zero initial condition $x(0) = 0,\ \dot{x}(0) = 0$:

Characteristic Equation $\rightarrow$ $s^2 + 0.6s + 2 = 0$

Characteristic Roots $\rightarrow$ $s_{1,2} = -0.3 \pm j1.38$ (Complex roots)

Solution $\rightarrow$ $$\boxed{x(t) = \underset{\text{Steady-state response}}{1} - \underbrace{e^{-0.3t}(0.217\sin(1.38\,t) - \cos(1.38\,t))}_{\text{Transient Response}}, \qquad t \geq 0}$$

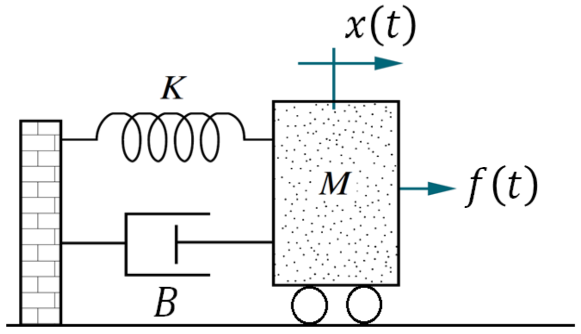


This solution shows that the mass oscillates with the damped frequency ω_d , and the transient response will disappear with time due to the exponential term.

Modeling of Translational Mechanical Systems

❑ Solving the Equation of Motion

○ Over-damped Response



In over-damped system, the damping force is large enough to limit the velocity and avoid the overshoot:

$$M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f(t)$$

Assume $M = 50\text{kg}$, $B = 300\ Ns/m$ and $K = 100N/m$ and the applied force is $f(t) = 100N$:

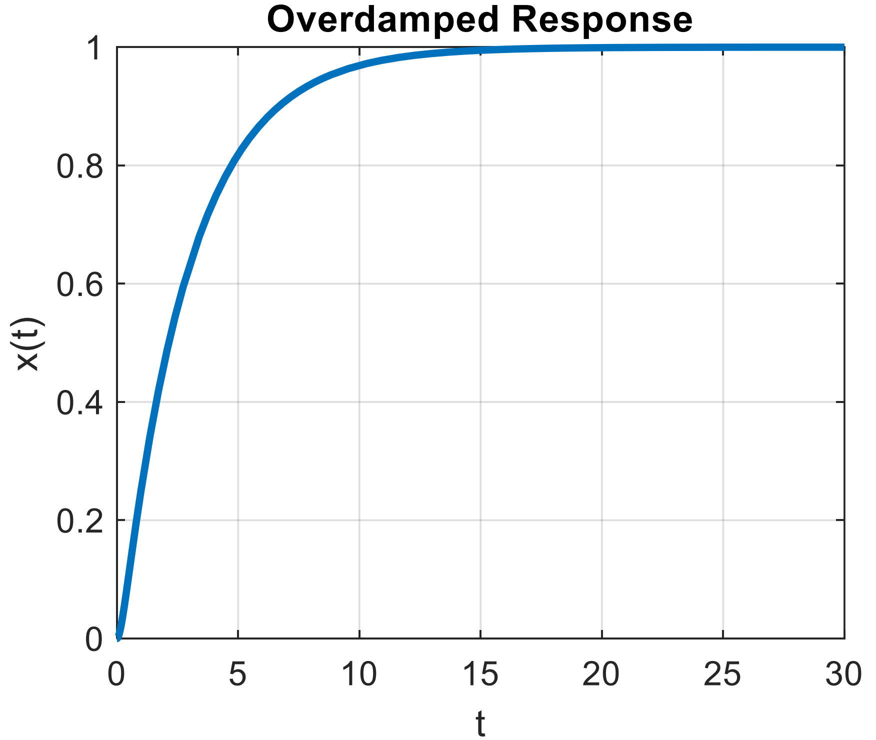
$$50\ddot{x}(t) + 300\dot{x}(t) + 100x(t) = 100 \quad \rightarrow \quad \ddot{x}(t) + 6\dot{x}(t) + 2x(t) = 2$$

- Solution of the equation of motion for zero initial condition $x(0) = 0,\ \dot{x}(0) = 0$:

Characteristic Equation $\rightarrow$ $s^2 + 6s + 2 = 0$

Characteristic Roots $\rightarrow$ $s_1 = -5.65,\ s_2 = -0.35$ (Distinct, real roots)

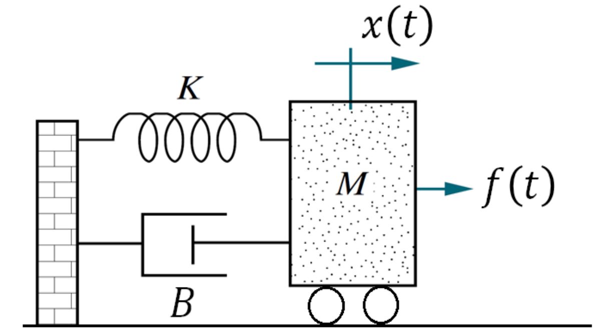
Solution $\rightarrow$ $$\boxed{x(t) = \underbrace{1}_{\text{Steady-state response}} + \underbrace{0.067e^{-5.65t} - 1.067e^{-0.35t}}_{\text{Transient Response}}, \qquad t \geq 0}$$



The transient response consists of exponential terms, and the steady-state response is 1.

Modeling of Translational Mechanical Systems

❑ Solving the Equation of Motion



○ No Damping (Undamped): B = 0

When there is no damping or friction, the system will be a simple mass-spring system:

$$M\ddot{x}(t) + Kx(t) = f(t)$$

Assume $M = 50\text{kg}$, $B = 0$ and $K = 100N/m$ and the applied force is $f(t) = 100N$:

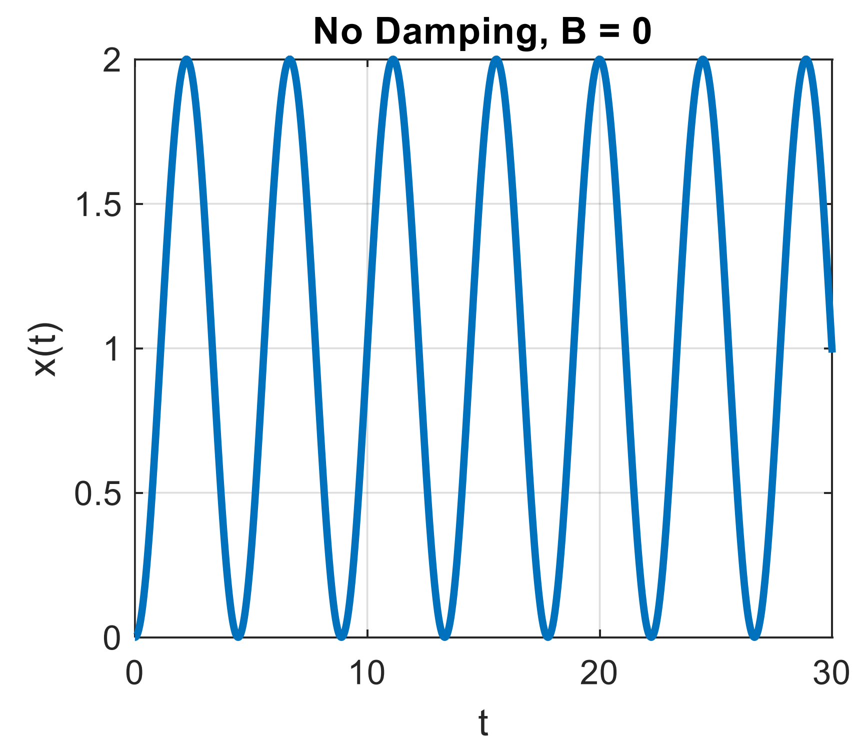
$$50\ddot{x}(t) + 100x(t) = 100 \quad \rightarrow \quad \ddot{x}(t) + 2x(t) = 2$$

Solution of the equation of motion for zero initial condition $x(0) = 0,\ \dot{x}(0) = 0$:

Characteristic Equation $\rightarrow \quad s^2 + 2 = 0$

Characteristic Roots $\rightarrow \quad s = \pm j\sqrt{2}$ (Imaginary roots)

Solution $\rightarrow \quad \boxed{x(t) = 1 - \cos(\sqrt{2}\,t)}$



This solution shows that the mass oscillates with the natural frequency ω_n , and the transient response will not disappear with time.

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{100}{50}} = \sqrt{2} \ \ \text{rad/s}$$

Modeling of Translational Mechanical Systems

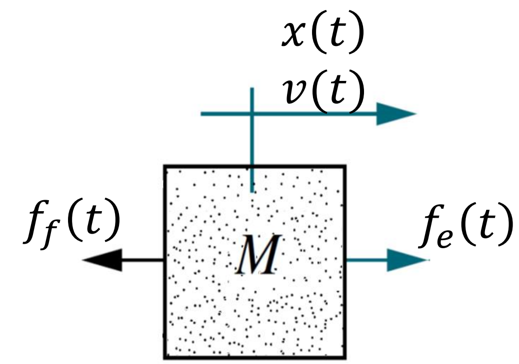
Example 4 Consider the following simple cruise system. Assume that the engine applies a forward force of $f_e(t)$ and air friction is proportional to the car's speed $v(t)$.

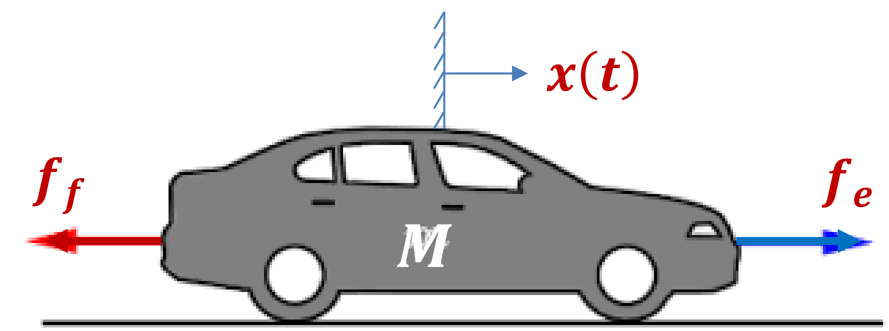
To obtain a system model we draw free-body diagrams

Apply Newton's second law as below

$$\sum f_{ext} = ma \quad \rightarrow \quad f_e(t) - f_f(t) = Ma(t)$$

$$f_e(t) - Bv(t) = Mv'(t)$$





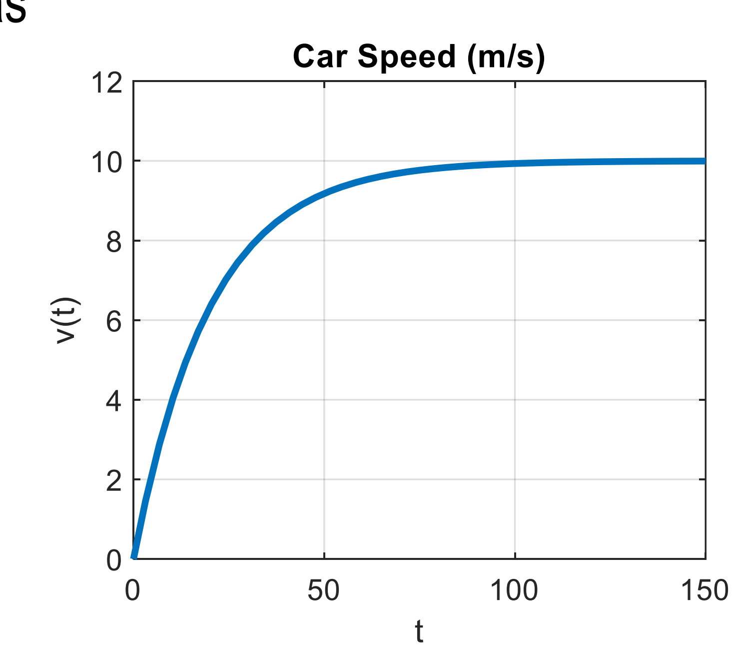
Input: Applied force $f_e(t)$

Output: Car speed $v(t)$

The differential equation relating speed of the car $v(t)$ to the engine force $f_e(t)$ is determined as

$$\boxed{f_e(t) = Mv'(t) + Bv(t)}$$

First-order differential equation



Assume $M = 1000\text{kg}$ and $B = 50Ns/m$ we have:

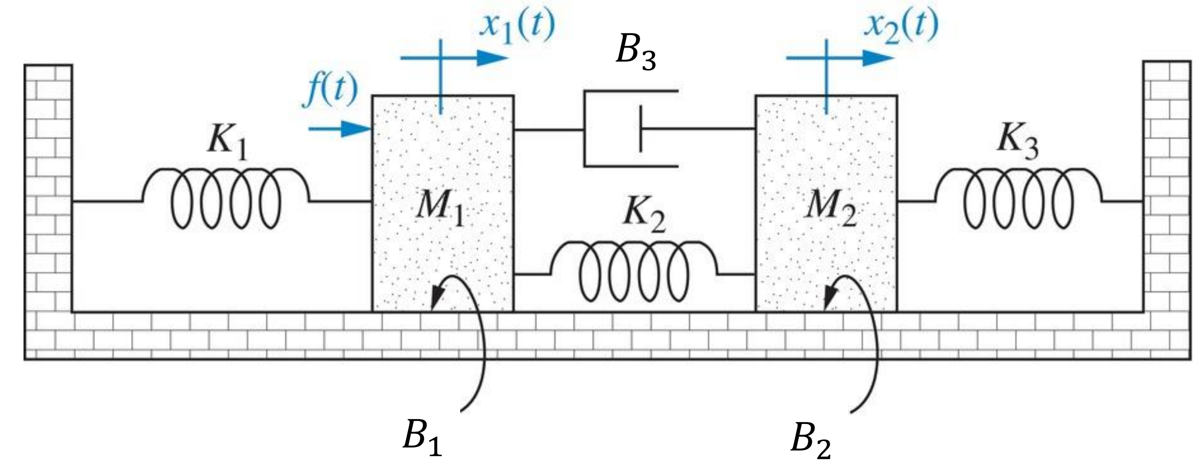
$$\boxed{f_e(t) = 1000v'(t) + 50v(t)}$$

Assume applied force is $f(t) = 500N$, and initial speed is zero. The solution of the first-order differential equation is:

$$\boxed{v(t) = 10(1 - e^{-t/20})}$$

Modeling of Translational Mechanical Systems

Example 5 Draw the free-body diagrams and use Newton's second law to write two modeling equations for the given two-mass system.

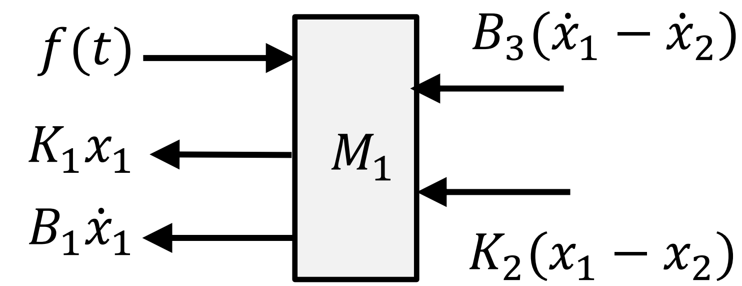


The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still.

Draw the free-body diagram of mass body M_1 and M_2.

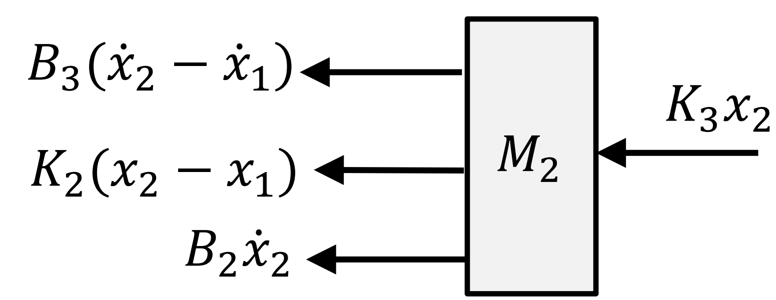
Mass M_1 $\rightarrow$ $f(t) - K_1 x_1 - B_1 \dot{x}_1 - B_3(\dot{x}_1 - \dot{x}_2) - K_2(x_1 - x_2) = M_1 \ddot{x}_1$

Mass M_2 $\rightarrow$ $- B_3(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) - B_2 \dot{x}_2 - K_3 x_2 = M_2 \ddot{x}_2$



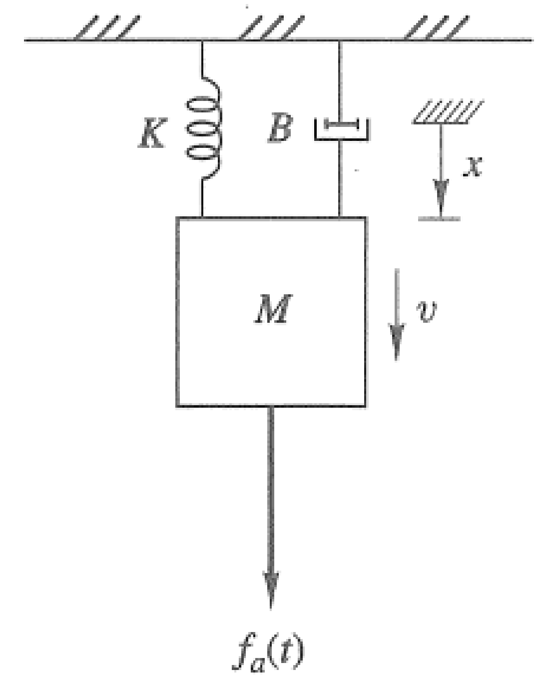
Rearrange the equations to find the equations of motion:

$$
\begin{cases}
\text{Mass } M_1 \rightarrow \quad M_1 \ddot{x}_1 + (B_1 + B_3)\dot{x}_1 + (K_1 + K_2)x_1 - B_3 \dot{x}_2 - K_2 x_2 = f(t) \\
\text{Mass } M_2 \rightarrow \quad M_2 \ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + (K_2 + K_3)x_2 - B_3 \dot{x}_1 - K_2 x_1 = 0
\end{cases}
$$



Modeling of Translational Mechanical Systems

❑ Choosing the Equilibrium Position as Coordinate Reference



- Assume the mass-spring-damper system in vertical motion.
- Draw the free-body diagram of mass body M including the gravitational force.

Mass M $\rightarrow$ $f_a(t) + Mg - Kx - B\dot{x} = M\ddot{x}$ $\quad\rightarrow\quad$ $f_a(t) + Mg = M\ddot{x} + B\dot{x} + Kx$

- Suppose that the applied force $f_a(t)$ is zero and the mass is not moving.

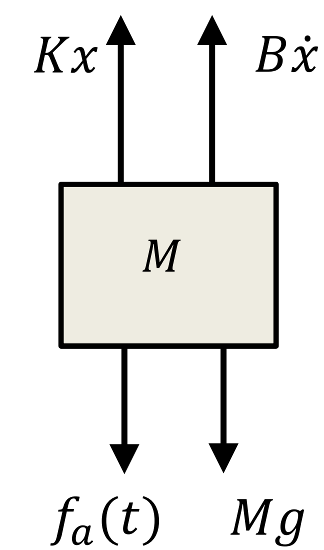
  Then $x = x_0$ is the constant displacement caused by the gravitational force.

  Since, $\dot{x}_0 = \ddot{x}_0 = 0$ the equation of motion is $\rightarrow$ $Mg = Kx_0$

- Now reconsider the case when $f_a(t)$ is nonzero and the mass is moving.

  Then $x = x_0 + z$, where z is the additional displacement caused by $f_a(t)$

$$f_a(t) + Mg = M\ddot{z} + B\dot{z} + K(x_0 + z) \quad \rightarrow \quad f_a(t) = M\ddot{z} + B\dot{z} + Kz$$



- For a mass connected to a spring element, the force due to gravity Mg is canceled out of the equation of motion by the force in the spring due to its static deflection, as long the displacement of the mass is measured from the equilibrium position.
- This means that if we are defined the vertical displacement from the static-equilibrium position caused by the gravitational force, then no need to show the Mg force in our free-body diagram.

Modeling of Translational Mechanical Systems

Displacement Input

Example 6

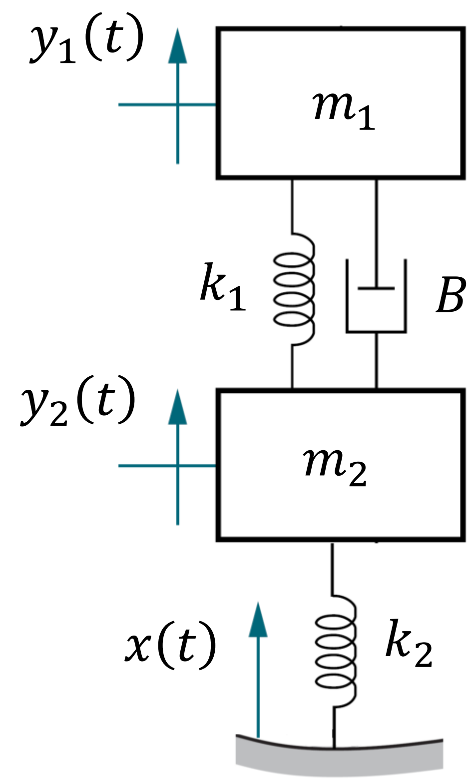
The suspension system for one wheel of an old-fashioned pickup truck is shown in the figure. The mass of the vehicle is m_1 and the mass of the wheel is m_2. The suspension spring has a spring constant k_1 and the tire has a spring constant k_2. The damping constant of the shock absorber is B. Derive the equations of motion for m_1 and m_2 in terms of the displacements from equilibrium, y_1 and y_2 with $x(t)$ as the input.

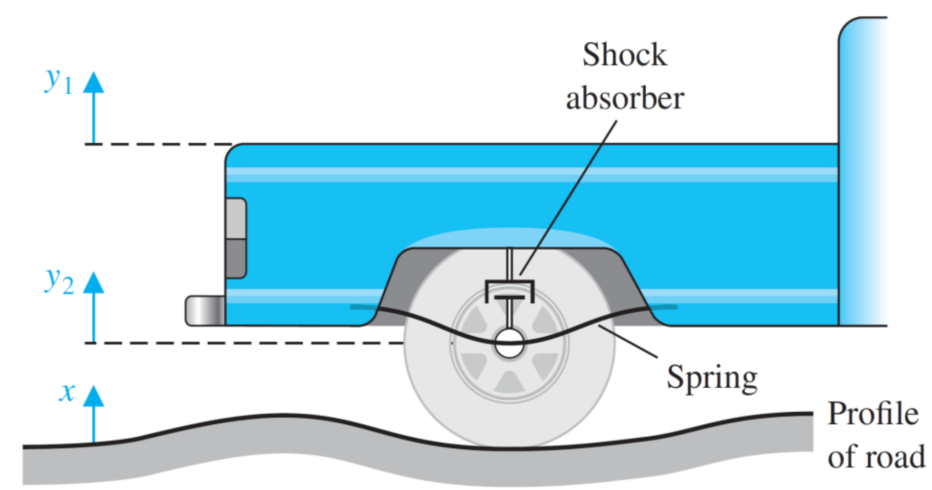
Assume that the mass is traveling <u>upward</u>, the suspension system can be modeled as a following mass-spring-damper system.

The system has two degrees of freedom, since each mass can be moved in the vertical direction while the other is held still.

Draw the free-body diagram of the system for each mass.

Place all the forces felt by the mass.

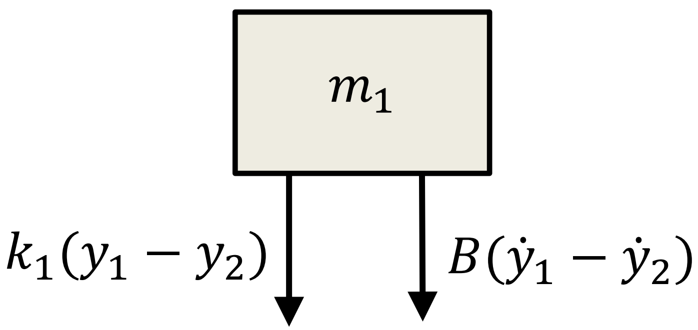


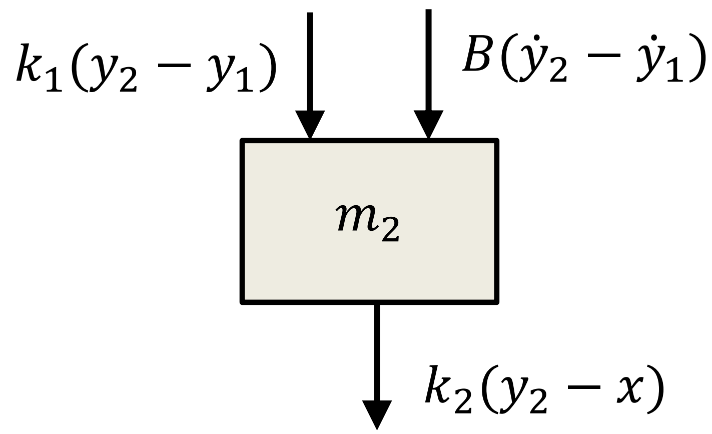


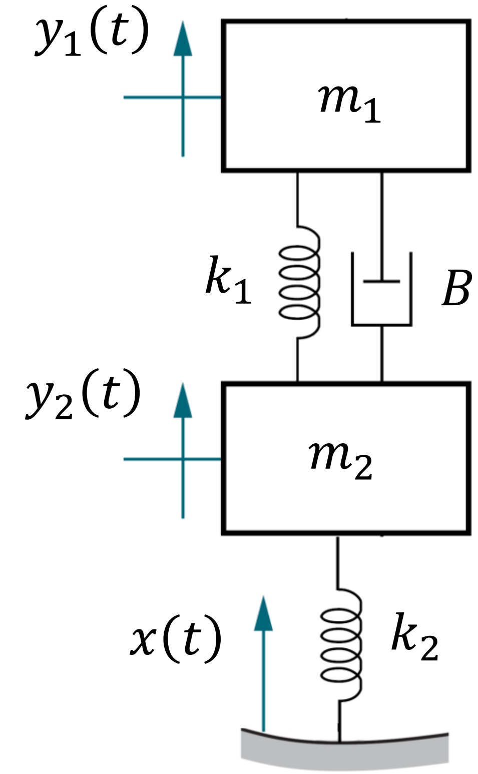
Modeling of Translational Mechanical Systems

Example 6

Draw the free-body diagram of mass body m_1 and m_2.







For mass m_1 $\rightarrow$ $-k_1(y_1 - y_2) - B(\dot{y}_1 - \dot{y}_2) = m_1\ddot{y}_1$ **Eqn.1**

For mass m_2 $\rightarrow$ $-k_1(y_2 - y_1) - B(\dot{y}_2 - \dot{y}_1) - k_2(y_2 - x) = m_2\ddot{y}_2$ **Eqn. 2**

Note that since the gravitational force is canceled out by the static spring force, no need to include gravity in the equations.

Rearrange the equations to find the equations of motion:

For mass m_1 $\rightarrow$ $m_1\ddot{y}_1 + B\dot{y}_1 + k_1 y_1 - B\dot{y}_2 - k_1 y_2 = 0$

For mass m_2 $\rightarrow$ $m_2\ddot{y}_2 + B\dot{y}_2 + (k_1 + k_2)y_2 - B\dot{y}_1 + k_1 y_1 - k_2 x = 0$

THANK YOU

HUMBER

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