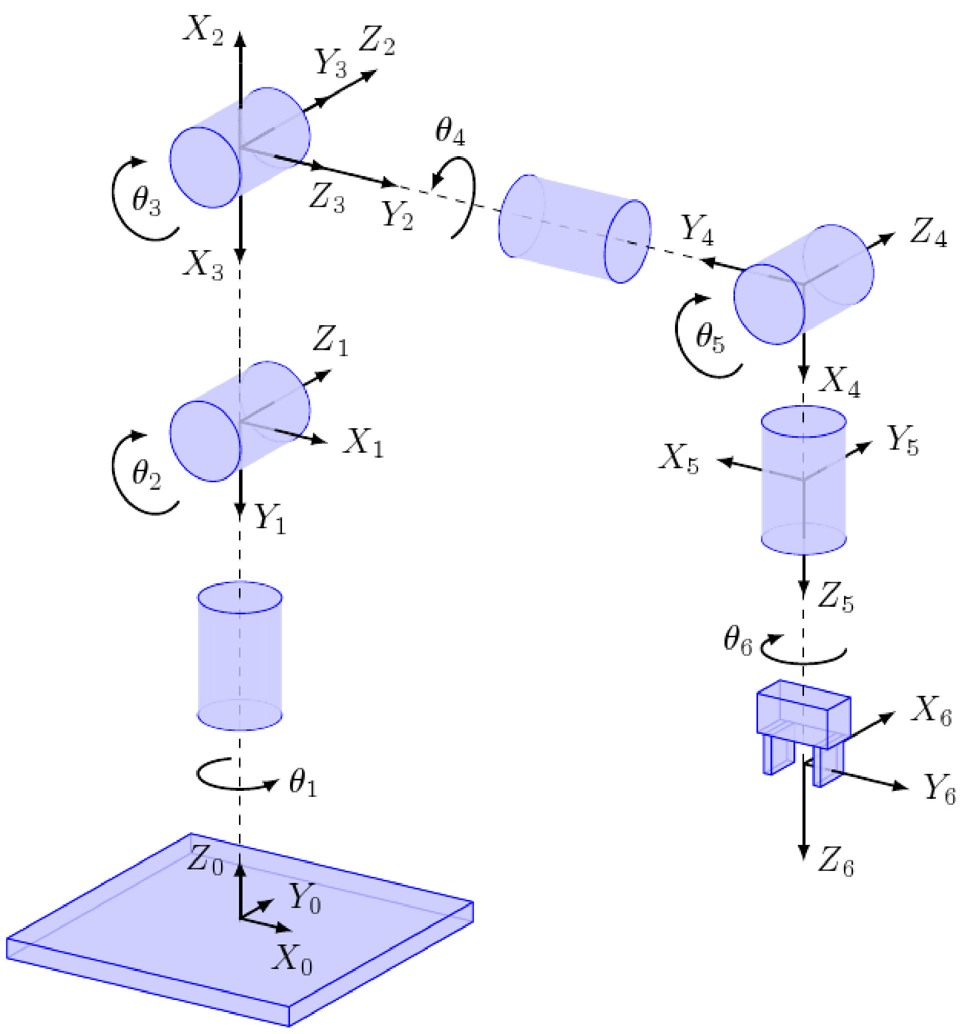
Kinematics and Dynamics of Robots

Module 12

Path-Planning and Trajectory Generation



Path-Planning and Trajectory Generation

- ***Path-Planning:*** Find out where in the space the end-effector move for its travel between two points which we call the ***Path***. Path Planning is the problem of planning collision free paths for the robot. The path planning problem is among the most difficult problems in computer science.
- Path planning provides a geometric description of robot motion, but it does not specify any dynamic aspects of the motion. For example, what should be the joint velocities and accelerations while traversing the path? These questions are addressed by a Trajectory Planning.
- ***Trajectory:*** Find the velocity components of the end-effector motion along the path.

Path-Planning and Trajectory Generation

- The trajectory planner computes a function *q(t)* that completely specifies the motion of the robot as it traverses the path.
- A trajectory is a function of time *q(t)* such that $q(t_0) = q_{init}$ and $q(t_f) = q_{final}$.
- In this case $t_f - t_0$ represents the amount of time taken to execute the trajectory. Since the trajectory is parameterized by time, we can compute velocities and accelerations along the trajectories by differentiation.
- We first consider **point to point** motion. In this case the task is to plan a trajectory from $q(t_0)$ to $q(t_f)$, i.e., the path is specified by its initial and final configurations. In some cases, there may be constraints on the trajectory (e.g., if the robot must start and end with zero velocity). Nevertheless, it is easy to realize that there are infinitely many trajectories that will satisfy a finite number of constraints on the endpoints. It is common practice therefore to choose trajectories from a finitely parameterizable family, for example, polynomials of degree *n*, with *n* dependent on the number of constraints to be satisfied. Once we have seen how to construct trajectories between two configurations, it is straightforward to generalize the method to the case of trajectories specified by multiple via points.

Parametric Equation

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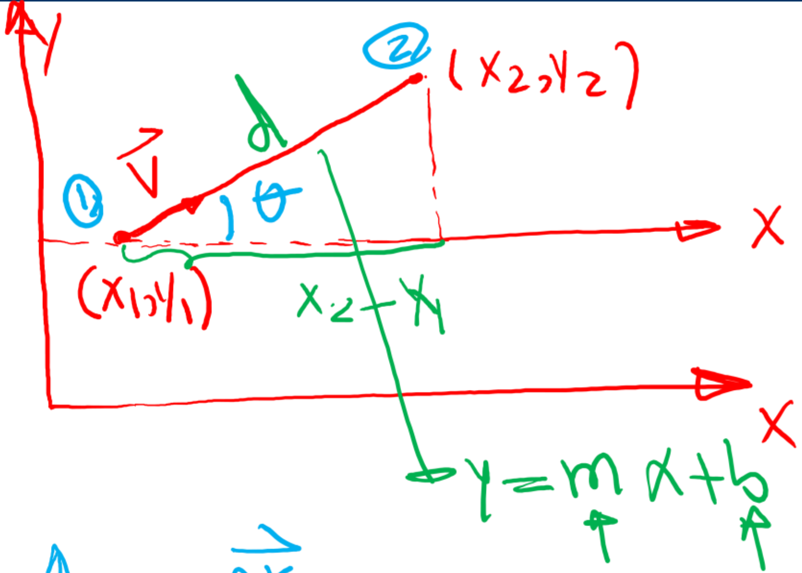

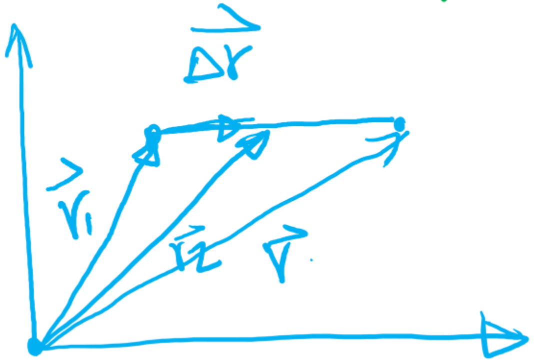

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d(xi + yj)}{dt} = \dot{x}i + \dot{y}j$$

The projection of $\vec{V}$ on x axis

The element of v along y

$$\vec{r} = xi + yj$$

$$|\dot{x}| = v\cos\theta$$

$$\cos\theta = \frac{x_2 - x_1}{d} = \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$$\sin\theta = \frac{y_2 - y_1}{d} = \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$$\dot{x} = V\cos\theta = \frac{V(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$$x = \dot{x}t + x_1 = \frac{V(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}t + x_1 = g_1(t)$$

$$\dot{y} = V\sin\theta = \frac{V(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$$y = \dot{y}t + y_1 = \frac{v(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}t + y_1 = g_2(t)$$

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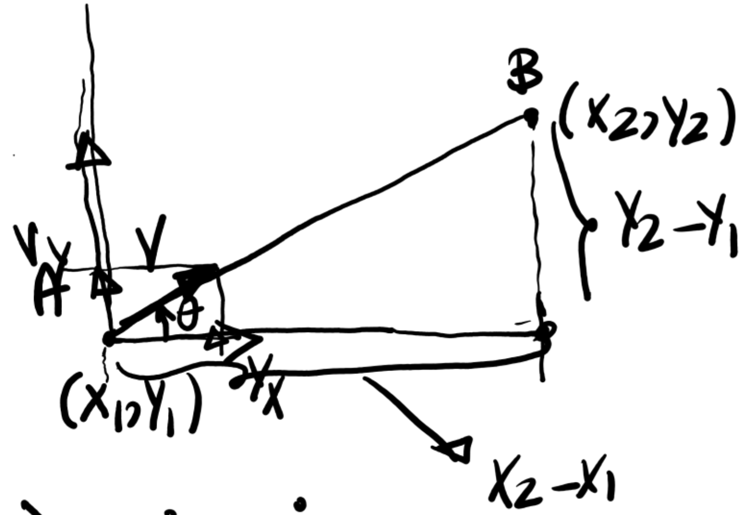


$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \Rightarrow y = mx + b$$

$$x = V_x t + x_0$$

$$\frac{dx}{dt} = V_x \Rightarrow dx = V_x dt \Rightarrow (x - x_0) = V_x (t - t_0)$$

$$x = V_x t + x_0 \quad ②$$

$$y = V_y t + y_0 \quad ④$$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\frac{d\vec{r}}{dt} = \dot{x}\,i + \dot{y}\,j = \frac{dx}{dt}\,i + \frac{dy}{dt}\,j$$

① $V_x = V \cdot \cos\theta = v \dfrac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$

① in ②: $x = \dfrac{(x_2 - x_1)\, v}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\, t + x_0 = x(t)$

③ in ④: $y = \dfrac{(y_2 - y_1)\, v}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\, t + y_0 = y(t)$

$\dot{x} = V_x =$

③ $V_y = V \cdot \sin\theta = v \dfrac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$

HUMBER

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J \begin{bmatrix} \dot{q}_1 = \dot{\theta}_1 \\ \dot{q}_2 = \dot{d}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Finding the end-effector velocity based on joint variables velocities

$$J^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J^{-1} J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = J^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$x = f(t) = x(t)$

$y = g(t) = y(t)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

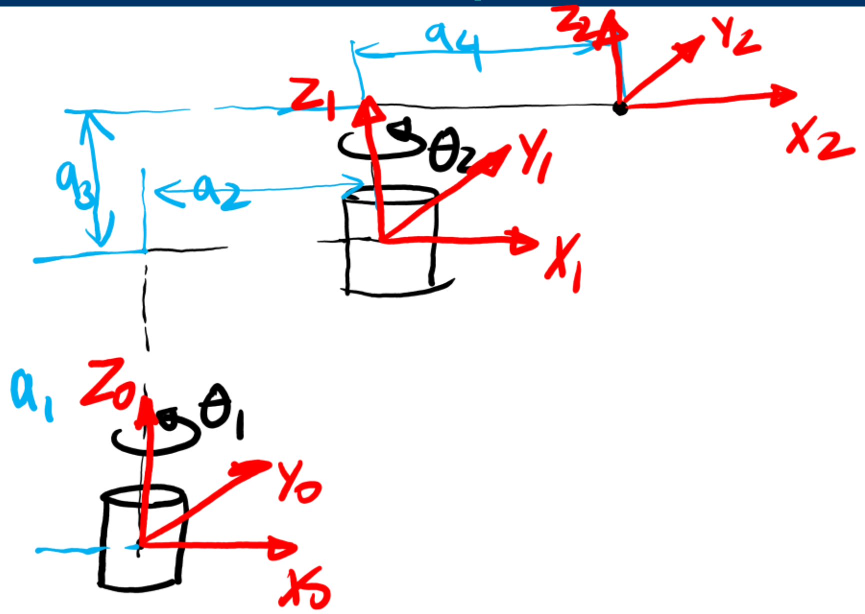
$\dot{q}_1 \leftarrow$ Joint variable velocities

End effector velocity components

The Jacobian matrix let us calculate the end-effector velocities when we know Joint variable velocities. The purpose is to find the reverse meaning that we need to find joint variables' velocities based on end-effector velocity.

$$J^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J^{-1} J \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \Rightarrow J^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Parametric Equation



$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{6\times 1} = \underbrace{\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ J_{31} & J_{32} \\ \square & \square \\ \square & \square \\ J_{61} & J_{62} \end{bmatrix}}_{6\times 2} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{2\times 1} \Rightarrow \begin{aligned} \dot{x} &= J_{11}\dot{\theta}_1 + J_{12}\dot{\theta}_2 \\ \dot{y} &= J_{21}\dot{\theta}_1 + J_{22}\dot{\theta}_2 \end{aligned}$$

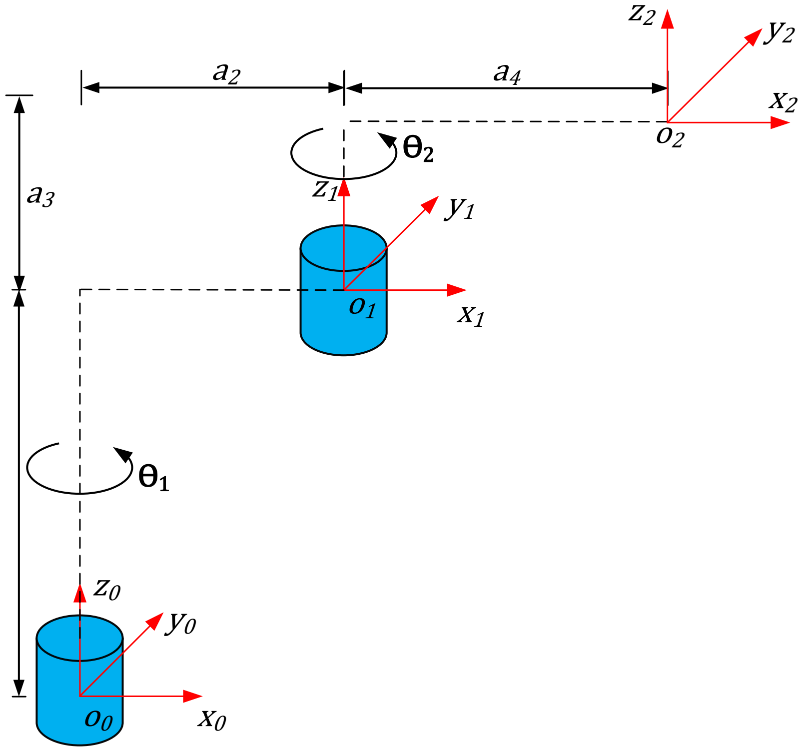
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}}_{J_{2\times 2}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \Rightarrow J^{-1}_{2\times 2} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J^{-1} J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = J^{-1}_{2\times 2} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

HUMBER

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(A) = ad - bc$$

Parametric Equation

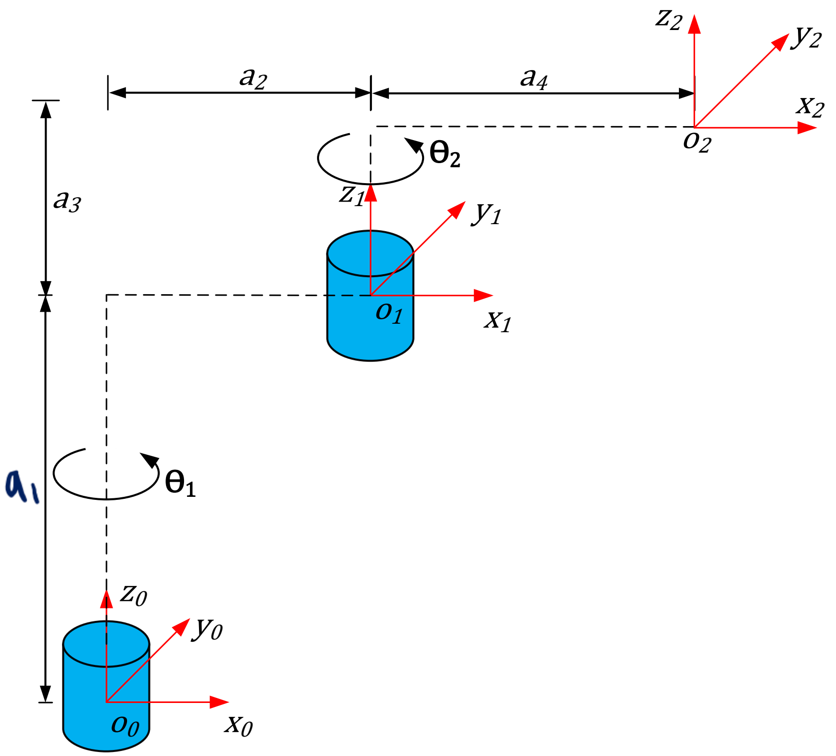


	Pris	Revolute
linear	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d^0_n - d^0_{i-1})$
Rot	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$

$$J_{2\times 6} = \begin{bmatrix} \underset{I}{R^0_0}\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d^0_2 - d^0_0) & R^0_1 \times (d^0_2 - d^0_1) \\ \underset{I}{R^0_0}\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \begin{matrix}0\\0\\1\end{matrix} & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\square & \square\\ \square & \square\\ \square & \square\\ \square & \square\\ \square & \square\\ \square & \square\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

Parametric Equation



$$R^0_1 = R_{z_0,\theta_1}\cdot \underbrace{R^0_1}_{I} = R_{z_0,\theta_1} = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d^0_1 = R_{z_0,\theta_1}\begin{bmatrix} a_2 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_2C\theta_1 \\ a_2S\theta_1 \\ a_1 \end{bmatrix}$$

$$H^0_1 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & a_2C\theta_1 \\ S\theta_1 & C\theta_1 & 0 & a_2S\theta_1 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R^0_1

d^0_1

$R^0_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$H^1_2 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_4C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_4S\theta_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H^0_2 = H^0_1 H^1_2 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & a_2C\theta_1 \\ S\theta_1 & C\theta_1 & 0 & a_2S\theta_1 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_4C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_4S\theta_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 H_2^1 = \begin{bmatrix} \overbrace{\begin{matrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{matrix}}^{R_1^0} & \overbrace{\begin{matrix} a_2 C\theta_1 \\ a_2 S\theta_1 \\ a_1 \end{matrix}}^{d_1^0} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} \overbrace{\begin{matrix} C\theta_2 & -S\theta_2 & 0 \\ S\theta_2 & C\theta_2 & 0 \\ 0 & 0 & 1 \end{matrix}}^{R_2^1} & \begin{matrix} a_4 C\theta_2 \\ a_4 S\theta_2 \\ a_3 \end{matrix} \rightarrow d_2^1 \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} \overbrace{\begin{matrix} C\theta_1 C\theta_2 - S\theta_1 S\theta_2 & -C\theta_1 S\theta_2 - S\theta_1 C\theta_2 & 0 \\ S\theta_1 C\theta_2 + C\theta_1 C\theta_2 & -S\theta_1 S\theta_2 + C\theta_1 C\theta_2 & 0 \\ 0 & 0 & 1 \end{matrix}}^{R_2^0} & \overbrace{\begin{matrix} a_4 C\theta_1 C\theta_2 - a_4 S\theta_1 S\theta_2 + a_2 C\theta_1 \\ a_4 S\theta_1 C\theta_2 + a_4 C\theta_1 S\theta_2 + a_2 S\theta_1 \\ a_3 + a_1 \end{matrix}}^{d_2^0} \\ 0 \qquad 0 \qquad 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} C(\theta_1+\theta_2) & -S(\theta_1+\theta_2) & 0 & a_4 C(\theta_1+\theta_2) + a_2 C\theta_1 \\ S(\theta_1+\theta_2) & C(\theta_1+\theta_2) & 0 & a_4 S(\theta_1+\theta_2) + a_2 S\theta_1 \\ 0 & 0 & 1 & a_3 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} a_4 C(\theta_1+\theta_2) + a_2 C\theta_1 \\ a_4 S(\theta_1+\theta_2) + a_2 S\theta_1 \\ a_3 + a_1 \end{matrix} \rightarrow d_2^0$$

$$\boxed{\begin{aligned} \sin(\theta_1 \pm \theta_2) &= \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2 \\ \cos(\theta_1 \pm \theta_2) &= \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2 \end{aligned}}$$

R_2^1, R_1^0, d_1^0, $\to d_2^1$

$$H_2^0 = \begin{bmatrix} C\theta_1 C\theta_2 - S\theta_1 S\theta_2 & -C\theta_1 S\theta_2 - S\theta_1 C\theta_2 & 0 & a_4 C\theta_1 C\theta_2 - a_4 S\theta_1 S\theta_2 + a_2 C\theta_1 \\ S\theta_1 C\theta_2 + C\theta_1 C\theta_2 & -S\theta_1 S\theta_2 + C\theta_1 C\theta_2 & 0 & a_4 S\theta_1 C\theta_2 + a_4 C\theta_1 S\theta_2 + a_2 S\theta_1 \\ 0 & 0 & 1 & a_3 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R_2^0, d_2^0

$$H_2^0 = \begin{bmatrix} C(\theta_1+\theta_2) & -S(\theta_1+\theta_2) & 0 & a_4 C(\theta_1+\theta_2) + a_2 C\theta_1 \\ S(\theta_1+\theta_2) & C(\theta_1+\theta_2) & 0 & a_4 S(\theta_1+\theta_2) + a_2 S\theta_1 \\ 0 & 0 & 1 & a_3 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

	Prismatic	Rev
linear	$R_{i-1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_n^0 - d_{i-1}^0)$
Rot	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix}$

i = joint number, n = total number of joints

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} R_{i-1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_2^0 - d_0^0) & R_{1}^0 \begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_2^0 - d_1^0) \\ R_0^0 \begin{bmatrix}0\\0\\1\end{bmatrix} & R_1^0 \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

joint 1 joint 2

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \times \underbrace{\begin{bmatrix} a_4 C(\theta_1+\theta_2) + a_2 C\theta_1 \\ a_4 S(\theta_1+\theta_2) + a_2 S\theta_1 \\ a_3 + a_1 \end{bmatrix}}_{d_2^0} & \begin{bmatrix}0\\0\\1\end{bmatrix} \times \left(\underbrace{\begin{bmatrix} a_4 C(\theta_1+\theta_2) + a_2 C\theta_1 \\ a_4 S(\theta_1+\theta_2) + a_2 S\theta_1 \\ a_3 + a_1 \end{bmatrix}}_{d_2^0} - \underbrace{\begin{bmatrix} a_2 C\theta_1 \\ a_2 S\theta_1 \\ a_1 \end{bmatrix}}_{d_1^0} \right) \\ \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix} a_4 C(\theta_1+\theta_2) + a_2 C\theta_1 \\ a_4 S(\theta_1+\theta_2) + a_2 S\theta_1 \\ a_3 + a_1 \end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix} a_4 C(\theta_1+\theta_2) \\ a_4 S(\theta_1+\theta_2) \\ a_3 \end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix}\dot x\\ \dot y\\ \dot z\\ w_x\\ w_y\\ w_z\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}a_4 C(\theta_1+\theta_2)+a_2 C\theta_1\\ a_4 S(\theta_1+\theta_2)+a_2 S\theta_1\\ a_3+a_1\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(\begin{bmatrix}a_4 C(\theta_1+\theta_2)+a_2 C\theta_1\\ a_4 S(\theta_1+\theta_2)+a_2 S\theta_1\\ a_3+a_1\end{bmatrix} - \begin{bmatrix}a_2 C\theta_1\\ a_2 S\theta_1\\ a_1\end{bmatrix}\right) \\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}a_4 C(\theta_1+\theta_2)+a_2 C\theta_1\\ a_4 S(\theta_1+\theta_2)+a_2 S\theta_1\\ a_3+a_1\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}a_4 C(\theta_1+\theta_2)\\ a_4 S(\theta_1+\theta_2)\\ a_3\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}$$

$$\begin{bmatrix}\dot x\\ \dot y\\ \dot z\\ w_x\\ w_y\\ w_z\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}a_4 C(\theta_1+\theta_2)+a_2 C\theta_1\\ a_4 S(\theta_1+\theta_2)+a_2 S\theta_1\\ a_3+a_1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}a_4 C(\theta_1+\theta_2)\\ a_4 S(\theta_1+\theta_2)\\ a_3\end{bmatrix}\\ \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix} \overset{J_{11}}{-a_4 S(\theta_1+\theta_2)-a_2 S\theta_1} & \overset{J_{12}}{-a_4 S(\theta_1+\theta_2)}\\ \underset{J_{21}}{a_4 C(\theta_1+\theta_2)+a_2 C\theta_1} & \underset{J_{22}}{a_4 C(\theta_1+\theta_2)}\\ 0&0\\0&0\\0&0\\1&1\end{bmatrix}$$

$$\vec a \times \vec b = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{vmatrix} i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = i(a_2b_3-b_2a_3)-j(a_1b_3-a_3b_1)+k(a_1b_2-b_1a_2)$$

$$= \begin{bmatrix} a_2b_3-b_2a_3\\ -(a_1b_3-a_3b_1)\\ a_1b_2-b_1a_2\end{bmatrix} \overset{a_1=0,\ a_2=0,\ a_3=1}{=} \begin{bmatrix}-b_2\\ +b_1\\ 0\end{bmatrix}$$

Parametric Equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ w_x \\ w_y \\ w_z \end{bmatrix} = \underbrace{\begin{bmatrix} \underbrace{-a_4 S(\theta_1+\theta_2) - a_2 S\theta_1}_{J_{11}} & \underbrace{-a_4 S(\theta_1+\theta_2)}_{J_{12}} \\ \underbrace{a_4 C(\theta_1+\theta_2) + a_2 C\theta_1}_{J_{21}} & \underbrace{a_4 C(\theta_1+\theta_2)}_{J_{22}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}_{J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}}_{J_2} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = J_2^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

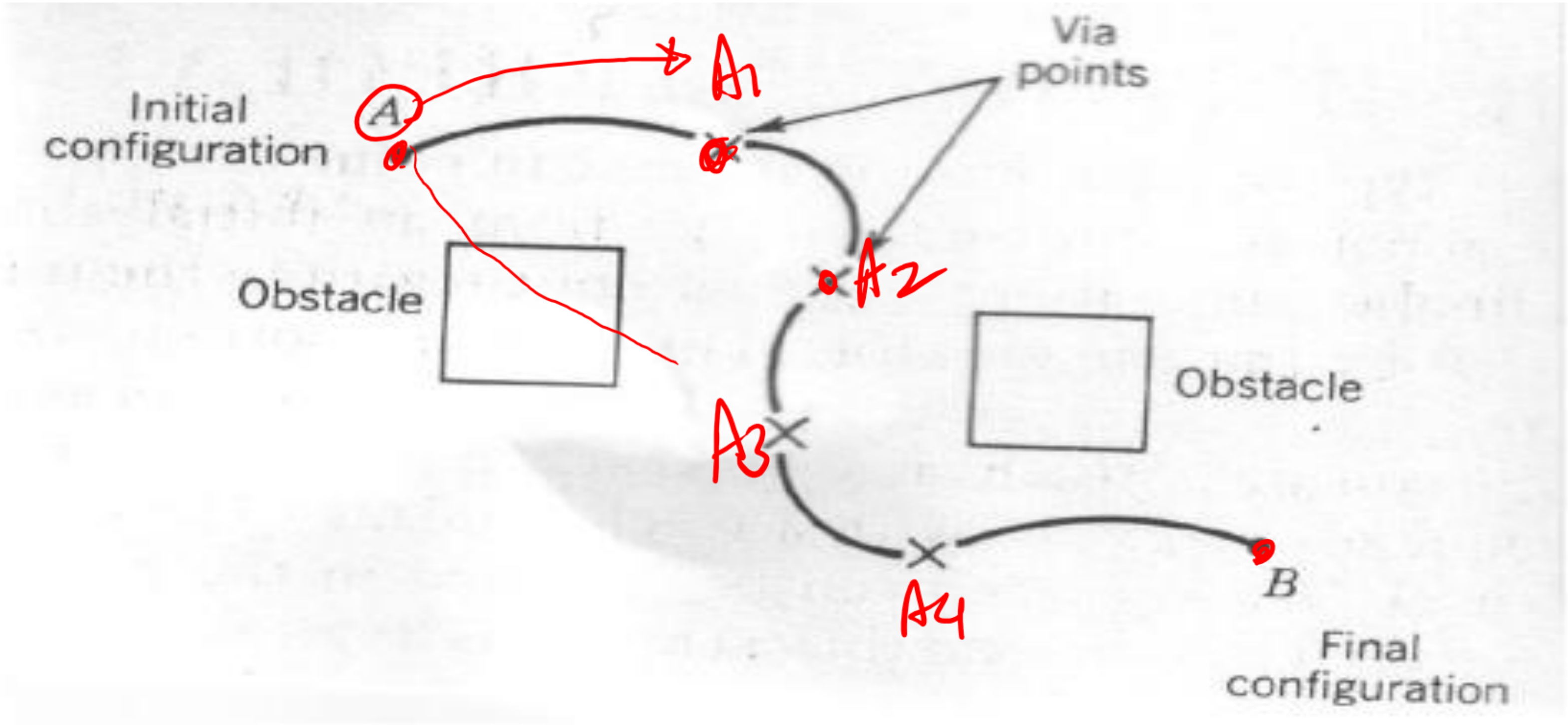
$$J_2^{-1} = \frac{1}{\det(J_2)} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = \frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{bmatrix} J_{22} & J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\begin{matrix}\dot{x} \\ \dot{y}\end{matrix}} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underbrace{\begin{bmatrix} -a_4 S(\theta_1+\theta_2) - a_2 S\theta_1 \; J_{11} & -a_4 S(\theta_1+\theta_2) \; J_{12} \\ a_4 C(\theta_1+\theta_2) + a_2 C\theta_1 \; J_{21} & a_4 C(\theta_1+\theta_2) \; J_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}_{J_2} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J_2^{-1} = \frac{1}{\det(J_2)} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = \frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = \frac{1}{(-a_4 S(\theta_1+\theta_2) - a_2 S\theta_1)(a_4 C(\theta_1+\theta_2)) - (-a_4 S(\theta_1+\theta_2))[a_4 C(\theta_1+\theta_2) + a_2 C\theta_1]}$$

$$\begin{bmatrix} a_4 C(\theta_1+\theta_2) & +a_4 S(\theta_1+\theta_2) \\ -a_4 C(\theta_1+\theta_2) - a_2 C\theta_1 & -a_4 S(\theta_1+\theta_2) - a_2 S\theta_1 \end{bmatrix}$$

f(t)


Initial configuration
A
A1
Via points
A2
Obstacle
A3
Obstacle
A4
B
Final configuration

Path-Planning and Trajectory Generation

HUMBER

- The problem here is to find a trajectory that connects an initial to a final configuration while satisfying other specified constraints at the endpoints (e.g., velocity and/or acceleration constraints).
- We will consider planning the trajectory for a single joint, since the trajectories for the remaining joints will be created independently and in exactly the same way.
- Thus, we will concern ourselves with the problem of determining *q(t),* where *q(t)* is a scalar joint variable.

A

B

$q(t)$

$\dot{q}(t)$

$\ddot{q}(t)$

$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$q(t_0) = \ldots$

$q(t_f) = \ldots$

$\dot{q}(t_0) = 0$

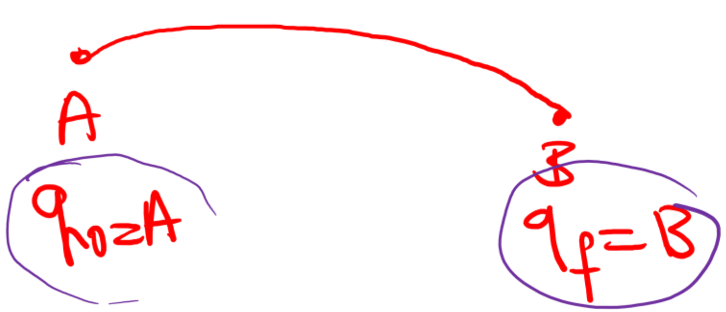
$\dot{q}(t_f) = 0$

$\ddot{q}(t_0) = \sim$

$\ddot{q}(t_f) \ldots$

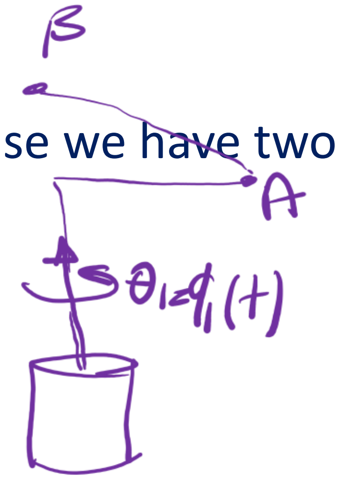
- Suppose that at time t_0 the joint variable the joint variable satisfies

$$
\begin{aligned}
&① \quad q(t_0) = q_0 \Rightarrow \text{intial Position @ point A} \\
&② \quad \dot{q}(t_0) = v_0 \Rightarrow \text{intial velocity @ poin A}
\end{aligned}
$$



- At time t_f

$$
\begin{aligned}
&③ \quad q(t_f) = q_f \Rightarrow \text{final position @ point B} \\
&④ \quad \dot{q}(t_f) = v_f = \text{final Velocity @ " "}
\end{aligned}
$$



- In addition, we may wish to specify the constraints on initial and final accelerations. In this case we have two the additional equations

$$
\begin{aligned}
&⑤ \quad \ddot{q}(t_0) = \alpha_0 \\
&⑥ \quad \ddot{q}(t_f) = \alpha_f
\end{aligned}
$$

Cubical Polynomial Approach

HUMBER

- In making a single smooth motion at least four constraints are required:
- Two constraints on the function value selection on the basis of initial and final position and the other two constraints are continuous velocity.

$$
\begin{aligned}
q(t_0) &= q_0 \\
\dot{q}(t_0) &= v_0 \\
q(t_f) &= q_f \\
\dot{q}(t_f) &= v_f
\end{aligned}
$$

- These four constrained can be satisfied by the polynomial of at least third degree.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \Rightarrow \dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$t_0 = 0$ $\quad q(0) = q_0 = a_0 + a_1 \times 0 + a_2 \times 0 + a_3 \times 0 = a_0 \Rightarrow a_0 = q_0$

$t_0 = 0$ $\quad \dot{q}(0) = v_0 = a_1 + 2a_2 \times 0 + 3a_3 \times 0 = a_1 \Rightarrow a_1 = v_0$

$t_f \Rightarrow q_f$ $\quad q(t_f) = q_f = q_0 + v_0 t_f + a_2 t_f^2 + a_3 t_f^3$

$t_f \Rightarrow v_f$ $\quad v_f = v_0 + 2a_2 t_f + 3a_3 t_f^2$

HUMBER

- Now consider a cubic trajectory of the form

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

- Then the desired velocity is given as

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

- Using these two equations with the four constraints yields four equations in four unknowns

$$\begin{aligned}
q_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \\
v_0 &= a_1 + 2a_2 t_0 + 3a_3 t_0^2 \\
q_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\
v_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2
\end{aligned}$$

HUMBER

$q(t) = \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ (a_0, a_1, a_2, a_3) unknown

$q(t_0) = \theta(t_0) = \theta_0$

$q(t_f) = \theta(t_f) = \theta_f$

$\dot{q}(t_0) = \dot{\theta}(t_0) = v_0$

$\dot{q}(t_f) = \dot{\theta}(t_f) = v_f$

$t_0 = 0$ $t_f = t_f - t_0 = T$ $t: \overset{t_0}{0} \longrightarrow \overset{t_f}{T}$

$v_0 = 0$ $v_f = 0$

$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$\dot{\theta}(t) = 0 + a_1 + 2a_2 t + 3a_3 t^2$

$\theta(0) = \theta_0 \Rightarrow$ ① $\theta(0) = a_0 + a_1 \times 0 + a_2 \times 0^2 + a_3 \times 0^3 = \boxed{a_0 = \theta_0}$

$\theta(T) = \theta_f \Rightarrow$ ② $\theta(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3 = \theta_f$

$\dot{\theta}(0) = 0 \Rightarrow$ ③ $\dot{\theta}(0) = a_1 + 2a_2 \times 0 + 3a_3 \times 0^2 = 0 \Rightarrow \boxed{a_1 = 0}$

$\dot{\theta}(T) = 0 \Rightarrow$ ④ $\dot{\theta}(T) = a_1 + 2a_2 \times T + 3a_3 \times T^2 = 0$

HUMBER

② $\theta(T) = \theta_0 + a_2 T^2 + a_3 T^3 = \theta_f$

④ $\dot{\theta}(T) = 2a_2 T + 3a_3 T^2 = 0 \Rightarrow T(2a_2 + 3a_3 T) = 0$

$\Rightarrow 2a_2 + 3a_3 T = 0 \Rightarrow \boxed{a_2 = \frac{-3a_3 T}{2}}$

Plug a_2 inside ②:

$\theta(T) = \theta_0 + \left(\frac{-3a_3}{2}\right) T \times T^2 + a_3 T^3 = \theta_f$

$\Rightarrow a_3 T^3 \left(-\frac{3}{2} + 1\right) = \theta_f - \theta_0 \Rightarrow a_3 = \frac{-2(\theta_f - \theta_0)}{T^3}$

$a_2 = \frac{-3}{2} T \times a_3 = \frac{-3\cancel{T}}{\cancel{2}} \times \frac{-\cancel{2}\,(\theta_f - \theta_0)}{T^{\cancel{3}\,2}} = \frac{3(\theta_f - \theta_0)}{T^2}$

$a_0 = \theta_0$

$a_1 = 0$

$a_2 = \frac{3}{T^2}(\theta_f - \theta_0)$

$a_3 = \frac{-2(\theta_f - \theta_0)}{T^3}$

- A single link robot with a rotary joint is motionless at Θ= 15 degree. It is desired to move the joint in a smooth manner to Θ= 75 degree in 3 seconds. Find the coefficients of a cubic polynomials which accomplishes this motion and brings the manipulator to rest at the goal. Plot the position, velocity, and acceleration of the joint as a function of time.

$q_0 = \theta_0 = 15$

$q_f = \theta_f = 75$

$t_0 = 0$

$t_f = 3$

$v_0 = 0$

$v_f = 0$

$q(t) = \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$\dot{q}(t) = \dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$

$q(0) = \theta(0) = a_0 = 15 = \theta_0$

$\dot{q}(0) = \dot{\theta}(0) = a_1 = 0 = v_0$

$q(t_f=3) = 15 + \alpha 3 + a_2 \times 3^2 + a_3 3^3 = 75 \Rightarrow 15 + 9a_2 + 27a_3 = 75$ ①

$9a_2 + 27a_3 = 60 \Rightarrow \boxed{3a_2 + 9a_3 = 20}$

$\dot{q}(t_f=3) = 0 + 2a_2 \times 3 + 3a_3 \times 3^2 = 0 \Rightarrow 6a_2 + 27a_3 = 0 \Rightarrow 2a_2 + 9a_3 = 0$ ②

② $2a_2 + 9a_3 = 0 \Rightarrow a_2 = \frac{-9}{2} a_3 \rightarrow$ Plugin ① : $3 \times \frac{-9}{2} a_3 + 9a_3 = 20 \Rightarrow \frac{(-27+18)a_3}{2} = 20$

$\Rightarrow a_3 = -\frac{40}{9} = -4.44$ $\quad a_2 = \frac{-9}{2} \times \frac{-40}{9} = 20$

Cubical Polynomial Approach

- We can determine the constrained by using the above equation.

$$a_0 = 15.0,$$

$$a_1 = 0.0,$$

$$a_2 = 20.0,$$

$$a_3 = -4.44.$$

- Equation of trajectory are

$$\theta(t) = 15.0 + 20.0t^2 - 4.44t^3,$$

$$\dot{\theta}(t) = 40.0t - 13.33t^2,$$

$$\ddot{\theta}(t) = 40.0 - 26.66t.$$

Cubical Polynomial Approach

$q(t) = \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$\theta(t_0) = \theta_0$ (1)

$\theta(t_f) = \theta_f$ (2)

$\dot\theta(t_0) = v_0$ (3)

$\dot\theta(t_f) = v_f$ (4)

a_0, a_1, a_2, a_3

$a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 = \theta_0$

$a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 = \theta_f$

$a_1 t_0 + 2a_2 t_0 + 3a_3 t_0^2 = v_0$

$a_1 t_f + 2a_2 t_f + 3a_3 t_f^2 = v_f$

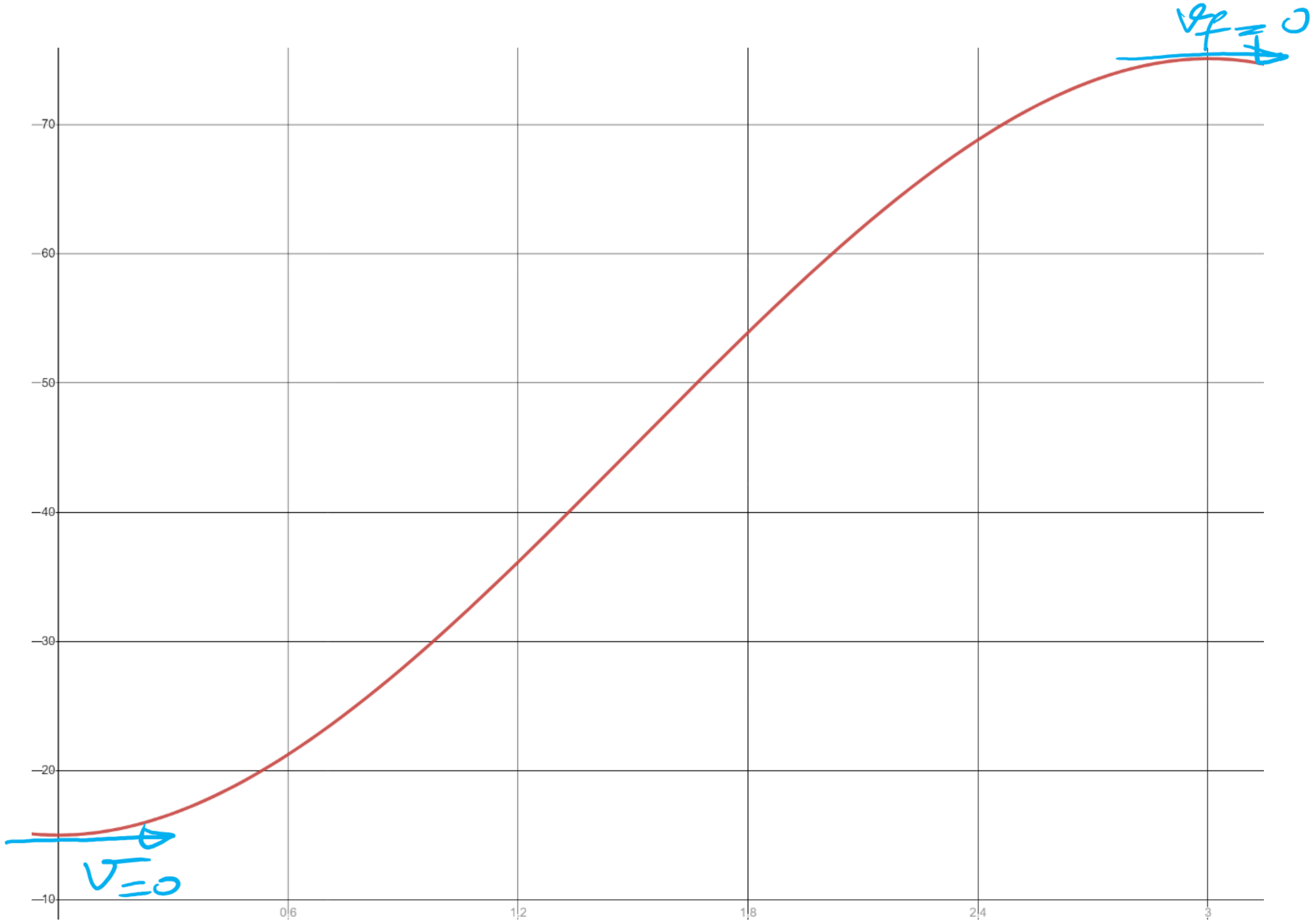
$$\underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & t_0 & 2t_0 & 2t_0^2 \\ 0 & t_f & 2t_f & 3t_f^2 \end{bmatrix}}_{4\times4} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_f \\ v_0 \\ v_f \end{bmatrix}$$

$M \quad A = B$

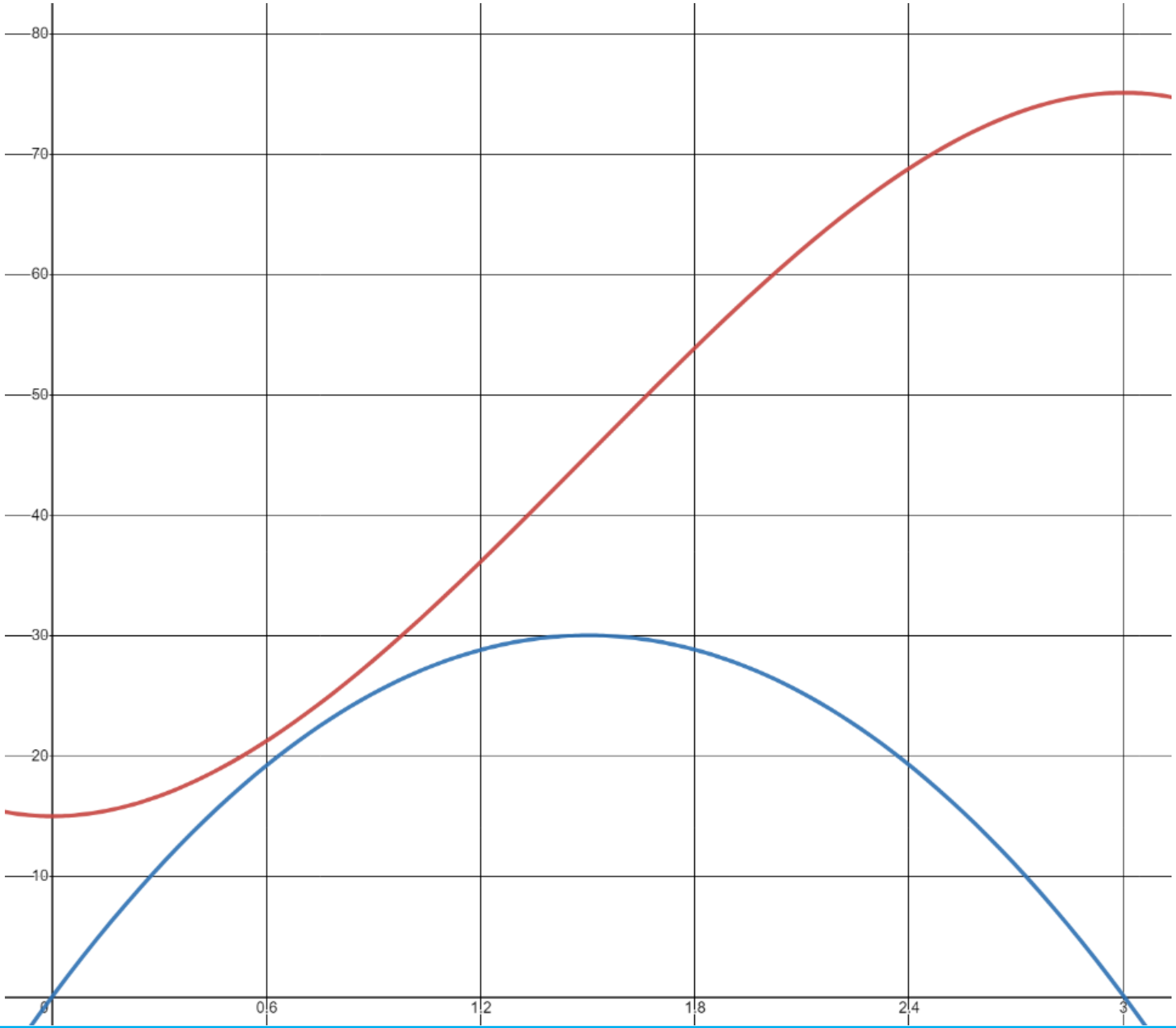
$MA = B \Rightarrow \boxed{A = M^{-1}B}$

$\boxed{A = B \cdot \cancel{M^{-1}}}$

Cubical Polynomial Approach



Cubical Polynomial Approach



Cubical Polynomial Approach