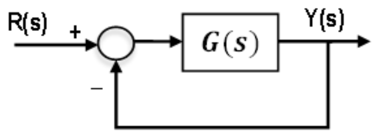
Worksheet 1 - Solution

1) Determine the type of the following unity-feedback systems for which the forward-path transfer function is given.



a) $G(s) = \frac{10}{(s+1)(10s+1)(20s+1)}$

Since the forward path system has no integrator (no pole at s=0), it is Type 0.

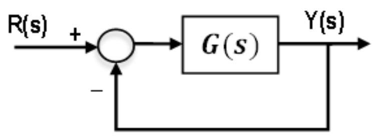
b) $G(s) = \frac{10(s+1)}{s(s+5)(s+6)^2}$

Since the forward path system has one integrator (one pole at s=0), it is Type 1.

c) $G(s) = \frac{100}{s^2(s^2+5s+5)}$

Since the forward path system has two integrator (two poles at s=0), it is Type 2.

2) Determine the step, ramp, and parabolic error constants and the correponding steady-state error of the following unity-feedback control systems. The forward path transfer functions are given.



a) $G(s) = \frac{100}{s^2(s^2+10s+100)}$

This is a Type 2 system.

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{100}{s^2(s^2+10s+100)} = \infty$$

$$e_{ss} = \frac{R}{1+k_p} = 0$$

Ramp-error constant and corresponding steady-state error:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{100}{s^2(s^2+10s+100)} = \infty$$

$$e_{ss} = \frac{R}{k_v} = 0$$

Parabolic-error constant and corresponding steady-state error:

$$k_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\frac{100}{s^2(s^2+10s+100)} = 1$$

$$e_{ss} = \frac{R}{k_a} = R$$

b) $G(s) = \frac{K}{s(1+0.1s)(1+0.5s)}$

This is a Type 1 system.

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K}{s(1+0.1s)(1+0.5s)} = \infty$$

$$e_{ss} = \frac{R}{1+k_p} = 0$$

Ramp-error constant and corresponding steady-state error:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{K}{s(1+0.1s)(1+0.5s)} = K$$

$$e_{ss} = \frac{R}{k_v} = \frac{R}{K}$$

Parabolic-error constant and corresponding steady-state error:

$$k_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\frac{K}{s(1+0.1s)(1+0.5s)} = 0$$

$$e_{ss} = \frac{R}{k_a} = \infty$$

c) $G(s) = \frac{1000}{(1+0.1s)(1+10s)}$

This is a Type 0 system.

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{1000}{(1+0.1s)(1+10s)} = 1000$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{R}{1001}$$

Ramp-error constant and corresponding steady-state error:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{1000}{(1+0.1s)(1+10s)} = 0$$

$$e_{ss} = \frac{R}{k_v} = \infty$$

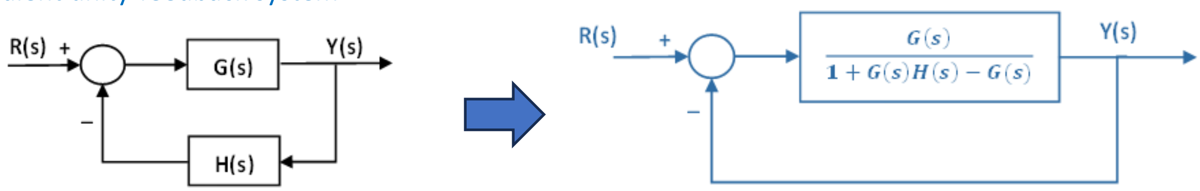
Parabolic-error constant and corresponding steady-state error:

$$k_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\frac{1000}{(1+0.1s)(1+10s)} = 0$$

$$e_{ss} = \frac{R}{k_a} = \infty$$

3) The following transfer functions are given for a single-loop nonunity-feedback control system. Find the steady-state error due to a unit-step input.

The equivalent unity-feedback system



a) $G(s) = \frac{1}{s^2+s+2}$, $H(s) = \frac{1}{s+1}$

First, find the equivalent unity-feedback system.

$$G_{eq}(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{\frac{1}{s^2+s+2}}{1+\left(\frac{1}{s^2+s+2}\right)\left(\frac{1}{s+1}\right)-\frac{1}{s^2+s+2}} = \frac{s+1}{s^3+2s^2+2s+2}$$

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G_{eq}(s) = \lim_{s\to 0} \frac{s+1}{s^3+2s^2+2s+2} = \frac{1}{2}$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

b) $G(s) = \frac{1}{s(s+5)}$, $H(s) = 5$

First, find the equivalent unity-feedback system.

$$G_{eq}(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{\frac{1}{s(s+5)}}{1+\left(\frac{1}{s(s+5)}\right)(5)-\frac{1}{s(s+5)}} = \frac{1}{s^2+5s+4}$$

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G_{eq}(s) = \lim_{s\to 0} \frac{1}{s^2+5s+4} = \frac{1}{4}$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$$

c) $G(s) = \frac{1}{s^2(s+10)}$, $H(s) = \frac{s+1}{s+5}$

First, find the equivalent unity-feedback system.

$$G_{eq}(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{\frac{1}{s^2(s+10)}}{1+\left(\frac{1}{s^2(s+10)}\right)\left(\frac{s+1}{s+5}\right)-\frac{1}{s^2(s+10)}} = \frac{s+5}{s^4+15s^3+50s^2-4}$$

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G_{eq}(s) = \lim_{s\to 0} \frac{s+5}{s^4+15s^3+50s^2-4} = \frac{-5}{4}$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{1-\frac{5}{4}} = -4$$

d) $G(s) = \frac{1}{s^2(s+12)}$, $H(s) = 5(s+2)$

First, find the equivalent unity-feedback system.

$$G_{eq}(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{\frac{1}{s^2(s+12)}}{1+\left(\frac{1}{s^2(s+12)}\right)(5(s+2)) - \frac{1}{s^2(s+12)}} = \frac{1}{s^3+12s^2+5s+9}$$

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G_{eq}(s) = \lim_{s\to 0} \frac{1}{s^3+12s^2+5s+9} = \frac{1}{9}$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{1+\frac{1}{9}} = \frac{9}{10} = 0.9$$

4) Find the position, velocity, and acceleration error constants for the following systems. Determine the steady-state error for a unit-step input, unit-ramp input and a parabolic input.

a)

The overall forward path transfer function is:

$$G(s) = \left(1-\frac{s+1}{s+2}\right)\left(1-\frac{4}{s^2}\right) = \frac{s^2-4}{s^2(s+2)}$$

This is a Type 2 system.

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{s^2-4}{s^2(s+2)} = \infty \quad \rightarrow \quad e_{ss} = \frac{R}{1+k_p} = 0$$

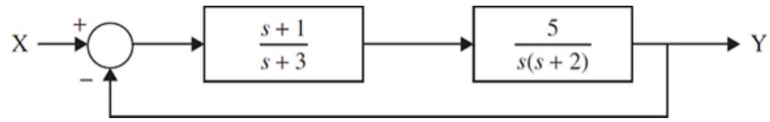
Ramp-error constant and corresponding steady-state error:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{s^2-4}{s^2(s+2)} = \infty \quad \rightarrow \quad e_{ss} = \frac{R}{k_v} = 0$$

Parabolic-error constant and corresponding steady-state error:

$$k_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\frac{s^2-4}{s^2(s+2)} = -2 \quad \rightarrow \quad e_{ss} = \frac{R}{k_a} = -\frac{1}{2}$$

b)



The overall forward path transfer function is:

$$G(s) = \left(\frac{s+1}{s+3}\right)\left(\frac{5}{s(s+2)}\right) = \frac{5(s+1)}{s(s+3)(s+2)}$$

This is a Type 1 system.

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{5(s+1)}{s(s+3)(s+2)} = \infty \quad \rightarrow \quad e_{ss} = \frac{R}{1+k_p} = 0$$

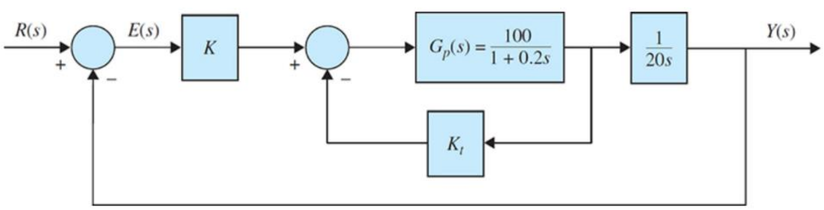
Ramp-error constant and corresponding steady-state error:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{5(s+1)}{s(s+3)(s+2)} = \frac{5}{6} \quad \rightarrow \quad e_{ss} = \frac{R}{k_v} = \frac{6}{5}$$

Parabolic-error constant and corresponding steady-state error:

$$k_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\frac{5(s+1)}{s(s+3)(s+2)} = 0 \quad \rightarrow \quad e_{ss} = \frac{R}{k_a} = \infty$$

5) The block diagram of a control system is shown below. Find the step, ramp, and parabolic error constants. The error signal is defined as $e(t)$. Find the steady-state errors in terms of K and K_t when the unit-step, unit-ramp, and parabolic inputs are applied. Assume that the system is stable.



The overall forward path transfer function is:

$$G(s) = K\left(\frac{100}{0.2s+1+100K_t}\right)\left(\frac{1}{20s}\right) = \frac{5K}{s(0.2s+1+100K_t)}$$

This is a Type 1 system.

Step-error constant and corresponding steady-state error:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{5K}{s(0.2s+1+100K_t)} = \infty \quad \rightarrow \quad e_{ss} = \frac{R}{1+k_p} = 0$$

Ramp-error constant and corresponding steady-state error:

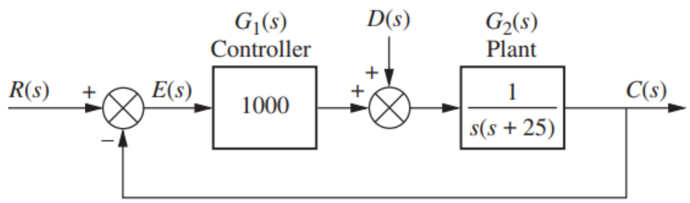
$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{5K}{s(0.2s+1+100K_t)} = \frac{5K}{1+100K_t} \quad \rightarrow \quad e_{ss} = \frac{R}{k_v} = \frac{1+100K_t}{5K}$$

Parabolic-error constant and corresponding steady-state error:

$$k_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\frac{5K}{s(0.2s+1+100K_t)} = 0 \quad \rightarrow \quad e_{ss} = \frac{R}{k_a} = \infty$$

6) Find the steady-state error component due to a unit-step disturbance for the following systems.

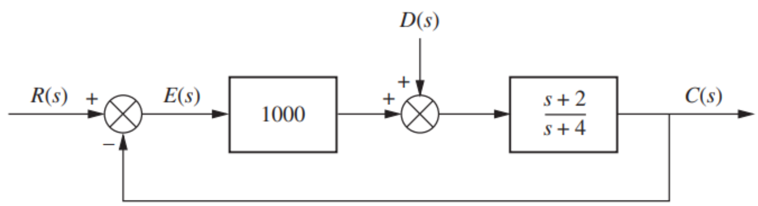
a)



Steady-state error due to unit-step disturbance is:

$$e_{ss,D} = \lim_{s\to 0}\frac{-sG_2(s)}{1+G_1(s)G_2(s)}D(s) = \lim_{s\to 0}\frac{-\frac{s}{s(s+25)}}{1+\frac{1000}{s(s+25)}}\left(\frac{1}{s}\right) = \lim_{s\to 0}\frac{-s}{s^2+25s+1000}\left(\frac{1}{s}\right) = \frac{-1}{1000} = -0.001$$

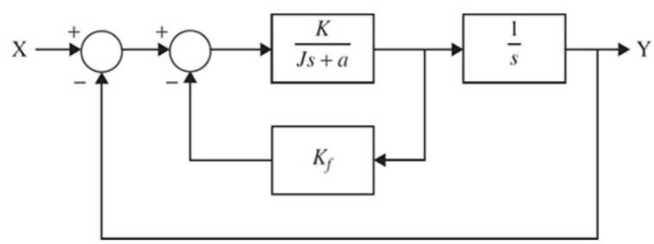
b)



Steady-state error due to unit-step disturbance is:

$$e_{ss,D} = \lim_{s\to 0}\frac{-sG_2(s)}{1+G_1(s)G_2(s)}D(s) = \lim_{s\to 0}\frac{-s\left(\frac{s+2}{s+4}\right)}{1+\frac{1000(s+2)}{s+4}}\left(\frac{1}{s}\right) = \lim_{s\to 0}\frac{-s(s+2)}{1001s+2004}\left(\frac{1}{s}\right) = -\frac{2}{2004} = -9.98\times 10^{-4}$$

7) Figure shows the block diagram of a servomotor.



Assume $J = 1\ kg.m^2$ and $a = 1\ \frac{N.m}{rad/sec}$. If the maximum overshoot of the unit-step input and the peak-time are 0.2 and 0.1 sec., respectively,

a) Find its damping ratio and natural frequency.

Find the damping ratio from the given maximum overshoot value:

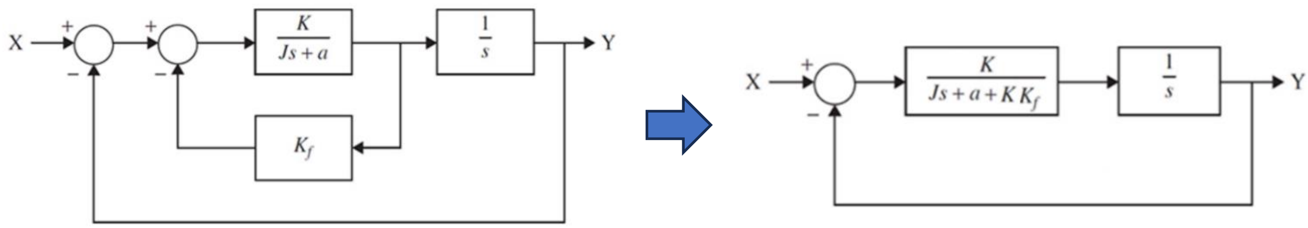
$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2+\ln^2(O.S.)}} = \frac{-\ln(0.2)}{\sqrt{\pi^2+\ln^2(0.2)}} \quad\rightarrow\quad \zeta = 0.456$$

Find the natural frequency from the given peak-time:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad\rightarrow\quad 0.1 = \frac{\pi}{\omega_n\sqrt{1-(0.456)^2}} \quad\rightarrow\quad \omega_n = 0.353\ rad/sec$$

b) Find the gain K and velocity feedback K_f. Also, calculate the rise-time and settling-time.

First find the overall closed-loop transfer function of the system by simplifying the block diagram:



Simplify the internal feedback loop:

$$\frac{\frac{K}{Js+a}}{1+\frac{KK_f}{Js+a}} = \frac{K}{Js+a+KK_f}$$

Then, find the overall transfer function:

$$\frac{Y(s)}{X(s)} = \frac{\frac{K}{s(Js+a+KK_f)}}{1+\frac{K}{s(Js+a+KK_f)}} = \frac{K}{Js^2+(a+KK_f)s+K} = \frac{K}{s^2+(1+KK_f)s+K}$$

Find the characteristic equation of the system based on the obtained damping ratio and natural frequency from Part (a) and match with the characteristic equation of the overall system to find the parameters K and K_f.

$$\boldsymbol{s^2+2\zeta\omega_n s+\omega_n^2} = s^2+2(0.456)(0.353)s+(0.353)^2 = s^2+0.322s+0.125$$

Matching the coefficients:

$$s^2+0.322s+0.125 = s^2+(1+KK_f)s+K$$

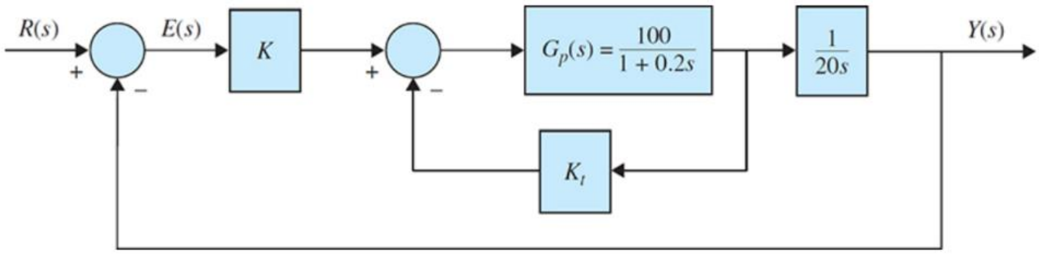
$$\begin{cases} 1+KK_f = 0.322 \\ K = 0.125 \end{cases} \quad \rightarrow \quad K = 0.125 \qquad K_f = -5.424$$

The rise-time and settling-time are obtained as:

$$\boldsymbol{t_r \cong \frac{0.8+2.5\zeta}{\omega_n}} \quad \rightarrow \quad t_r = \frac{0.8+2.5\times 0.456}{0.353} = 5.496\ sec$$

$$\boldsymbol{t_s \cong \frac{4}{\zeta\omega_n}} \quad \rightarrow \quad t_s = \frac{4}{0.456\times 0.353} = 24.850\ sec$$

9) For the controls system shown below, find the values of K and K_t so that the damping ration of the system is 0.6 and the settling-time of the unit-step response is 0.1 sec.



Find the undamped natural frequency from the given settling-time and the damping ratio:

$$\boldsymbol{t_s \cong \frac{4}{\zeta\omega_n}} \quad \rightarrow \quad 0.1 = \frac{4}{0.6\omega_n} \quad \rightarrow \quad \omega_n = 66.67 \text{ rad}/sec$$

The overall forward path transfer function is:

$$G(s) = K\left(\frac{100}{0.2s + 1 + 100K_t}\right)\left(\frac{1}{20s}\right) = \frac{5K}{s(0.2s + 1 + 100K_t)}$$

The closed-loop system is unity-feedback:

$$H(s) = 1$$

The overall closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{5K}{s(0.2s + 1 + 100K_t)}}{1 + \frac{5K}{s(0.2s + 1 + 100K_t)}} = \frac{5K}{0.2s^2 + (1 + 100K_t)s + 5K} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

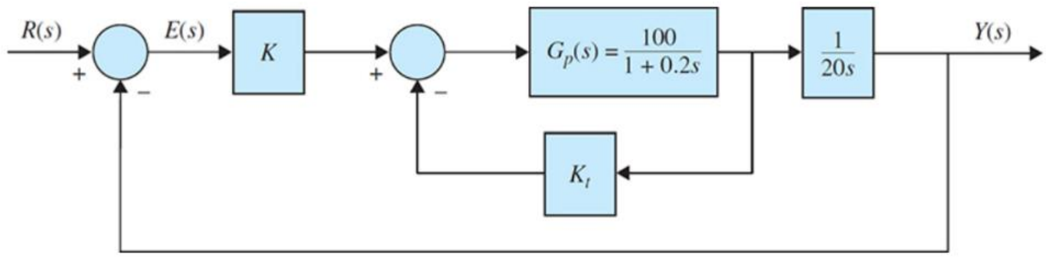
Matching the characteristic equation with the standard form:

$$\begin{cases} 2\zeta\omega_n = 5 + 500K_t \\ \omega_n^2 = 25K \end{cases} \quad \rightarrow \quad \begin{cases} 2(0.6)(66.67) = 5 + 500K_t \\ (66.67)^2 = 25K \end{cases} \quad \rightarrow \quad K = 117.79, \quad K_t = 0.15$$

The overall closed-loop system transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{2944.75}{s^2 + 80s + 2944.75}$$

10) For the control system shown below, find the values of K and K_t so that the maximum overshoot of the output is approximately 4.3% and the rise-time t_r is approximately 0.2sec.



Find the damping ratio from the given maximum overshoot value:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} = \frac{-\ln(0.43)}{\sqrt{\pi^2 + \ln^2(0.43)}} \quad \rightarrow \quad \zeta = 0.707$$

Find the natural frequency from the given rise-time:

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n} \quad \rightarrow \quad 0.2 = \frac{0.8 + 2.5 \times 0.707}{\omega_n} \quad \rightarrow \quad \omega_n = 12.84\ sec$$

The overall forward path transfer function is:

$$G(s) = K\left(\frac{100}{0.2s + 1 + 100K_t}\right)\left(\frac{1}{20s}\right) = \frac{5K}{s(0.2s + 1 + 100K_t)}$$

The closed-loop system is unity-feedback:

$$H(s) = 1$$

The overall closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{5K}{s(0.2s + 1 + 100K_t)}}{1 + \frac{5K}{s(0.2s + 1 + 100K_t)}} = \frac{5K}{0.2s^2 + (1 + 100K_t)s + 5K} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Matching the characteristic equation with the standard form:

$$\begin{cases} 2\zeta\omega_n = 5 + 500K_t \\ \omega_n^2 = 25K \end{cases} \quad \rightarrow \quad \begin{cases} 2(0.707)(12.84) = 5 + 500K_t \\ (12.84)^2 = 25K \end{cases} \quad \rightarrow \quad K = 6.59, \quad K_t = 0.026$$

The overall closed-loop system transfer function is:

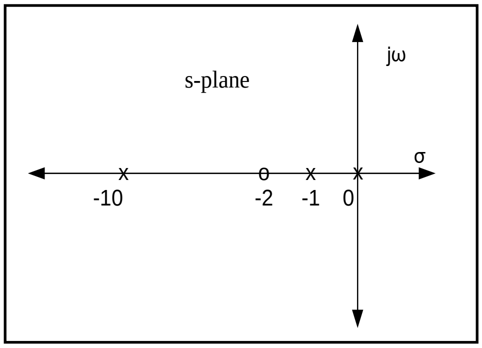
$$\frac{Y(s)}{R(s)} = \frac{164}{s^2 + 18s + 164}$$

11) Find the finite poles and zeros of the following functions. Mark the finite poles and zeros in the s-plane.

a) $G(s) = \frac{10(s+2)}{s^2(s+1)(s+10)}$

Poles: s = 0, 0, –1, –10

Zeros: s = –2

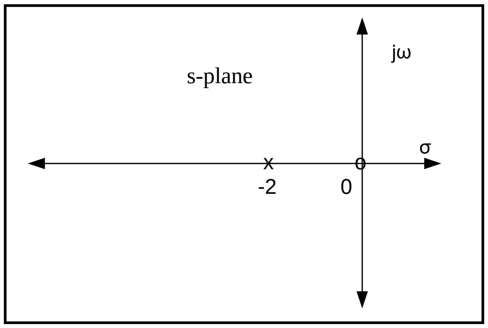


b) $G(s) = \frac{10s(s+1)}{(s+2)(s^2+3s+2)}$

$$G(s) = \frac{10s(s+1)}{(s+2)(s+2)(s+1)} = \frac{10s}{(s+2)^2}$$

Poles: s = –2, –2

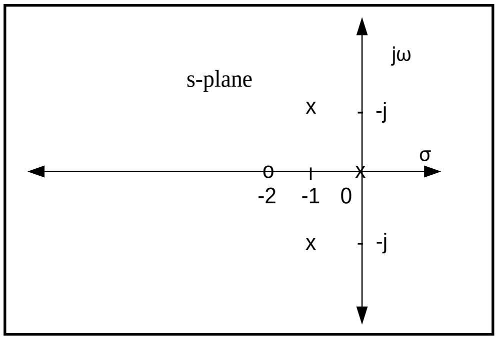
Zeros: s = 0



c) $G(s) = \frac{10(s+2)}{s(s^2+2s+2)}$

Poles: s = 0, –1 ± j1

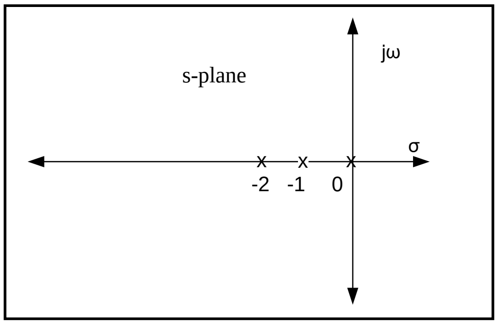
Zeros: s = –2



d) $G(s) = \frac{e^{-2s}}{10s(s+1)(s+2)}$

Poles: s = 0, –1, –2

No finite zeros.



11) The following differential equations represent LTI systems, where $r(t)$ denotes the input and $y(t)$ the output. Find the transfer function $Y(s)/R(s)$ for each system. (Assume zero initial conditions)

a) $\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 3\frac{dr(t)}{dt} + r(t)$

Taking Laplace transform assuming zero initial conditions and forming the transfer function model:

$$s^3Y(s) + 2s^2Y(s) + 5sY(s) + 6Y(s) = 3sR(s) + R(s) \quad \rightarrow \quad (s^3 + 2s^2 + 5s + 6)Y(s) = (3s + 1)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{3s + 1}{s^3 + 2s^2 + 5s + 6}$$

b) $\frac{d^4y(t)}{dt^4} + 10\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = 5r(t)$

Taking Laplace transform assuming zero initial conditions and forming the transfer function model:

$$s^4Y(s) + 10s^2Y(s) + sY(s) + 5Y(s) = 5R(s) \quad \rightarrow \quad (s^4 + 10s^2 + s + 5)Y(s) = 5R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{5}{s^4 + 10s^2 + s + 5}$$

c) $\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) + 2\int_0^t y(\tau)d\tau = \frac{dr(t)}{dt} + 2r(t)$

Taking Laplace transform assuming zero initial conditions and forming the transfer function model:

$$s^3Y(s) + 10s^2Y(s) + 2sY(s) + Y(s) + \frac{2}{s}Y(s) = sR(s) + 2R(s)$$

$$s^4Y(s) + 10s^3Y(s) + 2s^2Y(s) + sY(s) + 2Y(s) = s^2R(s) + 2sR(s)$$

$$(s^4 + 10s^3 + 2s^2 + s + 2)Y(s) = (s^2 + 2s)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{s(s + 2)}{s^4 + 10s^3 + 2s^2 + s + 2}$$

d) $2\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = r(t) + 2r(t - 1)$

Taking Laplace transform assuming zero initial conditions and forming the transfer function model:

$$2s^2Y(s) + sY(s) + 5Y(s) = R(s) + 2e^{-s}R(s) \quad \rightarrow \quad (2s^2 + s + 5)Y(s) = (1 + 2e^{-s})R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{1 + 2e^{-s}}{2s^2 + s + 5}$$

e) $\frac{d^2y(t+1)}{dt^2} + 4\frac{dy(t+1)}{dt} + 5y(t+1) = \frac{dr(t)}{dt} + 2r(t) + 2\int_{-\infty}^{t} r(\tau)d\tau$

Taking Laplace transform assuming zero initial conditions and forming the transfer function model:

$$s^2e^sY(s) + 4se^sY(s) + 5e^sY(s) = sR(s) + 2R(s) + \frac{2}{s}R(s)$$

$$s^3e^sY(s) + 4s^2e^sY(s) + 5se^sY(s) = s^2R(s) + 2sR(s) + 2R(s)$$

$$(s^3 + 4s^2 + 5s)e^sY(s) = (s^2 + 2s + 2)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{s^2 + 2s + 2}{(s^3 + 4s^2 + 5s)e^s} = \frac{(s^2 + 2s + 2)e^{-s}}{s(s^2 + 4s + 5)}$$

f) $\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 2y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = \frac{dr(t-1)}{dt} + 2r(t-1)$

Taking Laplace transform assuming zero initial conditions and forming the transfer function model:

$$s^3Y(s) + 2s^2Y(s) + sY(s) + 2Y(s) + \frac{2}{s}Y(s) = se^{-s}R(s) + 2e^{-s}R(s)$$

$$s^4Y(s) + 2s^3Y(s) + s^2Y(s) + 2sY(s) + 2Y(s) = s^2e^{-s}R(s) + 2se^{-s}R(s)$$

$$(s^4 + 2s^3 + s^2 + 2s + 2)Y(s) = (s^2 + 2s)e^{-s}R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)e^{-s}}{s^4 + 2s^3 + s^2 + 2s + 2}$$

12) Find the inverse Laplace transform of the following functions. First, perform partial fraction expansion on $G(s)$, then, use the Laplace transform table.

a) $G(s) = \frac{1}{s(s+2)(s+3)}$

$$G(s) = \frac{1/6}{s} + \frac{-1/2}{s+2} + \frac{1/3}{s+3} \quad \rightarrow \quad g(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}, \quad t \geq 0$$

b) $G(s) = \frac{10}{(s+1)^2(s+3)}$

$$G(s) = \frac{-10/4}{s+1} + \frac{5}{(s+1)^2} + \frac{10/4}{s+3} \quad \rightarrow \quad g(t) = -2.5e^{-t} + 5te^{-t} + 2.5e^{-3t}, \quad t \geq 0$$

c) $G(s) = \frac{100(s+2)}{s(s^2+4)(s+1)}e^{-s}$

$$G(s) = \left(\frac{50}{s} + \frac{-(30s+20)}{s^2+4} + \frac{-20}{s+1}\right)e^{-s} = \left(\frac{50}{s} + \frac{-30s}{s^2+4} + \frac{-20}{s^2+4} + \frac{-20}{s+1}\right)e^{-s}$$

$$g(t) = 50 - 30\cos 2(t-1) - 5\sin 2(t-1) - 20e^{-(t-1)}, \quad t \geq 1$$

d) $G(s) = \frac{1}{(s+1)^3}$

$$G(s) = \frac{1}{(s+1)^3} \quad \rightarrow \quad g(t) = \frac{1}{2}t^2e^{-t}, \quad t \geq 0$$

e) $G(s) = \frac{2(s^2+s+1)}{s(s+1)(s^2+5s+5)}$

$$G(s) = \frac{2(s^2+s+1)}{s(s+1)(s^2+5s+5)} = \frac{2(s^2+s+1)}{s(s+1)(s+3.6180)(s+1.3820)}$$

$$G(s) = \frac{2/5}{s} + \frac{-2}{s+1} + \frac{-0.9889}{s+3.6180} + \frac{2.5889}{s+1.3820} \quad \rightarrow \quad g(t) = 0.4 - 2e^{-t} - 0.9889e^{-3.618t} + 2.5889e^{-2.5889t}, \quad t \geq 0$$

f) $G(s) = \frac{2+2se^{-s}+4e^{-s}}{s^2+3s+2}$

$$G(s) = \frac{2+2se^{-s}+4e^{-s}}{s^2+3s+2} = \frac{2+2(s+2)e^{-s}}{(s+1)(s+2)} = \frac{2}{(s+1)(s+2)} + \frac{2e^{-s}}{s+1}$$

$$G(s) = \frac{2}{s+1} + \frac{-2}{s+2} + \frac{2e^{-s}}{s+1} \quad \rightarrow \quad g(t) = 2e^{-t} - 2e^{-2t} + 2e^{-(t-1)}, \quad t \geq 1$$

g) $G(s) = \frac{2s+1}{s^3+6s^2+11s+6}$

$$G(s) = \frac{2s+1}{s^3+6s^2+11s+6} = \frac{2s+1}{(s+1)(s+2)(s+3)}$$

$$G(s) = \frac{-1/2}{s+1} + \frac{3}{s+2} + \frac{-5/2}{s+3} \quad \rightarrow \quad g(t) = -0.5e^{-t} + 3e^{-2t} - 2.5e^{-3t}, \quad t \geq 0$$

h) $G(s) = \frac{3s^3+10s^2+8s+5}{s^4+5s^3+7s^2+5s+6}$

$$G(s) = \frac{3s^3+10s^2+8s+5}{s^4+5s^3+7s^2+5s+6} = \frac{3s^3+10s^2+8s+5}{(s+2)(s+3)(s^2+1)}$$

$$G(s) = \frac{1}{s+2} + \frac{1}{s+3} + \frac{s}{s^2+1} \quad \rightarrow \quad g(t) = e^{-2t} + e^{-3t} + \cos t, \quad t \geq 0$$

13) Solve the following differential equations by means of the Laplace transform.

a) $\frac{d^2f(t)}{dt^2}+5\frac{df(t)}{dt}+4f(t)=e^{-2t}u_s(t)$ (Assume zero initial conditions)

Take Laplace transform assuming zero initial conditions.

$$s^2F(s)+5sF(s)+4F(s)=\frac{1}{s+2} \quad \rightarrow \quad (s^2+5s+4)F(s)=\frac{1}{s+2}$$

$$F(s)=\frac{1}{(s+2)(s^2+5s+4)}=\frac{1}{(s+2)(s+4)(s+1)}$$

Apply partial fraction expansion:

$$F(s)=\frac{-1/2}{s+2}+\frac{1/6}{s+4}+\frac{1/3}{s+1}$$

Taking inverse Laplace transform:

$$f(t)=-\frac{1}{2}e^{-2t}+\frac{1}{6}e^{-4t}+\frac{1}{3}e^{-t}, \qquad t\geq 0$$

b) $\frac{d^3y(t)}{dt^3}+2\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+2y(t)=-e^{-t}u_s(t)$, $\frac{d^2y}{dt^2}(0)=-1,\ \frac{dy}{dt}(0)=1,\ y(0)=0$

Take Laplace transform assuming the given initial conditions.

$$s^3Y(s)-s^2y(0)-s\dot{y}(0)-\ddot{y}(0)+2\left(s^2Y(s)-sy(0)-\dot{y}(0)\right)+sY(s)-y(0)+2Y(s)=\frac{-1}{s+1}$$

$$s^3Y(s)-0-s+1+2(s^2Y(s)-0-1)+sY(s)-0+2Y(s)=\frac{-1}{s+1}$$

$$s^3Y(s)+2s^2Y(s)+sY(s)+2Y(s)-s-1=\frac{-1}{s+1}$$

$$(s^3+2s^2+s+2)Y(s)=s+1+\frac{-1}{s+1}$$

$$(s^3+2s^2+s+2)Y(s)=\frac{s^2+2s}{s+1}$$

$$Y(s)=\frac{s^2+2s}{(s+1)(s^3+2s^2+s+2)}=\frac{s(s+2)}{(s+1)(s+2)(s^2+1)}=\frac{s}{(s+1)(s^2+1)}$$

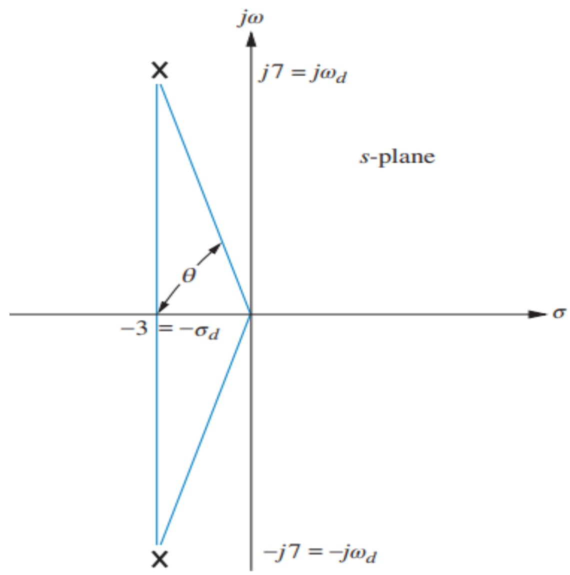
Apply partial fraction expansion:

$$Y(s)=\frac{-0.5}{s+1}+\frac{0.5s+0.5}{s^2+1}=\frac{-0.5}{s+1}+\frac{0.5s}{s^2+1}+\frac{0.5}{s^2+1}$$

Taking inverse Laplace transform:

$$y(t)=-0.5e^{-t}+0.5\cos t+0.5\sin t, \qquad t\geq 0$$

14) Given the pole plot, find ζ, $\omega_n, t_p, O.S.\%,$ and t_s.



The undamped natural frequency is the radial distance from the origin to the pole:

$$\omega_n = \sqrt{\omega_d^2 + \sigma^2} = \sqrt{7^2 + 3^2} = 7.616$$

The damping ratio is given by:

$$\zeta = \cos\theta = \frac{3}{7.616} = 0.394$$

The peak-time is:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ sec}$$

The percentage of overshoot is:

$$\%O.S. = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = e^{-\left(0.394\pi/\sqrt{1-0.394^2}\right)} \times 100 = 26\%$$

The settling-time is:

$$t_s \cong \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} = \frac{4}{3} = 1.333 \text{ sec}$$