Lecture 2

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Chapter 1

Systems of Equations

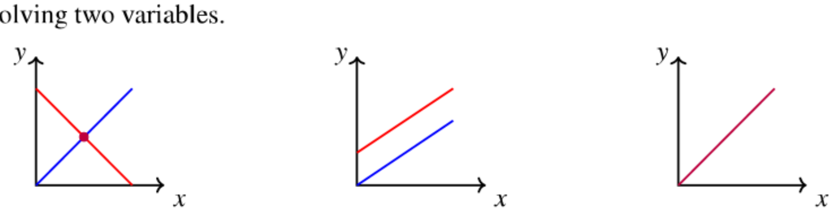
1.1 Systems of Equations, Geometry

Outcomes

A. *Relate the types of solution sets of a system of two (three) variables to the intersections of lines in a plane (the intersection of planes in three space)*

As you may remember, linear equations like $2x+3y=6$ can be graphed as straight lines in the coordinate plane. We say that this equation is in two variables, in this case x and y. Suppose you have two such equations, each of which can be graphed as a straight line, and consider the resulting graph of two lines. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives x and y values for which both equations are true. In other words, this point gives the ordered pair (x,y) that satisfy both equations. If the point (x,y) is a point of intersection, we say that (x,y) is a **solution** to the two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

When looking for the intersection of two lines in a graph, several situations may arise. The following picture demonstrates the possible situations when considering two equations (two lines in the graph) involving two variables.



One Solution   No Solutions   Infinitely Many Solutions

In the first diagram, there is a unique point of intersection, which means that there is only one (unique)

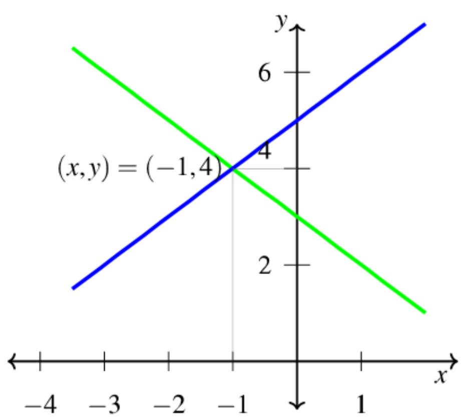
In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. When no solution exists, this means that the two lines are parallel and they never intersect. The third situation which can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x+y=1$ and $2x+2y=2$ are equations which when graphed yield the same line. In this case there are infinitely many points which are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations. When considering linear systems of equations, there are always three types of solutions possible; exactly one (unique) solution, infinitely many solutions, or no solution.

Example 1.1: A Graphical Solution

Use a graph to find the solution to the following system of equations

$$x+y=3$$
$$y-x=5$$

Solution. Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the (x,y) which satisfy both equations. In this case, there is one point of intersection at $(-1,4)$ which means we have one unique solution, $x=-1, y=4$.



♠

In the above example, we investigated the intersection point of two equations in two variables, x and y. Now we will consider the graphical solutions of three equations in two variables.

Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possible types of solutions; no solution, one solution, and infinitely many solutions. There are now more complex ways of achieving these situations, due to the presence of the third line. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below.

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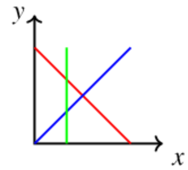
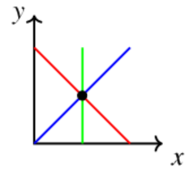


No Solution

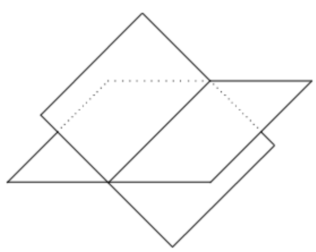


One Solution

Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the three equations. Remember, a solution is a point (x,y) which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

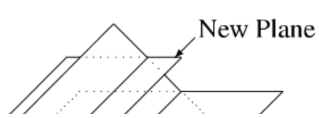
You may recall that equations in three variables, such as $2x+4y-5z=8$, form a plane. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the (x,y,z) that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables.

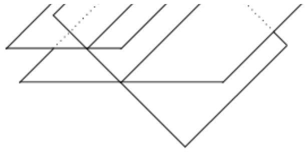


Notice how these two planes intersect in a line. This means that the points (x,y,z) on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the types of solutions possible for two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram.

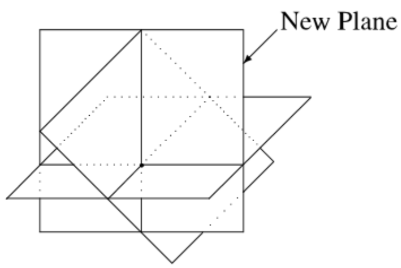




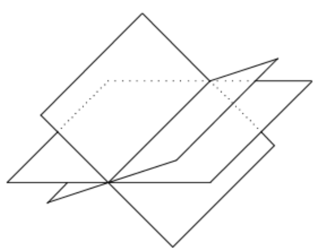
In this diagram, there is no point which lies in all three planes. There is no intersection between **all**

planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture.



In this case, the three planes have a single point of intersection. Can you think of other types of solutions possible? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram.



We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that the three equations graph the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more ways to achieve the different types of solutions than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x+y-2z+4w=8$ with more than three variables are often called **hyper-planes**. You may soon realize that it is tricky to draw the graphs of hyper-planes! Through the tools of linear algebra, we can algebraically examine these types of systems which are difficult to graph. In the following section, we will consider these algebraic tools.

Exercises

Exercise 1.1.1 *Graphically, find the point* (x_1, y_1) *which lies on both lines,* $x + 3y = 1$ *and* $4x - y = 3$*. That is, graph each line and see where they intersect.*

Exercise 1.1.2 *Graphically, find the point of intersection of the two lines* $3x + y = 3$ *and* $x + 2y = 1$*. That is, graph each line and see where they intersect.*

Exercise 1.1.3 *You have a system of* k *equations in two variables,* $k \geq 2$*. Explain the geometric significance of*

(a) No solution.

(b) A unique solution.

(c) An infinite number of solutions.

1.2 Systems of Equations, Algebraic Procedures

Outcomes

A. Use elementary operations to find the solution to a linear system of equations.

B. Find the row-echelon form and reduced row-echelon form of a matrix.

C. Determine whether a system of linear equations has no solution, a unique solution or an infinite number of solutions from its row-echelon form.

D. Solve a system of equations using Gaussian Elimination and Gauss-Jordan Elimination.

E. Model a physical system with linear equations and then solve.

We have taken an in depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

Definition 1.2: System of Linear Equations

A ***system of linear equations*** *is a list of equations,*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_j are real numbers. The above is a system of m equations in the n variables, $x_1, x_2 \cdots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, 2, 3, \cdots, m$$

The relative size of m and n is not important here. Notice that we have allowed a_{ij} and b_j to be any real number. We can also call these numbers **scalars** . We will use this term throughout the text, so keep in mind that the term **scalar** just means that we are working with real numbers.

Now, suppose we have a system where $b_i = 0$ for all i. In other words every equation equals 0. This is a special type of system.

Definition 1.3: Homogeneous System of Equations

A system of equations is called ***homogeneous*** *if each equation in the system is equal to 0. A homogeneous system has the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

where a_{ij} are scalars and x_i are variables.

Recall from the previous section that our goal when working with systems of linear equations was to find the point of intersection of the equations when graphed. In other words, we looked for the solutions to the system. We now wish to find these solutions algebraically. We want to find values for $x_1, \cdots, x_n$ which solve all of the equations. If such a set of values exists, we call $(x_1, \cdots, x_n)$ the **solution set**.

Recall the above discussions about the types of solutions possible. We will see that systems of linear equations will have one unique solution, infinitely many solutions, or no solution. Consider the following definition.

Definition 1.4: Consistent and Inconsistent Systems

A system of linear equations is called ***consistent*** *if there exists at least one solution. It is called* ***inconsistent*** *if there is no solution.*

If you think of each equation as a condition which must be satisfied by the variables, consistent would mean there is some choice of variables which can satisfy **all** the conditions. Inconsistent would mean there is no choice of the variables which can satisfy all of the conditions.

The following sections provide methods for determining if a system is consistent or inconsistent, and finding solutions if they exist.

1.2.1 Elementary Operations

We begin this section with an example.

Example 1.5: Verifying an Ordered Pair is a Solution

Algebraically verify that $(x, y) = (-1, 4)$ is a solution to the following system of equations.

$$\begin{aligned} x + y &= 3 \\ y - x &= 5 \end{aligned}$$

Solution. By graphing these two equations and identifying the point of intersection, we previously found that $(x, y) = (-1, 4)$ is the unique solution.

We can verify algebraically by substituting these values into the original equations, and ensuring that the equations hold. First, we substitute the values into the first equation and check that it equals 3.

$$x + y = (-1) + (4) = 3$$

This equals 3 as needed, so we see that $(-1, 4)$ is a solution to the first equation. Substituting the values into the second equation yields

$$y - x = (4) - (-1) = 4 + 1 = 5$$

which is true. For $(x, y) = (-1, 4)$ each equation is true and therefore, this is a solution to the system. ♠

Now, the interesting question is this: If you were not given these numbers to verify, how could you algebraically determine the solution? Linear algebra gives us the tools needed to answer this question. The following basic operations are important tools that we will utilize.

Definition 1.6: Elementary Operations

Elementary operations *are those operations consisting of the following.*

1. *Interchange the order in which the equations are listed.*
2. *Multiply any equation by a nonzero number.*
3. *Replace any equation with itself added to a multiple of another equation.*

It is important to note that none of these operations will change the set of solutions of the system of equations. In fact, elementary operations are the *key tool* we use in linear algebra to find solutions to systems of equations.

Consider the following example.

Example 1.7: Effects of an Elementary Operation

Show that the system

$$\begin{aligned} x+y &= 7 \\ 2x-y &= 8 \end{aligned}$$

has the same solution as the system

$$\begin{aligned} x+y &= 7 \\ -3y &= -6 \end{aligned}$$

Solution. Notice that the second system has been obtained by taking the second equation of the first system and adding -2 times the first equation, as follows:

$$2x-y+(-2)(x+y) = 8+(-2)(7)$$

By simplifying, we obtain

$$-3y = -6$$

which is the second equation in the second system. Now, from here we can solve for y and see that $y = 2$. Next, we substitute this value into the first equation as follows

$$x+y = x+2 = 7$$

Hence $x = 5$ and so $(x,y) = (5,2)$ is a solution to the second system. We want to check if $(5,2)$ is also a solution to the first system. We check this by substituting $(x,y) = (5,2)$ into the system and ensuring the equations are true.

$$\begin{aligned} x+y &= (5)+(2) = 7 \\ 2x-y &= 2(5)-(2) = 8 \end{aligned}$$

Hence, $(5,2)$ is also a solution to the first system. ♠

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the solution set. However, a linear system may involve many equations and many variables and there is no reason to limit our study to small systems. For any size of system in any number of variables, the solution set is still the collection of solutions to the equations. In every case, the above operations of Definition 1.6 do not change the set of solutions to the system of linear equations.

In the following theorem, we use the notation E_i to represent an expression, while b_i denotes a constant.

Theorem 1.8: Elementary Operations and Solutions

Theorem 1.8: Elementary Operations and Solutions

Suppose you have a system of two linear equations

$$\begin{aligned} E_1 &= b_1 \\ E_2 &= b_2 \end{aligned} \tag{1.1}$$

Then the following systems have the same solution set as 1.1:

1.
$$\begin{aligned} E_2 &= b_2 \\ E_1 &= b_1 \end{aligned} \tag{1.2}$$

2.
$$\begin{aligned} E_1 &= b_1 \\ kE_2 &= kb_2 \end{aligned} \tag{1.3}$$
for any scalar k, provided $k \neq 0$.

3.
$$\begin{aligned} E_1 &= b_1 \\ E_2 + kE_1 &= b_2 + kb_1 \end{aligned} \tag{1.4}$$
for any scalar k (including $k = 0$).

Before we proceed with the proof of Theorem 1.8, let us consider this theorem in context of Example 1.7. Then,

$$\begin{aligned} E_1 &= x + y, \quad b_1 = 7 \\ E_2 &= 2x - y, \quad b_2 = 8 \end{aligned}$$

Recall the elementary operations that we used to modify the system in the solution to the example. First, we added (-2) times the first equation to the second equation. In terms of Theorem 1.8, this action is given by

$$E_2 + (-2)E_1 = b_2 + (-2)b_1$$

or

$$2x - y + (-2)(x + y) = 8 + (-2)7$$

This gave us the second system in Example 1.7, given by

$$\begin{aligned} E_1 &= b_1 \\ E_2 + (-2)E_1 &= b_2 + (-2)b_1 \end{aligned}$$

From this point, we were able to find the solution to the system. Theorem 1.8 tells us that the solution we found is in fact a solution to the original system.

We will now prove Theorem 1.8.

Proof.

1. The proof that the systems 1.1 and 1.2 have the same solution set is as follows. Suppose that $(x_1, \cdots, x_n)$ is a solution to $E_1 = b_1, E_2 = b_2$. We want to show that this is a solution to the system in 1.2 above. This is clear, because the system in 1.2 is the original system, but listed in a different order. Changing the order does not effect the solution set, so $(x_1, \cdots, x_n)$ is a solution to 1.2.

2. Next we want to prove that the systems 1.1 and 1.3 have the same solution set. That is $E_1 = b_1, E_2 = b_2$ has the same solution set as the system $E_1 = b_1, kE_2 = kb_2$ provided $k \neq 0$. Let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, E_2 = b_2$,. We want to show that it is a solution to $E_1 = b_1, kE_2 = kb_2$. Notice that the only difference between these two systems is that the second involves multiplying the equation,

$E_2 = b_2$ by the scalar k. Recall that when you multiply both sides of an equation by the same number, the sides are still equal to each other. Hence if $(x_1, \cdots, x_n)$ is a solution to $E_2 = b_2$, then it will also be a solution to $kE_2 = kb_2$. Hence, $(x_1, \cdots, x_n)$ is also a solution to 1.3.

Similarly, let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, kE_2 = kb_2$. Then we can multiply the equation $kE_2 = kb_2$ by the scalar $1/k$, which is possible only because we have required that $k \neq 0$. Just as above, this action preserves equality and we obtain the equation $E_2 = b_2$. Hence $(x_1, \cdots, x_n)$ is also a solution to $E_1 = b_1, E_2 = b_2$.

3. Finally, we will prove that the systems 1.1 and 1.4 have the same solution set. We will show that any solution of $E_1 = b_1, E_2 = b_2$ is also a solution of 1.4. Then, we will show that any solution of 1.4 is also a solution of $E_1 = b_1, E_2 = b_2$. Let $(x_1, \cdots, x_n)$ be a solution to $E_1 = b_1, E_2 = b_2$. Then in particular it solves $E_1 = b_1$. Hence, it solves the first equation in 1.4. Similarly, it also solves $E_2 = b_2$. By our proof of 1.3, it also solves $kE_1 = kb_1$. Notice that if we add E_2 and kE_1, this is equal to $b_2 + kb_1$. Therefore, if $(x_1, \cdots, x_n)$ solves $E_1 = b_1, E_2 = b_2$ it must also solve $E_2 + kE_1 = b_2 + kb_1$.

   Now suppose $(x_1, \cdots, x_n)$ solves the system $E_1 = b_1, E_2 + kE_1 = b_2 + kb_1$. Then in particular it is a solution of $E_1 = b_1$. Again by our proof of 1.3, it is also a solution to $kE_1 = kb_1$. Now if we subtract these equal quantities from both sides of $E_2 + kE_1 = b_2 + kb_1$ we obtain $E_2 = b_2$, which shows that the solution also satisfies $E_1 = b_1, E_2 = b_2$.

♠

Stated simply, the above theorem shows that the elementary operations do not change the solution set of a system of equations.

We will now look at an example of a system of three equations and three variables. Similarly to the previous examples, the goal is to find values for x, y, z such that each of the given equations are satisfied when these values are substituted in.

Example 1.9: Solving a System of Equations with Elementary Operations

Find the solutions to the system,

$$\begin{array}{c} x + 3y + 6z = 25 \\ 2x + 7y + 14z = 58 \\ 2y + 5z = 19 \end{array} \tag{1.5}$$

Solution. We can relate this system to Theorem 1.8 above. In this case, we have

$$\begin{array}{ll} E_1 = x + 3y + 6z, & b_1 = 25 \\ E_2 = 2x + 7y + 14z, & b_2 = 58 \\ E_3 = 2y + 5z, & b_3 = 19 \end{array}$$

Theorem 1.8 claims that if we do elementary operations on this system, we will not change the solution set. Therefore, we can solve this system using the elementary operations given in Definition 1.6. First,

replace the second equation by (-2) times the first equation added to the second. This yields the system

$$\begin{array}{c} x + 3y + 6z = 25 \\ y + 2z = 8 \\ 2y + 5z = 19 \end{array} \tag{1.6}$$

Now, replace the third equation with (-2) times the second added to the third. This yields the system

$$
\begin{aligned}
x+3y+6z &= 25 \\
y+2z &= 8 \\
z &= 3
\end{aligned} \tag{1.7}
$$

At this point, we can easily find the solution. Simply take $z = 3$ and substitute this back into the previous equation to solve for y, and similarly to solve for x.

$$
\begin{aligned}
x+3y+6(3) = x+3y+18 &= 25 \\
y+2(3) = y+6 &= 8 \\
z &= 3
\end{aligned}
$$

The second equation is now

$$y+6=8$$

You can see from this equation that $y = 2$. Therefore, we can substitute this value into the first equation as follows:

$$x+3(2)+18=25$$

By simplifying this equation, we find that $x = 1$. Hence, the solution to this system is $(x,y,z) = (1,2,3)$. This process is called **back substitution**.

Alternatively, in 1.7 you could have continued as follows. Add (-2) times the third equation to the second and then add (-6) times the second to the first. This yields

$$
\begin{aligned}
x+3y &= 7 \\
y &= 2 \\
z &= 3
\end{aligned}
$$

Now add (-3) times the second to the first. This yields

$$
\begin{aligned}
x &= 1 \\
y &= 2 \\
z &= 3
\end{aligned}
$$

a system which has the same solution set as the original system. This avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x,y,z) = (1,2,3)$. ♠

1.2.2 Gaussian Elimination

The work we did in the previous section will always find the solution to the system. In this section, we will explore a less cumbersome way to find the solutions. First, we will represent a linear system with an **augmented matrix.** A **matrix** is simply a rectangular array of numbers. The size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. In order to construct an augmented matrix from a linear system, we create a **coefficient matrix** from the coefficients of the variables in the system, as well as a **constant matrix** from the constants. The coefficients from one equation of the system create one row of the augmented matrix.

For example, consider the linear system in Example 1.9

$$\begin{aligned} x+3y+6z &= 25 \\ 2x+7y+14z &= 58 \\ 2y+5z &= 19 \end{aligned}$$

This system can be written as an augmented matrix, as follows

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Notice that it has exactly the same information as the original system. Here it is understood that the first column contains the coefficients from x in each equation, in order, $\left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right]$. Similarly, we create a column from the coefficients on y in each equation, $\left[\begin{array}{c} 3 \\ 7 \\ 2 \end{array}\right]$ and a column from the coefficients on z in each equation, $\left[\begin{array}{c} 6 \\ 14 \\ 5 \end{array}\right]$. For a system of more than three variables, we would continue in this way constructing a column for each variable. Similarly, for a system of less than three variables, we simply construct a column for each variable.

Finally, we construct a column from the constants of the equations, $\left[\begin{array}{c} 25 \\ 58 \\ 19 \end{array}\right]$.

The rows of the augmented matrix correspond to the equations in the system. For example, the top row in the augmented matrix, $\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \end{array}\right]$ corresponds to the equation

$$x+3y+6z = 25.$$

Consider the following definition.

Definition 1.10: Augmented Matrix of a Linear System

For a linear system of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the x_i are variables and the a_{ij} and b_i are constants, the augmented matrix of this system is given by

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Now, consider elementary operations in the context of the augmented matrix. The elementary operations in Definition 1.6 can be used on the rows just as we used them on equations previously. Changes to a system of equations as a result of an elementary operation are equivalent to changes in the augmented matrix resulting from the corresponding row operation. Note that Theorem 1.8 implies that any elementary row operations used on an augmented matrix will not change the solution to the corresponding system of equations. We now formally define elementary row operations. These are the *key tool* we will use to find solutions to systems of equations.

Definition 1.11: Elementary Row Operations

The ***elementary row operations*** *(also known as* ***row operations****) consist of the following*

1. *Switch two rows.*
2. *Multiply a row by a nonzero number.*
3. *Replace a row by any multiple of another row added to it.*

Recall how we solved Example 1.9. We can do the exact same steps as above, except now in the context of an augmented matrix and using row operations. The augmented matrix of this system is

$$\left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Thus the first step in solving the system given by 1.5 would be to take (-2) times the first row of the augmented matrix and add it to the second row,

$$\left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Note how this corresponds to 1.6. Next take (-2) times the second row and add to the third,

$$\left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This augmented matrix corresponds to the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as 1.7. By back substitution you obtain the solution $x = 1, y = 2$, and $z = 3$.

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to **row-echelon form** or **reduced row-echelon form**, which we define next. These forms are used to find the solutions of the system of equations corresponding to the augmented matrix.

In the following definitions, the term **leading entry** refers to the first nonzero entry of a row when scanning the row from left to right.

Definition 1.12: Row-Echelon Form

An augmented matrix is in ***row-echelon form*** *if*

1. *All nonzero rows are above any rows of zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entries of any row above it.*
3. *Each leading entry of a row is equal to* 1*.*

We also consider another reduced form of the augmented matrix which has one further condition.

Definition 1.13: Reduced Row-Echelon Form

An augmented matrix is in ***reduced row-echelon form*** *if*

1. *All nonzero rows are above any rows of zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entries of any rows above it.*
3. *Each leading entry of a row is equal to* 1*.*
4. *All entries in a column above and below a leading entry are zero.*

Notice that the first three conditions on a reduced row-echelon form matrix are the same as those for row-echelon form.

Hence, every reduced row-echelon form matrix is also in row-echelon form. The converse is not necessarily true; we cannot assume that every matrix in row-echelon form is also in reduced row-echelon

form. However, it often happens that the row-echelon form is sufficient to provide information about the solution of a system.

The following examples describe matrices in these various forms. As an exercise, take the time to carefully verify that they are in the specified form.

Example 1.14: Not in Row-Echelon Form

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example 1.15: Matrices in Row-Echelon Form

The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that we could apply further row operations to these matrices to carry them to reduced row-echelon form. Take the time to try that on your own. Consider the following matrices, which are in reduced row-echelon form.

Example 1.16: Matrices in Reduced Row-Echelon Form

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

One way in which the row-echelon form of a matrix is useful is in identifying the pivot positions and pivot columns of the matrix.

Definition 1.17: Pivot Position and Pivot Column

A ***pivot position*** *in a matrix is the location of a leading entry in the row-echelon form of a matrix. A* ***pivot column*** *is a column that contains a pivot position.*

For example consider the following.

Example 1.18: Pivot Position

Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Where are the pivot positions and pivot columns of the augmented matrix A?

Solution. The row-echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is all we need in this example, but note that this matrix is not in reduced row-echelon form.

In order to identify the pivot positions in the original matrix, we look for the leading entries in the row-echelon form of the matrix. Here, the entry in the first row and first column, as well as the entry in the second row and second column are the leading entries. Hence, these locations are the pivot positions. We identify the pivot positions in the original matrix, as in the following:

$$\left[\begin{array}{rrr|r} \boxed{1} & 2 & 3 & 4 \\ 3 & \boxed{2} & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Thus the pivot columns in the matrix are the first two columns. ♠

The following is an algorithm for carrying a matrix to row-echelon form and reduced row-echelon form. You may wish to use this algorithm to carry the above matrix to row-echelon form or reduced row-echelon form yourself for practice.

Algorithm 1.19: Reduced Row-Echelon Form Algorithm

This algorithm provides a method for using row operations to take a matrix to its reduced row-echelon form. We begin with the matrix in its original form.

1. *Starting from the left, find the first nonzero column. This is the first pivot column, and the position at the top of this column is the first pivot position. Switch rows if necessary to place a nonzero number in the first pivot position.*
2. *Use row operations to make the entries below the first pivot position (in the first pivot column) equal to zero.*
3. *Ignoring the row containing the first pivot position, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more rows to modify.*
4. *Divide each nonzero row by the value of the leading entry, so that the leading entry becomes 1. The matrix will then be in row-echelon form.*

The following step will carry the matrix from row-echelon form to reduced row-echelon form.

5. *Moving from right to left, use row operations to create zeros in the entries of the pivot columns which are above the pivot positions. The result will be a matrix in reduced row-echelon form.*

Most often we will apply this algorithm to an augmented matrix in order to find the solution to a system of linear equations. However, we can use this algorithm to compute the reduced row-echelon form of any matrix which could be useful in other applications.

Consider the following example of Algorithm 1.19.

Example 1.20: Finding Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

Let

$$A = \begin{bmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix}$$

Find the row-echelon form of A. Then complete the process until A is in reduced row-echelon form.

Solution. In working through this example, we will use the steps outlined in Algorithm 1.19.

1. The first pivot column is the first column of the matrix, as this is the first nonzero column from the left. Hence the first pivot position is the one in the first row and first column. Switch the first two rows to obtain a nonzero entry in the first pivot position, outlined in a box below.

$$\begin{bmatrix} \boxed{1} & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix}$$

2. Step two involves creating zeros in the entries below the first pivot position. The first entry of the second row is already a zero. All we need to do is subtract 5 times the first row from the third row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & -10 & -8 \end{bmatrix}$$

3. Now ignore the top row. Apply steps 1 and 2 to the smaller matrix

$$\begin{bmatrix} -5 & -4 \\ -10 & -8 \end{bmatrix}$$

In this matrix, the first column is a pivot column, and -5 is in the first pivot position. Therefore, we need to create a zero below it. To do this, add -2 times the first row (of this matrix) to the second. The resulting matrix is

$$\begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix}$$

Our original matrix now looks like

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that there are no more rows to modify.

4. Now, we need to create leading 1s in each row. The first row already has a leading 1 so no work is needed here. Divide the second row by -5 to create a leading 1. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in row-echelon form.

5. Now create zeros in the entries above pivot positions in each column, in order to carry this matrix all the way to reduced row-echelon form. Notice that there is no pivot position in the third column so we do not need to create any zeros in this column! The column in which we need to create zeros is the second. To do so, subtract 4 times the second row from the first row. The resulting matrix is

$$\left[\begin{array}{ccc} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{array}\right]$$

This matrix is now in reduced row-echelon form. ♠

The above algorithm gives you a simple way to obtain the row-echelon form and reduced row-echelon form of a matrix. The main idea is to do row operations in such a way as to end up with a matrix in row-echelon form or reduced row-echelon form. This process is important because the resulting matrix will allow you to describe the solutions to the corresponding linear system of equations in a meaningful way.

In the next example, we look at how to solve a system of equations using the corresponding augmented matrix.

Example 1.21: Finding the Solution to a System

Give the complete solution to the following system of equations

$$\begin{array}{c} 2x+4y-3z=-1 \\ 5x+10y-7z=-2 \\ 3x+6y+5z=9 \end{array}$$

Solution. The augmented matrix for this system is

$$\left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array}\right]$$

In order to find the solution to this system, we wish to carry the augmented matrix to reduced row-echelon form. We will do so using Algorithm 1.19. Notice that the first column is nonzero, so this is our first pivot column. The first entry in the first row, 2, is the first leading entry and it is in the first pivot position. We will use row operations to create zeros in the entries below the 2. First, replace the second row with -5 times the first row plus 2 times the second row. This yields

$$\left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{array}\right]$$

Now, replace the third row with -3 times the first row plus to 2 times the third row. This yields

$$\left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 21 \end{array}\right]$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot position. We need to do just one row operation to create a zero below the 1.

Taking -1 times the second row and adding it to the third row yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

We could proceed with the algorithm to carry this matrix to row-echelon form or reduced row-echelon form. However, remember that we are looking for the solutions to the system of equations. Take another look at the third row of the matrix. Notice that it corresponds to the equation

$$0x + 0y + 0z = 20$$

There is no solution to this equation because for all x, y, z, the left side will equal 0 and $0 \neq 20$. This shows there is no solution to the given system of equations. In other words, this system is inconsistent. ♠

The following is another example of how to find the solution to a system of equations by carrying the corresponding augmented matrix to reduced row-echelon form.

Example 1.22: An Infinite Set of Solutions

Give the complete solution to the system of equations

$$\begin{array}{c} 3x - y - 5z = 9 \\ y - 10z = 0 \\ -2x + y = -6 \end{array} \tag{1.8}$$

Solution. The augmented matrix of this system is

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array}\right]$$

In order to find the solution to this system, we will carry the augmented matrix to reduced row-echelon form, using Algorithm 1.19. The first column is the first pivot column. We want to use row operations to create zeros beneath the first entry in this column, which is in the first pivot position. Replace the third row with 2 times the first row added to 3 times the third row. This gives

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{array}\right]$$

Now, we have created zeros beneath the 3 in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Take -1 times the second row and add to the third row.

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The entry below the pivot position in the second column is now a zero. Notice that we have no more pivot columns because we have only two leading entries.

At this stage, we also want the leading entries to be equal to one. To do so, divide the first row by 3.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is now in row-echelon form.

Let's continue with row operations until the matrix is in reduced row-echelon form. This involves creating zeros above the pivot positions in each pivot column. This requires only one step, which is to add

$\frac{1}{3}$ times the second row to the first row.

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is in reduced row-echelon form, which you should verify using Definition 1.13. The equations corresponding to this reduced row-echelon form are

$$\begin{aligned} x - 5z &= 3 \\ y - 10z &= 0 \end{aligned}$$

or

$$\begin{aligned} x &= 3 + 5z \\ y &= 10z \end{aligned}$$

Observe that z is not restrained by any equation. In fact, z can equal any number. For example, we can let $z = t$, where we can choose t to be any number. In this context t is called a **parameter** . Therefore, the solution set of this system is

$$\begin{aligned} x &= 3 + 5t \\ y &= 10t \\ z &= t \end{aligned}$$

where t is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of t we select, x, y, and z will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$\begin{aligned} x &= 3 + 5(4) = 23 \\ y &= 10(4) = 40 \\ z &= 4 \end{aligned}$$

♠

In Example 1.22 the solution involved one parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

Example 1.23: A Two Parameter Set of Solutions

Find the solution to the system

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned}$$

Solution. The augmented matrix is
$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right]$$
We wish to carry this matrix to row-echelon form. Here, we will outline the row operations used. However, make sure that you understand the steps in terms of Algorithm 1.19.

Take -1 times the first row and add to the second. Then take -1 times the first row and add to the third. This yields
$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]$$
Now add the second row to the third row and divide the second row by -1.
$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{1.9}$$

This matrix is in row-echelon form and we can see that x and y correspond to pivot columns, while z and w do not. Therefore, we will assign parameters to the variables z and w. Assign the parameter s to z and the parameter t to w. Then the first row yields the equation $x+2y-s+t=3$, while the second row yields the equation $y=2$. Since $y=2$, the first equation becomes $x+4-s+t=3$ showing that the solution is given by
$$\begin{aligned} x &= -1+s-t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned}$$
It is customary to write this solution in the form
$$\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} -1+s-t \\ 2 \\ s \\ t \end{array}\right] \tag{1.10}$$

♠

This example shows a system of equations with an infinite solution set which depends on two parameters. It can be less confusing in the case of an infinite solution set to first place the augmented matrix in reduced row-echelon form rather than just row-echelon form before seeking to write down the description of the solution.

In the above steps, this means we don't stop with the row-echelon form in equation 1.9. Instead we first place it in reduced row-echelon form as follows.
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
Then the solution is $y=2$ from the second row and $x=-1+z-w$ from the first. Thus letting $z=s$ and $w=t$, the solution is given by 1.10.

You can see here that there are two paths to the correct answer, which both yield the same answer. Hence, either approach may be used. The process which we first used in the above solution is called

Gaussian Elimination. This process involves carrying the matrix to row-echelon form, converting back to equations, and using back substitution to find the solution. When you do row operations until you obtain reduced row-echelon form, the process is called **Gauss-Jordan Elimination**.

We have now found solutions for systems of equations with no solution and infinitely many solutions, with one parameter as well as two parameters. Recall the three types of solution sets which we discussed in the previous section; no solution, one solution, and infinitely many solutions. Each of these types of solutions could be identified from the graph of the system. It turns out that we can also identify the type of solution from the reduced row-echelon form of the augmented matrix.

- *No Solution:* In the case where the system of equations has no solution, the reduced row-echelon form of the augmented matrix will have a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$$

  This row indicates that the system is inconsistent and has no solution.

- *One Solution:* In the case where the system of equations has one solution, every column of the coefficient matrix is a pivot column. The following is an example of an augmented matrix in reduced row-echelon form for a system of equations with one solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

- *Infinitely Many Solutions:* In the case where the system of equations has infinitely many solutions, the solution contains parameters. There will be columns of the coefficient matrix which are not pivot columns. The following are examples of augmented matrices in reduced row-echelon form for systems of equations with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  or

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \end{array}\right]$$

1.2.3 Uniqueness of the Reduced Row-Echelon Form

As we have seen in earlier sections, we know that every matrix can be brought into reduced row-echelon form by a sequence of elementary row operations. Here we will prove that the resulting matrix is unique; in other words, the resulting matrix in reduced row-echelon form does not depend upon the particular sequence of elementary row operations or the order in which they were performed.

Let A be the augmented matrix of a homogeneous system of linear equations in the variables $x_1, x_2, \cdots, x_n$ which is also in reduced row-echelon form. The matrix A divides the set of variables in two different types. We say that x_i is a **basic variable** whenever A has a leading 1 in column number i, in other words, when column i is a pivot column. Otherwise we say that x_i is a **free variable**.

Recall Example 1.23.

Example 1.24: Basic and Free Variables

Find the basic and free variables in the system

$$\begin{aligned} x+2y-z+w&=3\\ x+y-z+w&=1\\ x+3y-z+w&=5 \end{aligned}$$

Solution. Recall from the solution of Example 1.23 that the row-echelon form of the augmented matrix of this system is given by

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3\\ 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

You can see that columns 1 and 2 are pivot columns. These columns correspond to variables x and y, making these the basic variables. Columns 3 and 4 are not pivot columns, which means that z and w are free variables.

We can write the solution to this system as

$$\begin{aligned} x&=-1+s-t\\ y&=2\\ z&=s\\ w&=t \end{aligned}$$

Here the free variables are written as parameters, and the basic variables are given by linear functions of these parameters. ♠

In general, all solutions can be written in terms of the free variables. In such a description, the free variables can take any values (they become parameters), while the basic variables become simple linear functions of these parameters. Indeed, a basic variable x_i is a linear function of *only* those free variables x_j with $j>i$. This leads to the following observation.

Proposition 1.25: Basic and Free Variables

If x_i is a basic variable of a homogeneous system of linear equations, then any solution of the system with $x_j=0$ for all those free variables x_j with $j>i$ must also have $x_i=0$.

Using this proposition, we prove a lemma which will be used in the proof of the main result of this section below.

Lemma 1.26: Solutions and the Reduced Row-Echelon Form of a Matrix

Let A and B be two distinct augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced row-echelon form. Then, the two systems do not have exactly the same solutions.

Proof. With respect to the linear systems associated with the matrices A and B, there are two cases to consider:

- Case 1: the two systems have the same basic variables
- Case 2: the two systems do not have the same basic variables

In case 1, the two matrices will have exactly the same pivot positions. However, since A and B are not identical, there is some row of A which is different from the corresponding row of B and yet the rows each have a pivot in the same column position. Let i be the index of this column position. Since the matrices are in reduced row-echelon form, the two rows must differ at some entry in a column $j > i$. Let these entries be a in A and b in B, where $a \neq b$. Since A is in reduced row-echelon form, if x_j were a basic variable for its linear system, we would have $a = 0$. Similarly, if x_j were a basic variable for the linear system of the matrix B, we would have $b = 0$. Since a and b are unequal, they cannot both be equal to 0, and hence x_j cannot be a basic variable for both linear systems. However, since the systems have the same basic variables, x_j must then be a free variable for each system. We now look at the solutions of the systems in which x_j is set equal to 1 and all other free variables are set equal to 0. For this choice of parameters, the solution of the system for matrix A has $x_i = -a$, while the solution of the system for matrix B has $x_i = -b$, so that the two systems have different solutions.

In case 2, there is a variable x_i which is a basic variable for one matrix, let's say A, and a free variable for the other matrix B. The system for matrix B has a solution in which $x_i = 1$ and $x_j = 0$ for all other free variables x_j. However, by Proposition 1.25 this cannot be a solution of the system for the matrix A. This completes the proof of case 2. ♠

Now, we say that the matrix B is **equivalent** to the matrix A provided that B can be obtained from A by performing a sequence of elementary row operations beginning with A. The importance of this concept lies in the following result.

Theorem 1.27: Equivalent Matrices

The two linear systems of equations corresponding to two equivalent augmented matrices have exactly the same solutions.

The proof of this theorem is left as an exercise.

Now, we can use Lemma 1.26 and Theorem 1.27 to prove the main result of this section.

Theorem 1.28: Uniqueness of the Reduced Row-Echelon Form

Every matrix A is equivalent to a unique matrix in reduced row-echelon form.

Proof. Let A be an $m \times n$ matrix and let B and C be matrices in reduced row-echelon form, each equivalent to A. It suffices to show that $B = C$.

Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C each with a rightmost column of zeros to obtain B^+ and C^+. Note that B^+ and C^+ are matrices in reduced row-echelon form which are obtained from A^+ by respectively applying the same sequence of elementary row operations which were used to obtain B and C from A.

Now, A^+, B^+, and C^+ can all be considered as augmented matrices of homogeneous linear systems in the variables $x_1, x_2, \cdots, x_n$. Because B^+ and C^+ are each equivalent to A^+, Theorem 1.27 ensures that all three homogeneous linear systems have exactly the same solutions. By Lemma 1.26 we conclude that $B^+ = C^+$. By construction, we must also have $B = C$. ♠

According to this theorem we can say that each matrix A has a unique reduced row-echelon form.

1.2.4 Rank and Homogeneous Systems

There is a special type of system which requires additional study. This type of system is called a homogeneous system of equations, which we defined above in Definition 1.3. Our focus in this section is to consider what types of solutions are possible for a homogeneous system of equations.

Consider the following definition.

Definition 1.29: Trivial Solution

Consider the homogeneous system of equations given by

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

*Then, $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution to this system. We call this the **trivial solution**.*

If the system has a solution in which not all of the $x_1, \cdots, x_n$ are equal to zero, then we call this solution **nontrivial** . The trivial solution does not tell us much about the system, as it says that $0 = 0$! Therefore, when working with homogeneous systems of equations, we want to know when the system has a nontrivial solution.

Suppose we have a homogeneous system of m equations, using n variables, and suppose that $n > m$. In other words, there are more variables than equations. Then, it turns out that this system always has a nontrivial solution. Not only will the system have a nontrivial solution, but it also will have infinitely many solutions. It is also possible, but not required, to have a nontrivial solution if $n = m$ and $n < m$.

Consider the following example.

Example 1.30: Solutions to a Homogeneous System of Equations

Find the nontrivial solutions to the following homogeneous system of equations

$$\begin{array}{c} 2x + y - z = 0 \\ x + 2y - 2z = 0 \end{array}$$

Solution. Notice that this system has $m = 2$ equations and $n = 3$ variables, so $n > m$. Therefore by our previous discussion, we expect this system to have infinitely many solutions.

The process we use to find the solutions for a homogeneous system of equations is the same process

we used in the previous section. First, we construct the augmented matrix, given by we used in the previous section. First, we construct the augmented matrix, given by

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right]$$

Then, we carry this matrix to its reduced row-echelon form, given below.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x &= 0 \\ y - z &= 0 \end{aligned}$$

Since z is not restrained by any equation, we know that this variable will become our parameter. Let $z = t$ where t is any number. Therefore, our solution has the form

$$\begin{aligned} x &= 0 \\ y = z &= t \\ z &= t \end{aligned}$$

Hence this system has infinitely many solutions, with one parameter t. ♠

Suppose we were to write the solution to the previous example in another form. Specifically,

$$\begin{aligned} x &= 0 \\ y &= 0 + t \\ z &= 0 + t \end{aligned}$$

can be written as

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] + t\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]$$

Notice that we have constructed a column from the constants in the solution (all equal to 0), as well as a column corresponding to the coefficients on t in each equation. While we will discuss this form of solution more in further chapters, for now consider the column of coefficients of the parameter t. In this case, this is the column $\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]$.

There is a special name for this column, which is **basic solution**. The basic solutions of a system are columns constructed from the coefficients on parameters in the solution. We often denote basic solutions by X_1, X_2 etc., depending on how many solutions occur. Therefore, Example 1.30 has the basic solution $X_1 = \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]$.

We explore this further in the following example.

Example 1.31: Basic Solutions of a Homogeneous System

Consider the following homogeneous system of equations.

$$x + 4y + 3z = 0$$

$$3x+12y+9z=0$$

Find the basic solutions to this system.

Solution. The augmented matrix of this system and the resulting reduced row-echelon form are

$$\left[\begin{array}{rrr|r} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

When written in equations, this system is given by

$$x+4y+3z=0$$

Notice that only x corresponds to a pivot column. In this case, we will have two parameters, one for y and one for z. Let $y=s$ and $z=t$ for any numbers s and t. Then, our solution becomes

$$\begin{aligned} x &= -4s-3t \\ y &= s \\ z &= t \end{aligned}$$

which can be written as

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] + s\left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right] + t\left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right]$$

You can see here that we have two columns of coefficients corresponding to parameters, specifically one for s and one for t. Therefore, this system has two basic solutions! These are

$$X_1 = \left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right], X_2 = \left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right]$$

♠

We now present a new definition.

Definition 1.32: Linear Combination

*Let $X_1, \cdots, X_n, V$ be column matrices. Then V is said to be a **linear combination** of the columns $X_1, \cdots, X_n$ if there exist scalars, $a_1, \cdots, a_n$ such that*

$$V = a_1 X_1 + \cdots + a_n X_n$$

A remarkable result of this section is that a linear combination of the basic solutions is again a solution to the system. Even more remarkable is that every solution can be written as a linear combination of these

solutions. Therefore, if we take a linear combination of the two solutions to Example 1.31, this would also be a solution. For example, we could take the following linear combination

$$3\left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right] + 2\left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{r} -18 \\ 3 \\ 2 \end{array}\right]$$

You should take a moment to verify that

You should take a moment to verify that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is in fact a solution to the system in Example 1.31.

Another way in which we can find out more information about the solutions of a homogeneous system is to consider the **rank** of the associated coefficient matrix. We now define what is meant by the rank of a matrix.

Definition 1.33: Rank of a Matrix

Let A be a matrix and consider any row-echelon form of A. Then, the number r of leading entries of A does not depend on the row-echelon form you choose, and is called the **rank** *of A. We denote it by rank(A).*

Similarly, we could count the number of pivot positions (or pivot columns) to determine the rank of A.

Example 1.34: Finding the Rank of a Matrix

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

What is its rank?

Solution. First, we need to find the reduced row-echelon form of A. Through the usual algorithm, we find that this is

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here we have two leading entries, or two pivot positions, shown above in boxes.The rank of A is $r = 2$. ♠

Notice that we would have achieved the same answer if we had found the row-echelon form of A instead of the reduced row-echelon form.

Suppose we have a homogeneous system of m equations in n variables, and suppose that $n > m$. From our above discussion, we know that this system will have infinitely many solutions. If we consider the

rank of the coefficient matrix of this system, we can find out even more about the solution. Note that we are looking at just the coefficient matrix, not the entire augmented matrix.

Theorem 1.35: Rank and Solutions to a Homogeneous System

Let A be the $m \times n$ coefficient matrix corresponding to a homogeneous system of equations, and suppose A has rank r. Then, the solution to the corresponding system has $n - r$ parameters.

Consider our above Example 1.31 in the context of this theorem. The system in this example has $m = 2$ equations in $n = 3$ variables. First, because $n > m$, we know that the system has a nontrivial solution, and therefore infinitely many solutions. This tells us that the solution will contain at least one parameter. The rank of the coefficient matrix can tell us even more about the solution! The rank of the coefficient matrix of the system is 1, as it has one leading entry in row-echelon form. Theorem 1.35 tells us that the solution will have $n-r=3-1=2$ parameters. You can check that this is true in the solution to Example 1.31.

Notice that if $n=m$ or $n<m$, it is possible to have either a unique solution (which will be the trivial solution) or infinitely many solutions.

We are not limited to homogeneous systems of equations here. The rank of a matrix can be used to learn about the solutions of any system of linear equations. In the previous section, we discussed that a system of equations can have no solution, a unique solution, or infinitely many solutions. Suppose the system is consistent, whether it is homogeneous or not. The following theorem tells us how we can use the rank to learn about the type of solution we have.

Theorem 1.36: Rank and Solutions to a Consistent System of Equations

Let A be the $m\times(n+1)$ augmented matrix corresponding to a consistent system of equations in n variables, and suppose A has rank r. Then

1. *the system has a unique solution if $r=n$*
2. *the system has infinitely many solutions if $r<n$*

We will not present a formal proof of this, but consider the following discussions.

1. *No Solution* The above theorem assumes that the system is consistent, that is, that it has a solution. It turns out that it is possible for the augmented matrix of a system with no solution to have any rank r as long as $r>1$. Therefore, we must know that the system is consistent in order to use this theorem!
2. *Unique Solution* Suppose $r=n$. Then, there is a pivot position in every column of the coefficient matrix of A. Hence, there is a unique solution.
3. *Infinitely Many Solutions* Suppose $r<n$. Then there are infinitely many solutions. There are less pivot positions (and hence less leading entries) than columns, meaning that not every column is a pivot column. The columns which are *not* pivot columns correspond to parameters. In fact, in this case we have $n-r$ parameters.

1.2.5 Balancing Chemical Reactions

The tools of linear algebra can also be used in the subject area of Chemistry, specifically for balancing chemical reactions.

Consider the chemical reaction

$$SnO_2+H_2\rightarrow Sn+H_2O$$

Here the elements involved are tin (Sn), oxygen (O), and hydrogen (H). A chemical reaction occurs and the result is a combination of tin (Sn) and water (H_2O). When considering chemical reactions, we want to investigate how much of each element we began with and how much of each element is involved in the result.

An important theory we will use here is the mass balance theory. It tells us that we cannot create or delete elements within a chemical reaction. For example, in the above expression, we must have the same number of oxygen, tin, and hydrogen on both sides of the reaction. Notice that this is not currently the number of oxygen, tin, and hydrogen on both sides of the reaction. Notice that this is not currently the case. For example, there are two oxygen atoms on the left and only one on the right. In order to fix this, we want to find numbers x, y, z, w such that

$$xSnO_2 + yH_2 \rightarrow zSn + wH_2O$$

where both sides of the reaction have the same number of atoms of the various elements.

This is a familiar problem. We can solve it by setting up a system of equations in the variables x, y, z, w. Thus you need

$$\begin{array}{rl} Sn: & x = z \\ O: & 2x = w \\ H: & 2y = 2w \end{array}$$

We can rewrite these equations as

$$\begin{array}{rl} Sn: & x - z = 0 \\ O: & 2x - w = 0 \\ H: & 2y - 2w = 0 \end{array}$$

The augmented matrix for this system of equations is given by

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right]$$

The reduced row-echelon form of this matrix is

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right]$$

The solution is given by

$$\begin{array}{c} x - \frac{1}{2}w = 0 \\ y - w = 0 \\ z - \frac{1}{2}w = 0 \end{array}$$

which we can write as

$$\begin{array}{c} x = \frac{1}{2}t \\ y = t \\ z = \frac{1}{2}t \\ w = t \end{array}$$

For example, let $w = 2$ and this would yield $x = 1, y = 2$, and $z = 1$. We can put these values back into the expression for the reaction which yields

$$SnO_2 + 2H_2 \rightarrow Sn + 2H_2O$$

Observe that each side of the expression contains the same number of atoms of each element. This means that it preserves the total number of atoms, as required, and so the chemical reaction is balanced.

Consider another example.

Example 1.37: Balancing a Chemical Reaction

Potassium is denoted by K, oxygen by O, phosphorus by P and hydrogen by H. Consider the *reaction given by*

$$KOH + H_3PO_4 \rightarrow K_3PO_4 + H_2O$$

Balance this chemical reaction.

Solution. We will use the same procedure as above to solve this problem. We need to find values for x, y, z, w such that

$$xKOH + yH_3PO_4 \rightarrow zK_3PO_4 + wH_2O$$

preserves the total number of atoms of each element.

Finding these values can be done by finding the solution to the following system of equations.

$$\begin{array}{rl} K: & x = 3z \\ O: & x + 4y = 4z + w \\ H: & x + 3y = 2w \\ P: & y = z \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

and the reduced row-echelon form is

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is given by

$$\begin{aligned} x - w &= 0 \\ y - \tfrac{1}{3}w &= 0 \\ z - \tfrac{1}{3}w &= 0 \end{aligned}$$

which can be written as

$$\begin{aligned} x &= t \\ y &= \tfrac{1}{3}t \\ z &= \tfrac{1}{3}t \\ w &= t \end{aligned}$$

Choose a value for t, say 3. Then $w = 3$ and this yields $x = 3, y = 1, z = 1$. It follows that the balanced reaction is given by

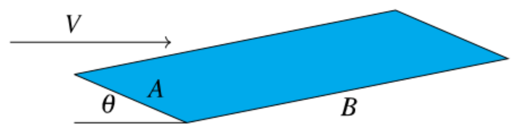
$$3KOH + 1H_3PO_4 \rightarrow 1K_3PO_4 + 3H_2O$$

Note that this results in the same number of atoms on both sides. ♠

Of course these numbers you are finding would typically be the number of moles of the molecules on each side. Thus three moles of KOH added to one mole of H_3PO_4 yields one mole of K_3PO_4 and three moles of H_2O.

1.2.6 Dimensionless Variables

This section shows how solving systems of equations can be used to determine appropriate dimensionless variables. It is only an introduction to this topic and considers a specific example of a simple airplane wing shown below. We assume for simplicity that it is a flat plane at an angle to the wind which is blowing against it with speed V as shown.



The angle θ is called the angle of incidence, B is the span of the wing and A is called the chord. Denote by l the lift. Then this should depend on various quantities like θ, V, B, A and so forth. Here is a table which indicates various quantities on which it is reasonable to expect l to depend.

Variable	Symbol	Units
chord	A	m
span	B	m
angle incidence	θ	$m^0 kg^0 \sec^0$
speed of wind	V	$m \sec^{-1}$
speed of sound	V_0	$m \sec^{-1}$
density of air	ρ	kgm^{-3}
viscosity	μ	$kg \sec^{-1} m^{-1}$
lift	l	$kg \sec^{-2} m$

Here m denotes meters, sec refers to seconds and kg refers to kilograms. All of these are likely familiar except for μ, which we will discuss in further detail now.

Viscosity is a measure of how much internal friction is experienced when the fluid moves. It is roughly a measure of how “sticky" the fluid is. Consider a piece of area parallel to the direction of motion of the fluid. To say that the viscosity is large is to say that the tangential force applied to this area must be large in order to achieve a given change in speed of the fluid in a direction normal to the tangential force. Thus

$$\mu \,(\text{area})\,(\text{velocity gradient}) = \text{ tangential force}$$

Hence

$$(\text{units on } \mu)\, m^2 \left(\frac{m}{\sec m}\right) = kg \sec^{-2} m$$

Thus the units on μ are

$$kg \sec^{-1} m^{-1}$$

as claimed above.

Returning to our original discussion, you may think that we would want

$$l = f(A, B, \theta, V, V_0, \rho, \mu)$$

This is very cumbersome because it depends on seven variables. Also, it is likely that without much care, a change in the units such as going from meters to feet would result in an incorrect value for l. The way to get around this problem is to look for l as a function of dimensionless variables multiplied by something which has units of force. It is helpful because first of all, you will likely have fewer independent variables and secondly, you could expect the formula to hold independent of the way of specifying length, mass and so forth. One looks for

$$l = f(g_1, \cdots, g_k)\,\rho V^2 AB$$

where the units on $\rho V^2 AB$ are

$$\frac{kg}{m^3}\left(\frac{m}{\sec}\right)^2 m^2 = \frac{kg \times m}{\sec^2}$$

which are the units of force. Each of these g_i is of the form

$$A^{x_1} B^{x_2} \theta^{x_3} V^{x_4} V_0^{x_5} \rho^{x_6} \mu^{x_7} \tag{1.11}$$

and each g_i is independent of the dimensions. That is, this expression must not depend on meters, kilograms, seconds, etc. Thus, placing in the units for each of these quantities, one needs

$$m^{x_1} m^{x_2} \left(m^{x_4} \sec^{-x_4}\right) \left(m^{x_5} \sec^{-x_5}\right) \left(kgm^{-3}\right)^{x_6} \left(kg \sec^{-1} m^{-1}\right)^{x_7} = m^0 kg^0 \sec^0$$

Notice that there are no units on θ because it is just the radian measure of an angle. Hence its dimensions consist of length divided by length, thus it is dimensionless. Then this leads to the following equations for the x_i.

$$\begin{array}{rl} m: & x_1 + x_2 + x_4 + x_5 - 3x_6 - x_7 = 0 \\ \sec: & -x_4 - x_5 - x_7 = 0 \\ kg: & x_6 + x_7 = 0 \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

The reduced row-echelon form is given by

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

and so the solutions are of the form

$$\begin{array}{rcl} x_1 & = & -x_2 - x_7 \\ x_3 & = & x_3 \\ x_4 & = & -x_5 - x_7 \\ x_6 & = & -x_7 \end{array}$$

Thus, in terms of vectors, the solution is

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array}\right] = \left[\begin{array}{c} -x_2 - x_7 \\ x_2 \\ x_3 \\ -x_5 - x_7 \\ x_5 \\ -x_7 \\ x_7 \end{array}\right]$$

Thus the free variables are x_2, x_3, x_5, x_7. By assigning values to these, we can obtain dimensionless variables by placing the values obtained for the x_i in the formula 1.11. For example, let $x_2 = 1$ and all the rest of the free variables are 0. This yields

$$x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0$$

The dimensionless variable is then $A^{-1}B^{1}$. This is the ratio between the span and the chord. It is called the aspect ratio, denoted as AR. Next let $x_3 = 1$ and all others equal zero. This gives for a dimensionless quantity the angle θ. Next let $x_5 = 1$ and all others equal zero. This gives

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 1, x_6 = 0, x_7 = 0$$

Then the dimensionless variable is $V^{-1}V_0^{1}$. However, it is written as V/V_0. This is called the Mach number $\mathscr{M}$. Finally, let $x_7 = 1$ and all the other free variables equal 0. Then

$$x_1 = -1, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 0, x_6 = -1, x_7 = 1$$

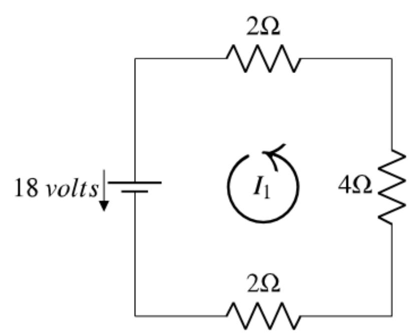
then the dimensionless variable which results from this is $A^{-1}V^{-1}\rho^{-1}\mu$. It is customary to write it as $\mathrm{Re} = (AV\rho)/\mu$. This one is called the Reynold's number. It is the one which involves viscosity. Thus we would look for

$$l = f(\mathrm{Re}, AR, \theta, \mathscr{M})\, kg \times m/\sec^2$$

This is quite interesting because it is easy to vary Re by simply adjusting the velocity or A but it is hard to vary things like μ or ρ. Note that all the quantities are easy to adjust. Now this could be used, along with wind tunnel experiments to get a formula for the lift which would be reasonable. You could also consider more variables and more complicated situations in the same way.

1.2.7 An Application to Resistor Networks

The tools of linear algebra can be used to study the application of resistor networks. An example of an electrical circuit is below.



The jagged lines ( ) denote resistors and the numbers next to them give their resistance in ohms, written as Ω. The voltage source ( ) causes the current to flow in the direction from the shorter of the two lines toward the longer (as indicated by the arrow). The current for a circuit is labelled I_k.

In the above figure, the current I_1 has been labelled with an arrow in the counter clockwise direction. This is an entirely arbitrary decision and we could have chosen to label the current in the clockwise direction. With our choice of direction here, we define a positive current to flow in the counter clockwise direction and a negative current to flow in the clockwise direction.

The goal of this section is to use the values of resistors and voltage sources in a circuit to determine the current. An essential theorem for this application is Kirchhoff's law.

Theorem 1.38: Kirchhoff's Law

The sum of the resistance (R) times the amps (I) in the counter clockwise direction around a loop

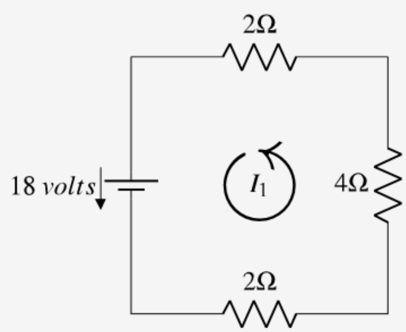
The sum of the resistance (R) times the amps (I) in the counter clockwise direction around a loop equals the sum of the voltage sources (V) in the same direction around the loop.

Kirchhoff's law allows us to set up a system of linear equations and solve for any unknown variables. When setting up this system, it is important to trace the circuit in the counter clockwise direction. If a resistor or voltage source is crossed against this direction, the related term must be given a negative sign.

We will explore this in the next example where we determine the value of the current in the initial diagram.

Example 1.39: Solving for Current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1.



Solution. Begin in the bottom left corner, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $2I_1$. Continuing in this way through all three resistors gives $2I_1 + 4I_1 + 2I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source matches the counter clockwise direction specified, so the voltage is positive.

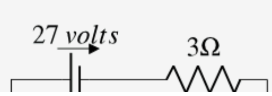
Therefore the equation and solution are given by

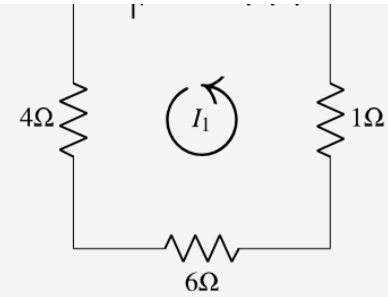
$$\begin{aligned} 2I_1 + 4I_1 + 2I_1 &= 18 \\ 8I_1 &= 18 \\ I_1 &= \frac{9}{4}A \end{aligned}$$

Since the answer is positive, this confirms that the current flows counter clockwise. ♠

Example 1.40: Solving for Current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1.





Solution. Begin in the top left corner this time, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $4I_1$. Continuing in this way through the four

resistors gives $4I_1 + 6I_1 + 1I_1 + 3I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source is opposite to the counter clockwise direction, so the voltage is negative.

Therefore the equation and solution are given by

$$\begin{aligned} 4I_1 + 6I_1 + 1I_1 + 3I_1 &= -27 \\ 14I_1 &= -27 \\ I_1 &= -\frac{27}{14}A \end{aligned}$$

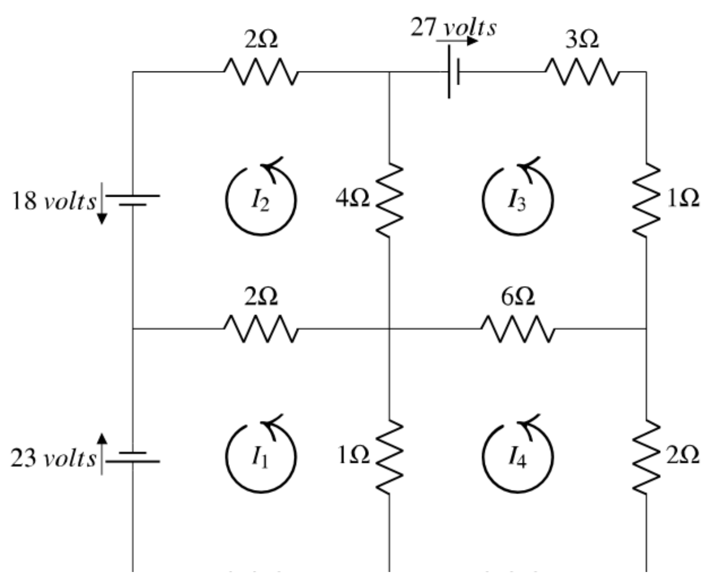
Since the answer is negative, this tells us that the current flows clockwise. ♠

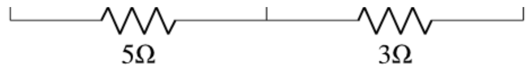
A more complicated example follows. Two of the circuits below may be familiar; they were examined in the examples above. However as they are now part of a larger system of circuits, the answers will differ.

Example 1.41: Unknown Currents

The diagram below consists of four circuits. The current (I_k) in the four circuits is denoted by I_1, I_2, I_3, I_4. Using Kirchhoff's Law, write an equation for each circuit and solve for each current.

Solution. The circuits are given in the following diagram.





Starting with the top left circuit, multiply the resistance by the amps and sum the resulting products. Specifically, consider the resistor labelled 2Ω that is part of the circuits of I_1 and I_2. Notice that current I_2 runs through this in a positive (counter clockwise) direction, and I_1 runs through in the opposite (negative) direction. The product of resistance and amps is then $2(I_2 - I_1) = 2I_2 - 2I_1$. Continue in this way for each resistor, and set the sum of the products equal to the voltage source to write the equation:

$$2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 = 18$$

The above process is used on each of the other three circuits, and the resulting equations are:

Upper right circuit:

$$4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + 3I_3 = -27$$

Lower right circuit:

$$3I_4 + 2I_4 + 6I_4 - 6I_3 + I_4 - I_1 = 0$$

Lower left circuit:

$$5I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -23$$

Notice that the voltage for the upper right and lower left circuits are negative due to the clockwise direction they indicate.

The resulting system of four equations in four unknowns is

$$\begin{array}{rcl} 2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 & = & 18 \\ 4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + I_3 & = & -27 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 & = & 0 \\ 5I_1 + I_1 - I_4 + 2I_1 - 2I_2 & = & -23 \end{array}$$

Simplifying and rearranging with variables in order, we have:

$$\begin{array}{rcl} -2I_1 + 8I_2 - 4I_3 & = & 18 \\ -4I_2 + 14I_3 - 6I_4 & = & -27 \\ -I_1 - 6I_3 + 12I_4 & = & 0 \\ 8I_1 - 2I_2 - I_4 & = & -23 \end{array}$$

The augmented matrix is

$$\left[\begin{array}{rrrr|r} -2 & 8 & -4 & 0 & 18 \\ 0 & -4 & 14 & -6 & -27 \\ -1 & 0 & -6 & 12 & 0 \\ 8 & -2 & 0 & -1 & -23 \end{array}\right]$$

The solution to this matrix is

$$\begin{array}{rcl} I_1 & = & -3A \\ I_2 & = & \frac{1}{4}A \\ I_3 & = & -\frac{5}{2}A \\ I_4 & = & -\frac{3}{2}A \end{array}$$

This tells us that currents I_1, I_3, and I_4 travel clockwise while I_2 travels counter clockwise. ♠