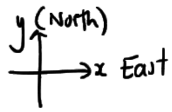
Day 2

July 4, 2023 10:03 AM

$$z = f(x,y) = 6 - xy^2$$



point $P(2,1,4)$

- rate of change of a function f at a point P if you move in a direction $\frac{\vec{v}}{\|\vec{v}\|}$ with speed $\|\vec{v}\|$

velocity $\vec{v}$

rate of change $= D_{\vec{v}} f(P)$

$= \nabla f(P) \cdot \vec{v}$, if f is $\boxed{\text{differentiable}}$

$= \left\langle \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P) \right\rangle \cdot \vec{v}$ first-order partials exist and are continuous

- rate of change of a function f at a point P if you move in a direction $\frac{\vec{v}}{\|\vec{v}\|}$ with speed $\cancel{\|\vec{v}\|}$ 1

velocity $\vec{v}$

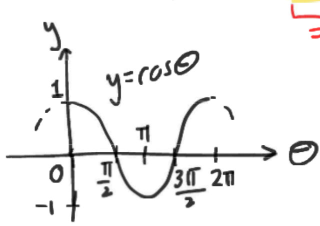
directional derivative

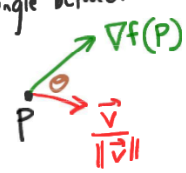
~~rate of change~~ $= D_{\frac{\vec{v}}{\|\vec{v}\|}} f(P)$

$= \nabla f(P) \cdot \frac{\vec{v}}{\|\vec{v}\|}$

$= \left\langle \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P) \right\rangle \cdot \frac{\vec{v}}{\|\vec{v}\|}$

$= \|\nabla f(P)\| \underbrace{\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\|}_{=1} \underbrace{\cos\theta}_{=1}$ angle between



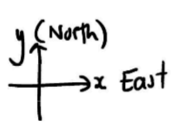


a) hiker should move in $-\frac{\nabla f(P)}{\|\nabla f(P)\|}$ to get steepest ~~ascent~~ descent

the maximal rate of ~~increase~~ decrease $= -\|\nabla f(P)\|$ (slope)

point $z = f(x,y) = 6 - xy^2$

$P(2,1,4)$ $z = f(2,1)$



$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -y^2, -2xy \rangle$$

$$\nabla f(2,1) = \langle -1^2, -2(2)(1) \rangle = \langle -1, -4 \rangle$$

$$\text{direction} = \frac{\nabla f(2,1)}{\|\nabla f(2,1)\|} = \frac{\langle -1,-4 \rangle}{\sqrt{(-1)^2+(-4)^2}} = \frac{1}{\sqrt{17}}\langle -1,-4\rangle$$

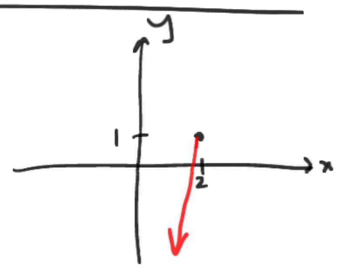
$$= \left\langle \frac{-1}{\sqrt{17}}, \frac{-4}{\sqrt{17}} \right\rangle$$

$$\text{Slope} = \text{maximal rate of increase} = \|\nabla f(2,1)\| = \sqrt{17}$$

$$\frac{1}{\sqrt{17}}\langle -1,-4\rangle$$

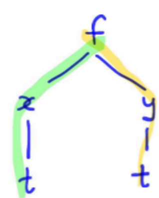
$$\frac{1}{\sqrt{17}}\langle -4, 1\rangle$$

or $\frac{1}{\sqrt{17}}\langle 4, -1\rangle$



§2.4, Q 16 $\quad z = f(x,y)$

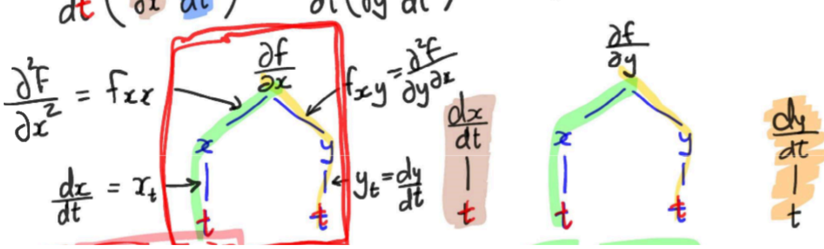
Find $\frac{d^2}{dt^2} z(x(t), y(t))$



$$\begin{tikzcd}
\mathbb{R} & \mathbb{R}^2 & \mathbb{R} \\
{[t]} \arrow[r, mapsto] \arrow[rr, bend right, "z = z(t)"'] & \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \arrow[r, mapsto, "f"] & {[f(x,y)]}
\end{tikzcd}$$

$$\frac{d^2}{dt^2} z = \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(\boxed{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}} \right)$$

$$= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\frac{dy}{dt}\right), \text{ by Sum Rule}$$



$$= \underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d}{dt}\left(\frac{dx}{dt}\right)}_{\text{by Product Rule}} + \underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d}{dt}\left(\frac{dy}{dt}\right)}_{\text{by Product Rule}}$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\right)\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \left(\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right)\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}$$

$\frac{\partial f}{\partial x}(2,1) = 5$, $\frac{\partial f}{\partial y}(2,1) = -2$, $f_{xy}(2,1) = 1$, $f_{xx}(2,1) = 2$, $f_{yy}(2,1) = -4$

$$x(t) = 2t^2 \longrightarrow \frac{dx}{dt} = 4t \longrightarrow \frac{d^2x}{dt^2} = 4$$

when $t=1$, $\left.\frac{dx}{dt}\right|_{t=1} = 4$

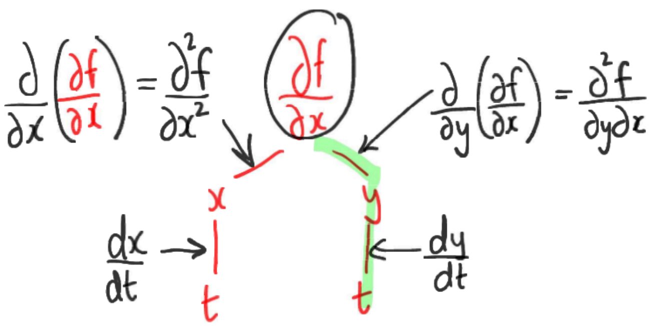
$$y(t) = t^3 \longrightarrow \frac{dy}{dt} = 3t^2 \longrightarrow \frac{d^2y}{dt^2} = 6t$$

$$\left.\frac{dy}{dt}\right|_{t=1} = 3(1)^2 = 3 \qquad \left.\frac{d^2y}{dt^2}\right|_{t=1} = 6(1) = 6$$

$$= (2\cdot 4 + 1\cdot 3)4 + 5\cdot 4 + (1\cdot 4 + (-4)3)3 + (-2)6$$

$$= 44 + 20 - 24 - 12$$

$$= 28$$



$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\cdot\frac{dy}{dt}$$

$$f_{xx}\, x_t + f_{xy}\, y_t$$

§2-6, Q22b Given $\boxed{z=f(x,y)}$, Surface is the graph of $f(x,y)$.

$\frac{\partial f}{\partial x}=3$, $\frac{\partial f}{\partial y}=-2$ when $(x,y,z)=(1,3,1)$.

Find eqn of tangent plane at $(\underset{x_0}{1},\underset{y_0}{3},\underset{z_0=f(x_0,y_0)=f(1,3)}{1})$.

Sol'n 1: $z=f(1,3)+\frac{\partial f}{\partial x}(1,3)(x-1)+\frac{\partial f}{\partial y}(1,3)(y-3)$

$z=1+3(x-1)+(-2)(y-3)$

$z=1+3x-3-2y+6$

$\boxed{z=3x-2y+4}$

Sol'n 2: $\underbrace{f(x,y)-z}_{G(x,y,z)}=0$

$\boxed{G(x,y,z)=K}$

$(x_0,y_0,z_0)=(1,3,1)$

eqn of tangent plane:

$\left\langle \frac{\partial f}{\partial x}(1,3), \frac{\partial f}{\partial y}(1,3), -1\right\rangle \cdot \langle x-1, y-3, z-1\rangle=0$

$\langle 3,-2,-1\rangle\cdot\langle x-1,y-3,z-1\rangle=0$

$3x-3-2y+6-z+1=0$

$\boxed{3x-2y+4=z}$

$\nabla G(x_0,y_0,z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle=0$

$\left\langle \frac{\partial G}{\partial x}(x_0,y_0,z_0), \frac{\partial G}{\partial y}(x_0,y_0,z_0), \frac{\partial G}{\partial z}(x_0,y_0,z_0)\right\rangle\cdot\langle x-x_0,y-y_0,z-z_0\rangle=0$

Q28

§3-3

Does $\displaystyle\sum_{n=3}^{\infty} \frac{1\;5}{n(\ln n)^{3/2}}$ converge?

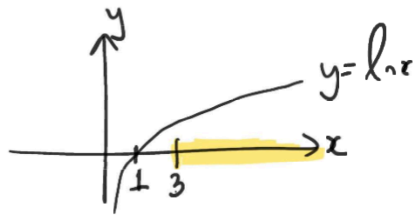
$\ln(n)$ has derivative $\frac{1}{n}$

Let $f : [3,\infty) \to \mathbb{R}$ be given by $f(x) = \dfrac{5}{x[\ln(x)]^{3/2}}$.

(Apply Integral Test)

(i) Since $x \geq 3 > 1$,

$\ln x > \ln(1) = 0$



So $(\ln x)^{3/2} > 0$ and, thus, $f(x)\ \dfrac{5}{x(\ln x)^{3/2}} > 0$ for $x \in [3,\infty)$

(ii) Suppose $x_1, x_2 \in [3,\infty)$ and $x_1 < x_2$.

(We want to show $f(x_2) < f(x_1)$.)

$\ln x_1 < \ln x_2 \Rightarrow [\ln x_1]^{3/2} < [\ln x_2]^{3/2}$

$x_1[\ln x_1]^{3/2} < x_2[\ln x_2]^{3/2}$ $\qquad \dfrac{5}{x[\ln(x)]^{3/2}}$.

$$\frac{1}{x_1[\ln x_1]^{3/2}} > \frac{1}{x_2[\ln x_2]^{3/2}}$$

$$f(x_1) = \frac{5}{x_1[\ln x_1]^{3/2}} > \frac{5}{x_2[\ln x_2]^{3/2}} = f(x_2)$$

(iii) for $n = 3, 4, \ldots$, $f(n) = \dfrac{5}{n(\ln n)^{3/2}}$

$$\int_3^\infty \frac{5}{x(\ln x)^{3/2}}\,dx = 5\int_{\ln 3}^\infty \frac{1}{u^{3/2}}\,du = \frac{5}{-\frac{3}{2}+1}\, u^{-\frac{3}{2}+1}\Big|_{\ln 3}^{\infty}$$

$u^{-3/2}$

Let $u = \ln x$

Then $du = \frac{1}{x}dx$

$$= \frac{-10}{\sqrt{u}}\Big|_{\ln 3}^{\infty}$$

$$= 0 + \frac{10}{\sqrt{\ln 3}}$$