HUMBER ENGINEERING

MENG-3020
SYSTEMS MODELING & SIMULATION

LECTURE 7

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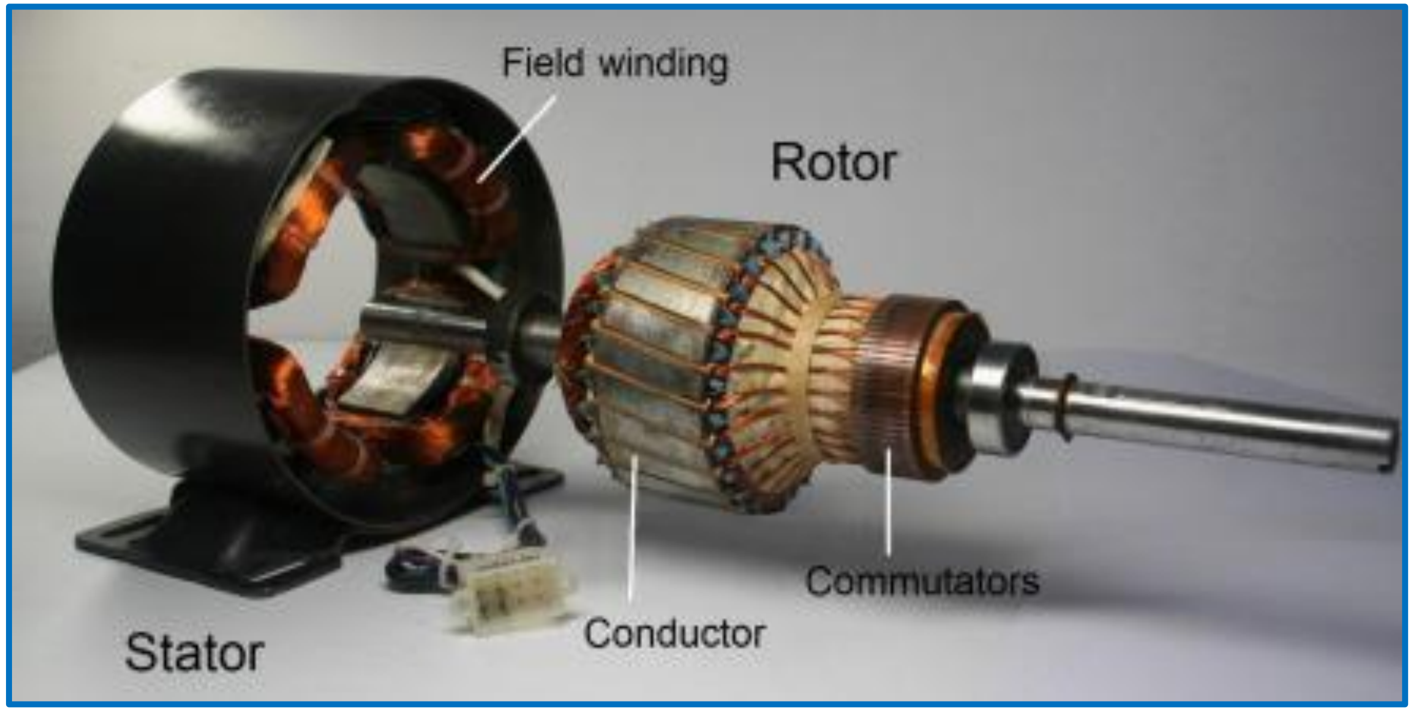
LECTURE 7
Modeling of Electromechanical Systems

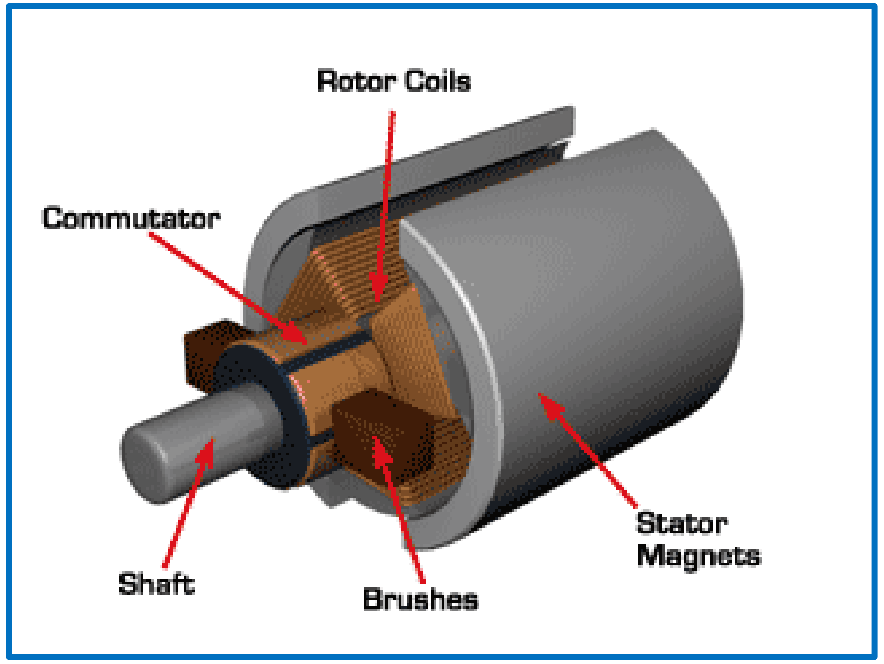
- Basic Characteristics of DC Motors
    - Armature-Controlled DC Motor
    - Field-Controlled DC Motor
        - Differential Equation Model
        - Block Diagram Model
        - Transfer Function Model
        - State-space Model

Modeling of Electromechanical Systems

❑ Basic DC Motor

- **DC motor** is basically an electro-mechanical conversion device that converts electric energy into mechanical energy.
- **DC motors** have five principal components:
  - Stator
  - Field System
  - Armature or Rotor
  - Commutator
  - Brushes





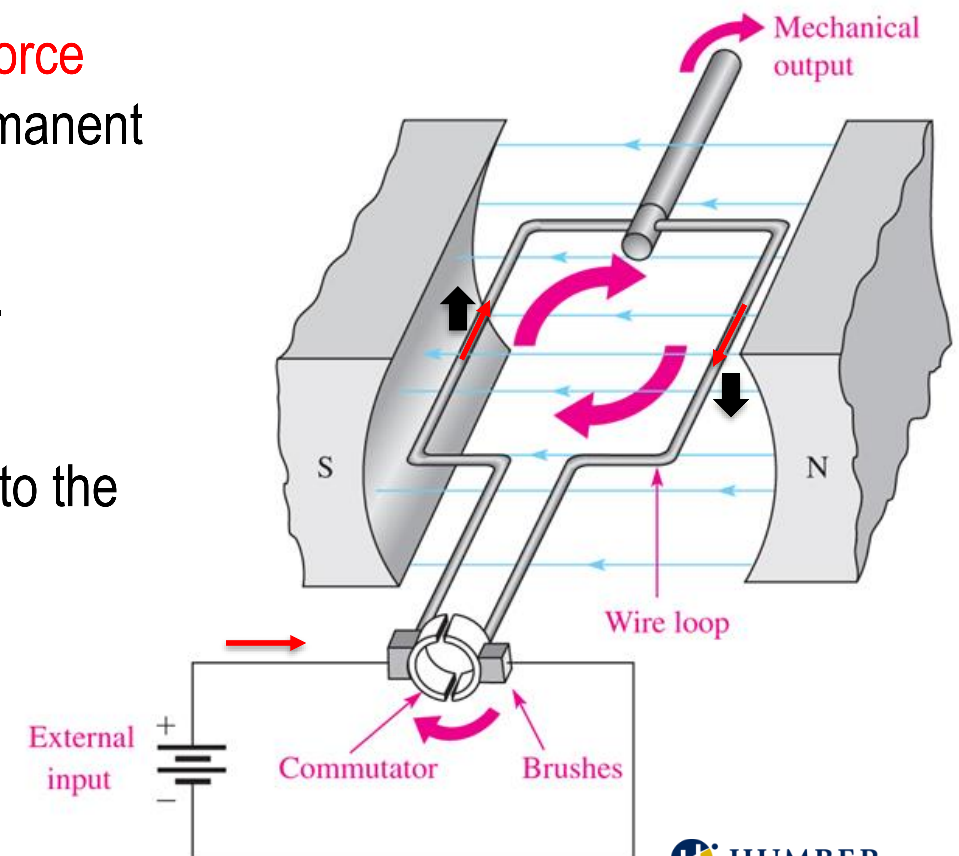
Modeling of Electromechanical Systems

❑ Operation of a Basic DC Motor

- A motor is a machine that converts electrical energy into mechanical energy by taking advantage of the force produced when a current-carrying conductor is in a magnetic field.
- A basic DC motor consist of a single loop of wire that is called **armature** or **rotor**, in a permanent magnet field. Each end of the wire loop is connected to a segment of **commutator**.
- As soon as the external input applies, a large current flows in the armature because its resistance is very low.
- The individual armature conductors are immediately subjected to a force because they are immersed in the magnetic field created by the permanent magnets.
- The force produce a powerful torque, causing the armature to rotate. Torque is noting but a twisting force acts on the armature to rotate it.
- The developed torque by the armature of a DC motor is proportional to the field flux and the armature current:

$$\boxed{\tau \propto \phi_f i_a}$$

Basic DC Motor



Modeling of Electromechanical Systems

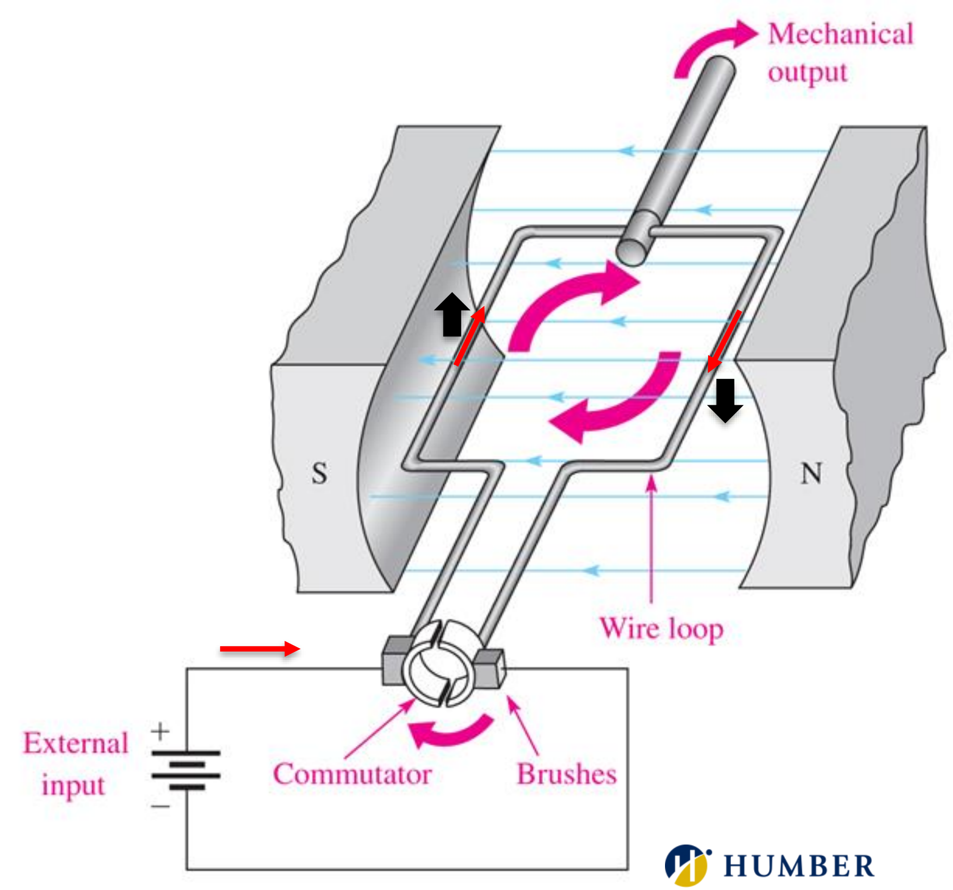
❑ Operation of a Basic DC Motor

- When the armature begins to rotate, a second phenomenon takes place: **The generator effect**.
- We know that when the armature conductors **cut** the magnetic field flux, a voltage is induced in the armature conductors. (*This is always true no matter what causes the rotation.*)
- The induced voltage is obtained as below.

$$\boxed{v_b \propto \phi_f \omega}$$

- In the case of a motor, based on Lenz's law, the direction of the induced voltage v_b is opposite to the applied external input voltage.
- Therefore, it is called back-electromotive-force (back-emf).

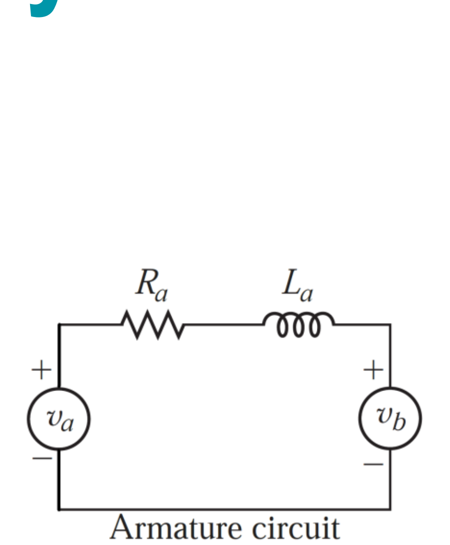
Basic DC Motor

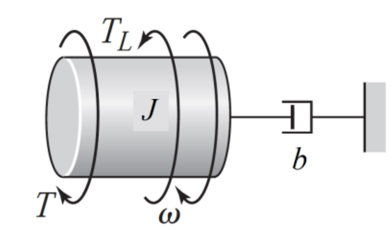


Modeling of Electromechanical Systems

❑ Model of DC Motors

- **DC motors** can be modeled by the following equivalent electro-mechanical model by considering two subsystems:
  - **Electrical Subsystem**
    - Armature circuit, including the armature winding <u>resistance</u> and <u>inductance</u>
    - Field circuit, including the field winding <u>resistance</u> and <u>inductance</u>
  - **Mechanical Subsystem**
    - Inertia due to the <u>load</u> as well as the <u>armature inertia</u>.
    - Damping can be present because of <u>shaft bearings</u> or <u>load damping</u>, such as with a fan or pump.





R_a = Armature Resistance, Ω

L_a = Armature Inductance, H

R_f = Field Resistance, Ω

L_f = Field Inductance, H

v_a = Applied Armature Voltage, V

v_b = Back-emf, V

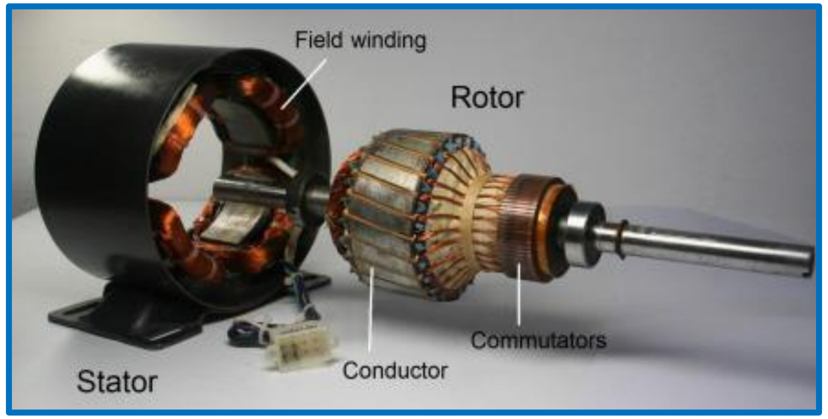
ω = Angular velocity, rad/sec

τ = Torque developed by the motor, $N.m$

J = Moment of inertia of the motor and load referred to the motor shaft,

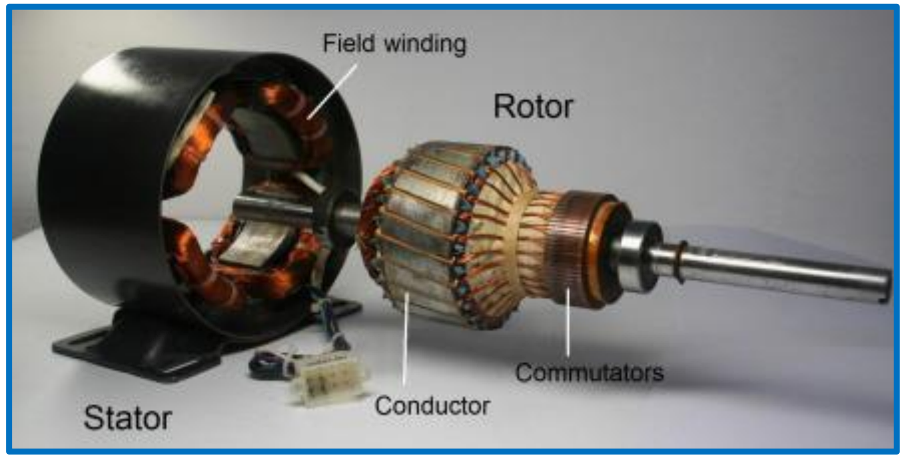
b = Viscous friction coefficient of the motor and load referred to the motor shaft,

- τ_L represents an additional torque acting on the load, other than the damping torque.
- The load torque τ_L opposes the motor torque in most applications, so we have shown it acting in the direction opposite that of τ.



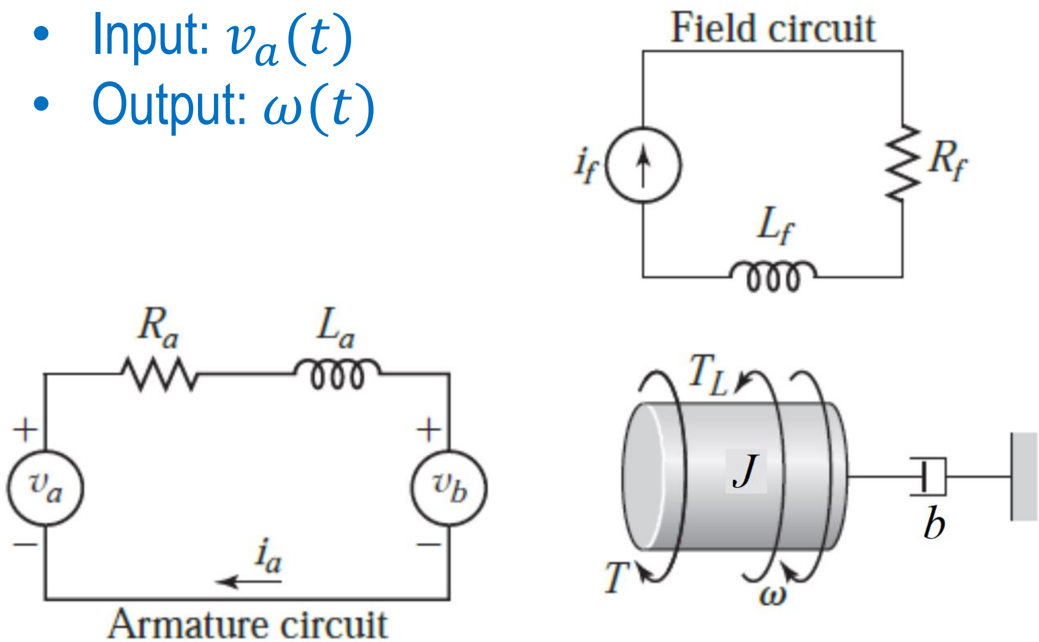
Modeling of Electromechanical Systems

❑ Control of DC Motor



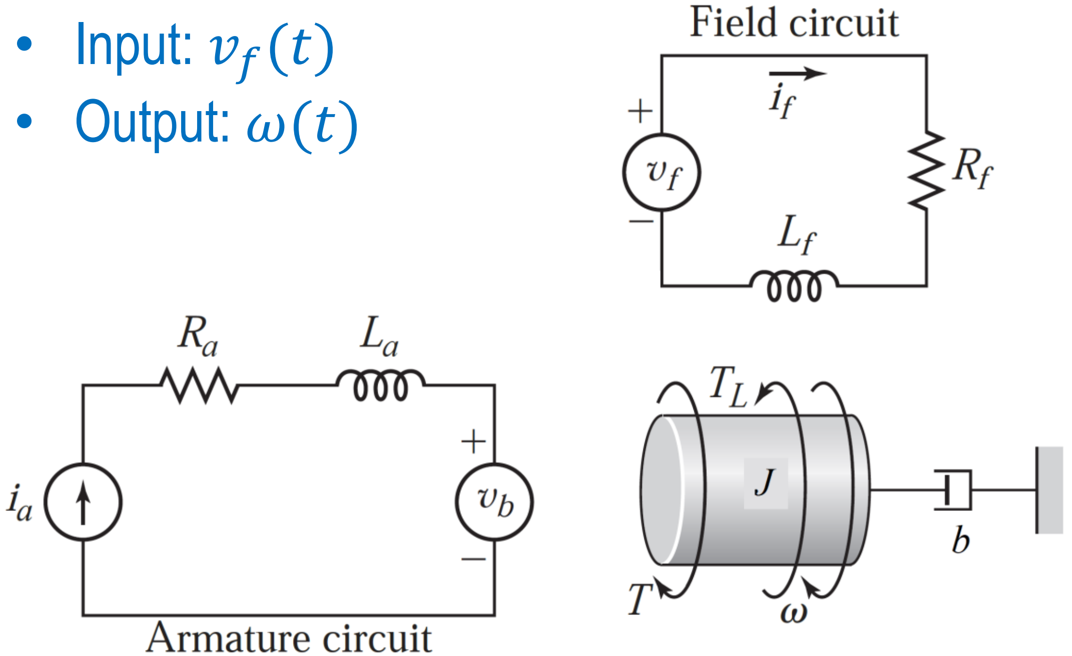
- Two general methods to control speed of DC motor:
  - **Armature-Controlled DC Motor**
    - Field current i_f and field flux ϕ_f is constant.
    - Using permanent magnet
  - **Field-Controlled DC Motor**
    - Armature current i_a is constant

- Input: $v_a(t)$
- Output: $\omega(t)$



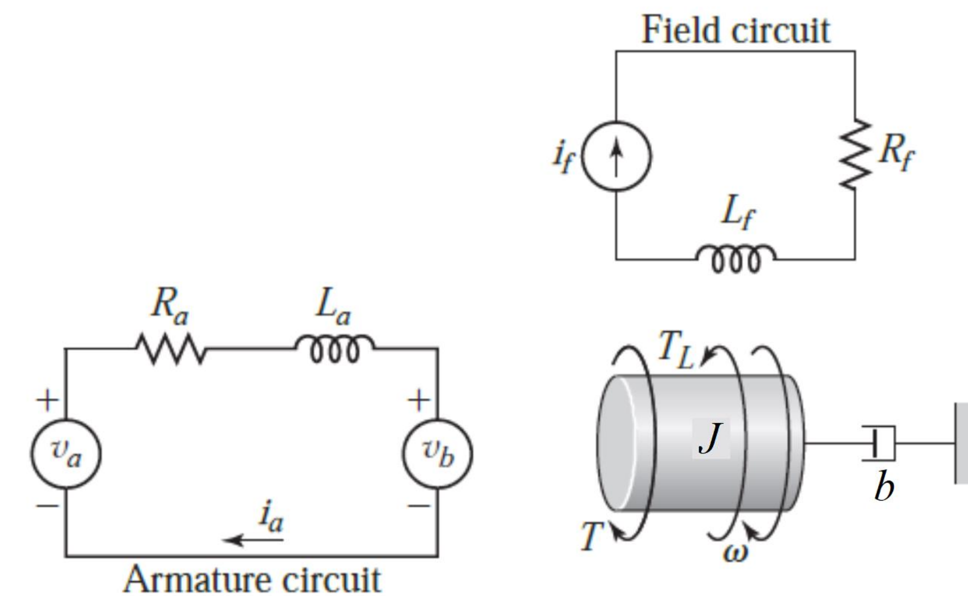
Armature-Controlled DC Motor

- Input: $v_f(t)$
- Output: $\omega(t)$



Field-Controlled DC Motor

Modeling of Armature-Controlled DC Motor



- The torque τ developed by the motor is proportional to the product of the armature current i_a and the magnetic flux ϕ_f.

$$\boldsymbol{\tau \propto \phi_f i_a}$$

- Since the field current i_f and the field flux ϕ_f are constant, the torque τ is directly proportional to the armature current i_a,

$$\boxed{\tau(t) = K_T i_a(t)}$$

where K_T is the motor's *torque constant*, the unit is $N.m/A$.

- When the armature is rotating, a voltage proportional to the product of the magnetic flux ϕ_f and angular velocity ω is induced in the armature.

$$\boldsymbol{v_b \propto \phi_f \omega}$$

- Since the field current i_f and the field flux ϕ_f are constant, the induced voltage v_b is directly proportional to the angular velocity ω,

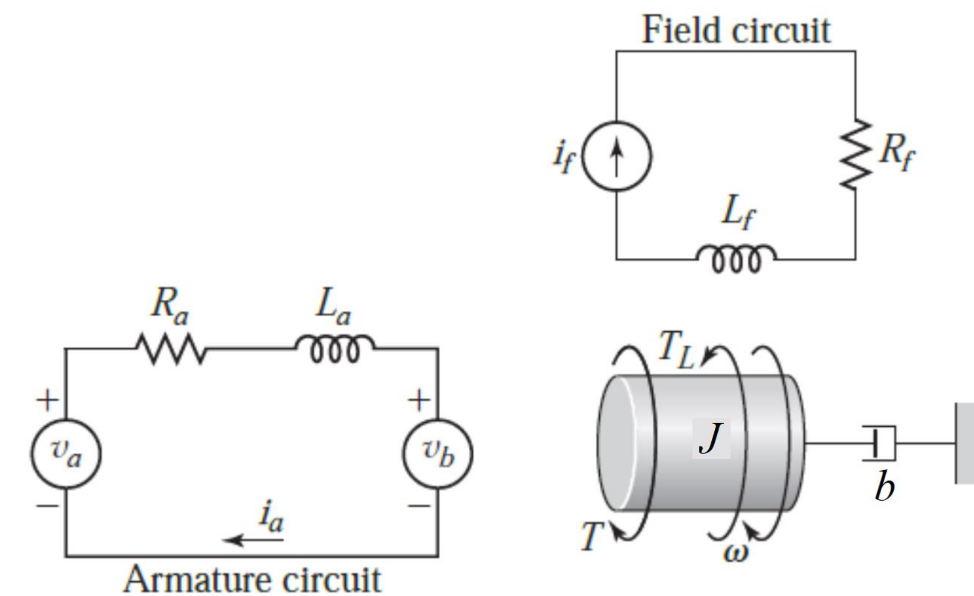
$$\boxed{v_b(t) = K_b \omega(t)}$$

where v_b is the *back-emf* K_b is the *back-emf constant*. the unit is $V.s/rad$.

Modeling of Armature-Controlled DC Motor

- **Relation between K_T and K_b**

$$\boxed{\tau(t) = K_T i_a(t)} \qquad \boxed{v_b(t) = K_b \omega(t)}$$



- For a given motor the relationship between the motor torque constant K_T and the back-emf constant K_b are obtained based on the mechanical power.
- The mechanical power developed in the armature is: $P(t) = v_b(t) i_a(t)$
- The mechanical power is also expressed as: $P(t) = \tau(t)\omega(t)$
- By substituting the K_T and K_b we have:

$$v_b(t) i_a(t) = \tau(t)\omega(t) \quad \rightarrow \quad K_b \omega(t) i_a(t) = K_T i_a(t) \omega(t)$$

- Thus, the values of K_b and K_T are identical if they represented in the following SI units:

$$K_b \left[\frac{V.s}{rad}\right] = K_T \left[\frac{N.m}{A}\right]$$

Modeling of Armature-Controlled DC Motor

❑ Differential Equation Model

- The speed of an armature-controlled DC motor is controlled by the armature voltage v_a.
- The differential equation for the armature circuit is obtained by applying a KVL in the armature :

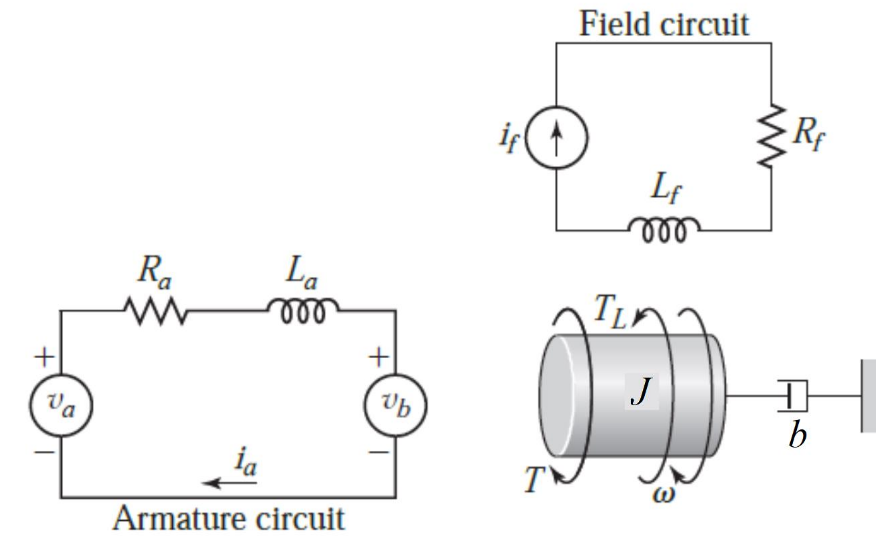
$$\boxed{v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t)}$$

- The armature current produces the torque that is applied to the inertia and friction. Hence, from the Newton's law applied to the inertia J,

$$\boxed{\tau(t) - \tau_L(t) = J\frac{d\omega\,(t)}{dt} + b\omega(t)}$$

- These two equations along with the previously obtained two equations constitute the system model.

$$\boxed{\tau(t) = K_T i_a(t)} \qquad \boxed{v_b(t) = K_b\omega(t)}$$



R_a = Armature Resistance, Ω

L_a = Armature Inductance, H

i_a = Armature Current, A

i_f = Field Current, A

v_a = Applied Armature Voltage, V

v_b = Back-emf, V

ω = Angular velocity of the motor shaft, rad/sec

θ = Angular displacement of the motor shaft, rad

τ = Torque developed by the motor, $N.m$

J = Moment of inertia of the motor and load referred to the motor shaft,

b = Viscous friction coefficient of the motor and load referred to the motor shaft,

Modeling of Armature-Controlled DC Motor

❑ Block Diagram Model

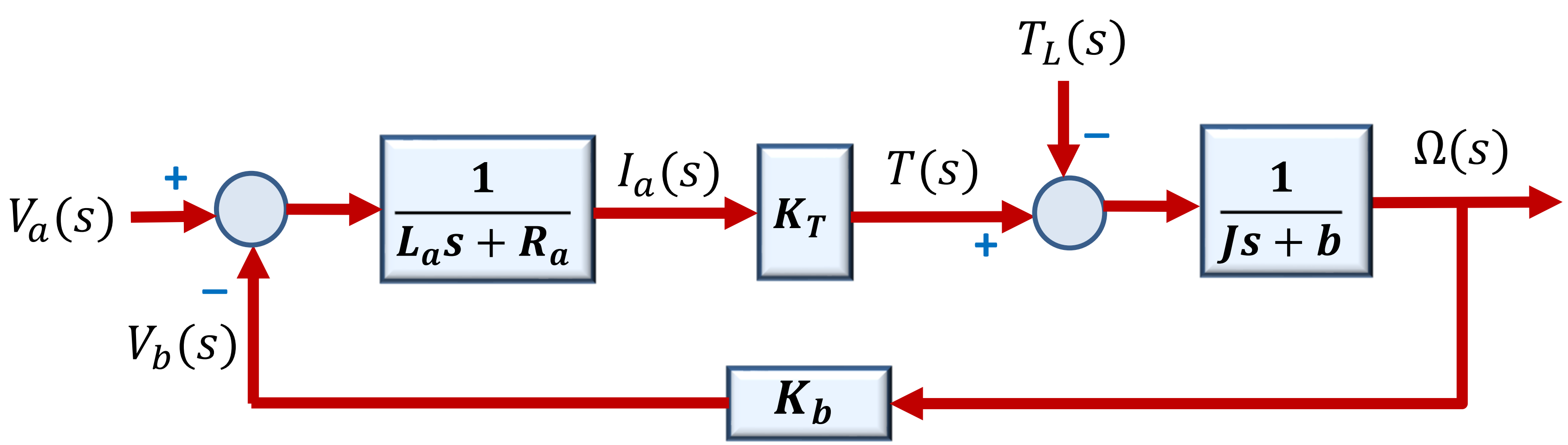
- Block diagram model of an armature-controlled DC motor with the applied voltage $v_a(t)$ as the input and the motor angular velocity $\omega(t)$ as the output is obtained as:

$$v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) \rightarrow V_a(s) = (L_a s + R_a) I_a(s) + V_b(s) \rightarrow \boxed{I_a(s) = \frac{1}{L_a s + R_a}\left(V_a(s) - V_b(s)\right)}$$

$$v_b(t) = K_b \omega(t) \rightarrow \boxed{V_b(s) = K_b \Omega(s)}$$

$$\tau(t) - \tau_L(t) = J\frac{d\omega\,(t)}{dt} + b\omega(t) \rightarrow T(s) - T_L(s) = (Js + b)\Omega(s) \rightarrow \boxed{\Omega(s) = \frac{1}{Js + b}\left(T(s) - T_L(s)\right)}$$

$$T(t) = K_T i_a(t) \rightarrow \boxed{T(s) = K_T I_a(s)}$$



Dynamics of the electrical and the mechanical subsystem are modeled as first-order systems.

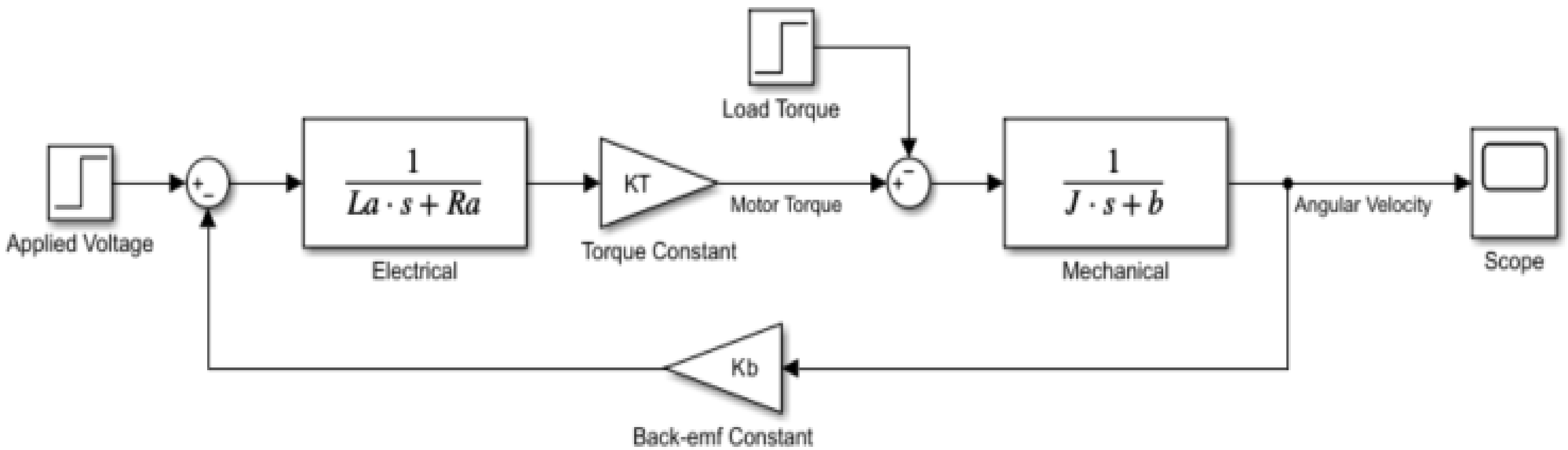
The back-emf acts as a negative feedback loop to slow down the motor's speed

Modeling of Armature-Controlled DC Motor

Example 1

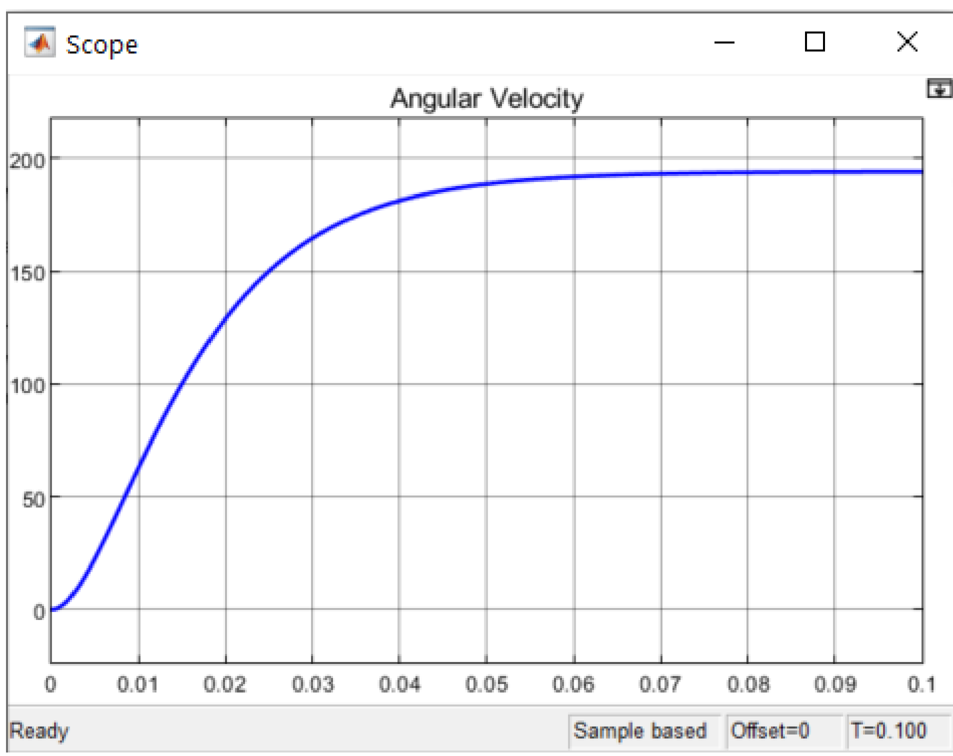
This example shows a Simulink implementation of the DC motor block diagram model.

If the applied voltage is 10V and the load torque is 0.01N.m, find the motor speed curve.

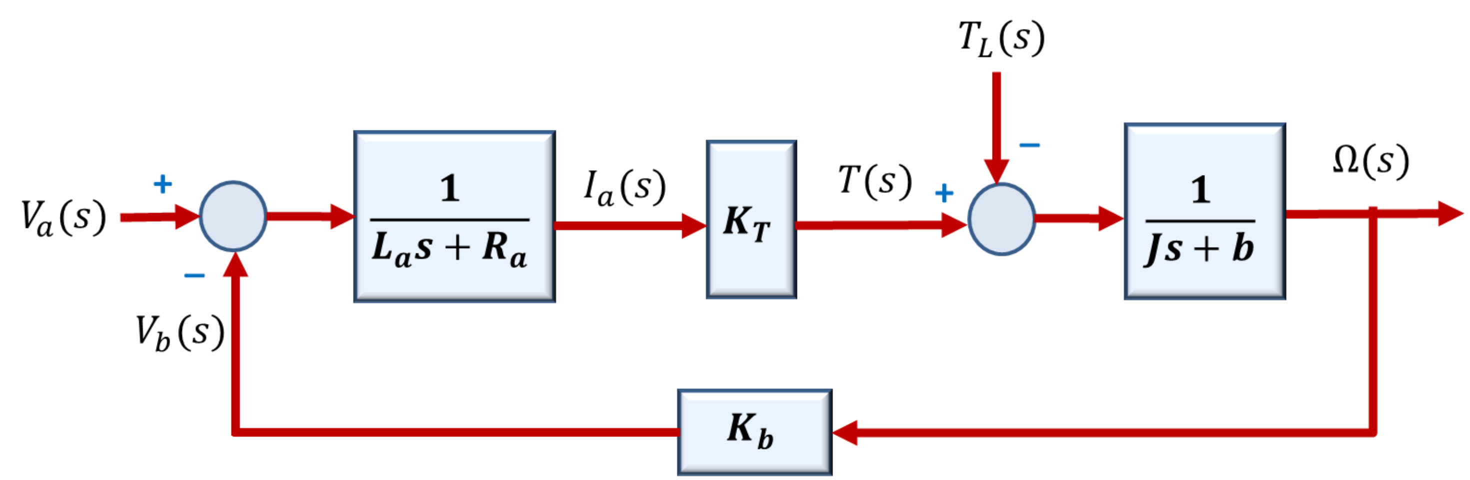



```
% DC Motor Parameters

KT = 0.05;
Kb = KT;
La = 2e-3;
Ra = 0.5;
J = 9e-5;
b = 1e-4;
Va = 10;
TL = 0.01;
```

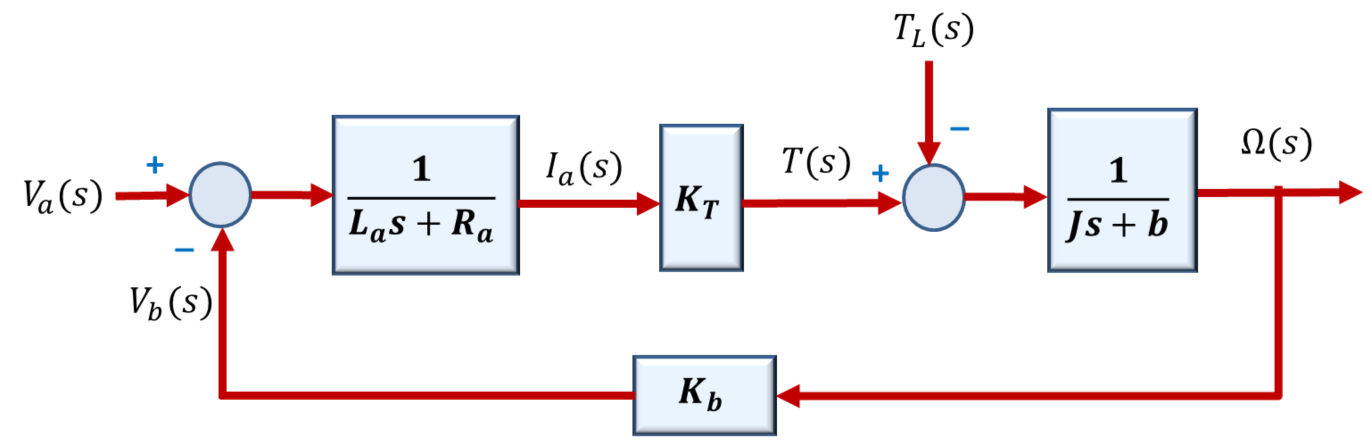


The final speed is about 194 rad/sec.

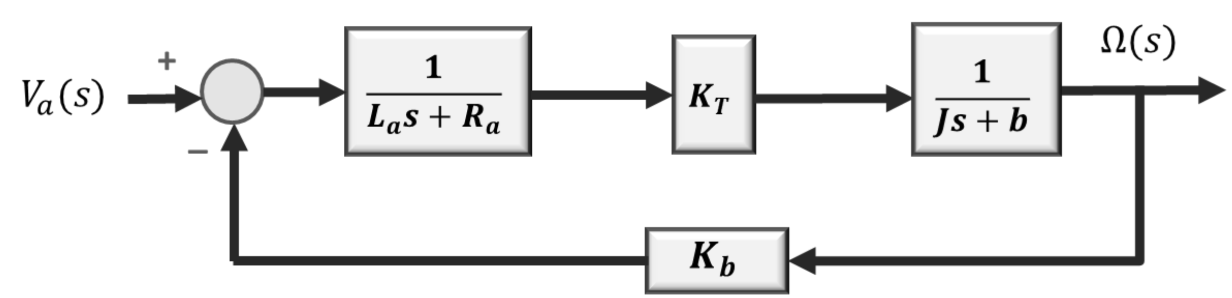


Modeling of Armature-Controlled DC Motor

❑ Transfer Function Model



- The transfer function model is obtained by finding the overall transfer function of the block diagram model.
- Since there are two independent inputs, we have to apply the superposition principle.
- Assume a **no-load** condition and set $T_L = 0$ to obtain the voltage-to-speed transfer function $\Omega(s)/V_a(s)$



$$\frac{\Omega(s)}{V_a(s)} = \frac{G}{1 + GH} = \frac{\frac{K_T}{(L_a s + R_a)(Js + b)}}{1 + \left(\frac{K_T}{(L_a s + R_a)(Js + b)}\right)(K_b)} = \frac{K_T}{(L_a s + R_a)(Js + b) + K_T K_b}$$

$$\boxed{\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_a J s^2 + (L_a b + J R_a)s + R_a b + K_T K_b}}$$

Voltage-to-Speed transfer function

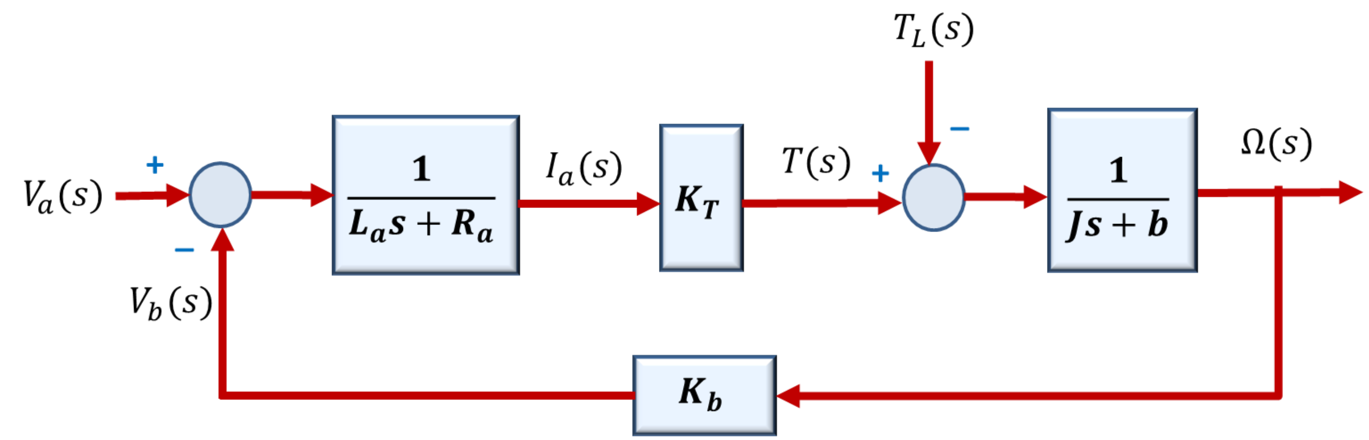
Modeling of Armature-Controlled DC Motor

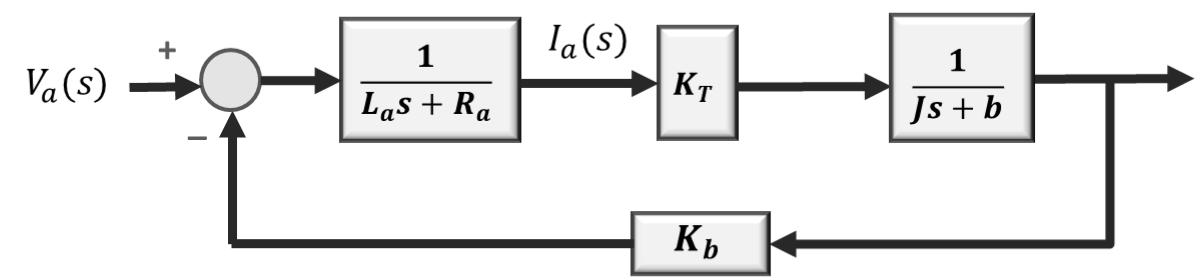
❑ Transfer Function Model

- The transfer function model is obtained by finding the overall transfer function of the block diagram model.

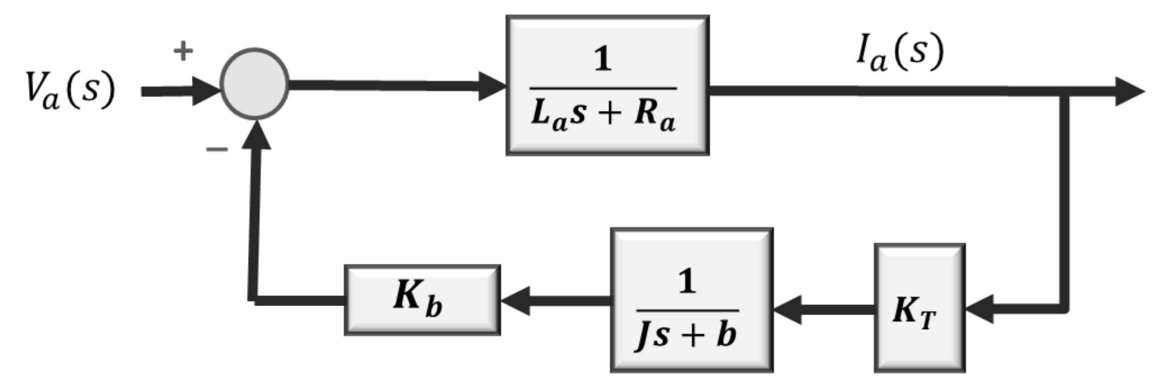
- Since there are two independent inputs, we have to apply the superposition principle.

- Assume a **no-load** condition and set $T_L = 0$ to obtain the voltage-to-current transfer function $I_a(s)/V_a(s)$





$$\frac{I_a(s)}{V_a(s)} = \frac{G}{1 + GH} = \frac{\dfrac{1}{L_a s + R_a}}{1 + \left(\dfrac{1}{L_a s + R_a}\right)\left(\dfrac{K_T K_b}{Js + b}\right)} = \frac{Js + b}{(L_a s + R_a)(Js + b) + K_T K_b}$$



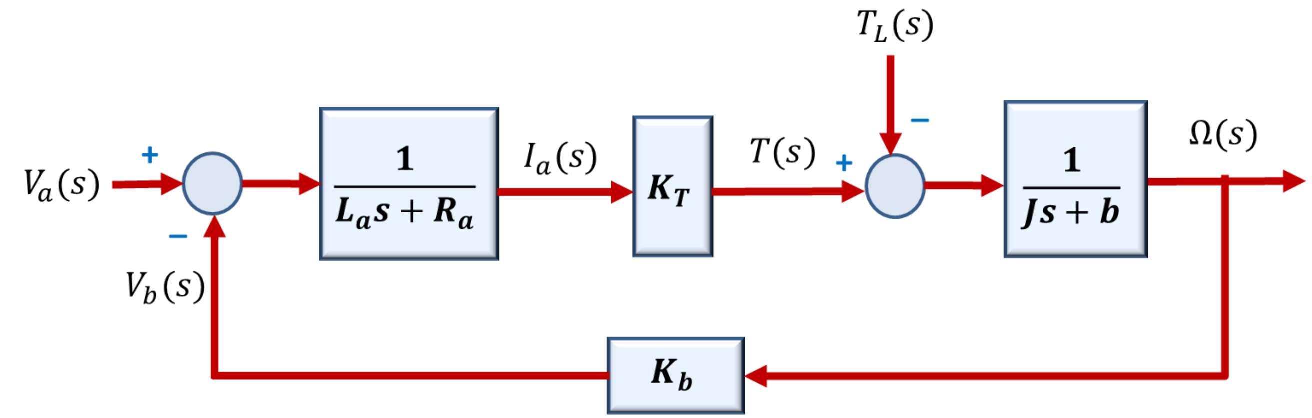
$$\boxed{\frac{I_a(s)}{V_a(s)} = \frac{Js + b}{L_a J s^2 + (L_a b + J R_a)s + R_a b + K_b K_T}}$$

Voltage-to-Current transfer function

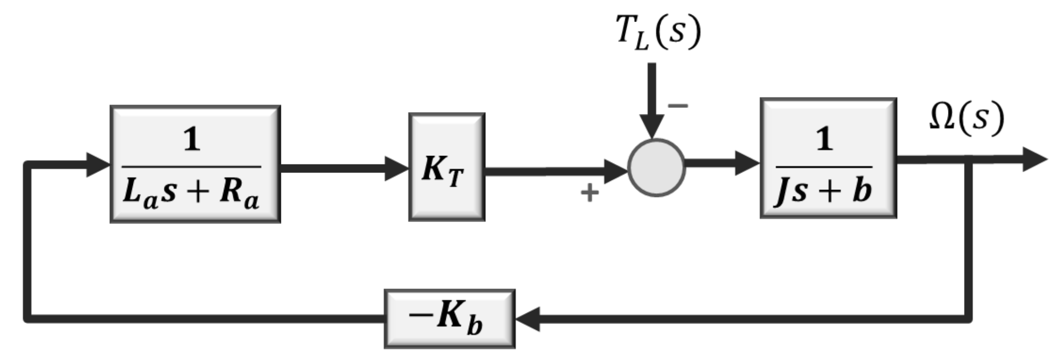
Modeling of Armature-Controlled DC Motor

❑ Transfer Function Model

- We can derive the load-torque-related transfer functions to analyze the effect of the load on motor speed and current.

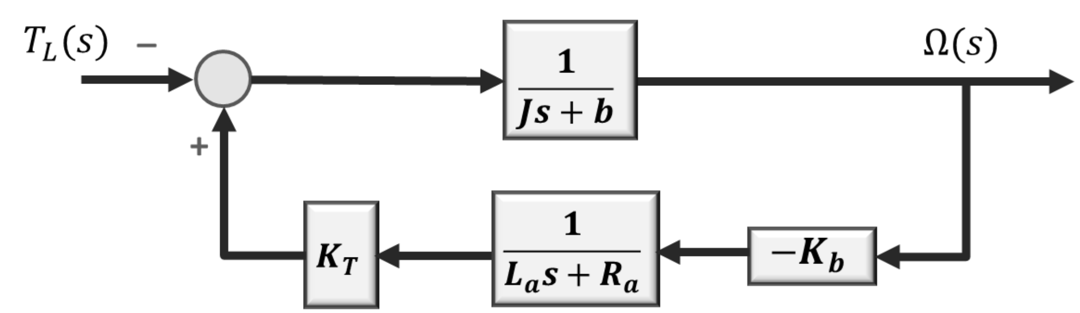


- Apply superposition principle and set $V_a = 0$ to consider the effect of load-torque only.



- The Load-torque-to-speed transfer function:

$$\frac{\Omega(s)}{T_L(s)} = -\frac{G}{1-GH} = -\frac{\frac{1}{Js+b}}{1-\left(\frac{1}{Js+b}\right)\left(\frac{-K_bK_T}{L_as+R_a}\right)} = -\frac{L_as+R_a}{(L_as+R_a)(Js+b)+K_TK_b}$$



$$\boxed{\frac{\Omega(s)}{T_L(s)} = -\frac{L_as+R_a}{L_aJs^2+(L_ab+JR_a)s+R_ab+K_bK_T}}$$

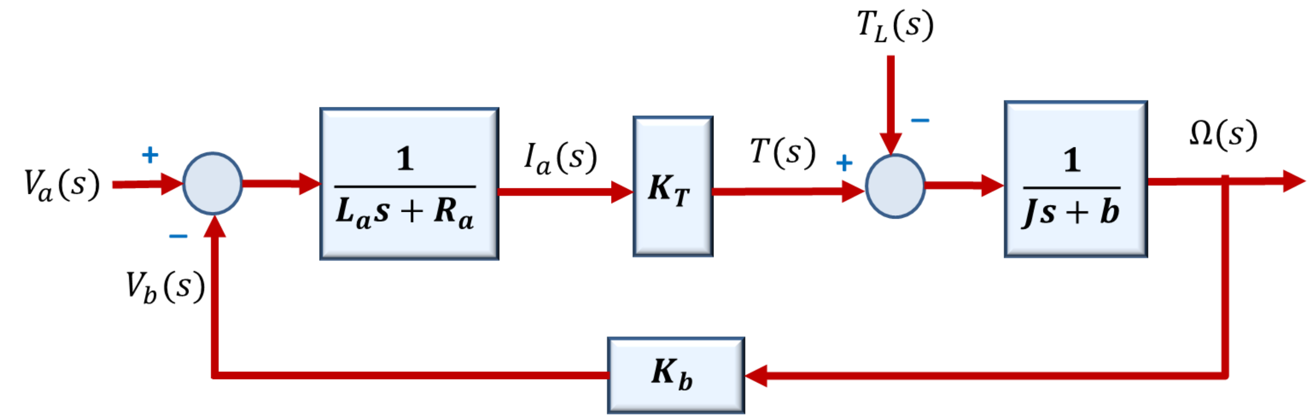
Load-Torque-to-Speed transfer function

- The minus sign indicates that the speed will <u>decrease</u> for a positive load torque.

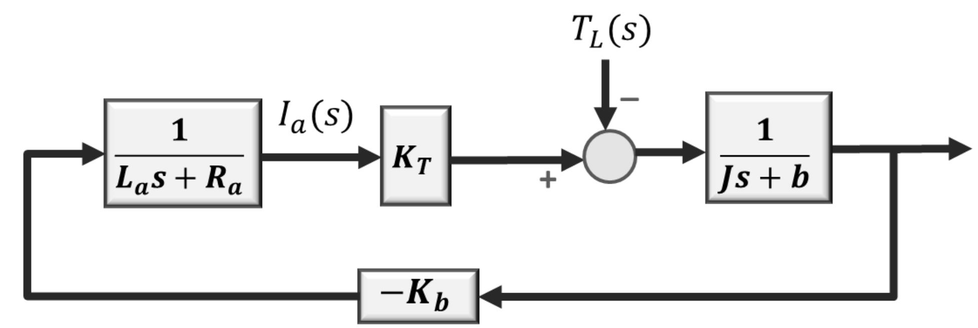
Modeling of Armature-Controlled DC Motor

❑ Transfer Function Model

- We can derive the load-torque-related transfer functions to analyze the effect of the load on motor speed and current.

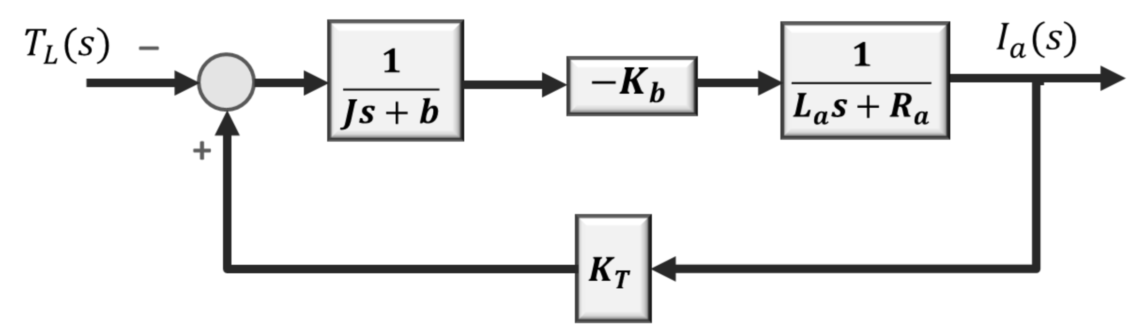


- Apply superposition principle and set $V_a = 0$ to consider the effect of load-torque only.



- The Load-torque-to-current transfer function:

$$\frac{I_a(s)}{T_L(s)} = -\frac{G}{1 - GH} = -\frac{\frac{-K_b}{(Js+b)(L_a s + R_a)}}{1 - \left(\frac{-K_b}{(Js+b)(L_a s + R_a)}\right)(K_T)} = \frac{K_b}{(L_a s + R_a)(Js + b) + K_T K_b}$$



$$\boxed{\frac{I_a(s)}{T_L(s)} = \frac{K_b}{L_a J s^2 + (L_a b + J R_a)s + R_a b + K_b K_T}}$$

Load-Torque-to-Current transfer function

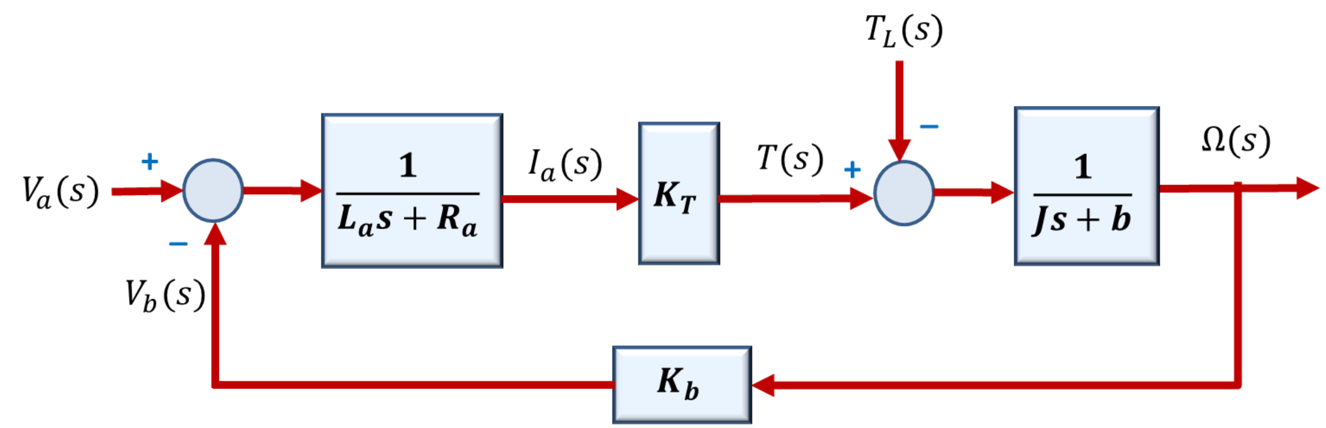
Modeling of Armature-Controlled DC Motor

❑ Transfer Function Model

- The four transfer function models of an armature-controlled DC motor are obtained as:

$$
\begin{cases}
\dfrac{\Omega(s)}{V_a(s)} = \dfrac{K_T}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T} \\[2ex]
\dfrac{I_a(s)}{V_a(s)} = \dfrac{Js + b}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}
\end{cases}
$$

$$
\begin{cases}
\dfrac{\Omega(s)}{T_L(s)} = -\dfrac{L_as + R_a}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T} \\[2ex]
\dfrac{I_a(s)}{T_L(s)} = \dfrac{K_b}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}
\end{cases}
$$



$$
\boxed{\Omega(s) = \frac{K_T}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}V_a(s) - \frac{L_as + R_a}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}T_L(s)}
$$

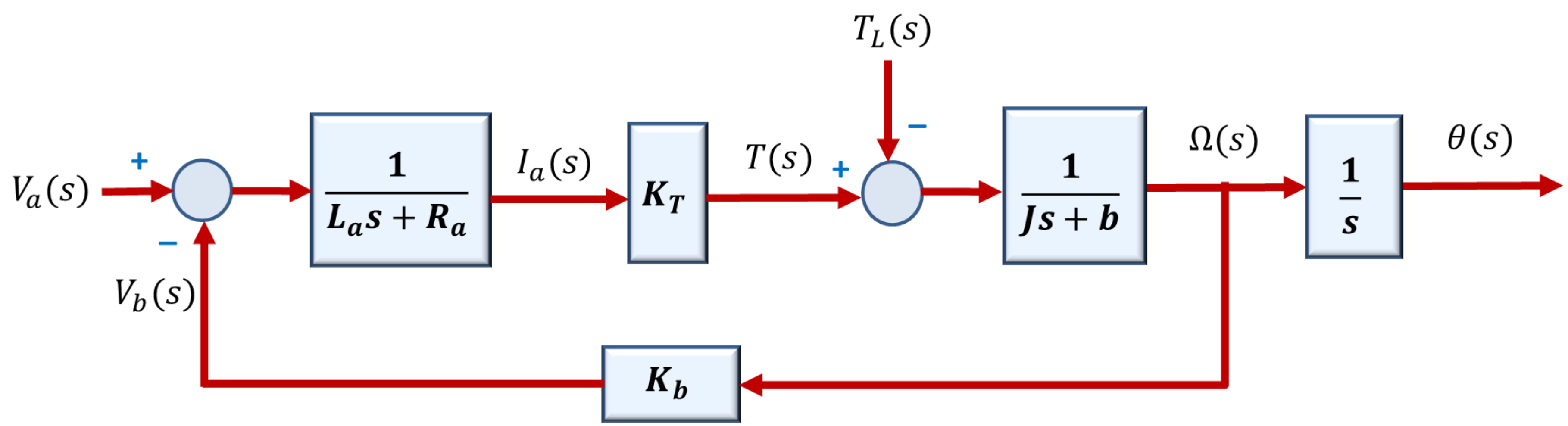
$$
\boxed{I_a(s) = \frac{Js + b}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}V_a(s) + \frac{K_b}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}T_L(s)}
$$

Modeling of Armature-Controlled DC Motor

❑ Transfer Function Model

- By having the angular velocity $\omega(t)$ of the motor shaft as an output we can easily derive the angular displacement $\theta(t)$ of the motor shaft and obtain the corresponding transfer function models.

$$\omega(t) = \frac{d\theta(t)}{dt} \quad \rightarrow \quad \Omega(s) = s\theta(s) \quad \rightarrow \quad \theta(s) = \frac{1}{s}\Omega(s)$$



- The voltage-to-displacement and load-torque-to-displacement transfer function models are obtained as:

$$\boxed{\frac{\theta(s)}{V_a(s)} = \frac{K_T}{s(L_a J s^2 + (L_a b + J R_a)s + R_a b + K_b K_T)}}$$

$$\boxed{\frac{\theta(s)}{T_L(s)} = -\frac{L_a s + R_a}{s(L_a J s^2 + (L_a b + J R_a)s + R_a b + K_b K_T)}}$$

Modeling of Armature-Controlled DC Motor

Example 2 This example shows how to find the dynamic response of a DC motor from its transfer function and a MATLAB implementation to plot the response.

Assume the applied voltage v_a is 10V and the load torque τ_L is zero.

$R_a = 0.5\Omega,$
$L_a = 2 \times 10^{-3} H,$
$J = 9 \times 10^{-5} kg.m^2,$
$b = 10^{-4} N.m.s/rad$
$K_T = K_b = 0.05\ N.m/A$

Substituting the given parameters in transfer functions:

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_a J s^2 + (L_a b + J R_a)s + R_a b + K_b K_T} = \frac{0.05}{18 \times 10^{-8} s^2 + 4.52 \times 10^{-5} s + 2.55 \times 10^{-3}}$$

$$\frac{I_a(s)}{V_a(s)} = \frac{Js + b}{L_a J s^2 + (L_a b + J R_a)s + R_a b + K_b K_T} = \frac{9 \times 10^{-5} s + 10^{-4}}{18 \times 10^{-8} s^2 + 4.52 \times 10^{-5} s + 2.55 \times 10^{-3}}$$

We can simplify the transfer functions to make it easier to find the dynamic response:

$$\frac{\Omega(s)}{V_a(s)} = \frac{2.77 \times 10^5}{s^2 + 2.51 \times 10^2 s + 1.416 \times 10^4} = \frac{2.77 \times 10^5}{(s + 165.52)(s + 85.59)}$$

$$\frac{I_a(s)}{V_a(s)} = \frac{5 \times 10^2 s + 5.555 \times 10^2}{s^2 + 2.51 \times 10^2 s + 1.416 \times 10^4} = \frac{5 \times 10^2 s + 5.55 \times 10^2}{(s + 165.52)(s + 85.59)}$$

Modeling of Armature-Controlled DC Motor

Example 2

This example shows how to find the dynamic response of a DC motor from its transfer function and a MATLAB implementation to plot the response.

Assume the applied voltage v_a is 10V and the load torque τ_L is zero.

$R_a = 0.5\Omega$,
$L_a = 2 \times 10^{-3} H$,
$J = 9 \times 10^{-5} kg.m^2$,
$b = 10^{-4} N.m.s/rad$
$K_T = K_b = 0.05\ N.m/A$

If v_a is a step function of magnitude 10V, we find the $\Omega(s)$ and $\mathrm{I_a}(s)$ applying partial-fraction expansion by hand or with MATLAB, then find the time response $\omega(t)$ and $i_a(t)$ taking inverse Laplace:

$$v_a(t) = 10 \quad \rightarrow \quad V_a(s) = \frac{10}{s}$$

$$\frac{\Omega(s)}{V_a(s)} = \frac{2.77 \times 10^5}{(s + 165.52)(s + 85.59)} \quad \rightarrow \quad \Omega(s) = \frac{2.77 \times 10^5}{(s + 165.52)(s + 85.59)} V_a(s) = \frac{2.77 \times 10^6}{s(s + 165.52)(s + 85.59)}$$

$$\Omega(s) = \frac{196}{s} + \frac{210}{s + 165.52} - \frac{406}{s + 85.59} \quad \rightarrow \quad \boxed{\omega(t) = 196 + 210e^{-165.52t} - 406e^{-85.59t}}$$

$$\frac{I_a(s)}{V_a(s)} = \frac{5 \times 10^2 s + 5.55 \times 10^2}{(s + 165.52)(s + 85.59)} \quad \rightarrow \quad I_a(s) = \frac{5 \times 10^2 s + 5.55 \times 10^2}{(s + 165.52)(s + 85.59)} V_a(s) = \frac{5 \times 10^3 s + 5.55 \times 10^3}{s(s + 165.52)(s + 85.59)}$$

$$I_a(s) = \frac{0.39}{s} - \frac{62.13}{s + 165.52} + \frac{61.74}{s + 85.59} \quad \rightarrow \quad \boxed{i_a(t) = 0.39 - 62.13e^{-165.52t} + 61.74e^{-85.59t}}$$

Modeling of Armature-Controlled DC Motor

Example 2

This example shows how to find the dynamic response of a DC motor from its transfer function and a MATLAB implementation to plot the response.

Assume the applied voltage v_a is 10V and the load torque τ_L is zero.

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$b = 10^{-4} N.m.s/rad$
$K_T = K_b = 0.05\ N.m/A$

$$\frac{I_a(s)}{V_a(s)} = \frac{Js + b}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}$$

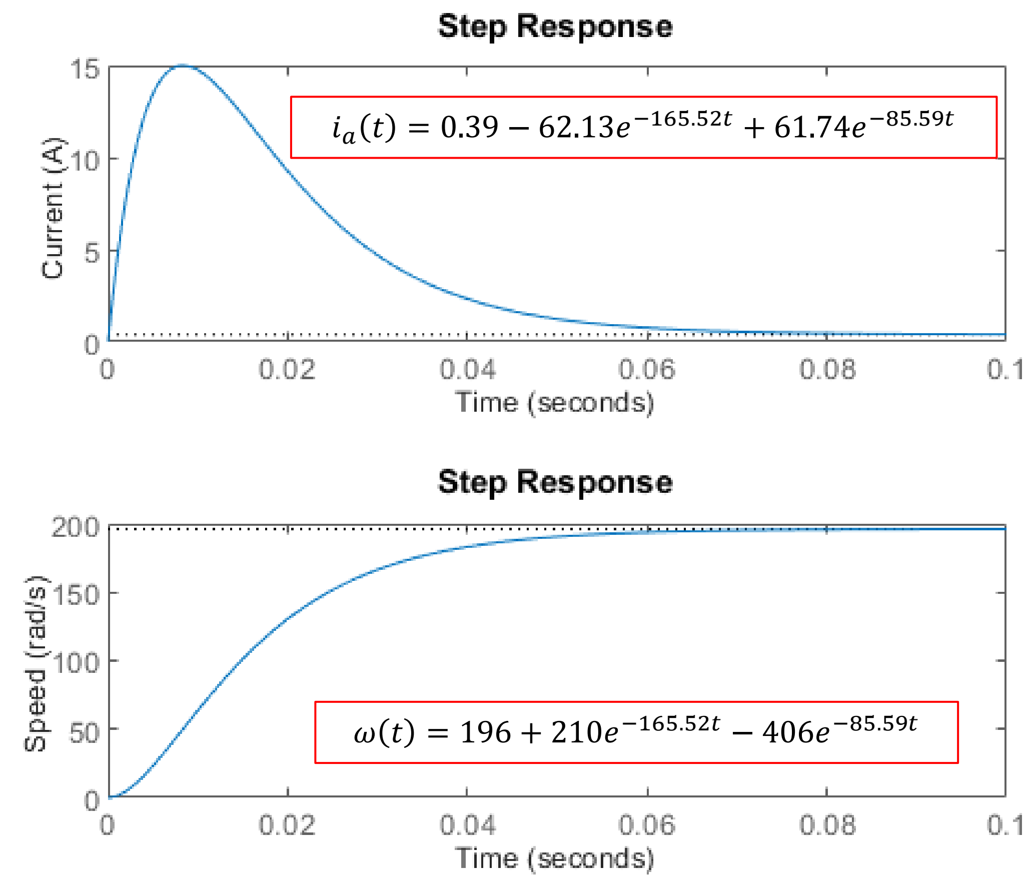
$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}$$

```
% DC Motor Parameters
KT = 0.05;    Kb = KT;
La = 2e-3;    Ra = 0.5;
J = 9e-5;     b = 1e-4;
Va = 10;      TL = 0;

% Current transfer function
current_tf = tf([J b],[La*J La*b+J*Ra Ra*b+Kb*KT]);

% Speed transfer function
speed_tf = tf([KT],[La*J La*b+J*Ra Ra*b+Kb*KT]);

% Plot step response with magnitude of 10
figure;
subplot(211), step(10*current_tf), ylabel('Current (A)')
subplot(212), step(10*speed_tf), ylabel('Speed (rad/s)')
```



Modeling of Armature-Controlled DC Motor

Example 2

This example shows how to find the dynamic response of a DC motor from its transfer function and a MATLAB implementation to plot the response.

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$R_a = 0.5\Omega,$
$L_a = 2 \times 10^{-3} H,$
$J = 9 \times 10^{-5} kg.m^2,$
$b = 10^{-4} N.m.s/rad$
$K_T = K_b = 0.05\ N.m/A$

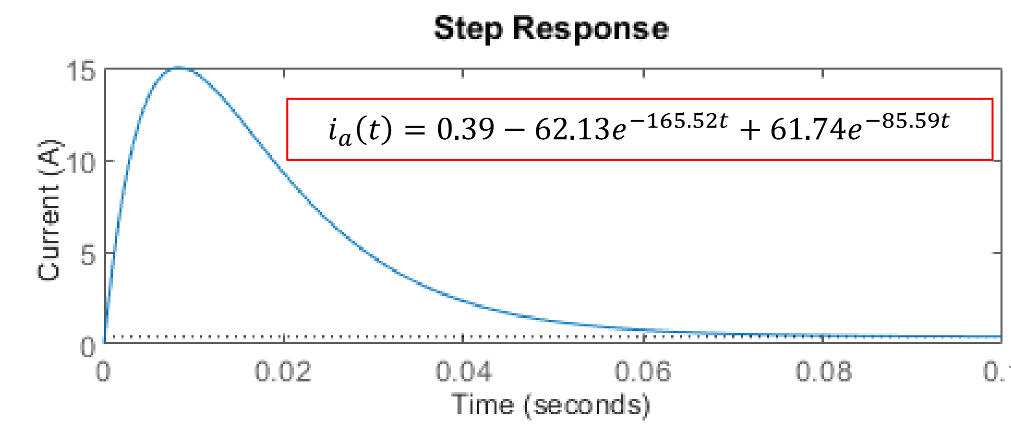
In this example, we can also determine the steady-state speed and steady-state current values from the time response and from the transfer function by applying Final-Value Theorem.

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

- Steady-state value of the armature current:

$$i_{a_{ss}} = \lim_{t\to\infty} i_a(t) = \lim_{t\to\infty}(0.39 - 62.13e^{-165.52t} + 61.74e^{-85.59t}) = 0.39\ A$$

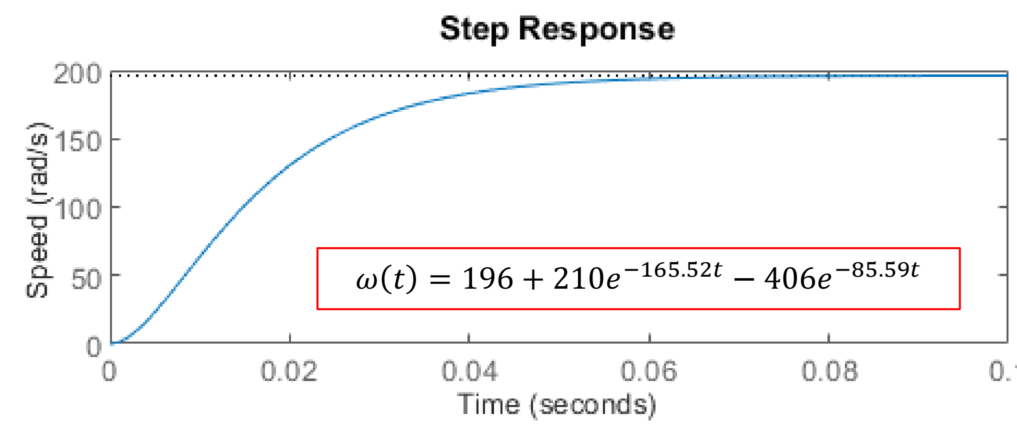
$$i_{a_{ss}} = \lim_{s\to 0} sI_a(s) = \lim_{s\to 0} s\left(\frac{5\times 10^3 s + 5.55\times 10^3}{s(s+165.52)(s+85.59)}\right) = 0.39\ A$$



- Steady-state value of the motor speed:

$$\omega_{ss} = \lim_{t\to\infty} \omega(t) = \lim_{t\to\infty}(196 + 210e^{-165.52t} - 406e^{-85.59t}) = 196\ rad/s$$

$$\omega_{ss} = \lim_{s\to 0} s\Omega(s) = \lim_{s\to 0} s\left(\frac{2.77\times 10^6}{s(s+165.52)(s+85.59)}\right) = 195.5 \approx 196\ rad/s$$



Modeling of Armature-Controlled DC Motor

State Space Model

- State space model is obtained by considering the applied voltage $v_a(t)$ and load torque $\tau_L(t)$ as the inputs and the armature current $i_a(t)$ and the angular velocity $\omega(t)$ as the outputs.

$$v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t)$$

$$\tau(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + b\omega(t)$$

$$v_b(t) = K_b \omega(t)$$

$$\tau(t) = K_T i_a(t)$$

- The state variables are defined as the armature current $i_a(t)$ and the motor speed $\omega(t)$:

$$\begin{cases} q_1(t) = i_a(t) \\ q_2(t) = \omega(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = \dfrac{di_a(t)}{dt} = \dfrac{1}{L_a}\left(v_a(t) - v_b(t) - R_a i_a(t)\right) \\ \dot{q}_2(t) = \dfrac{d\omega(t)}{dt} = \dfrac{1}{J}\left(\tau(t) - \tau_L(t) - b\omega(t)\right) \end{cases}$$

State-variable equations are obtained as:

$$\begin{cases} \dot{q}_1(t) = \dfrac{1}{L_a}\left(v_a(t) - K_b q_2(t) - R_a q_1(t)\right) \\ \dot{q}_2(t) = \dfrac{1}{J}\left(K_T q_1(t) - \tau_L(t) - b q_2(t)\right) \end{cases}$$

The **output equation** is obtained as:

$$y_1(t) = i_a(t) = q_1(t)$$

$$y_2(t) = \omega(t) = q_2(t)$$

Modeling of Armature-Controlled DC Motor

State Space Model

- We can represent the state and output equations in the standard matrix-vector form as below:

$$\begin{cases} \dot{q}_1(t) = \dfrac{1}{L_a}\left(v_a(t) - K_b q_2(t) - R_a q_1(t)\right) \\ \dot{q}_2(t) = \dfrac{1}{J}\left(K_T q_1(t) - \tau_L(t) - b q_2(t)\right) \end{cases} \qquad \begin{cases} y_1(t) = i_a(t) = q_1(t) \\ y_2(t) = \omega(t) = q_2(t) \end{cases}$$

$$\boxed{\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)}$$

State Equation

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{-R_a}{L_a} & \dfrac{-K_b}{L_a} \\ \dfrac{K_T}{J} & -\dfrac{b}{J} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_a} & 0 \\ 0 & -\dfrac{1}{J} \end{bmatrix} \begin{bmatrix} v_a(t) \\ \tau_L(t) \end{bmatrix}$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$$

Output Equation

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_a(t) \\ \tau_L(t) \end{bmatrix}$$

- The state-space representation is equivalent to the four transfer function models, which is more appropriate to model MIMO systems.

Modeling of Armature-Controlled DC Motor

Example 3

This example shows a MATLAB implementation of the DC motor by its state-space model.

Applied voltage is 10V and the load torque is 0.01N.m

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-R_a}{L_a} & \frac{-K_b}{L_a} \\ \frac{K_T}{J} & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{J} \end{bmatrix} \begin{bmatrix} v_a(t) \\ \tau_L(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_a(t) \\ \tau_L(t) \end{bmatrix}$$

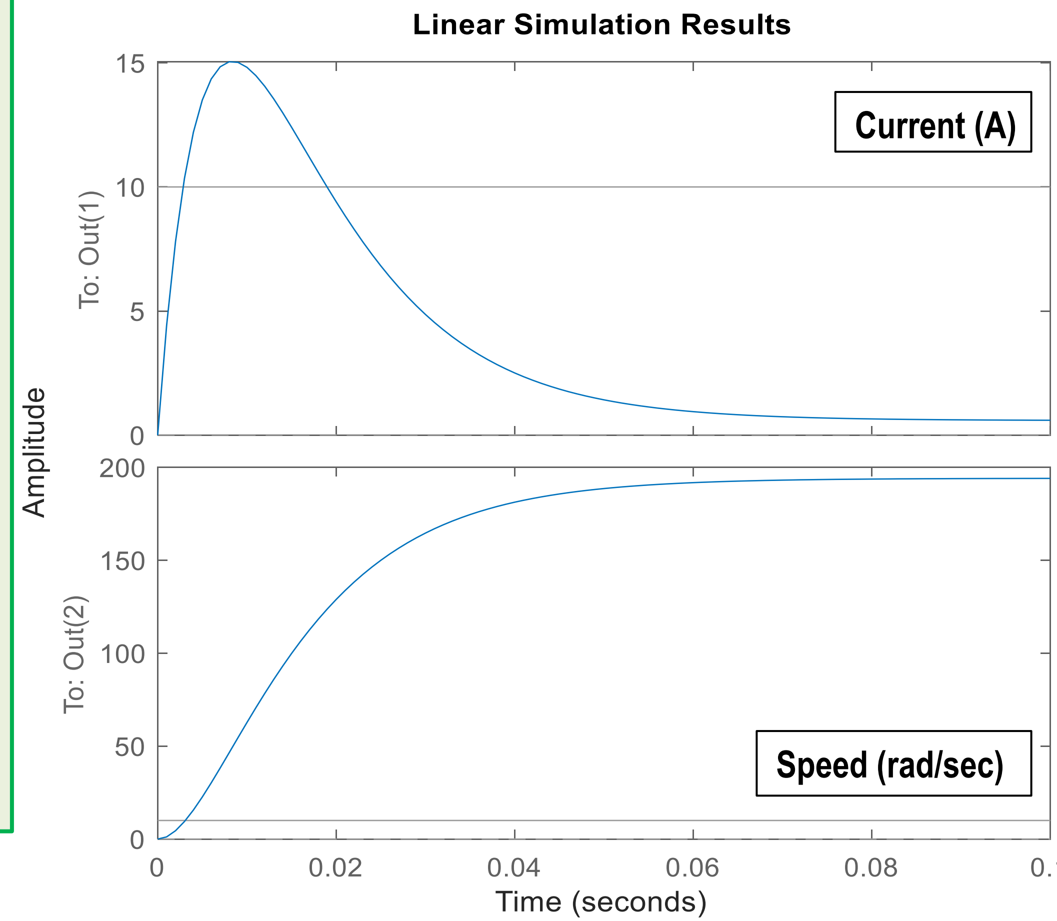
```
% DC Motor Parameters
KT = 0.05;     Kb = KT;
La = 2e-3;     Ra = 0.5;
J = 9e-5;      b = 1e-4;
Va = 10;       TL = 0.01;

% Define state space model matrices
A = [-Ra/La -Kb/La; KT/J -b/J];
B = [1/La 0; 0 -1/J];
C = [1 0; 0 1];
D = [0 0; 0 0];

% Create state space model
model_ss = ss(A,B,C,D);

% Plot the time response
t = 0:0.001:0.1;
u1 = Va*ones(size(t));
u2 = TL*ones(size(t));
u = [u1; u2];

figure;
lsim(model_ss,u,t)
```



Modeling of Armature-Controlled DC Motor

❑ Simplified First-order Model

- The inductance L_a in the armature circuit is usually **small** and may be neglected.
- If L_a is neglected, the previously obtained second-order transfer function models reduce to a **first-order model**.
- Consider the Voltage-to-Speed transfer function:

$$\boxed{\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_aJs^2 + (L_ab + JR_a)s + R_ab + K_bK_T}} \xrightarrow{L_a \approx 0} \boxed{\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{JR_as + R_ab + K_bK_T}}$$

Reduced-order Model

- The reduced order model is a first-order transfer function, and it can be written in the following standard form:

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{JR_as + (R_ab + K_bK_T)} = \frac{\dfrac{K_T}{R_ab + K_bK_T}}{\dfrac{JR_a}{R_ab + K_bK_T}s + 1} \to \boxed{\frac{\boldsymbol{\Omega(s)}}{\boldsymbol{V_a(s)}} = \frac{\boldsymbol{K_m}}{\boldsymbol{\tau_m s + 1}}} \to \boxed{\frac{\boldsymbol{\theta(s)}}{\boldsymbol{V_a(s)}} = \frac{\boldsymbol{K_m}}{\boldsymbol{s(\tau_m s + 1)}}}$$

where,

$$\boxed{K_m = \frac{K_T}{R_ab + K_bK_T}} \text{ motor gain} \qquad \boxed{\tau_m = \frac{JR_a}{R_ab + K_bK_T}} \text{ motor time constant}$$

Modeling of Armature-Controlled DC Motor

Example 4 This example compares time response of the first-order model and the second-order model of the DC motor. Applied voltage is 10V and the load torque is zero.

$R_a = 0.5\Omega,$
$L_a = 2 \times 10^{-3} H,$
$J = 9 \times 10^{-5} kg.m^2,$
$b = 10^{-4} N.m.s/rad$
$K_T = K_b = 0.05\ N.m/A$

If L_a is neglected ($L_a = 0$), the transfer function models are reduced to **first-order** models.

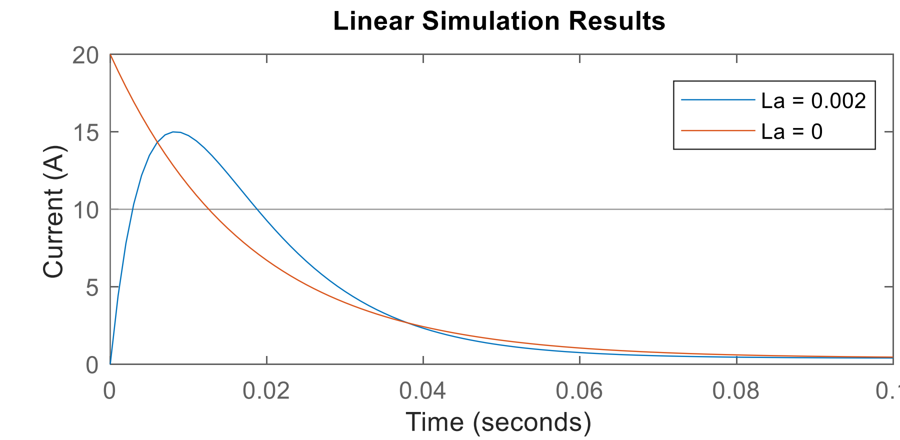
$$\frac{I_a(s)}{V_a(s)} = \frac{Js + b}{JR_a s + R_a b + K_b K_T} = \frac{9 \times 10^{-5} s + 10^{-4}}{4.5 \times 10^{-5} s + 2.55 \times 10^{-3}}$$

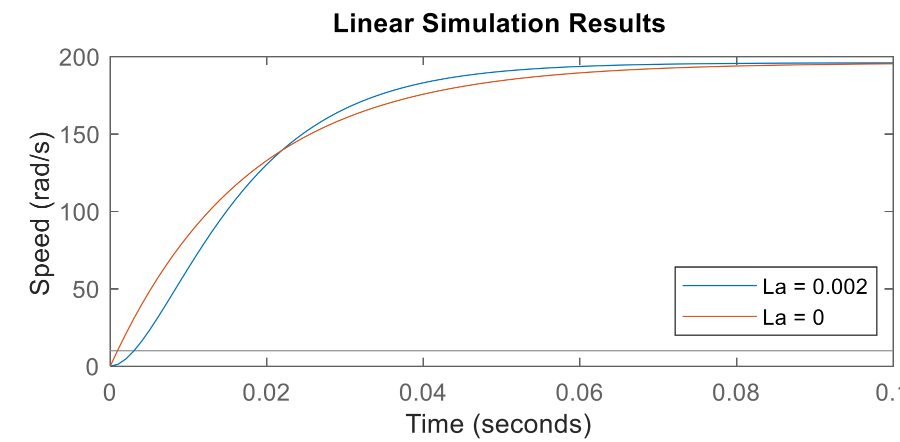
$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{JR_a s + R_a b + K_b K_T} = \frac{0.05}{4.5 \times 10^{-5} s + 2.55 \times 10^{-3}}$$

Applied voltage is 10V $\rightarrow \quad v_a(t) = 10 \quad \rightarrow \quad V_a(s) = \frac{10}{s}$

$$I_a(s) = \frac{9 \times 10^{-4} s + 10^{-3}}{s(4.5 \times 10^{-5} s + 2.55 \times 10^{-3})} \rightarrow \boxed{i_a(t) = 0.39 + 19.61 e^{-56.67t}}$$

$$\Omega(s) = \frac{0.5}{s(4.5 \times 10^{-5} s + 2.55 \times 10^{-3})} \rightarrow \boxed{\omega(t) = 196.1 - 196.1 e^{-56.67t}}$$



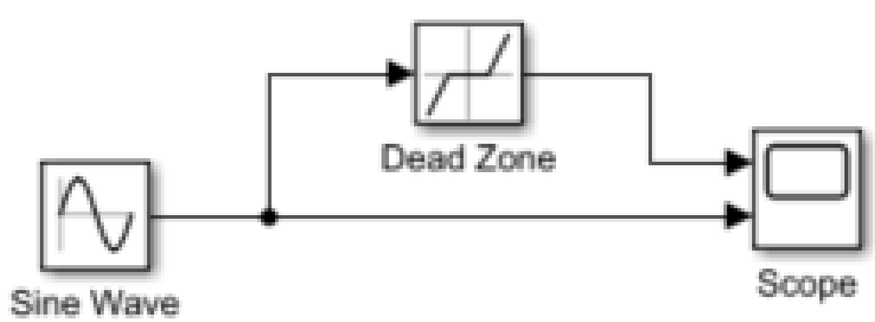


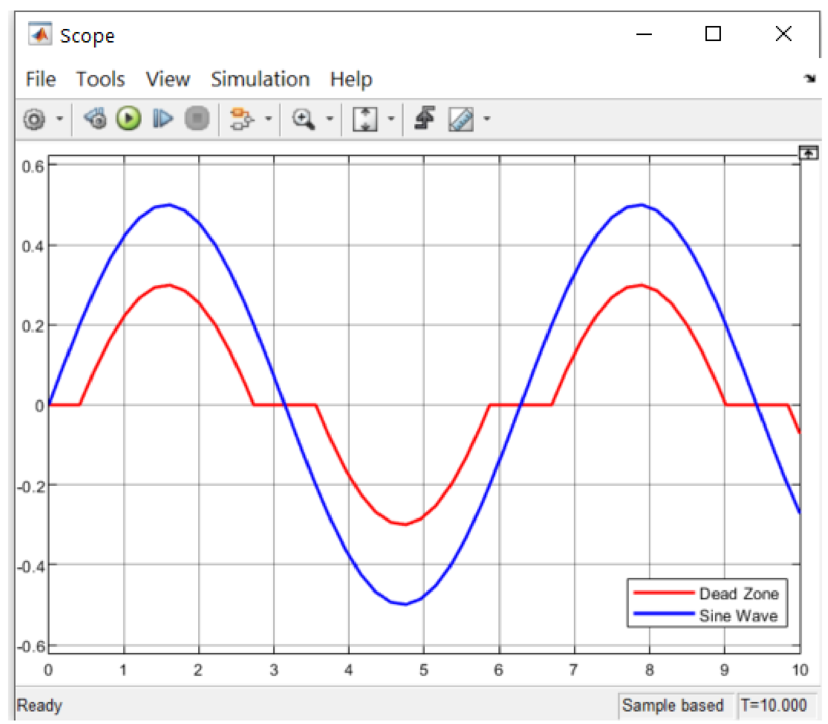
The results show that the simplified first-order model can provides us a good approximation of the voltage-to-speed transfer model of the DC motor.

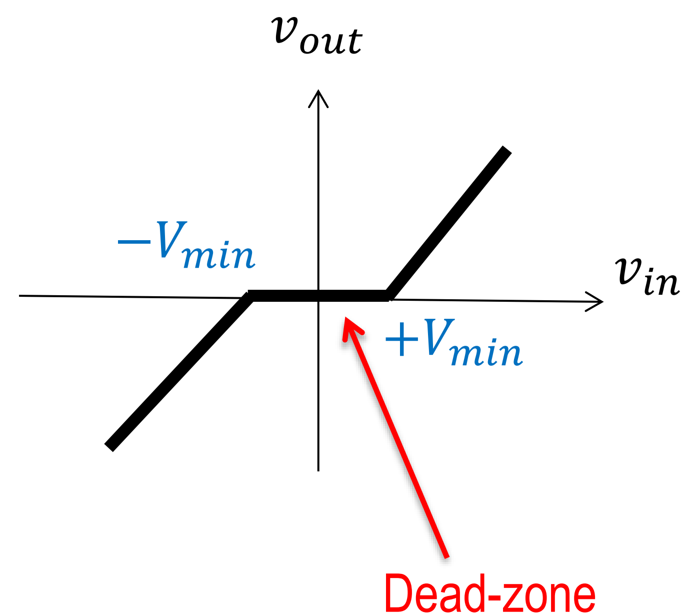
Modeling of Electromechanical Systems

❑ Dead-zone (Nonlinear Characteristics)

- **Dead-zone** nonlinearity refers to a condition in which output becomes zero when the input crosses a certain limiting value.
- For example, in electrical devices like DC servomotors a *minimum level of excitation voltage* is required to rotate the DC motor shaft, which is caused by the friction forces and the rotor inertia.
- The Dead-zone nonlinearity can be modeled in **Simulink** using the **Dead-zone** block.
- This example shows the effect of the dead-zone on the output of a DC motor control.
- Assume that the **dead-zone limits** are $\pm 0.2V$.



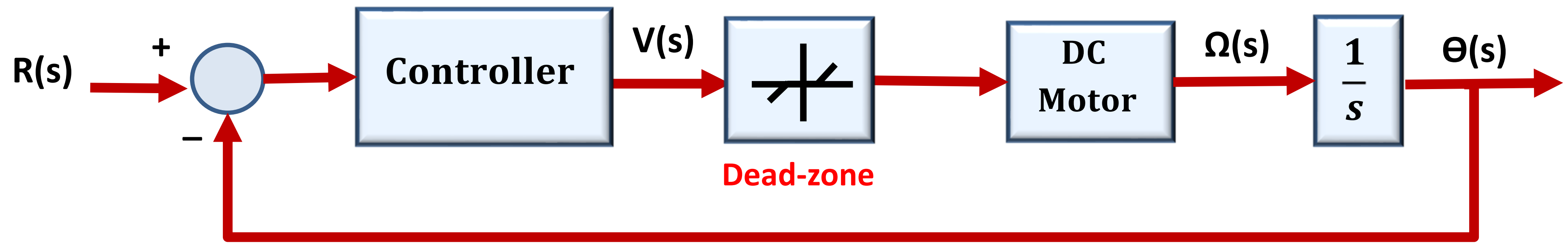


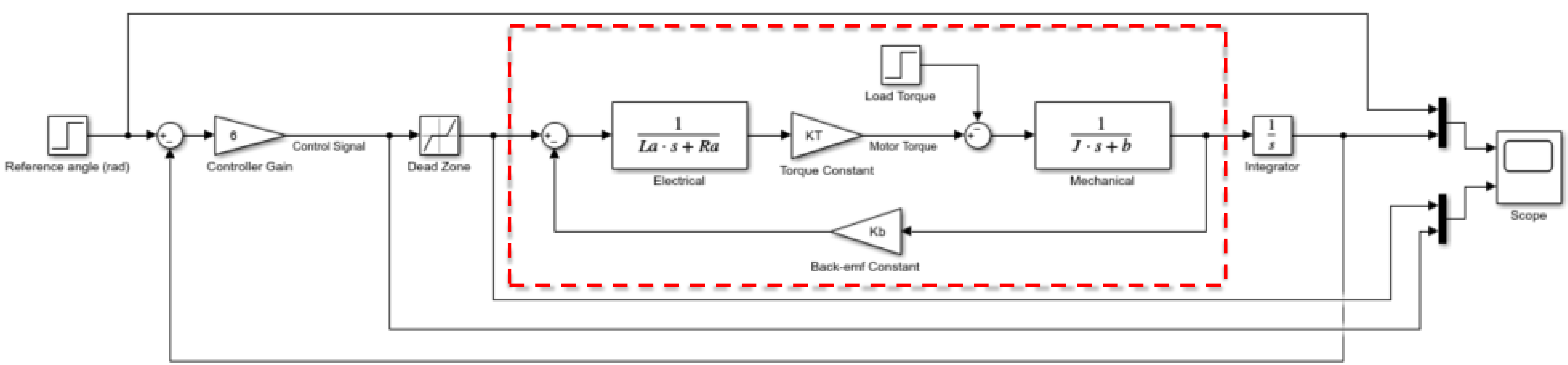


Modeling of Electromechanical Systems

Example 5

This example shows effect of the dead-zone nonlinearity of the DC motor on the position control. Assume that limit of the dead-zone is between $-0.2\ V\ to + 0.3\ V$.



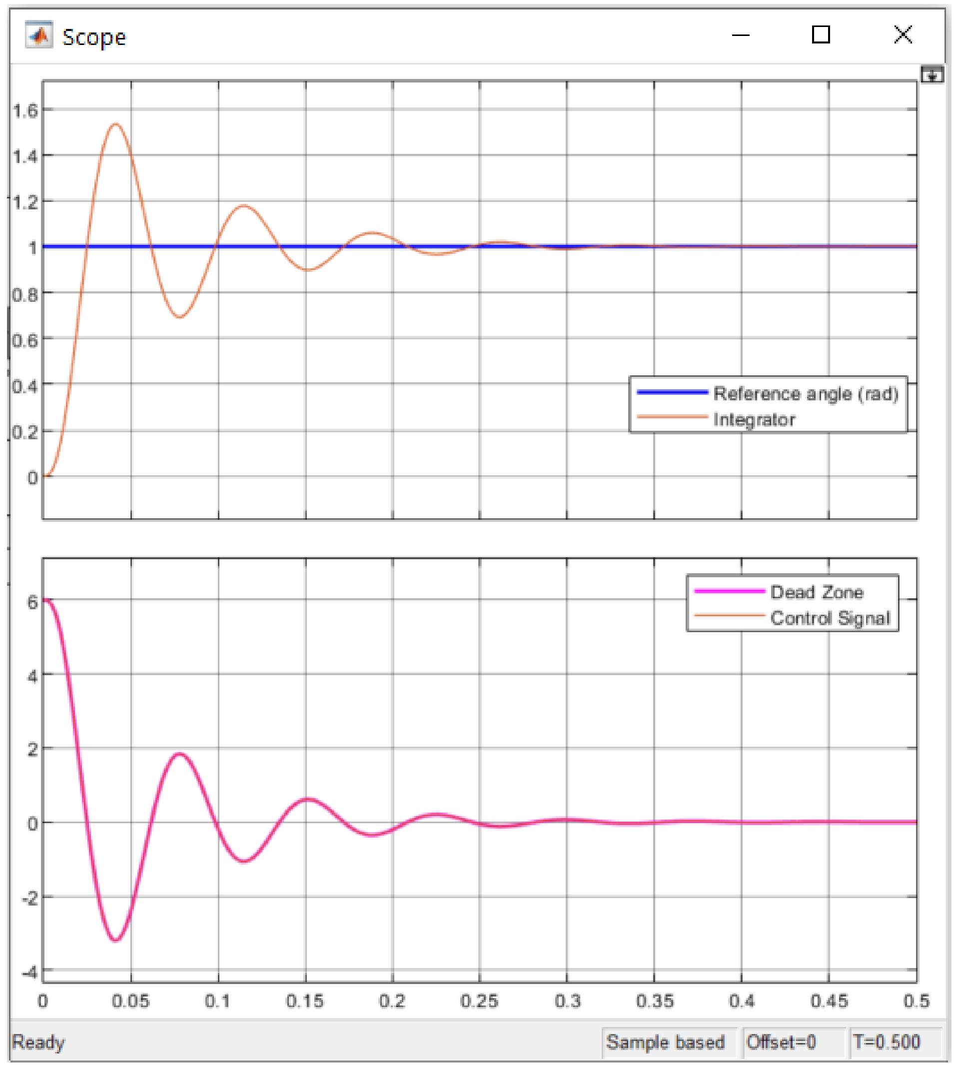


Modeling of Electromechanical Systems

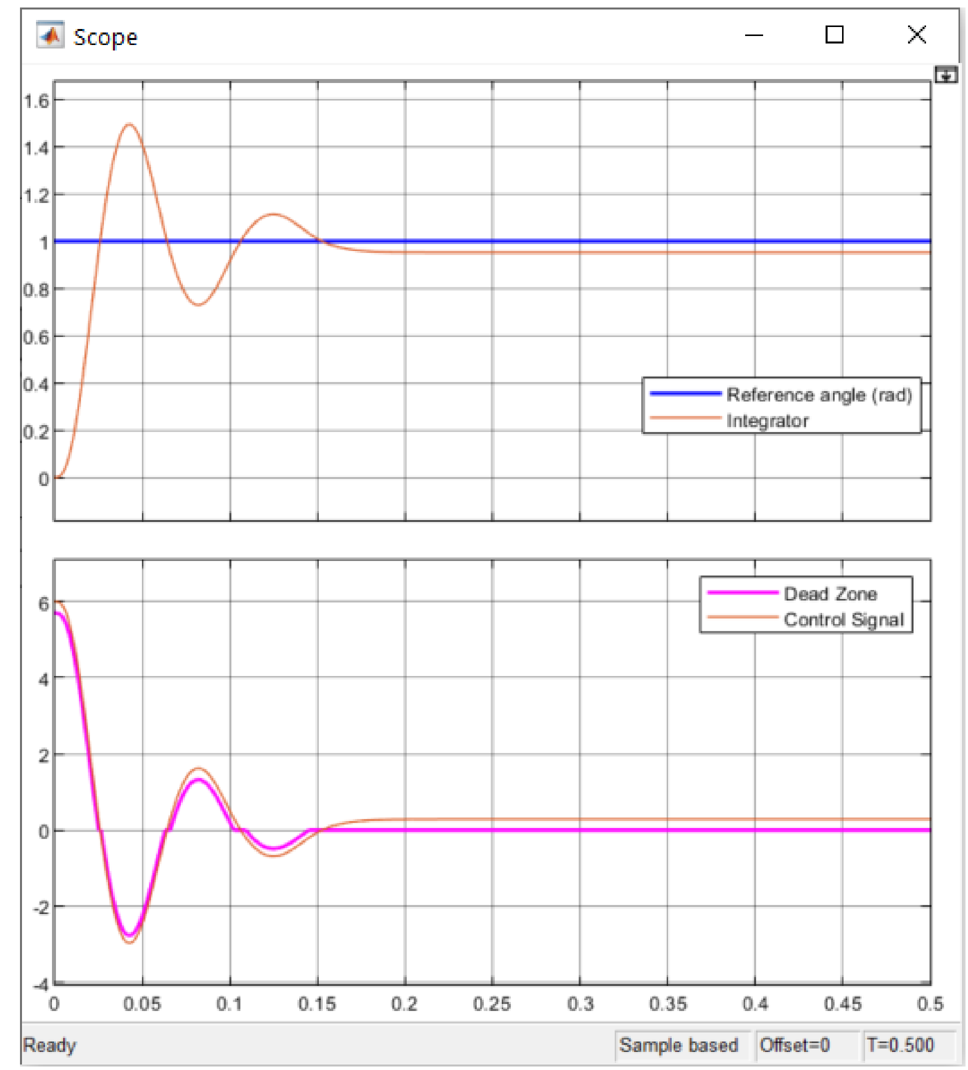
Example 5 This example shows effect of the dead-zone nonlinearity of the DC motor on the position control. Assume that limit of the dead-zone is between $-0.2\ V\ to + 0.3\ V$.

- In <u>control system design</u>, when the amplitude of the motor input voltage falls within the dead-zone, the motor does not respond, so the controller would not be able to correct the error in position or velocity if any exist.

Dead-zone is zero



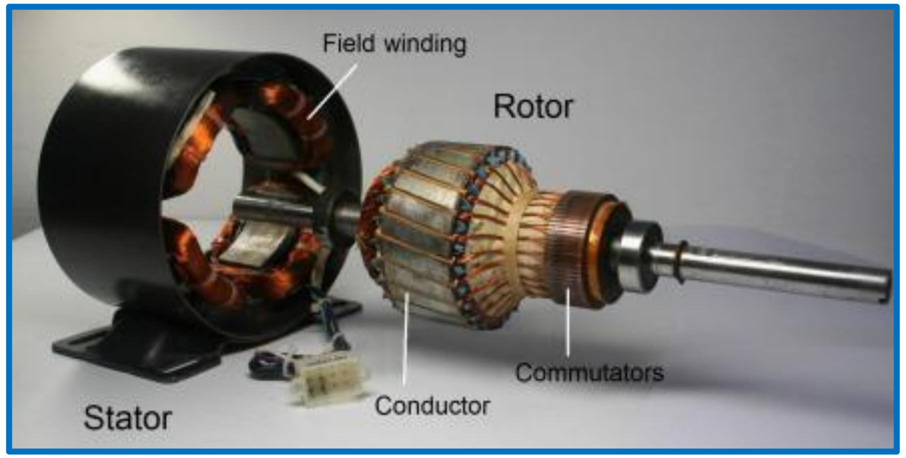
Dead-zone is between $-0.2\ V\ to + 0.3\ V$



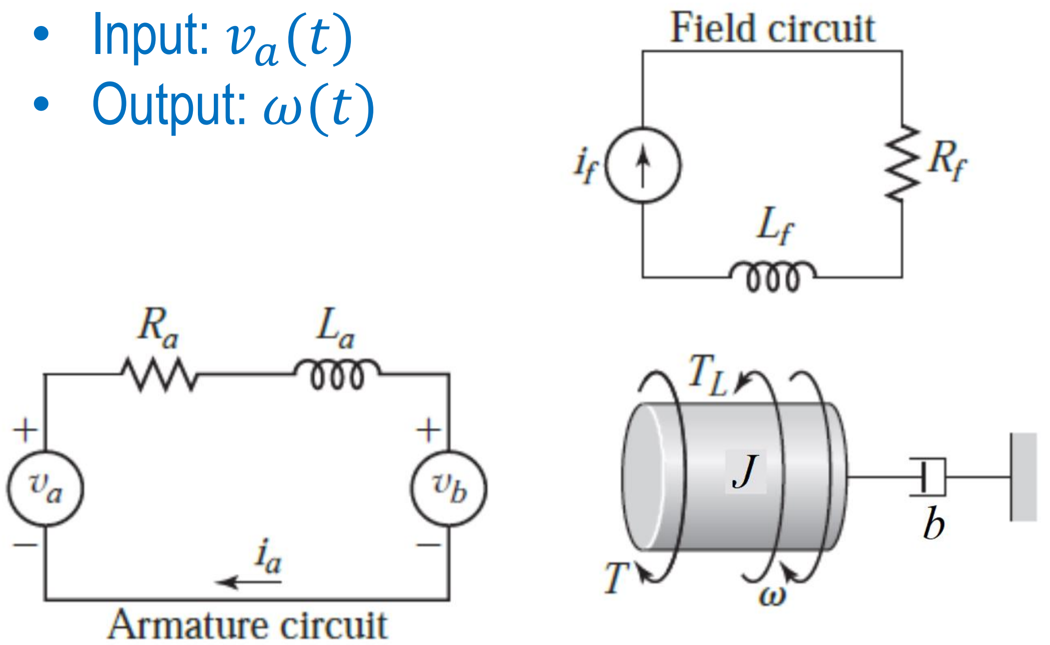
Modeling of Electromechanical Systems

❑ Control of DC Motor

- Two general methods to control speed of DC motor:
  - **Armature-Controlled DC Motor**
    - Field current i_f and field flux ϕ_f is constant.
    - Using permanent magnet
  - **Field-Controlled DC Motor**
    - Armature current i_a is constant

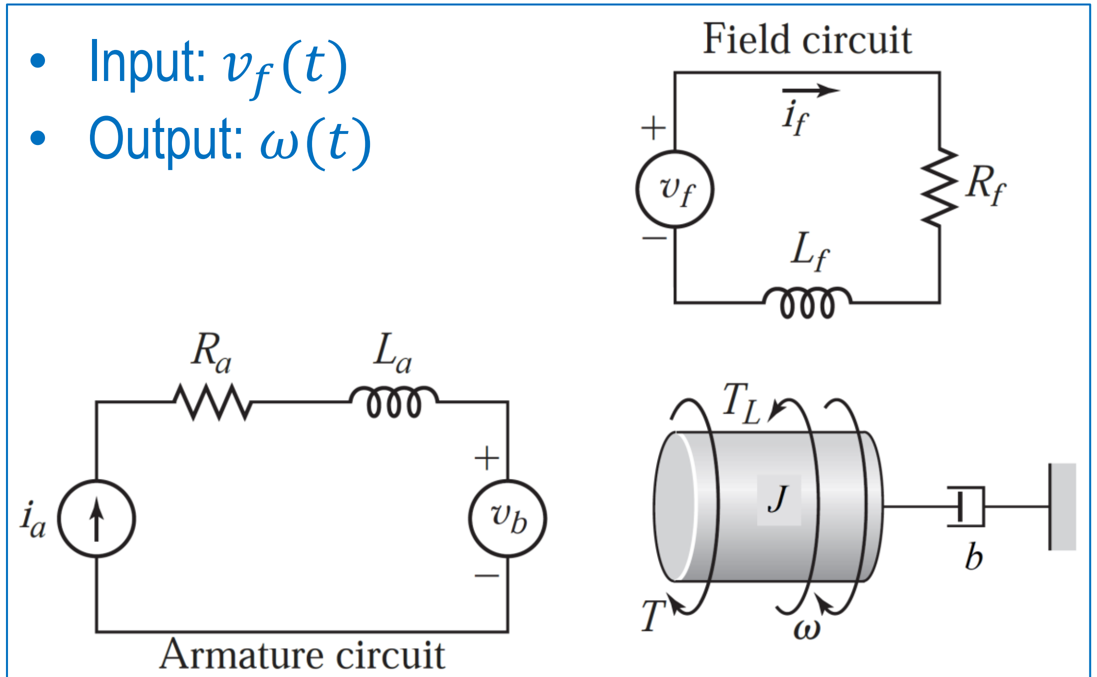


- Input: $v_a(t)$
- Output: $\omega(t)$



Armature-Controlled DC Motor

- Input: $v_f(t)$
- Output: $\omega(t)$



Field-Controlled DC Motor

Modeling of Field-Controlled DC Motor

❑ Differential Equation Model

- The armature current i_a is constant.
- The voltage v_f is applied to field circuit, whose inductance and resistance are L_f and R_f.
- The differential equation for the field circuit is obtained by applying a KVL:

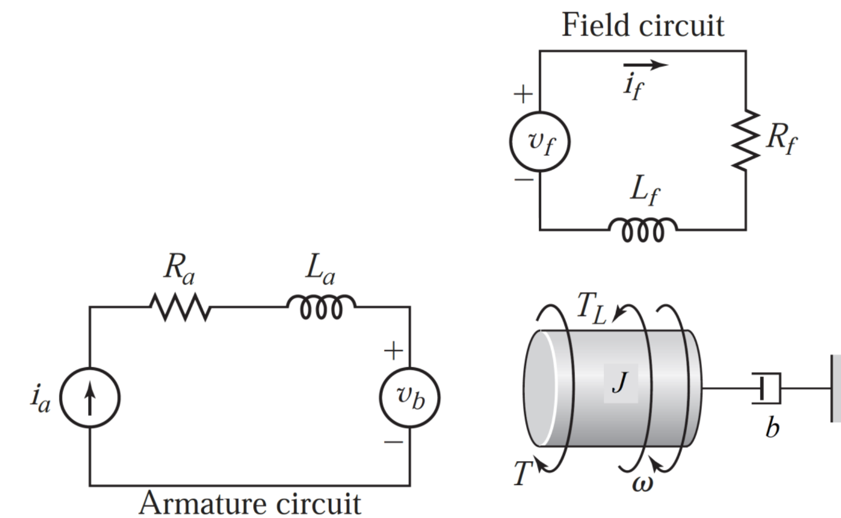
$$\boxed{v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt}}$$

- The torque is applied to the inertia and friction. Hence, from the Newton's law applied to the inertia J,

$$\boxed{\tau(t) - \tau_L(t) = J\frac{d\omega\ (t)}{dt} + b\omega(t)}$$

- These two equations along with the torque equation constitute the system model.

$$\boldsymbol{\tau \propto \phi_f i_a} \quad \rightarrow \quad \boxed{\tau(t) = K_T i_f(t)}$$



R_a = Armature Resistance, Ω

L_a = Armature Inductance, H

i_a = Armature Current, A

i_f = Field Current, A

v_f = Field Voltage, V

v_b = Back-emf, V

ω = Angular velocity of the motor shaft, rad/sec

θ = Angular displacement of the motor shaft, rad

τ = Torque developed by the motor, $N.m$

J = Moment of inertia of the motor and load referred to the motor shaft,

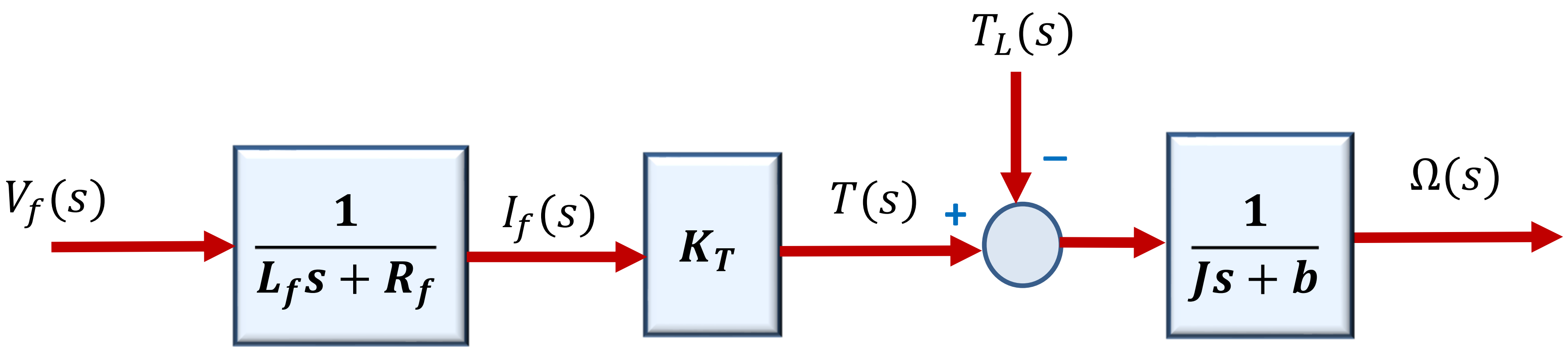
b = Viscous friction coefficient of the motor and load referred to the motor shaft,

Modeling of Field-Controlled DC Motor

❑ Block Diagram Model

- Block diagram model of a field-controlled DC motor with the field voltage $v_f(t)$ as the input and the motor angular velocity $\omega(t)$ as the output is obtained as:

$$
\begin{cases}
v_f(t) = R_f i_f(t) + L_f \dfrac{di_f(t)}{dt} \longrightarrow V_f(s) = (L_f s + R_f) I_f(s) \longrightarrow \boxed{I_f(s) = \dfrac{1}{L_f s + R_f} V_f(s)} \\[2ex]
\tau(t) - \tau_L(t) = J \dfrac{d\omega(t)}{dt} + b\omega(t) \longrightarrow T(s) - T_L(s) = (Js + b)\Omega(s) \longrightarrow \boxed{\Omega(s) = \dfrac{1}{Js + b}\big(T(s) - T_L(s)\big)} \\[2ex]
\tau(t) = K_T i_f(t) \longrightarrow \boxed{T(s) = K_T I_f(s)}
\end{cases}
$$



Dynamics of the electrical and the mechanical subsystem are modeled as first-order systems.

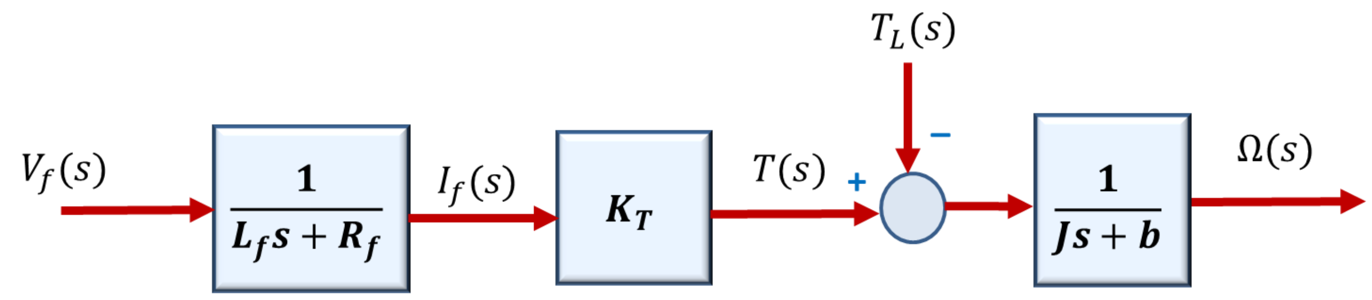
The model has no feedback loop because field circuit is stationary, and has no back-emf.

Modeling of Field-Controlled DC Motor

❑ Transfer Function Model

- Transfer function models can be determined from block diagram model by considering the field voltage $v_f(t)$ and load torque $\tau_L(t)$ as the inputs and the field current $i_f(t)$ and the angular velocity $\omega(t)$ as the outputs.

- The transfer function models are obtained as:



$$\frac{\Omega(s)}{V_f(s)} = \frac{K_T}{(L_f s + R_f)(Js + b)}$$

$$\frac{I_f(s)}{V_f(s)} = \frac{1}{L_f s + R_f}$$

$$\frac{\Omega(s)}{T_L(s)} = \frac{-1}{Js + b}$$

- Unlike the armature-controlled motor, the current in the field-controlled motor **is not affected** by the load torque.

Modeling of Field-Controlled DC Motor

State Space Model

- State space model is obtained by considering the applied voltage $v_f(t)$ and load torque $\tau_L(t)$ as the inputs and the field current $i_f(t)$ and the angular velocity $\omega(t)$ as the outputs.
- The state variables are defined as the field current $i_f(t)$ and the motor speed $\omega(t)$:

$$v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt}$$

$$\tau(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + b\omega(t)$$

$$\tau(t) = K_T i_f(t)$$

$$\begin{cases} q_1(t) = i_f(t) \\ q_2(t) = \omega(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = \dfrac{di_f(t)}{dt} = \dfrac{1}{L_f}\left(v_f(t) - R_f i_f(t)\right) \\ \dot{q}_2(t) = \dfrac{d\omega(t)}{dt} = \dfrac{1}{J}\left(\tau(t) - \tau_L(t) - b\omega(t)\right) \end{cases}$$

State-variable equations are obtained as:

$$\begin{cases} \dot{q}_1(t) = \dfrac{1}{L_f}\left(v_f(t) - R_f q_1(t)\right) \\ \dot{q}_2(t) = \dfrac{1}{J}\left(K_T q_1(t) - \tau_L(t) - b q_2(t)\right) \end{cases}$$

The **output equation** is obtained as:

$$y_1(t) = i_f(t) = q_1(t)$$

$$y_2(t) = \omega(t) = q_2(t)$$

Modeling of Field-Controlled DC Motor

State Space Model

- We can represent the state and output equations in the standard matrix-vector form as below:

$$\begin{cases} \dot{q}_1(t) = \dfrac{1}{L_f}\big(v_a(t) - R_f q_1(t)\big) \\ \dot{q}_2(t) = \dfrac{1}{J}\big(K_T q_1(t) - \tau_L(t) - b q_2(t)\big) \end{cases} \qquad \begin{cases} y_1(t) = i_f(t) = q_1(t) \\ y_2(t) = \omega(t) = q_2(t) \end{cases}$$

$$\boxed{\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)}$$

State Equation

$$\rightarrow \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{-R_f}{L_f} & 0 \\ \dfrac{K_T}{J} & -\dfrac{b}{J} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_f} & 0 \\ 0 & -\dfrac{1}{J} \end{bmatrix} \begin{bmatrix} v_f(t) \\ \tau_L(t) \end{bmatrix}$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$$

Output Equation

$$\rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_f(t) \\ \tau_L(t) \end{bmatrix}$$

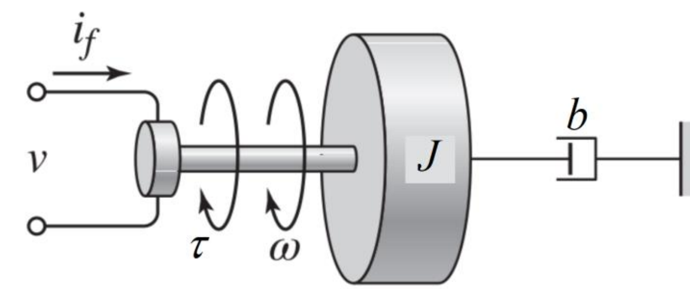
- The state-space representation is equivalent to the three transfer function models, which is more appropriate to model MIMO systems.

Modeling of Field-Controlled DC Motor

Example 6

A certain rotational system has an inertia $J = 50\text{kg.m}^2$ and a viscous damping constant $b = 10\text{Ns/m}$. The torque $\tau(t)$ is applied by an electric motor.
The equation of motion the mechanical subsystem is

$$50\omega'(t) + 10\omega(t) = \tau(t)$$



The voltage $v(t)$ is applied to the motor. The model of the motor's field current i_f in amperes is

$$0.001i_f'(t) + 5i_f(t) = v(t)$$

The torque-current relationship is $\tau(t) = 25i_f(t)$. Suppose the applied voltage is $v(t) = 10V$.

a) Determine the transfer function of mechanical and electrical subsystems and show the system in a block diagram model.

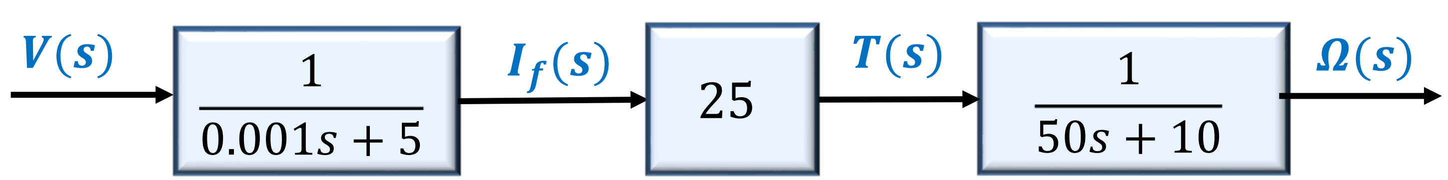
Transfer function of the mechanical and electrical subsystems are

$$50\omega'(t) + 10\omega(t) = \tau(t) \xrightarrow{\text{Laplace Transform}} 50s\Omega(s) + 10\Omega(s) = T(s) \longrightarrow \boxed{\frac{\Omega(s)}{T(s)} = \frac{1}{50s + 10}}$$

Mechanical Subsystem

$$0.001i_f'(t) + 5i_f(t) = v(t) \xrightarrow{\text{Laplace Transform}} 0.001sI_f(s) + 5I_f(s) = V(s) \longrightarrow \boxed{\frac{I_f(s)}{V(s)} = \frac{1}{0.001s + 5}}$$

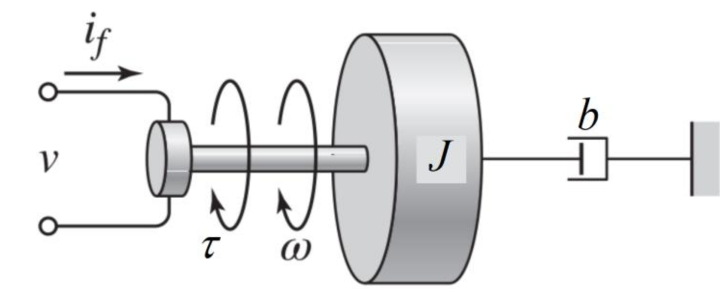
Electrical Subsystem



Modeling of Field-Controlled DC Motor

Example 6

A certain rotational system has an inertia $J = 50\text{kg.m}^2$ and a viscous damping constant $b = 10\text{Ns/m}$. The torque $\tau(t)$ is applied by an electric motor.
The equation of motion the mechanical subsystem is



$$50\omega'(t) + 10\omega(t) = \tau(t)$$

The voltage $v(t)$ is applied to the motor. The model of the motor's field current i_f in amperes is

$$0.001 i_f'(t) + 5 i_f(t) = v(t)$$

The torque-current relationship is $\tau(t) = 25 i_f(t)$. Suppose the applied voltage is $v(t) = 10V$.

b) Determine the steady-state speed of the inertia.

First find the overall transfer function:

$$V(s) \longrightarrow \boxed{\frac{1}{0.001s + 5}} \xrightarrow{I_f(s)} \boxed{25} \xrightarrow{T(s)} \boxed{\frac{1}{50s + 10}} \longrightarrow \Omega(s)$$

$$\frac{\Omega(s)}{V(s)} = \frac{25}{(0.001s + 5)(50s + 10)}$$

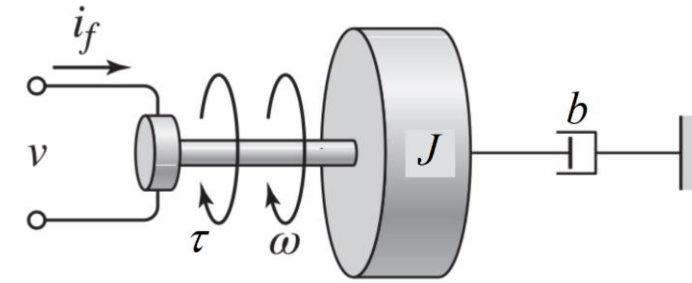
The steady-state speed is obtained using the final-value theorem:

$$\lim_{t\to\infty} \omega(t) = \lim_{s\to 0} s\Omega(s) \quad \rightarrow \quad \omega(\infty) = \lim_{s\to 0} s\left(\frac{25}{(0.001s + 5)(50s + 10)}\right)\left(\frac{10}{s}\right) = 5\, rad/s$$

Modeling of Field-Controlled DC Motor

Example 6

A certain rotational system has an inertia $J = 50$kg.m^2 and a viscous damping constant $b = 10$Ns/m. The torque $\tau(t)$ is applied by an electric motor.
The equation of motion the mechanical subsystem is



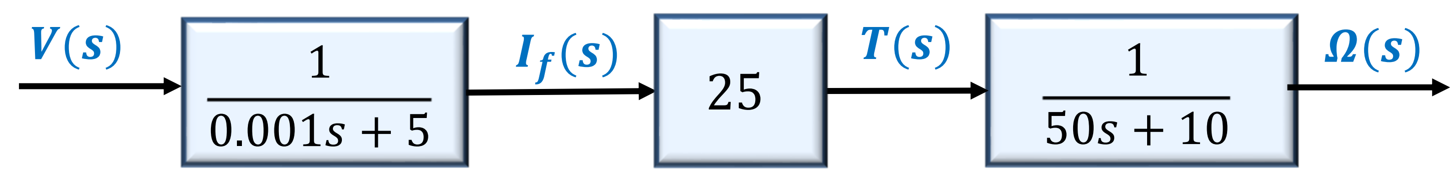
$$50\omega'(t) + 10\omega(t) = \tau(t)$$

The voltage $v(t)$ is applied to the motor. The model of the motor's field current i_f in amperes is

$$0.001 i_f'(t) + 5 i_f(t) = v(t)$$

The torque-current relationship is $\tau(t) = 25 i_f(t)$. Suppose the applied voltage is $v(t) = 10V$.

c) Determine the steady-state value of the current and the torque.



The steady-state value of the current is obtained using the final-value theorem:

$$\lim_{t\to\infty} i_f(t) = \lim_{s\to 0} sI_f(s) \quad \rightarrow \quad i_f(\infty) = \lim_{s\to 0} s\left(\frac{1}{0.001s+5}\right)\left(\frac{10}{s}\right) = 2\ A$$

The steady-state value of the torque:

$$\tau(t) = 25 i_f(t) \quad \rightarrow \quad \tau(\infty) = 25(2A) = 50\ N.m$$

THANK YOU

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