Chapter 2

Series Convergence Tests, Absolute vs Conditional Convergence

2.1 Readings

(1) CLP II §3.3 Convergence Tests

(2) CLP II §3.4 Absolute and Conditional Convergence

Recommended exercises

See course iDEAS site for a complete list of exercises.

2.2 Lesson Overview

2.2.1 Lesson goals

1. Use the Divergence Test to determine if a series fails to converge because its n^{th} term is not tending to zero as n tends to ∞.

2. Use the Integral Test to determine the convergence or divergence of a series $\sum a_n$ given the existence of a nonnegative, decreasing function f that agrees with the terms of the series at every integer.

3. Use the Comparison and Limit Comparison Tests to compare the convergence or divergence of a series with that of another series which we know more about.
4. Use the Alternating Series Test to deal with the convergence of series whose successive terms alternate in sign, are nonincreasing, and tend to zero as $n \to \infty$.
5. Use the Ratio Test to determine the convergence or divergence of series for which it is easy to compute the limit of the ratio of successive terms $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ such as series containing powers or factorials.
6. Determine whether a given series converges absolutely or conditionally.

2.3 The Divergence Test

FRY Theorem II.3.3.1, Divergence Test

Theorem 2.1. Let $\sum a_n$ be a series. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. That is, if the terms of the series are not converging to 0, then the series will not converge

Remark: The converse of this statement is not true because there exist series whose n^{th} term converges to 0, yet the series diverges. (The harmonic series $\sum_{n=1}^{\infty} 1/n$ is such an example: Its n^{th} term $\frac{1}{n}$ converges to 0, yet the series fails to converge.)

Example 2.2. (Like FRY Example II.3.3.2, an example of the divergence test in action)

Does the series $\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n}}_{a_n}$ converge? $= \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \cdots$

Here $a_n = \frac{n+1}{n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}(n+1)}{\frac{1}{n}(n)} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1}$$

$$= \lim_{n\to\infty} \left(1 + \cancel{\tfrac{1}{n}}^{\,0}\right) = 1 \neq 0$$

By the Divergence Test, $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges

Comment: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges. Divergence Theorem

If p, then q.

If $\neg q$, then $\neg p$. (Contrapositive)

Also TRUE If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If I live in Toronto, then I live in Canada.

If I live in Canada, then I live in Toronto.

WRONG $\rightarrow$ If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

Counterexample:

Consider harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (a_n)

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = 1.5$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$
$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$
$$= 2$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$
$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{1/2}$$
$$= 2.5$$

$\vdots$

$S_{16} > 3$

$S_{32} > 3.5$

$S_{64} > 4$

$\vdots$

$\lim_{n\to\infty} S_N = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Example 2.3. Can we use the Divergence Test to determine whether or not the series $\sum_{n=1}^{\infty} \cos(\pi n)$ diverges?

$$= \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) + \cdots$$
$$= -1 + 1 - 1 + 1 - \cdots$$

Since $\lim_{n\to\infty} \cos(\pi n) \neq 0$, by the Divergence Test, $\sum_{n=1}^{\infty} \cos(\pi n)$ diverges.

Example 2.4. Which of the following series can we conclude the divergence of because of the Divergence Test?

1. $\sum_{n=1}^{\infty} \frac{2n-1}{n+3}$ $= a_n$

   Since $\lim_{n\to\infty} \frac{2n-1}{n-3} = \lim_{n\to\infty} \frac{2-\cancelto{0}{\frac{1}{n}}}{1-\cancelto{0}{\frac{3}{n}}} = 2 \neq 0$, by the Divergence Test, this series diverges

2. $\sum_{n=1}^{\infty} \frac{1}{n}$ $= a_n$

   Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we cannot use the Divergence Test to say anything about its divergence.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{1}{n^2} = 0$. We cannot use the Divergence Series to conclude that the series diverges.

$$\sum_{n=0}^{\overset{N}{49}} \underbrace{1}_{a} \cdot \underbrace{\left(\frac{1}{5}\right)}_{r}^{n} = \boxed{\frac{a\left(1-r^{N+1}\right)}{1-r}} = \frac{1\left(1-\left(\frac{1}{5}\right)^{50}\right)}{1-1/5}$$

2.4 The Integral Test

FRY capture the sentiment behind this test well when they write, "In the integral test, we think of a series a_n, which we cannot evaluate explicitly, as the area of a union of rectangles with width one and height a_n. Then we compare that area with the area represented by an integral that we can evaluate explicitly."

FRY Theorem II.3.3.5, Integral Test

Theorem 2.5. Let $N_0 \in \mathbb{N}$. If $f : [N_0, \infty) \to \mathbb{R}$ is

* continuous,
- nonnegative,
- decreasing, and
- equals a_n at every integer $n \geq N_0$,

then

$$\sum_{n=1}^{\infty} a \quad \text{converges} \quad \iff \int_{N}^{\infty} f(x)\,dx \text{ converges.}$$

Furthermore, when the series converges, the truncation error is

$$\left| \sum_{n=1}^{\infty} a - \sum_{n=1}^{N} a \right| \leq \int_{N}^{\infty} f(x)\,dx \text{ for all } N \geq N_0.$$

Example 2.6. Determine whether the series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ converges or diverges?

Consider $f : [3,\infty) \to \mathbb{R}$ given by $f(x) = \dfrac{\ln x}{x}$.

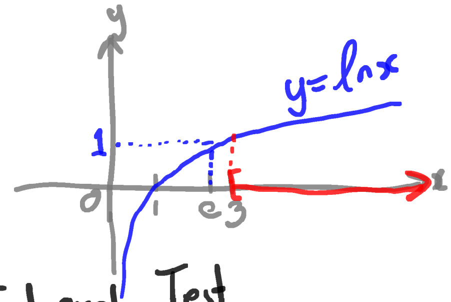
(i) Since $\ln x > 0$ for all $x \in [3,\infty)$
and $x > 0$ for all $x \in [3,\infty)$,
$f(x) = \dfrac{\ln x}{x} > 0$ for all $x \in [3,\infty)$.

(ii) $$f'(x) = \frac{(\ln x)'x - \ln x\,(x)'}{x^2} = \frac{\left(\frac{1}{x}\right)x - \ln x\,(1)}{x^2}$$

Quotient Rule

$$= \frac{1 - \ln x}{x^2}$$

For $x \in [3,\infty)$, since $x > e$, $1 - \ln x < 0$.
So $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$. Thus $f(x) = \dfrac{\ln x}{x}$ is decreasing on $[3,\infty)$.



(iii) For every integer $n \geq 3$,

$$f(n) = \frac{\ln(n)}{n} = a_n.$$

By The Integral Test,

$$\sum_{n=3}^{\infty} \underbrace{\frac{\ln(n)}{n}}_{= a_n} \text{ converges} \xLeftrightarrow{\text{if and only if}} \int_3^{\infty} \underbrace{\frac{\ln x}{x}}_{= f(x)} dx \text{ converges}$$

$$\int_3^{\infty} \frac{\ln x}{x} dx = \int_{\ln 3}^{\infty} u \, du = \lim_{b\to\infty} \boxed{\int_{\ln 3}^{b} u \, du} = \lim_{b\to\infty} \frac{1}{2}u^2 \Big|_{\ln 3}^{b}$$

Let $u = \ln x$

Then $du = \frac{1}{x} dx$

$$= \lim_{b\to\infty} \frac{1}{2}\left(b^2 - (\ln 3)^2\right)$$

$$= \infty$$

Since $\int_3^{\infty} \frac{\ln x}{x} dx$ diverges, $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ diverges.

Example 2.7. (FRY Example II.3.3.6, the p test)
For which values of p does the "p-series"

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

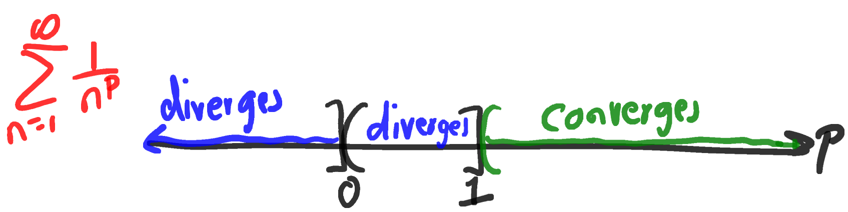
converge?[a]

[a]The p-test for integrals tells us that if $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ converges, and that if $0 < p \le 1$, then the $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

<u>p-test for integrals</u>

If $0 < p \le 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ diverges. $\xRightarrow[\text{by the Integral Test}]{\text{implies}}$ For $0<p\le 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

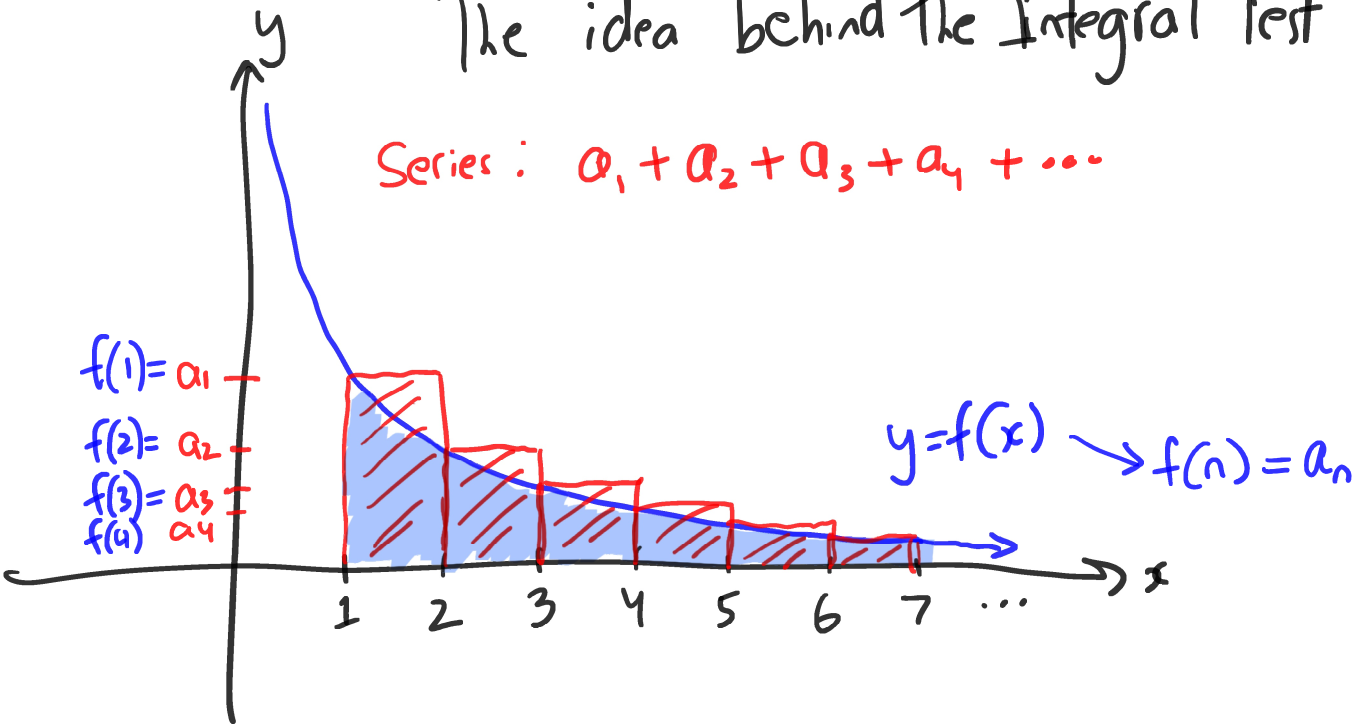
If $p>1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ converges. $\Longrightarrow$ For $p>1$. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges



Example 2.8. Which of the following p-series converges?

	$p =$	Consequence
1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	$1/2$	diverges
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$	1.01	converges
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2}}$	2	converges
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ $\quad \sqrt{n^3} = (n^3)^{1/2} = n^{\frac{3}{2}}$	$\frac{3}{2}$	converges
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.001}}$	$p = 0.001$	diverges
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$	$p = 1$	diverges

The idea behind the Integral Test



2.5 The Comparison Test

FRY Theorem II.3.3.8, Comparison Test

Theorem 2.9. Let $\sum c_n$ and $\sum d_n$ be nonnegative series, that is, $c_n \geq 0$ and $d_n \geq 0$ for all n.

(i) If there exists a natural number N_0 and a positive real number K such that $|a_n| \leq Kc_n$ for all $n \geq N_0$ and $\sum c_n$ converges, then $\sum a_n$ converges.

(ii) If there exists a natural number N_0 and a positive real number K such that $a_n \geq Kd_n$ for all $n \geq N_0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Example 2.10. (Like FRY Example II.3.3.9, comparing a series with $\sum n^{-p}$ to determine whether it converges)
Study the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{2}{n^2 + 2n + 3}.$$

Given $\sum a_n$ and Know $\boxed{\sum c_n \text{ converges}}$

If $|a_n| \leq K c_n$, Then $\sum a_n$ converge.
positive constant

$\sum_{n=1}^{\infty} \underbrace{\frac{2}{n^2}}_{a_n}$ converges?

Know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

c_n

Since, for every integer $n \geq 1$,

$$|a_n| = \left|\frac{2}{n^2}\right| = \frac{2}{n^2} \leq \underbrace{2}_{K} \cdot \underbrace{\frac{1}{n^2}}_{c_n},$$

And $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, our series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converge.

(ii) $\sum_{n=1}^{\infty} a_n$ converge? | $\sum_{n=1}^{\infty} d_n$ Know diverges

If $a_n \geq K d_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Eg $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ converges?

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Also, Here $K=1$

for every integer $n \geq 3$, $\frac{\ln(n)}{n} \geq \frac{1}{n}$,

N_0

by the Comparison Test, $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ diverges.

Eg 2.10 $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{2}{n^2+2n+3}}_{a_n}$?

Can I compare this series to $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{c_n}$?

Solution For $n \geq 1$

$$\Bigg|\underbrace{\frac{2}{n^2+2n+3}}_{a_n}\Bigg| = \frac{2}{n^2+2n+3} \leq \overset{K}{2} \cdot \underset{c_n}{\frac{1}{n^2}}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the Comparison Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+2n+3}$ converges.

If p is true

$$\frac{1}{n^2+2n+3} < \frac{1}{n^2}$$

then $p \vee q$ is true.

q: $\dfrac{1}{n^2+2n+3} = \dfrac{1}{n^2}$

$2<3$ TRUE	$3=3$ TRUE
$2 \leq 3$ TRUE	$3 \leq 3$ TRUE

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2-2n-3}$ Converge ?

$$\frac{2}{n^2-2n-3} \cancel{\leq} \frac{2}{n^2}$$

FRY Theorem II.3.3.11, Limit Comparison Test

Theorem 2.11. Let $\sum b_n$ be a positive series, that is, $b_n > 0$ for all n.

(i) If $\sum b_n$ converges and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$, then $\sum a_n$ converges.

(ii) If $\sum b_n$ diverges and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L \neq 0$, then $\sum a_n$ diverges.

Example 2.12. Determine whether or not the following series converges:

$$\sum_{n=5}^{\infty} \underbrace{\frac{\sqrt{n}}{n^2-1}}_{a_n}.$$

Thinking:

$$\frac{\sqrt{n}}{n^2-1} \underset{\text{for large } n}{\approx} \frac{\sqrt{n}}{n^2} = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$$

$2-\frac{1}{2} = 3/2$

Comparison Series:

Convergent series $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^{3/2}}}_{b_n}$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2-1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2-1} \cdot \frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2-1}$$

$$= \lim_{n\to\infty} \frac{1}{1-1/n^2}$$

$$= \frac{1}{1-0}$$

$$= 1 \leftarrow L$$

By the Limit Comparison Test, $\displaystyle\sum_{n=5}^{\infty} \frac{\sqrt{n}}{n^2-1}$ converges.