Class Note 2.2.3

September 16, 2022 9:06 AM

2.2.3 Derivatives of Elementary Functions

2.2.3A Derivatives of Trigonometric Functions

The trigonometric functions are defined as functions of the independent variable x, the input, that represents an *angle* measured in radians.

Basic Trigonometric Functions

Sine: $y = \sin x$

Cosine: $y = \cos x$

Tangent: $y = \tan x$

Reciprocal Trigonometric Functions

Cosecant: $y = \csc x$; $\csc x = \frac{1}{\sin x}$

Secant: $y = \sec x$; $\sec x = \frac{1}{\cos x}$

Cotangent: $y = \cot x$; $\cot x = \frac{1}{\tan x}$

The generalized derivative formulas as presented on the Formula Sheet

Assume that $u = u(x)$. The Chain Rule is embedded into the formulas

$$\frac{d}{dx}[\sin u] = \cos u \cdot u' \quad \frac{d}{dx}[\cos u] = -\sin u \cdot u' \quad \frac{d}{dx}[\tan u] = \sec^2 u \cdot u' \quad \frac{d}{dx}[\cot u] = -\csc^2 u \cdot u'$$

$$\frac{d}{dx}[\sec u] = \sec u \tan u \cdot u' \quad \frac{d}{dx}[\csc u] = -\csc u \cot u \cdot u'$$

Practice

1. Find the derivative of each function.

a. $y = 0.52 \cos x$, find y'

b. $y = \frac{\pi}{2} \sin \theta$, find $\frac{dy}{d\theta}$

c. $y = 7 \sin 2x - 3 \cos 4x$, find y'

2. Find the derivative of each function

a. 2. If $f(\theta) = 5 \sin(100\pi\theta - 0.40)$, find $f'(\theta)$.

b. $y = 0.05 \cos^5 x$

c. $y = 3 \sec(4x)$

d. If n is any integer, find $\frac{dy}{dx}$ for $y = \frac{\pi}{4} \cos(nx)$

Answers

1. a. $-0.52 \sin x$; b. $\frac{\pi}{2} \cos \theta$; c. $14 \cos 2x + 12 \sin 4x$

2. a. $500\pi \cos(100\pi\theta - 0.40)$; b. $-0.25 \sin x \cos^4 x$; c. $12 \sec(4x) \tan 4x$; d. $-\frac{n\pi}{4} \sin(nx)$

2.2.3B Derivatives of Exponential Functions

2a) $f(\theta) = 5 \sin(100\pi\theta - 0.4$

$f'(\theta) = \frac{df}{d\theta} = 5 \cos(100\pi\theta - 0.40) \cdot \frac{d}{d}$

$= 500\pi \cos(100\pi\theta - 0.40)$

2b) $y = 0.05 \cos^5 x = 0.05 [\cos x]$

$y' = 0.05(5)(\cos x)^4 \frac{d}{dx} \cos x = ($

$y' = -0.25 \sin x \cos^4 x$

2d) $y = \frac{\pi}{4} \cos(n$

$y' = \frac{\pi}{4} (-\sin($

2e) $y = \sin(\tan$

$y' = \cos(\tan$

$y' = 2x \cos$

Live poll

$f(x) = 2$

$f'(x) = 5$

$f(x) = 5 - 6$

$f'(-1) = 5 -$

$f'(-1) = (/$

$f(\pi) =$

$f'(x) = \quad ($

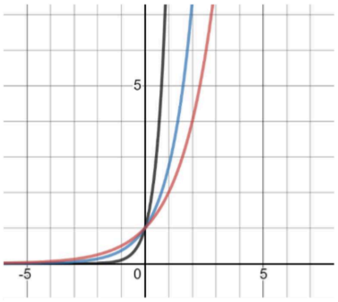
The exponential function is a power function with the constant Base and variable Exponent

$$POWER = BASE^{EXPONENT}$$

$$y = b^x,$$

where x- the independent variable, the INPUT

y – the dependent variable, the OUTPUT, is the value computed based on the input and a fixed parameter b.



https://www.desmos.com

Three exponential functions with $b > 1$ are shown on the left:

$y = 10^x$, $y = e^x$ and $y = 2^x$

Observe, that all three graphs

- rise very quickly
- pass through the point (0,1), the Y-int
- domain: $(-\infty, \infty)$
- range: $y > 0$
- Horizontal axis $y = 0$ is the horizontal asymptote:
  $\lim_{x \to -\infty} y = 0^+$

Exponential functions are continuous and smooth and used commonly in mathematical modelling

Derivative Formulas:

$$\frac{d}{dx}[b^x] = b^x \ln b$$

$$\frac{d}{dx}[e^x] = e^x$$

We start with basic examples that use the derivative formulas:

Derivative of the Exponential Function of Base b	**EXAMPLE 1**
$\frac{d}{dx}[b^x] = b^x \ln b$ *Verbally:* to differentiate the exponential function, copy the expression and adjust it by the $\ln b$ (ln of base). Use Chain Rule to handle any nested functions.	a. $y = 10^x$, then $b = 10$, $u = x$, $\frac{du}{dx} = 1$ $y' = \frac{d}{dx}[10^x] = 10^x \ln 10$ b. $y = 10^{3x}$, then $b = 10$, $u = 3x$, $\frac{du}{dx} = 3$ $y' = \frac{d}{dx}[10^{3x}] = 10^{3x} \ln 10\ (3)$ $= 3 \ln 10\ 10^{3x}$

EXAMPLE 2. (Self-Check) Find the derivative of $y = 2^{4x}$ and evaluate it at $x = 0.5$

$y' = 2^{4x} \cdot \ln 2 \cdot \frac{d}{dx}[4x] = 4 \ln 2\ 2^{4x}$

$y'(0.5) = 4 \ln 2\ 2^{4(0.5)}$

$y' = 4 \ln 2\ 4$

$y' = 16 \ln 2$

Derivative of the Exponential Function of Base e	**EXAMPLE 3**
$\frac{d}{dx}[e^x] = e^x$ $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$ Verbally: to differentiate the exponential function with the base e, just copy the expression. Use Chain Rule to handle any nested functions. There is no need for adjusting because	a. $y = e^x$, then $u = x$, $\frac{du}{dx} = 1$ $y' = \frac{d}{dx}[e^x] = e^x(1) = e^x$ b. $y = e^{5x}$, then $u = 5x$, $\frac{du}{dx} = 5$ $y' = \frac{d}{dx}[e^{5x}] = e^{5x}(5)$ $= 5(e^{5x})$

$\ln b = \ln e = 1$	

> The exponential function of base e: $y = e^x$ is very popular in applications for this very reason: the derivative of the function is the function itself

$x \to 3x^2+4 \to e(\ )$

EXAMPLE 4 Find the derivative of $y = e^{3x^2+4}$.

$y' = e^{3x^2+4}$

$y' = e^{3x^2+4} \cdot (6x)$ $\quad y' = 6xe^{3x^2+4}$

EXAMPLE 5 Find the derivative of $y = x^2e^{5x}$.

By the Product Rule: $y' = (2x)e^{5x} + x^2e^{5x}(5) = xe^{5x}(2 + 5x)$

EXAMPLE 6 (Self-Check) Find the derivative of each of the following functions:

a. $y = 5^{3-2x^2}$ b. $y = 8e^{\sqrt{x}}$ c. $y = t\,e^t$

$y' = 5^{3-2x^2} \ln 5$ $\quad y = 8e^{x^{\frac{1}{2}}}$

$y' = 5^{3-2x^2} \ln 5 \cdot 4x$ $\quad y' = 8e^{\sqrt{x}} \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{4e^{\sqrt{x}}}{\sqrt{x}}$

$y' = 4x(5^{3-2x^2}) \ln 5$

EXAMPLE 7 Find the derivative of $y = [\sin(e^x)]^3$.

This is a composite function. Break it up (decompose) to understand how to apply the Chain Rule

$$x \to e^x \to \sin(\ \ ) \to [\ \ ]^3$$

The chain rule is applied three times along with rules for differentiating the power function, sine and the exponential function:

$$y' = 3[\sin(e^x)]^2 \frac{d}{dx}[\sin(e^x)] = 3[\sin(e^x)]^2 \cos(e^x) \frac{d}{dx}[e^x]$$
$$= 3e^x[\sin(e^x)]^2 \cos(e^x)$$

EXAMPLE 8 Find the derivative of $\frac{dy}{d\theta}$ of $y = e^\theta \cos 2\theta$.

Using the product rule with $u = e^\theta$ and $v = \cos 2\theta$, we compute

$$y' = \frac{dy}{d\theta} = e^\theta \cos 2\theta + e^\theta(-\sin 2\theta)2$$
$$= e^\theta \cos 2\theta - 2e^\theta \sin 2\theta$$
$$= e^\theta(\cos 2\theta - 2\sin 2\theta)$$

Practice

Find the derivatives of the functions.

a. $y = 7^x \cos 3x$

b. $y = \frac{1}{2}(e^x + e^{-x})$

c. $y = \frac{1}{2}e^{\tan 2x}$

$y = \frac{1}{2}e^{\tan 2x}$

$y' = \frac{1}{2}e^{\tan 2x} \cdot \sec^2(2x) \cdot 2$

$\sec^2(2x)e^{\tan 2x}$

ANSWERS

Self-Check questions:

Self-Check questions:

Ex.2: $y' = 4\ln 2\ 2^{4x}$; $y'(0.5) = 16\ln 2 \cong 11.090$ to 3dp.

Ex.4 $y' = e^{3x^2+4}(6x) = 6xe^{3x^2+4}$

Ex 6. a. $-4x\ln 5\ 5^{3-2x^2}$; b. $\frac{4e^{\sqrt{x}}}{\sqrt{x}}$; c. $e^t(1+t)$

Practice

a. $7^x(\ln 7\cos 3x - 3\sin 3x)$; b. $\frac{1}{2}(e^x - e^{-x})$; c. $\sec^2(2x)e^{\tan 2x}$