Module 5

Methods of Integration. Further Topics

IDEAS work

- Assignment 3 is due on Sunday, Nov 20th
- Quiz 4 **Basics of Integration**; Available Fri 11/18/22 until Sun 11/20/22
  - Indefinite and definite integration; area under the curve; direct u-substitution
  - Contributes 5% towards the Final Grade
  - Two attempts
  - Time limit: 60 mins
- Bonus_1 Practice on Integration by Parts

Module 5 Further Integration Techniques

5.1 Integration by parts

5.2 Integrating by trigonometric substitution

5.3 Integrating rational functions by partial fraction decomposition(PFD)

Integration by Parts

Module 5.1

$$\int u\,dv = uv - \int v\,du + C$$

PARTS

or FACTORS

$$\boldsymbol{u} \times \boldsymbol{dv}$$

D I

[Textbook Link CLP 2, Section 1.7]

Formula and its Derivation Explained

Given two functions of the same input x:

$$u = u(x) \text{ and } v = v(x)$$

Integration by parts formula states that

$$\int u\,dv = uv - \int v\,du + C$$

BKM

Differential form

$$du = u'\,dx$$

$$dv = v'\,dx$$

Formula and its Derivation Explained

BKM

Given two functions of the same input x:

$u = u(x)$ and $v = v(x)$

Integration by parts formula : $\int udv = uv - \int vdu + C$

Product Rule for Differentiation

$$\frac{d}{dx}[uv] = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

$$d[uv] = du\ v + u\ dv$$

$$\int d[uv] = \int du\ v + \int u\ dv + C$$

$$uv = \int vdu + \int udv\ + C$$

$$\boxed{\int udv\ = uv - \int vdu + C}$$

Practice

$$\int \boldsymbol{u}\, \boldsymbol{dv} = \boldsymbol{uv} - \int \boldsymbol{v}\, \boldsymbol{du} + \boldsymbol{C}$$

Examples. Integrate the following

1. $\int xe^{2x}dx$
2. $\int xsinx\ dx$
3. $\int x^3 lnx\ dx$
4. $\int x\, cos(3x)\ dx$
5. $\int x\, sec^2(x)dx$

6. $\int e^x\, cos(x)\ dx$

Answers

1. $\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C;$
2. $-x\cos x + \sin x + C;$
3. $\frac{x^4}{4}\ln x - \frac{x^4}{16} + C;$
4. $\frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C;$
5. $x\tan x + \ln|\cos x| + C.$
6. $\frac{1}{2}e^{x}(\sin x + \cos x) + C$

$$\int u\,dv = uv - \int v\,du + C$$

LIATE

Examples. Integrate the following

1. $\int xe^{2x}dx$
2. $\int xsinx\ dx$
3. $\int x^3 lnx\ dx$
4. $\int x\ cos(3x)\ dx$
5. $\int x\ sec^2(x) dx$

6. $\int e^x\ cos(x)\ dx$

- Mnemonic
- provides a general guideline that might help in deciding which factor(part) to integrate and which to differentiate
- Stands for
  - Logarithmic
  - Inverse trigonometric
  - Algebraic(such as polynomial)
  - Trigonometric
  - Exponential
- If the two factors are coming from different classes above, then we differentiate (u) the factor that's nearer to the top of the list and integrate (dv) the factor that's closer to the bottom of the list

5. 1 Integration by Parts.

EXAMPLE 4 Solution

5. 1 Integration by Parts. Illustrative Example

$$\int u dv = uv - \int v du + C$$

EXAMPLE 4

$$\int x \cos(3x)\, dx$$

Identify $u \text{ and } dv$:

$u = x$ $\qquad$ $dv = \cos(3x)\, dx$

EXAMPLE 4. $\int x\cos(3x)\,dx$

$$\int u\,dv = uv - \int v\,du + C$$

Identify u and dv:

$u = x \qquad dv = \cos(3x)\,dx$

Work out du and v:

$u = x$, then $du = dx$

$dv = \cos(3x)\,dx$, then

$$v = \int \cos(3x)\,dx$$

EXAMPLE 4. $\int x\,cos(3x)\,dx$

$$\int udv = uv - \int vdu + C$$

Work out du and v:

$u = x$, then $du = dx$

$dv = cos(3x)\,dx$, then

$$v = \int cos(3x)\,dx$$

$$\int \cos u \; du = \sin u + C$$

$$v = \int cos(3x)\,dx = \int cos\,u\,\frac{du}{3} =$$

U-substitution

$u = 3x,$

$du = 3dx$

$\dfrac{du}{3} = dx$

$$= \frac{1}{3}\int cos\,u\;\,du = \frac{1}{3}\sin u = \frac{1}{3}\sin 3x$$

EXAMPLE 4. $\int x\, cos(3x)\, dx$

$$\int udv = uv - \int vdu + C$$

$$u = x,\ du = dx \qquad dv = cos(3x)\, dx,\ v = \frac{1}{3}\sin 3x$$

Apply the integration by parts formula:

$$\int x\, cos(3x)\, dx = x\left(\frac{1}{3}\sin(3x)\right) - \int \frac{1}{3}\sin(3x)\, dx$$

$$= \frac{1}{3}x\sin(3x) - \frac{1}{3}\int \sin(3x)\, dx$$

EXAMPLE 4. $\int x\, cos(3x)\, dx$

$$\int udv = uv - \int vdu + C$$

$$\int x\, cos(3x)\, dx = \cdots \quad = \frac{1}{3} x \sin(3x) - \frac{1}{3} \int \sin(3x)\, dx$$

$$\int \sin u \, du = -\cos u + C$$

U-substitution

$u = 3x,$

$du = 3dx$

$\frac{du}{3} = dx$

Integrate the "replacing" integral :

$$= \frac{1}{3} x \sin(3x) - \frac{1}{3} \int \sin(u) \frac{du}{3}$$

EXAMPLE 4. $\int x\,cos(3x)\,dx$

$$\int u\,dv = uv - \int v\,du + C$$

<u>U-substitution</u>

$u = 3x,$

$du = 3dx$

$\frac{du}{3} = dx$

$$\int \sin u\, du = -\cos u + C$$

$$\int x\,cos(3x)\,dx = \cdots = \frac{1}{3}x\sin(3x) - \frac{1}{3}\int \sin(u)\frac{du}{3}$$

$$= \frac{1}{3}x\sin(3x) - \frac{1}{3}\cdot\frac{1}{3}\int \sin(u)\,du = \frac{x}{3}\sin(3x) - \frac{1}{9}(-\cos u)$$

$$= \frac{x}{3}\sin(3x) + \frac{1}{9}\cos 3x + C$$ Answer.

EXAMPLE 5 Solution

5.1 Integration by Parts. Illustrative Example

$$\int u\,dv = uv - \int v\,du + C$$

EXAMPLE 5

$$\int x\,sec^2(x)dx$$

Identify $u\ and\ dv$:

$u = x$ $\qquad$ $dv = sec^2(x)dx$

EXAMPLE 5. $\int x\, sec^2(x)dx$

$$\int udv = uv - \int vdu + C$$

Identify $u\ and\ dv$:

$u = x \qquad dv = sec^2(x)dx$

Work out $du\ and\ v$:

$u = x$, then $du = dx$

$dv = sec^2(x)dx$, then

$$v = \int sec^2(x)dx$$

EXAMPLE 5. $\int x \, sec^2(x)dx$

$$\int udv = uv - \int vdu + C$$

Work out du and v:

$u = x$, then $du = dx$

$dv = sec^2(x)dx$, then

$$v = \int sec^2(x)dx$$

$$\int \sec^2 u \; du = \tan u + C$$

$$v = \tan x$$

EXAMPLE 5. $\int x \, sec^2(x)dx$

$$\int udv = uv - \int vdu + C$$

Work out $du \; and \; v$:

$u = x$, then $du = dx$

$dv = sec^2(x)dx$, then

$$v = \int sec^2(x)dx$$

$$v = \tan x$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

Apply the integration by parts formula:

$$\int x \, sec^2(x)dx = uv - \int vdu + C = x\tan x - \int \tan x dx + C$$

$$= x\tan x + \ln|\cos x| + C$$ Answer.