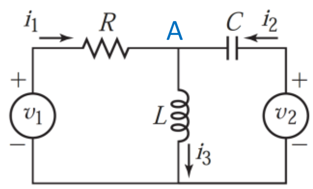
Worksheet 4 - Solution

Modeling of Electrcial Systems

1) The RLC circuit shown below has two input voltages v_1 and v_2. (a) Obtain the differential equation model for the current i_3. (b) Determine the transfer functions $I_3(s)/V_1(s)$ and $I_3(s)/V_2(s)$.



a) Apply KVL to the left-hand loop:

$$v_1 = v_R + v_L \quad \rightarrow \quad v_1 = Ri_1 + L\frac{di_3}{dt}$$

Apply KVL to the right-hand loop:

$$v_2 = v_C + v_L \quad \rightarrow \quad v_2 = \frac{1}{C}\int i_2\,dt + L\frac{di_3}{dt} \quad \rightarrow \quad \frac{dv_2}{dt} = \frac{1}{C}i_2 + L\frac{d^2i_3}{dt^2}$$

Apply KCL at node A and substitute for i_1 and i_2:

$$i_3 = i_1 + i_2 \quad \rightarrow \quad i_3 = \frac{1}{R}v_1 - \frac{L}{R}\frac{di_3}{dt} + C\frac{dv_2}{dt} - LC\frac{d^2i_3}{dt^2}$$

Rearrange this equation to obtain the answer:

$$RLC\frac{d^2i_3}{dt^2} + L\frac{di_3}{dt} + Ri_3 = v_1 + RC\frac{dv_2}{dt}$$

b) To obtain the transfer functions $I_3(s)/V_1(s)$ and $I_3(s)/V_2(s)$ take Laplace transform from this equation for zero initial conditions.

$$RLCs^2I_3(s) + LsI_3(s) + RI_3(s) = V_1(s) + RCsV_2(s)$$

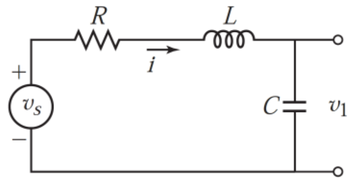
$$(RLCs^2 + Ls + R)I_3(s) = V_1(s) + RCsV_2(s)$$

$$I_3(s) = \frac{1}{RLCs^2 + Ls + R}V_1(s) + \frac{RCs}{RLCs^2 + Ls + R}V_2(s)$$

Then by applying the superposition theorem:

$$\frac{I_3(s)}{V_1(s)} = \frac{1}{RLCs^2 + Ls + R} \qquad \text{and} \qquad \frac{I_3(s)}{V_2(s)} = \frac{RCs}{RLCs^2 + Ls + R}$$

2) Consider the following series RLC circuit. Choose a suitable set of state variables, and obtain the state-space model of the circuit in matrix form. The input is the voltage v_s and the output is the voltage v_1.



In this circuit the energy is stored in the capacitor and in the inductor. Thus a suitable choice of state variables is voltage of the capacitor $v_1(t)$ and current of the inductor $i(t)$.

$q_1(t) = v_1(t)$

$q_2(t) = i(t)$

Apply KVL in the single loop,

$$v_s = v_R + v_L + v_C \quad \rightarrow \quad v_s = Ri + L\frac{di}{dt} + v_1 \quad \rightarrow \quad \frac{di}{dt} = \frac{1}{L}v_s - \frac{1}{L}v_1 - \frac{R}{L}i$$

Now find the state equations:

$$\dot{q}_1(t) = \dot{v}_1(t) \quad \rightarrow \quad \dot{q}_1(t) = \frac{1}{C}i(t) \quad \rightarrow \quad \dot{q}_1(t) = \frac{1}{C}q_2(t)$$

$$\dot{q}_2(t) = \frac{di}{dt} \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{L}v_s(t) - \frac{1}{L}v_1(t) - \frac{R}{L}i(t) \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{L}v_s(t) - \frac{1}{L}q_1(t) - \frac{R}{L}q_2(t)$$

The output equation is:

$$y(t) = v_1(t) \quad \rightarrow \quad y(t) = q_1(t)$$

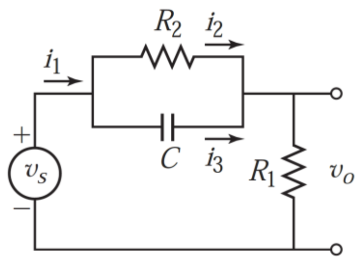
The system model has 2 state variables, 1 input, and 1 output.

Form the <u>state equation</u> and the <u>output equation</u> in the standard matrix-vector form.

$$\textit{State Equation} \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_s(t)$$

$$\textit{Output Equation} \quad \rightarrow \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0]v_s(t)$$

3) Use the impedance method to obtain the transfer function $V_o(s)/V_s(s)$ for the following circuit.



Note that R_2 and C are in parallel. Therefore their equivalent impedance $Z(s)$ is found from:

$$Z(s) = \frac{(R_2)\left(\frac{1}{Cs}\right)}{R_2 + \frac{1}{Cs}} \quad \rightarrow \quad Z(s) = \frac{R_2}{R_2Cs + 1}$$

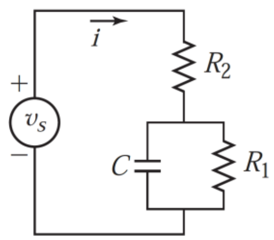
Applying voltage division formula:

$$V_o(s) = \frac{R_1}{R_1 + Z(s)} V_s(s)$$

which yields the desired transfer function:

$$\frac{V_o(s)}{V_s(s)} = \frac{R_1}{R_1 + Z(s)} = \frac{R_1R_2Cs + R_1}{R_1R_2Cs + R_1 + R_2}$$

4) Use the impedance method to obtain the transfer function $I(s)/V_s(s)$ for the following circuit.



In this circuit R_1 and C are in parallel. Therefore their equivalent impedance $Z(s)$ is found from:

$$Z(s) = \frac{(R_1)\left(\frac{1}{Cs}\right)}{R_1 + \frac{1}{Cs}} \quad \rightarrow \quad Z(s) = \frac{R_1}{R_1Cs + 1}$$

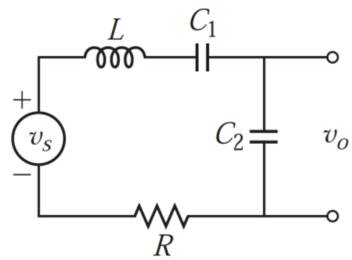
Applying Ohm's law:

$$I(s) = \frac{V_s(s)}{R_2 + Z(s)}$$

which yields the desired transfer function:

$$\frac{I(s)}{V_s(s)} = \frac{1}{R_2 + Z(s)} = \frac{R_1Cs + 1}{R_1R_2Cs + R_1 + R_2}$$

5) Use the impedance method to obtain the transfer function $V_o(s)/V_s(s)$ for the following circuit.



Note that R, L and C_1 are in series. Therefore their equivalent impedance $Z(s)$ is found from:

$$Z(s) = R + Ls + \frac{1}{C_1 s} \quad \rightarrow \quad Z(s) = \frac{LC_1 s^2 + RC_1 s + 1}{C_1 s}$$

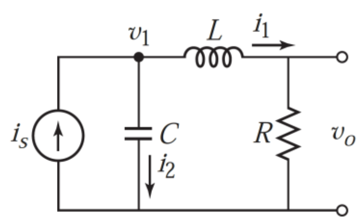
Applying voltage division formula:

$$V_o(s) = \frac{\frac{1}{C_2 s}}{\frac{1}{C_2 s} + Z(s)} V_s(s)$$

which yields the desired transfer function:

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{1 + C_2 s Z(s)} = \frac{C_1}{LC_1C_2 s^2 + RC_1C_2 s + C_1 + C_2}$$

6) For the following circuit, determine a suitable set of state variables, and obtain the state-space equations. Assume that the current source i_s is the input, and voltage v_o is the output. Draw the block diagram of the state-space model.



In this circuit the energy is stored in the capacitor and in the inductor. Thus a suitable choice of state variables is voltage of the capacitor $v_1(t)$ and current of the inductor $i_1(t)$.

$q_1(t) = v_1(t)$

$q_2(t) = i_1(t)$

Apply KCL in node v_1,

$$i_s = i_1 + i_2 \quad \rightarrow \quad i_s = i_1 + C\frac{dv_1}{dt} \quad \rightarrow \quad \frac{dv_1}{dt} = \frac{1}{C} i_s - \frac{1}{C} i_1$$

Apply KVL in the right-hand loop:

$$v_c = v_R + v_L \quad \rightarrow \quad v_1 - Ri_1 - L\frac{di_1}{dt} = 0 \quad \rightarrow \quad \frac{di_1}{dt} = \frac{1}{L} v_1 - \frac{R}{L} i_1$$

Now find the state equations:

$$\dot{q}_1(t) = \dot{v}_1(t) \quad \rightarrow \quad \dot{q}_1(t) = \frac{1}{C} i_s(t) - \frac{1}{C} i_1(t) \quad \rightarrow \quad \dot{q}_1(t) = \frac{1}{C} i_s(t) - \frac{1}{C} q_2(t)$$

$$\dot{q}_2(t) = \frac{di_1}{dt} \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{L} v_1(t) - \frac{R}{L} i_1(t) \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{L} q_1(t) - \frac{R}{L} q_2(t)$$

The output equation is:

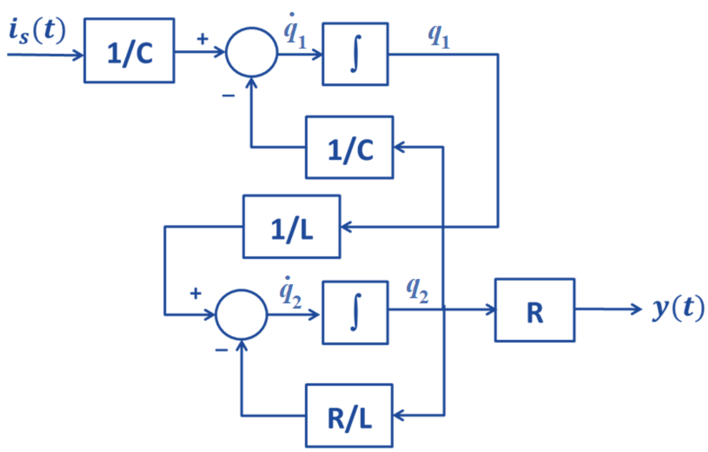
$$y(t) = v_o(t) \quad \rightarrow \quad y(t) = R i_1(t) \quad \rightarrow \quad y(t) = R q_2(t)$$

The system model has 2 state variables, 1 input, and 1 output.

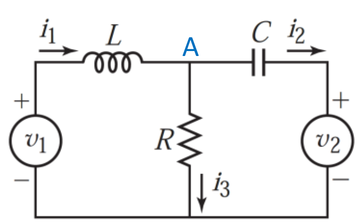
Form the state equation and the output equation in the standard matrix-vector form.

$$\textit{State Equation} \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_s(t)$$

$$\textit{Output Equation} \quad \rightarrow \quad y(t) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0] i_s(t)$$



7) For the following circuit, determine a suitable set of state variables, and obtain the state-space equations. Assume that the two inputs are v_1 and v_2 , and the two outputs are i_1 and i_2. Draw the block diagram of the state-space model.



In this circuit the energy is stored in the capacitor and in the inductor. Thus a suitable choice of state variables is voltage of the capacitor $v_c(t)$ and current of the inductor $i_1(t)$.

$$q_1(t) = v_c(t)$$

$$q_2(t) = i_1(t)$$

Apply KVL in the right-hand loop,

$$v_2 = v_R - v_C \quad \rightarrow \quad v_2 = Ri_3 - v_c \quad \rightarrow \quad i_3 = \frac{1}{R}v_2 + \frac{1}{R}v_c$$

Apply KVL in the left-hand loop,

$$v_1 = v_R + v_L \quad \rightarrow \quad v_1 = Ri_3 + L\frac{di_1}{dt} \quad \rightarrow \quad \frac{di_1}{dt} = \frac{1}{L}v_1 - \frac{R}{L}i_3 \quad \rightarrow \quad \frac{di_1}{dt} = \frac{1}{L}v_1 - \frac{1}{L}v_2 - \frac{1}{L}v_c$$

Apply KCL in node A,

$$i_1 = i_2 + i_3 \quad \rightarrow \quad i_2 = i_1 - i_3 = i_1 - \frac{1}{R}v_2 - \frac{1}{R}v_c$$

Now find the state equations:

$$\dot{q}_1(t) = \dot{v}_c(t) \quad \rightarrow \quad \dot{q}_1(t) = \frac{1}{C}i_2 = \frac{1}{C}\left(i_1 - \frac{1}{R}v_2 - \frac{1}{R}v_c\right) \quad \rightarrow \quad \dot{q}_1(t) = \frac{1}{C}q_2(t) - \frac{1}{RC}v_2(t) - \frac{1}{RC}q_1(t)$$

$$\dot{q}_2(t) = \frac{di_1}{dt} \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{L}v_1 - \frac{1}{L}v_2 - \frac{1}{L}v_c \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{L}v_1(t) - \frac{1}{L}v_2(t) - \frac{1}{L}q_1(t)$$

The output equation is:

$$y_1(t) = i_1(t) \quad \rightarrow \quad y_1(t) = q_2(t)$$

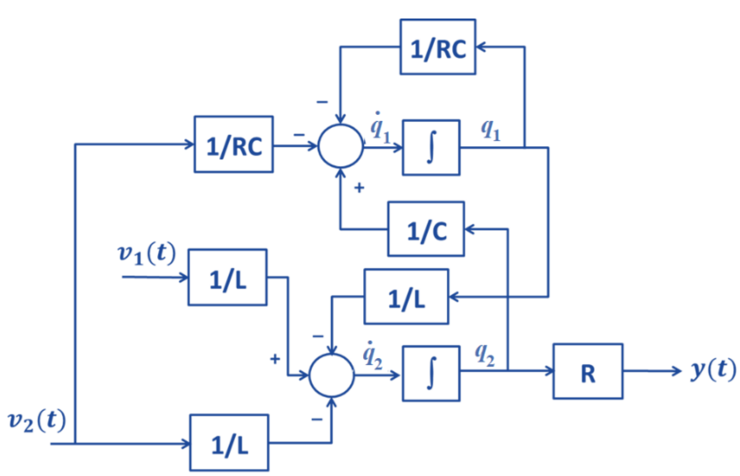
$$y_2(t) = i_2(t) \quad \rightarrow \quad y_2(t) = i_1 - \frac{1}{R}v_2 - \frac{1}{R}v_c = q_2(t) - \frac{1}{R}v_2(t) - \frac{1}{R}q_1(t)$$

The system model has 2 state variables, 2 input, and 2 output.

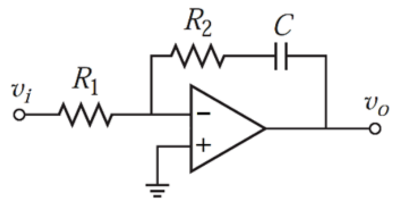
Form the state equation and the output equation in the standard matrix-vector form.

$$\textit{State Equation} \quad \rightarrow \quad \begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix} = \begin{bmatrix}-\frac{1}{RC} & \frac{1}{C}\\ -\frac{1}{L} & 0\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix} + \begin{bmatrix}0 & -\frac{1}{RC}\\ \frac{1}{L} & -\frac{1}{L}\end{bmatrix}\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}$$

$$\textit{Output Equation} \quad \rightarrow \quad \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{R} & 1\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -\frac{1}{R}\end{bmatrix}\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}$$



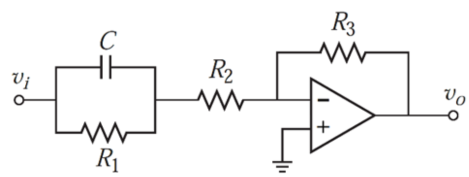
8) Obtain the transfer function $V_o(s)/V_i(s)$ for the following op-amp systems.



$$Z_1(s) = R_1$$

$$Z_2(s) = R_2 + \frac{1}{Cs} = \frac{R_2Cs+1}{Cs}$$

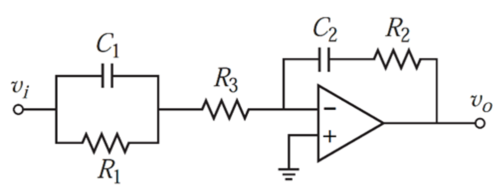
$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{\frac{R_2Cs+1}{Cs}}{R_1} = -\frac{R_2Cs+1}{R_1Cs}$$



$$Z_1(s) = R_2 + \frac{(R_1)\left(\frac{1}{Cs}\right)}{R_1 + \frac{1}{Cs}} = R_2 + \frac{R_1}{R_1Cs+1} = \frac{R_1R_2Cs + R_2 + R_1}{R_1Cs+1}$$

$$Z_2(s) = R_3$$

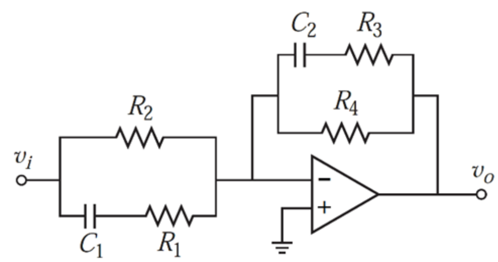
$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_3}{\frac{R_1R_2Cs+R_2+R_1}{R_1Cs+1}} = -\frac{R_3(R_1Cs+1)}{R_1R_2Cs+R_2+R_1}$$



$$Z_1(s) = R_3 + \frac{(R_1)\left(\frac{1}{C_1s}\right)}{R_1 + \frac{1}{C_1s}} = R_3 + \frac{R_1}{R_1C_1s+1} = \frac{R_1R_3C_1s + R_3 + R_1}{R_1C_1s+1}$$

$$Z_2(s) = R_2 + \frac{1}{C_2s} = \frac{R_2C_2s+1}{C_2s}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{\frac{R_2C_2s+1}{C_2s}}{\frac{R_1R_3C_1s+R_3+R_1}{R_1C_1s+1}} = -\frac{(R_2C_2s+1)(R_1Cs+1)}{C_2s(R_1R_3C_1s+R_3+R_1)}$$



$$Z_1(s) = \frac{(R_2)\left(R_1 + \frac{1}{C_1s}\right)}{R_2 + R_1 + \frac{1}{C_1s}} = \frac{R_2(R_1C_1s+1)}{(R_1+R_2)C_1s+1}$$

$$Z_2(s) = \frac{(R_4)\left(R_3 + \frac{1}{C_2s}\right)}{R_4 + R_3 + \frac{1}{C_2s}} = \frac{R_4(R_3C_2s+1)}{(R_3+R_4)C_2s+1}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{\frac{R_4(R_3C_2s+1)}{(R_3+R_4)C_2s+1}}{\frac{R_2(R_1C_1s+1)}{(R_1+R_2)C_1s+1}} = -\frac{R_4(R_3C_2s+1)((R_1+R_2)C_1s+1)}{R_2(R_1C_1s+1)((R_3+R_4)C_2s+1)}$$