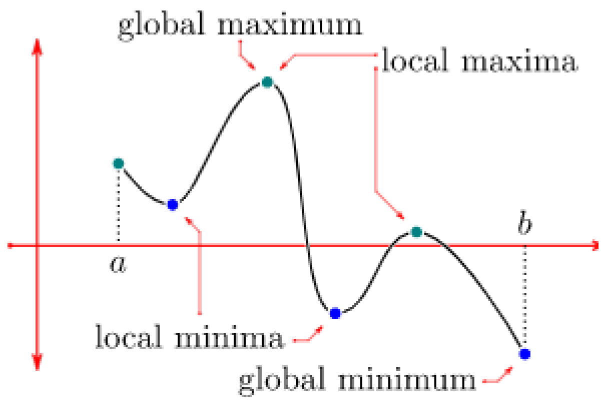
3.3 OPTIMIZATION: Applied Maximum and Minimum Problems

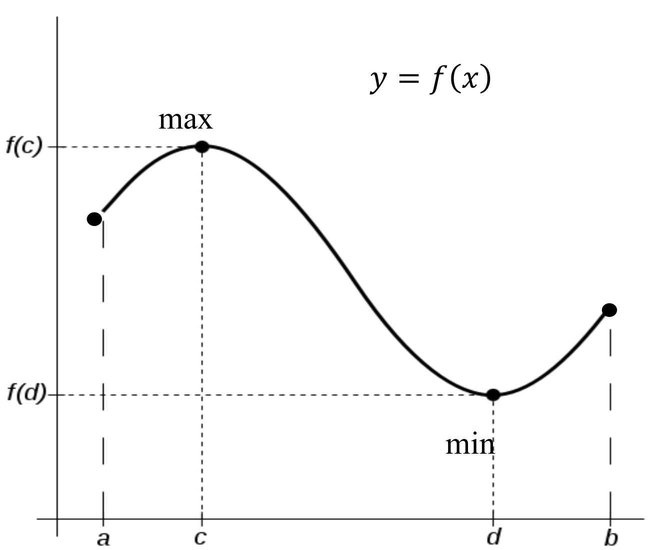
Extrema – max or min; the process of finding extreme values is referred to as *optimization.*



Textbook ref

EV Theorem
Continuous function on the finite closed interval attains its extreme values.

- The problems in this unit are concerned with finding the **absolute** maximum and **absolute** minimum.
- Problems are reduced to optimizing a continuous function $\boldsymbol{y = f(x)}$ over a finite closed interval $\boldsymbol{a \leq x \leq b}$ or $[a, b]$
- The methods used include:
  - The 1st derivative test(1DT) – tests critical values for **local** max/min
  - The 2nd derivative test(2DT) – tests critical values for **local** max/min
  - The Extreme Value Theorem (EVT) – identifies the **absolute** max/min NEW



Extreme values occur at the points where the tangent line to the curve is horizontal:

$$f'(x) = 0$$

Hence, $x = c$ and $x = d$ are critical values.

Example 1.

Finding the maximum efficiency.

An automobile manufacturer, in testing a new engine on one of its new models, found that the efficiency η (in %) of the engine as a function of the speed s (in km/h) of the car was given by $\eta = 0.768s - 0.00004s^3$. What is the maximum efficiency of the engine?

ANSWER. The maximum efficiency of the engine is about 41% and it is achieved at the speed of 80 km/h

Remarks on the use of the Extreme Value Theorem.

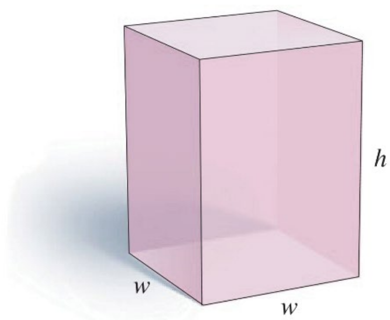
EVT: If a function f is continuous on a finite closed interval $a \le x \le b$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

To find the absolute extrema in this case

a) find the critical values of f, ie. such values of x that $f'(x) = 0$.
b) evaluate f at *all the critical points and the endpoints a and b*. (The value a is called the *left endpoint*, the value b is called the *right endpoint*)
c) The largest of the values in part a) is the absolute max and the smallest value is the absolute min.

Example 2.

Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?



Solution:
What is the function that we are maximizing? (**Objective Function**)

$$V = w^2 h$$

Is there any **constraint**?

$$2w + h = 64$$

Using constraint, the objective function can be written in terms of one variable

$$V = w^2(64 - 2w) = 64w^2 - 2w^3$$

Find critical points:

$$\frac{dV}{dw} = 0$$

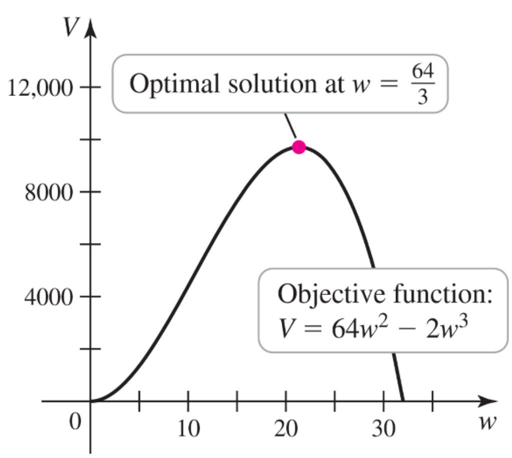
$$128w - 6w^2 = 0 \Rightarrow 2w(64 - 3w) = 0 \Rightarrow w = 0, \quad \frac{64}{3}$$

The endpoints are $w = 0$ and $w = 32$. Hence

$$V(0) = 0$$
$$V(32) = 0$$
$$V(64/3) \approx 9709\ in^3$$

We can check additionally that by the 2ndDT, $w = 64/3$ corresponds to a local maximum.

ANSWER



The dimensions of the optimal box are $w = 64/3\ in$ and $h = 64 - 2w = 64/3\ in$, so the optimal box is a **cube**. The graph shows the objective function and the optimum solution

General Strategy.

1. Draw a diagram
2. Label all given quantities, both constant and variable, on the sketch. Introduce letters to represent the variable quantities and the quantity to be optimized.
3. Find the formula for the quantity Q to be maximized or minimized.
4. Using the conditions stated in the problem, obtain a relationship between the variables involved (find formula relating the variables), so called **constraint**.
5. The quantity to be optimized (the dependent variable) should be expressed in terms of a <u>single</u> independent variable.
6. Find the interval of possible values for this variable from the physical restrictions in the problem, $a \leq x \leq b$.
7. Find maximum or minimum using the first derivative.
    a. Find the critical points of Q
    b. Evaluate Q at all critical points and at the endpoints a and b
8. It not possible to find the interval of possible values, then use the second derivative test or the first derivative test.
9. Interpret the results.

Example 3

A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running meters of chicken wire is available for the fence?

Pre - Solution:
Introduce the variables to represent the unknown quantities

Objective function:

<u>Objective function</u> is the quantity to be maximized (or minimized)

Constraints: The constraints are limitations that limit the degree to which we can pursue our OF

Solution:

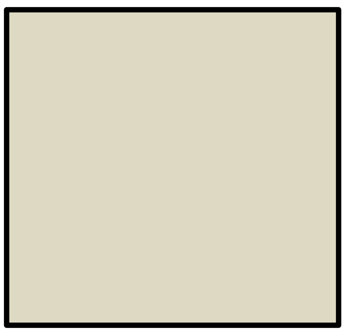
	x	$A(x) = 50x - x^2$
Left endpoint	0	
Critical value	25	
Right endpoint	50	

ANSWER
The rectangle with the greatest area and fixed perimeter is a square.

$$x = 25\text{m}, y = 25\text{m} \text{ and } Area = 625\ m^2$$

Example 4

An open-top box is to be made from a square of sheet metal 40 cm on a side by cutting a square from each corner and bending up the sides along the dashed lines. Find the dimension *x* of the cut-out that will result in a box of the greatest volume.



	x	**Volume:** $V(x) = 4x(20 - x)^2$ **(cm³)**
Right endpoint	0	
Critical value	$\frac{20}{3}$	
Left endpoint	20	

ANSWER

The volume of the resulting box is maximum when the cut-out piece $x = \frac{20}{3}$ cm. The respective maximum volume is 4741 (rounded) cm^3

Exercise 3.3

1. A person standing close to the edge on top of a 56-foot building throws a ball vertically upward. The quadratic function $h(t) = -16t^2 + 104t + 56$ models the ball's height about the ground, $h(t)$, in feet, t seconds after it was thrown.
    a. What is the maximum height of the ball? (Ans. 225 ft)
    b. How many seconds does it take until the ball hits the ground? (Ans. 7 s)
2. It is required to enclose a rectangular field by a fence and then divide it into two lots by a fence parallel to the short sides. If the area of the field is 2.50ha (1 ha = 10,000 m.sq) find the lengths of the sides so that the total length of fence will be a minimum.
   (Ans. 129 m and 194m.)
3. The power delivered to a load by a $3\bar{0}\ V$ source of internal resistance 2.0 Ω is $P(i) = 3\bar{0}i - 2.0i^2$ W, where i is the current in amperes. For what current will this source deliver the maximum power? (Ans. The power is max when $i = 7.5\ A$)
4. If 2700 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box. (Ans. x=30 cm and max V=15 500 cm^3).
5. Find the point on the curve $y = \sqrt{x}$ that is closest to the point (3,0) (Ans. (5/2, $\sqrt{5/2}$)
6. A rectangle is inscribed in a semicircle of a radius 2. Find the dimensions of the rectangle that has the maximum area. (Ans. $l = 2\sqrt{2}, h = \sqrt{2}$).
7. A rectangle is constructed under the graph of $f(x) = e^{7x}$ with one corner at (4,0) where $0 \le x \le 4$. Find the exact value of x that maximizes the area of the rectangle. (Ans. x=27/7)
8. Find the absolute extrema of $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$ on the interval $[-1,1]$ (Ans. xmin=1/8; xmax=-1)
9. Economics. Suppose the Sunglasses Hut Company has a profit function given by $P(q) = -0.02q^2 + 3q - 46$, where q is the number of thousands of pairs of sunglasses sold and produced, and $P(q)$ is the total profit, in 1000' of dollars, from selling and producing q pairs of sunglasses.
    a. How many pairs of sunglasses should be sold to maximize profits?
    b. What are the actual maximum profits that can be expected?
       Ans (75 K, $66.5K)