Tutorial 4: Uniform Circular Motion

Part A: Centripetal Acceleration

In Uniform Circular Motion, a particle moves in a **circular** path with constant **speed** .

In uniform circular motion the **velocity** vector at any instant is tangent to the circle.

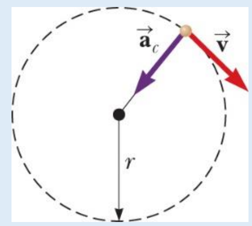
The type of acceleration that occurs in uniform motion is called **centripetal** acceleration, which is **perpendicular** to the instantaneous velocity and directed toward the **center of the circle**.

Centripetal acceleration happens due to the change in the **direction** the velocity.

Translational Velocity: $v = \frac{2\pi r}{T}$, $v = r\omega$

Angular Velocity: $\omega = \frac{2\pi}{T}$, $\omega = \frac{v}{r}$

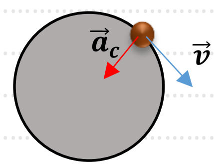
Centripetal Acceleration: $a_c = \frac{v^2}{r}$, $a_c = \frac{4\pi^2 r}{T^2}$, $a_c = r\omega^2$



1) A tire 0.500 m in radius rotates at a constant rate of 200 rev/min. Find the speed and acceleration of a small stone lodged in the tread of the tire on its outer edge.

First, find the period of the circular motion:

$$T = \frac{1}{200\ rev/min} \times \frac{60\ s}{1\ min} = 0.300\ s$$



Find the speed:

$$v = \frac{2\pi r}{T} = \frac{2\pi(0.500\ m)}{0.300\ s} = 10.47\ m/s$$

Direction is tangent to the circular path.

Find the acceleration:

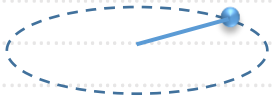
$$a_c = \frac{v^2}{r} = \frac{(10.47\ m/s)^2}{0.500\ m} = 219.24\ m/s^2$$

Direction is toward the center of the circle and perpendicular to the instantaneous velocity vector.

2) An athlete swings a ball, connected to the end of a chain, in a horizontal circle. The athlete is able to rotate the ball at the rate of $8.00\ rev/s$ when the length of the chain is $0.600\ m$. When he increases the length to $0.900\ m$, he is able to rotate the ball only $6.00\ rev/s$.

a) Which rate of rotation gives the greater speed for the ball?

First, find the period of the circular motion for each case:



$$T_1 = \frac{1}{8.00\ rev/s} = 0.125\ s$$

$$T_2 = \frac{1}{6.00\ rev/s} = 0.167\ s$$

Find the speed of the ball for each case:

$$v_1 = \frac{2\pi r_1}{T_1} = \frac{2\pi(0.600\ m)}{0.125\ s} = 30.16\ m/s$$

$$v_2 = \frac{2\pi r_2}{T_2} = \frac{2\pi(0.900\ m)}{0.167\ s} = 33.86\ m/s$$

Therefore, $6.00\ rev/s$ gives a greater speed to the ball.

b) What is the centripetal acceleration of the ball at 8.00 rev/s and 6.00 rev/s and which one gives the higher acceleration?

Find the acceleration for each case:

$$a_{c1} = \frac{v_1^2}{r_1} = \frac{(30.16\ m/s)^2}{0.600\ m} = 1.52 \times 10^3\ m/s^2$$

$$a_{c2} = \frac{v_2^2}{r_2} = \frac{(33.86\ m/s)^2}{0.900\ m} = 1.27 \times 10^3\ m/s^2$$

Therefore, $8.00\ rev/s$ gives a greater acceleration to the ball.

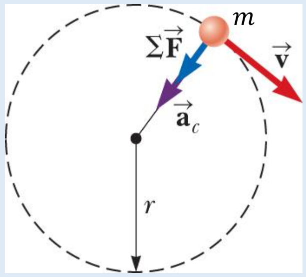
Part B: Centripetal Force

The net force causing the centripetal acceleration is called the ....**centripetal**.... force.

The centripetal force is acting in the same direction as the ....**centripetal acceleration**.... and it acts perpendicular to the ....**velocity**.... and directed inward towards the ....**center of the circle**....

Centripetal Force: $\sum F = ma_c$, $\sum F = m\frac{v^2}{r}$, $\sum F = mr\omega^2$

This net force might be gravity, friction, tension, a normal force, or a combination of two or more forces.

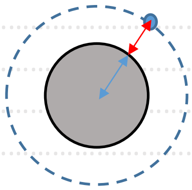


1) Assume an astronaut orbits the Moon. The path is circular and 100 km above the surface of the Moon, where the acceleration due to gravity is 1.52 m/s^2. The radius of the Moon is $1.70 \times 10^6\ m$.

(a) Determine the astronaut's orbital speed.

Gravitational force acts as the centripetal force.

The astronaut's orbital speed is found from Newton's second law:



$$\sum F = ma_c \rightarrow F_{g_m} = ma_c \rightarrow mg_m = m\frac{v^2}{r} \rightarrow v^2 = g_m r$$

$$v = \sqrt{g_m r}$$

$$v = \sqrt{(1.52\ m/s^2)(1.7 \times 10^6\ m + 100 \times 10^3\ m)}$$

$$v = \sqrt{(1.52\ m/s^2)(1.7 \times 10^6\ m + 0.1 \times 10^6\ m)}$$

$$v = \sqrt{(1.52\ m/s^2)(1.8 \times 10^6\ m)}$$

$v = 1.65 \times 10^3\ m/s$ **Tangent to the circular orbit.**

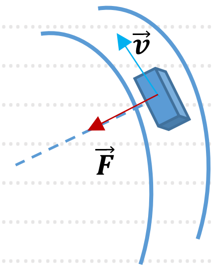
(b) Determine the period of the orbit.

To find the period, we use the following equation:

$$v = \frac{2\pi r}{T} \rightarrow T = \frac{2\pi r}{v} = \frac{2\pi(1.8 \times 10^6\ m)}{1.65 \times 10^3\ m/s} = 6.84 \times 10^3\ s$$

$$T = (6.84 \times 10^3\ s)\left(\frac{1\ hr}{3600\ s}\right) = 1.90\ hr$$

2) A curve in a road forms part of a horizontal circle. As a car goes around it at constant speed $14.0\ m/s$, the total horizontal force on the driver has magnitude $130\ N$. What is the total horizontal force on the driver if the speed on the same curve is $18.0\ m/s$ instead?



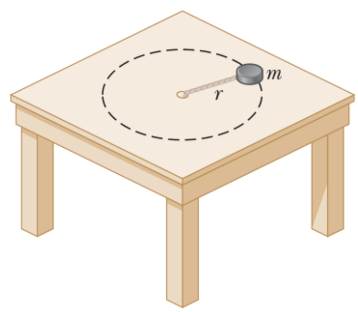
The total horizontal force is found from Newton's second law:

$$\sum F = ma_c$$

$$F_1 = m\frac{v_1^2}{r} \rightarrow 130\ N = \left(\frac{m}{r}\right)(14.0\ m/s)^2 \rightarrow \frac{m}{r} = \frac{130\ N}{(14.0\ m/s)^2}$$

$$F_2 = m\frac{v_2^2}{r} \rightarrow F_2 = \left(\frac{m}{r}\right)(18.0\ m/s)^2 \rightarrow F_2 = \left(\frac{130\ N}{(14.0\ m/s)^2}\right)(18.0\ m/s)^2 = 214.88\ N$$

3) A light string can support a stationary hanging load of $25.0\ kg$ before breaking. An object of mass $m = 3.00\ kg$ attached to the string rotates on a frictionless, horizontal table in a circle of radius $r = 0.800\ m$, and the other end of the string is held fixed as shown in figure. What range of speeds can the object have before the string breaks?

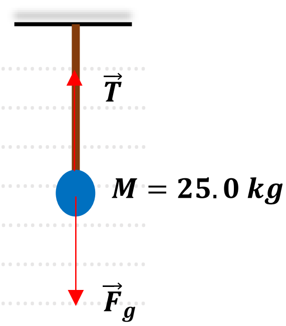


First, find the maximum tension force can be applied to the string before breaking.

$$T = F_g \rightarrow T_{max} = Mg \rightarrow T_{max} = (25.0\ kg)(9.8\ m/s^2) = 245\ N$$

Tension force acts as the centripetal force.

The maximum speed is found from Newton's second law:



$$\sum F = ma_c \rightarrow T = m\frac{v^2}{r} \rightarrow v^2 = \frac{Tr}{m} \rightarrow v_{max} = \sqrt{\frac{T_{max}\, r}{m}}$$

$$v_{max} = \sqrt{\frac{(245\ N)(0.800\ m)}{3.00\ kg}} = \sqrt{65.33} = 8.08\ m/s$$

Therefore, the range of speeds can the object have before the strings breaks is:

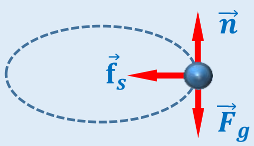
$$0 \le v \le 8.08\ m/s$$

Part C: Banked Curves and Safe Driving

Unbanked Flat Curve

Relies on the friction force: $f_s = F_c$

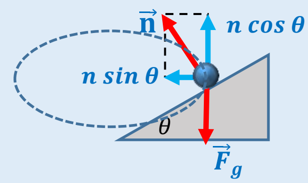
Minimum static friction: $\mu_{s,min} = \frac{v^2}{rg}$



Banked Curve

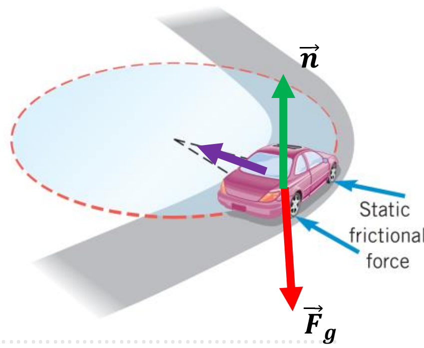
Relies on the normal force: $n \sin\theta = F_c$

Required angle: $\tan\theta = \frac{v^2}{rg}$



1) A 1200 kg car rounds an unbanked curve of radius 50.0 m at a constant speed of 80.0 km/h.

(a) What is the centripetal acceleration of the car?



First, convert the speed unit from km/h to m/s :

$$v = \frac{80\ km}{1\ h} \times \frac{1000\ m}{1\ km} \times \frac{1\ h}{3600\ s} = 22.22\ m/s$$

Then find the centripetal acceleration:

$$a_c = \frac{v^2}{r} \quad \rightarrow \quad a_c = \frac{(22.22\ m/s)^2}{50.0\ m} = 9.87\ m/s^2$$

Direction is toward the center of circle.

(b) How much centripetal force needed to cause this acceleration?

The total centripetal force needed is found from Newton's second law:

$$\sum F = ma_c \quad \rightarrow \quad F_c = (1200\ kg)(9.87\ m/s^2) = 11849.5\ N$$

Direction is toward the center of circle.

(c) If the coefficient of static friction between the road and the wheel of the car is 0.25, will the force of friction be enough to keep the car from skidding?

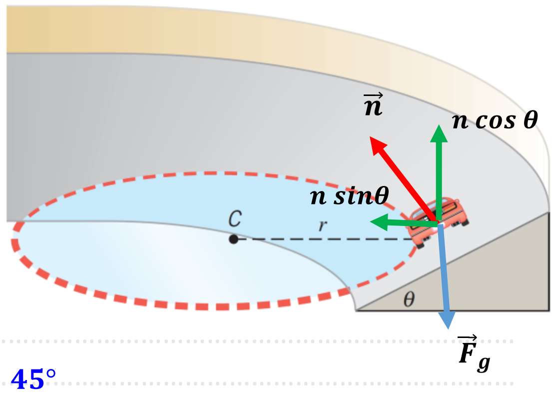
Find the force of static friction and compare with the required centripetal force from part (c)

$$f_s = \mu_s n \quad \rightarrow \quad f_s = \mu_s mg = (0.25)(1200\ kg)(9.8\ m/s^2) = 2940\ N$$

No. Since $f_s < F_c$, the force of friction is less than the required net centripetal force to keep the car on the road at the speed of 22.22 m/s.

Therefore, if the car moves with the speed of $v = 22.22\ m/s$, the force of friction will not be enough to keep the car from skidding.

(d) If the same car moves at the same speed around a banked curve of radius 50.0 m, find the required angle of banking for this car to travel safe without relying on friction.



The required angle is found from Newton's second law:

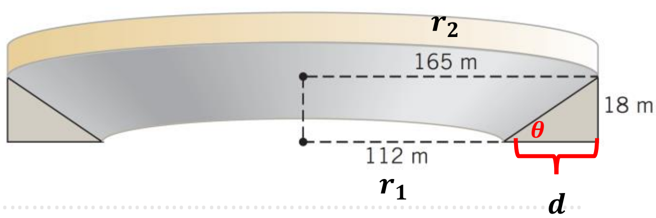
$$\sum F = ma_c \quad \rightarrow \quad n \sin\theta = m\frac{v^2}{r} \quad (1)$$

$$\sum F_y = 0 \quad \rightarrow \quad n\cos\theta - F_g = 0 \quad \rightarrow \quad n\cos\theta = mg \quad (2)$$

Divide the equation (1) by (2):

$$\tan\theta = \frac{v^2}{g\,r} \quad \rightarrow \quad \tan\theta = \frac{(22.22\ m/s)^2}{(9.8\ m/s^2)(50.0\ m)} \approx 1 \quad \rightarrow \quad \theta \approx 45^\circ$$

2) On a banked racetrack, the smallest circular path on which cars can move has a radius of 112 m, while the largest has a radius of 165 m as shown in figure. Find the smallest and the largest speed at which cars can move on this track without relying on friction.



First find the length of d as shown in the figure:

$$d = r_2 - r_1 = 165\ m - 112\ m = 53\ m$$

Use the following formula to obtain the speed in each case:

$$\tan\theta = \frac{v^2}{g\,r} \quad \rightarrow \quad v^2 = g\,r\tan\theta \quad \rightarrow \quad v = \sqrt{g\,r\tan\theta}$$

$$v_1 = \sqrt{g\,r_1\tan\theta} = \sqrt{(9.8\ m/s^2)(112\ m)\left(\frac{18\ m}{53\ m}\right)} = 19.31\ m/s$$ **smallest speed**

$$v_2 = \sqrt{g\,r_2\tan\theta} = \sqrt{(9.8\ m/s^2)(165\ m)\left(\frac{18\ m}{53\ m}\right)} = 23.43\ m/s$$ **largest speed**

Part D: Non-Uniform Circular Motion

In a non-uniform circular motion, a particle moves along a curved path, while its velocity changes both in ....magnitude.... and ....direction....

Radial acceleration is ....perpendicular.... to the instantaneous velocity.

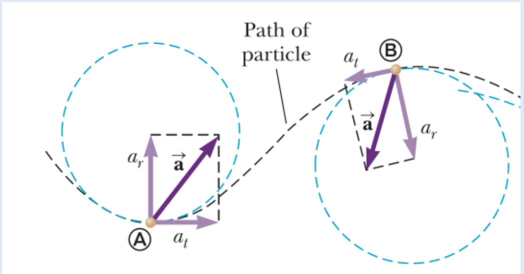
Tangential acceleration is ....parallel.... to the instantaneous velocity.

Total Acceleration: $\vec{a} = \vec{a}_r + \vec{a}_t \quad , \quad a = \sqrt{a_r{}^2 + a_t{}^2}$

Radial Acceleration: $a_r = \frac{v^2}{r}$

Tangential Acceleration: $a_t = \left|\frac{dv}{dt}\right|$

Total force acting on the particle: $\sum \vec{F} = \sum \vec{F}_r + \sum \vec{F}_t$



1) Assume a particle is moving with instantaneous speed $3.00\ m/s$ on a curved path with radius of curvature $2.00\ m$.

a) Can the particle have an acceleration of magnitude $6.00\ m/s^2$?

First, find the magnitude of the radial acceleration:

$$a_r = \frac{v^2}{r} = \frac{(3.00\ m/s)^2}{2.00\ m} = 4.50\ \ m/s$$

The magnitude of total acceleration can be larger than or equal to 4.50 m/s but not smaller.

Yes. Since 6.00 m/s > 4.50 m/s the particle can have a tangential acceleration component of:

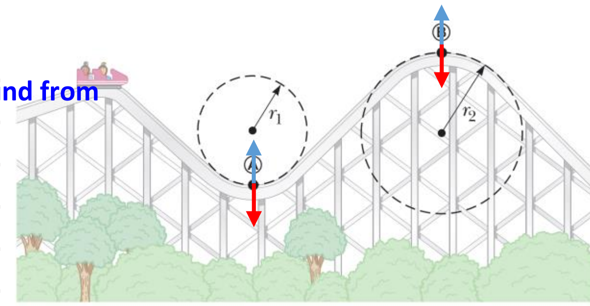
$$a_t = \sqrt{a^2 - a_r^2} = \sqrt{(6.00\ m/s^2)^2 - (4.50\ m/s^2)^2} = 3.97\ m/s^2$$

b) Can it have an acceleration of magnitude $4.00\ m/s^2$?

No. The magnitude of acceleration cannot be less than the radial component $a_r = 4.50\ \ m/s$

2) A roller-coaster car a mass of 500 kg when fully loaded with passengers. The path of the coaster from its initial point has only up-and-down motion, with no motion to the left or right. Assume the roller-coaster tracks at points A and B are parts of vertical circles of radius $r_1 = 10.0\ m$ and $r_2 = 15.0\ m$, respectively.

(a) If the vehicle has a speed of $20.0\ m/s$ at point A, what is the force exerted by the track on the car at this point?



The applied force by the track on the car at point A is found from Newton's second law:

$$\sum \boldsymbol{F} = \boldsymbol{m a_c} \quad \rightarrow \quad \boldsymbol{n_A - F_g = m a_c}$$

$$\boldsymbol{n_A = mg + m\frac{v_A^2}{r_1} = m\left(g + \frac{v_A^2}{r_1}\right)}$$

$$\boldsymbol{n_A = (500\ kg)\left((9.8\ m/s^2) + \frac{(20.0\ m/s)^2}{10.0\ m}\right) = 24900\ N}$$

(b) What is the maximum speed the vehicle can have at point B and still remain on the track.

The maximum speed can car have is found from Newton's second law:

$$\sum \boldsymbol{F} = \boldsymbol{m a_c} \quad \rightarrow \quad \boldsymbol{n_B - F_g = m(-a_c)} \quad \rightarrow \quad \boldsymbol{n_B - mg = m(-\frac{v_B^2}{r_2})}$$

$$\boldsymbol{v_B^2 = \frac{r_2}{m}(mg - n_B)} \quad \rightarrow \quad \boldsymbol{v_B = \sqrt{\frac{r_2}{m}(mg - n_B)}} \quad \boldsymbol{(1)}$$

From equation (1) , v_B becomes maximum if $n_B = 0$.

Therefore, insert the $n_B = 0$ in equation (1) to find the maximum speed at point B:

$$\boldsymbol{v_{B_{max}} = \sqrt{\frac{r_2}{m}(mg - 0)} = \sqrt{r_2\, g} = \sqrt{(15.0\ m)(9.8\ m/s^2)} = 12.12\ m/s}$$