HUMBER ENGINEERING

MENG 3510 – Control Systems
LECTURE 11

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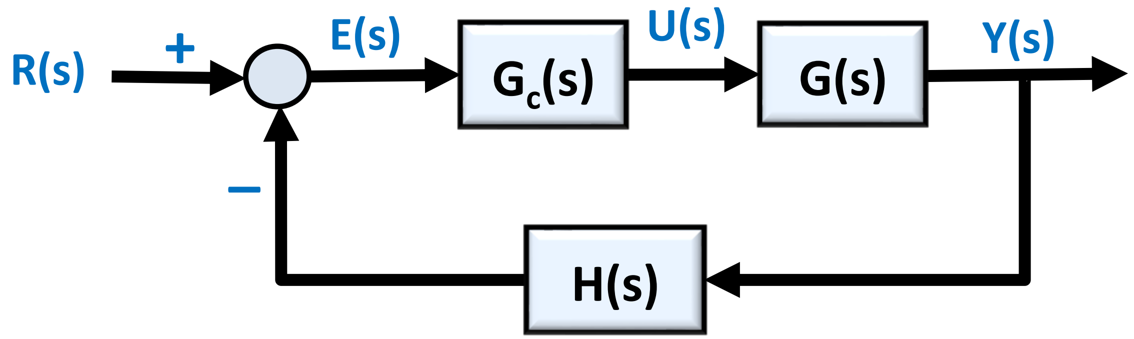
LECTURE 11
Controller Design via Frequency Response

- Control System Design via Bode Diagram
  - Frequency Response of P, PI and PD Controllers
  - Proportional Controller Design
  - PD Controller Design
  - PI Controller Design

Control System Design via Bode Diagram

- Consider the following closed-loop system with controller $G_c(s)$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}$$



- The goal is to design the control system using the Bode diagram of the open-loop system $G_c(s)G(s)H(s)$, to satisfy the desired performance criteria such as gain margin, phase margin, and steady-state error.
- We will focus on designing the PID family controllers via Bode plot.
  - **Proportional Controller**
  - **PI Controller**
  - **PD Controller**

$$G_c(s) = K_p$$

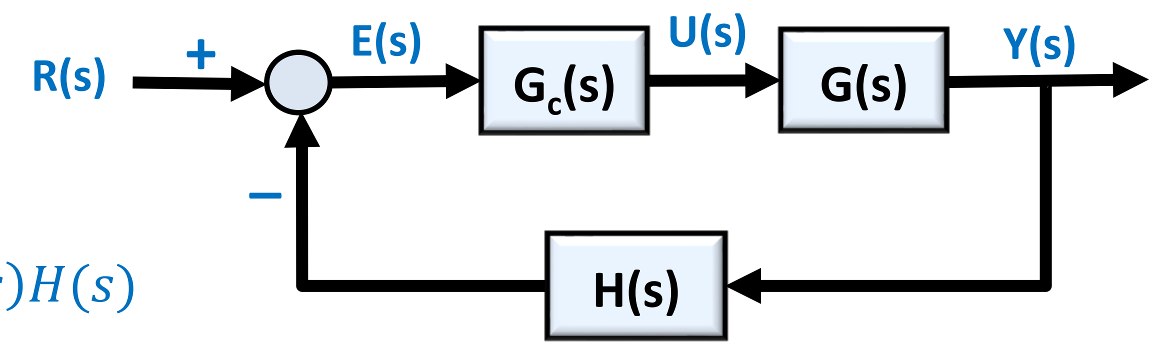
$$G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

$$G_c(s) = K_p\left(1 + \frac{T_d s}{\frac{T_d}{\beta} s + 1}\right)$$

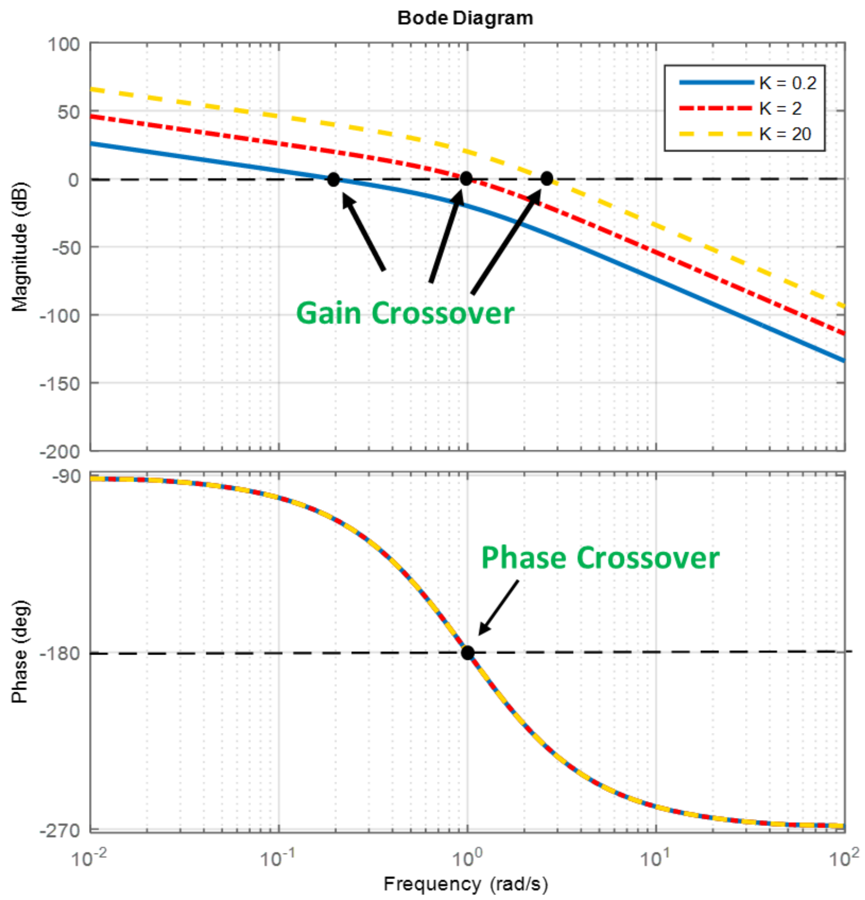
Proportional Controller

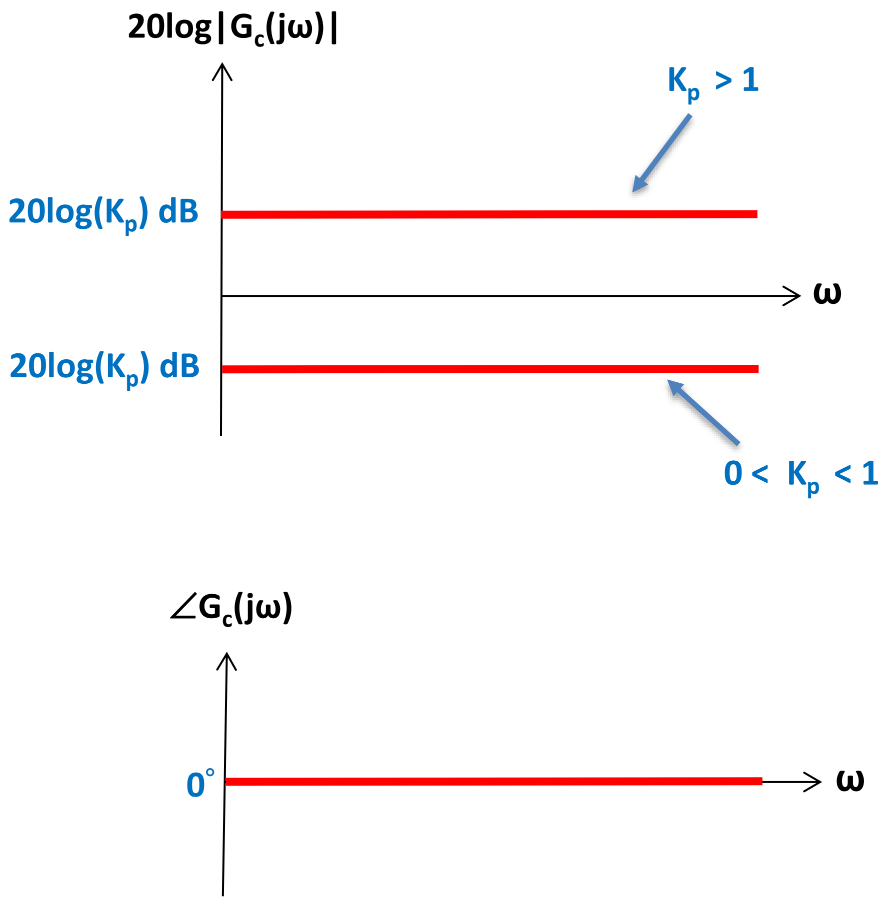
- The frequency response function of proportional controller

$$G_c(s) = K_p \quad \rightarrow \quad G_c(j\omega) = K_p$$



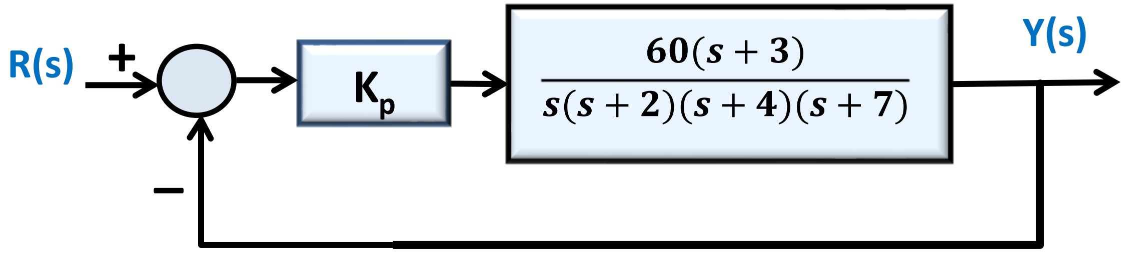
- Effect of a proportional controller on the Bode plot of the open-loop system $G_c(s)G(s)H(s)$
  - Shifts the Bode magnitude graph up or down
  - Does not affect the Bode phase graph





Proportional Controller Design

Example 1 Consider the following fourth-order system. Determine the K_p value so that the phase margin is at least 60° and the gain margin is at least 15dB.



Desired performance characteristics: $\boxed{PM \geq 60^\circ, \quad GM \geq 15\ dB}$

Step 1: Plot Bode diagram of the open-loop system $K_pG(s)H(s)$, and find PM and GM

- Plot the Bode diagram of open-loop system, $K_pG(s)H(s)$, for a convenient small value of the gain, $K_p = 1$.

$$K_pG(s)H(s) = \frac{K_p 60(s+3)}{s(s+2)(s+4)(s+7)}$$

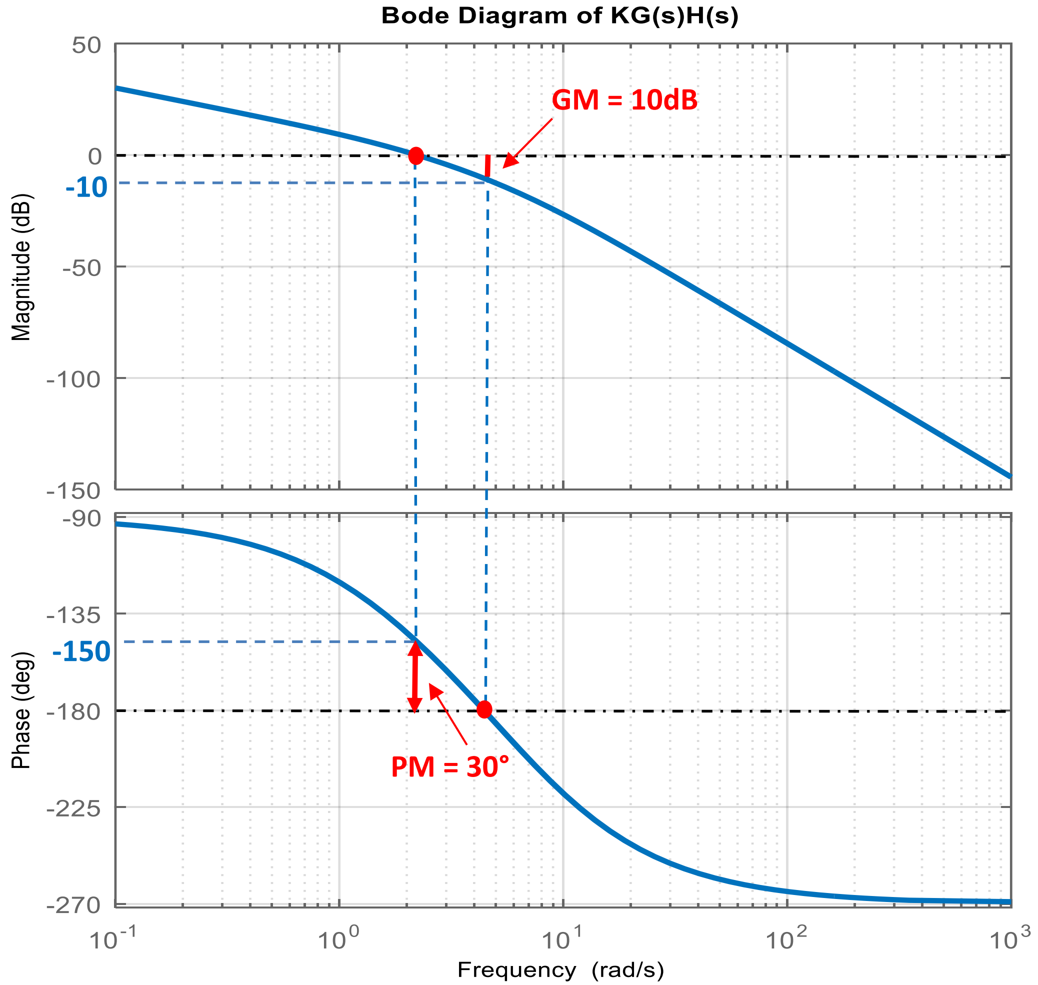
- Check the current Phase Margin and Gain Margin:

$$GM = 0\text{dB} - (-10\text{dB}) = 10\text{dB}$$

$$PM = 180^\circ + (-150^\circ) = 30^\circ$$

- Current GM and PM are not in the desired range.

$\boxed{\text{PM} = 30^\circ \quad \text{and} \quad \text{GM} = 10\text{ dB}}$



Proportional Controller Design

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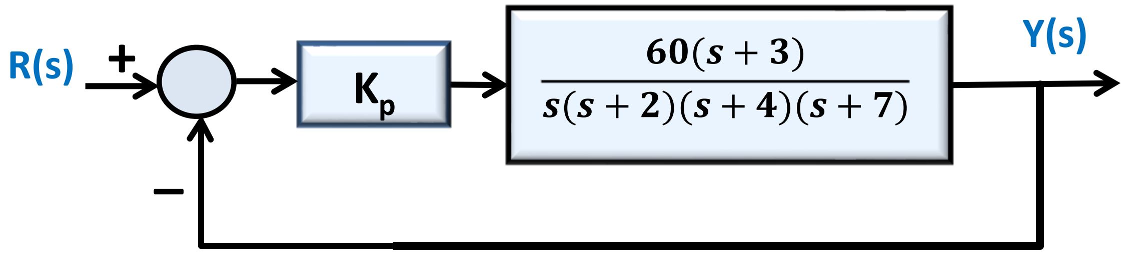


Desired performance characteristics: $\boxed{PM \geq 60^\circ, \quad GM \geq 15\ dB}$

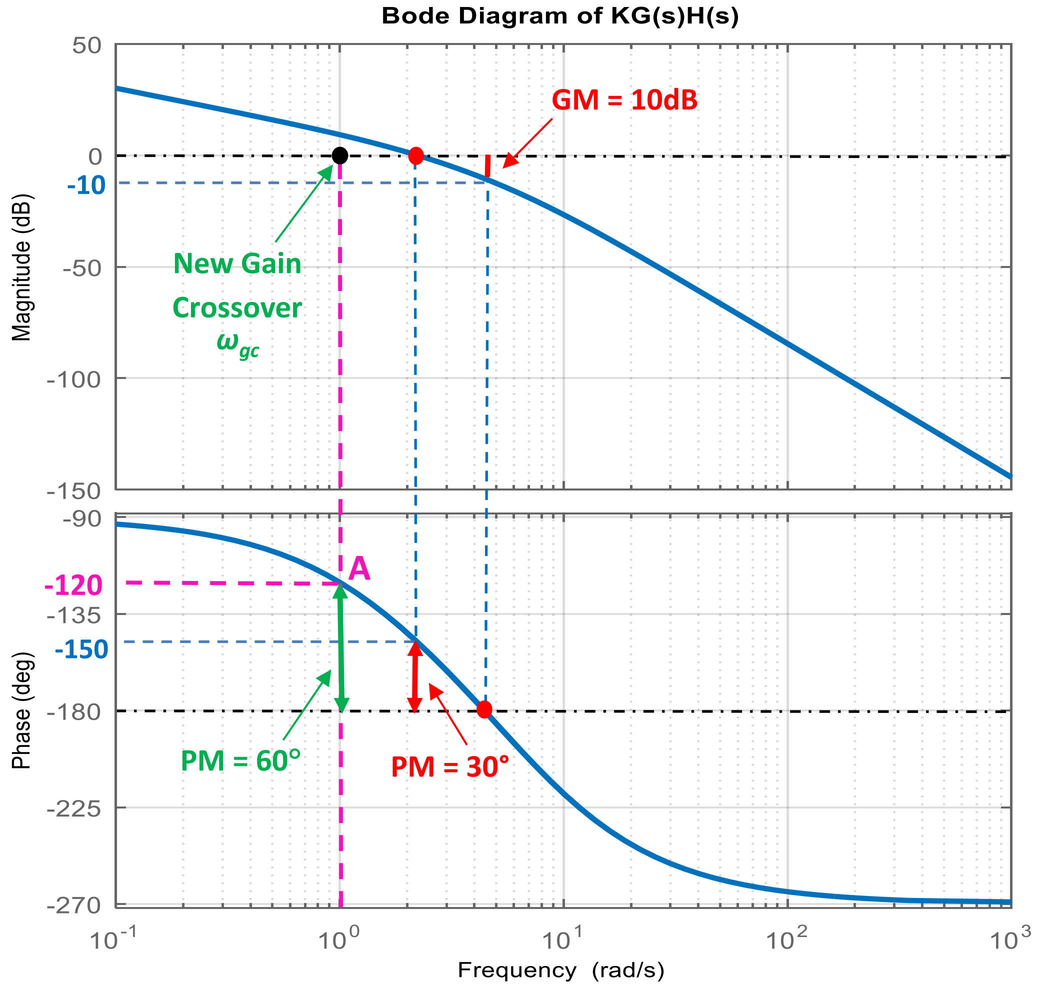
Step 2: Find the point on the phase plot to achieve the desired PM

$$PM_d = 180^\circ + \varphi \quad \rightarrow \quad \varphi = 60^\circ - 180^\circ = -120^\circ$$

- **Point A** on the Bode diagram shows the location which can give us the $PM = 60^\circ$
- Find the New Gain Crossover Frequency at the desired PM:

$$\boxed{PM = 60° \quad \rightarrow \quad \omega_g = 0.986\, rad/s}$$

- Therefore, we must decrease the current gain K_p to shift down the magnitude plot to achieve the desired gain crossover frequency and the desired PM.
- Note that by changing the gain value K_p, only the magnitude plot can be modified, and the phase plot is **not** affected.



Proportional Controller Design

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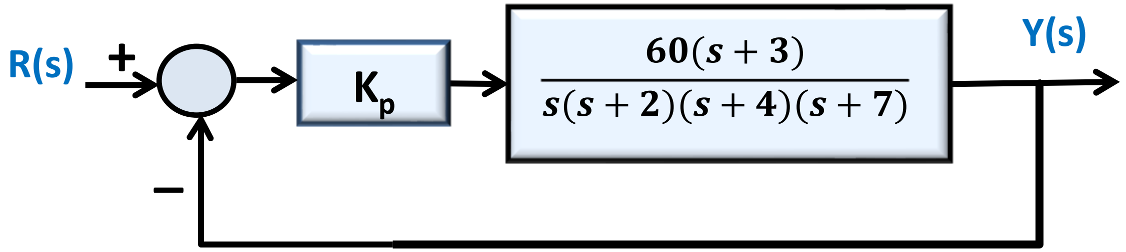


Desired performance characteristics: $\boxed{PM \geq 60^{\circ}, \quad GM \geq 15\,dB}$

Step 3: Find the gain value at the new gain crossover frequency

$$\boxed{\omega_{gc} = 0.986\, rad/s}$$

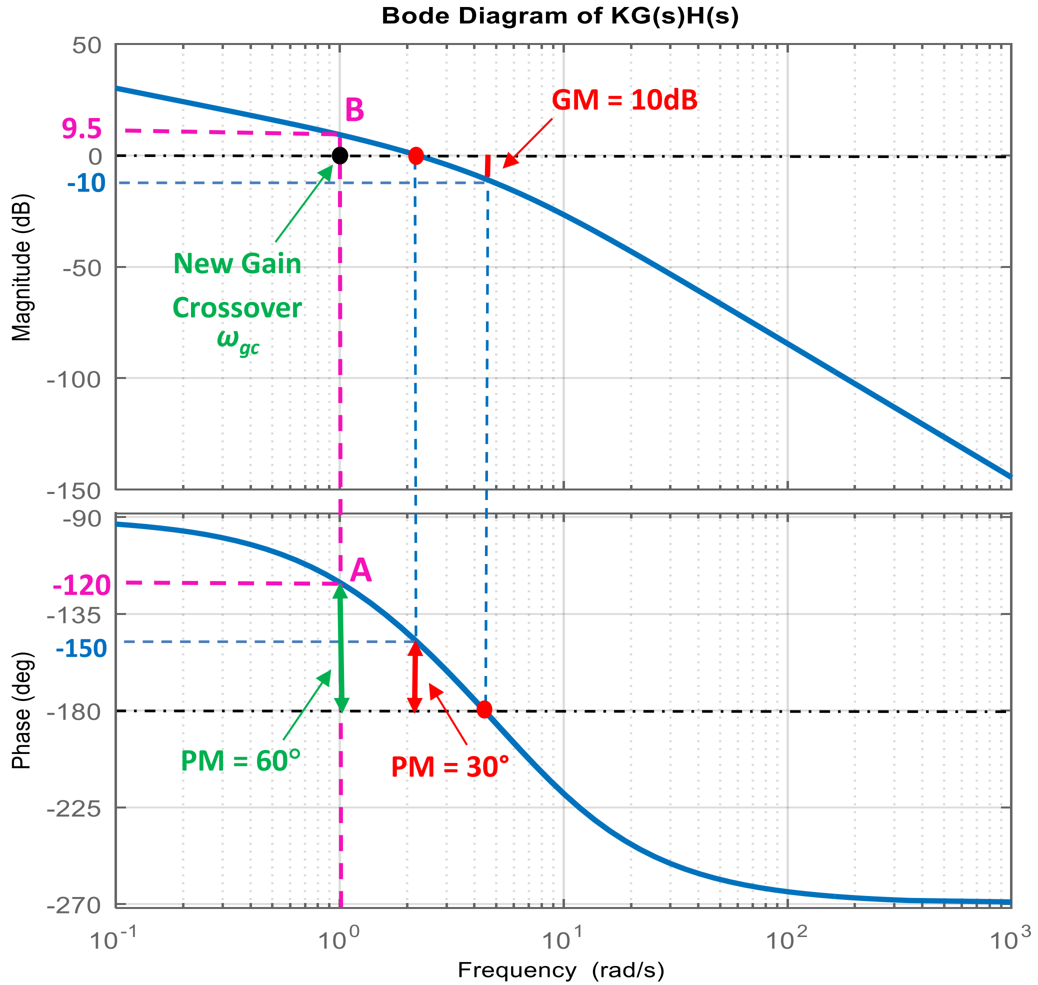
At point B $\rightarrow$ $20\log|K_pG(s)H(s)| = 9.5dB$

We want to change the gain K_p to make the magnitude at point B equal to $0dB$

Therefore, the new gain value must contribute the gain of $-9.5dB$

$$20\log|K_{new}| = -9.5dB \quad \rightarrow \quad \boxed{K_{new} = 10^{-9.5/20} = 0.335}$$

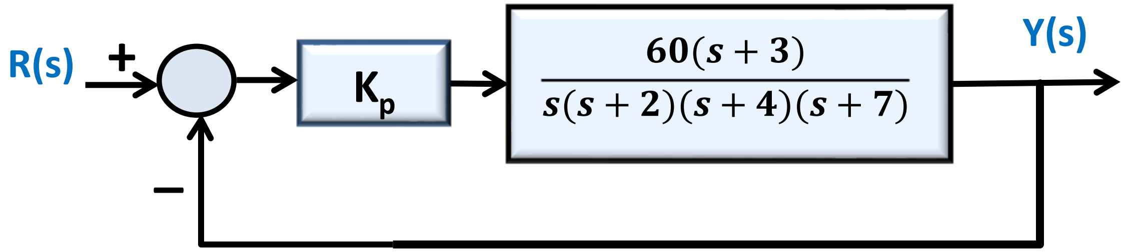
New Gain



Proportional Controller Design

Example 1 Consider the following fourth-order system. Determine the K_p value so that the phase margin is at least 60° and the gain margin is at least 15dB.

Desired performance characteristics: $\boldsymbol{PM \geq 60^{\circ}, \quad GM \geq 15\, dB}$

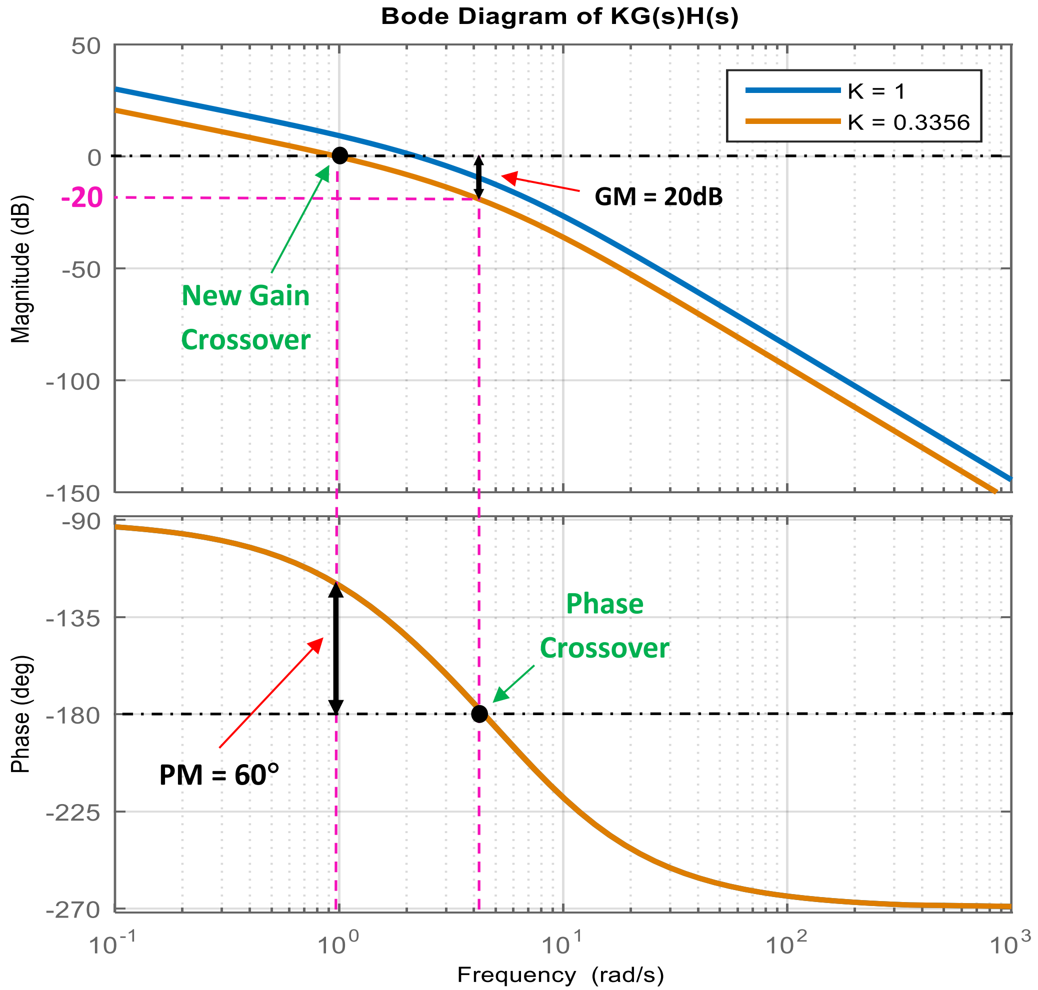


- We can plot the Bode diagram with the new gain to verify the result.

$$\boldsymbol{K_{new} = 0.335}$$

- The gain margin and phase margin satisfy the required criteria

$\mathbf{PM = 60^{\circ}}$ $\mathbf{GM = 20\ dB}$



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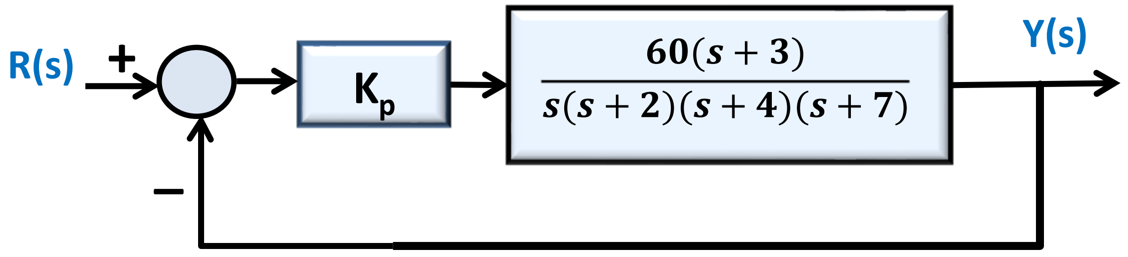
Proportional Controller Design

Example 1 Consider the following fourth-order system. Determine the K_p value so that the phase margin is at least 60° and the gain margin is at least 15dB.

Desired performance characteristics: $\boxed{PM \geq 60^{\circ}, \quad GM \geq 15\ dB}$

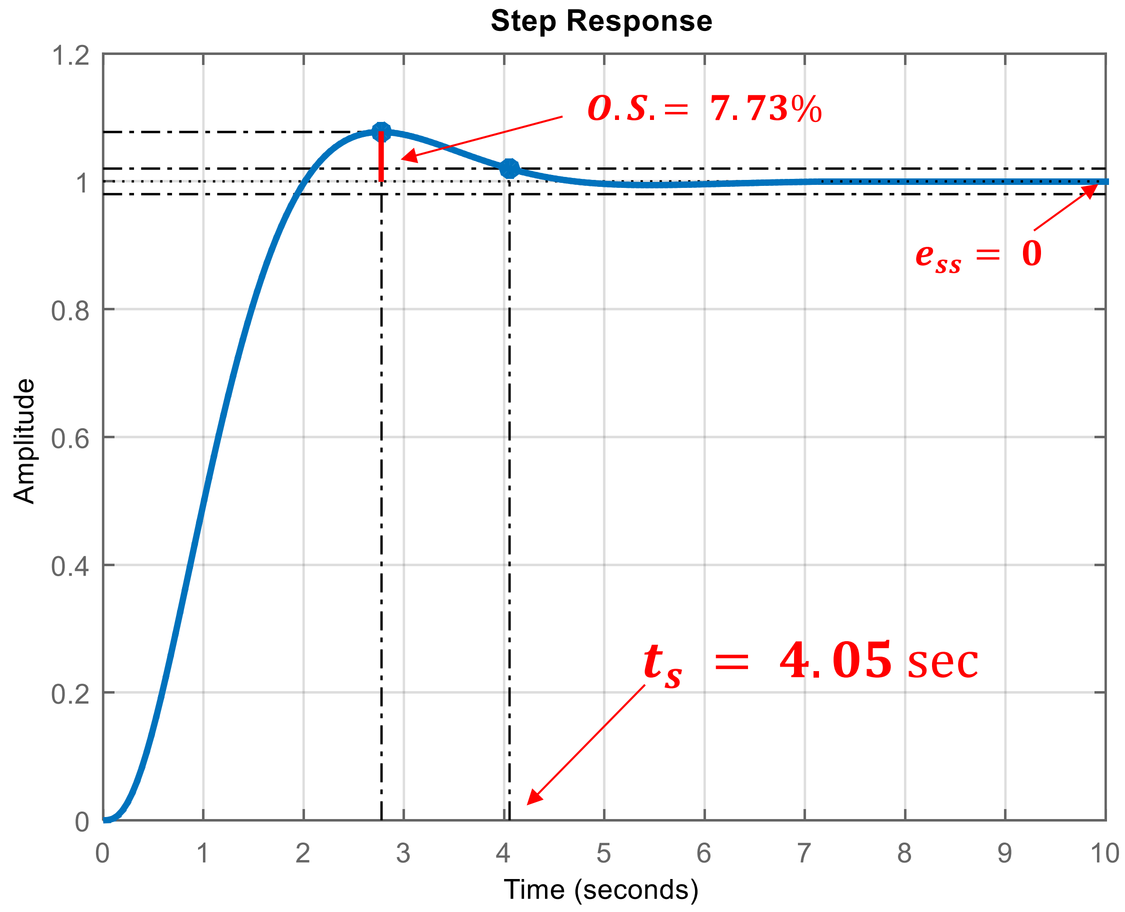


- We can also plot the unit-step response of the closed-loop system with $K_p = 0.335$ to check the performance of the system in time-domain.

$$\boxed{K_{new} = 0.335}$$

$\boxed{\text{PM} = 60^{\circ}}$ $\boxed{\text{GM} = 20\text{ dB}}$

$$\boxed{\begin{aligned} &\text{O.S.} = 7.73\% \\ &t_s = 4.05\ sec \\ &e_{ss} = 0 \end{aligned}}$$



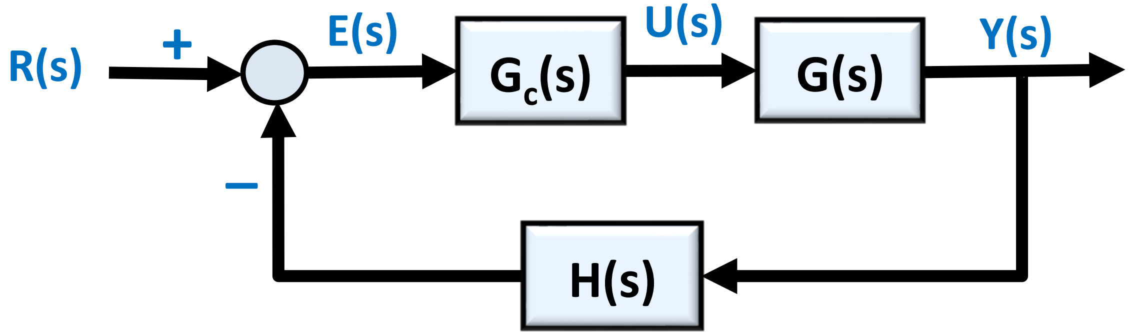
PI Controller Design via Bode Diagram

PI Controller

- The frequency response function of PI controller

$$G_c(s) = K_P\left(1 + \frac{1}{T_i s}\right) = K_p\left(\frac{T_i s + 1}{T_i s}\right) = K_p\left(\frac{1}{T_i s}\right)(T_i s + 1)$$

$$\boxed{G_c(j\omega) = K_p\left(\frac{1}{jT_i\omega}\right)(jT_i\omega + 1)}$$

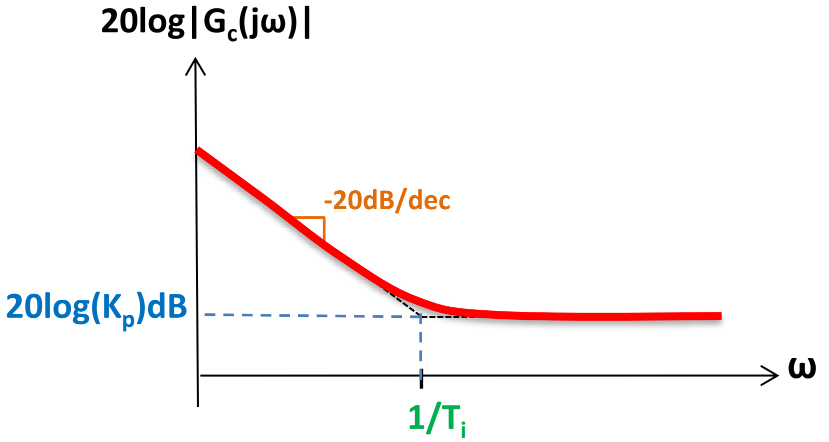


- The corner frequencies of single zero

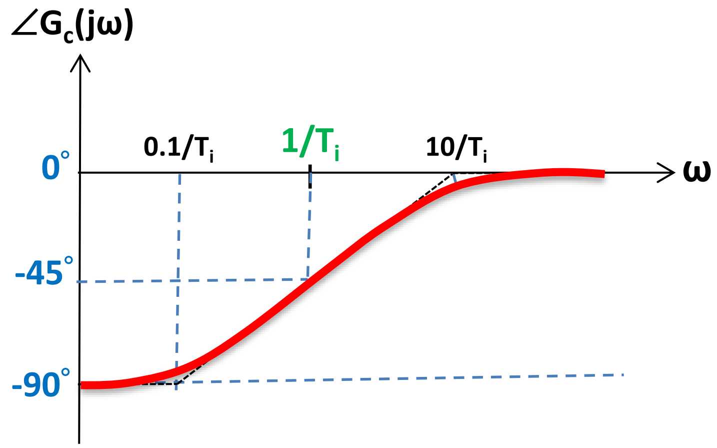
$$\omega_z = \frac{1}{T_i}$$

- The low-frequency gain and the high- frequency gain

$$G_c(j0) = \infty \quad \text{and} \quad G_c(j\infty) = K_p$$



- Effect of a PI controller on the Bode plot of the open-loop system $G_c(s)G(s)H(s)$
  - Eliminates steady-state error by applying infinite gain at low frequencies
  - Decreases the bandwidth by shifting the gain crossover frequency to the left, which makes slower transient response
  - Negative phase angle contribution (may increase phase lag)
- Negative phase must be kept at low frequencies far enough from the gain crossover frequency to **not effect** the PM.



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PI Controller Design

❑ PI Controller Design Steps via Bode Diagram

$$\boldsymbol{G_c(s) = K_P\left(1 + \frac{1}{T_i s}\right)}$$

Step 1: Plot Bode diagram of the open-loop system $KG(s)H(s)$, and find PM and GM

Step 2: Find the required phase margin, PM_{req} → $\boxed{\boldsymbol{PM_{req} = PM_d + \alpha^\circ}}$

Safety factor $(5^\circ \leq \alpha \leq 15^\circ)$

Step 3: Determine the frequency on the Bode diagram to achieve the required phase margin PM_{req}. Select this frequency as the new gain crossover frequency, ω_{gc}

Step 4: Find the corner frequency of zero → $\boxed{\boldsymbol{\omega_z = 0.1\omega_{gc}}}$

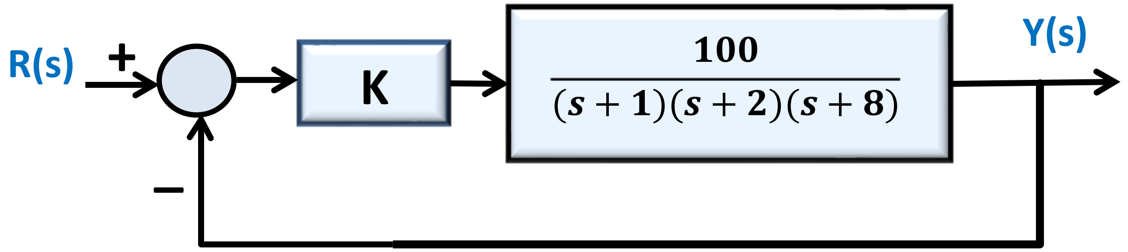
Step 5: Select the integral time constant T_i → $\boxed{\boldsymbol{T_i = \frac{1}{\omega_z}}}$

Step 6: Select the proportional gain K_p to bring down the magnitude plot to $0dB$ at the new crossover frequency ω_{gc}.

PI Controller Design

Example 2

Consider the following third-order system



It is desired to design a PI controller to achieve the following performance characteristics

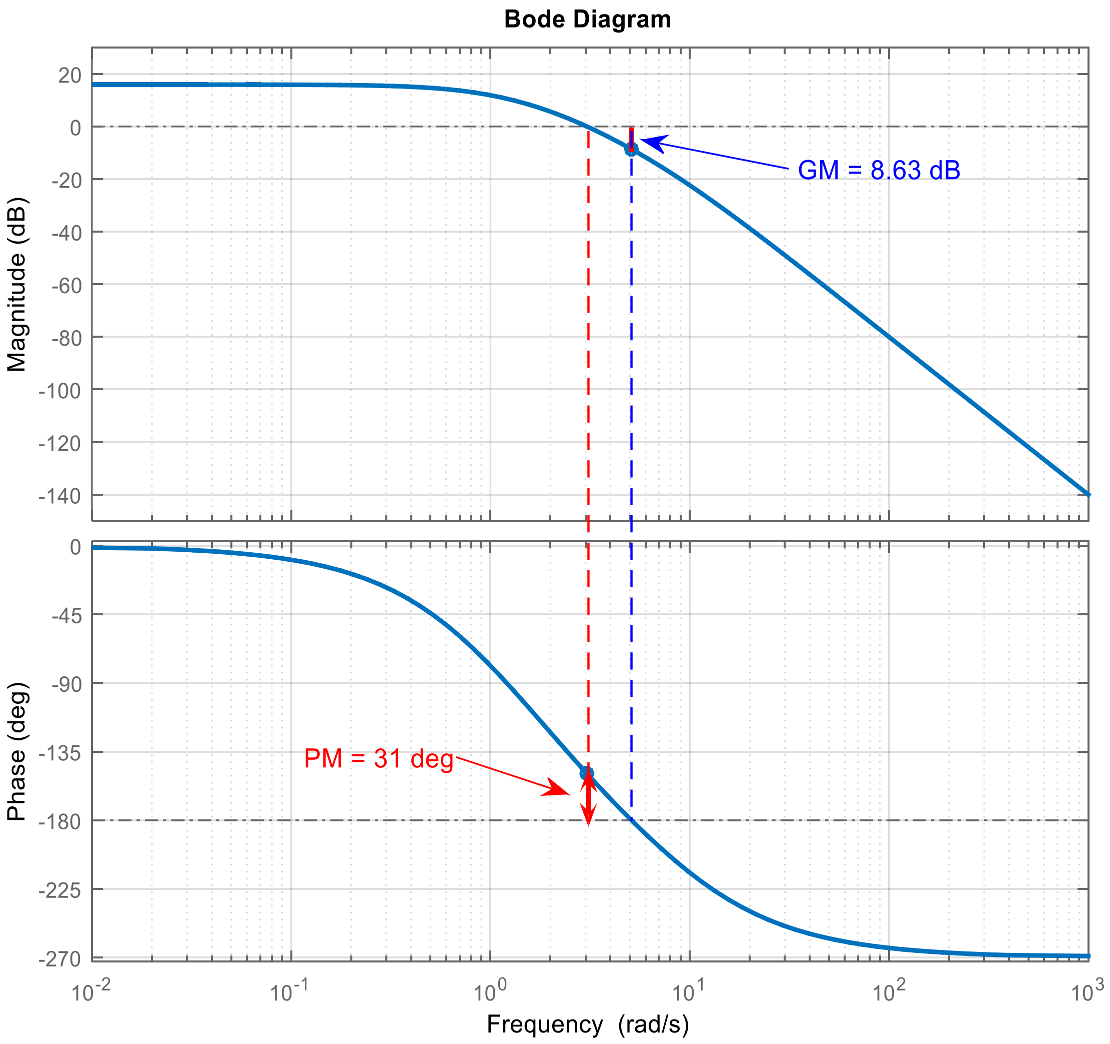
$$\boxed{e_{ss} = 0\,,\ PM > 70^\circ\,,\ GM\ > 10dB}$$

Step 1: Plot Bode diagram of the open-loop system $KG(s)H(s)$, and find PM and GM

From the Bode diagram of the uncompensated system with $K = 1$, we have the following relative stability margins

$$\boxed{PM = 31^\circ \qquad \text{and} \qquad GM = 8.63\text{dB}}$$

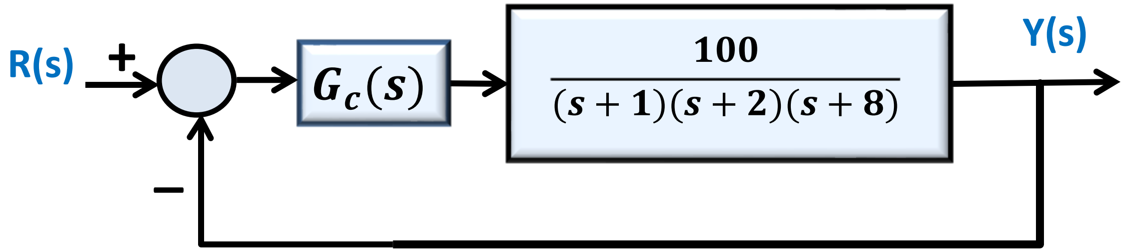
Current PM and GM are not in the desired range.



PI Controller Design

Example 2

Consider the following third-order system



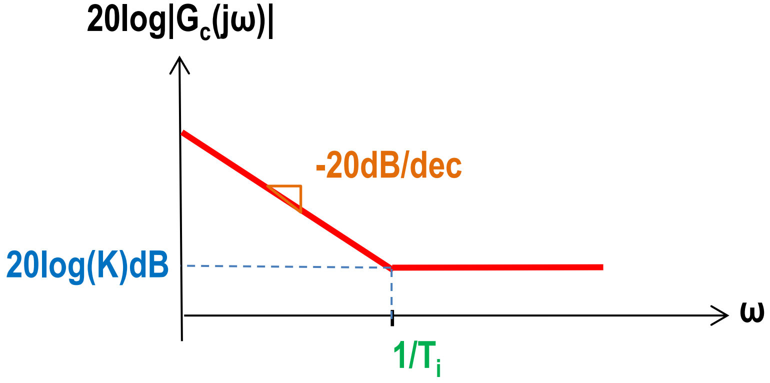
It is desired to design a PI controller to achieve the following performance characteristics

$$\boxed{e_{ss} = 0\,,\ PM > 70^{\circ}\,,\ GM > 10dB}$$

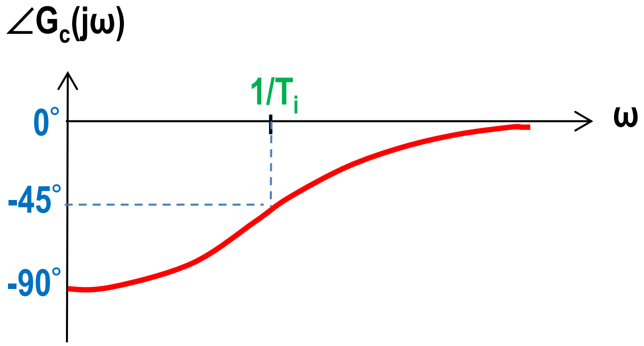
Next, design a PI Controller to achieve the desired PM.

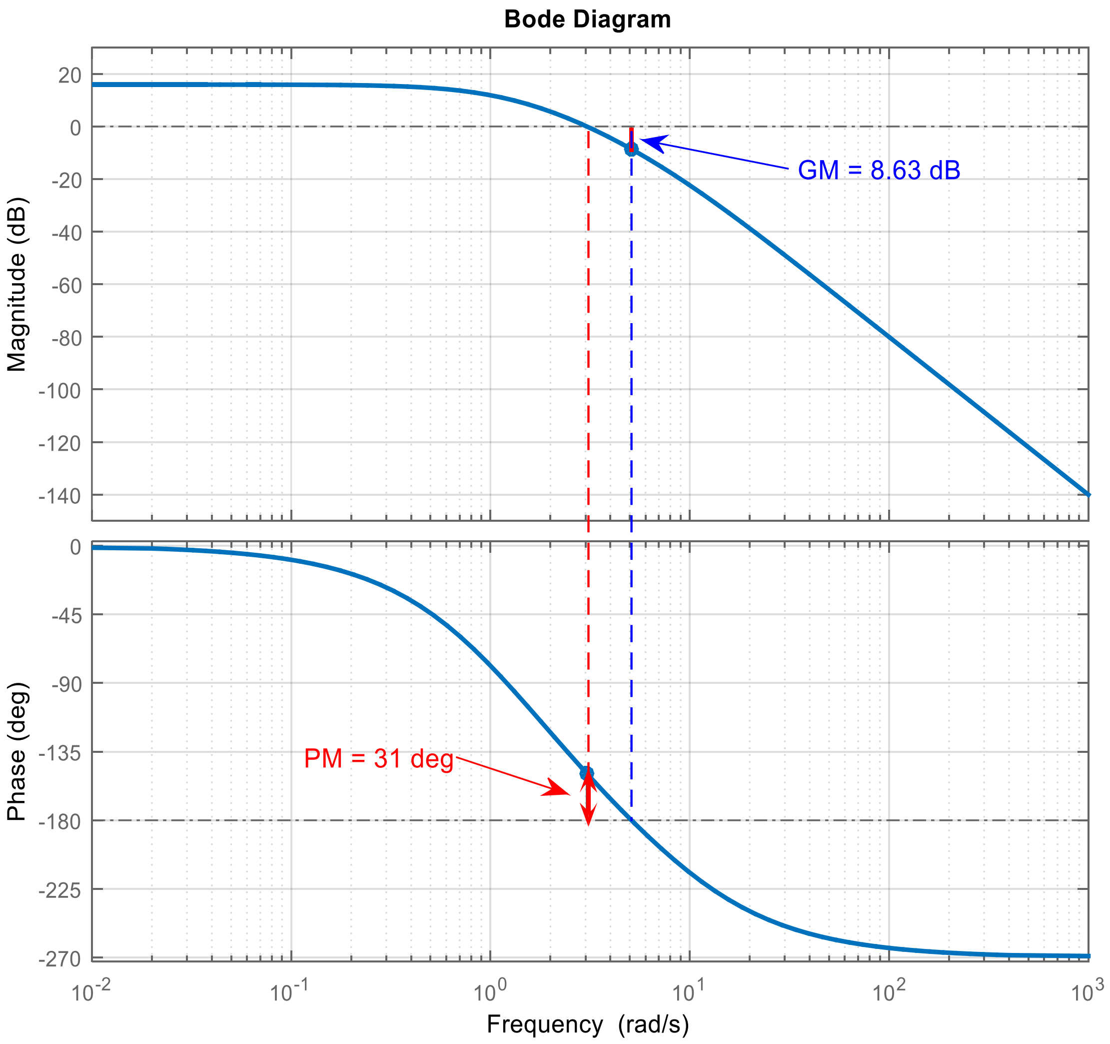
$$G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

- PI controller modifies the both magnitude and phase plots by shifting the gain crossover to the left.



- To avoid the effect of negative phase, zero of the PI controller must be selected one decade below the new gain cross over frequency

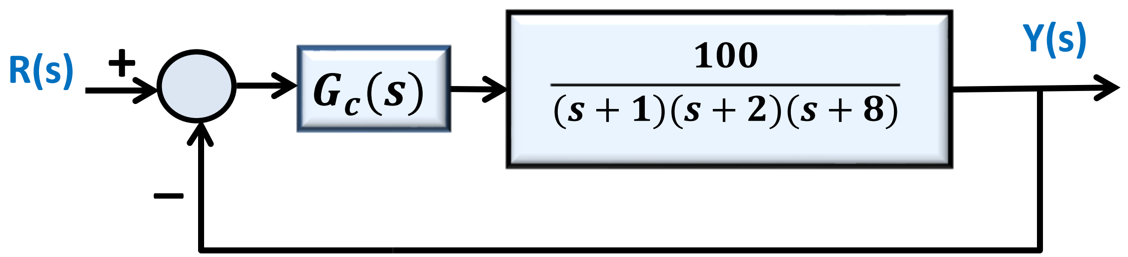




PI Controller Design

Example 2

Consider the following third-order system



It is desired to design a PI controller to achieve the following performance characteristics

$$\boxed{e_{ss} = 0\,,\ PM > 70^\circ\,, GM\ > 10dB}$$

Step 2: Find the required phase margin, PM_{req}

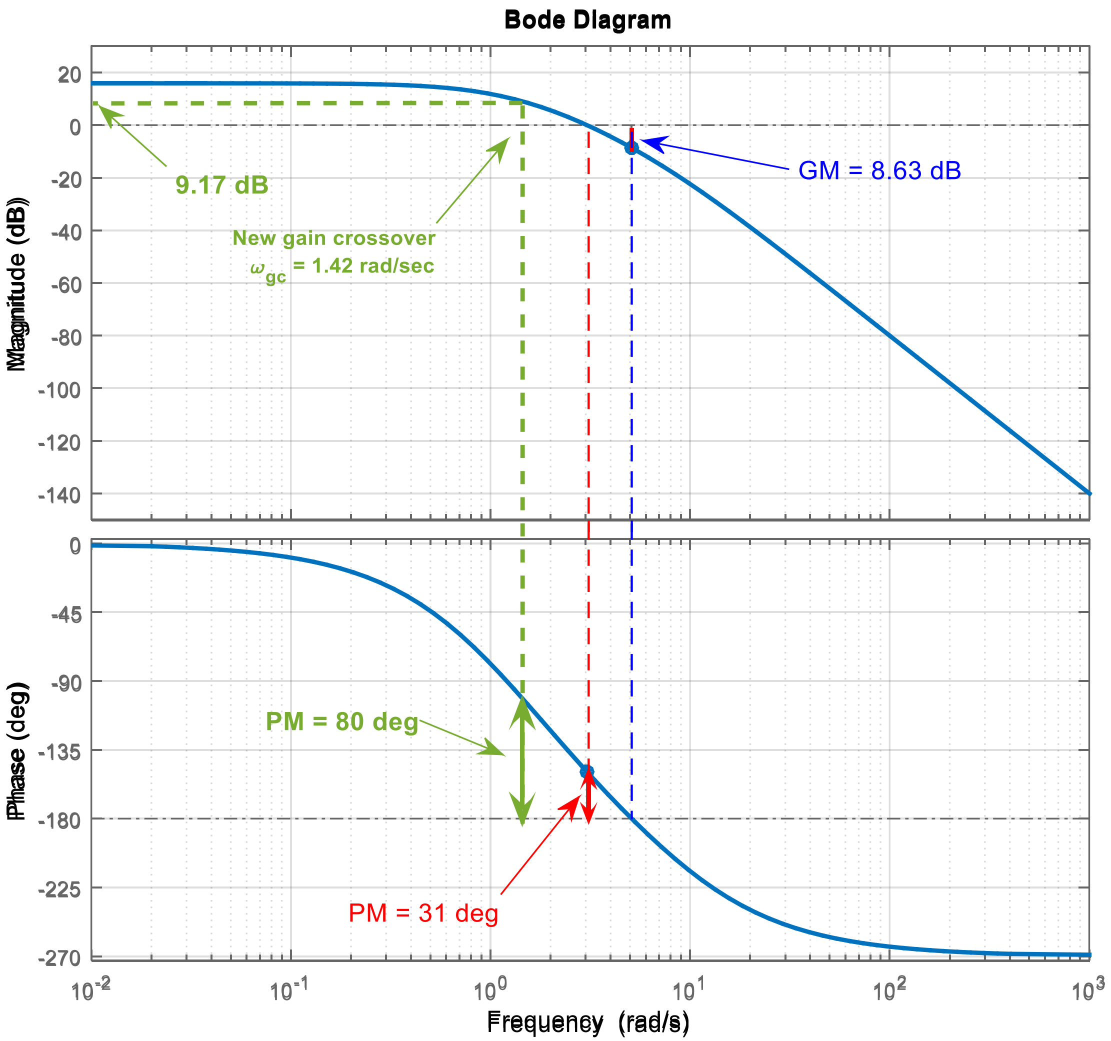
$$PM_{req} = PM_d + \alpha^\circ \ = \ 70^\circ + 10^\circ \longrightarrow \boxed{PM_{req} = 80^\circ}$$

- A 10° safety factor is added to compensate the shifting in the gain crossover frequency

Step 3: Determine the frequency on the Bode diagram to achieve the required phase margin PM_{req}. Select this frequency as the new gain crossover frequency, ω_{gc}

$$PM_{req} = 80^\circ \longrightarrow \boxed{\omega_{gc} = 1.42\ \mathrm{rad/s}}$$

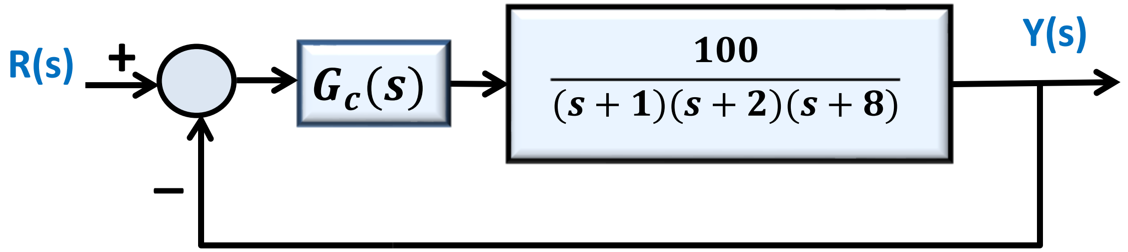
New Gain Crossover Frequency



PI Controller Design

Example 2

Consider the following third-order system



It is desired to design a PI controller to achieve the following performance characteristics

$$\boxed{e_{ss} = 0\,,\ PM > 70^{\circ}\,, GM\ > 10dB}$$

Step 4: Find the corner frequency of zero

$$\boxed{\omega_z = 0.1\omega_{gc} = 0.142\ rad/s}$$

Step 5: Select the integral time constant T_i

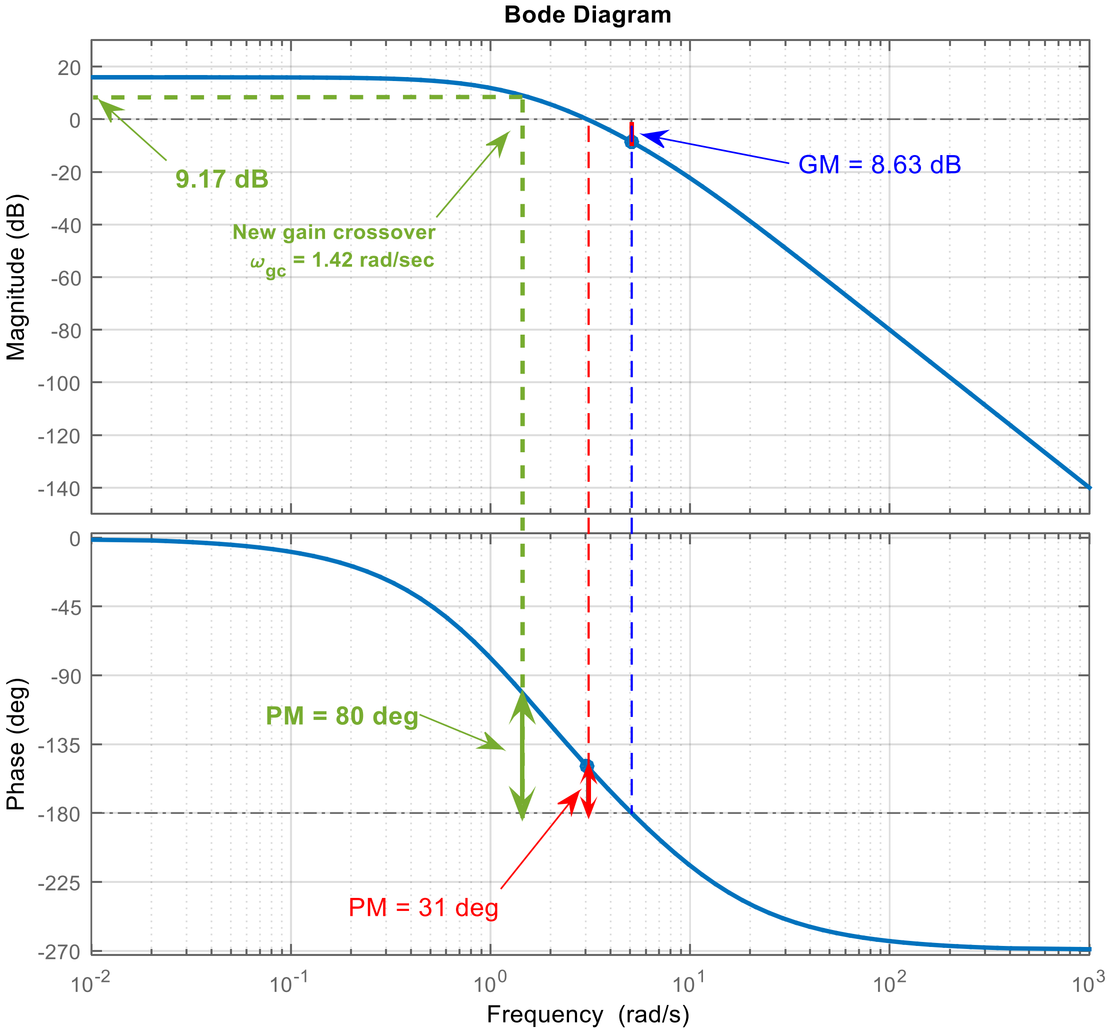
$$T_i = \frac{1}{\omega_z} \rightarrow \boxed{T_i = 7.04\ \ sec}$$

Step 6: Select the proportional gain K_p to bring down the magnitude plot to $0dB$ at the new crossover frequency.

- From the Bode plot the magnitude at the new gain crossover is

$$\boxed{\omega_{gc} = 1.42} \rightarrow \boxed{9.17\text{ dB}}$$

$$20\log(K_p) = -9.17\text{dB} \rightarrow \boxed{K_p = 10^{-9.17/20} = 0.35}$$

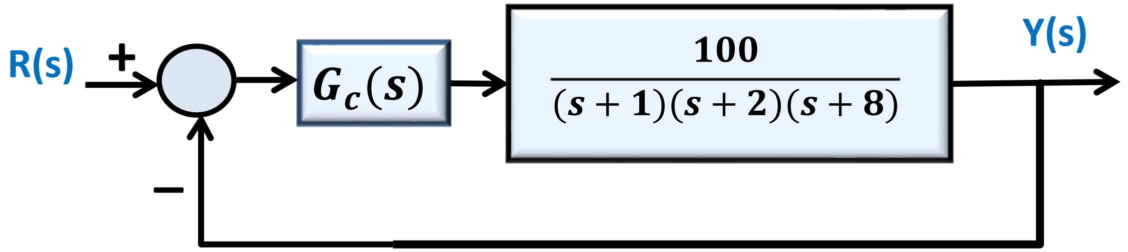


PI Controller Design

Example 2

Consider the following third-order system

It is desired to design a PI controller to achieve the following performance characteristics



$$e_{ss} = 0\ ,\ PM > 70^{\circ}\ , GM\ > 10dB$$

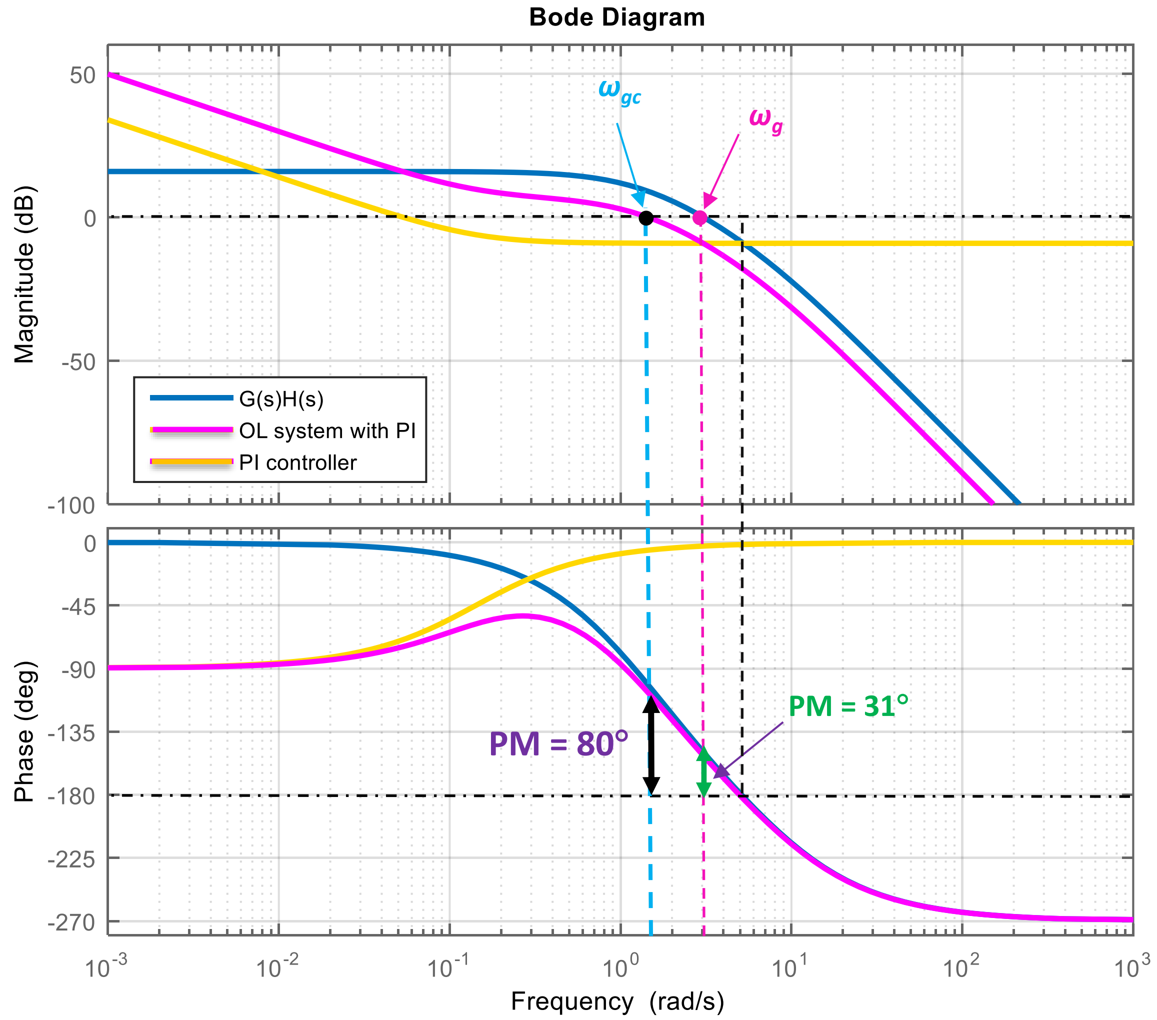
Designed PI Controller $G_c(s) = 0.35\left(1 + \frac{1}{7.04s}\right)$

- Graph shows Bode plot of the following systems

$$G(s)H(s) = \frac{100}{(s+1)(s+2)(s+8)}$$

$$G_c(s)G(s)H(s) = 0.35\left(1 + \frac{1}{7.04s}\right)\frac{100}{(s+1)(s+2)(s+8)}$$

$$G_c(s) = 0.35\left(1 + \frac{1}{7.04s}\right)$$



- The PI controller decreases the bandwidth of the system that results in slower transient response.
- The ω_z must be selected far enough from the new gain crossover frequency ω_{gc} to not effect the designed phase margin.

PD Controller Design via Bode Diagram

Ideal PD Controller

- The frequency response function of PD controller

$$\boxed{G_c(s) = K_P(1 + T_d s) \quad \rightarrow \quad G_c(j\omega) = K_p(1 + jT_d\omega)}$$

- The corner frequency of single zero

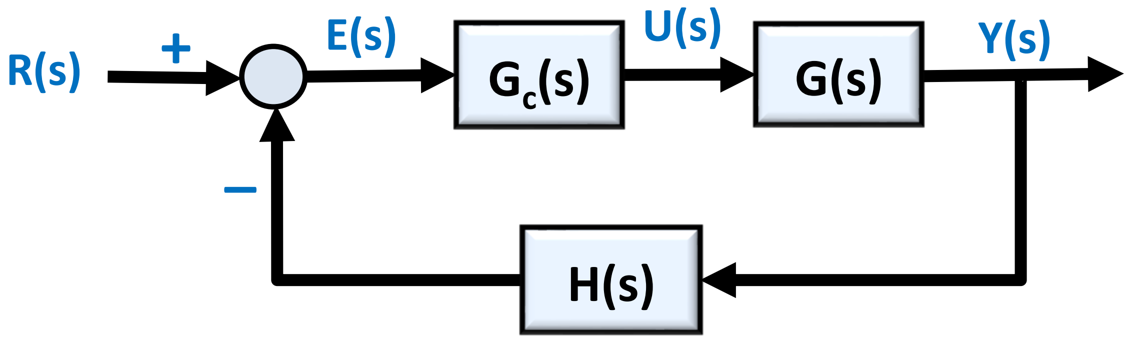
$$\omega_z = \frac{1}{T_d}$$

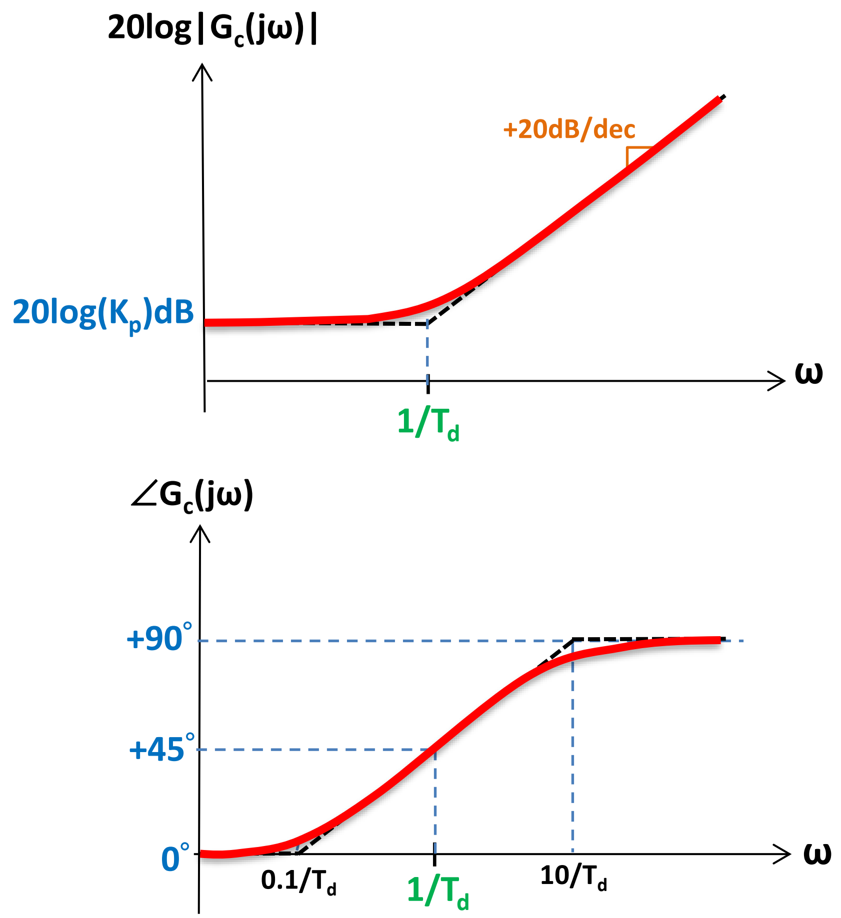
- The low-frequency gain and the high- frequency gain

$$G_c(j0) = K_p \quad \text{and} \quad G_c(j\infty) = \infty$$

- Effect of a PD controller on the Bode plot of the open-loop system $G_c(s)G(s)H(s)$
  - Increases the bandwidth by shifting the gain crossover frequency to the right, which makes faster transient response
  - Positive phase angle contribution, which helps to increase the PM, which makes less overshoot and enhances the stability

- Ideal PD controller **amplifies the high frequency noise.**

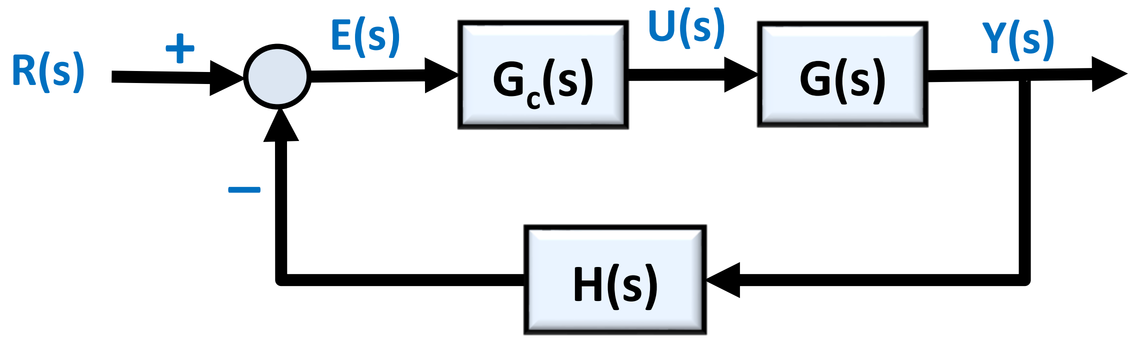




Practical PD Controller

- In practical applications, to avoid of the amplifying high frequency noise, the high frequency gain is limited by adding a **pole** to the controller

$$G_c(s) = K_P\left(1 + \frac{T_d s}{\frac{T_d}{\beta}s + 1}\right), \qquad 10 \le \beta \le 100$$



- The parameter β determines the distance between the pole and the zero.
- The corner frequency of the single pole and the single zero

$$\omega_p = \frac{\beta}{T_d} \quad \text{and} \quad \omega_z = \frac{\beta}{(1+\beta)T_d} \approx \frac{1}{T_d}$$

- The low-frequency gain and the high- frequency gain

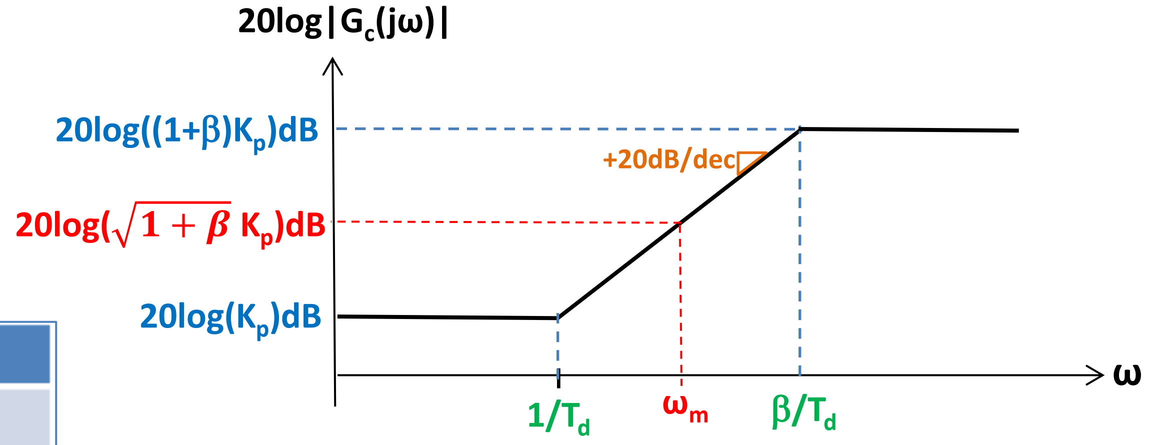
$$G_c(j0) = K_p \quad \text{and} \quad G_c(j\infty) = (1+\beta)K_p$$

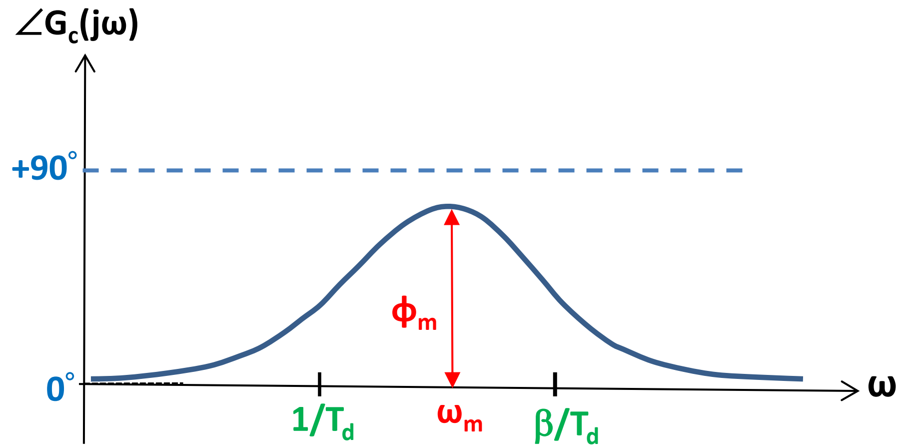
- The maximum phase frequency and maximum phase angle

$$\omega_m = \sqrt{\omega_p \cdot \omega_z} = \frac{\sqrt{\beta}}{T_d} \qquad \phi_m = \sin^{-1}\left(\frac{\beta - 1}{\beta + 1}\right)$$

Practical PD controller is also called phase-lead controller

β	ϕ_m
10	55°
20	65°
30	70°
40	72°
50	74°
60	75°
70	76°
80	77°
90	78°
100	78.5°





PD Controller Design

❑ PD Controller Design Steps via Bode Diagram

$$G_c(s) = K_P\left(1 + \frac{T_d s}{\frac{T_d}{\beta}s + 1}\right)$$

Step 1: Determine the proportional gain K_p to satisfy the desired steady-state error.

Step 2: Plot Bode diagram of the open-loop system with proportional gain $K_pG(s)H(s)$, and find PM and GM.

Step 3: Find the maximum phase lead angle, ϕ_m to be added to the system to achieve the desired PM criteria.

$$\boxed{\phi_m = PM_d - PM + \alpha^\circ}$$

Safety factor $(5^\circ \leq \alpha \leq 15^\circ)$

Step 4: Select the appropriate factor of β based on the ϕ_m value.

$$\phi_m = \sin^{-1}\left(\frac{\beta - 1}{\beta + 1}\right)$$

β	ϕ_m
10	55°
20	65°
30	70°
40	72°
50	74°
60	75°
70	76°
80	77°
90	78°
100	78.5°

Step 5: Find the new gain crossover frequency ω_{gc} where the magnitude is $-20\,log\sqrt{1+\beta}$

Step 6: Assign the maximum phase frequency ω_m at the new gain crossover frequency ω_{gc} value.

$$\boxed{\omega_m = \omega_{gc}}$$

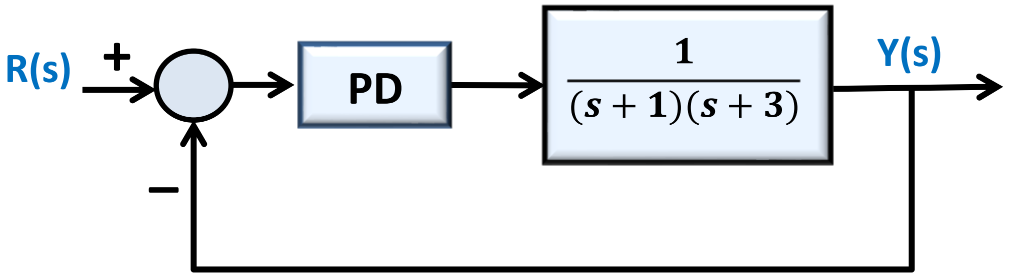
Step 7: Assign the derivative time constant T_d value as $\boxed{T_d = \frac{\sqrt{\beta}}{\omega_m}}$

PD Controller Design

Example 3

Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$\boxed{e_{ss} = 0.05,\ PM > 60^{\circ},\ \mathrm{GM} > 10\mathrm{dB}}$$



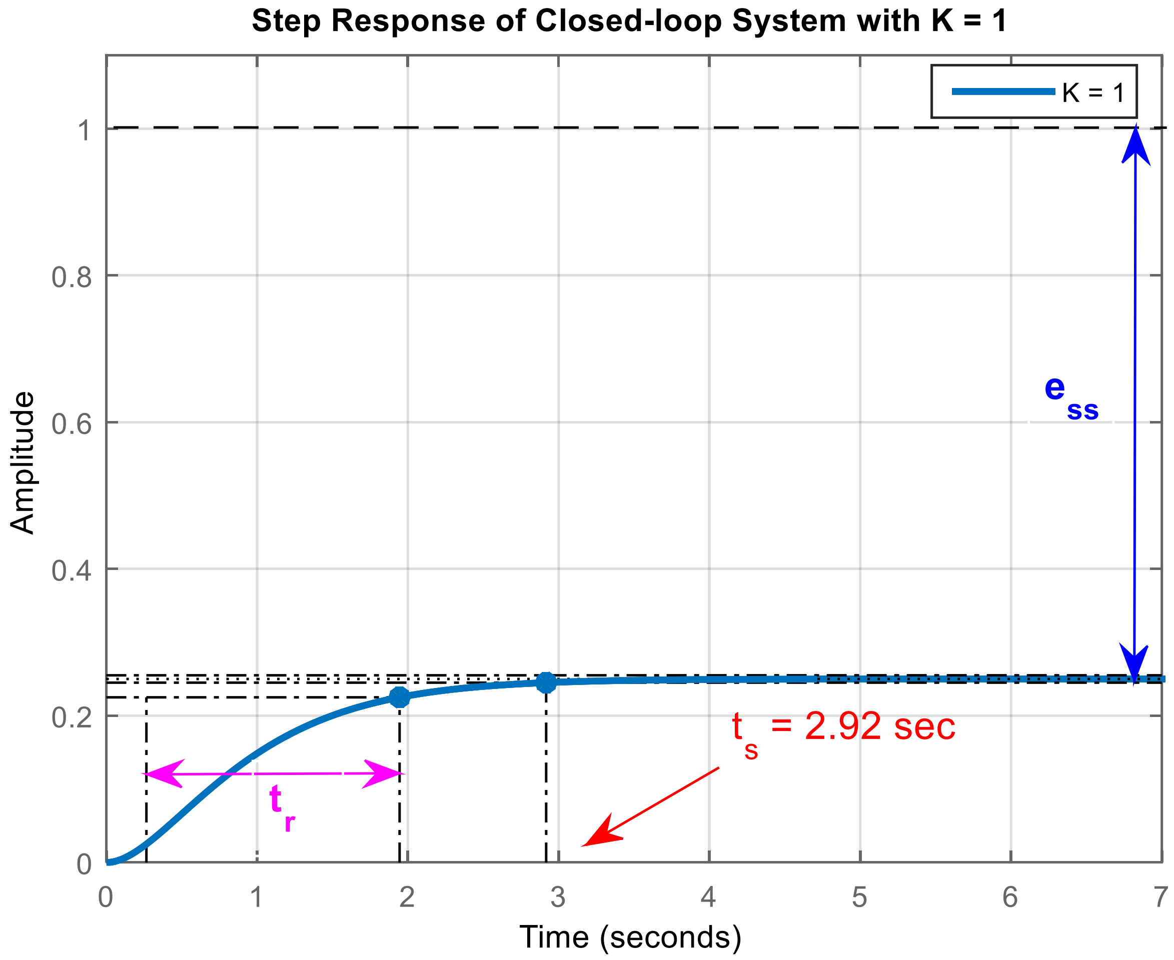
Predesign Performance Study

- First, check the unit-step response of the closed-loop system with proportional gain of $K_p = 1$.
- The results show that $G(s)$ is a slow system with a large steady-state error.

$$\boxed{t_r = 1.68\ \mathrm{sec}}$$

$$\boxed{t_s = 2.92\ \mathrm{sec}}$$

$$\boxed{e_{ss} = 0.75 = 75\%}$$

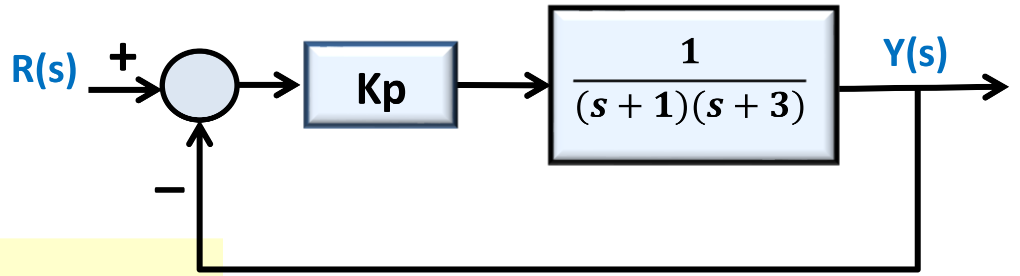


PD Controller Design

Example 3

Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$\boldsymbol{e_{ss} = 0.05,\ PM > 60^\circ,\ \mathrm{GM} > 10\mathrm{dB}}$$



Step 1: Determine the proportional gain K_p to satisfy the desired steady-state error of unit-step response

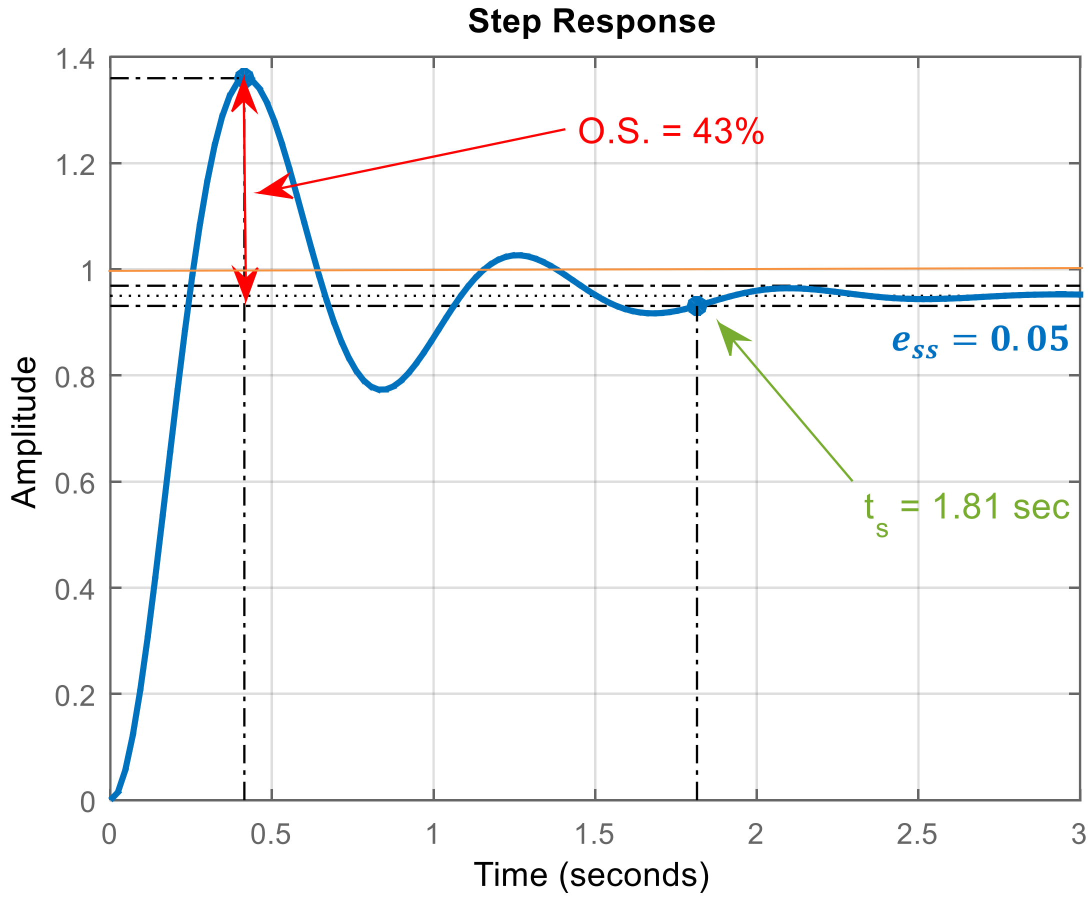
- First, find the step-error constant, k_p based on the desired steady-state error.

$$e_{ss} = \frac{R}{1+k_p} \quad \rightarrow \quad 0.05 = \frac{1}{1+k_p} \quad \rightarrow \quad k_p = 19$$

$$k_p = \lim_{s\to 0} K_p G(s)H(s) \quad \rightarrow \quad 19 = \lim_{s\to 0} K_p\left(\frac{1}{(s+1)(s+3)}\right)$$

$$19 = \frac{K_p}{3} \quad \rightarrow \quad K_p = 57$$

$$\boxed{K_p = 57}$$ **Desired Proportional Gain**

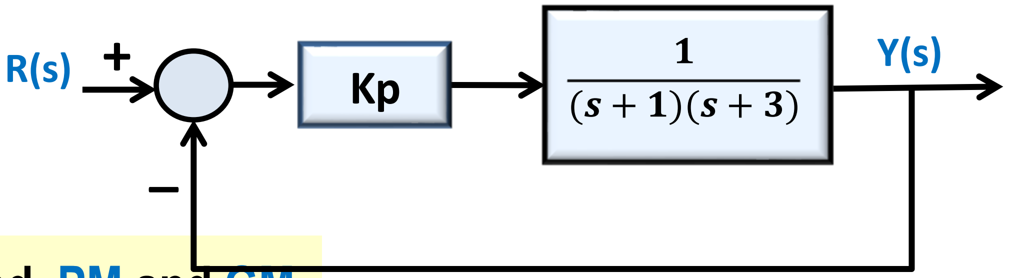


PD Controller Design

Example 3

Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$e_{ss} = 0.05,\ PM > 60^\circ,\ \mathrm{GM} > 10\mathrm{dB}$$



Step 2: Plot Bode diagram of the open-loop system with proportional gain $K_pG(s)H(s)$, and find PM and GM

- Open-loop system with desired proportional gain

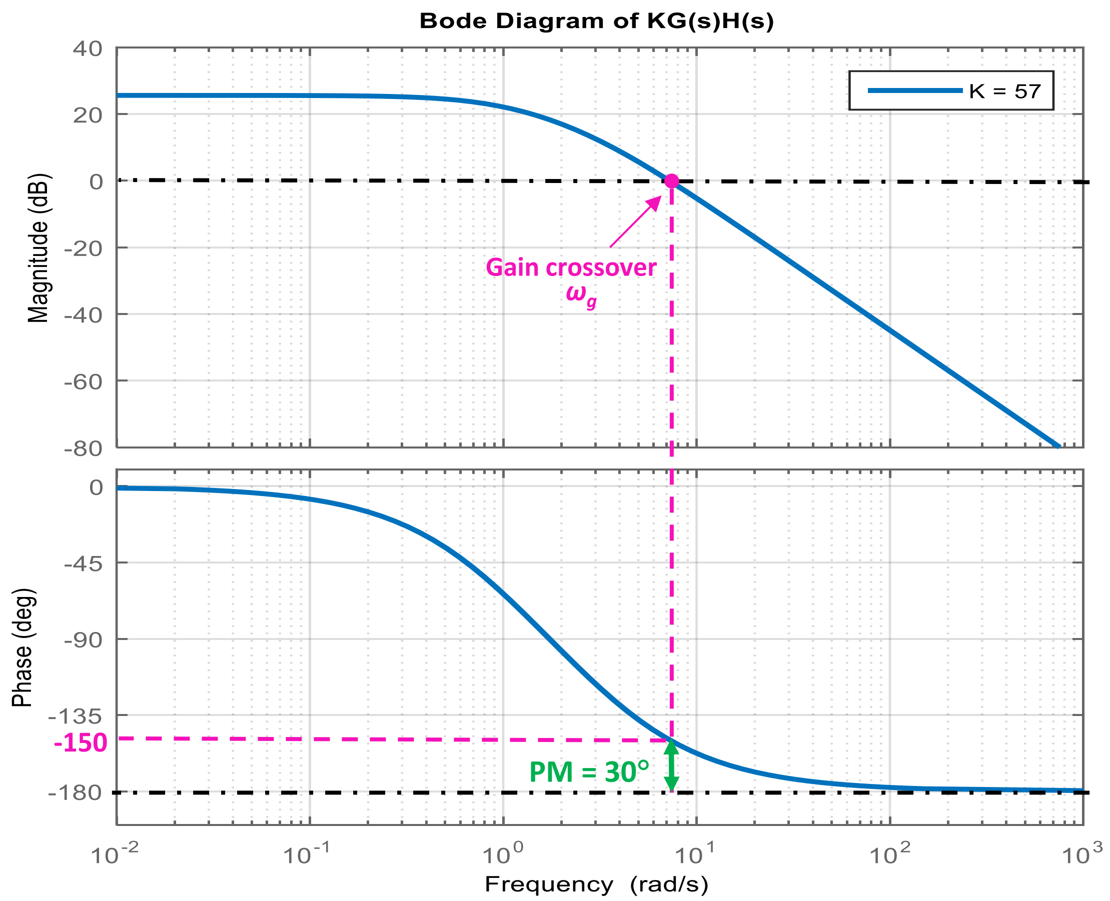
$$K_pG(s)H(s) = \frac{57}{(s+1)(s+3)}$$

- From the Bode diagram the gain crossover frequency, the phase margin and gain margin of the system with $K_p = 57$

$$\omega_g = 7.22\ rad/sec$$

$$PM = 30^\circ, GM = +\infty$$

- The desired PM is not satisfied by only proportional gain of $K_p = 57$. Therefore; we have to add the derivative part to increase the PM.

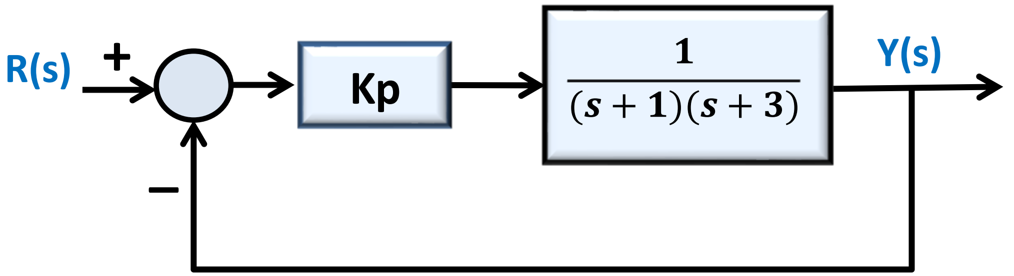


PD Controller Design

Example 3

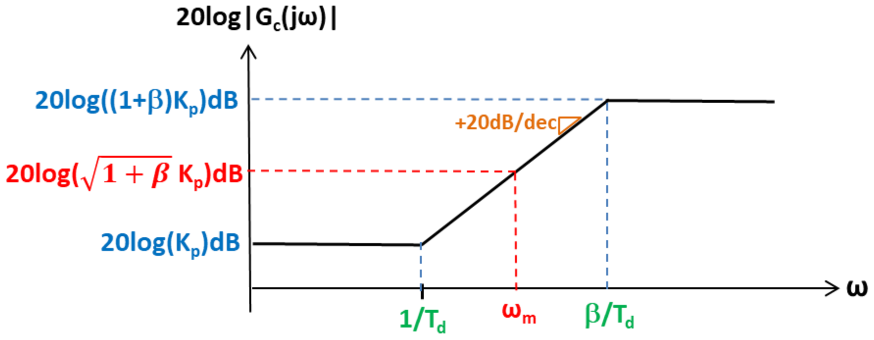
Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$e_{ss} = 0.05,\ PM > 60^\circ,\ \mathrm{GM} > 10\mathrm{dB}$$

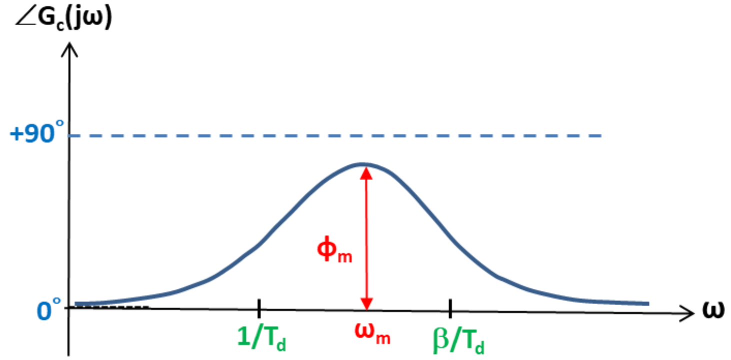


Next, design a PD Controller to achieve the desired PM.

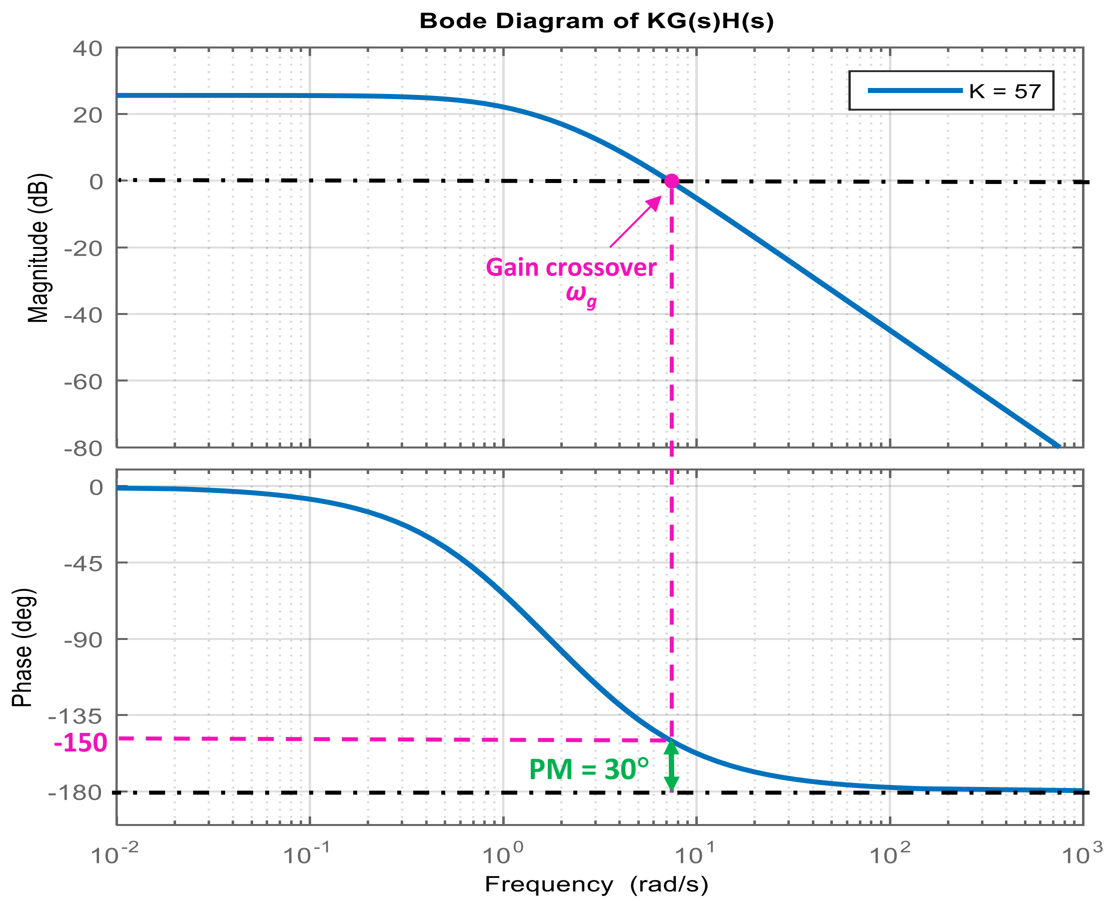
$$G_c(s) = K_P\left(1 + \frac{T_d s}{\frac{T_d}{\beta}s + 1}\right)$$



- Adding the lead compensator modifies both magnitude and phase plots.



- The magnitude plot shifts up, thus the gain crossover frequency ω_g will be shifted to the right.

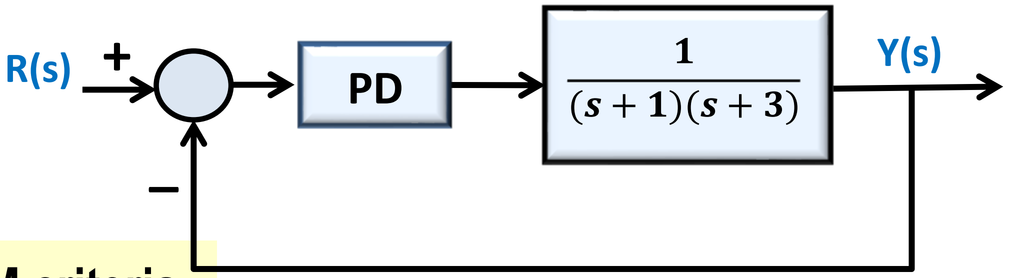


PD Controller Design

Example 3

Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$e_{ss} = 0.05,\ PM > 60^{\circ},\ \text{GM} > 10\text{dB}$$



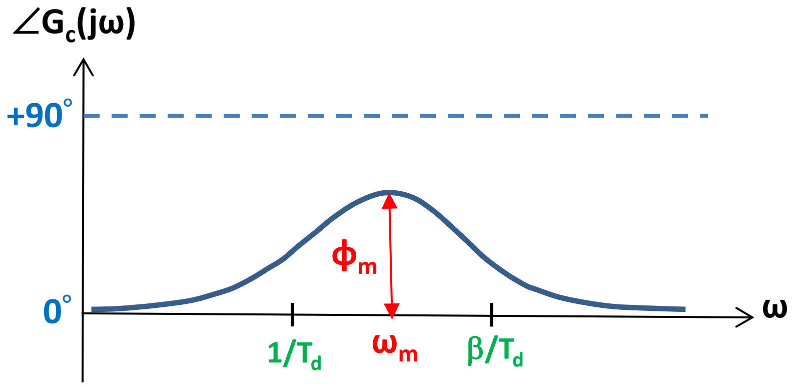
Step 3: Find the maximum phase angle, ϕ_m to be added to the system to achieve the desired PM criteria

$$\phi_m = PM_d - PM + 10^{\circ} = 60^{\circ} - 30^{\circ} + 10^{\circ} = 40^{\circ}$$

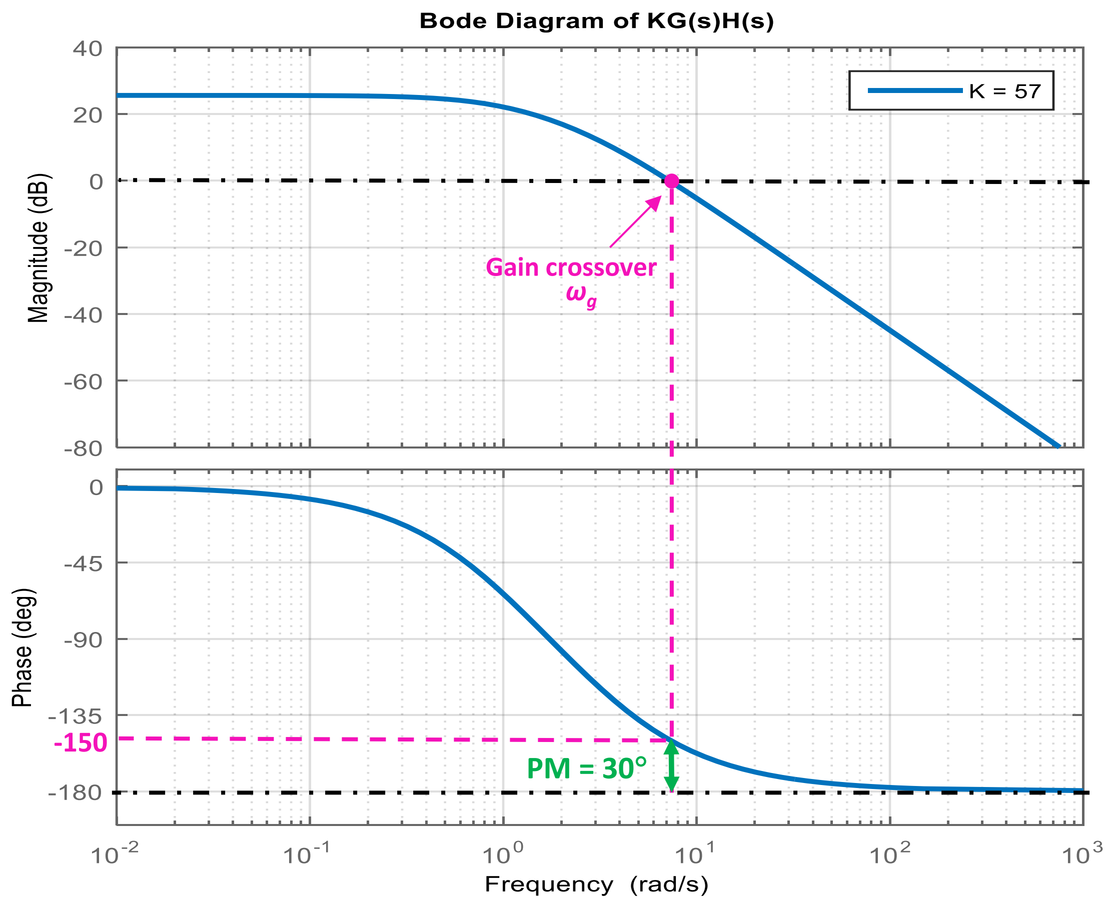
$$\boxed{\phi_m > 40^{\circ}}$$

Step 4: Select the appropriate factor of β based on the ϕ_m value

$$\boxed{\beta = 10} \rightarrow \boxed{\phi_m = 55^{\circ}}$$



β	ϕ_m
10	55°
20	65°
30	70°
40	72°
50	74°
60	75°
70	76°
80	77°
90	78°
100	78.5°

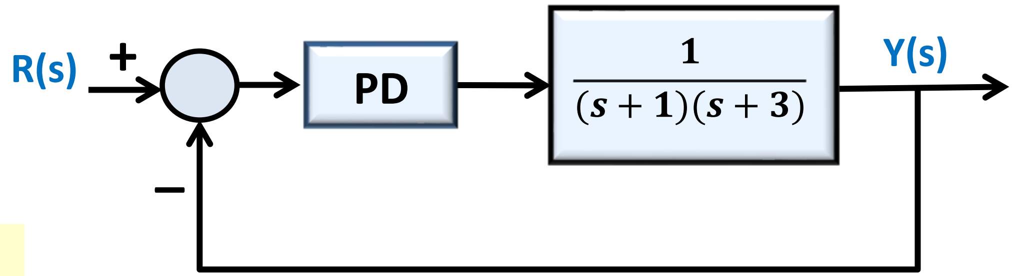


PD Controller Design

Example 3

Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$\boxed{e_{ss} = 0.05,\ PM > 60^{\circ},\ GM > 10\text{dB}}$$



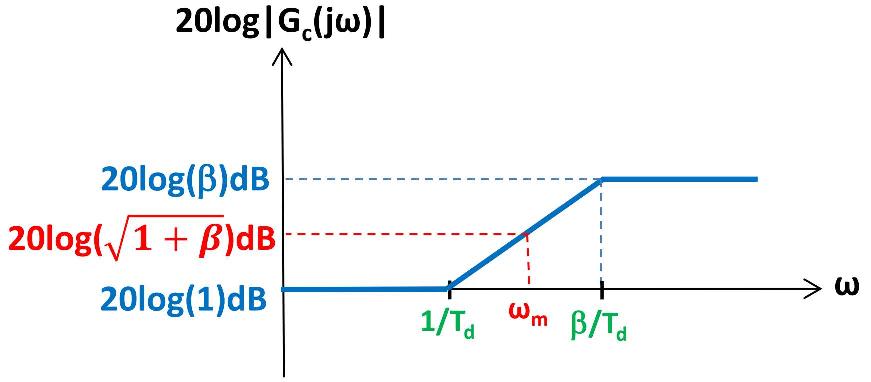
Step 5: Find the new gain crossover frequency ω_{gc} where the magnitude is $-20\log\sqrt{1+\beta}$

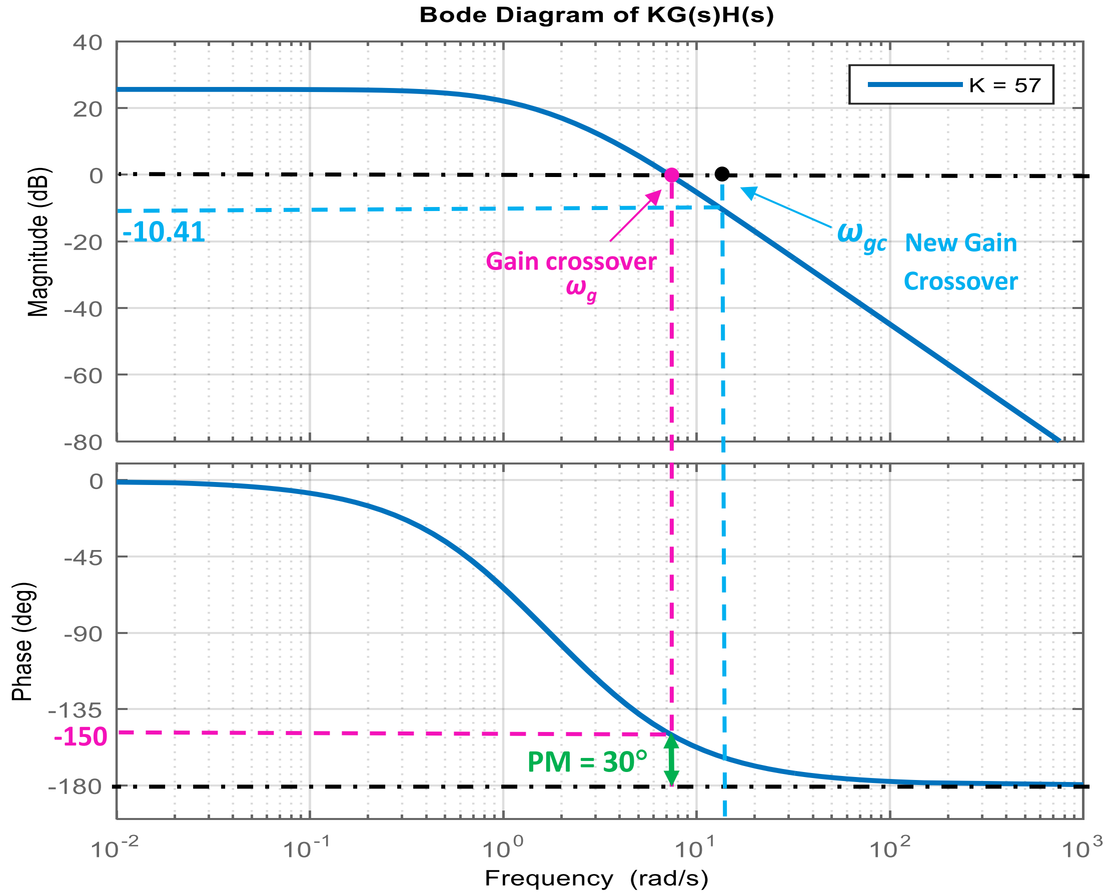
- The new gain crossover frequency, ω_{gc}, can be determined from the Bode diagram at the magnitude of $-20\log\sqrt{1+\beta}$.

$$-20\log\sqrt{1+\beta} = -20\log\sqrt{11} = -10.41\ \text{dB}$$

$$\boxed{\omega_{gc} = 13.5\ \text{rad/sec}}$$

New Gain Crossover Frequency



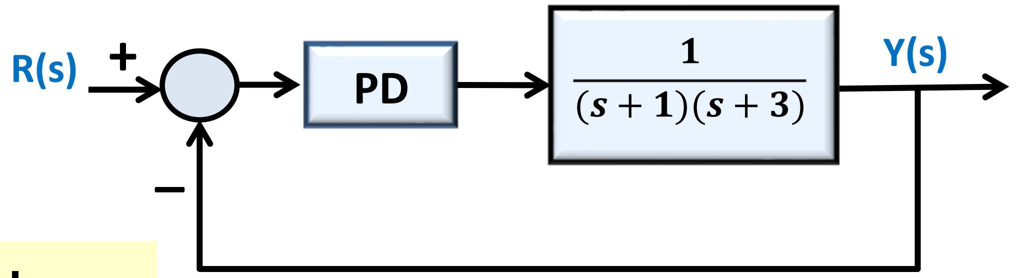


PD Controller Design

Example 3

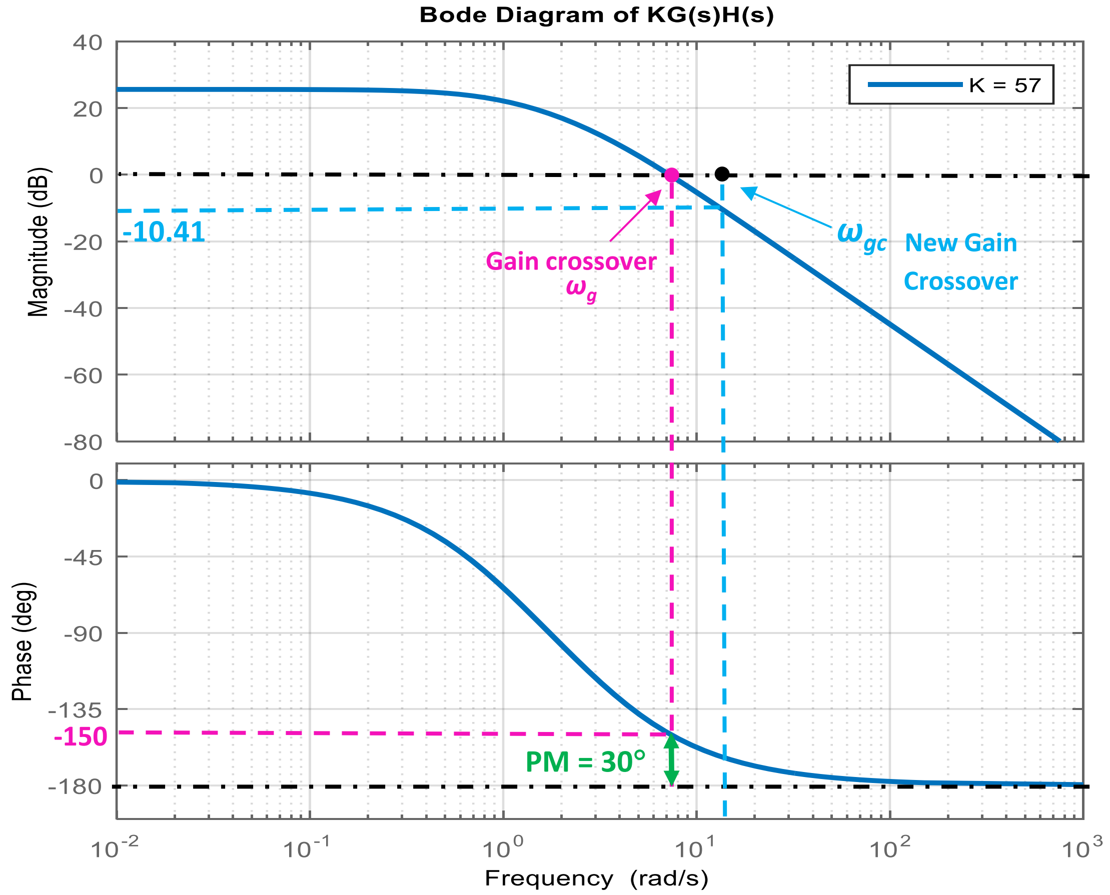
Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$e_{ss} = 0.05,\ PM > 60^{\circ},\ GM > 10\text{dB}$$



Step 6: Assign the maximum phase frequency ω_m at the new gain crossover frequency ω_{gc} value

$$\omega_m = \omega_{gc} = 13.5\ \text{rad/sec}$$



Step 7: Assign the derivative time constant T_d value

$$T_d = \frac{\sqrt{\beta}}{\omega_m} \rightarrow T_d = \frac{\sqrt{10}}{13.5} \longrightarrow T_d = 0.23$$

- The designed PD controller is obtained as

$$K_p = 57$$

$$T_d = 0.23$$

$$\beta = 10$$

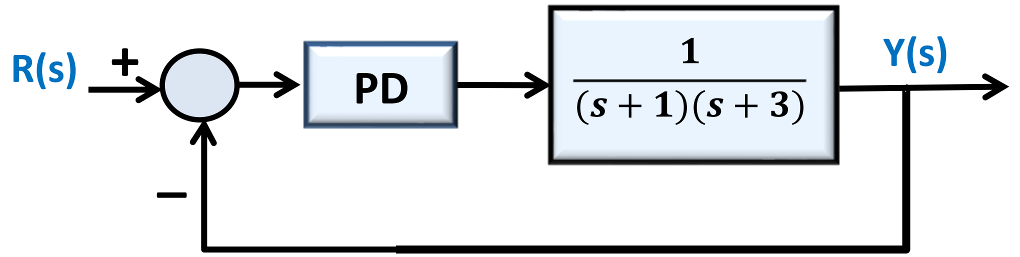
$$G_c(s) = 57\left(1 + \frac{0.23s}{0.023s + 1}\right)$$

PD Controller Design

Example 3

Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$e_{ss} = 0.05,\ PM > 60^{\circ},\ GM > 10\text{dB}$$

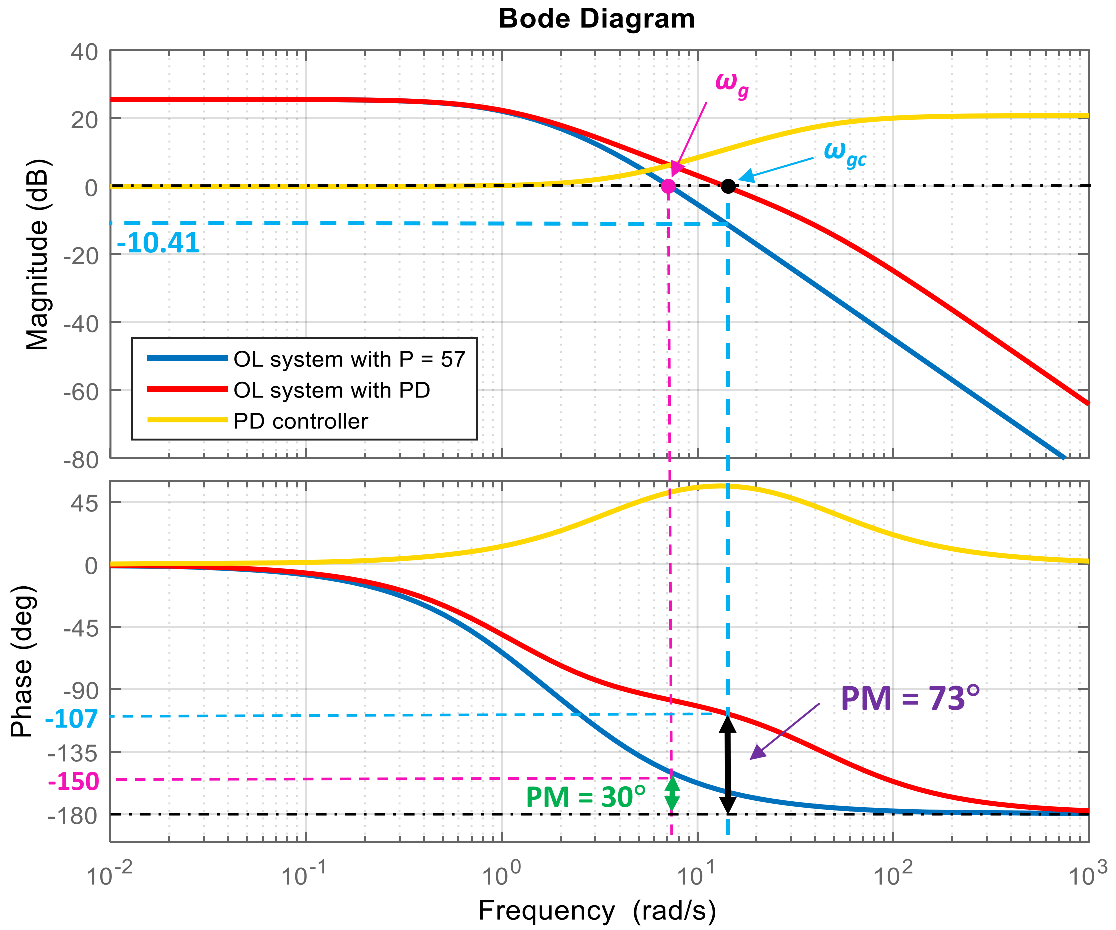


- Graph shows Bode plot of the open-loop system with proportional controller only, the open-loop system with PD controller, and the PD controller

$$K_pG(s)H(s) = \frac{57}{(s+1)(s+3)}$$

$$G_c(s)G(s)H(s) = 57\left(1 + \frac{0.23s}{0.023s+1}\right)\frac{1}{(s+1)(s+3)}$$

$$G_c(s) = 57\left(1 + \frac{0.23s}{0.023s+1}\right)$$



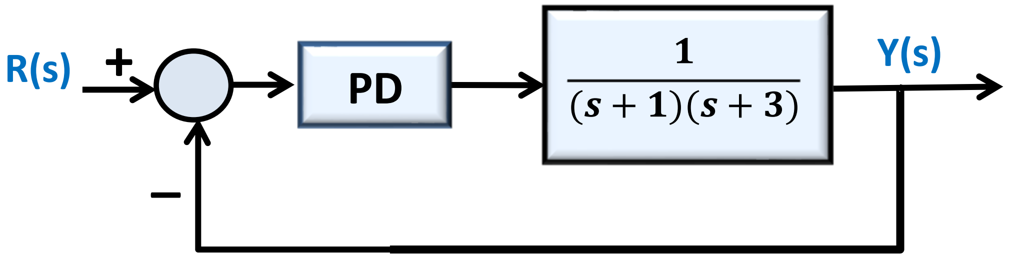
- The new gain crossover frequency is located at the frequency of the maximum peak of the designed lead compensator at ω_m.
- PD controller increases the phase margin of the system by contributing the positive phase.
- PD controller increases the bandwidth of the system that results in faster transient response.

PD Controller Design

Example 3

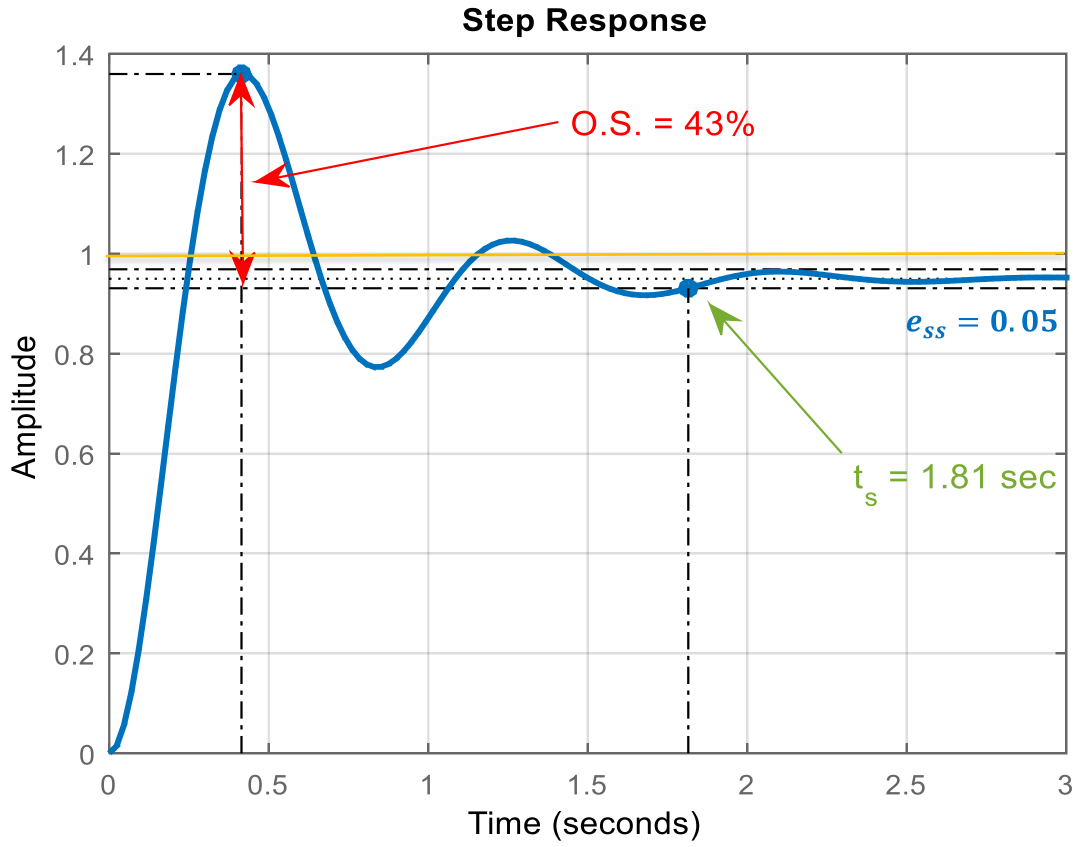
Consider the following second-order system. It is desired to design a PD controller to achieve the following performance characteristics

$$e_{ss} = 0.05,\ PM > 60^{\circ},\ GM > 10\mathrm{dB}$$

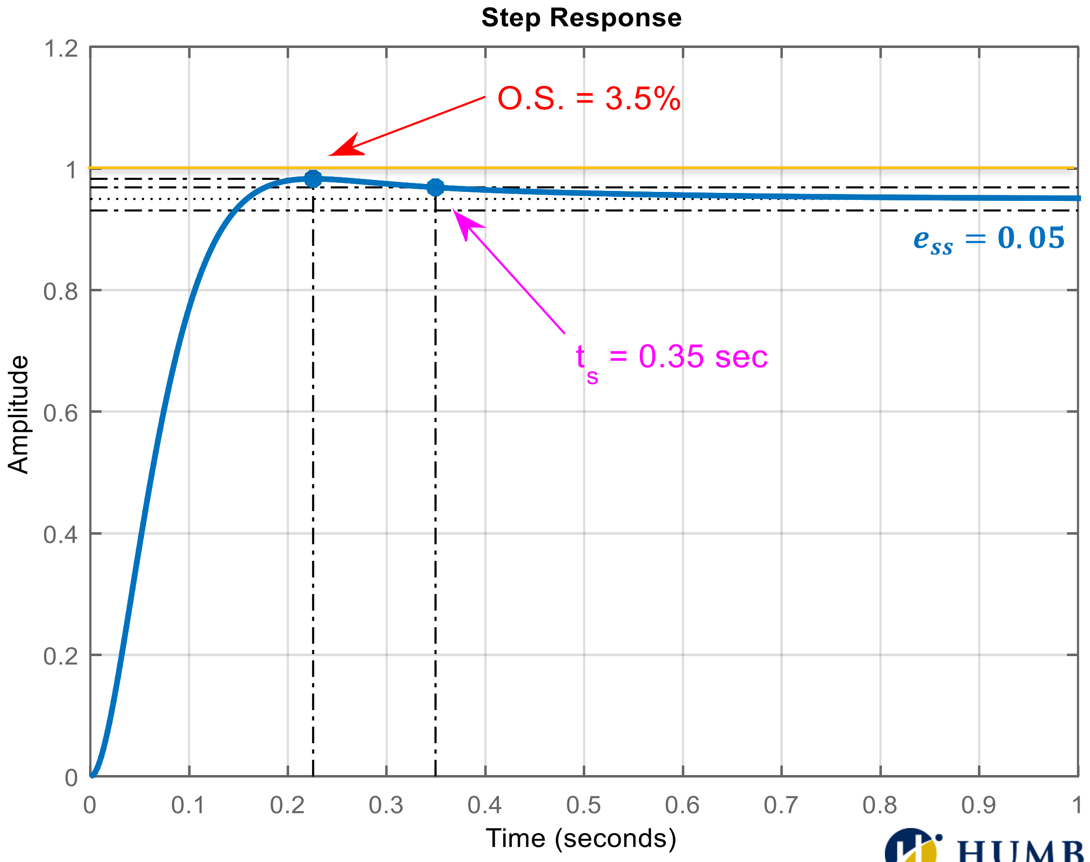


- Graphs show unit-step response of the closed-loop systems with P only controller $K_p = 57$ and with the designed PD controller

Closed-loop system with P only K_p = 57



Closed-loop system with the PD Controller



THANK YOU

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