

Lecture 3

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Chapter 2

Matrices

2.1 Matrix Arithmetic

Outcomes

- A. Perform the matrix operations of matrix addition, scalar multiplication, transposition and matrix multiplication. Identify when these operations are not defined. Represent these operations in terms of the entries of a matrix.
- B. Prove algebraic properties for matrix addition, scalar multiplication, transposition, and matrix multiplication. Apply these properties to manipulate an algebraic expression involving matrices.
- C. Compute the inverse of a matrix using row operations, and prove identities involving matrix inverses.
- E. Solve a linear system using matrix algebra.
- F. Use multiplication by an elementary matrix to apply row operations.
- G. Write a matrix as a product of elementary matrices.

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out that matrices are important not only for systems of equations but also in many applications.

Recall that a **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} \quad (2.1)$$

Recall that the size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns. You can remember the columns are like columns in a Greek temple. They stand upright while the rows lay flat like rows made by a tractor in a plowed field.

When specifying the size of a matrix, you always list the number of rows before the number of columns. You might remember that you always list the rows before the columns by using the phrase **Rowman Catholic**.

Consider the following definition.

Definition 2.1: Square Matrix

A matrix A which has size $n \times n$ is called a **square matrix**. In other words, A is a square matrix if it has the same number of rows and columns.

There is some notation specific to matrices which we now introduce. We denote the columns of a matrix A by A_j as follows

$$A = [A_1 \ A_2 \ \cdots \ A_n]$$

Therefore, A_j is the j^{th} column of A , when counted from left to right.

The individual elements of the matrix are called **entries** or **components** of A . Elements of the matrix are identified according to their position. The **(i,j)-entry** of a matrix is the entry in the i^{th} row and j^{th} column. For example, in the matrix 2.1 above, 8 is in position $(2,3)$ (and is called the $(2,3)$ -entry) because it is in the second row and the third column.

In order to remember which matrix we are speaking of, we will denote the entry in the i^{th} row and the j^{th} column of matrix A by a_{ij} . Then, we can write A in terms of its entries, as $A = [a_{ij}]$. Using this notation on the matrix in 2.1, $a_{23} = 8, a_{32} = -9, a_{12} = 2$, etc.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how matrix inverses play a similar role.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

Definition 2.2: The Zero Matrix

The $m \times n$ zero matrix is the $m \times n$ matrix having every entry equal to zero. It is denoted by 0.

One possible zero matrix is shown in the following example.

Example 2.3: The Zero Matrix

The 2×3 zero matrix is $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note there is a 2×3 zero matrix, a 3×4 zero matrix, etc. In fact there is a zero matrix for every size!

Definition 2.4: Equality of Matrices

Let A and B be two $m \times n$ matrices. Then $A = B$ means that for $A = [a_{ij}]$ and $B = [b_{ij}]$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

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In other words, two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

because they are different sizes. Also,

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

because, although they are the same size, their corresponding entries are not identical.

In the following section, we explore addition of matrices.

2.1.1 Addition of Matrices

When adding matrices, all matrices in the sum need have the same size. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 4 & 8 \\ 2 & 8 & 5 \end{bmatrix}$$

cannot be added, as one has size 3×2 while the other has size 2×3 .

However, the addition

$$\begin{bmatrix} 4 & 6 & 3 \\ 5 & 0 & 4 \\ 11 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 4 & -4 & 14 \\ 1 & 2 & 6 \end{bmatrix}$$

is possible.

The formal definition is as follows.

Definition 2.5: Addition of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where C is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

This definition tells us that when adding matrices, we simply add corresponding entries of the matrices. This is demonstrated in the next example.

Example 2.6: Addition of Matrices of Same Size

Add the following matrices, if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

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Solution. Notice that both A and B are of size 2×3 . Since A and B are of the same size, the addition is possible. Using Definition 2.5, the addition is done as follows.

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+(-6) & 0+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix}$$



Addition of matrices obeys very much the same properties as normal addition with numbers. Note that when we write for example $A + B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

Proposition 2.7: Properties of Matrix Addition

Let A, B and C be matrices. Then, the following properties hold.

- Commutative Law of Addition

$$A + B = B + A \quad (2.2)$$

- Associative Law of Addition

$$(A + B) + C = A + (B + C) \quad (2.3)$$

- Existence of an Additive Identity

$$\begin{aligned} \text{There exists a zero matrix } 0 \text{ such that} \\ A + 0 = A \end{aligned} \quad (2.4)$$

- Existence of an Additive Inverse

$$\begin{aligned} \text{There exists a matrix } -A \text{ such that} \\ A + (-A) = 0 \end{aligned} \quad (2.5)$$

Proof. Consider the Commutative Law of Addition given in 2.2. Let A, B, C , and D be matrices such that $A + B = C$ and $B + A = D$. We want to show that $D = C$. To do so, we will use the definition of matrix addition given in Definition 2.5. Now,

$$c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = d_{ij}$$

Therefore, $C = D$ because the ij^{th} entries are the same for all i and j . Note that the conclusion follows from the commutative law of addition of numbers, which says that if a and b are two numbers, then $a + b = b + a$. The proof of the other results are similar, and are left as an exercise.



We call the zero matrix in 2.4 the **additive identity**. Similarly, we call the matrix $-A$ in 2.5 the **additive inverse**. $-A$ is defined to equal $(-1)A = [-a_{ij}]$. In other words, every entry of A is multiplied by -1 . In the next section we will study scalar multiplication in more depth to understand what is meant by $(-1)A$.

2.1.2 Scalar Multiplication of Matrices

Recall that we use the word *scalar* when referring to numbers. Therefore, *scalar multiplication of a matrix* is the multiplication of a matrix by a number. To illustrate this concept, consider the following example in which a matrix is multiplied by the scalar 3.

$$3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{bmatrix}$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

The formal definition of scalar multiplication is as follows.

Definition 2.8: Scalar Multiplication of Matrices

If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Consider the following example.

Example 2.9: Effect of Multiplication by a Scalar

Find the result of multiplying the following matrix A by 7.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Solution. By Definition 2.8, we multiply each element of A by 7. Therefore,

$$7A = 7 \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$



Similarly to addition of matrices, there are several properties of scalar multiplication which hold.

Proposition 2.10: Properties of Scalar Multiplication

Let A, B be matrices, and k, p be scalars. Then, the following properties hold.

- Distributive Law over Matrix Addition

$$k(A + B) = kA + kB$$

- Distributive Law over Scalar Addition

$$(k + p)A = kA + pA$$

- Associative Law for Scalar Multiplication

$$k(pA) = (kp)A$$

- Rule for Multiplication by 1

The proof of this proposition is similar to the proof of Proposition 2.7 and is left an exercise to the reader.

2.1.3 Multiplication of Matrices

The next important matrix operation we will explore is multiplication of matrices. The operation of matrix multiplication is one of the most important and useful of the matrix operations. Throughout this section, we will also demonstrate how matrix multiplication relates to linear systems of equations.

First, we provide a formal definition of row and column vectors.

Definition 2.11: Row and Column Vectors

Matrices of size $n \times 1$ or $1 \times n$ are called **vectors**. If X is such a matrix, then we write x_i to denote the entry of X in the i^{th} row of a column matrix, or the i^{th} column of a row matrix.

The $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **column vector**. The $1 \times n$ matrix

$$X = [x_1 \ \cdots \ x_n]$$

is called a **row vector**.

We may simply use the term **vector** throughout this text to refer to either a column or row vector. If we do so, the context will make it clear which we are referring to.

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In this chapter, we will again use the notion of linear combination of vectors as in Definition 9.12. In this context, a linear combination is a sum consisting of vectors multiplied by scalars. For example,

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a linear combination of three vectors.

It turns out that we can express any system of linear equations as a linear combination of vectors. In fact, the vectors that we will use are just the columns of the corresponding augmented matrix!

Definition 2.12: The Vector Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can express this system in **vector form** which is as follows:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector.

The first important form of matrix multiplication is multiplying a matrix by a vector. Consider the product given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

We will soon see that this equals

$$7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

in general terms,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

Thus you take x_1 times the first column, add to x_2 times the second column, and finally x_3 times the third column. The above sum is a linear combination of the columns of the matrix. When you multiply a matrix

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on the left by a vector on the right, the numbers making up the vector are just the scalars to be used in the linear combination of the columns as illustrated above.

Here is the formal definition of how to multiply an $m \times n$ matrix by an $n \times 1$ column vector.

Definition 2.13: Multiplication of Vector by Matrix

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let X be an $n \times 1$ matrix given by

$$A = [A_1 \cdots A_n], X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product AX is the $m \times 1$ column vector which equals the following linear combination of the columns of A :

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \sum_{j=1}^n x_j A_j$$

If we write the columns of A in terms of their entries, they are of the form

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Then, we can write the product AX as

$$AX = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that multiplication of an $m \times n$ matrix and an $n \times 1$ vector produces an $m \times 1$ vector.

Here is an example.

Example 2.14: A Vector Multiplied by a Matrix

Compute the product AX for

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Solution. We will use Definition 2.13 to compute the product. Therefore, we compute the product AX as

follows.

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
& 1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}
\end{aligned}$$



Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

Definition 2.15: The Matrix Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Then we can express this system in **matrix form** as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The expression $AX = B$ is also known as the **Matrix Form** of the corresponding system of linear equations. The matrix A is simply the coefficient matrix of the system, the vector X is the column vector constructed from the variables of the system, and finally the vector B is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a homogeneous system of equations in matrix form, it would have the form $AX = 0$, for the zero vector 0 .

You can see from this definition that a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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will satisfy the equation $AX = B$ only when the entries x_1, x_2, \dots, x_n of the vector X are solutions to the original system.

Now that we have examined how to multiply a matrix by a vector, we wish to consider the case where we multiply two matrices of more general sizes, although these sizes still need to be appropriate as we will see. For example, in Example 2.14, we multiplied a 3×4 matrix by a 4×1 vector. We want to investigate how to multiply other sizes of matrices.

We have not yet given any conditions on when matrix multiplication is possible! For matrices A and B , in order to form the product AB , the number of columns of A must equal the number of rows of B . Consider a product AB where A has size $m \times n$ and B has size $n \times p$. Then, the product in terms of size of matrices is given by

$$(m \times \overbrace{n}^{\text{these must match!}}) (\overbrace{n \times p}{}^{}) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers don't match, you can't multiply the matrices!

When the number of columns of A equals the number of rows of B the two matrices are said to be **conformable** and the product AB is obtained as follows.

Definition 2.16: Multiplication of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix of the form

$$B = [B_1 \cdots B_n]$$

where B_1, \dots, B_p are the $n \times 1$ columns of B . Then the $m \times p$ matrix AB is defined as follows:

$$AB = A[B_1 \cdots B_p] = [(AB)_1 \cdots (AB)_p]$$

where $(AB)_k$ is an $m \times 1$ matrix or column vector which gives the k^{th} column of AB .

Consider the following example.

Example 2.17: Multiplying Two Matrices

Find AB if possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Solution. The first thing you need to verify when calculating a product is whether the multiplication is possible. The first matrix has size 2×3 and the second matrix has size 3×3 . The inside numbers are equal, so A and B are conformable matrices. According to the above discussion AB will be a 2×3 matrix.

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Definition 2.16 gives us a way to calculate each column of AB , as follows.

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}}_{\text{First column}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}}_{\text{Second column}} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}}_{\text{Third column}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

You know how to multiply a matrix times a vector, using Definition 2.13 for each of the three columns. Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$



Since vectors are simply $n \times 1$ or $1 \times m$ matrices, we can also multiply a vector by another vector.

Example 2.18: Vector Times Vector Multiplication

$$\text{Multiply if possible } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1 \ 2 \ 1 \ 0].$$

Solution. In this case we are multiplying a matrix of size 3×1 by a matrix of size 1×4 . The inside numbers match so the product is defined. Note that the product will be a matrix of size 3×4 . Using Definition 2.16, we can compute this product as follows

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1 \ 2 \ 1 \ 0] = \left[\underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{First column}} [1], \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{Second column}} [2], \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{Third column}} [1], \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{Fourth column}} [0] \right]$$

You can use Definition 2.13 to verify that this product is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$



Example 2.19: A Multiplication Which is Not Defined

Find BA if possible.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc} -2 & 1 & 1 \end{array} \right] \quad \left[\begin{array}{ccc} 0 & 2 & 1 \end{array} \right]$$

Solution. First check if it is possible. This product is of the form $(3 \times 3)(2 \times 3)$. The inside numbers do not match and so you can't do this multiplication. ♠

In this case, we say that the multiplication is not defined. Notice that these are the same matrices which we used in Example 2.17. In this example, we tried to calculate BA instead of AB . This demonstrates another property of matrix multiplication. While the product AB maybe be defined, we cannot assume that the product BA will be possible. Therefore, it is important to always check that the product is defined before carrying out any calculations.

Earlier, we defined the zero matrix 0 to be the matrix (of appropriate size) containing zeros in all entries. Consider the following example for multiplication by the zero matrix.

Example 2.20: Multiplication by the Zero Matrix

Compute the product $A0$ for the matrix

$$A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right]$$

and the 2×2 zero matrix given by

$$0 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$$

Solution. In this product, we compute

$$\left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right] \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$$

Hence, $A0 = 0$. ♠

Notice that we could also multiply A by the 2×1 zero vector given by $\left[\begin{array}{c} 0 \\ 0 \end{array} \right]$. The result would be the 2×1 zero vector. Therefore, it is always the case that $A0 = 0$, for an appropriately sized zero matrix or vector.

2.1.4 The ij^{th} Entry of a Product

In previous sections, we used the entries of a matrix to describe the action of matrix addition and scalar multiplication. We can also study matrix multiplication using the entries of matrices.

What is the ij^{th} entry of AB ? It is the entry in the i^{th} row and the j^{th} column of the product AB .

Now if A is $m \times n$ and B is $n \times p$, then we know that the product AB has the form

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right] \left[\begin{array}{cccc} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{array} \right]$$

The j^{th} column of AB is of the form

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right] \left[\begin{array}{c} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{array} \right]$$

which is an $m \times 1$ column vector. It is calculated by

$$\left[\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array} \right] \quad \left[\begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{array} \right] \quad \left[\begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{array} \right]$$

$$b_{1j} \begin{bmatrix} \cdots \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} \cdots \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{nj} \begin{bmatrix} \cdots \\ \vdots \\ a_{mn} \end{bmatrix}$$

Therefore, the i, j^{th} entry is the entry in row i of this vector. This is computed by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

The following is the formal definition for the i, j^{th} entry of a product of matrices.

Definition 2.21: The i, j^{th} Entry of a Product

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. Then AB is an $m \times p$ matrix and the (i, j) -entry of AB is defined as

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

Another way to write this is

$$(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, to find the (i, j) -entry of the product AB , or $(AB)_{ij}$, you multiply the i^{th} row of A , on the left by the j^{th} column of B . To express AB in terms of its entries, we write $AB = [(AB)_{ij}]$.

Consider the following example.

Example 2.22: The Entries of a Product

Compute AB if possible. If it is, find the $(3, 2)$ -entry of AB using Definition 2.21.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

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Solution. First check if the product is possible. It is of the form $(3 \times 2)(2 \times 3)$ and since the inside numbers match, it is possible to do the multiplication. The result should be a 3×3 matrix. We can first compute AB :

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\begin{bmatrix} 16 & 15 & 5 \\ 13 & 15 & 5 \\ 46 & 42 & 14 \end{bmatrix}$$

which is a 3×3 matrix as desired. Thus, the $(3, 2)$ -entry equals 42.

Now using Definition 2.21, we can find that the $(3, 2)$ -entry equals

$$\begin{aligned} \sum_{k=1}^2 a_{3k}b_{k2} &= a_{31}b_{12} + a_{32}b_{22} \\ &= 2 \times 3 + 6 \times 6 = 42 \end{aligned}$$

Consulting our result for AB above, this is correct!

You may wish to use this method to verify that the rest of the entries in AB are correct. ♠

Here is another example.

Example 2.23: Finding the Entries of a Product

Determine if the product AB is defined. If it is, find the $(2, 1)$ -entry of the product.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 0 & 0 & 0 \end{array} \right] \cdot \left[\begin{array}{cc} 2 & 6 \end{array} \right]$$

Solution. This product is of the form $(3 \times 3)(3 \times 2)$. The middle numbers match so the matrices are conformable and it is possible to compute the product.

We want to find the $(2, 1)$ -entry of AB , that is, the entry in the second row and first column of the product. We will use Definition 2.21, which states

$$(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

In this case, $n = 3$, $i = 2$ and $j = 1$. Hence the $(2, 1)$ -entry is found by computing

$$(AB)_{21} = \sum_{k=1}^3 a_{2k} b_{k1} = \left[\begin{array}{ccc} a_{21} & a_{22} & a_{23} \end{array} \right] \left[\begin{array}{c} b_{11} \\ b_{21} \\ b_{31} \end{array} \right]$$

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Substituting in the appropriate values, this product becomes

$$\left[\begin{array}{ccc} a_{21} & a_{22} & a_{23} \end{array} \right] \left[\begin{array}{c} b_{11} \\ b_{21} \\ b_{31} \end{array} \right] = \left[\begin{array}{ccc} 7 & 6 & 2 \end{array} \right] \left[\begin{array}{c} 1 \\ 3 \\ 2 \end{array} \right] = 1 \times 7 + 3 \times 6 + 2 \times 2 = 29$$

Hence, $(AB)_{21} = 29$.

You should take a moment to find a few other entries of AB . You can multiply the matrices to check that your answers are correct. The product AB is given by

$$AB = \left[\begin{array}{cc} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{array} \right]$$



2.1.5 Properties of Matrix Multiplication

As pointed out above, it is sometimes possible to multiply matrices in one order but not in the other order. However, even if both AB and BA are defined, they may not be equal.

Example 2.24: Matrix Multiplication is Not Commutative

Compare the products AB and BA , for matrices $A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right]$, $B = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$

Solution. First, notice that A and B are both of size 2×2 . Therefore, both products AB and BA are defined. The first product, AB is

$$AB = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right] \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] = \left[\begin{array}{cc} 2 & 1 \\ 4 & 3 \end{array} \right]$$

The second product, BA is

$$\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right] = \left[\begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array} \right]$$

Therefore, $AB \neq BA$.



This example illustrates that you cannot assume $AB = BA$ even when multiplication is defined in both orders. If for some matrices A and B it is true that $AB = BA$, then we say that A and B **commute**. This is one important property of matrix multiplication.

The following are other important properties of matrix multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

Proposition 2.25: Properties of Matrix Multiplication

The following hold for matrices A, B , and C and for scalars r and s ,

$$A(rB + sC) = r(AB) + s(AC) \quad (2.6)$$

$$(B + C)A = BA + CA \quad (2.7)$$

$$A(BC) = (AB)C \quad (2.8)$$

Proof. First we will prove 2.6. We will use Definition 2.21 and prove this statement using the i,j^{th} entries of a matrix. Therefore,

$$\begin{aligned} (A(rB + sC))_{ij} &= \sum_k a_{ik}(rB + sC)_{kj} = \sum_k a_{ik}(rb_{kj} + sc_{kj}) \\ &= r \sum_k a_{ik}b_{kj} + s \sum_k a_{ik}c_{kj} = r(AB)_{ij} + s(AC)_{ij} \\ &= (r(AB) + s(AC))_{ij} \end{aligned}$$

Thus $A(rB + sC) = r(AB) + s(AC)$ as claimed.

The proof of 2.7 follows the same pattern and is left as an exercise.

Statement 2.8 is the associative law of multiplication. Using Definition 2.21,

$$\begin{aligned} (A(BC))_{ij} &= \sum_k a_{ik}(BC)_{kj} = \sum_k a_{ik} \sum_l b_{kl}c_{lj} \\ &= \sum_l (AB)_{il}c_{lj} = ((AB)C)_{ij}. \end{aligned}$$

This proves 2.8. ♠

2.1.6 The Transpose

Another important operation on matrices is that of taking the **transpose**. For a matrix A , we denote the transpose of A by A^T . Before formally defining the transpose, we explore this operation on the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 4 was in the first row and the second column and it ended up in the second row and first column.

The definition of the transpose is as follows.

Definition 2.26: The Transpose of a Matrix

Let A be an $m \times n$ matrix. Then A^T , the **transpose** of A , denotes the $n \times m$ matrix given by

$$A^T = [a_{ij}]^T = [a_{ji}]$$

The (i, j) -entry of A becomes the (j, i) -entry of A^T .

Consider the following example.

Example 2.27: The Transpose of a Matrix

Calculate A^T for the following matrix

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 3 & 5 & 4 \end{bmatrix}$$

Solution. By Definition 2.26, we know that for $A = [a_{ij}]$, $A^T = [a_{ji}]$. In other words, we switch the row and column location of each entry. The $(1,2)$ -entry becomes the $(2,1)$ -entry.

Thus,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{bmatrix}$$

Notice that A is a 2×3 matrix, while A^T is a 3×2 matrix. ♠

The transpose of a matrix has the following important properties .

Lemma 2.28: Properties of the Transpose of a Matrix

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and r and s scalars. Then

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $(rA + sB)^T = rA^T + sB^T$

Proof. First we prove 2. From Definition 2.26,

$$\begin{aligned} (AB)^T &= [(AB)_{ij}]^T = [(AB)_{ji}] = \sum_k a_{jk}b_{ki} = \sum_k b_{ki}a_{jk} \\ &= \sum_k [b_{ik}]^T [a_{kj}]^T = [b_{ij}]^T [a_{ij}]^T = B^T A^T \end{aligned}$$

The proof of Formula 3 is left as an exercise. ♠

The transpose of a matrix is related to other important topics. Consider the following definition.

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Definition 2.29: Symmetric and Skew Symmetric Matrices

An $n \times n$ matrix A is said to be **symmetric** if $A = A^T$. It is said to be **skew symmetric** if $A = -A^T$.

We will explore these definitions in the following examples.

Example 2.30: Symmetric Matrices

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Use Definition 2.29 to show that A is symmetric.

Solution. By Definition 2.29, we need to show that $A = A^T$. Now, using Definition 2.26,

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so A is symmetric. ♠

Example 2.31: A Skew Symmetric Matrix

Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that A is skew symmetric.

Solution. By Definition 2.29,

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

You can see that each entry of A^T is equal to -1 times the same entry of A . Hence, $A^T = -A$ and so by Definition 2.29, A is skew symmetric. ▲

2.1.7 The Identity and Inverses

There is a special matrix, denoted I , which is called to as the **identity matrix**. The identity matrix is always a square matrix, and it has the property that there are ones down the main diagonal and zeroes elsewhere. Here are some identity matrices of various sizes.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first is the 1×1 identity matrix, the second is the 2×2 identity matrix, and so on. By extension, you can likely see what the $n \times n$ identity matrix would be. When it is necessary to distinguish which size of identity matrix is being discussed, we will use the notation I_n for the $n \times n$ identity matrix.

The identity matrix is so important that there is a special symbol to denote the $i j^{\text{th}}$ entry of the identity matrix. This symbol is given by $I_{ij} = \delta_{ij}$ where δ_{ij} is the **Kronecker symbol** defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

I_n is called the **identity matrix** because it is a **multiplicative identity** in the following sense.

Lemma 2.32: Multiplication by the Identity Matrix

Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix. Then $AI_n = A$. If I_m is the $m \times m$ identity matrix, it also follows that $I_mA = A$.

Proof. The (i, j) -entry of AI_n is given by:

$$\sum_k a_{ik} \delta_{kj} = a_{ij}$$

and so $AI_n = A$. The other case is left as an exercise for you. ♠

We now define the matrix operation which in some ways plays the role of division.

Definition 2.33: The Inverse of a Matrix

A square $n \times n$ matrix A is said to have an **inverse** A^{-1} if and only if

$$AA^{-1} = A^{-1}A = I_n$$

In this case, the matrix A is called **invertible**.

Such a matrix A^{-1} will have the same size as the matrix A . It is very important to observe that the inverse of a matrix, if it exists, is unique. Another way to think of this is that if it acts like the inverse, then it **is** the inverse.

Theorem 2.34: Uniqueness of Inverse

Suppose A is an $n \times n$ matrix such that an inverse A^{-1} exists. Then there is only one such inverse

matrix. That is, given any matrix B such that $AB = BA = I$, $B = A^{-1}$.

Proof. In this proof, it is assumed that I is the $n \times n$ identity matrix. Let A, B be $n \times n$ matrices such that A^{-1} exists and $AB = BA = I$. We want to show that $A^{-1} = B$. Now using properties we have seen, we get:

$$A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$$

Hence, $A^{-1} = B$ which tells us that the inverse is unique. ♠

The next example demonstrates how to check the inverse of a matrix.

Example 2.35: Verifying the Inverse of a Matrix

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Show $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of A .

Solution. To check this, multiply

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

showing that this matrix is indeed the inverse of A . ♠

Unlike ordinary multiplication of numbers, it can happen that $A \neq 0$ but A may fail to have an inverse. This is illustrated in the following example.

Example 2.36: A Nonzero Matrix With No Inverse

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that A does not have an inverse.

Solution. One might think A would have an inverse because it does not equal zero. However, note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If A^{-1} existed, we would have the following

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = A^{-1} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

$$\begin{aligned} &= A^{-1} \left(A \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \\ &= (A^{-1}A) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= I \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

This says that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is impossible! Therefore, A does not have an inverse. ♠

In the next section, we will explore how to find the inverse of a matrix, if it exists.

2.1.8 Finding the Inverse of a Matrix

In Example 2.35, we were given A^{-1} and asked to verify that this matrix was in fact the inverse of A . In this section, we explore how to find A^{-1} .

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & z \end{bmatrix}$$

as in Example 2.35. In order to find A^{-1} , we need to find a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can multiply these two matrices, and see that in order for this equation to be true, we must find the solution to the systems of equations,

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 0 \end{aligned}$$

and

$$\begin{aligned} z + w &= 0 \\ z + 2w &= 1 \end{aligned}$$

Writing the augmented matrix for these two systems gives

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array} \right]$$

for the first system and

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array} \right] \quad (2.9)$$

for the second.

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Let's solve the first system. Take -1 times the first row and add to the second to get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array} \right]$$

Now take -1 times the second row and add to the first to get

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array} \right]$$

Writing in terms of variables, this says $x = 2$ and $y = -1$.

Now solve the second system, 2.9 to find z and w . You will find that $z = -1$ and $w = 1$.

If we take the values found for x, y, z , and w and put them into our inverse matrix, we see that the inverse is

$$A^{-1} = \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

After taking the time to solve the second system, you may have noticed that exactly the same row operations were used to solve both systems. In each case, the end result was something of the form $[I|X]$ where I is the identity and X gave a column of the inverse. In the above,

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

the first column of the inverse was obtained by solving the first system and then the second column

$$\begin{bmatrix} z \\ w \end{bmatrix}$$

To simplify this procedure, we could have solved both systems at once! To do so, we could have written

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right]$$

and row reduced until we obtained

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

and read off the inverse as the 2×2 matrix on the right side.

This exploration motivates the following important algorithm.

Algorithm 2.37: Matrix Inverse Algorithm

Suppose A is an $n \times n$ matrix. To find A^{-1} if it exists, form the augmented $n \times 2n$ matrix

$$[A|I]$$

If possible do row operations until you obtain an $n \times 2n$ matrix of the form

$$[I|B]$$

When this has been done, $B = A^{-1}$. In this case, we say that A is **invertible**. If it is impossible to

row reduce to a matrix of the form $[I|B]$, then A has no inverse.

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This algorithm shows how to find the inverse if it exists. It will also tell you if A does not have an inverse.

Consider the following example.

Example 2.38: Finding the Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. Find A^{-1} if it exists.

Solution. Set up the augmented matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array} \right]$$

Now we row reduce, with the goal of obtaining the 3×3 identity matrix on the left hand side. First, take -1 times the first row and add to the second followed by -3 times the first row added to the third row. This yields

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array} \right]$$

Then take 5 times the second row and add to -2 times the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array} \right]$$

Next take the third row and add to -7 times the first row. This yields

$$\left[\begin{array}{ccc|ccc} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array} \right]$$

Now take $-\frac{7}{5}$ times the second row and add to the first row.

$$\left[\begin{array}{ccc|ccc} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array} \right]$$

Finally divide the first row by -7 , the second row by -10 and the third row by 14 which yields

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array} \right]$$

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Notice that the left hand side of this matrix is now the 3×3 identity matrix I_3 . Therefore, the inverse is the 3×3 matrix on the right hand side, given by

$$\left[\begin{array}{ccc} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array} \right]$$



It may happen that through this algorithm, you discover that the left hand side cannot be row reduced to the identity matrix. Consider the following example of this situation.

Example 2.39: A Matrix Which Has No Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find A^{-1} if it exists.

Solution. Write the augmented matrix $[A|I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array} \right]$$

and proceed to do row operations attempting to obtain $[I|A^{-1}]$. Take -1 times the first row and add to the second. Then take -2 times the first row and add to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array} \right]$$

Next add -1 times the second row to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array} \right]$$

At this point, you can see there will be no way to obtain I on the left side of this augmented matrix. Hence, there is no way to complete this algorithm, and therefore the inverse of A does not exist. In this case, we say that A is not invertible. ♠

If the algorithm provides an inverse for the original matrix, it is always possible to check your answer. To do so, use the method demonstrated in Example 2.35. Check that the products AA^{-1} and $A^{-1}A$ both equal the identity matrix. Through this method, you can always be sure that you have calculated A^{-1} properly!

One way in which the inverse of a matrix is useful is to find the solution of a system of linear equations. Recall from Definition 2.15 that we can write a system of equations in matrix form, which is of the form

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$AX = B$. Suppose you find the inverse of the matrix A^{-1} . Then you could multiply both sides of this equation on the left by A^{-1} and simplify to obtain

$$\begin{aligned} (A^{-1})AX &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Therefore we can find X , the solution to the system, by computing $X = A^{-1}B$. Note that once you have found A^{-1} , you can easily get the solution for different right hand sides (different B). It is always just $A^{-1}B$.

We will explore this method of finding the solution to a system in the following example.

Example 2.40: Using the Inverse to Solve a System of Equations

Consider the following system of equations. Use the inverse of a suitable matrix to give the solutions to this system.

$$\begin{aligned} x + z &= 1 \\ x - y + z &= 3 \\ x + y - z &= 2 \end{aligned}$$

Solution. First, we can write the system of equations in matrix form

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = B \quad (2.10)$$

The inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Verifying this inverse is left as an exercise.

From here, the solution to the given system 2.10 is found by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -\frac{3}{2} \end{bmatrix}$$



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What if the right side, B , of 2.10 had been $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$? In other words, what would be the solution to

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}?$$

By the above discussion, the solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

This illustrates that for a system $AX = B$ where A^{-1} exists, it is easy to find the solution when the vector B is changed.

We conclude this section with some important properties of the inverse.

Theorem 2.41: Inverses of Transposes and Products

Let A, B , and A_i for $i = 1, \dots, k$ be $n \times n$ matrices.

1. If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$
2. If A and B are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If A_1, A_2, \dots, A_k are invertible, then the product $A_1 A_2 \cdots A_k$ is invertible, and $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$

Consider the following theorem.

Theorem 2.42: Properties of the Inverse

Let A be an $n \times n$ matrix and I the usual identity matrix.

1. I is invertible and $I^{-1} = I$
2. If A is invertible then so is A^{-1} , and $(A^{-1})^{-1} = A$
3. If A is invertible then so is A^k , and $(A^k)^{-1} = (A^{-1})^k$
4. If A is invertible and p is a nonzero real number, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$

2.1.9 Elementary Matrices

We now turn our attention to a special type of matrix called an **elementary matrix**. An elementary matrix is always a square matrix. Recall the row operations given in Definition 1.11. Any elementary matrix, which we often denote by E , is obtained from applying *one* row operation to the identity matrix of the same size.

For example, the matrix

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the elementary matrix obtained from switching the two rows. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the elementary matrix obtained from multiplying the second row of the 3×3 identity matrix by 3. The matrix

$$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

is the elementary matrix obtained from adding -3 times the first row to the third row.

You may construct an elementary matrix from any row operation, but remember that you can only apply one operation.

Consider the following definition.

Definition 2.43: Elementary Matrices and Row Operations

Let E be an $n \times n$ matrix. Then E is an **elementary matrix** if it is the result of applying one row operation to the $n \times n$ identity matrix I_n .

Those which involve switching rows of the identity matrix are called **permutation matrices**.

Therefore, E constructed above by switching the two rows of I_2 is called a permutation matrix.

Elementary matrices can be used in place of row operations and therefore are very useful. It turns out that multiplying (on the left hand side) by an elementary matrix E will have the same effect as doing the row operation used to obtain E .

The following theorem is an important result which we will use throughout this text.

Theorem 2.44: Multiplication by an Elementary Matrix and Row Operations

To perform any of the three row operations on a matrix A it suffices to take the product EA , where E is the elementary matrix obtained by using the desired row operation on the identity matrix.

Therefore, instead of performing row operations on a matrix A , we can row reduce through matrix multiplication with the appropriate elementary matrix. We will examine this theorem in detail for each of the three row operations given in Definition 1.11.

First, consider the following lemma.

Lemma 2.45: Action of Permutation Matrix

Let P^{ij} denote the elementary matrix which involves switching the i^{th} and the j^{th} rows. Then P^{ij} is a permutation matrix and

$$P^{ij}A = B$$

where B is obtained from A by switching the i^{th} and the j^{th} rows.

We will explore this idea more in the following example.

Example 2.46: Switching Rows with an Elementary Matrix

Let

$$P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Find B where $B = P^{12}A$.

Solution. You can see that the matrix P^{12} is obtained by switching the first and second rows of the 3×3 identity matrix I .

Using our usual procedure, compute the product $P^{12}A = B$. The result is given by

$$B = \begin{bmatrix} c & d \\ a & b \\ e & f \end{bmatrix}$$

Notice that B is the matrix obtained by switching rows 1 and 2 of A . Therefore by multiplying A by P^{12} , the row operation which was applied to I to obtain P^{12} is applied to A to obtain B . ♠

Theorem 2.44 applies to all three row operations, and we now look at the row operation of multiplying a row by a scalar. Consider the following lemma.

Lemma 2.47: Multiplication by a Scalar and Elementary Matrices

Let $E(k, i)$ denote the elementary matrix corresponding to the row operation in which the i^{th} row is multiplied by the nonzero scalar, k . Then

$$E(k, i)A = B$$

where B is obtained from A by multiplying the i^{th} row of A by k .

We will explore this lemma further in the following example.

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Example 2.48: Multiplication of a Row by 5 Using Elementary Matrix

Let

$$E(5, 2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Find the matrix B where $B = E(5, 2)A$

Solution. You can see that $E(5, 2)$ is obtained by multiplying the second row of the identity matrix by 5.

Using our usual procedure for multiplication of matrices, we can compute the product $E(5, 2)A$. The resulting matrix is given by

$$B = \begin{bmatrix} a & b \\ 5c & 5d \\ e & f \end{bmatrix}$$

Notice that B is obtained by multiplying the second row of A by the scalar 5. ♠

There is one last row operation to consider. The following lemma discusses the final operation of adding a multiple of a row to another row.

Lemma 2.49: Adding Multiples of Rows and Elementary Matrices

Let $E(k \times i + j)$ denote the elementary matrix obtained from I by adding k times the i^{th} row to the j^{th} . Then

$$E(k \times i + j)A = B$$

where B is obtained from A by adding k times the i^{th} row to the j^{th} row of A .

Consider the following example.

Example 2.50: Adding Two Times the First Row to the Last

Let

$$E(2 \times 1 + 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Find B where $B = E(2 \times 1 + 3)A$.

Solution. You can see that the matrix E ($2 \times 1 + 3$) was obtained by adding 2 times the first row of I to the third row of I .

Using our usual procedure, we can compute the product $E(2 \times 1 + 3)A$. The resulting matrix B is given by

$$B = \begin{bmatrix} a & b \\ c & d \\ 2a+e & 2b+f \end{bmatrix}$$

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You can see that B is the matrix obtained by adding 2 times the first row of A to the third row. ♠

Suppose we have applied a row operation to a matrix A . Consider the row operation required to return A to its original form, to undo the row operation. It turns out that this action is how we find the inverse of an elementary matrix E .

Consider the following theorem.

Theorem 2.51: Elementary Matrices and Inverses

Every elementary matrix is invertible and its inverse is also an elementary matrix.

In fact, the inverse of an elementary matrix is constructed by doing the *reverse* row operation on I . E^{-1} will be obtained by performing the row operation which would carry E back to I .

- If E is obtained by switching rows i and j , then E^{-1} is also obtained by switching rows i and j .
- If E is obtained by multiplying row i by the scalar k , then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.
- If E is obtained by adding k times row i to row j , then E^{-1} is obtained by subtracting k times row i from row j .

Consider the following example.

Example 2.52: Inverse of an Elementary Matrix

Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Find E^{-1} .

Solution. Consider the elementary matrix E given by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Here, E is obtained from the 2×2 identity matrix by multiplying the second row by 2. In order to carry E back to the identity, we need to multiply the second row of E by $\frac{1}{2}$. Hence, E^{-1} is given by

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

We can verify that $EE^{-1} = I$. Take the product EE^{-1} , given by

$$EE^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This equals I so we know that we have computed E^{-1} properly. ♠

Suppose an $m \times n$ matrix A is row reduced to its reduced row-echelon form. By tracking each row

operation completed, this row reduction can be completed through multiplication by elementary matrices. Consider the following definition.

Definition 2.53: The Form $B = UA$

Let A be an $m \times n$ matrix and let B be the reduced row-echelon form of A . Then we can write $B = UA$ where U is the product of all elementary matrices representing the row operations done to A to obtain B .

Consider the following example.

Example 2.54: The Form $B = UA$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Find B , the reduced row-echelon form of A and write it in the form $B = UA$.

Solution. To find B , row reduce A . For each step, we will record the appropriate elementary matrix. First, switch rows 1 and 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

The resulting matrix is equivalent to finding the product of $P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and A .

Next, add (-2) times row 1 to row 3.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This is equivalent to multiplying by the matrix $E(-2 \times 1 + 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Notice that the resulting matrix is B , the required reduced row-echelon form of A .

We can then write

$$\begin{aligned} B &= E(-2 \times 1 + 3)(P^{12}A) \\ &= (E(-2 \times 1 + 3)P^{12})A \\ &= UA \end{aligned}$$

It remains to find the matrix U .

$$U = E(-2 \times 1 + 3)P^{12}$$

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$$\begin{aligned} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{aligned}$$

We can verify that $B = UA$ holds for this matrix U :

$$\begin{aligned} UA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= B \end{aligned}$$



While the process used in the above example is reliable and simple when only a few row operations are used, it becomes cumbersome in a case where many row operations are needed to carry A to B . The following theorem provides an alternate way to find the matrix U .

Theorem 2.55: Finding the Matrix U

Let A be an $m \times n$ matrix and let B be its reduced row-echelon form. Then $B = UA$ where U is an invertible $m \times m$ matrix found by forming the matrix $[A|I_m]$ and row reducing to $[B|U]$.

Let's revisit the above example using the process outlined in Theorem 2.55.

Example 2.56: The Form $B = UA$, Revisited

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Using the process outlined in Theorem 2.55, find U such that $B = UA$.

Solution. First, set up the matrix $[A|I_3]$.

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right]$$

Now, row reduce this matrix until the left side equals the reduced row-echelon form of A .

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array} \right]$$

The left side of this matrix is B , and the right side is U . Comparing this to the matrix U found above in Example 2.54, you can see that the same matrix is obtained regardless of which process is used. ♠

Recall from Algorithm 2.37 that an $n \times n$ matrix A is invertible if and only if A can be carried to the $n \times n$ identity matrix using the usual row operations. This leads to an important consequence related to the above discussion.

Suppose A is an $n \times n$ invertible matrix. Then, set up the matrix $[A|I_n]$ as done above, and row reduce until it is of the form $[B|U]$. In this case, $B = I_n$ because A is invertible.

$$\begin{aligned} B &= UA \\ I_n &= UA \\ U^{-1} &= A \end{aligned}$$

Now suppose that $U = E_1 E_2 \cdots E_k$ where each E_i is an elementary matrix representing a row operation used to carry A to I . Then,

$$U^{-1} = (E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}$$

Remember that if E_i is an elementary matrix, so too is E_i^{-1} . It follows that

$$\begin{aligned} A &= U^{-1} \\ &= E_k^{-1} \cdots E_2^{-1} E_1^{-1} \end{aligned}$$

and A can be written as a product of elementary matrices.

Theorem 2.57: Product of Elementary Matrices

Let A be an $n \times n$ matrix. Then A is invertible if and only if it can be written as a product of elementary matrices.

Consider the following example.

Example 2.58: Product of Elementary Matrices

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. Write A as a product of elementary matrices.

Solution. We will use the process outlined in Theorem 2.55 to write A as a product of elementary matrices. We will set up the matrix $[A|I]$ and row reduce, recording each row operation as an elementary matrix.

First:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right]$$

represented by the elementary matrix $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Secondly:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right]$$

represented by the elementary matrix $E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Finally:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

represented by the elementary matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Notice that the reduced row-echelon form of A is I . Hence $I = UA$ where U is the product of the above elementary matrices. It follows that $A = U^{-1}$. Since we want to write A as a product of elementary matrices, we wish to express U^{-1} as a product of elementary matrices.

$$\begin{aligned} U^{-1} &= (E_3 E_2 E_1)^{-1} \\ &= E_1^{-1} E_2^{-1} E_3^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &= A \end{aligned}$$

This gives A written as a product of elementary matrices. By Theorem 2.57 it follows that A is invertible. ♠

2.1.10 More on Matrix Inverses

In this section, we will prove three theorems which will clarify the concept of matrix inverses. In order to do this, first recall some important properties of elementary matrices.

Recall that an elementary matrix is a square matrix obtained by performing an elementary operation on an identity matrix. Each elementary matrix is invertible, and its inverse is also an elementary matrix. If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the product EA is the result of applying to A the same elementary row operation that was applied to the $m \times m$ identity matrix in order to obtain E .

Let R be the reduced row-echelon form of an $m \times n$ matrix A . R is obtained by iteratively applying

a sequence of elementary row operations to A . Denote by E_1, E_2, \dots, E_k the elementary matrices associated with the elementary row operations which were applied, in order, to the matrix A to obtain the resulting R . We then have that $R = (E_k \cdots (E_2(E_1A))) = E_k \cdots E_2 E_1 A$. Let E denote the product matrix $E_k \cdots E_2 E_1$ so that we can write $R = EA$ where E is an invertible matrix whose inverse is the product $(E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}$.

Now, we will consider some preliminary lemmas.

Lemma 2.59: Invertible Matrix and Zeros

Suppose that A and B are matrices such that the product AB is an identity matrix. Then the reduced row-echelon form of A does not have a row of zeros.

Proof. Let R be the reduced row-echelon form of A . Then $R = EA$ for some invertible square matrix E as described above. By hypothesis $AB = I$ where I is an identity matrix, so we have a chain of equalities

$$R(BE^{-1}) = (EA)(BE^{-1}) = E(AB)E^{-1} = EIE^{-1} = EE^{-1} = I$$

If R would have a row of zeros, then so would the product $R(BE^{-1})$. But since the identity matrix I does not have a row of zeros, neither can R have one. ♠

We now consider a second important lemma.

Lemma 2.60: Size of Invertible Matrix

Suppose that A and B are matrices such that the product AB is an identity matrix. Then A has at least as many columns as it has rows.

Proof. Let R be the reduced row-echelon form of A . By Lemma 2.59, we know that R does not have a row of zeros, and therefore each row of R has a leading 1. Since each column of R contains at most one of these leading 1s, R must have at least as many columns as it has rows. ♠

An important theorem follows from this lemma.

Theorem 2.61: Invertible Matrices are Square

Only square matrices can be invertible.

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Proof. Suppose that A and B are matrices such that both products AB and BA are identity matrices. We will show that A and B must be square matrices of the same size. Let the matrix A have m rows and n columns, so that A is an $m \times n$ matrix. Since the product AB exists, B must have n rows, and since the product BA exists, B must have m columns so that B is an $n \times m$ matrix. To finish the proof, we need only verify that $m = n$.

We first apply Lemma 2.60 with A and B , to obtain the inequality $m \leq n$. We then apply Lemma 2.60 again (switching the order of the matrices), to obtain the inequality $n \leq m$. It follows that $m = n$, as we wanted. ♠

Of course, not all square matrices are invertible. In particular, zero matrices are not invertible, along with many other square matrices.

The following proposition will be useful in proving the next theorem.

Proposition 2.62: Reduced Row-Echelon Form of a Square Matrix

If R is the reduced row-echelon form of a square matrix, then either R has a row of zeros or R is an identity matrix.

The proof of this proposition is left as an exercise to the reader. We now consider the second important theorem of this section.

Theorem 2.63: Unique Inverse of a Matrix

Suppose A and B are square matrices such that $AB = I$ where I is an identity matrix. Then it follows that $BA = I$. Further, both A and B are invertible and $B = A^{-1}$ and $A = B^{-1}$.

Proof. Let R be the reduced row-echelon form of a square matrix A . Then, $R = EA$ where E is an invertible matrix. Since $AB = I$, Lemma 2.59 gives us that R does not have a row of zeros. By noting that R is a square matrix and applying Proposition 2.62, we see that $R = I$. Hence, $EA = I$.

Using both that $EA = I$ and $AB = I$, we can finish the proof with a chain of equalities as given by

$$\begin{aligned}
BA = IBA &= (EA)B(E^{-1}E)A \\
&= E(AB)E^{-1}(EA) \\
&= EIE^{-1}I \\
&= EE^{-1} = I
\end{aligned}$$

It follows from the definition of the inverse of a matrix that $B = A^{-1}$ and $A = B^{-1}$. ♠

This theorem is very useful, since with it we need only test one of the products AB or BA in order to check that B is the inverse of A . The hypothesis that A and B are square matrices is very important, and without this the theorem does not hold.

We will now consider an example.

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Example 2.64: Non Square Matrices

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Show that $A^T A = I$ but $AA^T \neq I$.

Solution. Consider the product $A^T A$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $A^T A = I_2$, where I_2 is the 2×2 identity matrix. However, the product AA^T is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence AA^T is not the 3×3 identity matrix. This shows that for Theorem 2.63, it is essential that both matrices be square and of the same size. ♠

Is it possible to have matrices A and B such that $AB = I$, while $BA = 0$? This question is left to the reader to answer, and you should take a moment to consider the answer.

We conclude this section with an important theorem.

Theorem 2.65: The Reduced Row-Echelon Form of an Invertible Matrix

For any matrix A the following conditions are equivalent:

- A is invertible
- The reduced row-echelon form of A is an identity matrix

Proof. In order to prove this, we show that for any given matrix A , each condition implies the other. We first show that if A is invertible, then its reduced row-echelon form is an identity matrix, then we show that if the reduced row-echelon form of A is an identity matrix, then A is invertible.

If A is invertible, there is some matrix B such that $AB = I$. By Lemma 2.59, we get that the reduced row-echelon form of A does not have a row of zeros. Then by Theorem 2.61, it follows that A and the reduced row-echelon form of A are square matrices. Finally, by Proposition 2.62, this reduced row-echelon form of A must be an identity matrix. This proves the first implication.

Now suppose the reduced row-echelon form of A is an identity matrix I . Then $I = EA$ for some product E of elementary matrices. By Theorem 2.63, we can conclude that A is invertible. ♠

Theorem 2.65 corresponds to Algorithm 2.37, which claims that A^{-1} is found by row reducing the augmented matrix $[A|I]$ to the form $[I|A^{-1}]$. This will be a matrix product $E[A|I]$ where E is a product of elementary matrices. By the rules of matrix multiplication, we have that $E[A|I] = [EA|EI] = [EA|E]$.

It follows that the reduced row-echelon form of $[A|I]$ is $[EA|E]$, where EA gives the reduced row-echelon form of A . By Theorem 2.65, if $EA \neq I$, then A is not invertible, and if $EA = I$, A is invertible. If $EA = I$, then by Theorem 2.63, $E = A^{-1}$. This proves that Algorithm 2.37 does in fact find A^{-1} .

Exercises

Exercise 2.1.1 For the following pairs of matrices, determine if the sum $A + B$ is defined. If so, find the sum.

$$(a) A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(b) A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 4 \end{bmatrix}$$

Exercise 2.1.2 For each matrix A , find the matrix $-A$ such that $A + (-A) = 0$.

$$(a) A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(b) A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$$

Exercise 2.1.3 In the context of Proposition 2.7, describe $-A$ and 0 .

Exercise 2.1.4 For each matrix A , find the product $(-2)A$, $0A$, and $3A$.

$$(a) A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(b) A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$$