

Worksheet 9 – Solution

1) Consider the following second-order system transfer functions. Determine the corner frequency and the damping ratio of each system and match the transfer functions with the given Bode plots. Draw the asymptote lines of the magnitude plots and indicate the slope.

a) $G(s) = \frac{1}{s^2 + 1.8s + 1}$

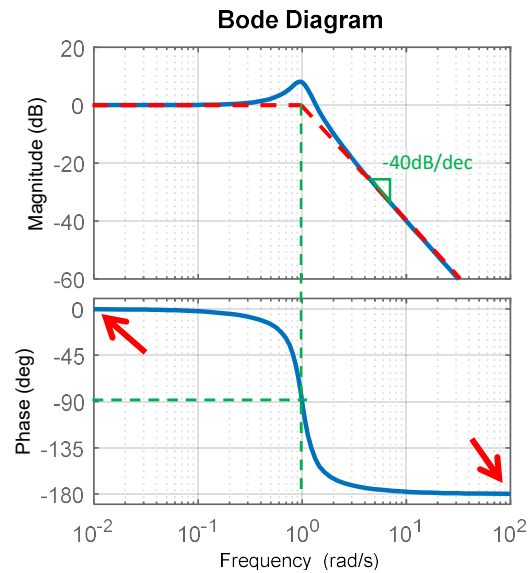
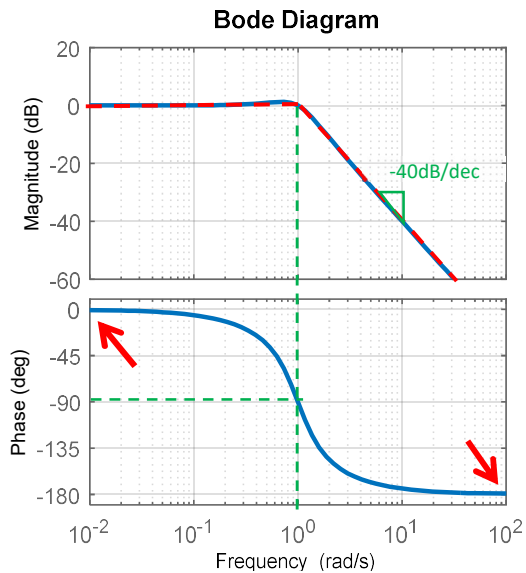
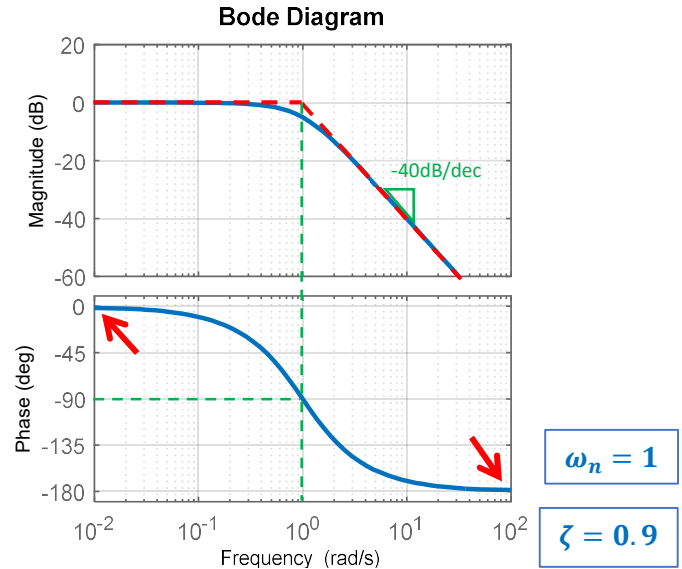
$$\begin{cases} \omega_n^2 = 1 \rightarrow \omega_n = 1 \\ 2\zeta\omega_n = 1.8 \rightarrow \zeta = \frac{1.8}{2\omega_n} = \frac{1.8}{2(1)} = 0.9 \end{cases}$$

b) $G(s) = \frac{1}{s^2 + 0.4s + 1}$

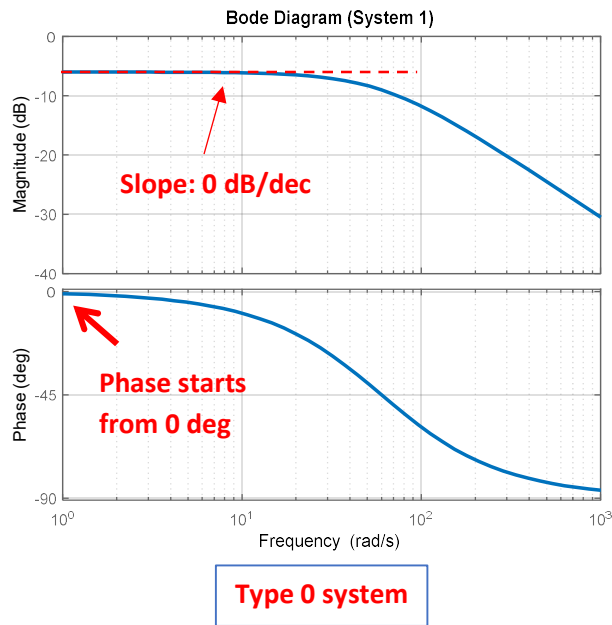
$$\begin{cases} \omega_n^2 = 1 \rightarrow \omega_n = 1 \\ 2\zeta\omega_n = 0.4 \rightarrow \zeta = \frac{0.4}{2\omega_n} = \frac{0.4}{2(1)} = 0.2 \end{cases}$$

c) $G(s) = \frac{1}{s^2 + s + 1}$

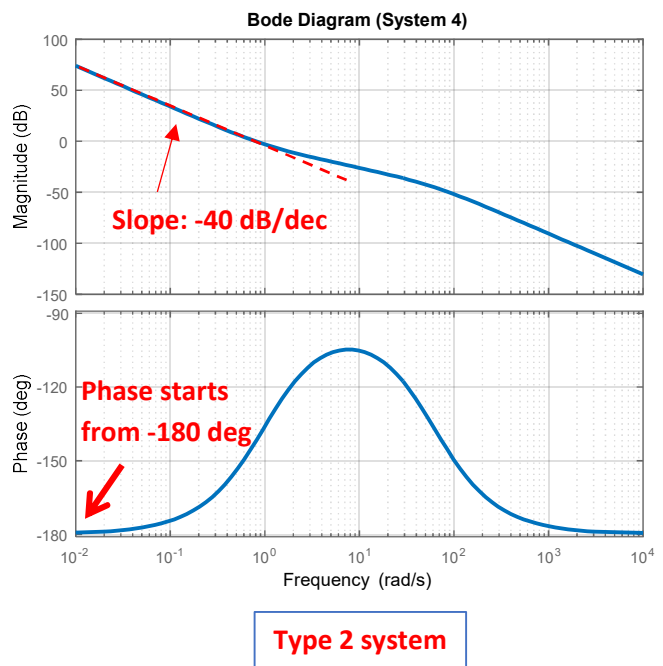
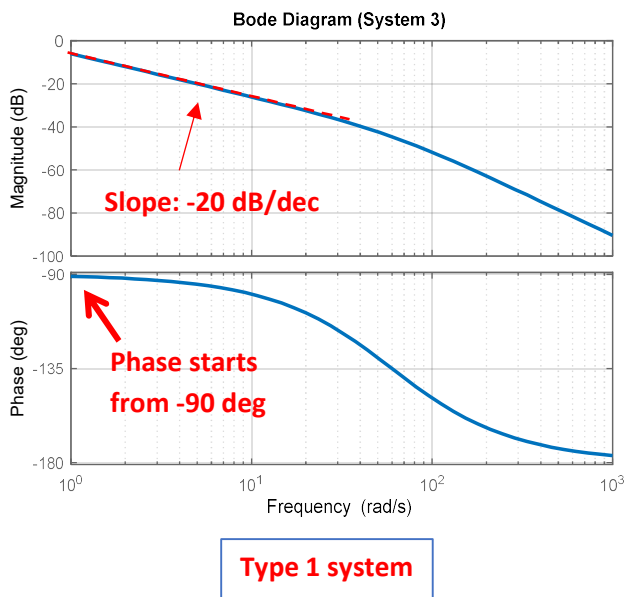
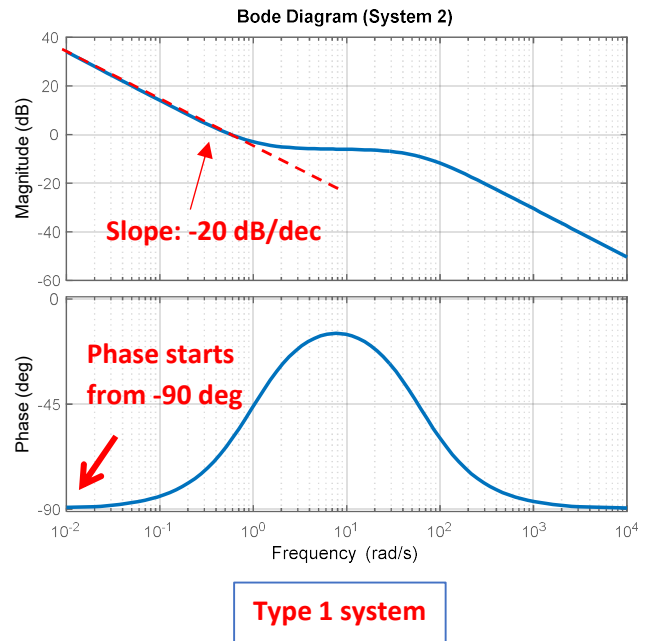
$$\begin{cases} \omega_n^2 = 1 \rightarrow \omega_n = 1 \\ 2\zeta\omega_n = 1 \rightarrow \zeta = \frac{1}{2\omega_n} = \frac{1}{2(1)} = 0.5 \end{cases}$$



2) Determine the type of the following systems from the given Bode diagrams.



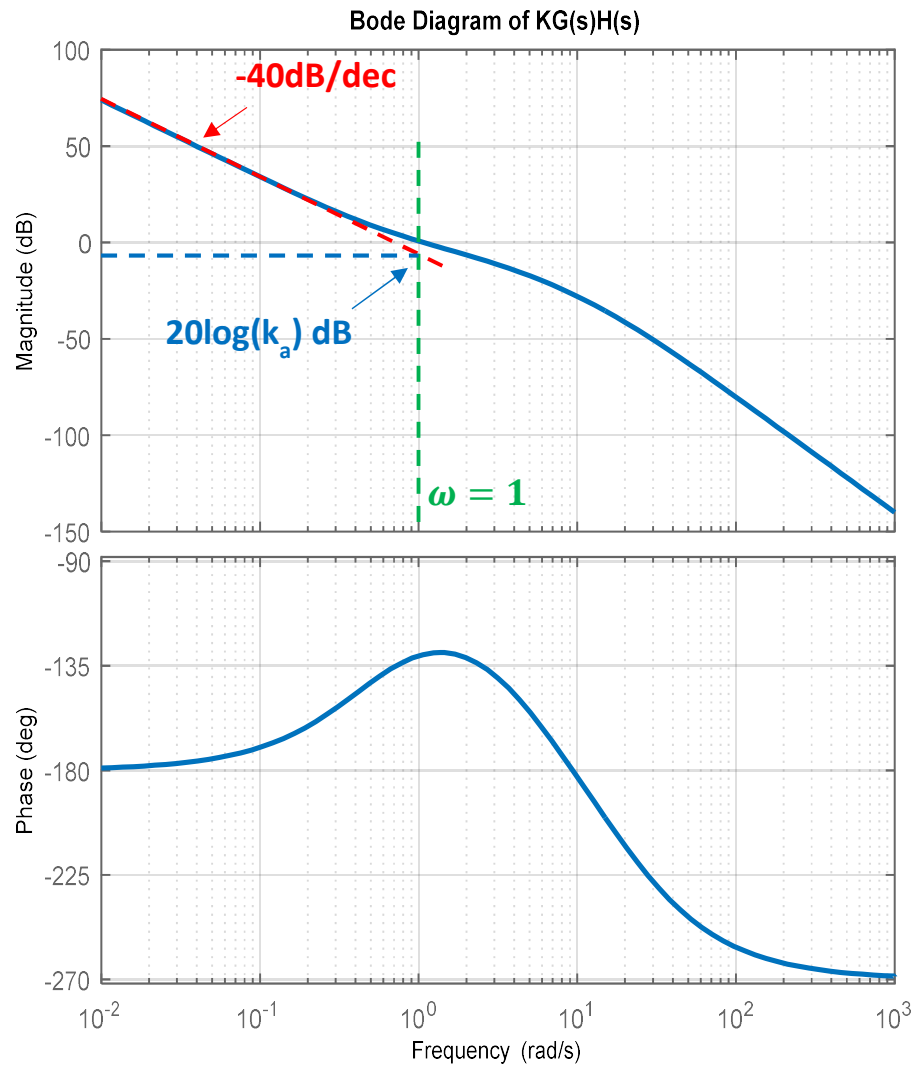
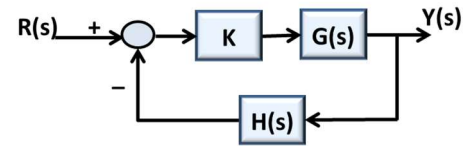
We can determine type of a system by checking the slope of Bode magnitude plot at low frequencies and starting phase angle at low frequencies.



3) Consider the Bode diagram of the open-loop system $KG(s)H(s)$.

- a) Determine Type of the open-loop system, corresponding error constant and steady-state error of the closed-loop system.

b) What is the steady-state error of the step-response?



The log magnitude plot starts at low frequencies with a slope of -40dB/dec

The system is Type 2

The parabolic-error constant:

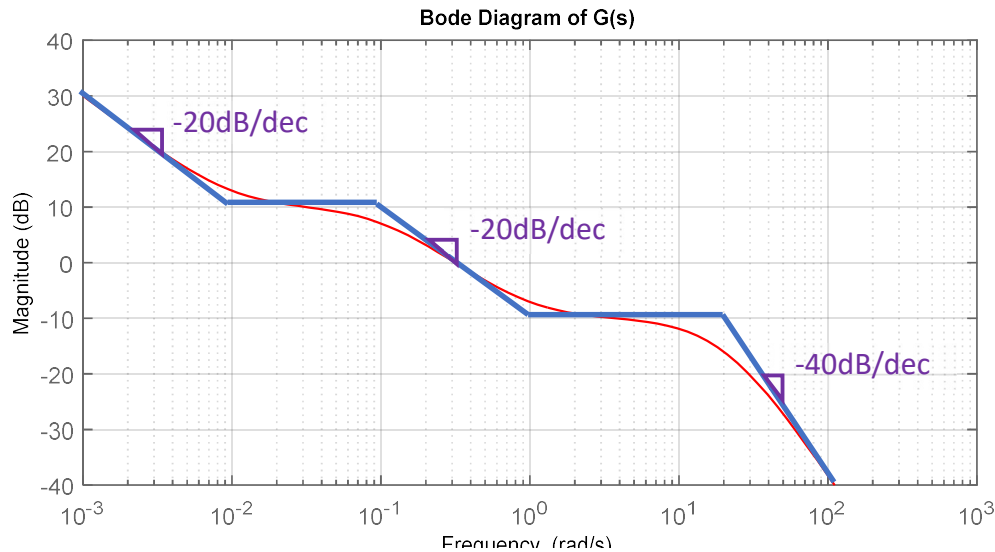
$$20\log(k_a)\text{dB} = -6\text{dB} \quad k_a = 10^{-6/20} \rightarrow k_a = 0.5$$

The steady-state error of parabolic-response:

$$e_{ss} = \frac{1}{k_a} = \frac{1}{0.5} \rightarrow e_{ss} = 2$$

Steady-state error of the step-response is **zero**.

4) Determine the transfer function $G(s)$ from the following Bode magnitude plot.



First, find the corner frequencies and the corresponding basic factor for each:

An integrator $\rightarrow \frac{1}{j\omega}$

A single zero at $\omega = 0.01 \rightarrow j\frac{\omega}{0.01} + 1$

A single pole at $\omega = 0.1 \rightarrow \frac{1}{j\frac{\omega}{0.1} + 1}$

A single zero at $\omega = 1 \rightarrow j\omega + 1$

Second-order pole at $\omega = 20 \rightarrow \frac{1}{\left(j\frac{\omega}{20} + 1\right)^2}$

Next, find the DC-gain from the starting point value:

The starting point at $\omega = 0.001 \text{ rad/sec}$ is 30dB

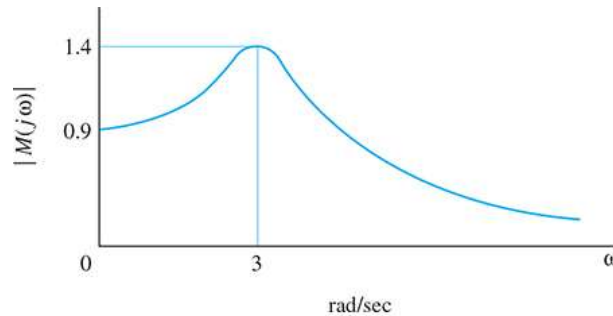
$$30\text{dB} = 20\log_{10} \left| \frac{K_B}{j0.001} \right| = 20\log_{10}|K_B| - 20\log_{10}|j0.001|$$

$$20\log_{10}|K_B| = 30\text{dB} - 60\text{dB} = -30\text{dB} \rightarrow K_B = 10^{-\frac{30}{20}} = 0.0316$$

Form the frequency response function $G(j\omega)$, then convert it to transfer function $G(s)$ by $s = j\omega$

$$G(j\omega) = 0.0316 \frac{\left(j\frac{\omega}{0.01} + 1\right)(j\omega + 1)}{j\omega \left(j\frac{\omega}{0.1} + 1\right) \left(j\frac{\omega}{20} + 1\right)^2} \rightarrow \boxed{G(s) = \frac{126.4(s + 0.01)(s + 1)}{s(s + 0.1)(s + 20)^2}}$$

5) The closed-loop frequency response magnitude graph of a second-order system is shown below. Sketch the corresponding unit-step response of the system, indicate the values of the maximum overshoot, peak-time, and the steady-state error due to a unit-step input.



We can determine the resonant peak M_r , DC-gain K_{dc} , and the resonance frequency ω_r from the given Bode magnitude graph:

$$M_r = 1.4, \quad K_{dc} = 0.9 \quad \text{and} \quad \omega_r = 3 \text{ rad/s}$$

Find the damping ratio from the resonant peak:

$$\frac{M_r}{K_{dc}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \rightarrow \frac{1.4}{0.9} = 1.56 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \rightarrow \zeta = 0.346$$

Determine the undamped natural frequency from resonance frequency:

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \rightarrow 3 = \omega_n\sqrt{1-2(0.346)^2} \rightarrow \omega_n = 3.44 \text{ rad/s}$$

Find the maximum overshoot and the peak-time:

$$O.S. = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \rightarrow O.S. = e^{-\pi(0.346)/\sqrt{1-(0.346)^2}} = 0.3139 \rightarrow O.S.\% = 31.39\%$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \rightarrow t_p = \frac{\pi}{3.44\sqrt{1-(0.346)^2}} = 0.97 \text{ sec}$$

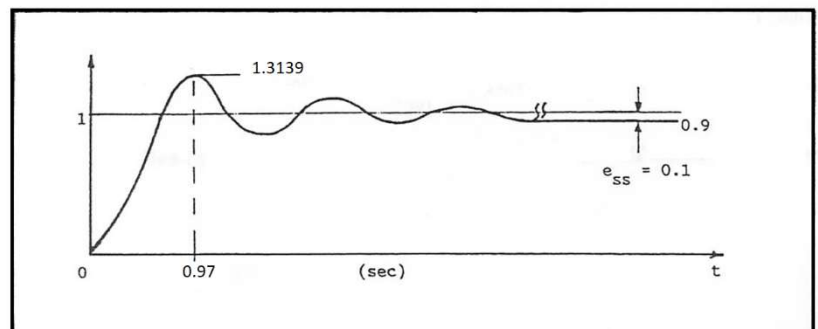
Find the steady-state error:

From the closed-loop system's Bode magnitude graph magnitude at $\omega = 0$ is 0.9. This indicates that the DC-gain of the closed-loop system or the steady-state value of the unit-step response.

Therefore, the steady-state error for unit-step input is:

$$e_{ss} = 1 - 0.9 = 0.1$$

A typical unit-step response graph can be:



6) The forward-path transfer function of a unity-feedback control system is

$$G(s) = \frac{K}{s(s + 6.54)}$$

Analytically, find the resonance peak M_r , resonant frequency ω_r , and bandwidth BW of the closed-loop system for the following values of K :

a) $K = 5$

First, find the closed-loop system transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K}{s^2 + 6.54s + K} = \frac{5}{s^2 + 6.54s + 5}$$

Find the damping ratio and the undamped natural frequency:

$$\omega_n^2 = 5 \rightarrow \omega_n = \sqrt{5} \text{ rad/s}$$

$$2\zeta\omega_n = 6.54 \rightarrow \zeta = \frac{6.54}{4.47} = 1.46$$

Determine the resonant peak, resonance frequency and the bandwidth:

Since $\zeta > 1$, closed-loop system is **overdamped**. Therefore, there is no resonant peak and resonant frequency for this system.

Bandwidth is the frequency, where the magnitude is $\frac{1}{\sqrt{2}} = 0.707$.

$$T(j\omega) = \frac{K}{(j\omega)^2 + 6.54(j\omega) + K} = \frac{K}{(K - \omega^2) + j(6.54\omega)} \rightarrow |T(j\omega)| = \frac{|K|}{|(K - \omega^2) + j(6.54\omega)|}$$

$$|T(j\omega)| = \frac{|K|}{\sqrt{(K - \omega^2)^2 + (6.54\omega)^2}} \rightarrow \frac{1}{\sqrt{2}} = \frac{|K|}{\sqrt{(K - \omega^2)^2 + (6.54\omega)^2}}$$

$$(K - \omega^2)^2 + (6.54\omega)^2 = 2K^2 \rightarrow \omega^4 + (6.54^2 - 2K)\omega^2 - K^2 = 0$$

$$\text{For } K = 5 \rightarrow \omega^4 + 32.77\omega^2 - 25 = 0 \rightarrow \omega = 0.864 \frac{\text{rad}}{\text{s}} \quad \text{Bandwidth}$$

b) $K = 21.38$

First, find the closed-loop system transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K}{s^2 + 6.54s + K} = \frac{21.38}{s^2 + 6.54s + 21.38}$$

Find the damping ratio and the undamped natural frequency:

$$\omega_n^2 = 21.38 \rightarrow \omega_n = \sqrt{21.38} \text{ rad/s}$$

$$2\zeta\omega_n = 6.54 \rightarrow \zeta = \frac{6.54}{9.25} = 0.707$$

Determine the resonant peak, resonance frequency and the bandwidth:

Since $\zeta < 1$, closed-loop system is **underdamped**. Since $\zeta > 0.5$ there is no resonant peak and resonant frequency for this system.

Bandwidth is the frequency, where the magnitude is $\frac{1}{\sqrt{2}} = 0.707$.

$$T(j\omega) = \frac{K}{(j\omega)^2 + 6.54(j\omega) + K} = \frac{K}{(K - \omega^2) + j(6.54\omega)} \rightarrow |T(j\omega)| = \frac{|K|}{|(K - \omega^2) + j(6.54\omega)|}$$

$$|T(j\omega)| = \frac{|K|}{\sqrt{(K - \omega^2)^2 + (6.54\omega)^2}} \rightarrow \frac{1}{\sqrt{2}} = \frac{|K|}{\sqrt{(K - \omega^2)^2 + (6.54\omega)^2}}$$

$$(K - \omega^2)^2 + (6.54\omega)^2 = 2K^2 \rightarrow \omega^4 + (6.54^2 - 2K)\omega^2 - K^2 = 0$$

For $K = 21.38 \rightarrow \omega^4 + 0.0116\omega^2 - 457.1 = 0 \rightarrow \omega = 4.62 \frac{\text{rad}}{\text{s}}$ Bandwidth

c) $K = 100$

First, find the closed-loop system transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K}{s^2 + 6.54s + K} = \frac{100}{s^2 + 6.54s + 100}$$

Find the damping ratio and the undamped natural frequency:

$$\omega_n^2 = 100 \rightarrow \omega_n = 10 \text{ rad/s}$$

$$2\zeta\omega_n = 6.54 \rightarrow \zeta = \frac{6.54}{20} = 0.327$$

Determine the resonant peak, resonance frequency and the bandwidth:

Since $\zeta < 1$, closed-loop system is underdamped. Since $\zeta < 0.5$ there is a resonant peak and resonant frequency for this system.

$$M_r = K_{dc} \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \rightarrow M_r = \frac{1}{2(0.327)\sqrt{1 - 0.327^2}} \rightarrow M_r = 1.618$$

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2} \rightarrow \omega_r = 10\sqrt{1 - 2(0.327)^2} \rightarrow \omega_r = 8.87 \text{ rad/s}$$

Bandwidth is the frequency, where the magnitude is $\frac{1}{\sqrt{2}} = 0.707$.

$$T(j\omega) = \frac{K}{(j\omega)^2 + 6.54(j\omega) + K} = \frac{K}{(K - \omega^2) + j(6.54\omega)} \rightarrow |T(j\omega)| = \frac{|K|}{|(K - \omega^2) + j(6.54\omega)|}$$

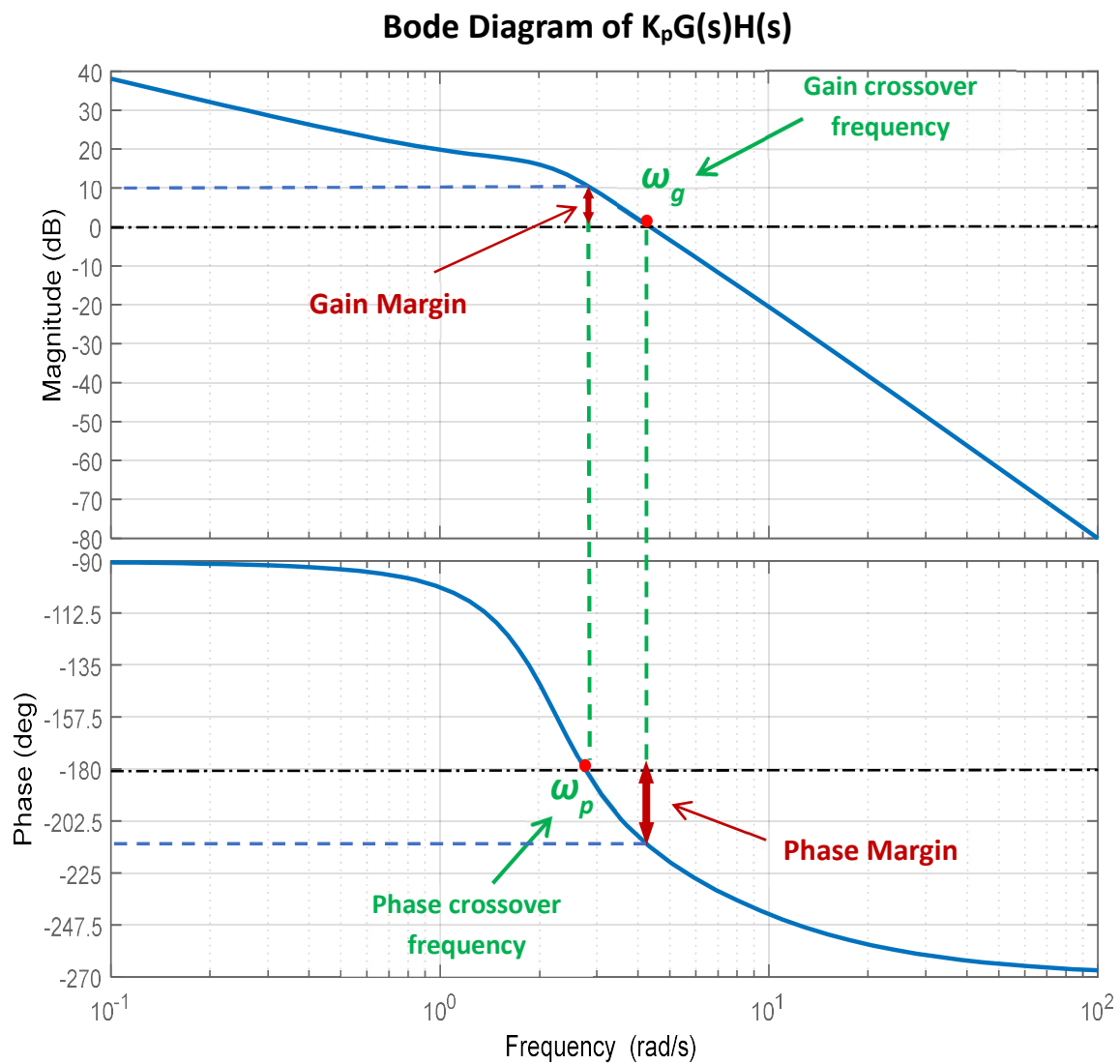
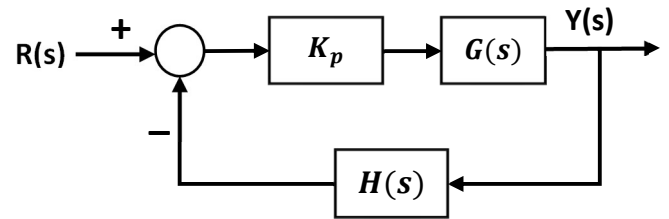
$$|T(j\omega)| = \frac{|K|}{\sqrt{(K - \omega^2)^2 + (6.54\omega)^2}} \rightarrow \frac{1}{\sqrt{2}} = \frac{|K|}{\sqrt{(K - \omega^2)^2 + (6.54\omega)^2}}$$

$$(K - \omega^2)^2 + (6.54\omega)^2 = 2K^2 \rightarrow \omega^4 + (6.54^2 - 2K)\omega^2 - K^2 = 0$$

For $K = 100 \rightarrow \omega^4 - 157.23\omega^2 - 10000 = 0 \rightarrow \omega = 14.35 \frac{\text{rad}}{\text{s}}$ Bandwidth

7) Consider the following closed-loop system with proportional control. Given the open-loop system Bode diagram, $K_p G(s)H(s)$.

- Find the gain crossover frequency ω_g and the phase crossover frequency ω_p and mark them on the Bode plot.
- Find the gain margin (GM) and phase margin (PM) and mark them on the Bode plot.
- Determine stability of the closed-loop system.
- Determine type of the open-loop system.
- What is the steady-state error of the step response of the closed-loop system?



From the Bode plot the crossover frequencies can be determined as

$$\omega_g \approx 4.5 \text{ rad/sec}$$

$$\omega_p \approx 2.9 \text{ rad/sec}$$

The gain margin and phase margin are obtained as

$$GM = 0\text{dB} - 10\text{dB} = -10\text{dB}$$

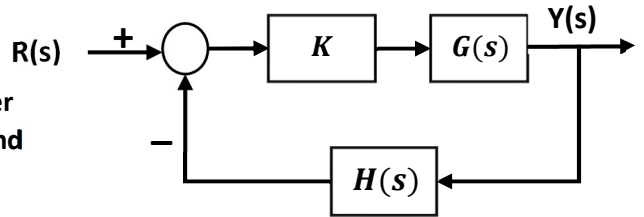
$$PM = 180^\circ + (-215^\circ) = -35^\circ$$

Since, $PM < 0$ and $GM < 0$, the closed-loop system is unstable.

Since, the slope of the log magnitude plot at low frequencies starts with -20dB/dec, and the phase plot starts at -90 degree, the open-loop transfer function is **Type 1**.

Since, the open-loop system is type 1, it has an integrator term ($\frac{1}{s}$) or pole at $s = 0$, therefore; step response of the closed-loop system is **zero**.

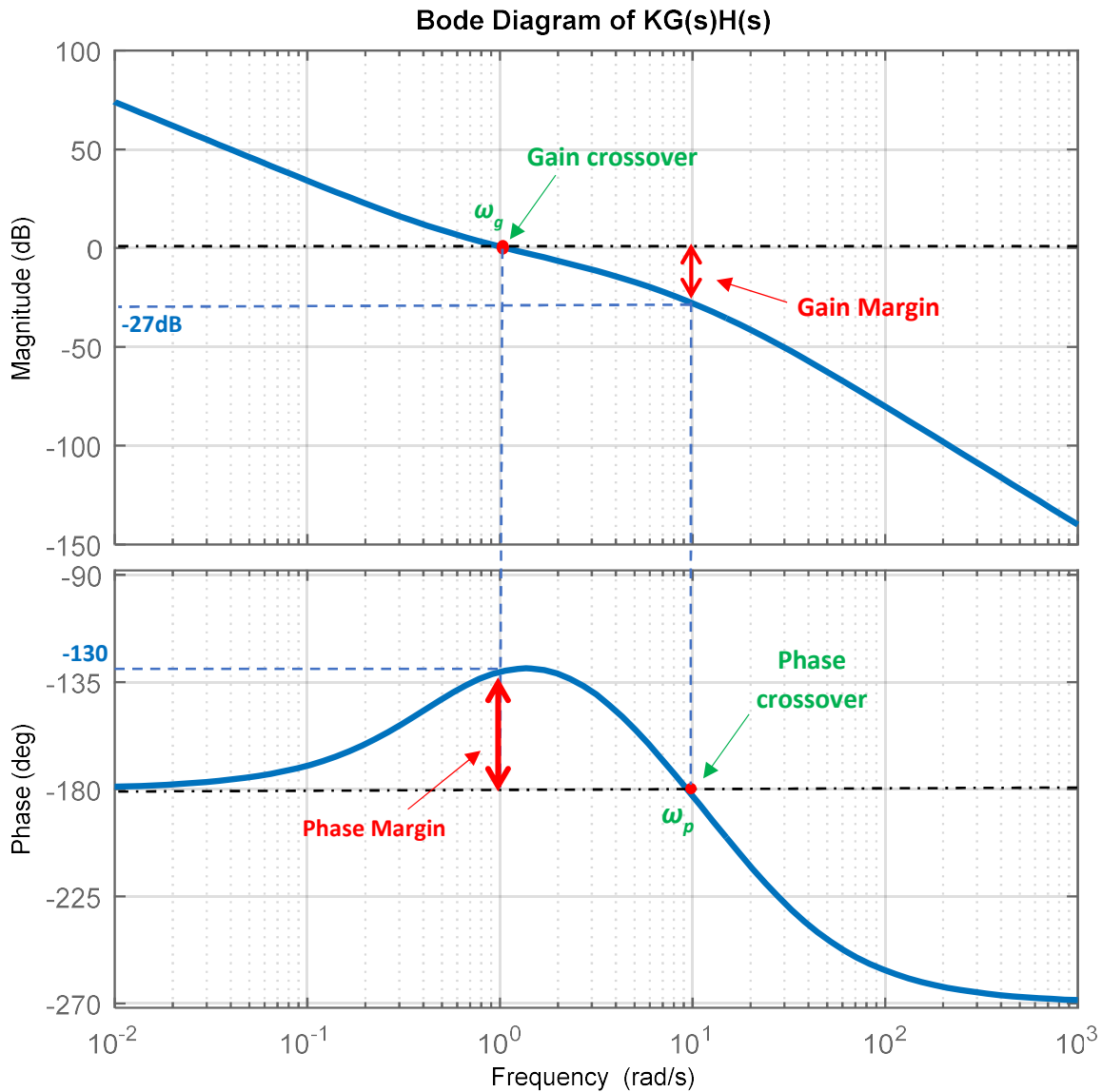
8) Consider Bode diagram of open-loop system $KG(s)H(s)$



a) Determine gain crossover frequency ω_g , phase crossover frequency ω_p , gain margin (GM) and phase margin (PM) and mark them on the graph.

b) Is the closed-loop system stable?

c) Determine type of the open-loop system.



From the Bode plot the crossover frequencies can be determined as

$$\omega_g = 1.08 \text{ rad/sec}$$

$$\omega_p = 9.35 \text{ rad/sec}$$

The gain margin and phase margin are obtained as

$$GM = 0\text{dB} - (-27\text{dB}) = 27\text{dB}$$

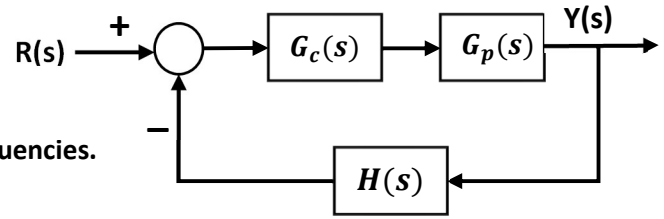
$$PM = 180^\circ + (-130^\circ) = 50^\circ$$

Since, $PM > 0$ and $GM > 0$, the closed-loop system is stable.

Since, the slope of the log magnitude plot at low frequencies starts with -40dB/dec, and the phase plot starts at -180 degree, the open-loop transfer function is **Type 2**.

9) Consider the following closed-loop system.

- Find the open-loop transfer function.
- Plot the open-loop system Bode diagram.
- Determine the gain crossover and phase crossover frequencies.
- Find the gain margin (GM) and phase margin (PM).
- Is the closed-loop system stable?



$$G_c(s) = \frac{2}{s}, \quad G_p(s) = \frac{1}{s+2}, \quad H(s) = \frac{1}{0.01s+1}$$

a) The open-loop transfer function is

$$G_c(s)G_p(s)H(s) = \left(\frac{2}{s}\right)\left(\frac{1}{s+2}\right)\left(\frac{1}{0.01s+1}\right)$$

b) Find the frequency response function, basic factors, and corner frequencies

$$G_c(s)G_p(s)H(s) = \left(\frac{2}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{0.5s+1}\right)\left(\frac{1}{0.01s+1}\right)$$

$$G_c(j\omega)G_p(j\omega)H(j\omega) = \left(\frac{1}{j\omega}\right)\left(\frac{1}{0.5j\omega+1}\right)\left(\frac{1}{0.01j\omega+1}\right)$$

The basic factors are:

- 1) first-order integrator
- 2) first-order pole with corner frequency of $\omega_c = 2$
- 3) first-order pole with corner frequency of $\omega_c = 100$

$$\text{Starting Slope: } -20\beta \frac{dB}{dec} = -20(1) \frac{dB}{dec} = -20 \frac{dB}{dec}$$

$$\text{Starting Point: } 20 \log \left| \frac{K_B}{(j\omega)^\beta} \right| = 20 \log \left| \frac{1}{(j0.01)^1} \right| = 40 \text{ dB}$$

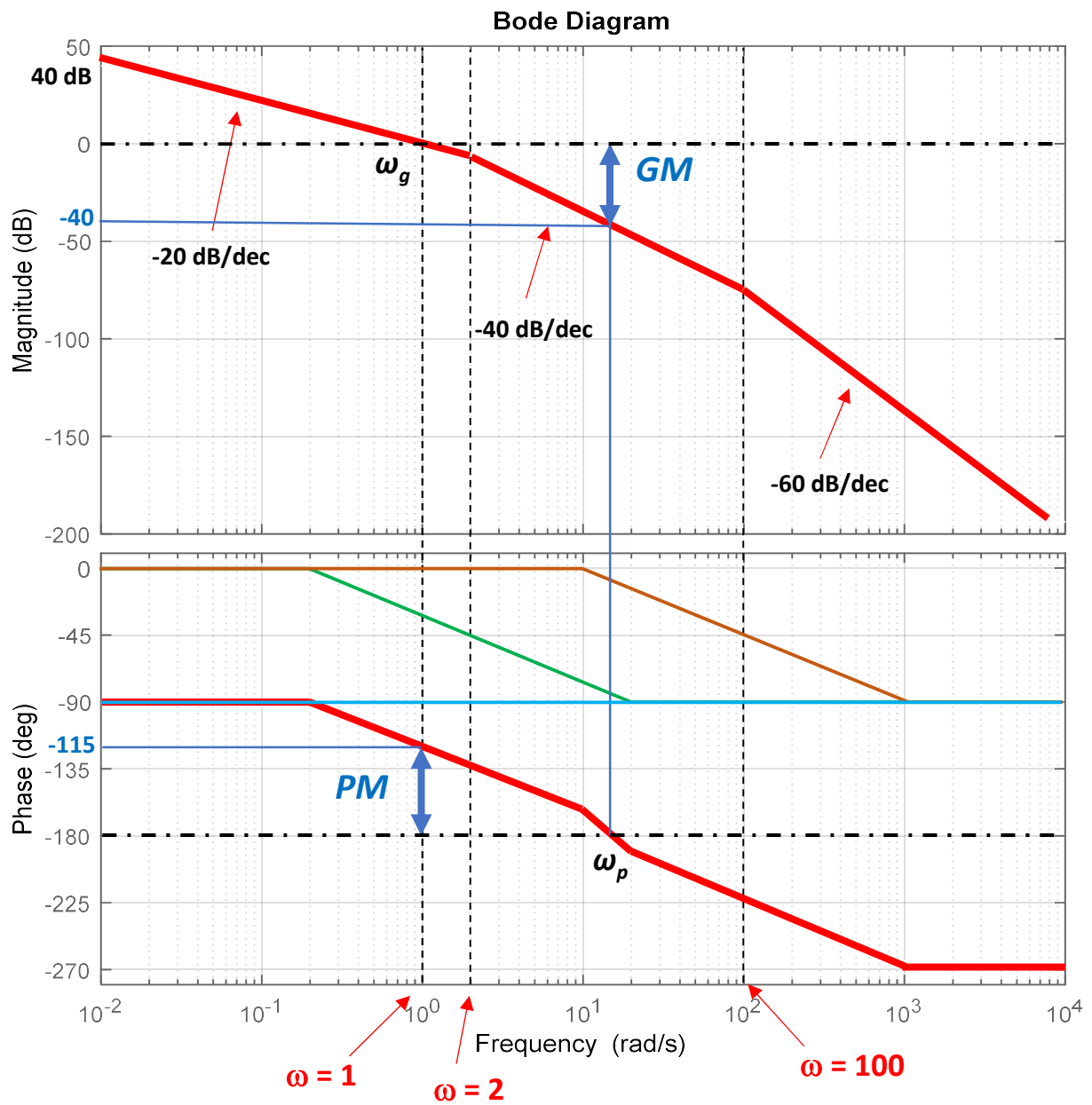
c) The **gain crossover frequency at $\omega_g = 1 \text{ rad/sec}$** and the **phase crossover frequency at $\omega_p = 14 \text{ rad/sec}$** are obtained from the graph.

d) The gain margin and phase margin are obtained as:

$$GM = 0\text{dB} - (-40\text{dB}) = 40\text{dB}$$

$$PM = 180^\circ + (-115^\circ) = 65^\circ$$

e) Since, **PM > 0** and **GM > 0**, the closed-loop system is **stable**.



10) The forward path transfer function of a system with an integral control $G_c(s) = \frac{K}{s}$ is

$$G(s) = \frac{1}{10s + 1}$$

a) Find K when the closed-loop resonance peak is 1.4.

The closed loop transfer function is:

$$\frac{Y(s)}{X(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{K}{10s^2 + s + K} = \frac{K}{s^2 + 0.1s + 0.1K}$$

The resonant peak:

$$M_r = K_{dc} \frac{1}{2\zeta\sqrt{1-\zeta^2}} \rightarrow 1.4 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \rightarrow \zeta = 0.387$$

According to the closed-loop transfer function:

$$2\zeta\omega_n = 0.1 \rightarrow \omega_n = \frac{0.1}{2(0.387)} = 0.129 \text{ rad/s}$$

$$\omega_n^2 = 0.1K \rightarrow K = \frac{(0.129)^2}{0.1} = 0.166$$

b) Determine the frequency at resonance and overshoot for step input.

The resonant peak is:

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \rightarrow \omega_r = 0.129\sqrt{1-2(0.387)^2} \rightarrow \omega_r = 0.108 \text{ rad/s}$$

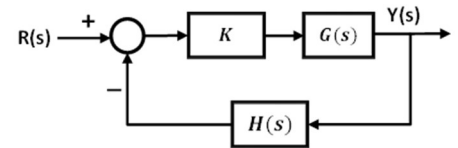
The overshoot for unit-step input is:

$$O.S. = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \rightarrow O.S. = e^{-\pi(0.387)/\sqrt{1-(0.387)^2}} = 0.268 \rightarrow O.S. \% = 26.8\%$$

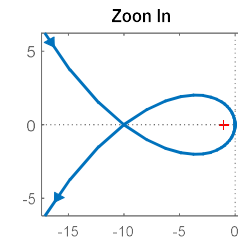
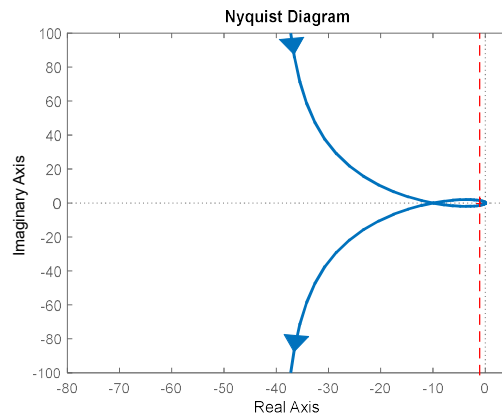
11) Consider the following closed-loop system

$$K = 10, \quad G(s) = \frac{s+3}{s(s-1)}, \quad H(s) = 1$$

Analyze stability of the closed-loop system using the Nyquist stability criterion.



$$Z = N + P$$



The open-loop transfer function is

$$KG(s)H(s) = \frac{10(s+3)}{s(s-1)}$$

Here, open-loop transfer function is **Type 1** and **unstable**. It has one pole in the right-half s-plane. It means $P = 1$.

Therefore, to have a stable closed-loop system, the Polar plot must encircle the $(-1, j0)$ point **one time counterclockwise**.

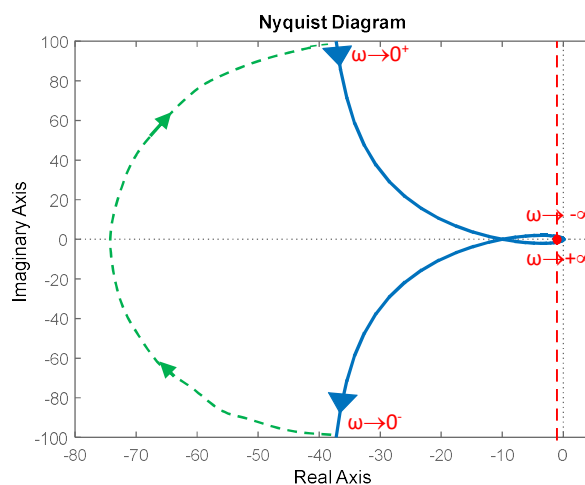
Here, we have the Polar plot the open-loop system $KG(s)H(s)$.

It can be seen that the Polar plot encircles the $(-1, j0)$ point **one time counterclockwise**. Therefore, $N = -1$.

According to the Nyquist stability criterion to have a stable closed-loop system we must have $Z = 0$.

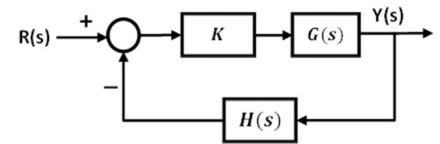
$$Z = N + P = -1 + 1 = 0$$

Therefore, the closed-loop system is **stable**, and the closed-loop system has **no pole** in the right-half s-plane.



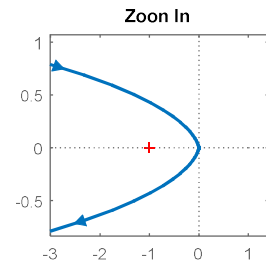
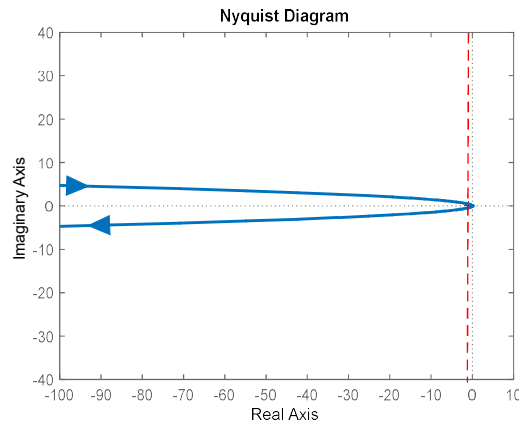
12) Consider the following closed-loop system

$$K = 6, \quad G(s) = \frac{1}{s^2(s+3)}, \quad H(s) = 1$$



Analyze stability of the closed-loop system using the Nyquist stability criterion.

$$Z = N + P$$



The open-loop transfer function is

$$KG(s)H(s) = \frac{6}{s^2(s+3)}$$

Here, the open-loop system is a **Type 2**, and it has no pole on the right-half s-plane. It means $P = 0$

Therefore, to have a stable closed-loop system, there must be **no encirclement** of the $(-1, j0)$ point.

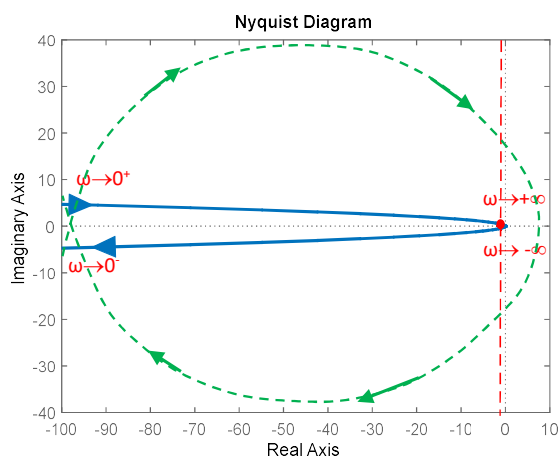
Here, we have the Polar plot the open-loop system $KG(s)H(s)$.

It can be seen that the Polar plot encircles the $(-1, j0)$ point **two times clockwise**. Therefore, $N = 2$.

According to the Nyquist criterion to have a stable closed-loop system we must have $Z = 0$.

$$Z = N + P = 2 + 0 = 2 \rightarrow Z \neq 0$$

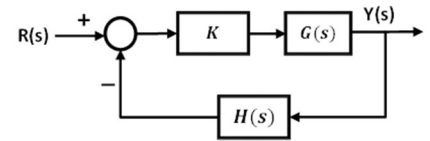
Therefore, the closed-loop system is **unstable**, and closed-loop system has **two poles** in the right-half s-plane.



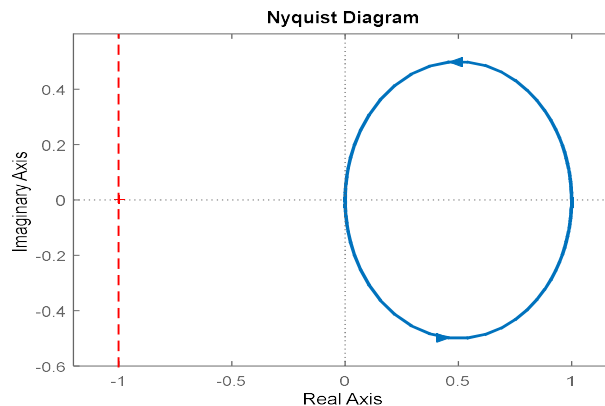
13) Consider the following closed-loop system

$$K = 6, \quad G(s) = \frac{1}{(s-1)(s-2)(s+3)}, \quad H(s) = 1$$

Analyze stability of the closed-loop system using the Nyquist stability criterion.



$$Z = N + P$$



The open-loop transfer function is

$$KG(s)H(s) = \frac{6}{(s-1)(s-2)(s+3)}$$

Here, the open-loop system is an **unstable** system, and it has **two poles** in the right-half s-plane. It means $P = 2$

Therefore, to have a stable closed-loop system, the Polar plot must encircle the $(-1, j0)$ point **two times counterclockwise**.

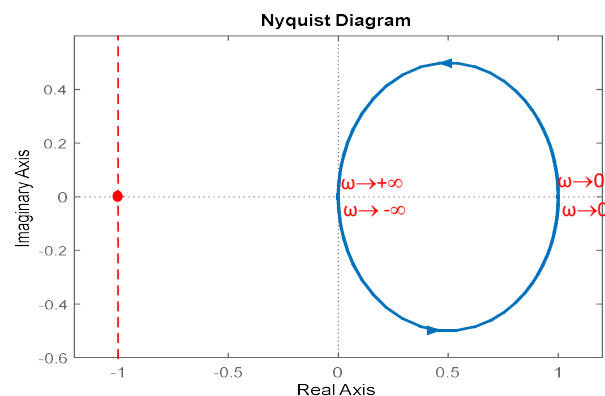
Here, we have the Polar plot of the open-loop system $KG(s)H(s)$.

It can be seen that there is no encirclement of the $(-1, j0)$ point. Therefore, $N = 0$.

According to the Nyquist criterion to have a stable closed-loop system we must have $Z = 0$.

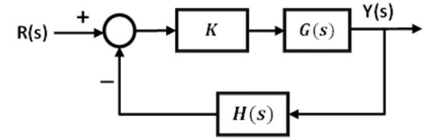
$$Z = N + P = 0 + 2 = 2 \rightarrow Z \neq 0$$

Therefore, the closed-loop system is **unstable**, and closed-loop system has **two poles** in the right-half s-plane.



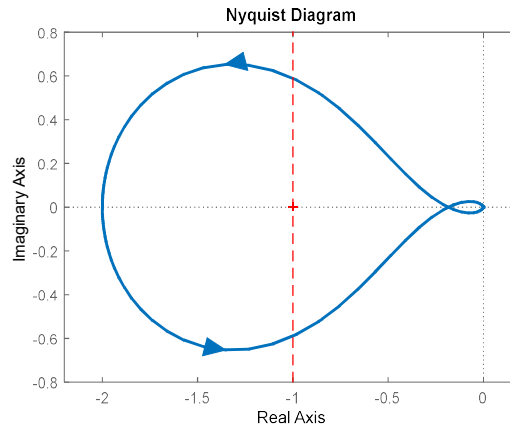
14) Consider the following closed-loop system

$$K = 10, \quad G(s) = \frac{10}{(s-1)(s+5)(s+10)}, \quad H(s) = 1$$



Analyze stability of the closed-loop system using the Nyquist stability criterion.

$$Z = N + P$$



The open-loop transfer function is

$$KG(s)H(s) = \frac{100}{(s-1)(s+5)(s+10)}$$

Here, the open-loop system is unstable, and it has **one pole** in the right-half s-plane. It means $P = 1$

Therefore, to have a stable closed-loop system, the Polar plot must encircle the $(-1, j0)$ point **one time counterclockwise**.

Here, we have the Polar plot of the open-loop system $KG(s)H(s)$.

It can be seen that the Polar plot encircles the $(-1, j0)$ point **one time counterclockwise**. Therefore, $N = -1$.

According to the Nyquist criterion to have a stable closed-loop system we must have $Z = 0$.

$$Z = N + P = -1 + 1 = 0$$

Therefore, the closed-loop system is **stable**, and the closed-loop system has **no pole** in the right-half s-plane.

