HUMBER ENGINEERING

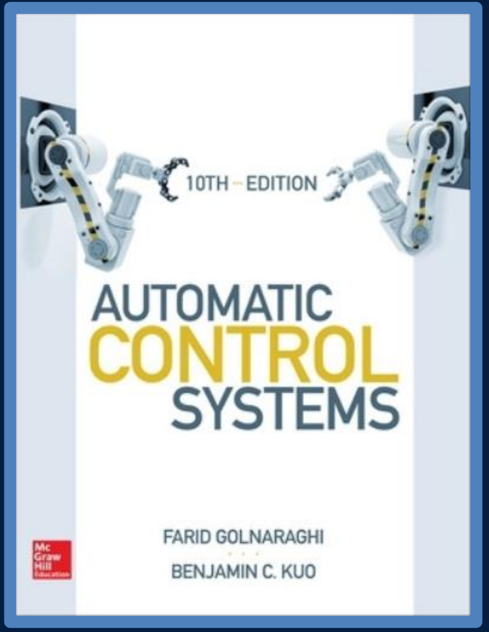
MENG 3510 – Control Systems
LECTURE 2

HUMBER

WE ARE HUMBER

LECTURE 2
Time-Domain Performance of Control Systems

- Typical Test Input Signals
- Review of Performance of First-Order & Second-Order Systems
  - Time Response Specification of Underdamped Systems
  - Time Response Specification and Pole Locations
- The Steady-State Error of Feedback Control Systems
  - Type of Control Systems
  - Error Constants
- Case Study: Antenna Control System



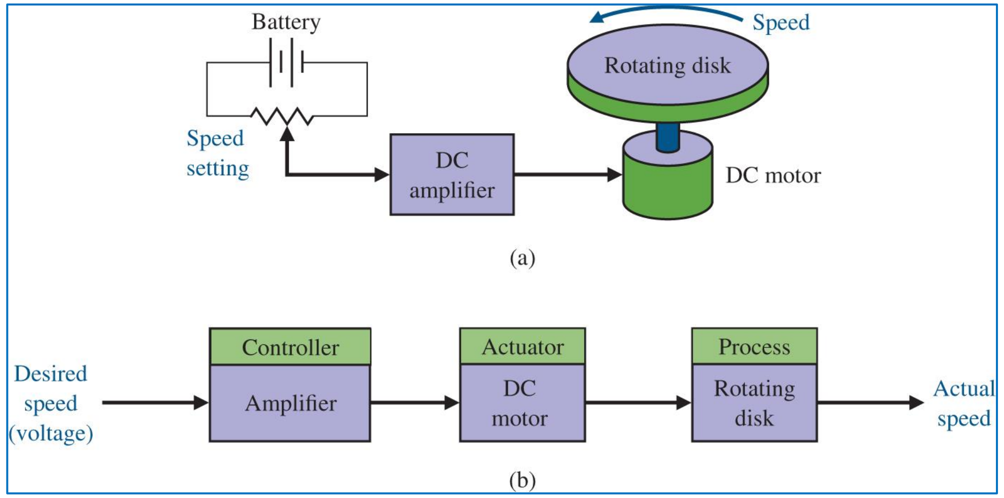
Chapters 7 & 3

What We Already Know?

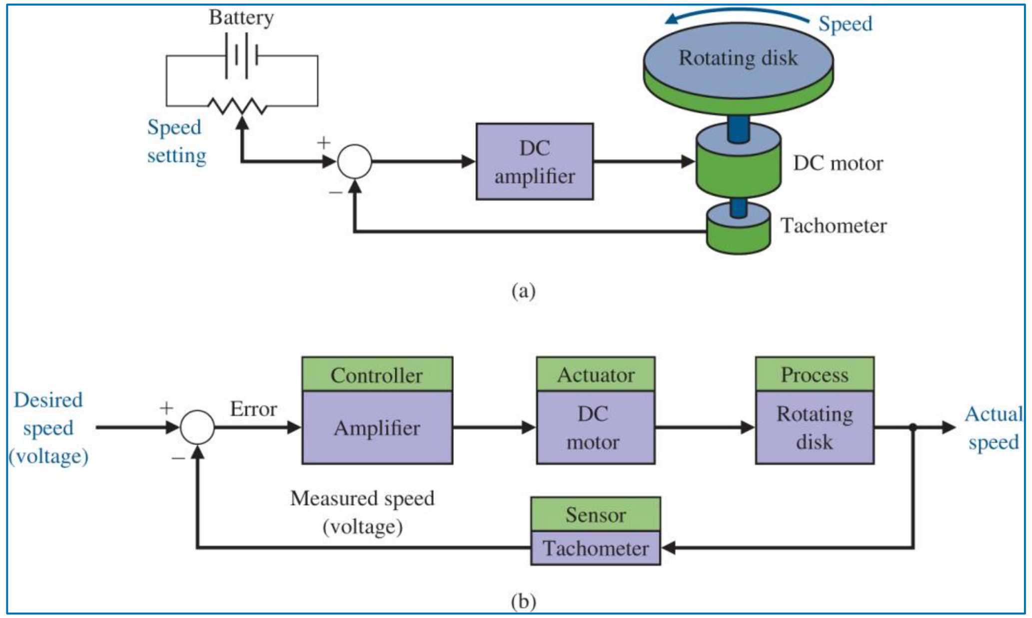
- Definition of Control Systems
  - Open-loop Control System
  - Closed-loop Control System
- Control Systems Block Diagrams
  - Basic elements and components of control system
- Control System Design Procedure

1. Establishment of goals, variables to be controlled, and performance specifications
2. System configuration and modeling
3. System analysis based on the model
4. Control system design, simulation, and verification

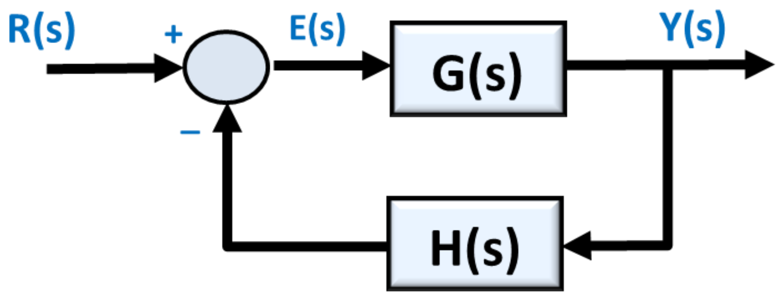
Open-loop Control System



Closed-loop Control System

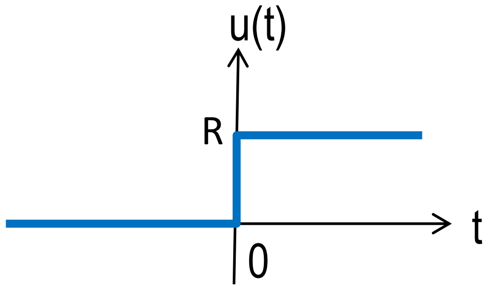


Typical Test Input Signals



- Following deterministic test signals are used as the input to evaluate the time-response performance of control systems.

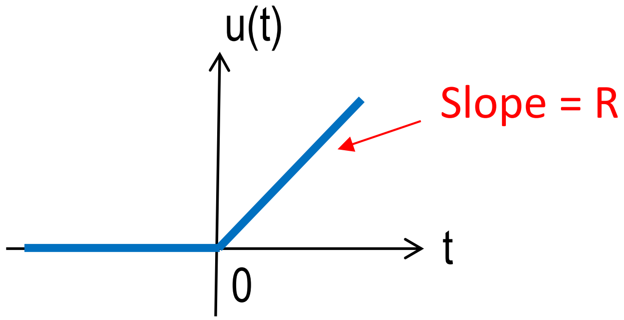
Step Function



$$u(t) = \begin{cases} R, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Represents an instantaneous change in the reference input.
- Useful in transient response and steady-state response analysis.
- System quickness in responding and relative stability.
- System's ability to follow constant-value inputs.

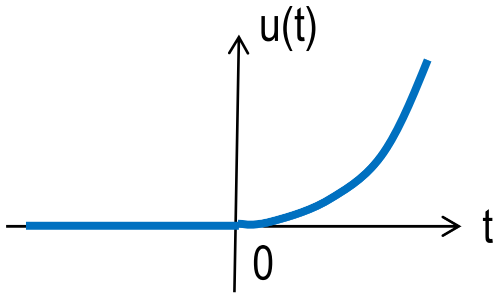
Ramp Function



$$u(t) = \begin{cases} Rt, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Represents a constant-rate-change reference input.
- Useful in steady-state response analysis.
- System's ability to follow a linearly increasing input, for example, position of constant-velocity target.

Parabolic Function



$$u(t) = \begin{cases} Rt^2, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Represents a signal one order faster than the ramp function.
- Useful in steady-state response analysis.
- For example, follow position of a constant-acceleration target.

Performance of First-Order Systems

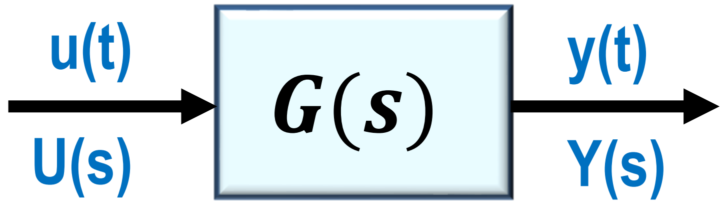
- First-order systems are systems whose input-output relationship is a first-order differential equation.
- Examples of systems that can be modeled as a first-order system
  - Cruise Control System
  - RC and RL Electric Circuits
  - Single-Tank Liquid Level Systems
  - Thermal Systems
  - Pressure Process Systems
  - Reduced-order DC Motor Speed Model

❑ Standard Form of a First-Order Transfer Function

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}}$$

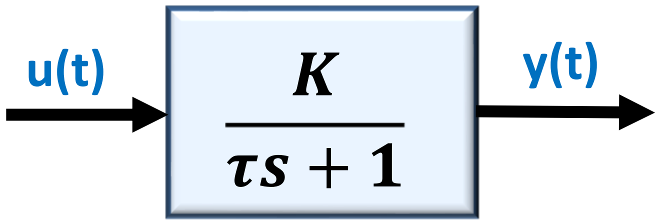
Steady-state gain
DC - gain

Time-constant



- Characteristic Equation $\rightarrow$ $\tau s + 1 = 0$
- First-order systems has one real pole $\rightarrow$ $s = -\frac{1}{\tau}$

Performance of First-Order Systems



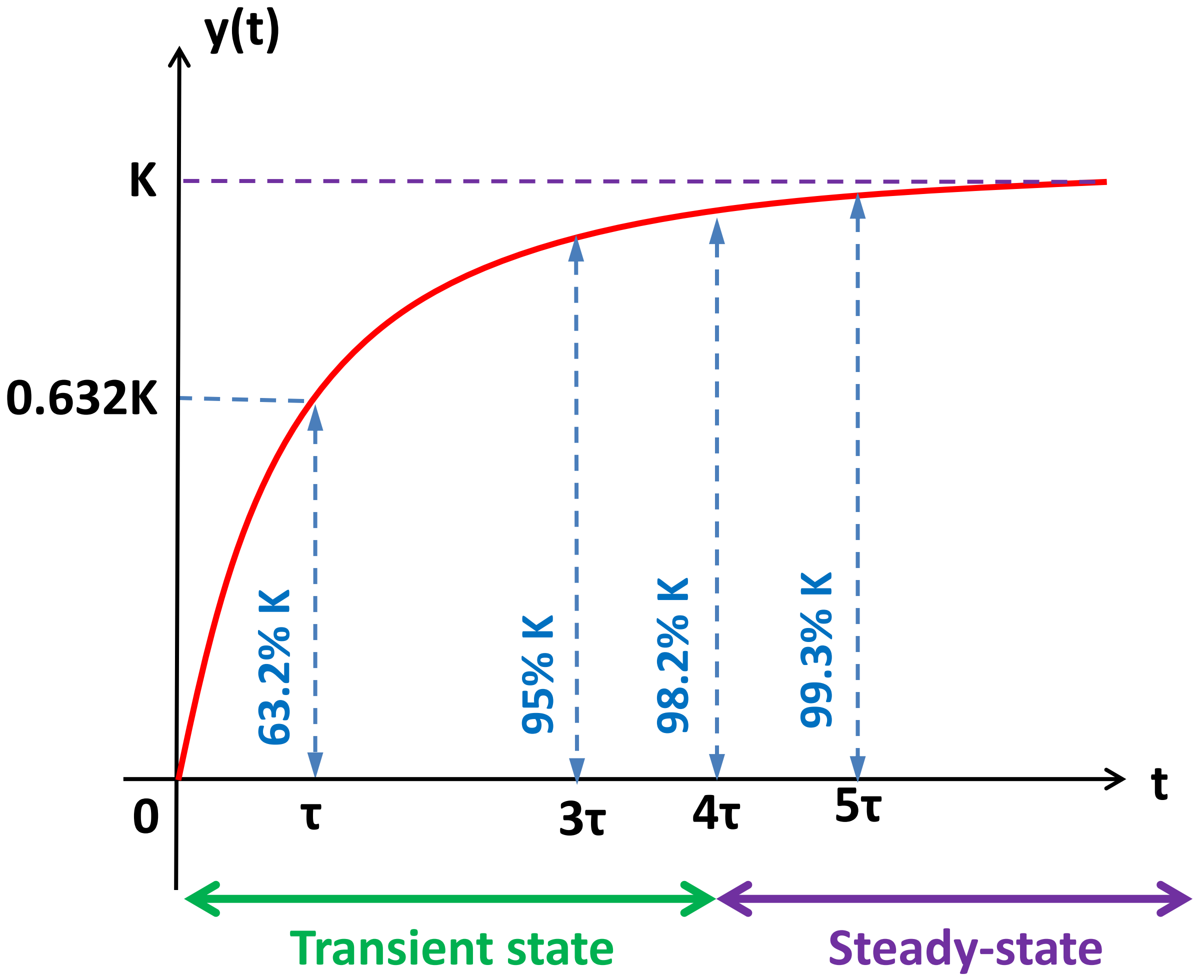
❑ Unit-Step Response

$$Y(s) = G(s)U(s) = \frac{K}{s(\tau s+1)}$$

$$\boxed{y(t) = K - Ke^{-t/\tau}, \qquad t \geq 0}$$

Steady-State Response $y_{ss}(t)$

Transient Response $y_t(t)$



- Time-Constant shows how fast a first-order system responds to the input. Smaller time-constant means faster response.

- Steady-state gain or DC-gain shows final value of the unit-step response in a stable system .

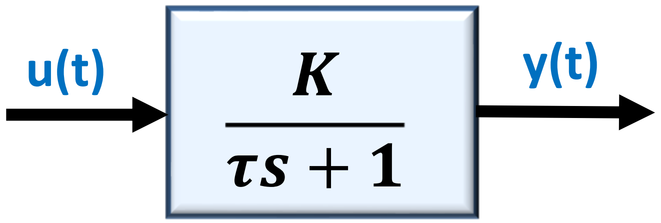
$$\boxed{\text{DC\_Gain} = \lim_{s\to 0} G(s)}$$

- 2% Settling-Time, t_s, is the time for the response to reach and stay within 2% of its final value. The time when $y(t) = 0.982K$.
- Note that 1% or 5% criteria can also be used.

$$\boxed{t_s = 4\tau}$$

Performance of First-Order Systems

❑ Pole Location & Stability



- Single real pole $\rightarrow$ $s = -\frac{1}{\tau}$

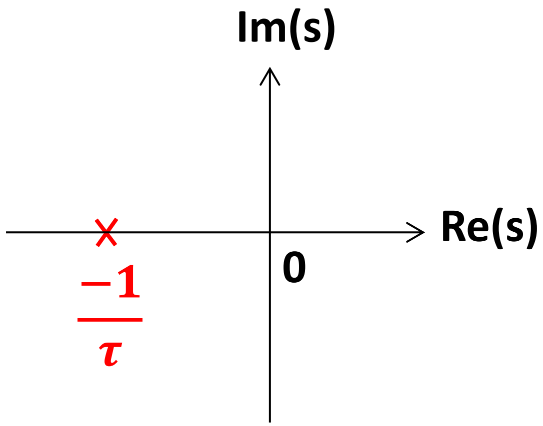
$$y(t) = K - Ke^{-t/\tau}, \qquad t \geq 0$$

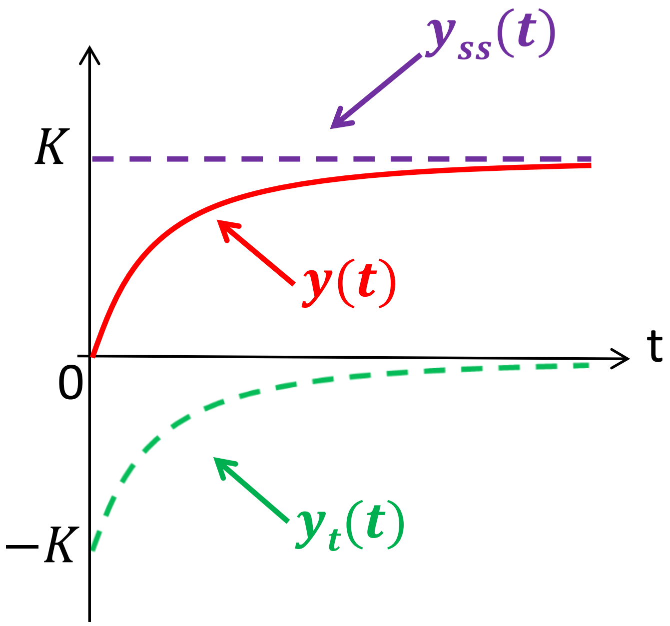
Steady-State Response $y_{ss}(t)$

Transient Response $y_t(t)$

Stable System

$\tau > 0$

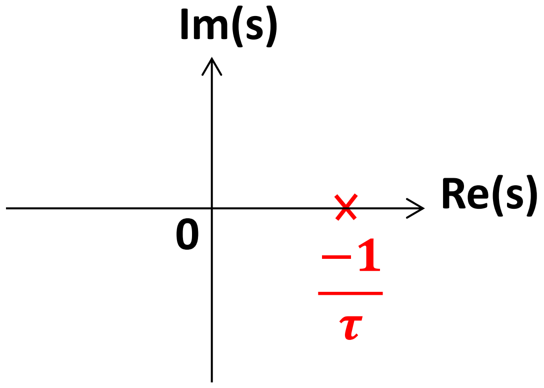


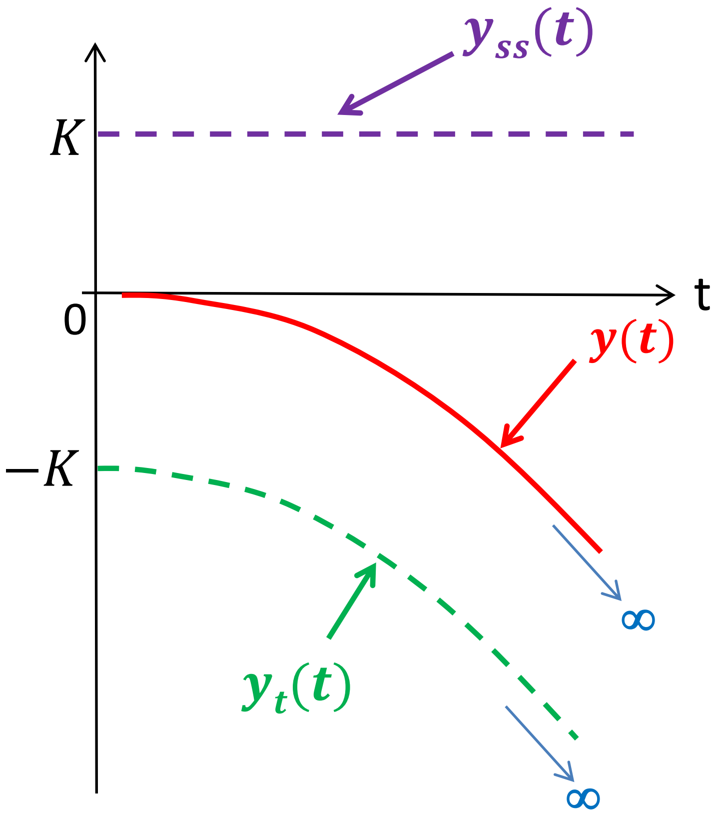


- The pole is on the left-half s-plane.
- From the **BIBO** stability the system is stable.

Unstable System

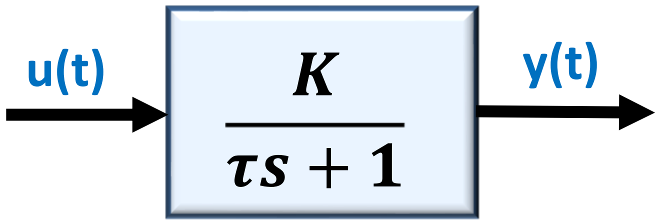
$\tau < 0$





- The pole is on the right-half s-plane.
- From the **BIBO** stability the system is unstable.

Performance of First-Order Systems

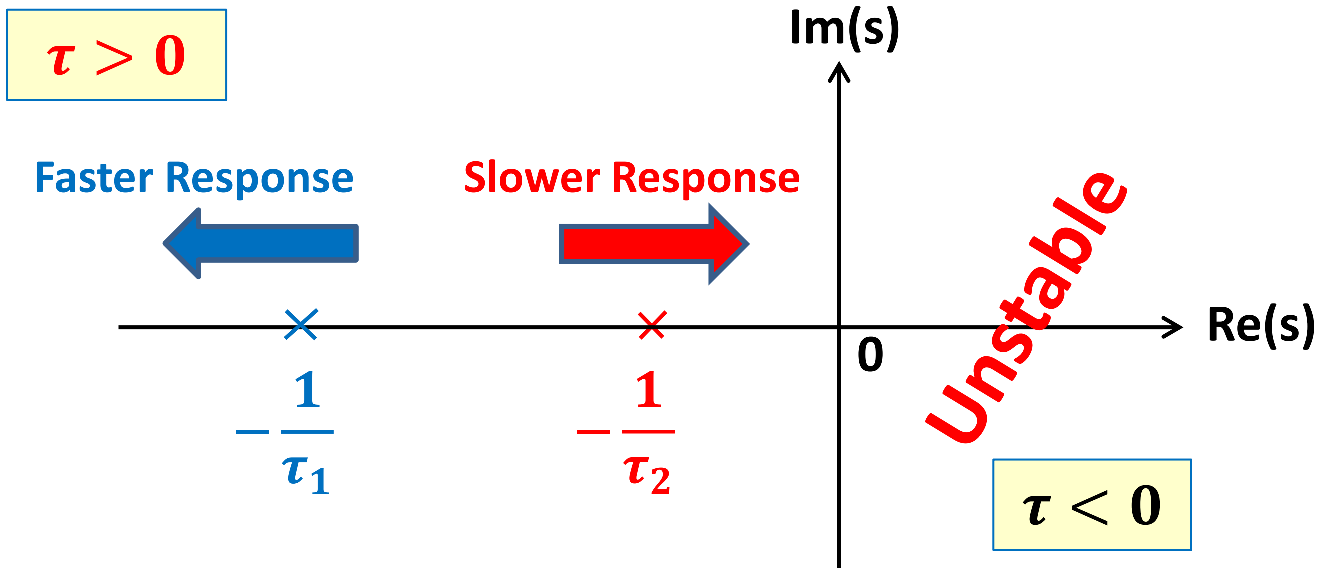


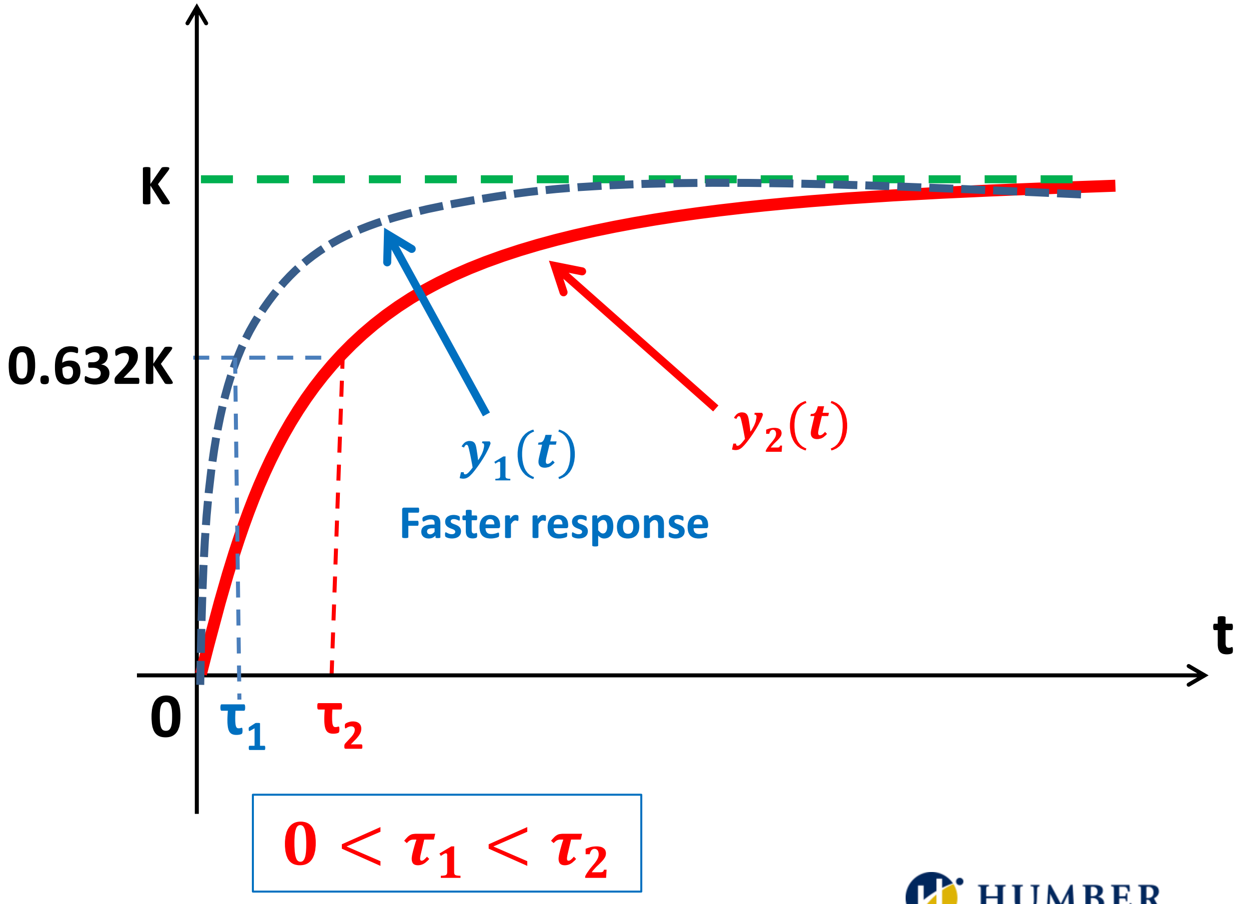
❑ Pole Location & Time Constant

- Time Constant shows how fast the first-order system responds to the input.
- The smaller the <u>time-constant</u>, the faster the system response.

$$G_1(s) = \frac{K}{\tau_1 s + 1} \quad \text{and} \quad G_2(s) = \frac{K}{\tau_2 s + 1}$$

- $G_1(s)$ has a faster response than $G_2(s)$
- Smaller time constant means the <u>pole</u> is more to the left and farther from the origin in the s-plane.





Performance of First-Order Systems

Example 1

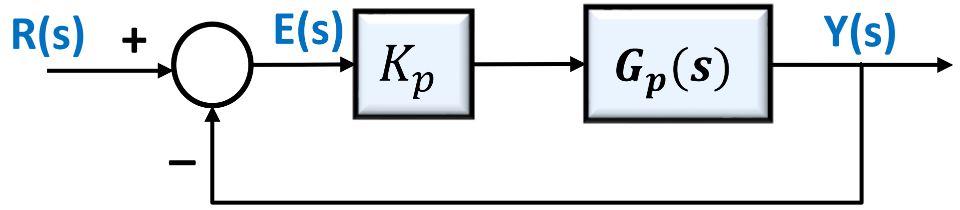
Consider the following transfer function model of a first-order system.

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{2.5}{35s+1}$$

a) Determine the time-constant and steady-state gain of system.

Time-constant $\rightarrow$ $\tau = 35\ sec$, Steady-state gain $\rightarrow$ $K = 2.5$

b) The following closed-loop system with proportional control gain K_p has been developed to increase the speed of the system. Determine the required gain K_p to increase the speed 10 times faster than the current value.



First find the closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_pG_p(s)}{1+K_pG_p(s)H(s)} = \frac{\frac{2.5\ K_p}{35s+1}}{1+\frac{2.5\ K_p}{35s+1}} = \frac{2.5\ K_p}{35s+1+2.5\ K_p}$$

Find the time-constant of the closed-loop transfer function and make it equal to the desired time-constant, then find the required gain K_p.

Time-constant of the closed-loop system is: $\tau_{cl} = \frac{35}{1+2.5\ K_p}$

The desired time-constant is 35/10 = 3.5 sec. $\rightarrow$ $3.5 = \frac{35}{1+2.5\ K_p}$ $\rightarrow$ $3.5 + 8.75K_p = 35$ $\rightarrow$ $\boxed{K_p = 3.6}$

Desired Proportional Gain

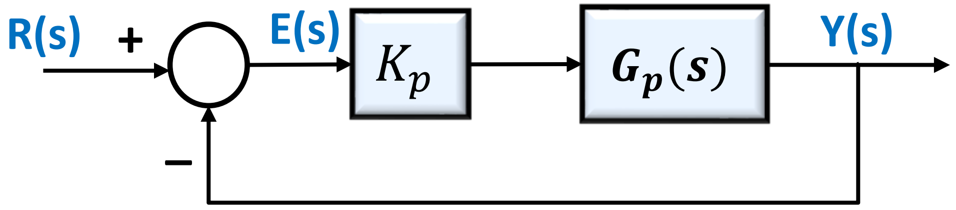
Performance of First-Order Systems

Example 1

Consider the following transfer function model of a first-order system.

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{2.5}{35s+1}$$

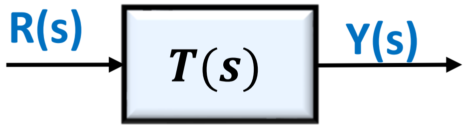
c) Find the closed-loop transfer function for the obtained proportional gain K_p.



For $K_p = 3.6$ the closed -loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2.5\,K_p}{35s+1+2.5\,K_p} = \frac{9}{35s+10}$$

d) The error signal is defined as $E(s) = R(s) - Y(s)$. Determine the steady-state tracking error e_{ss} due to a unit-step response, $R(s) = 1/s$ for the obtained proportional gain $K_p$using the Final−Value theorem.



$$E(s) = R(s) - Y(s) = R(s) - R(s)T(s) = R(s)[1 - T(s)] = \frac{1}{s}\left[1 - \frac{9}{35s+10}\right] = \frac{1}{s}\left(\frac{35s+1}{35s+10}\right) = \frac{35s+1}{s(35s+10)}$$

We can calculate e_{ss} using the Final-Value theorem:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left(\frac{35s+1}{s(35s+10)}\right) = 0.1 \quad \rightarrow \quad \boxed{e_{ss} = 10\%}$$

Steady-state Error

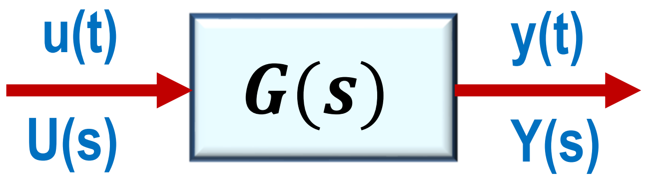
Performance of Second-Order Systems

- Second-order systems are systems whose input-output relationship is a **second-order differential equation**.
- Examples of systems that can be modeled as a second-order system.
  - Mass-Spring-Damper System
  - RLC Electric Circuits
  - Two-Tank Liquid Level System
  - Full-order DC Motor Speed Model

Standard Form of a Second-Order Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- K is the steady-state gain
- ζ is called Damping Ratio
- ω_n is called Natural Undamped Frequency
- **Characteristic Equation** $\rightarrow$ $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
- The system has two poles $\rightarrow$ $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- Stability and dynamic behavior of the second-order system can be described in terms of the damping ratio ζ and the natural frequency ω_n.



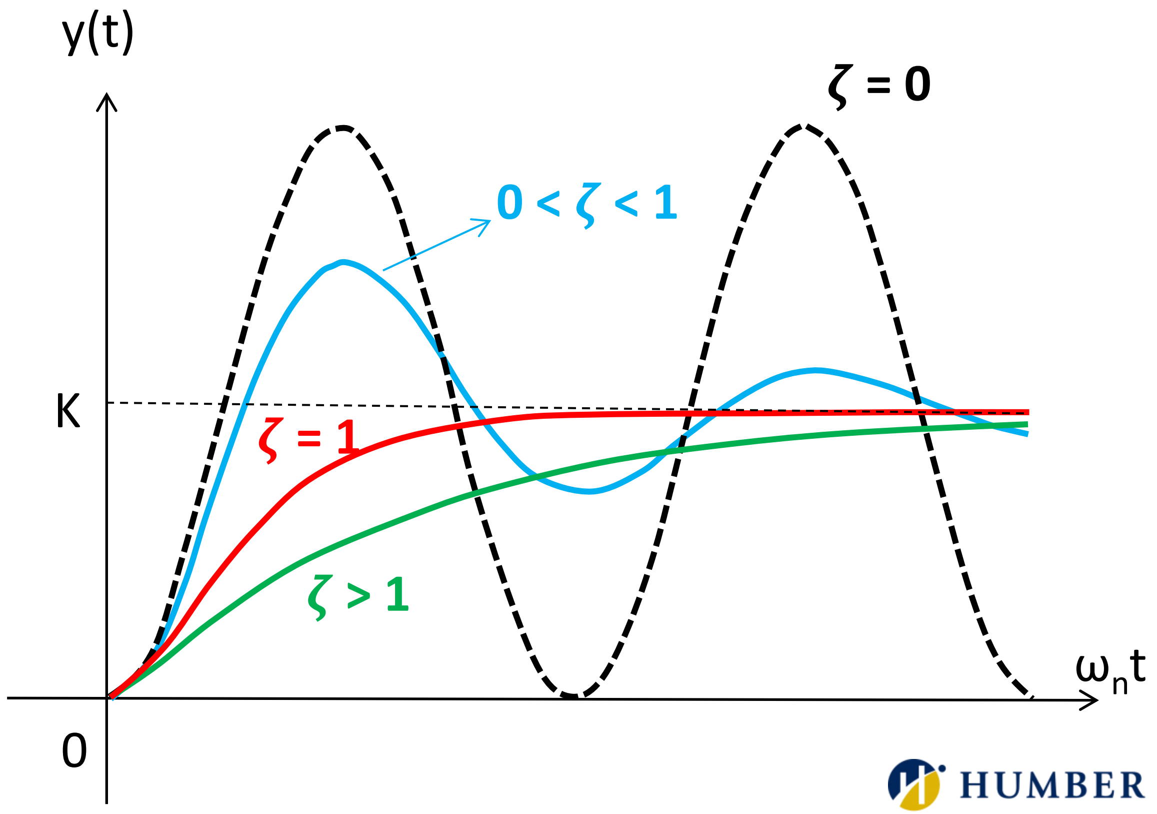
Performance of Second-Order Systems

❑ Unit-Step Response

- The pole locations and the step response $y(t)$ depend on the natural frequency ω_n and the damping ratio ζ.
  - $\zeta = 1 \rightarrow$ The poles are real and equal $\rightarrow \quad s_1 = s_2 = -\omega_n$
  - $\zeta > 1 \rightarrow$ The poless are real but not equal $\rightarrow \quad s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
  - $0 < \zeta < 1 \rightarrow$ The poles are complex conjugate $\rightarrow \quad s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
  - $\zeta = 0 \rightarrow$ The poles are imaginary. $\rightarrow \quad s_{1,2} = \pm j\omega_n$

$$\boxed{\begin{aligned} s_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\ s_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \end{aligned}}$$

- Step response of the second-order systems can be classified based on the damping ratio ζ
  - **Critically-damped Systems:** $\zeta = 1$
  - **Over-damped Systems:** $\zeta > 1$
  - **Under-damped Systems:** $0 < \zeta < 1$
  - **Undamped Systems:** $\zeta = 0$



- Note that negative damping ratio $\zeta < 0$ means growing magnitude of oscillations, which is called unstable system.

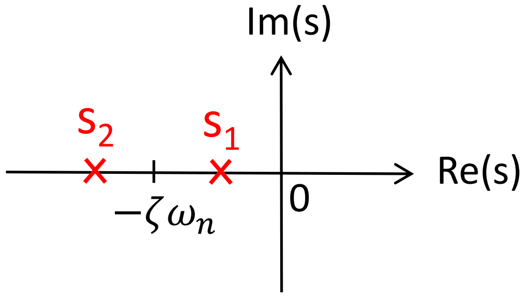
Performance of Second-Order Systems

❑ Unit-Step Response

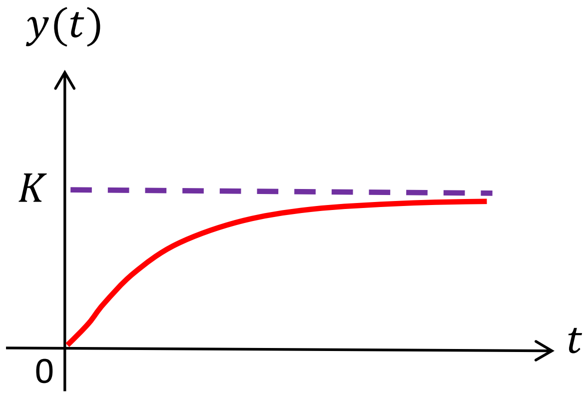
Over-damped System $\zeta > 1$

- System has two distinct real negative poles
- Output response is slow and is not oscillate
- Output response becomes slower by increasing the damping ratio ζ

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



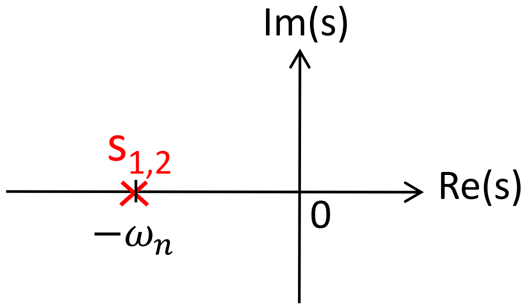
$$y(t) = K + C_1 e^{s_1 t} + C_2 e^{s_2 t}, \qquad t \geq 0$$



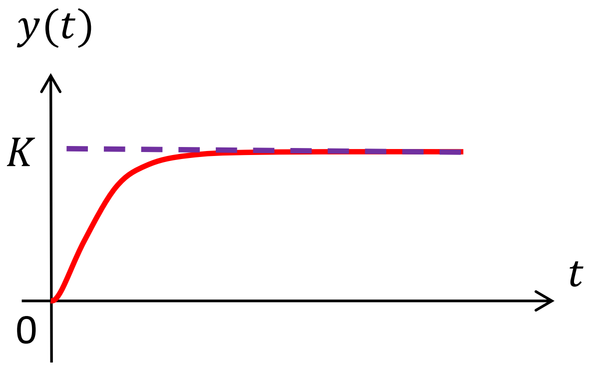
Critically-damped System $\zeta = 1$

- System has two repeated real negative poles
- Output response is not oscillated
- Fastest response without oscillation and overshoot

$$s_1 = s_2 = -\omega_n$$



$$y(t) = K - Ke^{-\omega_n t}(1 + \omega_n t), \qquad t \geq 0$$



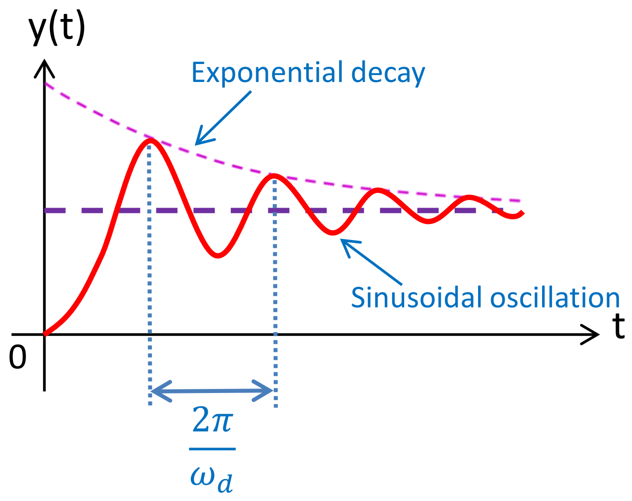
Performance of Second-Order Systems

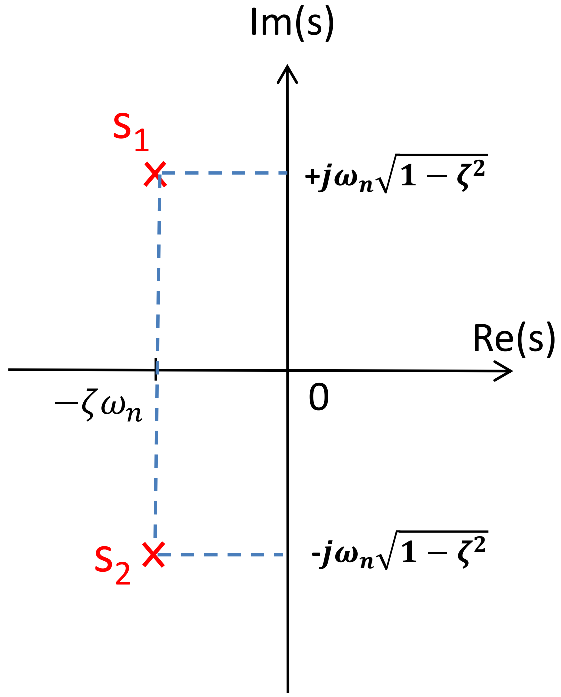
Unit-Step Response

Underdamped System $0 < \zeta < 1$

- System has one pair of complex conjugated poles
- Transient response of the system would oscillate, and it becomes more oscillatory with larger overshoot by decreasing the ζ
- Frequency of oscillations is $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- ω_d is called damped natural frequency

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



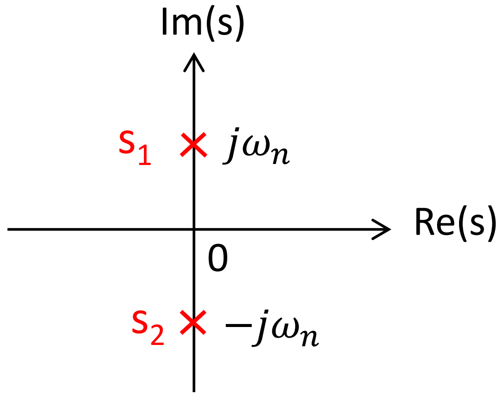


$$y(t) = K - \frac{Ke^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin(\omega_d t + \cos^{-1}\zeta)\right), \qquad t \ge 0$$

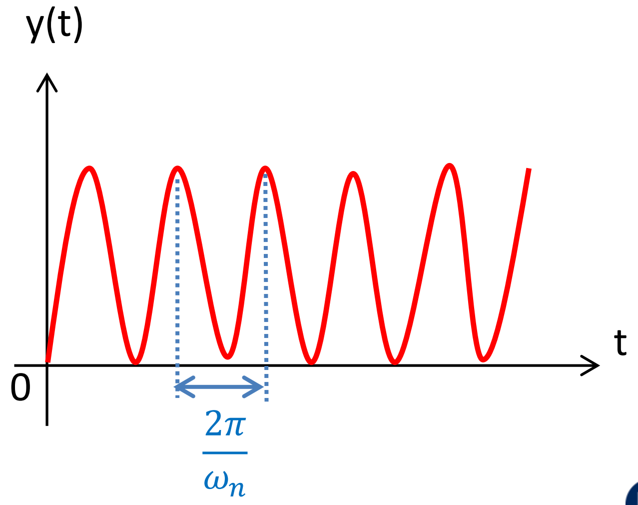
Undamped System $\zeta = 0$

- System has one pair of complex conjugate poles on the imaginary axes.
- The response has sustained oscillation with frequency of ω_n
- This is called marginally stable system.

$$s_{1,2} = \pm j\omega_n$$



$$y(t) = K - K\cos(\omega_n t), \qquad t \ge 0$$



Time Response Specification of Underdamped Systems

Rise time (t_r): The time required for the step response to rise from 10% to 90% of its final value.

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n}$$

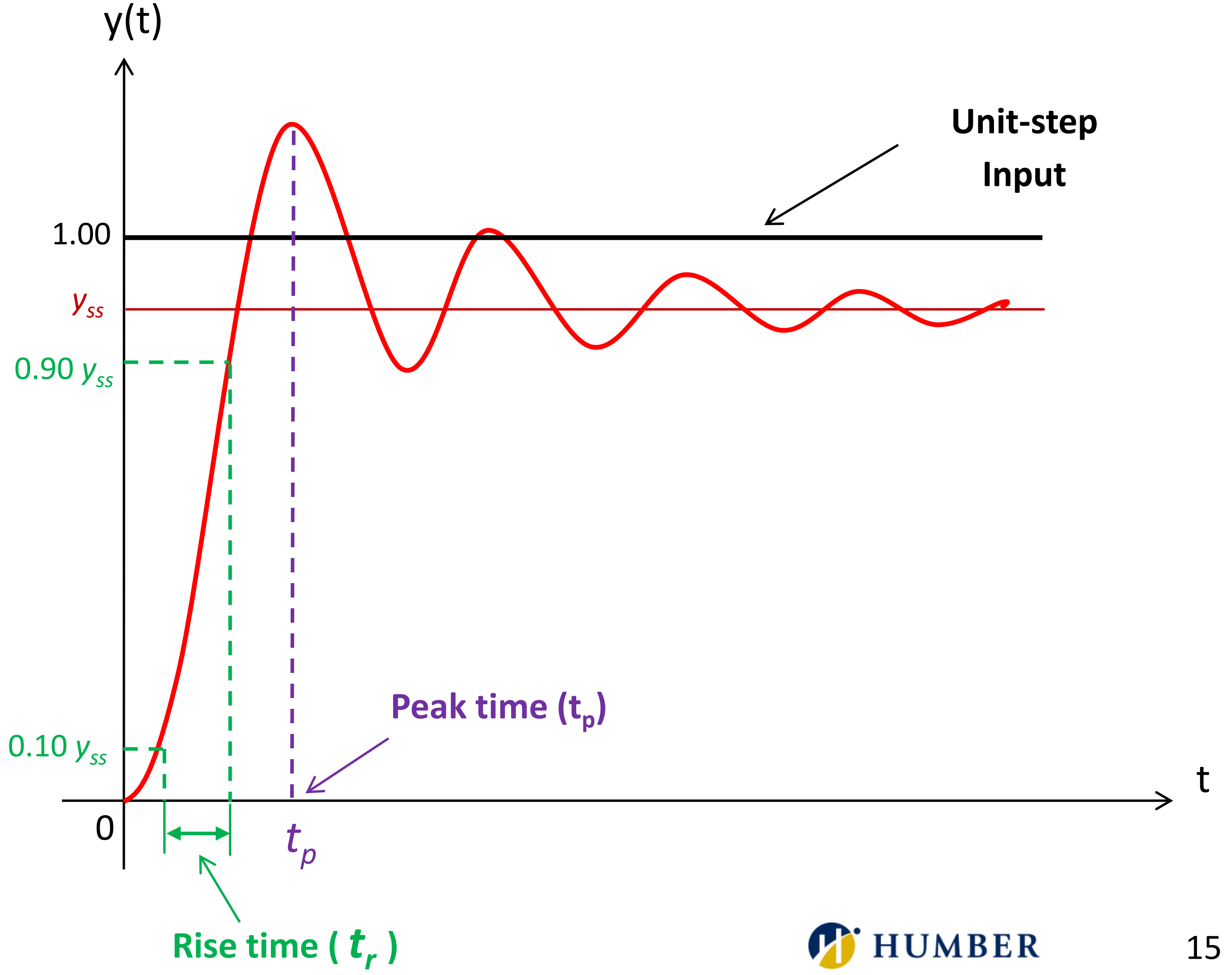
$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Rise-time shows how fast a system responds to an input.
- Rise-time is proportional to ζ and inversely proportional to ω_n, increasing the ω_n will reduce the rise-time.

Peak time (t_p): The time required for the step response to reach the first peak of the overshoot.

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

- Peak-time is inversely proportional to ω_n, increasing the ω_n will reduce the peak-time.



Time Response Specification of Underdamped Systems

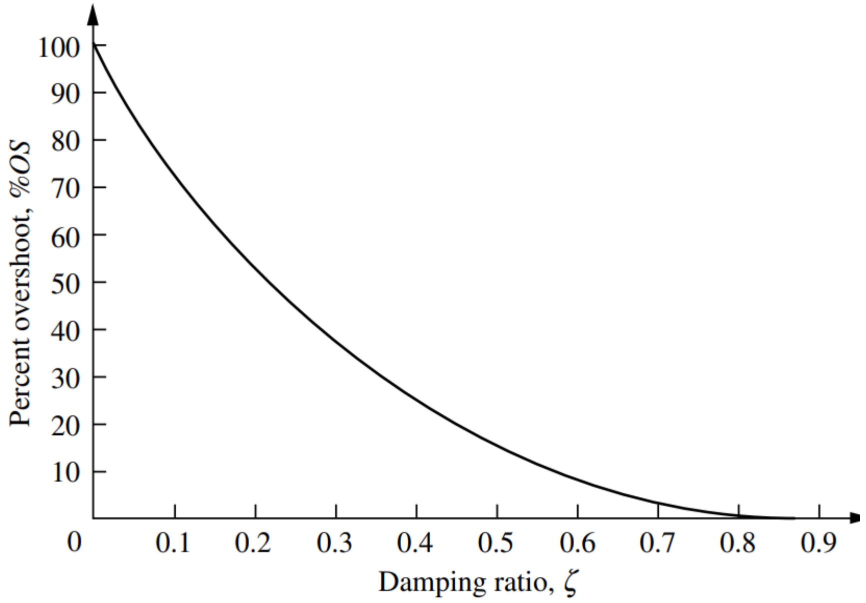
Maximum overshoot (M_p): The maximum peak value of the step response measured from the final value of the response.

$$M_p = y(t_p) - y_{ss} = y_{ss} e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\%O.S. = \frac{M_p}{y_{ss}} \times 100\%$$

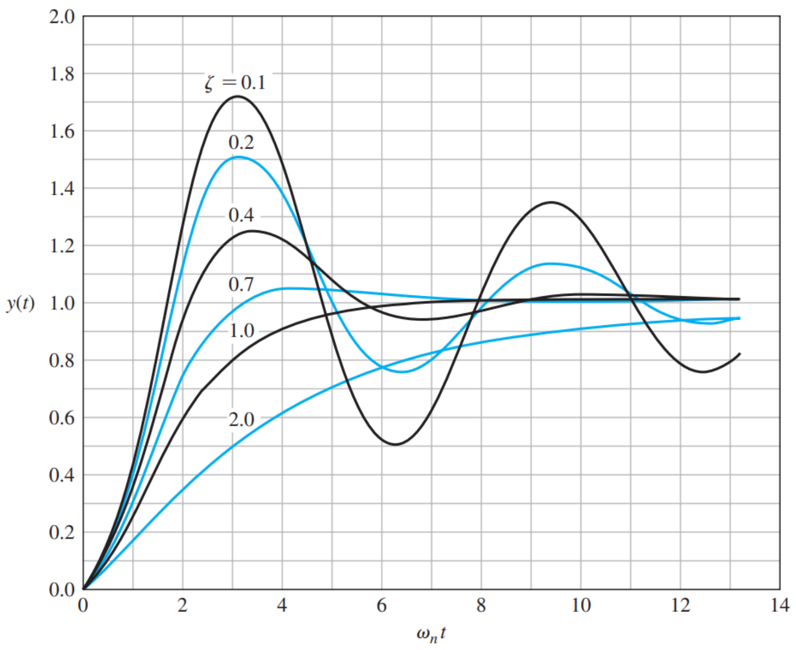
$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

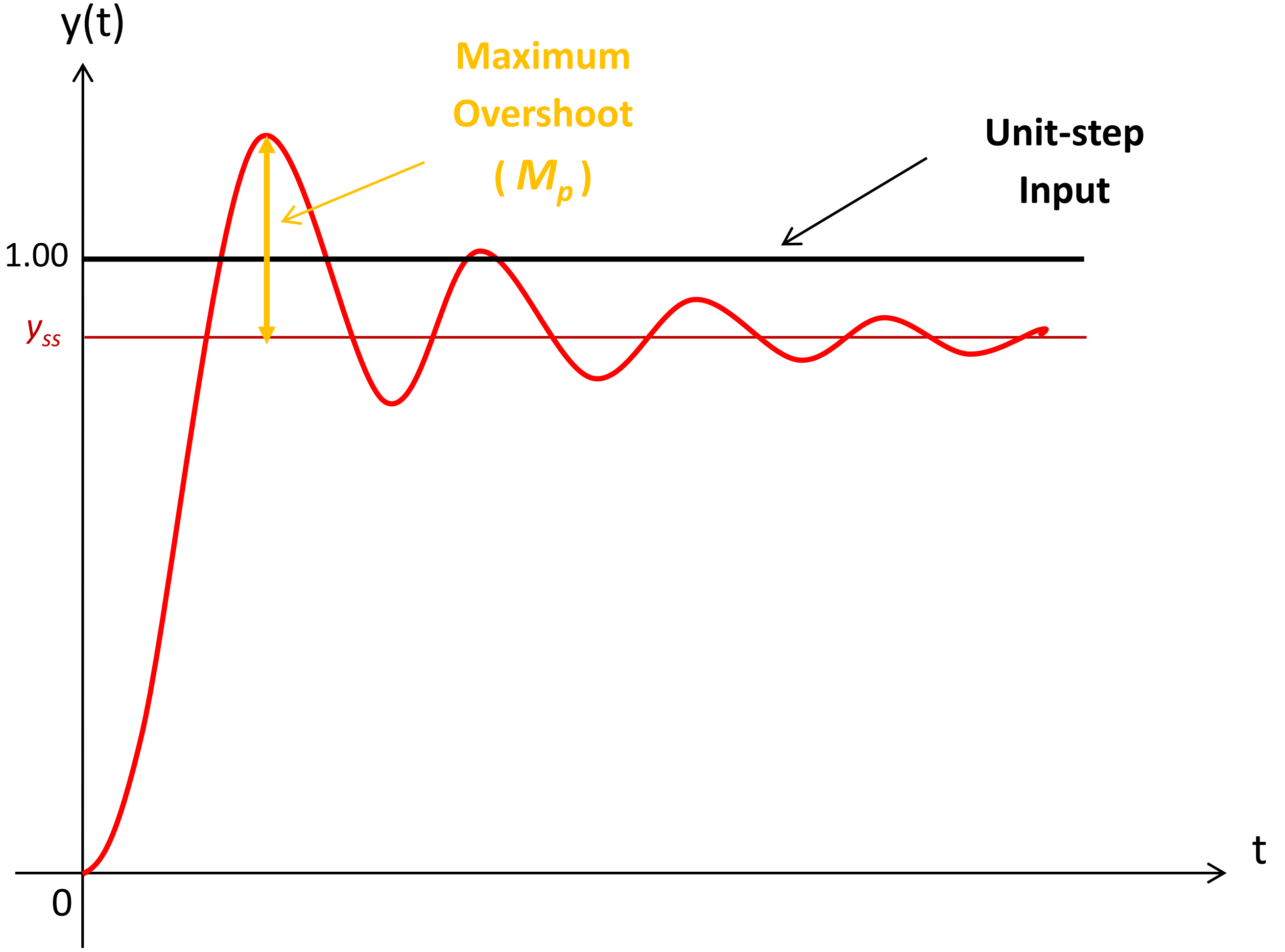
- Decreasing the damping ratio ζ will increase the overshoot.



$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}}$$

ζ	%O.S.
0.690	5%
0.591	10%
0.517	15%
0.456	20%





Time Response Specification of Underdamped Systems

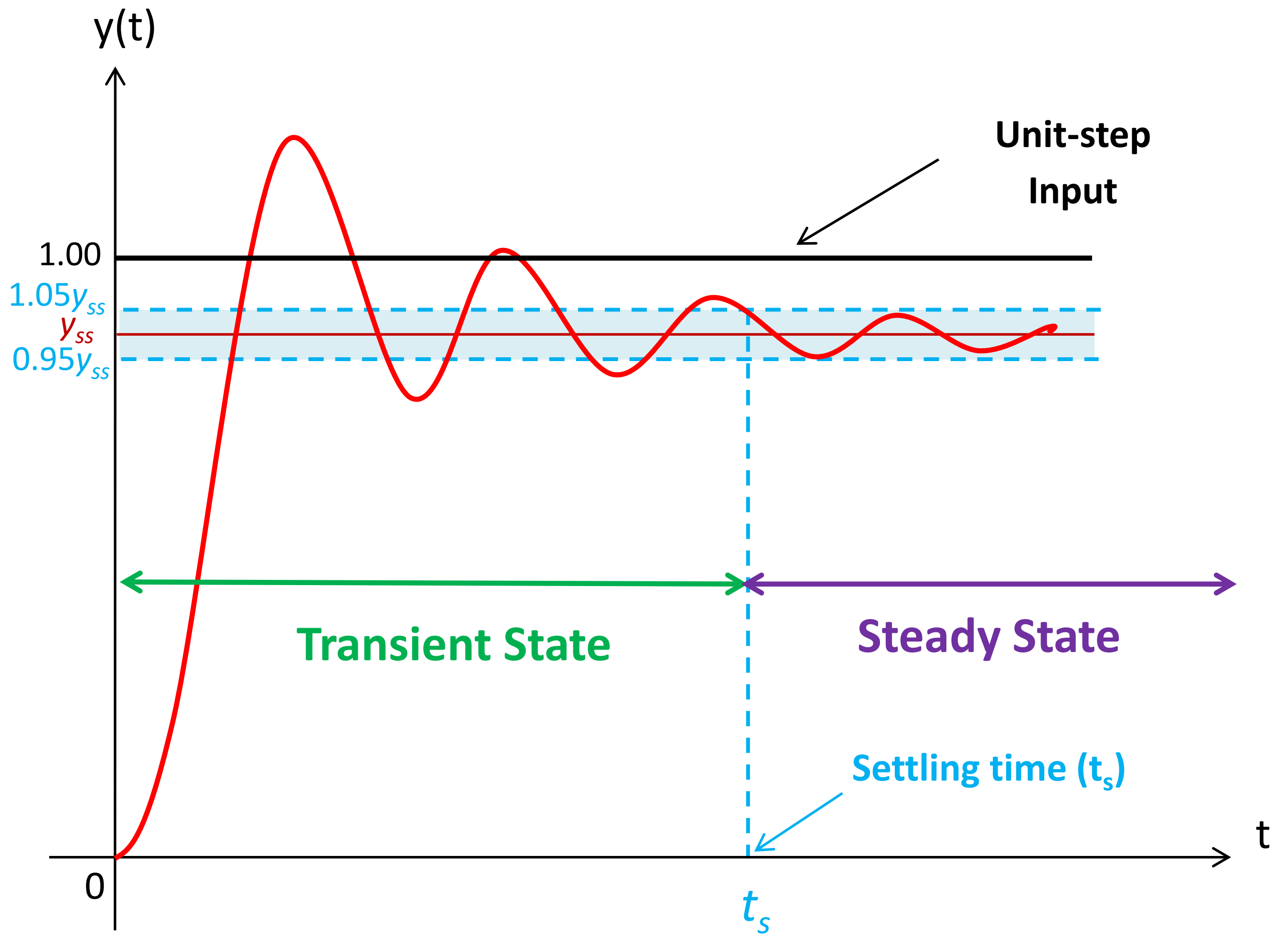
Settling time (t_s): The time required for the step response to reach and stay within the specified percentage of its final value (usually 2% or 5%)

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$2\%\ \text{criteria} \rightarrow \quad t_s \approx \frac{4}{\zeta\omega_n}\ , \qquad 0 < \zeta < 0.9$$

$$5\%\ \text{criteria} \rightarrow \begin{cases} t_s \approx \dfrac{3.2}{\zeta\omega_n}\ , & 0 < \zeta < 0.69 \\[2ex] t_s \approx \dfrac{4.5\zeta}{\omega_n}\ , & \zeta > 0.69 \end{cases}$$

- Settling-time shows how fast the step response settles to its final value.



Time Response Specification and Pole Locations

- Since the time response specifications (rise-time, peak-time, overshoot, settling-time) are given in terms of the ζ and ω_n, we can find a relation between the pole locations on the s-plane and the time response specifications.

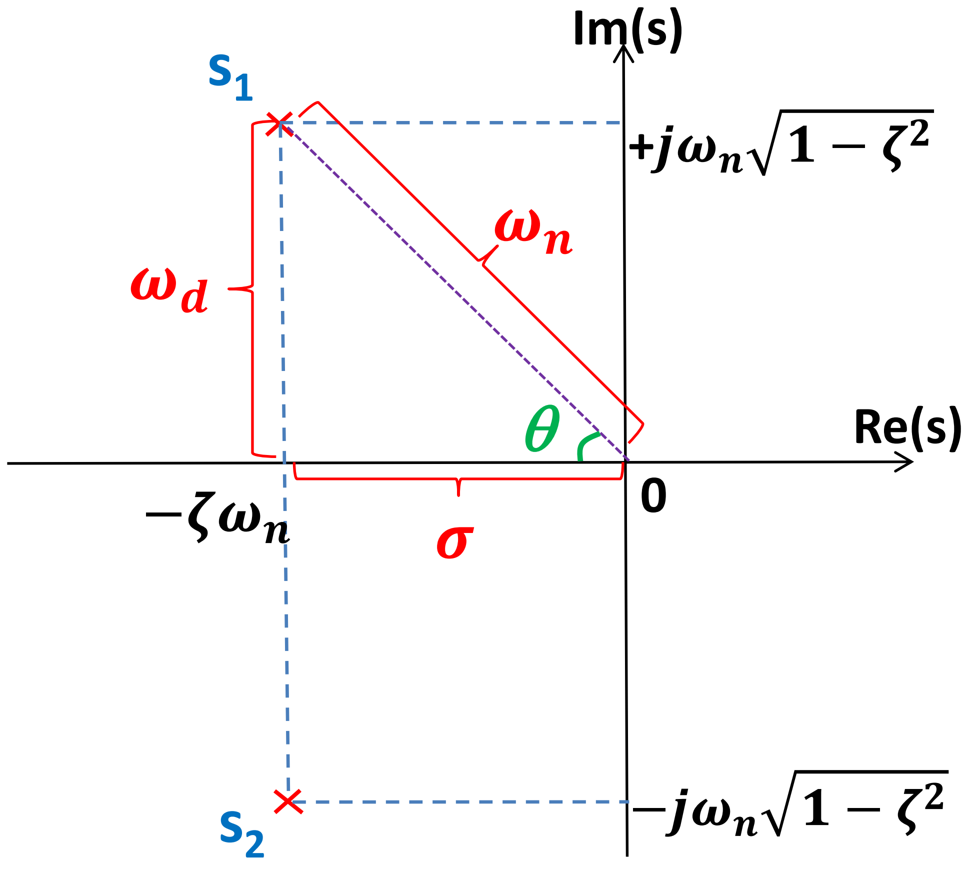
❑ Poles of Under-damped Systems ($0 < \zeta < 1$)

- Underdamped system has one pair of complex conjugated pole

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

$$\sigma = \zeta\omega_n \longrightarrow \textbf{Damping Factor}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \longrightarrow \textbf{Damped Natural Frequency}$$



$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

- Undamped natural frequency ω_n, determines the radial distance of poles to origin

$$cos\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta \quad \rightarrow \quad \theta = cos^{-1}\zeta$$

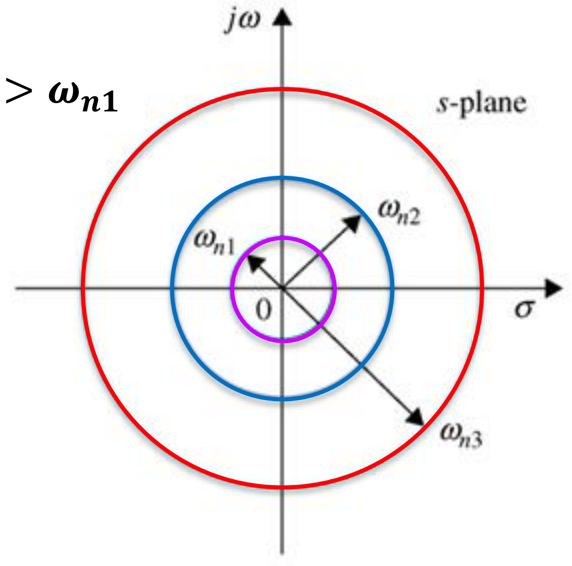
- Damping ratio ζ determines the cosine of the angle θ
- As ζ increases from 0 to 1, the θ decreases from 90° to 0°

Time Response Specification and Pole Locations

❑ Constant-Undamped-Natural-Frequency, ω_n Loci

- Constant natural frequency ω_n loci in s-plane are concentric circles with the center at s = 0 and radius of ω_n.

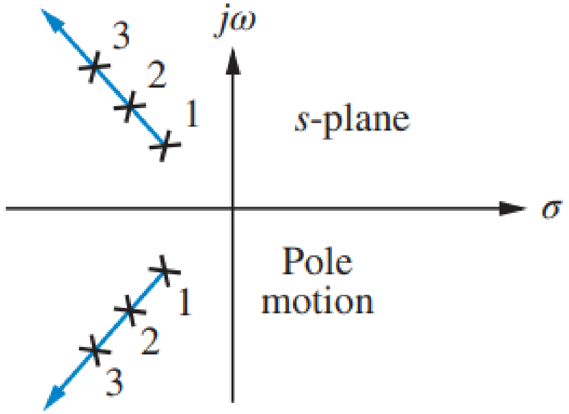
$\omega_{n3} > \omega_{n2} > \omega_{n1}$



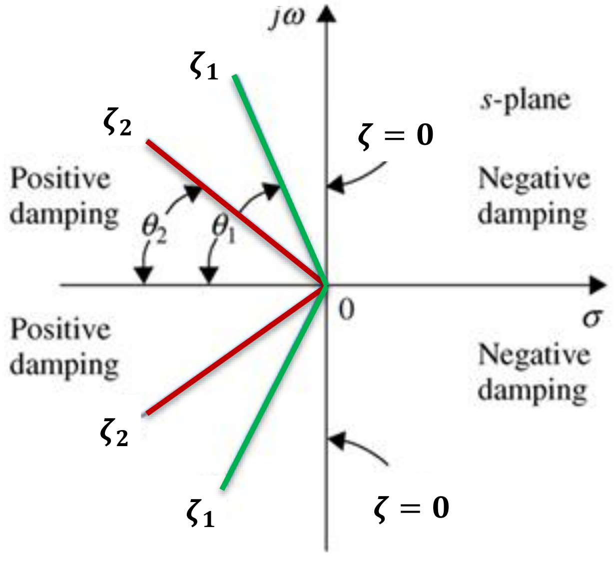
(a)

❑ Constant-Damping-Ratio, ζ Loci

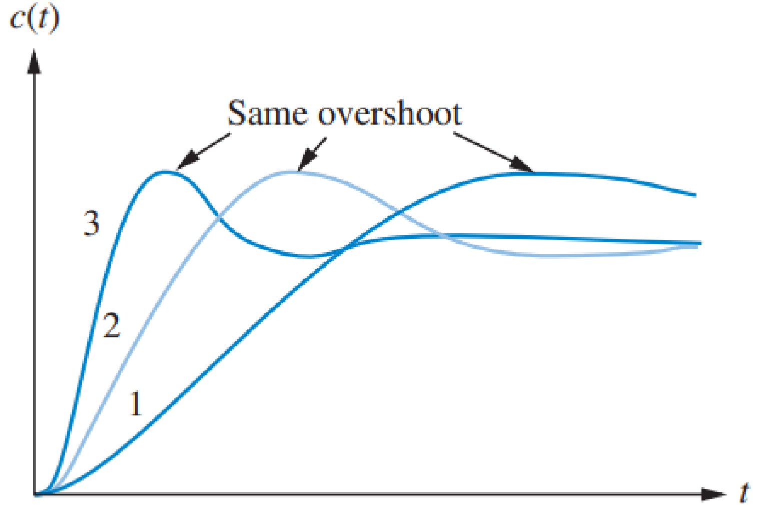
- Constant damping ratio ζ loci in the s-plane are radial lines passing through the origin.
- Poles moves along a constant radial line away from the origin
  - Damping ratio ζ remains constant
  - Natural frequency ω_n increases
  - Overshoot remains constant
  - Rise-time, peak-time and settling-time decrease



$\theta_1 > \theta_2 \rightarrow \zeta_1 < \zeta_2$



(b)



$$t_p = \frac{\pi}{\omega_d}$$

$$t_s \approx \frac{4}{\zeta\omega_n}$$

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n}$$

$$M_p = y_{ss}e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

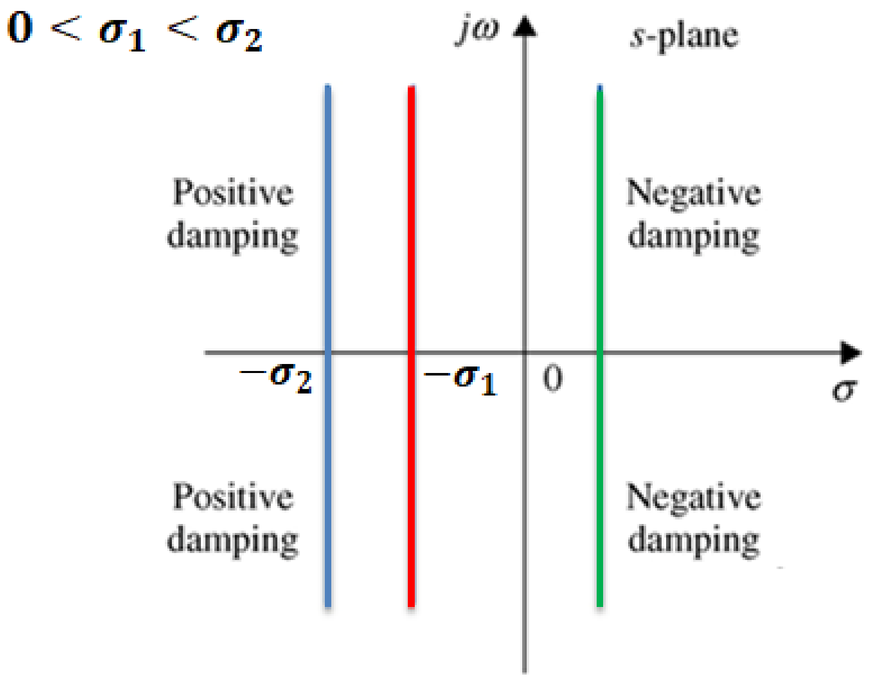
Time Response Specification and Pole Locations

❑ Constant-Damping-Factor, σ Loci

- Constant damping factor σ loci in s-plane are vertical lines parallel to imaginary axis.

- Poles moves vertically away from the origin
  - Real part of poles $\sigma = \zeta\omega_n$ remains constant
  - Damping ratio ζ decreases
  - Natural frequency ω_n increases
  - Overshoot increases
  - Rise-time decreases
  - Settling-time remains constant

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

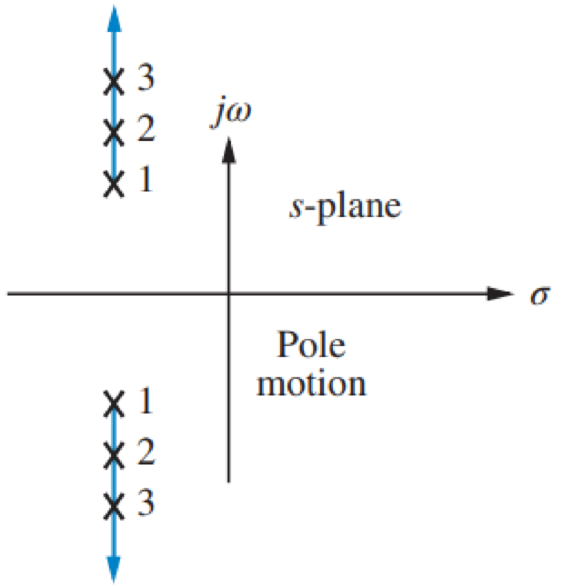


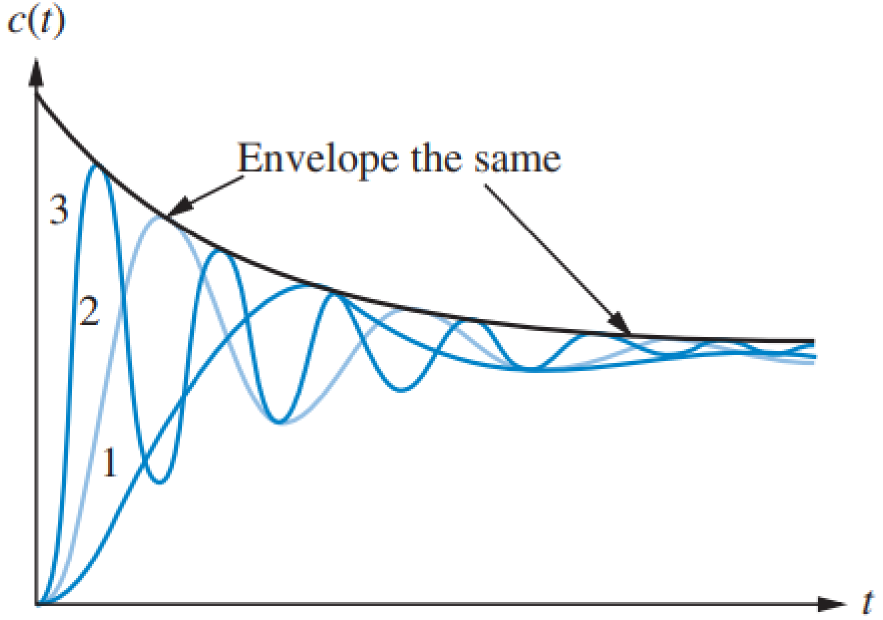
$$t_p = \frac{\pi}{\omega_d}$$

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n}$$

$$t_s \approx \frac{4}{\zeta\omega_n}$$

$$M_p = y_{ss}e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$





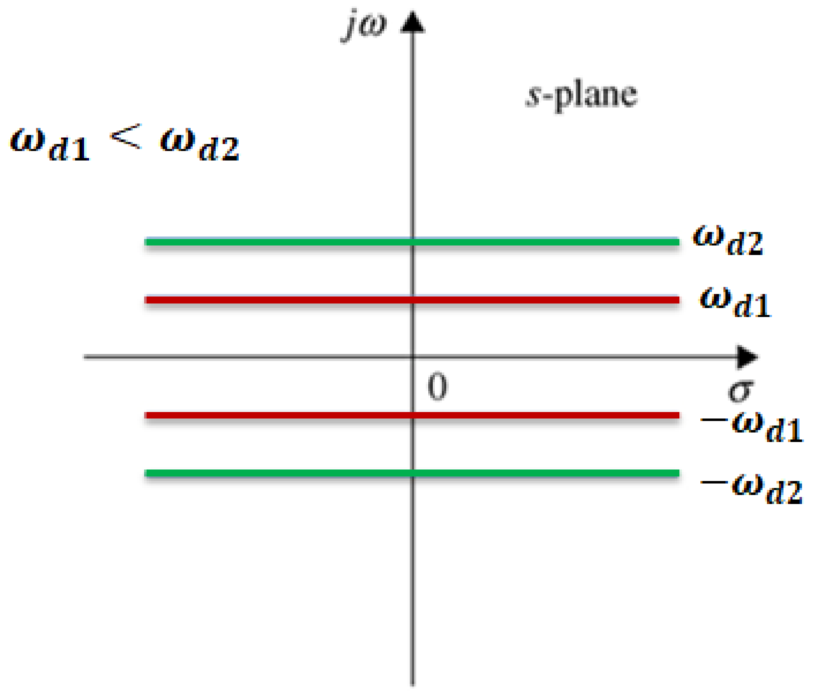
Time Response Specification and Pole Locations

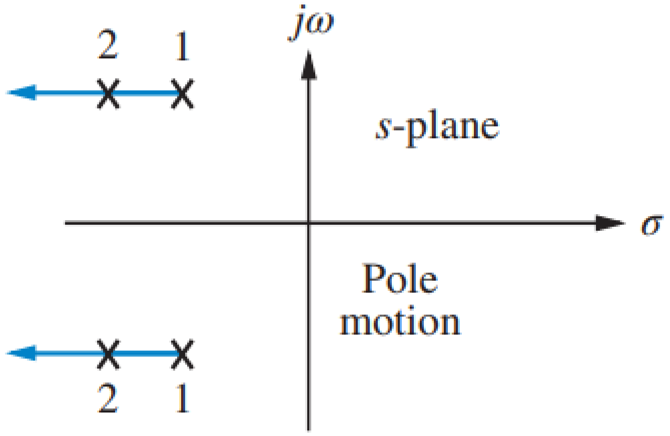
❑ Constant-Damped-Natural-Frequency , ω_d Loci

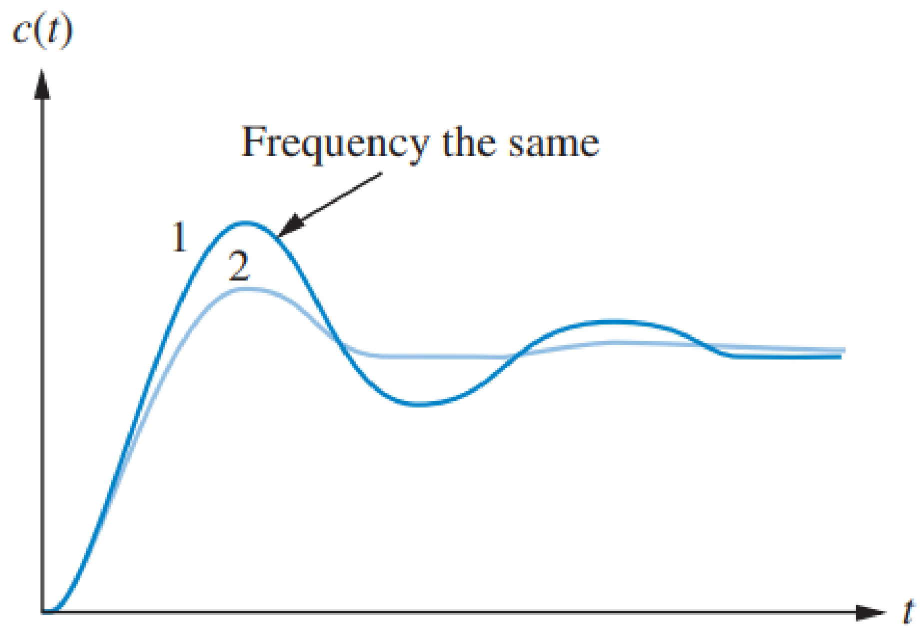
- Constant damped natural frequency ω_d loci in the s-plane are horizontal lines parallel to real axis.

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

- Poles moves horizontally away from the origin
  - Imaginary part of poles ω_d remains constant
  - Damping ratio ζ and natural frequency ω_n increase
  - Overshoot decreases
  - Rise-time and settling-time decreases
  - Peak-time remains constant







$$t_p = \frac{\pi}{\omega_d}$$

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n}$$

$$t_s \approx \frac{4}{\zeta\omega_n}$$

$$M_p = y_{ss}e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

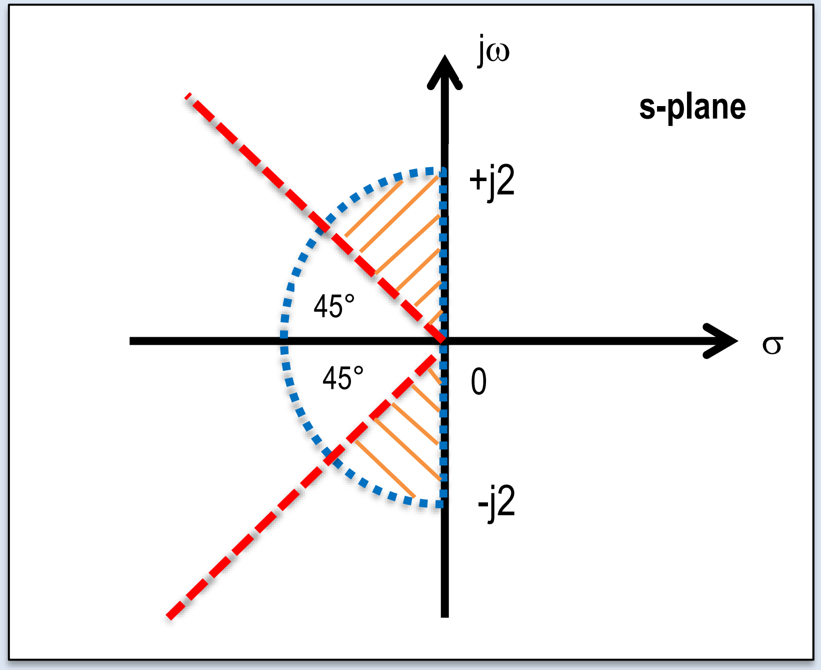
Quick Review

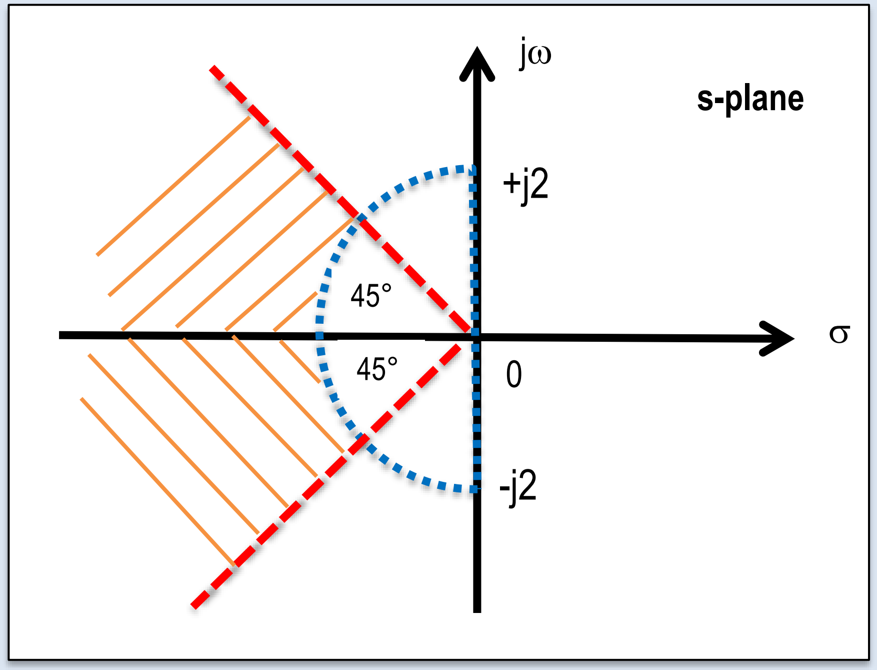
1. Match each specification with the given region in the s-plane in which the poles should be located.

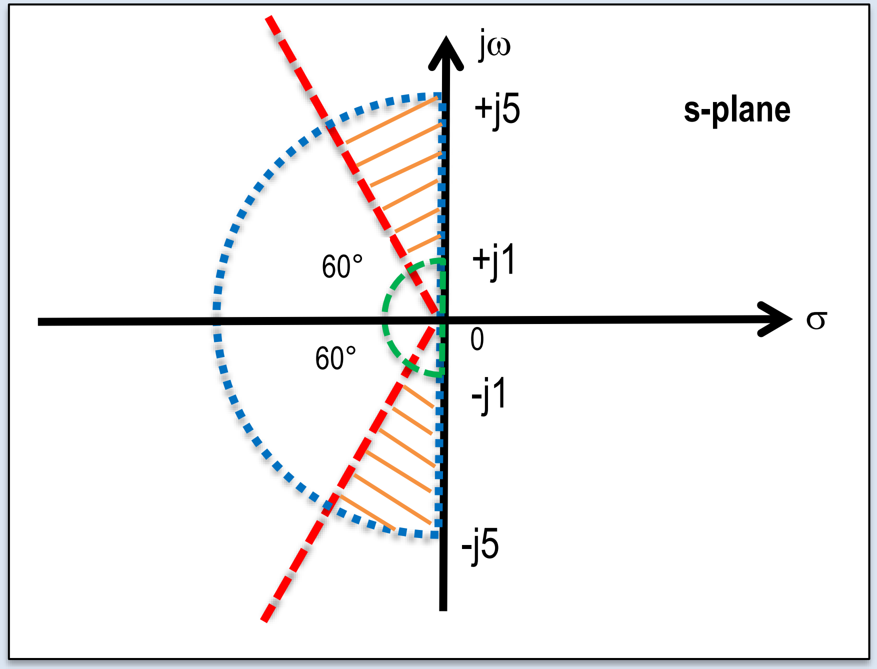
   a) $\zeta \geq 0.707, \omega_n \geq 2\ rad/s,$ positive damping

   b) $0 \leq \zeta \leq 0.707, \omega_n \leq 2\ rad/s,$ positive damping

   c) $\zeta \leq 0.5, 1 \leq \omega_n \leq 5\ rad/s,$ positive damping



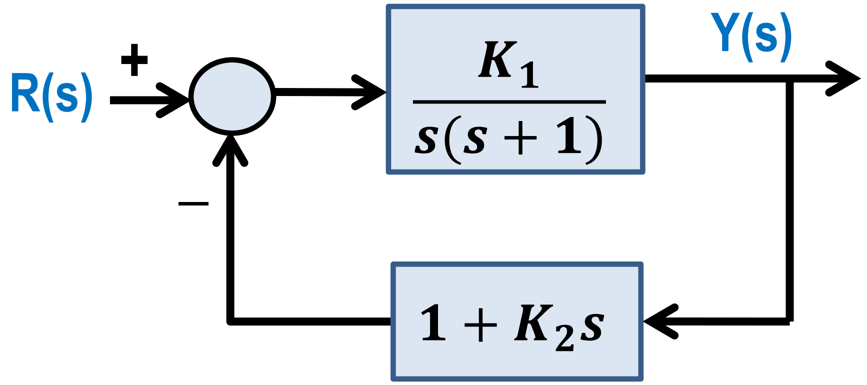




Performance of Second-Order Systems

Example 2

Consider the following closed-loop system



a) Determine the values of K_1 and K_2 so that the unit-step response has a maximum overshoot of 20% and the peak time is 1sec .

$$\boxed{\boldsymbol{O.S. = 20\% \quad \text{and} \quad t_p = 1\ sec}}$$

First calculate the damping ratio from the desired maximum overshoot value:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} = \frac{-\ln(0.2)}{\sqrt{\pi^2 + \ln^2(0.2)}} \rightarrow \boxed{\zeta = 0.456}$$

Desired Damping Ratio

Then, calculate the undamped natural frequency from the desired peak time value:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \rightarrow 1 = \frac{\pi}{\omega_n\sqrt{1-(0.456)^2}} \rightarrow \boxed{\omega_n = 3.53 \text{ rad/sec}}$$

Desired Natural Freq.

Next, having the desired damping ratio and natural frequency, determine the desired characteristic equation for this closed-loop system.

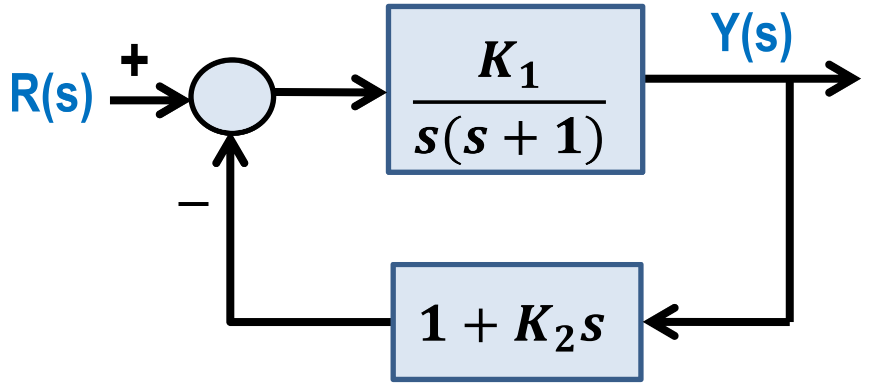
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = \boxed{s^2 + 3.2194s + 12.46}$$

Desired Characteristic Equation

Performance of Second-Order Systems

Example 2

Consider the following closed-loop system



a) Determine the values of K_1 and K_2 so that the unit-step response has a maximum overshoot of 20% and the peak time is $1 \sec$.

$$\boxed{\boldsymbol{O.S. = 20\% \quad \text{and} \quad t_p = 1\ sec}}$$

Find the characteristic equation of the closed-loop system in terms of the parameters K_1 and K_2 and compare with the desired characteristic equation to find the parameters.

The closed-loop transfer function is obtained as:

$$\boldsymbol{T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}} = \frac{\frac{K_1}{s(s+1)}}{1+\frac{K_1}{s(s+1)}(1+K_2 s)} = \frac{\frac{K_1}{s(s+1)}}{\frac{s(s+1)+K_1(1+K_2 s)}{s(s+1)}} = \frac{K_1}{s^2+(1+K_1K_2)s+K_1}$$

Compare the desired characteristic equation with the characteristic equation of the closed-loop system to find the parameters.

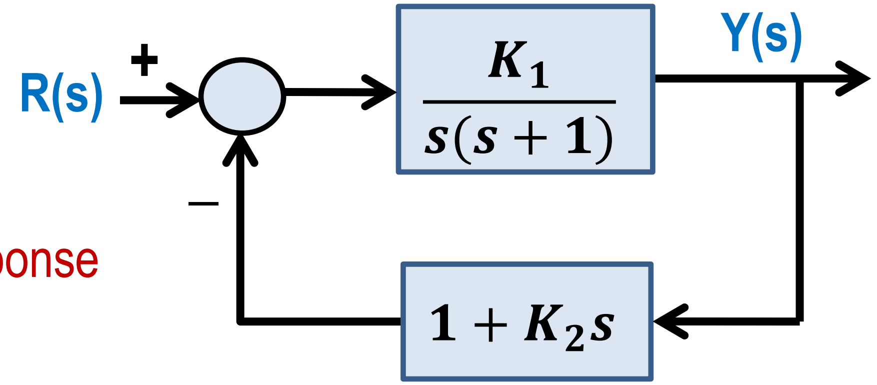
$$s^2 + 3.2194s + 12.46 = s^2 + (1+K_1K_2)s + K_1$$

$$\begin{cases} 1+K_1K_2 = 3.2194 \\ K_1 = 12.46 \end{cases} \quad \rightarrow \quad \boxed{K_1 = 12.46} \quad \boxed{K_2 = 0.1781}$$

Performance of Second-Order Systems

Example 2

Consider the following closed-loop system



b) Determine the closed-loop transfer function, rise time and settling time (2% criterion) of the unit-step response

Next, having the K_1 and K_2 values the closed-loop transfer function is:

$\boxed{\boldsymbol{K_1 = 12.46}}$

$\boxed{\boldsymbol{K_2 = 0.1781}}$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_1}{s^2 + (1 + K_1 K_2)s + K_1} = \frac{12.46}{s^2 + 3.2194s + 12.46}$$

The rise time and the settling time (2%) are obtained as below:

$\boxed{\boldsymbol{\omega_n = 3.53}}$

$\boxed{\boldsymbol{\zeta = 0.456}}$

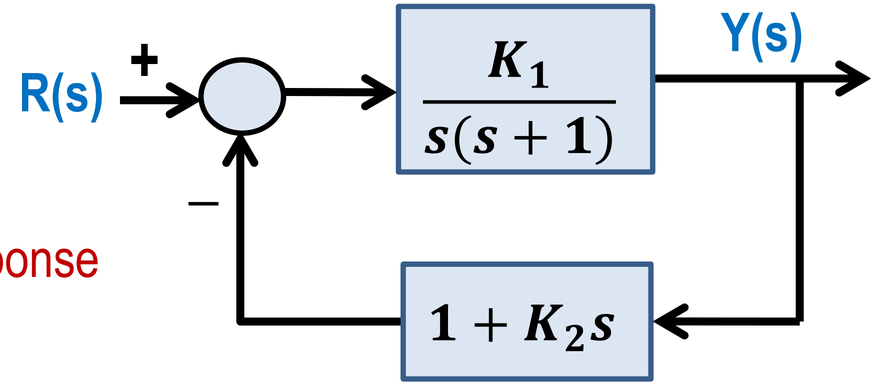
$$\boldsymbol{t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n}} \quad \rightarrow \quad t_r = \frac{0.8 + 2.5 \times 0.456}{3.53} = 0.5496\ sec$$ **Rise-time**

$$\boldsymbol{t_s \cong \frac{4}{\zeta\omega_n}} \quad \rightarrow \quad t_s = \frac{4}{0.456 \times 3.53} = 2.4850\ sec$$ **Settling-time**

Performance of Second-Order Systems

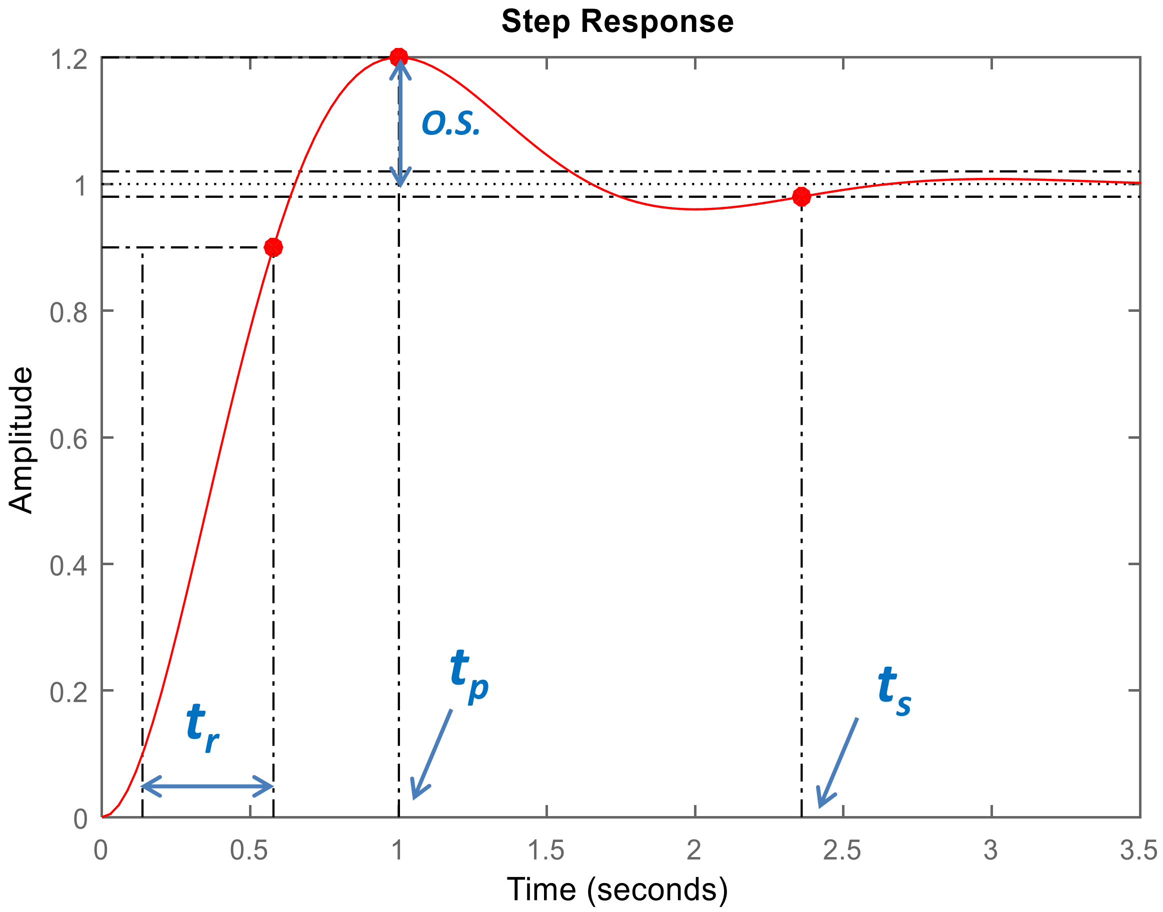
Example 2

Consider the following closed-loop system



b) Determine the closed-loop transfer function, rise time and settling time (2% criterion) of the unit-step response

We can also plot the step-response of the closed-loop system in MATLAB to check the results:



$$\frac{Y(s)}{R(s)} = \frac{12.46}{s^2 + 3.2194s + 12.46}$$

```
num = [12.46];
den = [1   3.2194   12.46];
sys = tf(num,den);
step(sys)
```

$t_p = 1\ sec$, $t_r = 0.5496\ sec$

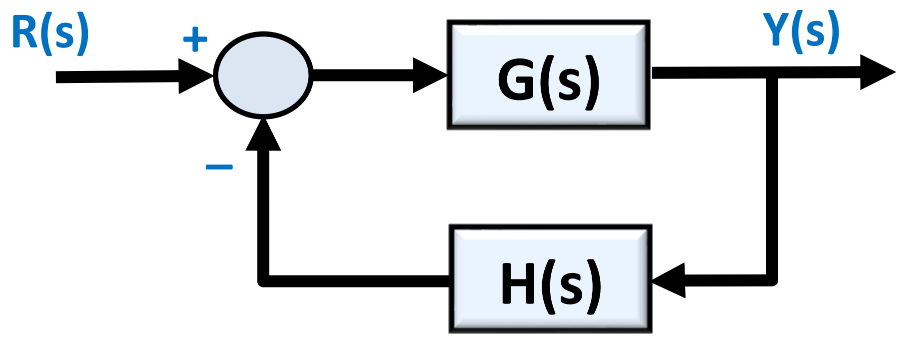
$O.S. = 0.2$, $t_s = 2.4850\ sec$

The Steady-State Error of Feedback Control Systems

Steady-State Error of Control Systems

- Consider the following stable closed-loop system
- **Steady-state Error** or **Tracking Error** is the error between reference input and actual output in a closed-loop system after the transient response has decayed.

Steady-state error is only defined if the system is STABLE, it shows the tracking capability of the control system.



Error = Reference Input – Actual Output

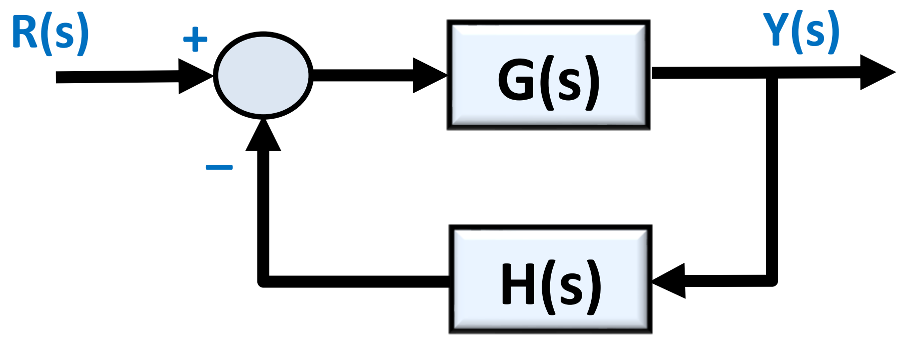
Error = R(s) – Y(s)

❑ Factors that affect the Steady-state Error:

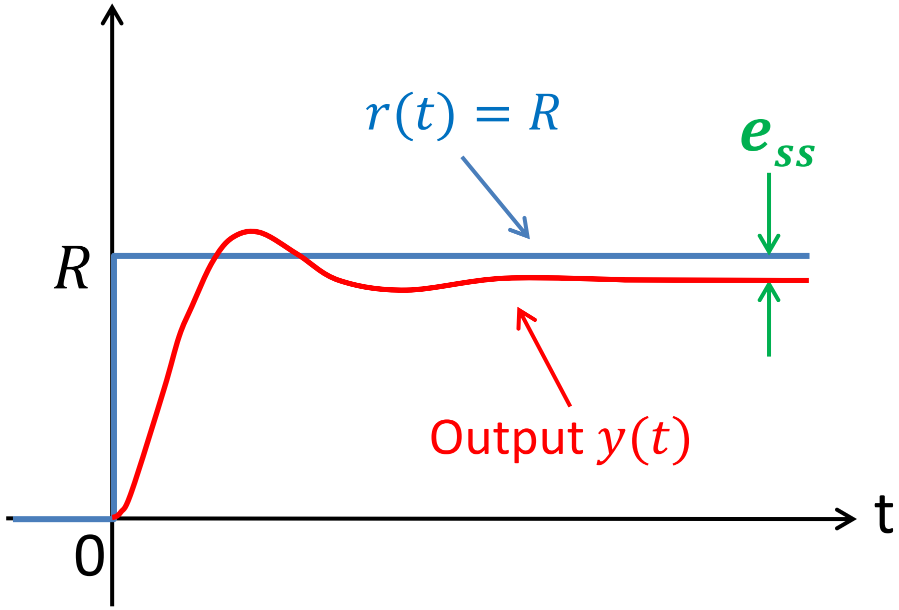
- Instrumentation & Measurement error
  - Sensors and Transducers
- System non-linearities
  - Dead-zone, static friction in DC motor
  - Saturation, amplifier system
  - Backlash, gear system
- External disturbances
  - Unwanted external inputs
  - Load changing
- **Type of the system transfer function**
- **Type of the input signal**

Steady-State Error of Control Systems

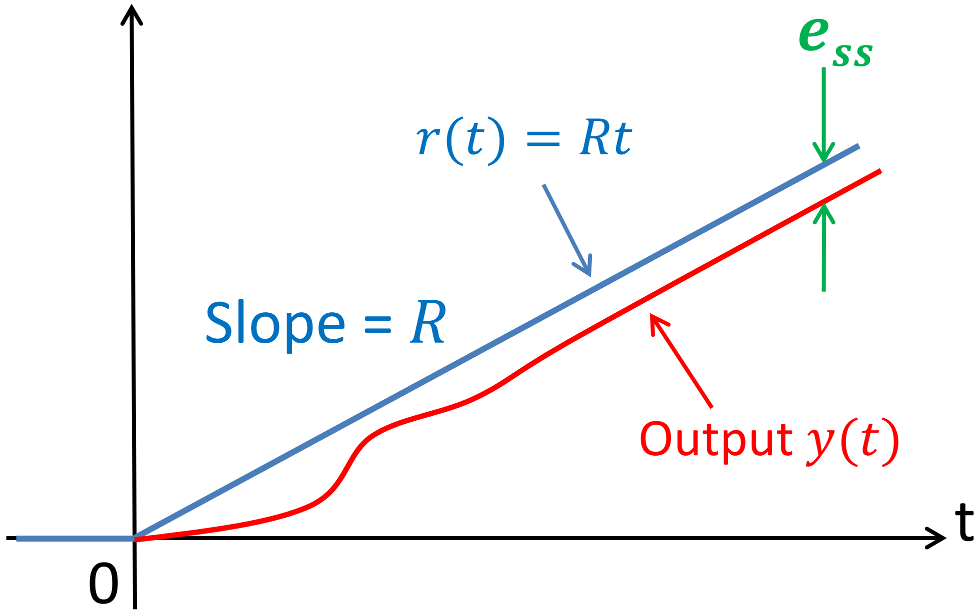
❑ Typical Test Signals for Steady-state Error



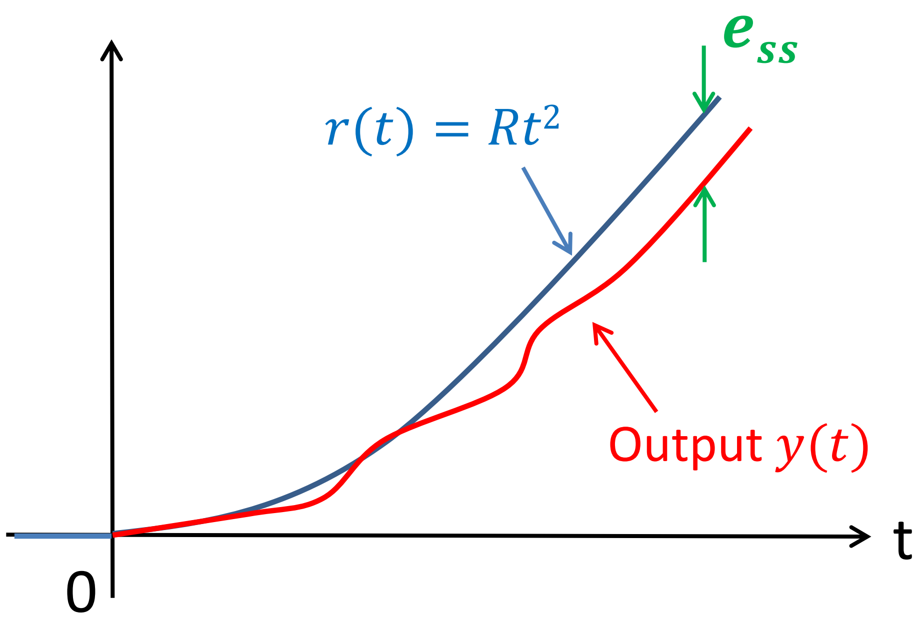
- **Step Input:** $r(t) = R\,, \quad t \geq 0$



- **Ramp Input:** $r(t) = Rt\,, \quad t \geq 0$



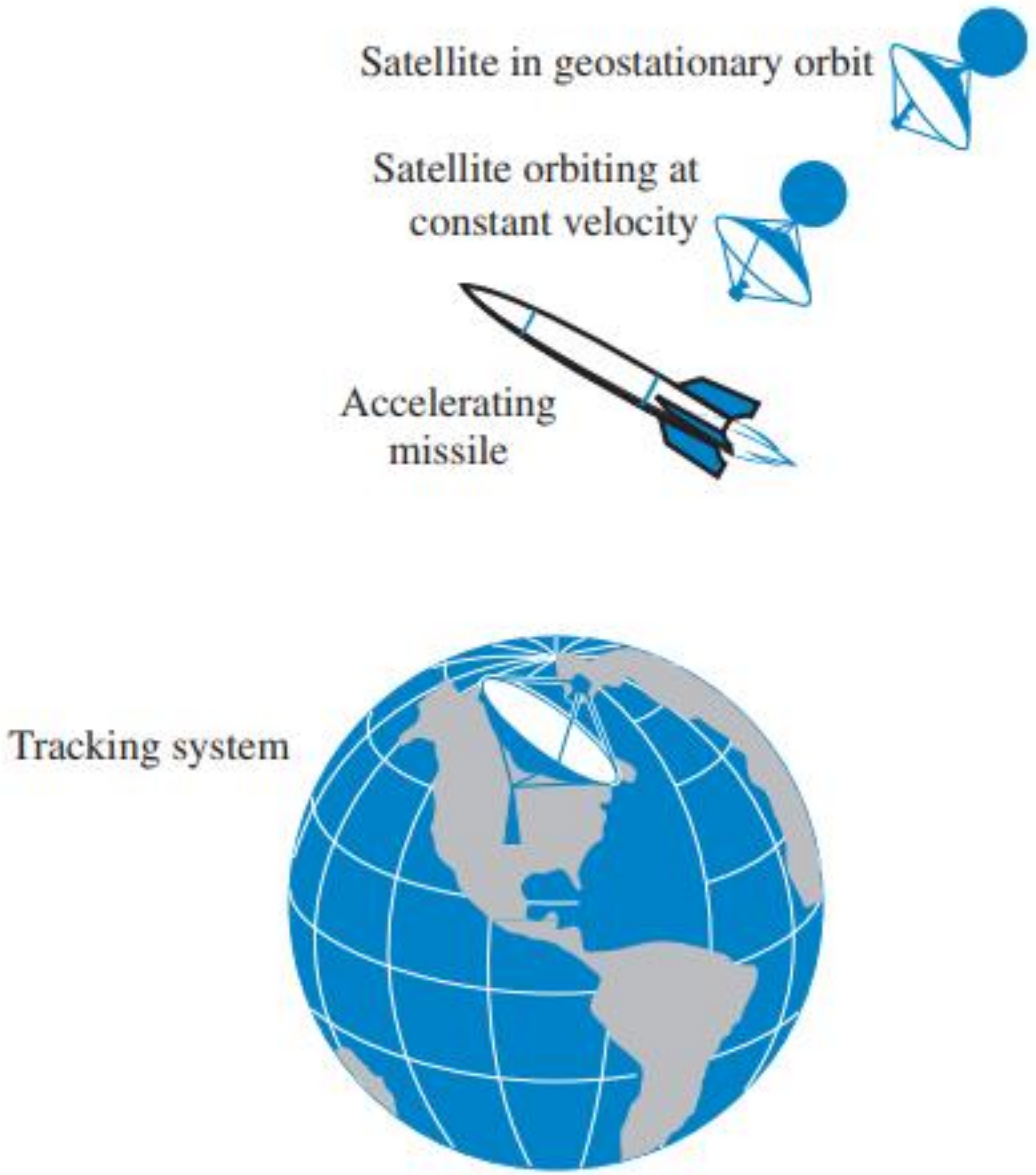
- **Parabolic Input:** $r(t) = Rt^2\,, \quad t \geq 0$



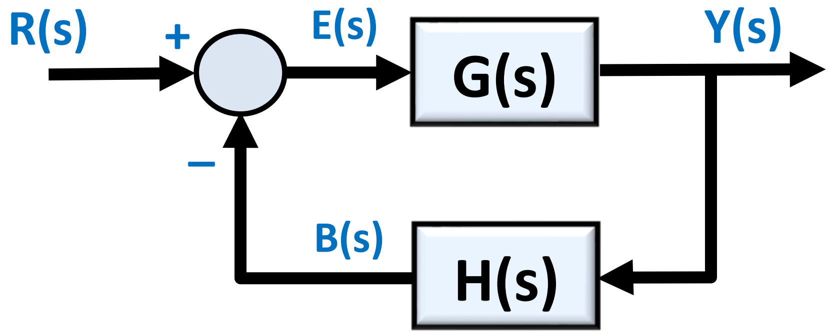
Steady-State Error of Control Systems

❑ Typical Test Signals for Steady-state Error

- Example in a positioning control system:
  - Step Input: Determining the ability of the control system to position itself with respect to a stationary target.
  - Ramp Input: Tracking a satellite that moves across the sky at a constant angular velocity.
  - Parabolic Input: Tracking an accelerating target such as a missile.



Steady-State Error of Control Systems



- Consider the following stable closed-loop system with the following closed-loop transfer function

$$\boxed{\boldsymbol{T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}}}$$

- The Error signal is determined as

$$E(s) = R(s) - B(s) \quad \rightarrow \quad E(s) = R(s) - H(s)Y(s) = R(s) - H(s)T(s)R(s)$$

$$E(s) = R(s) - \frac{G(s)H(s)}{1 + G(s)H(s)}R(s) = \underbrace{\frac{\mathbf{1}}{\mathbf{1 + G(s)H(s)}}}_{\text{System dependent}} \underbrace{\boldsymbol{R(s)}}_{\text{Input dependent}}$$

- Steady-state error is determined from the final-value theorem.

$$\boldsymbol{e_{ss} = e(\infty) = \lim_{t \to \infty} e(t)} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s}{1 + \boldsymbol{G(s)H(s)}} \boldsymbol{R(s)}$$

- If the closed-loop system has unity-feedback, the error signal $E(s)$ is truly represent the error between the reference input $R(s)$ and the output signal $Y(s)$.

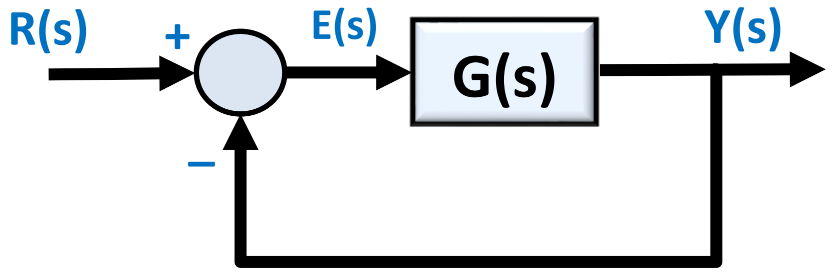
$$\text{If} \quad \boldsymbol{H(s) = 1} \quad \rightarrow \quad E(s) = R(s) - Y(s)$$

- In this case, the steady-state error formula can be simplified as,

$$\boxed{\boldsymbol{e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} R(s)}}$$

Steady-State Error of Unity-Feedback Systems

- The steady-state error formula can be simplified based on the test input signal.



$$\boxed{e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} R(s)}$$

Static Error Constants

Step Input

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)}\left(\frac{R}{s}\right) = \lim_{s\to 0} \frac{R}{1+G(s)} = \frac{R}{1+\underbrace{\lim_{s\to 0} G(s)}_{k_p}}$$

$$\boxed{k_p = \lim_{s\to 0} G(s)} \longrightarrow \boxed{e_{ss} = \frac{R}{1+k_p}}$$

Step-error Constant

Position-error Constant

Ramp input

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)}\left(\frac{R}{s^2}\right) = \lim_{s\to 0} \frac{R}{s+sG(s)} = \frac{R}{\underbrace{\lim_{s\to 0} sG(s)}_{k_v}}$$

$$\boxed{k_v = \lim_{s\to 0} sG(s)} \longrightarrow \boxed{e_{ss} = \frac{R}{k_v}}$$

Ramp-error Constant

Velocity-error Constant

Parabolic Input

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)}\left(\frac{R}{s^3}\right) = \lim_{s\to 0} \frac{R}{s^2+s^2G(s)} = \frac{R}{\underbrace{\lim_{s\to 0} s^2G(s)}_{k_a}}$$

$$\boxed{k_a = \lim_{s\to 0} s^2G(s)} \longrightarrow \boxed{e_{ss} = \frac{R}{k_a}}$$

Parabolic-error Constant

Acceleration-error Constant

- The steady-state error can be a finite or infinite value depends on the **type of the system $G(s)$.**

Steady-State Error of Unity-Feedback Systems

Type of a Transfer Function

- **Type of a transfer function** refers to the number of poles of the transfer function at the origin $s = 0$.
- **Type of a system** also indicates the number of integrators ($\frac{1}{s}$) in the system.

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

Example 3

Determine type of the following transfer functions.

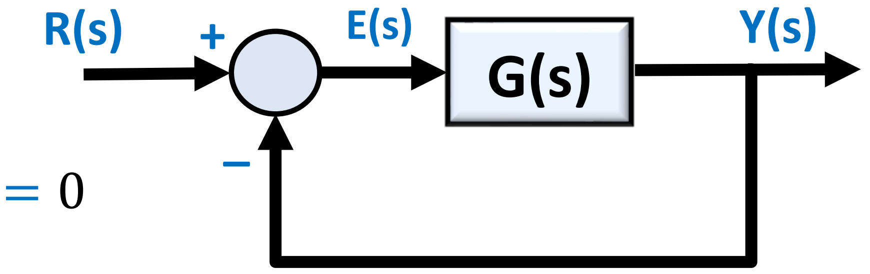
$G(s) = \frac{2s+1}{s^4(s+1)}$ $\rightarrow$ four poles at $s = 0$ $\rightarrow$ $Type\ 4$

$G(s) = \frac{2(s+5)}{s+3}$ $\rightarrow$ no pole at $s = 0$ $\rightarrow$ $Type\ 0$

$G(s) = \frac{10}{s\,(s+5)}$ $\rightarrow$ one pole at $s = 0$ $\rightarrow$ $Type\ 1$

$G(s) = \frac{2(s+1)}{s^2\,(s+5)}$ $\rightarrow$ two poles at $s = 0$ $\rightarrow$ $Type\ 2$

Steady-State Error of Unity-Feedback Systems



❑ Type 0 Systems

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$k_p = \lim_{s\to 0} G(s) = constant \qquad k_v = \lim_{s\to 0} sG(s) = 0 \qquad k_a = \lim_{s\to 0} s^2G(s) = 0$$

$$e_{ss} = \frac{R}{1+k_p} = constant \qquad e_{ss} = \frac{R}{k_v} = \infty \qquad e_{ss} = \frac{R}{k_a} = \infty$$

❑ Type 1 Systems

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{s(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$k_p = \lim_{s\to 0} G(s) = \infty \qquad k_v = \lim_{s\to 0} sG(s) = constant \qquad k_a = \lim_{s\to 0} s^2G(s) = 0$$

$$e_{ss} = \frac{R}{1+k_p} = 0 \qquad e_{ss} = \frac{R}{k_v} = constant \qquad e_{ss} = \frac{R}{k_a} = \infty$$

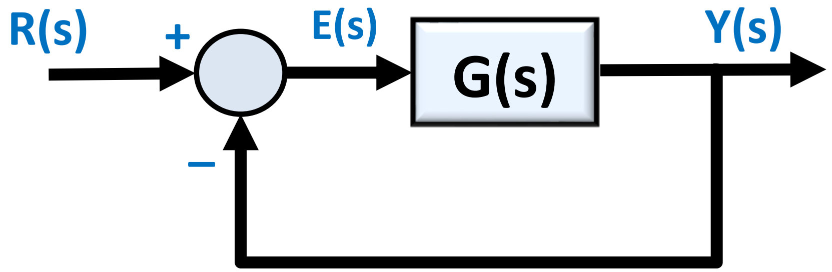
❑ Type 2 Systems

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{s^2(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$k_p = \lim_{s\to 0} G(s) = \infty \qquad k_v = \lim_{s\to 0} sG(s) = \infty \qquad k_a = \lim_{s\to 0} s^2G(s) = constant$$

$$e_{ss} = \frac{R}{1+k_p} = 0 \qquad e_{ss} = \frac{R}{k_v} = 0 \qquad e_{ss} = \frac{R}{k_a} = constant$$

Steady-State Error of Unity-Feedback Systems



- The values of the static error constants depend on the value and type of $G(s)$.

$$k_p = \lim_{s \to 0} G(s) \qquad k_v = \lim_{s \to 0} sG(s) \qquad k_a = \lim_{s \to 0} s^2 G(s)$$

$$e_{ss} = \frac{R}{1+k_p} \qquad e_{ss} = \frac{R}{k_v} \qquad e_{ss} = \frac{R}{k_a}$$

- The steady-state error can be a finite or infinite value depends on the type of the system $G(s)$.
- The following table summarize the relationships between input, system $G(s)$ type, static error constants, and steady-state errors.

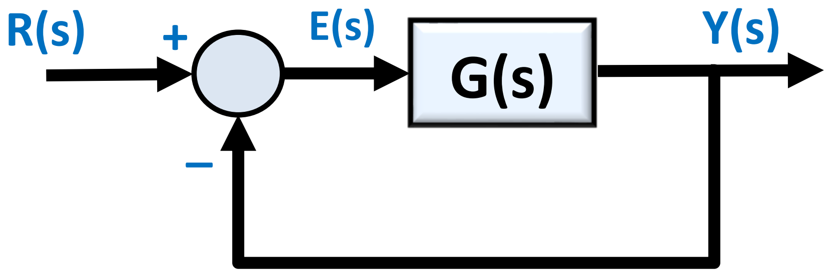
	G(s) is Type 0		G(s) is Type 1		G(s) is Type 2		G(s) is Type 3	
Step Input	k_p = constant	$e_{ss} = \frac{R}{1+k_p}$	$k_p = \infty$	$e_{ss} = 0$	$k_p = \infty$	$e_{ss} = 0$	$k_p = \infty$	$e_{ss} = 0$
Ramp Input	$k_v = 0$	$e_{ss} = \infty$	k_v = constant	$e_{ss} = \frac{R}{k_v}$	$k_v = \infty$	$e_{ss} = 0$	$k_v = \infty$	$e_{ss} = 0$
Parabolic Input	$k_a = 0$	$e_{ss} = \infty$	$k_a = 0$	$e_{ss} = \infty$	k_a = constant	$e_{ss} = \frac{R}{k_a}$	$k_a = \infty$	$e_{ss} = 0$

Steady-State Error of Unity-Feedback Systems

Example 4

Consider the following unity feedback system with the given system G(s)

$$G(s) = \frac{10}{5s+1}$$



a) Determine type of the open-loop system

$$\text{Open}-\text{loop} \quad \text{TF} \quad \rightarrow \quad G(s) = \frac{10}{5s+1} \quad \rightarrow \quad \text{Type } 0 \ \ (\text{No integrator})$$

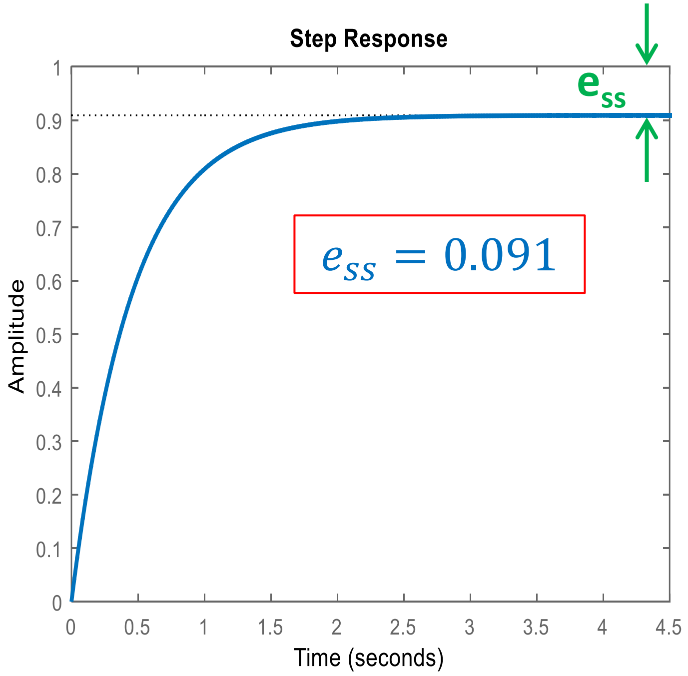
b) Determine steady-state error of the closed-loop system to unit-step and unit-ramp inputs.

Unit-step Input:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{10}{5s+1} = 10$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{11} = 0.091$$

Since the open-loop system is Type 0, the output signal can follow the step input with a finite steady-state error.

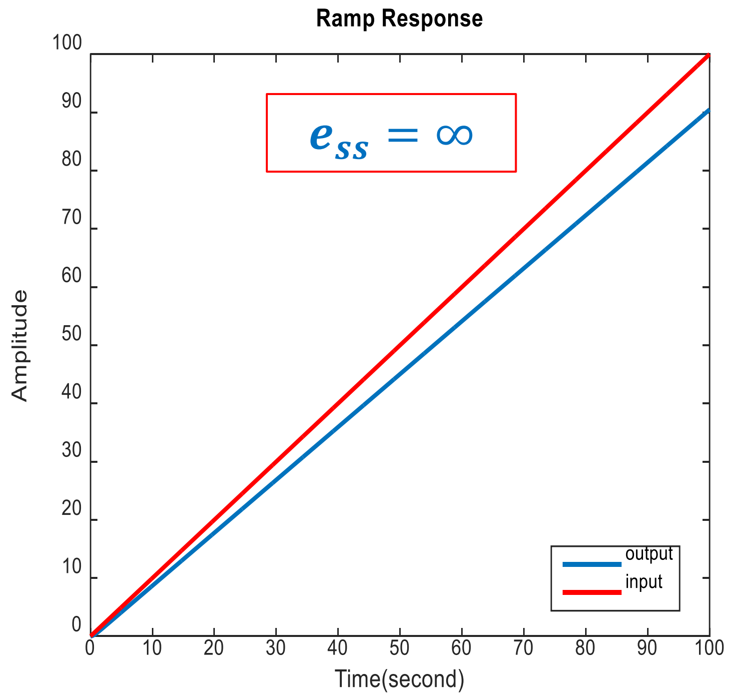


Unit-ramp Input:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{10s}{5s+1} = 0$$

$$e_{ss} = \frac{R}{k_v} = \frac{1}{0} = \infty$$

Since the open-loop system is Type 0, the output signal cannot follow the ramp input.

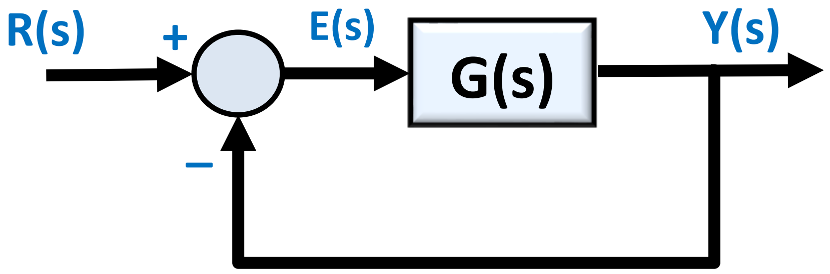


Steady-State Error of Unity-Feedback Systems

Example 5

Consider the following unity feedback system with the given system G(s)

$$G(s) = \frac{2}{s(s+1)}$$



a) Determine type of the open-loop system.

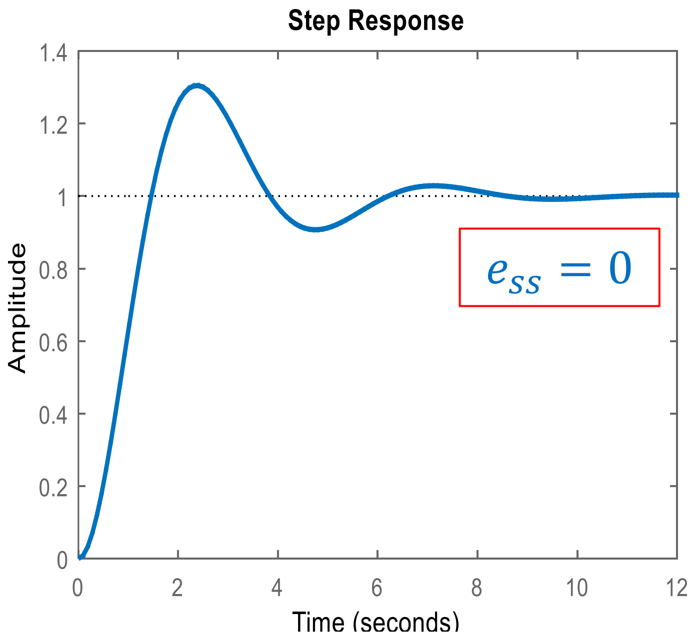
$$\text{Open} - \text{loop} \quad \text{TF} \quad \rightarrow \quad G(s) = \frac{2}{s(s+1)} \quad \rightarrow \quad \text{Type 1} \quad (\text{One integrator})$$

b) Determine steady-state error of the closed-loop system to unit-step and unit-ramp inputs.

Unit-step Input:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{2}{s(s+1)} = \infty$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{\infty} = 0$$

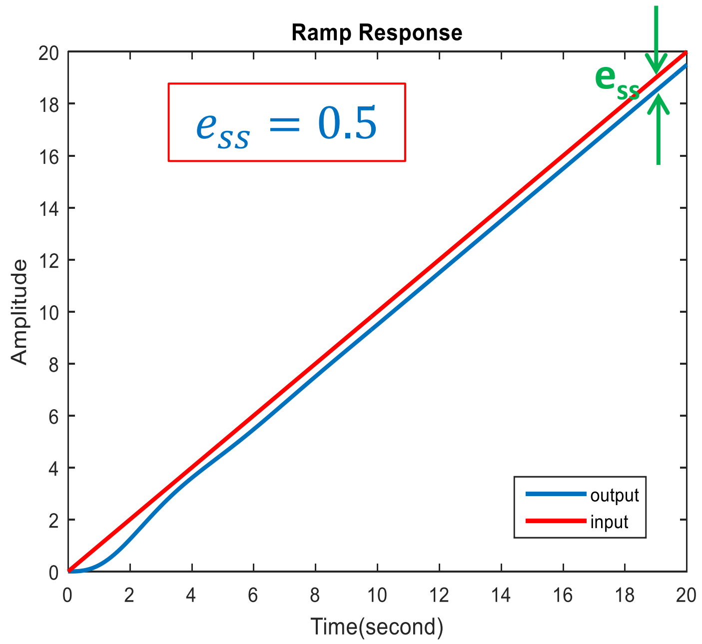


Since the open-loop system is Type 1, the output signal can follow the step input with zero steady-state error.

Unit-ramp Input:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{2}{s+1} = 2$$

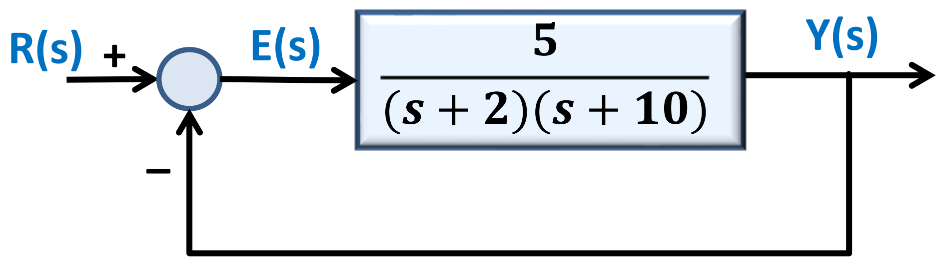
$$e_{ss} = \frac{R}{k_v} = \frac{1}{2} = 0.5$$



Since the open-loop system is Type 1, the output signal can follow the ramp input with a finite steady-state error.

Steady-State Error of Unity-Feedback Systems

Example 6 Consider the following unity-feedback closed-loop control system



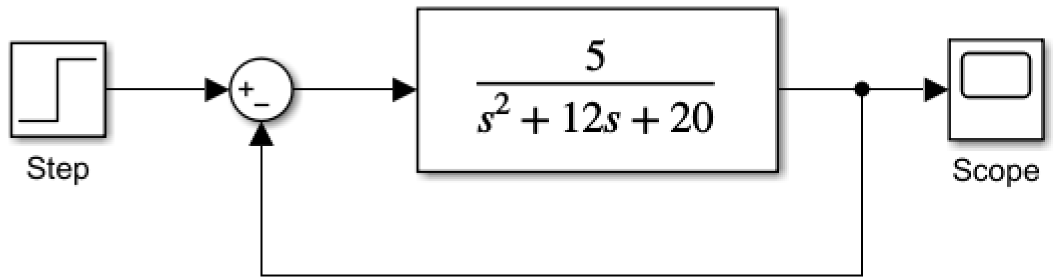
a) Find the steady-state error of the closed-loop system for unit-step input.

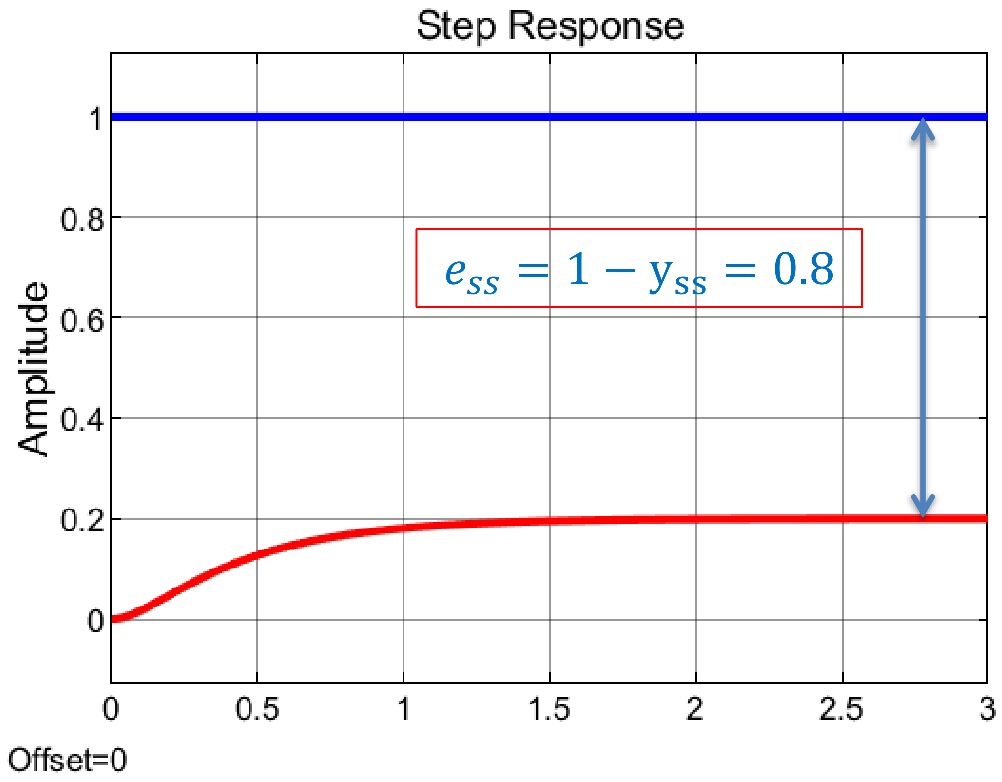
Since the system is type 0, first find the step-error constant, then calculate e_{ss}

$$k_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \left(\frac{5}{(s+2)(s+10)} \right) = 0.25$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{1+0.25} \quad \rightarrow \quad e_{ss} = \frac{1}{1.25} = 0.8 \quad \rightarrow \quad e_{ss} = 80\%$$

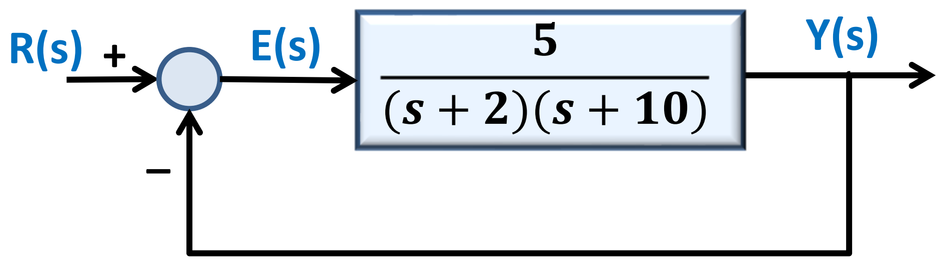
We can also simulate the system in **Simulink** and plot the unit-step response graph to check the steady-state error.



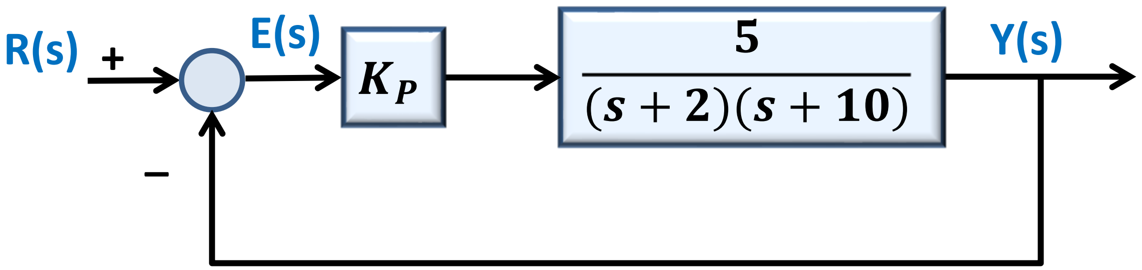


Steady-State Error of Unity-Feedback Systems

Example 6 Consider the following unity-feedback closed-loop control system



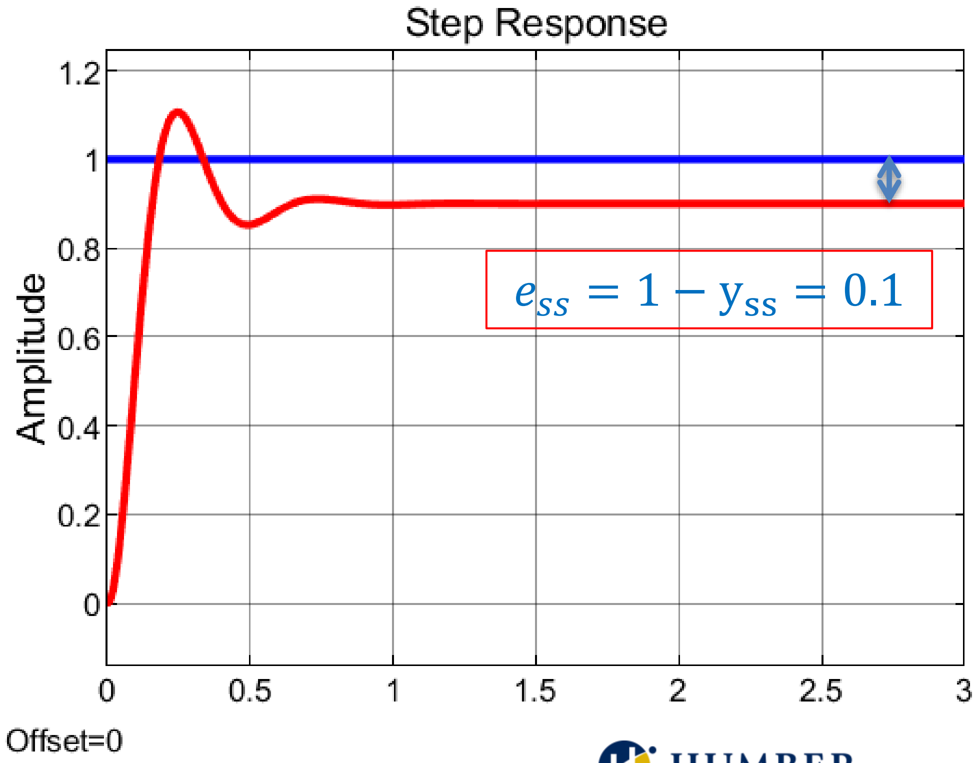
b) Determine the required proportional controller gain K_p to have a 10% steady-state error.



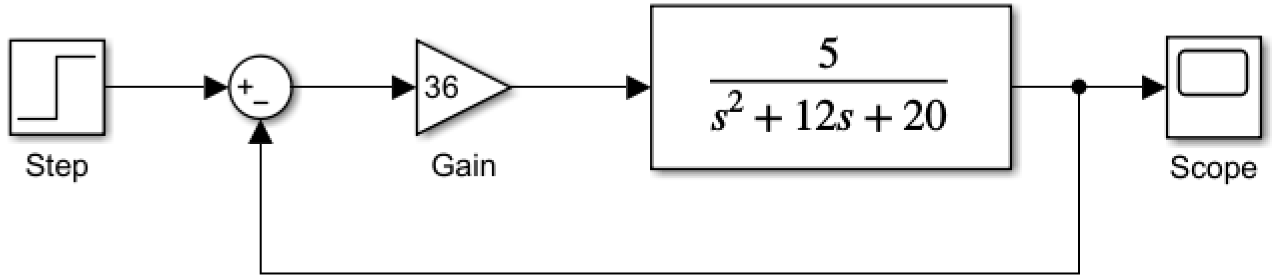
First find the step-error constant in terms of K_p then calculate e_{ss} to find the desired K_p value.

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0}\left(\frac{5K_p}{(s+2)(s+10)}\right) = 0.25K_p$$

$$e_{ss} = \frac{R}{1+k_p} = \frac{1}{1+0.25K_p} \quad\rightarrow\quad 0.1 = \frac{1}{1+0.25K_p} \quad\rightarrow\quad \boxed{K_p = 36}$$



We can plot the unit-step response graph in **Simulink**.

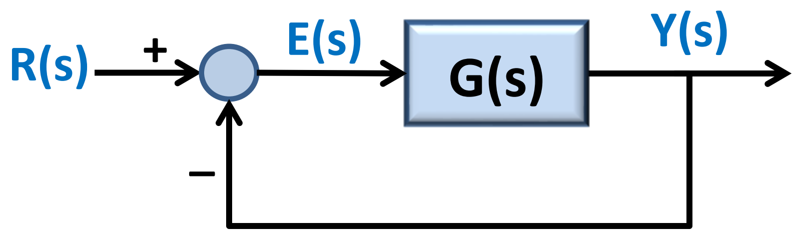


Steady-State Error of Unity-Feedback Systems

Example 7

Consider the unity-feedback control system whose open-loop transfer function is G(s)

$$G(s) = \frac{100}{s(0.1s+1)}$$



Determine the steady-state error when the input is: $r(t) = (1+t)u_s(t)$

Here, we have to use the general formula of the steady-state error for unity-feedback systems:

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} R(s)}$$

First, find the Laplace transform of the input signal

$$R(s) = \mathcal{L}[r(t)] \quad \rightarrow \quad R(s) = \mathcal{L}[1+t] = \frac{1}{s} + \frac{1}{s^2} = \frac{s+1}{s^2}$$

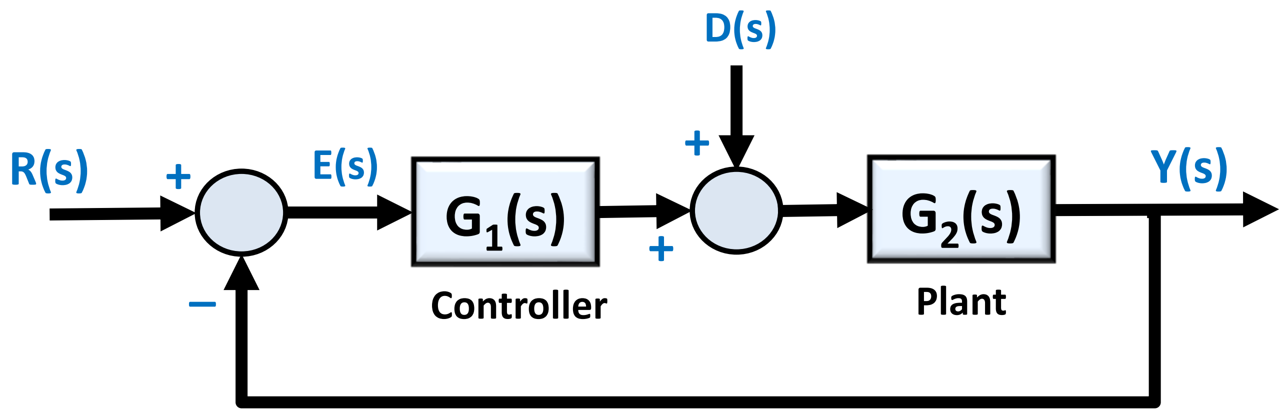
Next, determine the steady-state error

$$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} R(s) = \lim_{s \to 0} \left(\frac{s^2(0.1s+1)}{0.1s^2+s+100} \cdot \frac{s+1}{s^2} \right) \quad \rightarrow \quad \boxed{e_{ss} = 0.01}$$

Steady-State Error

Steady-State Error of Disturbances

- Consider the unity-feedback system with disturbance $D(s)$, and the reference input of $R(s)$.



- We can derive the Error signal expression with the disturbance included

$$E(s) = R(s) - Y(s)$$

$$E(s) = R(s) - \big(G_1(s)E(s) + D(s)\big)G_2(s) \quad \rightarrow \quad E(s) = R(s) - G_1(s)E(s)G_2(s) - D(s)G_2(s)$$

$$E(s) = \underbrace{\frac{1}{1 + G_1(s)G_2(s)}}_{\text{Error transfer function for reference input}} R(s) - \underbrace{\frac{G_2(s)}{1 + G_1(s)G_2(s)}}_{\text{Error transfer function for disturbance}} D(s)$$

- Steady-state error is determined from the final-value theorem.

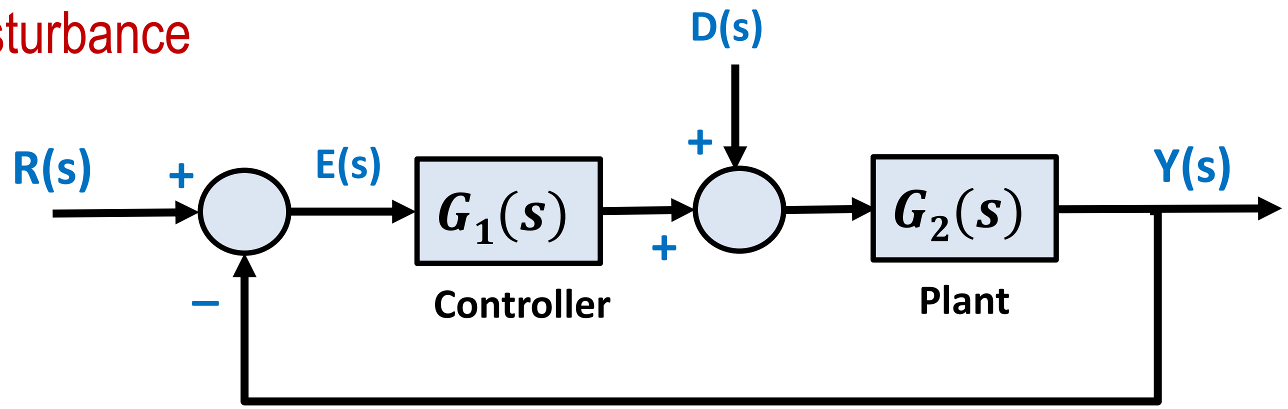
$$e_{ss} = e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \underbrace{\lim_{s\to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s)}_{\substack{\text{Steady-state error due to reference input R(s)} \\ e_{ss,R}}} + \underbrace{\lim_{s\to 0} \frac{-sG_2(s)}{1 + G_1(s)G_2(s)} D(s)}_{\substack{\text{Steady-state error due to the disturbance D(s)} \\ e_{ss,D}}}$$

Steady-State Error of Disturbances

Example 6

Consider the following unity-feedback system with disturbance

$$G_1(s) = K_p \qquad G_2(s) = \frac{1}{s+5}$$



a) Determine the controller gain, K_p, such that the steady-state error due to a unit-step disturbance is 1%.

Steady-state error due to disturbance is:

$$e_{ss,D} = \lim_{s\to 0}\frac{-sG_2(s)}{1+G_1(s)G_2(s)}D(s) = \lim_{s\to 0}\frac{-\frac{s}{s+5}}{1+\frac{K_p}{s+5}}\left(\frac{1}{s}\right) = \lim_{s\to 0}\frac{-s}{s+5+K_p}\left(\frac{1}{s}\right) = \frac{-1}{5+K_p}$$

We can calculate required gain K_p for disturbance error of 1%:

$$0.01 = \left|\frac{-1}{5+K_p}\right| \quad \rightarrow \quad \boxed{K_p = 95}$$

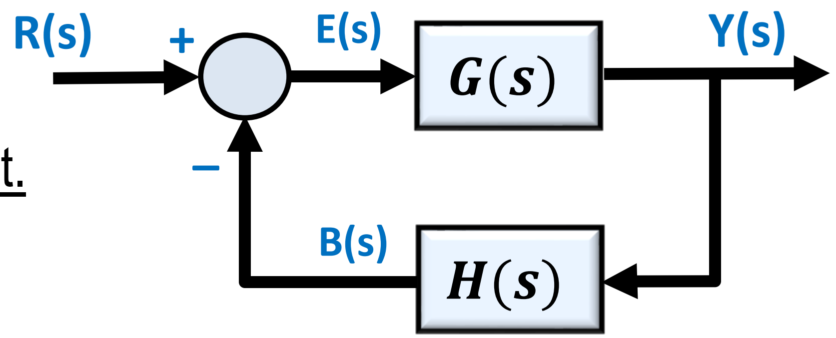
b) For this value of K_p, what is the steady-state error to a unit-step reference input?

Steady-state error due to reference input is:

$$e_{ss,R} = \lim_{s\to 0}\frac{s}{1+G_1(s)G_2(s)}R(s) = \lim_{s\to 0}\frac{s}{1+\frac{K_p}{s+5}}\left(\frac{1}{s}\right) = \lim_{s\to 0}\frac{s(s+5)}{s+5+K_p}\left(\frac{1}{s}\right) = \frac{5}{5+K_p} = \frac{5}{5+95} = 0.05 \quad \rightarrow \quad \boxed{e_{ss,R} = 5\%}$$

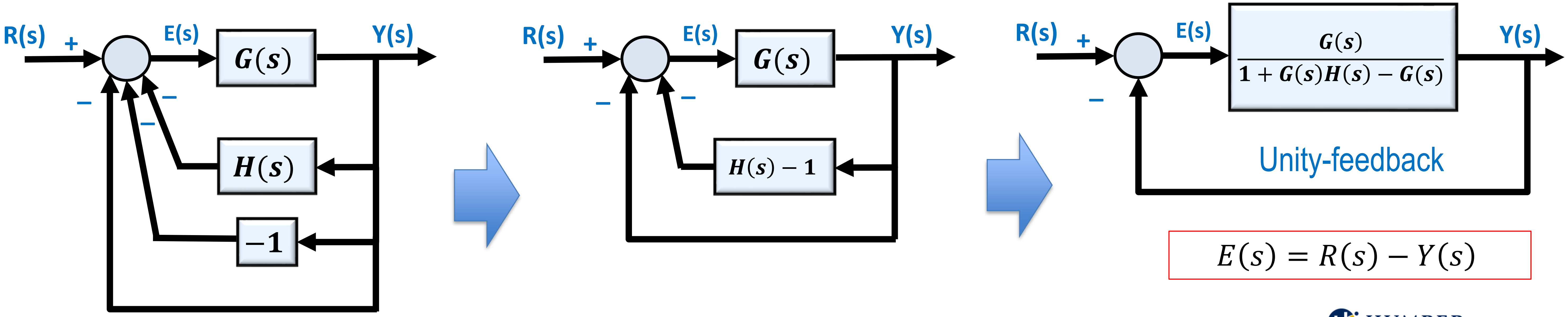
Steady-State Error of Non-Unity-Feedback Systems

- Control systems often do not have unity-feedback because of the compensation used to improve performance or because of the physical model for the system.
- The feedback path can be a pure gain other than unity or have some dynamic representation.



- Unlike a unity-feedback system, here the error is **not** the difference between the input and the output.
- It is possible to convert the general form to unity-feedback configuration and apply the previously learned methods to find the steady-state error:

1) Add and subtract unity-feedback paths
2) Combine $H(s)$ with negative one unity feedback path
3) Combine the feedback system consisting of $G(s)$ and $H(s)-1$

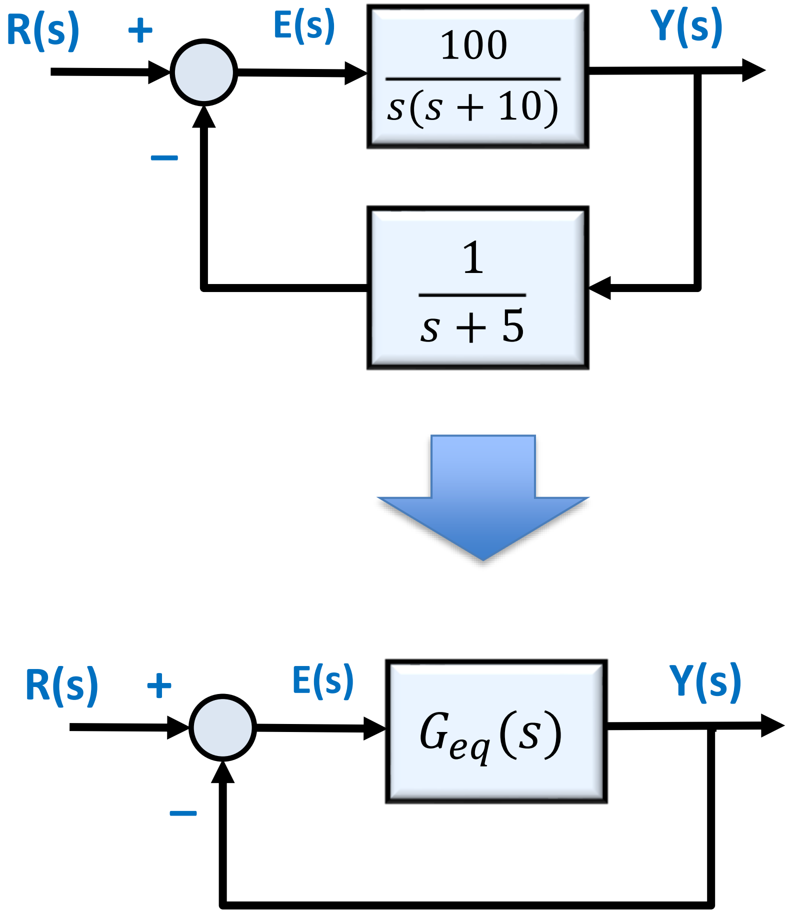


$$E(s) = R(s) - Y(s)$$

Steady-State Error of Non-Unity-Feedback Systems

Example 7

Given the closed-loop system, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input.



First, transform the given closed-loop system to a unity-feedback form.

$$G_{eq}(s) = \frac{\frac{100}{s(s+10)}}{1 + \frac{100}{s(s+10)(s+5)} - \frac{100}{s(s+10)}} = \frac{\frac{100}{s(s+10)}}{\frac{s(s+10)(s+5) + 100 - 100(s+5)}{s(s+10)(s+5)}}$$

$$G_{eq}(s) = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400} \quad \rightarrow \quad \textbf{Type 0}$$

$$\boldsymbol{G_{eq}(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}}$$

The appropriate error constant is step-error constant k_p:

$$k_p = \lim_{s \to 0} G_{eq}(s) = \lim_{s \to 0} \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400} = -\frac{500}{400} = -1.25$$

The steady-state error for unit-step input is:

$$e_{ss} = \frac{R}{1 + k_p} = \frac{1}{1 - 1.25} = -4$$

The negative value for steady-state error implies that the output step is larger than the input step.

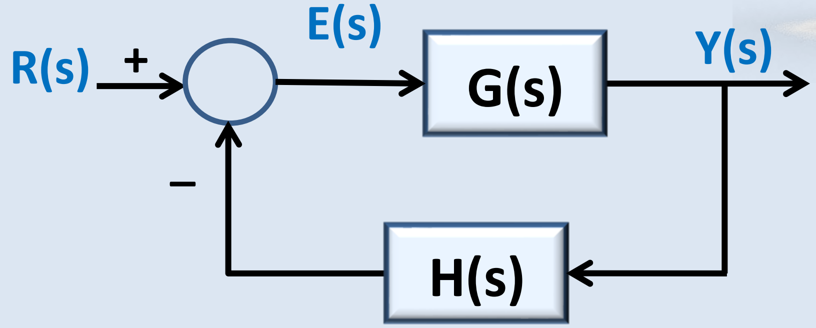
Quick Review

1) Determine steady-state error of the unity-feedback closed-loop system for unit-step input:

a) 0 b) 0.5

c) ∞ d) 0.67

$$G(s) = \frac{0.5}{s(s^2+s+1)}\ , \qquad H(s) = 1$$

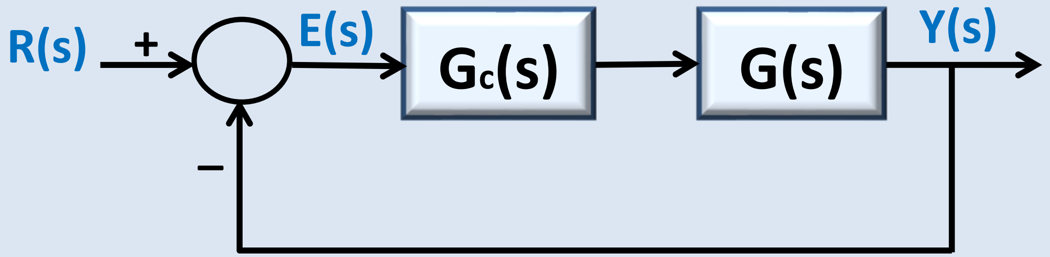


2) Which controller can provide the zero steady-state error for unit-ramp input:

a) $G_c(s) = 0.5$ b) $G_c(s) = \frac{0.5(s+0.1)}{s}$

c) $G_c(s) = \frac{0.5}{s+10}$ d) $G_c(s) = \frac{0.5s}{s+2}$

$$G(s) = \frac{1}{s(s+1)}$$



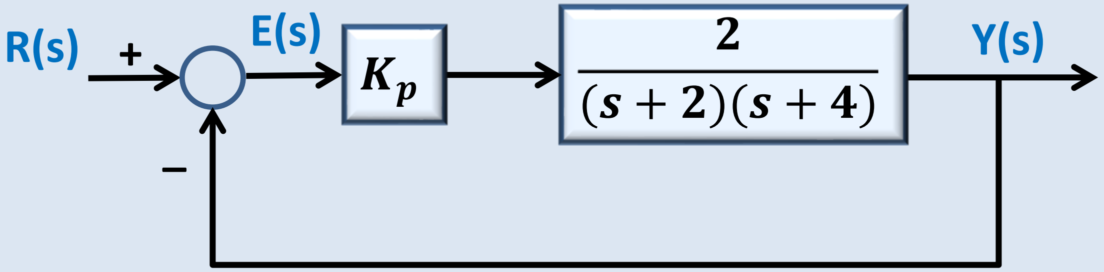
3) Which proportional controller gain can provide a 5% steady-state error for unit-step input signal?

a) Kp = 36

b) Kp = 56

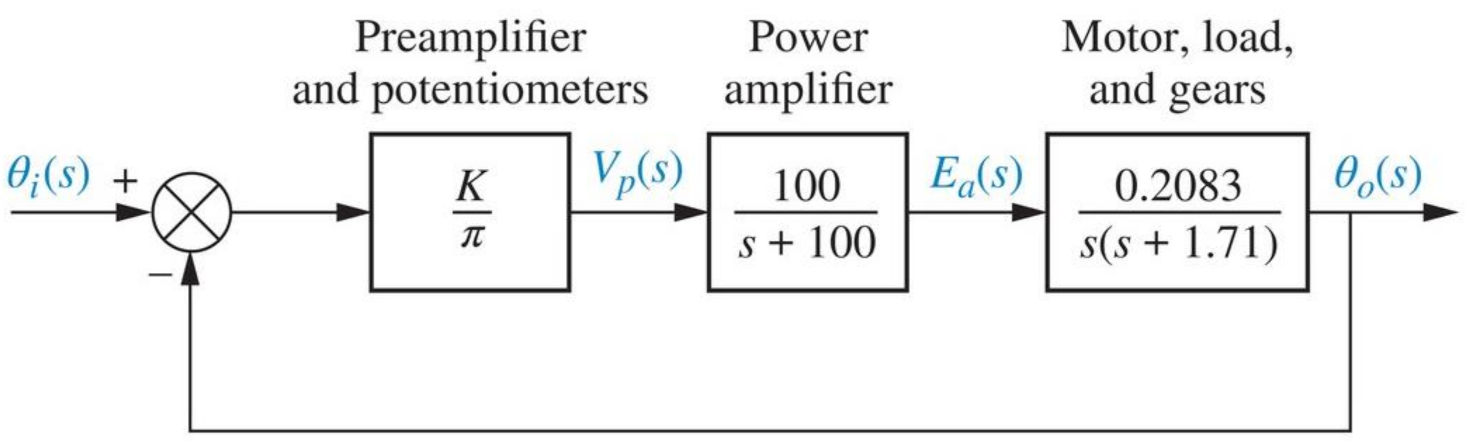
c) Kp = 76

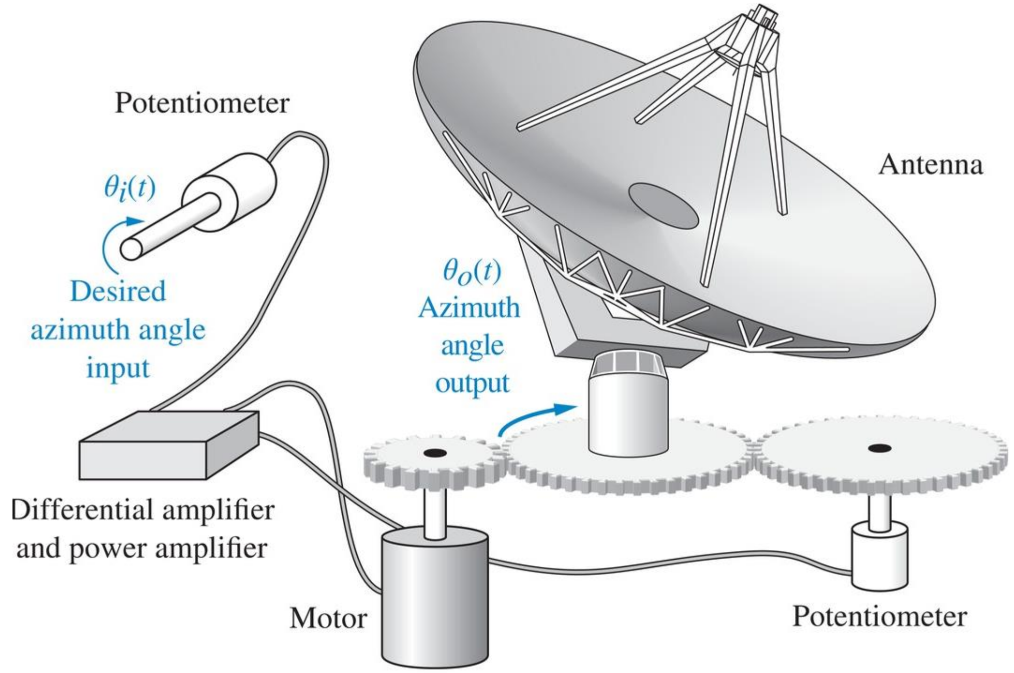
d) Kp = 96



Case Study: Antenna Control System

- Consider the ***motor-driven antenna azimuth position control system*** example from Lecture 1.
- We determined the block diagram of the control system as below:





- In this part, we will find the steady-state errors for step, ramp, and parabolic inputs to a closed-loop feedback control system.

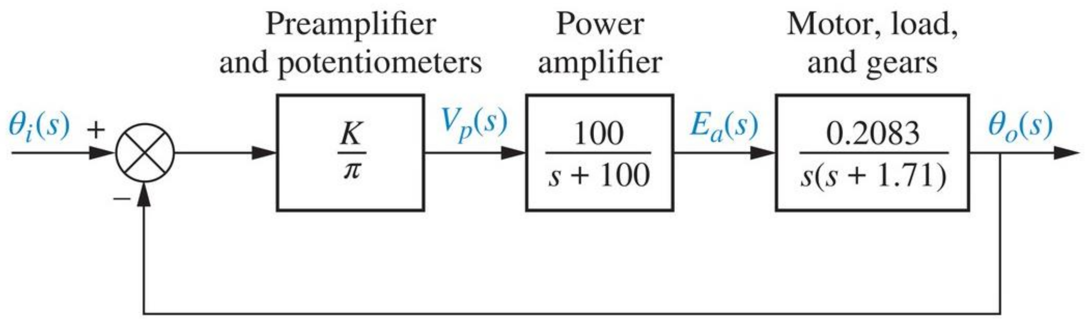
- We also evaluate the gain to meet a steady-state error requirement.

Case Study: Antenna Control System

For the antenna azimuth position control system with the given simplified unity – feedback block diagram model:

a) Find the steady-state error in terms of gain, K, for step, ramp, and parabolic inputs.

b) Find the value of gain, K, to yield a 10% error in the steady state



The equivalent forward path transfer function is Type 1:

$$G(s) = \frac{K(100)(0.2083)}{\pi s(s+100)(s+1.71)} = \frac{6.63K}{s(s+100)(s+1.71)}$$

Steady-state error for a unit-step input:

$$k_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{6.63K}{s(s+100)(s+1.71)} = \infty \quad \rightarrow \quad e_{ss} = \frac{R}{1+k_p} = 0$$

Steady-state error for a unit-ramp input:

$$k_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\frac{6.63K}{s(s+100)(s+1.71)} = 0.039K \quad \rightarrow \quad e_{ss} = \frac{R}{k_v} = \frac{1}{0.039K} = \frac{25.6}{K}$$

Steady-state error for a parabolic input:

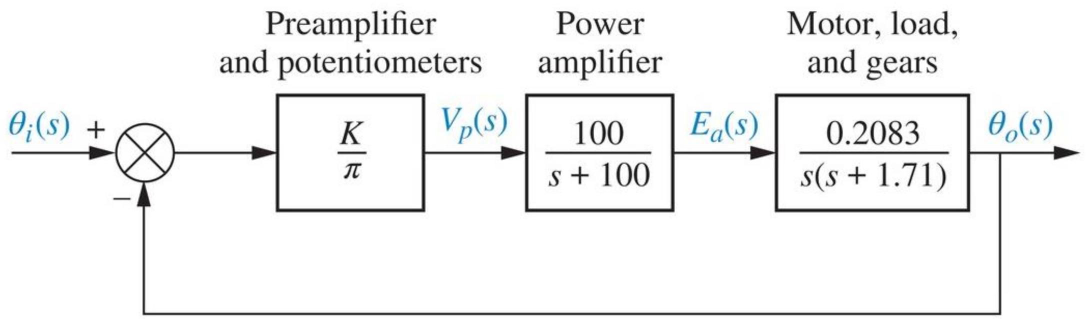
$$k_a = \lim_{s \to 0} s^2G(s) = \lim_{s \to 0} s^2\frac{6.63K}{s(s+100)(s+1.71)} = 0 \quad \rightarrow \quad e_{ss} = \frac{R}{k_a} = \infty$$

Case Study: Antenna Control System

For the antenna azimuth position control system with the given simplified unity – feedback block diagram model:

a) Find the steady-state error in terms of gain, K, for step, ramp, and parabolic inputs.

b) Find the value of gain, K, to yield a 10% error in the steady state



Since the open-loop system is Type 1, a 10% error in the steady state must refer to a ramp input.

This is the only input that yields a finite, nonzero error.

Hence, for a unit-ramp input:

$$e_{ss} = \frac{R}{k_v} = \frac{25.6}{K} \quad \rightarrow \quad 0.1 = \frac{25.6}{K} \quad \rightarrow \quad \boxed{K = 256}$$

Note: In general, we have to verify that the closed-loop system is stable for the obtained gain value.

We will discuss the stability analysis in Lecture 4.

THANK YOU

HUMBER

WE ARE HUMBER