HUMBER ENGINEERING

MENG-3020
SYSTEMS MODELING & SIMULATION

LECTURE 5

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LECTURE 5
Electrical Systems

- Modeling of Electrical Systems
  - Variables, Elements & Element Laws
  - Interconnection Laws
  - Complex Impedance Method
  - Operational Amplifiers
  - Loading Effect & Block Diagram Models

Electrical Systems: Variables & Elements

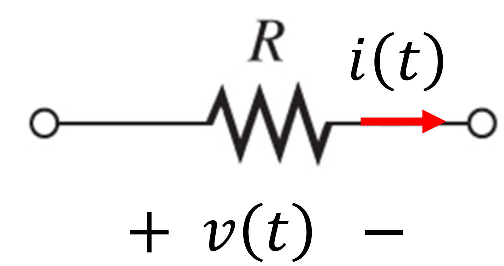
- The **variables** that are used to describe the electrical systems are:
  - $v(t)$: **Voltage (V)**
  - $i(t)$: **Current (A)**
- All these variables are function of time.
- Current is defined as the rate of change of the charge $Q(t)$ passing through an area: $i(t) = \frac{dQ(t)}{dt}$
- Voltage is defined as the required work or energy to move a charge between two points in a circuit.

- Electric circuit **elements** may be classified as **passive** and **active** elements.
  - The passive elements can store or dissipate energy that is already present in the circuit, but they cannot introduce additional energy into the circuit:
    - **Resistance Elements: Resistor**
    - **Capacitance Elements: Capacitor**
    - **Inductance Elements: Inductor**
  - The active elements can introduce energy into the circuit:
    - **Voltage Sources**
    - **Current Sources**
    - **Op-Amps**

Electrical Systems Element Laws

❑ Resistance Elements: Resistor

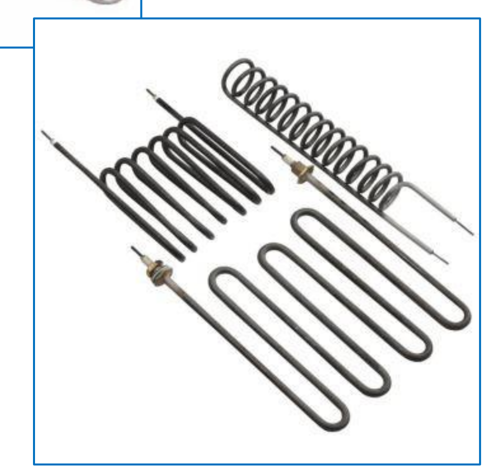
- A **resistor** is an element for which there is an algebraic relationship between the voltage across its terminals and the current through it.
- In a linear resistor the voltage and current are directly proportional to each other by **Ohm's law.**
- The R is the resistance. The unit is ohm (Ω).



$$\boxed{v(t) = Ri(t)}$$

- The resistance of a body of length l and constant cross-sectional area A made of a material with resistivity ρ is:

$$\boxed{R = \rho \frac{l}{A}}$$



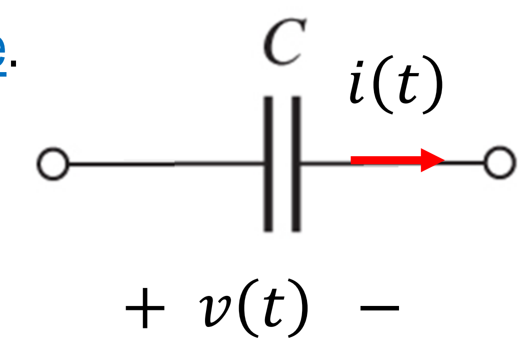
- Resistors do not store electric energy in any form, but instead dissipate it as heat.
- We can write the power dissipated by a linear resistor as:

$$\boxed{P = Ri^2 \quad or \quad P = \frac{v^2}{R}}$$

Electrical Systems Element Laws

❑ Capacitance Elements: Capacitor

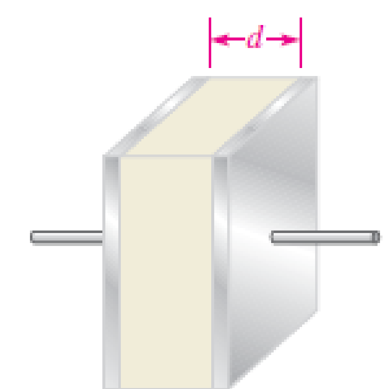
- A **capacitor** is an element that obeys an algebraic relationship between the voltage and the charge.
- For a linear capacitor, the charge and voltage are related as below
- The C is the capacitance. The unit is farad (F).



$$Q(t) = Cv(t) \quad \rightarrow \quad i(t) = C\frac{dv(t)}{dt} \quad \rightarrow \quad v(t) = \frac{1}{C}\int i(t)dt$$

- The capacitance of a capacitor depends on the plate area A, the distance between the plates d, and the permittivity of dielectric ε:

$$\boxed{C = \frac{\varepsilon A}{d}}$$



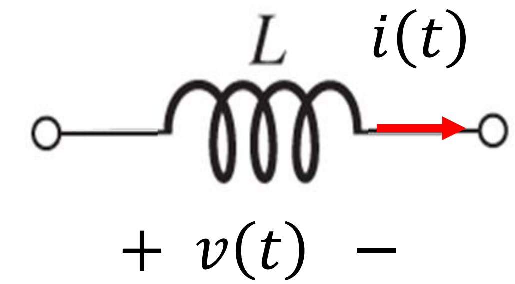
- The energy supplied to a capacitor is stored in its electrical field.
- For a fixed linear capacitor, the stored energy is:

$$\boxed{W = \frac{1}{2}Cv^2}$$

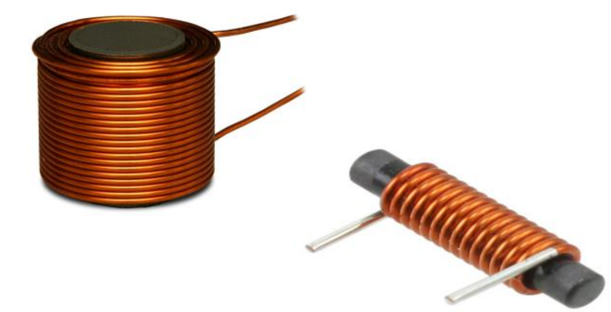
Electrical Systems Element Laws

❑ Inductance Elements: Inductor

- An **inductor** is an element for which there is an algebraic relationship between the voltage across its terminals and the derivative of the flux linkage.
- For a linear inductor, the current and voltage are related as below
- The L is the inductance. The unit is henry (H).



$$v(t) \propto \frac{d\phi}{dt} \quad \rightarrow \quad v(t) = L\frac{di(t)}{dt} \quad \rightarrow \quad i(t) = \frac{1}{L}\int v(t)dt$$



- The inductance of an inductor depends on the number of turns of wire N, the length of the core l, the cross-sectional area of the core A, and the permeability of the core material μ:

$$L = N^2\frac{\mu A}{l}$$

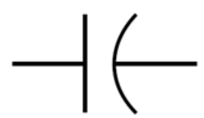
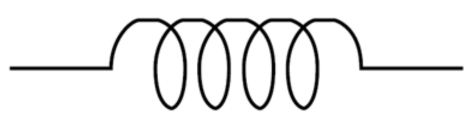
- The energy supplied to an inductor is stored in its magnetic field.
- For a fixed linear inductor, the stored energy is:

$$W = \frac{1}{2}Li^2$$

Electrical Systems Element Laws

❑ Summary

- Table shows summary of the voltage-current and current-voltage relationships for capacitor, resistor, and inductor.

Component	Voltage-current	Current-voltage
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$

Electrical Systems: Interconnection Laws

- Two interconnection laws are used in modeling electrical circuits.

❑ Kirchhoff's Voltage Law (KVL)

- The algebraic sum of the voltages around any loop in an electrical circuit is zero.

$$\boxed{\sum v_j = 0 \qquad \text{around any loop}}$$

where v_j denotes the voltage across the jth element in the loop.

❑ Kirchhoff's Current Law (KCL)

- The algebraic sum of all currents entering and leaving a node is zero.

$$\boxed{\sum i_j = 0 \qquad \text{at any node}}$$

where the summation is over the currents through all the elements joined to the node.

❑ Complex Impedance Method

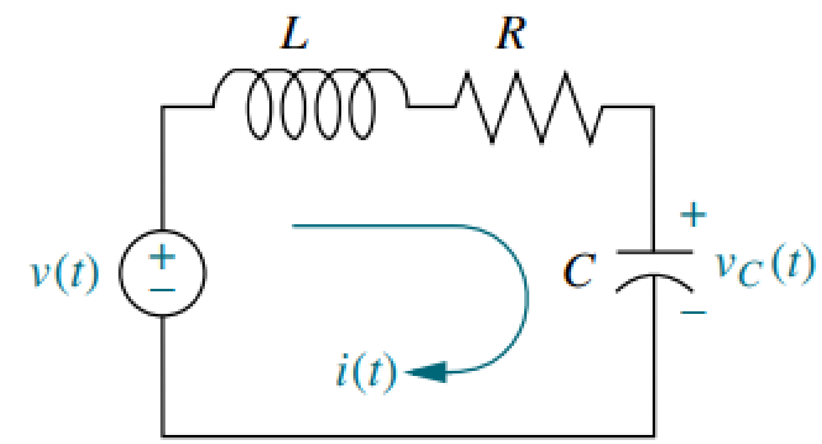
- Replace the component values with their impedance values in Laplace domain.
- Apply the circuit laws KVL and KCL in Laplace domain to find the TF model.

Modeling of Electrical Systems

Example 1

Given RLC network,

(a) Find the differential equation and the transfer function model relating the capacitor voltage $v_c(t)$ to the input voltage $v(t)$.



First, determine the input and output of the system.

The capacitor voltage $v_c(t)$ is the output. The applied voltage $v(t)$ is the input.

By using the Kirchhoff's Voltage Law (KVL) we have:

$$v(t) = v_L(t) + v_R(t) + v_C(t)$$

The differential equation relating $v(t)$ to $v_c(t)$ is determined as

$$v(t) = L\frac{di(t)}{dt} + Ri(t) + v_C(t) \xrightarrow{i(t)=C\frac{dv_c(t)}{dt}} \boxed{v(t) = LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t)}$$

Second-order Differential Equation

Taking Laplace transform of both side by assuming the **zero initial conditions**

$$V(s) = LCs^2V_c(s) + RCsV_c(s) + V_c(s) \quad \rightarrow \quad V(s) = (LCs^2 + RCs + 1)V_c(s)$$

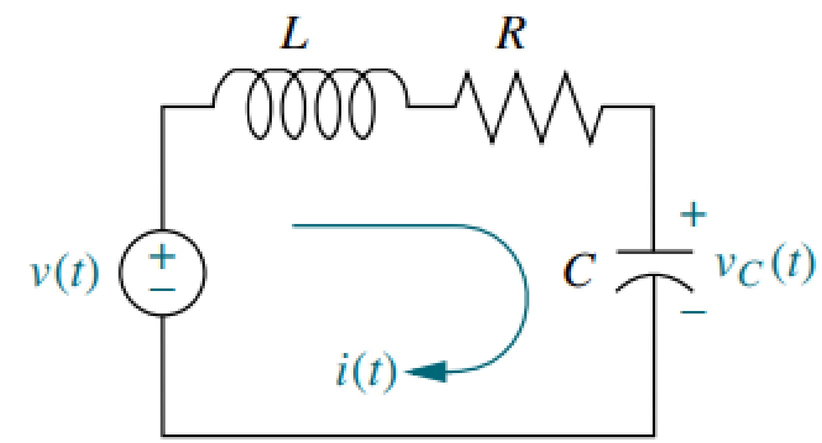
$$\boxed{\frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Transfer Function Model

Modeling of Electrical Systems

Example 1

Given RLC network,
(b) Find the differential equation and the transfer function model relating the current $i(t)$ to the input voltage $v(t)$.



In this part, current $i(t)$ is the output. The applied voltage $v(t)$ is the input.

By using the Kirchhoff's Voltage Law (KVL) we have:

$$v(t) = v_L(t) + v_R(t) + v_C(t)$$

The differential equation relating $v(t)$ to $i(t)$ is determined as

$$v(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt \quad \rightarrow \quad \boxed{\frac{dv(t)}{dt} = L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t)}$$

Second-order Differential Equation

Taking Laplace transform of both side by assuming the **zero initial conditions**

$$sV(s) = Ls^2I(s) + RsI(s) + \frac{1}{C}I(s) \quad \rightarrow \quad CsV(s) = (LCs^2 + RCs + 1)I(s)$$

$$\boxed{\frac{I(s)}{V(s)} = \frac{Cs}{LCs^2 + RCs + 1}}$$

Transfer Function Model

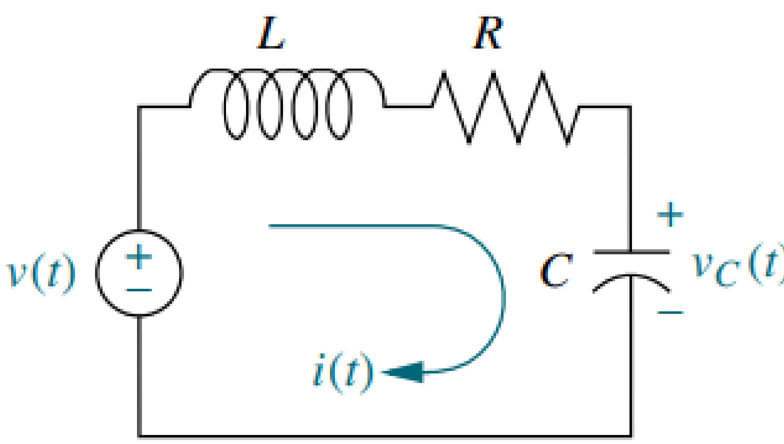
Modeling of Electrical Systems

Example 1

Given RLC network,

(c) Having the transfer function model and given component values find the loop current $i(t)$ if the applied voltage is $v(t) = 10V, \ \ t \geq 0.$

$$L = 1H, \qquad R = 3\Omega, \qquad C = 0.5F$$



$$\boxed{\frac{I(s)}{V(s)} = \frac{Cs}{LCs^2 + RCs + 1}}$$

First, find the transfer function from the given values: $\dfrac{I(s)}{V(s)} = \dfrac{0.5s}{0.5s^2 + 1.5s + 1}$

Next, find the loop current $I(s)$.

$$I(s) = \left(\frac{0.5s}{0.5s^2 + 1.5s + 1}\right)V(s) = \left(\frac{0.5s}{0.5s^2 + 1.5s + 1}\right)\left(\frac{10}{s}\right) = \frac{5}{0.5s^2 + 1.5s + 1} = \frac{10}{s^2 + 3s + 2} = \frac{10}{(s+1)(s+2)}$$

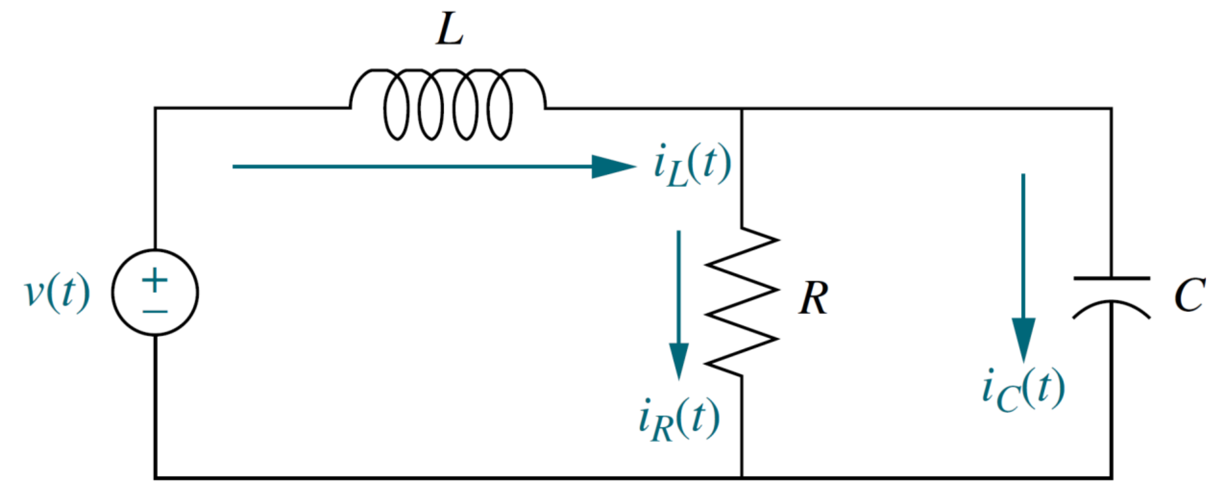
Apply partial fraction expansion method $\rightarrow \quad I(s) = \dfrac{10}{s+1} + \dfrac{-10}{s+2}$

Take the inverse Laplace transform to find $i(t)$ assuming zero initial conditions.

$$\boxed{i(t) = 10e^{-t} - 10e^{-2t} = 10(e^{-t} - e^{-2t}), \qquad t \geq 0}$$

Modeling of Electrical Systems

Example 2 Given the electric network, find a state-space model if the input is the applied voltage $v(t)$ and output is the current through the resistor $i_R(t)$.



The state variables q_1 and q_2 are selected as the inductor current $i_L(t)$ and the capacitor voltage $v_c(t)$.

$$q_1(t) = i_L(t) \quad \rightarrow \quad \dot{q}_1(t) = \dot{i}_L(t)$$

$$q_2(t) = v_c(t) \quad \rightarrow \quad \dot{q}_2(t) = \dot{v}_c(t)$$

By using a KCL and a KVL we have:

$$i_L(t) = i_R(t) + i_c(t) \;\rightarrow\; i_L(t) = \frac{v_R(t)}{R} + C\dot{v}_c(t) \rightarrow\; i_L(t) = \frac{1}{R}v_c(t) + C\dot{v}_c(t) \;\rightarrow\; \dot{v}_c(t) = \frac{1}{C}i_L(t) - \frac{1}{RC}v_c(t)$$

$$v(t) = v_L(t) + v_c(t) \;\rightarrow\; v(t) = L\dot{i}_L(t) + v_c(t) \;\rightarrow\; \dot{i}_L(t) = \frac{1}{L}v(t) - \frac{1}{L}v_c(t)$$

State-variable equations are obtained as:

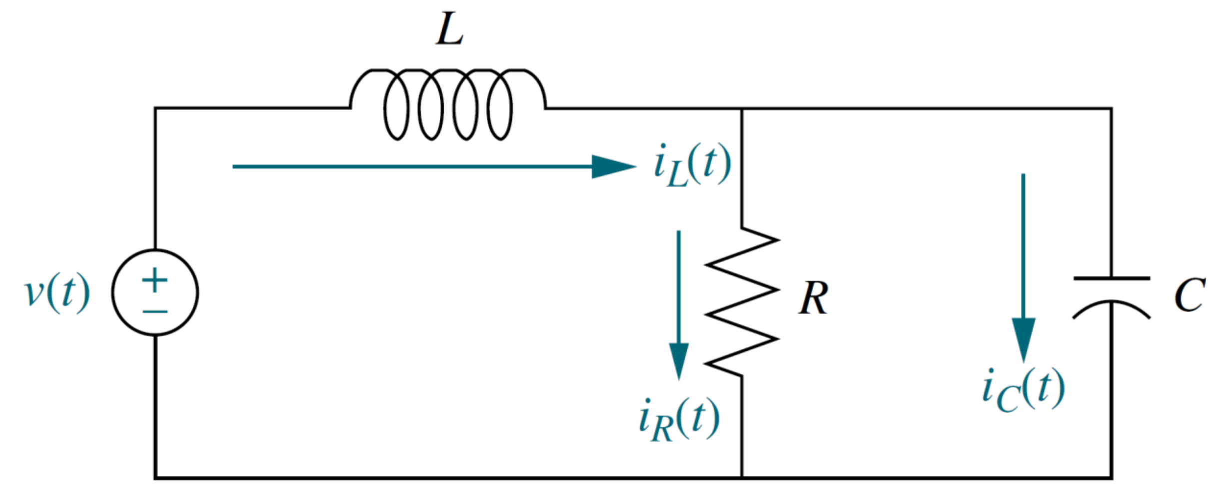
$$\begin{cases} \dot{q}_1(t) = \dfrac{1}{L}v(t) - \dfrac{1}{L}q_2(t) \\[2ex] \dot{q}_2(t) = \dfrac{1}{C}q_1(t) - \dfrac{1}{RC}q_2(t) \end{cases}$$

The **output equation** is obtained as:

$$y(t) = i_R(t) = \frac{v_R(t)}{R} = \frac{v_C(t)}{R} \;\rightarrow\; y(t) = \frac{1}{R}q_2(t)$$

Modeling of Electrical Systems

Example 2 Given the electric network, find a state-space model if the input is the applied voltage $v(t)$ and output is the current through the resistor $i_R(t)$.



Having the state and output equations:

$$\begin{cases} \dot{q}_1(t) = \frac{1}{L}v(t) - \frac{1}{L}q_2(t) \\ \dot{q}_2(t) = \frac{1}{C}q_1(t) - \frac{1}{RC}q_2(t) \\ y(t) = \frac{1}{R}q_2(t) \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

$$\boxed{\dot{\boldsymbol{q}}(t) = \mathbf{A}\,\boldsymbol{q}(t) + \mathbf{B}\,\mathbf{u}(t)}$$

State Equation

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v(t)$$

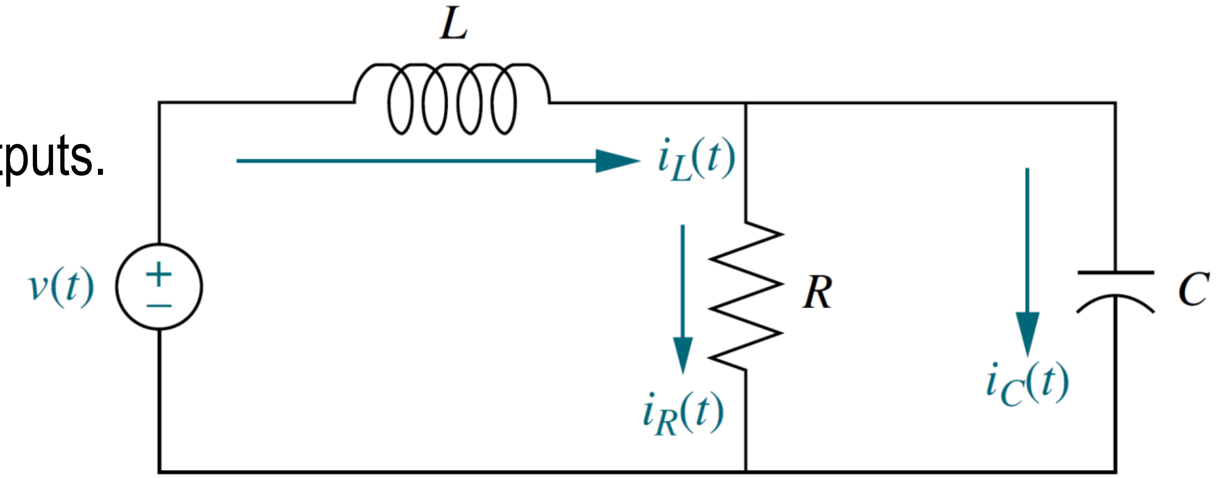
$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$$

Output Equation

$$y(t) = \begin{bmatrix} 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0]v(t)$$

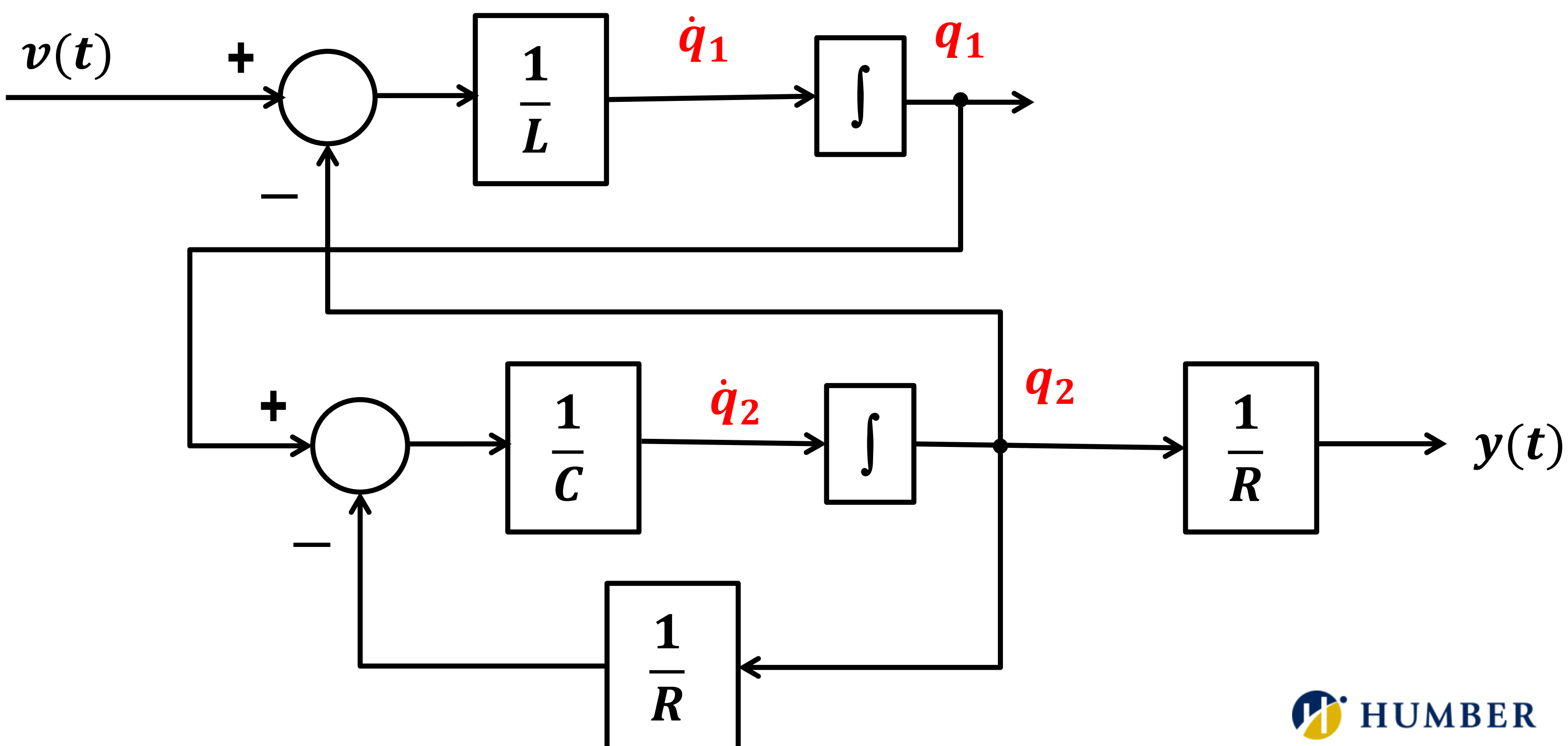
Modeling of Electrical Systems

Example 2 Given the electric network, find a state-space model if the input is the applied voltage $v(t)$ and output is the current through the resistor $i_R(t)$.



Following block diagram visualizes the state variables and the system outputs.

$$\begin{cases} \dot{q}_1(t) = \frac{1}{L}v(t) - \frac{1}{L}q_2(t) = \frac{1}{L}\big(v(t) - q_2(t)\big) \\ \dot{q}_2(t) = \frac{1}{C}q_1(t) - \frac{1}{RC}q_2(t) = \frac{1}{C}\left(q_1(t) - \frac{1}{R}q_2(t)\right) \\ y(t) = \frac{1}{R}q_2(t) \end{cases}$$



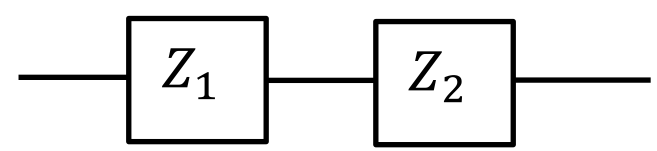
Modeling of Electrical Systems

❑ Complex Impedance Method

- In deriving transfer functions for electrical circuits, we can write the Laplace-transformed equations directly, without writing the differential equations, by using the complex impedance in Laplace domain.
- Recall the voltage-current relation of the electrical components, we can define the complex impedance $Z(s)$ in Laplace domain for each element as shown below:
  - For the capacitor : $i(t) = C\dfrac{dv(t)}{dt} \quad \rightarrow \quad I(s) = CsV(s) \quad \rightarrow \quad \boldsymbol{Z(s) = \dfrac{V(s)}{I(s)} = \dfrac{1}{Cs}}$
  - For the inductor : $v(t) = L\dfrac{di(t)}{dt} \quad \rightarrow \quad V(s) = LsI(s) \quad \rightarrow \quad \boldsymbol{Z(s) = \dfrac{V(s)}{I(s)} = Ls}$
  - For the resistor : $v(t) = Ri(t) \quad \rightarrow \quad V(s) = RI(s) \quad \rightarrow \quad \boldsymbol{Z(s) = \dfrac{V(s)}{I(s)} = R}$

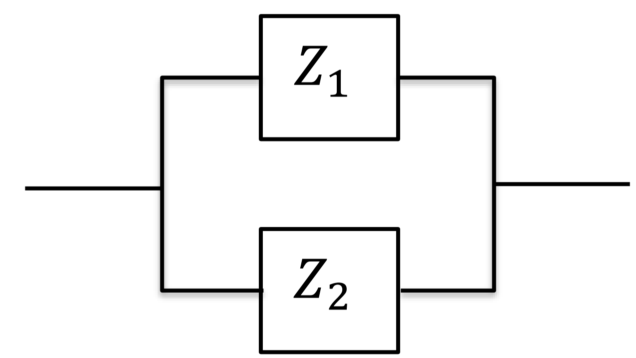
- **Series Impedances**

$$Z(s) = Z_1(s) + Z_2(s)$$



- **Parallel Impedances**

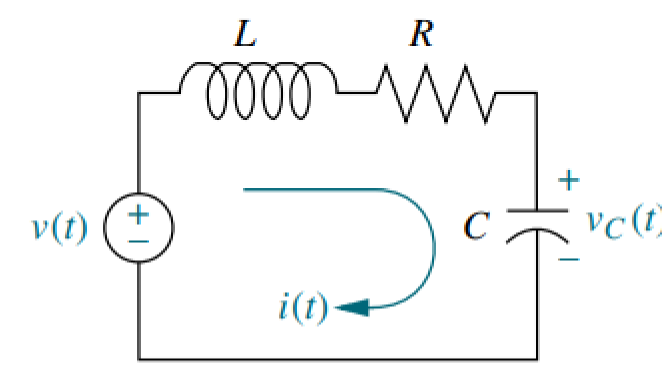
$$\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}$$

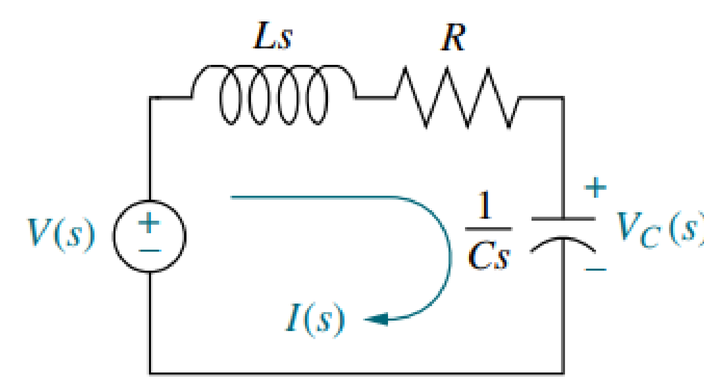


Modeling of Electrical Systems

❑ Complex Impedance Method

- To find the transfer function model of electrical networks using complex impedance method, we can perform the following steps:
  - Replace the passive element values with their impedances.
  - Replace all sources and time variables with their Laplace transform.
  - Apply circuit laws such as, KVL, KCL, voltage division or current division.
  - Solve the simultaneous equations for the output.
  - Form the transfer function.





- The following example shows how the concept of impedance simplifies the solution for the transfer function:

$$V(s) = LsI(s) + RI(s) + \frac{1}{Cs}I(s)$$

$$V(s) = \left(Ls + R + \frac{1}{Cs}\right)I(s)$$

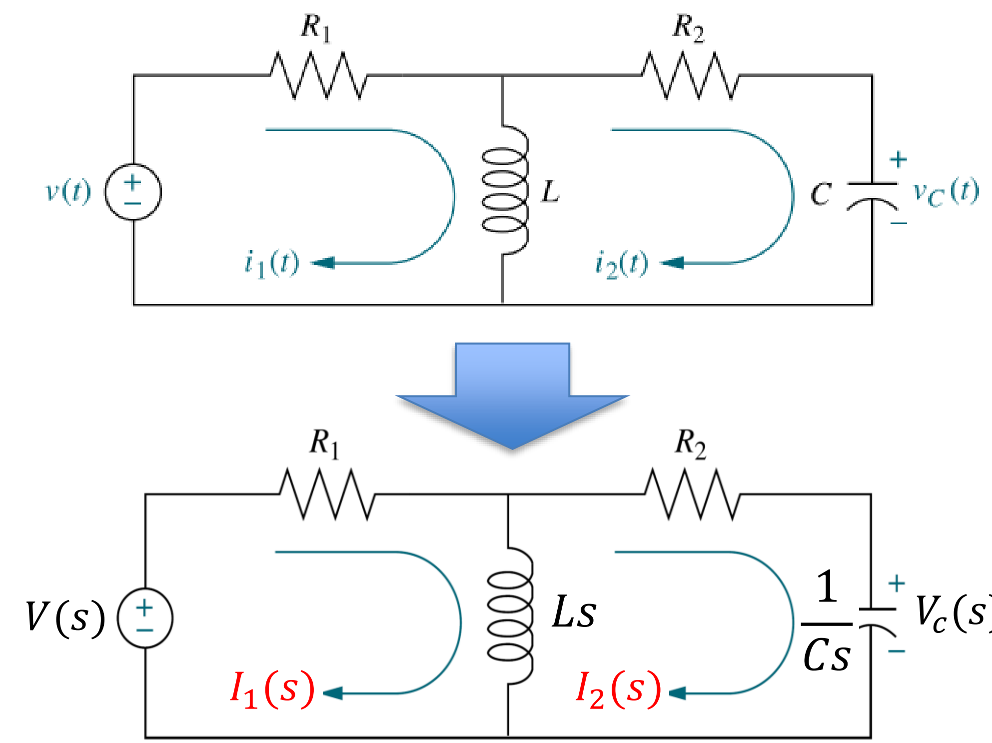
$$\boxed{\frac{I(s)}{V(s)} = \frac{Cs}{LCs^2 + RCs + 1}}$$

$$V_c(s) = \left(\frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}}\right)V(s)$$

$$\boxed{\frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Modeling of Electrical Systems

Example 3 Given the electric network, **(a)** find the transfer function $I_2(s)/V(s)$ applying the **impedance method**.



First step is to convert the circuit into Laplace domain for impedances and variables, assuming zero initial conditions.

Then apply the mesh analysis for each loop:

Mesh 1: $\quad R_1 I_1(s) + Ls\big(I_1(s) - I_2(s)\big) - V(s) = 0 \qquad$ Eqn. (1)

Mesh 2: $\quad R_2 I_2(s) + \dfrac{1}{Cs} I_2(s) + Ls\big(I_2(s) - I_1(s)\big) = 0 \qquad$ Eqn. (2)

From Eqn. (1) we have $\rightarrow \quad I_1(s) = \dfrac{V(s) + LsI_2(s)}{R_1 + Ls}$

Substitute in Eqn. (2) $\rightarrow \quad \left(R_2 + \dfrac{1}{Cs} + Ls\right) I_2(s) - Ls\left(\dfrac{V(s) + LsI_2(s)}{R_1 + Ls}\right) = 0$

Find the transfer function $I_2(s)/V(s)$

$$\left(\frac{R_2Cs + 1 + LCs^2}{Cs} - \frac{L^2s^2}{R_1 + Ls}\right) I_2(s) = \left(\frac{Ls}{R_1 + Ls}\right) V(s) \quad \rightarrow \quad \left(\frac{(R_2Cs + 1 + LCs^2)(R_1 + Ls) - L^2Cs^3}{Cs(R_1 + Ls)}\right) I_2(s) = \left(\frac{Ls}{R_1 + Ls}\right) V(s)$$

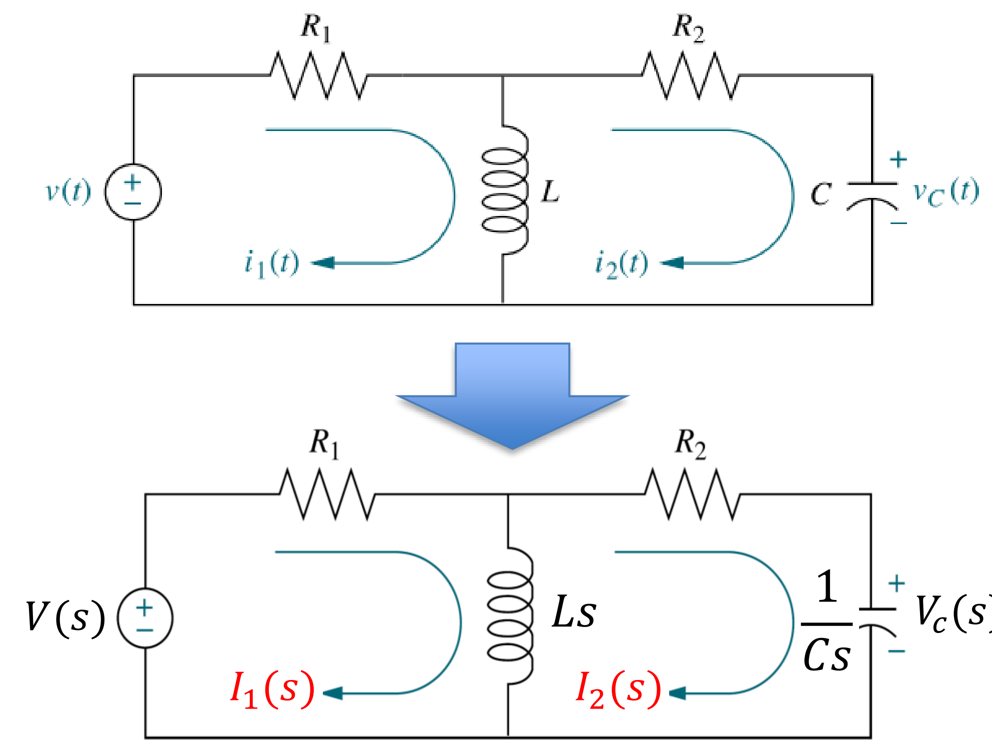
$$\boxed{\frac{I_2(s)}{V(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}}$$

Transfer Function Model

Modeling of Electrical Systems

Example 3 Given the electric network, **(b)** find the transfer function $V_c(s)/V(s)$.

$$\boxed{\frac{I_2(s)}{V(s)} = \frac{LCs^2}{(R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1}}$$



Having the transfer function $I_2(s)/V(s)$ and relation between $i_2(t)$ and $v_c(t)$:

$$i_2(t) = C\frac{dv_c(t)}{dt} \quad \rightarrow \quad I_2(s) = CsV_c(s)$$

Replace $I_2(s)$ in the transfer function of $I_2(s)/V(s)$ and simplify it to obtain the transfer function of $V_c(s)/V(s)$:

$$\frac{CsV_c(s)}{V(s)} = \frac{LCs^2}{(R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

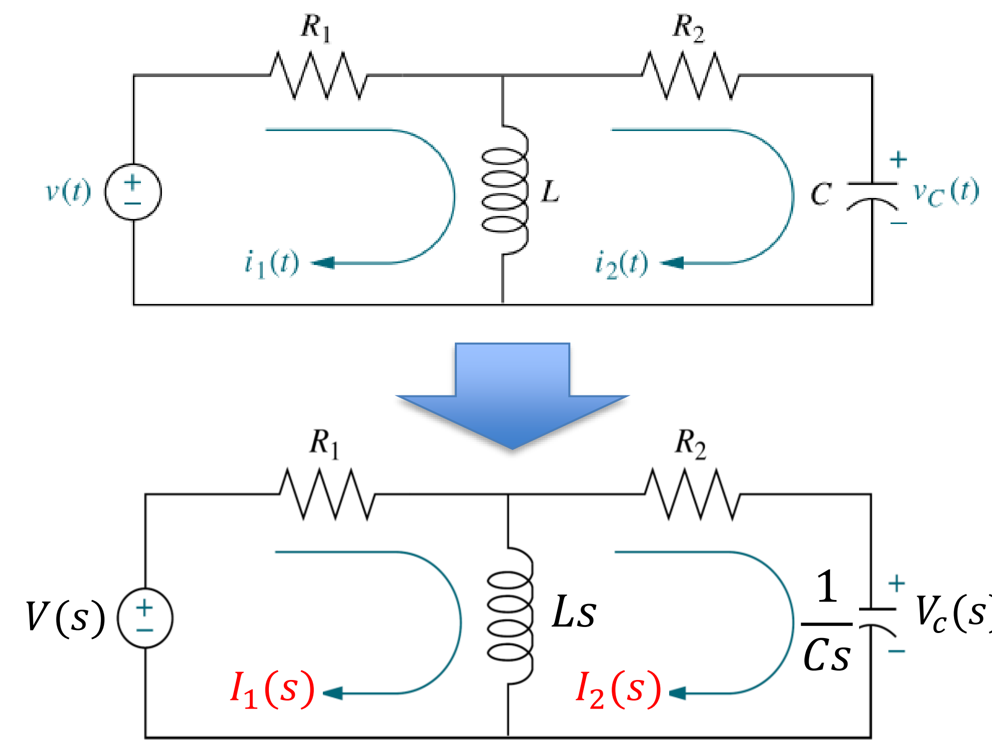
$$\boxed{\frac{V_c(s)}{V(s)} = \frac{Ls}{(R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1}}$$

Transfer Function Model

Modeling of Electrical Systems

Example 3 Given the electric network, **(c)** Having the transfer function $V_c(s)/V(s)$ find the differential equation model of the system.

$$\boxed{\frac{V_c(s)}{V(s)} = \frac{Ls}{(R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1}}$$



Reform the transfer function model:

$$\big((R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1\big)V_c(s) = LsV(s)$$

$$(R_1+R_2)LCs^2V_c(s) + (R_1R_2C + L)sV_c(s) + R_1V_c(s) = LsV(s)$$

Take inverse Laplace transform assuming zero initial conditions.

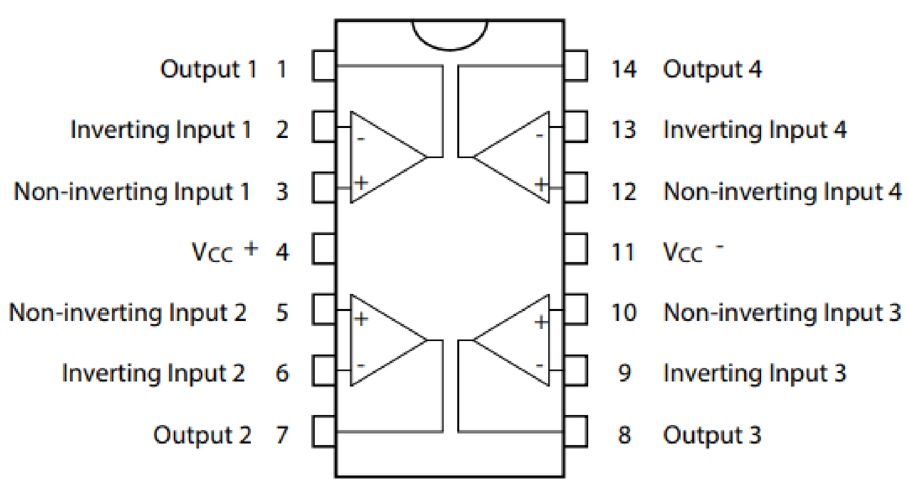
$$\boxed{(R_1+R_2)LC\ddot{v}_c(t) + (R_1R_2C + L)\dot{v}_c(t) + R_1v_c(t) = L\dot{v}(t)}$$

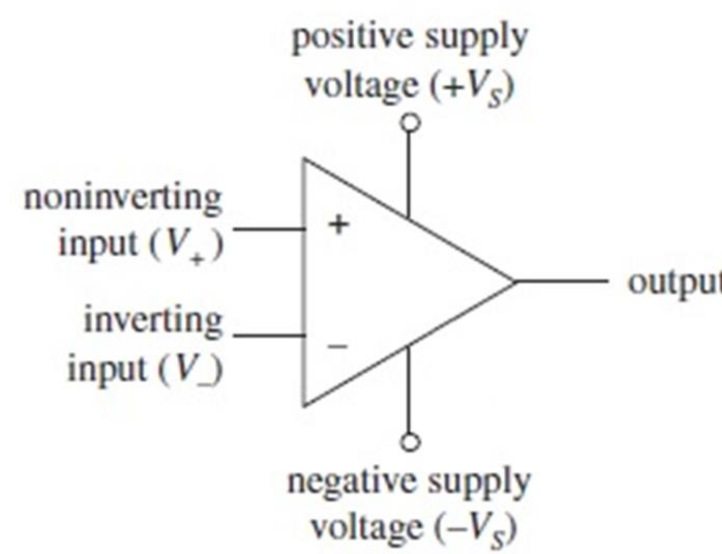
Differential Equation Model

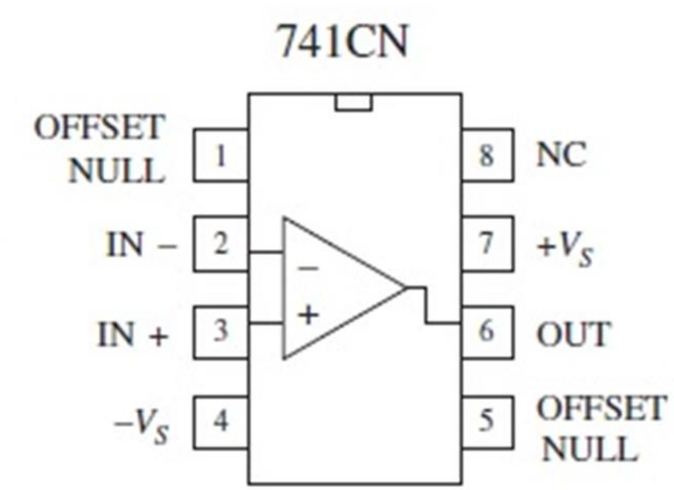
Electrical Systems Element

❑ Operational Amplifiers (Op-Amp)

- An operational amplifier (**op-amp**) is an active element, which is a **voltage amplifier** with a very large gain $(10^5 \text{ to } 10^6)$ when it is operating in its linear region.
- Typically, it consists many transistors plus a number of resistors and capacitors.
- It has two terminals on the input side, and one terminal on the output side.
- The input terminals are called the ***inverting*** (-) and ***noninverting*** (+) terminals.
- **IC 741** is a typical example of a commonly used op-amp IC chip.



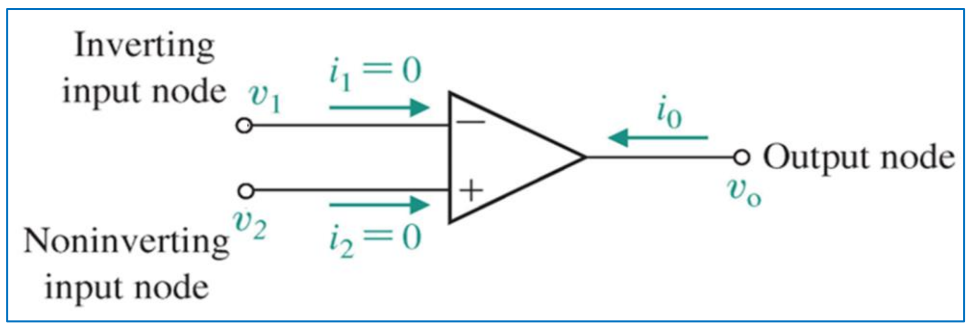


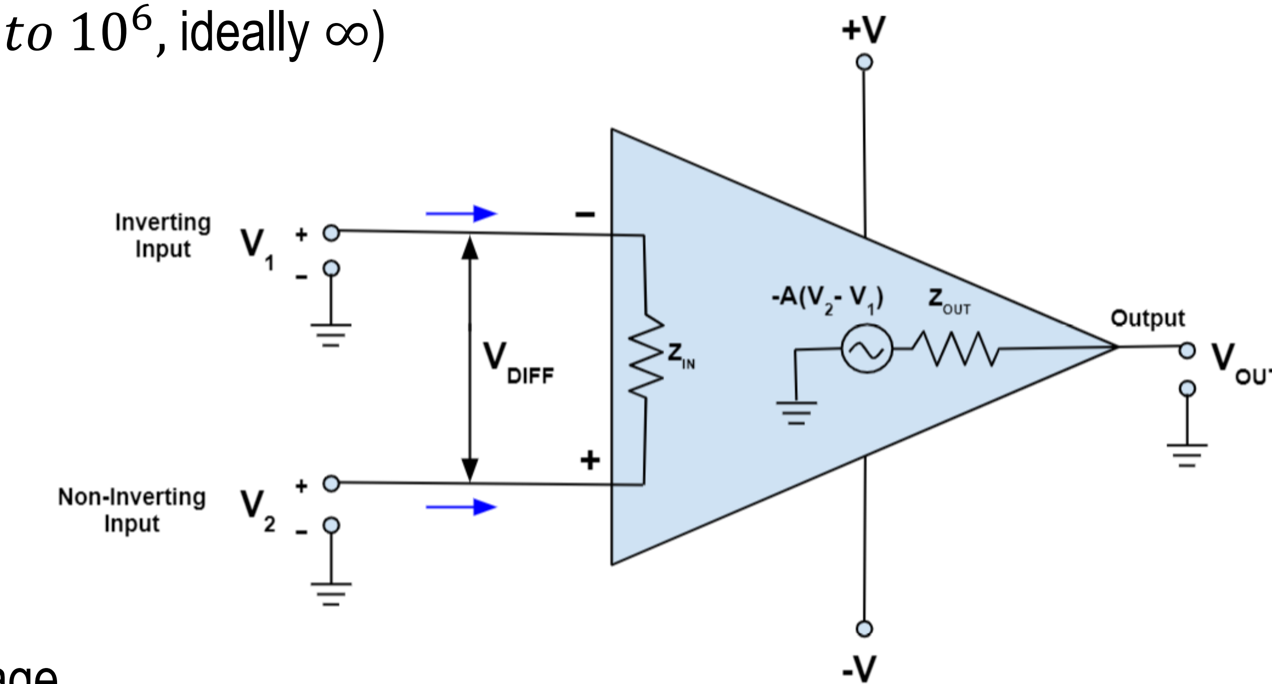


Electrical Systems Element Laws

❑ Ideal Operational Amplifiers (Op-Amp)

- Characteristics of an ideal op-amp is:
  1. High input impedance, ($Z_{in} = 10^5\ to\ 10^{11}\ \Omega$, ideally ∞), hence, $i_1 = 0$ and $i_2 = 0$
  2. Low output impedance, ($Z_{out} = 1\ to\ 10\Omega$, ideally 0),
  3. High constant open-loop gain amplification, ($A = 10^5\ to\ 10^6$, ideally ∞)
  4. The output is given by, $\boxed{v_o = A(v_2 - v_1)}$



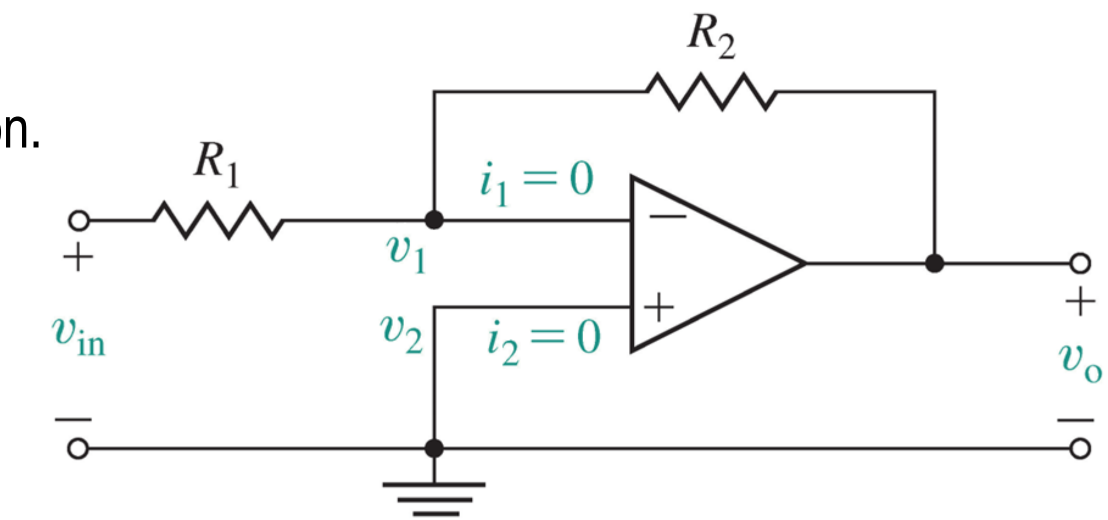


- **Note**: The output voltage cannot exceed the supply voltage.

- Op-Amps are the building blocks of analog control systems and electronic circuits. They are used to interface the subsystem connections, to implement amplifiers, filters and to add or subtract signals in a control system.

Modeling of Electrical Systems

❑ Inverting Operational Amplifier



- Consider the following **inverting amplifier** operating under ideal condition.
- Since $i_1 = 0$, from a KCL at node v_1 we have:

$$\frac{v_{in} - v_1}{R_1} = \frac{v_1 - v_o}{R_2}$$

- We know that,

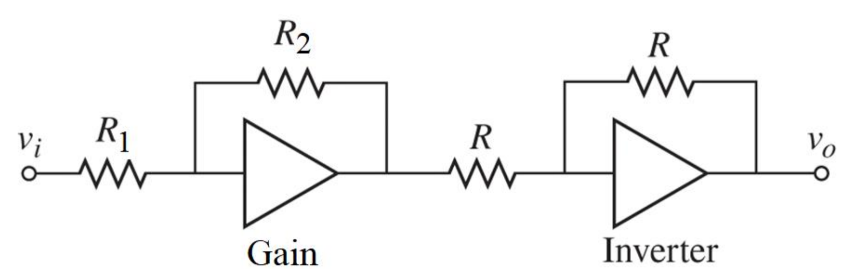
$$\boldsymbol{v_o = A(v_2 - v_1)} \xrightarrow{v_2 = 0} v_o = -Av_1 \rightarrow v_1 = -\frac{v_o}{A}$$

- Since $A = \infty$, we have $v_1 \approx 0$.

$$\frac{v_{in}}{R_1} = \frac{-v_o}{R_2} \rightarrow \frac{v_o}{v_{in}} = -\frac{R_2}{R_1} \rightarrow \boxed{\frac{V_o(s)}{V_{in}(s)} = -\frac{R_2}{R_1}}$$

Transfer Function Model

- Therefore, we can implement **static multiplier gains** to amplify the input voltage.
- If $\boldsymbol{R_1 = R_2}$, the ideal op-amp circuit will be an **inverter**, which inverts the sign of the input signal: $\boxed{v_o = -v_{in}}$
- Using an inverter in series with the multiplier gain eliminates the overall sign reversal.



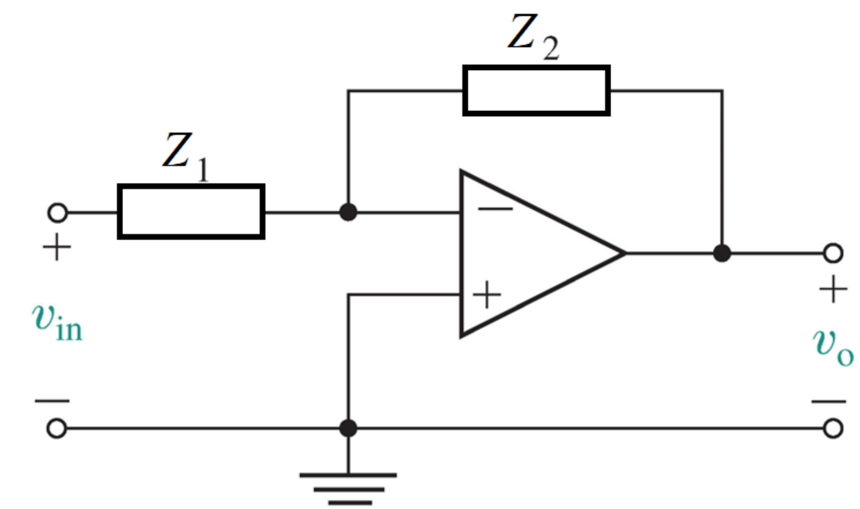
Modeling of Electrical Systems

❑ Inverting Operational Amplifier

- In general, the transfer function model is obtained in terms of the **impedances**:

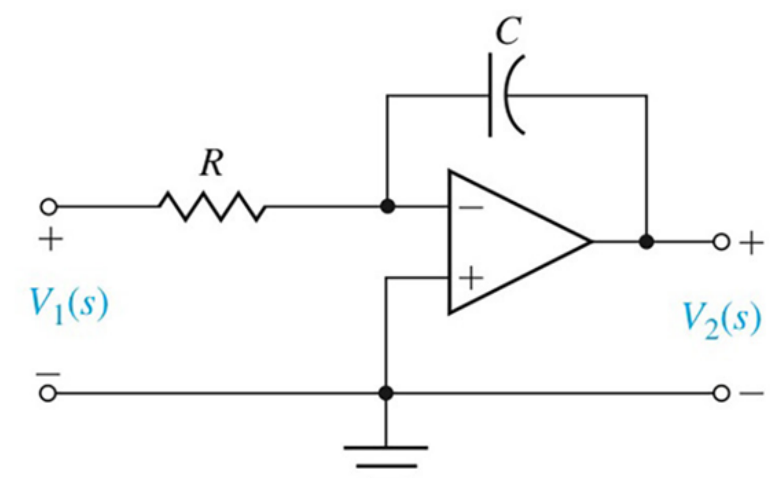
$$\boxed{\frac{V_o(s)}{V_{in}(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

Transfer Function Model



- Therefore, we can implement transfer function of dynamic elements as well.

❑ Integrating Circuit:



$$\frac{V_2(s)}{V_1(s)} = -\frac{\frac{1}{Cs}}{R} = -\frac{1}{RCs} \quad \rightarrow \quad v_2(t) = -\frac{1}{RC}\int_0^t v_1(t)dt$$

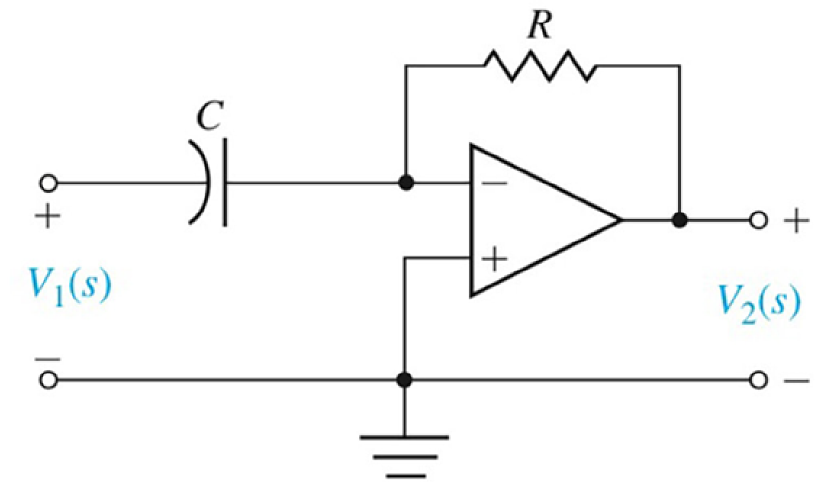
For example, selecting the $R = 1M\Omega$ and $C = 1\mu F$ we have $RC = (10^6\Omega)(10^{-6}F) = 1\ sec$:

$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{s} \quad \rightarrow \quad v_2(t) = -\int_0^t v_1(t)dt$$

Modeling of Electrical Systems

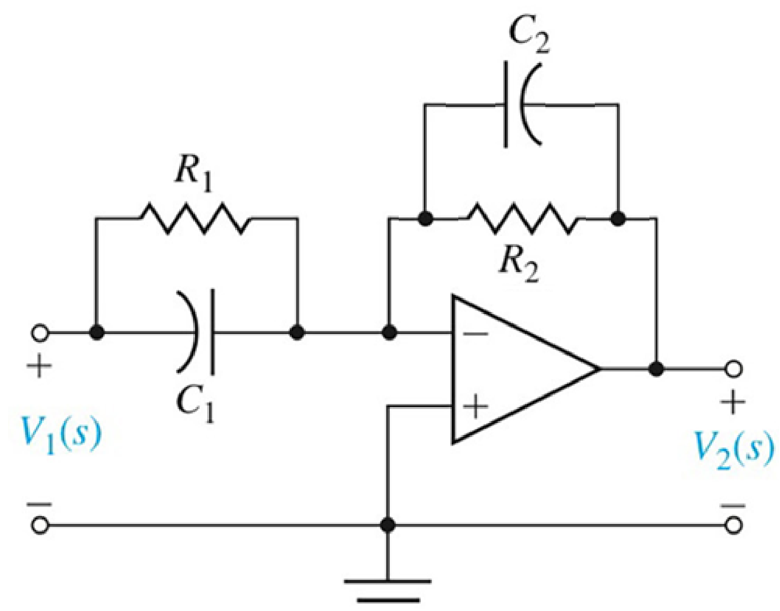
❑ Inverting Operational Amplifier

❑ Differentiating Circuit:



$$\frac{V_2(s)}{V_1(s)} = -\frac{R}{\frac{1}{Cs}} = -RCs \quad \rightarrow \quad v_2(t) = -RC\,\frac{dv_1(t)}{dt}$$

❑ First-order Filters (Transfer Functions):



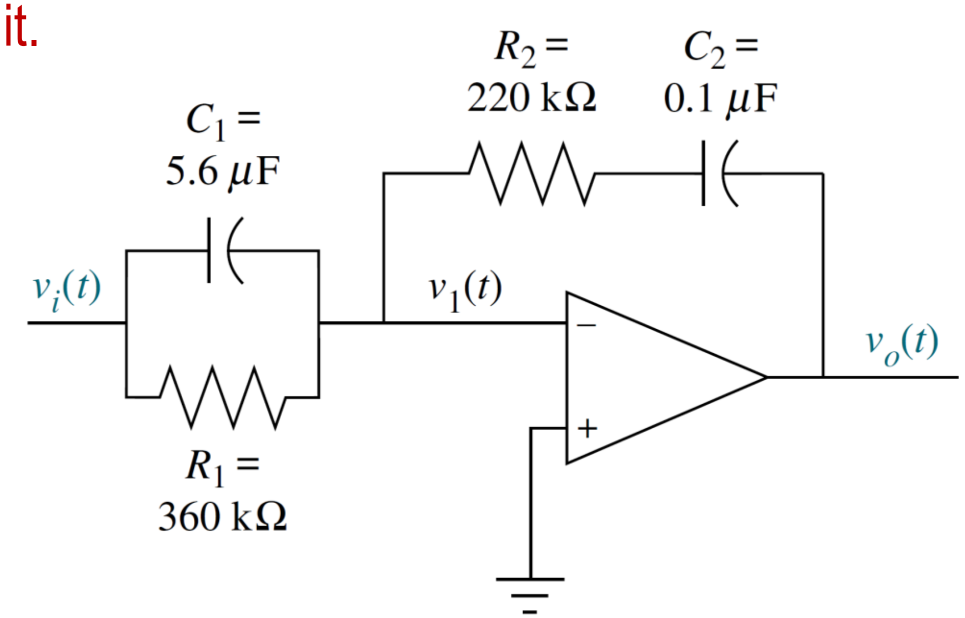
$$\frac{V_2(s)}{V_1(s)} = -\frac{Z_2}{Z_1} = -\frac{Y_1}{Y_2} = -\frac{\frac{1}{R_1} + C_1 s}{\frac{1}{R_2} + C_2 s} = -\frac{\frac{1 + R_1 C_1 s}{R_1}}{\frac{1 + R_2 C_2 s}{R_2}} = -\frac{R_2}{R_1}\left(\frac{1 + R_1 C_1 s}{1 + R_2 C_2 s}\right)$$

- For a low-pass filter set $C_1 = 0$: $$\frac{V_2(s)}{V_1(s)} = -\frac{R_2}{R_1}\left(\frac{1}{1 + R_2 C_2 s}\right)$$

- For a high-pass filter set $C_2 = 0$: $$\frac{V_2(s)}{V_1(s)} = -\frac{R_2}{R_1}(1 + R_1 C_1 s)$$

Modeling of Electrical Systems

Example 4 Find the transfer function $V_o(s)/V_i(s)$, for the following circuit.



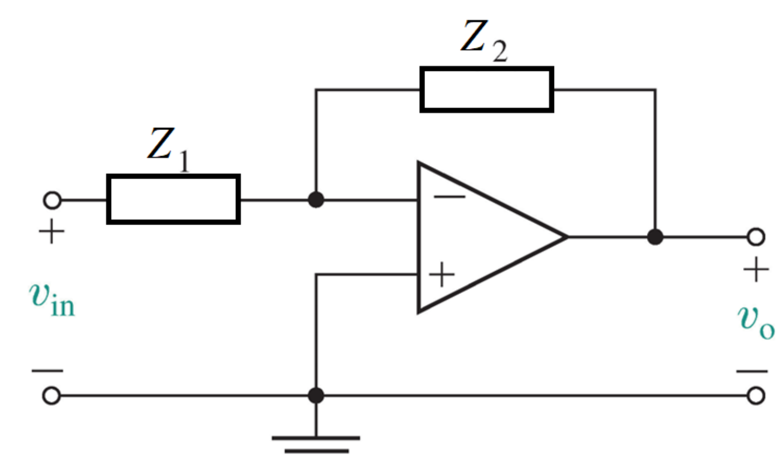
This is an **inverting** op-amp circuit. The general form of the transfer function is:

$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

First find the $Z_1(s)$ and $Z_2(s)$:

$$Z_1(s) = \frac{R_1\left(\frac{1}{C_1 s}\right)}{R_1 + \frac{1}{C_1 s}} = \frac{R_1}{R_1 C_1 s + 1} \qquad Z_2(s) = R_2 + \frac{1}{C_2 s} = \frac{R_2 C_2 s + 1}{C_2 s}$$

The transfer function is:



$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{\frac{R_2 C_2 s + 1}{C_2 s}}{\frac{R_1}{R_1 C_1 s + 1}} = -\frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_1 C_2 s}$$

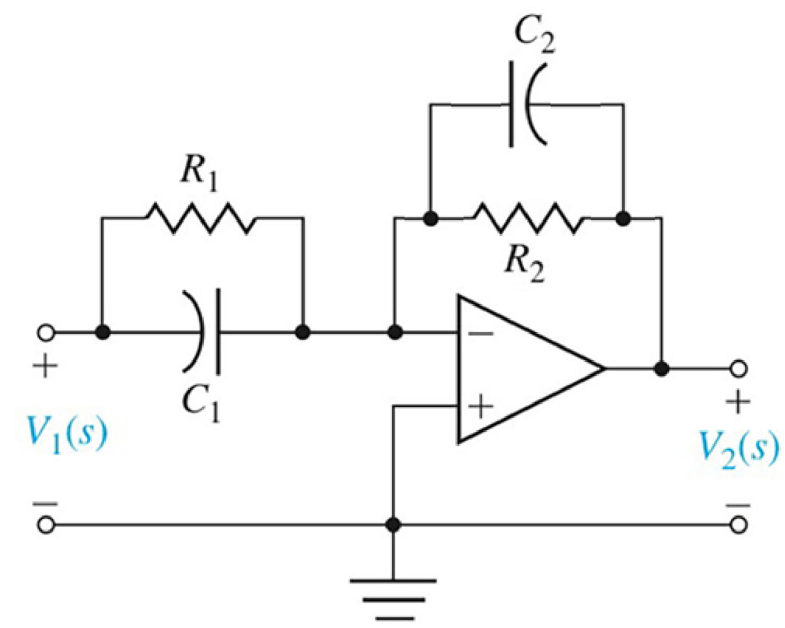
$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{(2.016s + 1)(0.022s + 1)}{0.036s}}$$

Modeling of Electrical Systems

Example 5 Determine the required components to implement the following transfer function.

$$\frac{V_2(s)}{V_1(s)} = -50\frac{s+5}{s+10} = -50\frac{5(0.2s+1)}{10(0.1s+1)} = -25\frac{0.2s+1}{0.1s+1}$$

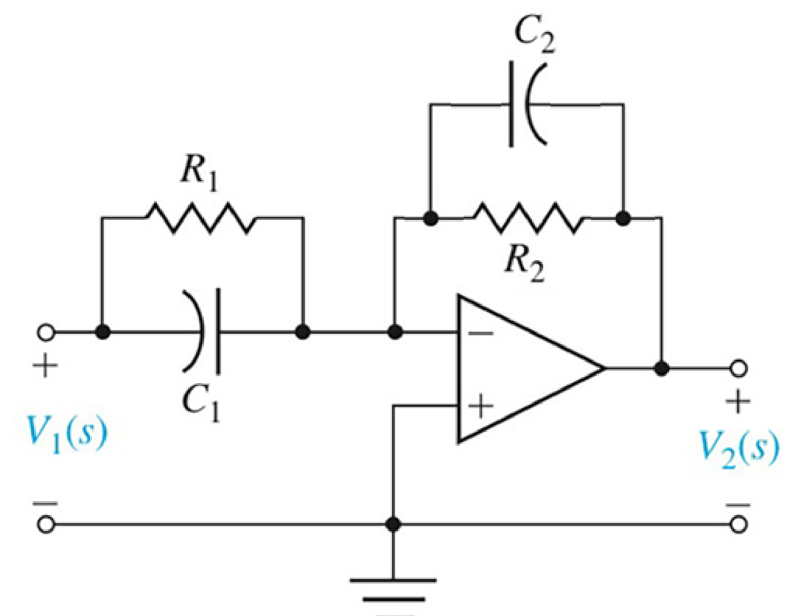
$$\rightarrow \left\{ \begin{array}{l} \dfrac{R_2}{R_1} = 25 \\ R_1C_1 = 0.2 \\ R_2C_2 = 0.1 \end{array} \right. \rightarrow \left\{ \begin{array}{ll} C_1 = 5\mu F, & R_1 = 40k\Omega \\ C_2 = 0.1\mu F, & R_2 = 1M\Omega \end{array} \right.$$



$$\boxed{\frac{V_2(s)}{V_1(s)} = -\frac{R_2}{R_1}\left(\frac{1+R_1C_1s}{1+R_2C_2s}\right)}$$

Modeling of Electrical Systems

Example 6 Determine the required components to implement the following transfer function.

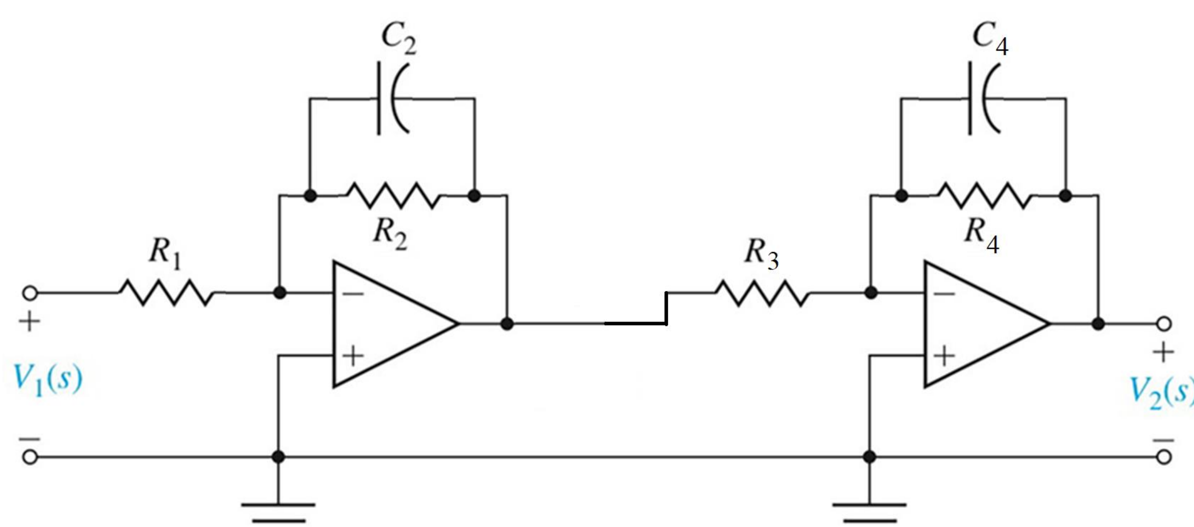


$$\frac{V_2(s)}{V_1(s)} = \frac{5}{(s+1)(s+2)} = \frac{5}{2(s+1)(0.5s+1)} = \left(\frac{2.5}{s+1}\right)\left(\frac{1}{0.5s+1}\right)$$

$$\boxed{\frac{V_2(s)}{V_1(s)} = -\frac{R_2}{R_1}\left(\frac{1+R_1C_1s}{1+R_2C_2s}\right)}$$

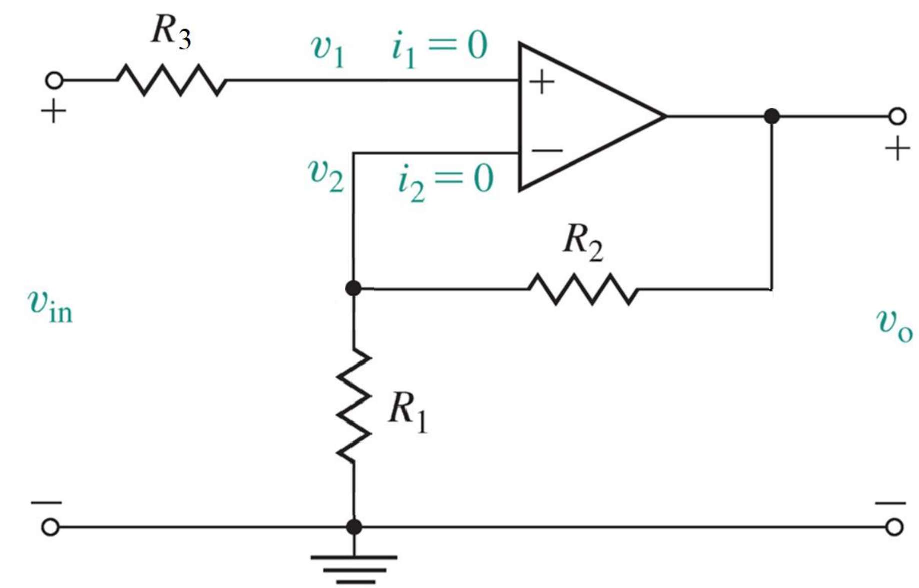
$$\begin{cases} \dfrac{R_2}{R_1} = 2.5 \\ R_1C_1 = 0 \\ R_2C_2 = 1 \end{cases} \qquad \begin{cases} \dfrac{R_4}{R_3} = 1 \\ R_3C_3 = 0 \\ R_4C_4 = 0.5 \end{cases}$$

$$\begin{cases} C_1 = 0, \quad R_1 = 400k\Omega \\ C_2 = 1\mu F, \quad R_2 = 1M\Omega \\ C_3 = 0, \quad R_3 = 500k\Omega \\ C_4 = 1\mu F, \quad R_4 = 500k\Omega \end{cases}$$



Modeling of Electrical Systems

❑ Noninverting Operational Amplifier



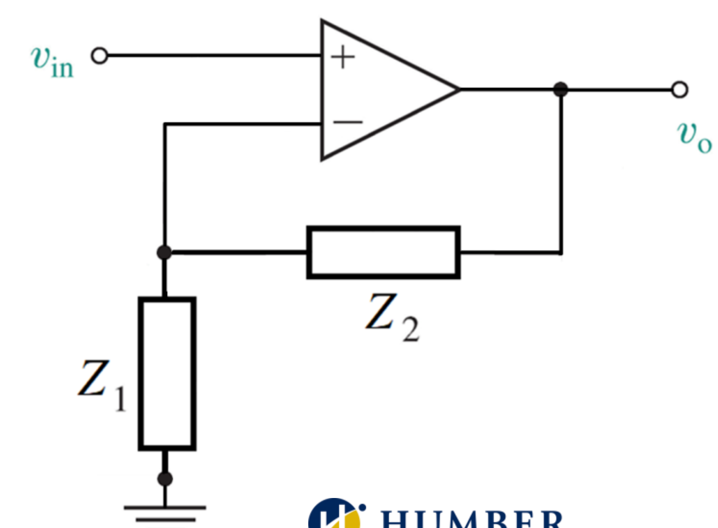
- Consider the following **noninverting amplifier** operating under the ideal condition.
- Since $i_1 = 0$, we have : $v_{in} = v_1$
- Since $i_2 = 0$, from a KCL at node v_2 we have: $v_2 = \dfrac{R_1}{R_1 + R_2} v_o$
- We know that,

$$\boldsymbol{v_o = A(v_1 - v_2)} \quad \rightarrow \quad v_o = A\left(v_{in} - \frac{R_1}{R_1 + R_2} v_o\right) \quad \rightarrow \quad v_o = \left(\frac{A(R_1 + R_2)}{AR_1 + R_1 + R_2}\right) v_{in} = \left(\frac{R_1 + R_2}{R_1 + \frac{1}{A}(R_1 + R_2)}\right) v_{in}$$

- Since $A = \infty$, we have,

$$v_o = \left(1 + \frac{R_2}{R_1}\right) v_{in} \quad \rightarrow \quad \boxed{\frac{V_o(s)}{V_{in}(s)} = 1 + \frac{R_2}{R_1}}$$

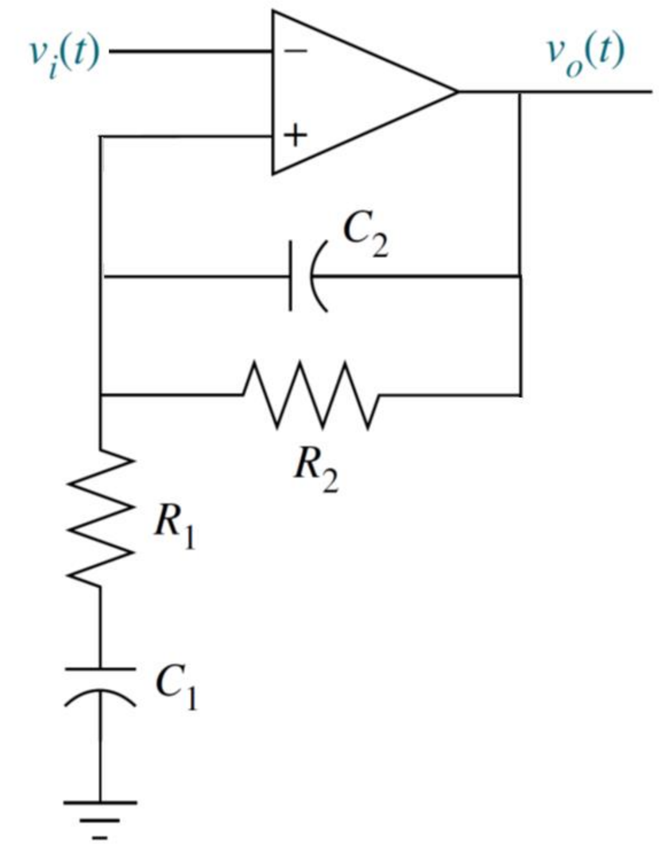
Transfer Function Model



General Form $\rightarrow$ $\boxed{\dfrac{V_o(s)}{V_{in}(s)} = 1 + \dfrac{Z_2(s)}{Z_1(s)}}$

Modeling of Electrical Systems

Example 7 Find the transfer function $V_o(s)/V_i(s)$, for the following circuit.



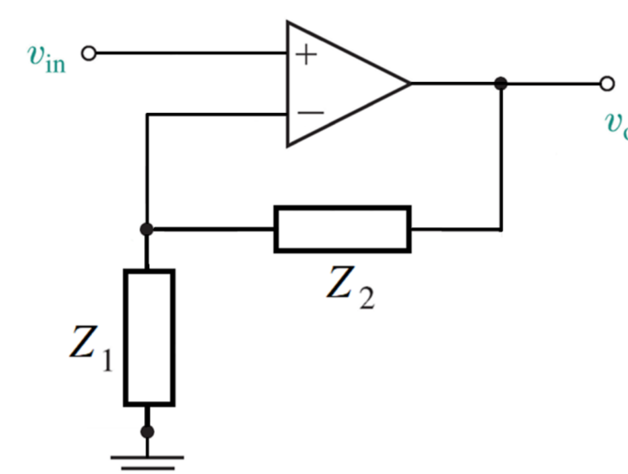
This is a **noninverting** op-amp circuit. The general form of the transfer function is:

$$\boxed{\frac{V_o(s)}{V_i(s)} = 1 + \frac{Z_2(s)}{Z_1(s)}}$$

First find the $Z_1(s)$ and $Z_2(s)$:

$$Z_1(s) = R_1 + \frac{1}{C_1 s} = \frac{R_1 C_1 s + 1}{C_1 s} \qquad Z_2(s) = \frac{R_2\left(\frac{1}{C_2 s}\right)}{R_2 + \frac{1}{C_2 s}} = \frac{R_2}{R_2 C_2 s + 1}$$

The transfer function is:



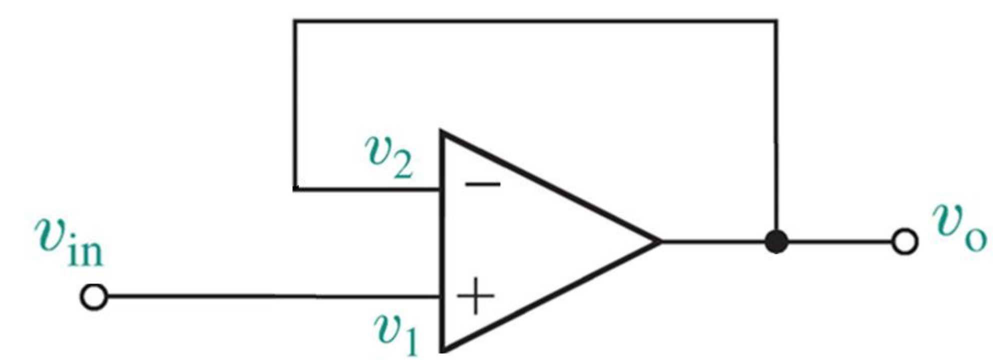
$$\frac{V_o(s)}{V_i(s)} = 1 + \frac{Z_2(s)}{Z_1(s)} = 1 + \frac{\frac{R_2}{R_2 C_2 s + 1}}{\frac{R_1 C_1 s + 1}{C_1 s}} = 1 + \frac{R_2 C_1 s}{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1}{R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_2)s + 1}}$$

Modeling of Electrical Systems

❑ Voltage Buffer (Voltage Follower)

- A **voltage buffer** is implemented using an op-amp in a negative feedback configuration. It means the output is connected to its inverting input.
- It is used to **avoid loading** of the signal source.
- Consider the following **voltage buffer** operating under the ideal condition with high input impedance and low output impedance.



- We know that,

$$\boldsymbol{v_o = A(v_1 - v_2)} = A(v_{in} - v_o) \quad \rightarrow \quad v_o(1 + A) = Av_{in} \quad \rightarrow \quad \frac{v_o}{v_{in}} = \frac{A}{1 + A}$$

- Since $A = \infty$, we have,

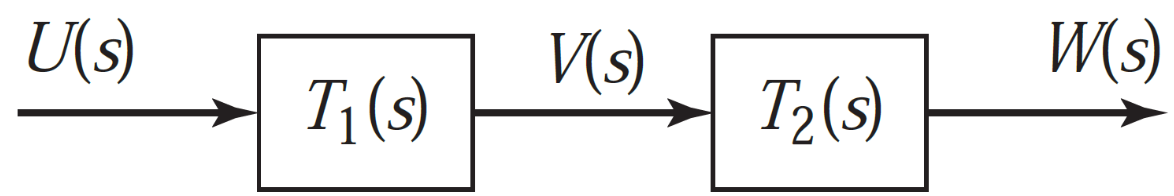
$$\frac{v_o}{v_{in}} = \lim_{A \to \infty} \frac{A}{1 + A} \approx 1 \quad \Rightarrow \quad \boxed{\boldsymbol{\frac{v_o}{v_{in}} \approx 1}}$$

Loading Effect & Block Diagram Models

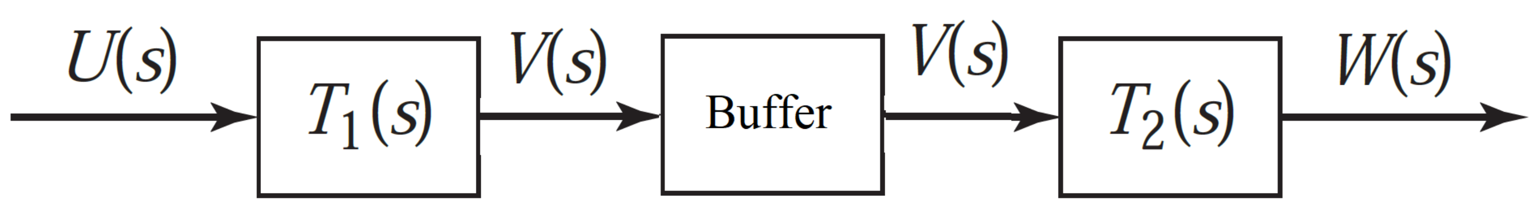
- Suppose two passive elements/circuits whose individual transfer functions are $T_1(s)$ and $T_2(s)$ are physically connected *end-to-end* so that the output of the lefthand element becomes the input to the righthand element.
- We can represent this connection by the block diagram shown as below ***only if*** the output w of the righthand element does ***not affect*** the inputs u and v or the behavior of the lefthand element.
- If it does, the righthand element is said to ***"load"*** the lefthand element.

$$T_1(s) = \frac{V(s)}{U(s)}$$

$$T_2(s) = \frac{W(s)}{V(s)}$$



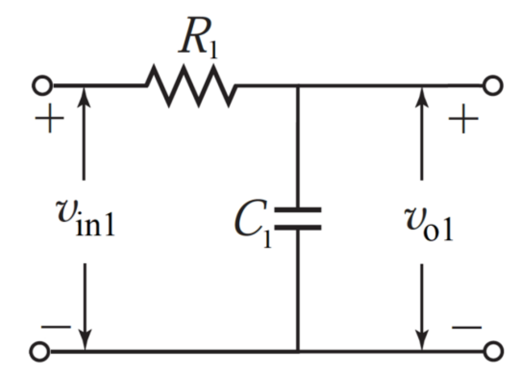
- We can avoid the loading effect by connecting the two stages via a **voltage buffer** to isolate the stages.

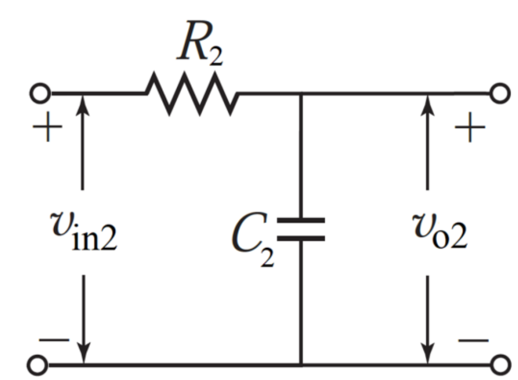


Loading Effect & Block Diagram Models

Example 8 Consider the following two electric circuits with their individual transfer functions $G_1(s)$ and $G_2(s)$.

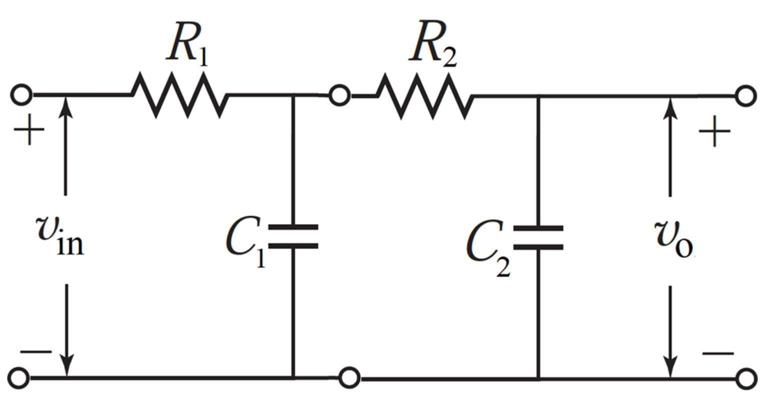
$$G_1(s) = \frac{V_{o1}(s)}{V_{in1}(s)} = \frac{1}{R_1C_1s + 1} \qquad G_2(s) = \frac{V_{o2}(s)}{V_{in2}(s)} = \frac{1}{R_2C_2s + 1}$$



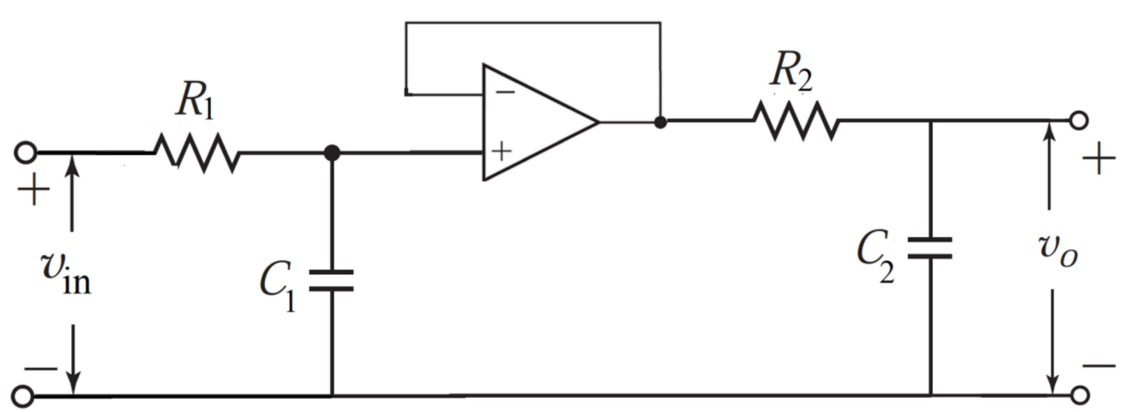


Find the overall transfer function $V_o(s)/V_{in}(s)$ in each cases.

(a) The circuits are connected end-to-end as shown below:

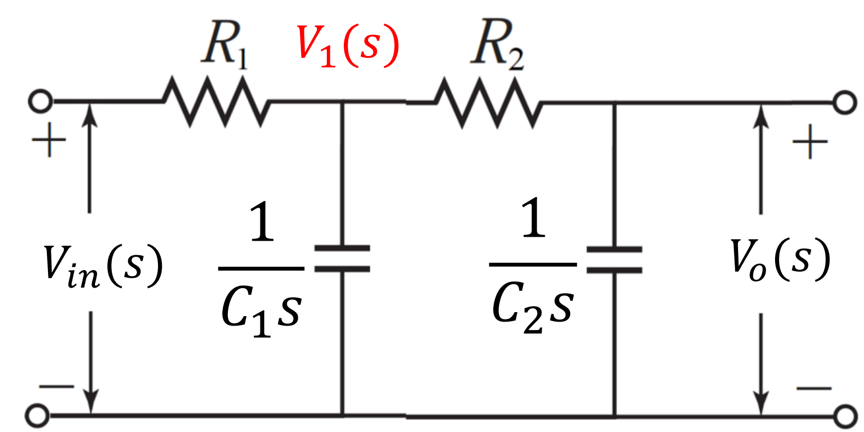


(b) The circuits are connected via a voltage buffer stage as shown below:



Loading Effect & Block Diagram Models

Example 8 (a) The circuits are connected end-to-end as shown below:



First, convert the circuit to Laplace domain based on the impedances of the elements:

$$V_o(s) = \frac{\frac{1}{C_2 s}}{R_2 + \frac{1}{C_2 s}} V_1(s) \quad \rightarrow \quad V_1(s) = (R_2 C_2 s + 1) V_o(s)$$

$$G_1(s) = \frac{1}{R_1 C_1 s + 1} \qquad G_2(s) = \frac{1}{R_2 C_2 s + 1}$$

Apply a KCL at node V_1:

$$\frac{V_{in}(s) - V_1(s)}{R_1} = \frac{V_1(s)}{\frac{1}{C_1 s}} + \frac{V_1(s) - V_o(s)}{R_2} \quad \rightarrow \quad R_2(V_{in}(s) - V_1(s)) - R_1 R_2 C_1 s V_1(s) - R_1(V_1(s) - V_o(s)) = 0$$

$$R_2 V_{in}(s) + (-R_2 - R_1 R_2 C_1 s - R_1) V_1(s) + R_1 V_o(s) = 0$$

$$R_2 V_{in}(s) + (-R_2 - R_1 R_2 C_1 s - R_1)(R_2 C_2 s + 1) V_o(s) + R_1 V_o(s) = 0$$

$$\boxed{\frac{V_o(s)}{V_{in}(s)} = \frac{1}{R_1 R_2 C_1 C_2 s^2 + (R_2 C_2 + 2 R_1 C_2) s + 1}}$$

$$R_2 V_{in}(s) + (-R_2^2 C_2 s - R_2 - R_1 R_2^2 C_1 C_2 s^2 - R_1 R_2 C_2 s - R_1 R_2 C_2 s - R_1 + R_1) V_o(s) = 0$$

$$V_{in}(s) + (-R_2 C_2 s - 1 - R_1 R_2 C_1 C_2 s^2 - 2 R_1 C_2 s) V_o(s) = 0$$

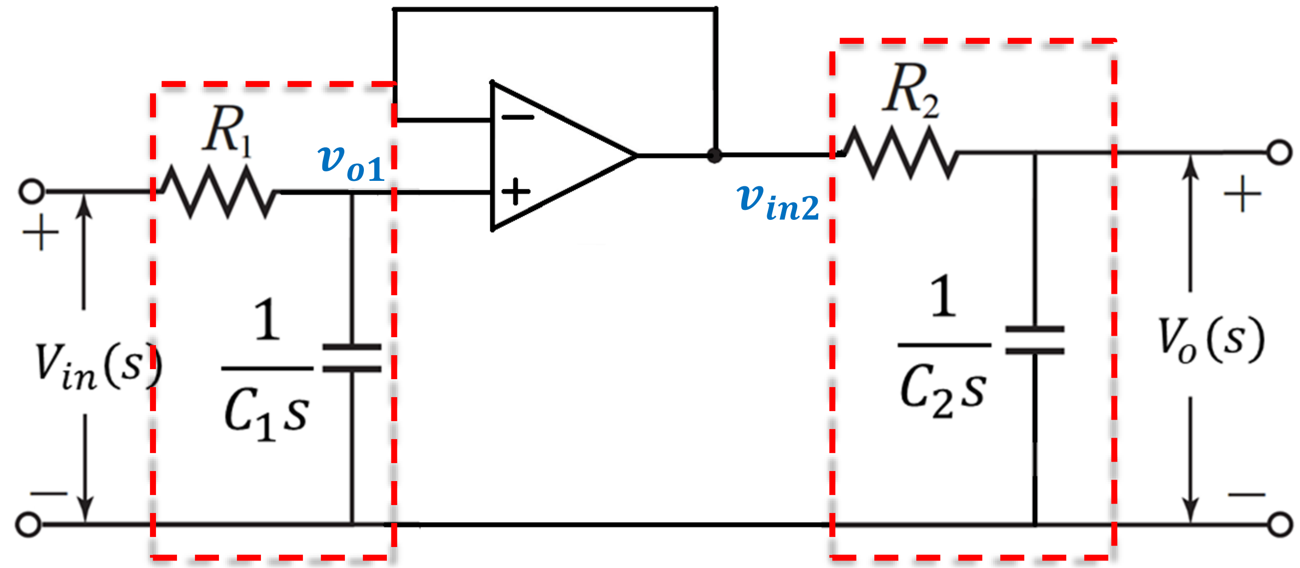
The overall transfer function is **not identical** with $G_1(s)G_2(s)$ because of the loading effect.

We cannot show the overall system as the series block diagram.

Loading Effect & Block Diagram Models

Example 8

(b) The circuits are connected via an op-amp buffer stage as shown below:

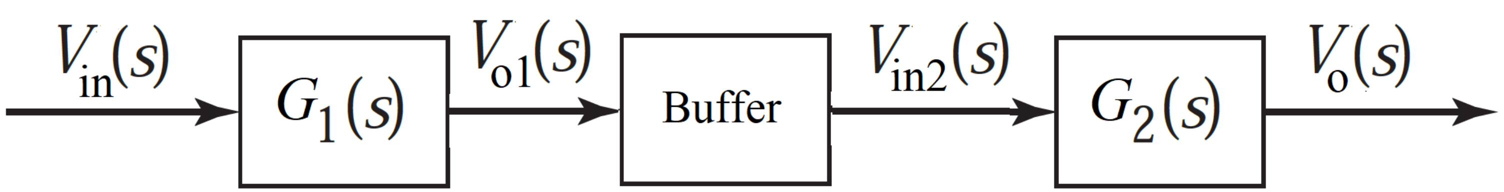


$$G_1(s) = \frac{1}{R_1 C_1 s + 1} \qquad G_2(s) = \frac{1}{R_2 C_2 s + 1}$$

Since the circuit stages are connected via an op-amp buffer we can show there is no loading effect between the stages.

$$v_{in2} = v_{o1}$$

The overall system can be modeled by the following block diagram model:

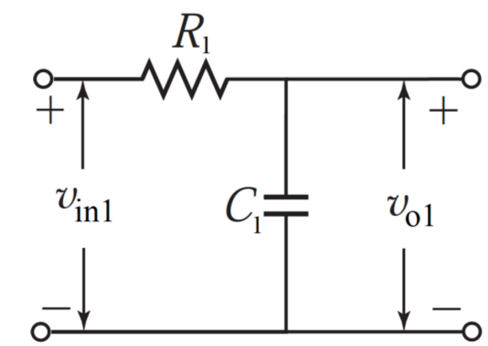


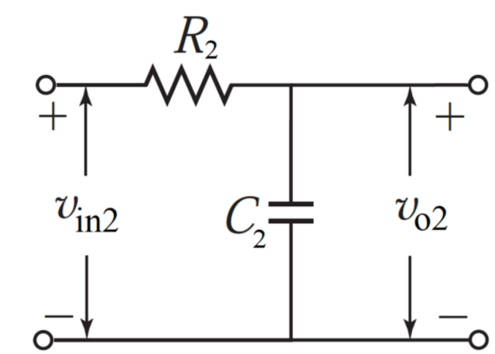
$$\frac{V_o(s)}{V_{in}(s)} = \frac{V_o(s)}{V_{in2}(s)} \cdot \frac{V_{in2}(s)}{V_{o1}(s)} \cdot \frac{V_{o1}(s)}{V_{in}(s)} = G_2(s) \cdot 1 \cdot G_1(s)$$

$$\boxed{\frac{V_o(s)}{V_{in}(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}}$$

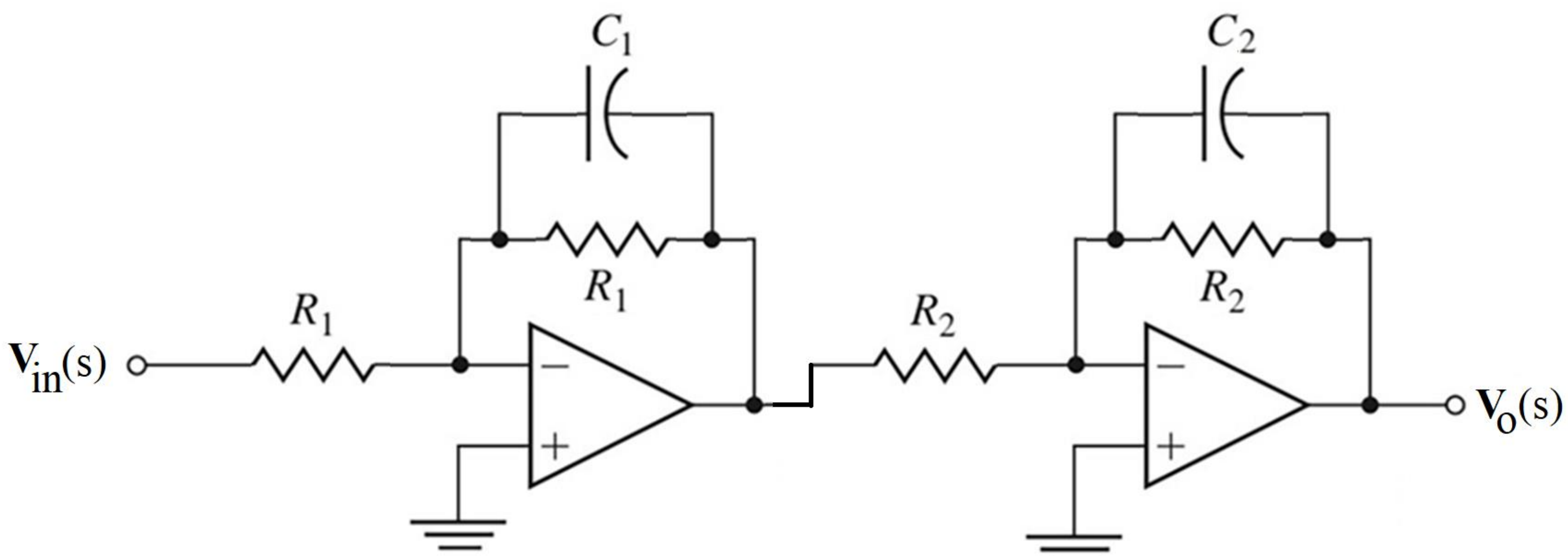
Loading Effect & Block Diagram Models

Example 8 (c) Implement each passive circuit using an op-amp and cascade them.



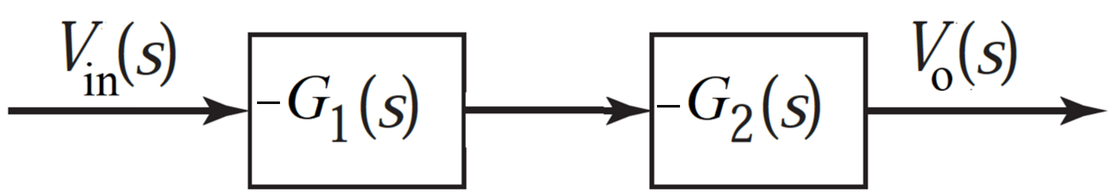


We can also implement each transfer function directly by an op-amp and them cascade them.



$$G_1(s) = \frac{1}{R_1C_1s + 1}$$

$$G_2(s) = \frac{1}{R_2C_2s + 1}$$



$$\frac{V_o(s)}{V_{in}(s)} = \left(\frac{-1}{R_1C_1s + 1}\right)\left(\frac{-1}{R_2C_2s + 1}\right)$$

$$\boxed{\frac{V_o(s)}{V_{in}(s)} = \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1)}}$$

THANK YOU

HUMBER

WE ARE HUMBER