Class Note 1.4 & 1.5

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Module 1: Limits and Continuity (continued)

$\lim_{x\to 0^+}\left(\frac{1}{x}\right) - \lim_{x\to 0^+}\left(\frac{1}{x^2}\right) = \infty - \infty \leftarrow$ indeterminate form

Solution: algebra simplification

$\lim_{x\to 0^+}\left(\frac{x-1}{x^2}\right) = \frac{-1}{0^+} = -\infty$

Warm Up

Show that

a. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right) = -\infty$ b. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right) = \frac{1}{2}$

Note, the indeterminate of the form $\infty - \infty$

1.4 Computing Limits and No So Basic Limit Laws

Theorem 1.4.17 (Squeeze theorem (or sandwich theorem or pinch theorem)).

Let $a \in \mathbb{R}$ and let f, g, h be three functions so that

$$f(x) \leqslant g(x) \leqslant h(x)$$

for all x in an interval around a, except possibly exactly at $x = a$. Then if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then it is also the case that

$$\lim_{x\to a} g(x) = L$$

Example. Prove that $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$

Limits Involving Radicals

Work out the following limits.

a. $\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{3x-6} = \frac{1}{3}$ b. $\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6} = -\frac{1}{3}$ c. $\lim_{x\to-\infty}\frac{\sqrt{3x^4+8}}{x^2-8} = \sqrt{3}$ d. $\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3} = 2$

e. $\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right) = 2$ f. $\lim_{x\to\infty}\left(\sqrt{x^6+5x^3}-x^3\right) = \frac{5}{2}$

Additional Problems

Works out the following limits:

a. $\lim_{t\to\infty}\frac{t-t\sqrt{t}}{2t^{3/2}+3t-5} = -\frac{1}{2}$

b. Find the horizontal and vertical asymptotes of the curve $y = \frac{2e^x}{e^x-5}$

1.5 Continuity

Definition of Continuity

A function f is said to be **continuous at $x = c$** provided the following conditions are satisfied:

1. $f(c)$ is defined.

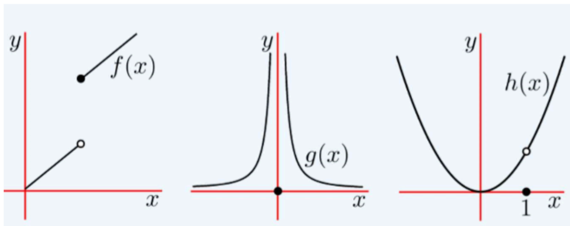
2. $\lim_{x\to c} f(x)$ exists.

3. $\lim f(x) = f(c)$.

Note:

- the third condition in the definition implies the first two. Thus, when we need to establish continuity at c, we need to verify the third condition only.
- Types of discontinuity: jump, infinite, removable.
- If a function f is continuous at each number in an open interval (a, b), then we say that f is **continuous on (a, b)**. This definition applies to infinite intervals as well.

Example 1



$$f(x) = \begin{cases} x & x < 1 \\ x+2 & x \geq 1 \end{cases}$$
$$g(x) = \begin{cases} 1/x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$
$$h(x) = \begin{cases} \frac{x^3-x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Types of discontinuity (Link)

a) Domain: $(-\infty, 1)$ and $[1, +\infty)$

$x < 1$, $f(x) = x$ continuous

$x > 1$, $f(x) = x+2$ continuous

$x = 1$ might be a problem

For f to be cont-s at $x=1$,

$$\lim_{x\to 1} f(x) = f(1)$$

$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} x = 1 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} x+2 = 2 \end{aligned}\right\}$$ hence $\lim_{x\to 1} f(x) = DNE$ and discontinuity occurs at $x=1$

Useful Properties of Continuous Functions

Theorem 1.6.5 Arithmetic of continuity.

Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a. Then the following functions are also continuous at $x = a$:

- $f(x) + g(x)$ and $f(x) - g(x)$,
- $cf(x)$ and $f(x)g(x)$, and
- $\frac{f(x)}{g(x)}$ provided $g(a) \neq 0$.

Theorem 1.6.8.

The following functions are continuous everywhere in their domains

- polynomials, rational functions
- roots and powers
- trig functions and their inverses
- exponential and the logarithm

A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.

Example 2 Verify the continuity of $f(x) = \frac{\sin x}{2+\cos x}$

Example 3 For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}$$

Example 4 Show that $|x|$ is continuous on $\mathbb{R}$.

Hint: we can write

$$|x| = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -x \text{ if } x < 0 \end{cases}$$

and consider the one-sided limits.

Continuity of Compositions
Limits and compositions work nicely together.

Theorem 1.6.10 Compositions and continuity.

If f is continuous at b and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(g(x)) = f(b)$. I.e.

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Hence if g is continuous at a and f is continuous at $g(a)$ then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

Note:

- the Theorem 1.6.10 states that a limit symbol can be moved through a function sigh provided the limit of the expression inside the function sign exists and the function is continuous at this limit.
- if the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f(g(x))$ is continuous everywhere.

Example 5 Where are the following functions continuous?

a. $f(x) = \sin(x^2 + \cos x)$
b. $g(x) = \sqrt{\sin(x)}$
c. $h(x) = |5 - x^2|$
d. $f(x) = \frac{x-2}{x^2+x+4}$
e. $u(x) = \frac{x-2}{x^2-3x-4}$

Additional Problems
1. Find a constant k so that the function

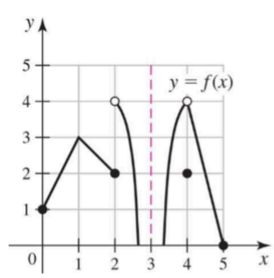
$$a(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{when } x \neq 0 \\ k & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$.

2. Where is the following function continuous?

$$f(x) = \begin{cases} \dfrac{x^2 + x}{x + 1} & \text{if } x \neq -1 \\ 2 & \text{if } x = -1 \end{cases}$$

3. Determine the values of x, if any, at which f is not continuous. Determine the types of discontinuities.



$x = 2$ Jump
$x = 3$ infinite
$x = 4$ removable, "hole"

4. (1.6.15) Describe all points for which this function is continuous: $\frac{1}{\sqrt{1+\cos x}}$.