Day 1

July 4, 2023 10:05 AM

Chapter 8

Maximum and Minimum Values

8.1 Readings

1. CLP III §2.9 Maximum and Minimum Values

8.2 Critical and Singular Points

8.2 Critical and Singular Points

FRY Defn III.2.9.1, Shifrin Defn 5.2

Definition 8.1. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in X$.

(i) f has an *absolute* or *global maximum* at $\mathbf{a}$ if $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in X$. In this case, we say that $\mathbf{a}$ is an absolute or global maximum point of f and that $f(\mathbf{a})$ is an absolute or global maximum value for f.

(ii) f has a *local maximum* at $\mathbf{a}$ if there exists a $\delta > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in B(\mathbf{a}, \delta) \cap X$[a]. In this case, we say that $\mathbf{a}$ is a local maximum point of f and that $f(\mathbf{a})$ is a local maximum value for f.

(iii) f has an *absolute* or *global minimum* at $\mathbf{a}$ if $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in X$. In this case, we say that $\mathbf{a}$ is an absolute or global minimum point of f and that $f(\mathbf{a})$ is an absolute or global minimum value for f.

(iv) f has a *local minimum* at $\mathbf{a}$ if there exists a $\delta > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in B(\mathbf{a}, \delta) \cap X$. In this case, we say that $\mathbf{a}$ is a local minimum point of f and that $f(\mathbf{a})$ is a local minimum value for f.

(v) If $\mathbf{a}$ is either a local maximum or local minimum point, we say that $\mathbf{a}$ is a *local extremum*.

[a] $B(\mathbf{a}, \delta)$ refers to a ball of radius $\delta > 0$ centred at $\mathbf{a}$, that is, $B(\mathbf{a}, \delta) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < \delta\}$. We also refer to $B(\mathbf{a}, \delta)$ as a neighbourhood of $\mathbf{a}$.

If a function is differentiable at a local maximum or minimum, then all of the partial derivatives of the function are zero there.

FRY Thm III.2.9.2, Shifrin Lemma 5.2.1: At extrema, the gradient is zero.

Lemma 8.2. Suppose $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ and $\mathbf{a}$ is a local extremum of f. Then $Df(\mathbf{a}) = \mathbf{0}$[a] or, equivalently, $\nabla f(\mathbf{a}) = \mathbf{0}$[b].

[a] $Df(\mathbf{a}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{a}) & \frac{\partial f}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f}{\partial x_n}(\mathbf{a}) \end{bmatrix}$.

[b] $\nabla f(\mathbf{a}) = \left\langle \frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \right\rangle$.

The points in the domain of a function where the gradient is zero or does not exist are given special names.

FRY Defn III.2.9.3, Shifrin § 5.2: What is a critical point of a function?

Definition 8.3. Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in X$. We say that $\mathbf{a}$ is a **critical point** if $Df(\mathbf{a}) = \mathbf{0}$, or, equivalently, $\nabla f(\mathbf{a}) = \mathbf{0}$. That is, $\frac{\partial f}{\partial x_1}(\mathbf{a}) = \frac{\partial f}{\partial x_2}(\mathbf{a}) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{a}) = 0$.

FRY Defn III.2.9.3: What is a singular point of a function?

Definition 8.4. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{s} \in X$. We say that $\mathbf{s}$ is a **singular point** of f if $Df(\mathbf{s})$, or, equivalently, $\nabla f(\mathbf{s})$, does not exist. That is, one or more of the partial derivatives of f does not exist at $\mathbf{s}$.

Example 8.5. (FRY Exercise 2.9.4.19a)
Let $f(x, y) = xy(x + y - 3)$. Find all the critical and singular points of f.

$\vec{a}$ s.t. $\nabla f(\vec{a}) = \vec{0}$ $\quad$ $\vec{s}$ at which $\nabla f(\vec{s})$ DNE

$$f(x,y) = xy(x+y-3) = x^2y + xy^2 - 3xy$$

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle = \left\langle 2xy + y^2 - 3y,\ x^2 + 2xy - 3x \right\rangle$$

$\nabla f(x,y) = \langle 0,0 \rangle$ iff

$$\begin{cases} 2xy + y^2 - 3y = 0 \\ x^2 + 2xy - 3x = 0 \end{cases} \longrightarrow \begin{cases} y(2x+y-3) = 0 & \text{— eqn(1)} \\ x(x+2y-3) = 0 & \text{— eqn(2)} \end{cases}$$

3

$$\begin{cases} y(2x+y-3) = 0 & \text{— eqn(1)} \\ x(x+2y-3) = 0 & \text{— eqn(2)} \end{cases}$$

From eqn(1), $y=0$ or $2x+y-3=0$

$\boxed{\text{If } y=0}$, eqn(2) tells us

$$x(x-3) = 0,$$

which occurs if $x=0$ or $x-3=0$, $x=3$

So, $(0,0)$ and $(3,0)$ are critical points of f.

From Eqn(1)

$\boxed{\text{If } 2x+y-3=0}$, then $y = 3-2x$. Substituting into eqn(2), $x(x+2y-3)=0$ — eqn(2)

$$x(x+2(3-2x)-3) = 0$$
$$x(x+6-4x-3) = 0$$
$$x(-3x+3) = 0$$

So, $x=0$ or $-3x+3=0$, $x=1$ $\quad \frac{3}{3} = \frac{3x}{3}$

$y = 3-2(0) = 3$ $\quad$ $y = 3-2(1) = 1$

So $(0,3)$ and $(1,1)$ are critical points of f.

Since f is a polynomial, so are its partial derivatives.

In particular, $\nabla f(x,y)$ is well-defined for all $(x,y) \in \mathbb{R}^2$

In particular $\nabla f(x,y)$ is well-defined for all $(x,y) \in$ in

So, f has no singular points.

Live Poll $f(x,y) = 3x^2 - 2y^2$

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$$

$$= \langle 6x, -4y \rangle$$

Critical points are (x,y) such that $\nabla f(x,y) = \langle 0,0 \rangle$

$$\begin{cases} 6x = 0 \\ -4y = 0 \end{cases} \longrightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

Critical point $(0,0)$.

8.3 Classifying Critical Points

Let U be an open subset of $\mathbb{R}^2$, and let $f : U \to \mathbb{R}$. We know that if f is $\mathcal{C}^1$ on a neighbourhood of a point $\mathbf{a}$ in U, then there is a linear transformation $Df(\mathbf{a}) : \mathbb{R}^2 \to \mathbb{R}$ (the derivative of f at $\mathbf{a}$) such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - \Big(f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{h})\Big)}{\|\mathbf{h}\|} = \mathbf{0}.$$

That is, the difference between $f(\mathbf{a}+\mathbf{h})$ and its linear approximation $f(\mathbf{a})+Df(\mathbf{a})(\mathbf{h})$ goes to $\mathbf{0}$ faster than $\|\mathbf{h}\|$ goes to 0. If f is $\mathcal{C}^2$ on a neighbourhood of $\mathbf{a}$ in U, then the second-degree Taylor polynomial $T^2_{f,\mathbf{a}}$ is an excellent quadratic approximation for $f(\mathbf{a}+\mathbf{h})$ in the sense that the error between the function's value at $\mathbf{a}+\mathbf{h}$ and the Taylor polynomial's value at $\mathbf{a}+\mathbf{h}$ goes to zero faster than $\|\mathbf{h}\|^2$ goes to zero.[1] That is,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - T^2_{f,\mathbf{a}}(\mathbf{a}+\mathbf{h})}{\|\mathbf{h}\|^2} = \mathbf{0}.$$

Recall that if $\mathbf{a} = (x_0, y_0)$ and $\mathbf{h} = (\Delta x, \Delta y)$, then $T^2_{f,\mathbf{a}}(\mathbf{a}+\mathbf{h}) = T^2_{f,(x_0,y_0)}(x_0 + \Delta x, y_0 + \Delta y)$ is given by

$$\begin{aligned} T^2_{f,(x_0,y_0)}(x_0 + \Delta x, y_0 + \Delta y) &= f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0, y_0)(\Delta x)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\Delta x\Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0, y_0)(\Delta y)^2. \end{aligned}$$

Using matrix notation,

$$\begin{aligned} &T^2_{f,(x_0,y_0)}(x_0 + \Delta x, y_0 + \Delta y) \\ &= f(x_0, y_0) + \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \end{aligned}$$

The matrix involving the second-order derivatives above is also special and given its own name.

[1] See Shifrin Proposition 5.3.2 for a proof.

Shifrin Defn 5.3.1, The Hessian matrix

Definition 8.6. Let U be an open subset of $\mathbb{R}^2$ and $\mathbf{a} \in U$. Let $f : U \to \mathbb{R}$ be of class $\mathcal{C}^2$ on U. Then the Hessian matrix of f at $\mathbf{a}$, denoted by $\mathcal{H}_f(\mathbf{a})$, is the symmetric matrix

$$\mathcal{H}_f(\mathbf{a}) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2}(\mathbf{a}) & \dfrac{\partial^2 f}{\partial y \partial x}(\mathbf{a}) \\ \\ \dfrac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) & \dfrac{\partial^2 f}{\partial y^2}(\mathbf{a}) \end{bmatrix}.$$

So,

$$\begin{aligned} &T^2_{f,(x_0,y_0)}(x_0 + \Delta x, y_0 + \Delta y) \\ &= f(x_0, y_0) + \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \mathcal{H}_f(x_0, y_0) \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}. \end{aligned}$$

If (x_0, y_0) is a critical point, then $\dfrac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\dfrac{\partial f}{\partial y}(x_0, y_0) = 0$. In this case, using the quadratic approximation $T^2_{f,(x_0,y_0)}$ for f,

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + \frac{1}{2}\begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \mathcal{H}_f(x_0,y_0) \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

Since the Hessian matrix is a symmetric matrix, there exists an orthonormal basis of eigenvectors of $\mathcal{H}_f(x_0,y_0)$ with corresponding orthogonal matrix U such that

$$\mathcal{H}_f(x_0,y_0) = UDU^T,$$

where D is a diagonal matrix with diagonal entries equal to the real eigenvalues of

$\mathcal{H}_f(x_0,y_0)$.[2] Thus,

$$\begin{aligned} f(x_0+\Delta x, y_0+\Delta y) &\approx f(x_0,y_0) + \frac{1}{2}\begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \mathcal{H}_f(x_0,y_0) \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\ &= f(x_0,y_0) + \frac{1}{2}\begin{bmatrix} \Delta x & \Delta y \end{bmatrix} UDU^T \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\ &= f(x_0,y_0) + \frac{1}{2}\left(U^T \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}\right)^T D \left(U^T \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}\right). \end{aligned}$$

If we let $\begin{bmatrix} \Delta x' \\ \Delta y' \end{bmatrix} = U^T \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$ and if $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1 and λ_2 are the eigenvalues of $\mathcal{H}_f(x_0,y_0)$, then

$$\begin{aligned} f(x_0+\Delta x, y_0+\Delta y) &\approx f(x_0,y_0) + \frac{1}{2}\begin{bmatrix} \Delta x' & \Delta y' \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \Delta x' \\ \Delta y' \end{bmatrix} \\ &= f(x_0,y_0) + \frac{1}{2}\lambda_1(\Delta x')^2 + \frac{1}{2}\lambda_2(\Delta y')^2. \end{aligned}$$

Observe that

- If both the eigenvalues λ_1 and λ_2 of the Hessian matrix $\mathcal{H}_f(x_0,y_0)$ are positive, then $\frac{1}{2}\lambda_1(\Delta x')^2 + \frac{1}{2}\lambda_2(\Delta y')^2 \geq 0$. Thus, in a neighbourhood of (x_0,y_0), $f(x_0 + \Delta x, y_0+\Delta y) \geq f(x_0,y_0)$. That is, (x_0,y_0) is a local minimum of f.
- If both the eigenvalues λ_1 and λ_2 of the Hessian matrix $\mathcal{H}_f(x_0,y_0)$ are negative, then $\frac{1}{2}\lambda_1(\Delta x')^2 + \frac{1}{2}\lambda_2(\Delta y')^2 \leq 0$. Thus, in a neighbourhood of (x_0,y_0), $f(x_0 + \Delta x, y_0+\Delta y) \leq f(x_0,y_0)$. That is, (x_0,y_0) is a local maximum of f.
- If the eigenvalue λ_1 is positive and the eigenvalue λ_2 is negative, then if we move in a direction such that $\Delta y' = 0$, then, because $\lambda_1 > 0$,

$$\begin{aligned} f(x_0+\Delta x, y_0+\Delta y) &\approx f(x_0,y_0) + \frac{1}{2}\lambda_1(\Delta x')^2 + \frac{1}{2}\lambda_2(\Delta y')^2 \\ &= f(x_0,y_0) + \frac{1}{2}\lambda_1(\Delta x')^2 + \frac{1}{2}\lambda_2(0)^2 \\ &\geq f(x_0,y_0). \end{aligned}$$

[2]Such a decomposition of the Hessian matrix follows from the Spectral Theorem (also known as the Symmetric Diagonalization Theorem or the Principal Axis Theorem). See Norman §8-1, Theorem 4.

On the other hand, if we move in a direction in which $\Delta x' = 0$, then, because $\lambda_2 < 0$,

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) &\approx f(x_0, y_0) + \frac{1}{2}\lambda_1(\Delta x')^2 + \frac{1}{2}\lambda_2(\Delta y')^2 \\ &= f(x_0, y_0) + \frac{1}{2}\lambda_1(0)^2 + \frac{1}{2}\lambda_2(\Delta y')^2 \\ &\leq f(x_0, y_0). \end{aligned}$$

That is, (x_0, y_0) is neither a local maximum nor a local minimum for f. We call such a point, where movement in one direction leads to the function increasing in value and movement in another direction leads to a decrease in the function's value, a **saddle point** of f. Similarly, if $\lambda_1 < 0$ and $\lambda_2 > 0$, then (x_0, y_0) is again a saddle point of f.

- If $\lambda_1 = 0$, then when we move in a direction in which $\Delta y' = 0$, we have to look at higher-order approximations than the quadratic to determine whether or not f is increasing or decreasing in value. The same is true if $\lambda_2 = 0$. That is, if either of the eigenvalues of $\mathcal{H}_f(x_0, y_0)$ is zero, then we do not have enough information to determine the nature of the critical point (x_0, y_0).

Note the following two additional facts:

(i) The determinant of a 2-by-2 matrix equals the product of its eigenvalues.

(ii) Suppose the determinant of a 2-by-2 symmetric matrix is positive. If its 1,1-entry is positive, then both its eigenvalues are positive. If its 1,1-entry is negative, then both its eigenvalues are negative.

We can summarize our work as follows:

FRY Thm III.2.9.16, Shifrin Corollary 5.3.4: The Second Derivative Test

Theorem 8.7. Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$. Suppose $(a, b) \in U$ is a critical point of f and f is $\mathcal{C}^2$ on a neighbourhood of (a, b).

If	*Then*
$\det \mathcal{H}_f(a, b)$ is positive and $f_{xx}(a, b)$ is positive	(a, b) is a local minimum
$\det \mathcal{H}_f(a, b)$ is positive and $f_{xx}(a, b)$ is negative	(a, b) is a local maximum
$\det \mathcal{H}_f(a, b)$ is negative	(a, b) is a saddle point
$\det \mathcal{H}_f(a, b)$ is zero	the test is inconclusive

First Derivative Test

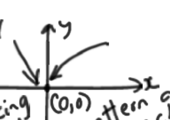
$f(x) = x^2$

set $f'(x) = 2x = 0 \implies x = 0$

sign of f': $- - -$ | $+ +$ x

f decreasing 0 f increasing

Since in higher dimensions there are infinitely many ways to approach a point, the strategy of looking at the pattern of sign changes "across" a point fails.



Second Derivative Test

$$\mathcal{H}_f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

uses the determinant of the Hessian matrix to help classify the critical points

Example 8.8. (FRY Exercise 2.9.4.19a)
Let $f(x, y) = xy(x + y - 3)$. Find and classify all the critical points of f.

$f(x,y) = xy(x+y-3) = x^2y + xy^2 - 3xy$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ $\quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$

$$H_f(x,y) = \begin{bmatrix} 2y & 2x+2y-3 \\ 2x+2y-3 & 2x \end{bmatrix}$$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

$\frac{\partial f}{\partial y}$

$= \langle 2xy + y^2 - 3y, \; x^2 + 2xy - 3x \rangle$ $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

We saw earlier the following four critical points $(0,0)$, $(3,0)$, $(0,3)$, $(1,1)$.

Look at $H(f)(x_0,y_0)$ and its determinant

det positive and $f_{xx}(x_0,y_0)$ is positive	determinant positive and $f_{xx}(x_0,y_0)$ negative	det negative	det zero
 local min $\lambda_1, \lambda_2 > 0$	$\lambda_1, \lambda_2 < 0$ local max 	Saddle pt 	2^{nd} Deriv. Test is inconclusive

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$$H(f)(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{bmatrix} = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$$= \begin{bmatrix} 2y & 2x+2y-3 \\ 2x+2y-3 & 2x \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$(0)(0) - (-3)(-3) = 0 - 9$

det $H_f(0,0)$

Since $|H_f(0,0)| = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$, $(0,0)$ is a saddle point.

$x=3$, $y=0$

Since $|H_f(3,0)| = \begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix} = -9 < 0$, $(3,0)$ is a saddle point.

Since $|H_f(0,3)| = \begin{vmatrix} 6 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0$, $(0,3)$ is a saddle point.

Since $|H_f(1,1)| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$ and $f_{xx}(1,1) = 2 > 0$, $(1,1)$ is a local minimum of f.

location

The minimum value here

Look at $H(f)(x_0,y_0)$ and its determinant

det positive and $f_{xx}(x_0,y_0)$ is positive | determinant positive and $f_{xx}(x_0,y_0)$ negative | det negative | det zero

2 Deriv. Test is inconclusive

is $f(1,1) = -1$.

$f(x,y) = xy(x+y-3)$

8.4 Absolute Maxima and Minima

Not all functions have an absolute (global) maximum or an absolute minimum. However, if our function is continuous on a closed and bounded domain, then it is guaranteed to have an absolute maximum and minimum.

FRY §2.9.3, Finding absolute max/min of cts function on compact domain

Proposition 8.9. Let D be an closed and bounded (i.e., compact) subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be $\mathcal{C}^1$ on the interior of D.

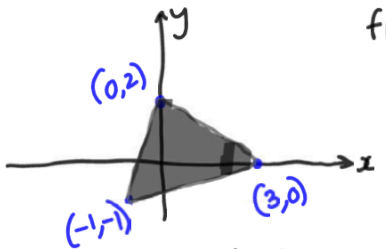
1. Build up a list of candidate points by finding all (a, b)'s for which
   - (a) (a, b) is critical point of f in the interior of D;
   - (b) (a, b) is a singular point of f in the interior of D; or
   - (c) (a, b) is a point on the boundary of D at which a maximum or minimum could occur.
2. Evaluate f at each (a, b) in the above list. The biggest value of $f(a, b)$ across the candidate points is the absolute maximum of f on D, and the smallest value is the absolute minimum of f on D.

$f(x,y) = x^2y + xy^2 - 3xy$

Continuous function

Example 8.10. (like FRY Exercise 2.9.4.19b)
Let $f(x, y) = xy(x + y - 3)$. Find the location and value of the absolute maximum and minimum of $f(x, y)$ restricted to the triangle with vertices $(0, 2)$, $(-1, -1)$ and $(3, 0)$ (and its interior).

$f(x,y) = x^2y + xy^2 - 3xy$



Closed and bounded domain

$f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

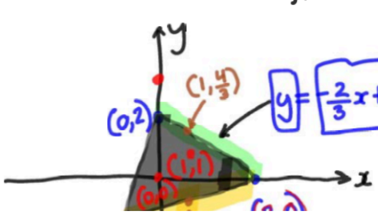
defined for all $(x,y) \in \mathbb{R}^2$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x, \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y \right\rangle$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$(0,0)$ is a Singular point

$f(x,y) = x^2\boxed{y} + x\boxed{y}^2 - 3xy$



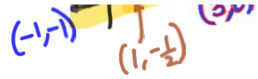
$\boxed{y} = \boxed{-\frac{2}{3}x + 2}$

$\overset{x_1\ y_1}{(0,2)}, \overset{x_2\ y_2}{(3,0)}$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-2}{3-0} = \frac{-2}{3}$

$y - y_1 = m(x - x_1)$ point-slope form

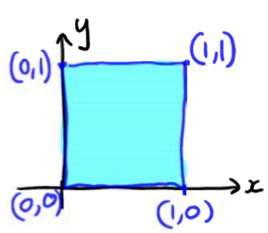
$y - 2 = -\frac{2}{3}(x - 0)$



$$\boxed{y=-\frac{2}{3}x+2}$$

Type	point (x,y)	$f(x,y)$
interior critical pt	$(0,0)$	
interior critical pt	$(1,1)$	
boundary pt	$(1,\frac{4}{3})$	
boundary pt	$(3,0)$	
boundary pt	$(1,-\frac{1}{2})$	
boundary pt.	$(-1,-1)$	
⋮		

Example 8.11. (FRY Example 2.9.23)
Find the maximum and minimum of $f(x,y)=xy-x^3y^2$ when (x,y) runs over the square $0\le x\le 1$, $0\le y\le 1$.



$f(x,y)=xy-x^3y^2$

$\nabla f(x,y)=\langle y-3x^2y^2,\ x-2x^3y\rangle=\langle 0,0\rangle$

when $\begin{cases} y-3x^2y^2=0 \\ x-2x^3y=0\end{cases}$ i.e., $\begin{cases} y(1-3x^2y)=0 & \text{eq(1)} \\ x(1-2x^2y)=0 & \text{eq(2)}\end{cases}$

From eq(1), $y=0$ or $1-3x^2y=0$

- If $y=0$, eq(2) gives us $x(\underbrace{1-2x^2(0)}_{1-0})=0$, i.e., $x=0$

  $(0,0)$ is a critical point of x but it's not in interior

• If $1-3x^2y=0$, then $1=3x^2y$. Both $\boxed{x\neq 0}$ and $y\neq 0$, and $y=\frac{1}{3x^2}$

Substituting into eq(2), $x\left(1-2\cancel{x^2}\left(\frac{1}{3\cancel{x^2}}\right)\right)=0$

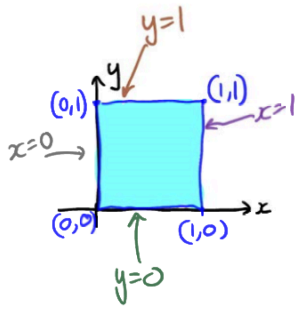
$$x\left(1-\frac{2}{3}\right)=0$$

$$\frac{1}{3}x=0$$

$$\boxed{x=0}$$

contradiction

Thus $1-3x^2y\neq 0$



$\boxed{x=0}$ $f(0,y)=0$

$\boxed{y=0}$ $f(x,0)=0$

11

$f(x,y)=xy-x^3y^2$

$\boxed{x=1}$ $g(y)=f(1,y)=y-y^2$

$g'(y)=1-2y=0 \Rightarrow y=\frac{1}{2}$

$\boxed{y=1}$ $h(x)=f(x,1)=x-x^3$

$\frac{dh}{dx}(x)=1-3x^2=0 \Rightarrow 3x^2=1 \Rightarrow x^2=\frac{1}{3}$

$\Rightarrow x=\frac{1}{\sqrt{3}}$ or $x=-\frac{1}{\sqrt{3}}$

$=\frac{\sqrt{3}}{3}$

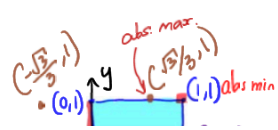
$\left(-\frac{1}{\sqrt{3}},1\right)$ is not in domain of f

Corner points on boundary	Value of f	
(0,0)	0	abs. min
(0,1)	0	abs. min
(1,0)	0	abs. min
(1,1)	0	abs. min
other boundary points of interest		
$\left(1,\frac{1}{2}\right)$	$\frac{1}{4}$	
$\left(\frac{\sqrt{3}}{3},1\right)$	$\frac{2\sqrt{3}}{9}\approx 0.385$	absolute max
$(x,0)$ for $0<x<1$	0	abs. min
$(0,y)$ for $0<y<1$	0	abs. min

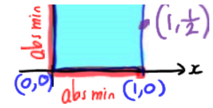
$f(x,y)=xy-x^3y^2$

$f\left(1,\frac{1}{2}\right)=1\cdot\frac{1}{2}-1^3\cdot\left(\frac{1}{2}\right)^2=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

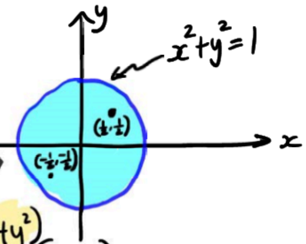
$f\left(\frac{\sqrt{3}}{3},1\right)=\frac{\sqrt{3}}{3}-\frac{\cancel{3}^{1}\sqrt{3}}{\cancel{27}_{9}}=\frac{2\sqrt{3}}{9}$



$(0,y)$, … abs. min



Example 8.12. (FRY Example 2.9.21)
Find the maximum and minimum of $T(x,y) = (x+y)e^{-x^2-y^2}$ on the unit disk $x^2+y^2 \le 1$.



$$T(x,y) = \boxed{(x+y)}\boxed{e^{-(x^2+y^2)}}$$

$$\nabla T(x,y) = \langle T_x(x,y), T_y(x,y) \rangle \overset{\text{set}}{=} \vec{0} = \langle 0,0 \rangle$$

$$T_x(x,y) = (1+0)e^{-(x^2+y^2)} + (x+y)e^{-(x^2+y^2)}(-2x)$$
$$= e^{-(x^2+y^2)}(1-2x(x+y))$$

$$T_y(x,y) = e^{-(x^2+y^2)}(1-2y(x+y))$$

$$\nabla T(x,y) = \langle 0,0 \rangle \text{ when } \begin{cases} e^{-(x^2+y^2)}(1-2x(x+y)) = 0 \\ e^{-(x^2+y^2)}(1-2y(x+y)) = 0 \end{cases}$$

Which occurs when

$$\begin{cases} 1-2x(x+y) = 0 \\ 1-2y(x+y) = 0 \end{cases}$$

$$1-2x(x+y) = 1-2y(x+y)$$
$$-2x(x+y) = -2y(x+y)$$
$$x(x+y) = y(x+y)$$
$$x(x+y) - y(x+y) = 0$$
$$(x-y)(x+y) = 0$$

$x-y=0$ or $x+y=0$

$x=y$ or $x=-y$

$\boxed{x=y}$: $1-2x(x+y)=0 \Rightarrow 1-2x(x+x)=0$

$\Rightarrow 1-2x(2x)=0$

$$\Rightarrow 1-4x^2=0$$
$$\Rightarrow 4x^2=1$$
$$\Rightarrow x^2=\frac{1}{4}$$
$$\Rightarrow x=\frac{1}{2} \text{ or } x=\frac{-1}{2}$$
$$y=\frac{1}{2} \qquad y=-\frac{1}{2}$$

So $(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$ or $\left(-\frac{1}{2},-\frac{1}{2}\right)$ are critical points on the interior of the disk.

$\boxed{x=-y}$ $\quad 1-2x(x+y)=0 \Rightarrow 1-2x(x-x=0$

$-x=y$

$$\Rightarrow 1-2x(0)=0$$
$$\Rightarrow 1-0=0$$
$$\Rightarrow 1=0$$

This can't be.

So $x\neq -y$

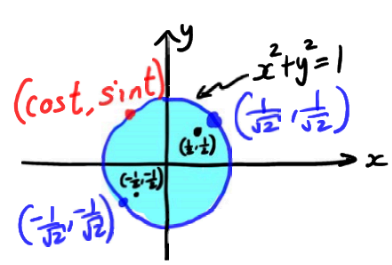
$$T(x,y)=(x+y)e^{-(x^2+y^2)}$$

Let $x=\cos t$, $y=\sin t$ where $0\leq t\leq 2\pi$

$x^2+y^2=\cos^2 t+\sin^2 t=1$

$$T(x,y)=T(\cos t,\sin t)$$
$$=(\cos t+\sin t)e^{-(\cos^2 t+\sin^2 t)}$$
$$=(\cos t+\sin t)e^{-1}$$
$$=\boxed{\frac{1}{e}\cos t+\frac{1}{e}\sin t}$$

$g(t)$



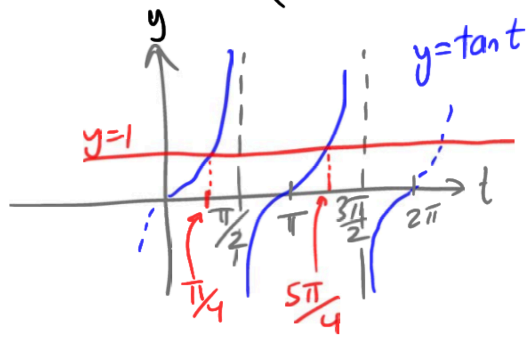
$$\frac{dg}{dt}(t)=-\frac{1}{e}\sin t+\frac{1}{e}\cos t=0$$
$$\Rightarrow -\sin t+\cos t=0$$
$$\Rightarrow \frac{\cos t}{\cos t}=\frac{\sin t}{\cos t}$$

$1=\tan t$

$$\Rightarrow t=\frac{\pi}{4} \text{ or } t=\frac{5\pi}{4}.$$

$$(x,y) = (\cos t, \sin t) = \left(\cos\tfrac{\pi}{4}, \sin\left(\tfrac{\pi}{4}\right)\right) = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right)$$

$$(x,y) = \left(\cos\tfrac{5\pi}{4}, \sin\tfrac{5\pi}{4}\right) = \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right) = \left(-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right)$$



$$T(x,y) = (x+y)e^{-(x^2+y^2)}$$

	Value of T	
interior critical points		
$\left(\frac{1}{2}, \frac{1}{2}\right)$ abs. max for T	$e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} \approx 0.607$	abs. max value
$\left(-\frac{1}{2}, -\frac{1}{2}\right)$ abs min for T	$-e^{-\frac{1}{2}} = \frac{-1}{\sqrt{e}} \approx -0.607$	abs min value
interior singular points		
none		
points of interest on boundary		
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\sqrt{2}e^{-1} = \frac{\sqrt{2}}{e} \approx 0.520$	
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$-\sqrt{2}e^{-1} = \frac{-\sqrt{2}}{e} \approx -0.520$	

8.5 References

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