Module 4 :

ANTIDERIVATIVES AND INTEGRALS

4.1 Introduction to antiderivatives; connection to differential equations and indefinite Integrals

4.2 Basic integration techniques (using Tables, rules, and U-substitution).

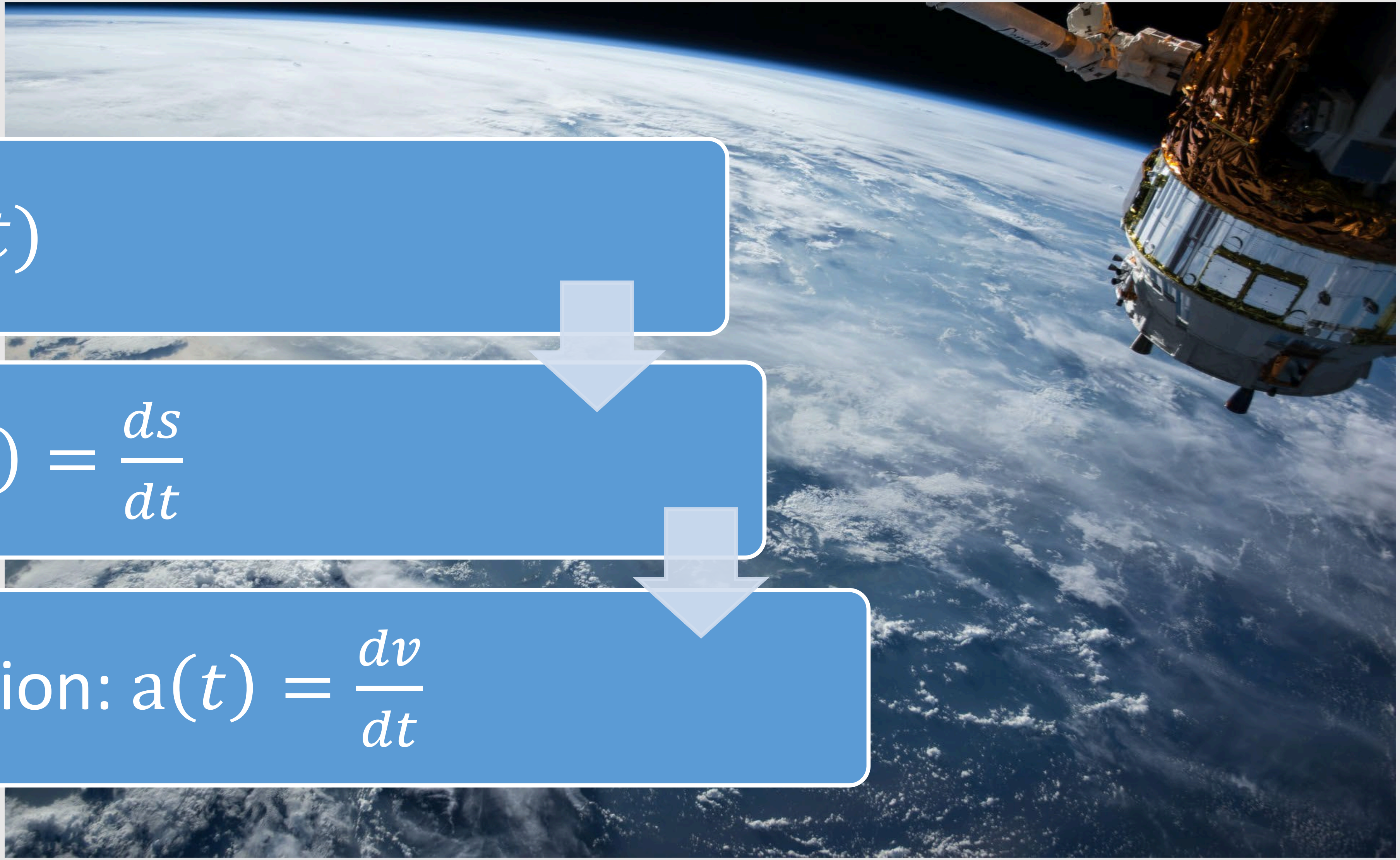
4.3 Definite integrals: definition and properties; The Evaluation Theorem (FTC, part 2)

4.4 The Fundamental Theorem of Calculus, part 1

Modeling Motion

is the primary application of the Calculus

BKM



Position: $s = s(t)$

Velocity: $v(t) = \frac{ds}{dt}$

Acceleration: $a(t) = \frac{dv}{dt}$

Source: https://unsplash.com/photos/yZygONrUBe8

Position: s=s(t)

velocity $v(t) = \frac{ds}{dt}$

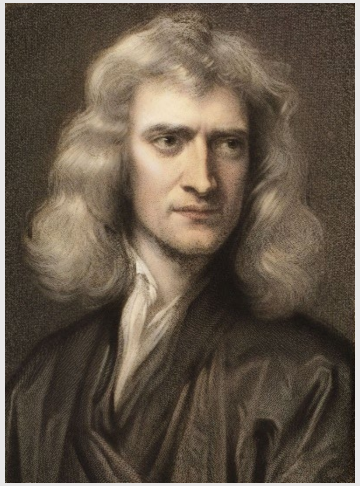
acceleration a$(t) = \frac{dv}{dt}$

Position: $s(t)$

Velocity: $v(t)$

Acceleration: a(t)

Most often the acceleration is **given**,



$$\vec{a} = \frac{\vec{F}}{m}$$

Acceleration

$\boldsymbol{a(t)}$

and there is a need to recover the velocity and trajectory of an object subjected to such acceleration.

Image source: https://www.nationalgeographic.org/article/isaac-newton-who-he-was-why-apples-are-falling/

Position: s=s(t)

velocity $v(t) = \frac{ds}{dt}$

acceleration $a(t) = \frac{dv}{dt}$

The process that reverses differentiation is called INTEGRATION

$$\int summa$$

Integration symbol

Position: s(t)= $\int v(t)dt + C$

Velocity: $v(t) = \int a(t)dt + C$

Acceleration: $a(t)$

ANTIDERIVATIVE

If $F'(x) = f(x)$,

then $F(x)$ is an antiderivative of $f(x)$

Derivative f(x)=F'(x)	Antiderivative F(x)
0	constant
1	x
10	$10x$
$2x$	x^2
$3x^2$	x^3
$5x^4$	x^5
$\frac{1}{x}$	$ln\|x\|$
e^t	e^t

Verify:

$$\frac{d}{dx}[10x] = 10$$

$$\frac{d}{dx}[x^2] = 2x$$

ANTIDERIVATIVE

If $F'(x) = f(x)$,

then $F(x)$ is an antiderivative of $f(x)$

Derivative f(x)=F'(x)	Antiderivative F(x)
$\cos x$	$\sin x$
$\sin t$	$-\cos t$
x^2	$\frac{1}{3}x^3$
$\sqrt{x}$	$\frac{2}{3}x^{3/2}$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\sec^2 x$	$\tan x$

Verify:

$$\frac{d}{dx}[F(x)] = f(x)$$

"Questionable Uniqueness" of an Antiderivative

Derivative f(x)=F'(x)	Antiderivative F(x)
$2x$	x^2

- Differentiation: for every function F(x), there is a UNIQUE derivative f(x)=F'(x);
- Integration: for every derivative $f(x)$, how many antiderivatives $F(x)$ can be found?

- Antiderivative $\leftarrow$ $F(x) = x^2 + 0$
- Antiderivative $\leftarrow$ $H(x) = x^2 + 3$
- Antiderivative $\leftarrow$ $L(x) = x^2 - 27.89$

Family of Antiderivatives

The three antiderivatives above differ only by a CONSTANT

More Formally

Definition 4.1.1.

A function $F(x)$ that satisfies

$$\frac{d}{dx}F(x) = f(x)$$

is called an antiderivative of $f(x)$.

Lemma 4.1.2.

Let $F(x)$ be an antiderivative of $f(x)$, then for any constant c, the function $F(x) + c$ is also an antiderivative of $f(x)$.

Just curious

Are there antiderivatives of a function f that cannot be obtained by adding some constant to a known antiderivative F, as described by Lemma 4.1.2 ?

(Assume that $F'(x) = f(x)$)

$\boxed{\text{Indefinite Integral}}$ is a Family of Antiderivatives

If $F'(x) = f(x)$, then

$$\boxed{\int f(x)dx = F(x) + C,} \quad \text{or} \int F'(x)\,dx = F(x) + C$$

where

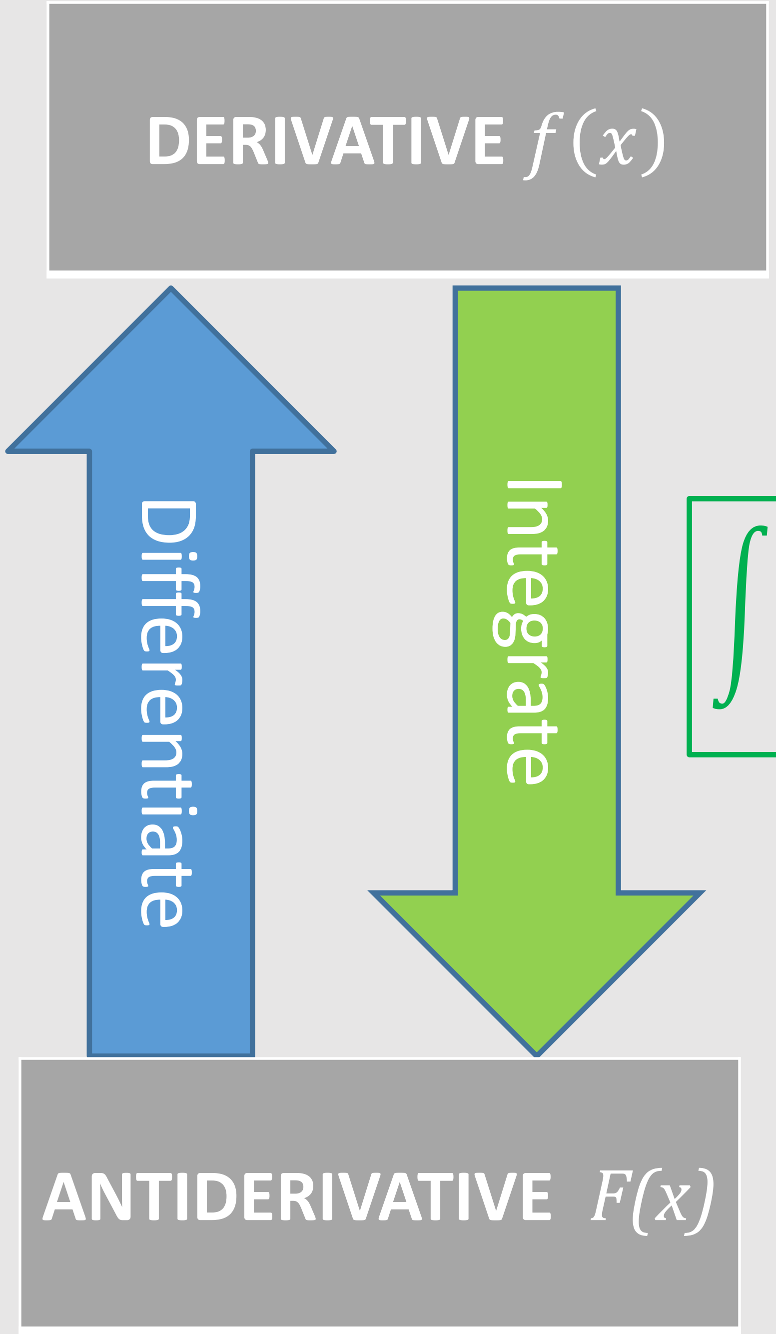
C is any real number/arbitrary **constant of integration**;

$f(x)$ is **integrand**;

dx is a differential of x,

x is the **integration variable**

$$f(x) = \mathrm{F}'(x)$$



INDEFINITE INTEGRAL

$$\boxed{\int f(x)dx = F(x) + C}$$

Tables of Known Integrals

https://en.wikipedia.org/wiki/List_of_integrals_of_trigonometric_functions

Integrands involving only sine [edit]

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax + C = \frac{x}{2} - \frac{1}{2a}\sin ax\cos ax + C$$

$$\int \sin^3 ax\,dx = \frac{\cos 3ax}{12a} - \frac{3\cos ax}{4a} + C$$

$$\int x\sin^2 ax\,dx = \frac{x^2}{4} - \frac{x}{4a}\sin 2ax - \frac{1}{8a^2}\cos 2ax + C$$

$$\int x^2\sin^2 ax\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x}{4a^2}\cos 2ax + C$$

$$\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + C$$

$$\int (\sin b_1x)(\sin b_2x)\,dx = \frac{\sin((b_2-b_1)x)}{2(b_2-b_1)} - \frac{\sin((b_1+b_2)x)}{2(b_1+b_2)} + C \qquad (\text{for } |b_1| \neq |b_2|)$$

$$\int \sin^n ax\,dx = -\frac{\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx \qquad (\text{for } n > 0)$$

$$\int \frac{dx}{\sin ax} = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$$

$$\int \frac{dx}{\sin^n ax} = \frac{\cos ax}{a(1-n)\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax} \qquad (\text{for } n > 1)$$

$$\int x^n\sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$$

BlackBoard:
download and review
Formula Sheets

Derive Your Own Integral!

Confirm that the differentiation formula is correct and state the appropriate integration formula

$$\frac{d}{dx}\left[\sqrt{1+x^2}\right] = \frac{x}{\sqrt{1+x^2}}$$

Solution:

$$\frac{d}{dx}\left[\sqrt{1+x^2}\right] = \frac{d}{dx}\left[(1+x^2)^{\frac{1}{2}}\right]$$

Chain rule

$$= \frac{1}{2}(1+x^2)^{-\frac{1}{2}}\,\frac{d}{dx}[1+x^2]$$

$$= \frac{1}{2}(1+x^2)^{-\frac{1}{2}}\,(2x) = \frac{x}{\sqrt{1+x^2}}$$

continued

Confirm that the formula is correct and state the appropriate integration formula

$$\frac{d}{dx}\left[\sqrt{1+x^2}\right] = \frac{x}{\sqrt{1+x^2}}$$

Solution(continued):

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + C$$

Now we can use this integral!

Integration "Inverses" Differentiation

Given:

$$\frac{d}{dx}\left[\sqrt{1+x^2}\right] = \frac{x}{\sqrt{1+x^2}}$$

Multiply both sides by *dx* and integrate both sides with respect to x:

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + C$$

Integration by Tables

Each integration formula works as long as the **variable of integration** matches with the **input** of the function being integrated.

$$\int \cos u \, du = \sin u + C$$

EXAMPLE.

$$\int \cos x dx = \sin x + C$$

$$\int \cos t dt = \sin t + C$$

Integration:

$\int u^n du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{u} du = \ln\|u\| + C$	$\int e^u du = e^u + C$
$\int \sin u \ du = -\cos u + C$	$\int \cos u \ du = \sin u + C$	$\int \tan u \ du = -\ln\|\cos u\| + C$
$\int \cot u \ du = \ln\|\sin u\| + C$	$\int \sec u \ du = \ln\|\sec u + \tan u\| + C$	$\int \csc u \ du = \ln\|\csc u - \cot u\| + C$
$\int \sin^2 u \ du = \frac{u}{2} - \frac{\sin 2u}{4} + C$	$\int \cos^2 u \ du = \frac{u}{2} + \frac{\sin 2u}{4} + C$	$\int \tan^2 u \ du = \tan u - u + C$
$\int \cot^2 u \ du = -\cot u - u + C$	$\int \sec^2 u \ du = \tan u + C$	$\int \csc^2 u \ du = -\cot u + C$

Integration by Tables

EXAMPLE.

Integration:

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1)$$

$$\int \frac{1}{u}\, du = \ln|u| + C$$

$$\int e^u du = e^u + C$$

$$\int \sin u\ du = -\cos u + C$$

$$\int \cos u\ du = \sin u + C$$

$$\int \tan u\ du = -\ln|\cos u| + C$$

$$\int \cot u\ du = \ln|\sin u| + C$$

$$\int \sec u\ du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\ du = \ln|\csc u - \cot u| + C$$

$$\int \sin^2 u\ du = \frac{u}{2} - \frac{\sin 2u}{4} + C$$

$$\int \cos^2 u\ du = \frac{u}{2} + \frac{\sin 2u}{4} + C$$

$$\int \tan^2 u\ du = \tan u - u + C$$

$$\int \cot^2 u\ du = -\cot u - u + C$$

$$\int \sec^2 u\ du = \tan u + C$$

$$\int \csc^2 u\ du = -\cot u + C$$

a. $\int \sec t dt =$

b. $\int sin^2 \theta\, d\theta =$

Integration by Tables

$$\int e^{\theta}\, d\theta =$$

EXAMPLE.

$$\int e^{\theta}\, d\theta =$$

$$\int e^{y}\, dy =$$

$$\int e^{sint}\, dsint =$$

$$\int e^{\omega t}\, d(\omega t) =$$

The Rules of Integration

1. The integral of the differential of a function, is the function itself.

$$\int du = u + C$$

This one is tricky. But what it says is: integration cancels differentiation;

2. The Constant Multiple Rule, where k is constant

$$\int kf(x)dx = k\int f(x)dx + C$$

3. The Sum and Difference Rules

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)\,dx + C$$

Remarks: The Rules of Integration.

2. The Constant Multiple Rule, where k is constant

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$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)\, dx + C$$

1.Limits and Derivatives also have properties like Rule 2 and 3. (True/False)

2.What term is used to refer to the property that combines rules 2 and 3?

3.Can Rule 3 be extended to three and more functions (True/False)

The Rules of Integration

Note, that Rule 5 deals with the exception for Rule 4

4. The Power Rule (applies to the power function)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \text{ where } n \neq -1$$

5. Case $n = -1$: $\int \frac{1}{x} dx = \int x^{-1} dx = ln|x| + C$

table integrals

Keep it simple

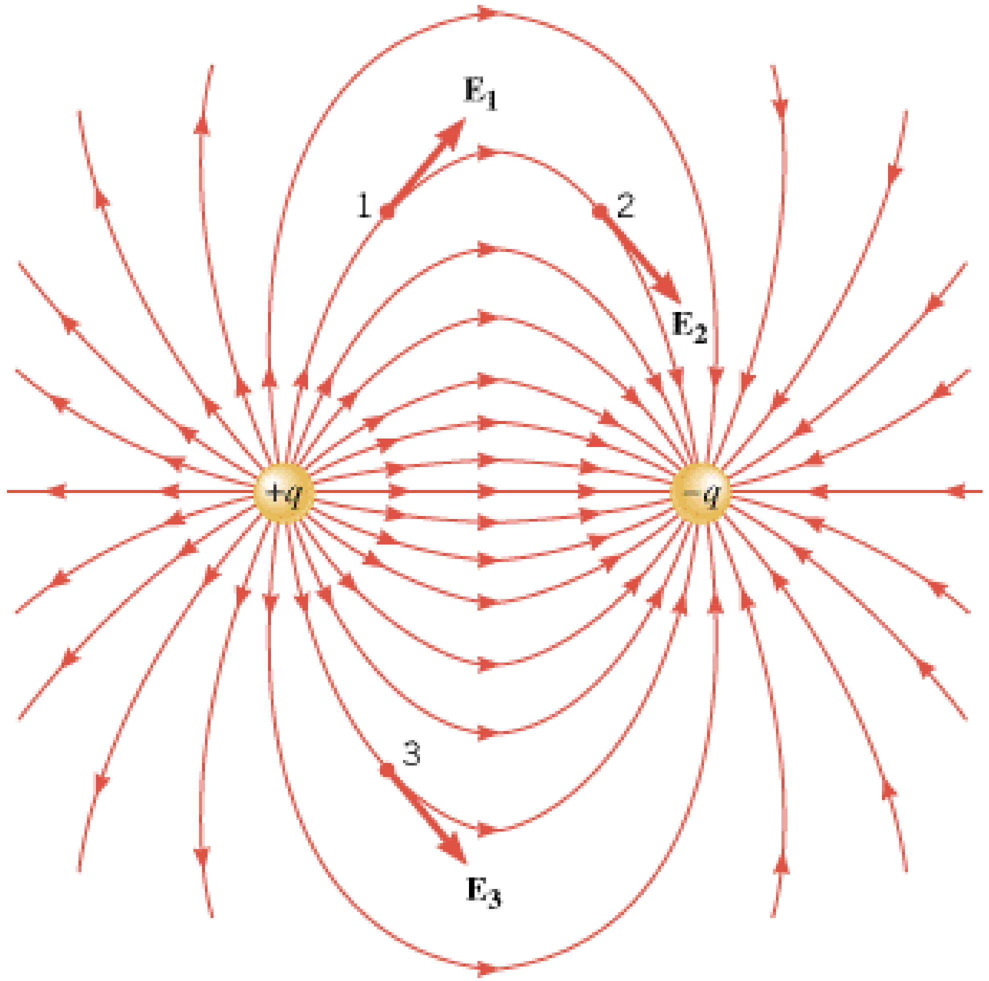
When integrating complex expressions try

- to break it down into simpler pieces using the rules of integration
- find the known integrals in the Table that seem to fit the given expression.
- insert the constant of integration in the final result rather than in intermediate calculations.

Proceed with the worksheet A

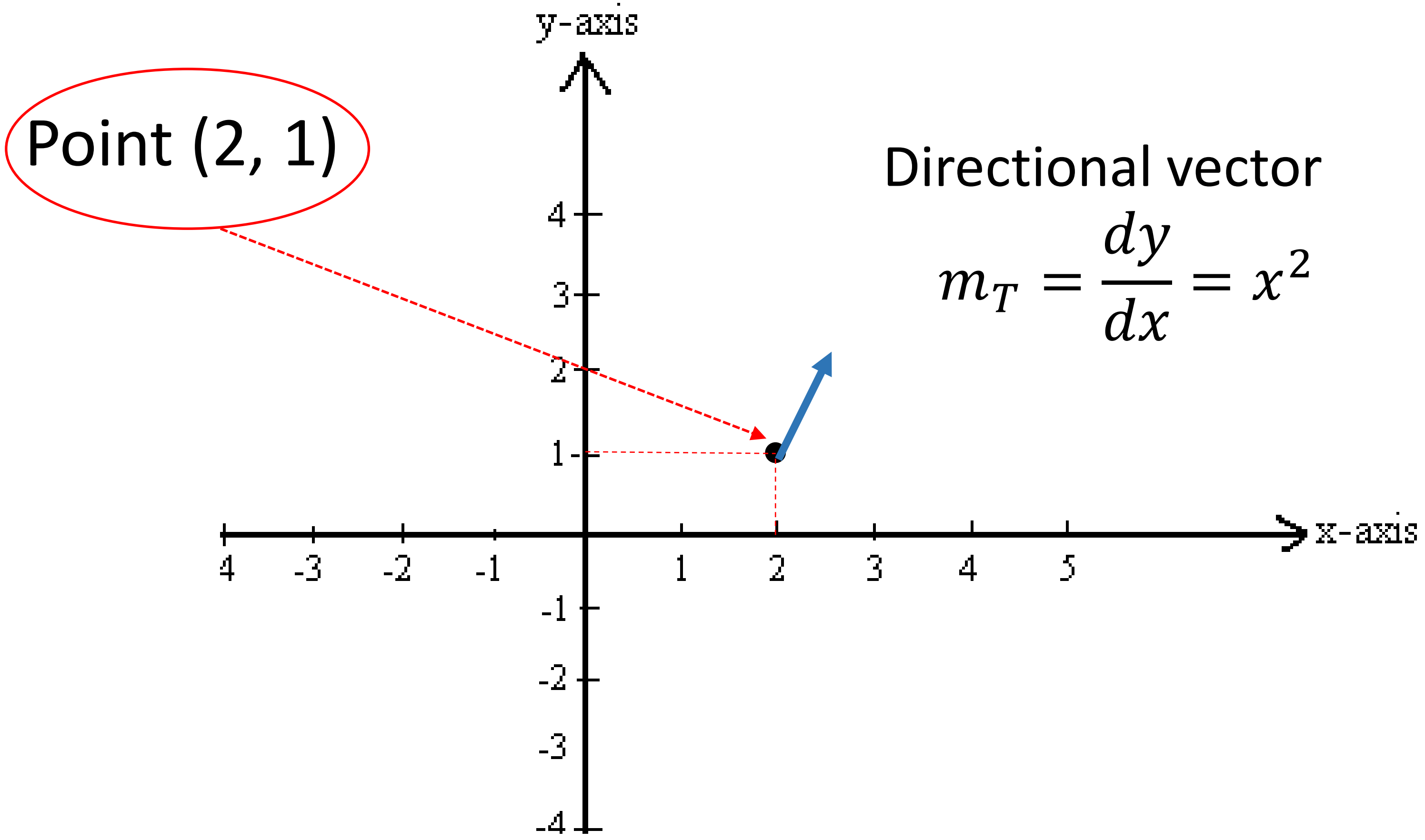
Find the Curve, Given its Variable Slope

Electric field lines of force visualized



<u>Example.</u>

Suppose that a curve y = f(x) in the xy-plane has the property that at each point (x, y) on the curve, the tangent line has the slope x^2.



Find an equation for the curve given that it passes through the point (2, 1).

Example.

Suppose that a curve y = f(x) in the xy-plane has the property that at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve given that it passes through the point (2, 1).

$m_T = \frac{dy}{dx} = x^2$; Point (2, 1)

Find the curve $y = f(x)$ that

✓ has a derivative $\frac{dy}{dx} = x^2$

✓ $y(2) = 1$ ⟵ Initial condition

When $x = 2$, then $y = 1$

IVP

IVP – initial value problem

- Solve the differential equation:

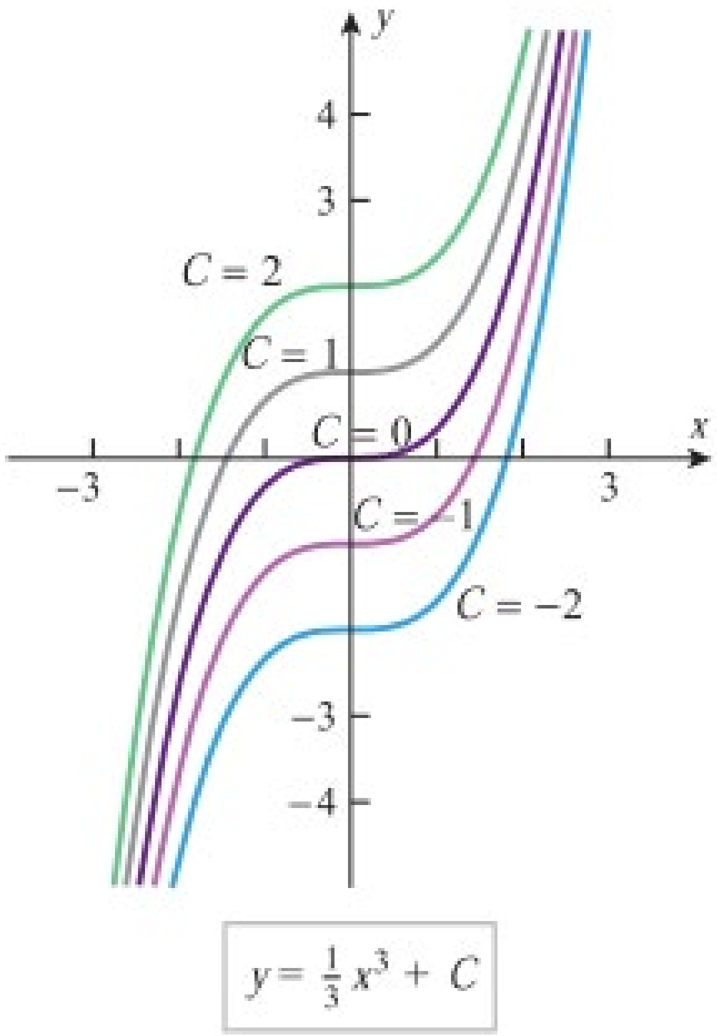
$$dy = x^2 dx$$

Integrating on both sides we obtain

$$\int dy = \int x^2 dx$$

$$y = \frac{1}{3}x^3 + C$$

- Integral curves – are graphs of the indefinite integral for different values of arbitrary constant of integration C.



- Select the single curve that passes through the given point (2, 1) or rephrasing "satisfies the given initial condition $y(2)$=1"

Select the single curve that passes through the given point (2, 1)

or rephrasing "satisfies the given initial conditions $y(2)$=1

$y(2)$=1 initial condition

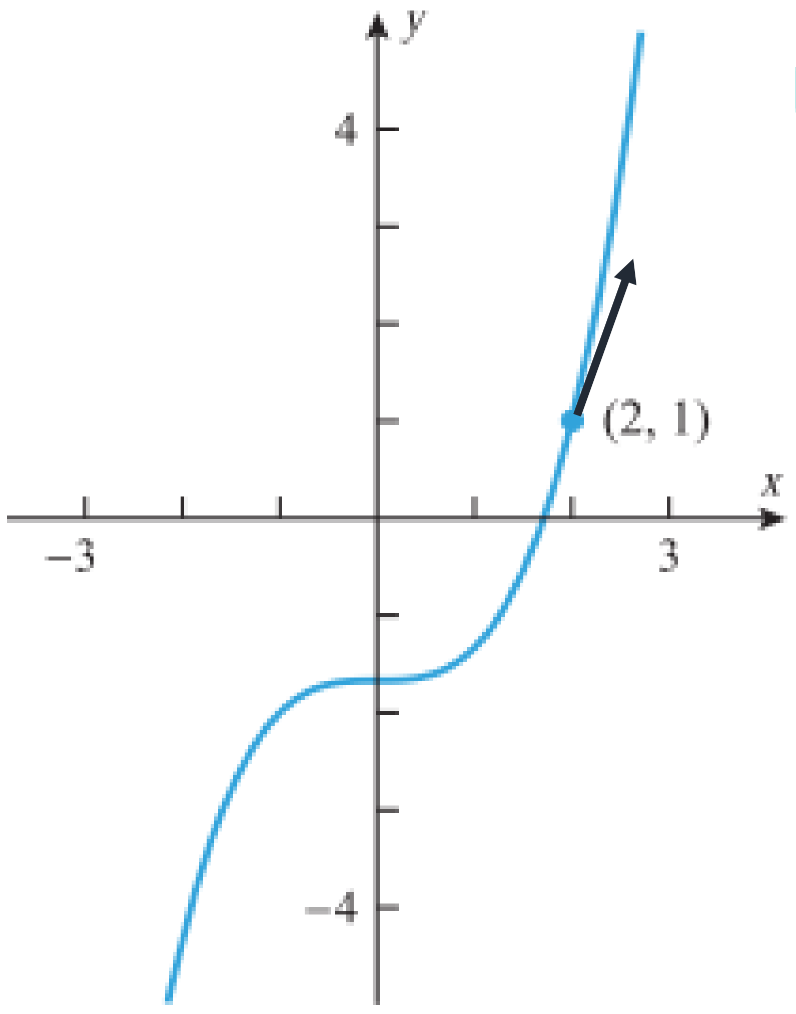
Substitute $x = 2$, y =1 into

$$y = \frac{1}{3}x^3 + C \quad \leftarrow \text{general solution(integral)}$$

$$1 = \frac{1}{3}\,(2^3) + C$$

Solving for C we obtain that $C = -\frac{5}{3}$ and hence, the curve is

$$y = \frac{1}{3}x^3 - \frac{5}{3}$$



$y = \frac{1}{3}x^3 - \frac{5}{3}$