Class Note 2.2.3 (2)

September 21, 2022 9:05 AM

Module 2.2.3C Derivatives of logarithmic functions

1

Logarithmic Functions

BKM

The logarithm y is the exponent to which base b is to be raised in order to get value x.

$$y = \log_b x$$

Note that

- x is the independent variable, the **input** of the logarithmic function**.** The input of any log functions is positive: $x > 0$
- $y = y(x)$ is the **output** , computed based on the input x and base b.
- Base b is the fixed parameter: $b > 0$ and $b \neq 1$

- The exponential b^x and logarithmic $\log_b x$ are inverse functions.

$$\log_{10} 100 = \mathbf{2} \Leftrightarrow 10^{\mathbf{2}} = 100$$

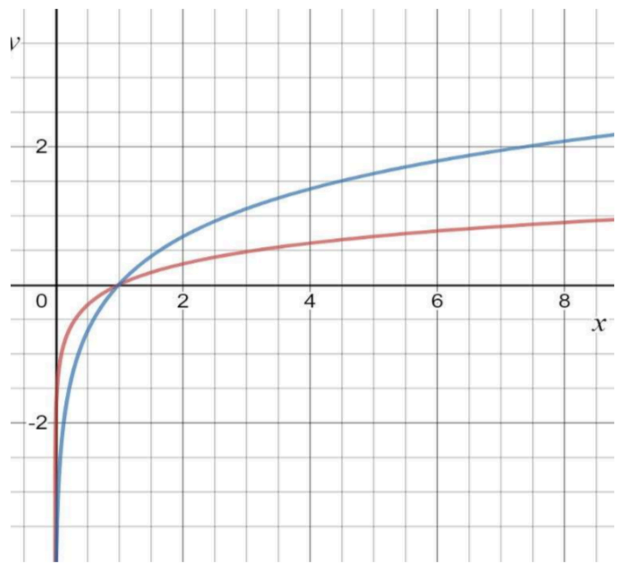
- $\log_b 1 = 0, \quad \log_b b = 1$

BKM

Natural Log: $b = e \rightarrow y = \ln x$

Common Log: $b = 10 \rightarrow y = \log x$

The two most commonly used log functions and their graphs.



https://www.desmos.com/calculator/rlv4zifac7

BKM

The Properties of Logarithms.

Refer to the Formula Sheet

*Properties of Logarithm:

$\log_b AB = \log_b A + \log_b B$

1. The logarithm of the product is the sum of the logarithms of the factors;

$$\log_b \frac{A}{B} = \log_b A - \log_b B$$

$$\log_b A^P = P \log_b A$$

2. The logarithm of the quotient is the difference of logarithms of numerator A and denominator B.

3. The logarithm of the power A^p can be expressed as the exponent of the power P multiplied by the log of the base of the power A

Note that, the rules 1-3 apply both ways: from left to right and from right to left.

Use properties of logarithms to simplify the following:

Example 1.

$$\ln\left[\sqrt{3x+1}\,(2x-5)^3\right] =$$

Example 2.

$$\ln\left[x^{\sin x}\right] =$$

1. $\frac{1}{2}\ln(3x+1) + 3\ln(2x-5)$; 2. $\sin x$

Example 3

$$\ln\left[\frac{e^{2x}\, x^{2/3}\sqrt{2x+1}}{(x-2)^3}\right] =$$

The Generalized Derivative Formulas

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad \frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$$

<u>Verbally:</u>

1. To differentiate the logarithm function take the reciprocal of the INPUT.
2. If there are nested functions, then apply the Chain Rule
3. If the base of the log is b, then adjust by multiplying by lnb in the denominator
4. For the natural logs, no such adjustment is needed because $\ln e = 1$

Find the derivatives of the given functions

EXAMPLE 1. $y = \ln x$, then

$$y' = \frac{1}{x}$$

To differentiate the ln function, take the reciprocal of the INPUT

EXAMPLE 2. $y = \log x$, then

$$y' = \frac{1}{x \ln 10}$$

To differentiate the log function with the base 10, take the reciprocal of the INPUT and adjust by $\ln 10$

Since in both examples $u = x$, then $\frac{du}{dx} = 1$

EXAMPLE 3.

Differentiate $y = \ln(2x^3 + 5x)$.

To differentiate the ln function, take the reciprocal of the INPUT .
If there is a nested function, use Chain Rule

$u = 2x^3 + 5x; \quad \frac{du}{dx} = 6x^2 + 5$

$$\frac{dy}{dx} = \frac{1}{2x^3 + 5x} \cdot (6x^2 + 5)$$

$$= \frac{6x^2 + 5}{2x^3 + 5x}$$

ANS: $\frac{dy}{dx} = \frac{6x^2+5}{x(2x^2+5)}$

EXAMPLE 4. $y = \ln x^3$,

Use the properties of logarithms to simplify the expression prior to differentiation

Simplify first $y = 3\ln x$.

Then differentiate

$$y' = \frac{d}{dx}[3\ln x] = 3\frac{d}{dx}[\ln x] = \frac{\mathbf{3}}{\boldsymbol{x}}$$

EXAMPLE 5. (Self-Check) $y = \ln x^{-2}$

Answers to self-check problems are on the last slide, or accessible by clicking on the blue Answer box

Answer

- **EXAMPLE 6.** $y = \log\left[\frac{(1+3x)}{x^2}\right]$

Rewrite the function as

$$y = \log(1+3x) - 2\log x.$$

Find the derivative now

$$y' = \frac{1}{\quad}(3) - 2\frac{1}{\quad} = \frac{1}{\quad}\left(\frac{\mathbf{3}}{\quad} - \frac{\mathbf{2}}{\quad}\right)$$

$$y' = \frac{\quad}{(1+3x)\ln 10}(3) - 2\frac{\quad}{x\ln 10} = \frac{\quad}{\ln 10}\left(\frac{\quad}{\mathbf{1+3x}} - \frac{\quad}{\boldsymbol{x}}\right)$$

EXAMPLE 7. $y = \log_a(x^2 - 3x)$. Use the derivative of the logarithm base b formula.

$$y' = \frac{1}{(x^2-3x)\ln a}(2x-3) = \frac{\mathbf{2x-3}}{\boldsymbol{(x^2-3x)\ln a}}$$

EXAMPLE 8. $y = \ln \sin x$,

This is a composite function, with the input of the *ln* function $u(x) = \sin x$. Use the Chain Rule along with the derivative for *ln* and derivative for *sin:*

$$y' = \frac{1}{\sin x}\frac{d}{dx}[\sin x] = \frac{\cos x}{\sin x} = \boldsymbol{\cot x}$$

Use the trigonometric identity: $\cot x = \frac{\cos x}{\sin x}$

Note that a reasonable effort is expected to be put into simplification and presentation of the final answer.

EXAMPLE 9. (Self-Check) $y = \ln(\sec x + \tan x)$

Answer

EXAMPLE 10. $y = x^3 \ln(5x+2)$

Using the product rule, with get

$$u = x^3 \rightarrow \qquad u' = 3x^2$$
$$v = \ln(5x+2) \rightarrow v' = \frac{5}{5x+2}$$

$$y' = 3x^2\ln(5x+2) + x^3\frac{5}{5x+2} = \mathbf{3x^2\, ln(5x+2) + \frac{5x^3}{5x+2}}$$

- **EXAMPLE 11.** $y = \sqrt[3]{\ln x}$

This is the composite (nested) function. Prepare it for the double chain rule:

$y = \sqrt[3]{\ln x} = (\ln x)^{\frac{1}{3}} \rightarrow$ the input/output sequence is as follows:

$$x \rightarrow \ln(\quad) \rightarrow (\quad)^{\frac{1}{3}}$$

$$y' = \frac{1}{3}(\ln x)^{-\frac{2}{3}}\frac{d}{dx}[\ln x] = \frac{1}{3}(\ln x)^{-\frac{2}{3}}\frac{1}{x} = \mathbf{\frac{1}{3x(ln\, x)^{\frac{2}{3}}}}$$

EXAMPLE 12. $y = \ln \frac{x+1}{\sqrt{x-2}}$,

Rewrite the function in simplified form taking advantage of the logs properties.

$$y = \ln(x+1) - \frac{1}{2}\ln(x-2)$$

Now differentiate

$$y' = \frac{1}{x+1} - \frac{1}{2}\,\frac{1}{x-2}$$

and algebraic simplification yields

$$y' = \boldsymbol{\frac{x-5}{2(x+1)(x-2)}}$$

Answers for Self-Check Examples

5. $y' = -\frac{2}{x}$
9. $y' = \sec x$

$\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

$\operatorname{arccsc}(x) = -\frac{1}{|x|}$

1

$$\arccos(x) = -\frac{\quad}{\sqrt{1-x^2}}$$

$$\operatorname{arcsec}(x) = \frac{\quad}{|x|\sqrt{\ }}$$

$$\arctan(x) = \frac{1}{1+x^2}$$

$$\operatorname{arccot}(x) = -\frac{\quad}{1}$$