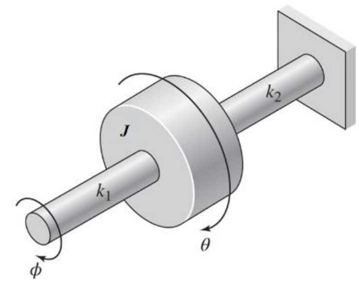
Worksheet 5 - Solution

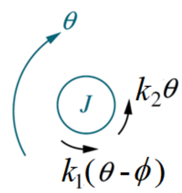
PART 1: Mass-Spring Systems

1) In the following system, the input is the angular displacement ϕ of the end of the shaft, and the output is the angular displacement θ of the inertia J. The shafts have torsional stiffness k_1 and k_2. The equilibrium position corresponds to $\phi = \theta = 0$. Derive the equation of motion and find the transfer function $\Theta(s)/\Phi(s)$.



From the free-body diagram, considering the CW as the positive direction, apply Newton's second law.

$$-k_1(\theta - \phi) - k_2\theta = J\ddot{\theta}$$



The differential equation model of system is obtained.

$$J\ddot{\theta} + (k_1 + k_2)\theta = k_1\phi$$

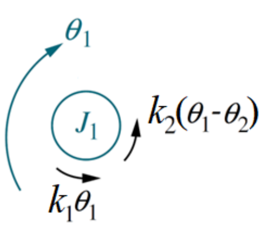
Take Laplace transform:

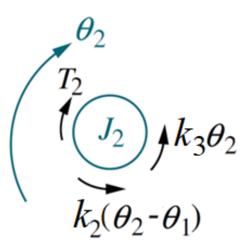
$$Js^2\theta(s) + (k_1 + k_2)\theta(s) = k_1\phi(s)$$

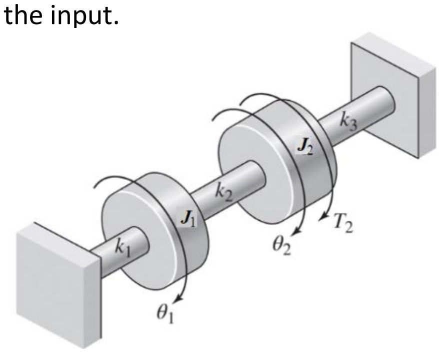
Form the equation as a transfer function model, where the input is the angular displacement ϕ of the end of the shaft, and the output is the angular displacement θ of the inertia J:

$$\frac{\theta(s)}{\phi(s)} = \frac{k_1}{Js^2 + k_1 + k_2}$$

2) The following figure models the three shafts as massless torsional springs. When $\theta_1 = \theta_2 = 0$, the springs are at their free lengths. Derive the equations of motion with the torque τ_2 as the input.







From the free-body diagram, considering the CW as the positive direction, apply Newton's second law:

Inertia J_1 $\rightarrow$ $-k_1\theta_1 - k_2(\theta_1 - \theta_2) = J_1\ddot{\theta}_1$

Inertia J_2 $\rightarrow$ $\tau_2(t) - k_2(\theta_2 - \theta_1) - k_3\theta_2 = J_2\ddot{\theta}_2$

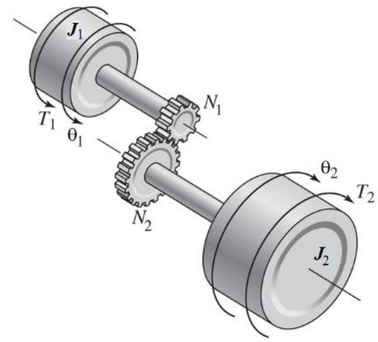
Simplify to obtain the equation of motion of the system

$$J_1\ddot{\theta}_1 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0$$

$$\tau_2(t) = J_2\ddot{\theta}_2 + (k_2 + k_3)\theta_2 - k_2\theta_1$$

3) Consider the spur gear shown in the following figure. The input shaft (shaft 1) is connected to a motor that produces a torque τ_1 at a speed ω_1, and drives the output load shaft (shaft 2). One use of such a system is to increase the effective motor torque. θ_1 is the input rotation and θ_2 is the output rotation.

a) Derive the expression for the equivalent inertia J_e felt on the input shaft, and obtain the equation of motion in terms of the displacement θ_1.



$$\boxed{\frac{\tau_1}{\tau_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1}}$$

The equivalent inertia felt on the input shaft is:

$$J_e = J_1 + J_2\left(\frac{N_1}{N_2}\right)^2$$

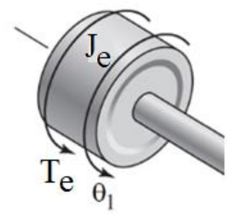
The equivalent torque in the input shaft is:

$$\tau_e = \tau_1 + \tau_2\left(\frac{N_1}{N_2}\right)$$

where, $\tau_2\left(\frac{N_1}{N_2}\right)$ is the reflected torque from shaft 2 to shaft 1.

The equation of motion in terms of θ_1 is:

$$\tau_e = J_e\ddot{\theta}_1 \quad \rightarrow \quad \tau_1 + \tau_2\left(\frac{N_1}{N_2}\right) = \left[J_1 + J_2\left(\frac{N_1}{N_2}\right)^2\right]\ddot{\theta}_1$$



b) If the inertias are $J_1 = 0.1\ kg.m^2$ for the motor shaft and $J_2 = 0.4\ kg.m^2$ for the load shaft. The motor speed ω_1 is five times faster than the load speed ω_2, so this device is called a *speed reducer*. Obtain the equation of motion in terms of ω_2. Assuming that the motor torque τ_1 and load torque τ_2 are given.

From the given speed information we have:

$$\frac{\omega_2}{\omega_1} = \frac{1}{5} \quad \rightarrow \quad \frac{N_1}{N_2} = \frac{1}{5}$$

The equivalent inertia is:

$$J_e = J_1 + J_2\left(\frac{N_1}{N_2}\right)^2 = 0.1 + 0.4\left(\frac{1}{5}\right)^2 = 0.116\ kg.m^2$$

Since $\ddot{\theta}_1 = \dot{\omega}_1$, the equation of motion in terms of ω_1 is:

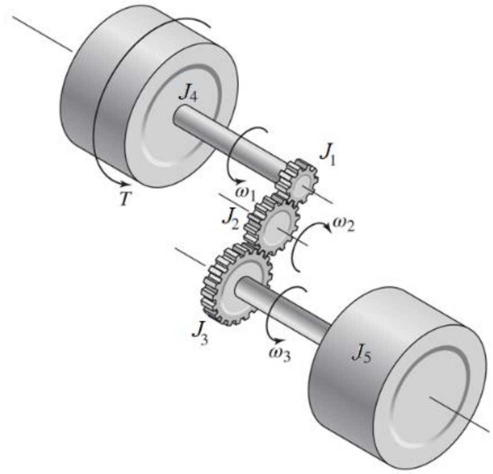
$$\tau_1 + \tau_2\left(\frac{N_1}{N_2}\right) = J_e\dot{\omega}_1 \quad \rightarrow \quad \tau_1 + \frac{\tau_2}{5} = 0.116\dot{\omega}_1$$

From the relation of $\omega_1 = 5\omega_2$, we can find the equation of motion in terms of ω_2:

$$\tau_1 + \frac{\tau_2}{5} = 0.116\dot{\omega}_1 \quad \rightarrow \quad \tau_1 + \frac{\tau_2}{5} = 0.58\dot{\omega}_2$$

4) For the system shown in the following figure, assume that the shaft inertias are small. The remaining inertias in $kg.m^2$ are $J_1 = 0.005,\ J_2 = 0.01,\ J_3 = 0.02,\ J_4 = 0.04$, and $J_5 = 0.2$. The speed ratios are:

$$\frac{\omega_1}{\omega_2} = \frac{3}{2}, \quad \frac{\omega_2}{\omega_3} = 2$$



Obtain the equation of motion in terms of ω_3. Assume that the torque τ is given.

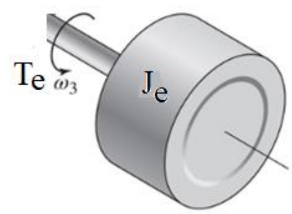
From the speed ratios we have:

$$\frac{N_2}{N_1} = \frac{\omega_1}{\omega_2} = \frac{3}{2} \quad and \quad \frac{N_3}{N_2} = \frac{\omega_2}{\omega_3} = 2$$

Reflect all inertias to shaft 3.

$$J_e = \left((J_4 + J_1)\left(\frac{N_2}{N_1}\right)^2 + J_2 \right)\left(\frac{N_3}{N_2}\right)^2 + J_3 + J_5$$

$$J_e = \left((0.04 + 0.005)\left(\frac{3}{2}\right)^2 + 0.01 \right)(2)^2 + 0.02 + 0.2 = 0.665\ kg.m^2$$



Reflected torque in shaft 3:

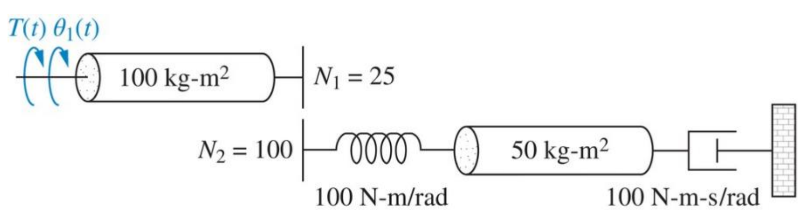
$$\tau_e = \tau\left(\frac{N_2}{N_1}\right)\left(\frac{N_3}{N_2}\right) = \tau\left(\frac{3}{2}\right)(2) = 3\tau$$

The equation of motion in terms of ω_3 is:

$$\tau_e = J_e\dot{\omega}_3 \quad \rightarrow \quad 3\tau = 0.665\dot{\omega}_3$$

PART 2: Mass-Spring-Damper Systems

5) Represent the rotational mechanical system shown below in the state-space form, where $\theta_1(t)$ is the output.



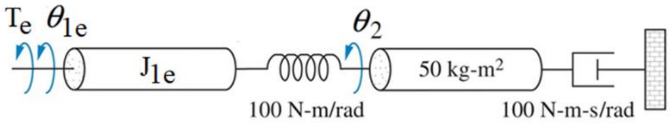
Reflect all components and the torque and displacement from input side 1 to side 2:

$$\tau_e(t) = \tau(t)\left(\frac{N_2}{N_1}\right) = \tau(t)\left(\frac{100}{25}\right) = 4\tau(t)$$

$$\theta_{1e}(t) = \theta_1(t)\left(\frac{N_1}{N_2}\right) = \theta_1(t)\left(\frac{25}{100}\right) = \frac{1}{4}\theta_1(t)$$

$$J_{1e} = J_1\left(\frac{N_2}{N_1}\right)^2 = (100)\left(\frac{100}{25}\right)^2 = 1600\ kg.m^2$$

$$\boxed{\frac{\tau_1}{\tau_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1}}$$



The equation of motion is:

Inertia J_{1e} $\rightarrow$ $\tau_e(t) - K\big(\theta_{1e}(t) - \theta_2(t)\big) = J_{1e}\ddot{\theta}_{1e}(t)$ $\rightarrow$ $4\tau(t) - 100\left(\frac{1}{4}\theta_1(t) - \theta_2(t)\right) = 1600\left(\frac{1}{4}\ddot{\theta}_1(t)\right)$

$Eqn.(1)$ $\rightarrow$ $400\ddot{\theta}_1(t) + 25\theta_1(t) - 100\theta_2(t) = 4\tau(t)$

Inertia J_2 $\rightarrow$ $-K\big(\theta_2(t) - \theta_{1e}(t)\big) - D\dot{\theta}_2(t) = J_2\ddot{\theta}_2(t)$ $\rightarrow$ $-100\left(\theta_2(t) - \frac{1}{4}\theta_1(t)\right) - 100\dot{\theta}_2(t) = 50\ddot{\theta}_2(t)$

$Eqn.(2)$ $\rightarrow$ $50\ddot{\theta}_2(t) - 25\theta_1(t) + 100\theta_2(t) + 100\dot{\theta}_2(t) = 0$

Define the state variables:

$q_1(t) = \theta_1(t)$

$q_2(t) = \dot{\theta}_1(t)$

$q_3(t) = \theta_2(t)$

$q_4(t) = \dot{\theta}_2(t)$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$\dot{q}_1(t) = \dot{\theta}_1(t)$ $\rightarrow$ $\dot{q}_1(t) = q_2(t)$

$\dot{q}_2(t) = \ddot{\theta}_1(t)$ $\rightarrow$ $\dot{q}_2(t) = \frac{1}{400}\big(4\tau(t) - 25\theta_1(t) + 100\theta_2(t)\big) = 0.01\tau(t) - 0.0625q_1(t) + 0.25q_3(t)$

$\dot{q}_3(t) = \dot{\theta}_2(t)$ $\rightarrow$ $\dot{q}_3(t) = q_4(t)$

$\dot{q}_4(t) = \ddot{\theta}_2(t)$ $\rightarrow$ $\dot{q}_4(t) = \frac{1}{50}\big(25\theta_1(t) - 100\theta_2(t) - 100\dot{\theta}_2(t)\big) = 0.5x_1(t) - 2q_3(t) - 2q_4(t)$

Find the output in terms of the state variables and the input.

$y(t) = \theta_1(t) = q_1(t)$

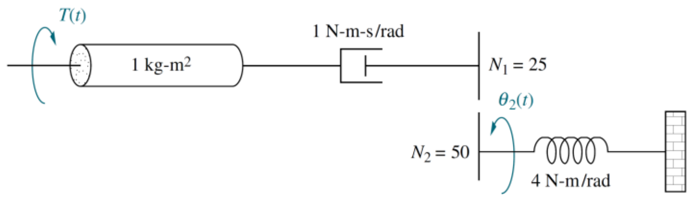
The system model has 4 state variables, 1 input, and 1 output.

Form the state equation and the output equation in the standard matrix-vector form.

$$State\ Equation \rightarrow \begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\\ \dot{q}_4(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -0.0625 & 0 & 0.25 & 0\\ 0 & 0 & 0 & 1\\ 0.5 & 0 & -2 & -2\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\\ q_4(t)\end{bmatrix} + \begin{bmatrix}0\\ 0.01\\ 0\\ 0\end{bmatrix}\tau(t)$$

$$Output\ Equation \rightarrow y(t) = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\\ q_4(t)\end{bmatrix} + [0]\tau(t)$$

6) Find the transfer function $\theta_2(s)/T(s)$, for the rotaional mechanical system with greas as shown below:

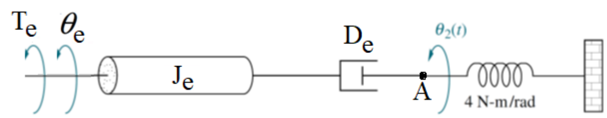


Reflect all components and the torque from side 1 to side 2:

$$\tau_e(t) = \tau(t)\left(\frac{N_2}{N_1}\right) = \tau(t)\left(\frac{50}{25}\right) = 2\tau(t)$$

$$J_e = J\left(\frac{N_2}{N_1}\right)^2 = (1)\left(\frac{50}{25}\right)^2 = 4\ kg.m^2$$

$$D_e = D\left(\frac{N_2}{N_1}\right)^2 = (1)\left(\frac{50}{25}\right)^2 = 4\ N.m.s/rad$$



Find the equation of motion:

Inertia $J_e \quad \rightarrow \quad \tau_e(t) - D_e\left(\dot{\theta}_e(t) - \dot{\theta}_2(t)\right) = J_e\ddot{\theta}_e(t) \quad \rightarrow \quad 2\tau(t) - 4\left(\dot{\theta}_e(t) - \dot{\theta}_2(t)\right) = 4\ddot{\theta}_e(t)$

Point $A \quad \rightarrow \quad -D_e\left(\dot{\theta}_2(t) - \dot{\theta}_e(t)\right) - K\left(\theta_2(t)\right) = 0 \quad \rightarrow \quad 4\left(\dot{\theta}_2(t) - \dot{\theta}_e(t)\right) + 4\theta_2(t) = 0$

Taking Laplace transform and form as a transfer function model:

$$2T(s) - 4s\theta_e(s) + 4s\theta_2(s) = 4s^2\theta_e(s) \qquad Eqn.(1)$$

$$4s\theta_2(s) - 4s\theta_e(s) + 4\theta_2(s) = 0 \qquad Eqn.(2)$$

Find $\theta_e(s)$ from Eqn. (2) and substitute in Eqn. (1):

$$From\ Eqn.\ (2) \quad \rightarrow \quad \theta_e(s) = \frac{(s+1)}{s}\theta_2(s)$$

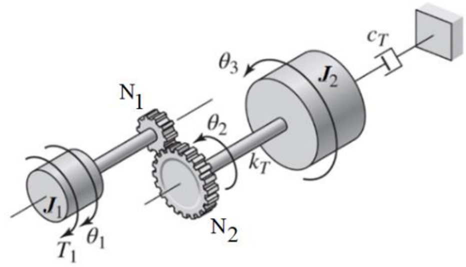
$$2T(s) - 4s\left(\frac{(s+1)}{s}\theta_2(s)\right) + 4s\theta_2(s) = 4s^2\left(\frac{(s+1)}{s}\theta_2(s)\right)$$

$$\rightarrow \quad 2T(s) - 4(s+1)\theta_2(s) + 4s\theta_2(s) = 4s(s+1)\theta_2(s)$$

$$\rightarrow \quad 2T(s) - 4s\theta_2(s) - 4\theta_2(s) + 4s\theta_2(s) = 4s^2\theta_2(s) + 4s\theta_2(s)$$

$$\rightarrow \quad \frac{\theta_2(s)}{T(s)} = \frac{2}{4s^2 + 4s + 4} = \frac{1/2}{s^2 + s + 1}$$

7) The following figure shows a drive train with a spur-gear pair. The first shaft turns N_2/N_1 times faster than the second shaft. Develop a model of the system including the elasticity of the second shaft. Assume the first shaft is rigid, and neglect the gear and shaft masses. The input is the applied torque τ_1. The outputs are the angles θ_1 and θ_3.



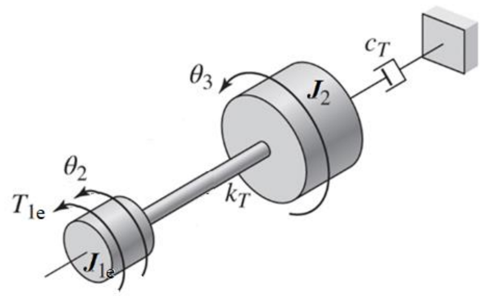
Assume that $\frac{N_2}{N_1} = N$

Reflect all components and the torque and the displacement from side 1 to side 2:

$$\tau_{1e}(t) = \tau_1(t)\left(\frac{N_2}{N_1}\right) = N\tau_1(t)$$

$$\theta_2(t) = \theta_1(t)\left(\frac{N_1}{N_2}\right) = \frac{1}{N}\theta_1(t)$$

$$J_{1e} = J_1\left(\frac{N_2}{N_1}\right)^2 = N^2 J_1$$



Find the equation of motion:

Inertia J_{1e} $\rightarrow$ $\tau_{1e}(t) - k_T\big(\theta_2(t) - \theta_3(t)\big) = J_{1e}\ddot{\theta}_2(t)$

$$\rightarrow \quad N\tau_1(t) - k_T\left(\frac{1}{N}\theta_1(t) - \theta_3(t)\right) = (N^2 J_1)\left(\frac{1}{N}\ddot{\theta}_1(t)\right)$$

$$\rightarrow \quad N^2\tau_1(t) - k_T\big(\theta_1(t) - N\theta_3(t)\big) = N^2 J_1\ddot{\theta}_1(t) \qquad Eqn.(1)$$

Inertia J_2 $\rightarrow$ $-k_T\big(\theta_3(t) - \theta_2(t)\big) - c_T\left(\dot{\theta}_3(t)\right) = J_2\ddot{\theta}_3(t)$

$$\rightarrow \quad -k_T\left(\theta_3(t) - \frac{1}{N}\theta_1(t)\right) - c_T\left(\dot{\theta}_3(t)\right) = J_2\ddot{\theta}_3(t)$$

$$\rightarrow \quad -k_T\left(\theta_3(t) - \frac{1}{N}\theta_1(t)\right) - c_T\left(\dot{\theta}_3(t)\right) = J_2\ddot{\theta}_3(t) \qquad Eqn.(2)$$

The system model consists of Eqn. (1) and Eqn. (2).

$$\begin{cases} N^2\tau_1(t) - k_T\big(\theta_1(t) - N\theta_3(t)\big) = N^2 J_1\ddot{\theta}_1(t) \\ -k_T\left(\theta_3(t) - \frac{1}{N}\theta_1(t)\right) - c_T\left(\dot{\theta}_3(t)\right) = J_2\ddot{\theta}_3(t) \end{cases} \rightarrow \begin{cases} N^2\tau_1(t) = N^2 J_1\ddot{\theta}_1(t) - N\theta_3(t) + k_T\theta_1(t) \\ J_2\ddot{\theta}_3(t) + c_T\dot{\theta}_3(t) + k_T\theta_3(t) - \frac{k_T}{N}\theta_1(t) = 0 \end{cases}$$