Worksheet 7 - Solution

1) Consider systems with the mathematical model given by the following differential equations:

Find the state-space representation of each system in Canonical Controllable form.

a) $\boldsymbol{\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = u(t)}$

First, find the required coefficients from the given differential equation:

$$a_0 = 2,\ \ a_1 = 5,\ \ a_2 = 10,\ \ b_0 = 1$$

This is a **strictly proper system with no zeros**. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ b_0\end{bmatrix}u(t) \ \rightarrow\ \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -5 & -10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + [0]u(t) \quad \rightarrow \quad y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + [0]u(t)$$

b) $\boldsymbol{5\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 20u(t)}$

First, find the required coefficients from the given differential equation:

$$\frac{d^3y(t)}{dt^3} + 0.8\frac{d^2y(t)}{dt^2} + 1.2\frac{dy(t)}{dt} + 1.6y(t) = 4u(t)$$

$$a_0 = 1.6,\ \ a_1 = 1.2,\ \ a_2 = 0.8,\ \ b_0 = 4$$

This is a **strictly proper system with no zeros**. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ b_0\end{bmatrix}u(t) \ \rightarrow\ \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1.6 & -1.2 & -0.8\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 4\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + [0]u(t) \quad \rightarrow \quad y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + [0]u(t)$$

c) $\frac{d^3y(t)}{dt^3} - 2\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 3y(t) = 5\frac{du(t)}{dt} + u(t)$

First, find the required coefficients from the given differential equation:

$$a_0 = 3,\ \ a_1 = 11,\ \ a_2 = -2,\ \ b_0 = 1,\ \ b_1 = 5$$

This is a **strictly proper system with zeros**. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u(t) \ \rightarrow\ \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -3 & -11 & 2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$$

$$y(t) = [b_0 \quad b_1 \quad b_2]\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + [0]u(t) \quad\rightarrow\quad y(t) = [1 \quad 5 \quad 0]\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + [0]u(t)$$

d) $\frac{d^2y(t)}{dt^2} - 5\frac{dy(t)}{dt} + 7y(t) = \frac{d^2u(t)}{dt^2} + \frac{du(t)}{dt} + 4u(t)$

First, find the required coefficients from the given differential equation:

$$a_0 = 7,\ \ a_1 = -5,\ \ b_0 = 4,\ \ b_1 = 1,\ \ b_2 = 1$$

This is a **proper** system. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u(t) \ \rightarrow\ \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -7 & 5\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u(t)$$

$$y(t) = [b_0 - b_2a_0 \quad b_1 - b_2a_1]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [b_2]u(t) \quad\rightarrow\quad y(t) = [-3 \quad 6]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [1]u(t)$$

2) Consider SISO systems with the following state-space representations. Find the transfer function $\frac{Y(s)}{U(s)}$ of each system.

a) $\dot{x}(t) = \begin{bmatrix}0 & 1\\ -5 & -10\end{bmatrix}x(t) + \begin{bmatrix}1\\0\end{bmatrix}u(t)$

$y(t) = [0 \quad 10]x(t) + 5u(t)$

First, find the $(s\mathbf{I} - \mathbf{A})^{-1}$,

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -5 & -10\end{bmatrix} = \begin{bmatrix}s & -1\\ 5 & s+10\end{bmatrix} \quad\rightarrow\quad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 10s + 5}\begin{bmatrix}s+10 & 1\\ -5 & s\end{bmatrix}$$

Find the transfer function model from the following equation,

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+10s+5}\begin{bmatrix}0 & 10\end{bmatrix}\begin{bmatrix}s+10 & 1\\ -5 & s\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + 5 = \frac{1}{s^2+10s+5}\begin{bmatrix}0 & 10\end{bmatrix}\begin{bmatrix}s+10\\ -5\end{bmatrix} + 5 = \frac{-50}{s^2+10s+5} + 5$$

$$G(s) = \frac{5s^2+50s-25}{s^2+10s+5}$$

b) $\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{-3} & \mathbf{-5}\end{bmatrix}\boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\boldsymbol{u}(\boldsymbol{t})$

$\boldsymbol{y}(\boldsymbol{t}) = [\mathbf{3} \quad \mathbf{0}]\boldsymbol{x}(\boldsymbol{t})$

First, find the $(s\mathbf{I}-\mathbf{A})^{-1}$,

$$s\mathbf{I}-\mathbf{A} = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -3 & -5\end{bmatrix} = \begin{bmatrix}s & -1\\ 3 & s+5\end{bmatrix} \quad \rightarrow \quad (s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s^2+5s+3}\begin{bmatrix}s+5 & 1\\ -3 & s\end{bmatrix}$$

Find the transfer function model from the following equation,

$$\boldsymbol{G}(\boldsymbol{s}) = \frac{\boldsymbol{Y}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})} = \mathbf{C}(\boldsymbol{s}\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+5s+3}\begin{bmatrix}3 & 0\end{bmatrix}\begin{bmatrix}s+5 & 1\\ -3 & s\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{s^2+5s+3}\begin{bmatrix}3 & 0\end{bmatrix}\begin{bmatrix}1\\ s\end{bmatrix} = \frac{3}{s^2+5s+3}$$

$$G(s) = \frac{3}{s^2+5s+3}$$

c) $\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{-1} & \mathbf{-2}\end{bmatrix}\boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix}\boldsymbol{u}(\boldsymbol{t})$

$\boldsymbol{y}(\boldsymbol{t}) = [\mathbf{0} \quad \mathbf{1}]\boldsymbol{x}(\boldsymbol{t}) - \mathbf{3}\boldsymbol{u}(\boldsymbol{t})$

First, find the $(s\mathbf{I}-\mathbf{A})^{-1}$,

$$s\mathbf{I}-\mathbf{A} = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix} = \begin{bmatrix}s & -1\\ 1 & s+2\end{bmatrix} \quad \rightarrow \quad (s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s^2+2s+1}\begin{bmatrix}s+2 & 1\\ -1 & s\end{bmatrix}$$

Find the transfer function model from the following equation,

$$\boldsymbol{G}(\boldsymbol{s}) = \frac{\boldsymbol{Y}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})} = \mathbf{C}(\boldsymbol{s}\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+2s+1}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+2 & 1\\ -1 & s\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + [-3] = \frac{1}{s^2+2s+1}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+2\\ -1\end{bmatrix} - 3 = \frac{-1}{s^2+2s+1} - 3$$

$$G(s) = \frac{-3s^2-6s-4}{s^2+2s+1}$$

d) $\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -\mathbf{1} & \mathbf{1} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u}(t)$

$\boldsymbol{y}(t) = [\mathbf{1} \quad \mathbf{1}]\boldsymbol{x}(t)$

First, find the $(s\mathbf{I} - \mathbf{A})^{-1}$,

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s+1 \end{bmatrix} \quad \rightarrow \quad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 2s + 1} \begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix}$$

Find the transfer function model from the following equation,

$$\boldsymbol{G}(\boldsymbol{s}) = \frac{\boldsymbol{Y}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})} = \mathbf{C}(\boldsymbol{s}\mathbf{I} - \mathbf{A})^{-\mathbf{1}}\mathbf{B} + \mathbf{D}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2s + 1} [1 \quad 1] \begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + 2s + 1} [1 \quad 1] \begin{bmatrix} 1 \\ s+1 \end{bmatrix} = \frac{s+2}{s^2 + 2s + 1}$$

$$G(s) = \frac{s+2}{s^2 + 2s + 1}$$

3) Consider the following systems that are represented by transfer function models.

Find the state-space representation of each system in Canonical Controllable form and draw its block diagram.

a) $\boldsymbol{G}(\boldsymbol{s}) = \frac{\boldsymbol{Y}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})} = \frac{\mathbf{5}}{\boldsymbol{s}^3 + \mathbf{10}\boldsymbol{s}^2 + \mathbf{10}\boldsymbol{s} + \mathbf{50}}$

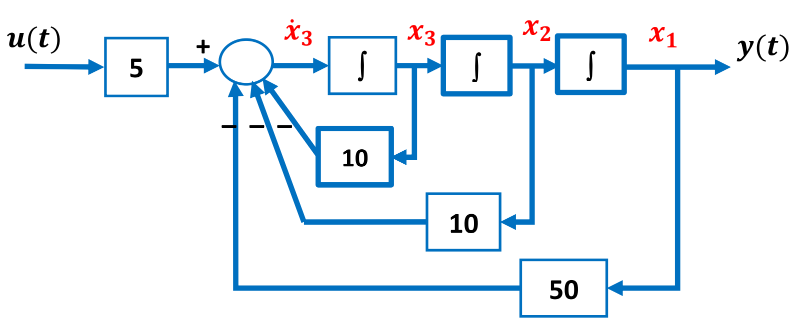
First, find the required coefficients from the given transfer function model:

$$a_0 = 50, \ \ a_1 = 10, \ \ a_2 = 10, \ \ b_0 = 5$$

This is a strictly proper system with no zeros. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathrm{a}_0 & -\mathrm{a}_1 & -\mathrm{a}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathrm{b}_0 \end{bmatrix} u(t) \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -50 & -10 & -10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t) \quad \rightarrow \quad y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$



b) $G(s) = \frac{Y(s)}{U(s)} = \frac{4(s+3)(s+1)}{(s+2)(s+6)}$

First, find the required coefficients from the given transfer function model:

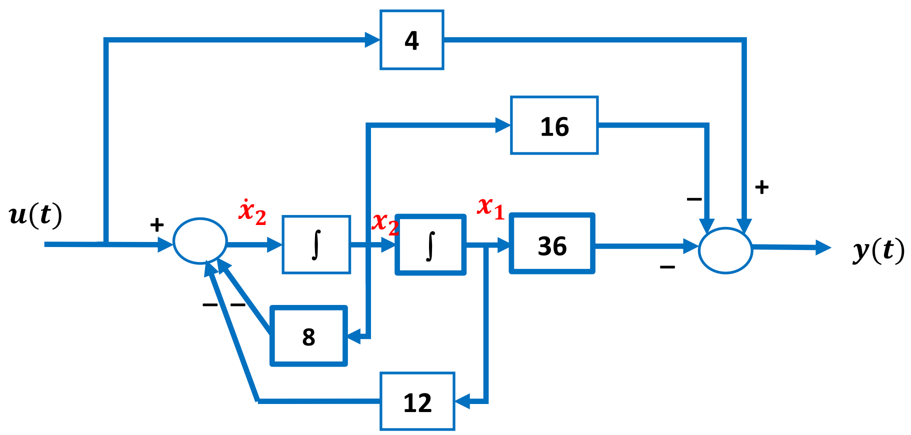
$$G(s) = \frac{4s^2 + 16s + 12}{s^2 + 8s + 12}$$

$$a_0 = 12, \; a_1 = 8, \; b_0 = 12, \; b_1 = 16, \; b_2 = 4$$

This is a **proper** system. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \;\rightarrow\; \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 - b_2 a_0 & b_1 - b_2 a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [b_2]u(t) \;\rightarrow\; y(t) = \begin{bmatrix} -36 & -16 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [4]u(t)$$



c) $G(s) = \frac{Y(s)}{U(s)} = \frac{s^2+2s+10}{s^3+4s^2+6s+10}$

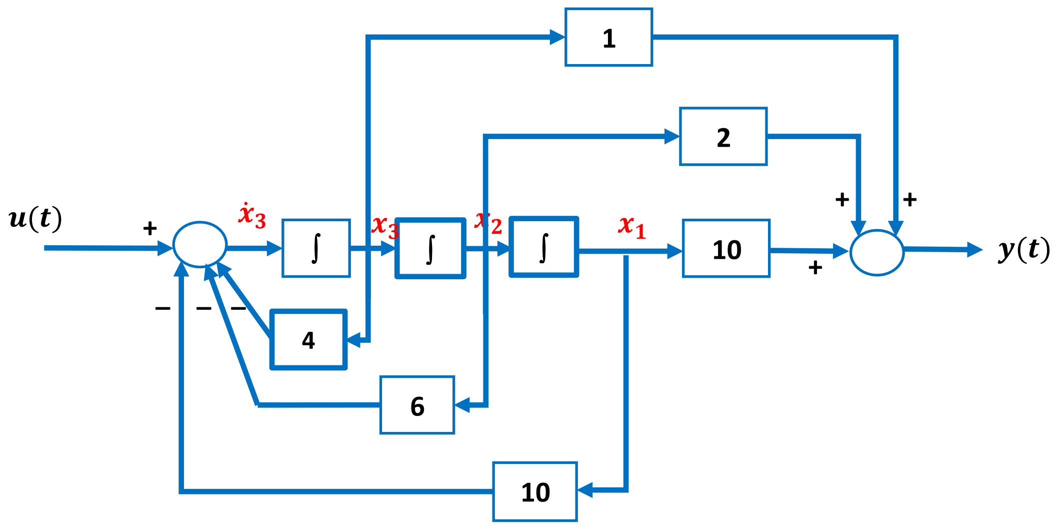
First, find the required coefficients from the given transfer function model:

$$a_0 = 10, \; a_1 = 6, \; a_2 = 4, \; b_0 = 10, \; b_1 = 2, \; b_2 = 1$$

This is a **strictly proper system with zeros**. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \;\rightarrow\; \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t) \;\rightarrow\; y(t) = \begin{bmatrix} 10 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$



d) $G(s) = \frac{Y(s)}{U(s)} = \frac{2s^2+s+5}{s^3+6s^2+11s+4}$

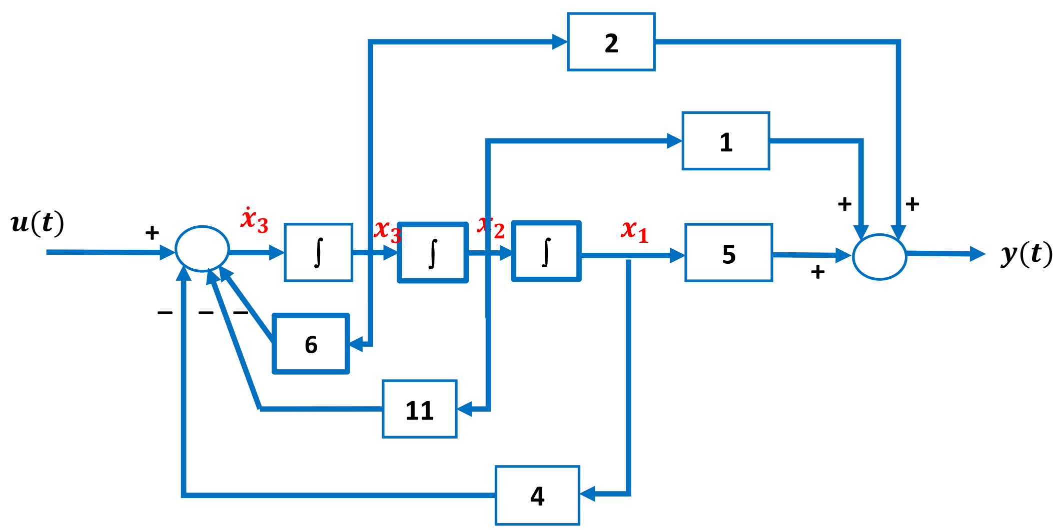
First, find the required coefficients from the given transfer function model:

$$a_0 = 4,\;\; a_1 = 11,\;\; a_2 = 6,\;\; b_0 = 5,\;\; b_1 = 1,\;\;\; b_2 = 2$$

This is a **strictly proper system with zeros**. Therefore, the Canonical Controllable form will be as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathrm{a}_0 & -\mathrm{a}_1 & -\mathrm{a}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \;\rightarrow\; \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [b_0 \quad b_1 \quad b_2] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t) \quad \rightarrow \quad y(t) = [5 \quad 1 \quad 2] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$



4) The state equations of an LTI are represented by

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t})$$

Find the characteristic equation, and the eigenvalues of matrix A for the following cases. Determine which system is stable.

a) $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{-1} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

First, create the matrix $\lambda\mathbf{I} - \mathbf{A}$ and find its determinant, which is the **Characteristic equation**.

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 1 \end{bmatrix}$$

$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 1 \end{vmatrix} = \lambda^2 + \lambda + 2$ → **Characteristic Equation**

Solve the characteristic equation to find the **eigenvalues**.

$$\lambda^2 + \lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = -0.5 \pm j1.32$$

Since the eigenvalues are in the left-half of the s-place, the system is **stable**.

b) $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-4} & \mathbf{-5} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$

First, create the matrix $\lambda\mathbf{I} - \mathbf{A}$ and find its determinant, which is the **Characteristic equation**.

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda + 5 \end{bmatrix}$$

$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda & -1 \\ 4 & \lambda + 5 \end{vmatrix} = \lambda^2 + 5\lambda + 4$ → **Characteristic Equation**

Solve the characteristic equation to find the **eigenvalues**.

$$\lambda^2 + 5\lambda + 4 = 0 \quad \rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = -4$$

Since the eigenvalues are in the left-half of the s-place, the system is **stable**.

c) $\boldsymbol{A} = \begin{bmatrix} \mathbf{-3} & \mathbf{0} \\ \mathbf{0} & \mathbf{-3} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

First, create the matrix $\lambda\mathbf{I} - \mathbf{A}$ and find its determinant, which is the **Characteristic equation**.

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \lambda + 3 & 0 \\ 0 & \lambda + 3 \end{bmatrix}$$

$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda + 3 & 0 \\ 0 & \lambda + 3 \end{vmatrix} = \lambda^2 + 6\lambda + 9$ $\rightarrow$ **Characteristic Equation**

Solve the characteristic equation to find the **eigenvalues**.

$$\lambda^2 + 6\lambda + 9 = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 = -3$$

Since the eigenvalues are in the left-half of the s-place, the system is <u>**stable**</u>.

d) $\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{-3} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

First, create the matrix $\lambda\mathbf{I} - \mathbf{A}$ and find its determinant, which is the **Characteristic equation**.

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & 0 \\ 0 & \lambda + 3 \end{bmatrix}$$

$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda + 3 \end{vmatrix} = \lambda^2 - 9$ $\rightarrow$ **Characteristic Equation**

Solve the characteristic equation to find the **eigenvalues**.

$$\lambda^2 + 9 = 0 \quad \rightarrow \quad \lambda_1 = 3, \quad \lambda_2 = -3$$

Since the one eigenvalue is in the right-half of the s-place, the system is <u>**unstable**</u>.

e) $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{-2} & \mathbf{0} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

First, create the matrix $\lambda\mathbf{I} - \mathbf{A}$ and find its determinant, which is the **Characteristic equation**.

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & -2 \\ 2 & \lambda \end{bmatrix}$$

$\det(\lambda\mathbf{I}-\mathbf{A}) = \begin{vmatrix} \lambda & -2 \\ 2 & \lambda \end{vmatrix} = \lambda^2 + 4$ → **Characteristic Equation**

Solve the characteristic equation to find the **eigenvalues**.

$\lambda^2 + 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm j2$

Since the eigenvalues are on the imaginary axis, the system is **marginally stable**.

f) $\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

First, create the matrix $\lambda\mathbf{I}-\mathbf{A}$ and find its determinant, which is the **Characteristic equation**.

$$\lambda\mathbf{I}-\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+2 & -1 \\ 0 & 0 & \lambda+2 \end{bmatrix}$$

$\det(\lambda\mathbf{I}-\mathbf{A}) = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+2 & -1 \\ 0 & 0 & \lambda+2 \end{vmatrix} = (\lambda+1)(\lambda+2)^2$ → **Characteristic Equation**

Solve the characteristic equation to find the **eigenvalues**.

$(\lambda+1)(\lambda+2)^2 = 0 \quad \rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = \lambda_3 = -2$

Since the eigenvalues are in the left-half of the s-place, the system is **stable**.

g) $\boldsymbol{A} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

First, create the matrix $\lambda\mathbf{I}-\mathbf{A}$ and find its determinant, which is the **Characteristic equation**.

$$\lambda\mathbf{I}-\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} \lambda+5 & -1 & 0 \\ 0 & \lambda+5 & -1 \\ 0 & 0 & \lambda+5 \end{bmatrix}$$

$\det(\lambda\mathbf{I}-\mathbf{A}) = \begin{vmatrix} \lambda+5 & -1 & 0 \\ 0 & \lambda+5 & -1 \\ 0 & 0 & \lambda+5 \end{vmatrix} = (\lambda+5)^3$ → **Characteristic Equation**

Solve the characteristic equation to find the **eigenvalues**.

$(\lambda+5)^3 = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = -5$

Since the eigenvalues are in the left-half of the s-place, the system is **stable**.

5) Check the controllability of the following systems:

a) $\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$

Find the controllability matrix of the system and check the determinant

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 2 & -2 \\ 5 & -10 \end{bmatrix} \quad \rightarrow \quad \det[\mathbf{Q_c}] = -10$$

Since the determinant is **non-zero**, the controllability matrix is full rank, so the system is **controllable**.

b) $\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$

Find the controllability matrix of the system and check the determinant

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \quad \rightarrow \quad \det[\mathbf{Q_c}] = 0$$

Since the determinant is **zero**, the controllability matrix is not full rank, so the system is **not controllable**.

c) $\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$

Find the controllability matrix of the system and check the determinant

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B}] = \begin{bmatrix} 4 & 2 & -4 & -2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -6 & 0 & 12 & 0 \end{bmatrix} \quad \rightarrow \quad \mathrm{rank}[\mathbf{Q_c}] = 2$$

Since the controllability matrix has one **full zero row**, it is not full rank, so the system is **not controllable**.

d) $\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$

Find the controllability matrix of the system and check the determinant

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B}] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix} \quad \rightarrow \quad \det[\mathbf{Q_c}] = -1$$

Since the determinant is **non-zero**, the controllability matrix is not full rank, so the system is **controllable**.

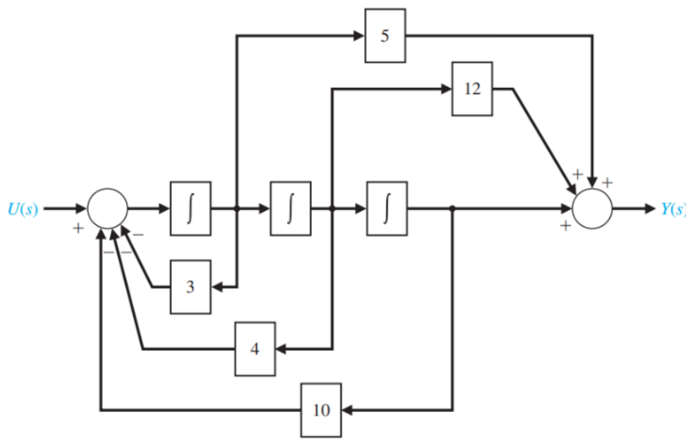
e) $\dot{x}(t) = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u(t)$

Find the controllability matrix of the system and check the determinant

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & -9 \\ 1 & -1 & 16 \end{bmatrix} \quad \rightarrow \quad \det[\mathbf{Q_c}] = 80$$

Since the determinant is **non-zero**, the controllability matrix is not full rank, so the system is **controllable**.

6) Consider the following block diagram model of a system.



a) Using the block diagram as a guide, obtain the state-space model of the system in the form of

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t})$$
$$\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t})$$

Since there are three integrator blocks, this is a third-order system. Since there is no direct connection between input and output, it is a strictly proper system.

Find the canonical controllable form parameters from the given block diagram model,

$$a_0 = 10, \qquad a_1 = 4, \qquad a_2 = 3, \qquad b_0 = 1, \qquad b_1 = 12, \qquad b_2 = 5$$

Therefore, the Canonical Controllable from will be as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathrm{a}_0 & -\mathrm{a}_1 & -\mathrm{a}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [b_0 \quad b_1 \quad b_2] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t) \quad \rightarrow \quad y(t) = [1 \quad 12 \quad 5] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

b) Using the state-space model as a guide, obtain a third-order differential equation model for the system.

The third-order differential equation of a strictly proper system have the following general form,

$$\dddot{y}(t) + a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_2\ddot{u}(t) + b_1\dot{u}(t) + b_0 u(t)$$

Therefore, the differential equation model will be,

$$\dddot{y}(t) + 3\ddot{y}(t) + 4\dot{y}(t) + 10y(t) = 5\ddot{u}(t) + 12\dot{u}(t) + u(t)$$

7) Given a system with the following transfer function, design a state feedback control to yield a step response with 15% overshoot and a settling time of 0.5 seconds.

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \frac{\mathbf{10}}{\boldsymbol{(s+1)(s+2)}}$$

First, find the canonical controllable form of the system,

$$\frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)} = \frac{10}{s^2 + 3s + 2}$$

This is a strictly proper system with no-zero. The required coefficients are,

$$a_0 = 2, \qquad a_1 = 3, \qquad b_0 = 10$$

Therefore, the Canonical Controllable from will be as,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ b_0\end{bmatrix}u(t) \quad\rightarrow\quad \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 10\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [0]u(t) \quad\rightarrow\quad y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [0]u(t)$$

Follow the steps to design the state feedback control,

Step 1: Check controllability of the open-loop system,

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix}0 & 10\\ 10 & -30\end{bmatrix} \quad\rightarrow\quad \det[\mathbf{Q_c}] = -100$$

Since the determinant is **non-zero**, the controllability matrix is full rank, so the system is **controllable**.

Step 2: Determine the desired characteristic polynomial,

First, calculate the desired damping ratio from the given desired maximum overshoot,

$$\boldsymbol{\zeta} = \frac{-\ln(\boldsymbol{O.S.})}{\sqrt{\boldsymbol{\pi}^2 + \ln^2(\boldsymbol{O.S.})}} \quad\rightarrow\quad \zeta = \frac{-\ln(0.15)}{\sqrt{\pi^2 + \ln^2(0.15)}} \quad\rightarrow\quad \zeta = 0.78$$

Then, calculate the undamped natural frequency from the given desired settling time:

$$\boldsymbol{t_s} = \frac{\mathbf{4}}{\boldsymbol{\zeta\omega_n}} \quad\rightarrow\quad 0.5 = \frac{4}{0.78\omega_n} \quad\rightarrow\quad \omega_n = 10.256 \text{ rad/sec}$$

The desired characteristic equation for the closed-loop system is,

$$\boldsymbol{s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 16s + 105.19}$$

Step 3: Obtain the closed-loop system matrix and determine the characteristic polynomial,

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 10 \end{bmatrix} [\mathrm{k}_1 \quad \mathrm{k}_2] = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 10\mathrm{k}_1 & 10\mathrm{k}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2-10\mathrm{k}_1 & -3-10\mathrm{k}_2 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A}_{cl} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2-10\mathrm{k}_1 & -3-10\mathrm{k}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{s} & -1 \\ 2+10\mathrm{k}_1 & \mathrm{s}+3+10\mathrm{k}_2 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}_{cl}) = \begin{vmatrix} \mathrm{s} & -1 \\ 2+10\mathrm{k}_1 & \mathrm{s}+3+10\mathrm{k}_2 \end{vmatrix} = s^2 + (3+10k_2)s + 2 + 10k_1$$

Step 4: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the state feedback gain value K,

$$s^2 + 16s + 105.19 \qquad \text{and} \qquad s^2 + (3+10k_2)s + 2 + 10k_1$$

$$\begin{cases} 3 + 10\mathrm{k}_2 = 16 \\ 2 + 10\mathrm{k}_1 = 105.19 \end{cases} \rightarrow \begin{cases} k_2 = 1.3 \\ k_1 = 10.319 \end{cases} \rightarrow \quad \mathbf{K} = [10.319 \quad 1.3] \quad \text{State-Feedback Gain}$$

8) Given a system with the following transfer function, design a state feedback control to yield a step response with 20.8% overshoot and a settling time of 4 seconds.

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \frac{\boldsymbol{s+4}}{\boldsymbol{(s+1)(s+2)(s+5)}}$$

First, find the canonical controllable form of the system,

$$\frac{Y(s)}{U(s)} = \frac{s+4}{(s+1)(s+2)(s+5)} = \frac{s+4}{s^3 + 8s^2 + 17s + 10}$$

This is a strictly proper system with zero. The required coefficients are,

$$a_0 = 10, \qquad a_1 = 17, \qquad a_2 = 8, \qquad b_0 = 4, \qquad b_1 = 1$$

Therefore, the Canonical Controllable from will be as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathrm{a}_0 & -\mathrm{a}_1 & -\mathrm{a}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [b_0 \quad b_1 \quad b_2] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t) \quad \rightarrow \quad y(t) = [4 \quad 1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

Follow the steps to design the state feedback control,

Step 1: Check controllability of the open-loop system,

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -8 \\ 1 & -8 & 47 \end{bmatrix} \quad \rightarrow \quad \det[\mathbf{Q_c}] = -1$$

Since the determinant is **non-zero**, the controllability matrix is full rank, so the system is **<u>controllable</u>**.

Step 2: Determine the desired characteristic polynomial,

First, calculate the desired damping ratio from the given desired maximum overshoot,

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \quad \rightarrow \quad \zeta = \frac{-\ln(0.208)}{\sqrt{\pi^2 + \ln^2(0.208)}} \quad \rightarrow \quad \zeta = 0.71$$

Then, calculate the undamped natural frequency from the given desired settling time:

$$t_s = \frac{4}{\zeta\omega_n} \quad \rightarrow \quad 4 = \frac{4}{0.71\omega_n} \quad \rightarrow \quad \omega_n = 1.41 \text{ rad/sec}$$

The desired dominant poles are determined based on the desired ζ and ω_n. The third pole will be selected far from the desired dominant poles at higher frequencies, for example at $s = -50$. Therefore, the desired characteristic equation for the closed-loop system is,

$$\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s + 50) = \left(s^2 + 2s + 2\right)(s + 50) = s^3 + 52s^2 + 102s + 100$$

Step 3: Obtain the closed-loop system matrix and determine the characteristic polynomial,

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 - k_1 & -17 - k_2 & -8 - k_3 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A}_{cl} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 - k_1 & -17 - k_2 & -8 - k_3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 10 + k_1 & 17 + k_2 & s + 8 + k_3 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}_{cl}) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 10 + k_1 & 17 + k_2 & s + 8 + k_3 \end{vmatrix} = s^3 + (8 + k_3)s^2 + (17 + k_2)s + 10 + k_1$$

Step 4: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the state feedback gain value K,

$$s^3 + 52s^2 + 102s + 100 \qquad \text{and} \qquad s^3 + (8 + k_3)s^2 + (17 + k_2)s + 10 + k_1$$

$$\begin{cases} 8 + k_3 = 52 \\ 17 + k_2 = 102 \\ 10 + k_1 = 100 \end{cases} \quad \rightarrow \quad \begin{cases} k_3 = 44 \\ k_2 = 85 \\ k_1 = 90 \end{cases} \quad \rightarrow \quad \mathbf{K} = \begin{bmatrix} 90 & 85 & 44 \end{bmatrix} \quad \text{State-Feedback Gain}$$

9) Given a system with the following transfer function, design a state feedback control to yield a step response with 20% overshoot and a settling time of 2 seconds.

$$\frac{Y(s)}{U(s)} = \frac{s+6}{(s+9)(s+8)(s+7)}$$

First, find the canonical controllable form of the system,

$$\frac{Y(s)}{U(s)} = \frac{s+6}{(s+9)(s+8)(s+7)} = \frac{s+6}{s^3+24s^2+191s+504}$$

This is a strictly proper system with zero. The required coefficients are,

$$a_0 = 504, \qquad a_1 = 191, \qquad a_2 = 24, \qquad b_0 = 6, \qquad b_1 = 1$$

Therefore, the Canonical Controllable from will be as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathrm{a}_0 & -\mathrm{a}_1 & -\mathrm{a}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \;\rightarrow\; \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504 & -191 & -24 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t) \;\rightarrow\; y(t) = \begin{bmatrix} 6 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

Follow the steps to design the state feedback control,

Step 1: Check controllability of the open-loop system,

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -24 \\ 1 & -24 & 385 \end{bmatrix} \;\rightarrow\; \det[\mathbf{Q_c}] = -1$$

Since the determinant is **non-zero**, the controllability matrix is full rank, so the system is **controllable**.

Step 2: Determine the desired characteristic polynomial,

First, calculate the desired damping ratio from the given desired maximum overshoot,

$$\boldsymbol{\zeta} = \frac{-\ln(\boldsymbol{O.S.})}{\sqrt{\boldsymbol{\pi}^2 + \ln^2(\boldsymbol{O.S.})}} \;\rightarrow\; \zeta = \frac{-\ln(0.20)}{\sqrt{\pi^2 + \ln^2(0.20)}} \;\rightarrow\; \zeta = 0.72$$

Then, calculate the undamped natural frequency from the given desired settling time:

$$\boldsymbol{t_s} = \frac{\mathbf{4}}{\boldsymbol{\zeta\omega_n}} \;\rightarrow\; 2 = \frac{4}{0.72\omega_n} \;\rightarrow\; \omega_n = 2.78 \text{ rad/sec}$$

The desired dominant poles are determined based on the desired ζ and ω_n. The third pole will be selected far from the desired dominant poles at higher frequencies, for example at $s = -50$. Therefore, the desired characteristic equation for the closed-loop system is,

$$(\boldsymbol{s^2 + 2\zeta\omega_n s + \omega_n^2})(\mathbf{s+50}) = (\boldsymbol{s^2 + 4s + 7.73})(\mathbf{s+50}) = \mathbf{s^3 + 54s^2 + 207.73s + 386}$$

Step 3: Obtain the closed-loop system matrix and determine the characteristic polynomial,

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504 & -191 & -24 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3]$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504 & -191 & -24 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504-k_1 & -191-k_2 & -24-k_3 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A}_{cl} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504-k_1 & -191-k_2 & -24-k_3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 504+k_1 & 191+k_2 & s+24+k_3 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}_{cl}) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 504+k_1 & 191+k_2 & s+24+k_3 \end{vmatrix} = s^3 + (24+k_3)s^2 + (191+k_2)s + 504 + k_1$$

Step 4: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the state feedback gain value K,

$$s^3 + 54s^2 + 207.73s + 386 \qquad \text{and} \qquad s^3 + (24+k_3)s^2 + (191+k_2)s + 504 + k_1$$

$$\begin{cases} 24 + \mathrm{k}_3 = 54 \\ 191 + \mathrm{k}_2 = 207.73 \\ 504 + k_1 = 386 \end{cases} \rightarrow \begin{cases} k_3 = 30 \\ k_2 = 16.73 \\ k_1 = -118 \end{cases} \rightarrow \quad \mathbf{K} = [30 \quad 16.73 \quad -118] \quad \text{State-Feedback Gain}$$

10) Design an integral controller for the following system to yield a step response with 10% overshoot, a peak time of 2 seconds, and zero steady-state error.

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{7} & -\mathbf{9} \end{bmatrix} \boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{y}(\boldsymbol{t}) = [\mathbf{4} \quad \mathbf{1}] \boldsymbol{x}(\boldsymbol{t})$$

Step 1: Determine the augmented open-loop system

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \\ y(t) = [\mathbf{C} \quad 0] \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \mathbf{D}u(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -7 & -9 & 0 \\ -4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = [4 \quad 1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix}$$

Step 2: Check controllability of the augmented open-loop system.

$$\bar{\text{A}} = \begin{bmatrix} 0 & 1 & 0 \\ -7 & -9 & 0 \\ -4 & -1 & 0 \end{bmatrix} \quad and \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{Q_c} = [\bar{\mathbf{B}} \quad \bar{\mathbf{A}}\bar{\mathbf{B}} \quad \bar{\mathbf{A}}^2\bar{\mathbf{B}}] = \begin{bmatrix} 0 & 1 & -9 \\ 1 & -9 & 74 \\ 0 & -1 & 5 \end{bmatrix} \quad \rightarrow \quad \det[\mathbf{Q_c}] = 4$$

Since the determinant is **non-zero**, the controllability matrix is full rank, so the system is **controllable**.

Step 3: Determine the desired characteristic polynomial

First, calculate the desired damping ratio from the given desired maximum overshoot,

$$\boldsymbol{\zeta} = \frac{-\ln(\boldsymbol{O.S.})}{\sqrt{\boldsymbol{\pi}^2 + \ln^2(\boldsymbol{O.S.})}} \quad \rightarrow \quad \zeta = \frac{-\ln(0.10)}{\sqrt{\pi^2 + \ln^2(0.10)}} \quad \rightarrow \quad \zeta = 0.5912$$

Then, calculate the undamped natural frequency from the given desired peak-time:

$$\boldsymbol{t_p} = \frac{\boldsymbol{\pi}}{\boldsymbol{\omega_n}\sqrt{\mathbf{1} - \boldsymbol{\zeta}^2}} \quad \rightarrow \quad 2 = \frac{\pi}{\omega_n\sqrt{1-(0.5912)^2}} \quad \rightarrow \quad \omega_n = 1.9475 \text{ rad/sec}$$

The desired dominant poles are determined based on the desired ζ and ω_n. The third pole will be selected far from the desired dominant poles at higher frequencies, for example at $s = -50$. Therefore, the desired characteristic equation for the closed-loop system is,

$$\left(\boldsymbol{s}^2 + 2\boldsymbol{\zeta}\boldsymbol{\omega}_n\boldsymbol{s} + \boldsymbol{\omega}_n^2\right)(\mathbf{s}+\mathbf{50}) = \left(\boldsymbol{s}^2 + \mathbf{2.3}\boldsymbol{s} + \mathbf{3.8}\right)(\mathbf{s}+\mathbf{50}) = \mathbf{s}^3 + \mathbf{52.3}\boldsymbol{s}^2 + \mathbf{118.8}\boldsymbol{s} + \mathbf{190}$$

Step 4: Obtain the closed-loop system matrix and determine the characteristic polynomial,

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -7 & -9 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ -7 & -9 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7-k_1 & -9-k_2 \end{bmatrix}$$

$$\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{B}k_i \\ -\mathbf{C} + \mathbf{DK} & -\mathbf{D}k_i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -7-k_1 & -9-k_2 & k_i \\ -4 & -1 & 0 \end{bmatrix}$$

$$\mathbf{sI} - \mathbf{A}_{cl} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -7-k_1 & -9-k_2 & k_i \\ -4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 7+k_1 & s+9+k_2 & -k_i \\ 4 & 1 & s \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}_{cl}) = \begin{vmatrix} s & -1 & 0 \\ 7+k_1 & s+9+k_2 & -k_i \\ 4 & 1 & s \end{vmatrix} = s^3 + (9+k_2)s^2 + (7+k_1+k_i)s + 4k_i$$

Step 5: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the gains K and k_i,

$$s^3 + 52.3s^2 + 118.8s + 190 \qquad \text{and} \qquad s^3 + (9 + k_2)s^2 + (7 + k_1 + k_i)s + 4k_i$$

$$\begin{cases} 9 + k_2 = 52.3 \\ 7 + k_1 + k_i = 118.8 \\ 4k_i = 190 \end{cases} \rightarrow \begin{cases} k_2 = 42.3 \\ k_1 = 64.3 \\ k_i = 47.5 \end{cases} \rightarrow \quad \begin{matrix} \boldsymbol{K} = [64.3 \quad 42.3] & \textit{State-Feedback Gain} \\ k_i = 47.5 & \text{Integrator Gain} \end{matrix}$$

11) Design an integral controller for the following system to yield a step response with 10% overshoot, a settling time of 0.5 seconds, and zero steady-state error.

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-3} & \mathbf{-5} \end{bmatrix} \boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{y}(\boldsymbol{t}) = [\mathbf{1} \quad \mathbf{0}] \boldsymbol{x}(\boldsymbol{t})$$

Step 1: Determine the augmented open-loop system

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \\ y(t) = [\mathbf{C} \quad 0] \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \mathbf{D} u(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix}$$

Step 2: Check controllability of the augmented open-loop system.

$$\bar{\mathrm{A}} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{Q_c} = [\bar{\mathbf{B}} \quad \bar{\mathbf{A}}\bar{\mathbf{B}} \quad \bar{\mathbf{A}}^2\bar{\mathbf{B}}] = \begin{bmatrix} 0 & 1 & -5 \\ 1 & -5 & 22 \\ 0 & 0 & -1 \end{bmatrix} \quad \rightarrow \quad \det[\mathbf{Q_c}] = 1$$

Since the determinant is **non-zero**, the controllability matrix is full rank, so the system is **controllable**.

Step 3: Determine the desired characteristic polynomial

First, calculate the desired damping ratio from the given desired maximum overshoot,

$$\boldsymbol{\zeta}=\frac{-\ln(\boldsymbol{O.S.})}{\sqrt{\boldsymbol{\pi}^2+\ln^2(\boldsymbol{O.S.})}} \quad \rightarrow \quad \zeta=\frac{-\ln(0.10)}{\sqrt{\pi^2+\ln^2(0.10)}} \quad \rightarrow \quad \zeta=0.5912$$

Then, calculate the undamped natural frequency from the given desired peak-time:

$$\boldsymbol{t_s}=\frac{\boldsymbol{4}}{\boldsymbol{\zeta\omega_n}} \quad \rightarrow \quad 0.5=\frac{4}{0.5912\omega_n} \quad \rightarrow \quad \omega_n=13.53\ \text{rad/sec}$$

The desired dominant poles are determined based on the desired ζ and ω_n. The third pole will be selected far from the desired dominant poles at higher frequencies, for example at $s=-50$. Therefore, the desired characteristic equation for the closed-loop system is,

$$\left(\boldsymbol{s^2+2\zeta\omega_n s+\omega_n^2}\right)(\mathbf{s}+\mathbf{50})=\left(\boldsymbol{s^2+16s+183.1}\right)(\mathbf{s}+\mathbf{50})=\mathbf{s}^3+\boldsymbol{66s^2+983.1s+9155}$$

Step 4: Obtain the closed-loop system matrix and determine the characteristic polynomial,

$$\mathbf{A}-\mathbf{BK}=\begin{bmatrix}0 & 1\\ -3 & -5\end{bmatrix}-\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -3 & -5\end{bmatrix}-\begin{bmatrix}0 & 0\\ k_1 & k_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -3-k_1 & -5-k_2\end{bmatrix}$$

$$\mathbf{A}_{cl}=\begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{B}k_i\\ -\mathbf{C}+\mathbf{DK} & -\mathbf{D}k_i\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -3-k_1 & -5-k_2 & k_i\\ -1 & 0 & 0\end{bmatrix}$$

$$\mathbf{sI}-\boldsymbol{A_{cl}}=\begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1 & 0\\ -3-k_1 & -5-k_2 & k_i\\ -1 & 0 & 0\end{bmatrix}=\begin{bmatrix}s & -1 & 0\\ 3+k_1 & s+5+k_2 & -k_i\\ 1 & 0 & s\end{bmatrix}$$

$$\det(s\mathbf{I}-\mathbf{A}_{cl})=\begin{vmatrix}s & -1 & 0\\ 3+k_1 & s+5+k_2 & -k_i\\ 1 & 0 & s\end{vmatrix}=s^3+(5+k_2)s^2+(3+k_1)s+k_i$$

Step 5: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the gains K and $\boldsymbol{k_i}$,

$$s^3+66s^2+983.1s+9155 \qquad \text{and} \qquad s^3+(5+k_2)s^2+(3+k_1)s+k_i$$

$$\begin{cases}5+k_2=66\\ 3+k_1=983.1\\ k_1=9155\end{cases} \rightarrow \begin{cases}k_2=61\\ k_1=980.1\\ k_i=9155\end{cases} \rightarrow \quad \boldsymbol{K}=\begin{bmatrix}980.1 & 61\end{bmatrix} \quad \textit{State-Feedback Gain}$$

$$k_i=9155 \qquad \text{Integrator Gain}$$