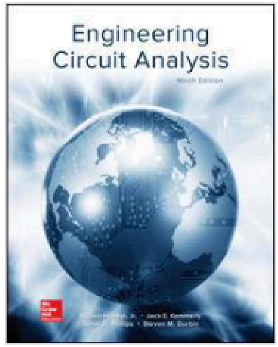
Lecture 8

November 24, 2023 12:15 AM

Sinusoids and Phasors



Sinusoids

Source — Circuit

DC source

- Sinusoids are interesting to us because there are a number of natural phenomenon that are sinusoidal in nature
- It is also a very easy signal to generate and transmit

  ← Sine / Cosine

- Also, through Fourier analysis, any practical periodic function can be made by adding sinusoids
- Lastly, they are very easy to handle mathematically

Sinusoids

$v(t) = V_m \sin \omega t$ time-domain form

Linear motion

velocity, v (m/sec)

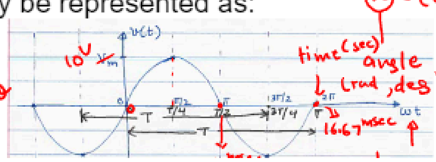
- A sinusoidal forcing function produces both a transient and a steady state response
- When the transient has died out, we say the circuit is in sinusoidal steady state
- A sinusoidal voltage may be represented as:

value of v at any time t

instantaneous value

$$v(t) = V_m \sin \omega t = V_m \sin\theta$$

$$= V_m \cos(\omega t)$$



v – instantaneous value

V_m – the amplitude of the sinusoid

ω - the angular frequency in rad/sec

t	v(t)
0	0
4	10
8	0
12	−10

rad (ω) ⇒ time (sec)

Sinusoids

$$v(t) = V_m \sin \omega t$$

→ takes to complete one cycle = Time period

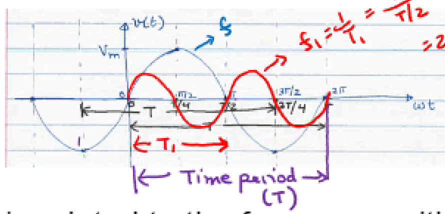
- The sinusoidal function repeats itself every T seconds
- This is called the period

$$T = \frac{2\pi}{\omega}$$

$\omega T = 2\pi$

$T = \frac{2\pi}{\omega}$ → rad / → rad/sec ⇒ sec

$f_1 = \frac{1}{T_1} = \frac{1}{T/2} = 2\frac{1}{T} = 2f$



- The period is inversely related to the frequency with units cycles per second, or Hertz (Hz)

$$f = \frac{1}{T}$$ ← sec, Hz

$f = \frac{1}{(2\pi/\omega)} \Rightarrow \frac{\omega}{2\pi} \Rightarrow \omega = 2\pi f$

$$\omega = 2\pi f$$ → Hz, rad/sec

Sinusoids

v(t) → time domain form of a sinusoid ⟨ amplitude, V_m; frequency, ω; phase, θ

Phase shift is used for expressing the relative timing of one wave versus another θ:

$v(t) = V_m \sin(\omega t \pm \theta)$

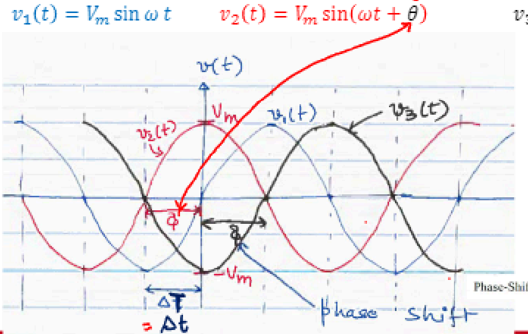
- Consider the three sinusoids:

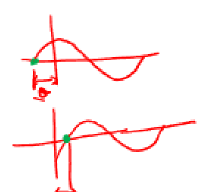
$v_1(t) = V_m \sin \omega t$ $\quad v_2(t) = V_m \sin(\omega t + \theta)$ $\quad v_3(t) = V_m sin(\omega t - \theta)$

θ = φ

↳ time

T → time period





Phase-Shift ⇒ $\theta = \frac{\Delta t}{T} \cdot 2\pi$ (radians) $= \frac{\Delta t}{T} \cdot 360°$(degrees)

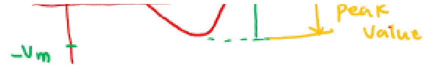
Sinusoids – Problem Solving

For a sinusoid $v(t) = 5sin(4\pi t - 60^0)^V$, calculate its amplitude, peak-to-peak value, phase, angular frequency, period and frequency.

$v(t) = V_m \sin(\omega t - \theta)$ $\qquad V_m = 5^V$

peak-to-peak value $= 10^V$

phase = 60°



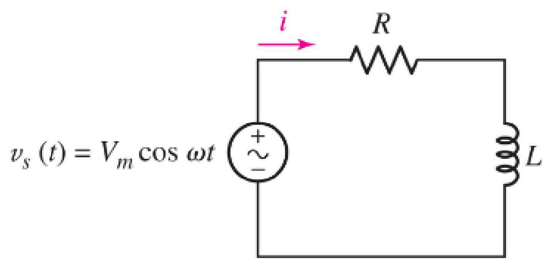
$\omega = 4\pi$ rad/sec

$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{4\pi} = \frac{1}{2}$ sec

$f = \frac{1}{T} = 2$ Hz

Steady State Response to Sinusoidal Sources

When the source is sinusoidal, the transient/natural response is often ignored and only the "steady-state" response is considered.



Source is assumed to exist forever: $-\infty < t < \infty$

$\bar{V}, \vec{V}, \vec{V}$ Complex Numbers

$v(t) = V_m \sin(\omega t \pm \theta)$

- A powerful method for representing sinusoids is the **phasor**
- **Phasor** is a complex number that represents the amplitude and phase of a Sinusoid $\bar{V} = V_m \angle \pm\theta$ at $\omega = 2$ rad/sec
- But in order to understand how they work, we need to cover some complex numbers first.
- A complex number z can be represented in rectangular form as:

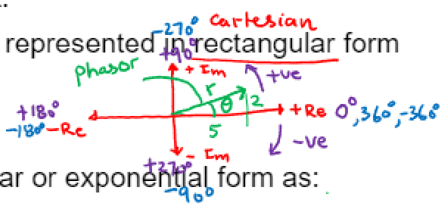
$$z = x + jy$$

$y = r\sin\theta$

$\bar{z} = 5 + 2j = 5 + j2 = 5 + 2i = 5 + i2$

$x = r\cos\theta$

cartesian



- It can also be written in polar or exponential form as:

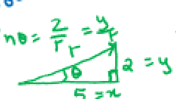
$$z = r\angle\phi = re^{j\phi}$$

$r = \sqrt{x^2 + y^2}$

$\cos\theta = \frac{5}{r}$ $\sin\theta = \frac{2}{r} = \frac{y}{r}$

$\tan\theta = \frac{2}{5} = \frac{y}{x}$

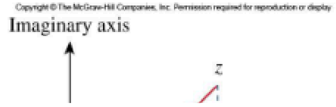
$\theta = \tan^{-1}(y/x)$



$r = \sqrt{5^2 + 2^2}$

Complex Numbers

- The different forms can be interconverted
- Starting with rectangular form, one can go to polar:

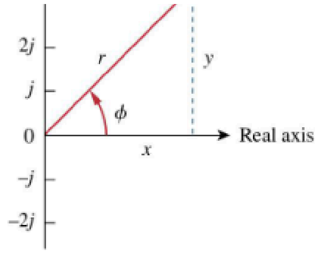


form, one can go to polar:

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x}$$

- Likewise, from polar to rectangular form goes as follows:

$$x = r\cos\phi \qquad y = r\sin\phi$$



Complex Numbers

- The following mathematical operations are important.

- Addition

$(x_1+jy_1)+(x_2+jy_2)$

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$x_1+x_2+jy_1+jy_2$

- Subtraction

$(x_1+jy_1)-(x_2+jy_2)$

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

$x_1-x_2+jy_1-jy_2$

- Multiplication

$(r_1\angle\phi_1)(r_2\angle\phi_2)$

$$z_1 z_2 = r_1 r_2\angle(\phi_1 + \phi_2)$$

- Division

$\frac{r_1\angle\phi_1}{r_2\angle\phi_2} = \frac{r_1}{r_2}\angle\phi_1-\phi_2$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

- Reciprocal

$$\frac{1}{z} = \frac{1}{r}\angle(-\phi)$$

$= \frac{1}{r\angle\phi} = \frac{1}{r}\angle-\phi$

- Square Root

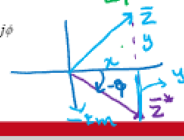
$$\sqrt{z} = \sqrt{r}\angle(\phi/2)$$

- Complex Conjugate

$$z^* = x - jy = r\angle-\phi = re^{-j\phi}$$

$\bar{z} = x+jy = r\angle\phi$

$\bar{z}^* = x-jy = r\angle-\phi$



Phasors

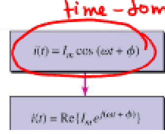
- A sinusoid can be represented as the real component of a vector in the complex plane
- The length of the vector is the amplitude of the sinusoid
- The vector, V, in polar form, is at an angle ϕ with respect to the positive real axis

Sinusoid-Phasor Transformation

$i(t) = I_m\cos(\omega t + \phi)$ → time-domain form

$$\bar{\mathbf{I}} = I_m e^{j\varphi} = I_m\angle\varphi$$ } phasor-domain, frequency-domain

time-domain



The phasor representation of a current (or voltage) is in the *frequency domain*

phasor domain

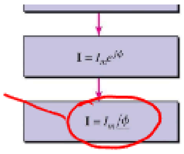


TABLE 9.1

$\cos(\omega t + \phi - 90°) = \sin(\omega t + \phi)$

Sinusoid-phasor transformation.

$$\cos\left(\omega t \pm 90^\circ\right) = \mp\sin\omega t$$

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle\phi$
$V_m \sin(\omega t + \phi)$ $= V_m \cos(\omega t + \phi - 90°)$	$V_m \angle\phi - 90°$
$I_m \cos(\omega t + \theta)$	$I_m \angle\theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle\theta - 90°$

A B

$\cos(\omega t + 90°) = \cos(A+B)$

$\cos A \cos B - \sin A \sin B$

$\cos\omega t \cancel{\cos 90}^{0} - \sin\omega t \sin 90°$ (1)

$0 - \sin\omega t$
$= -\sin\omega t$

Sinusoid-Phasor Transformation

10.7 Transform each of the following functions of time into phasor form: (*a*) $-5\sin(580t - 110°)$; (*b*) $3\cos 600t - \boxed{5\sin(600t + 110°)}$; (*c*) $8\cos(4t - 30°) + 4\sin(4t - 100°)$. *Hint:* First convert each into a single cosine function with a positive magnitude.

$\Rightarrow 5\cos(600t + 110° - 90°)$

$A + (-B)$ ✓
$A - (+B)$

Ans: $5\angle -20°$; $2.41\angle -134.8°$; $4.46\angle -47.9°$.

$$\cos\left(\omega t \pm 90^\circ\right) = \mp\sin\omega t$$

ⓐ $-5\sin(580t - 110°) = 5\left[-\sin(580t - 110°)\right]$

$= 5\cos(580t - 110° + 90°) = 5\cos(580t - 20°) = 5\angle -20°$

ⓑ $3\angle 0° - 5\angle 20° = 2.41\angle -134.8 = (1.698 - j1.71)$

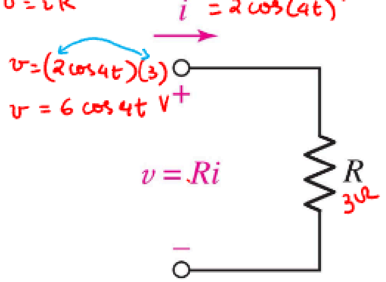
ⓒ $8\angle -30° + 4\angle -100° - 90° = 8\angle -30° + 4\angle -190° = 4.46\angle -47.9°$

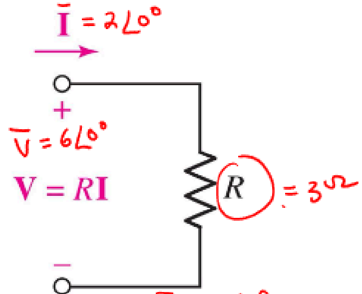
Phasors: The Resistor

In the frequency domain, Ohm's Law takes the same form:

Voltage and current are in phase of a pure resistor

$v = iR$





Ohm's Law $= \dfrac{\bar{V}}{\bar{I}} = \dfrac{6\angle 0°}{2\angle 0°} = 3\angle 0°$
$= 3\Omega$

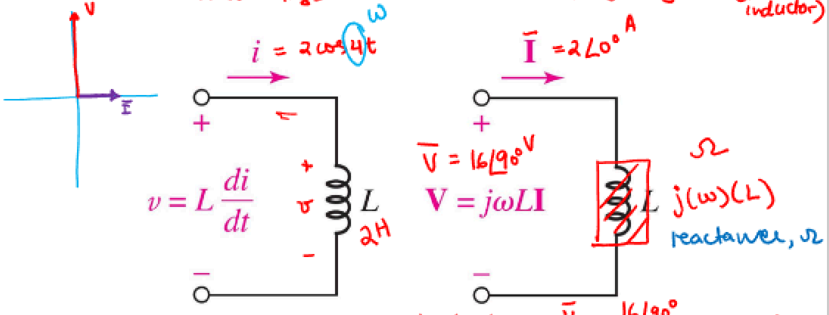
Phasors: The Inductor

i L

$v = L\frac{di}{dt}$

Differentiation in time becomes multiplication in phasor form: (calculus becomes algebra!)

→ current lags behind the voltage by 90° (pure inductor)



$v = 2\frac{d}{dt}(2\cos 4t)$
$= 16[-\sin 4t]$

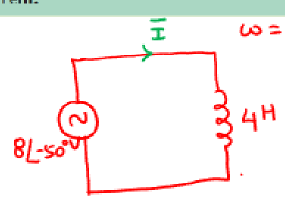
Ohm's law $\frac{\bar{V}}{\bar{I}} = \frac{16\angle 90°}{2\angle 0°} = 8\angle 90° = 8j$

Phasors: The Inductor

$\cos(\omega t \pm 90°) = \mp\sin\omega t$

Apply the voltage $8\angle -50°$ V at a frequency $\omega = 100$ rad/s to a 4 H inductor, and determine the phasor current and the time-domain current.

$\omega = 100$ rad/sec



$\bar{I} = 20\angle -140°$ mA

$i(t) = 0.02\cos(100t - 140°)$ A

$i(t) = 20\cos(100t - 140°)$ mA

$\frac{\bar{V}}{\bar{I}} = j\omega L$

$\frac{8\angle -50°}{\bar{I}} = j(100)(4)$

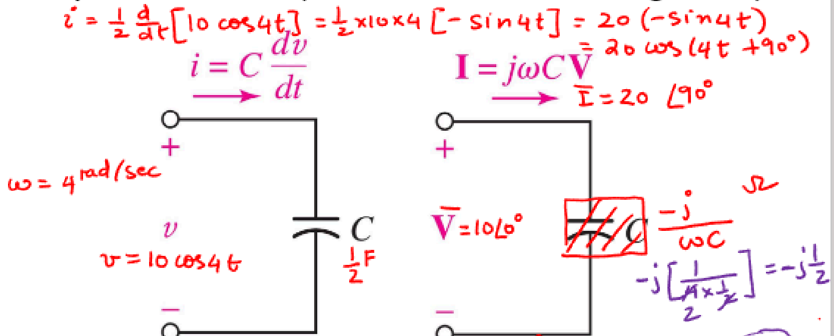
$\bar{I} = \frac{8\angle -50°}{400\angle 90°} = 0.02\angle -140°$ A

Phasors: The Capacitor

$i = C\frac{dv}{dt}$

Differentiation in time becomes multiplication in phasor form: (calculus becomes algebra!)

$i = \frac{1}{2}\frac{d}{dt}[10\cos 4t] = \frac{1}{2}\times 10\times 4[-\sin 4t] = 20(-\sin 4t)$
$= 20\cos(4t + 90°)$



$\frac{\bar{V}}{\bar{I}} = \frac{10\angle 0°}{20\angle 90°} = \frac{1}{2}\angle -90° = -j\frac{1}{2}$ Ω

Summary: Phasor Voltage/Current Relationships

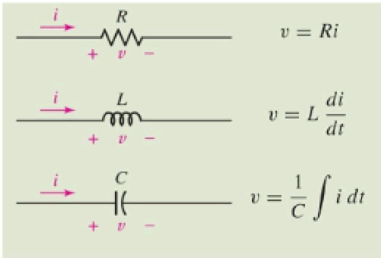
$v(t) = V_m \cos(\omega t + \phi)$

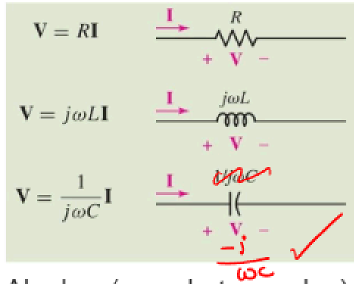
$v(t) = V_m \sin(\omega t + \phi)$

$\bar{V} = V_m \angle \phi$

$\bar{V} = V_m \angle \phi - 90^\circ$

Time Domain | Frequency Domain





Calculus (hard but real)

Algebra (easy but complex)

$j \times j = -1$ $\qquad \dfrac{-j \times j}{(\omega C) \times j} = \dfrac{-(-1)}{j \omega C} = \dfrac{1}{j\omega C}$