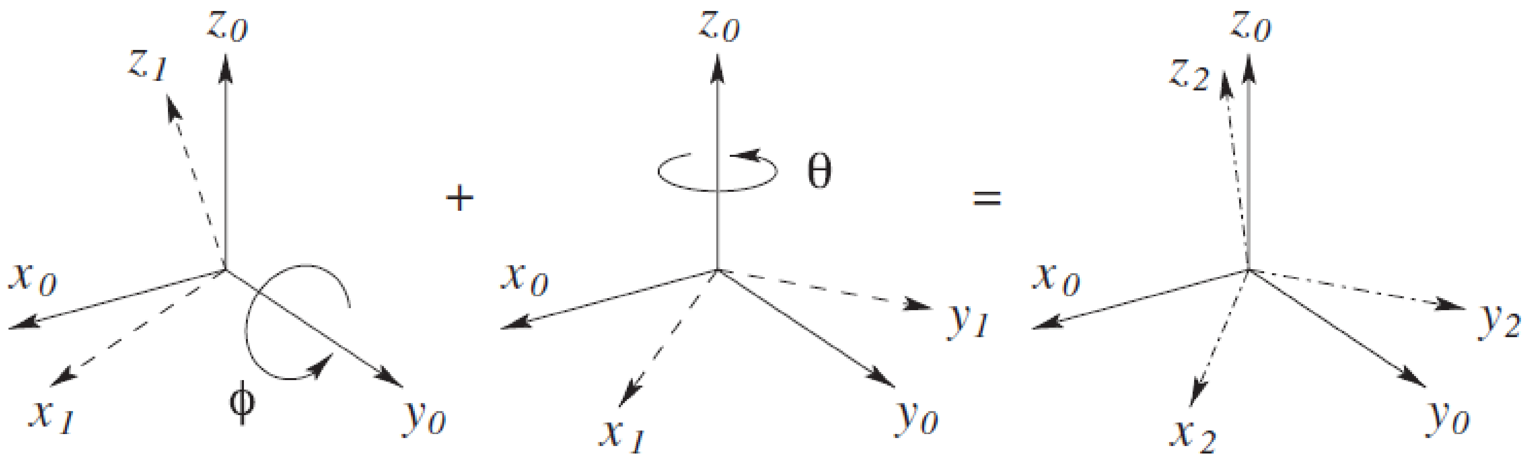
Kinematics and Dynamics of Robots

Module 4

Rotation with Respect to the Fixed Frame

- Many times it is desired to perform a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames. For example, we may wish to perform a rotation about y_0 followed by a rotation about z_0 (and not z_1!). We will refer to $o_0x_0y_0z_0$ as the **fixed frame**. In this case the composition law is different. It turns out that the correct composition law in this case is simply to multiply the successive rotation matrices **in the reverse order**. Note that the rotations themselves are not performed in reverse order. Rather they are performed about the fixed frame instead of about the current frame.



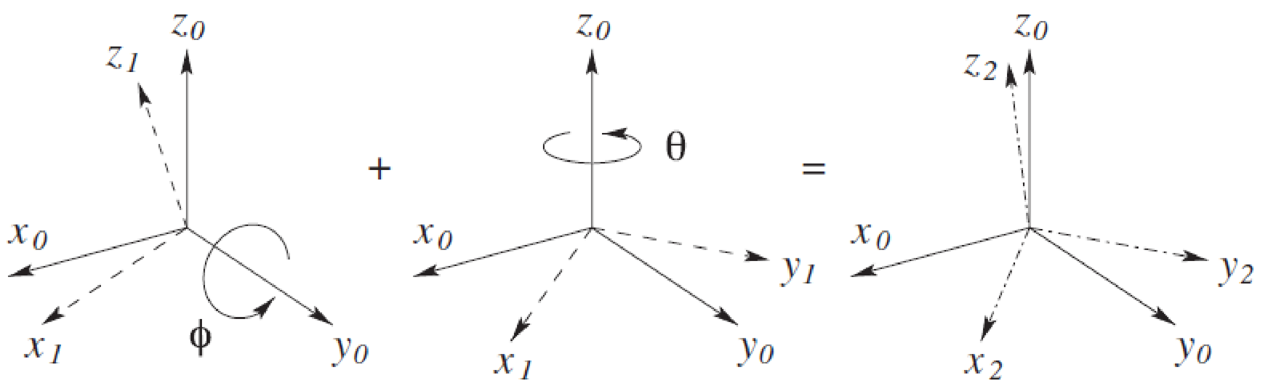
Composition of rotations about fixed axes.

Rotation with Respect to the Fixed Frame

HUMBER

- To see this, suppose we have two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ related by the rotational transformation R_1^0. If $R \in SO(3)$ represents a rotation relative to $o_0x_0y_0z_0$, then the representation for R in the **current** frame $o_1x_1y_1z_1$ is given by $(R_1^0)^{-1}\, R\, R_1^0$. Therefore, applying the composition law for rotations about the current axis yields

$$R_2^0 = R_1^0[(R_1^0)^{-1}\, R\, R_1^0] = R\, R_1^0$$



- Thus, when a rotation R is performed with respect to the world coordinate frame, the current rotation matrix is **premultiplied** by R to obtain the desired rotation matrix.

- *suppose that a rotation matrix R represents a rotation of angle ϕ about y_0 followed by a rotation of angle θ about the fixed z_0. The second rotation about the fixed axis is given by $R_{y,-\phi}R_{z,\theta}R_{y,\phi}$, which is the basic rotation about the z-axis expressed relative to the frame $o_1x_1y_1z_1$ using a similarity transformation. Therefore, the composition rule for rotational transformations gives us*

$$R = R_{y,\phi}\, [R_{y,-\phi}R_{z,\theta}\, R_{y,\phi}] = R_{z,\theta}R_{y,\phi}$$

Rules for Composition of Rotations

- We can summarize the rule of composition of rotational transformations by the following recipe. Given a fixed frame $o_0x_0y_0z_0$ and a current frame $o_1x_1y_1z_1$, together with rotation matrix R_1^0 relating them, if a third frame $o_2x_2y_2z_2$ is obtained by a rotation R performed relative to the **current frame** then **postmultiply** R_1^0 by R_2^1 to obtain

$$R_2^0 = R_1^0 R_2^1$$

- If the second rotation is to be performed relative to the **fixed frame** then it is both confusing and inappropriate to use the notation R_2^1 to represent this rotation. Therefore, if we represent the rotation by R, we **premultiply** R_1^0 by R to obtain

$$R_2^0 = R R_1^0$$

- In each case R_2^0 represents the transformation between the frames $o_0x_0y_0z_0$ and $o_2x_2y_2z_2$. The frame $o_2x_2y_2z_2$ that results from the first one is different from that resulting from the second one.
- Using the above rule for composition of rotations, it is an easy matter to determine the result of multiple sequential rotational transformations.

Rules for Composition of Rotations

- *Suppose R is defined by the following sequence of basic rotations in the order specified:*
  1. *A rotation of θ about the current x-axis*
  2. *A rotation of ϕ about the current z-axis*
  3. *A rotation of α about the fixed z-axis*
  4. *A rotation of β about the current y-axis*
  5. *A rotation of δ about the fixed x-axis*
- *Determine the cumulative effect of these rotations.*
- *Solution: In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ and pre- or postmultiply as the case may be to obtain*

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

Parameterizations of Rotations

- The nine elements r_{ij} in a general rotational transformation $R \in SO(3)$ are not independent quantities. Indeed, a rigid body possesses at most three rotational degrees of freedom, and thus at most three quantities are required to specify its orientation. This can be easily seen by examining the constraints that govern the matrices in SO(3):

  - *The columns of a rotation matrix are unit vectors, therefore:*

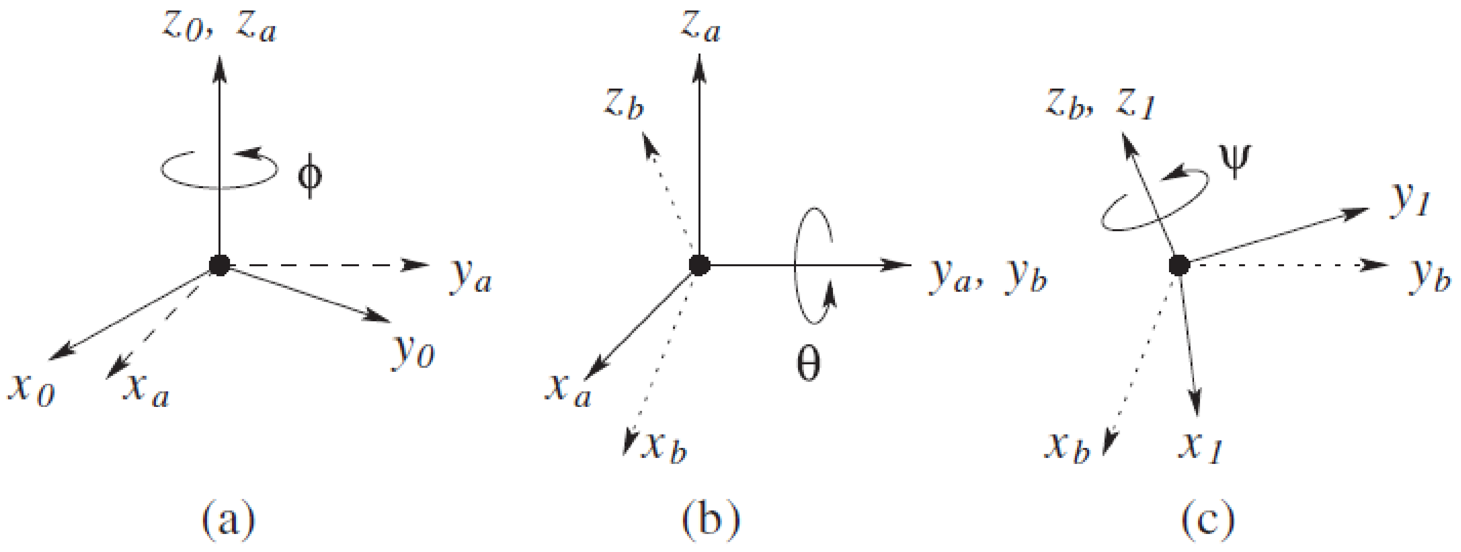
$$\sum_i r_{ij}^2 = 1, \quad j \in \{1,2,3\}$$

  - *Columns of a rotation matrix are mutually orthogonal*

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \quad i \neq j$$

- Together, these constraints define six independent equations with nine unknowns, which implies that there are three free variables. we derive three ways in which an arbitrary rotation can be represented using only three independent quantities: the **Euler angle** representation, the **roll-pitch-yaw** representation, and the **axis-angle** representation.

A common method of specifying a rotation matrix in terms of three independent quantities is to use the so-called **Euler angles**. Consider the fixed coordinate frame $o_0x_0y_0z_0$ and the rotated frame $o_1x_1y_1z_1$ shown here. We can specify the orientation of the frame $o_1x_1y_1z_1$ relative to the frame $o_0x_0y_0z_0$ by three angles $(\emptyset, \theta, \psi)$, known as Euler angles, and obtained by three successive rotations as follows. First rotate about the *z*-axis by the angle $\emptyset$. Next rotate about the current *y*-axis by the angle θ. Finally rotate about the current *z*-axis by the angle ψ. frame $o_ax_ay_az_a$ represents the new coordinate frame after the rotation by $\emptyset$, frame $o_bx_by_bz_b$ represents the new coordinate frame after the rotation by θ, and frame $o_1x_1y_1z_1$ represents the final frame, after the rotation by ψ. Frames $o_ax_ay_az_a$ and $o_bx_by_bz_b$ are shown in the figure only to help visualize the rotations.



Euler Angles

In terms of the basic rotation matrices the resulting rotational transformation can be generated as the product

$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

- The matrix R_{ZYZ} in is called the **ZYZ-Euler angle transformation**.

The more important and more difficult problem is to determine for a particular R = (r_{ij}) the set of Euler angles φ, θ, and ψ, that satisfy

$$R = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- To find a solution for this problem we break it down into two cases. First, suppose that not both of r_{13}, r_{23} are zero. Then we deduce that $s_\theta \neq 0$, and hence that not both of r_{31}, r_{32} are zero. If not both r_{13} and r_{23} are zero, then $r_{33} \neq \pm 1$, and we have $c_\theta = r_{33}$, $s_\theta = \pm\sqrt{1 - r_{33}^2}$ so

$$\theta = \operatorname{Atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right)$$

$$\theta = \operatorname{Atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right)$$

- where the function Atan2 is the **two-argument arctangent function** .

Euler Angles

If we choose the value for θ given by the first equation, then $s_\theta > 0$, and

$$\begin{aligned} \phi &= \text{Atan2}(r_{13}, r_{23}) \\ \psi &= \text{Atan2}(-r_{31}, r_{32}) \end{aligned}$$

If we choose the value for θ given by the second equation, then $s_\theta < 0$, and

$$\begin{aligned} \phi &= \text{Atan2}(-r_{13}, -r_{23}) \\ \psi &= \text{Atan2}(r_{31}, -r_{32}) \end{aligned}$$

- If $r_{13} = r_{23} = 0$, then the fact that R is orthogonal implies that $r_{33} = \pm 1$, and that $r_{31} = r_{32} = 0$. Thus, R has the form

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

If r_{33} = 1, then c_θ = 1 and s_θ = 0, so that θ = 0. In this case, Equation $\underline{R_{zyz}}$ becomes

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the sum $\phi + \psi$ can be determined as

$$\phi + \psi = \mathrm{Atan2}(r_{11}, r_{21}) = \mathrm{Atan2}(r_{11}, -r_{12})$$

Since only the sum $\phi + \psi$ can be determined in this case, there are infinitely many solutions. In this case, we may take ϕ = 0 by convention. If r_{33} = -1, then c_θ = -1 and s_θ = 0, so that θ = π. In this case Equation $\underline{R_{zyz}}$ becomes

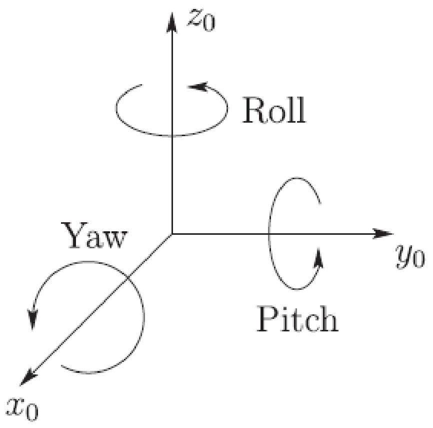
$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The solution is thus

$$\phi - \psi = \mathrm{Atan2}(-r_{11}, -r_{12})$$

As before there are infinitely many solutions.

A rotation matrix R can also be described as a product of successive rotations about the principal coordinate axes x_0, y_0, and z_0 taken in a specific order. These rotations define the **roll, pitch**, and **yaw** angles, which we shall also denote ϕ, θ, ψ.



We specify the order of rotation as x - y - z, in other words, first a yaw about x_0 through an angle ψ, then pitch about the y_0 by an angle θ, and finally roll about the z_0 by an angle ϕ. Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by

$$
\begin{aligned}
R &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
$$

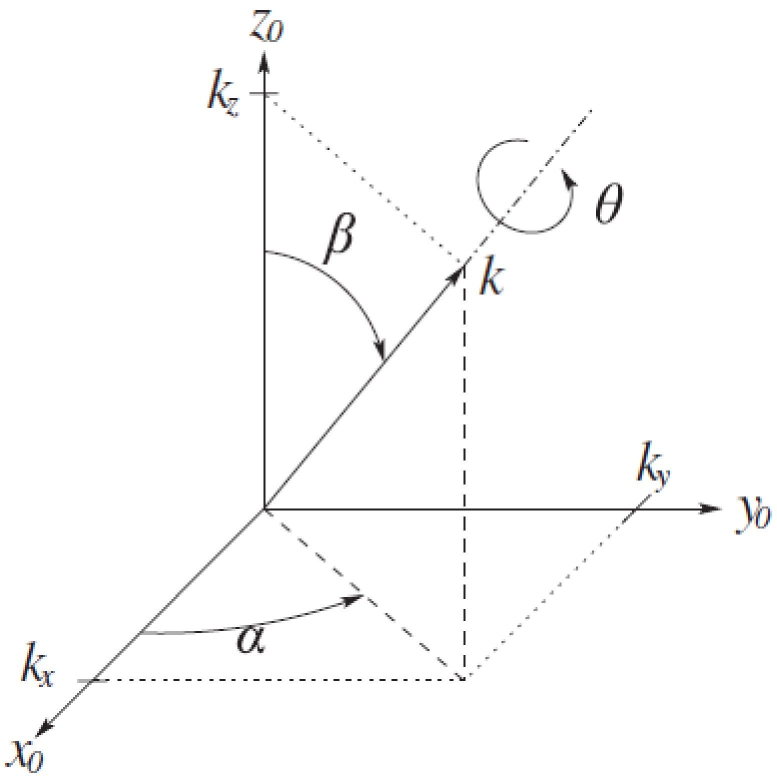
Roll, Pitch, Yaw Angles

Instead of yaw-pitch-roll relative to the fixed frames we could also interpret the above transformation as roll-pitch-yaw, in that order, each taken with respect to the current frame. The end result is the same matrix.

The three angles ϕ, θ, and ψ can be obtained for a given rotation matrix using a method that is similar to that used to derive the Euler angles.

Axis-Angle Representation

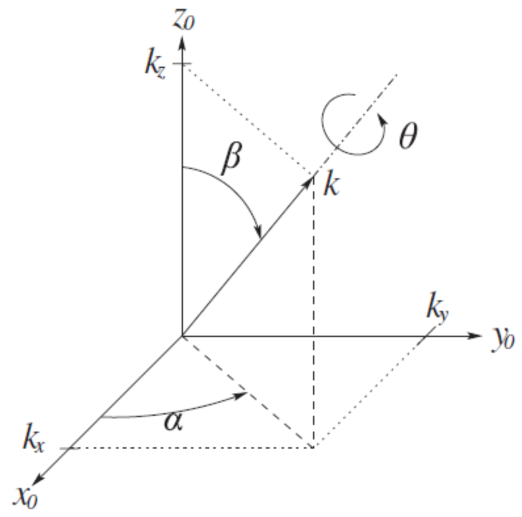
- Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices. Let $k = (k_x, k_y, k_z)$, expressed in the frame $o_0x_0y_0z_0$, be a unit vector defining an axis. We wish to derive the rotation matrix $R_{k,\theta}$ representing a rotation of θ about this axis.



- There are several ways in which the matrix $R_{k,\theta}$ can be derived. One approach is to note that the rotational transformation $R = R_{z,\alpha}R_{y,\beta}$ will bring the world z-axis into alignment with the vector k. Therefore, a rotation about the axis k can be computed using a similarity transformation as

$$R_{k,\theta} = R\, R_{z,\theta}\, R^{-1}$$

$$= R_{z,\alpha}R_{y,\beta}R_{z,\theta}R_{y,-\beta}R_{z,-\alpha} \qquad (1)$$



From we see that

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2+k_y^2}}, \quad \cos\alpha = \frac{k_x}{\sqrt{k_x^2+k_y^2}} \qquad (2)$$

$$\sin\beta = \sqrt{k_x^2+k_y^2}, \quad \cos\beta = k_z \qquad (3)$$

Note that the final two equations follow from the fact that k is a unit vector. Substituting Equations 2 and 3 into Equation 1, we obtain after some lengthy calculation

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right]$$

where v_θ = vers θ = 1 - c_θ.

Axis-Angle Representation

In fact, any rotation matrix $R \in SO(3)$ can be represented by a single rotation about a suitable axis in space by a suitable angle,

$$R = R_{k,\theta}$$

- where k is a unit vector defining the axis of rotation, and θ is the angle of rotation about k. The pair (k, θ) is called the **axis-angle representation** of R. Given an arbitrary rotation matrix R with components r_{ij}, the equivalent angle θ and equivalent axis k are given by the expressions

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

- These equations can be obtained by direct manipulation of the entries of the matrix given in Equation 2.6t. The axis-angle representation is not unique since a rotation of -θ about -k is the same as a rotation of θ about k, that is,

$$R_{k,\theta} = R_{-k,-\theta}$$

- If θ = 0 then R is the identity matrix and the axis of rotation is undefined.

Example

- *Suppose R is generated by a rotation of 90º about z0 followed by a rotation of 30º about y0 followed by a rotation of 60º about x0. Find R.*

HUMBER

Questions?