Day 2

July 6, 2023 9:37 AM

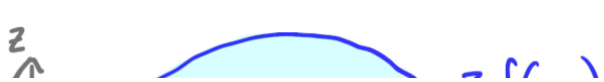
Chapter 9

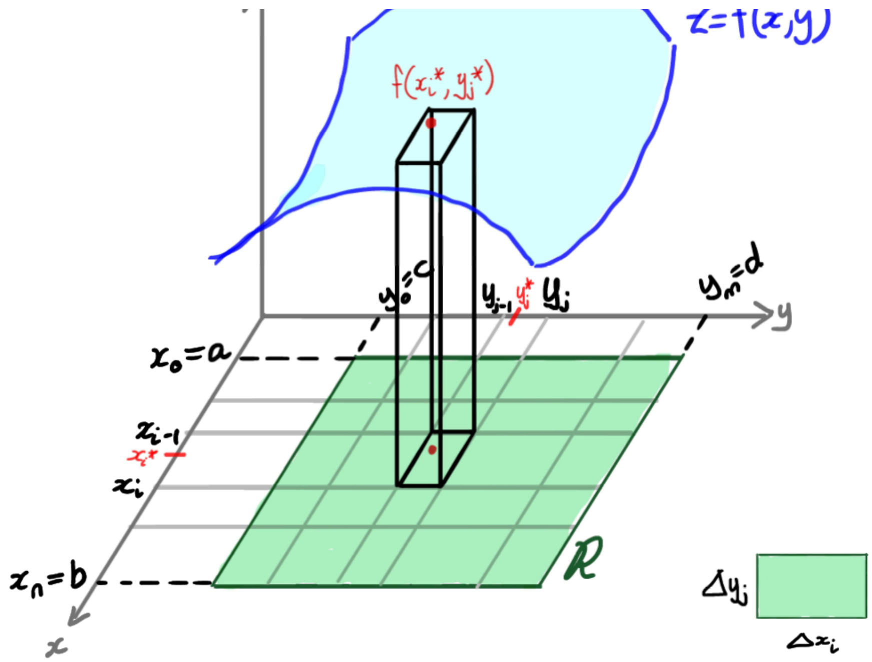
Double Integrals in Rectangular Coordinates

9.1 Readings

1. CLP III §3.1 Double Integrals

What is $\iint_R f(x,y)\,dA$?





area of ij^{th} subrectangle
$\Delta A_{ij} = \Delta x_i \Delta y_j$

$$\text{Total} \approx \sum_i \sum_j \underbrace{f(x_i^*, y_j^*)}_{\text{height}} \underbrace{\Delta x_i \Delta y_j}_{\text{area of base}}$$

When the limit exists, the double integral of f over $\mathcal{R}$ is defined to be

$$\iint_{\mathcal{R}} f(x,y)\, dA = \lim_{\substack{m,n\to\infty \\ \text{size of } ij^{th} \text{ rectangle} \to 0}} \sum_i \sum_j f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

and we say f is integrable over $\mathcal{R}$

9.2 Double Integrals in Rectangular Coordinates

Here are three applications of double integrals:

(i) If f is integrable over a region $\mathcal{R} \subset \mathbb{R}^2$, then

$$\textit{signed volume} \text{ of solid enclosed by graph } z = f(x,y) \text{ and } xy\text{-plane } = \iint_{\mathcal{R}} f\, dA.$$

(ii)

$$\textit{area} \text{ of region } \mathcal{R} \text{ lying in } xy\text{-plane } = \iint_{\mathcal{R}} 1\, dA = \iint_{\mathcal{R}} dA.$$

(iii) If $\rho(x,y)$ denotes the density at a point (x,y) of a thin lamina $\mathcal{L}$ lying in the xy-plane, then the

$$\textit{mass} \text{ of the thin lamina } \mathcal{L} = \iint_{\mathcal{L}} \rho(x,y)\, dA.$$

There are other applications of double integrals too.[1] These include:

- finding the area of a portion of the surface described by $z = f(x,y)$;

- calculating the *average value* of a function over a region $\mathcal{R}$;
- calculating the centre of mass $(\overline{x}, \overline{y})$ of a plate with varying density over a region $\mathcal{R} \subseteq \mathbb{R}^2$; and
- calculating the moment of inertia of a plate filling a region $\mathcal{R} \subseteq \mathbb{R}^2$ about a given axis.

[1]These are covered in §3.3 and §3.4 in the FRY CLP-3 textbook.

FRY Thm III.3.1.6, Double Integrals are "linear" functions

Theorem 9.1. Let f and g be integrable functions over $\mathcal{R} \subseteq \mathbb{R}^2$, and let $c \in \mathbb{R}$. Then

$$\iint_{\mathcal{R}} (f+g)\ dA = \iint_{\mathcal{R}} f\ dA + \iint_{\mathcal{R}} g\ dA,$$

and

$$\iint_{\mathcal{R}} (cf)\ dA = c \iint_{\mathcal{R}} f\ dA.$$

FRY Thm III.3.1.6, Splitting up the region of integration

Theorem 9.2. Let f be integrable over $\mathcal{R} \subseteq \mathbb{R}^2$, and suppose $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ do not overlap except possibly on their boundaries. Then

$$\iint_{\mathcal{R}} f\ dA = \iint_{\mathcal{R}_1} f\ dA + \iint_{\mathcal{R}2} f\ dA.$$

FRY Thm III.3.1.8, Inequalities for Integrals

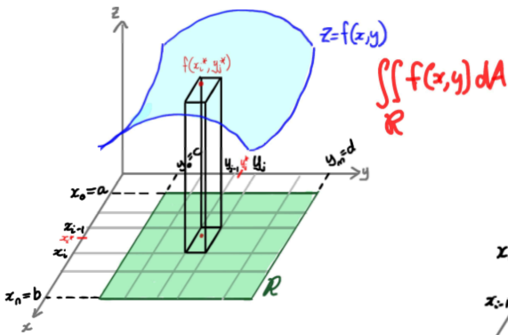
Theorem 9.3. Let f and g be integrable functions over a region $\mathcal{R} \subseteq \mathbb{R}^2$.

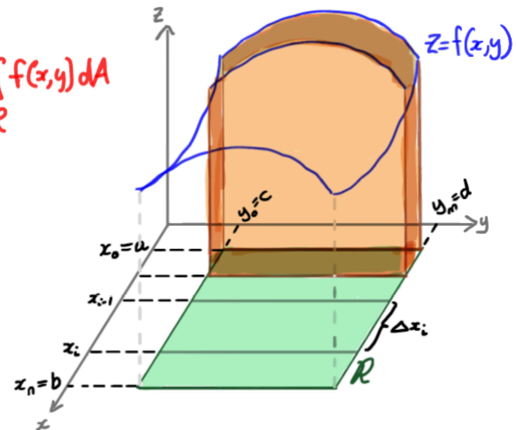
(a) If $f(x,y) \geq 0$ for all $(x,y) \in \mathcal{R}$, then $\iint_{\mathcal{R}} f(x,y)\ dA \geq 0$.

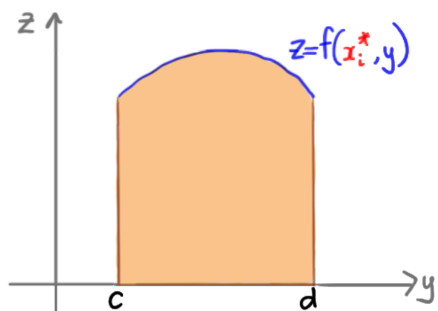
(b) If there are constants m and M such that $m \leq f(x,y) \leq M$ for all $(x,y) \in \mathcal{R}$, then $m \cdot \text{Area}(\mathcal{R}) \leq \iint_{\mathcal{R}} f(x,y) dA \leq M \cdot \text{Area}(\mathcal{R})$.

(c) If $f(x,y) \leq g(x,y)$ for all $(x,y) \in \mathcal{R}$, then $\iint_{\mathcal{R}} f(x,y)\ dA \leq \iint_{\mathcal{R}} g(x,y)\ dA$.

(d) Also, $\left| \iint_{\mathcal{R}} f(x,y)\ dA \right| \leq \iint_{\mathcal{R}} |f(x,y)|\ dA.$







i^{th} slice has area $\displaystyle\int_{y=c}^{y=d} f(x_i^*,y)\,dy$

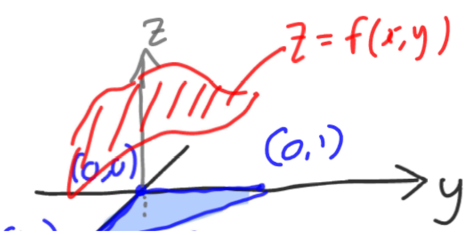
and width Δx_i

$$\text{So, total} \approx \sum_i \left(\int_{y=c}^{y=d} f(x_i^*,y)\,dy\right)\Delta x_i$$

$$\iint_R f\,dA = \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$$

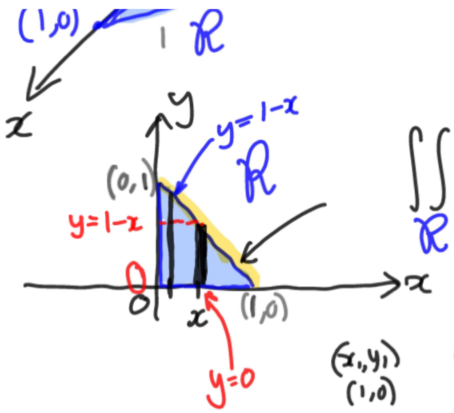
iterated integral through "vertical" slices

$$\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$



$f(x,y) = x^3 y^2$

Goal: Evaluate $\iint f(x,y)\,dA$



R

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1} \boxed{\int_{y=0}^{y=1-x} x^3 y^2\,dy}\,dx$$

iterated integral

$$\boxed{\int_0^1 \boxed{\int_0^{1-x} x^3 y^2\,dy}\,dx}$$

integrate with respect to y

(x_1,y_1) (x_2,y_2)

$(1,0)$ $(0,1)$

$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-0}{0-1}=\frac{1}{-1}=-1$

$y-y_1=m(x-x_1)$

$y-0=-1(x-1)$

$y=-x+1$

$$=\int_0^1 \frac{1}{3}x^3y^3\Big|_0^{1-x}\,dx$$

$$=\int_0^1\left[\frac{1}{3}x^3(1-x)^3-\cancel{\frac{1}{3}x^3\cdot 0^3}\right]dx$$

0

$$=\int_0^1 \frac{1}{3}x^3\left(1-3x+3x^2-x^3\right)dx$$

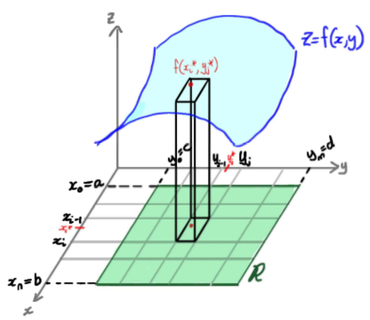
$$=\int_0^1\left(\frac{1}{3}x^3-x^4+x^5-\frac{1}{3}x^6\right)dx$$

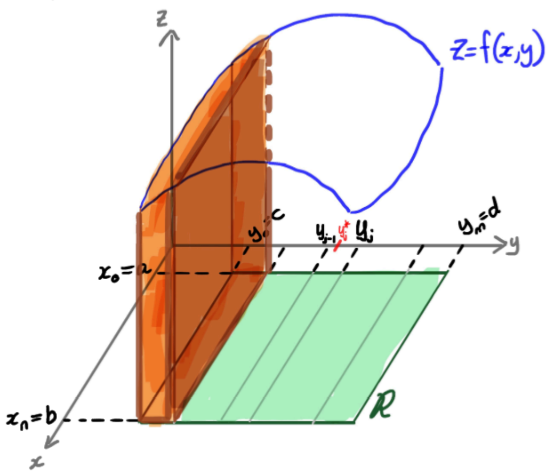
$$=\left[\frac{1}{3}\cdot\frac{1}{4}x^4-\frac{1}{5}x^5+\frac{1}{6}x^6-\frac{1}{3}\cdot\frac{1}{7}x^7\right]_0^1$$

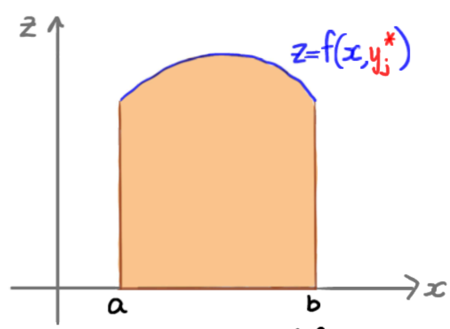
$$=\left[\frac{1}{12}(1)^4-\frac{1}{5}(1)^5+\frac{1}{6}(1)^6-\frac{1}{21}(1)^7\right]-\left[\frac{1}{12}(0)^4-\frac{1}{5}(0)^5+\frac{1}{6}(0)^6-\frac{1}{21}(0)^7\right]$$

$$=\frac{1}{12}-\frac{1}{5}+\frac{1}{6}-\frac{1}{21}$$

$$=\frac{1}{420}$$







j^{th} slice has area $\displaystyle\int_{x=b}^{x=a} f(x,y_j^*)\,dx$

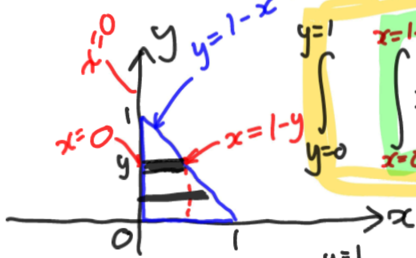
and width Δy_j

$$\text{So, total} \approx \sum_j \left(\int_{x=b}^{x=a} f(x,y_j^*)\,dx \right) \Delta y_j$$

$$\iint_R f\,dA = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\,dx \right) dy$$

iterated integral through "horizontal" slices

$$\iint f\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$



$\rightarrow$ So $x = 1-y$

$$\int_{y=0}^{y=1} \boxed{\int_{x=0}^{x=1-y} x^3y^2\,dx}\,dy = \int_0^1 \int_0^{1-y} x^3y^2\,dx\,dy$$

$$= \int_{y=0}^{y=1} \frac{1}{4}x^4y^2 \Big|_0^{1-y} dy$$

$$= \int_{y=0}^{y=1} \left[\frac{1}{4}(1-y)^4y^2 - 0\right] dy$$

$$= \frac{1}{4}\int_0^1 (1-4y+6y^2-4y^3+y^4)y^2\,dy$$

$$= \frac{1}{4}\int_0^1 \left(y^2-4y^3+6y^4-4y^5+y^6\right)dy$$

$$= \frac{1}{4}\left[\frac{1}{3}y^3 - y^4 + \frac{6}{5}y^5 - \underset{\uparrow\,\frac{4}{6}}{\frac{2}{3}}y^6 + \frac{1}{7}y^7\right]_0^1$$

$$= \frac{1}{4}\left(\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right)$$

$$= \frac{1}{420}$$

Shifrin Cor. 7.2.2, Fubini's Theorem

Theorem 9.4. Suppose f is continuous on the rectangle $\mathcal{R} = [a,b] \times [c,d]$. Then,

$$\iint_{\mathcal{R}} f\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Keep in mind that

$$\int_a^b \int_c^d f(x,y)\, dy\, dx \quad \text{is} \quad \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\, dy \right) dx,$$

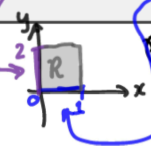
and

$$\int_c^d \int_a^b f(x,y)\, dx\, dy \quad \text{is} \quad \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\, dx \right) dy.$$

When computing such integrals, we work out the "inner" integral first.

Example 9.5. (FRY Exercise III.3.1.7.3a)
Evaluate $\iint_{\mathcal{R}} (x^2 + y^2)\, dA$ where $\mathcal{R} = [0,1] \times [0,2]$.

$$\iint_{R} (x^2+y^2)\, dA$$



$$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$$

$$= \int_0^1 \int_0^2 (x^2+y^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{1}{3} y^3 \right]_0^2 dx$$

$$= \int_0^1 \left(2x^2 + \frac{8}{3} \right) dx$$

$$= \left[\frac{2}{3} x^3 + \frac{8}{3} x \right]_0^1$$

$$= \frac{2}{3} + \frac{8}{3} = \frac{10}{3}$$

Same

$$\int_0^2 \int_0^1 (x^2+y^2)\, dx\, dy$$

$$= \int_0^2 \left[\frac{1}{3} x^3 + x y^2 \right]_0^1 dy$$

$$= \int_0^2 \left(\frac{1}{3} + y^2 \right) dy$$

$$= \left[\frac{1}{3} y + \frac{1}{3} y^3 \right]_0^2$$

$$= \left(\frac{2}{3} + \frac{8}{3} \right) - (0+0)$$

$$= \frac{10}{3}$$

$$R = \{ (x,y) : a \le x \le b,\ c \le y \le d \}$$

Note:

If the bounds of integration are constants, you can integrate in any $\boxed{\text{order}}$ you want.

$$\int_c^d \int_a^b f(x,y)\, \boxed{dx\, dy} = \int_a^b \int_c^d f(x,y)\, \boxed{dy\, dx}$$

The following result can help us compute double integrals quickly at times.

FRY Thm III.3.1.7

Theorem 9.6. Suppose

- $f(x,y)$ is continuous on the rectangle $\mathcal{R} = [a,b] \times [c,d] = \{(x,y): \ a \le x \le b,\ c \le y \le d\}$, and
- the function $f(x,y)$ can be written as the product $f(x,y) = g(x)h(y)$.

Then,

$$\iint_{\mathcal{R}} f(x,y)\ dA = \left(\int_a^b g(x)\ dx\right)\left(\int_c^d h(y)\ dy\right).$$

Example 9.7. Find the mass of a thin lamina described by the region $\mathcal{L} = \{(x,y): \ 0 \le x \le 1,\ 0 \le y \le 2\}$ whose density is given by the function $\rho(x,y) = 3x^2y$.