HUMBER ENGINEERING

MENG 3510 – Control Systems
LECTURE 3

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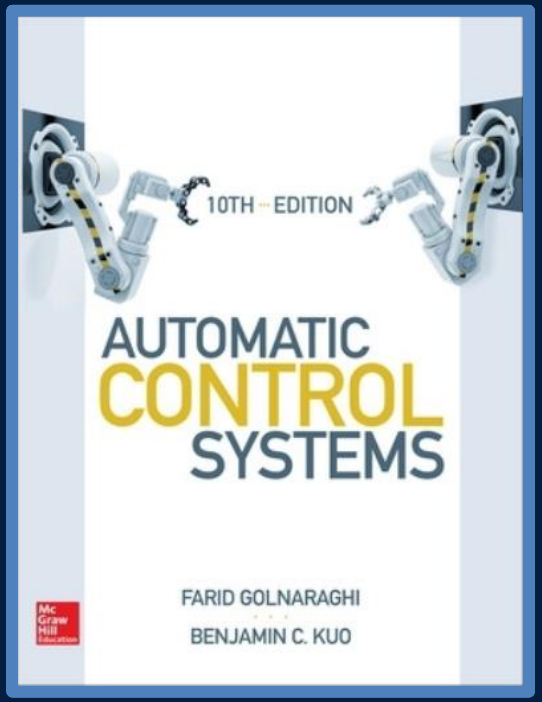
LECTURE 3
Block Diagrams & Signal Flow Graphs

- Block Diagram Representation
  - Series, Parallel & Feedback Connections
- Block Diagram Reduction Techniques
  - Moving a Comparator
  - Moving a Branch Point
- Signal Flow Graph
  - Definitions of Signal Flow Graph Terms
  - Mason’s Gain Formula
  - SFG & State-space Equations
- Case Study: Antenna Control System



Chapter 4

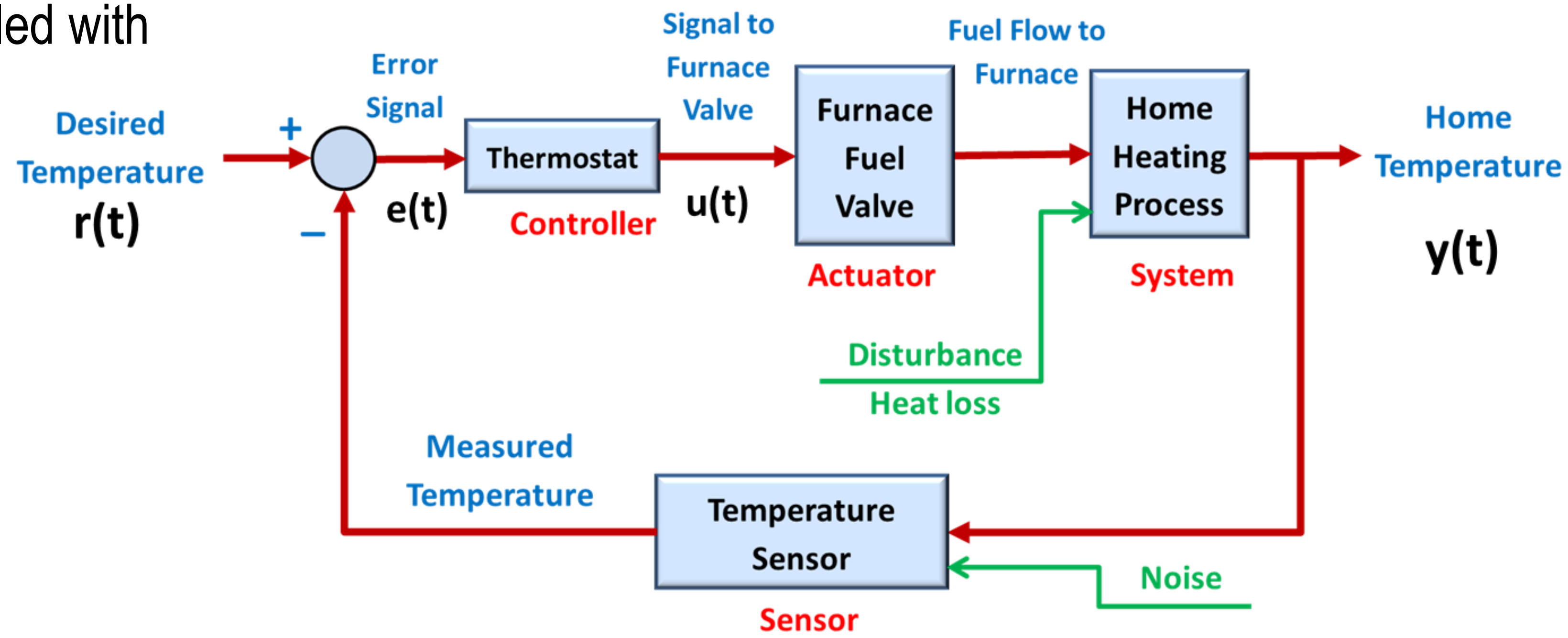
Block Diagram Models

- **Block diagram representation**, is usually used by control engineers to represent control systems because of its simplicity and versatility to show the interconnection of the system components.
- It provides a graphical approach to describe how components of a control system interact.

- **Basic elements** of block diagram representation
  - Rectangles $\rightarrow$ Subsystems transfer functions: G(s), H(s), ….
  - Arrows $\rightarrow$ Signal flow directions : u(t), e(t), y(t) …, U(s), E(s), Y(s), ….
  - Circles $\rightarrow$ Comparators to add or subtract signals

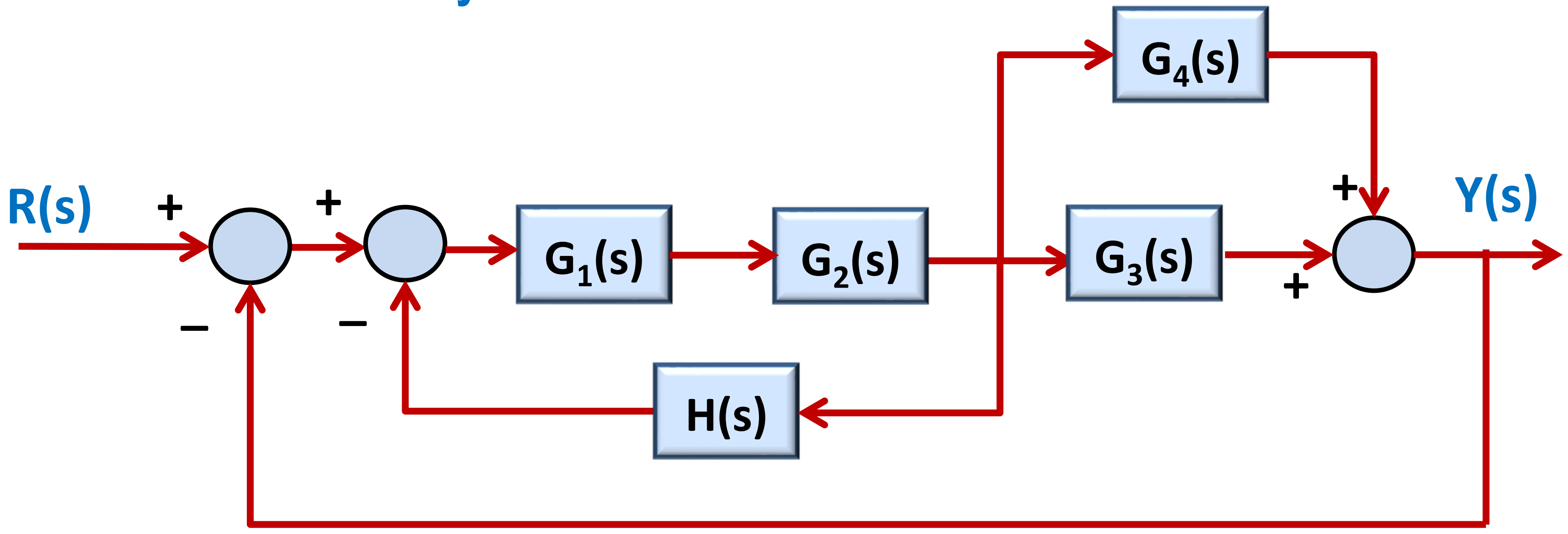
- Each subsystem is represented by a function block, labeled with the corresponding transfer function

$$\boxed{Y(s) = G(s)U(s)}$$



Block Diagram Models

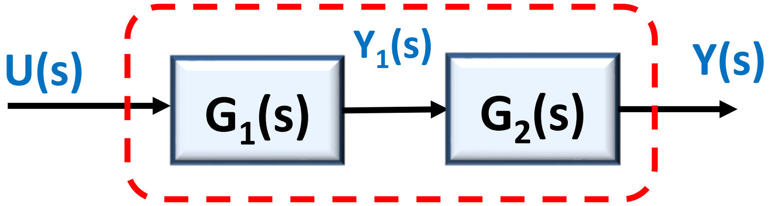
- Control systems may consist of several **interconnected subsystems.**
  - **Series Connection**
  - **Parallel Connection**
  - **Feedback Connection**

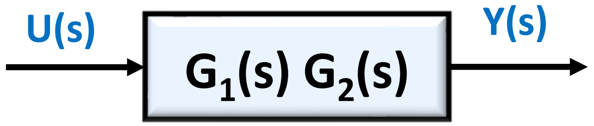


- How to simplify a complicated block diagram and determine the overall transfer function?
- There are two general approaches:
  - ✓ **Applying block diagram reduction techniques**
  - ✓ **Mason's Formula based on Signal Flow Graphs**

Block Diagram Models

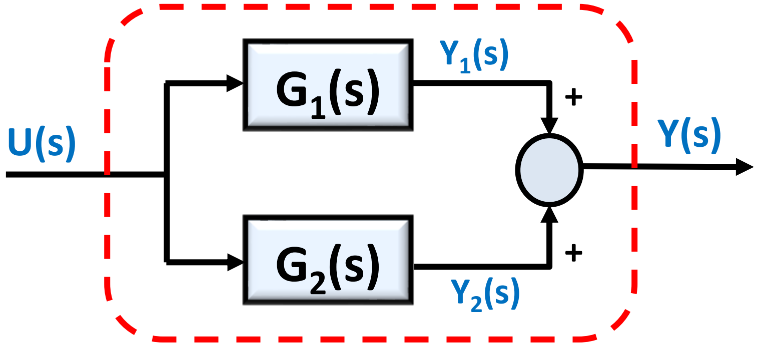
- **Series Connection:**



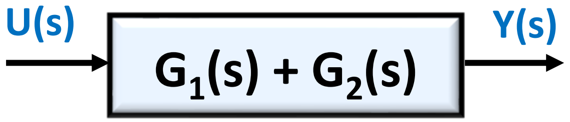


$$\frac{Y(s)}{U(s)} = G_1(s)G_2(s)$$

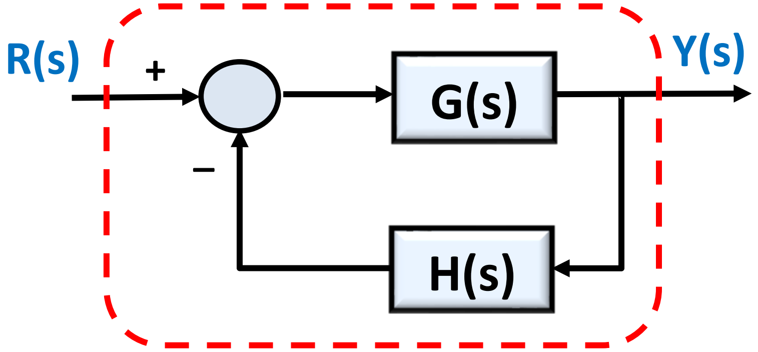
- **Parallel Connection:**



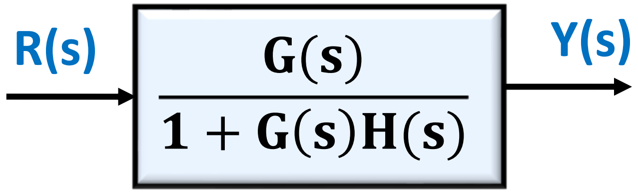
$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$



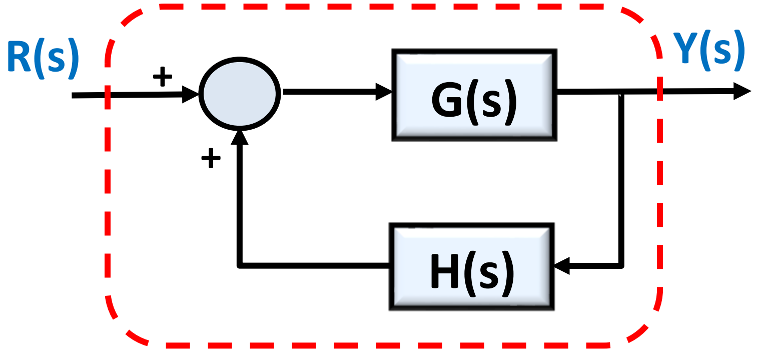
- **Feedback Connection (Negative Feedback):**



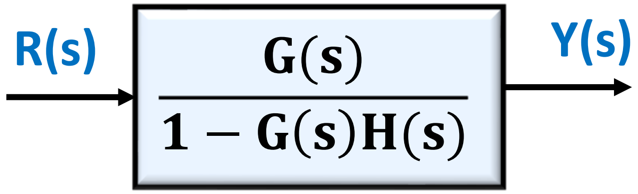
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$



- **Feedback Connection (Positive Feedback):**

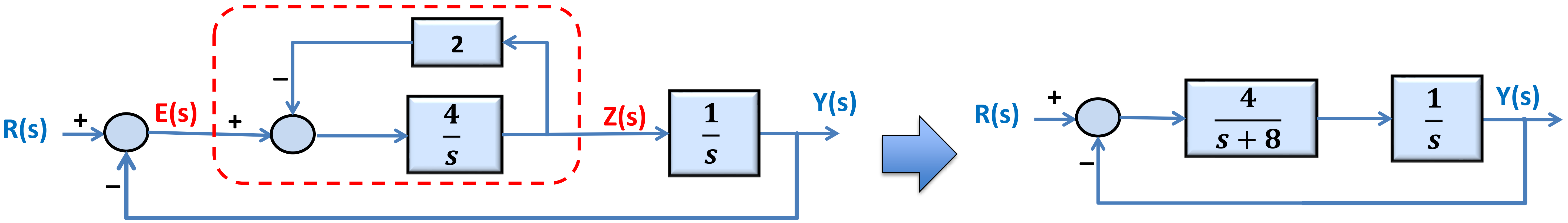


$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$



Block Diagram Models

Example 1 Find the closed-loop transfer function from Y(s) to R(s).

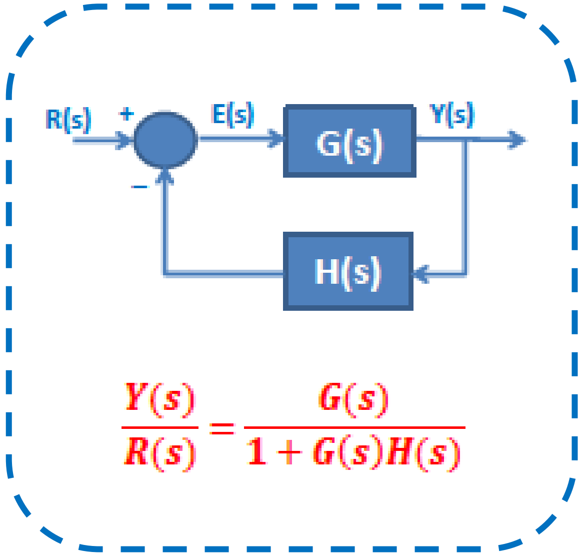


First determine the transfer function of internal feedback loop from Z(s) to E(s):

$$\frac{Z(s)}{E(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{4}{s}}{1 + \left(\frac{4}{s}\right)(2)} = \frac{\frac{4}{s}}{1 + \frac{8}{s}} = \frac{\frac{4}{s}}{\frac{s+8}{s}} = \frac{4}{s+8}$$

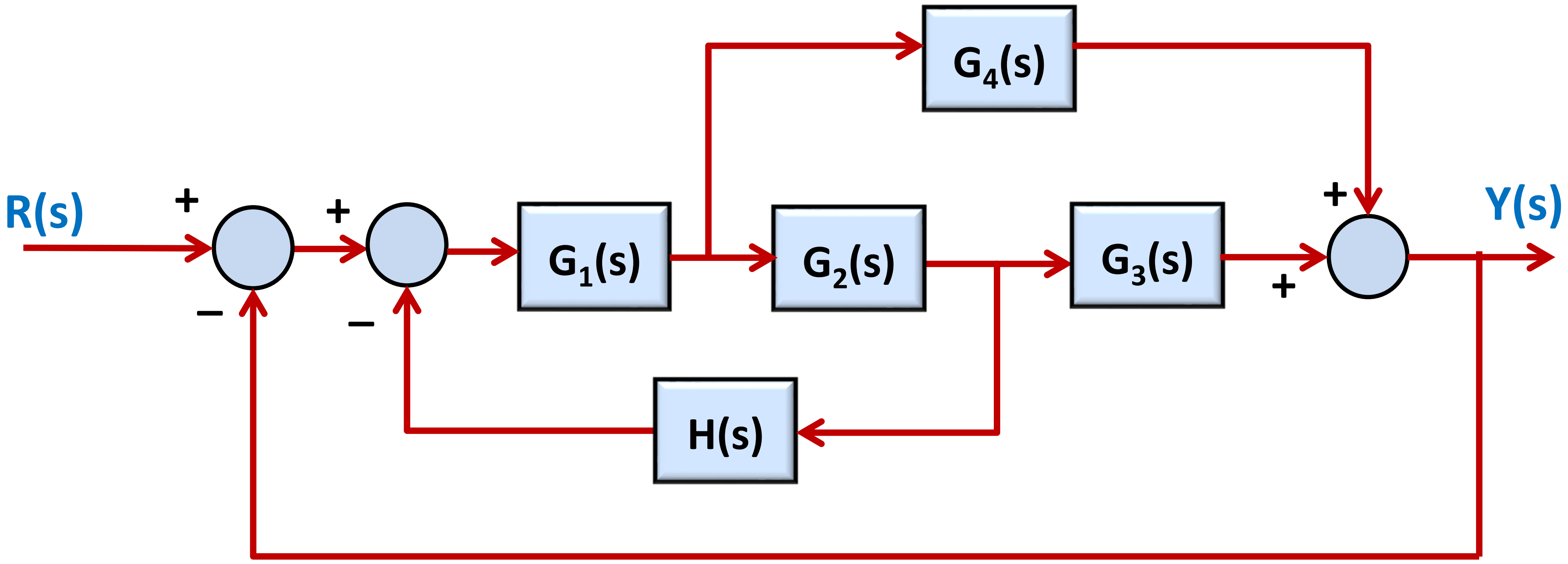
Thus, the overall transfer function from Y(s) to R(s) is:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{4}{s(s+8)}}{1 + \frac{4}{s(s+8)}(1)} = \frac{\frac{4}{s(s+8)}}{\frac{s(s+8)+4}{s(s+8)}} = \frac{4}{s^2 + 8s + 4}$$



Block Diagram Models

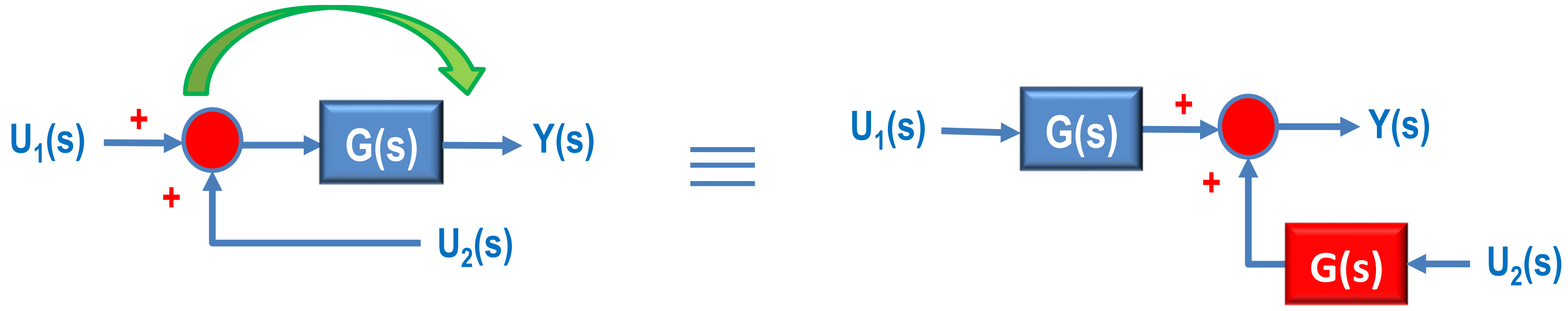
- In some cases, it is required to apply more block diagram transformation techniques to simplify the overall block diagram.



- There are some **block diagram transformation** techniques to simplify the topology of a block diagram.
  - ✓ **Moving a Comparator**
  - ✓ **Moving a Branch Point**

Block Diagram Models

❑ Moving a Comparator Behind a Block



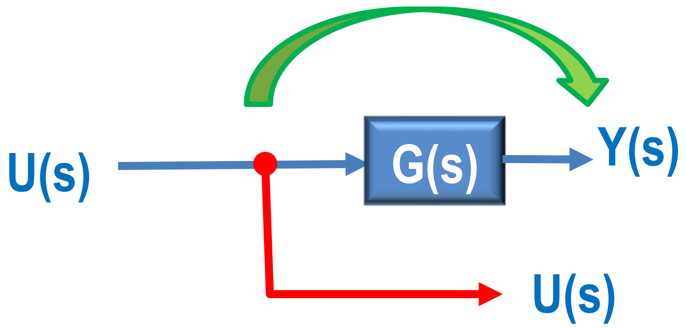
$$Y(s) = G(s)[U_1(s) + U_2(s)] = G(s)U_1(s) + G(s)U_2(s)$$

❑ Moving a Comparator Ahead of a Block

$$Y(s) = G(s)U_1(s) + U_2(s) = G(s)[U_1(s) + \frac{1}{G(s)}U_2(s)]$$

Block Diagram Models

❑ Moving a Branch Behind a Block



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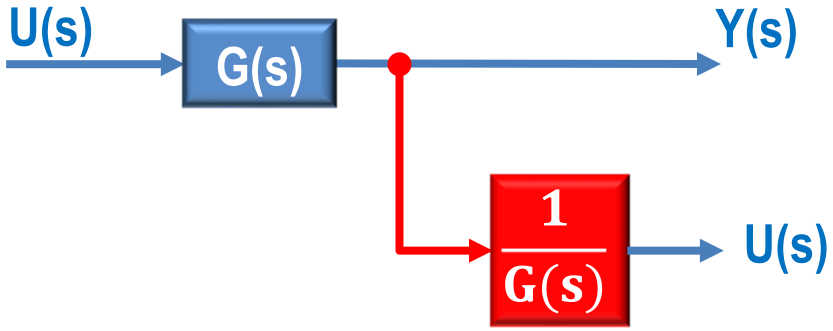
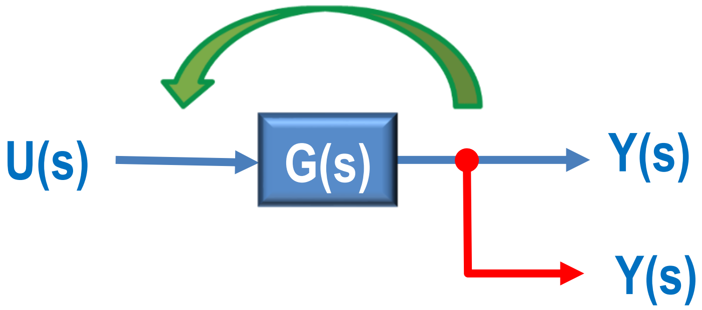


$$Y(s) = G(s)U(s) \;\rightarrow\; U(s) = \frac{1}{G(s)}Y(s)$$

❑ Moving a Branch Ahead of a Block



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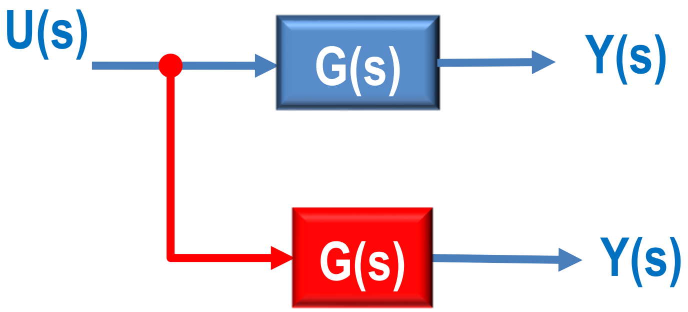
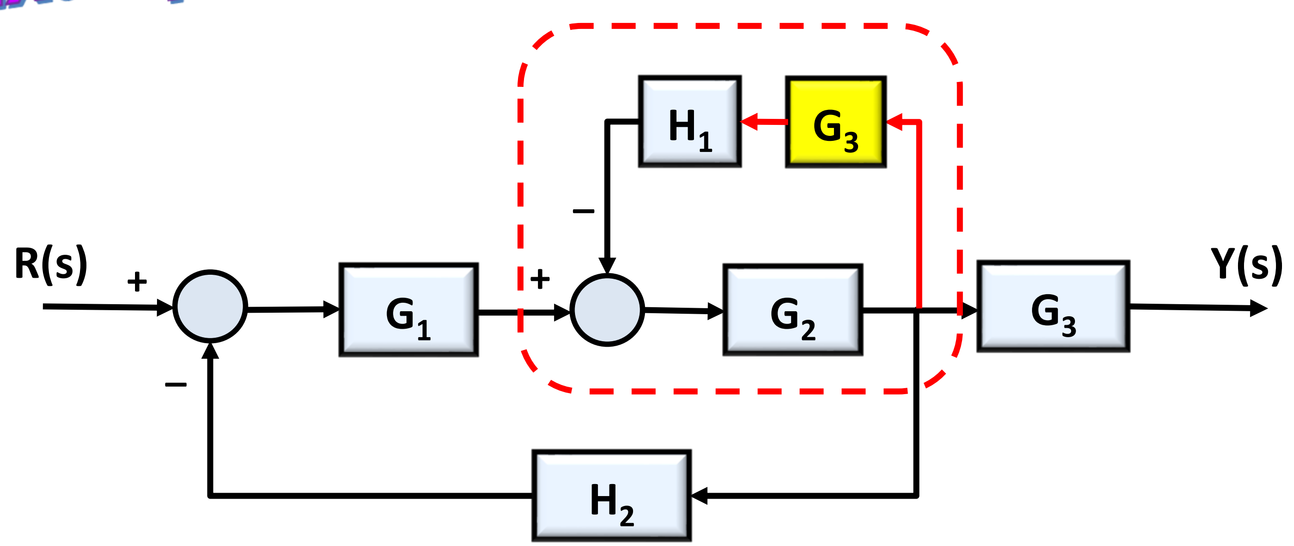


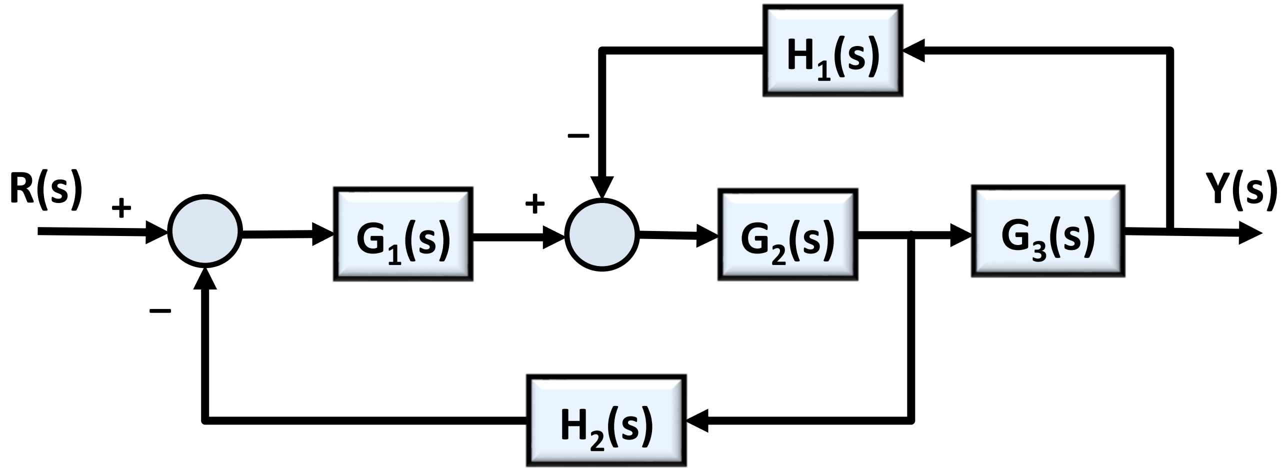
$$Y(s) = G(s)U(s)$$

Block Diagram Models

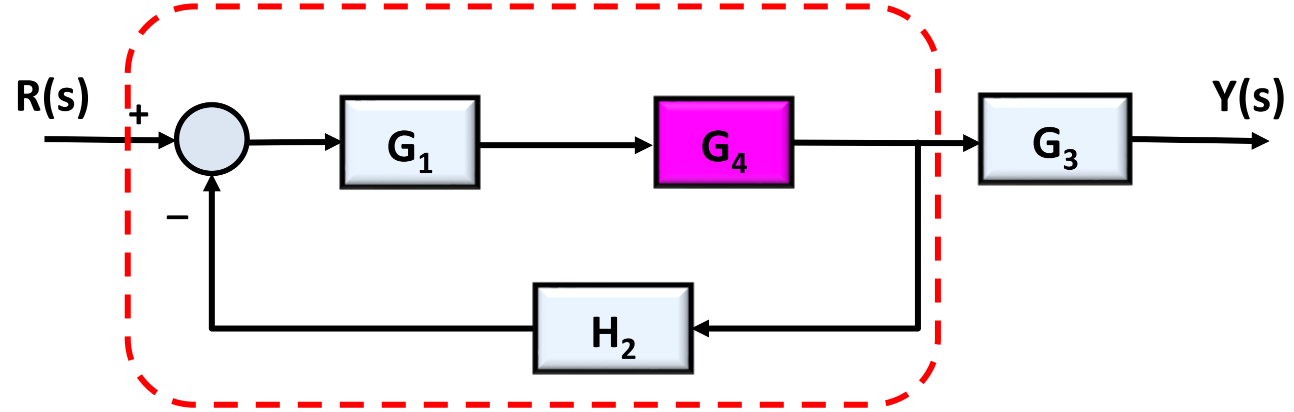
Example 2

Find the closed-loop transfer function utilizing the block diagram transformation techniques.

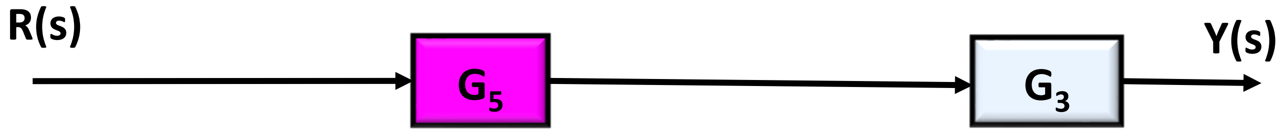




$$G_4 = \frac{G_2}{1 + G_2G_3H_1} \qquad G_5 = \frac{G_1G_4}{1 + G_1G_4H_2}$$



$$\frac{Y(s)}{R(s)} = G_5G_3 = \frac{G_1G_4G_3}{1 + G_1G_4H_2} = \frac{G_1\left(\frac{G_2}{1 + G_2G_3H_1}\right)G_3}{1 + G_1\left(\frac{G_2}{1 + G_2G_3H_1}\right)H_2}$$

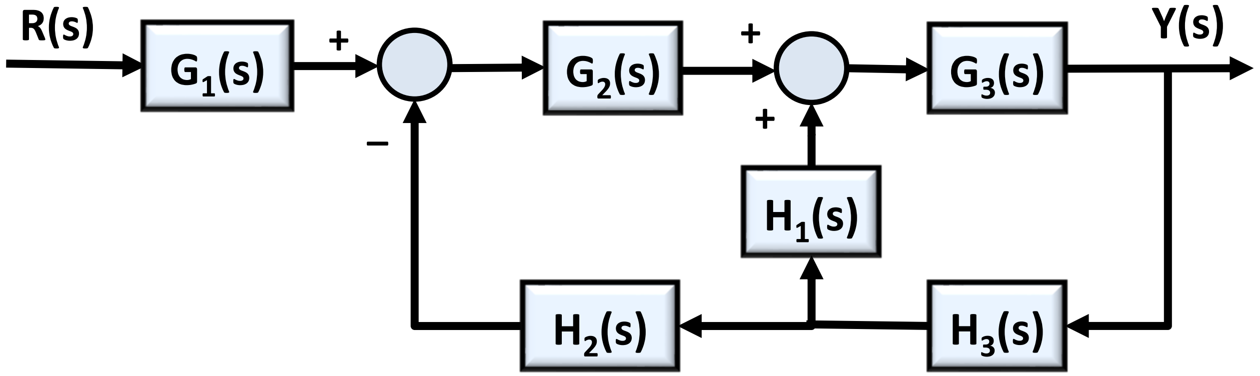


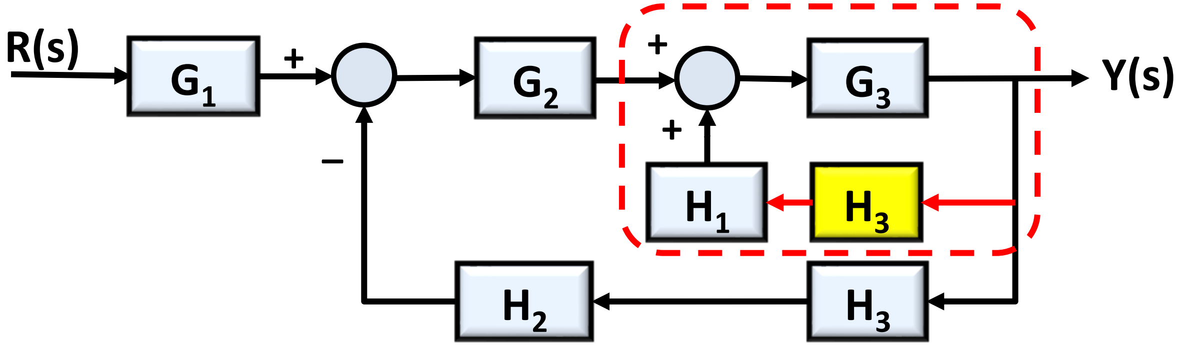
$$\boxed{\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3}{1 + G_2G_3H_1 + G_1G_2H_2}}$$

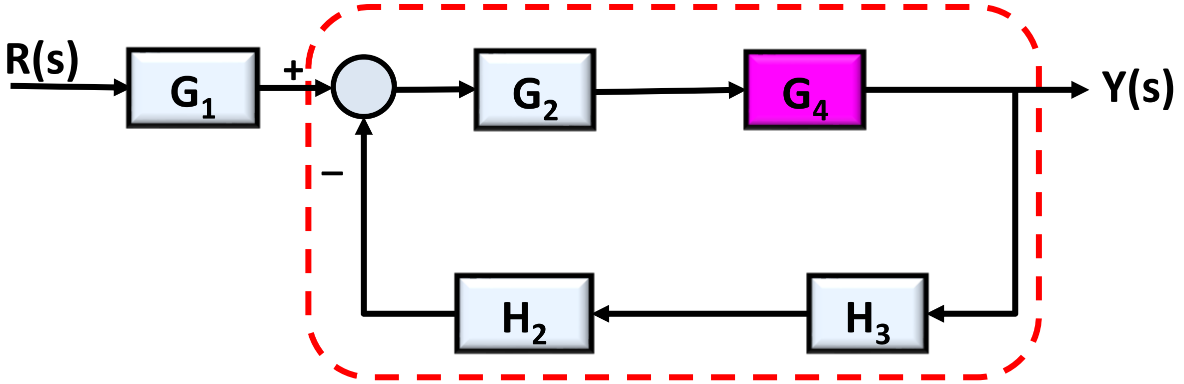
Block Diagram Models

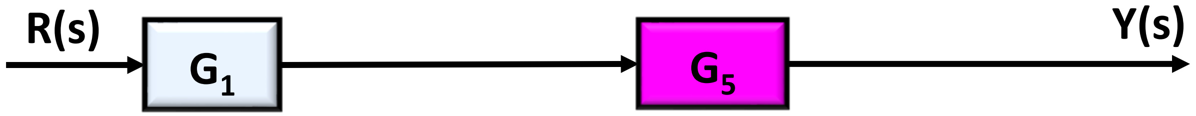
Example 3

Find the closed-loop transfer function utilizing the block diagram transformation techniques.









$$G_4 = \frac{G_3}{1 - G_3 H_1 H_3} \qquad G_5 = \frac{G_2 G_4}{1 + G_2 G_4 H_2 H_3}$$

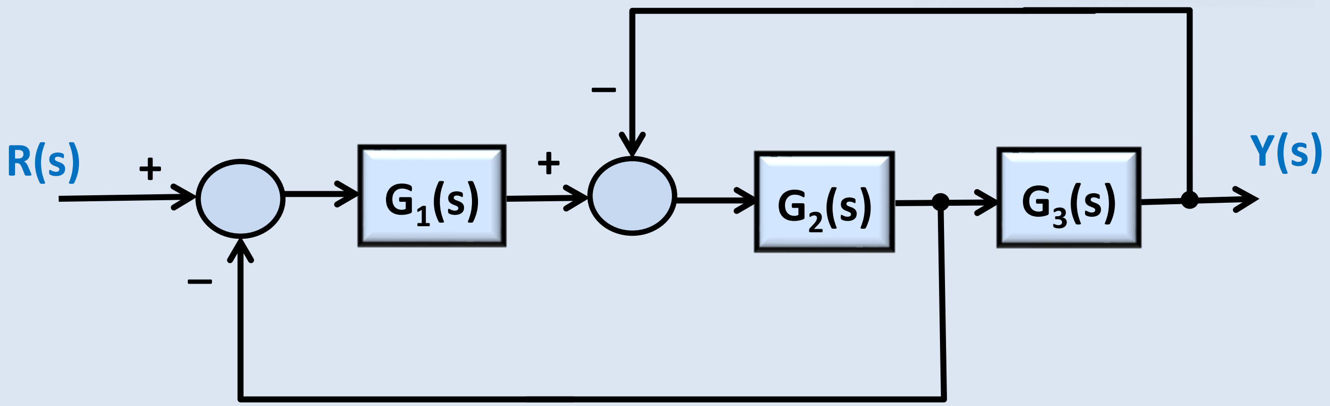
$$\frac{Y(s)}{R(s)} = G_1 G_5 = \frac{G_1 G_2 G_4}{1 + G_2 G_4 H_2 H_3} = \frac{G_1 G_2 \left(\frac{G_3}{1 - G_3 H_1 H_3}\right)}{1 + G_2 \left(\frac{G_3}{1 - G_3 H_1 H_3}\right) H_2 H_3}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3}{1 - G_3 H_1 H_3 + G_2 G_3 H_2 H_3}}$$

Quick Review

1) Find the overall closed-loop transfer function from Y(s) to R(s).

$$G_1(s) = 10, \qquad G_2(s) = \frac{1}{s+2}, \qquad G_3(s) = \frac{s+2}{s+10}$$



a) $\frac{Y(s)}{R(s)} = \frac{10(s+2)}{s^2+12s+20}$

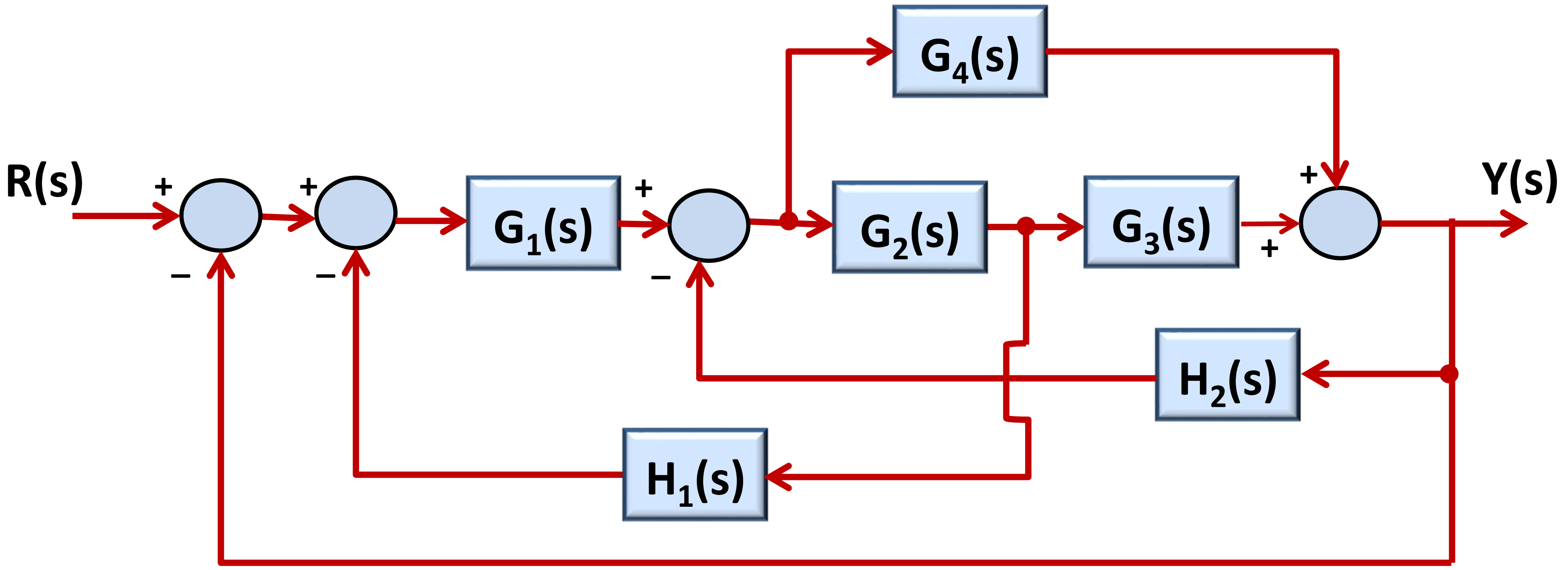
b) $\frac{Y(s)}{R(s)} = \frac{10(s+2)}{s^2+23s+122}$

c) $\frac{Y(s)}{R(s)} = \frac{5(s+10)}{s^2+5s+22}$

d) $\frac{Y(s)}{R(s)} = \frac{5(s+10)}{s^2+4s+100}$

Signal-Flow Graphs

- Block diagram reduction technique can be quite time-consuming for very complicated systems

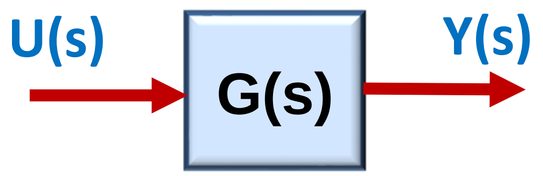


- **Mason's Gain Formula** is a systematic way to compute transfer function from any input to any output in the diagram
- The method is algorithmic and based on **Signal-Flow Graphs (SFG)**

Signal-Flow Graphs

- **Signal-Flow Graph (SFG)** is an alternative graphical approach to show the interconnection of a control system.
- **Basic elements of SFG:**
  - Nodes $\rightarrow$ Signals: U(s), Y(s), E(s) ….
  - Branches $\rightarrow$ Connects nodes. It has a gain and shows direction of signal flow
  - Transmittance $\rightarrow$ Gain between two nodes (Transfer functions): G(s), H(s), …

$$\boxed{\boldsymbol{Y(s) = G(s)U(s)}}$$



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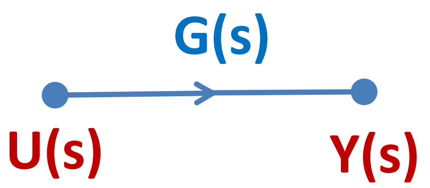
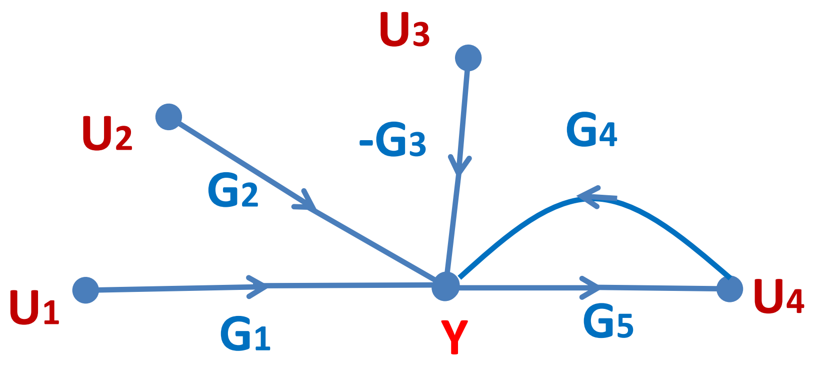


- **SGF Algebra:** The value of the variable in a node is equal to the sum of all signals **entering** the node.



$$Y = U_1G_1 + U_2G_2 - U_3G_3 + U_4G_4$$
$$U_4 = YG_5$$

Signal-Flow Graphs

❑ Series Connection

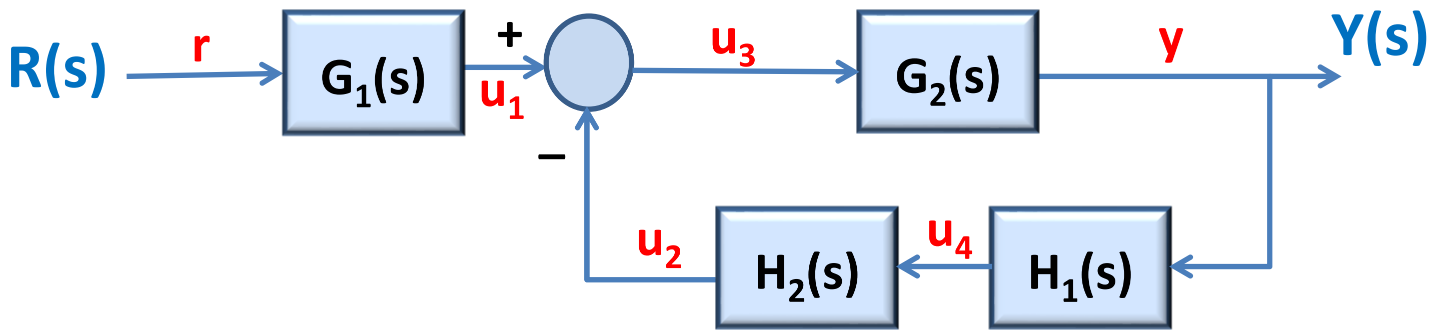
❑ Parallel Connection

❑ Feedback Connection

Signal-Flow Graphs

Example 4

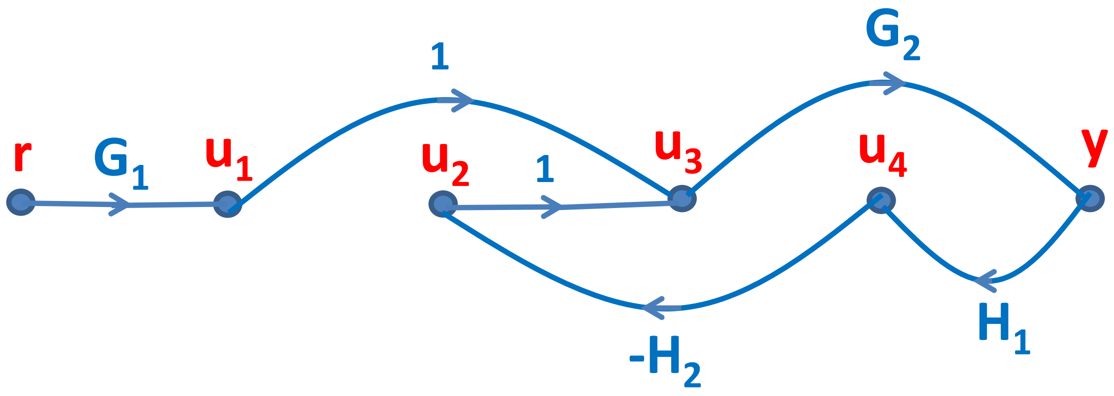
Draw Signal Flow Graph (SGF) of the following block diagram.



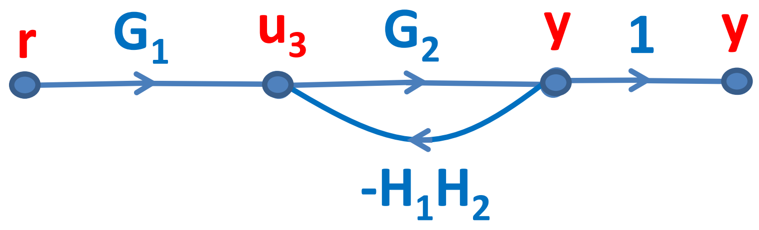
Step 1: Label all systems inputs-outputs

Step 2: Place all the nodes of SFG

Step 3: Draw the branches of SFG



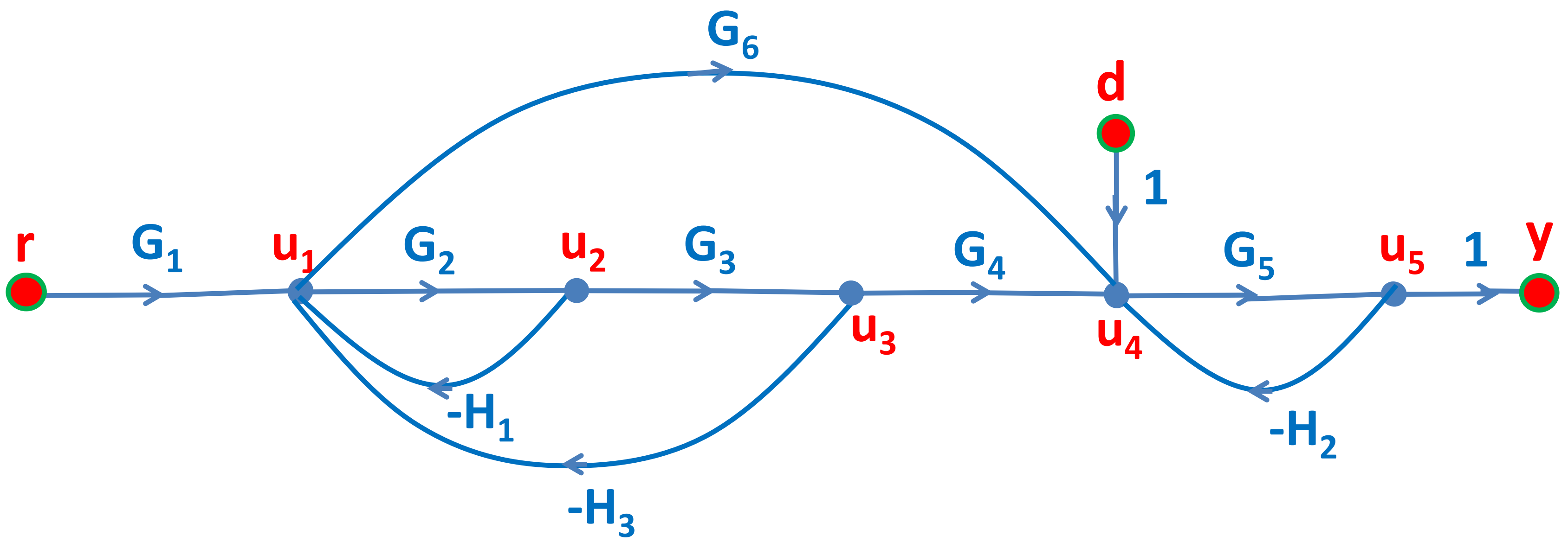
Step 4: Simplify the SFG



Signal-Flow Graphs

❑ Definitions of SFG Terms

- The following terms are useful for the purpose of identification and execution of the SFG algebra.
  - **Input Node (Source)** $\rightarrow$ A node that has only outgoing branches.
  - **Output Node (Sink)** $\rightarrow$ A node that has only incoming branches.

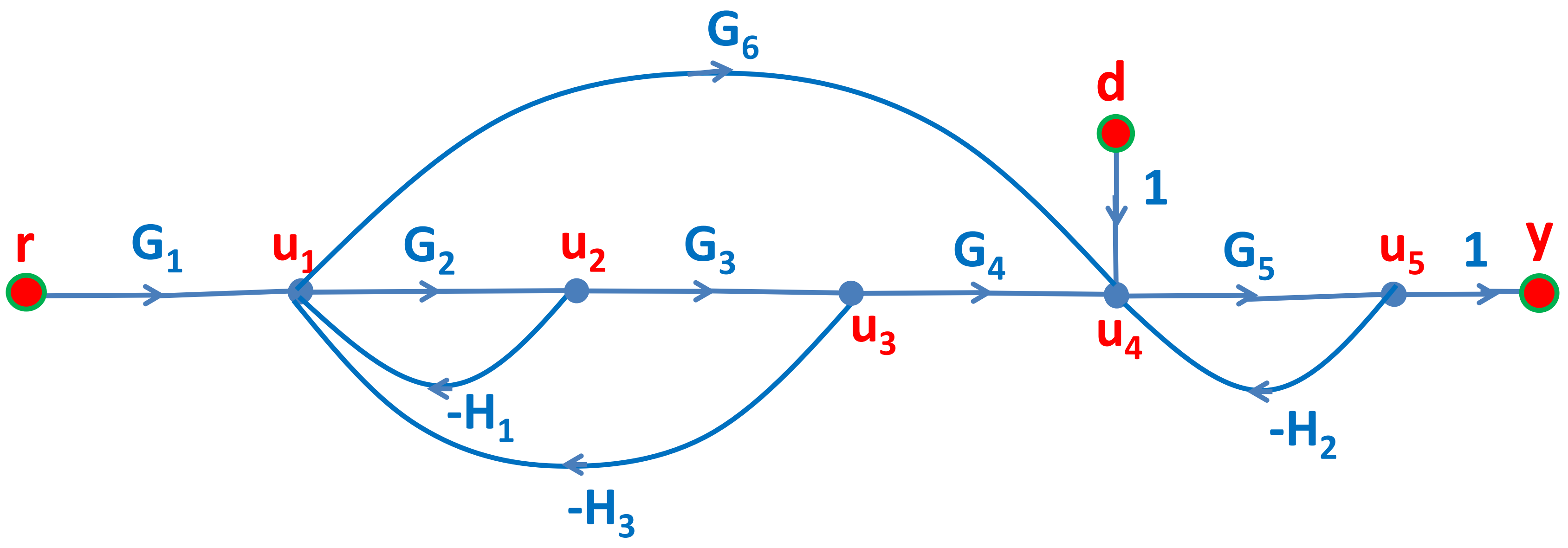


- **Input nodes:** r , d
- **Output node:** y

Signal-Flow Graphs

❑ Definitions of SFG Terms

- The following terms are useful for the purpose of identification and execution of the SFG algebra.
  - **Forward Path** → A path from an input-node (source) and to an output-node (sink) that does not cross any nodes more than once. The path must be in the same direction of branches.
  - **Forward Path Gain** → Product of the branch gains of a forward path.

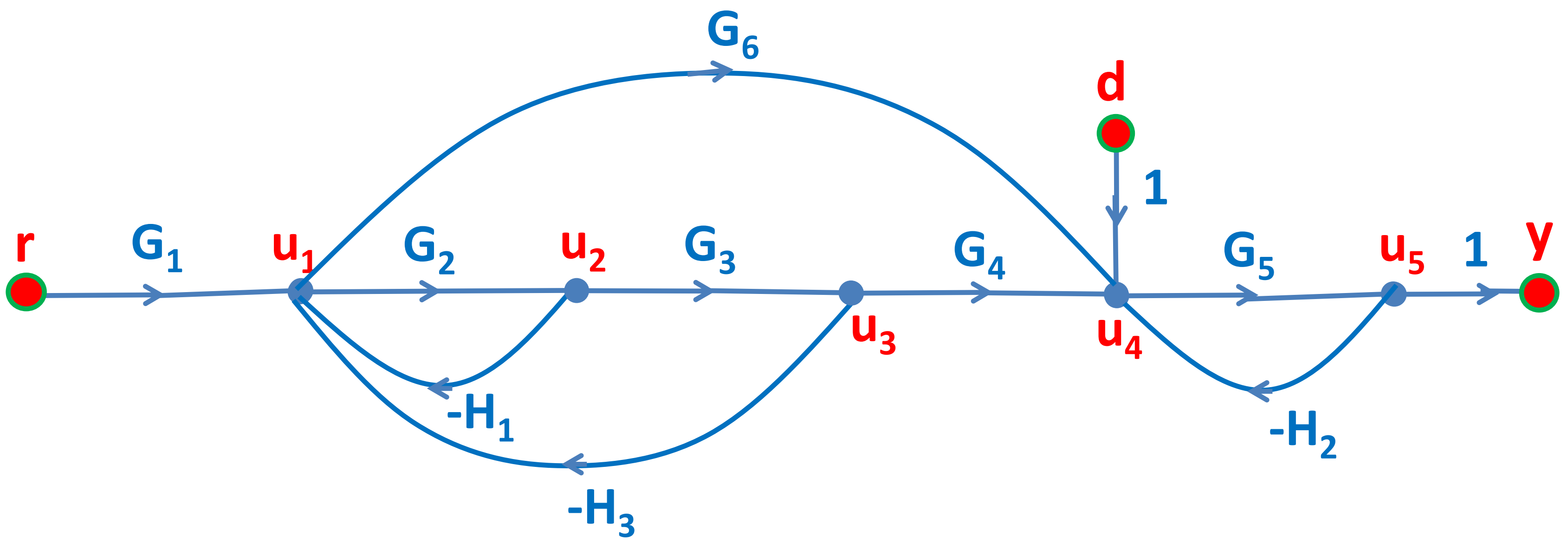


- **Forward paths gain from r to y:** $G_1G_2G_3G_4G_5$, $G_1G_6G_5$
- **Forward path gain from d to y:** G_5

Signal-Flow Graphs

❑ Definitions of SFG Terms

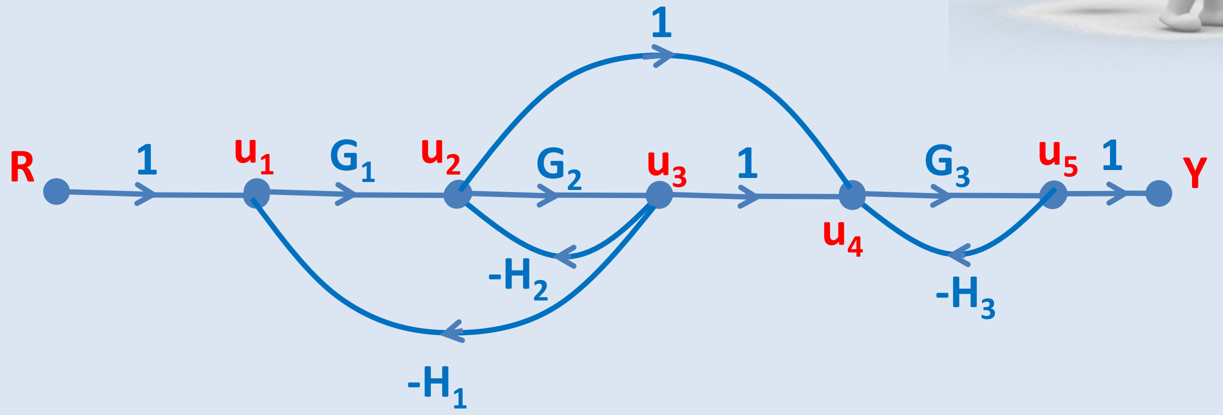
- The following terms are useful for the purpose of identification and execution of the SFG algebra.
  - **Loop** $\rightarrow$ A path that originates and terminates on the same node, and along which no other node is encountered more than once. The path must be in the same direction of branches.
  - **Loop Gain** $\rightarrow$ Product of the branch gains (transmittances) of a loop.
  - **Non-touching Loops** $\rightarrow$ Loops with no common node.



- **Loop gains:** $-G_2H_1$, $-G_5H_2$, $-G_2G_3H_3$
- **Non-touching loops:** $(-G_2H_1$ and $-G_5H_2)$, $(-G_2G_3H_3$ and $-G_5H_2)$

Quick Review

1) Given the SFG determine the following terms:



- **Input Node:**
- **Output Node:**
- **Number of Forward Paths:**
- **Forward Path Gains:**
- **Number of Loops:**
- **Loop Gains:**
- **Non-touching Loops:**

Mason's Gain Formula

- **Mason's Gain Formula** is a systematic method based on SFG to determine the overall transfer function or gain between input node and output node without applying the block diagram reduction techniques.

$$\boldsymbol{M = \frac{Y_{out}}{Y_{in}} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k \Delta_k}$$

M : Transfer function or Gain

Y_{in} : Input node variable/signal

Y_{out} : Output node variable/signal

N : Total number of forward paths between Y_{in} and Y_{out}

M_k : Gain of the kth forward path between Y_{in} and Y_{out}

Δ : Determinant of SFG

Δ_k : Cofactor of path M_k

Δ and Δ_k are determined as below:

Δ = 1 – (sum of all loop gains)
+ (sum of products of all combinations of two non-touching loops)
- (sum of products of all combinations of three non-touching loops)
+ (sum of products of all combinations of four non-touching loops)
- ....

Δ_k = the Δ of the SFG non-touching with the forward path M_k when M_k has been removed

Mason's Gain Formula

❑ Steps to Calculate Mason's Gain Formula

$$\boldsymbol{M = \frac{Y_{out}}{Y_{in}} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k \Delta_k}$$

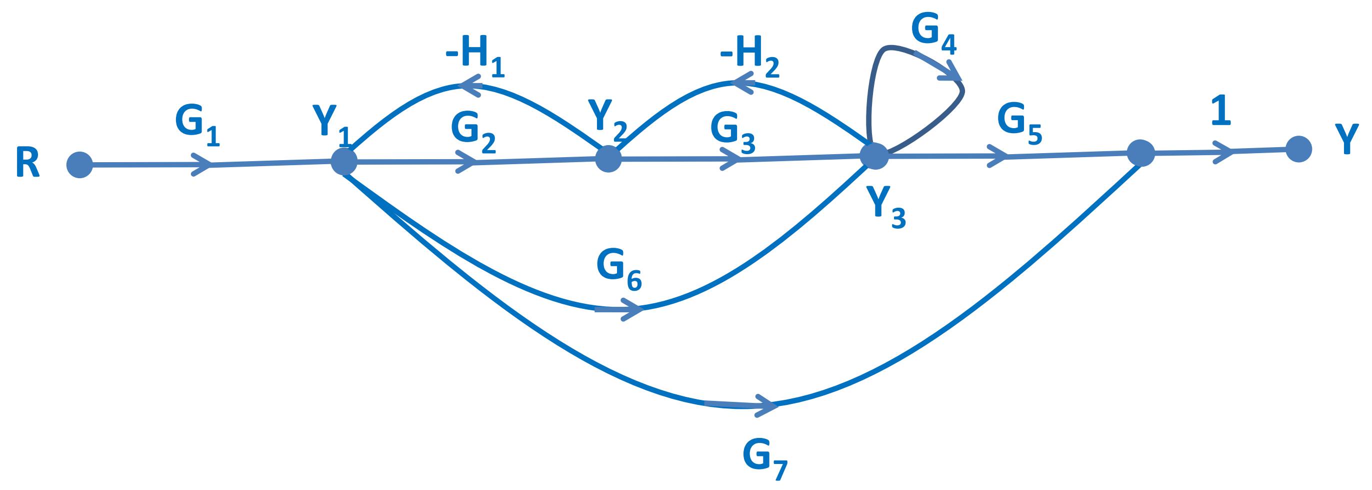
1) Determine input node and output node of the SFG $\rightarrow Y_{in}$ and Y_{out}
2) Calculate all forward path gains from the input node to output node $\rightarrow M_k$ and N
3) Calculate all loop gains of the SFG (if any) $\rightarrow L_i$
4) Calculate all non-touching loops of the SFG (if any) $\rightarrow L_{ij}$
5) Calculate determinant of the SFG $\rightarrow \Delta$
6) Calculate cofactors of path M_k $\rightarrow \Delta_k$

Mason's Gain Formula

Example 5

Find the system transfer function from Y to R for the following SFG

$$\frac{Y}{R} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k \Delta_k$$



R : Input node

Y : Output node

N : Total number of forward paths between R and Y

M_k : Gain of the kth forward path between R and Y

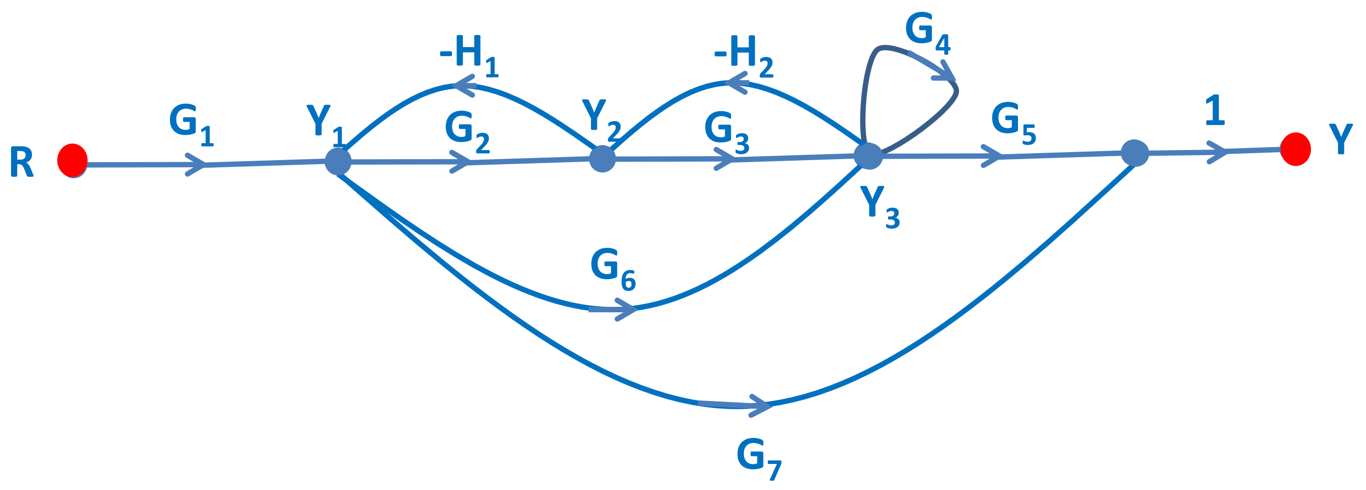
Δ : Determinant of SFG

Δ_k : Cofactor of path M_k

Mason's Gain Formula

Example 5

Find the system transfer function from Y to R for the following SFG



$$\frac{Y}{R} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k \Delta_k$$

Step 1: Determine the input node and output node

R : Input node

Y : Output node

Step 2: Calculate all forward path gains between R and Y

N : Total number of forward paths between R and Y

M_k : Gain of the kth forward path between R and Y

Mason's Gain Formula

Example 5

Find the system transfer function from Y to R for the following SFG

Step 2: Calculate all forward path gains between R and Y

There are three forward paths, so $N = 3$

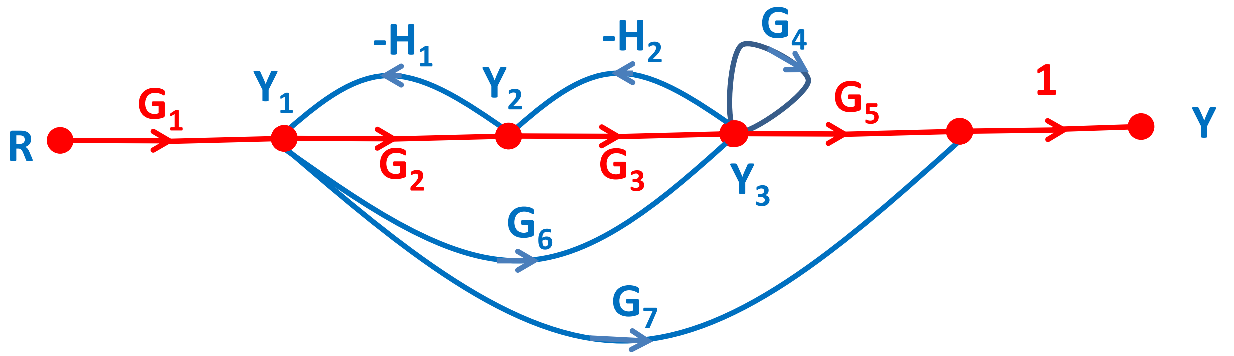
$$M_1 = G_1 G_2 G_3 G_5$$

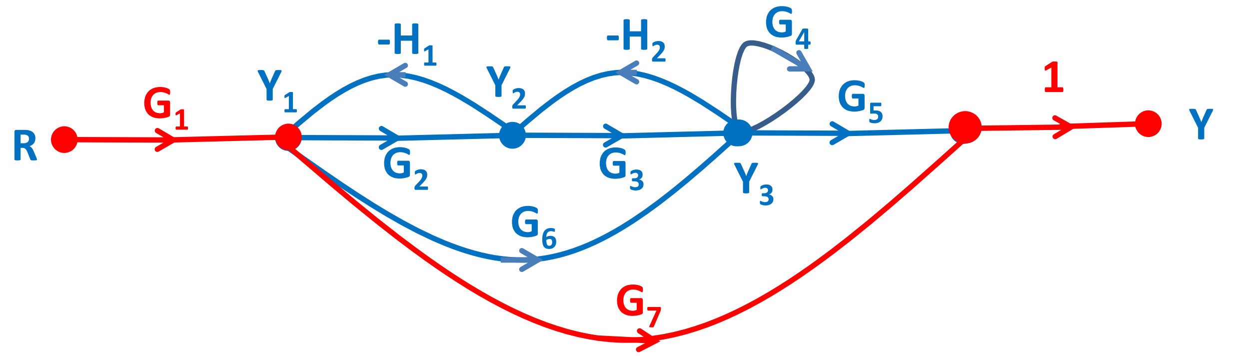
$$M_2 = G_1 G_7$$

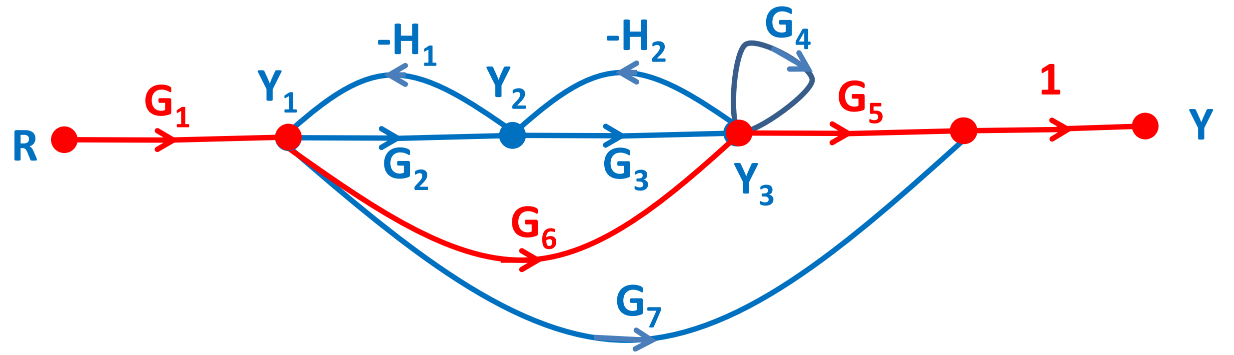
$$M_3 = G_1 G_6 G_5$$

$$\frac{Y}{R} = \frac{1}{\Delta}\sum_{k=1}^{3} M_k \Delta_k$$

$$\frac{Y}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$





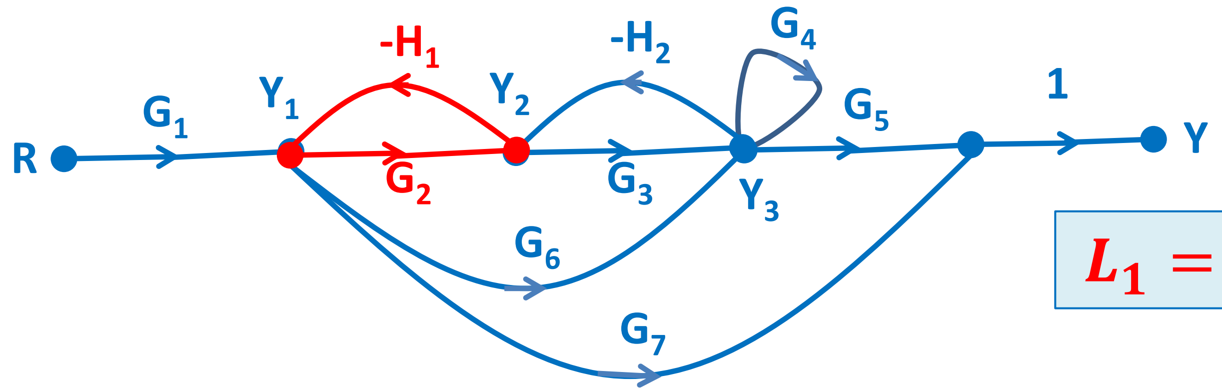


Mason's Gain Formula

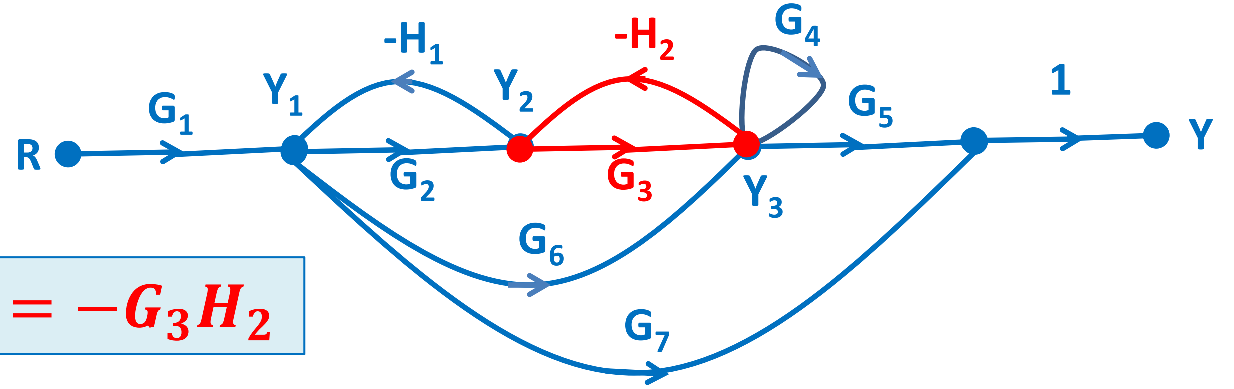
Example 5

Find the system transfer function from Y to R for the following SFG

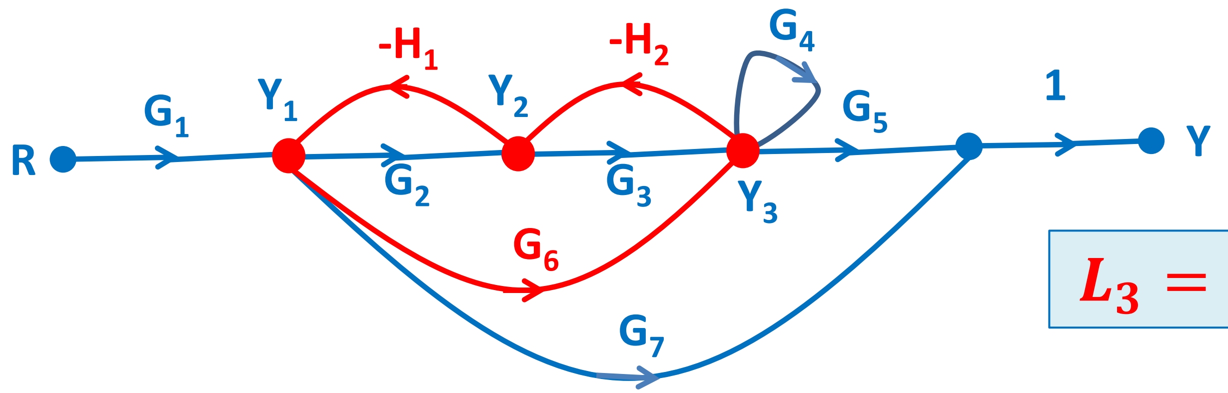
Step 3: Calculate all loop gains



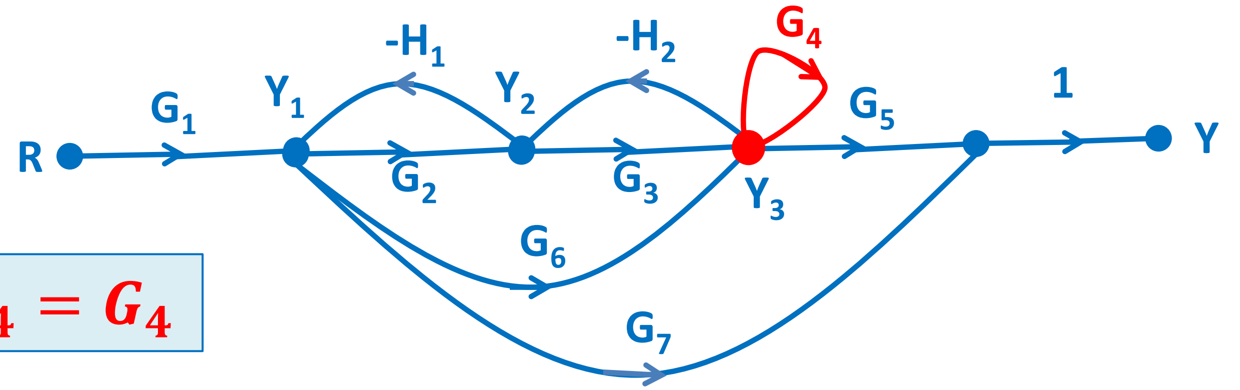
$$L_1 = -G_2H_1$$



$$L_2 = -G_3H_2$$



$$L_3 = G_6H_1H_2$$



$$L_4 = G_4$$

Step 4: Determine the non-touching loops

Non-touching loops

$$L_1 = -G_2H_1 \qquad L_4 = G_4$$

Mason's Gain Formula

Example 5

Find the system transfer function from Y to R for the following SFG

Step 5: Calculate determinant of the SFG $\rightarrow \Delta$

Δ = 1 – (sum of all loop gains)
+ (sum of products of all combinations of two non-touching loops)
- (sum of products of all combinations of three non-touching loops)
+ (sum of products of all combinations of four non-touching loops)
- ....

All loop gains

$$L_1 = -G_2H_1 \qquad L_3 = G_6H_1H_2$$
$$L_2 = -G_3H_2 \qquad L_4 = G_4$$

Non-touching loops

$$L_1 = -G_2H_1 \qquad L_4 = G_4$$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_4)$$

$$\Delta = 1 - (-G_1H_1 - G_3H_2 + G_6H_1H_2 + G_4) + (-G_4G_2H_1)$$

Mason's Gain Formula

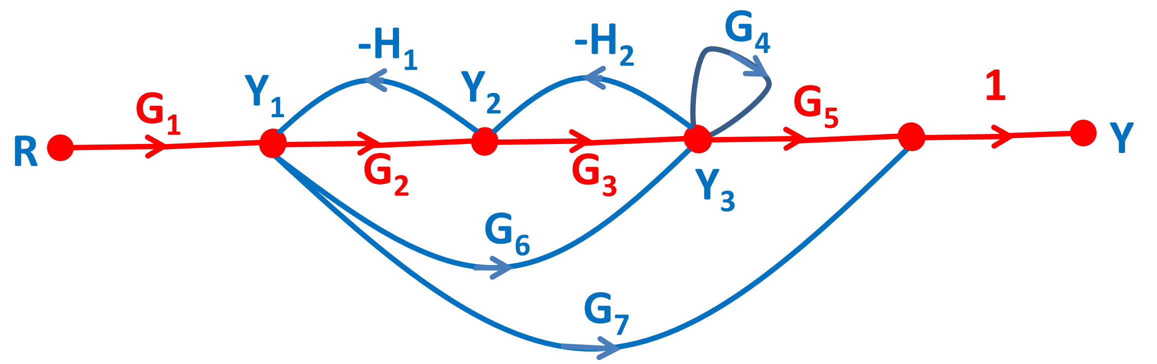
Example 5 Find the system transfer function from Y to R for the following SFG

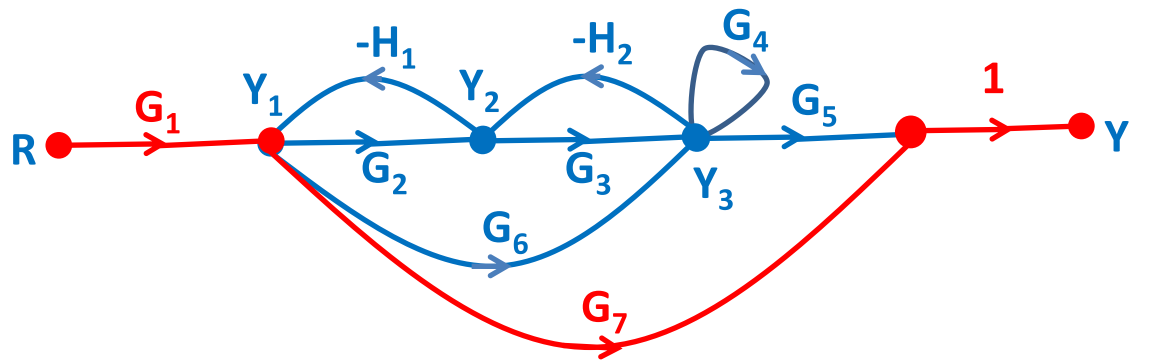
Step 6: Calculate cofactors of path $M_k \rightarrow \Delta_k$

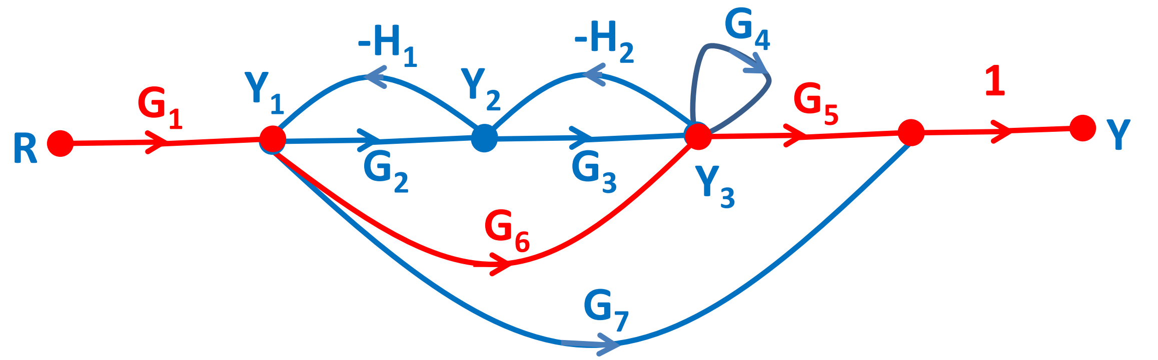
Δ_k = the Δ of the SFG non-touching with the forward path M_k when M_k has been removed

$$\boxed{\frac{Y}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}}$$

For Δ_1 remove $\boldsymbol{M_1 = G_1 G_2 G_3 G_5}$







Mason’s Gain Formula

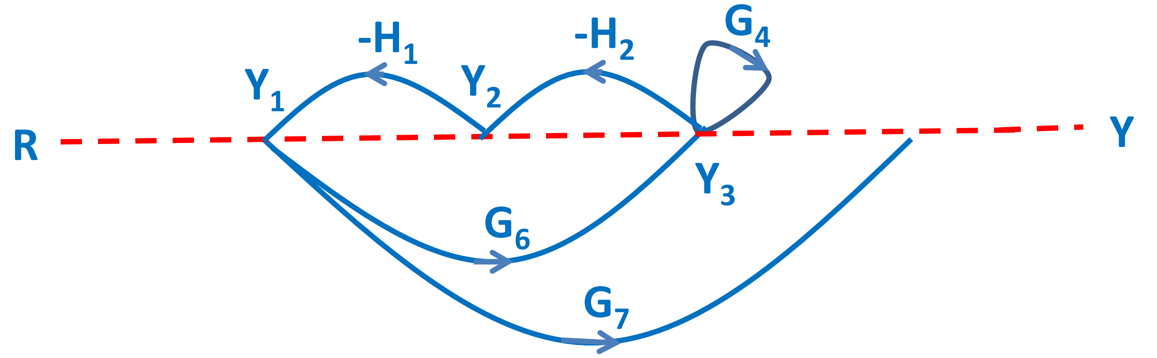
Example 5

Find the system transfer function from Y to R for the following SFG

Step 6: Calculate cofactors of path $M_k \rightarrow \Delta_k$

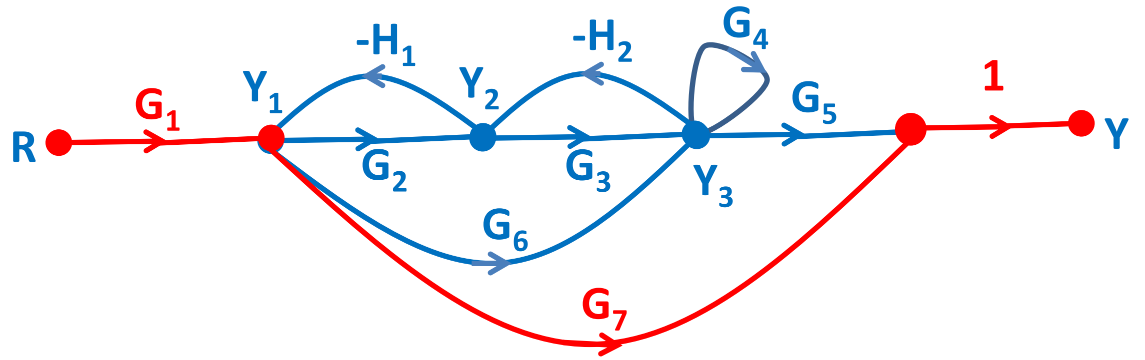
Δ_k = the Δ of the SFG non-touching with the forward path M_k when M_k has been removed

$$\frac{Y}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

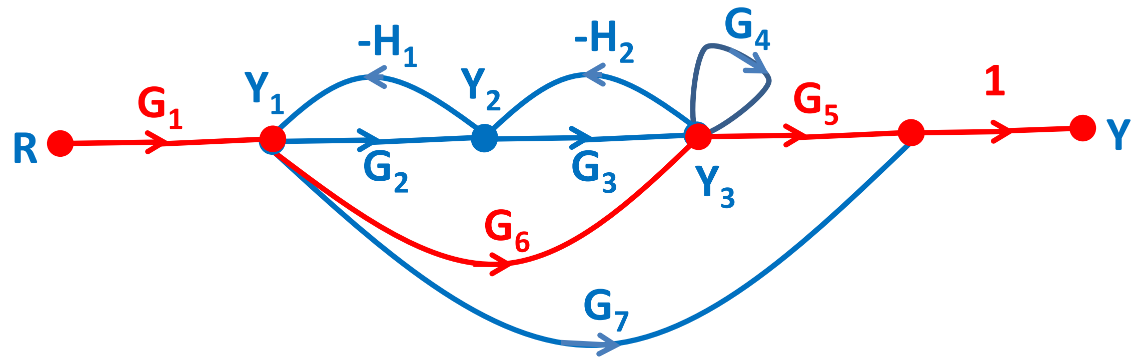


For Δ_1 remove $\boldsymbol{M_1 = G_1G_2G_3G_5}$

$$\boxed{\boldsymbol{\Delta_1 = 1}}$$



For Δ_2 remove $\boldsymbol{M_2 = G_1G_7}$



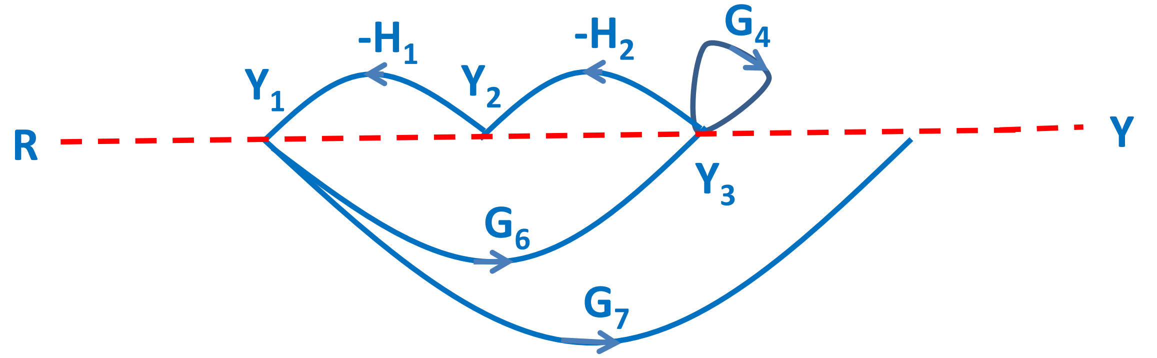
Mason's Gain Formula

Example 5

Find the system transfer function from Y to R for the following SFG

Step 6: Calculate cofactors of path $M_k \;\rightarrow\; \Delta_k$

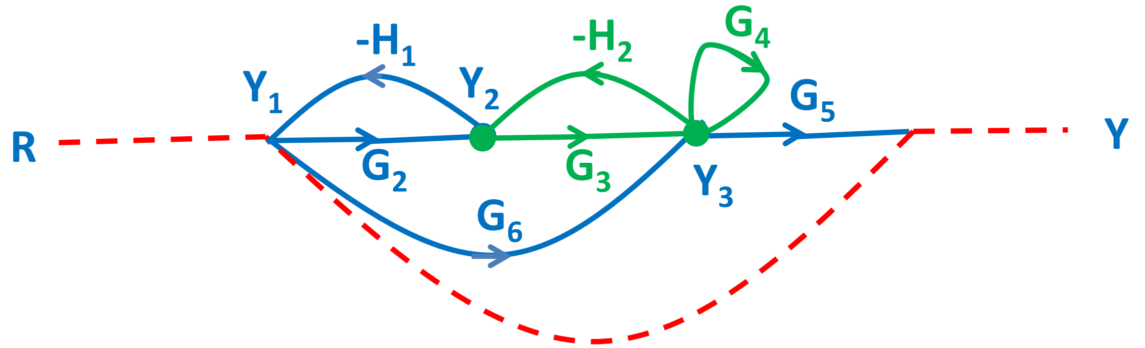
Δ_k = the Δ of the SFG non-touching with the forward path M_k when M_k has been removed



$$\boxed{\frac{Y}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}}$$

For Δ_1 remove $M_1 = G_1G_2G_3G_5$

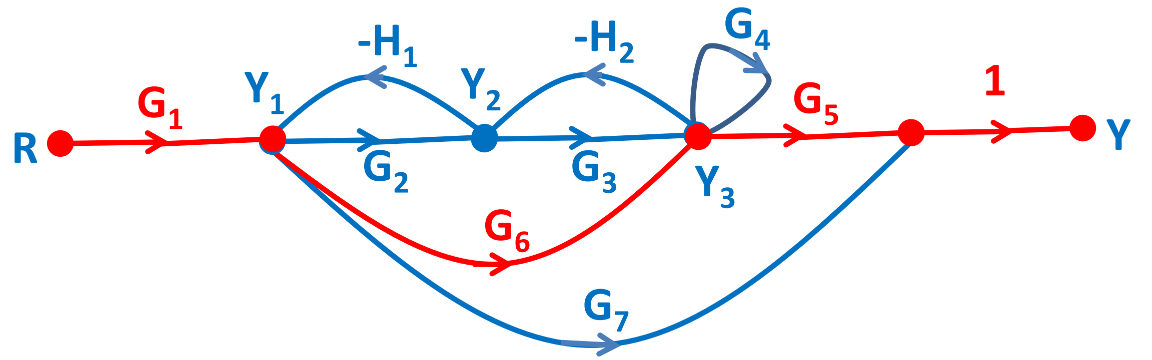
$$\boxed{\Delta_1 = 1}$$



For Δ_2 remove $M_2 = G_1G_7$

$$\boxed{\Delta_2 = 1 - (L_2 + L_4) = 1 - (-G_3H_2 + G_4)}$$

For Δ_3 remove $M_3 = G_1G_6G_5$



Mason's Gain Formula

Example 5

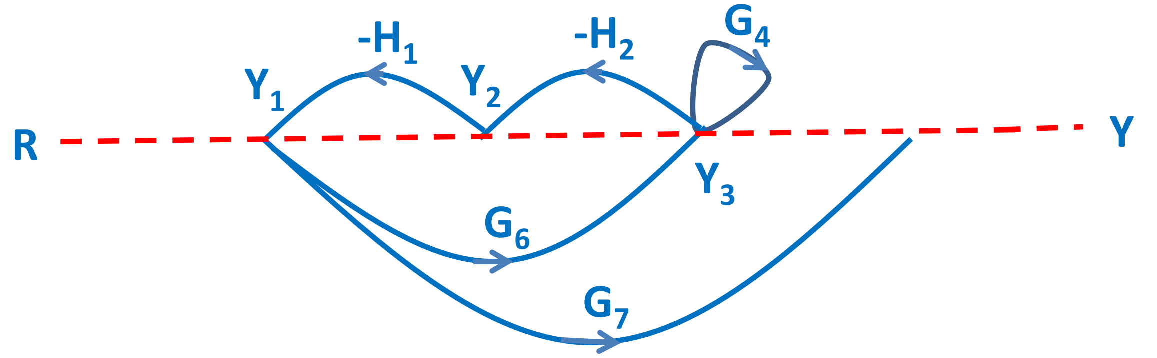
Find the system transfer function from Y to R for the following SFG

Step 6: Calculate cofactors of path $M_k \rightarrow \Delta_k$

Δ_k = the Δ of the SFG non-touching with the forward path M_k when M_k has been removed

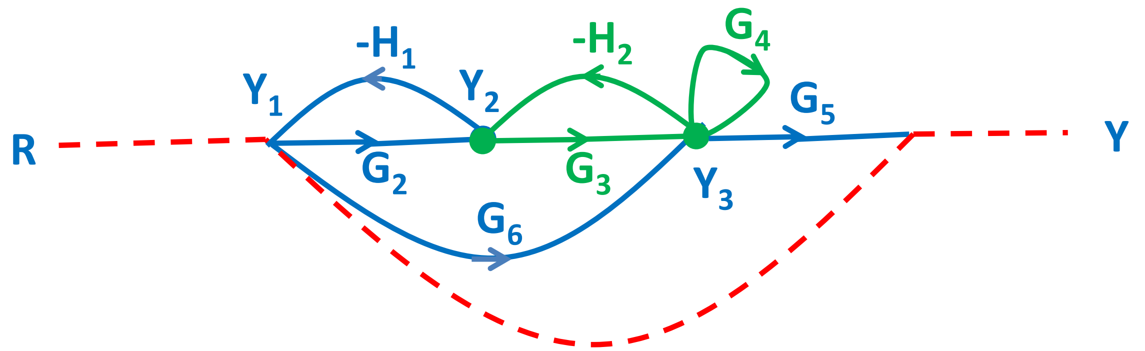
$$\boxed{\frac{Y}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}}$$

For Δ_1 remove $\boldsymbol{M_1 = G_1G_2G_3G_5}$



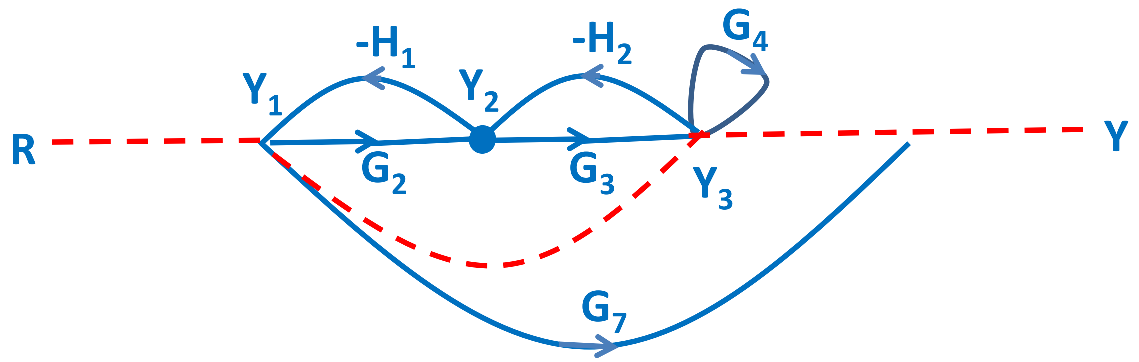
$$\boxed{\boldsymbol{\Delta_1 = 1}}$$

For Δ_2 remove $\boldsymbol{M_2 = G_1G_7}$



$$\boxed{\boldsymbol{\Delta_2 = 1 - (L_2 + L_4) = 1 - (-G_3H_2 + G_4)}}$$

For Δ_3 remove $\boldsymbol{M_3 = G_1G_6G_5}$



$$\boxed{\boldsymbol{\Delta_3 = 1}}$$

Mason’s Gain Formula

Example 5 Find the system transfer function from Y to R for the following SFG

Step 7: Calculate the overall transfer function

$$M_1 = G_1G_2G_3G_5$$
$$M_2 = G_1G_7$$
$$M_3 = G_1G_6G_5$$

$$L_1 = -G_2H_1$$
$$L_2 = -G_3H_2$$
$$L_3 = G_6H_1H_2$$
$$L_4 = G_4$$

$$\Delta_1 = 1$$
$$\Delta_2 = 1 + G_3H_2 - G_4$$
$$\Delta_3 = 1$$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_4)$$

$$\frac{Y}{R} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k\Delta_k$$

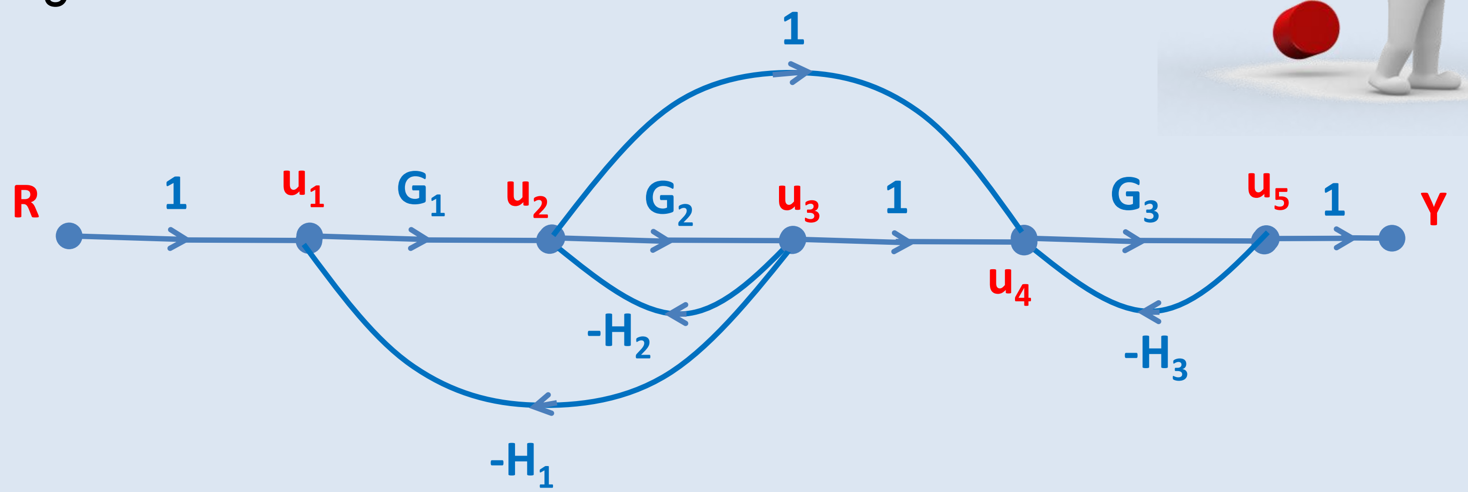
$N = 3$

$$\frac{Y}{R} = \frac{1}{\Delta}\sum_{k=1}^{3} M_k\Delta_k = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

$$\frac{Y}{R} = \frac{G_1G_2G_3G_5 + G_1G_7(1 + G_3H_2 - G_4) + G_1G_6G_5}{1 - (-G_1H_1 - G_3H_2 + G_6H_1H_2 + G_4) + (-G_4G_2H_1)}$$

Quick Review

Determine the transfer function from Y to R for the following SFG.



Step 1: Determine the input node and output node

Step 2: Calculate all forward path gains between R and Y

Step 3: Calculate all loop gains

Step 4: Determine the non-touching loops

Step 5: Calculate determinant of the SFG

Step 6: Calculate the cofactors of each forward path

Step 7: Calculate the overall transfer function

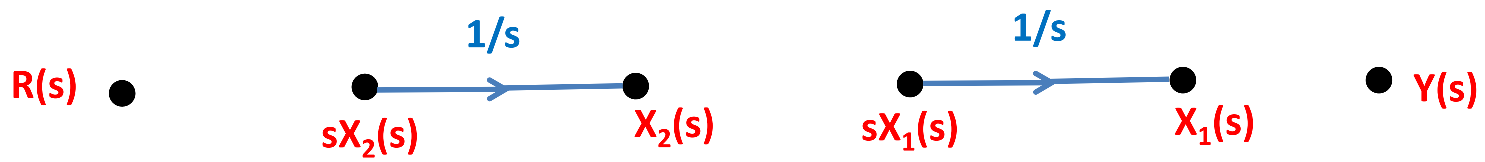
$$\boxed{\frac{Y}{R} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k \Delta_k}$$

Signal-Flow Graphs & State-Space Equations

- **State Diagram**, is an extension of the SFG to portray state equations and differential equations.
- A state diagram is constructed following all the rules of the SFG using the Laplace-transformed state equations.
- The basic elements of a state diagram are similar to the conventional SFG, except for the integration operation.
- Consider the following state and output equations:

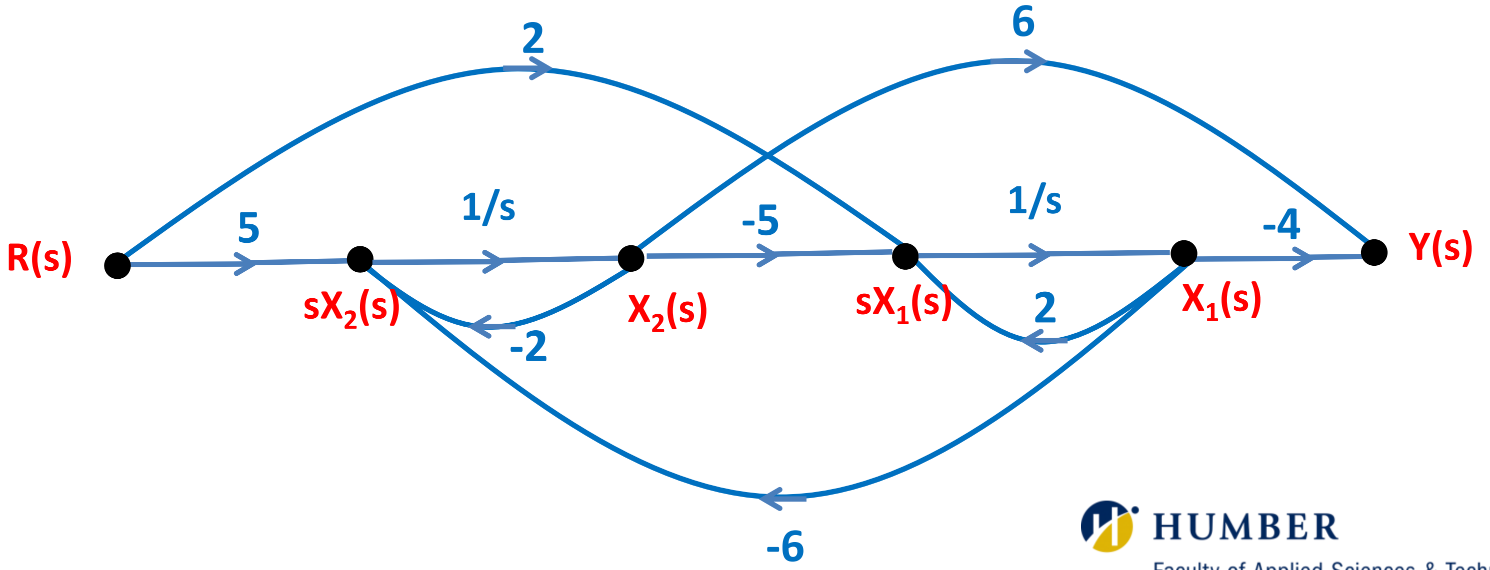
$$\begin{cases} \dot{x}_1 = 2x_1 - 5x_2 + 2r \\ \dot{x}_2 = -6x_1 - 2x_2 + 5r \\ y = -4x_1 + 6x_2 \end{cases}$$

1) First identify the following nodes:
   - Input node and output node,
   - One node for each state variable
   - One node for derivative of state variables



2) Next connect the state variables and their derivatives with the defining integration $1/s$.

3) Then using the state and output equations, feed to each node the indicated signals.



Signal-Flow Graphs & State-Space Equations

Example 6 Draw a signal-flow graph for the following state and output equations.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = [6 \quad 8 \quad 2]\mathbf{x}(t)$$

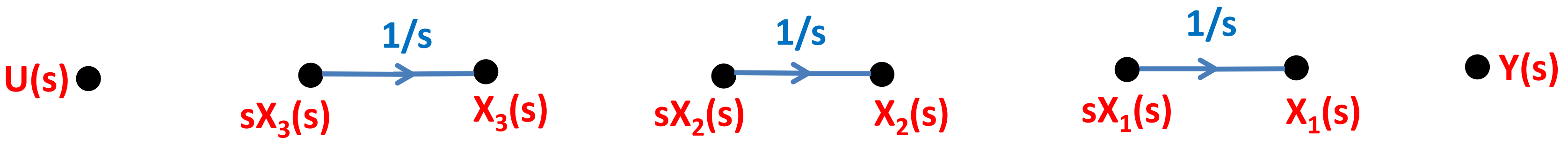
First rewrite the state and output equations.

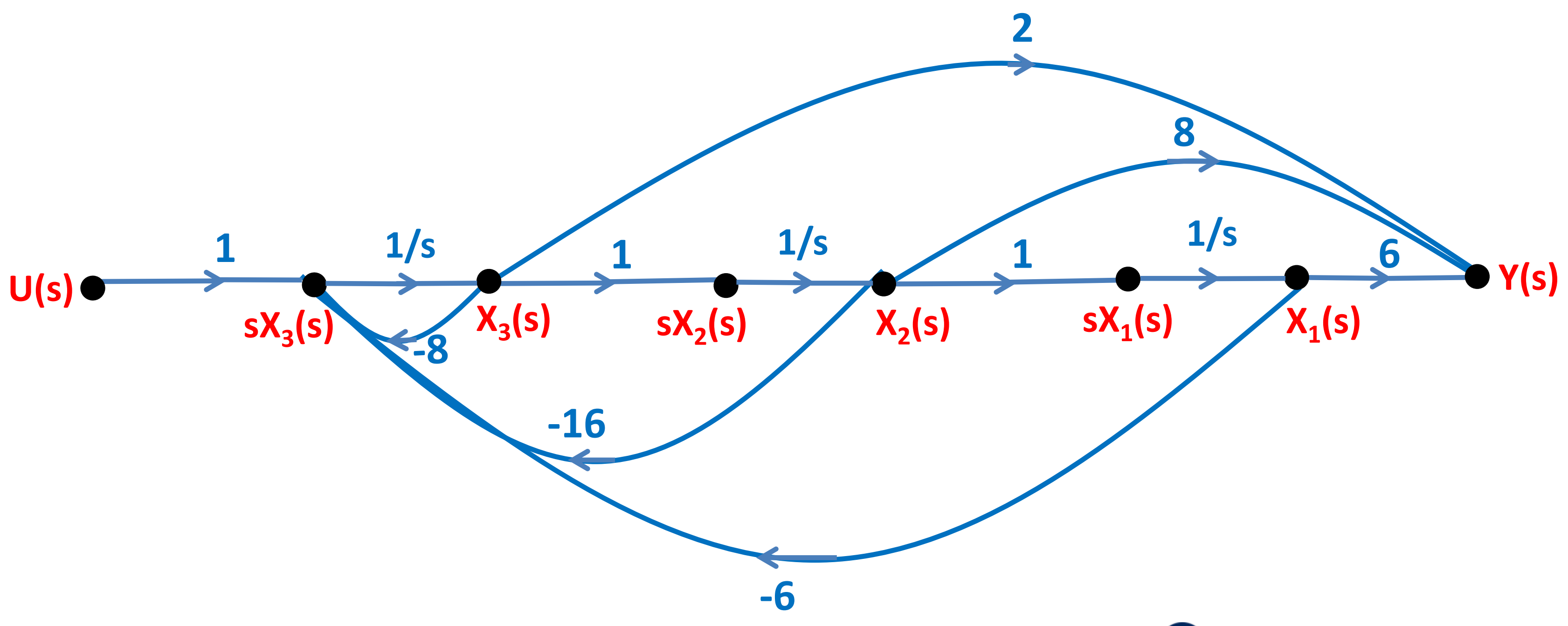
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -6x_1 - 16x_2 - 8x_3 + u \\ y = 6x_1 + 8x_2 + 2x_3 \end{cases}$$

1) Identify the following nodes:
   - Input node and output node,
   - One node for each state variable
   - One node for derivative of state variables

2) Connect the state variables and their derivatives with the defining integration $1/s$.

3) Using the state and output equations, feed to each node the indicated signals.



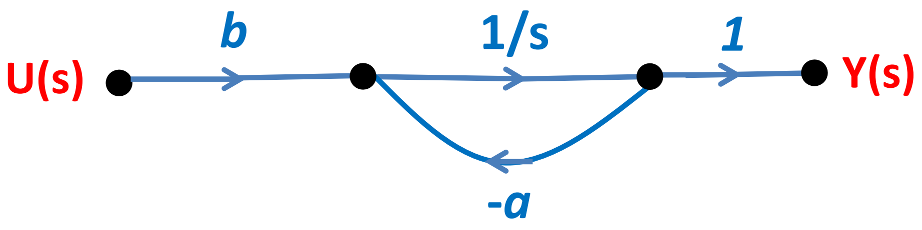


Signal-Flow Graphs & State-Space Equations

- SFG and the state-space representation can be derived directly from the transfer function model, which is helpful to find SFG of simple control system block diagrams.

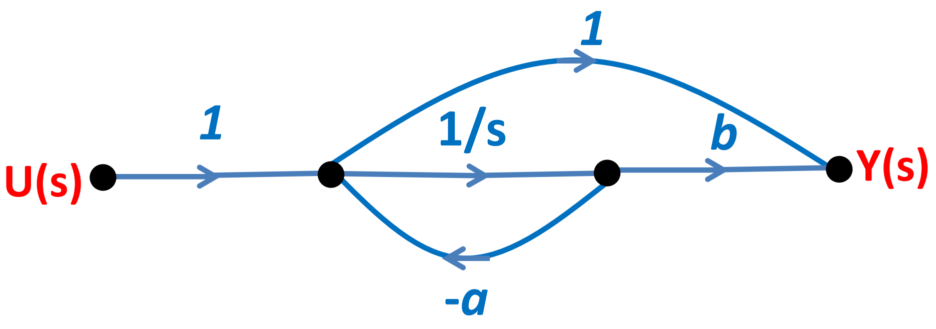
❑ First-order Transfer Function with no Zero:

$$\frac{Y(s)}{U(s)} = \frac{b}{s+a}$$



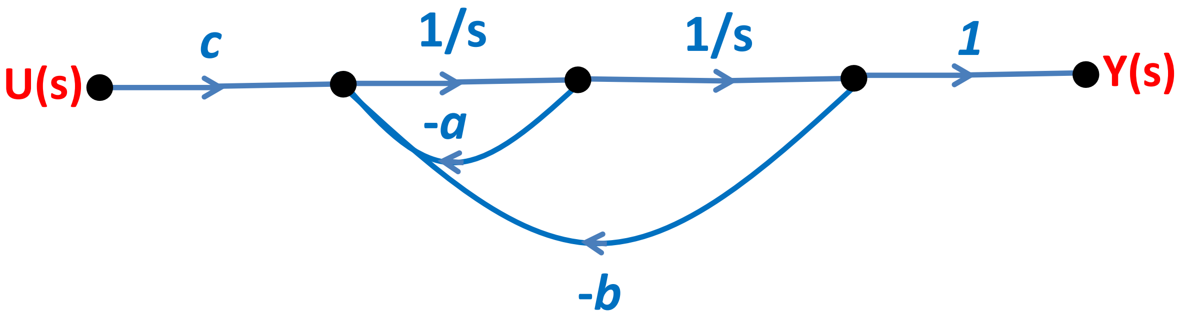
❑ First-order Transfer Function with a single Zero:

$$\frac{Y(s)}{U(s)} = \frac{s+b}{s+a}$$



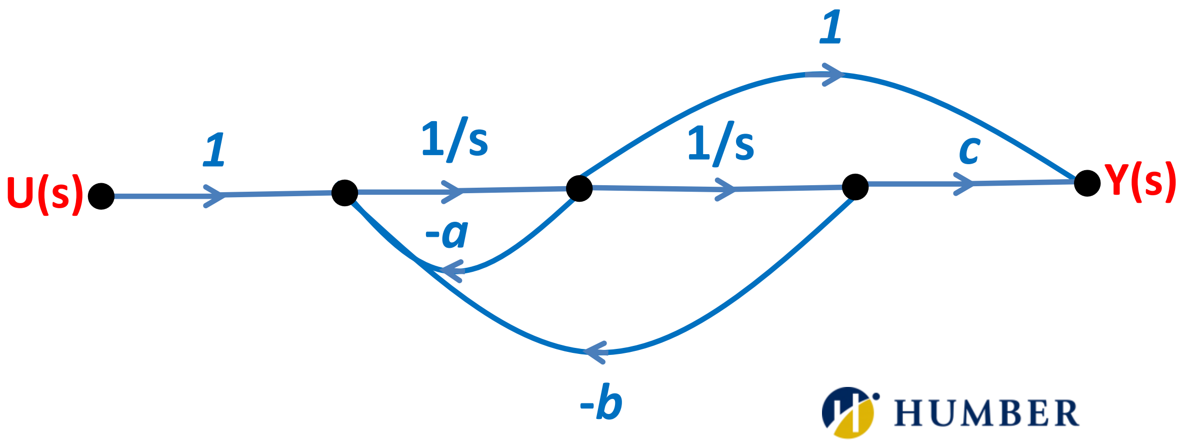
❑ Second-order Transfer Function with no Zero:

$$\frac{Y(s)}{U(s)} = \frac{c}{s^2+as+b}$$



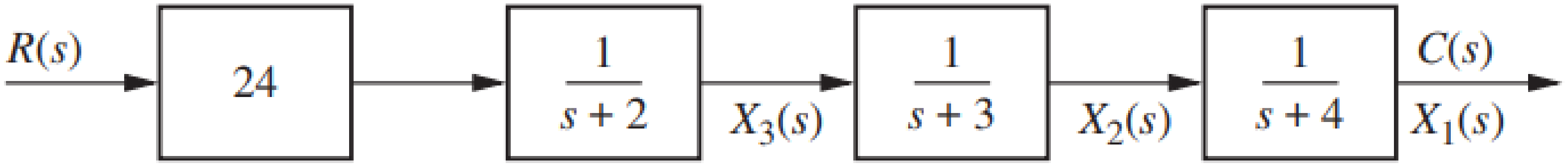
❑ Second-order Transfer Function with a single Zero:

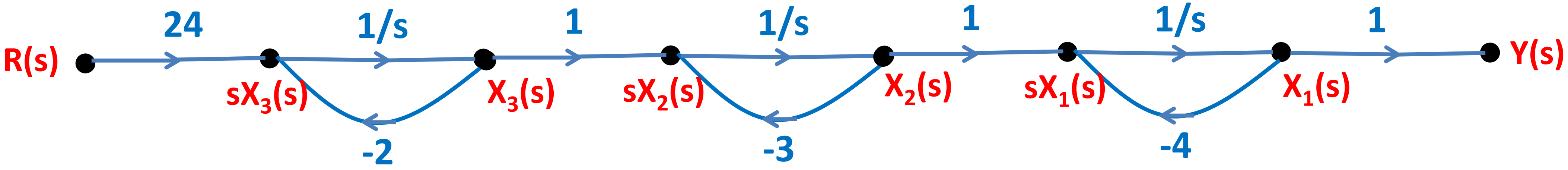
$$\frac{Y(s)}{U(s)} = \frac{s+c}{s^2+as+b}$$



Signal-Flow Graphs & State-Space Equations

Example 7 Draw a signal-flow graph for the following cascade system. Given the state variables as $x_1,\ x_2$ and x_3 derive a state-space representation from the SFG.





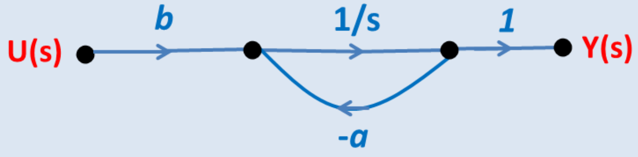
Defining the state variables as $x_1,\ x_2$ and x_3.

$$\begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = -3x_2 + x_3 \\ \dot{x}_3 = -2x_3 + 24r \\ y = x_1 \end{cases}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} u(t)$$

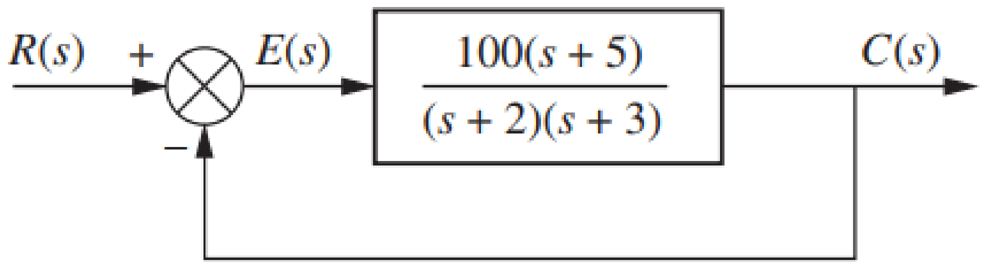
$$y(t) = [1 \quad 0 \quad 0]\mathbf{x}(t)$$

$$\frac{Y(s)}{U(s)} = \frac{b}{s+a}$$



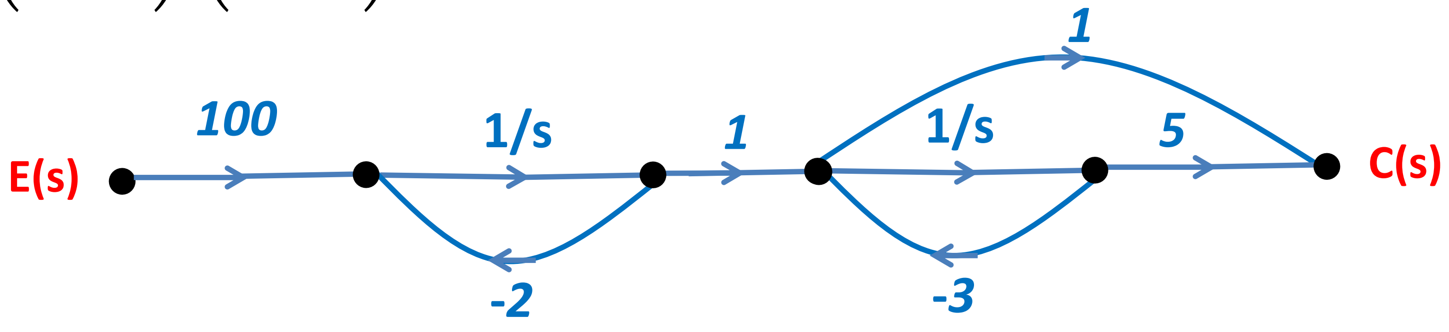
Signal-Flow Graphs & State-Space Equations

Example 8 Draw a signal-flow graph for the following feedback control system. Define appropriate state variables and derive a state-space model.

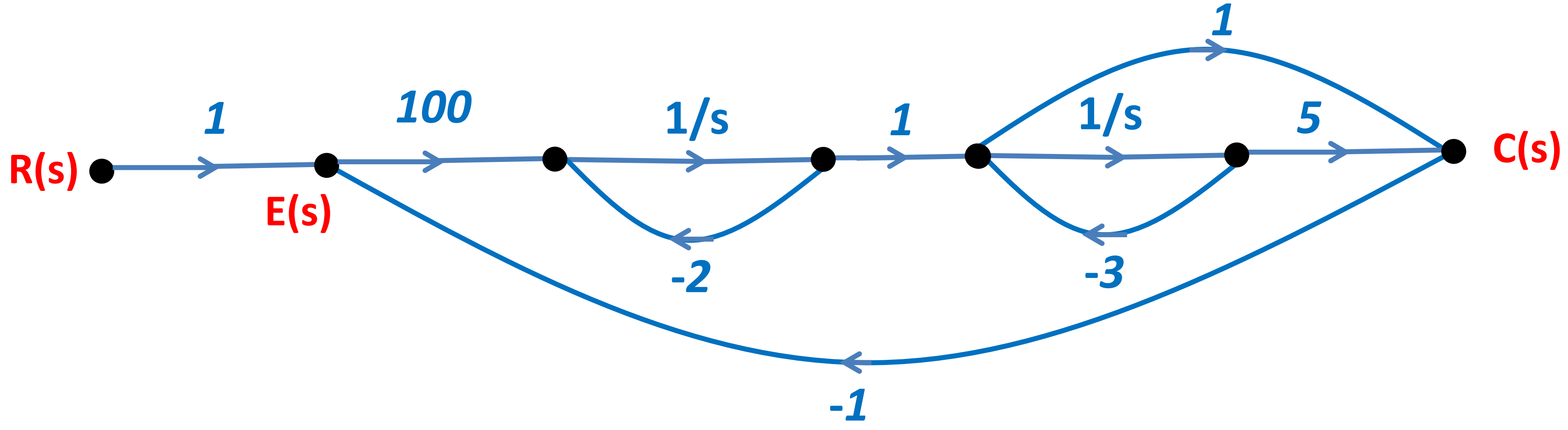


First, model the forward path transfer function in cascade form.

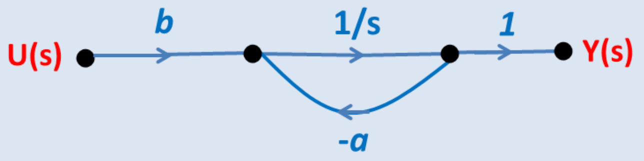
$$\frac{C(s)}{E(s)} = \left(\frac{100}{s+2}\right)\left(\frac{s+5}{s+3}\right)$$



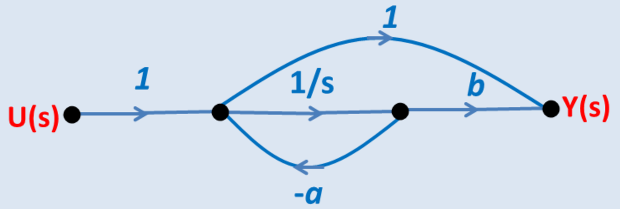
Next add the feedback and input paths to complete the SFG.



$$\frac{Y(s)}{U(s)} = \frac{b}{s+a}$$

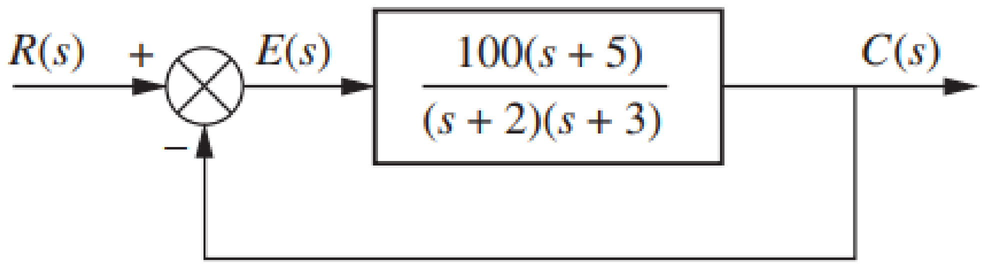


$$\frac{Y(s)}{U(s)} = \frac{s+b}{s+a}$$

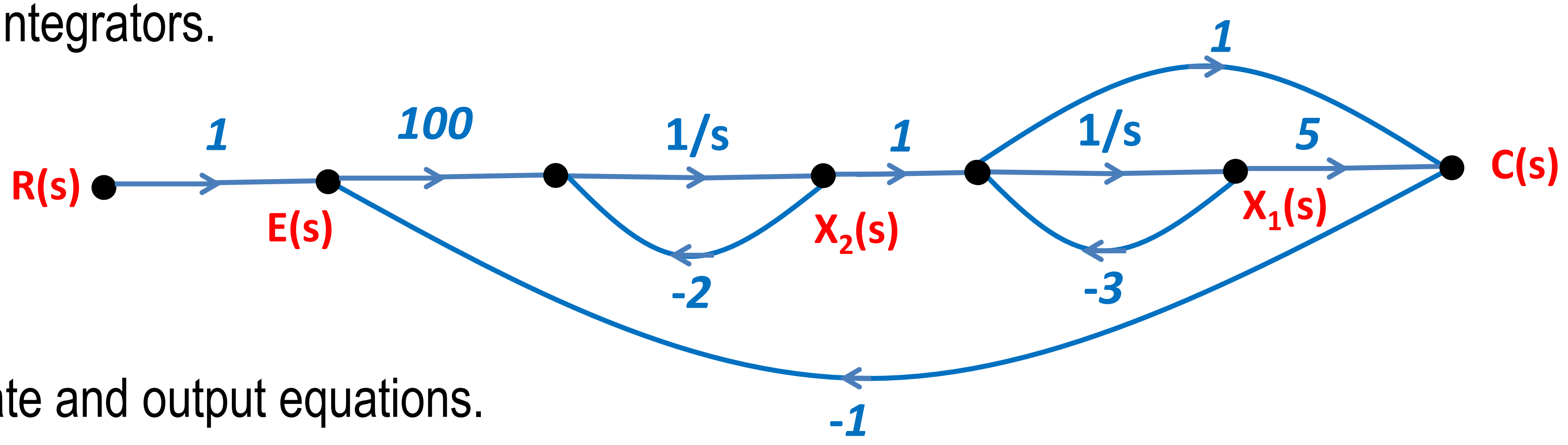


Signal-Flow Graphs & State-Space Equations

Example 8 Draw a signal-flow graph for the following feedback control system. Define appropriate state variables and derive a state-space model.



To find the state-scape model from SFG, first, define the state variables as the output node of the integrators.



Next, derive the state and output equations.

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 \\ \dot{x}_2 = -2x_2 + 100(r - c) \quad \rightarrow \quad \dot{x}_2 = -2x_2 + 100(r - 2x_1 - x_2) = -200x_1 - 102x_2 + 100r \\ y = c \quad \rightarrow \quad y = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2 \end{cases}$$

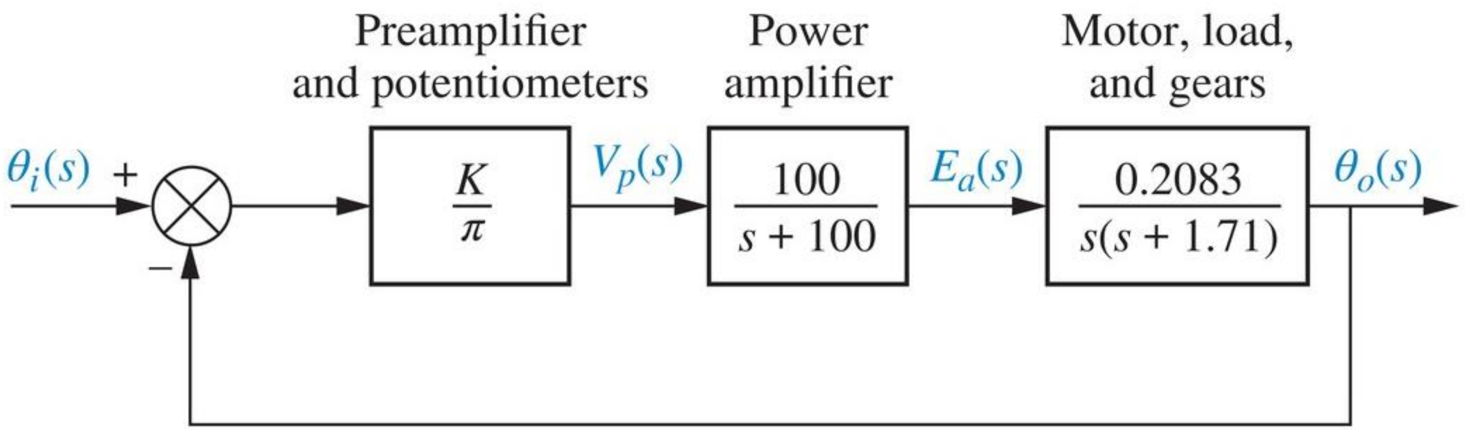
State-space model in vector-matrix form.

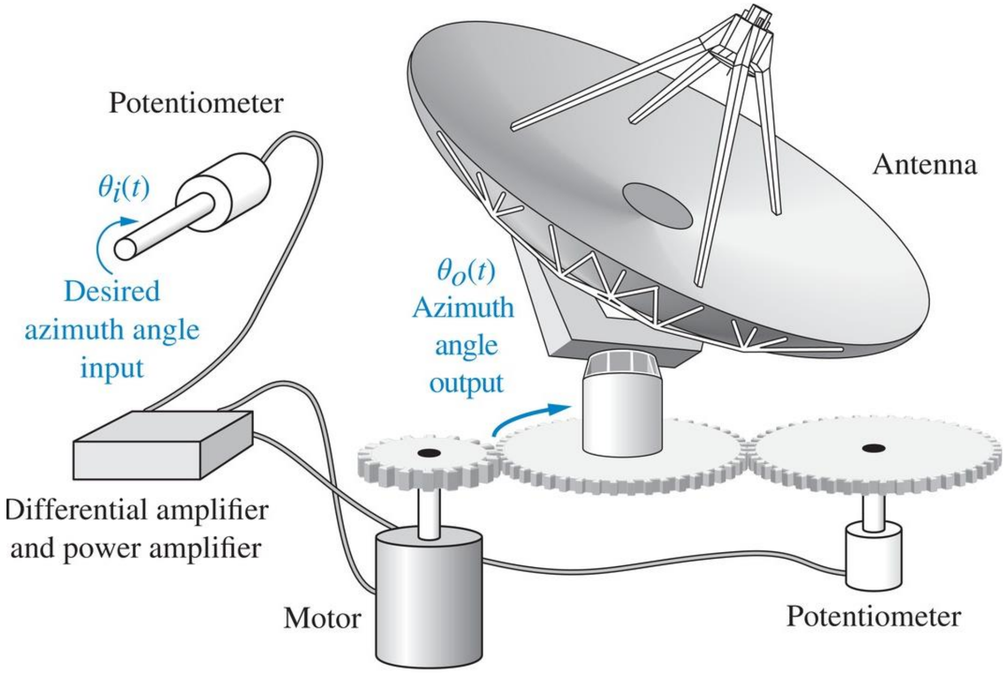
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \mathbf{x}(t)$$

Case Study: Antenna Control System

- Consider the ***motor-driven antenna azimuth position control system*** example from Lecture 1.
- We determined the block diagram of the control system as below:





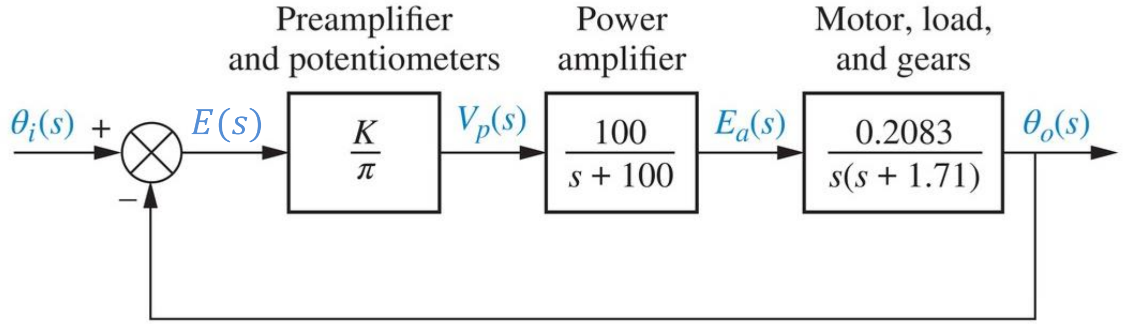
- In this part, we will represent the SFG of overall closed-loop system and find the state-space representation of the closed-loop system.

- We also evaluate the overall transfer function model by applying Mason's gain formula.

Case Study: Antenna Control System

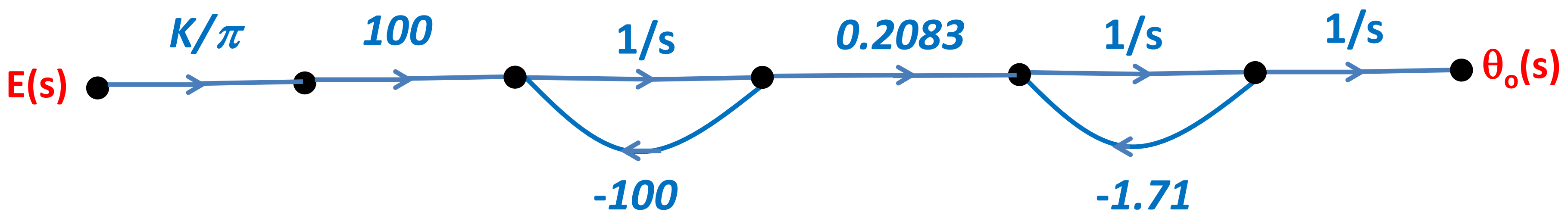
For the antenna azimuth position control system with the given simplified unity-feedback block diagram model:

a) Represent each subsystem with a signal-flow graph and find the state-space representation of the closed-loop system from the signal-flow graph.

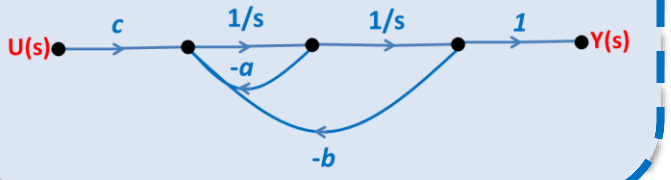


First, model the forward path transfer function in cascade form.

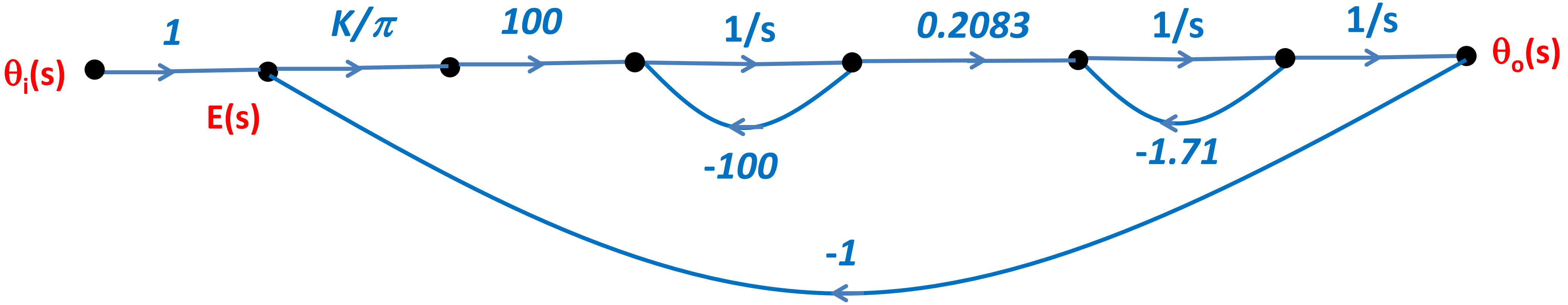
$$\frac{\theta_o(s)}{E(s)} = \left(\frac{K}{\pi}\right)\left(\frac{100}{s+100}\right)\left(\frac{0.2083}{s(s+1.71)}\right)$$



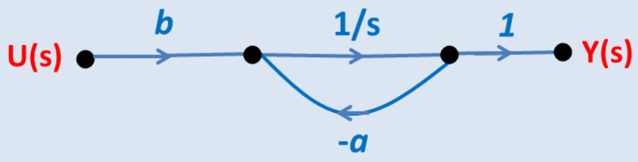
$$\frac{Y(s)}{U(s)} = \frac{c}{s^2 + as + b}$$



Next add the feedback and input paths to complete the SFG.



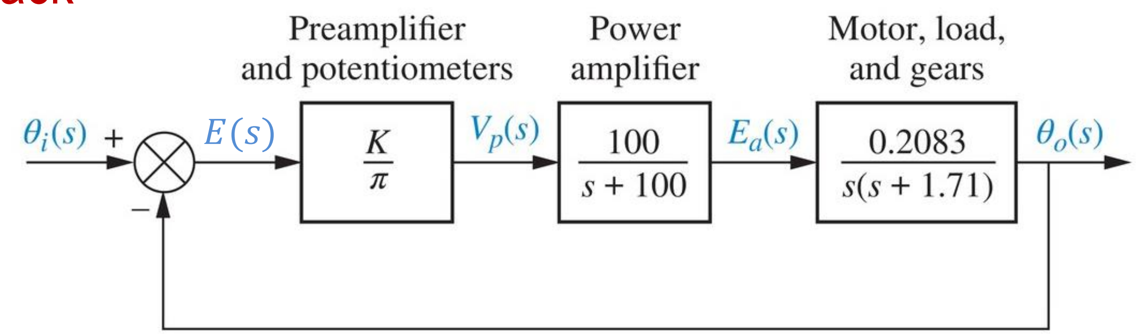
$$\frac{Y(s)}{U(s)} = \frac{b}{s+a}$$



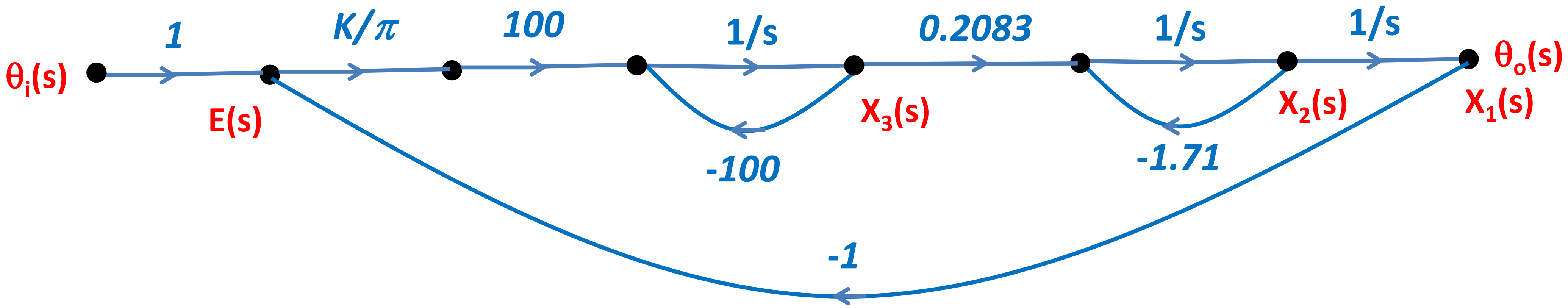
Case Study: Antenna Control System

For the antenna azimuth position control system with the given simplified unity-feedback block diagram model:

a) Represent each subsystem with a signal-flow graph and find the state-space representation of the closed-loop system from the signal-flow graph.



To find the state equations, first, define the state variables as the outputs of the integrators, then write the state equations by inspection of the SFG.



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -1.71x_2 + 0.2083x_3 \\ \dot{x}_3 = -100x_3 + \dfrac{100K}{\pi}(\theta_i - \theta_o) = -100x_3 + 31.83K\theta_i - 31.83Kx_1 \\ y = \theta_o \quad \rightarrow \quad y = x_1 \end{cases}$$

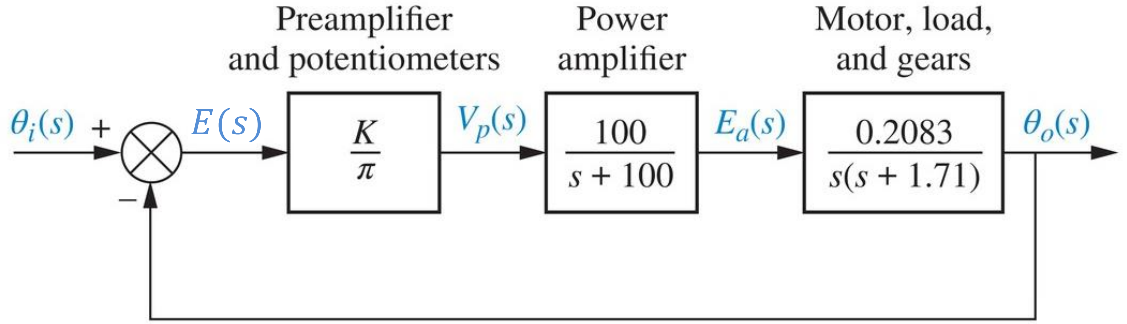
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1.71 & 0.2083 \\ -31.83K & 0 & -100 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ -31.83K \end{bmatrix} \theta_i(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t)$$

Case Study: Antenna Control System

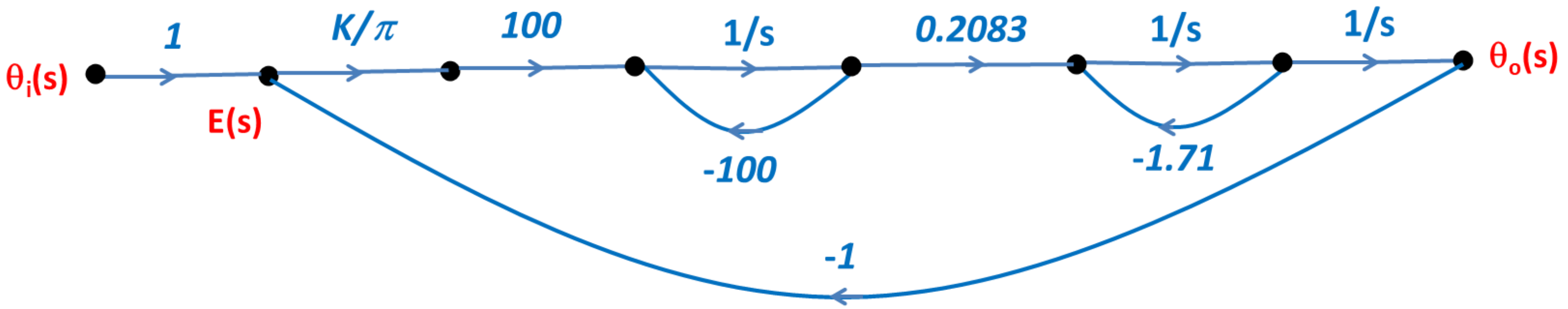
For the antenna azimuth position control system with the given simplified unity-feedback block diagram model:

b) Use the signal-flow graph found in part (a) along with Mason's gain formula to find the closed-loop transfer function.



Step 1: Determine the input node and output node

Input Node: θ_i Output Node: θ_o



Step 2: Calculate all forward path gains between input and output

$$M_1 = \left(\frac{K}{\pi}\right)(100)\left(\frac{1}{s}\right)(0.2083)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right) = \frac{6.63K}{s^3}$$

Step 3: Calculate all loop gains

$$L_1 = -\frac{100}{s}, \qquad L_2 = -\frac{1.71}{s}, \qquad L_3 = -\frac{6.63K}{s^3}$$

Step 4: Determine the non-touching loops

L_1 and L_2 are non-touching loops

Step 5: Calculate determinant of the SFG

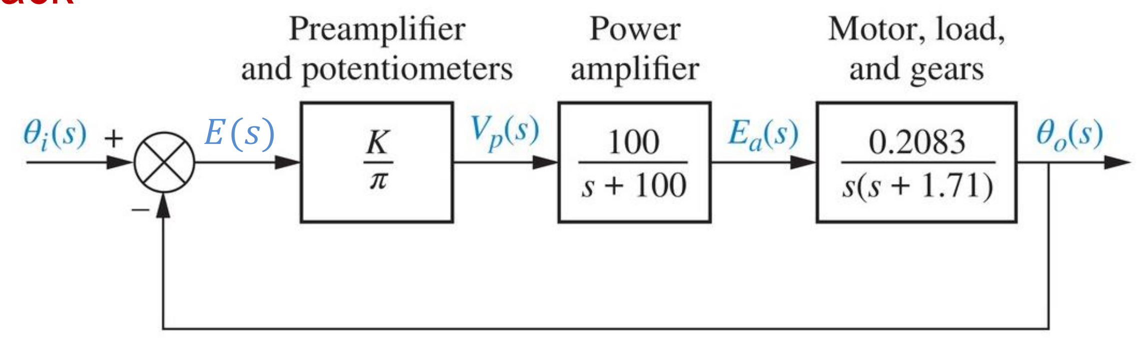
$$\Delta = 1 - (L_1 + L_2 + L_3) + (L_1 L_2) = 1 + \frac{100}{s} + \frac{1.71}{s} + \frac{6.63K}{s^3} + \frac{171}{s^2} = \frac{s^3 + 101.71s^2 + 171s + 6.63K}{s^3}$$

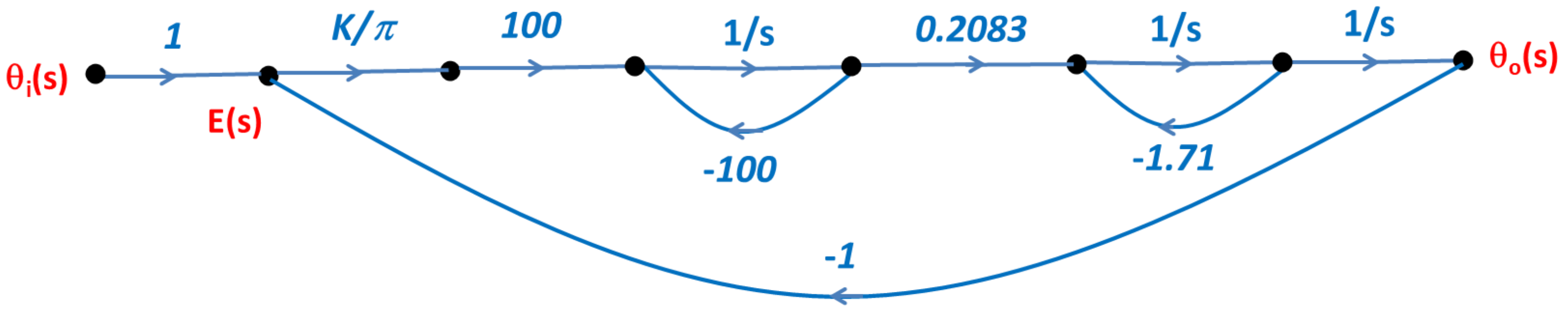
$$\boxed{\frac{Y}{R} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k \Delta_k}$$

Case Study: Antenna Control System

For the antenna azimuth position control system with the given simplified unity-feedback block diagram model:

b) Use the signal-flow graph found in part (a) along with Mason's gain formula to find the closed-loop transfer function.





Step 6: Calculate the cofactors of each forward path

$$\Delta_1 = 1$$

Step 7: Calculate the overall transfer function

$$\boxed{\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{M_1\Delta_1}}{\boldsymbol{\Delta}}}$$

$$\frac{Y(s)}{R(s)} = \frac{M_1\Delta_1}{\Delta} = \frac{\left(\frac{6.63K}{s^3}\right)(1)}{\frac{s^3+101.71s^2+171s+6.63K}{s^3}} = \frac{6.63K}{s^3+101.71s^2+171s+6.63K}$$

$$\boxed{\frac{\boldsymbol{Y}}{\boldsymbol{R}} = \frac{1}{\boldsymbol{\Delta}}\sum_{\boldsymbol{k}=1}^{\boldsymbol{N}} \boldsymbol{M_k\Delta_k}}$$

THANK YOU

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