HUMBER ENGINEERING

MENG-3020
SYSTEMS MODELING & SIMULATION

FINAL EXAM REVIEW

HUMBER

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LECTURE 12
Final Exam Review

- Least-Squares Estimation & Linear Regression Models
- Model Validation
- Time Response Modeling and Analysis
- Modeling of Electromechanical Systems
- Block-Diagram Modeling of Dynamic Systems
- Modeling of Mechanical Systems
- Modeling of Electrical Systems

Least-Squares Estimation & Linear Regression Models

Example 1 Find a least-squares solution of the following equation and determine the Mean Square Error (MSE) of the estimation.

$$A\mathbf{x} = \mathbf{y} \quad \rightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Ty$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 1 & 2 \end{bmatrix}$$

We can determine the least-squares solution by solving the Normal Equation.

$$\boxed{A^TA\hat{\mathbf{x}} = A^T\mathbf{y}}$$

Normal Equation

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{cases} 2\hat{x}_1 + \hat{x}_2 = 5 \\ \hat{x}_1 + 5\hat{x}_2 = 7 \end{cases}$$

$$\rightarrow \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \rightarrow \hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Least-Squares Estimation of the parameters

The MSE of the estimation is obtained as:

$$\boldsymbol{\varepsilon} = \mathbf{y} - \hat{\mathbf{y}} \rightarrow \boldsymbol{\varepsilon} = \mathbf{y} - A\hat{\mathbf{x}} \rightarrow \boldsymbol{\varepsilon} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}}_{y} - \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$MSE = \frac{1}{N}\sum_{k=1}^{N} \varepsilon^2(k) \rightarrow MSE = \frac{1}{3}\left((0)^2 + (0)^2 + (2)^2\right) = 1.33$$

MSE of the estimation

$$\boxed{MSE = \frac{1}{N}\sum_{k=1}^{N} \varepsilon^2(k)}$$

Least-Squares Estimation & Linear Regression Models

Example 2 Assume that the following data points of a static system are available. Estimate a polynomial model by the Least-Squares method to model the data. Determine mean squared error (MSE) of the estimation.

a) Linear Model $\rightarrow\ y = a_1x + a_0$

b) Quadratic Model $\rightarrow\ y = a_2x^2 + a_1x + a_0$

x	-3	1	-7	5
y	70	21	110	-35

a) Linear Model $\rightarrow\ y = a_1x + a_0$

First, convert the data in matrix-vector form and solve the Normal equation.

$$\begin{cases} -3a_1 + a_0 = 70 \\ a_1 + a_0 = 21 \\ -7a_1 + a_0 = 110 \\ 5a_1 + a_0 = -35 \end{cases} \rightarrow \overbrace{\begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix}}^{A} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \overbrace{\begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}}^{y} \rightarrow A\mathbf{x} = \mathbf{y} \xrightarrow{\text{Normal Equation}} A^T A\hat{\mathbf{x}} = A^T\mathbf{y}$$

$$\begin{bmatrix} -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix} \rightarrow \begin{bmatrix} 84 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} -1134 \\ 166 \end{bmatrix}$$

$$\boxed{\hat{a}_1 = -12.1, \qquad \hat{a}_0 = 29.4} \longrightarrow \hat{\mathrm{x}} = \begin{bmatrix} -12.1 \\ 29.4 \end{bmatrix}$$

Estimated linear model $\longrightarrow$ $\boxed{y = -12.1x + 29.4}$

Least-Squares Estimation & Linear Regression Models

Example 2 Assume that the following data points of a static system are available. Estimate a polynomial model by the Least-Squares method to model the data. Determine mean squared error (MSE) of the estimation.

a) Linear Model $\rightarrow$ $y = a_1 x + a_0$

b) Quadratic Model $\rightarrow$ $y = a_2 x^2 + a_1 x + a_0$

x	-3	1	-7	5
y	70	21	110	-35

a) Linear Model $\rightarrow$ $y = a_1 x + a_0$

The MSE of the estimation is obtained as:

$$\boldsymbol{\varepsilon} = \mathbf{y} - \hat{\mathbf{y}} \quad \rightarrow \quad \boldsymbol{\varepsilon} = \mathbf{y} - A\hat{\mathbf{x}} \quad \rightarrow \quad \boldsymbol{\varepsilon} = \overset{Y}{\begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}} - \overset{A}{\begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix}} \overset{\hat{X}}{\begin{bmatrix} -12.1 \\ 29.4 \end{bmatrix}} = \overset{\downarrow}{\begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}} - \overset{\downarrow}{\begin{bmatrix} 65.7 \\ 17.3 \\ 114.1 \\ -31.1 \end{bmatrix}} = \begin{bmatrix} 4.3 \\ 3.7 \\ -4.1 \\ -3.9 \end{bmatrix} \leftarrow$$

$$MSE = \frac{1}{N}\sum_{k=1}^{N} \varepsilon^2(k) \quad \rightarrow \quad MSE = \frac{1}{4}\left((4.3)^2 + (3.7)^2 + (-4.1)^2 + (-3.9)^2\right) = 16.05$$

MSE of the estimation

$$\boxed{MSE = \frac{1}{N}\sum_{k=1}^{N} \varepsilon^2(k)}$$

Least-Squares Estimation & Linear Regression Models

Example 2 Assume that the following data points of a static system are available. Estimate a polynomial model by the Least-Squares method to model the data. Determine mean squared error (MSE) of the estimation.

a) Linear Model $\rightarrow$ $y = a_1 x + a_0$

b) Quadratic Model $\rightarrow$ $y = a_2 x^2 + a_1 x + a_0$

x	-3	1	-7	5
y	70	21	110	-35

b) Quadratic Model $\rightarrow$ $y = a_2 x^2 + a_1 x + a_0$

First, convert the data in matrix-vector form and solve the Normal equation.

$$\begin{cases} 9a_2 - 3a_1 + a_0 = 70 \\ a_2 + a_1 + a_0 = 21 \\ 49a_2 - 7a_1 + a_0 = 110 \\ 25a_2 + 5a_1 + a_0 = -35 \end{cases} \rightarrow \begin{bmatrix} 9 & -3 & 1 \\ 1 & 1 & 1 \\ 49 & -7 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix} \rightarrow A\mathbf{x} = \mathbf{y} \xrightarrow{\text{Normal Equation}} A^T A \hat{\mathbf{x}} = A^T \mathbf{y}$$

$$\begin{bmatrix} 9 & 1 & 49 & 25 \\ -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & -3 & 1 \\ 1 & 1 & 1 \\ 49 & -7 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_2 \\ \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 9 & 1 & 49 & 25 \\ -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix} \rightarrow \begin{bmatrix} 3108 & -244 & 84 \\ -244 & 84 & -4 \\ 84 & -4 & 4 \end{bmatrix} \begin{bmatrix} \hat{a}_2 \\ \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 5166 \\ -1134 \\ 166 \end{bmatrix}$$

$$\hat{a}_2 = -0.25, \qquad \hat{a}_1 = -12.6, \qquad \hat{a}_0 = 34.15 \quad \rightarrow \quad \hat{\mathrm{x}} = \begin{bmatrix} -0.25 \\ -12.6 \\ 34.15 \end{bmatrix}$$

Estimated quadratic model $\longrightarrow$ $\boxed{y = -0.25x^2 - 12.6x + 34.15}$

Least-Squares Estimation & Linear Regression Models

Example 2 Assume that the following data points of a static system are available. Estimate a polynomial model by the Least-Squares method to model the data. Determine mean squared error (MSE) of the estimation.

a) Linear Model $\rightarrow$ $y = a_1 x + a_0$

b) Quadratic Model $\rightarrow$ $y = a_2 x^2 + a_1 x + a_0$

x	-3	1	-7	5
y	70	21	110	-35

b) Quadratic Model $\rightarrow$ $y = a_2 x^2 + a_1 x + a_0$

The MSE of the estimation is obtained as:

$$\boldsymbol{\varepsilon} = \mathbf{y} - \hat{\mathbf{y}} \rightarrow \boldsymbol{\varepsilon} = \mathbf{y} - A\hat{\mathbf{x}} \rightarrow \boldsymbol{\varepsilon} = \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix} - \begin{bmatrix} 9 & -3 & 1 \\ 1 & 1 & 1 \\ 49 & -7 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} -0.25 \\ -12.6 \\ 34.15 \end{bmatrix} = \overset{y}{\begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}} - \overset{\hat{y}}{\begin{bmatrix} 69.7 \\ 21.3 \\ 110.1 \\ -35.1 \end{bmatrix}} = \begin{bmatrix} 0.3 \\ -0.3 \\ -0.1 \\ 0.1 \end{bmatrix}$$

$$MSE = \frac{1}{N}\sum_{k=1}^{N} \varepsilon^2(k) \rightarrow MSE = \frac{1}{4}\left((0.3)^2 + (-0.3)^2 + (-0.1)^2 + (0.1)^2\right) = 0.05$$

MSE of the estimation

$$\boxed{MSE = \frac{1}{N}\sum_{k=1}^{N} \varepsilon^2(k)}$$

Least-Squares Estimation & Linear Regression Models

Example 3 Consider a dynamic system with sampled input $u(k)$, output $y(k)$ and $e(k)$ is white noise. According to the identification experiments we assume the following ARX model for this system.

The a and b are unknown parameters.

$$y(k) = \underbrace{\boxed{\frac{\overbrace{bq^{-1}}^{B(q)}}{\underbrace{1 + aq^{-1}}_{A(q)}}}}_{G(q)} u(k) + \underbrace{\boxed{\frac{1}{\underbrace{1 + aq^{-1}}_{A(q)}}}}_{H(q)} e(k)$$

a) Obtain the one-step ahead optimal predictor of $y(k)$. Write the predictor $\hat{y}(k|k-1)$ as a function of $u(k-1),\ u(k-2), \cdots, y(k-1), y(k-2), \cdots$

$$\boxed{\hat{y}(k|k-1) = \frac{G(q)}{H(q)} u(k) + \left[1 - \frac{1}{H(q)}\right] y(k)}$$

Optimal Predictor

$y(k) = G(q)u(k) + H(q)e(k)$

Here, the model has an ARX structure. Therefore, the general form of the optimal predictor is:

$$\boxed{\hat{y}(k|k-1) = B(q)u(k) + [1 - A(q)]y(k)}$$

$$\begin{cases} A(q) = 1 + aq^{-1} \\ B(q) = bq^{-1} \end{cases}$$

$$\hat{y}(k|k-1) = bq^{-1}u(k) + [1 - (1 + aq^{-1})]y(k)$$

$$\hat{y}(k|k-1) = bq^{-1}u(k) - aq^{-1}y(k)$$

The optimal predictor for the given ARX model is obtained as:

$$\boxed{\hat{y}(k|k-1) = bu(k-1) - ay(k-1)}$$

Least-Squares Estimation & Linear Regression Models

Example 3 Consider a dynamic system with sampled input $u(k)$, output $y(k)$ and $e(k)$ is white noise. According to the identification experiments we assume the following ARX model for this system.

The a and b are unknown parameters.

$$y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k)$$

b) Determine the linear regression form of the given model, where θ is the unknown parameters vector.

$$\rightarrow \quad y(k) = \varphi^T(k)\theta + e(k)$$

Find the input-output equation from the given ARX structure model:

$$y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k) \longrightarrow (1+aq^{-1})y(k) = bq^{-1}u(k) + e(k)$$

$$y(k) + aq^{-1}y(k) = bq^{-1}u(k) + e(k)$$

$$y(k) = -aq^{-1}y(k) + bq^{-1}u(k) + e(k)$$

$$\boxed{y(k) = -ay(k-1) + bu(k-1) + e(k)}$$

Input-output Equation

The linear regression form is:

$$y(k) = \underbrace{[-y(k-1) \quad u(k-1)]}_{\varphi^T(k)} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\theta} + e(k)$$

Least-Squares Estimation & Linear Regression Models

Example 3 Consider a dynamic system with sampled input $u(k)$, output $y(k)$ and $e(k)$ is white noise. According to the identification experiments we assume the following ARX model for this system.

The a and b are unknown parameters.

$$y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k)$$

c) Assume that the following numerical input-output data are available for samples $k = 1, 2, 3, 4$.
Determine a least-squares estimation of the unknown parameters a and b.

k	1	2	3	4
u	2	-2	-2	2
y	0.00	0.00	1.00	1.50

We have the input-output equation of the model from Step (b):

$$y(k) = -ay(k-1) + bu(k-1) + e(t)$$

First, find the Normal Equation by given I/O data samples $k = 1, 2, 3, 4$.

$$\begin{cases} y(2) = -ay(1) + bu(1) + e(2) \\ y(3) = -ay(2) + bu(2) + e(3) \\ y(4) = -ay(3) + bu(3) + e(4) \end{cases}$$

Vector-matrix form

$$\underbrace{\begin{bmatrix} y(2) \\ y(3) \\ y(4) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} -y(1) & u(1) \\ -y(2) & u(2) \\ -y(3) & u(3) \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\theta} + \underbrace{\begin{bmatrix} e(2) \\ e(3) \\ e(4) \end{bmatrix}}_{\epsilon} \quad \Rightarrow \quad Y = \Phi\theta + \epsilon$$

The Normal Equation and its Least-squares solution is obtained as:

$$Y = \Phi\hat{\theta} \rightarrow \quad \Phi^T Y = \Phi^T \Phi \hat{\theta} \quad \rightarrow \quad \hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T Y$$

Least-Squares Estimation & Linear Regression Models

Example 3 Consider a dynamic system with sampled input $u(k)$, output $y(k)$ and $e(k)$ is white noise. According to the identification experiments we assume the following ARX model for this system.

The a and b are unknown parameters.

$$y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k)$$

c) Assume that the following numerical input-output data are available for samples $k = 1, 2, 3, 4$.
Determine a least-squares estimation of the unknown parameters a and b.

k	1	2	3	4
u	2	-2	-2	2
y	0.00	0.00	1.00	1.50

The least squares estimation of unknown parameters a and b are obtained as below

$$\boxed{\hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T Y} \longrightarrow \begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix} = (\Phi^T\Phi)^{-1}\Phi^T Y$$

$$\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix} = \left(\begin{bmatrix}-y(1) & -y(2) & -y(3)\\ u(1) & u(2) & u(3)\end{bmatrix}\begin{bmatrix}-y(1) & u(1)\\ -y(2) & u(2)\\ -y(3) & u(3)\end{bmatrix}\right)^{-1}\begin{bmatrix}-y(1) & -y(2) & -y(3)\\ u(1) & u(2) & u(3)\end{bmatrix}\begin{bmatrix}y(2)\\ y(3)\\ y(4)\end{bmatrix}$$

$$\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix} = \left(\begin{bmatrix}0 & 0 & -1\\ 2 & -2 & -2\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & -2\\ -1 & -2\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 0 & -1\\ 2 & -2 & -2\end{bmatrix}\begin{bmatrix}0\\ 1\\ 1.5\end{bmatrix} \longrightarrow \begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix} = \begin{bmatrix}1 & 2\\ 2 & 12\end{bmatrix}^{-1}\begin{bmatrix}-1.5\\ -3\end{bmatrix}$$

Least-Squares Estimation of the Parameters $\begin{bmatrix}\hat{\boldsymbol{a}}\\ \hat{\boldsymbol{b}}\end{bmatrix} = \begin{bmatrix}\mathbf{-1}\\ \mathbf{-0.25}\end{bmatrix}$

Least-Squares Estimation & Linear Regression Models

Example 3 Consider a dynamic system with sampled input $u(k)$, output $y(k)$ and $e(k)$ is white noise. According to the identification experiments we assume the following ARX model for this system.

The a and b are unknown parameters.

$$y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k)$$

d) Determine MSE of the estimation.

$$\boldsymbol{\varepsilon} = \mathbf{Y} - \widehat{Y} \rightarrow \boldsymbol{\varepsilon} = \mathbf{Y} - \Phi\widehat{\boldsymbol{\theta}} \rightarrow \boldsymbol{\varepsilon} = \begin{bmatrix} 0 \\ 1 \\ 1.5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 0 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1.5 \end{bmatrix} - \begin{bmatrix} -0.5 \\ 0.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

$$\boldsymbol{MSE} = \frac{\mathbf{1}}{\boldsymbol{N}} \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{N}} \boldsymbol{\varepsilon}^{\mathbf{2}}(\boldsymbol{k}) \rightarrow MSE = \frac{1}{3}\left((0.5)^2 + (0.5)^2 + (0)^2\right) = 0.167$$

MSE of the estimation

e) Determine the estimated ARX model based on the estimated model parameters.

From the estimated parameters $\hat{a} = -1$ and $\hat{b} = -0.25$ we have the following ARX model

$$\boxed{y(k) = \frac{-0.25q^{-1}}{1-q^{-1}}u(k) + \frac{1}{1-q^{-1}}e(k)}$$

Estimated ARX Model from LS Method

Model Validation

Example 4 Assume that an identification experiment is performed on a process and following model structure is selected to identify the system.

$$y(k)+ay(k-2)=bu(k-1)+e(k)$$

After modeling the system residual analysis (Normality Test) is performed to validate the identified model.

a) Determine the structure and order of the model. The deterministic part (system dynamics) and stochastic part (noise filter) of the model.

G(q) H(q)

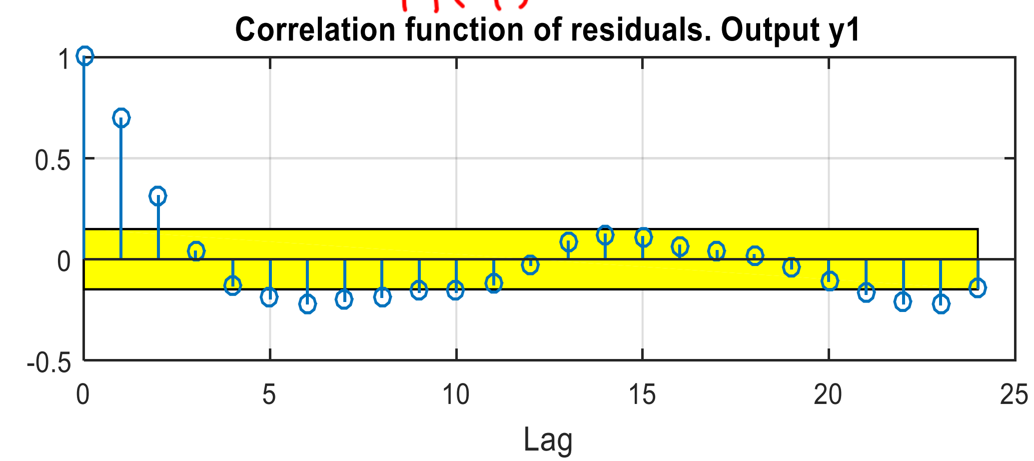
$$y(k)+ay(k-2)=bu(k-1)+e(k)$$

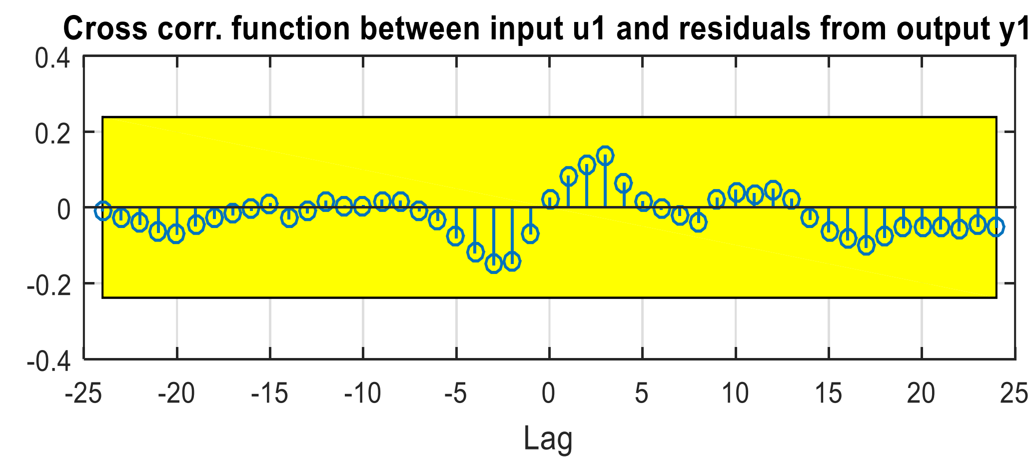
$$y(k)+aq^{-2}y(k)=bq^{-1}u(k)+e(k)$$

$$\underbrace{(1+aq^{-2})}y(k)=\underbrace{bq^{-1}}u(k)+e(k)$$

$$\boxed{y(k)=\frac{bq^{-1}}{1+aq^{-2}}u(k)+\frac{1}{1+aq^{-2}}e(k)}$$

Second-order ARX Model





$$G(q)=\frac{bq^{-1}}{1+aq^{-2}}$$

Deterministic part
System dynamics

$$H(q)=\frac{1}{1+aq^{-2}}$$

Stochastic part
Noise filter

Model Validation

Example 4 Assume that an identification experiment is performed on a process and following model structure is selected to identify the system.

$$y(k) + ay(k-2) = bu(k-1) + e(k)$$

After modeling the system residual analysis (Normality Test) is performed to validate the identified model.

b) What conclusions can you draw from the residual analysis?

Whiteness Test → **FAIL**

Residuals does not have a white noise profile, which means the noise filter model $H(q)$ is not acceptable.

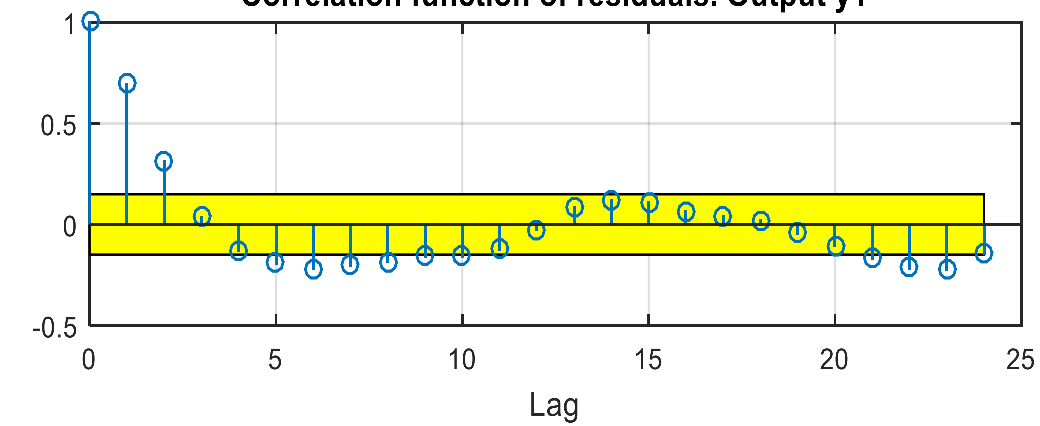
Independence Test → **PASS**

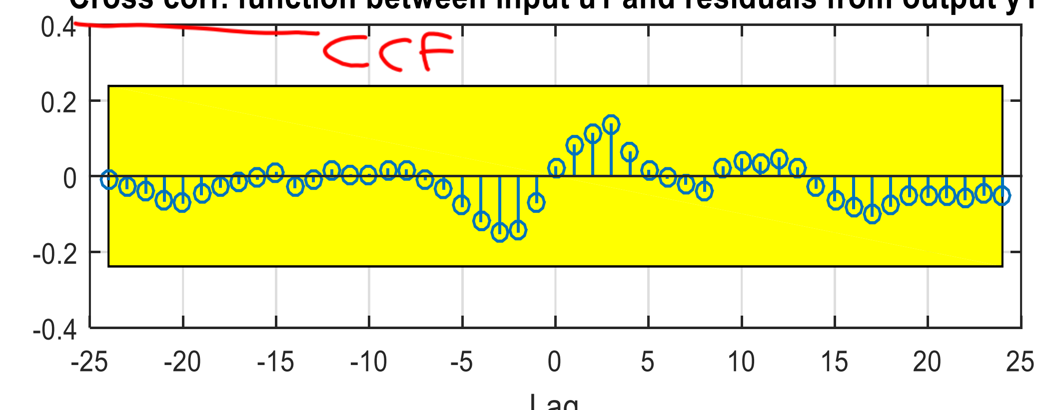
Input signal and residuals are uncorrelated, which means the dynamic system model $G(q)$ is correct.

c) What would be the next step in the identification procedure?

Next step is to determine the noise filter profile or $H(q)$.

ACF





Model Validation

Example 5 We collected input-output data of an unknown system and split the data to identification and validation datasets. The system has been identified by OE model structure by the identification dataset:

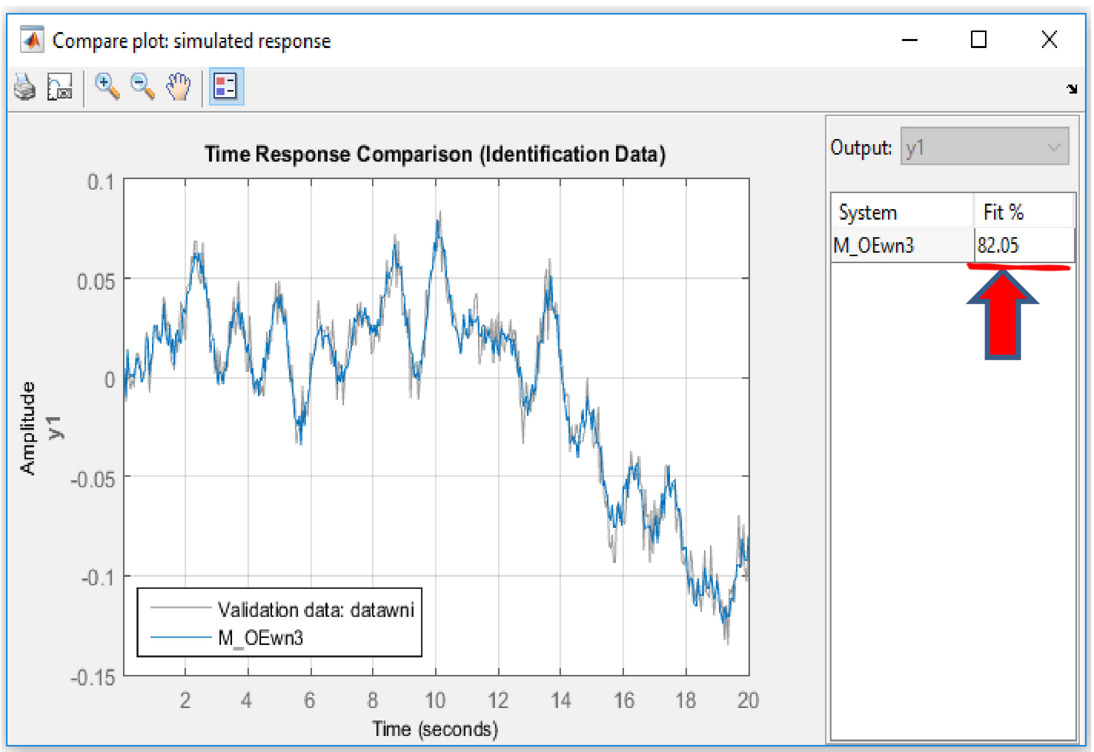
$$y(k) = \frac{B(q)}{F(q)}u(k) + e(k)$$

$H(q) = 1$

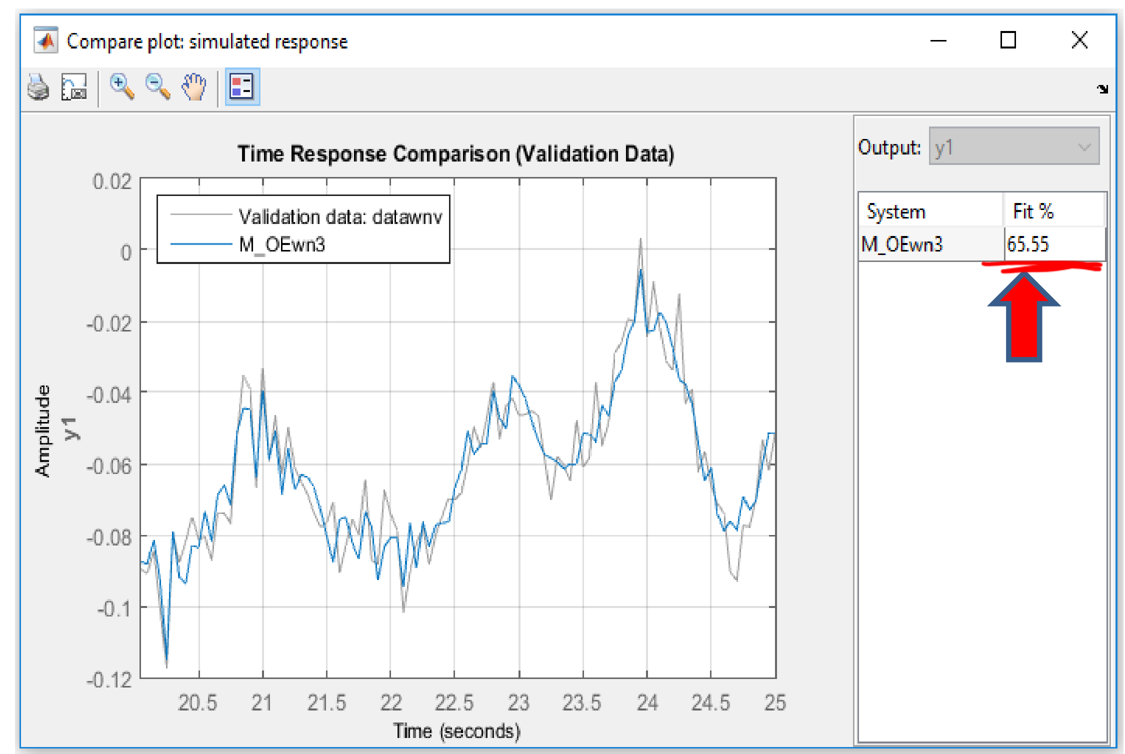
What conclusions can you draw from the validation results?

Why the percentage of fits are different for identification and validation datasets?

Validation by Identification Data



Validation by Validation Data



Since the percentage of fit using validation dataset (65.55%) is drastically less than the percentage of fit using identification dataset (82.05%), we can conclude that the OE model has over-fitting issue.

Model Validation

Example 6 We collected input-output data on an unknown system and identified the system by the following two model structures:

OE

First model $\rightarrow$ $y(k) = \frac{bq^{-1}}{1+fq^{-1}}u(k) + e(k)$

$H(q)=1$

ARX

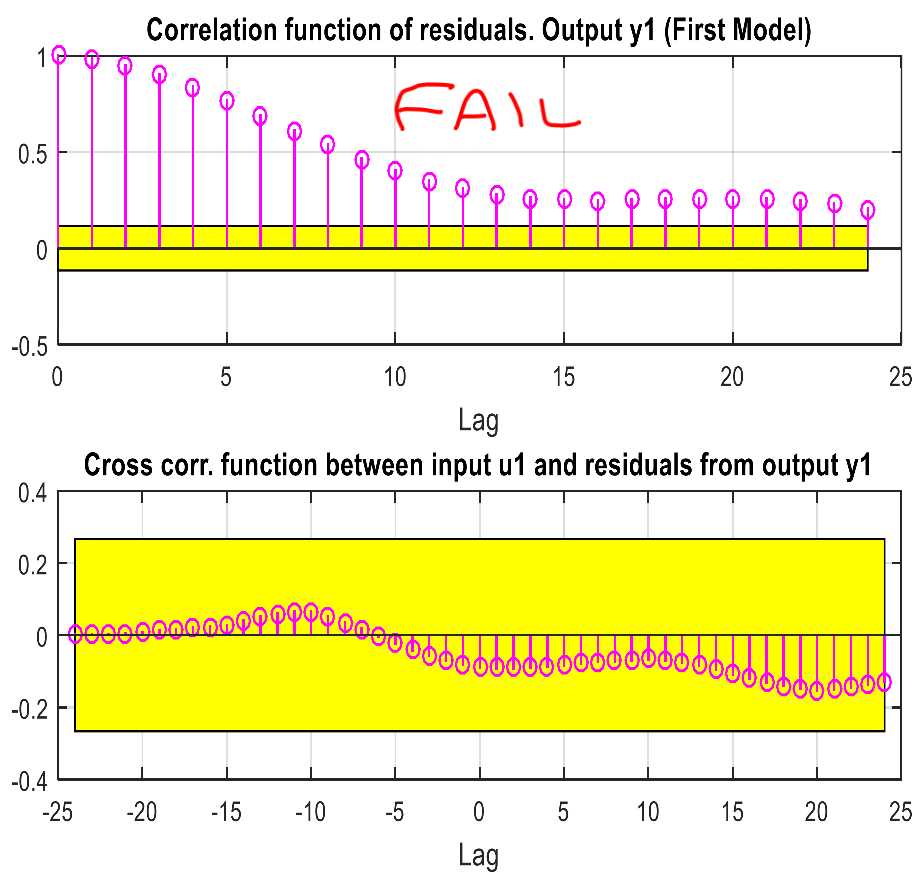
Second model $\rightarrow$ $y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k)$

$\frac{B(q)}{A(q)}$ $\frac{1}{A(q)}$

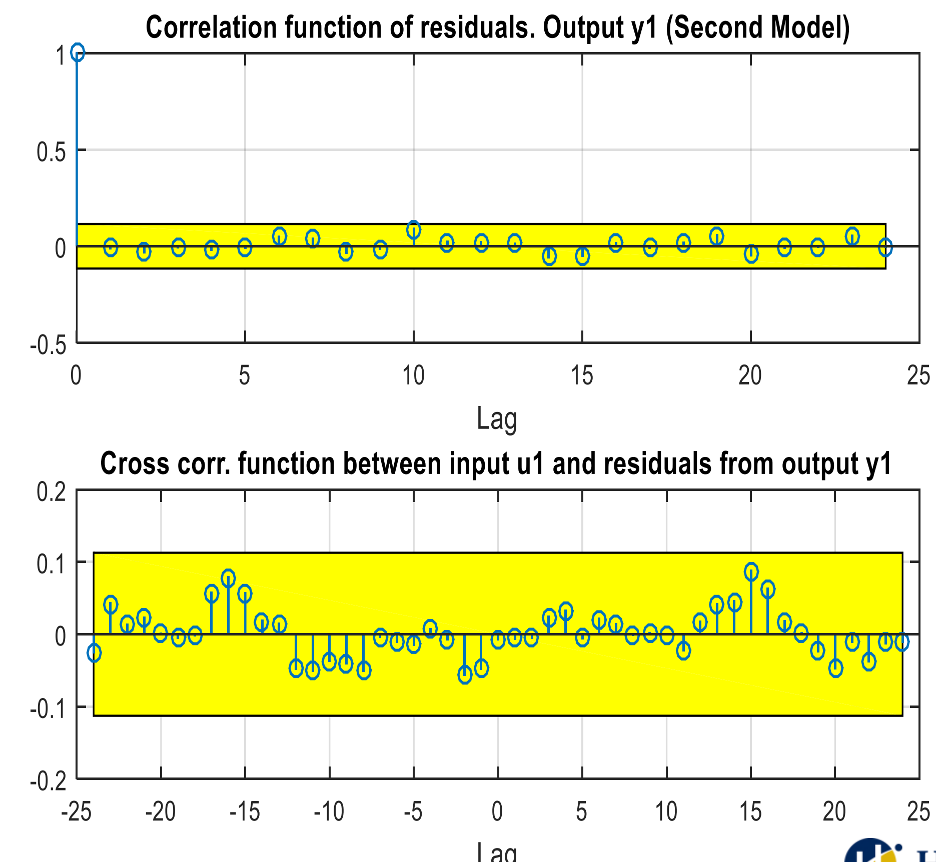
After modeling the system residual analysis (Normality Test) is performed to validate the identified model.

What conclusions can you draw from the residual analysis of each model?

First Model



Second Model



Model Validation

Example 6 We collected input-output data on an unknown system and identified the system by the following two model structures:

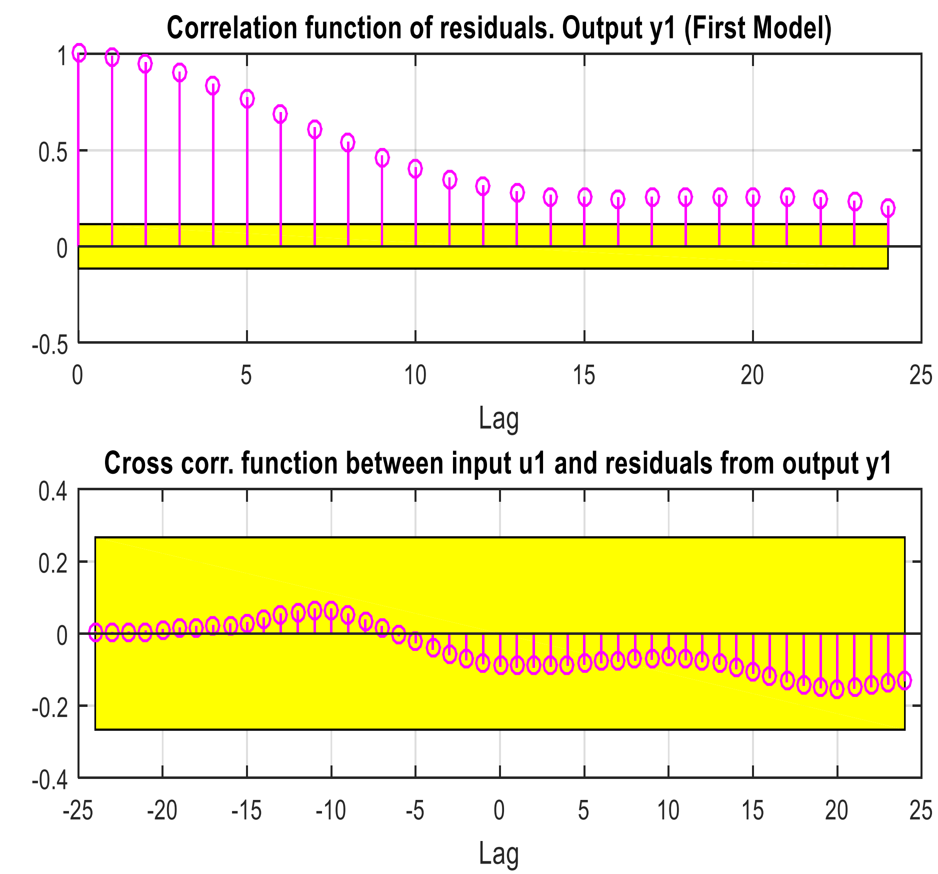
First model $\rightarrow$ $y(k) = \frac{bq^{-1}}{1+fq^{-1}}u(k) + e(k)$

Second model $\rightarrow$ $y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k)$

After modeling the system residual analysis (Normality Test) is performed to validate the identified model.

What conclusions can you draw from the residual analysis of each model?

First Model



First model is a first-order OE model.

The residual analysis of first model shows that:

1) ACF fails the Whiteness Test of residuals. It means assumption of white noise is not correct, and noise model of $H(q) = 1$ is not acceptable, which means the noise is not white noise.

2) CCF passes the Independency Test. In means there is no correlation between the residuals and the input data.

Model Validation

Example 6 We collected input-output data on an unknown system and identified the system by the following two model structures:

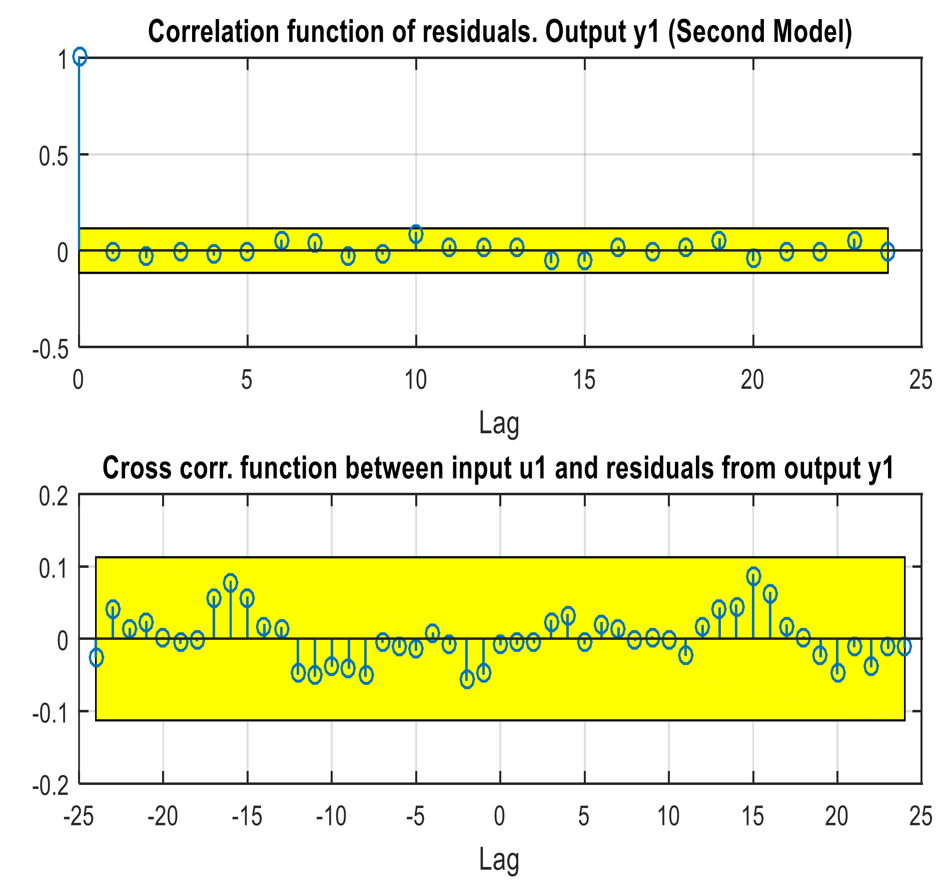
First model $\rightarrow$ $y(k) = \frac{bq^{-1}}{1+fq^{-1}}u(k) + e(k)$

Second model $\rightarrow$ $y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + \frac{1}{1+aq^{-1}}e(k)$

After modeling the system residual analysis (Normality Test) is performed to validate the identified model.

What conclusions can you draw from the residual analysis of each model?

Second Model



Second model is a first-order ARX model.

The residual analysis of second model shows that:

1) ACF passes the Whiteness Test of residuals. It means noise model $H(q) = \frac{1}{1+aq^{-1}}$ is correct, and the noise is a process noise.

2) CCF passes the Independency Test. In means there is no correlation between the residuals and the input data.

Examples of Conceptual Questions

1) What is system identification? How is it used in engineering applications?

System identification is the process of building mathematical models of dynamic systems from observed data. It is used to model real-world systems for control, prediction, and analysis when the system dynamics are unknown or complex.

2) Difference between the parametric and non-parametric models in system identification. Provide examples of each.

Parametric models involve defining a model structure with unknown parameters, for example, transfer function, state-space, ARX model, OE model, BJ model, while non-parametric models do not assume a specific structure but instead estimate the system response directly from the data, step response, impulse response, frequency response.

3) Why is it important to ensure proper data quality during system identification? Discuss at least two preprocessing techniques.

*Good data quality ensures the accuracy and reliability of the identified model. Preprocessing techniques include **detrending** (removing trends or DC-offset) and **removing outliers** (removing bad data due to measurement errors).*

4) What is the least squares estimation method used in system identification? Provide the basic steps involved in the process.

The least squares method estimates the parameters of a model by minimizing the sum of squared errors between the predicted and actual outputs.

$$LSE = \sum_{k=1}^{N} \varepsilon^2(k)$$

Steps include choosing a model structure which can be shown in linear regression form, collecting input-output data, forming the Normal equation and solving for the parameters that minimize the error.

Examples of Conceptual Questions

5) After identifying a system model, how would you validate the accuracy of your model? Describe at least two methods of validation.

Model validation can be done by:

- ***Cross-validation,*** *comparing the model output to new, unseen data to check the generalizability.*
- ***Residual analysis****, which involves checking the error between the model prediction and actual output to ensure no patterns remain.*

6) What is the difference between overfitting and underfitting in the context of system identification, and how can they be avoided?

Overfitting *occurs when a model is too complex, capturing noise in the data, while* ***underfitting*** *happens when the model is too simple to capture the system's dynamics.*
These can be avoided by choosing an appropriate model structure, model order, and using cross-validation.

7) How are non-parametric time-domain methods used for system identification? Give an example of a simple time-domain test.

Non-parametric time-domain methods involve analyzing system output in response to an input signal over time.
An example is using a step response to identify the time constants and DC-gain of a system.

8) What is a state-space model, and why is it useful in system identification?

A state-space model represents a system using a set of first-order differential equations.
It is useful in system identification because it can model multi-input, multi-output systems and can describe both linear and nonlinear dynamics.

9) How does system identification play a role in designing a controller for an unknown system?

System identification helps by providing an estimated model of the system dynamics, which can be used to design controllers and improving the system's performance.

Examples of Conceptual Questions

10) What is cross-validation in the context of system identification, and why is it important?

Cross-validation involves splitting the data into identification (training) and validation (testing) datasets to evaluate the model's performance on unseen data. This helps ensure the model does not overfit and generalizes well to new data.

11) Given a set of noisy input-output data from a temperature control system, describe the steps you would take to identify a model for this system.

- *Preprocess the data to remove any trends or noise.*
- *Choose a model structure such as ARX, OE, CT transfer function, state-space.*
- *Estimate the parameters using techniques such as least squares estimation.*
- *Validate the model by comparing the predicted output with actual measurements using cross-validation and checking whiteness in ACF of the residuals and independency in CCF of the residuals and input data using residual analysis.*
- *Refine the model by adjusting the structure.*

12) What is the difference between overfitting and over-parametrizing in the context of system identification, and how can they be avoided?

Overfitting *occurs when a model is too complex and fits the training data too closely, capturing not only the underlying system dynamics but also the noise and random fluctuations in the data. This results in a model that performs well on the training data but poorly on new, unseen data because it has "learned" the specific characteristics of the training set rather than generalizable patterns. Overfitting can be avoided by cross-validation, which splitting data into training and test sets to ensure the model performs well on unseen data.*

Over-parametrizing *refers to the situation where a model has more parameters than necessary to represent the system's true dynamics, which can occur even in noise-free dataset. This can lead to overfitting, but it also means that the model may be unnecessarily complicated (extra poles and zeros) and difficult to estimate accurately, especially if the available data is limited. It can be avoided by correct order selection and pole-zero cancellation.*

Examples of Conceptual Questions

13) What is the potential risk of choosing too high or too low sampling frequency when doing system identification?

Choosing a Sampling Frequency That Is Too Low

- ***Aliasing****: If the sampling frequency is too low (below the* **Nyquist rate***, which is twice the highest frequency component of the system being modeled),* ***aliasing*** *can occur. This means that higher-frequency signals may be misrepresented as lower-frequency components, leading to distorted or incorrect data that doesn't accurately reflect the real system behavior.*
- ***Loss of Information:*** *A low sampling frequency might not capture high-frequency dynamics of the system, leading to a loss of important details. This can result in an inaccurate model, particularly when the system has fast dynamics (high-frequency behavior).*
- ***Inaccurate Estimation of System Parameters:*** *Low sampling frequencies may prevent the accurate estimation of system parameters (e.g., time constants, resonance frequencies) because not enough data points are available to capture fast changes.*

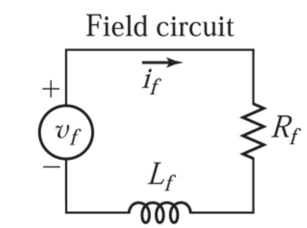
Choosing a Sampling Frequency That Is Too High

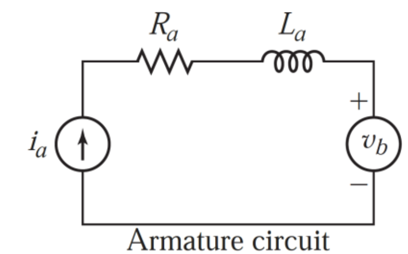
- ***Increased Computational Load:*** *A higher sampling frequency means more data points are collected per unit of time, which leads to larger datasets. This increases computational requirements for both storage and processing, making system identification more time-consuming and resource-intensive.*
- ***Noise Sensitivity:*** *Higher sampling frequencies can also amplify noise. If the system's noise level is high or if there is sensor noise, a higher sampling frequency might capture more noise, which can degrade the quality of the system identification and make it harder to distinguish between the true system dynamics and noise.*
- ***Overfitting Risk:*** *When data is sampled at a very high frequency, the model may "overfit" to the noise or small variations in the data that are irrelevant to the system's true behavior. This means that the identified model may capture the noise or artifacts in the data rather than the actual system dynamics.*
- ***Redundancy in Data:*** *Sampling at a frequency much higher than necessary might produce redundant information, which doesn't add value to the system identification but increases the time required to process and analyze the data.*

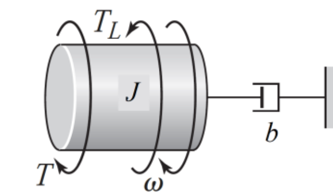
Modeling of Electromechanical Systems

Example 7

Consider the following filed-controlled DC motor, where L_f and R_f are the field circuit inductance and resistance, L_a and R_a are the armature circuit inductance and resistance. The torque developed by the motor is assumed to be related linearly to the field current by $\tau(t) = k_T i_f(t)$.







a) Derive the set of differential equations to model the DC motor.

We have to derive the equations for electrical and the mechanical subsystems of the DC motor.

The differential equation model of electrical subsystem is obtained by applying a KVL to the field circuit :

$$v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt}$$

The mechanical subsystem model is obtained by applying Newton's second law to the inertia J,

$$\tau(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + b\omega(t)$$

The set of differential equation to model the DC motor is:

$$\begin{cases} v_f(t) = R_f i_f(t) + L_f \dfrac{di_f(t)}{dt} \\ k_T i_f(t) - \tau_L(t) = J\dfrac{d\omega(t)}{dt} + b\omega(t) \\ \tau(t) = k_T i_f(t) \end{cases}$$

Modeling of Electromechanical Systems

Example 7

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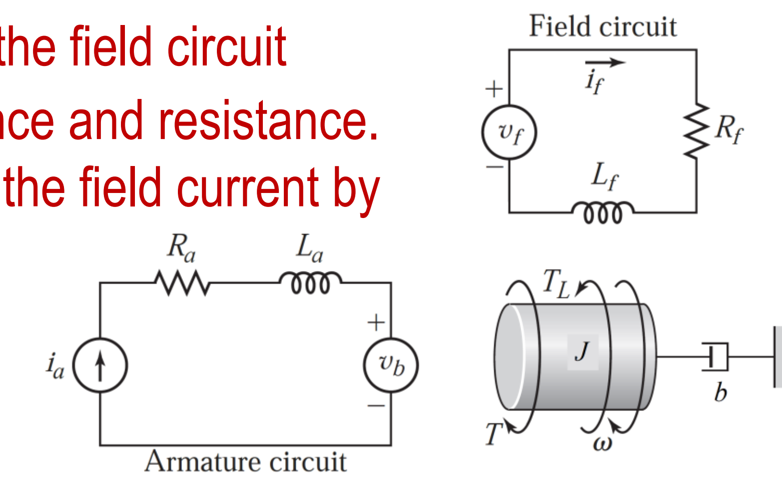


b) Find the block diagram model of the field-controlled DC motor for $v_f(t)$ and $\tau_L(t)$ as inputs and $\theta(t)$ as output.

Take Laplace transform of the differential equation with zero initial conditions :

$$v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt} \rightarrow V_f(s) = R_f I_f(s) + L_f s I_f(s) = (R_f + L_f s) I_f(s) \rightarrow I_f(s) = \frac{1}{R_f + L_f s} V_f(s)$$

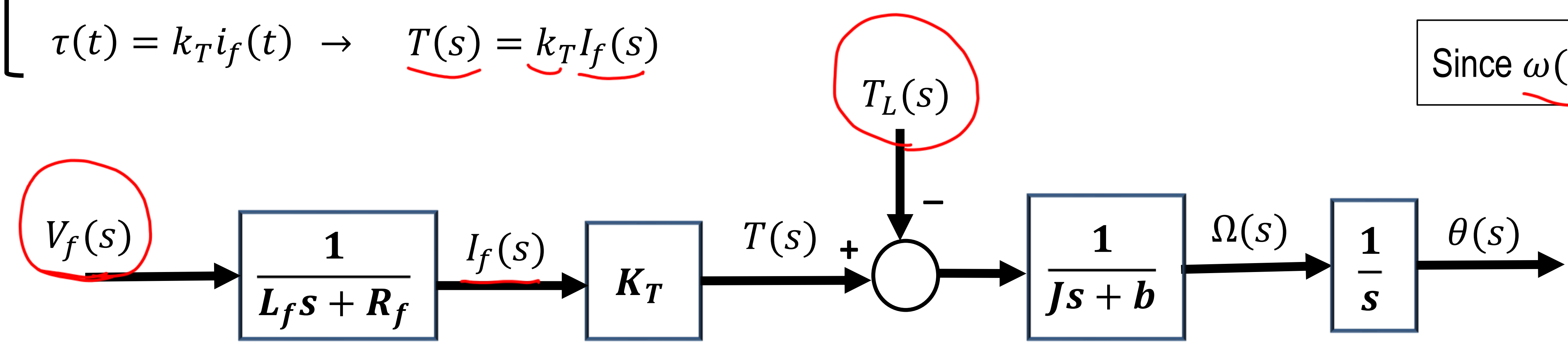
electrical subsystem

$$\tau(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + b\omega(t) \rightarrow T(s) - T_L(s) = Js\Omega(s) + b\Omega(s) \rightarrow \Omega(s) = \frac{1}{Js + b}(T(s) - T_L(s))$$

mechanical subsystem

$$\tau(t) = k_T i_f(t) \rightarrow T(s) = k_T I_f(s)$$

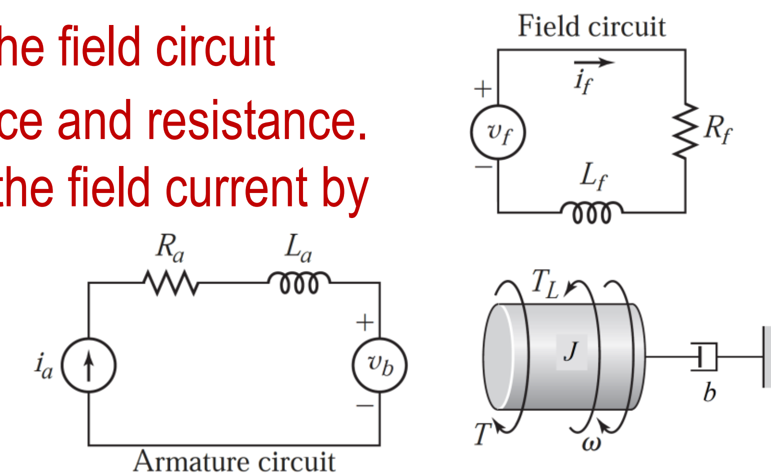
Since $\omega(t) = \frac{d\theta(t)}{dt}$ then $\Omega(s) = s\theta(s)$



Modeling of Electromechanical Systems

Example 7

Consider the following filed-controlled DC motor, where L_f and R_f are the field circuit inductance and resistance, L_a and R_a are the armature circuit inductance and resistance. The torque developed by the motor is assumed to be related linearly to the field current by $\tau(t) = k_T i_f(t)$.

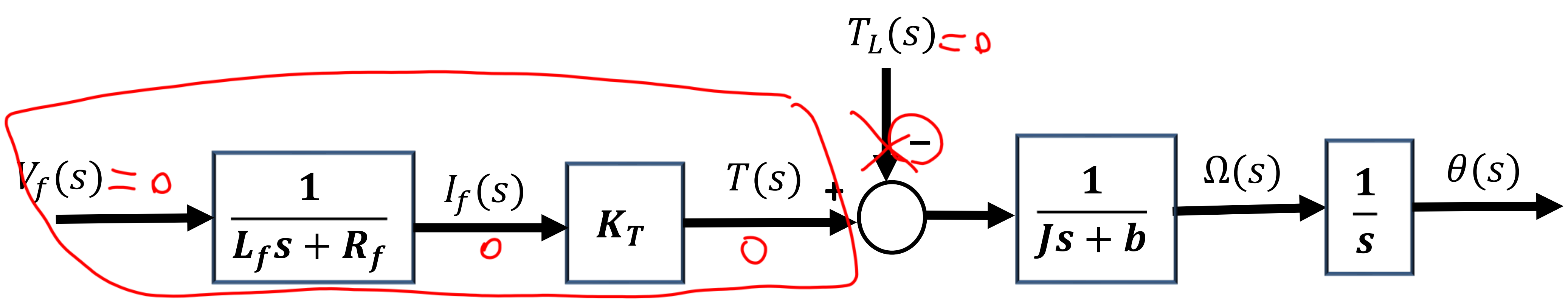


c) Find the transfer function of the field-controlled DC motor for $v_f(t)$ and $\tau_L(t)$ as inputs and $\theta(t)$ as output.

The transfer function model is obtained by applying superposition principle and by simplifying the block diagram model:

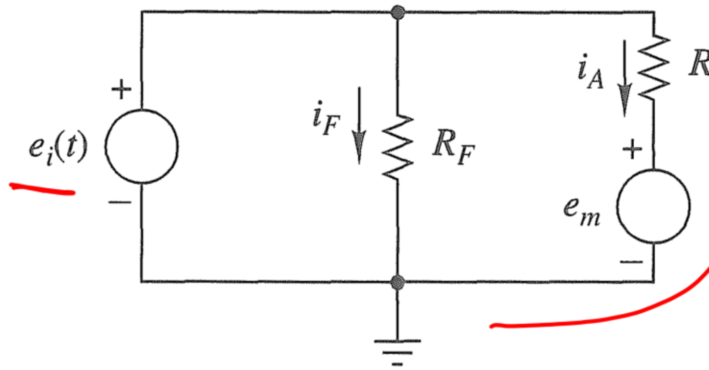
- Assume $T_L = 0$: $\rightarrow \quad \theta(s) = \dfrac{k_T}{s(R_f + L_f s)(Js + b)} V_f(s)$
- Assume $V_f = 0$: $\rightarrow \quad \theta(s) = -\dfrac{1}{s(Js + b)} T_L(s)$

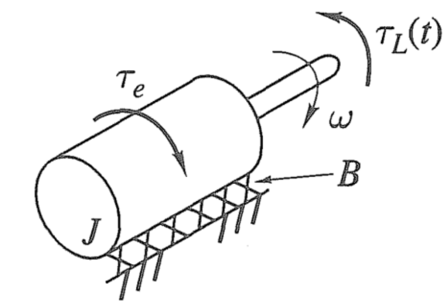
$$\boxed{\theta(s) = \frac{k_T}{s(R_f + L_f s)(Js + b)} V_f(s) - \frac{1}{s(Js + b)} T_L(s)}$$



Modeling of Electromechanical Systems

Example 8 The field and armature windings of an electric motor are connected in parallel (shunt) directly across a voltage source $e_i(t)$. The resistance of field and armature windings are R_F and R_A, respectively, and the inductances of both windings are negligible.





Assume that the torque developed by the motor is related to the field and armature currents by $\tau_e(t) = k_T i_F(t) i_A(t)$ and the back-emf voltage is related to the field current and the motor speed by $e_m(t) = k_m i_F(t)\omega(t)$.

Find a differential equation relating $\omega(t)$ to $e_i(t)$ and $\tau_L(t)$.

From the mechanical subsystem we have:

$$\tau_e(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + B\omega(t) \quad \rightarrow \quad k_T i_F(t) i_A(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + B\omega(t)$$

From the field circuit we have:

$$e_i(t) = R_F i_F(t) \quad \rightarrow \quad i_F(t) = \frac{e_i(t)}{R_F}$$

From the armature circuit we have:

$$e_i(t) = R_A i_A(t) + e_m(t) \quad \rightarrow \quad i_A(t) = \frac{e_i(t) - e_m(t)}{R_A} = \frac{e_i(t) - k_m i_F(t)\omega(t)}{R_A} = \frac{e_i(t) - k_m\left(\frac{e_i(t)}{R_F}\right)\omega(t)}{R_A} = \left(\frac{R_F - k_m\omega(t)}{R_F R_A}\right)e_i(t)$$

Modeling of Electromechanical Systems

Example 8

The field and armature windings of an electric motor are connected in parallel (shunt) directly across a voltage source $e_i(t)$. The resistance of field and armature windings are R_F and R_A, respectively, and the inductances of both windings are negligible.

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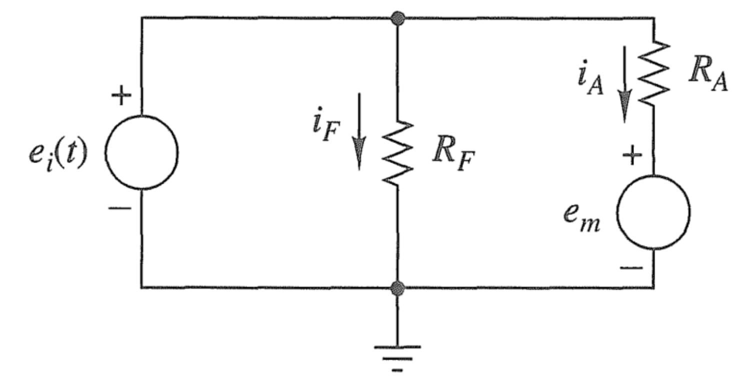
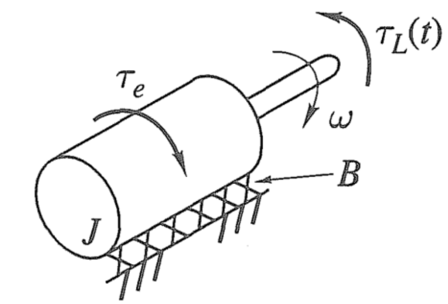
Find a differential equation relating $\omega(t)$ to $e_i(t)$ and $\tau_L(t)$.





$$\begin{cases} i_A(t) = \left(\dfrac{R_F - k_m\omega(t)}{R_F R_A}\right) e_i(t) \\[2ex] i_F(t) = \dfrac{e_i(t)}{R_F} \end{cases}$$

Substitute the $i_F(t)$ and $i_A(t)$ into the mechanical subsystem equation:

$$k_T i_F(t) i_A(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + B\omega(t)$$

$$k_T\left(\frac{e_i(t)}{R_F}\right)\left(\frac{R_F - k_m\omega(t)}{R_F R_A}\right)e_i(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + B\omega(t)$$

$$\left(\frac{k_T e_i^2(t)}{R_F}\right)\left(\frac{1}{R_A} - \frac{k_m\omega(t)}{R_F R_A}\right) - \tau_L(t) = J\frac{d\omega(t)}{dt} + B\omega(t)$$

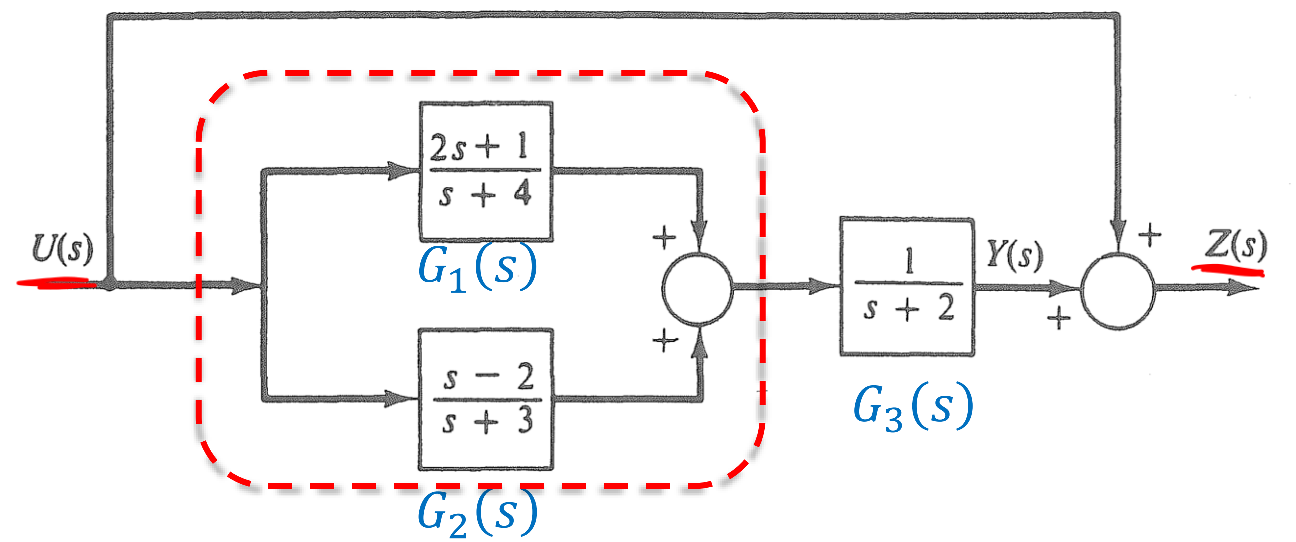
$$\frac{k_T e_i^2(t)}{R_F R_A} - \frac{k_T k_m e_i^2(t)}{R_F^2 R_A}\omega(t) - \tau_L(t) = J\frac{d\omega(t)}{dt} + B\omega(t)$$

The differential equation model is:

$$\boxed{J\frac{d\omega(t)}{dt} + \left(B + \frac{k_T k_m e_i^2(t)}{R_F^2 R_A}\right)\omega(t) = \frac{k_T}{R_F R_A}e_i^2(t) - \tau_L(t)}$$

Block Diagram Modeling of Dynamic Systems

Example 9 Evaluate the transfer functions $Y(s)/U(s)$ and $Z(s)/U(s)$ for the block diagram shown below.



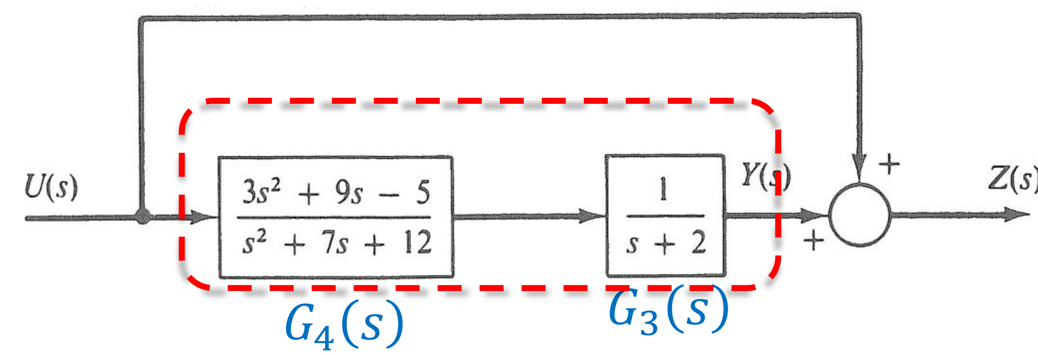
We can name the blocks as $G_1(s), G_2(s)$ and $G_3(s)$.

First, simplify the parallel combination of G_1 and G_2:

$$G_4(s) = G_1(s) + G_2(s) = \frac{2s+1}{s+4} + \frac{s-2}{s+3} = \frac{3s^2+9s-5}{s^2+7s+12}$$

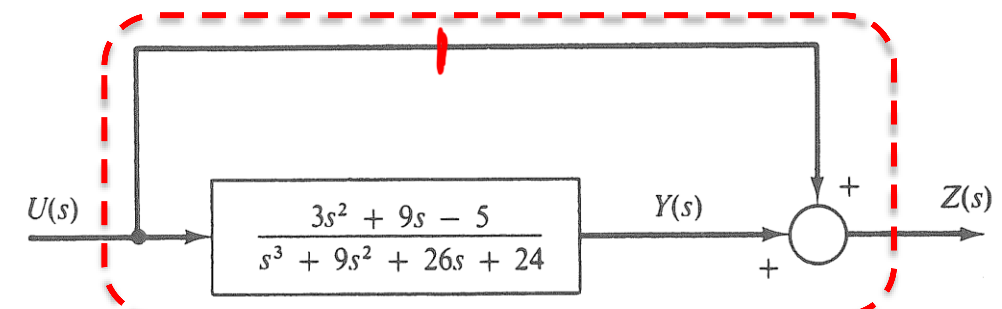
The series combination of G_4 and G_3 gives us the $Y(s)/U(s)$:

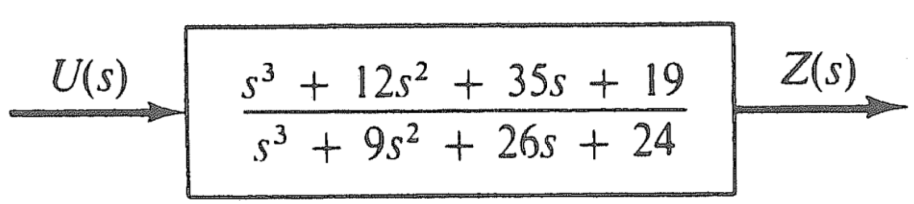
$$\frac{Y(s)}{U(s)} = G_4(s)G_3(s) = \left(\frac{3s^2+9s-5}{s^2+7s+12}\right)\left(\frac{1}{s+2}\right) = \frac{3s^2+9s-5}{s^3+9s^2+26s+24}$$



We can reduce the final parallel combination to obtain the $Z(s)/U(s)$:

$$\frac{Z(s)}{U(s)} = 1 + \frac{Y(s)}{U(s)} = 1 + \frac{3s^2+9s-5}{s^3+9s^2+26s+24} = \frac{s^3+12s^2+35s+19}{s^3+9s^2+26s+24}$$

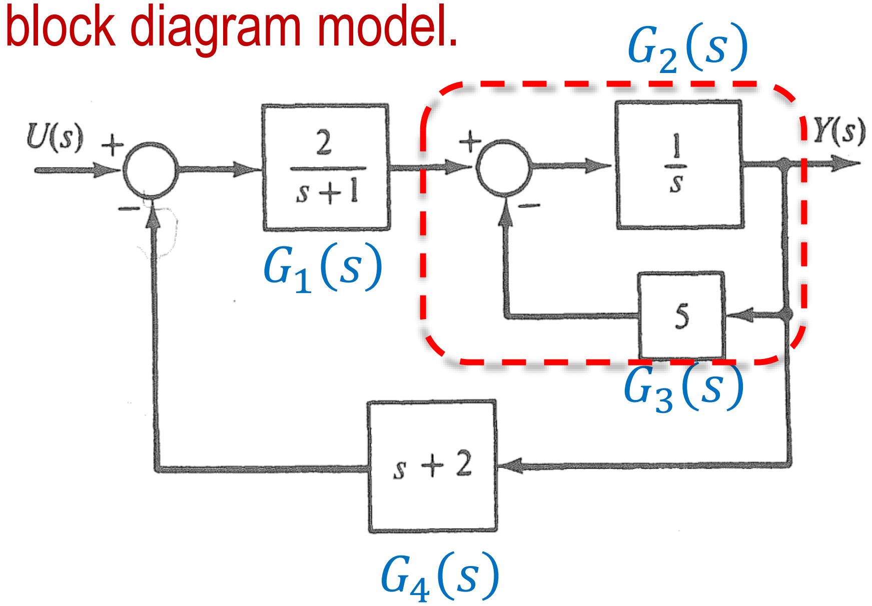




Block Diagram Modeling of Dynamic Systems

Example 10

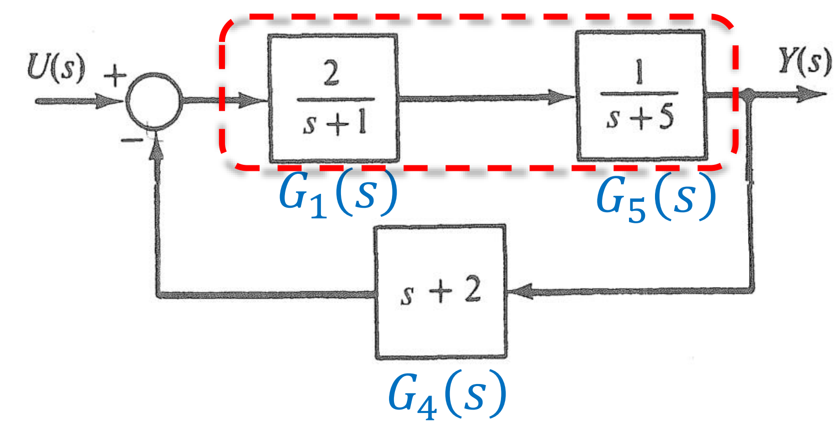
Determine the closed-loop transfer function $Y(s)/U(s)$ for each block diagram model.



We can name the blocks as $G_1(s), G_2(s), G_3(s)$ and $G_4(s)$.

First, simplify the negative feedback combination of G_2 and G_3:

$$G_5(s) = \frac{G_2(s)}{1 + G_2(s)G_3(s)} = \frac{\frac{1}{s}}{1 + \left(\frac{1}{s}\right)(5)} = \frac{\frac{1}{s}}{\frac{s+5}{s}} = \frac{1}{s+5}$$

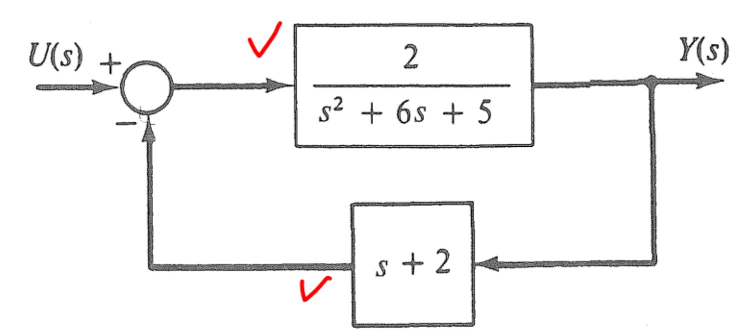


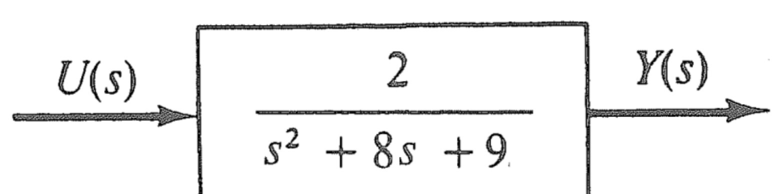
Next, simplify the series combination of G_1 and G_5:

$$G_6(s) = G_1(s)G_5(s) = \left(\frac{2}{s+1}\right)\left(\frac{1}{s+5}\right) = \frac{2}{s^2 + 6s + 5}$$

We can reduce the final negative feedback combination to obtain the $Y(s)/U(s)$:

$$\frac{Y(s)}{U(s)} = \frac{G_6(s)}{1 + G_6(s)G_4(s)} = \frac{\frac{2}{s^2+6s+5}}{1 + \left(\frac{2}{s^2+6s+5}\right)(s+2)} = \frac{\frac{2}{s^2+6s+5}}{\frac{s^2+8s+9}{s^2+6s+5}} = \frac{2}{s^2+8s+9}$$





Time Response Modeling & Analysis

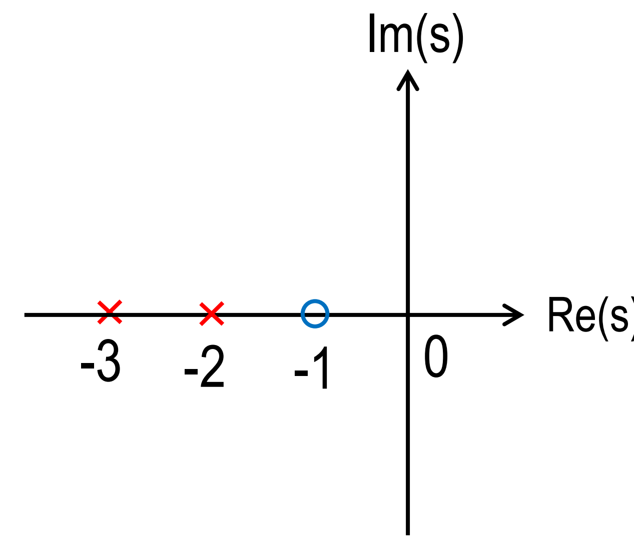
Example 11 Consider the following transfer function model of a dynamic system.

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2+5s+6}$$

a) Find the poles and zeroes and sketch the pole-zero map for the following transfer function. Judge the stability of the system.

First, find the poles and zeros of the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)} \rightarrow \begin{cases} \textbf{zeros} \rightarrow s = -1 \\ \textbf{poles} \rightarrow s = -2, \quad s = -3 \end{cases}$$



Complex s-plane

Since, all poles are located at the left-half of the s-plane, the system is stable.

b) Find the system's input-output differential equation.

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2+5s+6} \rightarrow s^2Y(s) + 5sY(s) + 6Y(s) = sU(s) + U(s)$$

Take inverse Laplace transform to find the differential equation:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = \dot{u}(t) + u(t)$$

Example 12

Consider the following transfer function model of a dynamic system.

$$\frac{Y(s)}{U(s)} = \underbrace{\frac{s+1}{s^2+5s+6}}$$

c) Use the Initial-value and Final-value theorems to find the initial value and final value of the system response to a step input of $u(t) = 6, \quad t \geq 0$.

From the initial-value theorem:

$$y(0) = \lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\underbrace{\left(\frac{s+1}{s^2+5s+6}\right)U(s)}_{Y(s)} = \lim_{s\to\infty} \cancel{s}\left(\frac{s+1}{s^2+5s+6}\right)\left(\frac{6}{\cancel{s}}\right) = \lim_{s\to\infty}\left(\frac{6s+6}{s^2+5s+6}\right) = 0$$

From the final-value theorem:

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\underbrace{\left(\frac{s+1}{s^2+5s+6}\right)U(s)} = \lim_{s\to 0} s\left(\frac{s+1}{s^2+5s+6}\right)\left(\frac{6}{s}\right) = \lim_{s\to 0}\left(\frac{\overset{0}{6\cancel{s}}+6}{\underset{0}{\cancel{s^2}}+\underset{0}{5\cancel{s}}+6}\right) = 1$$

Time Response Modeling & Analysis

Example 12 Consider the following transfer function model of a dynamic system.

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2+5s+6}$$

y(t)

d) Find the system response to step input $u(t) = 6, \quad t \geq 0$. Determine the transient part and the steady-state part of the response.

The step response is obtained as:

$$\underline{Y(s)} = \underbrace{\left(\frac{s+1}{s^2+5s+6}\right)}\underbrace{U(s)} = \left(\frac{s+1}{s^2+5s+6}\right)\underbrace{\left(\frac{6}{s}\right)} = \frac{6(s+1)}{s(s+2)(s+3)} = \frac{1}{s} + \frac{3}{s+2} + \frac{-4}{s+3}$$

Taking inverse Laplace transform, we can find the system response:

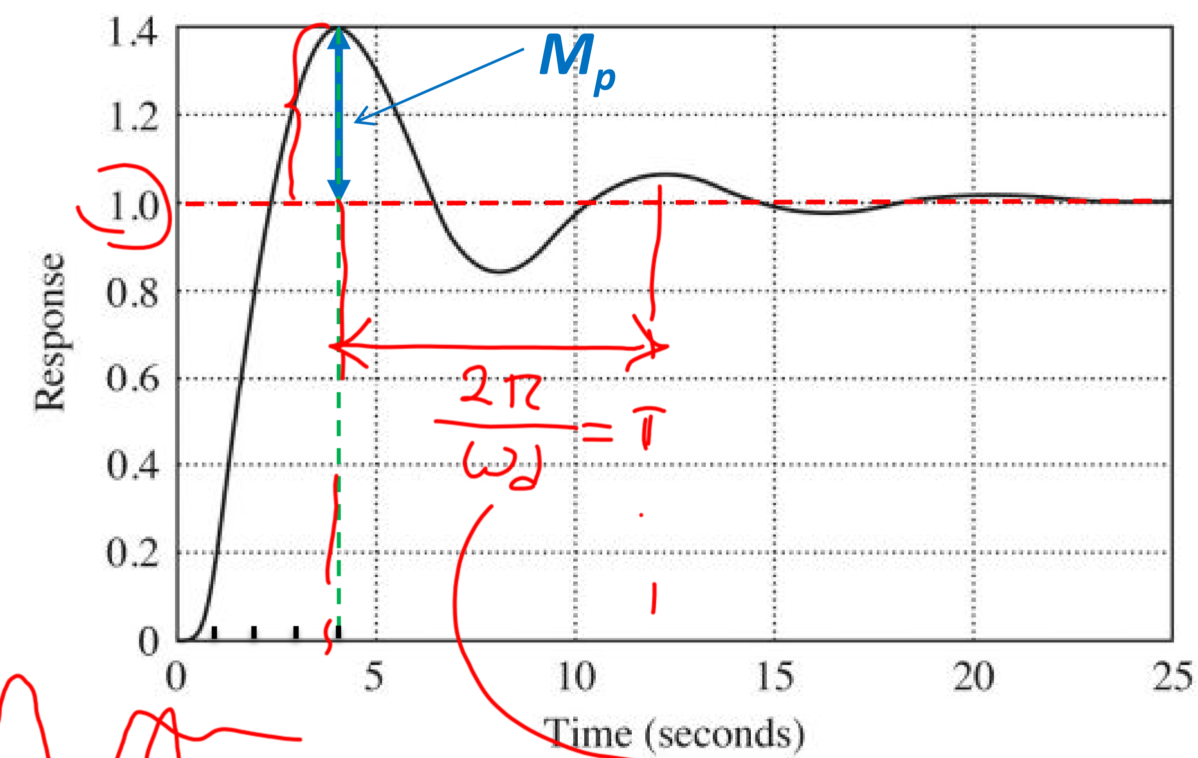
$$\rightarrow \boxed{y(t) = 1 + 3e^{-2t} - 4e^{-3t}, \qquad t \geq 0}$$

Transient part:	$3e^{-2t} - 4e^{-3t}$
Steady-state part:	1

Time Response Modeling & Analysis

Example 13 For the unit step response shown in figure below, find a second-order transfer function model.

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



From the unit-step response graph we have:

$\boxed{K = 1.0}$ $\boxed{t_p \approx 4sec, \quad M_p = 1.4 - 1.0 = 0.4}$

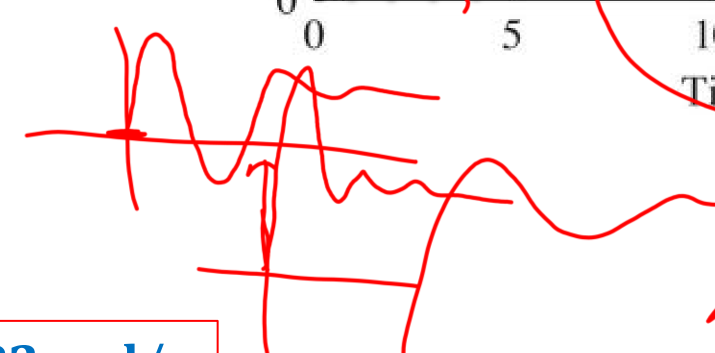
The damping ratio is determined as below

$$O.S. = \frac{M_p}{y_{ss}} = \frac{0.4}{1} = 0.4$$

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} = \frac{-\ln(0.4)}{\sqrt{\pi^2 + \ln^2(0.4)}} \rightarrow \boxed{\zeta = 0.28}$$

The undamped natural frequency is calculated as below

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \rightarrow 4 = \frac{\pi}{\omega_n\sqrt{1-(0.28)^2}} \rightarrow \boxed{\omega_n = 0.82 \text{ rad/s}}$$



$\omega_d = \omega_n\sqrt{1-\zeta^2}$

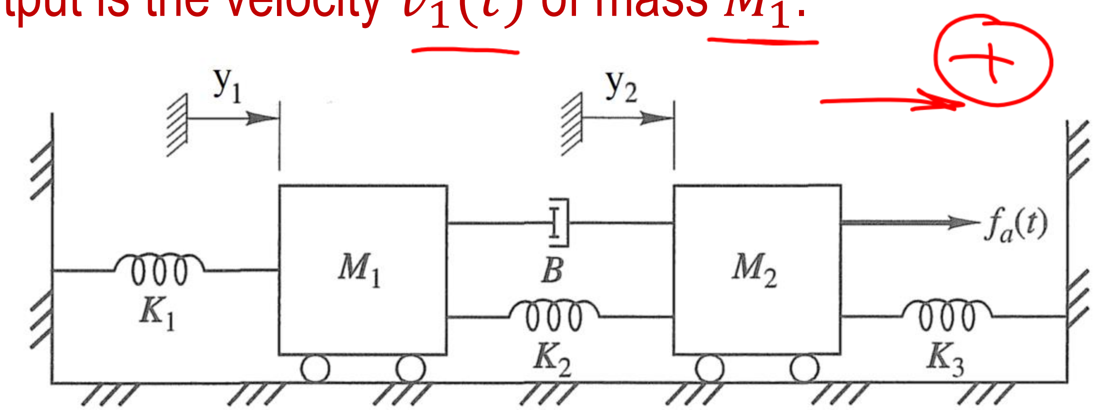
$4\zeta\omega_n < a < 5\zeta\omega_n$

The second-order transfer function model is obtained as:

$$\boxed{G(s) = \frac{0.67}{s^2 + 0.46s + 0.67}}$$

Modeling of Mechanical Systems

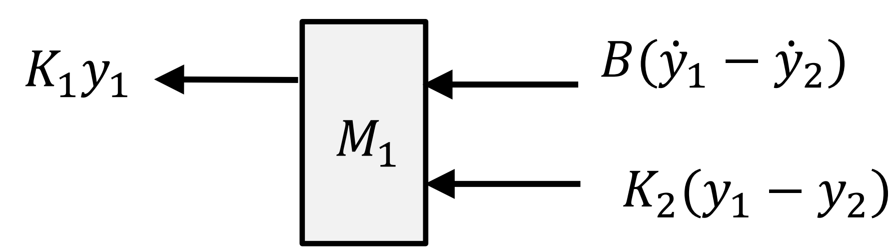
Example 14 Consider the following translational mechanical system. The springs are undeflected when $y_1 = y_2 = 0$. The input is the applied force $f_a(t)$, and the system output is the velocity $v_1(t)$ of mass M_1.



a) Draw the free-body diagrams of the system and write a set of ordinary differential equations.

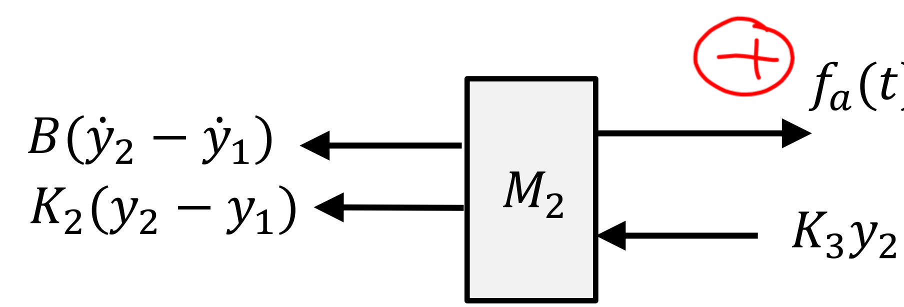
Find the equation of motion for each mass from the free-body diagrams. We can apply Newton's second law:

$$\begin{cases} -K_1 y_1 - B(\dot{y}_1 - \dot{y}_2) - K_2(y_1 - y_2) = M_1 \ddot{y}_1 \\ f_a(t) - B(\dot{y}_2 - \dot{y}_1) - K_2(y_2 - y_1) - K_3 y_2 = M_2 \ddot{y}_2 \end{cases}$$



Simplifying the equations, we have:

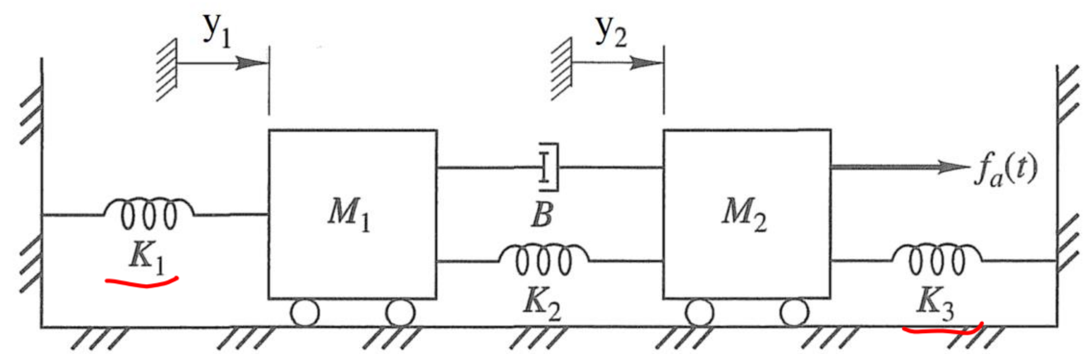
$$\begin{cases} M_1 \ddot{y}_1 + B\dot{y}_1 + (K_1 + K_2) y_1 - B\dot{y}_2 - K_2 y_2 = 0 \\ M_2 \ddot{y}_2 + B\dot{y}_2 + (K_2 + K_3) y_2 - B\dot{y}_1 - K_2 y_1 = f_a(t) \end{cases}$$



Modeling of Mechanical Systems

Example 14 Consider the following translational mechanical system. The springs are undeflected when $y_1 = y_2 = 0$. The input is the applied force $f_a(t)$, and the system output is the velocity $v_1(t)$ of mass M_1.

b) Define the state variables as the displacement of the springs K_1 and K_3 and the velocity of the masses M_1 and M_2. Find a state-space model for the system and write the equations in matrix-vector form



$$\begin{cases} M_1\ddot{y}_1 + B\dot{y}_1 + (K_1+K_2)y_1 - B\dot{y}_2 - K_2y_2 = 0 \\ M_2\ddot{y}_2 + B\dot{y}_2 + (K_2+K_3)y_2 - B\dot{y}_1 - K_2y_1 = f_a(t) \end{cases}$$

$$\boldsymbol{q_1(t) = y_1(t)} \quad \rightarrow \quad \dot{q}_1(t) = \dot{y}_1(t) \rightarrow \dot{q}_1(t) = q_2(t) \qquad \color{red}{Eqn.\ (1)}$$

$$\boldsymbol{q_2(t) = \dot{y}_1(t)} \quad \rightarrow \quad \dot{q}_2(t) = \ddot{y}_1(t) \rightarrow \dot{q}_2(t) = \frac{1}{M_1}(-B\dot{y}_1 - (K_1+K_2)y_1 + B\dot{y}_2 + K_2y_2)$$

$$\rightarrow \dot{q}_2(t) = \frac{1}{M_1}(-Bq_2 - (K_1+K_2)q_1 + Bq_4 + K_2q_3) \qquad \color{red}{Eqn.\ (2)}$$

$$\boldsymbol{q_3(t) = y_2(t)} \quad \rightarrow \quad \dot{q}_3(t) = \dot{y}_2(t) \rightarrow \dot{q}_3(t) = q_4(t) \qquad \color{red}{Eqn.\ (3)}$$

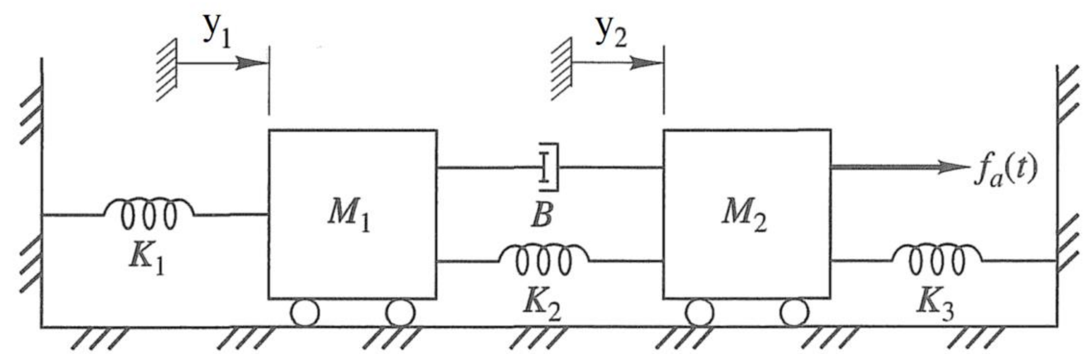
$$\boldsymbol{q_4(t) = \dot{y}_2(t)} \quad \rightarrow \quad \dot{q}_4(t) = \ddot{y}_2(t) \rightarrow \dot{q}_4(t) = \frac{1}{M_2}(f_a(t) - B\dot{y}_2 - (K_2+K_3)y_2 + B\dot{y}_1 + K_2y_1)$$

$$\rightarrow \dot{q}_4(t) = \frac{1}{M_2}(f_a(t) - Bq_4 - (K_2+K_3)q_3 + Bq_2 + K_2q_1) \qquad \color{red}{Eqn.\ (4)}$$

Modeling of Mechanical Systems

Example 14 Consider the following translational mechanical system. The springs are undeflected when $y_1 = y_2 = 0$. The input is the applied force $f_a(t)$, and the system output is the velocity $v_1(t)$ of mass M_1.

b) Define the state variables as the displacement of the springs K_1 and K_3 and the velocity of the masses M_1 and M_2. Find a state-space model for the system and write the equations in matrix-vector form



State-variable equations are obtained as:

$$
\begin{cases}
\dot{q}_1(t) = q_2(t) \\
\dot{q}_2(t) = \dfrac{1}{M_1}\left(-Bq_2 - (K_1+K_2)q_1 + Bq_4 + K_2 q_3\right) \\
\dot{q}_3(t) = q_4(t) \\
\dot{q}_4(t) = \dfrac{1}{M_2}\left(f_a(t) - Bq_4 - (K_2+K_3)q_3 + Bq_2 + K_2 q_1\right)
\end{cases}
$$

State Equation $\boxed{\dot{\boldsymbol{q}}(t) = \mathbf{A}\,\boldsymbol{q}(t) + \mathbf{B}\,\mathbf{u}(t)}$

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
\dfrac{-(K_1+K_2)}{M_1} & -\dfrac{B}{M_1} & \dfrac{K_2}{M_1} & \dfrac{B}{M_1} \\
0 & 0 & 0 & 1 \\
\dfrac{K_2}{M_2} & \dfrac{B}{M_2} & -\dfrac{(K_2+K_3)}{M_2} & -\dfrac{B}{M_2}
\end{bmatrix}
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} +
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{M_2} \end{bmatrix} f_a(t)
$$

The **output equation** is obtained as:

$$y(t) = \dot{y}_1(t) \quad \rightarrow \quad y(t) = q_2(t)$$

Output Equation $\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$

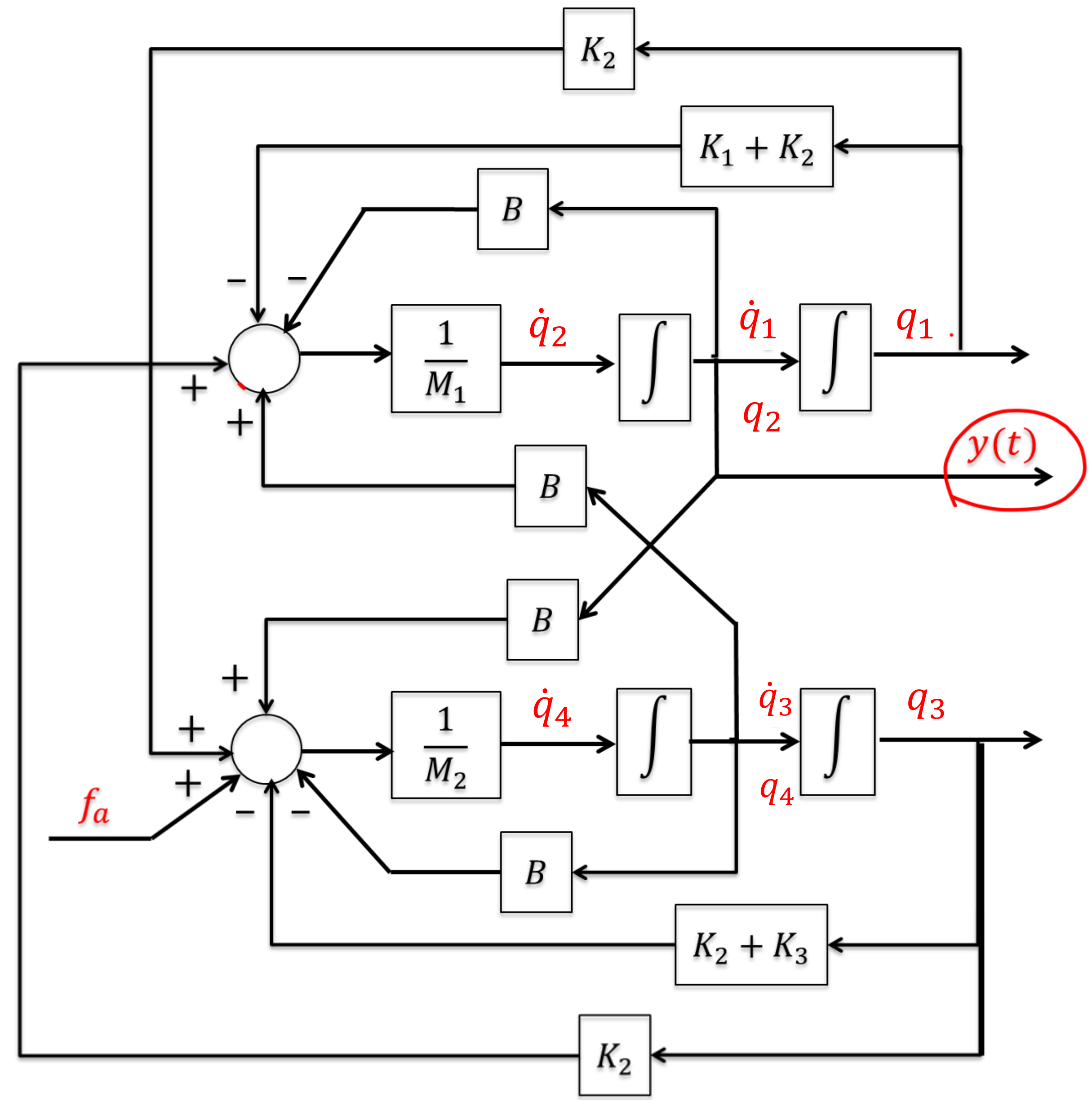
$$
y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + [0] f_a(t)
$$

Modeling of Mechanical Systems

Example 14 Consider the following translational mechanical system. The springs are undeflected when $y_1 = y_2 = 0$. The input is the applied force $f_a(t)$, and the system output is the velocity $v_1(t)$ of mass M_1.

c) Draw the block diagram model of the system.

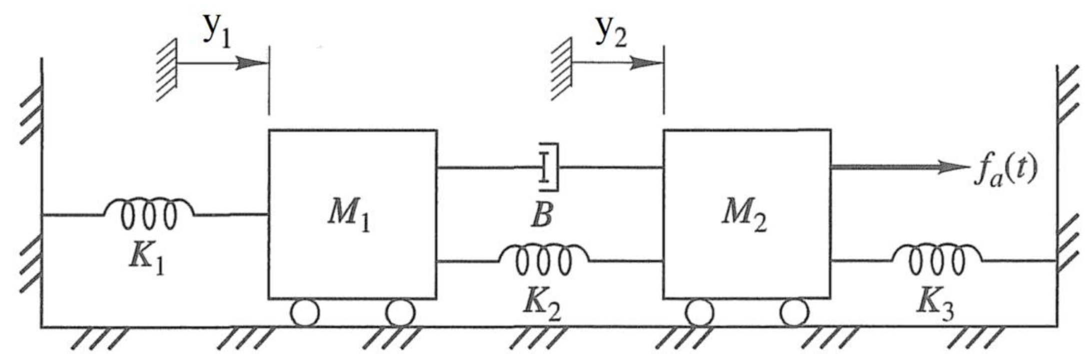
$$
\begin{cases}
\dot{q}_1(t) = q_2(t) \\
\dot{q}_2(t) = \dfrac{1}{M_1}\left(-Bq_2 - (K_1+K_2)q_1 + Bq_4 + K_2 q_3\right) \\
\dot{q}_3(t) = q_4(t) \\
\dot{q}_4(t) = \dfrac{1}{M_2}\left(f_a(t) - Bq_4 - (K_2+K_3)q_3 + Bq_2 + K_2 q_1\right) \\
y(t) = q_2(t)
\end{cases}
$$



Modeling of Mechanical Systems

Example 14 Consider the following translational mechanical system. The springs are undeflected when $y_1 = y_2 = 0$. The input is the applied force $f_a(t)$, and the system output is the velocity $v_1(t)$ of mass M_1.

d) Assume all the coefficients are equal to 1, with appropriate units, develop a transfer function model $V_1(s)/F_a(s)$ for the system.



$$\begin{cases} M_1\ddot{y}_1 + B\dot{y}_1 + (K_1 + K_2)y_1 - B\dot{y}_2 - K_2y_2 = 0 \\ M_2\ddot{y}_2 + B\dot{y}_2 + (K_2 + K_3)y_2 - B\dot{y}_1 - K_2y_1 = f_a(t) \end{cases}$$

Setting the coefficients equal to 1, we have:

$$\begin{cases} \ddot{y}_1 + \dot{y}_1 + 2y_1 - \dot{y}_2 - y_2 = 0 \\ \ddot{y}_2 + \dot{y}_2 + 2y_2 - \dot{y}_1 - y_1 = f_a(t) \end{cases}$$

Taking Laplace transform:

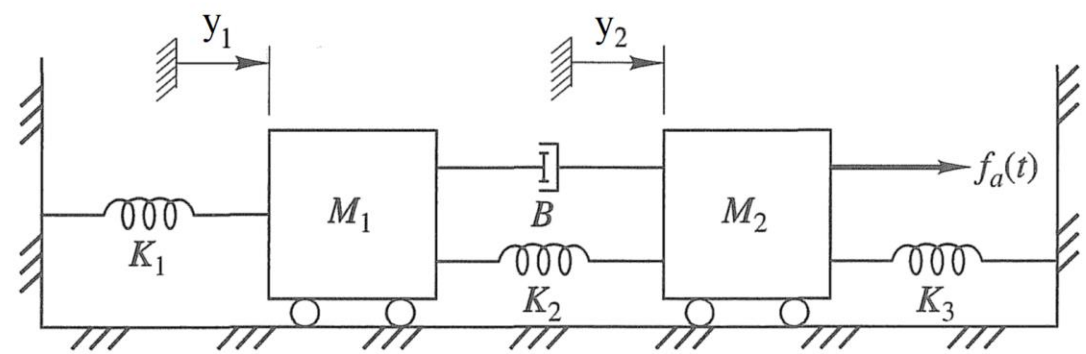
$$\begin{cases} s^2Y_1(s) + sY_1(s) + 2Y_1(s) - sY_2(s) - Y_2(s) = 0 \quad \rightarrow \quad (s^2 + s + 2)Y_1(s) - (s+1)Y_2(s) = 0 \qquad Eqn.(1) \\ s^2Y_2(s) + sY_2(s) + 2Y_2(s) - sY_1(s) - Y_1(s) = F_a(s) \quad \rightarrow \quad (s^2 + s + 2)Y_2(s) - (s+1)Y_1(s) = F_a(s) \qquad Eqn.(2) \end{cases}$$

Find $Y_2(s)$ from Eqn. (1) and substitute into Eqn. (2).

Modeling of Mechanical Systems

Example 14 Consider the following translational mechanical system. The springs are undeflected when $y_1 = y_2 = 0$. The input is the applied force $f_a(t)$, and the system output is the velocity $v_1(t)$ of mass M_1.

d) Assume all the coefficients are equal to 1, with appropriate units, develop a transfer function model $V_1(s)/F_a(s)$ for the system.



$$\begin{cases} (s^2+s+2)Y_1(s) - (s+1)Y_2(s) = 0 & \textit{Eqn. (1)} \\ (s^2+s+2)Y_2(s) - (s+1)Y_1(s) = F_a(s) & \textit{Eqn. (2)} \end{cases}$$

Find $Y_2(s)$ from Eqn. (1) and substitute into Eqn. (2).

$$\textit{Eqn. (1)} \rightarrow \quad Y_2(s) = \frac{s^2+s+2}{s+1} Y_1(s)$$

$V_1(s)$

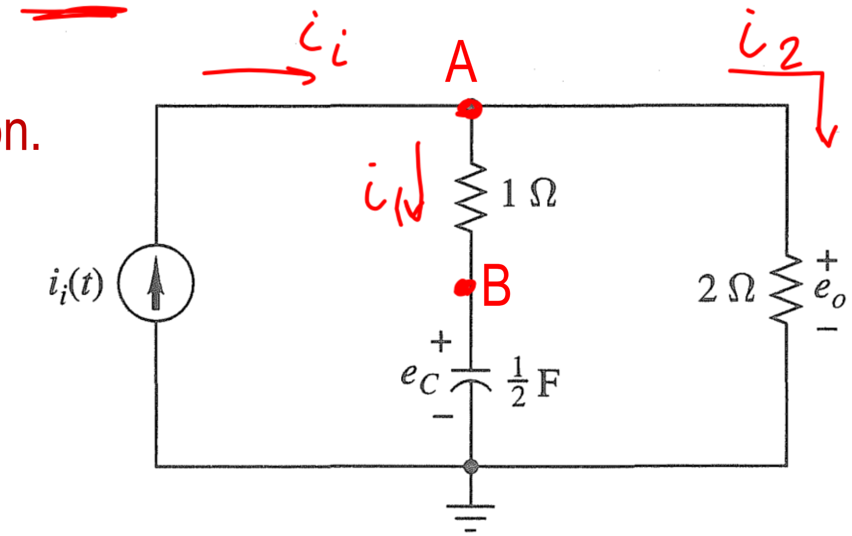
$$\textit{Eqn. (2)} \rightarrow \quad \frac{(s^2+s+2)^2}{s+1} Y_1(s) - (s+1)Y_1(s) = F_a(s) \quad \rightarrow \frac{sY_1(s)}{F_a(s)} = \frac{s(s+1)}{(s^2+2s+3)(s^2+1)}$$

Since $v_1(t) = \dot{y}_1(t)$ and $V_1(s) = sY_1(s)$, the transfer function model $V_1(s)/F_a(s)$ is obtained as:

$$\boxed{\frac{V_1(s)}{F_a(s)} = \frac{s(s+1)}{(s^2+2s+3)(s^2+1)}}$$

Modeling of Electrical Systems

Example 15 Consider the following electrical system with $i_i(t)$ as an input and $e_o(t)$ as the output.



a) Assume that the capacitor is initially uncharged. Find the input-output differential equation.

Applying KCL at node A and B.

Node A: $i_i(t) = i_1(t) + i_2(t) \quad \rightarrow \quad i_i(t) = \dfrac{e_o(t) - e_c(t)}{1\Omega} + \dfrac{e_o(t)}{2\Omega}$

$$i_i(t) = 1.5e_o(t) - e_c(t) \qquad Eqn.\,(1)$$

Node B: $\dfrac{e_o(t) - e_c(t)}{1\Omega} = \dfrac{1}{2}\dfrac{de_c(t)}{dt} \quad \rightarrow \quad e_o(t) - e_c(t) = 0.5\dfrac{de_c(t)}{dt} \qquad Eqn.\,(2)$

Find $e_c(t)$ from Eqn. (1) and substitute into Eqn. (2)

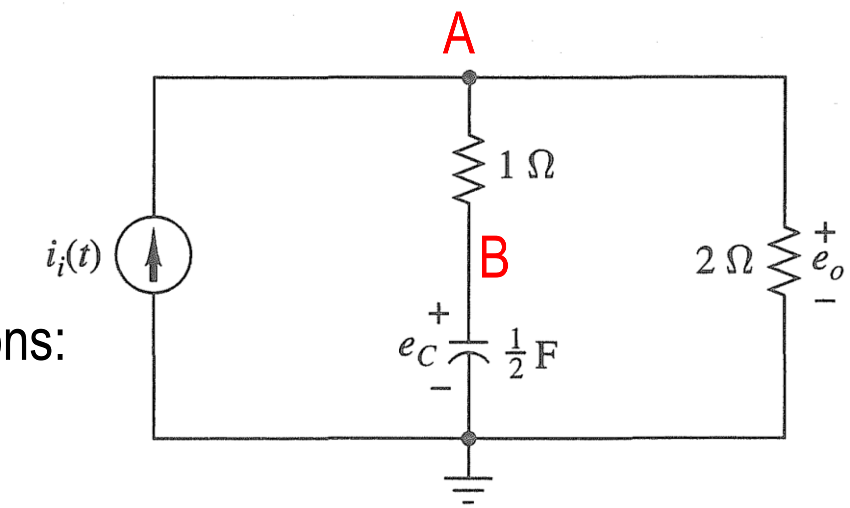
$$e_o(t) - \big(1.5e_o(t) - i_i(\mathrm{t})\big) = 0.5\left(1.5\frac{de_o(t)}{dt} - \frac{di_i(t)}{dt}\right)$$

The differential equation is:

$$\boxed{0.75\frac{de_o(t)}{dt} + 0.5e_o(t) = 0.5\frac{di_i(t)}{dt} + i_i(t)}$$

Modeling of Electrical Systems

Example 15 Consider the following electrical system with $i_i(t)$ as an input and $e_o(t)$ as the output.



b) Find the transfer function model of the system.

$$\frac{3}{4}\frac{de_o(t)}{dt} + \frac{1}{2}e_o(t) = \frac{1}{2}\frac{di_i(t)}{dt} + i_i(t)$$

Taking Laplace transform from the differential equation model assuming zero initial conditions:

output

$$\frac{3}{4}sE_o(s) + \frac{1}{2}E_o(s) = \frac{1}{2}sI_i(s) + I_i(s) \quad \rightarrow \quad \boxed{\frac{E_o(s)}{I_i(s)} = \frac{\frac{1}{2}s + 1}{\frac{3}{4}s + \frac{1}{2}}}$$

input

c) Determine the voltage $e_o(t)$ if the input is, $i_i(t) = 2A,\ t \geq 0.$

The $e_o(t)$ is obtained as:

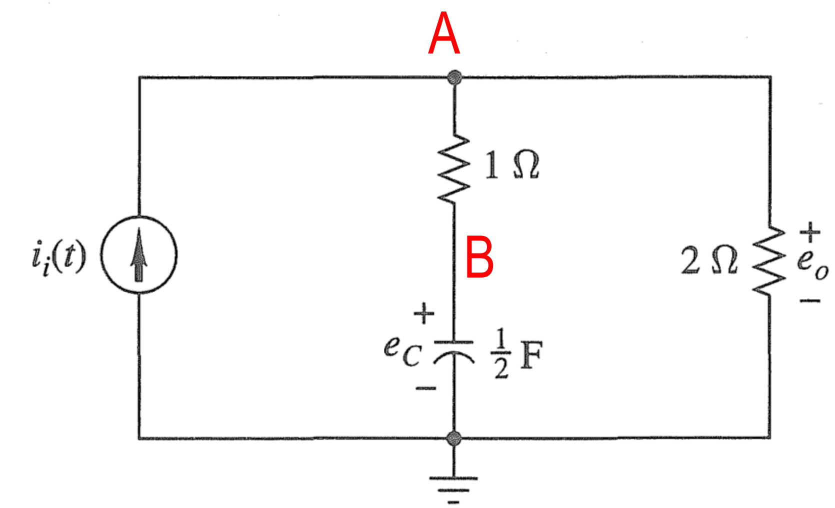
$$E_o(s) = \left(\frac{\frac{1}{2}s + 1}{\frac{3}{4}s + \frac{1}{2}}\right) I_i(s) = \left(\frac{\frac{1}{2}s + 1}{\frac{3}{4}s + \frac{1}{2}}\right)\left(\frac{2}{s}\right) = \frac{s + 2}{\frac{3}{4}s\left(s + \frac{2}{3}\right)} = \frac{4}{s} + \frac{-8/3}{s + 2/3}$$

Take inverse Laplace transform to find the $e_o(t)$:

$$\boxed{e_o(t) = 4 - \frac{8}{3}e^{-\frac{2t}{3}}, \qquad t \geq 0}$$

Modeling of Electrical Systems

Example 15 Consider the following electrical system with $i_i(t)$ as an input and $e_o(t)$ as the output.



d) Then sketch the voltage $e_o(t)$ from Step (c).

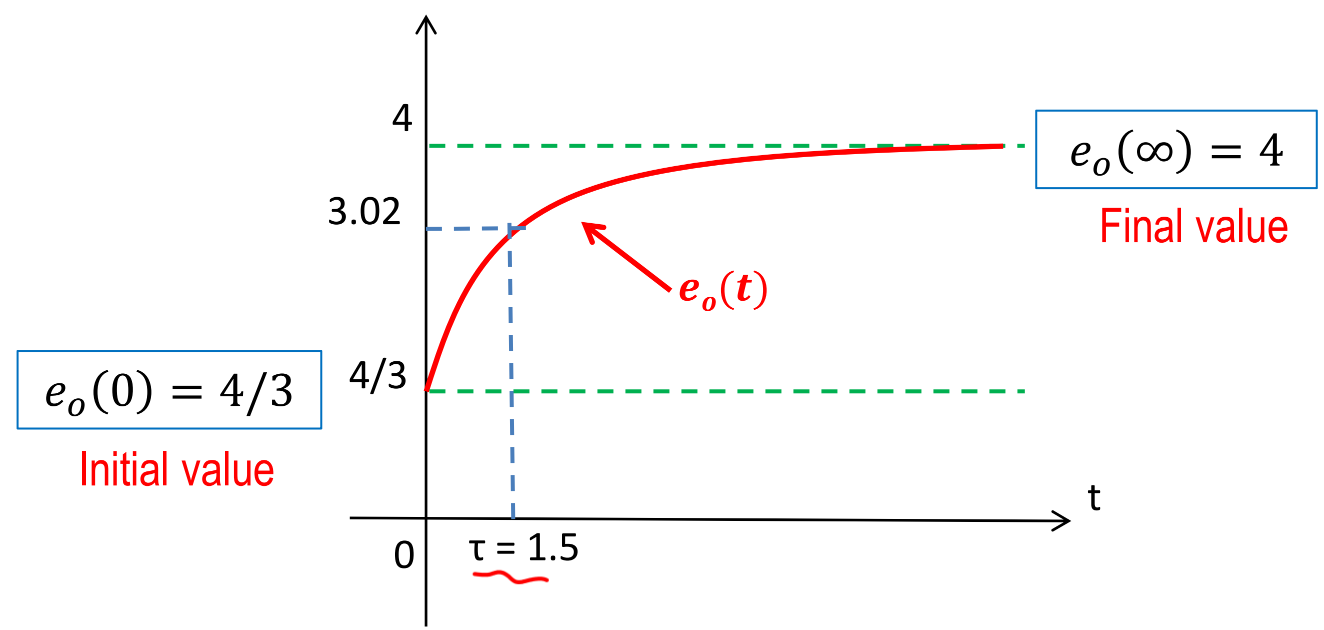
$$\boxed{e_o(t) = 4 - \frac{8}{3}e^{-\frac{2t}{3}}, \qquad t \geq 0}$$

The initial value of voltage $e_o(t)$ is:

$$e_o(0) = \lim_{t\to 0} e_o(t) = \lim_{t\to 0}\left(4 - \frac{8}{3}e^{-\frac{2t}{3}}\right) = 4 - \frac{8}{3} = \frac{4}{3}$$

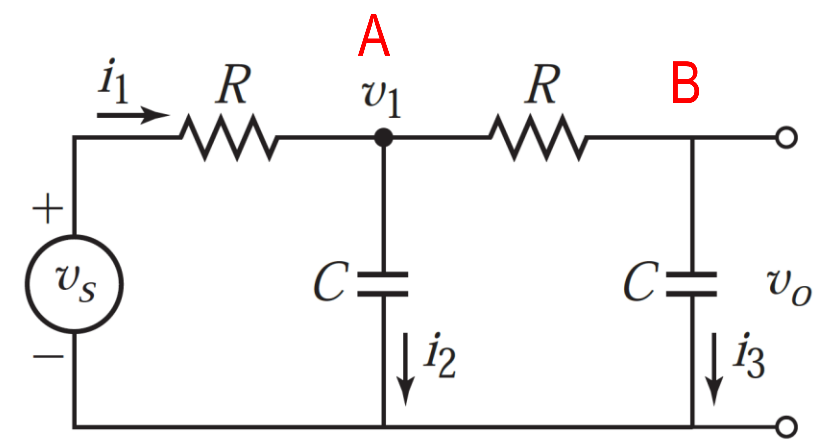
The final value of voltage $e_o(t)$ is:

$$e_o(\infty) = \lim_{t\to \infty} e_o(t) = \lim_{t\to \infty}\left(4 - \frac{8}{3}e^{-\frac{2t}{3}}\right) = 4$$



Modeling of Electrical Systems

Example 16 Consider the following dual RC circuit shown below.



a) Derive a differential equation for the output voltage v_o as a function of the input voltage v_s.

Applying KCL at node A.

$$\text{Node } v_1\text{:}\quad i_1(t) = i_2(t) + i_3(t) \;\rightarrow\; \frac{v_s - v_1}{R} = C\frac{dv_1}{dt} + \frac{v_1 - v_o}{R} \;\rightarrow\; v_s = RC\frac{dv_1}{dt} + 2v_1 - v_o \quad Eqn.(1)$$

Applying KCL at node B.

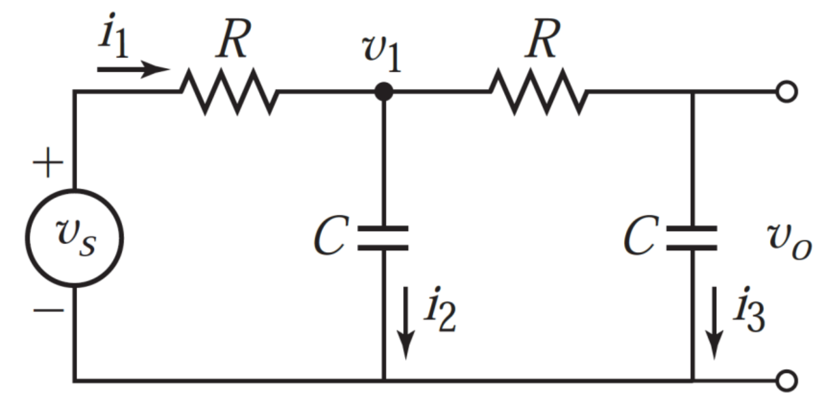
$$\text{Node } v_o\text{:}\quad i_3(t) = C\frac{dv_o}{dt} = \frac{v_1 - v_o}{R} \;\rightarrow\; v_1 = RC\frac{dv_o}{dt} + v_o \quad Eqn.(2)$$

Substitute v_1 from Eqn. (2) into Eqn. (1):

$$v_s = RC\frac{d}{dt}\left(RC\frac{dv_o}{dt} + v_o\right) + 2\left(RC\frac{dv_o}{dt} + v_o\right) - v_o \;\rightarrow\; \boxed{v_s = R^2C^2\frac{d^2v_o}{dt^2} + 3RC\frac{dv_o}{dt} + v_o}$$

Modeling of Electrical Systems

Example 16 Consider the following dual RC circuit shown below.



b) Derive the transfer function relating to the output voltage $V_o(s)$ as a function of the input voltage $V_s(s)$.

$$v_s = R^2C^2\frac{d^2v_o}{dt^2} + 3RC\frac{dv_o}{dt} + v_o$$

Take Laplace transform of the differential equation:

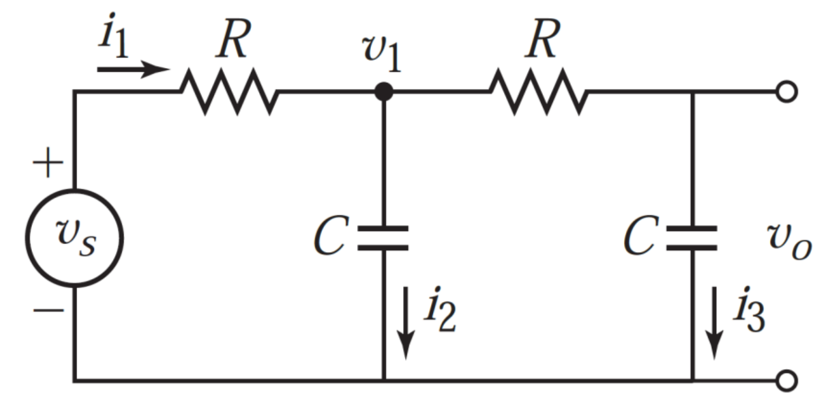
$$V_s(s) = R^2C^2s^2V_o(s) + 3RCsV_o(s) + V_o(s) \quad \rightarrow \quad V_s(s) = (R^2C^2s^2 + 3RCs + 1)V_o(s)$$

The transfer function is obtained as:

$$\boxed{\frac{V_o(s)}{V_s(s)} = \frac{1}{R^2C^2s^2 + 3RCs + 1}}$$

Modeling of Electrical Systems

Example 16 Consider the following dual RC circuit shown below.



c) Determine the voltage $v_o(t)$ for $R = 10\Omega$ and $C = 0.1F$ if the applied input is $v_s(t) = 5V, \quad t \geq 0$. Judge if $v_o(t)$ is over-damped or under-damped response?

$$\frac{V_o(s)}{V_s(s)} = \frac{\frac{1}{R^2C^2}}{s^2 + \frac{3}{RC}s + \frac{1}{R^2C^2}} = \frac{1}{s^2 + 3s + 1}$$

The $v_o(t)$ is obtained as:

$$V_o(s) = \left(\frac{1}{s^2 + 3s + 1}\right)V_s(s) = \left(\frac{1}{s^2 + 3s + 1}\right)\left(\frac{5}{s}\right) = \frac{5}{s(s^2 + 3s + 1)} = \frac{5}{s} + \frac{-5(s+3)}{s^2 + 3s + 1}$$

$$V_o(s) = \frac{5}{s} + \frac{-5(s+3)}{(s+0.38)(s+2.62)} = \frac{5}{s} + \frac{-5.85}{s+0.38} + \frac{0.85}{s+2.62}$$

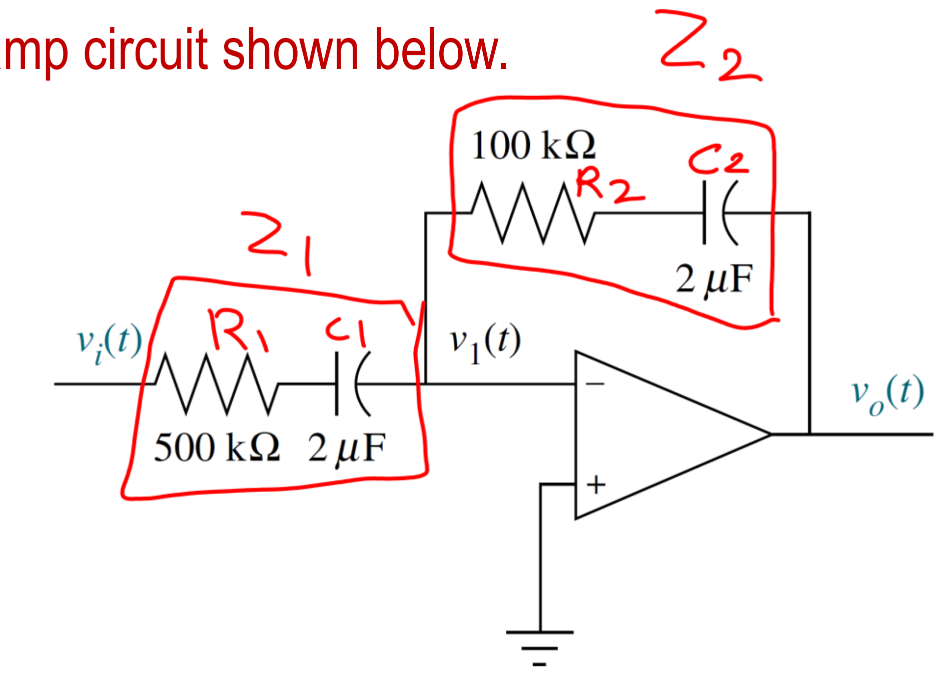
Taking inverse Laplace transform to find the$v_o(t)$:

$$\boxed{v_o(t) = 5 - 5.85e^{-0.38t} + 0.85e^{-2.62t}, \qquad t \geq 0}$$

Since the poles of the transfer function are all real, the $v_o(t)$ has over-damped response.

Modeling of Electrical Systems

Example 17 Find the transfer function $G(s) = V_o(s)/V_i(s)$ for the op-amp circuit shown below.



This is an **inverting** op-amp circuit. The general form of the transfer function is:

$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

First find the $Z_1(s)$ and $Z_2(s)$:

$$Z_1(s) = R_1 + \frac{1}{C_1 s} = \frac{R_1 C_1 s + 1}{C_1 s} \qquad Z_2(s) = R_2 + \frac{1}{C_2 s} = \frac{R_2 C_2 s + 1}{C_2 s}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{\dfrac{R_2 C_2 s + 1}{C_2 s}}{\dfrac{R_1 C_1 s + 1}{C_1 s}} = -\frac{C_1(R_2 C_2 s + 1)}{C_2(R_1 C_1 s + 1)} = -\frac{(2\times 10^{-6})\big((100\times 10^3)(2\times 10^{-6})s + 1\big)}{(2\times 10^{-6})\big((500\times 10^3)(2\times 10^{-6}) + 1\big)}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{0.2s + 1}{s + 1}}$$

THANK YOU

HUMBER

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