Chapter 1

Sequences and Series

1.1 Readings

(1) CLP II §3.1 Sequences

(2) CLP II §3.2 Series

Recommended exercises

See course iDEAS site for complete list of exercises.

1.2 Lesson Overview

Lesson goals:

1. Define what a sequence of numbers is and what it means for a sequence to converge or diverge.

2. Determine whether a given sequence converges or not using the knowledge of associated functions, the arithmetic of sequences, and the Squeeze Theorem.

3. Use the knowledge that continuous functions preserve convergence to find the limit of a convergent sequence under the action of a continuous function.

4. Define a series and its associated sequence of partial sums.

5. Define what it means for a series to converge or diverge.

6. Evaluate finite geometric series.
7. Determine whether a given infinite geometric series converges or not and, if it converges, to what value.
8. Recognize and work with telescoping series.
9. Use the arithmetic of series to evaluate the sum, difference, or scalar product of convergent series.

1.3 Sequences

Rosen Definition 2.4.2.1, What is a sequence?

Definition 1.1. A sequence of real numbers is a function from a subset of the set of integers, often the set $\{0, 1, 2, 3, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$, to the set of real numbers $\mathbb{R}$. Letting a_n denote the image of the integer n, we call a_n the n^{th} term (or member) of the sequence and refer to n as the index for this n^{th} term. More informally, a sequence is an ordered list of numbers

$$a_0, a_1, a_2, a_3, \ldots,$$

and we denote the sequence by

$$\{a_n\}_{n=0}^{\infty} \quad \text{or} \quad (a_n)_{n=0}^{\infty}.$$

More generally, if the sequence begins at an index N as in

$$a_N, a_{N+1}, a_{N+2}, a_{N+3}, \ldots,$$

then we denote it by

$$\{a_n\}_{n=N}^{\infty} \quad \text{or} \quad (a_n)_{n=N}^{\infty}.$$

When our discussion of the sequence does not depend on the starting index or the index is known from the context, we simply write

$$\{a_n\} \quad \text{or} \quad (a_n).$$

Example 1.2. (FRY Example II.3.1.2, Examples of sequences)

(i) $\{a_n = \frac{1}{n}\}_{n=1}^{\infty}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$a_1 \quad a_2 \quad a_3 \quad a_4$

(ii) $(b_n)_{n=0}^{\infty}$ where $b_n = n$ for $n = 0, 1, 2, \ldots$.

$0, 1, 2, 3, \ldots$

(iii) $\{c_m = 1\}$

$1, 1, 1, 1, \ldots$

(iv) $\{e_n\}$ where $e_n = \left(1 + \frac{1}{n}\right)^n$ for $n = 1, 2, 3, \ldots$

$\left(1+\frac{1}{1}\right)^1, \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{3}\right)^3, \left(1+\frac{1}{4}\right)^4, \ldots, \left(1+\frac{1}{20}\right)^{20}, \ldots, \left(1+\frac{1}{50}\right)^{50}, \ldots$

$2, \quad 2.25, \quad 2.370, \quad 2.441, \ldots, 2.653, \ldots, 2.691, \ldots, 2.718268237, \ldots$

the sequence converges to

$e = 2.718281828\ldots$

FRY II.3.1.3, What does it mean for a sequence to converge?

Definition 1.3. A sequence $\{a_n\}$ is said to converge to a limit A if a_n approaches A as n tends to infinity, that is, if

$$\lim_{n\to\infty} a_n = A \qquad \text{i.e.,} \qquad a_n \to A \text{ as } n \to \infty.$$

A sequence is said to converge if it converges to some limit. Otherwise, it is said to diverge.

It turns out that if a sequence converges to a limiting value, then that limiting value is unique. Also the limit, when it exists, is determined by the "long run" behaviour of its terms: Changing any finite number of beginning terms of the sequence does not affect the limiting value that it converges to.

Example 1.4. (FRY Example II.3.1.4, Does the sequence converge? If yes, to what? If not, why not?)

(i) $\{a_n = \frac{1}{n}\}_{n=1}^{\infty}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$a_n \to 0$ since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

(ii) $(b_n)_{n=0}^{\infty}$ where $b_n = n$ for $n = 0, 1, 2, \ldots$.

$0, 1, 2, 3, \ldots$

$b_n \to \infty$ since $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} n = \infty$

(iii) $\{c_m = 1\}$

$1, 1, 1, 1, \ldots$

$\lim\limits_{m\to\infty} c_m = \lim\limits_{m\to\infty} 1 = 1$

It takes some work, but it can be shown that the sequence $e_n = \left(1 + \frac{1}{n}\right)^n$ in Example Eg. 1.2 converges to the number e.

Example 1.5. (Like FRY Example II.3.1.5, determine whether or not the given sequence converges)
Does the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{n}{2n^2+1}$ converge? If yes, to what limit? If not, why not?

$$\lim_{n\to\infty} \frac{n}{2n^2+1} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)(n)}{\left(\frac{1}{n^2}\right)(2n^2+1)} = \lim_{n\to\infty} \frac{\cancel{\frac{1}{n}}^{\,0}}{2+\cancel{\frac{1}{n^2}}^{\,0}}$$

$$= \frac{0}{2+0} = 0$$

So $a_n \to 0$

1.3.1 Tools for assessing the convergence of a sequence

Use an associated function

FRY Thm II.3.1.6, Using functions to determine whether a sequence converges

Theorem 1.6. Let $f(x)$ be a function such that

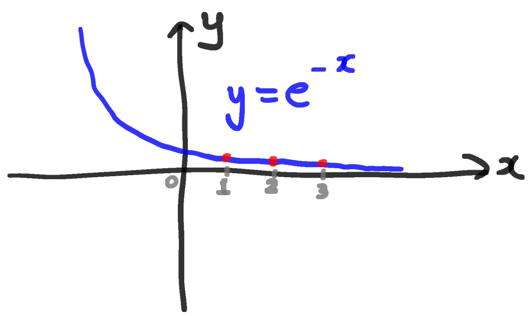
$$\lim_{x\to\infty} f(x) = L.$$

Then the sequence $\{a_n = f(n)\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

Example 1.7. (FRY Example II.3.1.7, use knowledge of corresponding function to determine whether or not the given sequence converges)
Does the sequence $\{e^{-n}\}_{n=1}^{\infty}$ converge? If yes, to what limit? If not, why not?

$e^{-1}, e^{-2}, e^{-3}, \ldots$



Consider $f(x)=e^{-x}$

Since $\lim_{x\to\infty} e^{-x} = 0$,

and $f(n)=e^{-n}$,

the sequence $\{e^{-n}\}_{n=1}^{\infty}$ also converges to 0.

Use the arithmetic of sequences

FRY Theorem II.3.1.8, arithmetic of limits and sequences

Theorem 1.8. Suppose the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to the real numbers A and B, respectively, that is,

$$\lim_{n\to\infty} a_n = A \qquad \text{and} \qquad \lim_{n\to\infty} b_n = B.$$

Then

(i) The limit of the sum is the sum of the limits: $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(ii) The limit of the difference is the difference of the limits: $\lim_{n\to\infty}(a_n - b_n) = A - B$.

(iii) The limit of the product is the product of the limits: $\lim_{n\to\infty}(a_n b_n) = AB$.

(iv) The limit of the quotient is the quotient of the limits provided that the terms of the quotient sequence are well-defined and the limit of the denominator sequence is nonzero: $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$.

Example 1.9. (Like FRY Example II.3.1.9, use the arithmetic of sequences to determine the convergence of a given sequence)
Determine whether or not the sequence

$$\left\{\frac{n}{2n^2+1} + 7e^{-n}\right\}_{n=1}^{\infty}$$

converges.

We saw earlier that

$$\frac{n}{2n^2+1} \to 0 \longleftrightarrow \lim_{n\to\infty} \frac{n}{2n^2+1} = 0$$

and

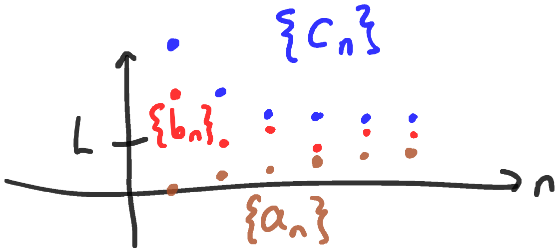
$$e^{-n} \to 0. \longleftrightarrow \lim_{n\to\infty} e^{-n} = 0$$

Also, since $7 \to 7 \longleftrightarrow \lim_{n\to\infty} 7 = 7$,

$$\frac{n}{2n^2+1} + 7e^{-n} \to 0 + 7\cdot 0 = 0 + 0 = 0.$$

times

We've used arithmetic of sequences



Use the Squeeze Theorem for sequences

FRY Theorem II.3.1.10, Squeeze Theorem for sequences

Theorem 1.10. If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

and

$$a_n \le b_n \le c_n \qquad \text{for all } n,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Example 1.11. Determine $\lim_{n\to\infty} b_n$ where $b_n = \dfrac{\cos n}{n}$.

$$\frac{\cos(1)}{1}, \frac{\cos(2)}{2}, \frac{\cos(3)}{3}, \ldots$$

Note $-1 \le \cos n \le 1$

So $\underbrace{\dfrac{-1}{n}}_{a_n} \le \underbrace{\dfrac{\cos n}{n}}_{b_n} \le \underbrace{\dfrac{1}{n}}_{c_n}$ for all n.

Since $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, by the Squeeze Theorem,

$$\frac{\cos n}{n} \to 0.$$

Use the property that continuous functions preserve convergence

FRY Thm II.3.1.12, Continuous functions preserve convergence

Theorem 1.12. Let $\{a_n\}$ be a sequence that converges to L. Let f be a function whose domain contains all the members of the sequence $\{a_n\}$ and that is continuous at L. Then the sequence $\{f(a_n)\}$ converges and

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example 1.13. (FRY Example II.3.1.13, using the continuity of a function to determine the convergence of a given sequence)
Determine whether or not the sequence

$$\left\{\sin\left(\frac{\pi n}{2n+1}\right)\right\}_{n=0}^{\infty}$$

converges.

Consider $\left\{\frac{\pi n}{2n+1}\right\}_{n=0}^{\infty}$

$\{a_n\}$

$$\lim_{n\to\infty}\frac{\pi n}{2n+1} = \lim_{n\to\infty}\frac{\pi}{2+\cancel{\frac{1}{n}}^{\,0}} = \frac{\pi}{2+0} = \frac{\pi}{2}$$

divide every term by n

So $$\lim_{n\to\infty}\sin\left(\frac{\pi n}{2n+1}\right) = \sin\left(\lim_{n\to\infty}\frac{\pi n}{2n+1}\right),$$ because sine is a continuous function

$$= \sin\left(\frac{\pi}{2}\right)$$

$$= 1$$

1.4 Series

Fry Section II.3.2.1, What is a series?

Definition 1.14. A series is a sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

of infinitely many terms. We write the sum more compactly using summation notation:

$$\sum_{n=1}^{\infty} a_n.$$

When it is clear from the context where the summation starts or when we wish to talk about the series without focusing on where the summation starts, we write $\sum a_n$.

The index of summation, n, is a "dummy" index and we may use a different letter (given that it is not already being used in describing the terms of the series). For example, we may use the notation $\displaystyle\sum_{m=1}^{\infty} \frac{3}{10^m}$ in place of $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n}$. We can also manipulate the index of summation to start at a different integer. For example, consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n}$. If we let $j = n + 1$, then as n runs from 1 to ∞, the index j runs from 2 to ∞. So, using j, we may start our summation at 2 instead of 1. Since $j = n + 1$ implies that $n = j - 1$, we may rewrite the power 10^n as 10^{j-1}. Hence,

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \sum_{j=2}^{\infty} \frac{3}{10^{j-1}}.$$

FRY Definition II.3.2.3, What does it mean for a series to converge?

Definition 1.15. (The N^{th} partial sum S_N of a series. The sequence $\{S_N\}$ of partial sums. What does it mean to say that a series converges? What does it means for a series to converge to a number S? What does it mean to say that a series diverges?)
Consider the series

$$\sum_{n=1}^{\infty} a_n.$$

The $N^{\mathbf{th}}$ **partial sum** of the series, denoted by S_N, is the sum of the first N terms of the series:

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

The partial sums themselves form a sequence called the **sequence of partial sums**:

$$\{S_N\}_{N=1}^{\infty} = \left\{ \sum_{n=1}^{N} a_n \right\}_{N=1}^{\infty}.$$

We say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if its sequence of partial sums, $\{S_N\}_{N=1}^{\infty}$, converges. If the sequence of partial sums converges to S, we say that the series $\displaystyle\sum_{n=1}^{N} a_n$ converges to S and write

$$\sum_{n=1}^{\infty} a_n = S.$$

Note that

$$\sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N = S.$$

If the sequence of partial sums diverges, then we say that the series $\sum a_n$ diverges.

We began the index of summation at 1 in the above discussion. However, the above definitions still hold if we start it at any other well-defined value, like 0 or 3 or 1500.

Example 1.16. (Getting practice with summation notation, partial sums, and the sequence of partial sums)

(i) Write the first four terms of the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$.

(ii) Write out the first four terms of the sequence of partial sums for the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$.

(i) $\frac{3}{10}, \frac{3}{10^2}, \frac{3}{10^3}, \frac{3}{10^4}$

a_1 a_2 a_3 a_4

i.e., $0.3, 0.03, 0.003, 0.0003$

(ii) $\{S_N\}_{N=1}^{\infty}$ sequence of partial sums

$S_N = \sum_{n=1}^{N} a_n$

$S_1 = \sum_{n=1}^{1} a_n = a_1 = \boxed{0.3}$

$S_2 = \sum_{n=1}^{2} a_n = a_1 + a_2 = 0.3 + 0.03 = \boxed{0.33}$

$S_3 = a_1 + a_2 + a_3 = 0.3 + 0.03 + 0.003 = \boxed{0.333}$

$S_4 = a_1 + a_2 + a_3 + a_4 = \dots = \boxed{0.3333}$

$\boxed{\text{Ans: } 0.3,\ 033,\ 0.333,\ 0.3333}$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Example 1.17. (FRY Exercise II.3.2.2, Getting practice with summation notation, partial sums, and the sequence of partial sums)

Write out the first four terms of the sequence of partial sums for the 'harmonic" series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$S_1 = \frac{1}{1} = 1$$

$$S_2 = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \approx 1.833$$

$$S_4 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.083$$

Example 1.18. (Getting practice with the definition of convergence of a series)

Determine whether or not the series $\displaystyle\sum_{n=5}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges. Find the sum if it converges.

$\{S_n\}_{n=5}^{\infty}$

$S_5 = \frac{1}{5} - \frac{1}{6}$

The series in Example Eg. 1.18 is an example of a "telescoping series," which is a series in which nearly all the terms cancel each other out. Also note that, since $\frac{1}{n}-\frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n^2+n}$, the analysis above applies to the series $\displaystyle\sum_{n=5}^{\infty}\frac{1}{n^2+n}$ once we recognize that $\frac{1}{n^2+n}$ and $\frac{1}{n}-\frac{1}{n+1}$ are equivalent.