Signal Processing (MENG3520)

Module 8

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MODULE 8

FOURIER METHODS – PART 2

FOURIER TRANSFORM FAMILY

The following four types are all part of the Fourier transform family:

- Continuous-time Fourier series (CTFS) – periodic continuous-time signals
- Continuous-time Fourier transform (CTFT) – aperiodic continuous-time signals
- Discrete-time Fourier transform (DTFT) – aperiodic discrete-time signals
- Discrete Fourier transform (DFT) – periodic discrete-time signals.

Module Outline

- 8.1 Continuous-Time Fourier Transform (CTFT)
- 8.2 Convergence of CT Fourier Transform
- 8.3 Properties of CT Fourier Transform
- 8.4 Practice

8.1

Continuous-Time Fourier Transform (CTFT)

Why Continuous-time Fourier Transform?

- Periodic signals are often used in solving Engineering problems.
- Fourier transform is a more general tool that is able to represent both periodic and aperiodic signals.
- Fourier transform can be derived from Fourier series through a limiting process, both share some important similarities.

How to Transition From CTFS to CTFT?

- Consider an aperiodic signal as the limiting case of periodic signal with period $T \to \infty$, a more general signal representation is developed, which is called the Fourier Transform.

Recall: the continuous-time Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \qquad c_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

As $T \to \infty$, the Fourier series coefficients become:

$$c_n = \lim_{T\to\infty}\frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jn\omega_0 t}dt$$

If we define $X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$, compare $X(\omega)$ with c_n, we have:

$$c_n = \frac{1}{T}X(\omega)\Big|_{\omega = n\omega_0}$$

Conclusion: $\frac{1}{T}X(\omega)$ is the envelop of c_n.

Definition of CT Fourier Transform (CTFT)

The **continuous-time Fourier transform** of signal $x(t)$ is defined as:

$$X(\omega) \triangleq \mathcal{F}(x(t)) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

This transform is also called forward Fourier transform equation or Fourier transform analysis equation.

The **CT inverse Fourier transform** of $X(\omega)$ is defined as:

$$x(t) \triangleq \mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

This transform is called inverse Fourier transform equation or Fourier transform synthesis equation.

The **Fourier transform pair:** $x(t) \xleftrightarrow{CTFT} X(\omega)$

Frequency Spectrum of a Signal

- Fourier transform provides us with a frequency-domain perspective on signals, describing how information is distributed at different frequencies.
- The Fourier transform of a signal $x(t)$ quantifies how much information $x(t)$ has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the frequency spectrum of the signal.

Modifying Frequency Spectrum

Since the definition of filtering is to process signals in a frequency dependent manner, filtering can be considered as:

- Modifying the spectrum of a signal by going through a system.
- Fourier transform is a widely use tool in filter design and analysis.

8.2

Convergence of CT Fourier Transform

Convergence of Fourier Transform

- Meaning of Fourier transform convergence: validate that the synthesized signal $\tilde{x}(t)$ in the time domain is a true representation of the original signal $x(t)$.

$$\tilde{x}(t) \triangleq \mathcal{F}^{-1}\left(X(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

- Where,

$$X(\omega) \triangleq \mathcal{F}\left(x(t)\right) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

Convergence of Fourier Transform

- Convergence general case: If CT signal $x(t)$ is continuous and absolutely integrable, and the Fourier transform of $x(t)$ is also absolutely integrable, then then the Fourier transform representation of $x(t)$ converges pointwise.
- However, this is a strong condition that most practical signals do not meet. Thus, this case is of little use in practice.

- In Engineering, since Fourier transform is such an important tool, we need to develop simpler criteria to determine the convergence of such transform so that we can comfortably apply the transform tools.
- As the result, two convergence special cases are presented. Transform convergence can be quickly determined if signals satisfy one of the two cases.

Convergence special case 1: If signal $x(t)$ is of finite energy :

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

Then the Fourier transform of this signal $X(j\omega)$ is MSE convergent.

Convergence special case 2 (Dirichlet conditions): If CT signal $x(t)$ satisfies:

- Absolutely integrable $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$;
- Has finite number of maxima and minima over any finite period.
- Has finite number of discontinuities over a finite interval, each is finite.

Then this signal is pointwise convergent everywhere except at the points of discontinuities of $x(t)$.

Converges at the average of the left- and right-hand sides of the values of $x(t)$, at the points of discontinuities of $x(t)$.

8.3

PROPERTIES OF CT FOURIER TRANSFORM

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \qquad y(t) \overset{CTFT}{\longleftrightarrow} Y(\omega)$$

Property	**Time Domain**	**Fourier Domain**
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\omega) + \beta Y(\omega)$
Translation / time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Conjugation	$x^* (t)$	$X^* (\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Time and frequency scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \qquad y(t) \overset{CTFT}{\longleftrightarrow} Y(\omega)$$

Property	**Time Domain**	**Fourier Domain Coefficients**
Differentiation in time	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Differentiation in frequency	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Even Symmetry	$x(t)$ real and even	$X(\omega)$ even and real
Odd Symmetry	$x(t)$ real and odd	$X(\omega)$ odd and imaginary
Conjugate symmetry	$x(t)$ real	$X(\omega) = X^*(-\omega)$

DUALITY OF THE FOURIER TRANSFORM

- If $x(t) \overset{CTFT}{\longleftrightarrow} X(\omega)$, then:
- $X(\omega) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$
- $x(t) = \mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$
- These two transforms are similar but not identical.

- Let’s use two examples to see the relation between the Fourier pair of $x(t)$ and $X(\omega)$.

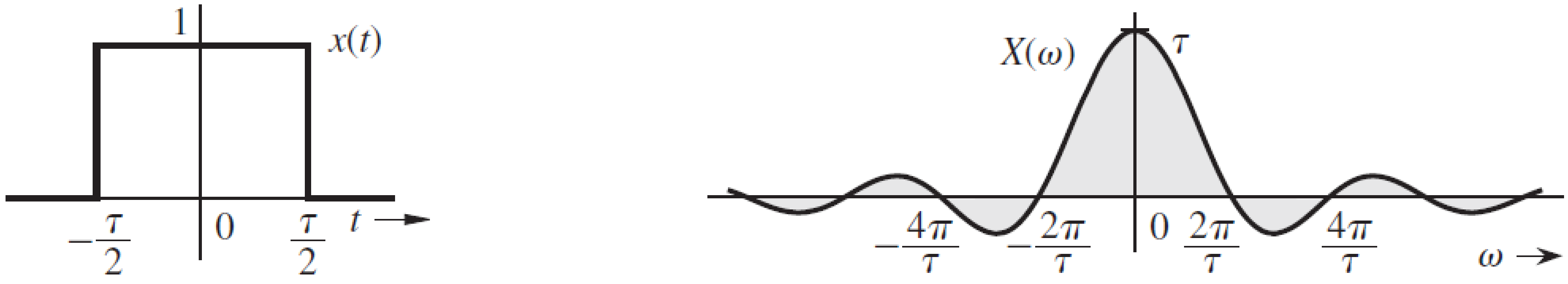
Example 1. Find the Fourier transform of $x(t) = rect(t/\tau)$:

Solution: $X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t}dt$

$$= \frac{1}{-j\omega}e^{-j\omega t}\Bigg|_{-\tau/2}^{\tau/2} = \frac{1}{j\omega}\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right) = \frac{2\sin(\omega\tau/2)}{\omega} = \tau\frac{\sin(\pi\frac{\omega\tau}{2\pi})}{\left(\pi\frac{\omega\tau}{2\pi}\right)} = \tau\mathrm{sinc}(\omega\tau/2\pi)$$

Here: $sinc(x) \triangleq \frac{\sin(\pi x)}{\pi x}$ is the normalized $sinc()$ function.

Conclusion: $\boldsymbol{rect(t/\tau) \overset{CTFT}{\longleftrightarrow} \tau sinc(\omega\tau/2\pi)}$



(a) (b)

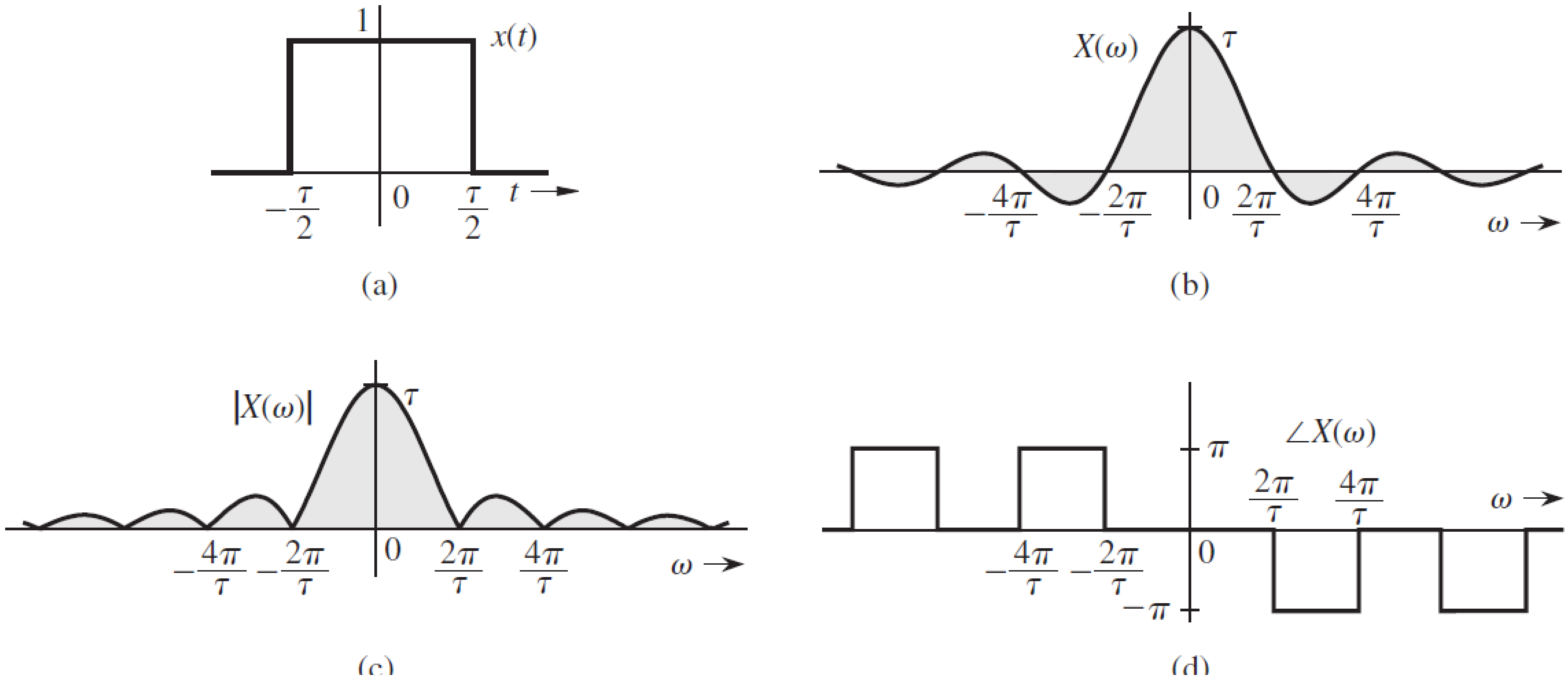
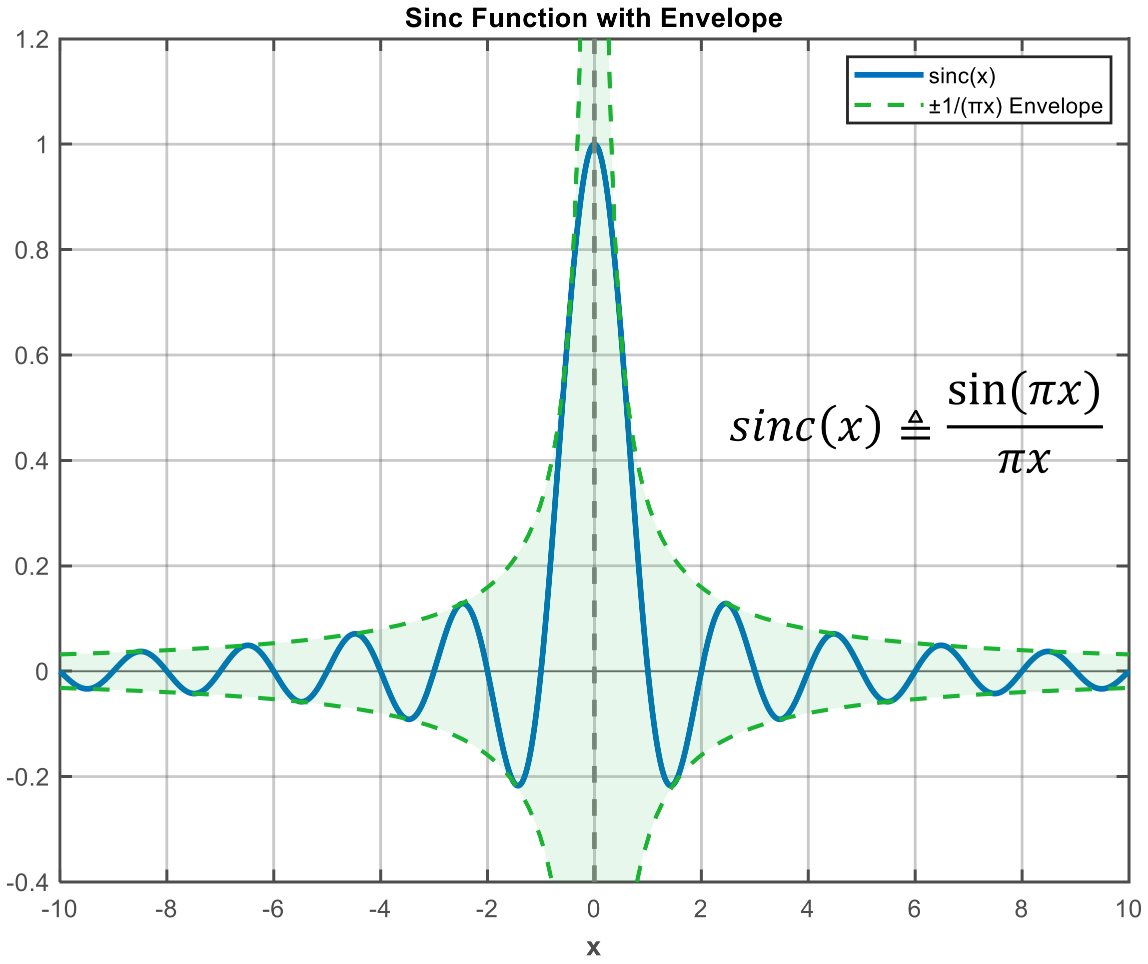


Figure 7.10 **(a)** A gate pulse $x(t)$, **(b)** its Fourier spectrum $X(\omega)$, **(c)** its amplitude spectrum $|X(\omega)|$, and **(d)** its phase spectrum $\angle X(\omega)$.

The function $sinc(x) = \frac{\sin(\pi x)}{\pi x}$ is the normalized sinc function. It is very important in signal processing. It is also known as the filtering or interpolating function.



1. $sinc(x)$ is an even function.
2. $sinc(x) = 0$ when $\sin(x) = 0$ except at $x = 0$. This means that $sinc(\pi x) = 0$ for integer x.
3. Using L'Hôpital's rule, $sinc$ (0) = 1.
4. $sinc(x) = 0$ is the product of an oscillating signal $\sin(\pi x)$ (of period 2) and a monotonically decreasing function $1/\pi x$. Therefore, $sinc(x)$ exhibits damped oscillations of period 2π, with amplitude decreasing continuously as $1/\pi x$.

Example 2. Let $X(\omega) = rect(\omega/2W) = \begin{cases} 1, |\omega| < W \\ 0, |\omega| > W \end{cases}$, then use synthesis equation to compute $x(t)$:

$$x(t) = \mathcal{F}^{-1}\big(X(\omega)\big) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-W}^{+W} e^{j\omega t}d\omega = \frac{1}{2\pi jt}e^{j\omega t}\Bigg|_{-W}^{W}$$

$$= \frac{1}{2\pi jt}\left(e^{jWt} - e^{-jWt}\right) = \frac{\sin(Wt)}{\pi t} = \frac{\sin\left(\pi\left(\frac{Wt}{\pi}\right)\right)}{\frac{\pi}{W}\left(\pi\left(\frac{Wt}{\pi}\right)\right)} = \frac{W}{\pi}\,\mathrm{sinc}(Wt/\pi)$$

Conclusion: $\boldsymbol{\frac{W}{\pi} sinc(Wt/\pi) \overset{CTFT}{\longleftrightarrow} rect(\omega/2W)}$

Homework:

Review: in-class examples, textbook chapter 7.

Example: 7.2-7.8

Problems: 7.1-5,7.1-6, 7.2-1,7.2-3, 7.3-9, 7.3-11