The tools of one-variable calculus can help find partial derivatives (Implicit Differentiation)

Let $z(x, y)$ be defined by the equation

$$zy - y + x = \ln(xyz).$$

Evaluate z_x at $(x, y, z) = (-1, -2, 1/2)$.

Take the derivative with respect to x:

$$yz_x - 0 + 1 = \frac{1}{xyz} \cdot \frac{\partial}{\partial x}(xyz)$$

$$yz_x + 1 = \frac{1}{x\cancel{y}z} \cdot \cancel{y}\, \frac{\partial}{\partial x}(xz)$$

$$\frac{\partial z}{\partial x}$$

$$yz_x + 1 = \frac{1}{xz} \cdot \left(1z + x z_x\right)$$

$$yz_x + 1 = \frac{1}{x} + \frac{z_x}{z}$$

$$\frac{1}{xz} \cdot z = \frac{\cancel{z}}{x\cancel{z}}$$

$$y z_x - \frac{z_x}{z} = \frac{1}{x} - 1 \qquad \left| \frac{1}{\cancel{x} z} \cdot \cancel{x} z_x \right.$$

$$\left(y - \frac{1}{z}\right) z_x = \frac{1}{x} - 1$$

$$\boxed{z_x = \frac{\frac{1}{x} - 1}{y - \frac{1}{z}}}$$

$$z_x\left(\underset{x}{-1}, \underset{y}{-2}, \underset{z}{\tfrac{1}{2}}\right) = \frac{\frac{1}{-1} - 1}{-2 - \frac{1}{1/2}} = \frac{-2}{-4} = \frac{1}{2}$$

We can take higher-order partial derivatives

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, there are four possible second derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) = f_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = f_{xy}(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = f_{yx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = f_{yy}(x,y)$$

Example: Calculating Higher-Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^2y^3.$$

Find all the second-order partial derivatives of f.

$$\frac{\partial f}{\partial x} = f_x = 2xy^3, \qquad \frac{\partial f}{\partial y} = f_y = 3x^2y^2$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \overset{\text{def'n}}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy^3\right) = 2y^3$$

mixed partial

$$f_{xy} = \frac{\partial^2 f}{\partial y\, \partial x} \overset{\text{def'n}}{=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy^3\right) = \boxed{6xy^2}$$

same!

derivatives

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2y^2\right) = \boxed{6xy^2}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3x^2y^2\right) = 6x^2y$$

$$f(x,y) = \sin(xy)$$

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$$\frac{\partial f}{\partial x} = \underbrace{\cos(xy) \cdot \frac{\partial}{\partial x}(xy)}_{\text{Chain Rule}} = \cos(xy) \cdot y = y\cos(xy)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\cos(xy)\right) = y \cdot \underbrace{-\sin(xy) \cdot \boxed{\overset{y}{\frac{\partial}{\partial x}(xy)}}}_{\text{Chain Rule}}$$

times

$$= -y^2 \sin(xy)$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\boxed{y}\,\boxed{\cos(xy)}\right)$$

Product Rule

$$= 1\cos(xy) + y\cdot -\sin(xy)\cdot \boxed{\overbrace{\frac{\partial}{\partial y}(xy)}^{x}}$$

$$= \cos(xy) - xy\sin(xy)$$

$$\lim_{x\to 0} x^3\cdot\frac{1}{x^2}$$

$$\frac{a}{a+b}\leq 1 \qquad 0 \leq r^2\sin^4\theta \leq r^2$$

$$\frac{\cos^2\theta}{\cos^2\theta + r^2\sin^4\theta}$$

Clairaut's Theorem on the equality of mixed partial derivatives

If the partial derivatives $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ exist and are continuous at (x_0, y_0), then

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0),$$

or, stated in another way,

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

Clairaut's Theorem can be applied to yet higher order partial derivatives

means

(FRY Exercise III.2.3.3.1)
Suppose f is of **class C^3 on** $\mathbb{R}^3$, that is, all of its first-, second-, and third-order partial derivatives exist and are continuous at every point in $\mathbb{R}^3$. Show that

$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z)$$
$$= f_{zxy}(x,y,z) = f_{zyx}(x,y,z)$$

Note: (i) f is of class C^1 if its first-order derivatives exist and are continuous.

(ii) f is of class C^2 if its first- and second-order partial derivatives exist and are continuous.

(iii) f is of class C^∞ if all of its higher-order derivatives exist and are continuous

Example: Using Clairaut's Theorem

(FRY Exercise III.2.3.3.3a)
Let $f(x, y) = x^2y^3$. Find $\boxed{f_{xyy}(1, -1)}$ and $f_{yxy}(1, -1)$.

f is a polynomial.
Thus, it has partial derivatives of every order and these derivatives, being polynomials, are themselves continuous.

By Clairaut's Theorem, $\boxed{f_{xyy}(1,-1)} = f_{yxy}(1,-1)$.

$$f(x,y) = x^2y^3$$

$$f_x = 2xy^3$$

$$f_{xy} = (f_x)_y = (2xy^3)_y = 6xy^2$$

$$f_{xyy} = (f_{xy})_y = (6xy^2)_y = 12xy$$

$$f_{yxy}(1,-1) = f_{xyy}(1,-1) = 12(1)(-1) = -12$$

by Clairaut's Theorem

Eg:

$$f(r,\theta) = r^3\sin\theta$$

Clairaut's Theorem also applies in other coordinate systems like Polar Coordinates.

$$f_r = 3r^2\sin\theta \qquad f_\theta = r^3\cos\theta$$

$$f_{r\theta} = 3r^2\cos\theta \qquad f_{\theta r} = 3r^2\cos\theta$$

$$\boxed{f_{r\theta r} = 6r\cos\theta \qquad f_{\theta rr} = 6r\cos\theta}$$

$$f_{\theta r\theta} = \left(f_{\theta r}\right)_\theta = \left(3r^2\cos\theta\right)_\theta$$

$$= -3r^2\sin\theta$$