Chapter 3

Power Series and Taylor Series

3.1 Readings

(1) CLP II §3.5 Power Series

(2) CLP II §3.6 Taylor Series

Recommended exercises

See course iDEAS site for a complete list of exercises.

3.2 Lesson Overview

Lesson goals

(i) find the radius and interval of convergence for a given power series

(ii) construct power series representations of functions using operations like multiplication by finite polynomials, differentiation, and integration on known power series

(iii) know the power series representations for common functions such as $\frac{1}{1-x}$, $\frac{1}{(1-x)^2}$, $\log(1+x)$, $\arctan x$, e^x, $\sin x$, $\cos x$

(iv) construct Taylor and Maclaurin series for a function at a given point

(v) create Taylor series using Taylor series for known functions

(vi) use known Taylor series to exactly evaluate an infinite series

(vii) use Taylor series to evaluate limits

3.3 Power Series

FRY Definition II.3.5.1, What is a Power Series?

Definition 3.1. A series of the form

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

is called a power series centred at c. The numbers A_n are called the coefficients of the power series.

Example 3.2. (FRY Example II.3.5.4, An example of a power series)
Let $a \neq 0$. Consider the following power series which has constant coefficients and is centred at 0:

$$\sum_{n=0}^{\infty} ax^n = ax^0 + ax^1 + ax^2 + ax^3 + \cdots$$
$$= a + ax + ax^2 + ax^3 + \cdots .$$

(a) Does the series converge when $x = 1/2$? When $x = 1$? When $x = 3/2$?

(b) Find a closed-form representation for the above power series. (Make a special note of the case when $a = 1$.)

Assume $a \neq 0$.

Power Series

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \cdots$$

x is a variable

(a) When $x = \frac{1}{2}$,

geom. series with $r = \frac{1}{2}$

$|r| = \frac{1}{2} < 1$

$$a + a\cdot\frac{1}{2} + a\cdot\left(\frac{1}{2}\right)^2 + a\cdot\left(\frac{1}{2}\right)^3 + \cdots = \frac{a}{1-\frac{1}{2}} = 2a$$

first term

When $x = 1$,

$$a + a(1) + a(1)^2 + a(1)^3 + \cdots$$
$$= a + a + a + a + \cdots$$

diverges

because $r = 1$ and this is a geometric series

because $\lim_{n\to\infty} a = a \neq 0$ (Divergence Test)

When $x=\frac{3}{2}$:

$a + a\left(\frac{3}{2}\right) + a\left(\frac{3}{2}\right)^2 + a\left(\frac{3}{2}\right)^3 + \cdots$ diverges

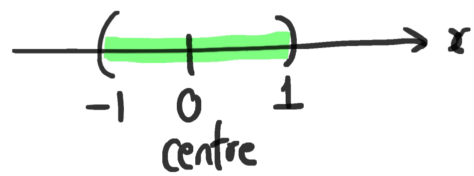
for the same reasons as when $x=1$.

(b) $\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \cdots$

$= \boxed{\dfrac{a}{1-x}}$, when $|x|<1$

think of x as the being ratio in a geometric series

Closed-form representation



radius of convergence = 1

interval of convergence = $(-1,1)$

not including -1

Example 3.3. (FRY Example II.3.5.15)
Find a power series representation for $\dfrac{1}{1+x^2}$.

Recall $\dfrac{a}{1-x} = a + ax + ax^2 + ax^3 + \cdots$, for $|x|<1$

When $a=1$: $\boxed{\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots}$, for $|x|<1$

$$\frac{1}{1-\text{☺}} = 1 + \text{☺} + \text{☺}^2 + \text{☺}^3 + \cdots \quad \text{for } |\text{☺}| < 1$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots, \text{ for } |-x^2|<1$$

$$= 1 - x^2 + x^4 - x^6 + \cdots, \quad \text{for } |x|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Example 3.4. Find a closed-form representation of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$ where $|x| < 1$.

$$\sum_{n=0}^{\infty} (-1)^n x^n = (-1)^0 x^0 + (-1)^1 x^1 + (-1)^2 x^2 + (-1)^3 x^3 + \cdots, \quad |x|<1$$

$$= 1 - x + x^2 - x^3 + \cdots, \quad |x|<1$$

$$= 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

$$= \frac{1}{1-(-x)}$$

$$= \frac{1}{1+x}$$, [4] where $|x|<1$

FRY Theorem II.3.5.9, every power series has a radius of convergence

Theorem 3.5. Let

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

be a power series. Then exactly one of the following must hold:

(a) The power series converges for every number x. In this case, we say that the **radius of convergence** is ∞ and the **interval of convergence** is $\mathbb{R}$.

(b) There is a finite nonzero number R such that the power series converges on the interval $(c-R, c+R)$ and diverges at every x that is farther than R away from c. (Depending on the coefficients, the series may or may not converge at the interval endpoints $c-R$ and $c+R$.) In this case, the radius of convergence is R and the interval of convergence is $(c-R, c+R)$.

(c) The power series converges for $x = c$ and diverges for all $x \neq c$. In this case, we say that the radius of convergence is 0 and the interval of convergence is $\{c\}$.

Notes:

(i) We adopt the symbol R even in the cases where the radius of convergence is ∞ or 0; that is, we write $R = \infty$ or $R = 0$, respectively.

In practice, we often try using the Ratio Test to help us determine the radius of convergence. Here's how. Suppose we are working with the power series

$$\sum_{n=0}^{\infty} A_n(x-c)^n,$$

where, eventually, all of the A_n are nonzero. The Ratio test tells us that

$$\text{If } \lim_{n\to\infty}\left|\frac{A_{n+1}(x-c)^{n+1}}{A_n(x-c)^n}\right| < 1, \text{ then } \sum_{n=0}^{\infty} A_n(x-c)^n \text{ converges.}$$

$$\text{I.e., if } \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| |x-c| < 1, \text{ then } \sum_{n=0}^{\infty} A_n(x-c)^n \text{ converges.}$$

$$\text{I.e., if } |x-c|\cdot\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| < 1, \text{ then } \sum_{n=0}^{\infty} A_n(x-c)^n \text{ converges.}$$

<u>*Case 1:*</u> $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = 0.$

In this case, $|x-c|\cdot\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = |x-c|\cdot 0 = 0$, and, no matter the x, $|x-c|\cdot\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$ will always be less than 1. So, $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for all values of x.

<u>*Case 2:*</u> $0 < \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| < \infty.$

In this case, $|x-c|\cdot\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| < 1$ when $|x-c| < \dfrac{1}{\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|}$. Defining R to be $\dfrac{1}{\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|}$, we see that $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges when $|x-c| < R$, that is, for all x values that are within a distance R of c.

Case 3: $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \infty.$

In this case, $|x-c|\cdot\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$ is zero when $x=c$ and ∞ when $x\neq c$. So, $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges only on the set $\{c\}$.

In summary,

	Radius of Convergence R	Interval of Convergence (*check convergence at endpoints)
$\lim_{n\to\infty}\left\lvert\frac{A_{n+1}}{A_n}\right\rvert = \infty$	0	$\{c\}$
$\lim_{n\to\infty}\left\lvert\frac{A_{n+1}}{A_n}\right\rvert = 0$	∞	$\mathbb{R}$
$0 < \lim_{k\to\infty}\left\lvert\frac{A_{n+1}}{A_n}\right\rvert < \infty$	$\frac{1}{\lim_{n\to\infty}\left\lvert\frac{A_{n+1}}{A_n}\right\rvert}$	one of $(c-R, c+R)$, $[c-R, c+R)$, $(c-R, c+R]$, or $[c-R, c+R]$

Table 3.1: The Ratio Test can help us determine the radius of convergence and the interval of convergence of a given power series $\sum_{n=0}^{\infty} A_n(x-c)^n$.

FRY Definition II.3.5.10, Interval of convergence for a power series

Definition 3.6. The set of x for which the power series $\sum_{n=0}^{\infty} A_n(x-c)^n$ is called the interval of convergence of the series.

Example 3.7. (FRY Example II.3.5.5)
What is the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

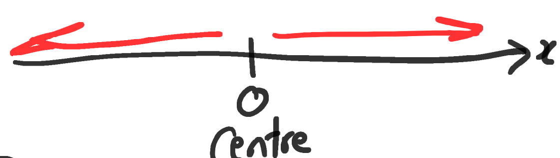
Also note that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ may be written as $\sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n$. That is,

- the series is centred at 0, and
- the coefficients of the series are given by $A_n = \frac{1}{n!}$, where $n = 0, 1, 2, \ldots$.

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$$\frac{n!}{(n+1)!} \qquad \frac{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{2\cdot 1}}{(n+1)\cancel{(n)}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{2\cdot 1}}$$

So, radius of convergence $R = \infty$.



The interval of convergence is $\mathbb{R}$.

$(-\infty, \infty)$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Example 3.8. (FRY Example II.3.5.6)
What is the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} n!\, x^n$?

$$\sum_{n=0}^{\infty} n!\, x^n \;=\; 0!\; x^0 + 1!\; x^1 + 2!\; x^2 + 3!\; x^3 + \cdots \;=\; 1 + x + 2x^2 + 6x^3 + \cdots .$$

Also note that $\sum_{n=0}^{\infty} n!\; x^n$ may be written as $\sum_{n=0}^{\infty} n!\; (x-0)^n$. That is,

- the series is centred at 0, and
- the coefficients of the series are given by $A_n = n!$, where $n = 0, 1, 2, \ldots$.

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty} n+1 = \infty$$

$$\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{n!}^{\,1}}{{}_{1}\cancel{n!}}$$

Radius of convergence $= 0$

Interval of convergence $= \{0\}$

The set consisting of the single element 0

Example 3.9. (FRY Exercise II.3.5.3.9)
Find the radius of convergence and the interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \cdot \frac{(x+1)^n}{3^n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{-1}{3}\right)^n (x+1)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{-1}{3}\right)^n \left(x-(-1)\right)^n
\end{aligned}$$

$$\sum_{n=0}^{\infty} \underbrace{A_n}_{\text{coefficients}} (x-\underbrace{c}_{\text{centre}})^n$$

Thus,

- the series is centred at -1, and
- the coefficients of the series are given by $A_n = \frac{1}{n+1}\left(\frac{-1}{3}\right)^n$, where $n = 0, 1, 2, \ldots$.

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n+2}\left(\frac{-1}{3}\right)^{n+1}}{\frac{1}{n+1}\left(\frac{-1}{3}\right)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{1}{3}\cdot\frac{n+1}{n+2}\right|$$

$$= \lim_{n\to\infty}\left(\frac{1}{3}\right)\frac{n+1}{n+2}$$

$$= \frac{1}{3}\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{1}{3}\cdot 1$$

$$= \frac{1}{3}$$

$$\lim_{n\to\infty}\frac{n+1}{n+2} = \lim_{n\to\infty}\frac{\frac{1}{n}(n+1)}{\frac{1}{n}(n+2)} = \lim_{n\to\infty}\frac{1+\cancel{1/n}^{\to 0}}{1+\cancel{2/n}^{\to 0}} = \frac{1+0}{1+0} = 1$$

$$|x-c|\cdot\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| < 1$$

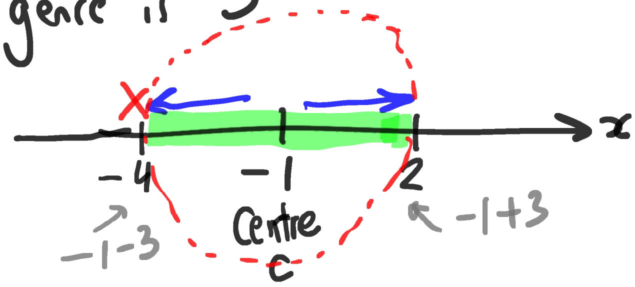
$$|x-c|\cdot\frac{1}{3}<1$$

$$|x-c|<\frac{1}{1/3}=3$$

radius of convergence

$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$

So radius of convergence is 3



When $x=-4$:

power series

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{-1}{3}\right)^n(x+1)^n=\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{-1}{3}\right)^n(-3)^n$$

$x+1$ equals $-4+1$

$$=\sum_{n=0}^{\infty}\frac{1}{n+1}\Big(\underbrace{-\frac{1}{3}\cdot -3}_{+1}\Big)^n$$

$$=\sum_{n=0}^{\infty}\frac{1}{n+1}=\underbrace{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots}_{\text{harmonic series}}$$

diverges

When $x=2$:

power series

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{-1}{3}\right)^n(x+1)^n=\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{-1}{3}\right)^n(3)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$$

$$\left(\frac{-1}{3}\cdot 3\right)^n$$

Look at $\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$.

(i) $\frac{1}{n+1} \geq 0$ for all $n=0,1,2,3,\ldots$

$\left\{\frac{1}{n+1}\right\}$ is a nonnegative sequence

(ii) Since $n+2 > n+1$ for every $n=0,1,2,\ldots$

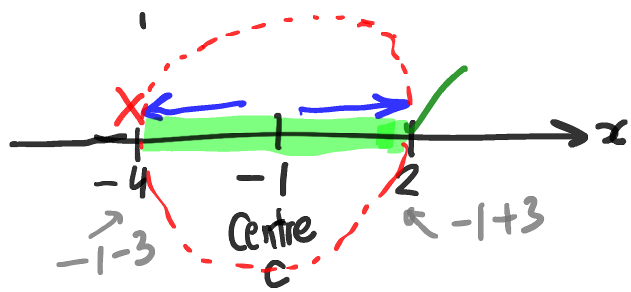
$\frac{1}{n+2} < \frac{1}{n+1}$ for every $n=0,1,2,\ldots$

$\left\{\frac{1}{n+1}\right\}$ is a decreasing sequence

(iii) $\lim_{n\to\infty} \frac{1}{n+1} = 0$

By the Alternating Series Test, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges.

I.e., the power series converges when $x=2$.



Interval of convergence: $[-4,2]$

3.4 Operations on Power Series

FRY Theorem 3.5.13, operations on power series

Theorem 3.10. Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x - c| < R$. That is, we are assuming that both power series have a radius of convergence at least R. Let α be a constant. Then

$$\begin{aligned}
f(x) + g(x) &= \sum_{n=0}^{\infty} (A_n + B_n)(x-c)^n \\
\alpha f(x) &= \sum_{n=0}^{\infty} \alpha A_n (x-c)^n \\
(x-c)^N f(x) &= \sum_{n=0}^{\infty} A_n (x-c)^{n+N} = \sum_{k=N}^{\infty} A_{k-N} (x-c)^k \quad \text{for any integer } N \geq 1 \\
f'(x) &= \sum_{n=0}^{\infty} A_n n (x-c)^{n-1} = \sum_{n=1}^{\infty} A_n n (x-c)^{n-1} \\
\int_c^x f(t)\, dt &= \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \\
\int f(x)\, dx &= \left(\sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \right) + C
\end{aligned}$$

for all x obeying $|x - c| < R$. Moreover, the radius of convergence for each of the power series on the right is at least R: It is exactly R for all with the possible exceptions of the sum $\sum_{n=0}^{\infty} (A_n + B_n)(x-c)^n$ and the scalar product $\sum_{n=0}^{\infty} \alpha A_n (x-c)^n$.

Another operation that we can carry out on power series is *composition* on the

x-values where it is appropriate to do so.

Example 3.11. (FRY Example II.3.5.20)
Find a power series representation for $\log(1+x)$. (Exercise: What is the radius and interval of convergence for this power series?)

For $x>-1$, $\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$

$$= \int_0^x (1-t+t^2-t^3+t^4-\dots)\,dt$$

$$= \left[t-\tfrac{1}{2}t^2+\tfrac{1}{3}t^3-\tfrac{1}{4}t^4+\tfrac{1}{5}t^5+\dots\right]_0^x$$

$$= x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\tfrac{1}{5}x^6+\dots$$

$$\boxed{= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}x^{n+1}} \quad = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}x^m$$

$$\int_0^x \frac{1}{1+t}\,dt$$
$$= \ln|1+t|\Big|_0^x$$
$$= \ln|1+x| - \ln|1+0|$$
$$= \ln|1+x| - 0$$
$$= \ln|1+x|$$
if $x>-1$ $\;= \ln(1+x)$

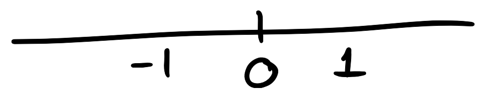
That is, $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ (centre $c=0$)

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

Radius of convergence:

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}/_{n+2}}{(-1)^n/_{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n+2} = 1$$

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|} = \frac{1}{1} = 1$$



When $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(-1)^{n+1} = \sum_{n=0}^{\infty} \frac{\cancel{(-1)^{2n}}^{\,1}(-1)}{n+1}$

$$= \sum_{n=0}^{\infty} \frac{-1}{n+1}$$

$$= -1 \cdot \sum_{n=0}^{\infty} \frac{1}{n+1}$$

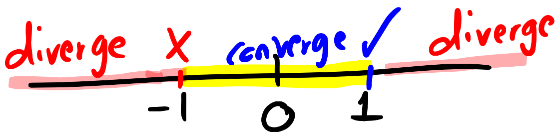
Let $m = n+1$

$$\underbrace{= -1 \cdot \underbrace{\sum_{m=1}^{\infty} \frac{1}{m}}_{\text{diverges}}}_{\text{diverges}}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

When $x=1$ the power series becomes

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(1)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by the Alternating Series Test



So the interval of convergence for $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ is $(-1, 1]$.

Example 3.12. (FRY Example II.3.5.21)
Find a power series representation for $\arctan x$. (Exercise: What is the radius and interval of convergence for this power series?)

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \int_0^x \left(1 - t^2 + t^4 - t^6 + \cdots\right) dt$$

$$= \left[t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \cdots\right]_0^x$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

Integrating a power series does not change the radius of convergence. So, since the radius of convergence of $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ was $R=1$,

the radius of convergence for $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ is also $R=1$.

What about the interval of convergence?

When $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (-1)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancelto{1}{(-1)^{2n}} (-1)}{2n+1}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

which converges by the Alternating Series Test

When $x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (1)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

which converges by the Alternating Series Test

So, the interval of convergence is $[-1,1]$.