HUMBER

Kinematics and Dynamics of Robots

Module 2

Rigid Motions

- A large part of robot kinematics is concerned with establishing various coordinate frames to represent the positions and orientations of rigid objects, and with transformations among these coordinate frames. Indeed, the geometry of three-dimensional space and of rigid motions plays a central role in all aspects of robotic manipulation.
- We begin by examining representations of points and vectors in a Euclidean space equipped with multiple coordinate frames. Following this, we introduce the concept of a rotation matrix to represent relative orientations among coordinate frames. We then combine these two concepts to build homogeneous transformation matrices, which can be used to simultaneously represent the position and orientation of one coordinate frame relative to another. Furthermore, homogeneous transformation matrices can be used to perform coordinate transformations. Such transformations allow us to represent various quantities in different coordinate frames, which we discuss later.

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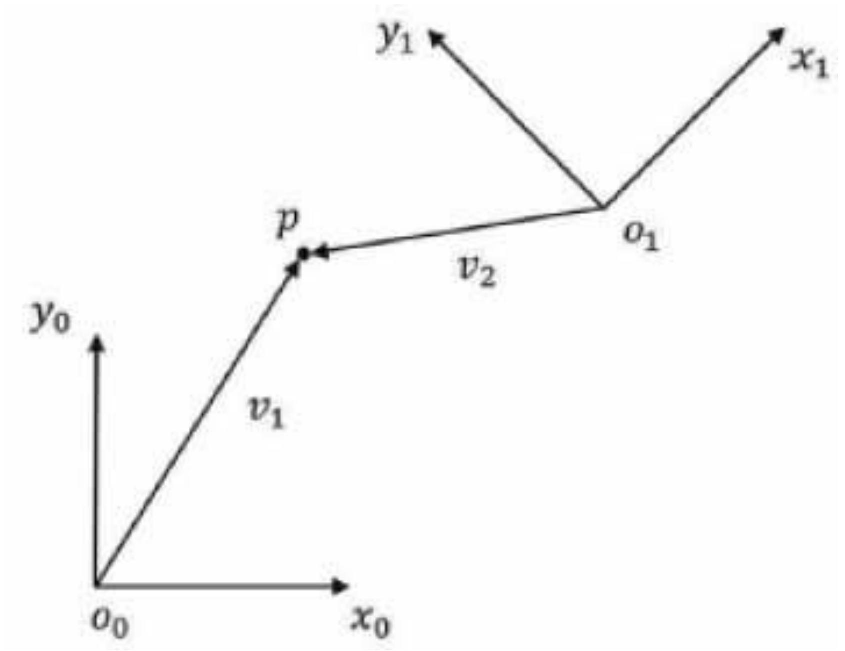
- We could specify the coordinates of the point *p* with respect to either frame $o_0x_0y_0$ or frame $o_1x_1y_1$. In the former case we might assign to p the coordinate vector (5, 6) and in the latter case (-3, 3). So that the reference frame will always be clear, we will adopt a notation in which a superscript is used to denote the reference frame. Thus, we write

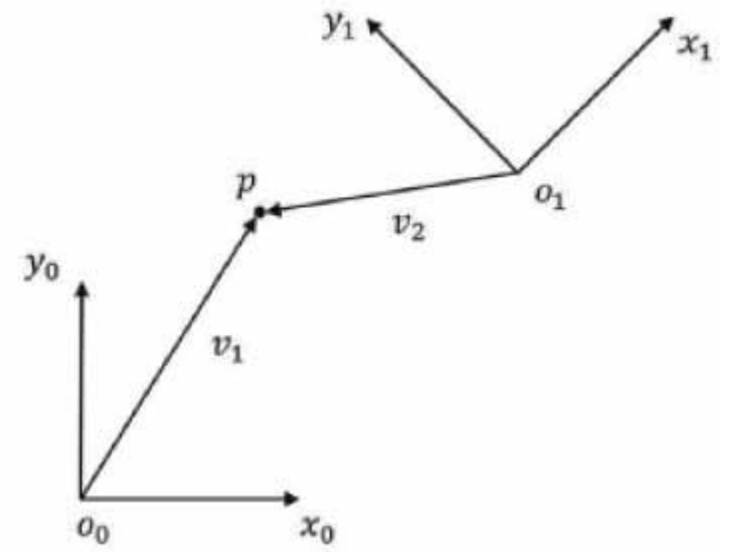
$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad p^1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$



Representing Positions

- Geometrically, a point corresponds to a specefic location in space. We stress here that p is a geometric entity, a point in space, while both p^0 and p^1 are coordinate vectors that represent the location of this point in space with respect to coordinate frames $o_0x_0y_0$ and $o_1x_1y_1$, respectively. When no confusion can arise, we may simply refer to these coordinate frames as frame 0 and frame 1, respectively.

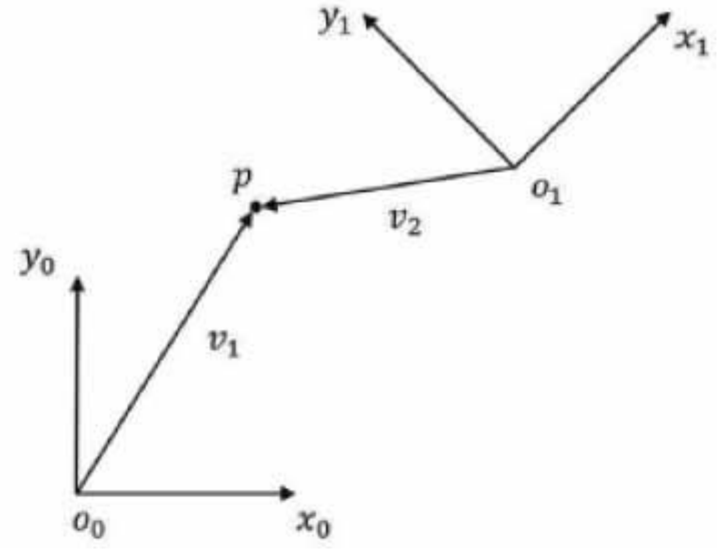
$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad p^1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$



- it is instructive to distinguish between the two fundamental approaches to geometric reasoning: the **synthetic** approach and the **analytic** approach. In the former, one reasons directly about geometric entities (e.g., points or lines), while in the latter, one represents these entities using coordinates or equations, and reasoning is performed via algebraic manipulations.

- Since the origin of a coordinate frame is also a point in space, we can assign coordinates that represent the position of the origin of one coordinate frame with respect to another. for example, we may have

$$o_1^0 = \begin{bmatrix} 12 \\ 8 \end{bmatrix}, \quad o_0^1 = \begin{bmatrix} -16 \\ 3 \end{bmatrix}$$



- Thus, o^0_1 specifies the coordinates of the point o_1 relative to frame 0 and o^1_0 specifies the coordinates of the point 0_0 relative to frame 1. In cases where there is only a single coordinate frame, or in which the reference frame is obvious, we will often omit the superscript. This is a slight abuse of notation, and you are advised to bear in mind the difference between the geometric entity called p and any particular coordinate vector that is assigned to represent p. The former is independent of the choice of coordinate frames, while the latter obviously depends on the choice of coordinate frames.

- While a point corresponds to a specific location in space, a vector specifies a direction and a magnitude. Vectors can be used, for example, to represent displacements or forces. Therefore, while the point p is not equivalent to the vector v_1, the displacement from the origin o_0 to the point p is given by the vector v_1. we will use the term vector to refer to what are sometimes called free vectors, that is, vectors that are not constrained to be located at a particular point in space. Under this convention, it is clear that points and vectors are not equivalent, since points refer to specific locations in space, but a free vector can be moved to any location in space. Thus, two vectors are equal if they have the same **direction** and the same **magnitude**.
- When assigning coordinates to vectors, we use the same notational convention that we used when assigning coordinates to points. Thus, v_1 and v_2 are geometric entities that are invariant with respect to the choice of coordinate frames, but the representation by coordinates of these vectors depends directly on the choice of reference coordinate frame.

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad v_1^1 = \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \quad v_2^0 = \begin{bmatrix} -6 \\ 2 \end{bmatrix}, \quad v_2^1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

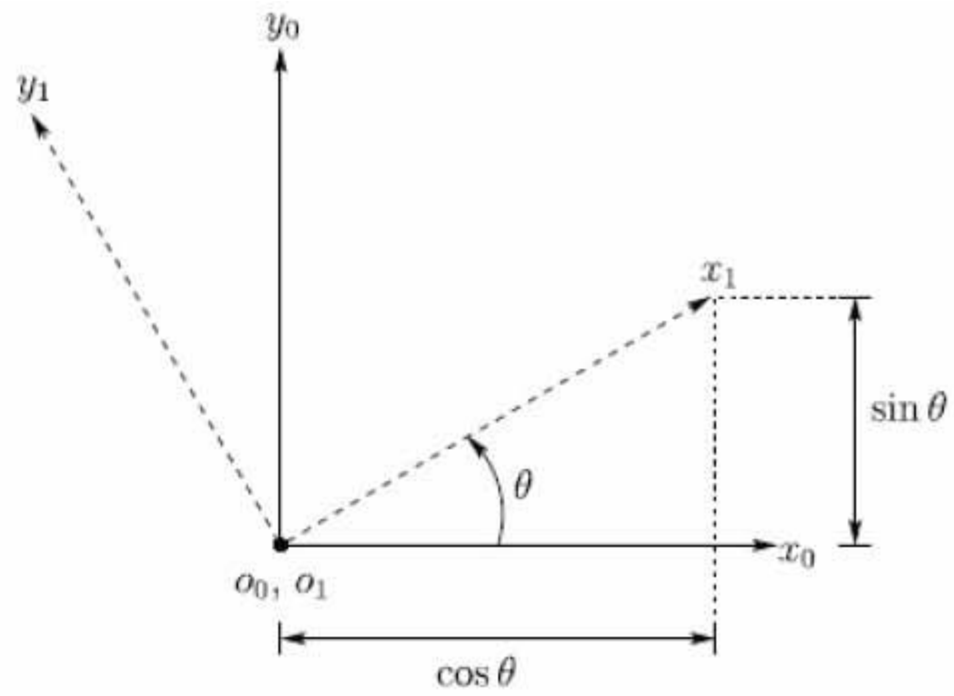
- In order to perform algebraic manipulations using coordinates, it is essential that all coordinate vectors **be defined with respect to the same coordinate frame**. In the case of free vectors, it is enough that they be defined with respect to "parallel" coordinate frames, that is, frames whose respective coordinate axes are parallel, since only their magnitude and direction are specified and not their absolute locations in space.
- Thus, we see a clear need not only for a representation system that allows points to be expressed with respect to various coordinate frames, but also for a mechanism that allows us to transform the coordinates of points from one coordinate frame to another.

Representing Rotations

- In order to represent the relative position and orientation of one rigid body with respect to another, we attach coordinate frames to each body, and then specify the geometric relationship between these coordinate frames.
- we saw how one can represent the position of the origin of one frame with respect to another frame. In this section, we address the problem of describing the orientation of one coordinate frame relative to another frame. We begin with the case of rotations in the plane, and then generalize our results to the case of rotations in a three-dimensional space

Rotation in the plane

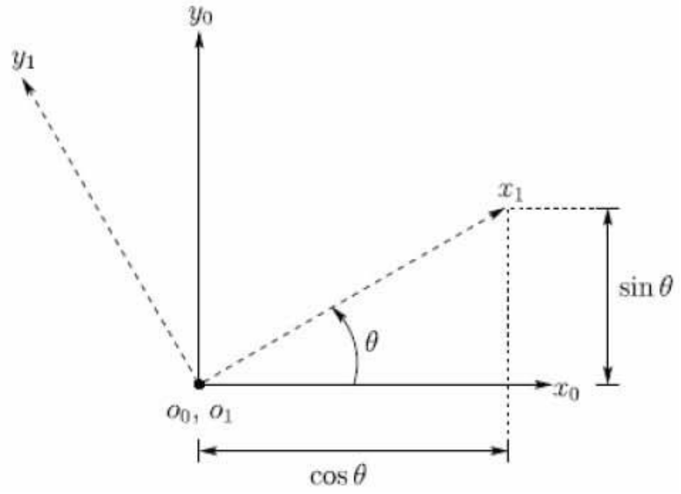
- Assume two coordinate frames shown here, with frame $o_1x_1y_1$ obtained by rotating frame $o_0x_0y_0$ by an angle θ. Perhaps the most obvious way to represent the relative orientation of these two frames is merely to specify the angle of rotation θ. This choice of representation does not scale well to the three-dimensional case.



Rotation in the plane

- A slightly less obvious way to specify the orientation is to specify the coordinate vectors for the axes of frame $o_1x_1y_1$ with respect to coordinate frame $o_0x_0y_0$:

$$R_1^0 = [x_1^0 | y_1^0]$$



- in which x_1^0 and y_1^0 are the coordinates in frame $o_0x_0y_0$ of unit vectors x_1 and y_1, respectively. A matrix in this form is called a rotation matrix. Rotation matrices have a number of special properties that we will discuss below.

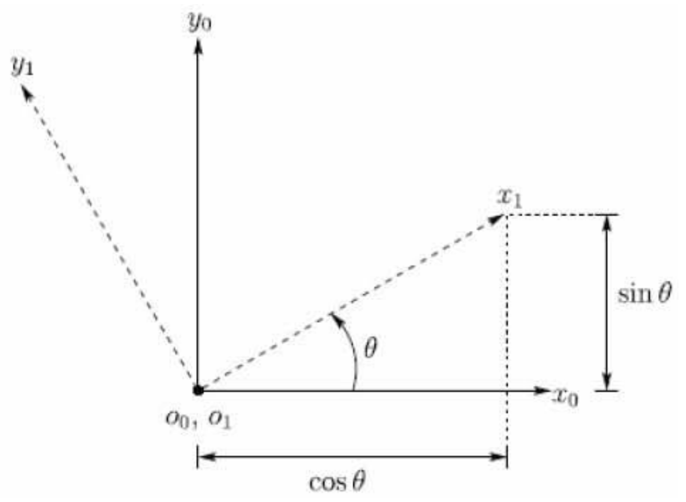
In the two-dimensional case, it is straightforward to compute the entries of this matrix.

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

which gives

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



- Note that we have continued to use the notational convention of allowing the superscript to denote the reference frame. Thus, is a matrix whose column vectors are the coordinates of the unit vectors along the axes of frame $o_1x_1y_1$ expressed relative to frame $o_0x_0y_0$.
- Although we have derived the entries for in terms of the angle θ, it is not necessary that we do so. An alternative approach, and one that scales nicely to the three-dimensional case, is to build the rotation matrix by projecting the axes of frame $o_1x_1y_1$ onto the coordinate axes of frame $o_0x_0y_0$.

- Recalling that the dot product of two unit vectors gives the projection of one onto the other, we obtain

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix}$$

- which can be combined to obtain the rotation matrix

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$

- Thus, the columns of specify the direction cosines of the coordinate axes of o1x1y1 relative to the coordinate axes of o0x0y0. For example, the first column (x1 · x0, x1 · y0) of specifies the direction of x1 relative to the frame o0x0y0. Note that the right-hand sides of these equations are defined in terms of geometric entities, and not in terms of their coordinates. Examining the Figure it can be seen that this method of defining the rotation matrix by projection gives the same result as we obtained in Equation 2.3a.
- If we desired instead to describe the orientation of frame o0x0y0 with respect to the frame o1x1y1 (that is, if we desired to use the frame o1x1y1 as the reference frame), we would construct a rotation matrix of the form

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y & y_0 \cdot y_1 \end{bmatrix}$$

Rotation in the plane

- Since the dot product is commutative, (that is, xi · yj = yj · xi), we see that

$$R_0^1 = \left(R_1^0\right)^T$$

- In a geometric sense, the orientation of o0x0y0 with respect to the frame o1x1y1 is the inverse of the orientation of o1x1y1 with respect to the frame o0x0y0. Algebraically, using the fact that coordinate axes are mutually orthogonal, it can readily be seen that

$$\left(R_1^0\right)^T = \left(R_1^0\right)^{-1}$$

- The above relationship implies that

$$\left(R_1^0\right)^T R_1^0 = I$$

- and it is easily shown that the column vectors of R_1^0 are of unit length and mutually orthogonal.
- Thus R_1^0 is an orthogonal matrix. It also follows from the above that det R_1^0 =±1. If we restrict ourselves to right-handed coordinate frames, then det R_1^0 =1
- More generally, these properties extend to higher dimensions, which can be formalized as the so-called special orthogonal group of order n. R_1^0

Rotation in the plane

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- More generally, these properties extend to higher dimensions, which can be formalized as the so-called special orthogonal group of order *n*.

Definition 2.1. *The* **special orthogonal group of order** n, *denoted* $SO(n)$, *is the set of* $n \times n$ *real-valued matrices*

$$SO(n) = \{R \in \mathbb{R}^{n \times n} \mid R^T R = RR^T = I \text{ and } \det R = +1\} \tag{2.2}$$

Thus, for any $R \in SO(n)$ *the following properties hold*

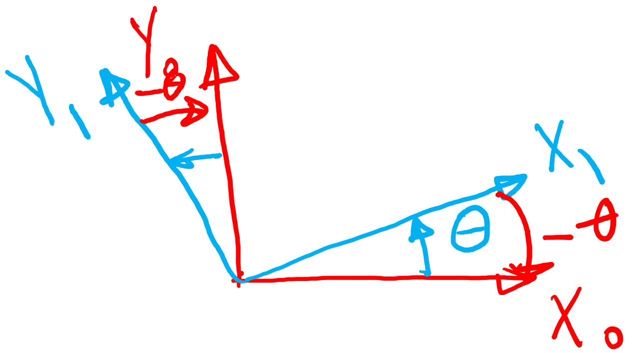
- $R^T = R^{-1} \in SO(n)$
- *The columns (and therefore the rows) of* R *are mutually orthogonal*
- *Each column (and therefore each row) of* R *is a unit vector*
- $\det R = 1$

The special case, $SO(2)$, *respectively,* $SO(3)$, *is called the* **rotation group** *of order* 2, *respectively* 3.

Rotation in the plane

- To provide further geometric intuition for the notion of the inverse of a rotation matrix, note that in the two-dimensional case, the inverse of the rotation matrix corresponding to a rotation by angle θ can also be easily computed simply by constructing the rotation matrix for a rotation by the angle -θ:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T$$



$\emptyset \to 1$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$1 \to \emptyset$

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

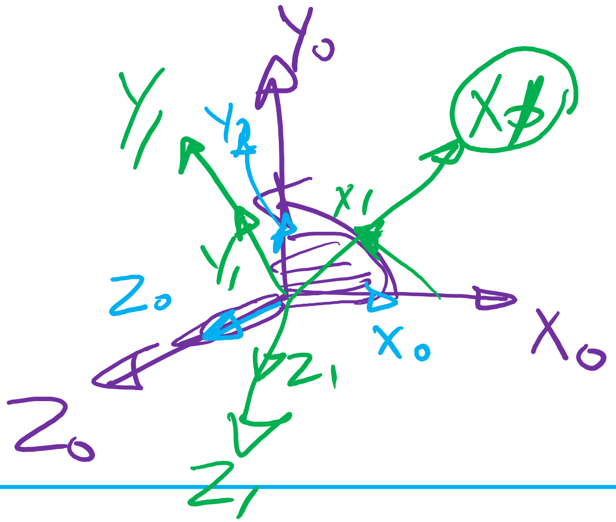
$$R_{-\theta} = R_\theta^T$$

Rotations in Three Dimensions

- The projection technique described above scales nicely to the three-dimensional case. In three dimensions, each axis of the frame o1x1y1z1 is projected onto coordinate frame o0x0y0z0. The resulting rotation matrix $R \in SO(3)$ is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

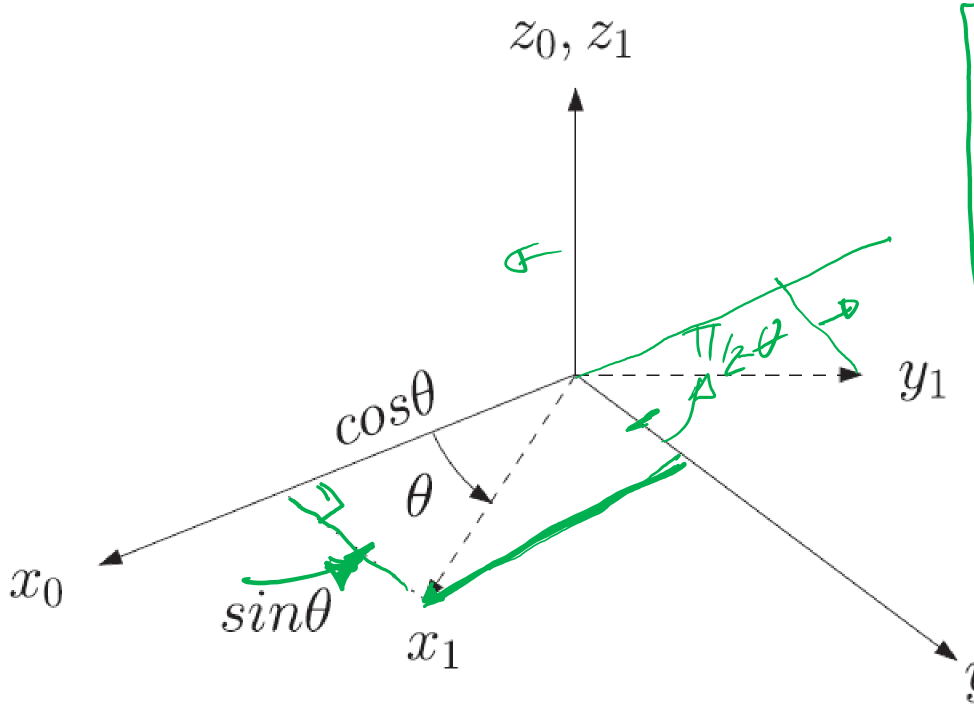
- As was the case for rotations matrices in two dimensions, matrices in this form are orthogonal, with determinant equal to 1 and therefore elements of SO(3).



$$R_1^0 \text{ wrt} = \begin{bmatrix} \vec{X}_1 \cdot \vec{X}_0 & \vec{Y}_1 \cdot \vec{X}_0 & \vec{Z}_1 \cdot \vec{X}_0 \\ \vec{X}_1 \cdot \vec{Y}_0 & \vec{Y}_1 \cdot \vec{Y}_0 & \vec{Z}_1 \cdot \vec{Y}_0 \\ \vec{X}_1 \cdot \vec{Z}_0 & \vec{Y}_1 \cdot \vec{Z}_0 & \vec{Z}_1 \cdot \vec{Z}_0 \end{bmatrix}$$

- Suppose the frame o1x1y1z1 is rotated through an angle θ about the z0-axis, and we wish to find the resulting transformation matrix. By convention, the right-hand rule defines the positive sense for the angle θ to be such that rotation by θ about the z-axis would advance a right-hand threaded screw along the positive z-axis. we see that

$R_1^0 = ?$



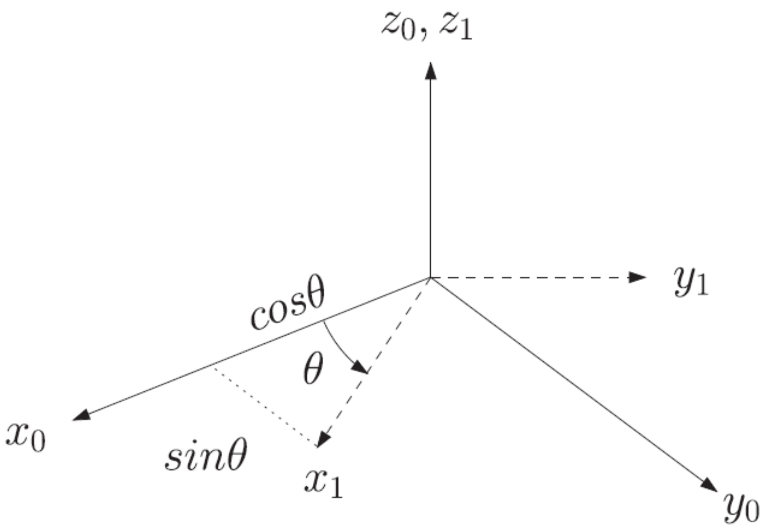
$$\begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

R_1^0 (2d)

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 \cdot x_0 = \cos\theta, \qquad y_1 \cdot x_0 = -\sin\theta,$$
$$x_1 \cdot y_0 = \sin\theta, \qquad y_1 \cdot y_0 = \cos\theta$$

$$z_0 \cdot z_1 \;=\; 1$$



- while all other dot products are zero. Thus, the rotation matrix R_1^0 has a particularly simple form in this case, namely

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations in Three Dimensions

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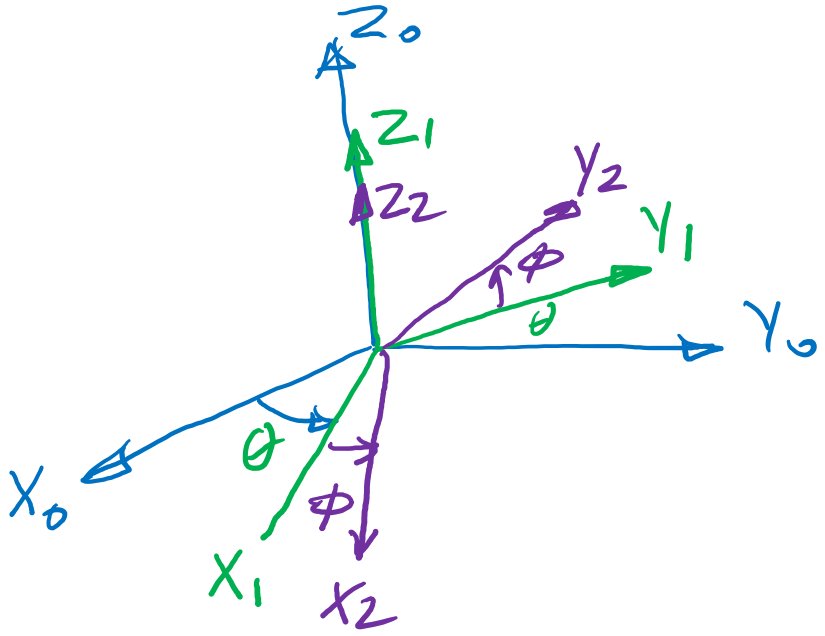
- The rotation matrix given in Equation 2.3c is called a basic rotation matrix (about the z-axis). In this case we find it useful to use the more descriptive notation $R_{z,\theta}$ instead of R_1^0 to denote the matrix. It is easy to verify that the basic rotation matrix has the properties

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R_{z,0} &= I \\ \underbrace{R_{z,\theta}}\underbrace{R_{z,\phi}} &= \underbrace{R_{z,\theta+\phi}} \end{aligned}$$

- which together imply

$$(R_{z,\theta})^{-1} = R_{z,-\theta}$$



$$0 \longrightarrow 2$$

$$R_{z,(\theta+\phi)}$$

$$0 \longrightarrow 1 \longrightarrow 2$$

$$R_{z,\theta} \quad R_{z,\phi}$$

$$R_{z\theta} \cdot R_{z\phi} = R_{z,\theta+\phi}$$

Rotations in Three Dimensions

- Similarly, the basic rotation matrices representing rotations about the x and y-axes are given as

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

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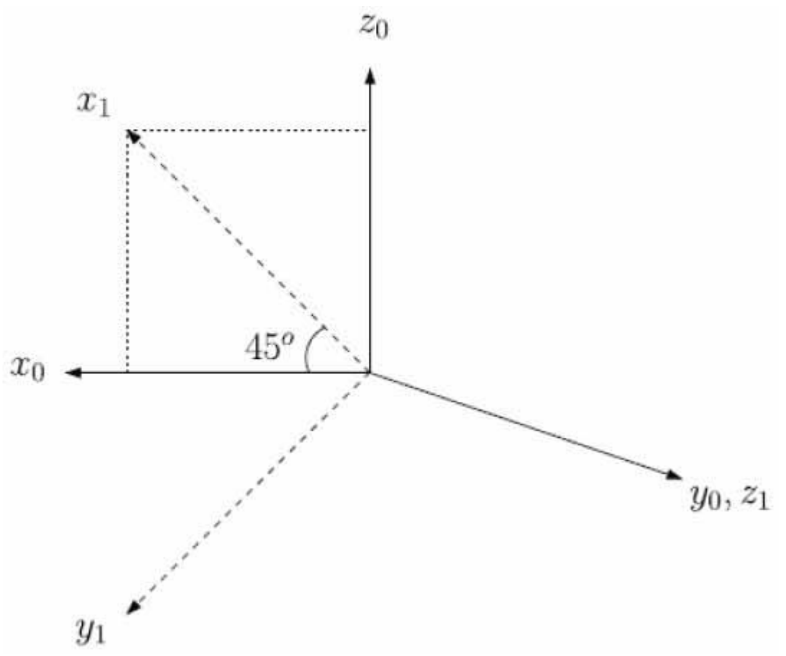
- Example: Find the Rotation matrix for the following:



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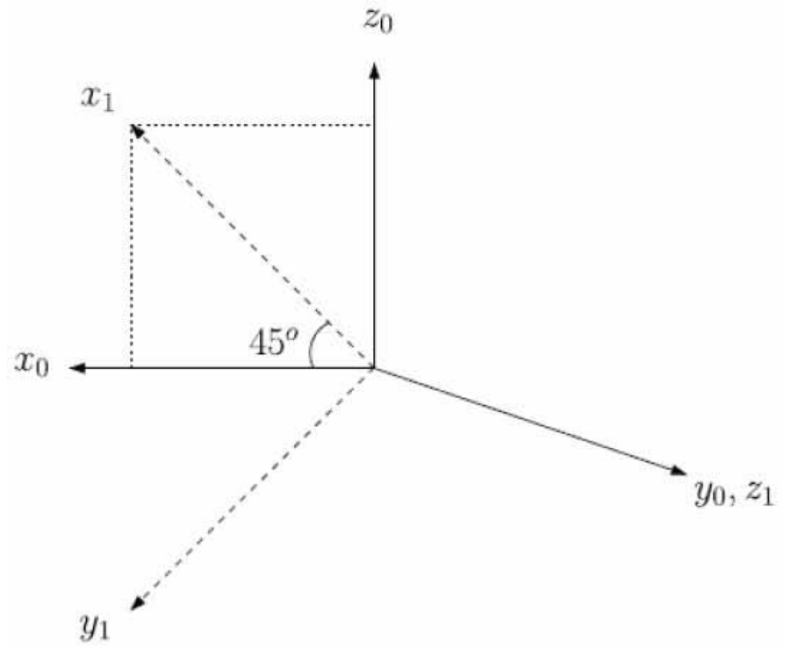
- Example: Find the Rotation matrix for the following:

Consider the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$. *Projecting the unit vectors* x_1, y_1, z_1 onto x_0, y_0, z_0 gives the coordinates of x_1, y_1, z_1 in the $o_0x_0y_0z_0$ *frame as*

$$x_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad z_1^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



The rotation matrix R_1^0 *specifying the orientation of* $o_1x_1y_1z_1$ *relative to* $o_0x_0y_0z_0$ *has these as its column vectors, that is,*

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

Questions?