Signal Processing (MENG3520)

Module 4

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Exercise

Consider the following RLC circuit, let $R = 3\Omega, L = 1H \ and \ C = 0.5F$.

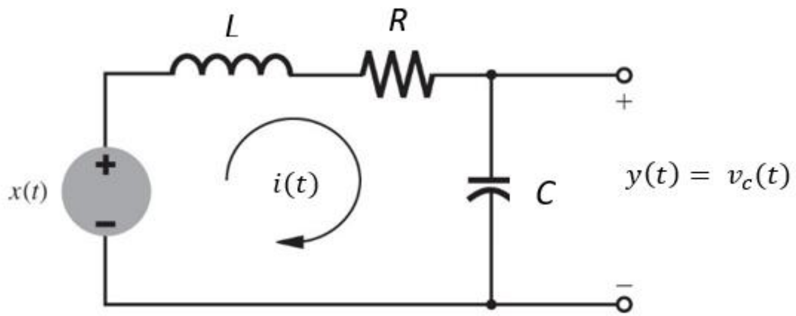
a. Determine the characteristic equation and find its roots analytically.

b. Find the zero-input response with the given initial conditions. $v_c(0^-) = 5V \ and \ i_L(0^-) = 0A$.

c. Determine the unit impulse response of the system.

d. Calculate the zero-state response for the specified input signals.

- $x(t) = u(t)$
- $x(t) = 10e^{-2t}u(t)$



Exercise

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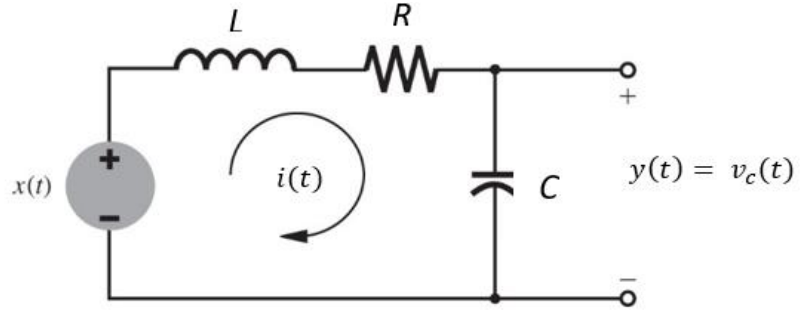
a. Determine the characteristic equation and find its roots analytically.

Answer:



According to KVL: $x(t) = v_L(t) + v_R(t) + v_C(t)$

Since $i(t) = C\dfrac{dv_C}{dt}$:

$$v_L(t) = L\frac{di(t)}{dt} = LC\frac{d^2v_C}{dt}, v_R(t) = Ri(t) = RC\frac{dv_C}{dt}$$

$$x(t) = LC\frac{d^2v_C}{dt} + RC\frac{dv_C}{dt} + v_c(t)$$

$$x(t) = 0.5\frac{d^2v_C}{dt} + 3\frac{dv_C}{dt} + 2v_c(t)$$

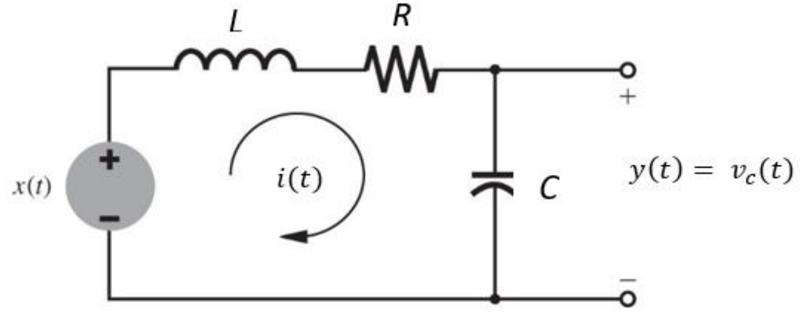
Let $y(t) = v_c(t)$, the system equation is: $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y(t) = 2x(t)$

Characteristic equation is: $\lambda^2 + 3\lambda + 2 = 0$, roots: $\lambda_1 = -1, \lambda_2 = -2$.

Exercise

Consider the following RLC circuit, let $R = 3\Omega, L = 1H \; and \; C = 0.5F$.

b. Find the zero-input response with the given initial conditions. $v_c(0^-) = 5V \; and \; i_L(0^-) = 0A$.



Answer:

Let zero-input response: $y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$y_0(0) v_c(0^-) = y_0(0) = 5$

$i_L(0^-) = C \frac{dy_0}{dt}\bigg|_{t\,=\,0^-} = 0$, thus: $\dot{y}_0(0) = 0$.

Two initial conditions become: $y_0(0) = 5, \dot{y}_0(0) = 0$.

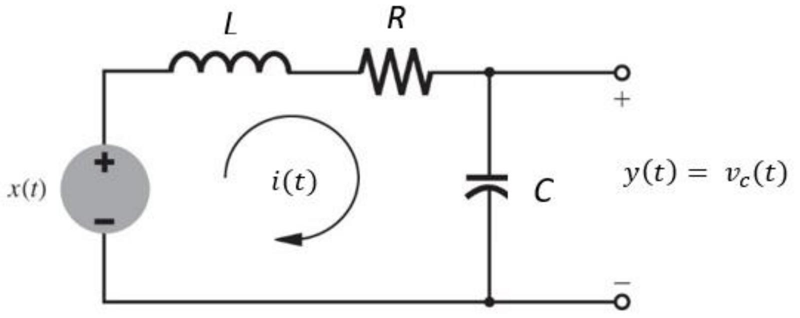
Solve equations: $\begin{cases} c_1 + c_2 = 5 \\ -c_1 - 2c_2 = 0 \end{cases}$ We have: $c_1 = 10, c_2 = -5$,

Zero input response $y_0(t) = 10e^{-t} + 5e^{-2t}$

Exercise

Consider the following RLC circuit, let $R = 3\Omega, L = 1H \; and \; C = 0.5F$.

c. Determine the unit impulse response of the system.



Answer:

System equation is: $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 2x(t)$

Order of the system $N = 2, M < N, b_0 = 0.$

Thus, the impulse response is in the form: $h(t) = [P(D)\, y_n\,(t)]u(t)$.

$y_n(t) = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}$ satisfies the simplified initial conditions:

$$y_n(0) = 0 \; and \; \dot{y}_n(0) = 1$$

Solve equations: $\begin{cases} c_3 + c_4 = 0 \\ -c_3 - 2c_4 = 1 \end{cases}$ We have: $c_1 = 1, c_4 = -1,$

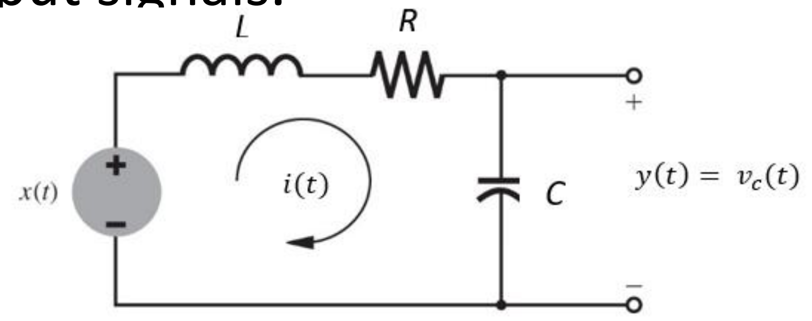
Impulse response $h(t) = [P(D) y_n\,(t)]u(t) = 2y_n\,(t)u(t) = (2e^{-t} - 2e^{-2t})u(t)$

Exercise

Consider the following RLC circuit, let $R = 3\Omega, L = 1H\ and\ C = 0.5F$.

d. Calculate the zero-state response for the specified input signals.

- $x(t) = u(t)$
- $x(t) = 10e^{-2t}u(t)$



Answer:

$x(t) = u(t) \rightarrow$

$$y_{ZSR}(t) = x(t) * h(t) = h(t) * x(t) = \left((2e^{-t} - 2e^{-2t})u(t)\right) * u(t)$$

$$= \left(2e^{-t}u(t)\right) * u(t) - \left(2e^{-2t}u(t)\right) * u(t)$$

$$= \int_{-\infty}^{t} 2e^{-\tau}u(\tau)u(t-\tau)d\tau - \int_{-\infty}^{t} 2e^{-2\tau}u(\tau)u(t-\tau)d\tau$$

$$= \int_{0}^{t} 2e^{-\tau}d\tau - \int_{0}^{t} 2e^{-2\tau}d\tau$$

$$= 2(1 - e^{-t})u(t) - 2\left(\frac{1-e^{-2t}}{2}\right)u(t) = (1 - 2e^{-t} + e^{-2t})u(t)$$

TABLE 2.1 Select Convolution Integrals

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
1	$x(t)$	$\delta(t-T)$	$x(t-T)$
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1 - e^{\lambda t}}{-\lambda}u(t)$
3	$u(t)$	$u(t)$	$tu(t)$

Answer:

$x(t) = u(t) \rightarrow$

$$y_{ZSR}(t) = x(t) * h(t) = h(t) * x(t) = \left((2e^{-t} - 2e^{-2t})u(t)\right) * u(t)$$

$$= \left(2e^{-t}u(t)\right) * u(t) - \left(2e^{-2t}u(t)\right) * u(t)$$

$$= \int_{-\infty}^{t} 2e^{-\tau}u(\tau)u(t-\tau)d\tau - \int_{-\infty}^{t} 2e^{-2\tau}u(\tau)u(t-\tau)d\tau$$

$$= 2(1 - e^{-t})u(t) - 2\left(\frac{1-e^{-2t}}{2}\right)u(t) = (1 - 2e^{-t} + e^{-2t})u(t)$$

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Consider the following RLC circuit, let $R = 3\Omega, L = 1H\ and\ C = 0.5F$.

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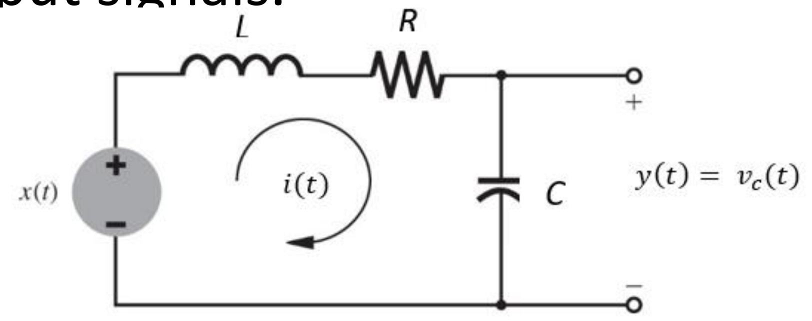


Answer:

$x(t) = 10e^{-2t}u(t) \rightarrow$

$y_{ZSR}(t) = x(t) * h(t) = h(t) * x(t) = (2e^{-t} - 2e^{-2t})u(t) * 10e^{-2t}u(t)$

$= 20[\,[e^{-t}u(t) * e^{-2t}u(t)] - [e^{-2t}u(t) * e^{-2t}u(t)]]$

$= 20\left[[e^{-t} - e^{-2t}]u(t) - te^{-2t}u(t)\right] = 20(e^{-t} - e^{-2t} - te^{-2t})u(t)$

MODULE 4

LAPLACE TRANSFORM

Overview

- Important tools for frequency domain analysis of continuous-time (CT) systems: Laplace transform, and Fourier transform
- The Laplace transform is the more general form.
- The Fourier transform can be considered a special case of the Laplace transform.
- For this module, we will explore the Laplace transform and how it is used to analyze CT LTI systems.

Module Outline

- 4.1 Eigenfunctions of CT LTI systems
- 4.2 Definition of Laplace Transform and Inverse Laplace Transform
- 4.3 ROC, Poles, and Zeros.
- 4.4 Properties of the Laplace transform
- 4.5 Transfer Functions
- 4.6 Analog filters
- 4.7 Frequency response of CT LTI systems

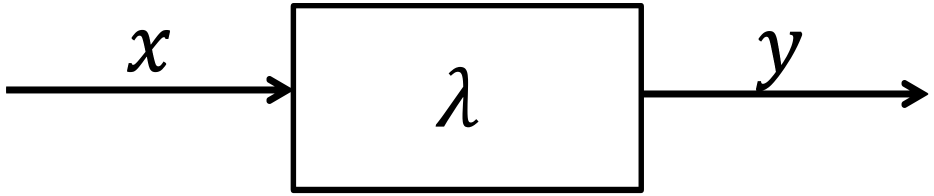
4.1

Eigenfunctions of CT LTI Systems

- Input x is an **eigenfunction** of the system H if the corresponding output y is:

$$y = \lambda x$$

- Where λ is a complex constant called the **eigenvalue**.
- When input is an eigenfunction of the system H, the system acts as an ideal amplifier with the amplifier gain defined by λ.



- Eigenfunctions and eigenvalues are important concepts in linear algebra, differential equations and now in signal processing.
- Different types of systems will have different types of eigenfunctions – we are interested in learning the form of the eigenfunctions for the systems that we are interested in – LTI systems.

Conclusion first: complex exponentials e^{st} are the eigenfunctions of LTI systems:

- In our previous modules we have spent time discussing the nature of complex exponentials.
- In fact, the main reason why complex exponentials are extremely important in the context of signal processing is because of it being the eigenfunctions of LTI systems.

- Consider a CT LTI system with impulse response $h(t)$.
- Let $x(t) = e^{st}$ be a complex exponential excitation / input to the system.
- Output:

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$y(t) = e^{st} \boxed{\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau}$$

$$\underbrace{e^{st}}_{x(t)} \qquad \underbrace{\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau}_{H(s)}$$

$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$ is a complex constant.

$$y(t) = H(s)x(t)$$

- **Important conclusion**: complex exponentials e^{st} are the eigenfunctions of CT LTI systems.

$$\xrightarrow{x(t)=e^{st}} \boxed{H(s)} \xrightarrow{y=x(t)H(s)}$$

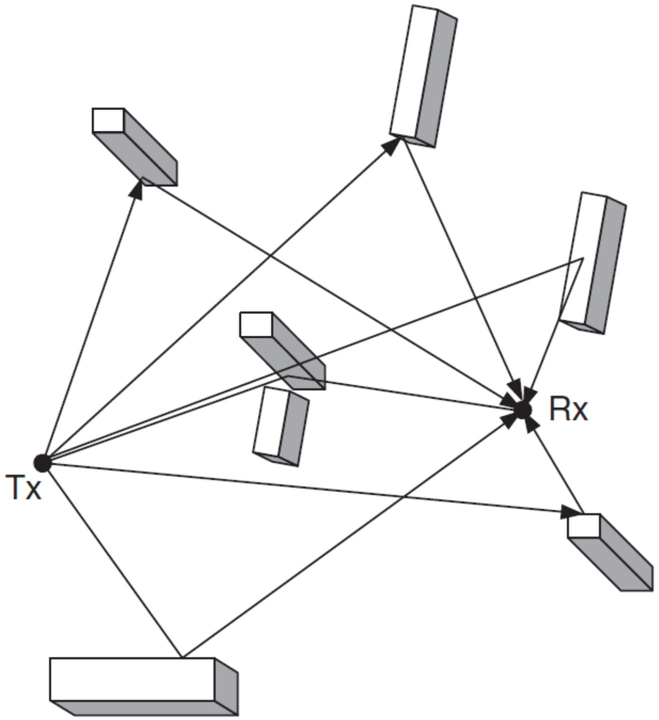
- This property is only valid for LTI systems, not time varying or non-linear systems.

- Suppose for the same CT LTI system, input $x(t)$ can be expressed as:
- $x(t) = \sum_k a_k e^{s_k t}$, where a_k and s_k are complex constant.
- Because LTI,
- $y(t) = \sum_k x(t)H(s_k) = \sum_k a_k H(s_k) e^{s_k t}$

- Important conclusion: if an input to a LTI system is a linear combination of complex exponentials, the output can be expressed as a linear combination of the same complex exponentials.

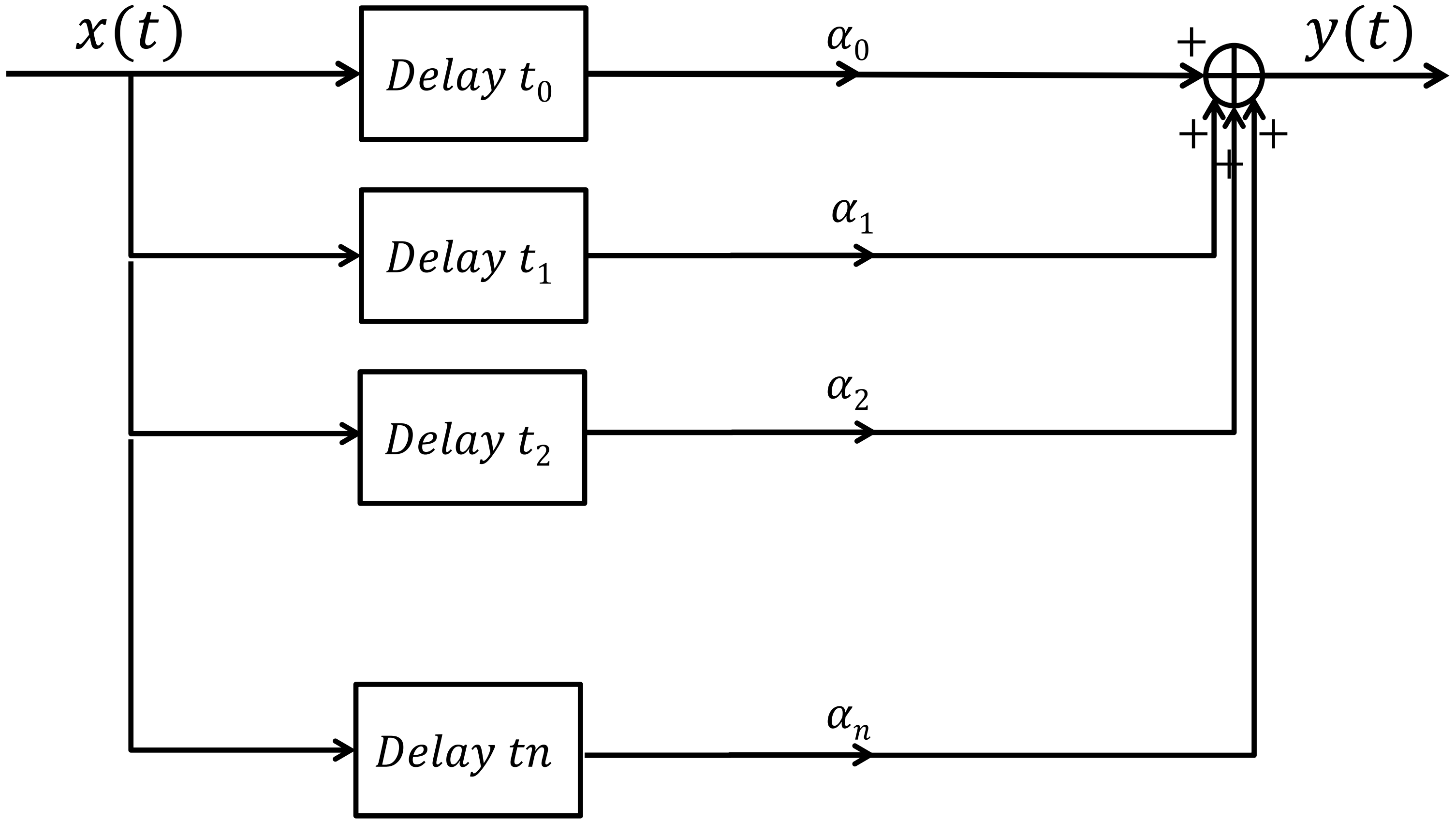
Example: EM propagation

Real wireless environment is a very “harsh” environment: different paths will have different propagation loss - multipath fading



EM propagation in wireless environment

Find the system function of the channel causing the multipath fading:



EM propagation in wireless environment

Find the system function $H(s)$ of the channel causing the multipath fading:

$$y(t) = \sum_{k=0}^{n} \alpha_k x(t - t_k)$$

Let $x(t) = e^{st}$, then:

$$y(t) = H(s)x(t) = \sum_{k=0}^{n} \alpha_k x(t - t_k)$$

$$H(s) = \frac{y(t)}{x(t)} = \sum_{k=0}^{n} \alpha_k e^{-st_k}$$

4.2

DEFINITION OF LAPLACE TRANSFORM AND INVERSE LAPLACE TRANSFORM

- For general values of complex variable $s = \sigma + j\omega$, with σ and ω being the real and imaginary parts, the Laplace transform of a general function $x(t)$ is defined as:

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

- The inverse Laplace transform, is defined as:

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

- Where c is a constant chosen to ensure the convergence of the integral.

- In practice, we often refer $x(t)$ and $X(s)$ a Laplace transform pair.
- The Laplace transform is denoted using the Laplace symbol $\mathcal{L}$:

$$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$$

Or

$$X(s) = \mathcal{L}\{x(t)\}$$

- Note: $s = \sigma + j\omega$ is a variable representing the complex frequency
- $\sigma = Re(s)$, indicates the rate of decay.
- $\omega = Im(s)$, indicates the rate of oscillation.

- This Laplace transform is also called bilateral Laplace transform due to integral from $-\infty$ to $+\infty$, differentiating from the unilateral Laplace transform.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0.
- Unilateral Laplace transform can be considered a special case of bilateral transform.

4.3

ROC, POLES AND ZEROS

Region of Convergence (ROC), also referred to as the region of existence, for the Laplace transform $F(s)$, is the set of values of s (the region in the complex plane) for which the integral $F(s) \triangleq \int_{-\infty}^{+\infty} f(t)e^{-st}dt$ converges.

- If $F(s) \triangleq \int_{-\infty}^{+\infty} f(t)e^{-st}dt$ does not converge, then the Laplace transform of function $f(t)$ does not exist.
- For the Laplace transform of $f(t)$ to exist:
- $\left|\int_{-\infty}^{+\infty} f(t)e^{-st}dt\right| = \left|\int_{-\infty}^{+\infty} f(t)e^{-(\sigma+j\omega)t}dt\right| \leq \int_{-\infty}^{+\infty}|f(t)e^{-\sigma t}|dt < \infty$
- σ needs to be chosen appropriately while ω does not affect the ROC.

Activity. Compute the Laplace transform of the following signals and determine ROC.

- (a) $f(t) = -e^{-at}u(-t)$

Activity: compute the Laplace transform of the following signals and determine ROC.

- (b) $f(t) = e^{-at}u(t)$

Activity: compute the Laplace transform of the following signals and determine ROC.

- (c) $f(t) = e^{-t}u(t) + e^{3t}u(-t)$

- For any rational function $F(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}dt = L\{f(t)\} = N(s)/D(s)$.
- Zeros: points on the s-plane where the values of *s* that make the function $F(s) = 0$. indicated on s-plane as “o”.
- Poles: points on the s-plane where the values of *s* that make the function $F(s) \to \infty$. Indicated on s-plane as “x”.
- Usually only finite zeros and poles are considered, infinite zeros and poles are also possible.

Relation between ROC and Poles:

- Property 1: The ROC consists of strips parallel to the $j\omega$ (axis, which means that it is the damping σ that defines the ROC, not frequency ω.
- The ROC is the values of σ such that $\left|\int_{-\infty}^{+\infty} f(t)e^{-st}dt\right| \leq \int_{-\infty}^{+\infty}|f(t)e^{-\sigma t}|dt \ < \infty$, this condition is independent of frequency ω.

Relation between ROC and Poles:

- Property 2: For rational Laplace transforms, no poles are included in the ROC.
- The ROC is the region where the Laplace transform is defined, whilst the poles are where the transform becomes non-convergent.

Relation between ROC and Poles:

- Property 3: if $f(t)$ is of finite duration and is absolutely integrable, then ROC is the entire s-plane.
- Since in this case,

$$\int_{T_1}^{T_2} |f(t)|dt < \infty$$

$$F(s) = \int_{T_1}^{T_2} f(t)e^{-st}dt \leq \int_{T_1}^{T_2} |f(t)||e^{-st}|dt = \int_{T_1}^{T_2} |f(t)|e^{-\sigma t}dt$$

$$< \max(e^{-\sigma t}) \int_{T_1}^{T_2} |f(t)|dt < \infty$$

Relation between ROC and Poles:

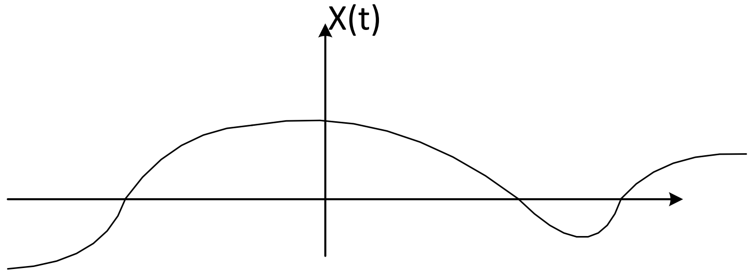
- Property 4: if $f(t)$ is right-sided, and if the line $Re\{s\} = \sigma_0$ is also in the ROC, then all values of s for which $Re\{s\} > \sigma_0$ are also in the ROC.
- This means that for right-sided $f(t)$, if there exists a real value $Re\{s\} = \sigma_0$ where the transform converges, all the points to the right of that point are also in the ROC.
- $e^{-\sigma t}$ is decaying faster toward $+\infty$ than $e^{-\sigma_0 t}$ for $\sigma > \sigma_0$

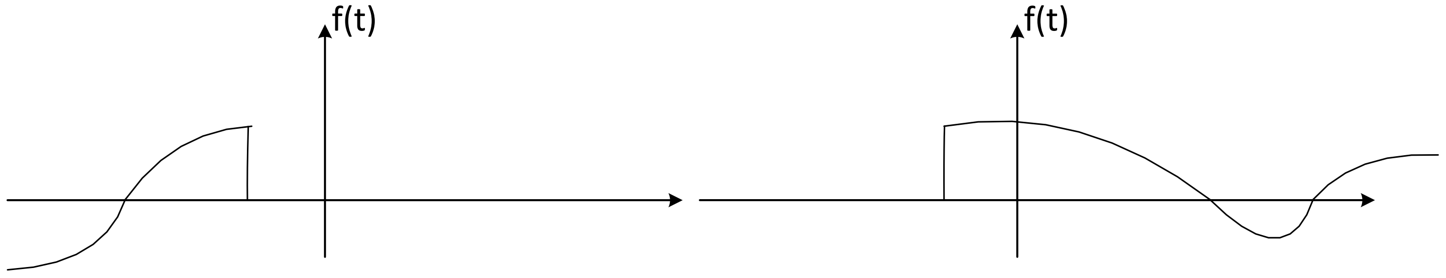
Relation between ROC and Poles:

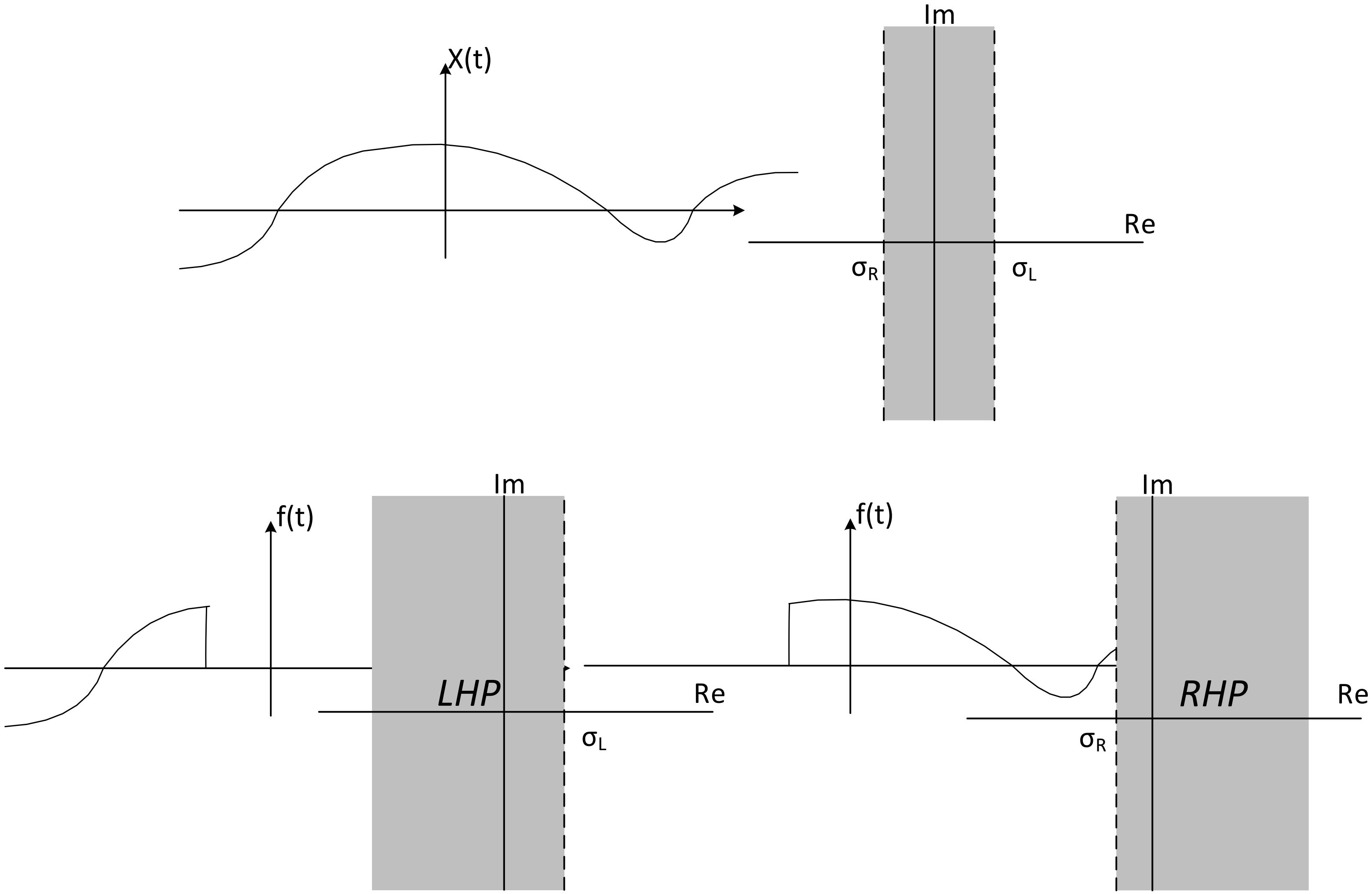
- Property 5: if $f(t)$ is left-sided, and if the line $Re\{s\} = \sigma_0$ is also in the ROC, then all values of s for which $Re\{s\} < \sigma_0$ are also in the ROC.
- This means that for left-sided $f(t)$, if there exists a real value $Re\{s\} = \sigma_0$ where the transform converges, all the points to the left of that point are also in the ROC.
- $e^{-\sigma t}$ is decaying faster toward $-\infty$ than $e^{-\sigma_0 t}$ for $\sigma < \sigma_0$

Relation between ROC and Poles:

- Property 6: if $f(t)$ is two-sided, and if the line $Re\{s\} = \sigma_0$ is also in the ROC, then the ROC will consists of a strip in the s-plane that includes the line $Re\{s\} = \sigma_0$.
- Break $f(t)$ into the sum of a right-sided and a left-sided signals.


X(t)


f(t)
f(t)


X(t)
Im
Re
σ_R
σ_L
f(t)
Im
LHP
Re
σ_L
f(t)
Im
RHP
Re
σ_R

Relation between ROC and Poles:

- Property 7: if the Laplace transform $F(s) = L\{f(t)\}$ is rational, then its ROC bounded by poles or extends to infinity.

Relation between ROC and Poles:

- Property 8: if the Laplace transform $F(s) = L\{f(t)\}$ is rational, then:
- If $f(t)$ is right-sided, then the ROC is the region in the s-plane to the right of the rightmost pole.
- If $f(t)$ is left-sided, then the ROC is the region in the s-plane to the left of the leftmost pole.

Let Laplace transform of $f(t)$ to be:

$F(s) = \frac{1}{(s+1)(s+2)}$, determine the ROC:

- (a) If $f(t)$ is right-sided;
- (b) If $f(t)$ is left-sided;
- (c) If $f(t)$ is two-sided.

4.4

Properties of Laplace Transform

- Linearity of the Laplace transform: if
- $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$, ROC=$R_1$
- $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$, ROC=$R_2$

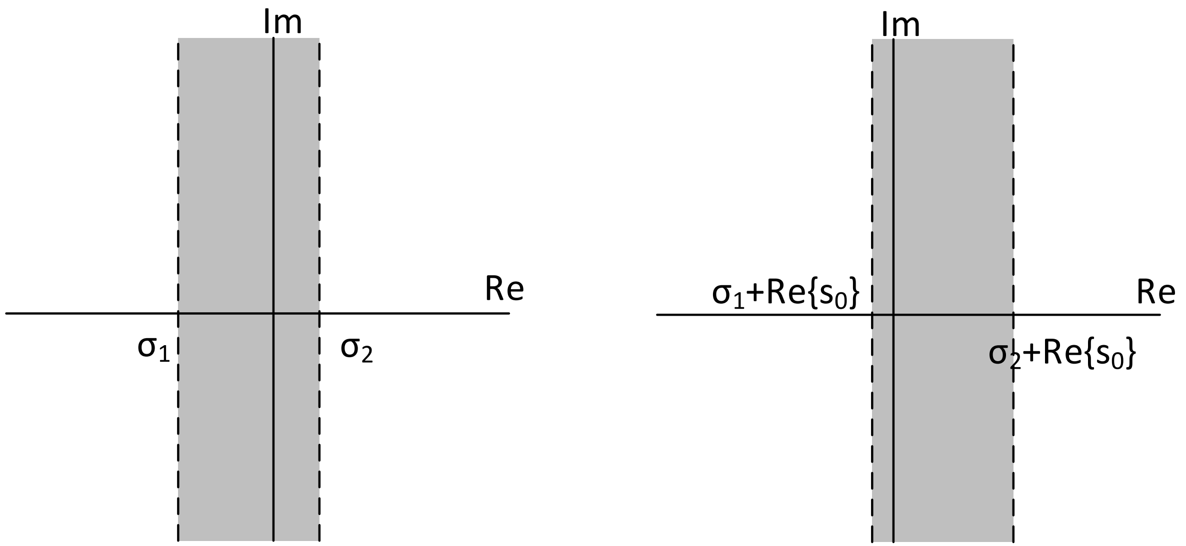
- $ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\leftrightarrow} aX_1(s) + bX_2(s)$
- ROC containing $R_1 \cap R_2$

Example: compute the Laplace transform of the following signal:
$g(t) = A\cos(\Omega_0 t)u(t)$

Solution: $g(t) = A\frac{e^{j\Omega_0 t}}{2}u(t) + A\frac{e^{-j\Omega_0 t}}{2}u(t)$

- Time shifting:
- If: $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC=R
- Then: $x(t - t_0) \overset{\mathcal{L}}{\leftrightarrow} e^{-st_0}X(s)$, ROC=R

- s-domain shifting:
- If: $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC=R
- Then:$e^{s_0 t}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s - s_0)$,
- ROC=R+Re{s_0}



Example: compute the modulated periodic complex exponential
$g(t) = e^{j\omega_0 t} x(t)$

Solution: because if $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC=R

Then:$e^{s_0 t} x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s - s_0)$, ROC=R+Re{$s_0$}

Thus using this s-domain shifting property, let:

$$s_0 = j\omega_0$$

Then

$$G(s) = X(s - j\omega_0), \mathrm{ROC} = \mathrm{R}$$

- Time scaling:
- If: $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC = R
- Then:$x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right),$ ROC = aR

Example: prove that if $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC=R

Then:$x(-t) \overset{\mathcal{L}}{\leftrightarrow} X(-s)$, ROC=-R

- Conjugation:
- If: $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC=R
- Then:$x^*(t) \overset{\mathcal{L}}{\leftrightarrow} X^*(s^*)$, ROC=R

- Convolution Property: if
- $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$, ROC = R_1
- $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$, ROC = R_2

- $x_1(t) * x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)X_2(s)$
- ROC contains $R_1 \cap R_2$

- Differentiation in the time domain:
- If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC = R
- Then $\frac{d}{dt}x(t) \overset{\mathcal{L}}{\leftrightarrow} sX(s)$, ROC contains R

- Generalization of the derivative property of the Laplace transform:
- If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC = R
- Then $\frac{d^N}{dt^N} x(t) \overset{\mathcal{L}}{\leftrightarrow} s^N X(s)$, ROC contains R

- *Application of the linearity and the derivative properties of the Laplace transform makes solving differential equations an algebraic problem.*

- Differentiation in the s-domain:
- If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC = R
- Then $-tx(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{d}{ds}X(s)$, ROC = R

Example: if $u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}$, σ > 0, find the inverse Laplace transform of ${}^{1}/_{s^2}$

Solution: $-tx(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{d}{ds}X(s), \sigma > 0$

$$-tu(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{1}{s^2}, \sigma > 0$$

$$tu(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}, \sigma > 0$$

- Time domain integration property:
- If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC = R
- Then $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{X(s)}{s}$, ROC = R ∩ Re(s)>0

4.5

Transfer Functions

- (Recall) For general values of complex variable $s = \sigma + j\omega$, with σ and ω being the real and imaginary parts, the Laplace transform of a general function $f(t)$ is defined as:

$$F(s) \triangleq \int_{-\infty}^{+\infty} f(t)e^{-st}dt, s \,\epsilon\, \mathrm{ROC}$$

- The inverse Laplace transform, is defined as:

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds\,, s \,\epsilon\, \mathrm{ROC}$$

- Let the impulse response of the system $h(t)$, then the Laplace transform of $h(t)$ is defined as:

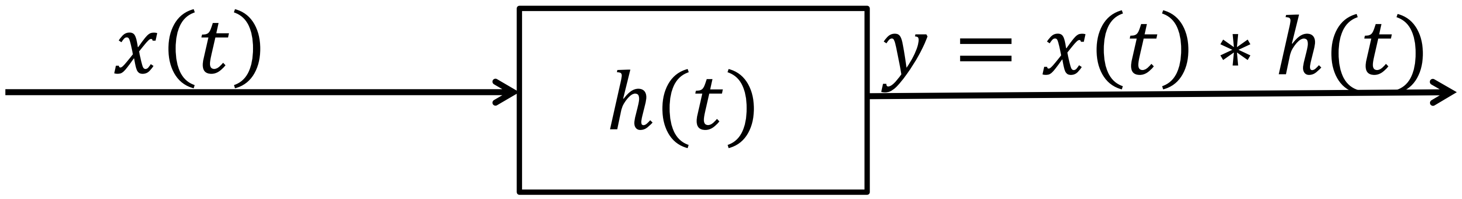
$$H(s) \triangleq \int_{-\infty}^{+\infty} h(t)e^{-st}dt, s\epsilon ROC$$

- For LTI systems, $h(t)$ completely characterizes the system in the time domain, relating its input with its corresponding output. Similarly, $H(s)$ completely characterizes the system in the s domain.

- What type of insights do you get from studying $H(s)$?

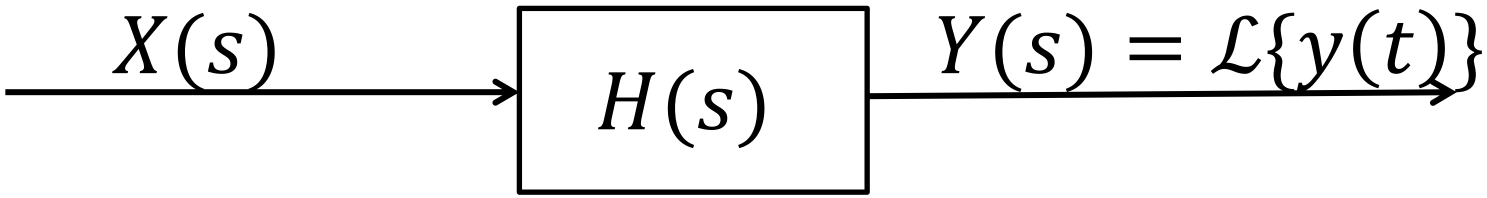
Transfer Function

- Input signal $x(t)$
- LTI system function $h(t)$
- Output signal $y(t) = x(t) * h(t)$



Transfer Function

- Input signal $X(s) = \mathcal{L}\{x(t)\}$
- LTI system function $H(s) = \mathcal{L}\{h(t)\}$
- Output signal $Y(s) = \mathcal{L}\{y(t)\}$



- $Y(s) = \mathcal{L}\{y(t)\} = \int_{-\infty}^{+\infty}\big(h(t) * x(t)\big)e^{-st}dt$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right)e^{-st}dt$$

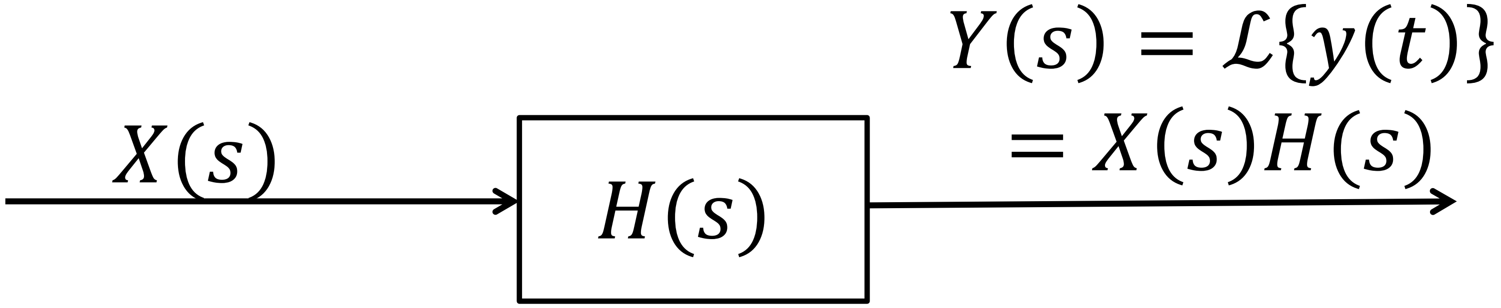
$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} x(t-\tau)\, e^{-st}dt\right)h(\tau)d\tau$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} x(t-\tau)\, e^{-s(t-\tau)}d(t-\tau)\right)e^{-s\tau}h(\tau)d\tau$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} x(\lambda)\, e^{-s\lambda}d\lambda\right)e^{-s\tau}h(\tau)d\tau = X(s)\left(\int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau}d\tau\right)$$

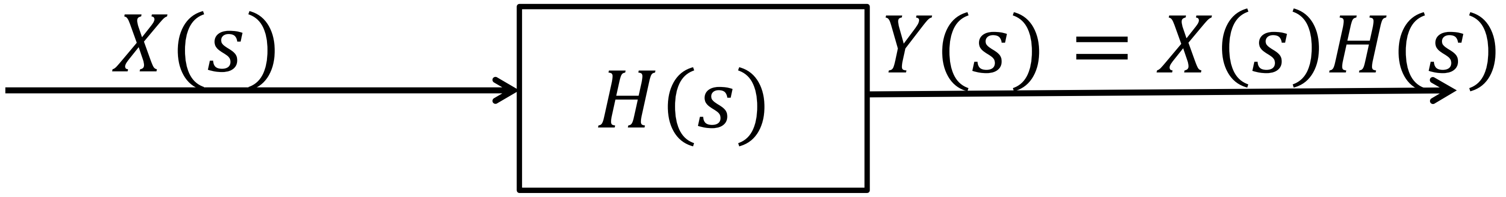
$$= X(s)H(s)$$

- Conclusion: when you analyse the system in the s domain, the output y becomes: $Y(s) = \mathcal{L}\{y(t)\} = X(s)H(s)$



Transfer Function $H(s)$: describes how the system "transfers" the excitation to the response. Through the Laplace transform, time-domain convolution becomes *s*-domain multiplication.

$$H(s) = \frac{Y(s)}{X(s)}$$



Transfer Function

$$H(s) = \frac{Y(s)}{X(s)} = |H(s)|e^{j\Phi}$$

Connecting Systems with Different Transfer Functions

Series (cascade): $H(s) = H_1(s)H_2(s)$

Parallel: $H(s) = H_1(s) + H_2(s)$

4.6

Analog Filters

- Filters: systems that process signals in a frequency dependent manner.
- Filtering: to change the relative amplitudes of the frequency components in a signal or eliminate altogether.
- Filter can be carried out by analog or digital means.
- Filters can be divided into frequency shaping filters and frequency selective filters.

- Frequency-shaping filters: Systems that designed to change the shape of the frequency spectrum
- Applications: equalizer

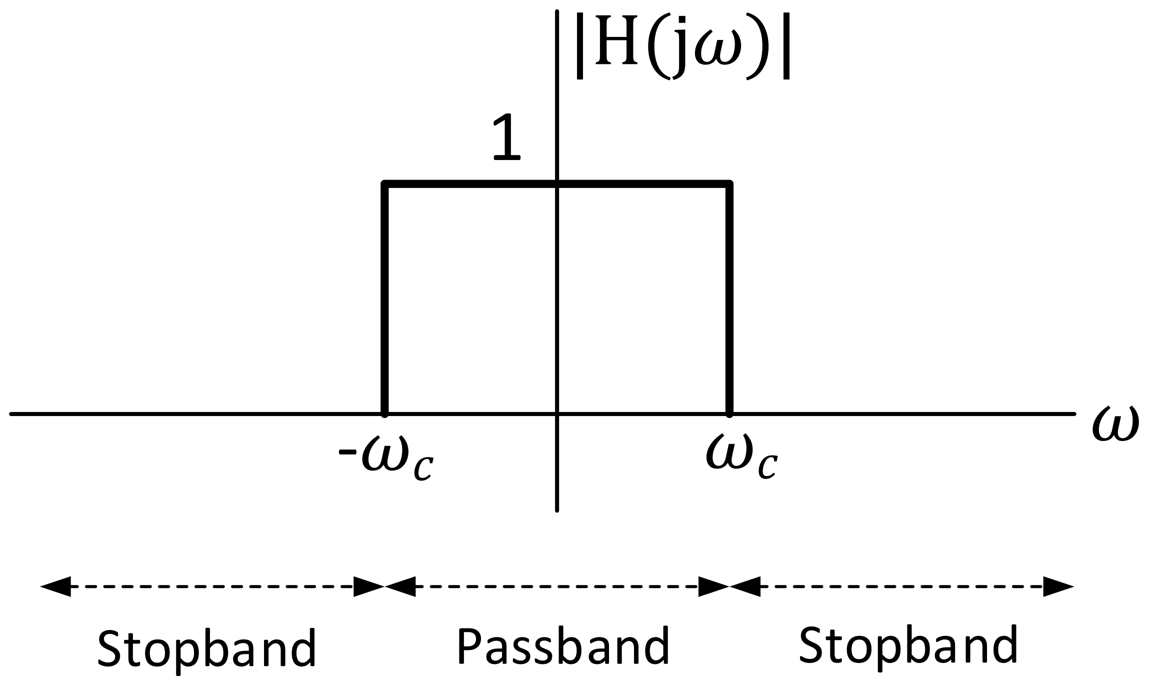
- Frequency-shaping filters: Systems that designed to change the shape of the frequency spectrum
- Applications:

- Frequency-selective filters: systems that designed to pass some frequencies undistorted and eliminate others completely.
- Applications:
- Communications: modulations and demodulation
- Manufacturing: common safety and harmonic pollution removal.
- General signal processing operations: speech synthesis, images processing, etc.

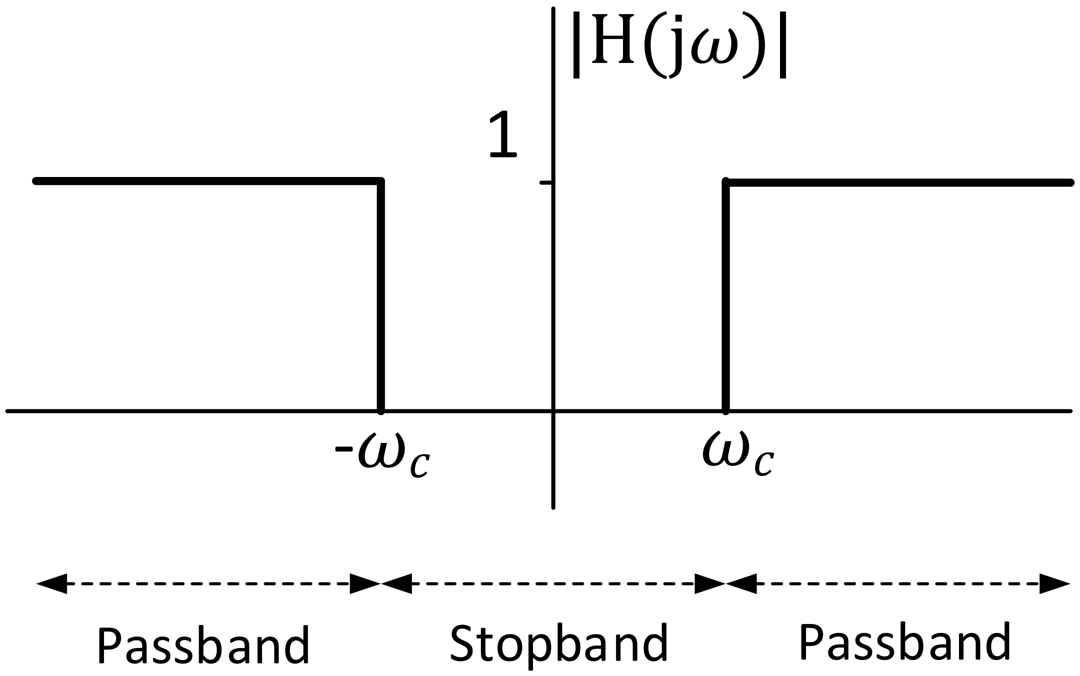
Types of frequency-selective filters:

- Low-pass filters: used to pass a band of preferred low frequencies and reject undesirable high frequencies.
- High-pass filters: used to pass a band of preferred high frequencies and reject undesirable low frequencies.
- Band-pass filters: used to pass a band of frequencies and reject low- and high-frequency bands.

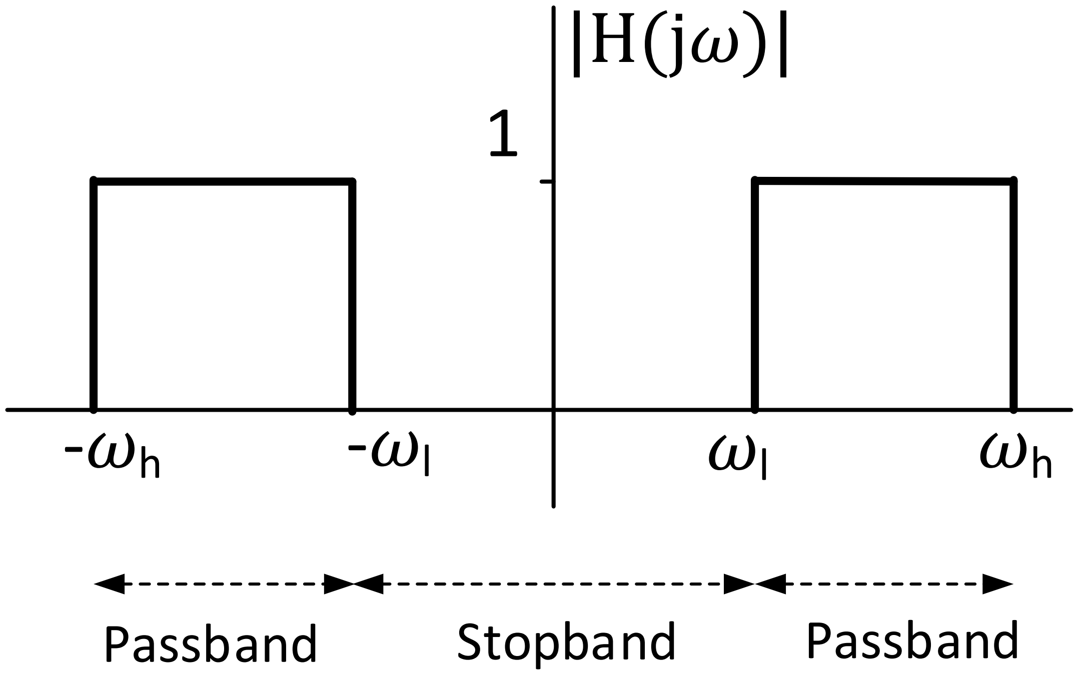
- Ideal low-pass and its corresponding frequency response, here ω_c is the cutoff frequency



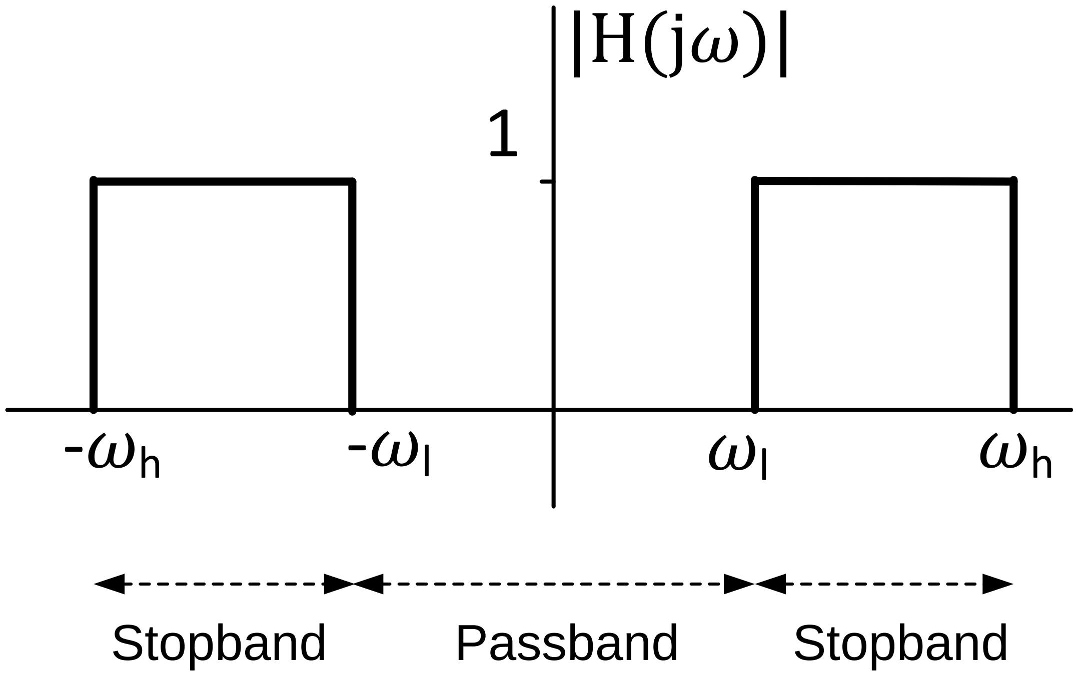
- Ideal high-pass and its corresponding frequency response, here ω_c is the cutoff frequency



- Ideal band-pass and its corresponding frequency response, here ω_l and ω_h is the lower and upper cutoff frequency



- Ideal band-stop filter and its corresponding frequency response, here ω_l and ω_h is the lower and upper stopband frequency



- Ideal filters vs non-ideal filters
- Ideal filters are used to describe idealized systems in certain circumstances.
- Implementation of ideal filters are often limited or non-practical.

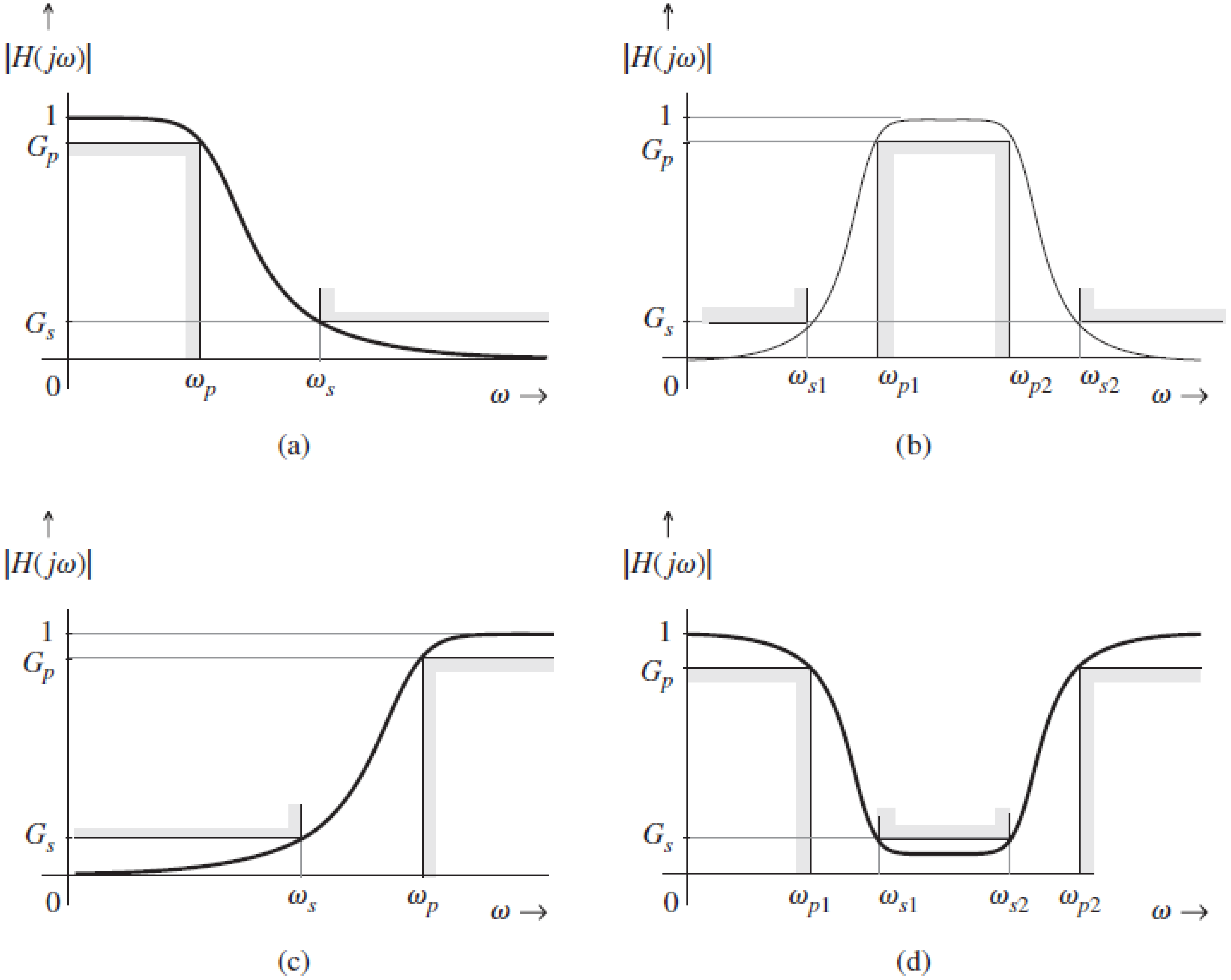


Figure 4.55 Passband, stopband, and transition band in filters of various types.

- Question: How to achieve frequency selective filters?
- Answer: through the use of LTI systems described by linear constant-coefficient differential or difference equations.

- Reasons:
- Physical systems are often modeled as such.
- The resulting systems are easy to implement both digitally or analog since h.

Common analog filter designs.

- Butterworth: flat in the passband and the stopband, however, with a bigger transition band between the pass- and the stopband.
- Chebyshev I: reduces the transition band (a steeper roll-off) at the expense of ripples in the passband.
- Chebyshev II: also known as the inverse Chebyshev filters, reduces the transition band at the expense of ripples in the stopband.
- Elliptic: with equalized ripple (equiripple) in both the passband and the stopband.

Analog Filters	Advantages	Disadvantages
	• Processing speed: usually much faster than digital filters. • Amplitude dynamic range: much higher ratio between the highest process-able signal amplitude and the lowest process-able signal amplitude. • Frequency dynamic range: much higher ratio between the highest process-able signal frequency and the lowest process-able signal frequency. • Peripheral interfacing hardware support unnecessary: usually directly interfacing with the physical analog quantities both as inputs and outputs.	• Component accuracy: The achievable accuracy is limited by the accuracy and linearity of the resistors and capacitors. • Higher cost of construction for complex designs: the limited accuracy and linearity significantly complicates designs with high number of components. • Less flexibility and adaptability: hardware based prototyping which is hard to design, test and troubleshoot.

	Advantages	Disadvantages
Digital Filters	• Compact design: main implementation unit usually only requires a microprocessor, which can be used to complete other DSP tasks. • Flexible and adaptive design: software programmable and easier to prototype and troubleshoot. • Component accuracy: the achievable accuracy is limited by the round-off error in digital calculator. • Noise resistance: less prone to thermal noise compared to analog filters. Better achievable signal to noise ratio (SNR). • Able to achieve linear phase (FIR).	• Processing speed: slower than analog filters with extra latency. • Peripheral interfacing hardware necessary: requires analog to digital converter (ADC) and digital to analog converter (DAC) to interface with the physical analog quantities as inputs and outputs. • Computation must be complete in a sampling period – limits real-time operations.

4.7

FREQUENCY RESPONSE OF LTI SYSTEMS

Derive Frequency Response From Transfer Function

Consider a LTI system:

$$H(s) = \frac{Y(s)}{X(s)} = |H(s)|e^{j\Phi}, \mathrm{s} = \sigma + j\omega$$

How to examine the impact of this transfer function at different frequencies?

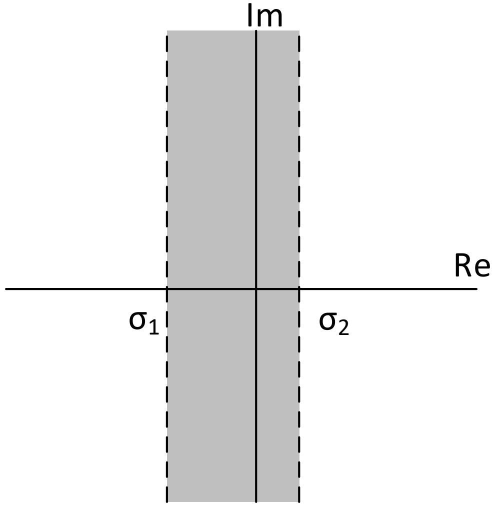
Investigate the frequency response function $H(j\omega)$:

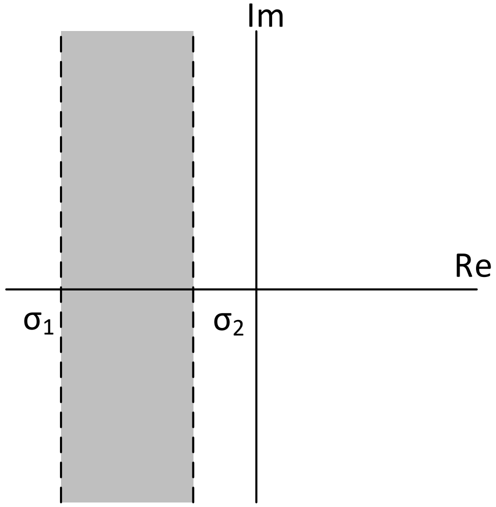
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = H(s)|_{\sigma=0}$$

Derive Frequency Response From Transfer Function

Note: the frequency response function $H(j\omega)$ only exists if $\sigma = 0$ is part of the ROC

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = H(s)\big|_{\sigma=0}$$





Using Transfer Function to Analyse Frequency Response

Many LTI systems of practical interest can be represented by linear differential equations with constant coefficients as follows:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} k_k \frac{d^k}{dt^k} x(t)$$

Transform into the *s*-domain, use linearity and derivative properties of the Laplace transform.

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Thus for systems represented by linear constant-coefficient differential equations:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Zeros: the solutions of $\sum_{k=0}^{M} b_k s^k = 0$

Poles: the solutions to $\sum_{k=0}^{N} a_k s^k = 0$

If only interested in the frequency response of such a system:

$$H(j\omega) = H(s)|_{\sigma=0} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Zeros ($z_1, \ldots, z_M$):

the solutions of $\sum_{k=0}^{M} b_k (j\omega)^k = 0$

Poles ($p_1, \ldots, p_N$):

the solutions to $\sum_{k=0}^{N} a_k (j\omega)^k = 0$

Example: given a LTI system $H(s) = \frac{s}{s+3}$, its frequency response: $H(j\omega) = \frac{j\omega}{j\omega+3}$. Analyze the magnitude and phase of the frequency response of this system.

$$|H(j\omega)| = \left|\frac{j\omega}{j\omega + 3}\right|$$

$$\arg\big(H(j\omega)\big) = \arg(j\omega) - \arg(j\omega + 3)$$

$\omega \to 0$:

$\omega \to \infty$:

Homework:

Review: in-class examples, textbook chapter 4.

Textbook examples: 4.1, 4.6, 4.7,4.10, 4.27

Problems: 4.1-2, 4.3-11, 4.3-14