Lecture 4

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Chapter 5

Linear Transformations

5.1 Linear Transformations

Outcomes

A. *Understand the definition of a linear transformation, and that all linear transformations are determined by matrix multiplication.*

Recall that when we multiply an $m \times n$ matrix by an $n \times 1$ column vector, the result is an $m \times 1$ column vector. In this section we will discuss how, through matrix multiplication, an $m \times n$ matrix **transforms** an $n \times 1$ column vector into an $m \times 1$ column vector.

Recall that the $n \times 1$ vector given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to belong to $\mathbb{R}^n$, which is the set of all $n \times 1$ vectors. In this section, we will discuss transformations of vectors in $\mathbb{R}^n$.

Consider the following example.

Example 5.1: A Function Which Transforms Vectors

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. *Show that by matrix multiplication A transforms vectors in* $\mathbb{R}^3$ *into vectors in* $\mathbb{R}^2$.

Solution. First, recall that vectors in $\mathbb{R}^3$ are vectors of size 3×1, while vectors in $\mathbb{R}^2$ are of size 2×1. If we multiply A, which is a 2×3 matrix, by a 3×1 vector, the result will be a 2×1 vector. This what we mean when we say that A *transforms* vectors.

Now, for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, multiply on the left by the given matrix to obtain the new vector. This product looks like

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y \\ 2x+y \end{bmatrix}$$

The resulting product is a 2×1 vector which is determined by the choice of x and y. Here are some

numerical examples.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Here, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$ was transformed by the matrix into the vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$.

Here is another example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \end{bmatrix}$$

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The idea is to define a function which takes vectors in $\mathbb{R}^3$ and delivers new vectors in $\mathbb{R}^2$. In this case, that function is multiplication by the matrix A.

Let T denote such a function. The notation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ means that the function T transforms vectors in $\mathbb{R}^n$ into vectors in $\mathbb{R}^m$. The notation $T(\vec{x})$ means the transformation T applied to the vector $\vec{x}$. The above example demonstrated a transformation achieved by matrix multiplication. In this case, we often write

$$T_A(\vec{x}) = A\vec{x}$$

Therefore, T_A is the transformation determined by the matrix A. In this case we say that T is a matrix transformation.

Recall the property of matrix multiplication that states that for k and p scalars,

$$A(kB + pC) = kAB + pAC$$

In particular, for A an $m \times n$ matrix and B and C, $n \times 1$ vectors in $\mathbb{R}^n$, this formula holds.

In other words, this means that matrix multiplication gives an example of a linear transformation, which we will now define.

Definition 5.2: Linear Transformation

*Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function, where for each $\vec{x} \in \mathbb{R}^n, T(\vec{x}) \in \mathbb{R}^m$. Then T is a **linear transformation** if whenever k, p are scalars and $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$ ($n \times 1$ vectors),*

$$T(k\vec{x}_1 + p\vec{x}_2) = kT(\vec{x}_1) + pT(\vec{x}_2)$$

Consider the following example.

Example 5.3: Linear Transformation

Let T be a transformation defined by $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-z \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

Show that T is a linear transformation.

Solution. By Definition 9.55 we need to show that $T(k\vec{x}_1 + p\vec{x}_2) = kT(\vec{x}_1) + pT(\vec{x}_2)$ for all scalars k, p and vectors $\vec{x}_1, \vec{x}_2$. Let

$$\vec{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Then

$$\begin{aligned}
T(k\vec{x}_1 + p\vec{x}_2) &= T\left(k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix} + \begin{bmatrix} px_2 \\ py_2 \\ pz_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 + px_2 \\ ky_1 + py_2 \\ kz_1 + pz_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (kz_1 + pz_2) \end{bmatrix} \\
&= \begin{bmatrix} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - kz_1) + (px_2 - pz_2) \end{bmatrix} \\
&= \begin{bmatrix} kx_1 + ky_1 \\ kx_1 - kz_1 \end{bmatrix} + \begin{bmatrix} px_2 + py_2 \\ px_2 - pz_2 \end{bmatrix} \\
&= k\begin{bmatrix} x_1 + y_1 \\ x_1 - z_1 \end{bmatrix} + p\begin{bmatrix} x_2 + y_2 \\ x_2 - z_2 \end{bmatrix} \\
&= kT(\vec{x}_1) + pT(\vec{x}_2)
\end{aligned}$$

Therefore T is a linear transformation. ♠

Two important examples of linear transformations are the zero transformation and identity transformation. The zero transformation defined by $T(\vec{x}) = \vec{0}$ for all $\vec{x}$ is an example of a linear transformation. Similarly the identity transformation defined by $T(\vec{x}) = \vec{x}$ is also linear. Take the time to prove these using the method demonstrated in Example 5.3.

We began this section by discussing matrix transformations, where multiplication by a matrix transforms vectors. These matrix transformations are in fact linear transformations.

Theorem 5.4: Matrix Transformations are Linear Transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a transformation defined by $T(\vec{x}) = A\vec{x}$. Then T is a linear transformation.

It turns out that every linear transformation can be expressed as a matrix transformation, and thus linear transformations are exactly the same as matrix transformations.

Exercises

Exercise 5.1.1 *Show the map $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ where A is an $m \times n$ matrix and $\vec{x}$ is an $m \times 1$ column vector is a linear transformation.*

Exercise 5.1.2 *Show that the function $T_{\vec{u}}$ defined by $T_{\vec{u}}(\vec{v}) = \vec{v} - proj_{\vec{u}}(\vec{v})$ is also a linear transformation.*

Exercise 5.1.3 *Let $\vec{u}$ be a fixed vector. The function $T_{\vec{u}}$ defined by $T_{\vec{u}}\vec{v} = \vec{u} + \vec{v}$ has the effect of translating all vectors by adding $\vec{u} \neq \vec{0}$. Show this is not a linear transformation. Explain why it is not possible to represent $T_{\vec{u}}$ in $\mathbb{R}^3$ by multiplying by a 3×3 matrix.*

5.2 The Matrix of a Linear Transformation I

Outcomes

A. *Find the matrix of a linear transformation with respect to the standard basis.*

B. *Determine the action of a linear transformation on a vector in $\mathbb{R}^n$.*

In the above examples, the action of the linear transformations was to multiply by a matrix. It turns out that this is always the case for linear transformations. If T is **any** linear transformation which maps $\mathbb{R}^n$ to $\mathbb{R}^m$, there is **always** an $m \times n$ matrix A with the property that

$$T(\vec{x}) = A\vec{x} \tag{5.1}$$

for all $\vec{x} \in \mathbb{R}^n$.

Theorem 5.5: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then we can find a matrix A such that $T(\vec{x}) = A\vec{x}$. In this case, we say that T is determined *or* induced *by the matrix A.*

Here is why. Suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation and you want to find the matrix defined by this linear transformation as described in 5.1. Note that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} x_i \vec{e}_i$$

where $\vec{e}_i$ is the i^{th} column of I_n, that is the $n \times 1$ vector which has zeros in every slot but the i^{th} and a 1 in this slot.

Then since T is linear,

$$T(\vec{x}) = \sum_{i=1}^{n} x_i T(\vec{e}_i)$$

$$= \left[\begin{array}{ccc} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{array}\right]\left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array}\right]$$

$$= A\left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array}\right]$$

The desired matrix is obtained from constructing the i^{th} column as $T(\vec{e}_i)$. Recall that the set $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is called the standard basis of $\mathbb{R}^n$. Therefore the matrix of T is found by applying T to the standard basis. We state this formally as the following theorem.

Theorem 5.6: Matrix of a Linear Transformation

Let $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then the matrix A satisfying $T(\vec{x}) = A\vec{x}$ is given by

$$A = \left[\begin{array}{ccc} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{array}\right]$$

where $\vec{e}_i$ is the i^{th} column of I_n, and then $T(\vec{e}_i)$ is the i^{th} column of A.

The following Corollary is an essential result.

Corollary 5.7: Matrix and Linear Transformation

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Consider the following example.

Example 5.8: The Matrix of a Linear Transformation

Suppose T is a linear transformation, $T: \mathbb{R}^3 \to \mathbb{R}^2$ where

$$T\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right], T\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 9 \\ -3 \end{array}\right], T\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Find the matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

Solution. By Theorem 5.6 we construct A as follows:

$$A = \left[\begin{array}{ccc} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{array}\right]$$

In this case, A will be a 2×3 matrix, so we need to find $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Luckily, we have been given these values so we can fill in A as needed, using these vectors as the columns of A. Hence,

$$A = \left[\begin{array}{rrr} 1 & 9 & 1 \\ 2 & -3 & 1 \end{array}\right]$$

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In this example, we were given the resulting vectors of $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Constructing the matrix A was simple, as we could simply use these vectors as the columns of A. The next example shows how to find A when we are not given the $T(\vec{e}_i)$ so clearly.

Example 5.9: The Matrix of Linear Transformation: Inconveniently Defined

Suppose T is a linear transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2$ and

$$T\left[\begin{array}{c} 1 \\ \end{array}\right] = \left[\begin{array}{c} 1 \\ \end{array}\right], T\left[\begin{array}{c} 0 \\ \end{array}\right] = \left[\begin{array}{c} 3 \\ \end{array}\right]$$

$$T\left[\begin{array}{c} 1 \end{array}\right] = \left[\begin{array}{c} 2 \end{array}\right], T\left[\begin{array}{c} -1 \end{array}\right] = \left[\begin{array}{c} 2 \end{array}\right]$$

Find the matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

Solution. By Theorem 5.6 to find this matrix, we need to determine the action of T on $\vec{e}_1$ and $\vec{e}_2$. In Example 9.91, we were given these resulting vectors. However, in this example, we have been given T of two different vectors. How can we find out the action of T on $\vec{e}_1$ and $\vec{e}_2$? In particular for $\vec{e}_1$, suppose there exist x and y such that

$$\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = x\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + y\left[\begin{array}{c} 0 \\ -1 \end{array}\right] \tag{5.2}$$

Then, since T is linear,

$$T\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = xT\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + yT\left[\begin{array}{c} 0 \\ -1 \end{array}\right]$$

Substituting in values, this sum becomes

$$T\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = x\left[\begin{array}{c} 1 \\ 2 \end{array}\right] + y\left[\begin{array}{c} 3 \\ 2 \end{array}\right] \tag{5.3}$$

Therefore, if we know the values of x and y which satisfy 5.2, we can substitute these into equation 5.3. By doing so, we find $T(\vec{e}_1)$ which is the first column of the matrix A.

We proceed to find x and y. We do so by solving 5.2, which can be done by solving the system

$$\begin{array}{c} x = 1 \\ x - y = 0 \end{array}$$

We see that $x = 1$ and $y = 1$ is the solution to this system. Substituting these values into equation 5.3, we have

$$T\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = 1\left[\begin{array}{c} 1 \\ 2 \end{array}\right] + 1\left[\begin{array}{c} 3 \\ 2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right] + \left[\begin{array}{c} 3 \\ 2 \end{array}\right] = \left[\begin{array}{c} 4 \\ 4 \end{array}\right]$$

Therefore $\left[\begin{array}{c} 4 \\ 4 \end{array}\right]$ is the first column of A.

Computing the second column is done in the same way, and is left as an exercise.

The resulting matrix A is given by

$$A = \left[\begin{array}{cc} 4 & -3 \\ 4 & -2 \end{array}\right]$$

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This example illustrates a very long procedure for finding the matrix of A. While this method is reliable and will always result in the correct matrix A, the following procedure provides an alternative method.

Procedure 5.10: Finding the Matrix of Inconveniently Defined Linear Transformation

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Suppose there exist vectors $\{\vec{a}_1, \cdots, \vec{a}_n\}$ in $\mathbb{R}^n$ such that $\left[\begin{array}{ccc} \vec{a}_1 & \cdots & \vec{a}_n \end{array}\right]^{-1}$ exists, and

$$T(\vec{a}_i) = \vec{b}_i$$

Then the matrix of T must be of the form

$$\left[\begin{array}{ccc} \vec{b}_1 & \cdots & \vec{b}_n \end{array}\right]\left[\begin{array}{ccc} \vec{a}_1 & \cdots & \vec{a}_n \end{array}\right]^{-1}$$

We will illustrate this procedure in the following example. You may also find it useful to work through Example 5.9 using this procedure.

Example 5.11: Matrix of a Linear Transformation Given Inconveniently

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and

$$T\left[\begin{array}{c} 1 \\ 3 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right], T\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 2 \\ 1 \\ 3 \end{array}\right], T\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

Find the matrix of this linear transformation.

Solution. By Procedure 5.10, $A = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 3 & 1 & 1 \end{array}\right]^{-1}$ and $B = \left[\begin{array}{ccc} 0 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$\lfloor\ 1 \quad 1 \quad 0\ \rfloor \qquad\qquad \lfloor\ 1 \quad 3 \quad 1\ \rfloor$

Then, Procedure 5.10 claims that the matrix of T is

$$C = BA^{-1} = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}$$

Indeed you can first verify that $T(\vec{x}) = C\vec{x}$ for the 3 vectors above:

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

But more generally $T(\vec{x}) = C\vec{x}$ for any $\vec{x}$. To see this, let $\vec{y} = A^{-1}\vec{x}$ and then using linearity of T:

$$T(\vec{x}) = T(A\vec{y}) = T\left(\sum_i \vec{y}_i \vec{a}_i\right) = \sum \vec{y}_i T(\vec{a}_i) \sum \vec{y}_i \vec{b}_i = B\vec{y} = BA^{-1}\vec{x} = C\vec{x}$$

$\spadesuit$

Recall the dot product discussed earlier. Consider the map $\vec{v} \mapsto \mathrm{proj}_{\vec{u}}(\vec{v})$ which takes a vector a transforms it to its projection onto a given vector $\vec{u}$. It turns out that this map is linear, a result which follows from the properties of the dot product. This is shown as follows.

$$\begin{aligned} \mathrm{proj}_{\vec{u}}(k\vec{v} + p\vec{w}) &= \left(\frac{(k\vec{v} + p\vec{w}) \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u} \\ &= k\left(\frac{\vec{v} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u} + p\left(\frac{\vec{w} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u} \\ &= k\,\mathrm{proj}_{\vec{u}}(\vec{v}) + p\,\mathrm{proj}_{\vec{u}}(\vec{w}) \end{aligned}$$

Consider the following example.

Example 5.12: Matrix of a Projection Map

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and let* T *be the projection map* $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *defined by*

$$T(\vec{v}) = \mathit{proj}_{\vec{u}}(\vec{v})$$

for any $\vec{v} \in \mathbb{R}^3$.

1. *Does this transformation come from multiplication by a matrix?*
2. *If so, what is the matrix?*

Solution.

1. First, we have just seen that $T(\vec{v}) = \mathrm{proj}_{\vec{u}}(\vec{v})$ is linear. Therefore by Theorem 5.5, we can find a matrix A such that $T(\vec{x}) = A\vec{x}$.

2. The columns of the matrix for T are defined above as $T(\vec{e}_i)$. It follows that $T(\vec{e}_i) = \mathrm{proj}_{\vec{u}}(\vec{e}_i)$ gives the i^{th} column of the desired matrix. Therefore, we need to find

$$\text{proj}_{\vec{u}}(\vec{e}_i) = \left(\frac{\vec{e}_i \bullet \vec{u}}{\vec{u} \bullet \vec{u}} \right) \vec{u}$$

For the given vector $\vec{u}$, this implies the columns of the desired matrix are

$$\frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{2}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{3}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which you can verify. Hence the matrix of T is

$$\frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

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Exercises

Exercise 5.2.1 *Consider the following functions which map $\mathbb{R}^n$ to $\mathbb{R}^n$.*

(a) *T multiplies the j^{th} component of $\vec{x}$ by a nonzero number b.*

(b) *T replaces the i^{th} component of $\vec{x}$ with b times the j^{th} component added to the i^{th} component.*

(c) *T switches the i^{th} and j^{th} components.*

Show these functions are linear transformations and describe their matrices A such that $T(\vec{x}) = A\vec{x}$.

Exercise 5.2.2 *You are given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and you know that*

$$T(A_i) = B_i$$

where $\begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}^{-1}$ exists. Show that the matrix of T is of the form

$$\begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix} \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}^{-1}$$

Exercise 5.2.3 *Suppose T is a linear transformation such that*

$$T\begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Exercise 5.2.4 *Suppose T is a linear transformation such that*

$$T\begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Exercise 5.2.5 *Suppose T is a linear transformation such that*

$$T\begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}$$
$$T\begin{bmatrix} -1 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$$
$$T\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -3 \end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Exercise 5.2.6 *Suppose T is a linear transformation such that*

$$T\begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$
$$T\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Exercise 5.2.7 *Suppose T is a linear transformation such that*

$$T\begin{bmatrix} 1 \\ 2 \\ -18 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$$
$$T\begin{bmatrix} -1 \\ -1 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$
$$T\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Exercise 5.2.8 *Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. Show that each is a linear transformation and determine for each the matrix A such that $T(\vec{x}) = A\vec{x}$.*

(a) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 2y-3x+z \end{bmatrix}$

(b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7x+2y+z \\ 3x-11y+2z \end{bmatrix}$

(c) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2y+z \\ x+2y+6z \end{bmatrix}$

(d) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y-5x+z \\ x+y+z \end{bmatrix}$

Exercise 5.2.9 *Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. Explain why each of these functions T is not linear.*

(a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y+3z+1 \end{bmatrix}$

$$\begin{bmatrix} z \end{bmatrix} = \begin{bmatrix} 2y-3x+z \end{bmatrix}$$

(b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y^2+3z \\ 2y+3x+z \end{bmatrix}$

(c) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin x+2y+3z \\ 2y+3x+z \end{bmatrix}$

(d) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 2y+3x-\ln z \end{bmatrix}$

Exercise 5.2.10 *Suppose*

$$\begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}^{-1}$$

*exists where each $A_j \in \mathbb{R}^n$ and let vectors $\{B_1, \cdots, B_n\}$ in $\mathbb{R}^m$ be given. Show that there **always** exists a linear transformation T such that $T(A_i) = B_i$.*

Exercise 5.2.11 *Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$.*

Exercise 5.2.12 *Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 5 & 3 \end{bmatrix}^T$.*

Exercise 5.2.13 *Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}^T$.*

5.3 Properties of Linear Transformations

Outcomes

A. Use properties of linear transformations to solve problems.

B. Find the composite of transformations and the inverse of a transformation.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there are some important properties of T which will be examined in this section. Consider the following theorem.

Theorem 5.13: Properties of Linear Transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation and let $\vec{x} \in \mathbb{R}^n$.

- *T preserves the zero vector.*

$$T(0\vec{x}) = 0T(\vec{x}). \text{ Hence } T(\vec{0}) = \vec{0}$$

- *T preserves the negative of a vector:*

$$T((-1)\vec{x}) = (-1)T(\vec{x}). \text{ Hence } T(-\vec{x}) = -T(\vec{x})$$

$T((-1)\vec{x}) = (-1)T(\vec{x})$. Hence $T(-\vec{x}) = -T(\vec{x})$.

- *T preserves linear combinations:*

$$\text{Let } \vec{x}_1,\dots,\vec{x}_k \in \mathbb{R}^n \text{ and } a_1,\dots,a_k \in \mathbb{R}.$$

$$\text{Then if } \vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_k\vec{x}_k, \text{ it follows that}$$

$$T(\vec{y}) = T(a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_k\vec{x}_k) = a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \dots + a_kT(\vec{x}_k).$$

These properties are useful in determining the action of a transformation on a given vector. Consider the following example.

Example 5.14: Linear Combination

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix}, T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$.

Solution. Using the third property in Theorem 9.57, we can find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ by writing $\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$.

Therefore we want to find $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

The necessary augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{cc|c} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $a = 1, b = -2$ and

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

Now, using the third property above, we have

$$\begin{aligned} T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} &= T\left(1\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right) \\ &= 1T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix} \end{aligned}$$

Therefore, $T\begin{bmatrix} -7 \\ 3 \\ \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ \end{bmatrix}$. ♠

$$\begin{bmatrix} -9 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

Suppose two linear transformations act in the same way on $\vec{x}$ for all vectors. Then we say that these transformations are equal.

Definition 5.15: Equal Transformations

Let S and T be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $S = T$ if and only if for every $\vec{x} \in \mathbb{R}^n$,

$$S(\vec{x}) = T(\vec{x})$$

Suppose two linear transformations act on the same vector $\vec{x}$, first the transformation T and then a second transformation given by S. We can find the **composite** transformation that results from applying both transformations.

Definition 5.16: Composition of Linear Transformations

*Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Then the **composite** of S and T is*

$$S \circ T : \mathbb{R}^k \mapsto \mathbb{R}^m$$

The action of $S \circ T$ is given by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k$$

Notice that the resulting vector will be in $\mathbb{R}^m$. Be careful to observe the order of transformations. We write $S \circ T$ but apply the transformation T first, followed by S.

Theorem 5.17: Composition of Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations such that T is induced by the matrix A and S is induced by the matrix B. Then $S \circ T$ is a linear transformation which is induced by the matrix BA.

Consider the following example.

Example 5.18: Composition of Transformations

Let T be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

and S a linear transformation induced by the matrix

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find the matrix of the composite transformation $S \circ T$. Then, find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution. By Theorem 5.17, the matrix of $S \circ T$ is given by BA.

$$BA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix}$$

To find $(S \circ T)(\vec{x})$, multiply $\vec{x}$ by BA as follows

$$\begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

To check, first determine $T(\vec{x})$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

Then, compute $S(T(\vec{x}))$ as follows:

$$\left[\begin{array}{cc} 2 & 3 \\ 0 & 1 \end{array}\right]\left[\begin{array}{c} 9 \\ 2 \end{array}\right] = \left[\begin{array}{c} 24 \\ 2 \end{array}\right]$$

♠

Consider a composite transformation $S \circ T$, and suppose that this transformation acted such that $(S \circ T)(\vec{x}) = \vec{x}$. That is, the transformation S took the vector $T(\vec{x})$ and returned it to $\vec{x}$. In this case, S and T are inverses of each other. Consider the following definition.

Definition 5.19: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^n$ be linear transformations. Suppose that for each $\vec{x} \in \mathbb{R}^n$,

$$(S \circ T)(\vec{x}) = \vec{x}$$

and

$$(T \circ S)(\vec{x}) = \vec{x}$$

Then, S is called an inverse of T and T is called an inverse of S. Geometrically, they reverse the action of each other.

The following theorem is crucial, as it claims that the above inverse transformations are unique.

Theorem 5.20: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation induced by the matrix A. Then T has an inverse transformation if and only if the matrix A is invertible. In this case, the inverse transformation is unique and denoted $T^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n$. T^{-1} is induced by the matrix A^{-1}.

Consider the following example.

Example 5.21: Inverse of a Transformation

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation induced by the matrix

$$A = \left[\begin{array}{cc} 2 & 3 \\ 3 & 4 \end{array}\right]$$

Show that T^{-1} exists and find the matrix B which it is induced by.

Solution. Since the matrix A is invertible, it follows that the transformation T is invertible. Therefore, T^{-1} exists.

You can verify that A^{-1} is given by:

$$A^{-1} = \left[\begin{array}{cc} -4 & 3 \\ 3 & -2 \end{array}\right]$$

Therefore the linear transformation T^{-1} is induced by the matrix A^{-1}. ♠

Exercises

Exercise 5.3.1 *Show that if a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then it is* ***always*** *the case that $T\left(\vec{0}\right) = \vec{0}$.*

Exercise 5.3.2 *Let T be a linear transformation induced by the matrix $A = \left[\begin{array}{cc} 3 & 1 \\ -1 & 2 \end{array}\right]$ and S a linear transformation induced by $B = \left[\begin{array}{cc} 0 & -2 \\ 4 & 2 \end{array}\right]$. Find matrix of $S \circ T$ and find $(S \circ T)(\vec{x})$ for $\vec{x} = \left[\begin{array}{c} 2 \\ -1 \end{array}\right]$.*

Exercise 5.3.3 *Let T be a linear transformation and suppose $T\left(\left[\begin{array}{c} 1 \\ -4 \end{array}\right]\right) = \left[\begin{array}{c} 2 \\ -3 \end{array}\right]$. Suppose S is a linear transformation induced by the matrix $B = \left[\begin{array}{cc} 1 & 2 \\ -1 & 3 \end{array}\right]$. Find $(S \circ T)(\vec{x})$ for $\vec{x} = \left[\begin{array}{c} 1 \\ -4 \end{array}\right]$.*

Exercise 5.3.4 *Let T be a linear transformation induced by the matrix $A = \left[\begin{array}{cc} 2 & 3 \end{array}\right.$ and S a linear* *transformation induced by* $B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$. *Find matrix of* $S \circ T$ *and find* $(S \circ T)(\vec{x})$ *for* $\vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Exercise 5.3.5 *Let* T *be a linear transformation induced by the matrix* $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$. *Find the matrix of* T^{-1}.

Exercise 5.3.6 *Let* T *be a linear transformation induced by the matrix* $A = \begin{bmatrix} 4 & -3 \\ 2 & -2 \end{bmatrix}$. *Find the matrix of* T^{-1}.

Exercise 5.3.7 *Let* T *be a linear transformation and suppose* $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. *Find the matrix of* T^{-1}.

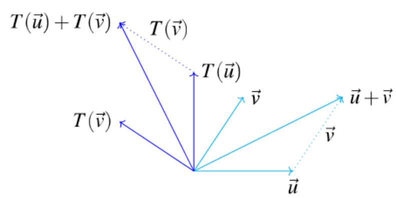
5.4 Special Linear Transformations in $\mathbb{R}^2$

Outcomes

A. *Find the matrix of rotations and reflections in* $\mathbb{R}^2$ *and determine the action of each on a vector in* $\mathbb{R}^2$.

In this section, we will examine some special examples of linear transformations in $\mathbb{R}^2$ including rotations and reflections. We will use the geometric descriptions of vector addition and scalar multiplication discussed earlier to show that a rotation of vectors through an angle and reflection of a vector across a line are examples of linear transformations.

More generally, denote a transformation given by a rotation by T. Why is such a transformation linear? Consider the following picture which illustrates a rotation. Let $\vec{u}, \vec{v}$ denote vectors.



Let's consider how to obtain $T(\vec{u}+\vec{v})$. Simply, you add $T(\vec{u})$ and $T(\vec{v})$. Here is why. If you add $T(\vec{u})$ to $T(\vec{v})$ you get the diagonal of the parallelogram determined by $T(\vec{u})$ and $T(\vec{v})$, as this action is our usual vector addition. Now, suppose we first add $\vec{u}$ and $\vec{v}$, and then apply the transformation T to $\vec{u}+\vec{v}$. Hence, we find $T(\vec{u}+\vec{v})$. As shown in the diagram, this will result in the same vector. In other words, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.

This is because the rotation preserves all angles between the vectors as well as their lengths. In particular, it preserves the shape of this parallelogram. Thus both $T(\vec{u}) + T(\vec{v})$ and $T(\vec{u}+\vec{v})$ give the same vector. It follows that T distributes across addition of the vectors of $\mathbb{R}^2$.

Similarly, if k is a scalar, it follows that $T(k\vec{u}) = kT(\vec{u})$. Thus rotations are an example of a linear transformation by Definition 9.55.

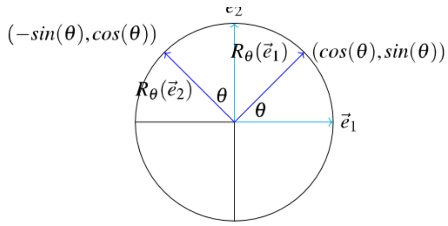
The following theorem gives the matrix of a linear transformation which rotates all vectors through an angle of θ.

Theorem 5.22: Rotation

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation given by rotating vectors through an angle of* θ. *Then the matrix* A *of* R_θ *is given by*

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Proof. Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These identify the geometric vectors which point along the positive x axis and positive y axis as shown.



From Theorem 5.6, we need to find $R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$, and use these as the columns of the matrix A of T. We can use $\cos, \sin$ of the angle θ to find the coordinates of $R_\theta(\vec{e}_1)$ as shown in the above picture. The coordinates of $R_\theta(\vec{e}_2)$ also follow from trigonometry. Thus

$$R_\theta(\vec{e}_1) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right], R_\theta(\vec{e}_2) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right]$$

Therefore, from Theorem 5.6,

$$A = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

We can also prove this algebraically without the use of the above picture. The definition of $(\cos(\theta), \sin(\theta))$ is as the coordinates of the point of $R_\theta(\vec{e}_1)$. Now the point of the vector $\vec{e}_2$ is exactly $\pi/2$ further along the unit circle from the point of $\vec{e}_1$, and therefore after rotation through an angle of θ the coordinates x and y of the point of $R_\theta(\vec{e}_2)$ are given by

$$(x, y) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2)) = (-\sin\theta, \cos\theta)$$

♠

Consider the following example.

Example 5.23: Rotation in $\mathbb{R}^2$

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation through $\pi/2$. Find the matrix of $R_{\frac{\pi}{2}}$. Then, find $R_{\frac{\pi}{2}}(\vec{x})$ where $\vec{x} = \left[\begin{array}{c} 1 \\ -2 \end{array}\right]$.

Solution. By Theorem 5.22, the matrix of $R_{\frac{\pi}{2}}$ is given by

$$\left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right] = \left[\begin{array}{cc} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{array}\right] = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]$$

To find $R_{\frac{\pi}{2}}(\vec{x})$, we multiply the matrix of $R_{\frac{\pi}{2}}$ by $\vec{x}$ as follows

$$\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{c} 1 \\ -2 \end{array}\right] = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

♠

We now look at an example of a linear transformation involving two angles.

Example 5.24: The Rotation Matrix of the Sum of Two Angles

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of ϕ and then through an angle θ. Hence the linear transformation rotates all vectors through an angle of $\theta + \phi$.

Solution. Let $R_{\theta+\phi}$ denote the linear transformation which rotates every vector through an angle of $\theta + \phi$. Then to obtain $R_{\theta+\phi}$, we first apply R_ϕ and then R_θ where R_ϕ is the linear transformation which rotates through an angle of ϕ and R_θ is the linear transformation which rotates through an angle of θ. Denoting the corresponding matrices by $A_{\theta+\phi}$, A_ϕ, and A_θ, it follows that for every $\vec{u}$

$$R_{\theta+\phi}(\vec{u}) = A_{\theta+\phi}\vec{u} = A_\theta A_\phi \vec{u} = R_\theta R_\phi(\vec{u})$$

Notice the order of the matrices here!

Consequently, you must have

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = A_\theta A_\phi \end{aligned}$$

The usual matrix multiplication yields

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} \\ &= A_\theta A_\phi \end{aligned}$$

Don't these look familiar? They are the usual trigonometric identities for the sum of two angles derived here using linear algebra concepts.

♠

Here we have focused on rotations in two dimensions. However, you can consider rotations and other geometric concepts in any number of dimensions. This is one of the major advantages of linear algebra. You can break down a difficult geometrical procedure into small steps, each corresponding to multiplication by an appropriate matrix. Then by multiplying the matrices, you can obtain a single matrix which can give you numerical information on the results of applying the given sequence of simple procedures.

Linear transformations which reflect vectors across a line are a second important type of transformations in $\mathbb{R}^2$. Consider the following theorem.

Theorem 5.25: Reflection

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by reflecting vectors over the line $\vec{y} = m\vec{x}$. Then the matrix of Q_m is given by

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Consider the following example.

Example 5.26: Reflection in $\mathbb{R}^2$

Let $Q_2 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection over the line $\vec{y} = 2\vec{x}$. Then Q_2 is a linear transformation. Find the matrix of Q_2. Then, find $Q_2(\vec{x})$ where $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution. By Theorem 5.25, the matrix of Q_2 is given by

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(2)^2}\begin{bmatrix} 1-(2)^2 & 2(2) \\ 2(2) & (2)^2-1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}$$

To find $Q_2(\vec{x})$ we multiply $\vec{x}$ by the matrix of Q_2 as follows:

$$\frac{1}{5}\begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{19}{5} \\ \frac{2}{5} \end{bmatrix}$$

♠

Consider the following example which incorporates a reflection as well as a rotation of vectors.

Example 5.27: Rotation Followed by a Reflection

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of $\pi/6$ and then reflecting through the x axis.

an angle of $\pi/6$ and then reflecting through the x axis.

Solution. By Theorem 5.22, the matrix of the transformation which involves rotating through an angle of $\pi/6$ is

$$\left[\begin{array}{cc} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{array}\right] = \left[\begin{array}{cc} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{array}\right]$$

Reflecting across the x axis is the same action as reflecting vectors over the line $\vec{y} = m\vec{x}$ with $m = 0$. By Theorem 5.25, the matrix for the transformation which reflects all vectors through the x axis is

$$\frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right] = \frac{1}{1+(0)^2}\left[\begin{array}{cc} 1-(0)^2 & 2(0) \\ 2(0) & (0)^2-1 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]$$

Therefore, the matrix of the linear transformation which first rotates through $\pi/6$ and then reflects through the x axis is given by

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{cc} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{array}\right] = \left[\begin{array}{cc} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{array}\right]$$

♠

Exercises

Exercise 5.4.1 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$.*

Exercise 5.4.2 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$.*

Exercise 5.4.3 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $-\pi/3$.*

Exercise 5.4.4 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$.*

Exercise 5.4.5 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/12$.* ***Hint:*** *Note that $\pi/12 = \pi/3 - \pi/4$.*

Exercise 5.4.6 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$ and then reflects across the x axis.*

Exercise 5.4.7 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$ and then reflects across the x axis.*

Exercise 5.4.8 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$ and then reflects across the x axis.*

Exercise 5.4.9 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.*

Exercise 5.4.10 *Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/4$.*

Exercise 5.4.11 *Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/4$.*

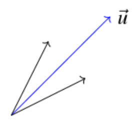
Exercise 5.4.12 *Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/6$.*

Exercise 5.4.13 *Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/6$.*

Exercise 5.4.14 *Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $5\pi/12$.* ***Hint:*** *Note that $5\pi/12 = 2\pi/3 - \pi/4$.*

Exercise 5.4.15 *Find the matrix of the linear transformation which rotates every vector in $\mathbb{R}^3$ counter clockwise about the z axis when viewed from the positive z axis through an angle of 30° and then reflects through the xy plane.*

Exercise 5.4.16 *Let $\vec{u} = \left[\begin{array}{c} a \\ b \end{array} \right]$ be a unit vector in $\mathbb{R}^2$. Find the matrix which reflects all vectors across this vector, as shown in the following picture.*



Hint: *Notice that $\left[\begin{array}{c} a \\ b \end{array} \right] = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array} \right]$ for some θ. First rotate through $-\theta$. Next reflect through the x axis. Finally rotate through θ.*

5.5 One to One and Onto Transformations

Outcomes

A. *Determine if a linear transformation is onto or one to one.*

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. We define the **range** or **image** of T as the set of vectors of $\mathbb{R}^m$ which are of the form $T(\vec{x})$ (equivalently, $A\vec{x}$) for some $\vec{x} \in \mathbb{R}^n$. It is common to write $T\mathbb{R}^n$, $T(\mathbb{R}^n)$, or $\mathrm{Im}(T)$ to denote these vectors.

Lemma 5.28: Range of a Matrix Transformation

Let A be an $m \times n$ matrix where $A_1, \cdots, A_n$ denote the columns of A. Then, for a vector $\vec{x} = \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right]$ in $\mathbb{R}^n$,

$$A\vec{x} = \sum_{k=1}^{n} x_k A_k$$

Therefore, $A(\mathbb{R}^n)$ is the collection of all linear combinations of these products.

Proof. This follows from the definition of matrix multiplication. ♠

This section is devoted to studying two important characterizations of linear transformations, called one to one and onto. We define them now.

Definition 5.29: One to One

*Suppose $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$. A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is called **one to one** (often written as $1-1$) if whenever $\vec{x}_1 \neq \vec{x}_2$ it follows that :*

$$T(\vec{x}_1) \neq T(\vec{x}_2)$$

Equivalently, if $T(\vec{x}_1) = T(\vec{x}_2)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is one to one if it never takes two different vectors to the same vector.

The second important characterization is called onto.

Definition 5.30: Onto

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is called **onto** *if whenever $\vec{x}_2 \in \mathbb{R}^m$ there exists $\vec{x}_1 \in \mathbb{R}^n$ such that $T(\vec{x}_1) = \vec{x}_2$.*

We often call a linear transformation which is one-to-one an **injection**. Similarly, a linear transformation which is onto is often called a **surjection**.

The following proposition is an important result.

Proposition 5.31: One to One

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

Proof. We need to prove two things here. First, we will prove that if T is one to one, then $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$. Second, we will show that if $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$, then it follows that T is one to one. Recall that a linear transformation has the property that $T(\vec{0}) = \vec{0}$.

Suppose first that T is one to one and consider $T(\vec{0})$.

$$T(\vec{0}) = T\left(\vec{0} + \vec{0}\right) = T(\vec{0}) + T(\vec{0})$$

and so, adding the additive inverse of $T(\vec{0})$ to both sides, one sees that $T(\vec{0}) = \vec{0}$. If $T(\vec{x}) = \vec{0}$ it must be the case that $\vec{x} = \vec{0}$ because it was just shown that $T(\vec{0}) = \vec{0}$ and T is assumed to be one to one.

Now assume that if $T(\vec{x}) = \vec{0}$, then it follows that $\vec{x} = \vec{0}$. If $T(\vec{v}) = T(\vec{u})$, then

$$T(\vec{v}) - T(\vec{u}) = T(\vec{v} - \vec{u}) = \vec{0}$$

which shows that $\vec{v} - \vec{u} = 0$. In other words, $\vec{v} = \vec{u}$, and T is one to one. ♠

Note that this proposition says that if $A = \left[\begin{array}{ccc} A_1 & \cdots & A_n \end{array}\right]$ then A is one to one if and only if whenever

$$0 = \sum_{k=1}^{n} c_k A_k$$

it follows that each scalar $c_k = 0$.

We will now take a look at an example of a one to one and onto linear transformation.

Example 5.32: A One to One and Onto Linear Transformation

Suppose

$$T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]$$

Then, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Is T onto? Is it one to one?

Solution. Recall that because T can be expressed as matrix multiplication, we know that T is a linear transformation. We will start by looking at onto. So suppose $\left[\begin{array}{c} a \\ b \end{array}\right] \in \mathbb{R}^2$. Does there exist $\left[\begin{array}{c} x \\ y \end{array}\right] \in \mathbb{R}^2$ such that $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$? If so, then since $\left[\begin{array}{c} a \\ b \end{array}\right]$ is an arbitrary vector in $\mathbb{R}^2$, it will follow that T is onto.

This question is familiar to you. It is asking whether there is a solution to the equation

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$$

This is the same thing as asking for a solution to the following system of equations.

$$\begin{array}{c} x + y = a \\ x + 2y = b \end{array}$$

Set up the augmented matrix and row reduce.

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2a - b \\ 0 & 1 & b - a \end{array}\right] \tag{5.4}$$

You can see from this point that the system has a solution. Therefore, we have shown that for any a, b, there is a $\left[\begin{array}{c} x \\ y \end{array}\right]$ such that $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$. Thus T is onto.

Now we want to know if T is one to one. By Proposition 5.31 it is enough to show that $A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$. Consider the system $A\vec{x} = \vec{0}$ given by:

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

This is the same as the system given by

$$\begin{array}{c} x+y=0 \\ x+2y=0 \end{array}$$

We need to show that the solution to this system is $x = 0$ and $y = 0$. By setting up the augmented matrix and row reducing, we end up with

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

This tells us that $x = 0$ and $y = 0$. Returning to the original system, this says that if

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

then

$$\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

In other words, $A\vec{x} = \vec{0}$ implies that $\vec{x} = \vec{0}$. By Proposition 5.31, A is one to one, and so T is also one to one.

We also could have seen that T is one to one from our above solution for onto. By looking at the matrix given by 5.4, you can see that there is a **unique** solution given by $x = 2a-b$ and $y = b-a$. Therefore, there is only one vector, specifically $\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 2a-b \\ b-a \end{array}\right]$ such that $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$. Hence by Definition 5.29, T is one to one. ♠

Example 5.33: An Onto Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T\left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right] = \left[\begin{array}{c} a+d \\ b+c \end{array}\right] \text{ for all } \left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right] \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution. You can prove that T is in fact linear.

To show that T is onto, let $\left[\begin{array}{c} x \\ y \end{array}\right]$ be an arbitrary vector in $\mathbb{R}^2$. Taking the vector $\left[\begin{array}{c} x \\ y \\ 0 \\ 0 \end{array}\right] \in \mathbb{R}^4$ we have

$$T\left[\begin{array}{c} x \\ y \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} x+0 \\ y+0 \end{array}\right] = \left[\begin{array}{c} x \\ y \end{array}\right]$$

This shows that T is onto.

By Proposition 5.31 T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$. Observe that

$$T\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{c} 1+-1 \\ 0+0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

There exists a nonzero vector $\vec{x}$ in $\mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. It follows that T is not one to one. ♠

The above examples demonstrate a method to determine if a linear transformation T is one to one or onto. It turns out that the matrix A of T can provide this information.

Theorem 5.34: Matrix of a One to One or Onto Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation induced by the $m \times n$ matrix A. Then T is one to one if and only if the rank of A is n. T is onto if and only if the rank of A is m.

Consider Example 5.33. Above we showed that T was onto but not one to one. We can now use this theorem to determine this fact about T.

Example 5.35: An Onto Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution. Using Theorem 5.34 we can show that T is onto but not one to one from the matrix of T. Recall that to find the matrix A of T, we apply T to each of the standard basis vectors $\vec{e}_i$ of $\mathbb{R}^4$. The result is the 2×4 matrix A given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Fortunately, this matrix is already in reduced row-echelon form. The rank of A is 2. Therefore by the above theorem T is onto but not one to one. ♠

Recall that if S and T are linear transformations, we can discuss their composite denoted $S \circ T$. The following examines what happens if both S and T are onto.

Example 5.36: Composite of Onto Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. If T and S are onto, then $S \circ T$ is onto.

Solution. Let $\vec{z} \in \mathbb{R}^m$. Since S is onto, there exists a vector $\vec{y} \in \mathbb{R}^n$ such that $S(\vec{y}) = \vec{z}$. Furthermore, since T is onto, there exists a vector $\vec{x} \in \mathbb{R}^k$ such that $T(\vec{x}) = \vec{y}$. Thus

$$\vec{z} = S(\vec{y}) = S(T(\vec{x})) = (ST)(\vec{x}),$$

showing that for each $\vec{z} \in \mathbb{R}^m$ there exists and $\vec{x} \in \mathbb{R}^k$ such that $(ST)(\vec{x}) = \vec{z}$. Therefore, $S \circ T$ is onto. ♠

The next example shows the same concept with regards to one-to-one transformations.

Example 5.37: Composite of One to One Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Prove that if T and S are one to one, then $S \circ T$ is one-to-one.

Solution. To prove that $S \circ T$ is one to one, we need to show that if $S(T(\vec{v})) = \vec{0}$ it follows that $\vec{v} = \vec{0}$. Suppose that $S(T(\vec{v})) = \vec{0}$. Since S is one to one, it follows that $T(\vec{v}) = \vec{0}$. Similarly, since T is one to one, it follows that $\vec{v} = \vec{0}$. Hence $S \circ T$ is one to one. ♠

Exercises

Exercise 5.5.1 *Let T be a linear transformation given by*

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Is T one to one? Is T onto?

Is T one to one? Is T onto?

Exercise 5.5.2 *Let T be a linear transformation given by*

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise 5.5.3 *Let T be a linear transformation given by*

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise 5.5.4 *Let T be a linear transformation given by*

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 0 & 2 \\ 2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise 5.5.5 *Give an example of a 3×2 matrix with the property that the linear transformation determined by this matrix is one to one but not onto.*

Exercise 5.5.6 *Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that the transformation T determined by A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.* ***Hint:*** *The vectors $\vec{e}_1, \cdots, \vec{e}_m$ occur as columns in the reduced row-echelon form for A.*

Exercise 5.5.7 *Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one.* ***Hint:*** *If not, there exists a vector, $\vec{x}$ such that $A\vec{x} = 0$, and this implies at least one column of A is a linear combination of the others. Show this would require the rank to be less than n.*

Exercise 5.5.8 *Explain why an $n \times n$ matrix A is both one to one and onto if and only if its rank is n.*

5.6 Isomorphisms

Outcomes

A. Determine if a linear transformation is an isomorphism.

B. Determine if two subspaces of $\mathbb{R}^n$ are isomorphic.

Recall the definition of a linear transformation. Let V and W be two subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A mapping $T : V \to W$ is called a **linear transformation** or **linear map** if it preserves the algebraic operations of addition and scalar multiplication. Specifically, if a, b are scalars and $\vec{x}, \vec{y}$ are vectors,

$$T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

Consider the following important definition.

Definition 5.38: Isomorphism

A linear map T is called an ***isomorphism*** *if the following two conditions are satisfied.*

- *T is one to one. That is, if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.*
- *T is onto. That is, if $\vec{w} \in W$, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.*

Two such subspaces which have an isomorphism as described above are said to be ***isomorphic.***

Consider the following example of an isomorphism.

Example 5.39: Isomorphism

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

Show that T is an isomorphism.

Solution. To prove that T is an isomorphism we must show

1. T is a linear transformation;

2. T is one to one;
3. T is onto.

We proceed as follows.

1. T is a linear transformation:

   Let k, p be scalars.

$$\begin{aligned}
T\left(k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} + \begin{bmatrix} px_2 \\ py_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 + px_2 \\ ky_1 + py_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (ky_1 + py_2) \end{bmatrix} \\
&= \begin{bmatrix} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - ky_1) + (px_2 - py_2) \end{bmatrix} \\
&= \begin{bmatrix} kx_1 + ky_1 \\ kx_1 - ky_1 \end{bmatrix} + \begin{bmatrix} px_2 + py_2 \\ px_2 - py_2 \end{bmatrix} \\
&= k\begin{bmatrix} x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + p\begin{bmatrix} x_2 + y_2 \\ x_2 - y_2 \end{bmatrix} \\
&= kT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + pT\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)
\end{aligned}$$

   Therefore T is linear.

2. T is one to one:

   We need to show that if $T(\vec{x}) = \vec{0}$ for a vector $\vec{x} \in \mathbb{R}^2$, then it follows that $\vec{x} = \vec{0}$. Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   This provides a system of equations given by

$$\begin{aligned} x+y &= 0 \\ x-y &= 0 \end{aligned}$$

   You can verify that the solution to this system if $x = y = 0$. Therefore

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   and T is one to one.

3. T is onto:

   Let a, b be scalars. We want to check if there is always a solution to

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right)=\left[\begin{array}{c} x+y \\ x-y \end{array}\right]=\left[\begin{array}{c} a \\ b \end{array}\right]$$

This can be represented as the system of equations

$$\begin{array}{rcl} x+y &=& a \\ x-y &=& b \end{array}$$

Setting up the augmented matrix and row reducing gives

$$\left[\begin{array}{rr|r} 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & \frac{a+b}{2} \\ 0 & 1 & \frac{a-b}{2} \end{array}\right]$$

This has a solution for all a, b and therefore T is onto.

Therefore T is an isomorphism. ♠

An important property of isomorphisms is that its inverse is also an isomorphism.

Proposition 5.40: Inverse of an Isomorphism

Let $T : V \rightarrow W$ be an isomorphism and V, W be subspaces of $\mathbb{R}^n$. Then $T^{-1} : W \rightarrow V$ is also an isomorphism.

Proof. Let T be an isomorphism. Since T is onto, a typical vector in W is of the form $T(\vec{v})$ where $\vec{v} \in V$. Consider then for a, b scalars,

$$T^{-1}\left(aT(\vec{v}_1)+bT(\vec{v}_2)\right)$$

where $\vec{v}_1, \vec{v}_2 \in V$. Is this equal to

$$aT^{-1}\left(T(\vec{v}_1)\right)+bT^{-1}\left(T(\vec{v}_2)\right)=a\vec{v}_1+b\vec{v}_2?$$

Since T is one to one, this will be so if

$$T\left(a\vec{v}_1+b\vec{v}_2\right)=T\left(T^{-1}\left(aT(\vec{v}_1)+bT(\vec{v}_2)\right)\right)=aT(\vec{v}_1)+bT(\vec{v}_2).$$

However, the above statement is just the condition that T is a linear map. Thus T^{-1} is indeed a linear map. If $\vec{v} \in V$ is given, then $\vec{v}=T^{-1}\left(T(\vec{v})\right)$ and so T^{-1} is onto. If $T^{-1}(\vec{v})=0$, then

$$\vec{v}=T\left(T^{-1}(\vec{v})\right)=T(\vec{0})=\vec{0}$$

and so T^{-1} is one to one. ♠

Another important result is that the composition of multiple isomorphisms is also an isomorphism.

Proposition 5.41: Composition of Isomorphisms

Let $T : V \rightarrow W$ and $S : W \rightarrow Z$ be isomorphisms where V, W, Z are subspaces of $\mathbb{R}^n$. Then $S \circ T$ defined by $(S \circ T)(\vec{v})=S(T(\vec{v}))$ is also an isomorphism.

Proof. Suppose $T : V \rightarrow W$ and $S : W \rightarrow Z$ are isomorphisms. Why is $S \circ T$ a linear map? For a, b scalars,

$$\begin{aligned} S \circ T\left(a\vec{v}_1+b(\vec{v}_2)\right) &= S\left(T\left(a\vec{v}_1+b\vec{v}_2\right)\right)=S\left(aT\vec{v}_1+bT\vec{v}_2\right) \\ &= aS(T\vec{v}_1)+bS(T\vec{v}_2)=a(S \circ T)(\vec{v}_1)+b(S \circ T)(\vec{v}_2) \end{aligned}$$

Hence $S \circ T$ is a linear map. If $(S \circ T)(\vec{v})=\vec{0}$, then $S(T(\vec{v}))=\vec{0}$ and it follows that $T(\vec{v})=\vec{0}$ and hence by this lemma again, $\vec{v}=\vec{0}$. Thus $S \circ T$ is one to one. It remains to verify that it is onto. Let $\vec{z} \in Z$. Then since S is onto, there exists $\vec{w} \in W$ such that $S(\vec{w})=\vec{z}$. Also, since T is onto, there exists $\vec{v} \in V$ such that $T(\vec{v})=\vec{w}$. It follows that $S(T(\vec{v}))=\vec{z}$ and so $S \circ T$ is also onto. ♠

Consider two subspaces V and W, and suppose there exists an isomorphism mapping one to the other. In this way the two subspaces are related, which we can write as $V \sim W$. Then the previous two propositions together claim that $\sim$ is an equivalence relation. That is: $\sim$ satisfies the following conditions:

- $V \sim V$
- If $V \sim W$, it follows that $W \sim V$
- If $V \sim W$ and $W \sim Z$, then $V \sim Z$

We leave the verification of these conditions as an exercise.

We leave the verification of these conditions as an exercise.

Consider the following example.

Example 5.42: Matrix Isomorphism

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. Then T is an isomorphism.

Solution. The reason for this is that, since A is invertible, the only vector it sends to $\vec{0}$ is the zero vector. Hence if $A(\vec{x}) = A(\vec{y})$, then $A(\vec{x} - \vec{y}) = \vec{0}$ and so $\vec{x} = \vec{y}$. It is onto because if $\vec{y} \in \mathbb{R}^n, A\left(A^{-1}(\vec{y})\right) = \left(AA^{-1}\right)(\vec{y})$ $= \vec{y}$. ♠

In fact, all isomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be expressed as $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. One simply considers the matrix whose i^{th} column is $T\vec{e}_i$.

Recall that a basis of a subspace V is a set of linearly independent vectors which span V. The following fundamental lemma describes the relation between bases and isomorphisms.

Lemma 5.43: Mapping Bases

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. If T is one to one, then it has the property that if $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent, so is $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$.
More generally, T is an isomorphism if and only if whenever $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W.

Proof. First suppose that T is a linear transformation and is one to one and $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent. It is required to show that $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$ is also linearly independent. Suppose then that

$$\sum_{i=1}^{k} c_i T(\vec{u}_i) = \vec{0}$$

Then, since T is linear,

$$T\left(\sum_{i=1}^{n} c_i \vec{u}_i\right) = \vec{0}$$

Since T is one to one, it follows that

$$\sum_{i=1}^{n} c_i \vec{u}_i = 0$$

Now the fact that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent implies that each $c_i = 0$. Hence $\{T(\vec{u}_1), \cdots, T(\vec{u}_n)\}$ is linearly independent.

Now suppose that T is an isomorphism and $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. It was just shown that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. It remains to verify that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$. If $\vec{w} \in W$, then since T is onto there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, it follows that there exists scalars $\{c_i\}_{i=1}^{n}$ such that

$$\sum_{i=1}^{n} c_i \vec{v}_i = \vec{v}.$$

Hence,

$$\vec{w} = T(\vec{v}) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T(\vec{v}_i)$$

It follows that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ showing that this set of vectors is a basis for W.

Next suppose that T is a linear transformation which takes a basis to a basis. This means that if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W. Then if $w \in W$, there exist scalars c_i such that $w = \sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$ showing that T is onto. If $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \vec{0}$ then $\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$ and since the vectors $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ are linearly independent, it follows that each $c_i = 0$. Since $\sum_{i=1}^{n} c_i \vec{v}_i$ is a typical vector in V, this has shown that if $T(\vec{v}) = \vec{0}$ then $\vec{v} = \vec{0}$ and so T is also one to one. Thus T is an isomorphism. ♠

The following theorem illustrates a very useful idea for defining an isomorphism. Basically, if you know what it does to a basis, then you can construct the isomorphism.

Theorem 5.44: Isomorphic Subspaces

Suppose V and W are two subspaces of $\mathbb{R}^n$. Then the two subspaces are isomorphic if and only if they have the same dimension. In the case that the two subspaces have the same dimension, then for a linear map $T : V \to W$, the following are equivalent.

1. *T is one to one.*
2. *T is onto.*

3. T is an isomorphism.

Proof. Suppose first that these two subspaces have the same dimension. Let a basis for V be $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. Now define T as follows.

$$T(\vec{v}_i) = \vec{w}_i$$

for $\sum_{i=1}^n c_i \vec{v}_i$ an arbitrary vector of V,

$$T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \sum_{i=1}^n c_i T \vec{v}_i = \sum_{i=1}^n c_i \vec{w}_i.$$

It is necessary to verify that this is well defined. Suppose then that

$$\sum_{i=1}^n c_i \vec{v}_i = \sum_{i=1}^n \hat{c}_i \vec{v}_i$$

Then

$$\sum_{i=1}^n (c_i - \hat{c}_i) \vec{v}_i = \vec{0}$$

and since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, $c_i = \hat{c}_i$ for each i. Hence

$$\sum_{i=1}^n c_i \vec{w}_i = \sum_{i=1}^n \hat{c}_i \vec{w}_i$$

and so the mapping is well defined. Also if a, b are scalars,

$$\begin{aligned} T\left(a\sum_{i=1}^n c_i \vec{v}_i + b\sum_{i=1}^n \hat{c}_i \vec{v}_i\right) &= T\left(\sum_{i=1}^n (ac_i + b\hat{c}_i) \vec{v}_i\right) = \sum_{i=1}^n (ac_i + b\hat{c}_i) \vec{w}_i \\ &= a\sum_{i=1}^n c_i \vec{w}_i + b\sum_{i=1}^n \hat{c}_i \vec{w}_i \\ &= aT\left(\sum_{i=1}^n c_i \vec{v}_i\right) + bT\left(\sum_{i=1}^n \hat{c}_i \vec{v}_i\right) \end{aligned}$$

Thus T is a linear transformation.

Now if

$$T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \sum_{i=1}^n c_i \vec{w}_i = \vec{0},$$

then since the $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are independent, each $c_i = 0$ and so $\sum_{i=1}^n c_i \vec{v}_i = \vec{0}$ also. Hence T is one to one. If $\sum_{i=1}^n c_i \vec{w}_i$ is a vector in W, then it equals

$$\sum_{i=1}^n c_i T(\vec{v}_i) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$$

showing that T is also onto. Hence T is an isomorphism and so V and W are isomorphic.

Next suppose $T : V \mapsto W$ is an isomorphism, so these two subspaces are isomorphic. Then for $\{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis for V, it follows that a basis for W is $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ showing that the two subspaces have the same dimension.

Now suppose the two subspaces have the same dimension. Consider the three claimed equivalences.

First consider the claim that 1. $\Rightarrow$ 2. If T is one to one and if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, then $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. If it is not a basis, then it must fail to span W. But then there would exist $\vec{w} \notin \operatorname{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n), \vec{w}\}$ would be linearly independent which is impossible because there exists a basis for W of n vectors.

Hence $\operatorname{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ and so $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis. If $\vec{w} \in W$, there exist scalars c_i such that

$$\vec{w} = \sum_{i=1}^n c_i T(\vec{v}_i) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$$

showing that T is onto. This shows that $1. \Rightarrow 2.$

Next consider the claim that $2. \Rightarrow 3.$ Since 2. holds, it follows that T is onto. It remains to verify that T is one to one. Since T is onto, there exists a basis of the form $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$. Then it follows that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent. Suppose

$$\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$$

Then

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

Hence each $c_i = 0$ and so, $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. Now it follows that a typical vector in V is of the form $\sum_{i=1}^{n} c_i \vec{v}_i$. If $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \vec{0}$, it follows that

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

and so, since $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$ is independent, it follows each $c_i = 0$ and hence $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$. Thus T is one to one as well as onto and so it is an isomorphism.

If T is an isomorphism, it is both one to one and onto by definition so 3. implies both 1. and 2. $\spadesuit$

Note the interesting way of defining a linear transformation in the first part of the argument by describing what it does to a basis and then "extending it linearly" to the entire subspace.

Example 5.45: Isomorphic Subspaces

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Show that V and W are isomorphic.

Solution. First observe that these subspaces are both of dimension 3 and so they are isomorphic by Theorem 5.44. The three vectors which span W are easily seen to be linearly independent by making them the columns of a matrix and row reducing to the reduced row-echelon form.

You can exhibit an isomorphism of these two spaces as follows.

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

and extend linearly. Recall that the matrix of this linear transformation is just the matrix having these vectors as columns. Thus the matrix of this isomorphism is

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

You should check that multiplication on the left by this matrix does reproduce the claimed effect resulting from an application by T. $\spadesuit$

Consider the following example.

Example 5.46: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Let $T : V \mapsto W$ be defined as follows.

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Find the matrix of this isomorphism T.

Solution. First note that the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are indeed a basis for $\mathbb{R}^3$ as can be seen by making them the columns of a matrix and using the reduced row-echelon form.

Now recall the matrix of T is a 4×3 matrix A which gives the same effect as T. Thus, from the way we multiply matrices,

$$A \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

Note how the span of the columns of this new matrix must be the same as the span of the vectors defining W. ♠

This idea of defining a linear transformation by what it does on a basis works for linear maps which are not necessarily isomorphisms.

Example 5.47: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Let $T : V \mapsto W$ be defined as follows.

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Find the matrix of this linear transformation.

Solution. Note that in this case, the three vectors which span W are not linearly independent. Nevertheless the above procedure will still work. The reasoning is the same as before. If A is this matrix, then

$$A \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and so

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The columns of this last matrix are obviously not linearly independent. ♠

Exercises

Exercise 5.6.1 *Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $T : V \to W$ be a linear transformation. Suppose that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is linearly independent. Show that it must be the case that $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is also linearly independent.*

Exercise 5.6.2 *Let*

$$V = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Let $T\vec{x} = A\vec{x}$ where A is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Give a basis for $\operatorname{im}(T)$.

Exercise 5.6.3 *Let*

$$V = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Let $T\vec{x} = A\vec{x}$ where A is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Find a basis for $\operatorname{im}(T)$. *In this case, the original vectors do not form an independent set.*

Exercise 5.6.4 *If $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent and T is a one to one linear transformation, show that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is also linearly independent. Give an example which shows that if T is only linear, it can happen that, although $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent, $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is not. In fact, show that it can happen that each of the $T\vec{v}_j$ equals 0.*

Exercise 5.6.5 *Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $T : V \to W$ be a linear transformation. Show that if T is onto W and if $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V, then $span\{T\vec{v}_1, \cdots, T\vec{v}_r\} = W$.*

Exercise 5.6.6 *Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ as follows.*

$$T\vec{x} = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 2 & 2 & -2 & 6 \\ 1 & 1 & -1 & 3 \end{bmatrix} \vec{x}$$

Find a basis for $\operatorname{im}(T)$. *Also find a basis for* $\ker(T)$.

Exercise 5.6.7 *Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ as follows.*

$$T\vec{x} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$$

where on the right, it is just matrix multiplication of the vector $\vec{x}$ which is meant. Explain why T is an isomorphism of $\mathbb{R}^3$ to $\mathbb{R}^3$.

Exercise 5.6.8 *Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation given by*

$$T\vec{x} = A\vec{x}$$

where A is a 3×3 matrix. Show that T is an isomorphism if and only if A is invertible.

Exercise 5.6.9 *Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by*

$$T\vec{x} = A\vec{x}$$

where A is an $m \times n$ matrix. Show that T is never an isomorphism if $m \neq n$. In particular, show that if $m > n$, T cannot be onto and if $m < n$, then T cannot be one to one.

Exercise 5.6.10 *Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ as follows.*

$$T\vec{x} = \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{array}\right] \vec{x}$$

where on the right, it is just matrix multiplication of the vector $\vec{x}$ which is meant. Show that T is one to one. Next let $W = \mathrm{im}(T)$. Show that T is an isomorphism of $\mathbb{R}^2$ and im (T).

Exercise 5.6.11 *In the above problem, find a 2×3 matrix A such that the restriction of A to $\mathrm{im}(T)$ gives the same result as T^{-1} on $\mathrm{im}(T)$.* ***Hint:*** *You might let A be such that*

$$A \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \end{array}\right], \ A \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

now find another vector $\vec{v} \in \mathbb{R}^3$ such that

$$\left\{ \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right], \vec{v} \right\}$$

is a basis. You could pick

$$\vec{v} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

for example. Explain why this one works or one of your choice works. Then you could define $A\vec{v}$ to equal some vector in $\mathbb{R}^2$. Explain why there will be more than one such matrix A which will deliver the inverse isomorphism T^{-1} on $\mathrm{im}(T)$.

Exercise 5.6.12 *Now let V equal span $\left\{ \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right] \right\}$ and let $T : V \to W$ be a linear transformation where*

$$W = span \left\{ \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array}\right] \right\}$$

and

$$T \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], T \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

Explain why T is an isomorphism. Determine a matrix A which, when multiplied on the left gives the same result as T on V and a matrix B which delivers T^{-1} on W. ***Hint:*** *You need to have*

$$A \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{array}\right]$$

Now enlarge $\left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]$ to obtain a basis for $\mathbb{R}^3$. You could add in $\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$ for example, and then pick another vector in $\mathbb{R}^4$ and let $A \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$ equal this other vector. Then you would have

$$A \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

This would involve picking for the new vector in $\mathbb{R}^4$ the vector $\left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right]^T$. Then you could find A.

You can do something similar to find a matrix for T^{-1} denoted as B.

5.7 The Kernel And Image Of A Linear Map

Outcomes

A. *Describe the kernel and image of a linear transformation, and find a basis for each.*

In this section we will consider the case where the linear transformation is not necessarily an isomorphism. First consider the following important definition.

Definition 5.48: Kernel and Image

Let V and W be subspaces of $\mathbb{R}^n$ and let $T : V \mapsto W$ be a linear transformation. Then the image of T denoted as $\operatorname{im}(T)$ is defined to be the set

$$\operatorname{im}(T) = \{T(\vec{v}) : \vec{v} \in V\}$$

In words, it consists of all vectors in W which equal $T(\vec{v})$ for some $\vec{v} \in V$.
The kernel of T, written $\ker(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. That is,

$$\ker(T) = \left\{\vec{v} \in V : T(\vec{v}) = \vec{0}\right\}$$

It follows that $\operatorname{im}(T)$ and $\ker(T)$ are subspaces of W and V respectively.

Proposition 5.49: Kernel and Image as Subspaces

Let V, W be subspaces of $\mathbb{R}^n$ and let $T : V \to W$ be a linear transformation. Then $\ker(T)$ is a subspace of V and $\operatorname{im}(T)$ is a subspace of W.

Proof. First consider $\ker(T)$. It is necessary to show that if $\vec{v}_1, \vec{v}_2$ are vectors in $\ker(T)$ and if a, b are scalars, then $a\vec{v}_1 + b\vec{v}_2$ is also in $\ker(T)$. But

$$T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + bT(\vec{v}_2) = a\vec{0} + b\vec{0} = \vec{0}$$

Thus $\ker(T)$ is a subspace of V.

Next suppose $T(\vec{v}_1), T(\vec{v}_2)$ are two vectors in $\operatorname{im}(T)$. Then if a, b are scalars,

$$aT(\vec{v}_2) + bT(\vec{v}_2) = T(a\vec{v}_1 + b\vec{v}_2)$$

and this last vector is in $\operatorname{im}(T)$ by definition. ♠

We will now examine how to find the kernel and image of a linear transformation and describe the basis of each.

Example 5.50: Kernel and Image of a Linear Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be defined by

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\operatorname{im}(T)$.

Solution. You can verify that T is a linear transformation.

First we will find a basis for $\ker(T)$. To do so, we want to find a way to describe all vectors $\vec{x} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be such a vector. Then

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The values of a,b,c,d that make this true are given by solutions to the system

$$\begin{array}{rcl} a-b & = & 0 \\ c+d & = & 0 \end{array}$$

The solution to this system is $a = s, b = s, c = t, d = -t$ where s,t are scalars. We can describe $\ker(T)$ as follows.

$$\ker(T) = \left\{ \begin{bmatrix} s \\ s \\ t \\ -t \end{bmatrix} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Notice that this set is linearly independent and therefore forms a basis for $\ker(T)$.

We move on to finding a basis for $\operatorname{im}(T)$. We can write the image of T as

$$\operatorname{im}(T) = \left\{ \begin{bmatrix} a-b \\ c+d \end{bmatrix} \right\}$$

We can write this in the form

$$\operatorname{span} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

This set is clearly not linearly independent. By removing unnecessary vectors from the set we can create a linearly independent set with the same span. This gives a basis for $\operatorname{im}(T)$ as

$$\operatorname{im}(T) = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

♠

Recall that a linear transformation T is called one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$. Using the concept of kernel, we can state this theorem in another way.

Theorem 5.51: One to One and Kernel

*Let T be a linear transformation where $\ker(T)$ is the kernel of T. Then T is one to one if and only if $\ker(T)$ consists of **only** the zero vector.*

A major result is the relation between the dimension of the kernel and dimension of the image of a linear transformation. In the previous example $\ker(T)$ had dimension 2, and $\operatorname{im}(T)$ also had dimension of 2. Consider the following theorem.

Theorem 5.52: Dimension of Kernel and Image

Let $T : V \to W$ be a linear transformation where V,W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$m = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

Proof. From Proposition 5.49, $\operatorname{im}(T)$ is a subspace of W. We know that there exists a basis for $\operatorname{im}(T)$, written $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$. Similarly, there is a basis for $\ker(T), \{\vec{u}_1, \cdots, \vec{u}_s\}$. Then if $\vec{v} \in V$, there exist scalars c_i such that scalars c_i such that

$$T(\vec{v}) = \sum_{i=1}^{r} c_i T(\vec{v}_i)$$

Hence $T\left(\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i\right) = 0$. It follows that $\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i$ is in $\ker(T)$. Hence there are scalars a_i such that

$$\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i = \sum_{j=1}^{s} a_j \vec{u}_j$$

Hence $\vec{v} = \sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j$. Since $\vec{v}$ is arbitrary, it follows that

$$V = \operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$$

If the vectors $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ are linearly independent, then it will follow that this set is a basis. Suppose then that

$$\sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j = 0$$

Apply T to both sides to obtain

$$\sum_{i=1}^{r} c_i T(\vec{v}_i) + \sum_{j=1}^{s} a_j T(\vec{u})_j = \sum_{i=1}^{r} c_i T(\vec{v}_i) = 0$$

Since $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$ is linearly independent, it follows that each $c_i = 0$. Hence $\sum_{j=1}^{s} a_j \vec{u}_j = 0$ and so, since the $\{\vec{u}_1, \cdots, \vec{u}_s\}$ are linearly independent, it follows that each $a_j = 0$ also. Therefore $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V and so

$$n = s + r = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

♠

The above theorem leads to the next corollary.

Corollary 5.53:

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$\dim(\ker(T)) \leq m$$
$$\dim(\operatorname{im}(T)) \leq m$$

This follows directly from the fact that $n = \dim(\ker(T)) + \dim(\operatorname{im}(T))$.

Consider the following example.

Example 5.54:

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

Then $\operatorname{im}(T) = V$ is a subspace of $\mathbb{R}^3$ and T is an isomorphism of $\mathbb{R}^2$ and V. Find a 2×3 matrix A such that the restriction of multiplication by A to $V = \operatorname{im}(T)$ equals T^{-1}.

Solution. Since the two columns of the above matrix are linearly independent, we conclude that $\dim(\operatorname{im}(T)) = 2$ and therefore $\dim(\ker(T)) = 2 - \dim(\operatorname{im}(T)) = 2 - 2 = 0$ by Theorem 5.52. Then by Theorem 5.51 it follows that T is one to one.

Thus T is an isomorphism of $\mathbb{R}^2$ and the two dimensional subspace of $\mathbb{R}^3$ which is the span of the columns of the given matrix. Now in particular,

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; T(\vec{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$T^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_1, \; T^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_2$$

Extend T^{-1} to all of $\mathbb{R}^3$ by defining

$$T^{-1} \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix} = \vec{z}$$

Notice that the vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

are linearly independent so T^{-1} can be extended linearly to yield a linear transformation defined on $\mathbb{R}^3$. The matrix of T^{-1} denoted as A needs to satisfy

$$A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and so

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so the restriction to V of matrix multiplication by this matrix yields T^{-1}. ♠

Exercises

Exercise 5.7.1 *Let $V = \mathbb{R}^3$ and let*

$$W = span(S), \text{ where } S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

Find a basis of W consisting of vectors in S.

Exercise 5.7.2 *Let T be a linear transformation given by*

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ *and* $\operatorname{im}(T)$.

Exercise 5.7.3 *Let T be a linear transformation given by*

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ *and* $\operatorname{im}(T)$.

Exercise 5.7.4 *Let $V = \mathbb{R}^3$ and let*

$$W = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

Extend this basis of W to a basis of V.

Exercise 5.7.5 *Let T be a linear transformation given by*

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

What is $\dim(\ker(T))$*?*

5.8 The Matrix of a Linear Transformation II

Outcomes

A. *Find the matrix of a linear transformation with respect to general bases.*

We begin this section with an important lemma.

Lemma 5.55: Mapping of a Basis

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be an isomorphism. Then T maps any basis of $\mathbb{R}^n$ to another basis for $\mathbb{R}^n$.
Conversely, if $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation which maps a basis of $\mathbb{R}^n$ to another basis of $\mathbb{R}^n$, then it is an isomorphism.

Proof. First, suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation which is one to one and onto. Let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. We wish to show that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is also a basis for $\mathbb{R}^n$.

First consider why it is linearly independent. Suppose $\sum_{k=1}^n a_k T(\vec{v}_k) = \vec{0}$. Then by linearity we have $T\left(\sum_{k=1}^n a_k \vec{v}_k\right) = \vec{0}$ and since T is one to one, it follows that $\sum_{k=1}^n a_k \vec{v}_k = \vec{0}$. This requires that each $a_k = 0$ because $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is independent, and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent.

Next take $\vec{w} \in \mathbb{R}^n$. Since T is onto, there exists $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, in particular it is a spanning set and there are scalars b_k such that $T\left(\sum_{k=1}^n b_k \vec{v}_k\right) = T(\vec{v}) = \vec{w}$. Therefore $\vec{w} = \sum_{k=1}^n b_k T(\vec{v}_k)$ which is in the span $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$. Therefore, $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis as claimed.

Suppose now that $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation such that $T(\vec{v}_i) = \vec{w}_i$ where $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are two bases for $\mathbb{R}^n$.

To show that T is one to one, let $T\left(\sum_{k=1}^n c_k \vec{v}_k\right) = \vec{0}$. Then $\sum_{k=1}^n c_k T(\vec{v}_k) = \sum_{k=1}^n c_k \vec{w}_k = \vec{0}$. It follows that each $c_k = 0$ because it is given that $\{\vec{w}_1, \cdots, \vec{w}_n\}$ is linearly independent. Hence $T\left(\sum_{k=1}^n c_k \vec{v}_k\right) = \vec{0}$ implies that $\sum_{k=1}^n c_k \vec{v}_k = \vec{0}$ and so T is one to one.

To show that T is onto, let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^n$. This vector can be written as $\vec{w} = \sum_{k=1}^n d_k \vec{w}_k = \sum_{k=1}^n d_k T(\vec{v}_k) = T\left(\sum_{k=1}^n d_k \vec{v}_k\right)$. Therefore, T is also onto. ♠

Consider now an important definition.

Definition 5.56: Coordinate Vector

Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $\vec{x}$ be an arbitrary vector in $\mathbb{R}^n$. Then $\vec{x}$ is uniquely represented as $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ for scalars $a_1, \cdots, a_n$.
*The **coordinate vector** of $\vec{x}$ with respect to the basis B, written $C_B(\vec{x})$ or $[\vec{x}]_B$, is given by*

$$C_B(\vec{x}) = C_B(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Consider the following example.

Example 5.57: Coordinate Vector

Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and let $\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Solution. First, note the order of the basis is important so label the vectors in the basis B as

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2\}$$

Now we need to find a_1, a_2 such that $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2$, that is:

$$\left[\begin{array}{r} 3 \\ -1 \end{array}\right] = a_1 \left[\begin{array}{r} 1 \\ 0 \end{array}\right] + a_2 \left[\begin{array}{r} -1 \\ 1 \end{array}\right]$$

Solving this system gives $a_1 = 2, a_2 = -1$. Therefore the coordinate vector of $\vec{x}$ with respect to the basis B is

$$C_B(\vec{x}) = \left[\begin{array}{c} a_1 \\ a_2 \end{array}\right] = \left[\begin{array}{r} 2 \\ -1 \end{array}\right]$$

$\spadesuit$

Given any basis B, one can easily verify that the coordinate function is actually an isomorphism.

Theorem 5.58: C_B is a Linear Transformation

For any basis B of $\mathbb{R}^n$, the coordinate function

$$C_B : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear transformation, and moreover an isomorphism.

We now discuss the main result of this section, that is how to represent a linear transformation with respect to different bases.

Theorem 5.59: The Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation, and let B_1 and B_2 be bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then the following holds

$$C_{B_2}T = M_{B_2B_1}C_{B_1} \tag{5.5}$$

where $M_{B_2B_1}$ is a unique $m \times n$ matrix.
If the basis B_1 is given by $B_1 = \{\vec{v}_1, \cdots, \vec{v}_n\}$ in this order, then

$$M_{B_2B_1} = [C_{B_2}(T(\vec{v}_1))\ C_{B_2}(T(\vec{v}_2))\ \cdots C_{B_2}(T(\vec{v}_n))]$$

Proof. The above equation 5.5 can be represented by the following diagram.

$$\begin{array}{ccc} & T & \\ \mathbb{R}^n & \to & \mathbb{R}^m \\ C_{B_1} \downarrow & \circ & \downarrow C_{B_2} \\ \mathbb{R}^n & \to & \mathbb{R}^m \\ & M_{B_2B_1} & \end{array}$$

Since C_{B_1} is an isomorphism, then the matrix we are looking for is the matrix of the linear transformation

$$C_{B_2}TC_{B_1}^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^m.$$

By Theorem 5.6, the columns are given by the image of the standard basis $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$. But since $C_{B_1}^{-1}(\vec{e}_i) = \vec{v}_i$, we readily obtain that

$$\begin{aligned} M_{B_2B_1} &= \left[C_{B_2}TC_{B_1}^{-1}(\vec{e}_1)\ C_{B_2}TC_{B_1}^{-1}(\vec{2}_2)\ \cdots\ C_{B_2}TC_{B_1}^{-1}(\vec{e}_n)\right] \\ &= [C_{B_2}(T(\vec{v}_1))\ C_{B_2}(T(\vec{v}_2))\ \cdots\ C_{B_2}(T(\vec{v}_n))] \end{aligned}$$

and this completes the proof. $\spadesuit$

Consider the following example.

Example 5.60: Matrix of a Linear Transformation

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *be a linear transformation defined by* $T\left(\left[\begin{array}{c} a \\ b \end{array}\right]\right) = \left[\begin{array}{c} b \\ a \end{array}\right]$.

Consider the two bases

$$B_1 = \{\vec{v}_1, \vec{v}_2\} = \left\{ \left[\begin{array}{c} 1 \\ 0 \end{array}\right], \left[\begin{array}{r} -1 \\ 1 \end{array}\right] \right\}$$

and

$$B_2 = \left\{ \left[\begin{array}{c} 1 \\ 1 \end{array}\right], \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \right\}$$

Find the matrix M_{B_2,B_1} *of* T *with respect to the bases* B_1 *and* B_2.

Solution. By Theorem 5.59, the columns of $M_{B_2B_1}$ are the coordinate vectors of $T(\vec{v}_1), T(\vec{v}_2)$ with respect to B_2.

Since

$$T\left(\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right) = \left[\begin{array}{c} 0 \\ 1 \end{array}\right],$$

a standard calculation yields

$$\left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \left(\frac{1}{2}\right)\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \left(-\frac{1}{2}\right)\left[\begin{array}{r} 1 \\ -1 \end{array}\right],$$

the first column of $M_{B_2B_1}$ is $\left[\begin{array}{r} \frac{1}{2} \\ -\frac{1}{2} \end{array}\right]$.

The second column is found in a similar way. We have

$$T\left(\left[\begin{array}{r} -1 \\ 1 \end{array}\right]\right) = \left[\begin{array}{r} 1 \\ -1 \end{array}\right],$$

and with respect to B_2 calculate:

$$\left[\begin{array}{r} 1 \\ -1 \end{array}\right] = 0\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + 1\left[\begin{array}{r} 1 \\ -1 \end{array}\right]$$

Hence the second column of $M_{B_2B_1}$ is given by $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$. We thus obtain

$$M_{B_2B_1} = \left[\begin{array}{rc} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{array}\right]$$

We can verify that this is the correct matrix $M_{B_2B_1}$ on the specific example

$$\vec{v} = \left[\begin{array}{r} 3 \\ -1 \end{array}\right]$$

First applying T gives

$$T(\vec{v}) = T\left(\left[\begin{array}{r} 3 \\ -1 \end{array}\right]\right) = \left[\begin{array}{r} -1 \\ 3 \end{array}\right]$$

and one can compute that

$$C_{B_2}\left(\left[\begin{array}{r} -1 \\ 3 \end{array}\right]\right) = \left[\begin{array}{r} 1 \\ -2 \end{array}\right].$$

On the other hand, one compute $C_{B_1}(\vec{v})$ as

$$C_{B_1}\left(\left[\begin{array}{r} 3 \\ -1 \end{array}\right]\right) = \left[\begin{array}{r} 2 \\ -1 \end{array}\right],$$

and finally applying $M_{B_1B_2}$ gives

$$\left[\begin{array}{rc} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{array}\right]\left[\begin{array}{r} 2 \\ -1 \end{array}\right] = \left[\begin{array}{r} 1 \\ -2 \end{array}\right]$$

as above.

We see that the same vector results from either method, as suggested by Theorem 5.59. ♠

If the bases B_1 and B_2 are equal, say B, then we write M_B instead of M_{BB}. The following example illustrates how to compute such a matrix. Note that this is what we did earlier when we considered only $B_1 = B_2$ to be the standard basis.

Example 5.61: Matrix of a Linear Transformation with respect to an Arbitrary Basis

Consider the basis B of $\mathbb{R}^3$ given by

$$B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

And let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be the linear transformation defined on B as:

$$T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, T \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

1. *Find the matrix M_B of T relative to the basis B.*
2. *Then find the usual matrix of T with respect to the standard basis of $\mathbb{R}^3$.*

Solution.

Equation 5.5 gives $C_B T = M_B C_B$, and thus $M_B = C_B T C_B^{-1}$.

Now $C_B(\vec{v}_i) = \vec{e}_i$, so the matrix of C_B^{-1} (with respect to the standard basis) is given by

$$\left[C_B^{-1}(\vec{e}_1) \; C_B^{-1}(\vec{e}_2) \; C_B^{-1}(\vec{e}_2) \right] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Moreover the matrix of TC_B^{-1} is given by

$$\left[TC_B^{-1}(\vec{e}_1) \; TC_B^{-1}(\vec{e}_2) \; TC_B^{-1}(\vec{e}_2) \right] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Thus

$$\begin{aligned} M_B &= C_B T C_B^{-1} = [C_B^{-1}]^{-1} [TC_B^{-1}] \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -5 & 1 \\ -1 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{aligned}$$

Consider how this works. Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$.

Apply C_B^{-1} to $\vec{b}$ to get

$$b_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Apply T to this linear combination to obtain

$$b_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 \\ -b_1 + 2b_2 + b_3 \\ b_1 - b_2 + b_3 \end{bmatrix}$$

Now take the matrix M_B of the transformation (as found above) and multiply it by $\vec{b}$.

$$\begin{bmatrix} 2 & -5 & 1 \\ -1 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 + b_3 \\ -b_1 + 4b_2 \\ -2b_2 + b_3 \end{bmatrix}$$

Is this the coordinate vector of the above relative to the given basis? We check as follows.

$$\begin{aligned} &(2b_1 - 5b_2 + b_3) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-b_1 + 4b_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2b_2 + b_3) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} b_1 + b_2 \\ -b_1 + 2b_2 + b_3 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} b_1 - b_2 + b_3 \end{bmatrix}$$

You see it is the same thing.

Now lets find the matrix of T with respect to the standard basis. Let A be this matrix. That is, multiplication by A is the same as doing T. Thus

$$A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Hence

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & -3 \\ -3 & -2 & 4 \end{bmatrix}$$

Of course this is a very different matrix than the matrix of the linear transformation with respect to the non standard basis. ♠

Exercises

Exercise 5.8.1 *Let* $B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ *be a basis of* $\mathbb{R}^2$ *and let* $\vec{x} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ *be a vector in* $\mathbb{R}^2$. *Find* $C_B(\vec{x})$.

Exercise 5.8.2 *Let* $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ *be a basis of* $\mathbb{R}^3$ *and let* $\vec{x} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ *be a vector in* $\mathbb{R}^2$. *Find* $C_B(\vec{x})$.

Exercise 5.8.3 *Let* $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *be a linear transformation defined by* $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$.

Consider the two bases

$$B_1 = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find the matrix M_{B_2,B_1} *of* T *with respect to the bases* B_1 *and* B_2.

5.9 The General Solution of a Linear System

Outcomes

A. *Use linear transformations to determine the particular solution and general solution to a system of equations.*

B. *Find the kernel of a linear transformation.*

Recall the definition of a linear transformation discussed above. T is a **linear transformation** if whenever $\vec{x}, \vec{y}$ are vectors and k, p are scalars,

$$T(k\vec{x} + p\vec{y}) = kT(\vec{x}) + pT(\vec{y})$$

Thus linear transformations distribute across addition and pass scalars to the outside.

It turns out that we can use linear transformations to solve linear systems of equations. Indeed given a system of linear equations of the form $A\vec{x} = \vec{b}$, one may rephrase this as $T(\vec{x}) = \vec{b}$ where T is the linear transformation T_A induced by the coefficient matrix A. With this in mind consider the following definition.

Definition 5.62: Particular Solution of a System of Equations

Suppose a linear system of equations can be written in the form

$$T(\vec{x}) = \vec{b}$$

If $T(\vec{x}_p) = \vec{b}$*, then* $\vec{x}_p$ *is called a* ***particular solution*** *of the linear system.*

Recall that a system is called homogeneous if every equation in the system is equal to 0. Suppose we represent a homogeneous system of equations by $T(\vec{x}) = \vec{0}$. It turns out that the $\vec{x}$ for which $T(\vec{x}) = \vec{0}$ are part of a special set called the **null space** of T. We may also refer to the null space as the **kernel** of T, and we write $ker(T)$.

Consider the following definition.

Definition 5.63: Null Space or Kernel of a Linear Transformation

Let T *be a linear transformation. Define*

$$\ker(T) = \left\{ \vec{x} : T(\vec{x}) = \vec{0} \right\}$$

The kernel, $\ker(T)$ *consists of the set of all vectors* $\vec{x}$ *for which* $T(\vec{x}) = \vec{0}$*. This is also called the* ***null space*** *of* T*.*

We may also refer to the kernel of T as the **solution space** of the equation $T(\vec{x}) = \vec{0}$.

Consider the following example.

Example 5.64: The Kernel of the Derivative

Let $\frac{d}{dx}$ *denote the linear transformation defined on* f*, the functions which are defined on* $\mathbb{R}$ *and have a continuous derivative. Find* $\ker\left(\frac{d}{dx}\right)$*.*

Solution. The example asks for functions f which the property that $\frac{df}{dx} = 0$. As you may know from calculus, these functions are the constant functions. Thus $\ker\left(\frac{d}{dx}\right)$ is the set of constant functions. ♠

Definition 5.63 states that $\ker(T)$ is the set of solutions to the equation,

$$T(\vec{x}) = \vec{0}$$

Since we can write $T(\vec{x})$ as $A\vec{x}$, you have been solving such equations for quite some time.

We have spent a lot of time finding solutions to systems of equations in general, as well as homogeneous systems. Suppose we look at a system given by $A\vec{x} = \vec{b}$, and consider the related homogeneous

system. By this, we mean that we replace $\vec{b}$ by $\vec{0}$ and look at $A\vec{x} = \vec{0}$. It turns out that there is a very important relationship between the solutions of the original system and the solutions of the associated homogeneous system. In the following theorem, we use linear transformations to denote a system of equations. Remember that $T(\vec{x}) = A\vec{x}$.

Theorem 5.65: Particular Solution and General Solution

Suppose $\vec{x}_p$ *is a solution to the linear system given by ,*

$$T(\vec{x}) = \vec{b}$$

Then if $\vec{y}$ *is any other solution to* $T(\vec{x}) = \vec{b}$*, there exists* $\vec{x}_0 \in \ker(T)$ *such that*

$$\vec{y} = \vec{x}_p + \vec{x}_0$$

Hence, every solution to the linear system can be written as a sum of a particular solution, $\vec{x}_p$*, and a solution* $\vec{x}_0$ *to the associated homogeneous system given by* $T(\vec{x}) = \vec{0}$*.*

Proof. Consider $\vec{y} - \vec{x}_p = \vec{y} + (-1)\vec{x}_p$. Then $T(\vec{y} - \vec{x}_p) = T(\vec{y}) - T(\vec{x}_p)$. Since $\vec{y}$ and $\vec{x}_p$ are both solutions to the system, it follows that $T(\vec{y}) = \vec{b}$ and $T(\vec{x}_p) = \vec{b}$.

Hence, $T(\vec{y}) - T(\vec{x}_p) = \vec{b} - \vec{b} = \vec{0}$. Let $\vec{x}_0 = \vec{y} - \vec{x}_p$. Then, $T(\vec{x}_0) = \vec{0}$ so $\vec{x}_0$ is a solution to the associated homogeneous system and so is in $\ker(T)$. ♠

Sometimes people remember the above theorem in the following form. The solutions to the system

$T(x) = b$ are given by $x_p + \ker(T)$ where x_p is a particular solution to $T(x) = b$.

For now, we have been speaking about the kernel or null space of a linear transformation T. However, we know that every linear transformation T is determined by some matrix A. Therefore, we can also speak about the null space of a matrix. Consider the following example.

Example 5.66: The Null Space of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix}$$

Find $\mathrm{null}(A)$*. Equivalently, find the solutions to the system of equations* $A\vec{x} = \vec{0}$*.*

Solution. We are asked to find $\left\{ \vec{x} : A\vec{x} = \vec{0} \right\}$. In other words we want to solve the system, $A\vec{x} = \vec{0}$. Let

$\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$. Then this amounts to solving

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the linear system

$$\begin{array}{c} x + 2y + 3z = 0 \\ 2x + y + z + 2w = 0 \\ 4x + 5y + 7z + 2w = 0 \end{array}$$

To solve, set up the augmented matrix and row reduce to find the reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 4 & 5 & 7 & 2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This yields $x = \frac{1}{3}z - \frac{4}{3}w$ and $y = \frac{2}{3}w - \frac{5}{3}z$. Since $\mathrm{null}(A)$ consists of the solutions to this system, it consists vectors of the form,

$$\begin{bmatrix} \frac{1}{3}z - \frac{4}{3}w \\ \frac{2}{3}w - \frac{5}{3}z \\ z \\ w \end{bmatrix} = z \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

♠

Consider the following example.

Example 5.67: A General Solution

The ***general solution*** *of a linear system of equations is the set of all possible solutions. Find the general solution to the linear system,*

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 25 \end{bmatrix}$$

given that $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ *is one solution.*

Solution. Note the matrix of this system is the same as the matrix in Example 5.66. Therefore, from Theorem 5.65, you will obtain all solutions to the above linear system by adding a particular solution $\vec{x}_p$ to the solutions of the associated homogeneous system, $\vec{x}$. One particular solution is given above by

$$\vec{x}_p = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \tag{5.6}$$

Using this particular solution along with the solutions found in Example 5.66, we obtain the following solutions,

$$z \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Hence, any solution to the above linear system is of this form. ♠

Exercises

Exercise 5.9.1 *Write the solution set of the following system as a linear combination of vectors*

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.2 *Using Problem 5.9.1 find the general solution to the following linear system.*

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Exercise 5.9.3 *Write the solution set of the following system as a linear combination of vectors*

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.4 *Using Problem 5.9.3 find the general solution to the following linear system.*

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Exercise 5.9.5 *Write the solution set of the following system as a linear combination of vectors.*

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.6 *Using Problem 5.9.5 find the general solution to the following linear system.*

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Exercise 5.9.7 *Write the solution set of the following system as a linear combination of vectors*

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.8 *Using Problem 5.9.7 find the general solution to the following linear system.*

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Exercise 5.9.9 *Write the solution set of the following system as a linear combination of vectors*

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.10 *Using Problem 5.9.9 find the general solution to the following linear system.*

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

Exercise 5.9.11 *Write the solution set of the following system as a linear combination of vectors*

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.12 *Using Problem 5.9.11 find the general solution to the following linear system.*

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} r = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 0 \end{bmatrix}$$

Exercise 5.9.13 *Write the solution set of the following system as a linear combination of vectors*

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.14 *Using Problem 5.9.13 find the general solution to the following linear system.*

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

Exercise 5.9.15 *Write the solution set of the following system as a linear combination of vectors*

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5.9.16 *Using Problem 5.9.15 find the general solution to the following linear system.*

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

Exercise 5.9.17 *Suppose $A\vec{x} = \vec{b}$ has a solution. Explain why the solution is unique precisely when $A\vec{x} = \vec{0}$ has only the trivial solution.*