HUMBER ENGINEERING

MENG 3510 – Control Systems
LECTURE 8

HUMBER

WE ARE HUMBER

LECTURE 8
Stability Analysis & Controller Design via State-Space

- Review of State-Space Representation of LTI Systems
  - Converting from Transfer Function to State-Space
  - Converting form State-Space to Transfer Function
- Stability Analysis via State-Space Equations
- Control System Design via State-Space Equations
  - Controllability
  - State Feedback Control Design
  - State Feedback with Integral Control Design

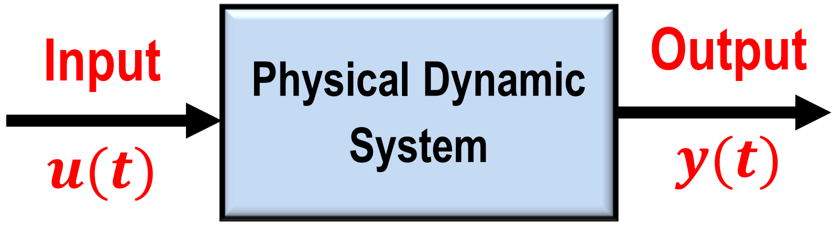
Introduction

- In general, there are two approaches to analyze and design the control systems:
  - Classical Control Methods
  - Modern Control Methods

	Classical Control Methods	Modern Control Methods
Modeling	○ Transfer function model ○ Provides a model in Laplace domain ○ Provides only input-output description (External description) ○ Limited to LTI and SISO systems	○ State space representation ○ Provides a model in time domain ○ Provides internal description of the system via state variables ○ Applicable for time-varying, nonlinear, and MIMO systems
Analysis	○ Based on graphical techniques such as Root-locus plots, Bode plots and Nyquist plots	○ Based on analytical formula, not graphical methods
Design	○ Classical design techniques are not applicable to design Optimal and Adaptive control systems	○ Applicable to design Optimal control systems, Robust control, Adaptive control, Predictive control and …

Review of State-Space Representation of LTI Systems

- In State Space Representation, dynamic model of the system is described by a set of first-order differential equations in terms of the variables called the **state variables**.



- General form of a State Space Representation for a nth order LTI system with m input and p output is:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) & \longleftarrow \quad \textbf{State Equation} \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) & \longleftarrow \quad \textbf{Output Equation} \end{cases}$$

$\mathbf{A}$: system matrix $(n \times n)$	$\mathbf{B}$: input matrix $(n \times m)$	$\mathbf{x}(t)$: state vector $(n \times 1)$	
$\mathbf{C}$: output matrix $(p \times n)$	$\mathbf{D}$: feed-forward matrix $(p \times m)$	$\mathbf{u}(t)$: input vector $(m \times 1)$	$\mathbf{y}(t)$: output vector $(p \times 1)$

- The **state variables** of a dynamic system are the minimum set of linearly independent variables that describes the effect of the history of the system (past inputs and dynamics) on its response in the future.

- The minimum number of required state variables equals the **order** of the differential equation describing the system.

- There is **no unique** set of state variables that describe any given system; many different sets of state variables may be selected to obtain a complete system description.

- The state variables may be selected based on physical and measurable variables, or in terms of variables that are not directly measurable.

Review of State-Space Representation of LTI Systems

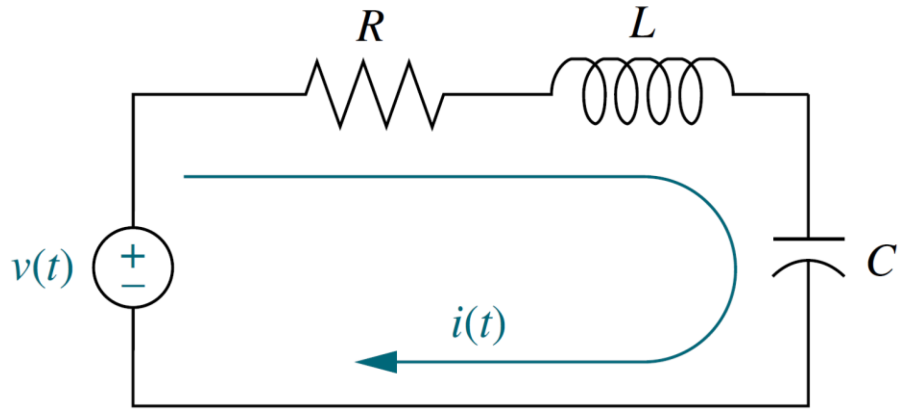
- In physical dynamic systems it is often convenient to associate the state variables with the energy storage elements in the system. Because any energy that is initially stored in these elements can affect the response of the system at a later time.

System	Element	Energy	Physical Variable
Electrical Systems	Capacitor C	$\frac{1}{2}Cv^2$	Voltage $v(t)$
	Inductor L	$\frac{1}{2}Li^2$	Current $i(t)$
Translational Mechanical Systems	Mass M	$\frac{1}{2}Mv^2$	Translational Velocity $v(t)$
	Translational Spring K	$\frac{1}{2}Kx^2$	Translational Displacement $x(t)$
Rotational Mechanical Systems	Moment of Inertia J	$\frac{1}{2}J\omega^2$	Angular Velocity $\omega(t)$
	Tortional Spring K	$\frac{1}{2}K\theta^2$	Angular Displacement $\theta(t)$

Review of State-Space Representation of LTI Systems

Example 1

Find the state-space model for the given RLC network.
Assume the applied voltage $v(t)$ as the input, and the capacitor voltage $v_c(t)$ is the output



Apply KVL to find the differential equation describes dynamics of the system

$$v(t) = v_R(t) + v_L(t) + v_C(t) \quad \rightarrow \quad v(t) = Ri(t) + L\dot{i}\,(t) + v_C(t)$$

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$$v_L(t) = L\frac{di_L(t)}{dt}$$

The state variables x_1 and x_2 are selected as the inductor current $i_L(t)$ and capacitor voltage $\mathrm{v_c}(t)$.

$$x_1(t) = i_L(t) \;\rightarrow\; \dot{x}_1(t) = \dot{i}_L(t) \;\rightarrow\; \dot{x}_1(t) = \frac{1}{L}v(t) - \frac{R}{L}i(t) - \frac{1}{L}v_c(t)$$

$$\rightarrow\; \dot{x}_1(t) = \frac{1}{L}v(t) - \frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \qquad \text{Eqn. (1)}$$

$$x_2(t) = v_c(t) \;\rightarrow\; \dot{x}_2(t) = \dot{v}_c(t) \;\rightarrow\; \dot{x}_2(t) = \frac{1}{C}i(t) \;\rightarrow\; \dot{x}_2(t) = \frac{1}{C}x_1(t) \qquad \text{Eqn. (2)}$$

State-variable equations are obtained as:

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{L}v(t) - \dfrac{R}{L}x_1(t) - \dfrac{1}{L}x_2(t) \\ \dot{x}_2(t) = \dfrac{1}{C}x_1(t) \end{cases}$$

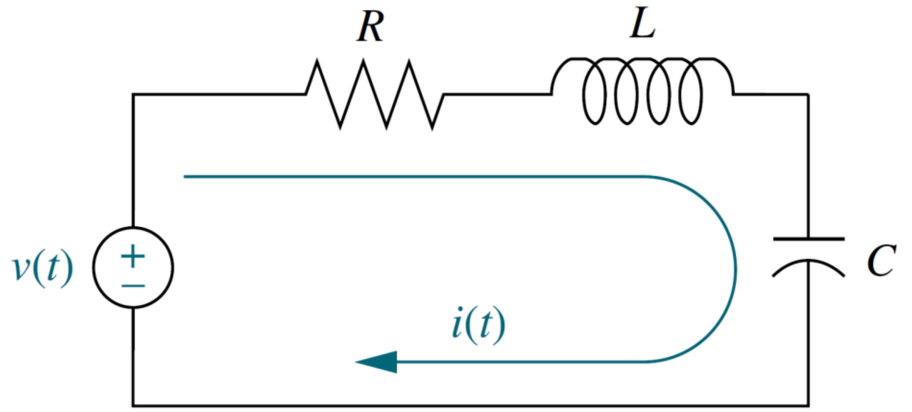
The **output equation** is obtained as:

$$v_c(t) = x_2(t)$$

Review of State-Space Representation of LTI Systems

Example 1

Find the state-space model for the given RLC network.
Assume the applied voltage $v(t)$ as the input, and the capacitor voltage $v_c(t)$ is the output



Having the state and output equations:

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{L}v(t) - \dfrac{R}{L}x_1(t) - \dfrac{1}{L}x_2(t) \\ \dot{x}_2(t) = \dfrac{1}{C}x_1(t) \\ v_c(t) = x_2(t) \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)} \rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t)$$

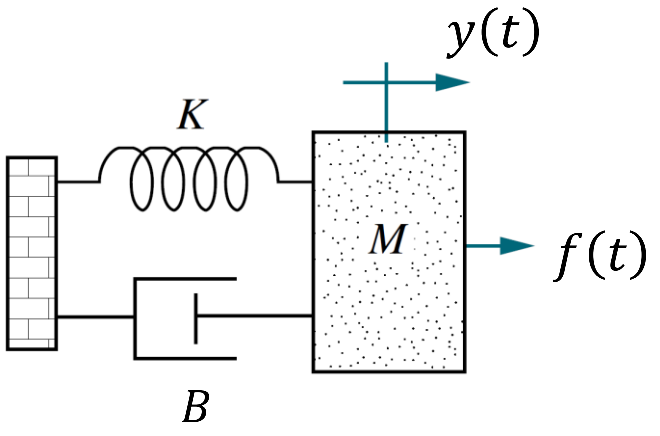
State Equation

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)} \rightarrow y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

Output Equation

Review of State-Space Representation of LTI Systems

Example 2 Find the state-variable equations for the given mass-spring-damper system. Assume the applied force $f(t)$ as the input, and the displacement $y(t)$ is the output



Draw the free-body diagram of the system and write the differential equation of the system.

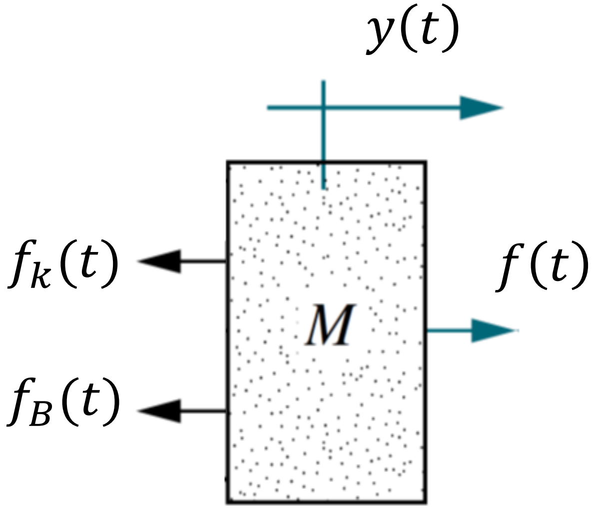
$$f(t) - f_k(t) - f_B(t) = Ma(t) \quad \rightarrow \quad f(t) - Ky(t) - B\dot{y}(t) = M\ddot{y}(t)$$

The state variables x_1 and x_2 are selected as the displacement of the spring $y(t)$ and velocity of the mass $\dot{y}(t)$.

$$x_1(t) = y(t) \quad \rightarrow \quad \dot{x}_1(t) = \dot{y}(t) \quad \rightarrow \quad \dot{x}_1(t) = x_2(t) \quad \text{Eqn. (1)}$$

$$x_2(t) = \dot{y}(t) \quad \rightarrow \quad \dot{x}_2(t) = \ddot{y}(t) \quad \rightarrow \quad \dot{x}_2(t) = \frac{1}{M}f(t) - \frac{K}{M}y(t) - \frac{B}{M}\dot{y}(t)$$

$$\rightarrow \quad \dot{x}_2(t) = \frac{1}{M}f(t) - \frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) \quad \text{Eqn. (2)}$$



State-variable equations are obtained as:

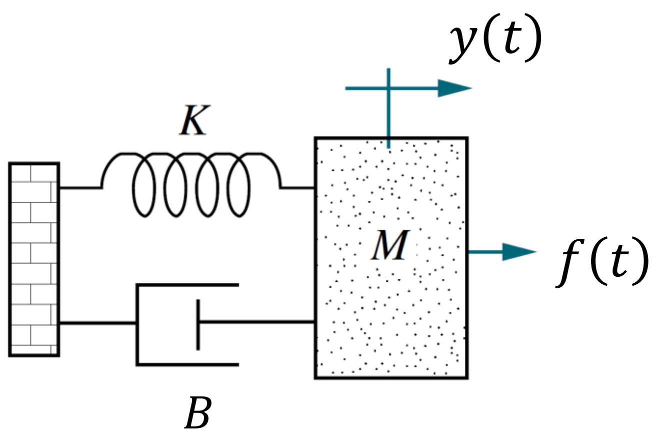
$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{1}{M}f(t) - \frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) \end{cases}$$

The **output equation** is obtained as:

$$y(t) = x_1(t)$$

Review of State-Space Representation of LTI Systems

Example 2 Find the state-variable equations for the given mass-spring-damper system.
Assume the applied force $f(t)$ as the input, and the displacement $y(t)$ is the output



Having the state and output equations::

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{1}{M} f(t) - \frac{K}{M} x_1(t) - \frac{B}{M} x_2(t) \\ y(t) = x_1(t) \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

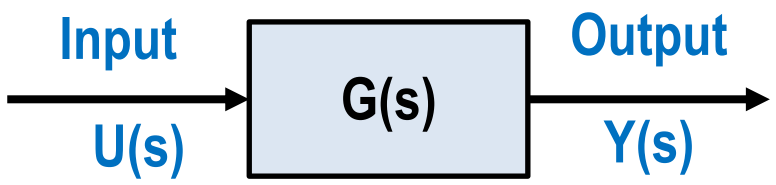
$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)} \rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t)$$

State Equation

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)} \rightarrow y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

Output Equation

Converting From Transfer Function to State-Space



- Consider an LTI, SISO system with the transfer function of G(s):

$$G(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

- Determining the state space representation from the transfer function is called **realization**.
- A transfer function is realizable if and only if the transfer function is proper or strictly proper.

Strictly Proper Systems ($m < n$)

- G(s) can be realized with minimum of *n* state variables as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

In this case $\rightarrow$ $\mathbf{D} = \mathbf{0}$

Proper Systems ($m = n$)

- G(s) can be realized with minimum of *n* state variables as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

In this case $\rightarrow$ $\mathbf{D} = \lim_{s \to \infty} G(s)$

- General idea is deriving the differential equation from the given transfer function and then realizing the state space equations from the differential equation.

Converting From Transfer Function to State-Space

Example 3 Determine the state space representation of the following transfer function. Draw a block diagram to visualize the state variables.

$$\boldsymbol{G(s) = \frac{Y(s)}{U(s)} = \frac{12}{s^3 + 5s^2 + 11s + 8}}$$

First, find the associated differential equation

$$s^3Y(s) + 5s^2Y(s) + 11sY(s) + 8Y(s) = 12U(s) \longrightarrow \dddot{y}(t) + 5\ddot{y}(t) + 11\dot{y}(t) + 8y(t) = 12u(t)$$

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{y}(t) \\ x_3(t) = \ddot{y}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = \dot{y}(t) \\ \dot{x}_2(t) = \ddot{y}(t) \\ \dot{x}_3(t) = \dddot{y}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = -8x_1(t) - 11x_2(t) - 5x_3(t) + 12u(t) \\ \text{Output} \rightarrow \quad y(t) = x_1(t) \end{cases}$$

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)} \longrightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -11 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} u(t)$$

State Equation

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)} \longrightarrow y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

Output Equation

Since G(s) is a strictly proper transfer function, $\mathbf{D} = \mathbf{0}$

Converting From Transfer Function to State-Space

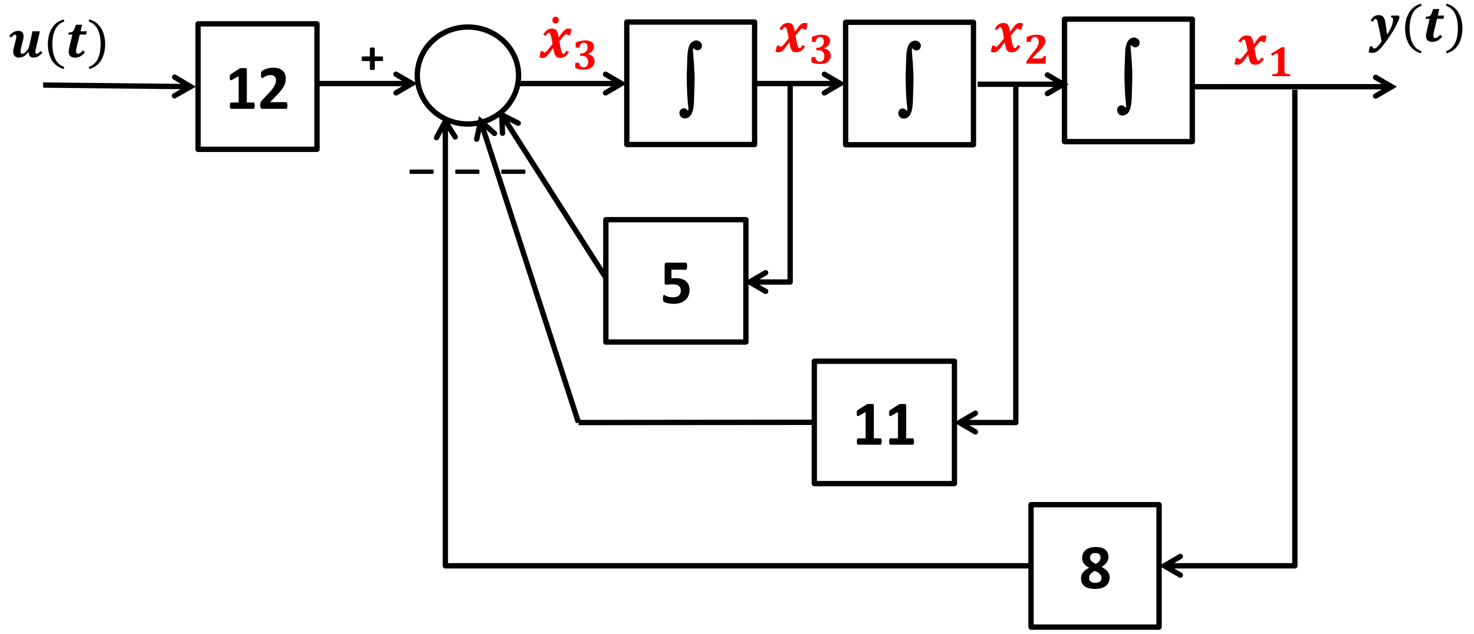
Example 3 Determine the state space representation of the following transfer function.
Draw a block diagram to visualize the state variables.

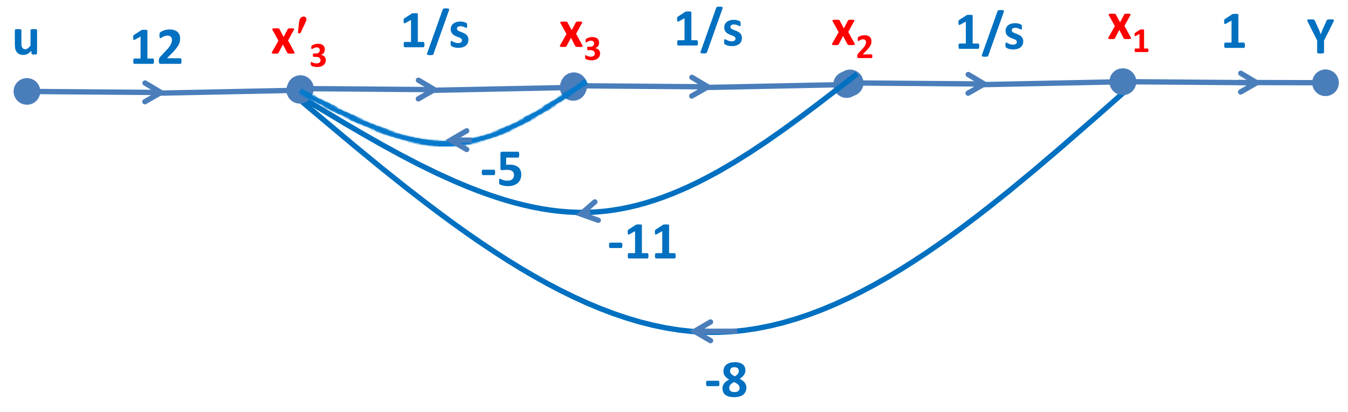
$$G(s) = \frac{Y(s)}{U(s)} = \frac{12}{s^3 + 5s^2 + 11s + 8}$$

The following block diagram and signal flow graph visualize the state variables.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = -8x_1(t) - 11x_2(t) - 5x_3(t) + 12u(t) \end{cases}$$

Output → $y(t) = x_1(t)$





This signal flow graph is called State Diagram.

Converting From Transfer Function to State-Space

❑ Strictly Proper Transfer Function with No Zeros

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

- In this case numerator of the transfer function is a constant number.
- The nth-order differential equation does not include input signal derivatives

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t)$$

- The nth-order state space equation is obtained as

$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$ →

State Equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u(t)$$

$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$ →

Output Equation

$$y(t) = [1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [0]u(t)$$

This state variables are called Phase Variables.

Converting From Transfer Function to State-Space

❑ Strictly Proper Transfer Function with Zeros

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

- In this case the numerator is a mth-order polynomial with $m < n$
- The nth-order differential equation includes input signal derivatives

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

- The nth-order state space equation is obtained as

$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$ →

State Equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$ →

Output Equation

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [0]u(t)$$

Converting From Transfer Function to State-Space

Example 4 Determine the state space representation of the following transfer function. Draw a block diagram to visualize the state variables.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{6s + 4}{s^3 + 2s^2 + 10s + 9}$$

This is a third order system. The state and output equations are obtained based on the general format:

State Equation $\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \longrightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -10 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Output Equation $\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t) \longrightarrow y(t) = \begin{bmatrix} 4 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

Since G(s) is a strictly proper transfer function $\rightarrow$ $\mathbf{D} = \mathbf{0}$

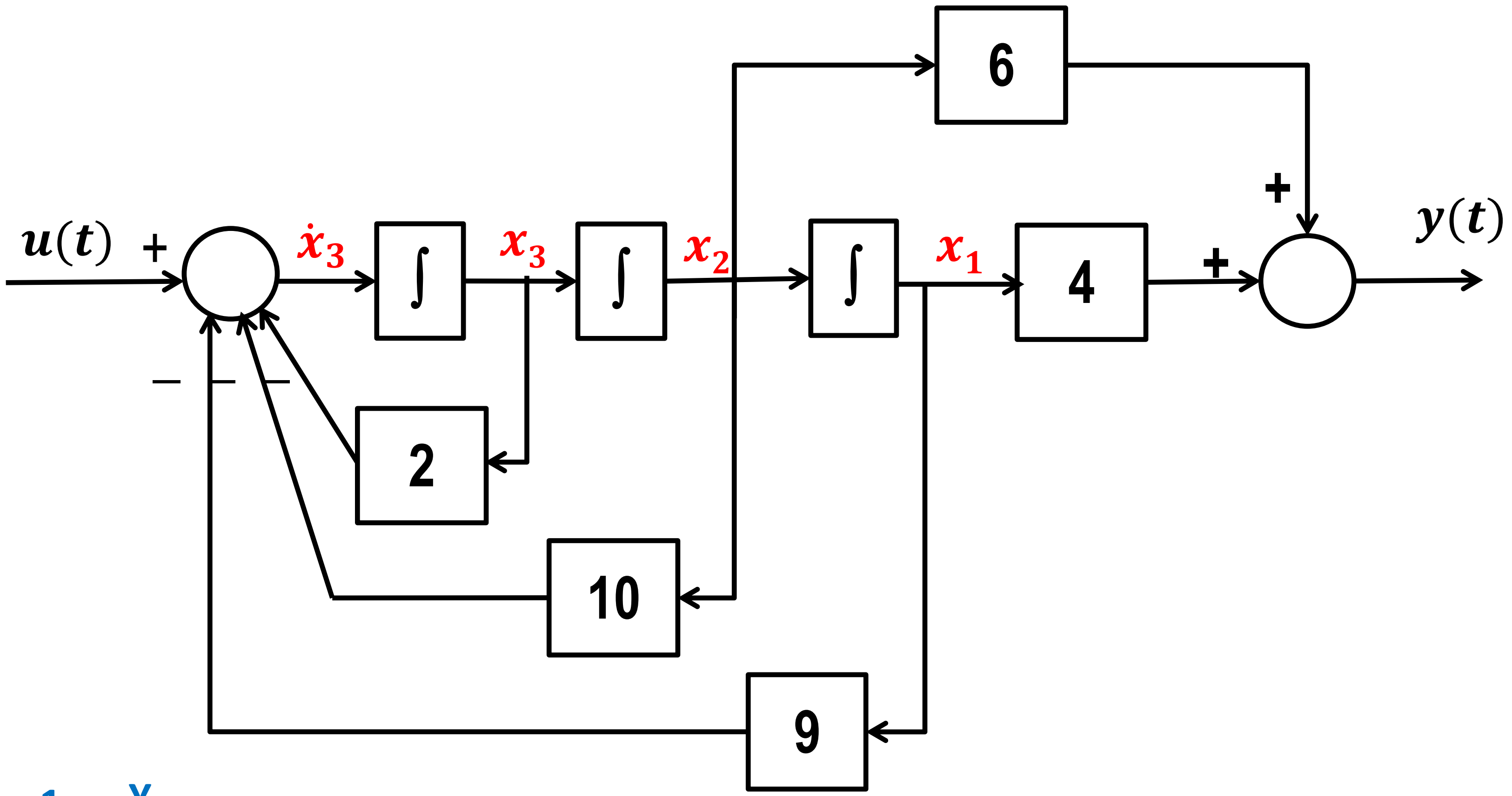
Converting From Transfer Function to State-Space

Example 4 Determine the state space representation of the following transfer function. Draw a block diagram to visualize the state variables.

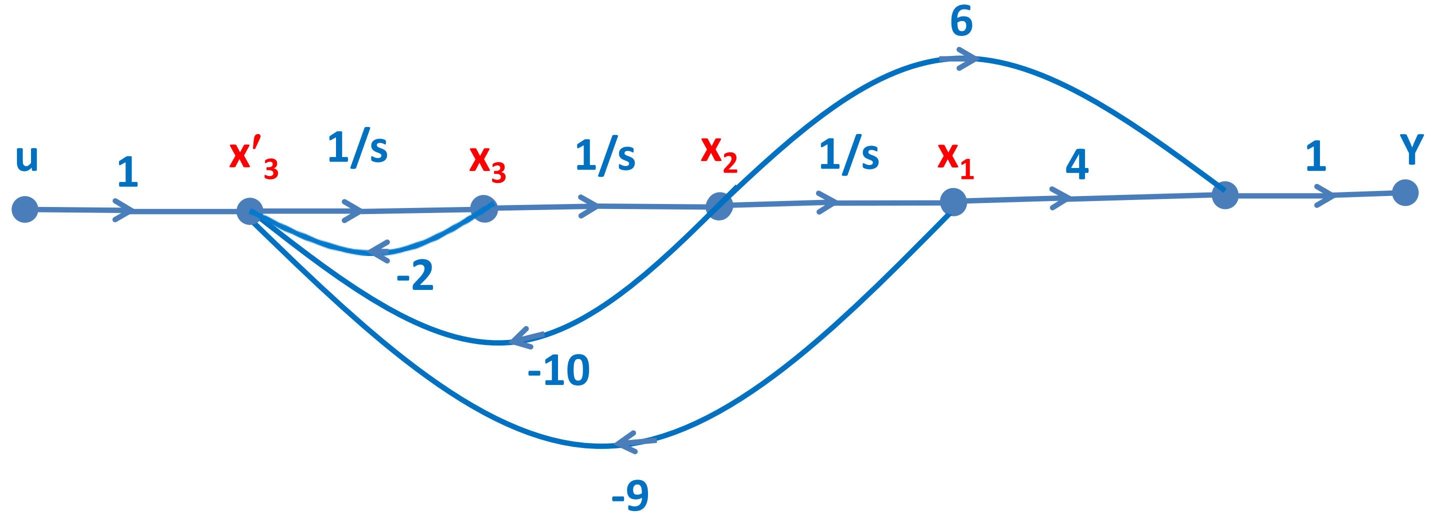
$$G(s) = \frac{Y(s)}{U(s)} = \frac{6s+4}{s^3+2s^2+10s+9}$$

Following block diagram visualizes the state variables.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = -9x_1(t) - 10x_2(t) - 2x_3(t) + u(t) \\ \text{Output} \rightarrow \quad y(t) = 6x_2(t) + 4x_1(t) \end{cases}$$



The state diagram of the system is



Converting From Transfer Function to State-Space

❑ Proper Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

- In this case the numerator is a mth-order polynomial with $m = n$
- The nth-order differential equation includes input signal derivatives

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

- The nth-order state space equation is obtained as

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

State Equation

$$\rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$$

Output Equation

$$\rightarrow y(t) = \begin{bmatrix} b_0 - b_m a_0 & b_1 - b_m a_1 & \cdots & b_{m-1} - b_m a_{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [b_m]u(t)$$

$$\boxed{D = \lim_{s\to\infty} G(s) = b_m}$$

Converting From Transfer Function to State-Space

Example 5 Determine the state space representation of the following transfer function.
Draw a block diagram to visualize the state variables.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 12s^2 + 6s + 3}{s^3 + 4s^2 + 15s + 7}$$

This is a third order system. The state and output equations are obtained based on the general format:

State Equation $\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \longrightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -15 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Output Equation $\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$

$$y(t) = [b_0 - b_3 a_0 \quad b_1 - b_3 a_1 \quad b_2 - b_3 a_2] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [b_3] u(t) \longrightarrow y(t) = [-11 \quad -24 \quad 4] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [2] u(t)$$

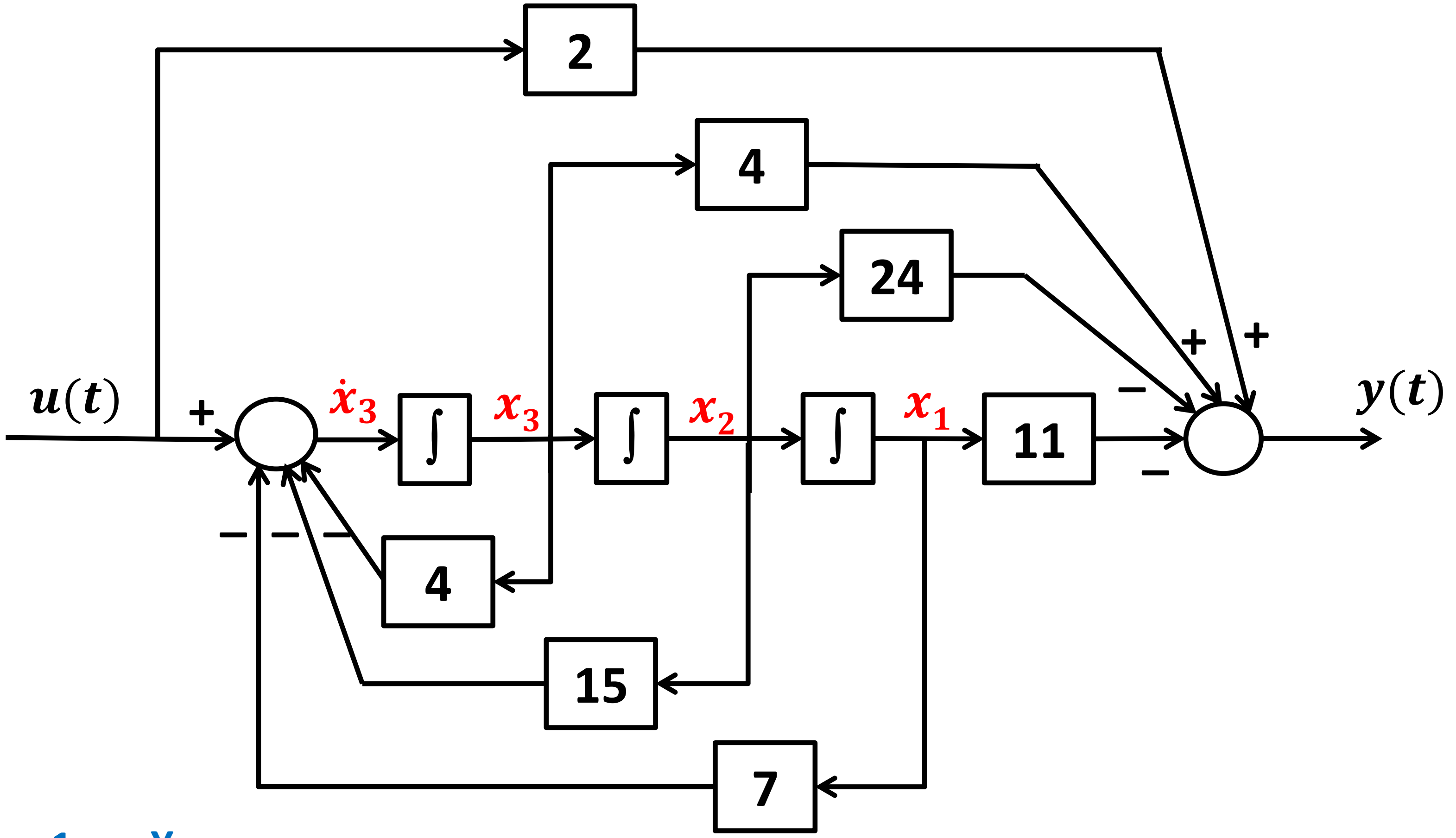
Converting From Transfer Function to State-Space

Example 5 Determine the state space representation of the following transfer function. Draw a block diagram to visualize the state variables.

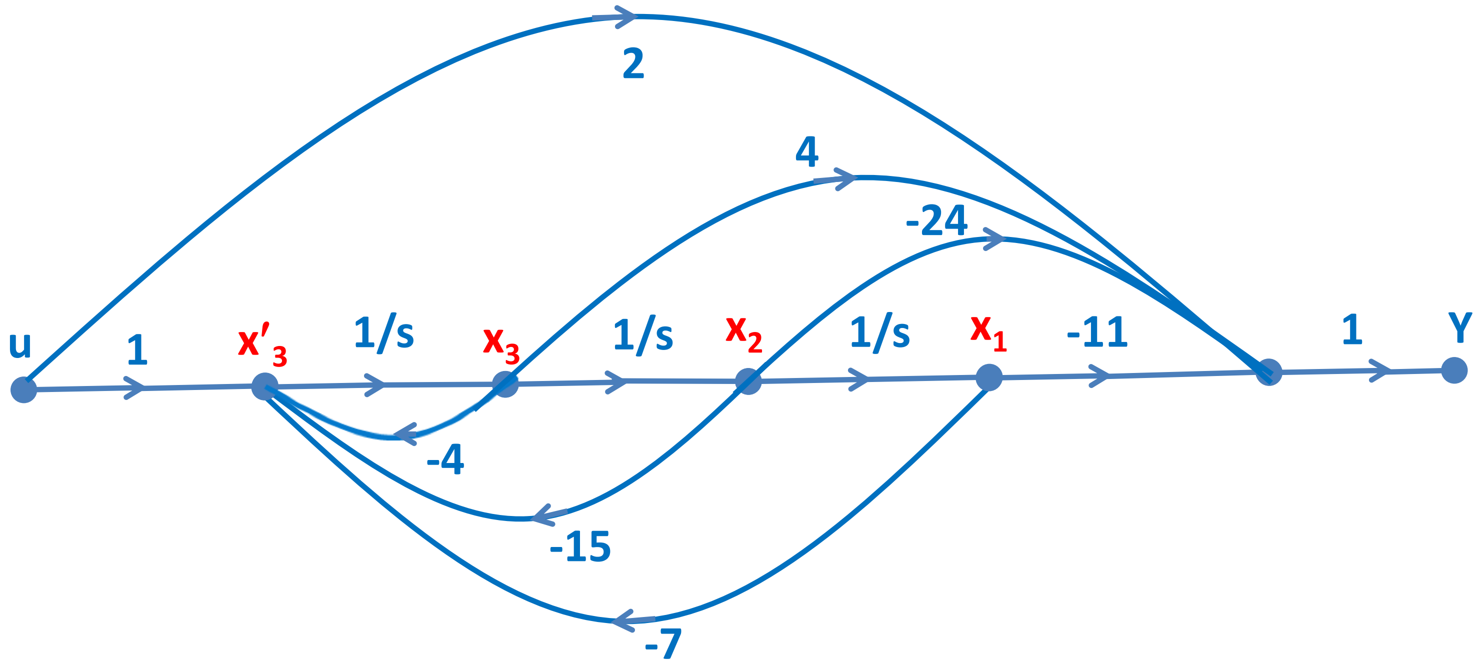
$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 12s^2 + 6s + 3}{s^3 + 4s^2 + 15s + 7}$$

The following block diagram visualizes the state variables.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = -7x_1(t) - 15x_2(t) - 4x_3(t) + u(t) \\ y(t) = 4x_3(t) - 24x_2(t) - 11x_1(t) + 2u(t) \end{cases}$$



The state diagram of the system is



Converting From State-Space to Transfer Function

- Determining the transfer function from the state space representation is called **reconstruction**.
- Consider a LTI system with the state space representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

- Take Laplace transform of the state space equations assuming the zero initial condition

$$\begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \longrightarrow (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \longrightarrow \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \longrightarrow \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathbf{Y}(\mathbf{s}) = \underbrace{[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]}_{\text{Transfer Function Matrix}}\ \mathbf{U}(s)$$

- In **MIMO** LTI systems Transfer function matrix is a matrix array, which relates the output vector $\mathbf{Y}(s)$ to the input vector $\mathbf{U}(s)$
- For **SISO** LTI systems the transfer function is obtained as below

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

Converting From State-Space to Transfer Function

Reminder: Matrix Inverse

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$$

For 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \rightarrow \quad A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

For 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\mathrm{adj}(A) = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -(a_{12}a_{33} - a_{13}a_{32}) & a_{12}a_{23} - a_{13}a_{22} \\ -(a_{21}a_{33} - a_{23}a_{31}) & a_{11}a_{33} - a_{13}a_{31} & -(a_{11}a_{23} - a_{21}a_{13}) \\ a_{21}a_{32} - a_{22}a_{31} & -(a_{11}a_{32} - a_{12}a_{31}) & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

Converting From State-Space to Transfer Function

Example 6

Consider the following state space representation of a system

Determine transfer function of the system.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-4 & -3\\ 1 & -5\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}3\\ 6\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [0]u(t)$$

The transfer function is determined by the following formula:

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

First, find the $(s\mathbf{I} - \mathbf{A})^{-1}$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}-4 & -3\\ 1 & -5\end{bmatrix} = \begin{bmatrix}s+4 & 3\\ -1 & s+5\end{bmatrix} \rightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2+9s+23}\begin{bmatrix}s+5 & -3\\ 1 & s+4\end{bmatrix}$$

Substitute the $(s\mathbf{I} - \mathbf{A})^{-1}$, $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{D}$ in the transfer function formula

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+9s+23}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}s+5 & -3\\ 1 & s+4\end{bmatrix}\begin{bmatrix}3\\ 6\end{bmatrix} = \frac{1}{s^2+9s+23}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}3s-3\\ 6s+27\end{bmatrix} = \frac{1}{s^2+9s+23}(3s-3+12s+54)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{15s+51}{s^2+9s+23}}$$

Transfer Function

Converting From State-Space to Transfer Function

Example 7

Consider the following state space representation of a system

Determine transfer function of the system.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -15 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -11 & -24 & 4 \end{bmatrix} \mathbf{x}(t) + [2] u(t)$$

The transfer function is determined by the following formula

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \mathrm{C}(s\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D}}$$

$$(s\mathrm{I} - \mathrm{A})^{-1} = \frac{\mathrm{adj}(s\mathrm{I} - \mathrm{A})}{\det(s\mathrm{I} - \mathrm{A})}$$

First, find the $(s\mathrm{I} - \mathrm{A})^{-1}$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -15 & -4 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 7 & 15 & s+4 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3 + 4s^2 + 15s + 7} \begin{bmatrix} s^2 + 4s + 15 & s+4 & 1 \\ -7 & s(s+4) & s \\ -7s & -(15s+7) & s^2 \end{bmatrix}$$

Converting From State-Space to Transfer Function

Example 7

Consider the following state space representation of a system

Determine transfer function of the system.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -15 & -4 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} -11 & -24 & 4 \end{bmatrix}\mathbf{x}(t) + [2]u(t)$$

Substitute the $(s\mathbf{I}-\mathbf{A})^{-1}, \mathbf{C}, \mathbf{B}$ and $\mathbf{D}$ in the transfer function formula:

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3+4s^2+15s+7}\begin{bmatrix} -11 & -24 & 4 \end{bmatrix}\begin{bmatrix} s^2+4s+15 & s+4 & 1 \\ -7 & s(s+4) & s \\ -7s & -(15s+7) & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 2$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3+4s^2+15s+7}\begin{bmatrix} -11 & -24 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} + 2 = \frac{4s^2-24s-11}{s^3+4s^2+15s+7} + 2 = \frac{4s^2-24s-11+2s^3+8s^2+30s+14}{s^3+4s^2+15s+7}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{2s^3+12s^2+6s+3}{s^3+4s^2+15s+7}}$$

Transfer Function

Stability Analysis via State-Space Equations

Characteristic Polynomial and Eigenvalues

- Consider a LTI system with the following state space equations

$$\begin{cases} \dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) = \mathbf{C}\mathbf{x}(\mathbf{t}) + \mathbf{D}\mathbf{u}(\mathbf{t}) \end{cases}$$

- **Characteristic polynomial** of the system matrix is obtained as below

Characteristic Polynomial $\boxed{\det(\lambda\mathbf{I} - \mathbf{A}) = 0}$

- Given matrix $\mathbf{A}_{n\times n}$ with real arrays, the characteristic polynomial is a nth order monic polynomial with real coefficients

$$\boxed{\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0}$$

- The roots of the characteristic equation are called **eigenvalues** of the matrix **A**.
- The eigenvalues are always **real** or **complex conjugate** numbers.

Stability Analysis via State-Space Equations

- Consider a LTI system with the following state-space equations and characteristic polynomial

$$\begin{cases} \dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) = \mathbf{C}\mathbf{x}(\mathbf{t}) + \mathbf{D}\mathbf{u}(\mathbf{t}) \end{cases}$$

$$\boxed{\det(\lambda \mathbf{I} - \mathbf{A}) = 0}$$

Characteristic Polynomial

- Recall the reconstruction formula to obtain the transfer function:

$$\boxed{\frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(\frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right)\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A})}{\underbrace{\det(s\mathbf{I} - \mathbf{A})}_{\text{Characteristic Equation}}}$$

- Therefore, $\det(s\mathbf{I} - \mathbf{A}) = 0$ is the characteristic equation of the system, which is identical to the characteristic polynomial of the matrix **A**

- Therefore, the eigenvalues of the matrix **A** are identical to the system's poles with no pole-zero cancellation.

Stability of the system is analyzed by checking the eigenvalues of matrix A

Stability Analysis via State-Space Equations

Example 8 Determine stability of the following system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

First, find the characteristic polynomial of the matrix **A**

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{vmatrix} = \boxed{s^3 + 6s^2 + 11s + 6}$$ Characteristic Polynomial

Next, create the Routh-Hurwitz table for the characteristic equation.

s^3	1	11
s^2	6	6
s^1	10	0
s^0	6	0

Since there is no sign change in the first column all the eigenvalues are located on the left-half of the s-plane.
Therefore, the system is stable.

Stability Analysis via State-Space Equations

Example 8

Determine stability of the following system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

We can also determine the characteristic polynomial and the eigenvalues in MATLAB

```
A = [0 1 0;0 0 1;-6 -11 -6];
poly(A)
ans =

     1.0000     6.0000    11.0000     6.0000


eig(A)
ans =

   -1.0000
   -2.0000
   -3.0000
```

Characteristic Polynomial:

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda + 1)(\lambda + 2)(\lambda + 3)$$

Eigenvalues:

$$\boxed{\begin{aligned} \lambda_1 &= -1, \\ \lambda_2 &= -2, \\ \lambda_3 &= -3 \end{aligned}}$$

Control System Design
via
State Space Equations

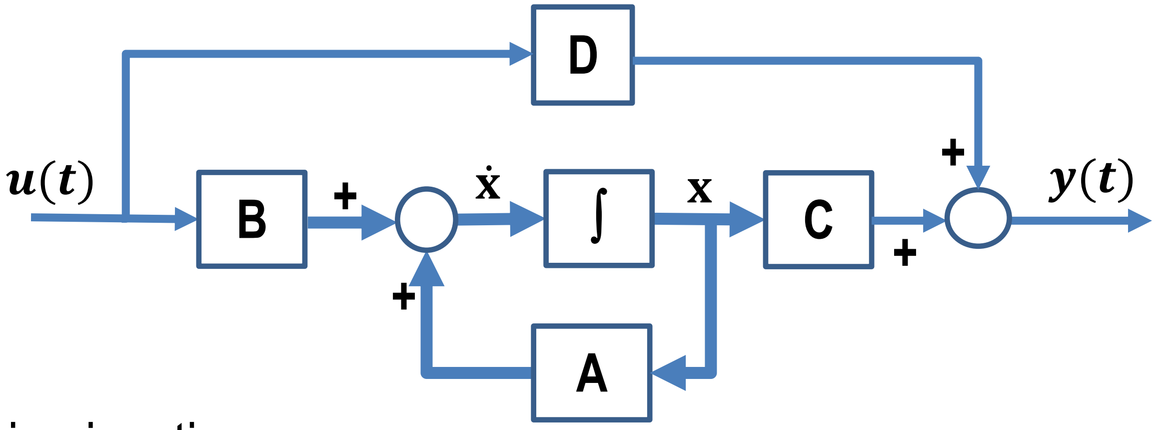
Control System Design via State-Space Equations

- One of the attractive features of the state-space design method is that under the certain conditions it is possible to assign the system eigenvalues to arbitrary values by designing appropriate feedback of the system **states**.
- The design technique is called **Pole-placement.**
- In Classics Control Design techniques, the idea is to locate the dominant poles at the desired locations, but the Pole-placement method specifies **all** closed-loop poles at the desired locations.
- There are two main approaches for Pole-placement method:
  - **State-Feedback Control**
    - The idea is to design a regulator to stabilize and regulate the system.
  - **State-Feedback with Integral Control**
    - The idea is to design a controller for tracking purposes.

State Feedback Control Design

- Consider the state-space representation of a LTI system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

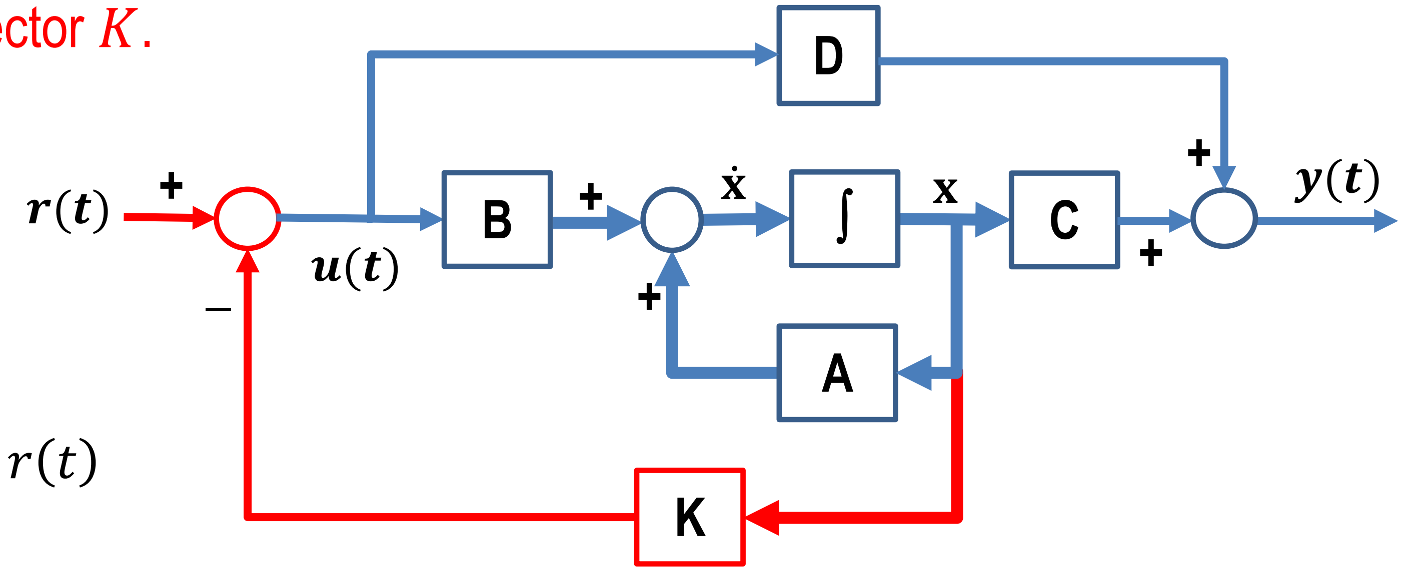


- In a typical feedback control system, the output, $y(t)$, is fed back to the summing junction.
- In a state feedback control instead of feeding back $y(t)$, we feed back **all** the state variables.
- If each state variable is fed back to the control, $u(t)$, through a gain, k, there would be n gains, k, that could be adjusted to yield the required closed-loop pole values.
- The feedback through the gains, k, is represented by a feedback vector K.
- The **State Feedback Control Law** is defined as below:

$$\boxed{u(t) = -\mathbf{K}\mathbf{x}(t) + r(t)}$$

State Feedback Gain

$$u(t) = -\underbrace{\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}}_{1 \times n \text{ vector}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + r(t)$$

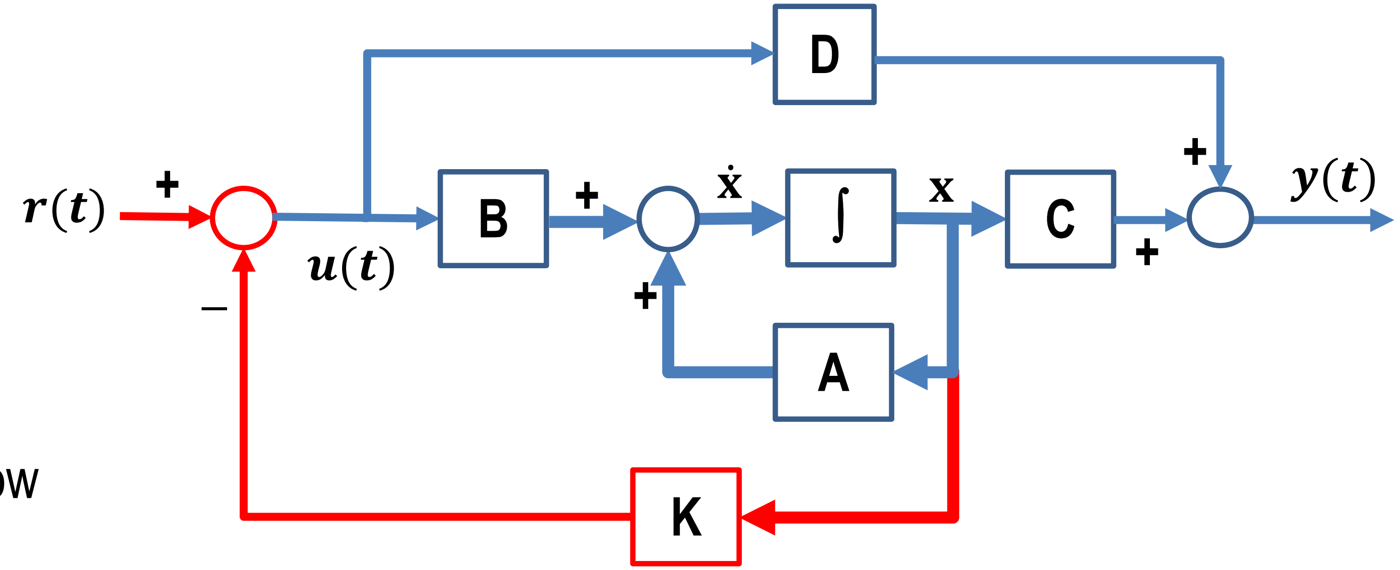


State Feedback Control Design

State Feedback Closed-Loop System

- Recall the State Feedback Control Law :

$$\boxed{u(t) = -\mathbf{K}\mathbf{x}(t) + r(t)}$$



- State space equations of the closed-loop system is determined as below

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{cases} \xrightarrow{u(t) = -\mathbf{K}\mathbf{x}(t) + r(t)} \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}[-\mathbf{K}\mathbf{x}(t) + r(t)] \\ y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}[-\mathbf{K}\mathbf{x}(t) + r(t)] \end{cases}$$

Closed-loop System

$$\boxed{\begin{cases} \dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}r(t) \\ y(t) = (\mathbf{C} - \mathbf{D}\mathbf{K})\mathbf{x}(t) + \mathbf{D}r(t) \end{cases}}$$

$$\boxed{\det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = 0}$$

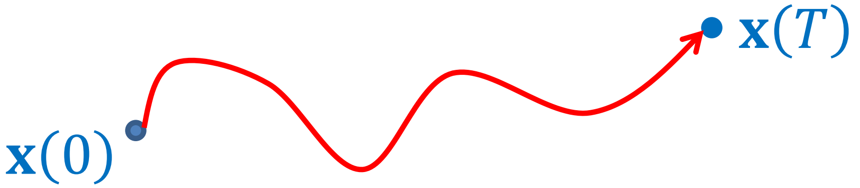
Closed-loop Characteristic Equation

- In state feedback control we can locate the system eigenvalues/poles in any arbitrary location in the s-plane, if the system is **Controllable**.

State Feedback Control Design

❑ Controllability

- Controllability means we can move the state variables in any desired direction in a finite time T by a suitable choice of control signal. In this case the system is called **Controllable.**



- Equivalently, in a controllable system, the system **eigenvalues/poles** can be moved to any desired locations by the state feedback control design.

- Controllability of the system in state-space model of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ can be determined by checking rank of the following matrix:

Controllability Matrix $(\boldsymbol{n} \times \boldsymbol{n})$

$$\boxed{\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \mathbf{A}^3\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]}$$

- The pair $[\mathbf{A} \quad \mathbf{B}]$ is called controllable, if $\mathbf{Q_c}$ is a full rank matrix, (has rank *n*). Full rank means $\mathbf{det}[\mathbf{Q_c}] \neq \mathbf{0}$
- **Definition:** The **rank** of a matrix is the maximum number of linearly independent column (or row) vectors in the matrix.

State Feedback Control Design

Example 9 Consider the following state space representation of a third order system

Determine the controllability of the system.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Find the controllability matrix of the system and check the determinant

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B}] \longrightarrow \mathbf{Q_c} = \begin{bmatrix} 0 & -1 & -4 \\ 0 & 0 & 0 \\ 1 & 3 & 8 \end{bmatrix} \longrightarrow \det(\mathbf{Q_c}) = 0$$

Controllability matrix is **not** a full rank matrix.

Therefore, the system is not controllable.

The matrix $\mathbf{Q_c}$ has only two linearly independent row vectors.

$$\mathrm{rank}(\mathbf{Q_c}) = 2$$

- We can also determine the controllability matrix by using the `ctrb` function in MATLAB and checking **rank** of the matrix.

```
A = [1  2  -1;0  1  0;1  -4  3];
B = [0; 0; 1];
Qc = ctrb(A,B)
Qc =
     0    -1    -4
     0     0     0
     1     3     8

rank(Qc)
ans =
     2
```

State Feedback Control Design

Example 9 Consider the following state space representation of a third order system

Determine the controllability of the system.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}2 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

We can determine the transfer function model and partial fraction expansion of it to better understand the uncontrollable modes of the system.

```
A = [1 2 -1;0 1 0;1 -4 3];
B = [0;0;1];
C = [2 0 1];
D = 0;
[num,den] = ss2tf(A,B,C,D);
G = tf(num,den)
G =

      s^2 - 4 s + 3
  ---------------------
  s^3 - 5 s^2 + 8 s - 4
eig(A)
ans =
    2.0000
    2.0000
    1.0000
```

Applying partial fraction expansion:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^2 - 4s + 3}{s^3 - 5s^2 + 8s - 4} = \frac{(s-1)(s-3)}{(s-1)(s-2)^2} = \frac{0}{s-1} + \frac{1}{s-2} + \frac{-1}{(s-2)^2}$$

- The mode $s-1$ is **not controllable** by the control signal $u(t)$, which means the pole at $s=1$ cannot be relocated or controlled by $u(t)$. Thus, the system is uncontrollable.
- The uncontrollable modes appear due to the pole-zero cancellation.

State Feedback Control Design

❑ State Feedback Controller Design Procedure

$$\begin{cases} \dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{B}r(t) \\ y(t) = (\mathbf{C} - \mathbf{DK})\mathbf{x}(t) + \mathbf{D}r(t) \end{cases}$$

Step 1: Check controllability of the open-loop system using the controllability matrix.

$$\boxed{\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \mathbf{A}^3\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]}$$

If $\mathbf{Q_c}$ is a full rank matrix ($\det[\mathbf{Q_c}] \neq 0$), the open-loop system is controllable.

Step 2: Determine the desired eigenvalue/pole locations and the desired characteristic polynomial, based on the desired performance criteria.

Step 3: Obtain the closed-loop system matrix and determine the characteristic polynomial

$$\boxed{\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK}} \quad \longrightarrow \quad \boxed{\det(s\mathbf{I} - \mathbf{A}_{cl})}$$

where $\mathbf{K} = [k_1 \quad k_2 \quad \cdots \quad k_n]$ is the state feedback gain vector

Step 4: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the state feedback gain value $\mathbf{K}$.

State Feedback Control Design

Example 10 Consider the following state space representation of a third-order system in canonical controllable form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 30 & -31 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 30 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

a) Determine the open-loop system poles and check stability of the system.

First, find the characteristic polynomial of the matrix A, and the eigenvalues/poles

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 30 & -31 & 10 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -30 & 31 & s-10 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -30 & 31 & s-10 \end{vmatrix} = \boxed{\boldsymbol{s^3 - 10s^2 + 31s - 30}}$$

Open-loop Characteristic Polynomial

$$s^3 - 10s^2 + 31s - 30 = 0 \quad \rightarrow \quad \boxed{\boldsymbol{s_1 = 5, \quad s_2 = 3, \quad s_3 = 2}}$$

Open-loop Poles

The eigenvalues/poles are located on the right-half of the s-plane.

Therefore, the system is **unstable**.

State Feedback Control Design

Example 10

Consider the following state space representation of a third-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30 & -31 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}30 & 10 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

b) Design a state feedback control to locate the poles at $s_1 = -1,\ s_2 = -5,\ s_3 = -7$

Step 1: Check controllability of the open-loop system.

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]$$

$$\mathbf{Q_c} = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 10\\ 1 & 10 & 69\end{bmatrix} \rightarrow \boxed{\det(\mathbf{Q_c}) = -1}$$

Determinant is non-zero, means the controllability matrix is a full rank matrix. **Therefore, the system is controllable.**

Step 2: Determine the desired characteristic polynomial.

$$\boxed{(s+1)(s+5)(s+7) = s^3 + 13s^2 + 47s + 35}$$

Desired Characteristic Polynomial

State Feedback Control Design

Example 10

Consider the following state space representation of a third-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30 & -31 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}30 & 10 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

b) Design a state feedback control to locate the poles at $s_1=-1,\ s_2=-5,\ s_3=-7$

Step 3: Obtain the closed-loop system matrix and determine the characteristic polynomial

$$\mathbf{A}_{cl}=\mathbf{A}-\mathbf{BK}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30 & -31 & 10\end{bmatrix}-\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30-k_1 & -31-k_2 & 10-k_3\end{bmatrix}$$

$$s\mathbf{I}-\mathbf{A}_{cl}=\begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30-k_1 & -31-k_2 & 10-k_3\end{bmatrix}=\begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ -30+k_1 & 31+k_2 & s-10+k_3\end{bmatrix}$$

$$\det(s\mathbf{I}-\mathbf{A}_{cl})=\begin{vmatrix}s & -1 & 0\\ 0 & s & -1\\ -30+k_1 & 31+k_2 & s-10+k_3\end{vmatrix}=\boxed{s^3+(-10+k_3)s^2+(31+k_2)s+(-30+k_1)}$$

Closed-loop characteristic polynomial

State Feedback Control Design

Example 10 Consider the following state space representation of a third-order system in canonical controllable form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 30 & -31 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 30 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

b) Design a state feedback control to locate the poles at $s_1 = -1,\ s_2 = -5,\ s_3 = -7$

Step 4: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the state feedback gain value $\mathbf{K}$.

Desired Characteristic Polynomial $\rightarrow$ $\boldsymbol{s^3 + 13s^2 + 47s + 35}$

Closed-loop Characteristic Polynomial $\rightarrow$ $\boldsymbol{s^3 + (-10 + k_3)s^2 + (31 + k_2)s + (-30 + k_1)}$

$$\begin{cases} -10 + k_3 = 13 \\ 31 + k_2 = 47 \\ -30 + k_1 = 35 \end{cases} \rightarrow \begin{cases} k_3 = 23 \\ k_2 = 16 \\ k_1 = 65 \end{cases} \rightarrow \boxed{\mathbf{K} = \begin{bmatrix} 65 & 16 & 23 \end{bmatrix}}$$

State Feedback Gain

State Feedback Control Design

Example 10 Consider the following state space representation of a third-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30 & -31 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}30 & 10 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

c) Determine state-space equations of the closed-loop system and check the closed-loop pole locations.

$$\mathbf{A}-\mathbf{BK} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30 & -31 & 10\end{bmatrix} - \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\begin{bmatrix}65 & 16 & 23\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -35 & -47 & -13\end{bmatrix}$$

$$\boxed{\begin{cases}\dot{\mathbf{x}}(t) = (\mathbf{A}-\mathbf{BK})\mathbf{x}(t) + \mathbf{B}r(t)\\ y(t) = (\mathbf{C}-\mathbf{DK})\mathbf{x}(t) + \mathbf{D}r(t)\end{cases}}$$

Closed-loop System

Closed-loop System $\rightarrow$

$$\begin{cases}\dot{\mathbf{x}}(t) = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -35 & -47 & -13\end{bmatrix}\mathbf{x}(t) + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r(t)\\ y(t) = \begin{bmatrix}30 & 10 & 0\end{bmatrix}\mathbf{x}(t)\end{cases}$$

$$\boxed{\mathbf{K} = \begin{bmatrix}65 & 16 & 23\end{bmatrix}}$$

State Feedback Gain

$$\det(s\mathbf{I}-\mathbf{A}_{cl}) = \begin{vmatrix}s & -1 & 0\\ 0 & s & -1\\ 35 & 47 & s+13\end{vmatrix} = \boldsymbol{s^3 + 13s^2 + 47s + 35}$$

$$s^3 + 13s^2 + 47s + 35 = 0 \quad \rightarrow \quad \boxed{\boldsymbol{s_1 = -1, \quad s_2 = -5, \quad s_3 = -7}}$$

Closed-loop poles

The closed-loop eigenvalues/poles are located at the desired locations.

State Feedback Control Design

Example 10 Consider the following state space representation of a third-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30 & -31 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}30 & 10 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

We can also design the state feedback gain using `place` function in MATLAB

```
A = [0 1 0;0 0 1;30 -31 10];
B = [0; 0; 1];
eig(A)
ans =
    2.0000
    3.0000
    5.0000

Qc = ctrb(A,B)
Qc =
     0     0     1
     0     1    10
     1    10    69

det(Qc)
ans =
    -1

rank(Qc)
ans =
     3
```

```
Pcl = [-1; -5; -7];
K = place(A,B,Pcl)
K =
   65.0000   16.0000   23.0000

Acl = A-B*K
Acl =
         0    1.0000         0
         0         0    1.0000
  -35.0000  -47.0000  -13.0000

eig(Acl)
ans =
   -1.0000
   -5.0000
   -7.0000
```

$$\begin{cases}\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{B}r(t)\\ y(t) = (\mathbf{C} - \mathbf{DK})\mathbf{x}(t) + \mathbf{D}r(t)\end{cases}$$

Closed-loop System

$$\mathbf{K} = \begin{bmatrix}65 & 16 & 23\end{bmatrix}$$

State Feedback Gain

State Feedback Control Design

Example 10

Consider the following state space representation of a third-order system in canonical controllable form

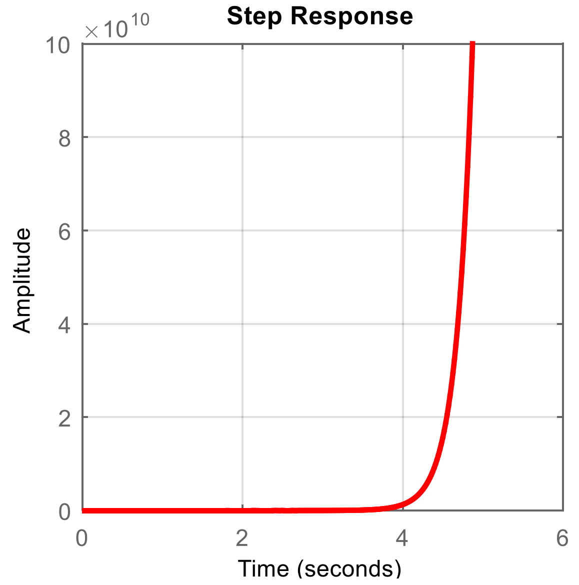
$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 30 & -31 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}30 & 10 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

Following figures show the unit-step response of the open-loop and closed-loop systems

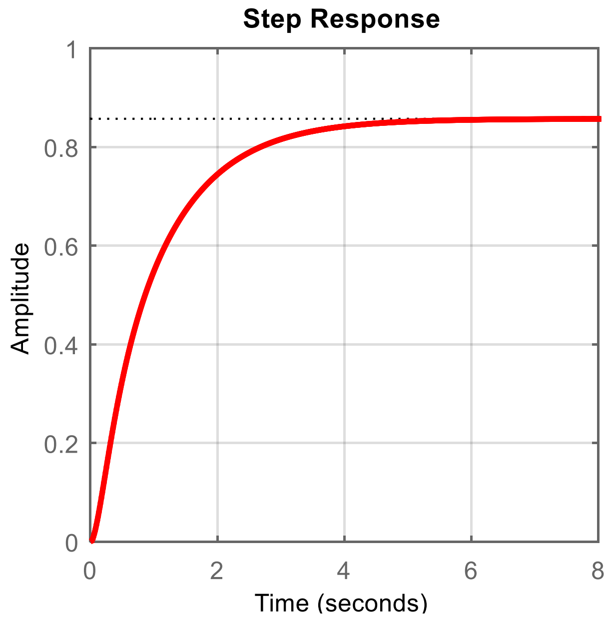
```
A = [0 1 0;0 0 1;30 -31 10];
B = [0; 0; 1];
C = [30 10 0];
D = [0];

OLsys = ss(A,B,C,D);

stepplot(OLsys)


K = [65 16 23];
Acl = A-B*K;
Bcl = B;
Ccl = C-D*K;
Dcl = D;

CLsys = ss(Acl,Bcl,Ccl,Dcl);

stepplot(CLsys)
```

Open-loop System



The open-loop system is unstable, and the unit-step response is not a bounded signal.

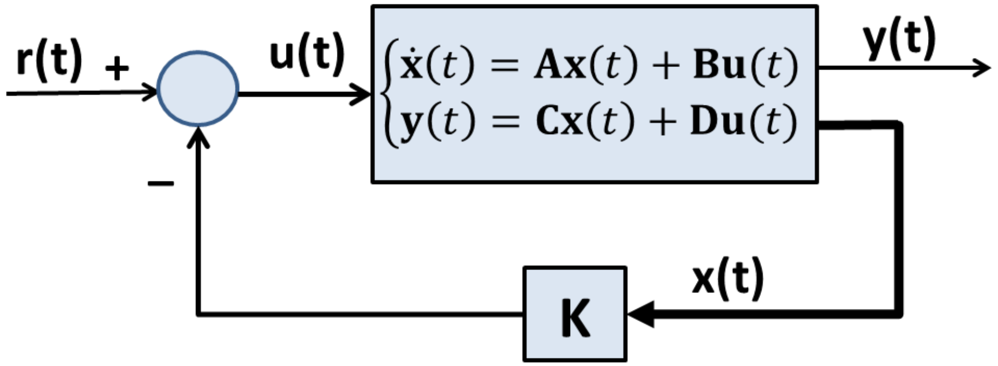
Closed-loop System

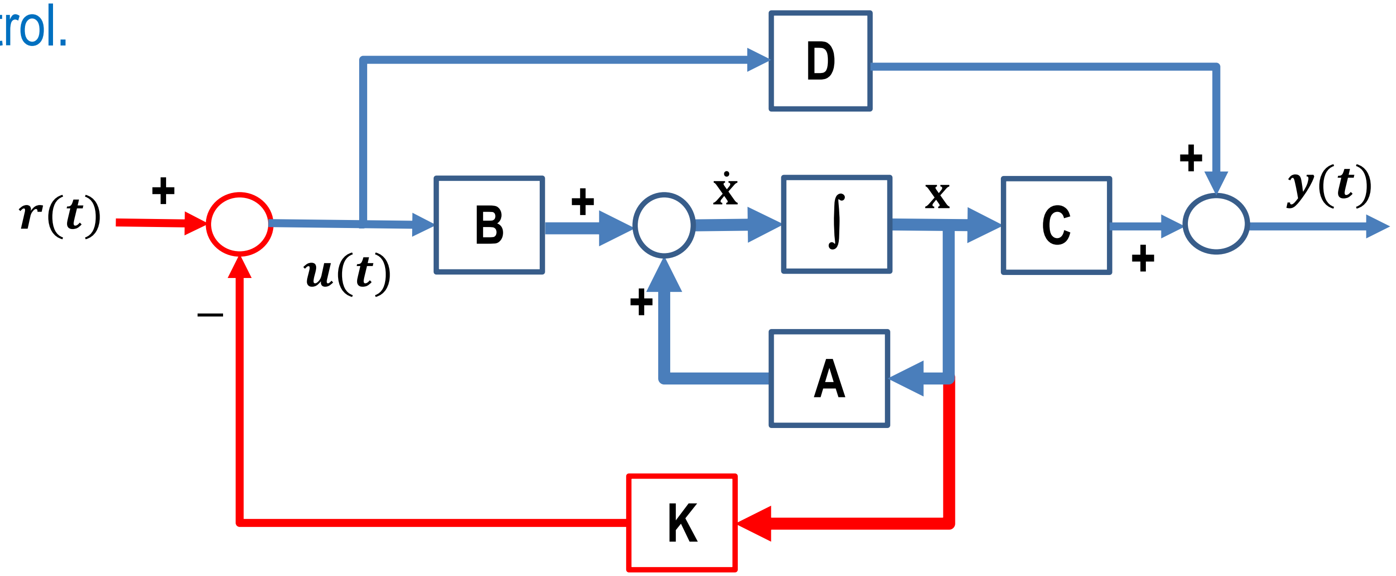


The closed-loop system is stable; however, it does not track the reference input.

State Feedback with Integral Control

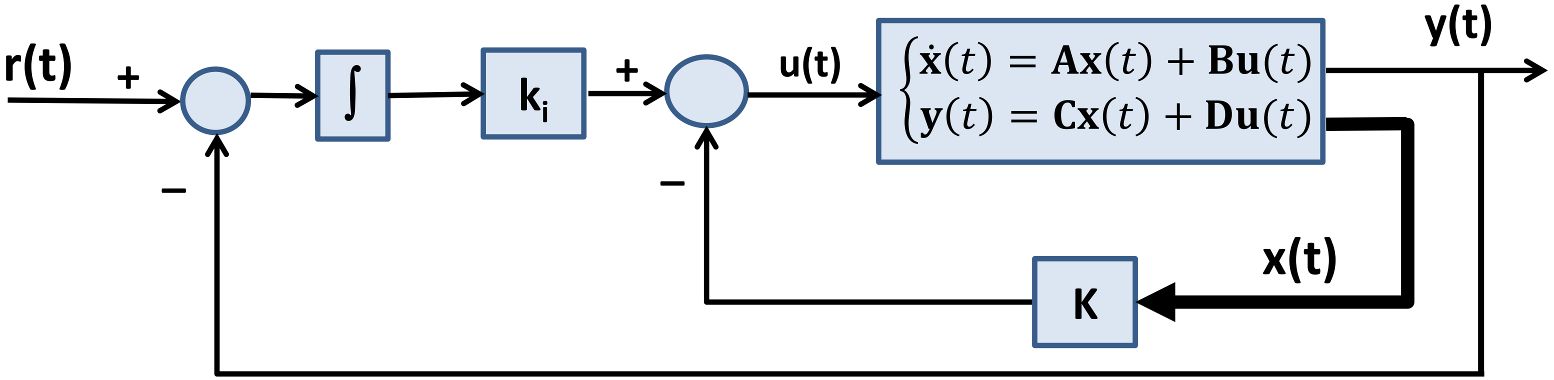
- Consider the following closed-loop system with state feedback control.





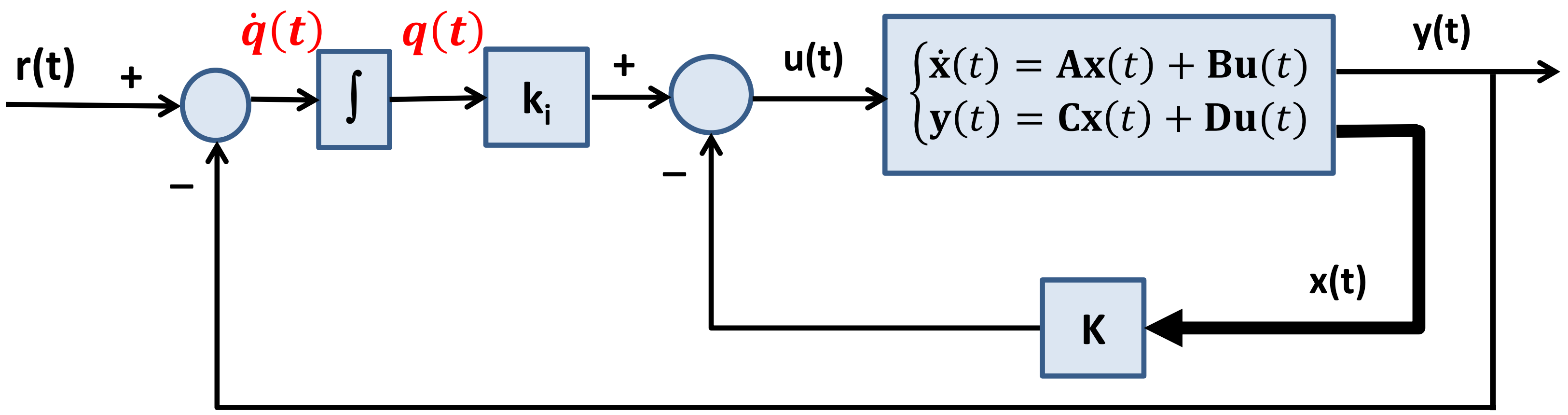
- The state feedback control can stabilize the unstable systems and locate the closed-loop poles at the desired locations, if the system is controllable.
- However, state feedback **does not** guarantee the tracking of reference input.
- The tracking capability of the closed-loop system can be guaranteed by adding an **integrator** to the closed-loop system.

State Feedback with Integral Control



State Feedback with Integral Control

- The tracking capability of the closed-loop system can be guaranteed by adding an **integrator** to the closed-loop system.



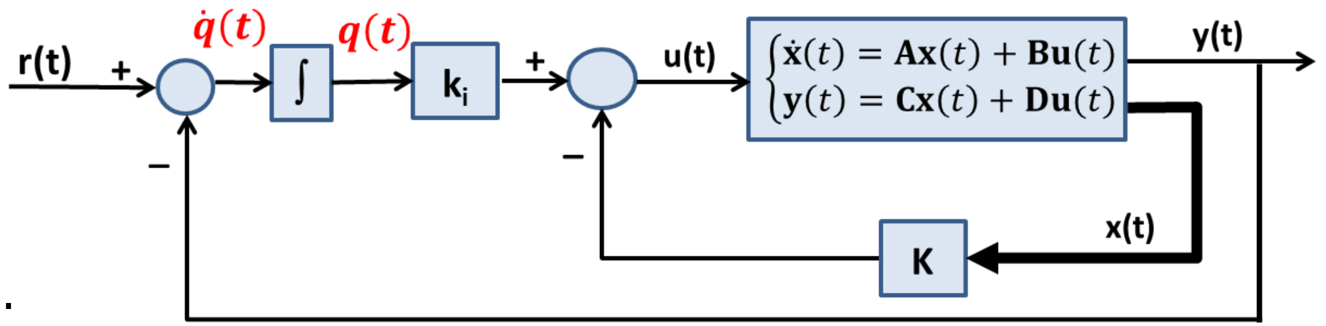
- In this case, an additional state variable, $q(t)$, is defined for the integrator.
- The **State Feedback with Integral Control Law** is obtained as,

$$\boxed{u(t) = -\underbrace{\mathbf{K}}_{\text{State Feedback Gain}}\mathbf{x}(t) + \underbrace{k_i}_{\text{Integrator Gain}} q(t) \quad \rightarrow \quad u(t) = -[\mathbf{K} \quad -k_i]\begin{bmatrix}\mathbf{x}(t) \\ q(t)\end{bmatrix}}$$

$$u(t) = -\underbrace{[k_1 \quad k_2 \quad \cdots \quad k_n \quad -k_i\,]}_{\mathbf{1}\times(\boldsymbol{n}+\mathbf{1})\text{ vector}}\begin{bmatrix}x_1 \\ x_2 \\ \vdots \\ x_n \\ q\end{bmatrix}$$

State Feedback with Integral Control

State Feedback with Integral Closed-Loop System



- The Augmented open-loop and closed-loop system equations are determined as below,
- First, the additional state variable, $q(t)$ is added to the open-loop system.

$$\dot{q}(t) = r(t) - y(t) \quad \rightarrow \quad \dot{q}(t) = r(t) - \mathbf{C}\mathbf{x}(t) - \mathbf{D}u(t)$$

- Considering the new state variable $q(t)$, the open-loop system can be rewritten as augmented matrices and vectors as below

Augmented Open-loop System

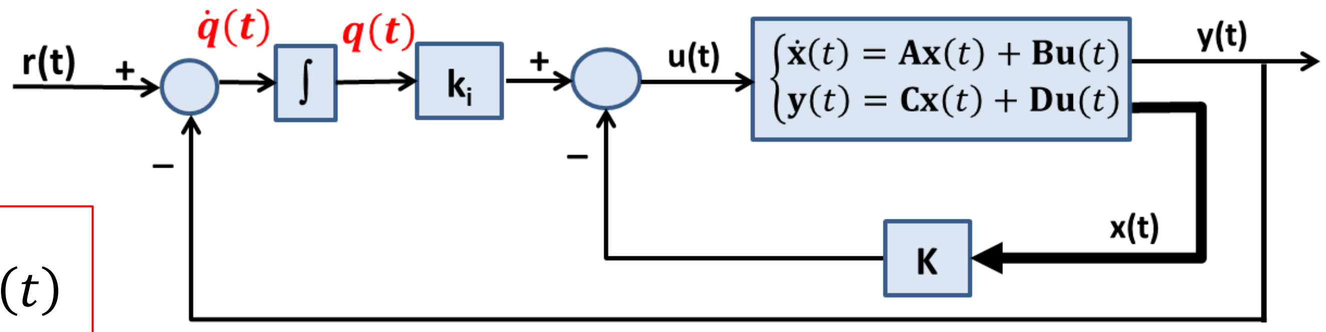
$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \\ y(t) = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \mathbf{D}u(t) \end{cases}$$

- In designing the state feedback with integrator control the augmented open-loop system must be **Controllable**.

State Feedback with Integral Control

❑ State Feedback with Integral Closed-Loop System

- The Augmented open-loop and closed-loop system equations are determined as below,



Augmented Open-loop System

$$\begin{cases}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{q}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{B}\\ -\mathbf{D}\end{bmatrix}u(t) + \begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)\\ y(t) = [\mathbf{C} \quad 0]\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \mathbf{D}u(t)\end{cases}$$

- The augmented state space equations of the closed-loop system is determined by substituting the control signal formula, $u(t)$, into the state space equations of the augmented open-loop system.

$$u(t) = -[\mathbf{K} \quad -k_i]\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix}$$

$$\begin{cases}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{q}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{B}\\ -\mathbf{D}\end{bmatrix}[-\mathbf{K} \quad k_i]\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)\\ y(t) = [\mathbf{C} \quad 0]\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \mathbf{D}[-\mathbf{K} \quad k_i]\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix}\end{cases}$$

Augmented Closed-loop System

$$\begin{cases}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{q}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{B}k_i\\ -\mathbf{C}+\mathbf{DK} & -\mathbf{D}k_i\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)\\ y(t) = [\mathbf{C}-\mathbf{DK} \quad \mathbf{D}k_i]\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix}\end{cases}$$

State Feedback with Integral Control

State Feedback with Integral Control Design Procedure

Step 1: Given state space representation, determine the augmented open-loop system.

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{cases} \Rightarrow \begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \\ y(t) = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \mathbf{D}u(t) \end{cases}$$

Step 2: Check controllability of the augmented open-loop system.

Step 3: Determine the desired eigenvalue/pole locations and the desired characteristic polynomial, based on the desired performance criteria.

Step 4: Obtain the augmented closed-loop system matrix and determine the characteristic polynomial

$$\boxed{\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}k_i \\ -\mathbf{C} + \mathbf{D}\mathbf{K} & -\mathbf{D}k_i \end{bmatrix}} \rightarrow \boxed{\det(s\mathbf{I} - \mathbf{A}_{cl})}$$

where $\mathbf{K} = [k_1 \quad k_2 \quad \cdots \quad k_n]$ is the state feedback gain vector and k_i is the integrator gain.

Step 5: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the $\mathbf{K}$ and k_i values .

State Feedback with Integral Control

Example 11

Consider the following state space representation of a second-order system in canonical controllable form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [50 \quad 10] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

a) Determine the open-loop system poles and check stability of the system.

First, find the characteristic polynomial of the matrix A, and the eigenvalues/poles

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 3 & s-4 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}) = \begin{vmatrix} s & -1 \\ 3 & s-4 \end{vmatrix} = \boxed{\boldsymbol{s^2 - 4s + 3}}$$

Open-loop Characteristic Polynomial

$$s^2 - 4s + 3 = 0 \quad \rightarrow \quad \boxed{\boldsymbol{s_1 = 5, \qquad s_2 = 1}}$$

Open-loop Poles

The eigenvalues/poles are located on the right-half of the s-plane.

Therefore, the system is **unstable**.

State Feedback with Integral Control

Example 11 Consider the following state space representation of a second-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

b) Design a state feedback with integral control to locate the poles at $s_1 = -3,\ s_2 = -5$ and guarantee the tracking capability for step input.

$$y(t) = \begin{bmatrix}50 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Step 1: Determine the augmented open-loop system.

$$\begin{cases}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{q}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{B}\\ -\mathbf{D}\end{bmatrix}u(t) + \begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)\\ y(t) = \begin{bmatrix}\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \mathbf{D}u(t)\end{cases}$$

$$\Rightarrow \begin{cases}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{q}(t)\end{bmatrix} = \left[\begin{array}{cc|c}0 & 1 & 0\\ -3 & 4 & 0\\ \hline -50 & -10 & 0\end{array}\right]\begin{bmatrix}x_1(t)\\ x_2(t)\\ q(t)\end{bmatrix} + \left[\begin{array}{c}0\\ 1\\ \hline 0\end{array}\right]u(t) + \left[\begin{array}{c}0\\ 0\\ \hline 1\end{array}\right]r(t)\\ y(t) = \left[\begin{array}{cc|c}50 & 10 & 0\end{array}\right]\begin{bmatrix}x_1(t)\\ x_2(t)\\ q(t)\end{bmatrix}\end{cases}$$

Step 2: Check controllability of the augmented open-loop system.

$$\bar{\mathbf{A}} = \begin{bmatrix}0 & 1 & 0\\ -3 & 4 & 0\\ -50 & -10 & 0\end{bmatrix} \qquad \bar{\mathbf{B}} = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix} \rightarrow \mathbf{Q_c} = \begin{bmatrix}\bar{\mathbf{B}} & \bar{\mathbf{A}}\bar{\mathbf{B}} & \bar{\mathbf{A}}^2\bar{\mathbf{B}}\end{bmatrix}$$

$$\mathbf{Q_c} = \begin{bmatrix}0 & 1 & 4\\ 1 & 4 & 13\\ 0 & -10 & -90\end{bmatrix} \rightarrow \boxed{\det(\mathbf{Q_c}) = 50} \rightarrow$$ **The system is controllable**

State Feedback with Integral Control

Example 11

Consider the following state space representation of a second-order system in canonical controllable form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 50 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

b) Design a state feedback with integral control to locate the poles at $s_1 = -3,\ s_2 = -5$ and guarantee the tracking capability for step input.

Step 3: Determine the desired characteristic polynomial.

The desired characteristic equation is determined from the location of the desired closed-loop poles and considering the third pole more than ten times far from the desired poles at the higher frequencies.

$$\boxed{(s+3)(s+5)(s+100) = s^3 + 108s^2 + 815s + 1500}$$

Desired Characteristic Polynomial

State Feedback with Integral Control

Example 11 Consider the following state space representation of a second-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}50 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

b) Design a state feedback with integral control to locate the poles at $s_1 = -3,\ s_2 = -5$ and guarantee the tracking capability for step input.

Step 4: Obtain the augmented closed-loop system matrix and determine the characteristic polynomial

$$\boxed{\mathbf{K} = [k_1 \quad k_2]}$$

$$\boxed{\mathbf{A}_{cl} = \begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{B}k_i\\ -\mathbf{C}+\mathbf{DK} & -\mathbf{D}k_i\end{bmatrix}}$$

$$\boxed{\det(s\mathbf{I}-\mathbf{A}_{cl})}$$

$$\mathbf{A}-\mathbf{BK} = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix} - \begin{bmatrix}0\\ 1\end{bmatrix}[k_1 \quad k_2] = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix} - \begin{bmatrix}0 & 0\\ k_1 & k_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3-k_1 & 4-k_2\end{bmatrix}$$

$$\mathbf{A}_{cl} = \begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{B}k_i\\ -\mathbf{C}+\mathbf{DK} & -\mathbf{D}k_i\end{bmatrix} = \left[\begin{array}{cc|c}0 & 1 & 0\\ -3-k_1 & 4-k_2 & k_i\\ \hline -50 & -10 & 0\end{array}\right]$$

$$s\mathbf{I}-\mathbf{A}_{cl} = \begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ -3-k_1 & 4-k_2 & k_i\\ -50 & -10 & 0\end{bmatrix} = \begin{bmatrix}s & -1 & 0\\ 3+k_1 & s-4+k_2 & -k_i\\ 50 & 10 & s\end{bmatrix}$$

$$\det(s\mathbf{I}-\mathbf{A}_{cl}) = \begin{vmatrix}s & -1 & 0\\ 3+k_1 & s-4+k_2 & -k_i\\ 50 & 10 & s\end{vmatrix} = \boxed{s^3 + (-4+k_2)s^2 + (3+k_1+10k_i)s + 50k_i}$$

Closed-loop characteristic polynomial

State Feedback with Integral Control

Example 11

Consider the following state space representation of a second-order system in canonical controllable form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 50 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

b) Design a state feedback with integral control to locate the poles at $s_1 = -3,\ s_2 = -5$ and guarantee the tracking capability for step input.

Step 5: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the $\mathbf{K}$ and k_i values .

Desired Characteristic Polynomial $\rightarrow$ $\boldsymbol{s^3 + 108s^2 + 815s + 1500}$

Closed-loop Characteristic Polynomial $\rightarrow$ $\boldsymbol{s^3 + (-4 + k_2)s^2 + (3 + k_1 + 10k_i)s + 50k_i}$

$$\begin{cases} -4 + k_2 = 108 \\ 3 + k_1 + 10k_i = 815 \\ 50k_i = 1500 \end{cases} \rightarrow \begin{cases} k_2 = 112 \\ k_1 = 512 \\ k_i = 30 \end{cases}$$

Therefore, the state feedback gain vector and the integrator gain are obtained as below

$$\boxed{\mathbf{K} = \begin{bmatrix} 512 & 112 \end{bmatrix}}$$

State Feedback Gain

$$\boxed{k_i = 30}$$

Integrator Gain

State Feedback with Integral Control

Example 11

Consider the following state space representation of a second-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}50 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

c) Determine state space equations of the closed-loop system and check the closed-loop pole locations.

Closed-loop System

$$\begin{cases}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{q}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{B}k_i\\ -\mathbf{C}+\mathbf{DK} & -\mathbf{D}k_i\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)\\ y(t) = \begin{bmatrix}\mathbf{C}-\mathbf{DK} & \mathbf{D}k_i\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix}\end{cases}$$

$$\boxed{\mathbf{K} = \begin{bmatrix}512 & 112\end{bmatrix}}$$

State Feedback Gain

$$\boxed{k_i = 30}$$

Integrator Gain

$$\mathbf{A}-\mathbf{BK} = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix} - \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}512 & 112\end{bmatrix} = \begin{bmatrix}0 & 1\\ -515 & -108\end{bmatrix}$$

$$\begin{cases}\dot{\mathbf{x}}(t) = \begin{bmatrix}0 & 1 & 0\\ -515 & -108 & 30\\ -50 & -10 & 0\end{bmatrix}\mathbf{x}(t) + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r(t)\\ y(t) = \begin{bmatrix}50 & 10 & 0\end{bmatrix}\mathbf{x}(t)\end{cases}$$

Closed-loop System

$$\det(s\mathbf{I}-\mathbf{A}_{cl}) = \begin{vmatrix}s & -1 & 0\\ 515 & s+108 & -30\\ 50 & 10 & s\end{vmatrix} = \boldsymbol{s^3 + 108s^2 + 815s + 1500}$$

$$s^3 + 108s^2 + 815s + 1500 = 0 \quad \rightarrow \quad \boxed{s_1 = -3, \quad s_2 = -5, \quad s_3 = -100}$$

Closed-loop poles

The dominant closed-loop eigenvalues/poles are located at the desired locations. The third pole located far from the dominant poles and has no effect o

State Feedback with Integral Control

Example 11 Consider the following state space representation of a second-order system in canonical controllable form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}50 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

We can also design the state feedback with integral gain using `place` function in MATLAB

```
A = [0 1;-3 4];
B = [0; 1];
C = [50 10];
D = [0];

Abar = [A zeros(size(B)); -C 0];
Bbar = [B;-D];
Cbar = [C 0];
Dbar = D;

Qc = ctrb(Abar,Bbar)
Qc =
     0     1     4
     1     4    13
     0   -10   -90

det(Qc)
ans =
     50

rank(Qc)
ans =
     3
```

```
Pcl = [-3; -5; -100];
K = place(Abar,Bbar,Pcl)
K =
   512.0000   112.0000   -30.0000
```

k_1 $\quad k_2$ $\quad -k_i$

```
K = [512 112];
ki = 30;
Acl = [A-B*K B*ki;-C+D*K -D*ki];
Acl =
        0      1      0
     -515   -108     30
      -50    -10      0

eig(Acl)
ans =
   -100.0000
     -5.0000
     -3.0000
```

$$\begin{cases}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{q}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{B}\\ -\mathbf{D}\end{bmatrix}u(t) + \begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)\\ y(t) = \begin{bmatrix}\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \mathbf{D}u(t)\end{cases}$$

Augmented Open-loop System

$$\begin{cases}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{q}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{B}k_i\\ -\mathbf{C}+\mathbf{DK} & -\mathbf{D}k_i\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)\\ y(t) = \begin{bmatrix}\mathbf{C}-\mathbf{DK} & \mathbf{D}k_i\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ q(t)\end{bmatrix}\end{cases}$$

Closed-loop System

State Feedback with Integral Control

Example 11

Consider the following state space representation of a second-order system in canonical controllable form

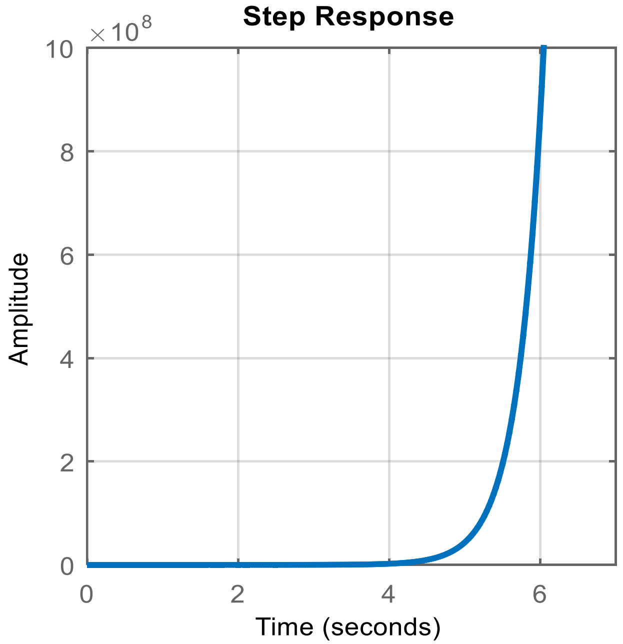
$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}50 & 10\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Following figures show unit-step response of the open-loop and closed-loop systems

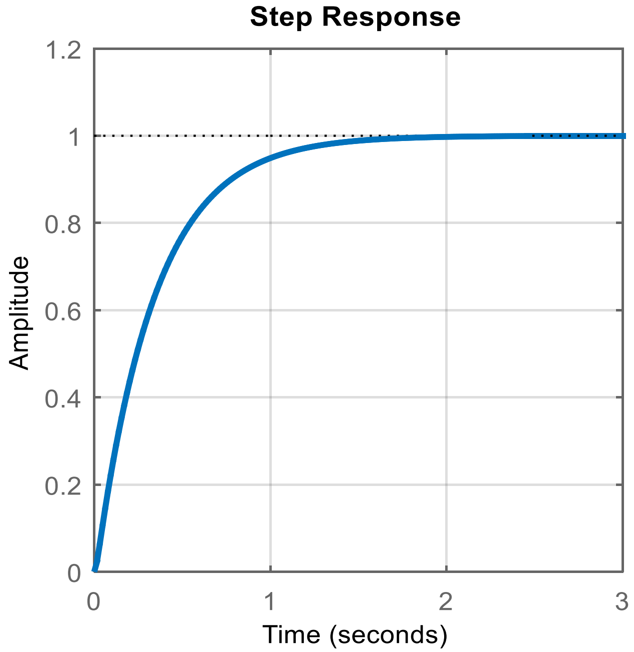
```
A = [0 1;-3 4];
B = [0; 1];
C = [50 10];
D = [0];

OLsys = ss(A,B,C,D);

stepplot(OLsys)


K = [512 112];
ki = 30;
Acl = [A-B*K B*ki;-C+D*K -D*ki];
Bcl = [zeros(size(B)); 1];
Ccl = [C-D*K D*ki];
Dcl = [0];

CLsys = ss(Acl,Bcl,Ccl,Dcl);

stepplot(CLsys)
```

Open-loop System



The open-loop system is unstable, and the unit-step response is not a bounded signal.

Closed-loop System



The closed-loop system is stable, and output signal tracks the reference input with zero steady-state error.

THANK YOU

HUMBER

WE ARE HUMBER