Chapter 10

Triple Integrals

10.1 Triple Integrals in Cylindrical Coordinates

FRY Defn III.3.6.1, Cylindrical coordinates

Definition 10.1. The cylindrical coordinates of a point (x, y, z) in three-dimensional space are denoted by r, θ, and z, where

(i) r is the distance from $(x, y, 0)$ to $(0, 0, 0)$ or, equivalently, the distance from (x, y, z) to the z-axis;

(ii) θ is the (counterclockwise) angle from the positive x-axis to the line segment joining $(x, y, 0)$ to $(0, 0, 0)$; and

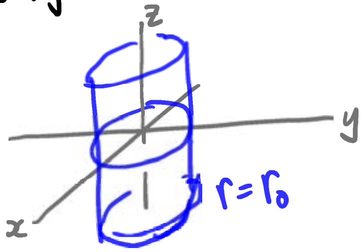
(iii) z is the signed distance from (x, y, z) to the xy-plane.

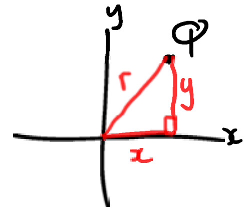
The equations

- $r = r_0$ describes a cylinder (of constant radius r_0);
- $\theta = \theta_0$ describes a plane that contains (passes through) the z-axis and makes an angle of θ_0 with the positive x-axis; and
- $z = z_0$ describes a plane parallel to the xy-plane that is steady at a height of z_0; and
- $z = r$ describes a cone.

If $r=2$ $\quad (0\le\theta\le 2\pi,\ -\infty<z<\infty)$

then $\sqrt{x^2+y^2}=2$ and $x^2+y^2=2^2$

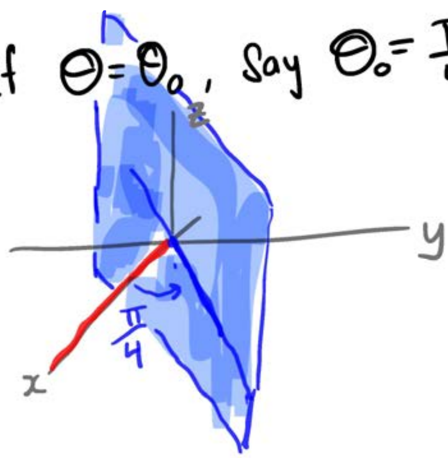




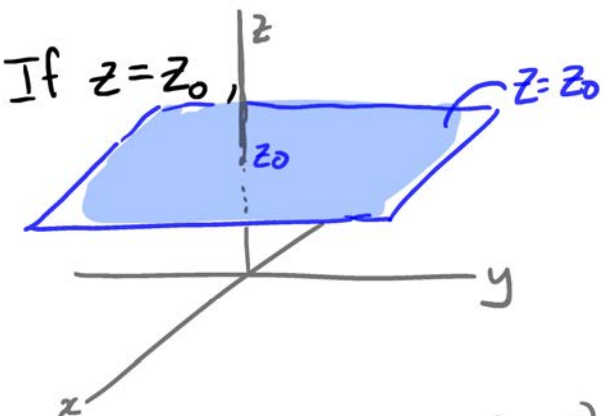
$r^2=x^2+y^2$

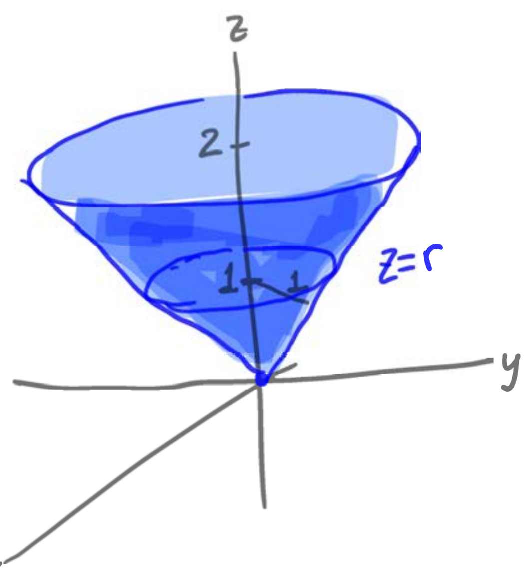
$r=\sqrt{x^2+y^2}$

If $\theta=\theta_0$, say $\theta_0=\frac{\pi}{4}$, $\quad$ (Here $-\infty<r<\infty$, $-\infty<z<\infty$.)

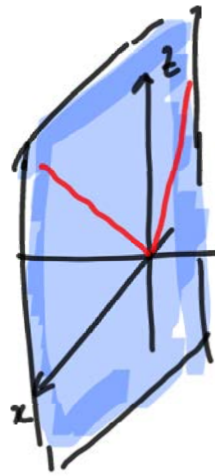


If $z=z_0$,





$\boxed{z=r}=\sqrt{x^2+y^2}$ ← distance between $(x,y,0)$ and $(0,0,0)$



$z=\sqrt{x^2+y^2}$

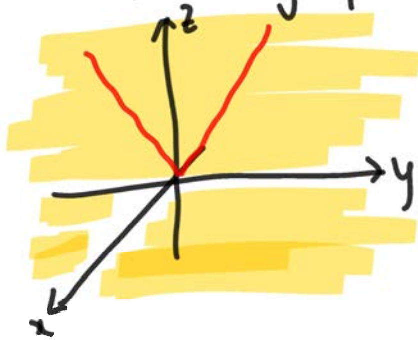
in the xz-plane

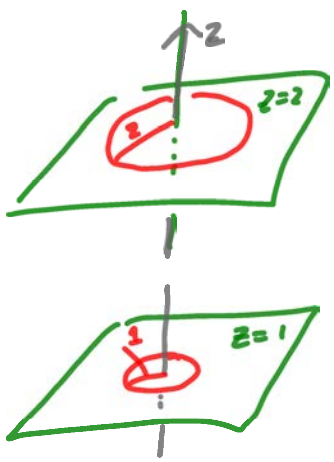
y is 0

so

$z=\sqrt{x^2+0^2}=|x|$

In the yz-plane, x is 0 and $z=\sqrt{x^2+y^2}=\sqrt{0^2+y^2}=|y|$





$$\iiint_{\mathcal{R}} f(x,y,z)\,\underbrace{dV}_{m^3} = \iiint_{\mathcal{R}} f(r\cos\theta, r\sin\theta, z)\,\underbrace{r\; dr\,d\theta\,dz}_{dV}$$

$|\det D\vec{g}|$

$(x,y,z) \overset{\vec{g}}{\longmapsto} (r,\theta,z)$

Given (r, θ, z),

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Given (x, y, z),

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad z = z.$$

Note that we need to add π to $\arctan(y/x)$ to get the correct value for θ if the x- and y-coordinates are such that $(x, y, 0)$ lies in the second or third quadrant of the xy-plane.

If $\mathbf{g}$ denotes the change of variable transformation from (x, y, z)-coordinates into (r, θ, z)-coordinates, then

$$D\mathbf{g}(r,\theta,z) = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\[2ex] \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant of the derivative matrix of the change of variables transformation $\mathbf{g}$, through cofactor expansion along the third row, is

$$\det D\mathbf{g}(\rho, \theta, \phi) = r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r.$$

Thus,

$$|\det D\mathbf{g}(r, \theta, z)| = |r| = r.$$

We use this information to adjust the volume element dV when changing from Cartesian to spherical coordinates:

$$\iiint_{\mathcal{R}} f(x,y,z)\; dV = \iiint_{\mathcal{R}} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta,$$

though we may use a different order of integration depending on the domain of integration $\mathcal{R}$.

$$\text{Vol} = \boxed{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}} dz\, dy\, dx$$

What is this region?

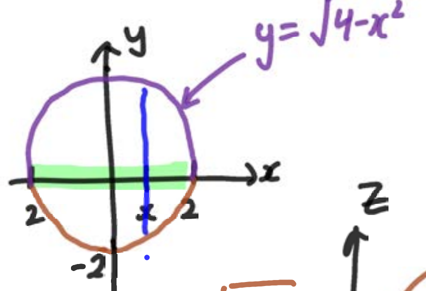
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{\sqrt{x^2+y^2}}^{2} dy\, dx$$

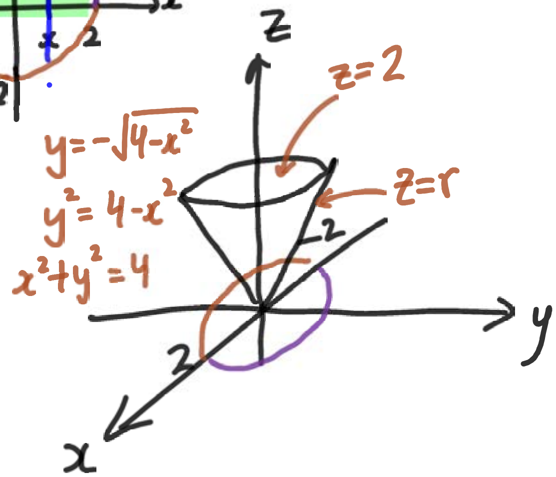
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(2 - \sqrt{x^2+y^2}\right) dy\, dx$$

$\downarrow$ difficult? (try trig. substitution)

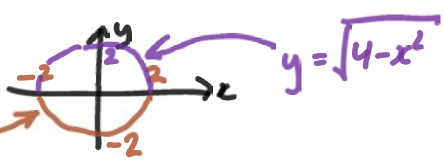
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2y -$$

What is this region?

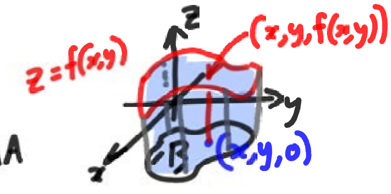




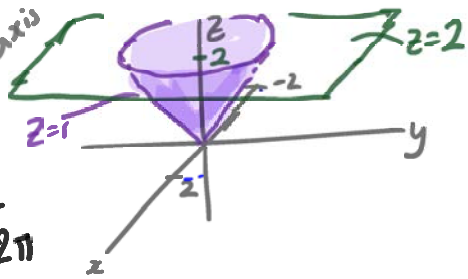
$\boxed{y = -\sqrt{4-x^2}}$



$$\iint_R f(x,y)\,dA$$



Example 10.2. Evaluate the volume described by the triple integral $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\underbrace{\sqrt{x^2+y^2}}_{=r}}^{2} dz\,dy\,dx.$



distance from z-axis

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$r \le z \le 2$

$$-\sqrt{4-x^2} \le y \le \underbrace{\sqrt{4-x^2}}$$

$$y = \sqrt{4-x^2}$$

$$y^2 = 4-x^2$$

$$\underbrace{x^2+y^2 = 2^2}_{\text{circle}}$$

$$\underbrace{\sqrt{x^2+y^2}}_{r} \le z \le 2$$

$$\text{Vol} = \int_0^2\int_0^{2\pi}\int_r^2 \underbrace{r\,dz\,d\theta\,dr}_{dV}$$

$$= \int_0^2\int_0^{2\pi} rz\Big|_r^2\,d\theta\,dr$$

$$= \int_0^2\int_0^{2\pi}(2r-r^2)\,d\theta\,dr$$

$$= \left(\int_0^2(2r-r^2)\,dr\right)\left(\int_0^{2\pi}d\theta\right)$$

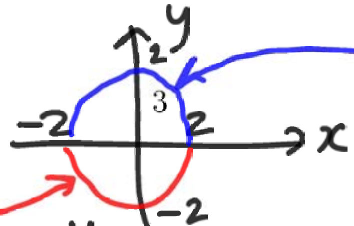
When we integrate 1 (over a single variable) we get the length of the interval that we are integrating over.

$$= \left[r^2-\frac{1}{3}r^3\right]_0^2(2\pi)$$

$$= \left(4-\frac{8}{3}\right)(2\pi) = \frac{8}{3}\pi$$

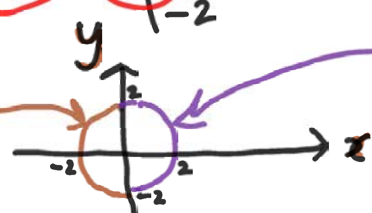
Note

$y = -\sqrt{4-x^2}$



$y = \sqrt{4-x^2}$

$x = -\sqrt{4-y^2}$



$x = \sqrt{4-y^2}$

If we try using rectangular coordinates $\text{mass} = \iiint_{\mathcal{B}} \rho(x,y,z)\,dV = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-y^2}}\int_0^{2} e^{-x^2-y^2}\,dz\,dx\,dy = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} z e^{-x^2-y^2}\Big|_0^2\,dx\,dy$

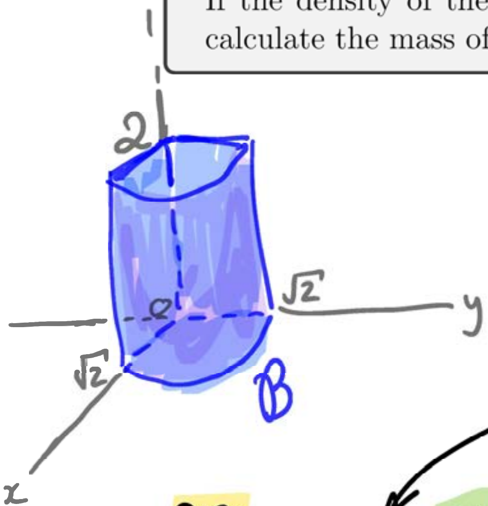
$$= \int_0^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} 2e^{-x^2-y^2}\,dx\,dy$$

we can't proceed ...

Example 10.3. A body $\mathcal{B}$ occupies the region

$$\left\{(x,y,z):\ 0 \le y \le \sqrt{2},\ 0 \le x \le \sqrt{2-y^2},\ 0 \le z \le 2\right\}.$$

If the density of the body is described by the function $\rho(x,y,z) = e^{-x^2-y^2}$, calculate the mass of $\mathcal{B}$.

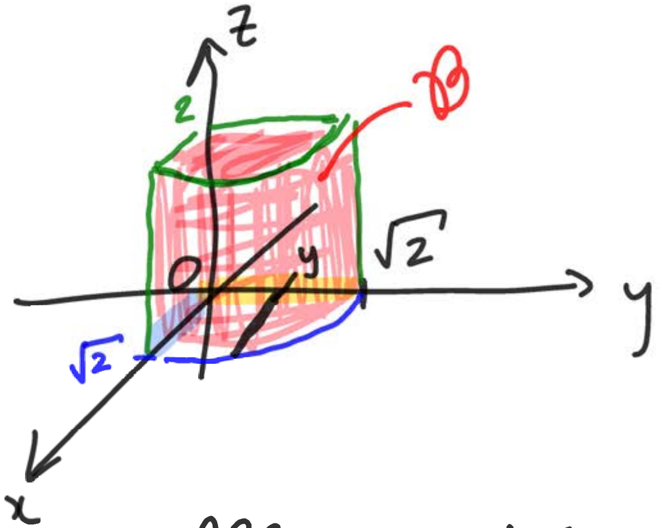


$\text{mass} = \underline{\text{density}} \cdot \text{volume}$

$\text{g/cm}^3 \cdot \text{cm}^3$

$$\text{mass} = \iiint_{\mathcal{B}} \rho(x,y,z)\,dV$$

$$\mathcal{B} = \{(x,y,z) : 0 \le y \le \sqrt{2},\ 0 \le x \le \sqrt{2-y^2},\ 0 \le z \le 2\}$$



$x = \sqrt{2-y^2}$

$x^2 = 2 - y^2$

$x^2 + y^2 = 2$

radius-squared

radius $= \sqrt{2}$

distance from the z-axis

$0 \le r \le \sqrt{2}$

$0 \le \theta \le \pi/2$

$0 \le z \le 2$

$$\text{mass} = \iiint_{\mathcal{B}} \rho(x,y,z)\, dV$$

$$= \int_0^{\sqrt{2}} \int_0^{\pi/2} \int_0^{2} e^{-r^2} \underbrace{r\, dz\, d\theta\, dr}_{dV}$$

$\rho(x,y,z)$

$= e^{-x^2-y^2}$

$= e^{-(x^2+y^2)}$

$= e^{-r^2}$, since $x^2+y^2=r^2$

$$= \left(\int_0^{\sqrt{2}} r e^{-r^2}\, dr\right)\left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{2} dz\right)$$

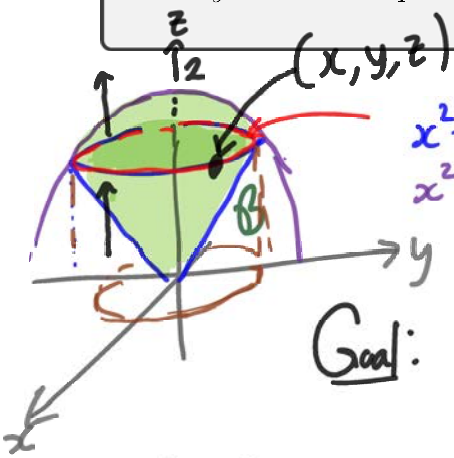
$$= \left[-\frac{1}{2} e^{-r^2}\right]_0^{\sqrt{2}} \left(\pi/2\right)(2)$$

$$\int_a^b dx = \text{length of } [a,b] = b - a$$

$$= \left(-\frac{1}{2} e^{-2} + \frac{1}{2}\right)\pi$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right) \approx 1.358$$

Example 10.4. (FRY Exercise III.3.7.5.16)
Let B denote the region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $x^2 + y^2 = z^2$. Compute $\iiint_B z^2 \, dV$.



$x^2+y^2=z^2$ (cone) $\quad z=\sqrt{x^2+y^2}=r$

$x^2+y^2+z^2=4$ (sphere)

radius of sphere $=\sqrt{4}=2$

Goal: Find $\iiint_B z^2 \, dV$

When the cone and sphere intersect, $\begin{cases} \boxed{x^2+y^2}=z^2 & \text{—eqn(1)} \\ x^2+y^2+z^2=4 & \text{—eqn(2)} \end{cases}$

$z^2+z^2=4$

$2z^2=4$

$z^2=2$

$z=\sqrt{2}$ or $\cancel{z=-\sqrt{2}}$

$x^2+y^2=(\sqrt{2})^2=2$

So $0 \le r \le \sqrt{2}$

$0 \le \theta \le 2\pi$

$r \le z \le \sqrt{4-r^2}$

Sphere: $\underbrace{x^2+y^2}+z^2=4$

$r^2+z^2=4$

$z^2=4-r^2$

$z=\sqrt{4-r^2}$

Since $z \ge 0$

$$\iiint_B z^2\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} z^2\, r\,dz\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} z^2\, r\,dz\,dr\right)$$

$$= 2\pi \int_0^{\sqrt{2}} \frac{1}{3} r z^3 \Big|_r^{\sqrt{4-r^2}} dr$$

$$= \frac{2\pi}{3}\int_0^{\sqrt{2}} \left(r(4-r^2)^{3/2} - r^4\right) dr$$

$$= \frac{2}{3}\pi \left[\frac{-1}{5}(4-r^2)^{5/2} - \frac{1}{5} r^5\right]_0^{\sqrt{2}}$$

$$= \frac{-2\pi}{15}\left[(4-r^2)^{5/2} + r^5\right]_0^{\sqrt{2}}$$

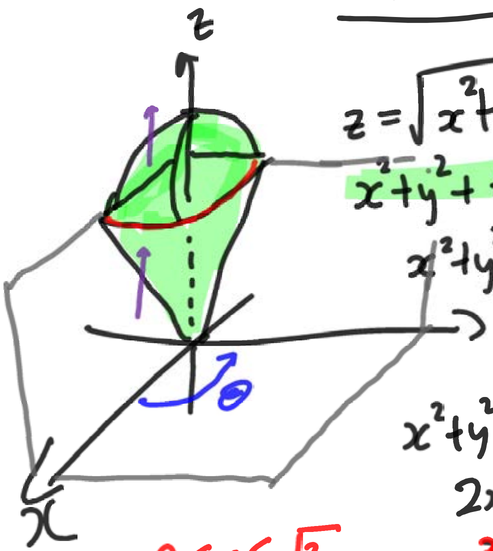
$$= \frac{-2\pi}{15}\left[\,2^{5/2} - 4^{5/2} + 2^{5/2}\,\right]$$

$$= -\frac{2\pi}{15}\left[8\sqrt{2} - 32\right]$$

$$= \frac{16\pi}{15}\left(4-\sqrt{2}\right)$$

$$\approx 8.665$$

Live Poll



$z=\sqrt{x^2+y^2}$ cone

$x^2+y^2+z^2=6$ sphere

$z^2=6-x^2-y^2$

$z=\sqrt{6-x^2-y^2}$

$=\sqrt{6-(x^2+y^2)}$

$=\sqrt{6-r^2}$

$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=6$

$x^2+y^2+x^2+y^2=6$

$2x^2+2y^2=6$

$x^2+y^2=3=(\sqrt{3})^2$

$0\le r\le\sqrt{3}$

$0\le\theta\le\pi/2$

$r\le z\le\sqrt{6-r^2}$

10.2 Triple Integrals in Spherical Coordinates

FRY Defn III.3.7.1, Spherical coordinates

Definition 10.5. The spherical coordinates of a point in three-dimensional space are denoted by ρ, θ, and ϕ, where

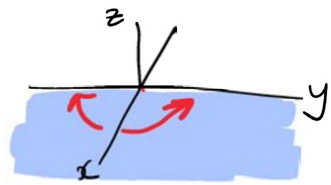
(i) ρ represent the distance from the origin $(0, 0, 0)$ to the point,

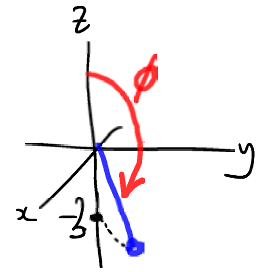
(ii) θ is the angle between the positive x-axis and the line segment from the origin to the projection of the point onto the xy-plane, and

(iii) ϕ is the angle between the z-axis and the line segment from the origin to the point.

The equations

- $\rho = \rho_0$, where ρ_0 is a constant, describes a sphere;
- $\theta = \theta_0$, where θ_0 is a constant, describes a plane; and
- $\phi = \phi_0$, where ϕ_0 is a constant, describes a cone.





Given (ρ, θ, ϕ),

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

Given (x, y, z),

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$

Notes:

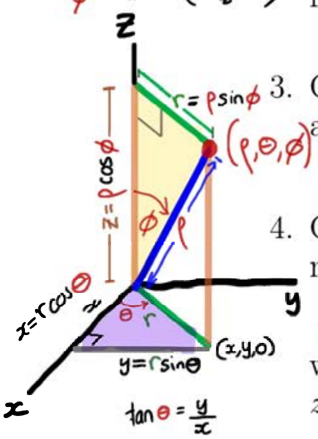
1. If the x- and y-coordinates are such that $(x, y, 0)$ lies in the second or third quadrant of the xy-plane, then we add π to $\arctan\left(\frac{y}{x}\right)$ to get the correct value for θ .

$r = \sqrt{x^2+y^2}$

$\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$

$\phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

2. Similarly, when $z < 0$, in order to get ϕ to lie in the interval $[0, \pi]$, we add π to the above formula for ϕ. (Alternatively, if we don't want to worry about making such adjustments to get the correct value of ϕ, we could simply use the formula $\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$).



3. Given the spherical coordinates (ρ, θ, ϕ), the cylindrical coordinates of the point are
$$r = \rho\sin\phi \quad \qquad z = \rho\cos\phi$$
$$\cancel{r = \rho \sin\theta}, \quad \theta = \theta, \quad \cancel{z = \rho\cos\theta}.$$

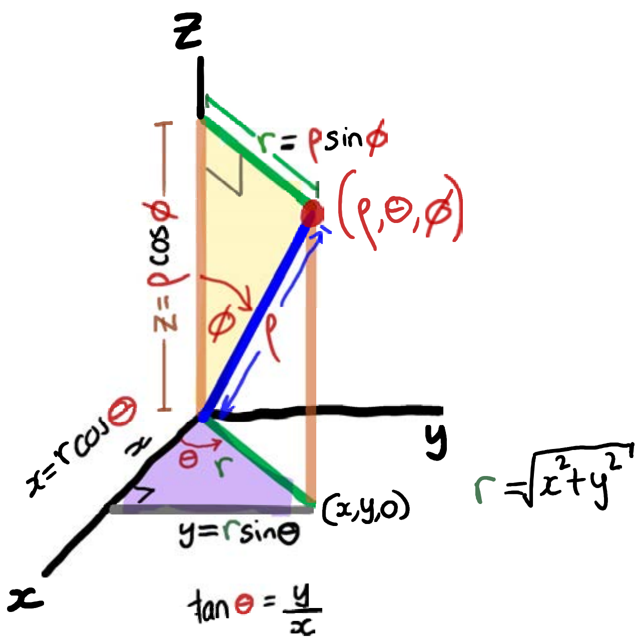
4. Given the cylindrical coordinates (r, θ, z), the corresponding spherical coordinates are
$r = \sqrt{x^2+y^2}$
$$\rho = \sqrt{r^2+z^2}, \quad \theta = \theta, \quad \phi = \arctan\left(\frac{r}{z}\right),$$
with the adjustment referred to above made to the arctan computation when $z < 0$.

If $\mathbf{g}$ denotes the change of variable transformation from (x, y, z)-coordinates into (ρ, θ, ϕ)-coordinates, then

$x = r\cos\theta$
$= \rho\sin\phi\cos\theta$

$y = r\sin\theta$
$= \rho\sin\phi\sin\theta$

$$D\mathbf{g}(\rho, \phi, \theta) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}.$$



$$x = r\cos\theta$$
$$= \rho\sin\phi\cos\theta$$

$$y = r\sin\theta$$
$$= \rho\sin\phi\sin\theta$$

The determinant of the derivative matrix of the change of variables transformation $\mathbf{g}$, through cofactor expansion along the third row, is

$$
\begin{aligned}
&\det D\mathbf{g}(\rho,\theta,\phi)\\
&= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta +\right)\\
&\qquad - \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)\\
&= \cos\phi\left(-\rho^2\sin\phi\cos\phi\right) - \rho\sin\phi\left(\rho\sin^2\phi\right)\\
&= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi\\
&= -\rho^2\sin\phi.
\end{aligned}
$$

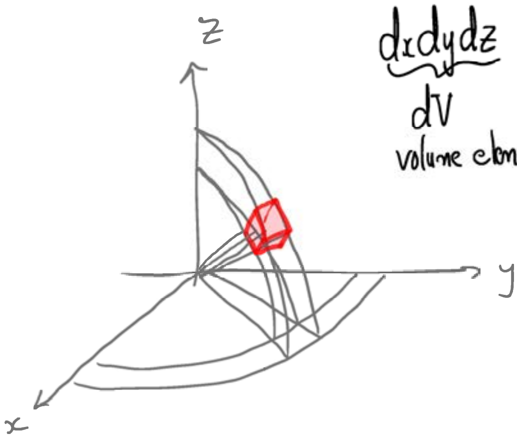
Thus,

$$|\det D\mathbf{g}(\rho,\theta,\phi)| = \overset{|-\rho^2\sin\phi|}{\cancel{|\rho^2\sin\phi|}} = \rho^2\sin\phi,$$

where we have dropped the absolute sign because both ρ^2 and $\sin\phi$ are nonnegative, the latter since $\phi \in [0,\pi]$. We use this information to adjust the volume element dV when changing from Cartesian to spherical coordinates:

$$\iiint_{\mathcal{R}} f(x,y,z)\ dV = \iiint_{\mathcal{R}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \underbrace{\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}_{dV},$$

though we may use a different order of integration depending on the domain of integration $\mathcal{R}$.



$\underbrace{dxdydz}_{dV}$

volume element

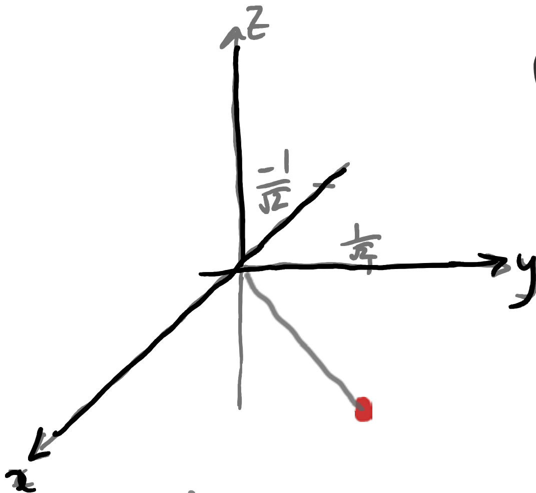
$(x,y,z) \xmapsto{\vec{g}} (\rho,\theta,\phi)$

$|\det D\vec{g}| = \rho^2\sin\phi$

$$\rho = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\left(\frac{y}{x}\right) \qquad \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

Example 10.6. Convert from the Cartesian coordinates $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{3}\right)$ to spherical coordinates.

$\quad x \quad y \quad z$



$$\rho = \sqrt{\frac{1}{2}+\frac{1}{2}+3} = 2$$

$$\theta = \arctan\left(\frac{\frac{1}{\sqrt{2}}}{-1/\sqrt{2}}\right) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

$$\phi = \arctan\left(\frac{\sqrt{\frac{1}{2}+\frac{1}{2}}}{-\sqrt{3}}\right) = \frac{-\pi}{6} + \pi = \frac{5\pi}{6}$$