3.1 Related Rates

Introduction.

In a related rates problem, we are given the rate of change of one quantity, and we are asked to find the rate of change of a related quantity. To do this, we find an equation that relates the two quantities and use the Chain Rule to differentiate both sides of the equation with respect to time.

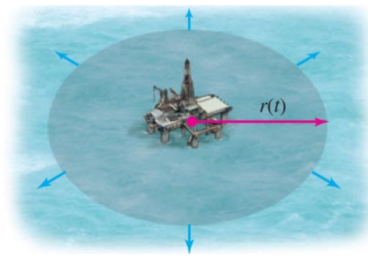
The volume V of a sphere, for instance, is related to its radius r through the formula $V(r) = \frac{4}{3}\pi r^3$. If the radius changes, then the volume changes accordingly. Knowing how V is related to r makes it possible to find out how the rate of change of volume $\frac{dV}{dt}$ is related to the rate of change of radius $\frac{dr}{dt}$. We may not know in detail how the dimensions of an object vary with time, but we should treat both the radius and the volume as implicit functions of t.

Steps for Solving Related Rate Problems:

1. Read the problem carefully. Identify and name the variables.
2. Draw a diagram, if possible, to illustrate the problem and label the appropriate parts
3. Introduce notation. Assign symbols to all quantities that are functions of time
4. Express the given information and the required rate in terms of derivatives
5. Write an equation that related the various quantities of the problem. If necessary, use geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t
7. Substitute the given information into the resulting equation and solve for the unknown rate

Exercise 3.1 Related Rates

1. An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/hr, how fast is the area of the patch increasing when the patch has a radius of 100 meters?



Solution: Two variables change simultaneously: the radius of the circle ($\frac{dr}{dt}$) and its area ($\frac{dA}{dt}$). The key relationship between the radius and area is $A = \pi r^2$. Since the area is related to the changing radius, the Chain Rule yields:

$$\frac{dA}{dt} = \frac{dA}{dr} . \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r . \frac{dr}{dt}$$

Substituting the given values $r = 100\ m$ and $\frac{dr}{dt} = 30\ m/hr$,

$$\frac{dA}{dt} = 2\pi(100)(30) = 6000\pi\ m^2/hr$$

2. When a small pebble is dropped in a pool of still water, it produces a circular wave that travels outward at a constant speed of 15 cm/s. At what rate is the area inside the wave increasing:

a. When the radius is 5 cm?

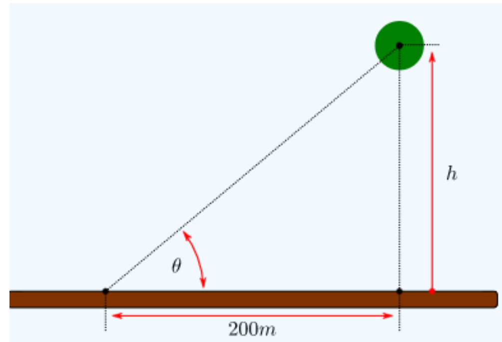
b. When the area is 400π cm^2?

c. When 4 seconds have elapsed?

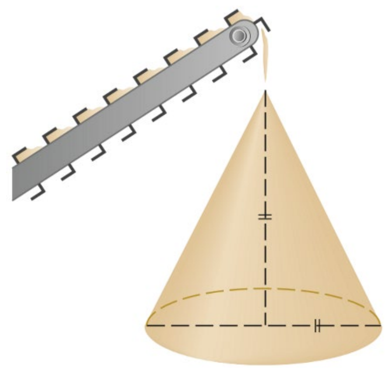
3. A water tank is built in the shape of a circular cone with height 5 m and diameter 6m at the top. Water is being pumped into the tank at a rate of 1.6 m^3/min. Find the rate at which the water level is rising when the water is 2 m deep. ($V = \frac{1}{3}\pi r^2 h$)

4. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing? ($V = \pi r^2 h$)

5. An observer stands 200 meters from the launch site of a hot-air balloon. The balloon rises vertically at a constant rate of 4 m$/s$. How fast is the angle of elevation of the balloon increasing 30 seconds after the launch? (The angle of elevation is the angle between the ground and the observer's line of sight to the balloon.)



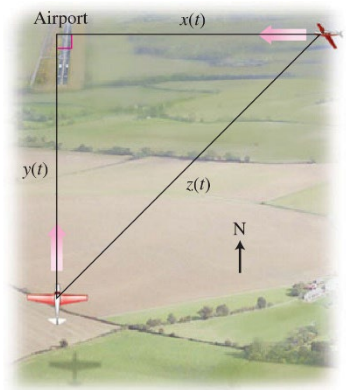
6. Gravel is being dumped from a conveyor belt at a rate of 3 ft^3/min , and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 *ft* high?



7. A conveyor belt system at a gravel pit pours washed sand onto the ground at the rate of 180 m^3/h. The sand forms a conical pile with height of one third of the diameter of the base. Find out how fast the height of the pile is increasing at the instant when the radius of the base is 6m.

8. (Text 3.2.5) Two particles move in the Cartesian plane. Particle A travels on the x-axis starting at (10,0) and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at (0,12) and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point (4,0)?

9. Two small planes approach an airport, one flying due west at 120 mi/hr and the other flying due north at 150 mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 225 miles from the airport?



10. A man 6ft tall walks with a speed of 8 ft/s away from a street light that is atop an 18ft tall pole. Haw fast is the tip of his shadow moving along the ground when he is 100 ft from the light pole?

11. A ladder 6 meters long leaning against a wall begins to slide. How fast is the top of the ladder falling, when the bottom of the ladder is 4 meters from the wall and sliding at a speed of 1 m/s.

12. A person standing close to the edge on top of a 56-foot building throws a ball vertically upward. The quadratic function $h(t) = -16t^2 + 104t + 56$ models the ball's height about the ground, $h(t)$, in feet, t seconds after it was thrown.

a. What is the maximum height of the ball? (Ans. 225 ft)

b. How many seconds does it take until the ball hits the ground? (Ans. 7 s)

Answers.

2. a) area increases at a rate of 150π cm^2/s; area increases at a rate of 600π cm^2/s; 1800π cm^2/s;
3. $\frac{10}{9\pi}$ m/min
4. $\frac{3}{25\pi}$ m/min
5. The angle of elevation increases at a rate of $\frac{1}{68}$ rad/s thirty seconds after the launch.
6. 0.12 ft/m
7. $\frac{5}{\pi}$ m/hr
8. The distance between A and B shortens at a rate of $\frac{17}{5}$ units/s when particle A is in position (4,0).
9. The distance between converging planes decreases at a rate of $30\sqrt{41}$ mi/hr when planes are 180 mi and 225 mi from the airport.

10. The tip of his shadow is moving at a constant speed of 12 ft/s away from the base of the streetlight.
11. The top of the ladder is falling at a rate of $\frac{4}{\sqrt{20}}$ m/s.
12. a. 225 ft; b. 7s