Signal Processing (MENG3520)

Module 5

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Midterm Exam

- Date: February 25, 2025
- Scope: All modules and labs covered up to the midterm
- Weight: 20% of the final grade
- Duration: 120 minutes
- Permitted Aids: Basic calculator, provided formula sheet (closed book)
- Format:
  - Multiple-choice questions (3%×10 = 30%)
  - Short-answer questions (5%×5 = 25%)
  - Problem-solving questions (15%×3 = 45%)

Example

Problem 4.1-2. By direct integration, compute the Laplace transform of the following signals.

(a) $e^{-2t}u(t-5)+\delta(t-1)$

Answer:

$$X(s)=\mathcal{L}\{x(t)\}=\int_{-\infty}^{+\infty}x(t)e^{-st}dt=\int_{-\infty}^{+\infty}(e^{-2t}u(t-5)+\delta(t-1))e^{-st}dt$$

$$=\int_{-\infty}^{+\infty}e^{-2t}u(t-5)e^{-st}dt+\int_{-\infty}^{+\infty}\delta(t-1)e^{-st}dt=\int_{5}^{+\infty}e^{-(s+2)t}dt+e^{-s}$$

$$=\frac{-1}{(s+2)}e^{-(s+2)t}\Big|_{t=5}^{t=+\infty}+e^{-s}=\frac{e^{-5(s+2)}}{(s+2)}+e^{-s}$$

ROC is $Re\{s+2\}>0$, $Re\{s\}>-2$.

Example

Problem 4.1-2. By direct integration, compute the Laplace transform of the following signals.

(b) $\pi e^{3t}u(t+5) - \delta(2t)$

Answer:

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_{-\infty}^{+\infty} (\pi e^{3t}u(t+5) - \delta(2t))e^{-st}dt$$

$$= \int_{-\infty}^{+\infty} \pi e^{3t}u(t+5)e^{-st}dt - \int_{-\infty}^{+\infty} \delta(2t)e^{-st}dt = \int_{-5}^{+\infty} \pi e^{(-s+3)t}dt - \frac{1}{2}\int_{-\infty}^{+\infty} \delta(2t)e^{-\frac{s}{2}(2t)}d2t$$

$$= \frac{1}{(-s+3)}\pi e^{(-s+3)t}\Big|_{t=-5}^{t=+\infty} - \frac{1}{2}\int_{-\infty}^{+\infty} \delta(\tau)e^{-\frac{s}{2}(\tau)}d\tau = \frac{\pi e^{5s-15}}{(s-3)} - \frac{1}{2}$$

ROC is $Re\{-s+3\} < 0$, $Re\{s\} > 3$.

Example

Problem 4.1-2. By direct integration, compute the Laplace transform of the following signals.

(c) $\sum_{k=0}^{\infty} \delta(t-kT)\,, T>0$

Answer:

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_{-\infty}^{+\infty} \left(\sum_{k=0}^{\infty} \delta(t-kT) \right) e^{-st}dt$$

$$= \sum_{k=0}^{\infty} \left(\int_{-\infty}^{+\infty} \delta(t-kT)e^{-st}dt \right) = \sum_{k=0}^{\infty} e^{-kTs} = \frac{1}{1-e^{-Ts}}$$

For $T>0$, ROC is $Re\{s\}>0$ so that $\sum_{k=0}^{\infty} e^{-kTs}$ converges.

Example

Problem 4.3-11. For a system with transfer function $H(s) = \frac{2s+3}{s^2+2s+5}$.

(a) Find the zero-state response for inputs $x_1(t) = 10u(t)$ and $x_2(t) = u(t-5)$.

Answer:

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)e^{-st}dt = \int_{-\infty}^{+\infty} 10u(t)e^{-st}dt = \int_{0}^{+\infty} 10\, e^{-st}dt = \frac{10}{s}$$

ROC is $Re\{s\} > 0$ so that $\int_0^{+\infty} 10\, e^{-st}dt$ converges.

Zero-state response is:

$$Y_1(s) = X_1(s)H(s) = \frac{10(2s+3)}{s(s^2+2s+5)} = \frac{6}{s} + \frac{-6s+8}{(s^2+2s+5)} = \frac{6}{s} - \frac{6(s+1)}{(s+1)^2+4} + \frac{14}{(s+1)^2+4}$$

Since Laplace pairs:

$$e^{-at}\cos(bt)\,u(t) \leftrightarrow \frac{(s+a)}{(s+a)^2+b^2}, \qquad e^{-at}\sin(bt)\,u(t) \leftrightarrow \frac{b}{(s+a)^2+b^2}, \qquad u(t) \leftrightarrow \frac{1}{s}$$

$$y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\} = 6u(t) - 6e^{-t}\cos(2t)\,u(t) + 7e^{-t}\sin(2t)\,u(t)$$

Example

Problem 4.3-11. For a system with transfer function $H(s) = \frac{2s+3}{s^2+2s+5}$.

(a) Find the zero-state response for inputs $x_1(t) = 10u(t)$ and $x_2(t) = u(t-5)$.

Answer: if input $x_2(t) = u(t-5)$, by using time shifting property $x(t-t_0) \overset{\mathcal{L}}{\leftrightarrow} e^{-st_0}X(s)$.

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \frac{e^{-5s}}{s}$$

ROC is $Re\{s\} > 0$.

Zero-state response is:

$$Y_2(s) = X_2(s)H(s) = \frac{e^{-5s}(2s+3)}{s(s^2+2s+5)} = \frac{e^{-5s}}{10}Y_1(s)$$

Again, using time shifting property:

$$y_2(t) = \frac{1}{10}y_1(t-5)$$
$$= 0.6u(t-5) - 0.6e^{-(t-5)}\cos(2t-10)\,u(t-5) + 0.7e^{-(t-5)}\sin(2t-10)\,u(t-5)$$

Example

Problem 4.3-11. For a system with transfer function $H(s) = \frac{2s+3}{s^2+2s+5}$.

(b) Write the differential equation relation the out $y(t)$ to the input $x(t)$.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s+3}{s^2+2s+5}$$

$$(s^2+2s+5)Y(s) = (2s+3)X(s)$$

$$s^2Y(s) + 2sY(s) + Y(s) = 2sX(s) + 3X(s)$$

Because of the differentiation property: $\frac{d}{dt}x(t) \overset{\mathcal{L}}{\leftrightarrow} sX(s)$, transform the above equation into the time domain:

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = 2\frac{d}{dt}x(t) + 3x(t)$$

Example

Problem 4.3-14. An LTI system with a step response given by $s(t) = e^{-t}u(t) - e^{-2t}u(t)$. Determine the output of this system $y(t)$ given an input $x(t) = \delta(t-\pi) - \cos(\sqrt{3})\,u(t)$.

Answer: unit impulse response is the first order derivative of the step response, thus the system impulse response $h(t)$ is:

$$h(t) = \frac{d}{dt}s(t) = \frac{d}{dt}(e^{-t}u(t) - e^{-2t}u(t)) = e^{-t}\frac{d}{dt}\big(u(t)\big) - e^{-t}u(t) - e^{-2t}\frac{d}{dt}\big(u(t)\big) + 2e^{-2t}u(t)$$
$$= e^{-t}\delta(t) - e^{-t}u(t) - e^{-2t}\delta(t) + 2e^{-2t}u(t) = \delta(t) - e^{-t}u(t) - \delta(t) + 2e^{-2t}u(t)$$
$$= -e^{-t}u(t) + 2e^{-2t}u(t)$$

Now the given input $x(t)$ is a linear combination of a time-shifted impulse function and a unit step function, thus the response $y(t)$ to this input $x(t)$ is the linear combination of the time-shifted impulse response and a step response.

$$y(t) = h(t-\pi) - \cos(\sqrt{3})\,s(t)$$
$$= -e^{-(t-\pi)}u(t-\pi) + 2e^{-2(t-\pi)}u(t-\pi) - \cos(\sqrt{3})\,(e^{-t}u(t) - e^{-2t}u(t))$$

Overview

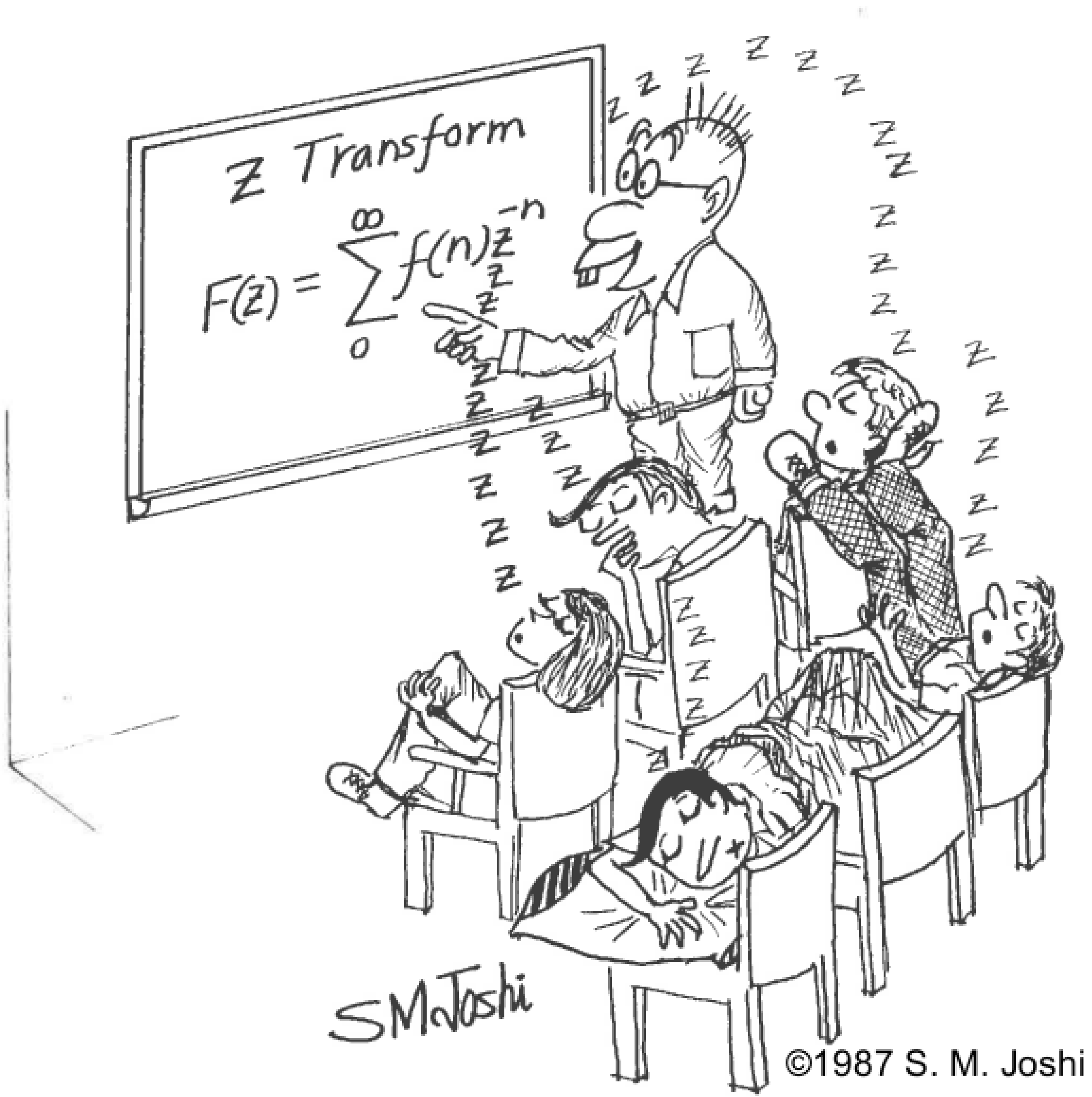
- Every analysis method used in continuous time has a corresponding discrete time counterpart.
- The counterpart of the Laplace transform is the z-transform.
- The z-transform expresses DT signals as linear combination of DT complex exponential.
- The z-transform is critical in modern digital signal processing and system analysis because of the widely adaption of digital signals and systems.

Module Outline

- 5.1 Eigen-sequences of DT LTI systems
- 5.2 Definition of z transform and inverse z transform
- 5.3 Relation between the Laplace transform and z transform
- 5.4 Z plane, poles, and zeros.
- 5.5 Properties of the z-transform

MODULE 5

Z TRANSFORM



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5.1

Eigen-Sequences of DT LTI Systems

- Recall how we derived the Laplace transform.
- If a CT system $h(t)$ is LTI, its Laplace transform is $H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$. The complex exponentials e^{st} are its eigenfunctions.

$$x(t) = e^{st} \longrightarrow \boxed{H(s)} \longrightarrow y(t) = x(t)H(s)$$

- This means that if the input is a complex exponential, the system behaves as an ideal amplifier.

- Now we are hoping to establish similar transform for the discrete time signals. Instead of eigen-functions, now we need to find certain eigen-sequences $x[n]$ such that for a system:

$$y[n] = H\{x[n]\} = \lambda x[n]$$

- The question is, what are the eigen-sequences for an DT LTI system?

Activity: consider a DT system

$$H\{x[n]\} = x[n] - x[n-1]$$

Prove that $x[n] = n$ is NOT an eigen-sequence for this system.

Proof:

$$H\{x[n]\} = x[n] - x[n-1]$$
$$= n - (n-1) = 1$$
$$= \frac{1}{n}x[n] \neq \lambda x[n]$$

- Consider a DT LTI system with impulse response $h[n]$.
- Let $x[n] = z^n$ be a complex exponential excitation / input to the system.
- Output:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Since $x[n] = z^n$:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

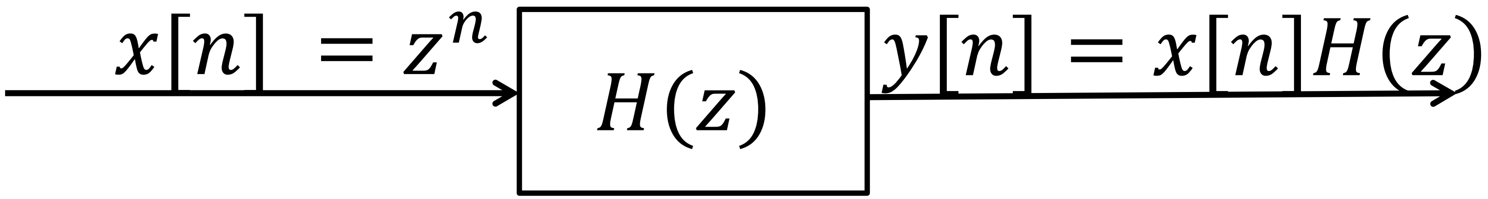
$$= x[n] \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

$$y[n] = \underbrace{z^n}_{x[n]} \underbrace{\boxed{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}}_{H(z)}$$

z^n is the eigen-sequence of this DT LTI system, and $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ is the corresponding eigenvalue for this eigen-sequence.

$$y[n] = H(z)x[n]$$

- Conclusion: complex exponentials z^n are the eigen-sequences of DT LTI systems.



- This property is only valid for LTI systems, not time varying or non-linear systems.
- This property only holds if $\sum_{k=-\infty}^{\infty} h[k]z^{-k}$ converges.

Applications of eigen-sequences: similar to CT, we can use the property of eigen-sequences to avoid convolution.

- For a DT LTI system, for input $x[n] = \sum_k a_k z_k^n$, where a_k and z_k are complex constants.
- Because it's LTI, and z_k^n are the eigen-sequences of the system, the response to each $a_k z_k^n$ is $a_k z_k^n H(z_k)$.
- Thus, $y[n] = \sum_k a_k z_k^n H(z_k)$
- Conclusion: if an input to a DT LTI system is a linear combination of complex exponentials, the output is also a linear combination of the same complex exponentials.

- Activity: Consider a DT LTI system with impulse response $h[n] = \delta[n-1]$. Find its system function and compute the response $y[n]$ to input $x[n] = 2e^{n}\cos(\pi n)$.
- Answer: the z transform of the impulse response is:

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} = \sum_{k=-\infty}^{\infty} \delta[k-1]z^{-k} = z^{-1}$$

Input sequence is:

$$x[n] = 2e^n\cos(\pi n) = e^n\left(e^{j\pi n} + e^{-j\pi n}\right) = e^{(1+j\pi)n} + e^{(1-j\pi)n}$$

Let $z_1 = e^{(1+j\pi)}$, $x_1[n] = e^{(1+j\pi)n} = z_1^n$

Let $z_2 = e^{(1-j\pi)}$, $x_2[n] = e^{(1-j\pi)n} = z_2^n$

Since it is a LTI system with $H(z) = z^{-1}$, the output is:

$$y[n] = \sum_k a_k z_k^n H(z_k) = z_1^n H(z_1) + z_2^n H(z_2)$$
$$= z_1^n z_1^{-1} + z_2^n z_2^{-1} = z_1^{n-1} + z_2^{n-1} = e^{(1+j\pi)(n-1)} + e^{(1-j\pi)(n-1)}$$
$$= e^{n-1}\left(e^{j\pi(n-1)} + e^{-j\pi(n-1)}\right) = 2e^{n-1}\cos(\pi(n-1))$$

5.2

DEFINITION OF z-TRANSFORM

The bilateral z-transform of any DT sequence $x[n]$ is:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

The inverse z-transform is defined as:

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz$$

Where Γ is the counter-clockwise closed circular contour centered at the origin with radius *r.*

In practice, we often refer $x[n]$ and $X(z)$ a z-transform pair.

- The z-transform is denoted using the z-transform symbol $\mathcal{Z}$:

$$x[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)$$

Or

$$X(z) = \mathcal{Z}\{x[\mathrm{n}]\}$$

Two types of z-transforms:

- The unilateral z-transform (summation from 0 to $+\infty$) is for solving differential equations with non-zero initial conditions.
- The bilateral z-transform (summation from $-\infty$ to $+\infty$) offers insight into the nature of system characteristics such as stability, causality, and frequency response.
- The difference between the two is the lower bound of the summation.
- The unilateral z-transform is a special case of the bilateral z-transform.

Example of z-Transforms

Compute the z-transform of $x[n] = a^n u[n]$

Compute the z-transform of $x[n] = a^n u[n]$

Solution: $X(z) = \mathcal{Z}\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$

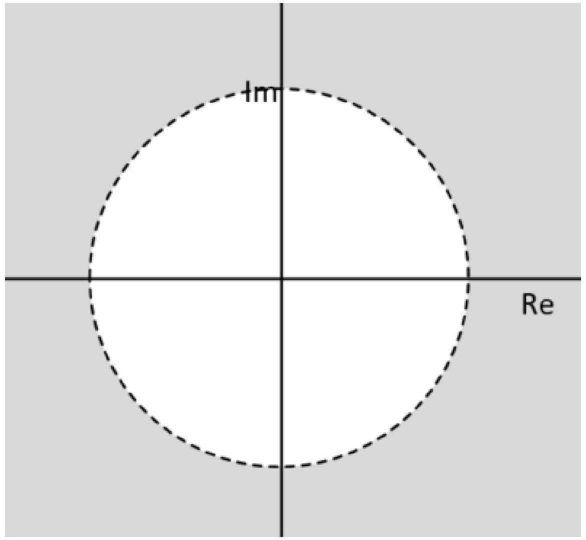
$$= \sum_{k=0}^{\infty} a^k z^{-k}$$

Converges when, $\left|\frac{a}{z}\right| < 1.$

Thus:

$$X(z) = \frac{z}{z-a}$$

With ROC $|z|>|a|$.



Example of z-Transforms

Compute the z-transform:

$$x[n] = -a^n u[-n-1]$$

Compute the z-transform:

$$x[n] = -a^n u[-n-1]$$

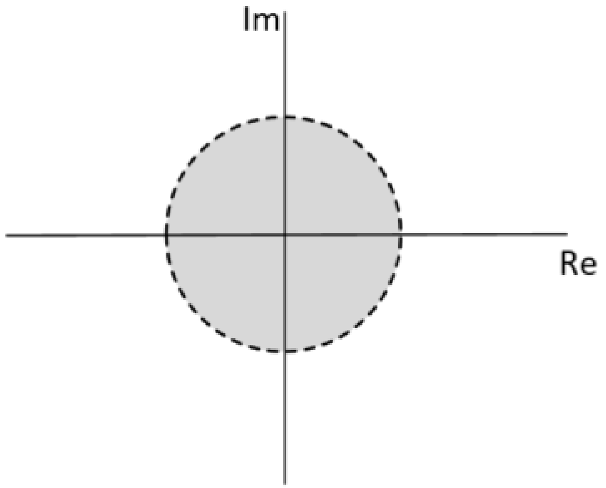
Solution: $X(z) = \mathcal{Z}\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$

$$= -\sum_{k=-\infty}^{-1} a^k z^{-k} = -\sum_{k=1}^{\infty} a^{-k} z^k$$

Converges when, $\left|\frac{z}{a}\right| < 1$.

Thus:

$X(z) = \frac{z}{z-a}$, ROC is $|z|<|a|$.



5.3

Relation Between the Laplace Transform and the z-Transform

Let us examine the z-transform from a different angle, starting from how the DT signal is generated. It can be viewed as a sampled signal:

$$x_s(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

Its Laplace transform is:

$$X(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)] = \sum_n \left(x(nT_s) \int_{-\infty}^{+\infty} \delta(t - nT_s)e^{-st}dt \right)$$

$$X(s) = \sum_n x(nT_s)e^{-nsT_s}$$

Let us compare the Laplace transform of the sampled signal and its intended z-transform:

$$X(s) = \sum_{n} x(nT_s) e^{-nsT_s}$$

$$X(z) = \mathcal{Z}\{x[nT_s]\} = \sum_{n} x[nT_s] z^{-n}$$

Let $z = e^{sT_s}$, then both equations are equivalent.

Conclusions:

- The relation $z = e^{sT_s}$ provide the connection between the s-plane and the z-plane.

  Considering $s = \sigma + j\Omega$,

  $$z = e^{sT_s} = e^{(\sigma + j\Omega)T_s} \triangleq re^{j\omega}$$

  Then $r = e^{\sigma T_s}$ and $\omega = \Omega T_s$.

- Here ω is the digital frequency, and Ω is the analog frequency.
- Sampling period T_s is a critical parameter that ties the mapping between the s-plane and the z-plane.

5.4

The z-Plane, Poles and Zeros

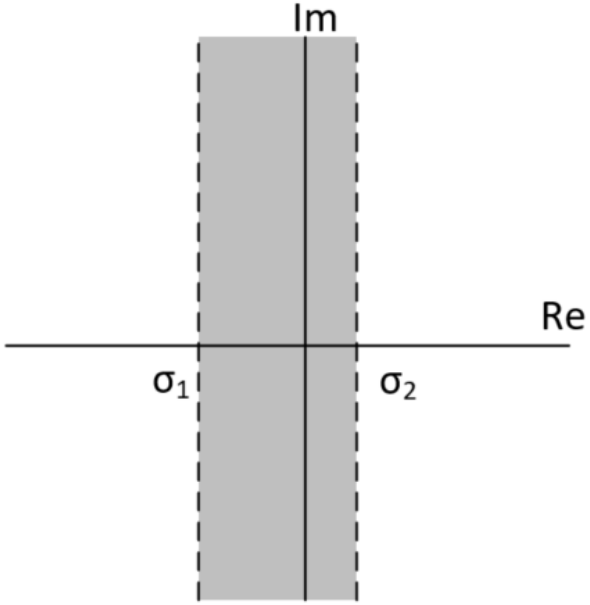
The z-transform convert the DT domain signals and system into the complex z-plane.

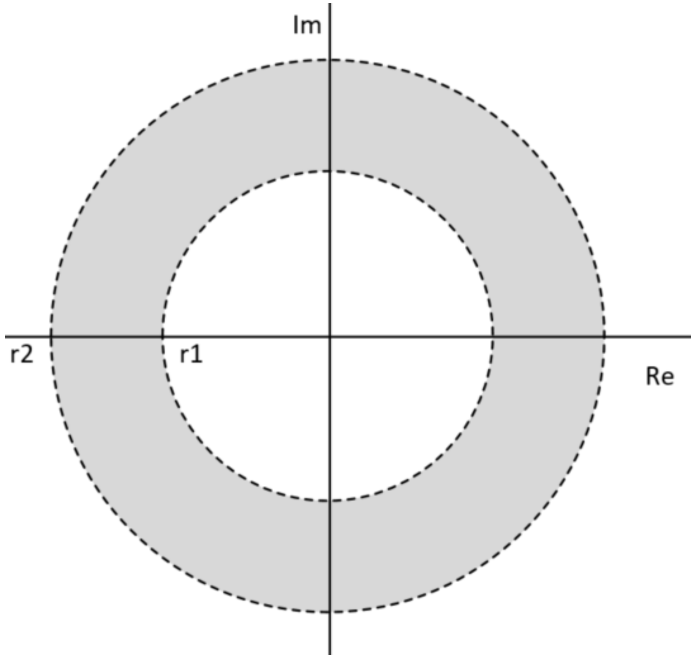
$$z = re^{j\omega}$$

Compared to the Laplace transform:

$$s = \sigma + \Omega$$

$r = e^{\sigma T_s}$ and $\omega = \Omega T_s$.





For any rational function $F(z) = N(z)/D(z)$.

- Zeros: points on the z-plane where the values of *z* that make the function $F(z) = 0$. indicated on z-plane as “o”.
- Poles: points on the z-plane where the values of *z* that make the function $F(z) \to \infty$. Indicated on z-plane as “x”.

Example: compute the z-transform and find its zeros and poles:

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

Solution: $X(z) = \mathcal{Z}\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$

$$= 7 \sum_{k=-0}^{\infty} \left(\frac{1}{3}\right)^k z^{-k} - 6 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k} = \frac{7}{1 - \frac{1}{3z}} - \frac{6}{1 - \frac{1}{2z}}$$

$$= \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}$$

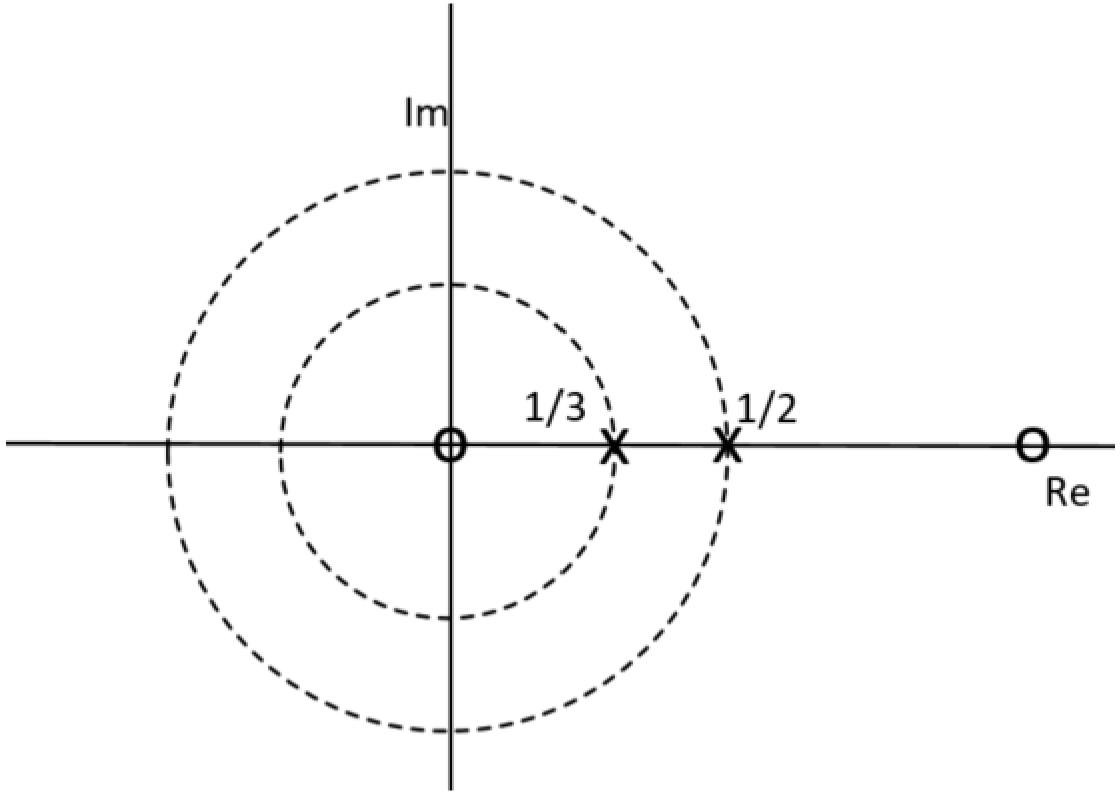
$|z| > 1/3$ $\quad\quad$ $|z| > 1/2$

Zeros: z = 0, z = 3/2

Poles: z = 1/3, z = 1/2

Solution:

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, |z| > 1/2$$



Relation between ROC and Poles:

- Property 1: The ROC consists of a ring in the z-plane centered about the origin.
- This means that it is the damping r that defines the ROC, not frequency ω.

Relation between ROC and Poles:

- Property 2: For rational z-transforms, no poles are included in the ROC.
- The ROC is the region where the z-transform is defined, whilst the poles are where the transform becomes non-convergent.

Relation between ROC and Poles:

- Property 3: if $x[n]$ is of finite duration and is absolutely summable, then ROC is the entire z-plane, except either $z = 0$ or $|z| = \infty$.
- $X(z) = \sum_{k=m1}^{m2} x[k]z^{-k}$
- $X(z)$ will converge, if each term of $x[k]$ is finite.

Relation between ROC and Poles:

- Property 4: if $x[n]$ is right-sided, and if the circle $|z| = r_0$ is part of the ROC, then all values of z for which $|z| > r_0$ are also in the ROC.
- This means that for right-sided $x[n]$, if there exists a real value $|z| = r_0$ where the transform converges, all the points to the outside of that ring are also in the ROC.
- r^{-n} is decaying faster toward $+\infty$ than ${r_0}^{-n}$ for $r > r_0$

Relation between ROC and Poles:

- Property 5: if $x[n]$ is left-sided, and if the circle $|z| = r_0$ is part of the ROC, then all values of z for which $0 < |z| < r_0$ are also in the ROC.
- This means that for left-sided $x[n]$, if there exists a real value $|z| = r_0$ where the transform converges, all the points to the inside of that ring (except for the origin) are also in the ROC.
- The origin needs special attention because if the summation contains a negative power z term, it is not converging at the origin.

Relation between ROC and Poles:

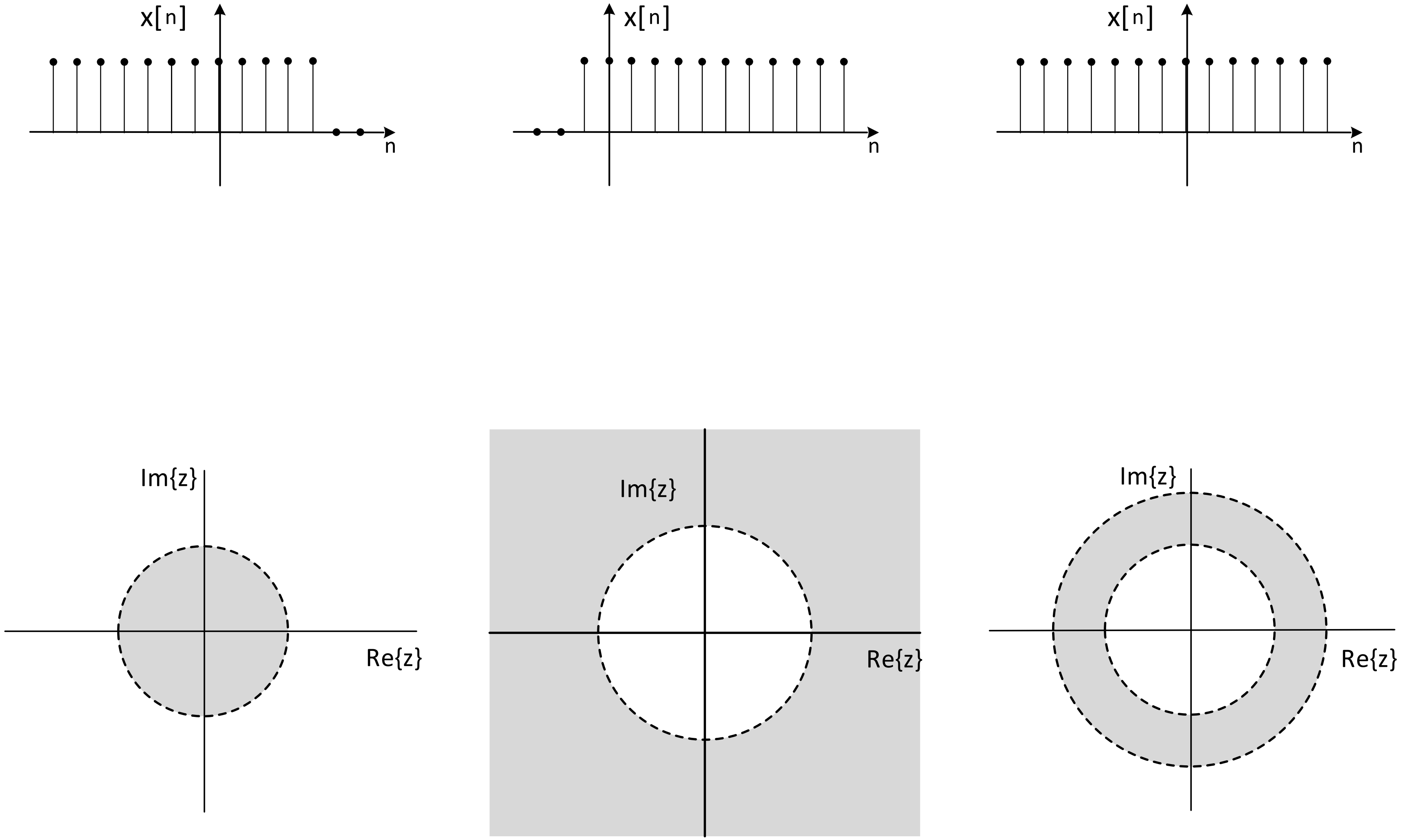
- Property 6: if $x[n]$ is two-sided, and if the line $|z| = r_0$ is part of the ROC, then the ROC will consists of a ring in the z-plane that includes the circle $|z| = r_0$.
- Break $x[n]$ into the sum of a right-sided and a left-sided signals.

Relation between ROC and Poles:

- Property 7: if the z-transform $X(z) = \mathcal{Z}\{x[n]\}$ is rational, then its ROC is bounded by poles or extends to infinity.

Relation between ROC and Poles:

- Property 8: if the z-transform $X(z) = \mathcal{Z}\{x[n]\}$ is rational, then:
- If $x[n]$ is right-sided, then the ROC is the region in the z-plane outside the outermost pole.
- If $x[n]$ is left-sided, then the ROC is the region in the z-plane inside the innermost non-zero pole except the origin.
- If $x[n]$ is two-sided, then you need to break the signal into right-sided and left-sided, the ROC is the region inside the innermost non-zero pole (r-) except the origin corresponding to the left-sided signal, and outside the outermost pole (r+) corresponding to the right-sided signal. $r+ < |z| < r-$


x[n]
n
x[n]
n
x[n]
n
Im{z}
Re{z}
Im{z}
Re{z}
Im{z}
Re{z}

Activity: what is the ROC of the z-transform of the following?

(a) $x_1[n] = \left(-\frac{1}{2}\right)^n u[-n] + \left(\frac{1}{3}\right)^n u[-n]$

(b) $x_2[n] = \left(-\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

(c) $x_3[n] = \left(-\frac{1}{2}\right)^n u[-n] + \left(\frac{1}{3}\right)^n u[n]$

(a) $x_1[n] = \left(-\frac{1}{2}\right)^n u[-n] + \left(\frac{1}{3}\right)^n u[-n]$

Sol: $|z| < \frac{1}{3}$

(b) $x_2[n] = \left(-\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

Sol: $|z| > \frac{1}{2}$

(c) $x_3[n] = \left(-\frac{1}{2}\right)^n u[-n] + \left(\frac{1}{3}\right)^n u[n]$

Sol: $\frac{1}{3} < |z| < \frac{1}{2}$

(d) $x_4[n] = \left(-\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[-n]$

Sol: doesn't exist.

5.5

PROPERTIES OF z-TRANSFORM

Most properties of the z-transform are analogous to its CT counterpart, the Laplace transform.

- **Linearity of the z-transform**: if
- $x_1[n] \overset{z}{\leftrightarrow} X_1(z)$, ROC=$R_1$
- $x_2[n] \overset{z}{\leftrightarrow} X_2(z)$, ROC=$R_2$

- $ax_1[n] + bx_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z)$
- ROC containing $R_1 \cap R_2$

- **Time shift**:
- If: $x[n] \overset{z}{\leftrightarrow} X(z)$, ROC=R
- Then: $x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}X(z)$, ROC=R, with possible addition of origin or $|z| = \infty$
- Multiplication of z^{-n_0} may:
  - introduce a pole at the origin if $n_0 > 0$
  - introduce a pole at the ∞ if $n_0 < 0$

- **Scaling in the z-domain:**
- If: $x[n] \overset{z}{\leftrightarrow} X(z)$, ROC=R
- Then:

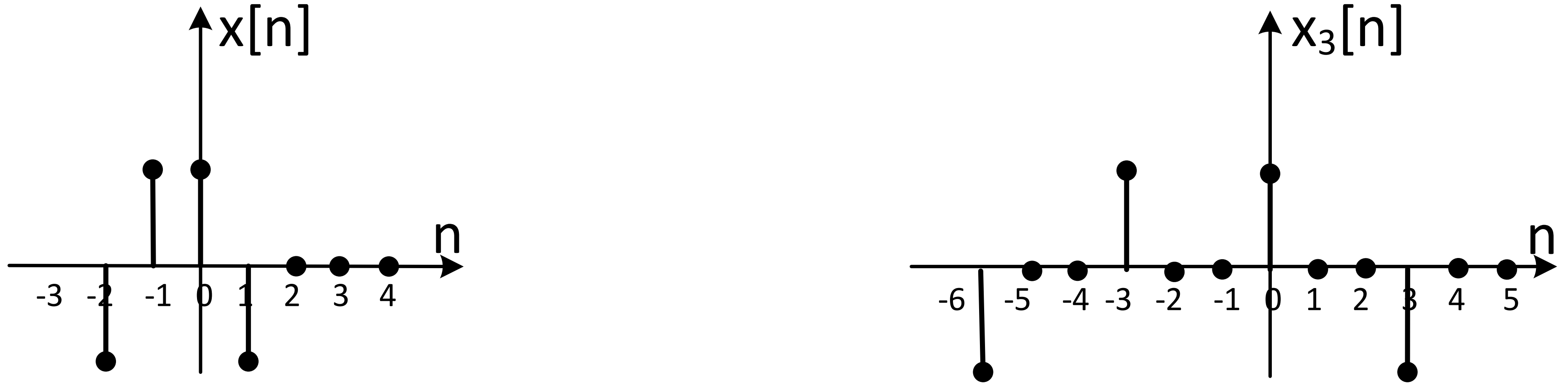
$$z_0^n x[n] \overset{z}{\leftrightarrow} X\left(\frac{z}{z_0}\right)$$

- ROC=$|z_0|$R.

- **Time reversal**:
- If: $x[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)$, ROC=R
- Then: $x[-n] \overset{\mathcal{Z}}{\leftrightarrow} X\left(\frac{1}{z}\right)$
- ROC=$\frac{1}{R}$, i.e. if z_0 belong to the ROC of x[n], then $1/z_0$ belongs to the ROC of x[-n]

- **Time expansion (scaling)**:
- If: $x[n] \overset{z}{\leftrightarrow} X(z)$, ROC=R
- $x_k[n] = \begin{cases} x\left[\frac{n}{k}\right], if\ n\ is\ multiple\ of\ k \\ 0, if\ n\ is\ not\ multiple\ of\ k \end{cases}$
- $x_k[n] \overset{z}{\leftrightarrow} X\left(z^k\right)$
- ROC=$R^{1/k}$.

- **Time expansion (scaling)**:
- $x_k[n] = \begin{cases} x\left[\frac{n}{k}\right], if\ n\ is\ multiple\ of\ k \\ 0, if\ n\ is\ not\ multiple\ of\ k \end{cases}$
- Example, let $k = 3$, if $x[n]$ is defined as the figure on the left, then, $x_3[n]$ is the signal on the right.



- **Time expansion (scaling)**:
- To prove $x_k[n] \overset{z}{\leftrightarrow} X\left(z^k\right)$, let's write the z transform of $x_k[n]$:
- $X_k(z) = \sum_{n=-\infty}^{\infty} x_k[n] z^{-n} = \sum_{m=-\infty}^{\infty} x[mk/k] z^{-mk} = \sum_{m=-\infty}^{\infty} x[m] \left(z^k\right)^{-mk} = X\left(z^k\right)$

- **Conjugation**:
- If: $x[n] \overset{z}{\leftrightarrow} X(z)$, ROC=R
- Then:$x^*[n] \overset{z}{\leftrightarrow} X^*(z^*)$, ROC=R

- If $x[n]$ is real, then $X(z) = X(z^*)$
- This means for real $x[n]$, its poles and zeros in the z-domain must be either real-valued or occur in conjugate pairs.

- **Convolution Property**: if
- $x_1[n] \overset{z}{\leftrightarrow} X_1(z)$, ROC=$R_1$
- $x_2[n] \overset{z}{\leftrightarrow} X_2(z)$, ROC=$R_2$

- $x_1[n] * x_2[n] \overset{z}{\leftrightarrow} X_1(z)X_2(z)$
- ROC contains $R_1 \cap R_2$

- **Differentiation in the z-domain**:
- If: $x[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)$, ROC=R
- Then:$nx[n] \overset{\mathcal{Z}}{\leftrightarrow} -z\frac{d}{dz}X(z)$, ROC=R

- Multiplication by *n* in the time domain corresponds to differentiation with respect to *z* and multiplication of the result by -*z* in the *z*-domain. The ROC remains unchanged.

- **Accumulation in the time domain**:
- If: $x[n] \overset{z}{\leftrightarrow} X(z)$, ROC=R
- Then:

$$\sum_{k=-\infty}^{n} x[k] \overset{z}{\leftrightarrow} \frac{1}{1 - z^{-1}} X(z)$$

- ROC contains R and |z|>1

The Initial-value Theorem

- For causal signals $x[n] = 0$ for $n < 0$, and $x[n]$:
- $x[0] = \lim_{z \to \infty} X(z)$

- Left-hand side: discrete time-domain
- Right-hand side: z-domain

Why do we want to learn these properties?

Because inverse z-transform is very mathematically demanding and we want to avoid this, so usually we break the X(z) into terms with known time function.

Please refer to tables in the next slide for the list of z-transform properties and list of commonly used z-transform pairs.

Homework:

Review: in-class examples, textbook chapter 5. Section 5.1, 5.2.

Textbook examples: 5.2, 5.3, 5.4.

Problems: 5.1-3, 5.1-4, 5.1-7, 5.2-4

Textbook Example 5.2.c

Find the z-transforms of $\cos(\beta n)u[n]$.

Answer:

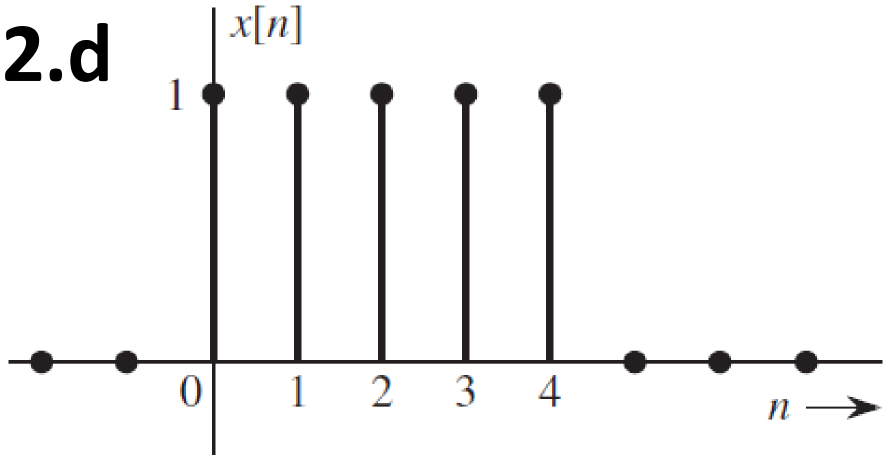
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=0}^{+\infty} \cos(\beta n)u[n]z^{-n} = \sum_{n=0}^{+\infty} \cos(\beta n)z^{-n}$$

$$= \sum_{n=0}^{+\infty} \frac{1}{2}\left(e^{j\beta n} + e^{-j\beta n}\right)z^{-n} = \sum_{n=0}^{+\infty} \frac{1}{2} e^{j\beta n} z^{-n} + \sum_{n=0}^{+\infty} \frac{1}{2} e^{-j\beta n} z^{-n}$$

$$= \frac{1}{2}\frac{z}{z - e^{j\beta}} + \frac{1}{2}\frac{z}{z - e^{-j\beta}}$$

In order to converge, $\left|\frac{e^{j\beta}}{z}\right| < 1$, $\left|\frac{1}{e^{j\beta}z}\right| < 1$, thus $|z| > 1$.

Textbook Example 5.2.d



Find the z-transforms of $x[n]$ shown in the figure.

Answer:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=0}^{4} z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} = \frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

In order to converge, ROC is all $z \neq 0$.

Textbook Example 5.3.a

Find the inverse z-transforms of: $X(z) = \frac{8z-19}{(z-2)(z-3)}$ by partial fraction expansion and tables.

Answer: first use partial fraction expansion to expand $X(z) = \frac{8z-19}{(z-2)(z-3)} = \frac{a}{z-2} + \frac{b}{z-3}$.

Here we need to use equality to find coefficients a and b.

Since $a(z-3) + b(z-2) \; = \; 8z - 19$

We have: $a + b = 8, -3a - 2b = -19$

Thus $a = 3, \mathrm{b} = 5$.

From Table 5.1, we know that $x[n] = \gamma^{n-1}u[n-1] \xleftrightarrow{z\,transform} \frac{1}{z-\gamma}$.

Therefore, $x[n] = (3(2)^{n-1} + 5(3)^{n-1})u[n-1]$.

Textbook Example 5.3.c

Find the inverse z-transforms of: $X(z) = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$ by partial fraction expansion and tables.

Answer: first expand $\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{a}{z-1} + \frac{bz+c}{z^2-6z+25}$.

Here we need to use equalities to find coefficients a, b and c. We can use Heaviside coverup method to find coefficient a first to reduce the workload.

Multiply both side by $(z-1)$ and let $z = 1$, we have $a = 2$.

Since: $a(z^2 - 6z + 25) + (bz + c)(z - 1) = a(z^2 - 6z + 25) + bz^2 + (c - b)z - c = 6z + 34$

We have: $a + b = 0, \quad -6a + (c - b) = 6, \quad 25a - c = 34$

Thus $b = -2, c = 16$.

$$X(z) = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2 - 6z + 25}$$

Textbook Example 5.3.c

Find the inverse z-transforms of: $X(z) = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$ by partial fraction expansion and tables.

Answer:

$$X(z) = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25} = \frac{2z}{z-1} - \frac{2z(z-3)}{z^2-6z+25} + \frac{10z}{z^2-6z+25}$$

From Table 5.1, we have: $\beta = \cos^{-1} 0.6$, $|\gamma| = 5$, $\sin\beta = 0.8$.

$|\gamma|^n \sin(\beta n)\, u[n] \leftrightarrow \frac{z|\gamma| \sin\beta}{z^2-(2|\gamma| \cos\beta)z+|\gamma|^2}$, $|\gamma|^n \cos(\beta n)\, u[n] \leftrightarrow \frac{z(z-|\gamma| \cos\beta)}{z^2-(2|\gamma| \cos\beta)z+|\gamma|^2}$

Therefore, $x[n] = 2u[n-1] - 2(5)^n \cos\beta n\, u[n] + 2.5(5)^n \sin\beta n\, u[n]$

Textbook Problem 5.1-4.d

Use the definition of the z-transform to find the z-transforms of $x[n] = \gamma^n \cos\frac{\pi n}{2} u[n]$ and the corresponding ROC.

Answer: $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=0}^{+\infty} \gamma^n \cos\frac{\pi n}{2} u[n]z^{-n} = \sum_{n=0}^{+\infty} \gamma^n \cos\frac{\pi n}{2} z^{-n} = \sum_{n=0}^{+\infty} \gamma^{2n} (\cos \pi n) z^{-2n} = \sum_{n=0}^{+\infty} \gamma^{2n} (-1)^n z^{-2n} = \sum_{n=0}^{+\infty} (-\frac{\gamma^2}{z^2})^n = \frac{1}{1+\frac{\gamma^2}{z^2}} == \frac{z^2}{z^2+\gamma^2}$

It converges when $\left|\frac{\gamma^2}{z^2}\right| < 1$, thus ROC is $|z| > |\gamma|$

Textbook Problem 5.1-4.j

Use the definition of the z-transform to find the z-transforms of $x[n] = \frac{\gamma^n}{n!}u[n]$ and the corresponding ROC.

Answer: $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=-\infty}^{+\infty} \frac{\gamma^n}{n!}u[n]z^{-n} = \sum_{n=0}^{+\infty} \frac{\gamma^n}{n!} z^{-n} = \sum_{n=0}^{+\infty} \frac{\left(\frac{\gamma}{z}\right)^n}{n!}$

Since we know that $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$, then:

$$X(z) = \sum_{n=0}^{+\infty} \frac{\left(\frac{\gamma}{z}\right)^n}{n!} = e^{\left(\frac{\gamma}{z}\right)}$$

It converges when$z \neq 0$, thus ROC is $|z| > 0$

Textbook Problem 5.1-7.l

Find the inverse unilateral z-transform for $X(z) = \frac{z^2(-2z^2+8z-7)}{(z-1)(z-2)^3}$.

Answer: $\frac{X(z)}{z} = \frac{z(-2z^2+8z-7)}{(z-1)(z-2)^3} = \frac{a}{z-1} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + \frac{d}{(z-2)^3}$

Here we need to find coefficients a, b, c and d.

Step 1. First, we use Heaviside coverup to find a and d:

To find a, multiply both sides with $(z-1)$ and let $z = 1$, we have $a = 1$.

To find d, multiple both side with $(z-2)^3$ and let $z = 2$, we have $d = 2$.

Step 2. Now we have:

$$\frac{X(z)}{z} = \frac{z(-2z^2+8z-7)}{(z-1)(z-2)^3} = \frac{1}{z-1} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + \frac{2}{(z-2)^3}$$

$$(z-2)^3 + b(z-1)(z-2)^2 + c(z-1)(z-2) - (z-1) = -2z^2 + 8z - 7$$

Textbook Problem 5.1-7.l

Find the inverse unilateral z-transform for $X(z) = \frac{z^2(-2z^2+8z-7)}{(z-1)(z-2)^3}$.

Answer: $\frac{X(z)}{z} = \frac{z(-2z^2+8z-7)}{(z-1)(z-2)^3} = \frac{1}{z-1} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + \frac{2}{(z-2)^3}$

Step 2. Now we have:

$$\begin{aligned}&(z-2)^3 + b(z-1)(z-2)^2 + c(z-1)(z-2) - (z-1)\\ &= (z^3 - 6z^2 + 12z - 8) + (bz - 2b + c)(z-1)(z-2) + 2(z-1)\\ &= -2z^3 + 8z^2 - 7z\end{aligned}$$

We have: $1 + b = -2, -8 + 2(-2b + c) - 2 = 0,$

Thus $b = -3, c = -1.$

$$X(z) = \frac{1}{z-1} - \frac{3}{z-2} - \frac{1}{(z-2)^2} + \frac{2}{(z-2)^3}$$

It converges when$z \neq 0$, thus ROC is $|z| > 0$

Textbook Problem 5.1-7.l

Find the inverse unilateral z-transform for $X(z) = \frac{z^2(-2z^2+8z-7)}{(z-1)(z-2)^3}$.

Answer: $\frac{X(z)}{z} = \frac{z(-2z^2+8z-7)}{(z-1)(z-2)^3} = \frac{1}{z-1} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + \frac{2}{(z-2)^3}$

Step 2. Now we have:

$$\begin{aligned}
&(z-2)^3 + b(z-1)(z-2)^2 + c(z-1)(z-2) - (z-1) \\
&= (z^3 - 6z^2 + 12z - 8) + (bz - 2b + c)(z-1)(z-2) + 2(z-1) \\
&= -2z^3 + 8z^2 - 7z
\end{aligned}$$

We have: $1 + b = -2, -8 + 2(-2b + c) - 2 = 0,$ thus $b = -3, c = -1.$

$$X(z) = \frac{z}{z-1} - \frac{3z}{z-2} - \frac{z}{(z-2)^2} + \frac{2z}{(z-2)^3}$$

It converges when$z \neq 0$, thus ROC is $|z| > 0$

Textbook Problem 5.1-7.l

Find the inverse unilateral z-transform for $X(z) = \frac{z^2(-2z^2+8z-7)}{(z-1)(z-2)^3}$.

Answer: $\frac{X(z)}{z} = \frac{z(-2z^2+8z-7)}{(z-1)(z-2)^3} = \frac{1}{z-1} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + \frac{2}{(z-2)^3}$

Step 2. Now we have:

$$(z-2)^3 + b(z-1)(z-2)^2 + c(z-1)(z-2) - (z-1)$$
$$= (z^3 - 6z^2 + 12z - 8) + (bz - 2b + c)(z-1)(z-2) + 2(z-1) = -2z^3 + 8z^2 - 7z$$

We have: $1 + b = -2, -8 + 2(-2b + c) - 2 = 0,$ thus $b = -3, c = -1.$

$$X(z) = \frac{z}{z-1} - \frac{3z}{z-2} - \frac{z}{(z-2)^2} + \frac{2z}{(z-2)^3}$$

Then you can use the table to find :

$$x(n) = \left[1 - 3(2)^n - \frac{n}{2}(2)^{n-1} + \frac{n(n-1)}{4}(2)^{n-2}\right]u[n]$$