arclength of $\mathcal{C}$ $= \displaystyle\int_{\mathcal{C}} ds = \int_{t\in[a,b]} \|\vec{r}\,'(t)\|\,dt$

Surface area of $\mathcal{S}$ $= \displaystyle\iint_{\mathcal{S}} dS = \iint_{(u,v)\in D} \|\vec{r}_u \times \vec{r}_v\|\,dA$

mass of $\mathcal{C}$ $= \displaystyle\int_{\mathcal{C}} \delta\, ds$ ← density

mass of surface $\mathcal{S}$ $= \displaystyle\iint_{\mathcal{S}} \delta\, dS$

$$\int_{\mathcal{C}} f\,ds \qquad\qquad \iint_{\mathcal{S}} f\,dS$$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} \qquad\qquad \iint_{\mathcal{S}} \vec{F}\cdot\hat{n}\,dS$$

Surface Area

Let $\mathbf{r}(u, v)$ be a parametrization of a surface $\mathcal{S}$ that lies in $\mathbb{R}^3$. Then

$$\text{surface area}(\mathcal{S}) = \iint_{\mathcal{S}} dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA,$$

where $\mathbf{r}_u$ and $\mathbf{r}_v$ are the functions we get by taking partial derivatives of $\mathbf{r}$ with respect to u and v, respectively.

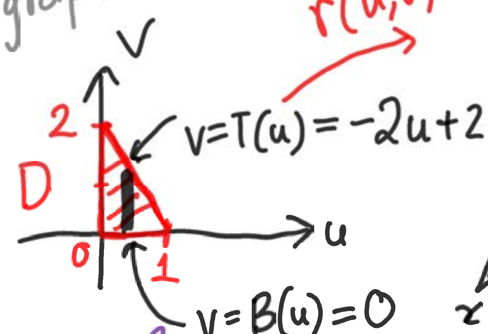
Example: Surface Area

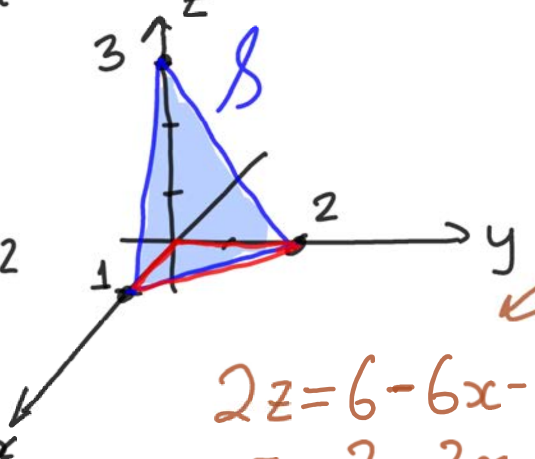
Find the surface area of the portion $\mathcal{S}$ of the plane $6x + 3y + 2z = 6$ that lies in the first octant.

Surface area of $\mathcal{S}$ $= \iint_{\mathcal{S}} dS$ ← upper case S $= \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA$

↑ calligraphic $\mathcal{S}$

$\vec{r}(u,v)$





$2z = 6 - 6x - 3y$

$z = 3 - 3x - \frac{3}{2}y$

Let $u=x$

Let $v=y$

1st: Parametrize $\mathcal{S}$

Let $\vec{r}(u,v) = \langle u, v, 3-3u-\frac{3}{2}v \rangle$

where $(u,v) \in D$, where

$D = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le -2u+2\}$

If S is graph $z = f(x,y)$
then $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$.

2nd: Work out $\|\vec{r}_u \times \vec{r}_v\|$

$$\vec{r}(u,v) = \langle u, v, 3 - 3u - \tfrac{3}{2}v \rangle$$

$$\vec{r}_u(u,v) = \langle 1, 0, -3 \rangle$$

$$\vec{r}_v(u,v) = \langle 0, 1, -\tfrac{3}{2} \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 1 & -3/2 \end{vmatrix}$$

$$\vec{r}_u(u,v) \times \vec{r}_v(u,v) = \langle 3, \tfrac{3}{2}, 1 \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{3^2 + \left(\tfrac{3}{2}\right)^2 + 1^2} = \frac{7}{2}$$

3rd: Calculate surface area

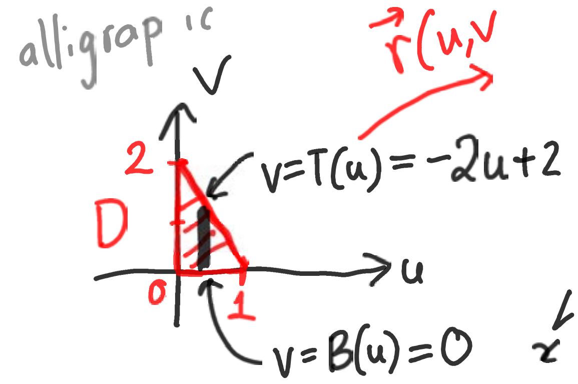
$$\text{Surface area}(S) = \iint_S dS = \iint_D \|\vec{r}_u \times \vec{r}_v\|\, dA$$

$$= \int_{u=0}^{1} \int_{v=0}^{-2u+2} \frac{7}{2} \, dv\,du$$

$$= \frac{7}{2} \int_{u=0}^{1} v \Big|_{0}^{-2u+2} du$$

$$= \frac{7}{2} \int_{u=0}^{1} (-2u+2)\, du$$

$$= \frac{7}{2} \left[-u^2 + 2u \right]_0^1 = \frac{7}{2}\left(\underbrace{[-1+2]}_{=1} - \underbrace{[-0+0]}_{=0} \right) = \frac{7}{2}$$



Surface Integral of Scalar-Valued Functions

Let

- $\mathbf{r} : D \to \mathbb{R}^3$ be parametrization of surface $\mathcal{S}$ in $\mathbb{R}^3$, and
- f be a continuous real-valued function.

Then the surface integral of the scalar function f over the surface $\mathcal{S}$ is given by

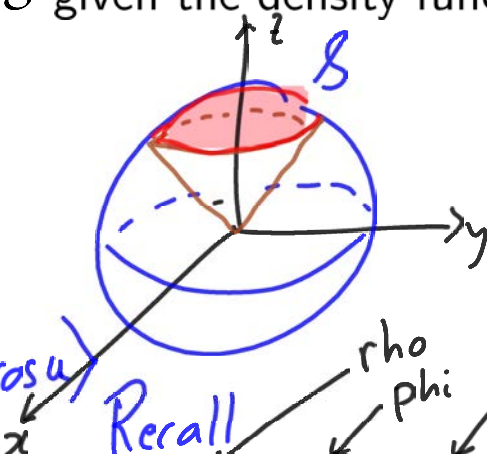
$$\iint_{\mathcal{S}} f \; dS = \iint_{D} f(\mathbf{r}(u, v)) \, \|\mathbf{r}_u \times \mathbf{r}_v\| \; dA.$$

Example: Surface Integral of Scalar-Valued Function

radius 2

Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = 4$ for which $z \geq \sqrt{x^2 + y^2}$. Find the mass of $\mathcal{S}$ given the density function $\delta(x, y, z) = z$.

$z = \sqrt{x^2 + y^2}$ cone



1st Parametrize S

$\vec{r}(u,v) = \langle 2\sin u\cos v, 2\sin u\sin v, 2\cos u\rangle$

where $(u,v) \in D$

where $D = \{(u,v) : 0 \leq u \leq \frac{\pi}{4}, 0 \leq v \leq 2\pi\}$

Work at $\|\vec{r}_u \times \vec{r}_v\|$

Recall

rho phi theta

$x = 2\sin\phi\cos\theta$

$y = 2\sin\phi\sin\theta$

$z = 2\cos\phi$

Let $u = \phi$ and $v = \theta$