HUMBER ENGINEERING

MENG-3020
SYSTEMS MODELING & SIMULATION

LECTURE 8

HUMBER

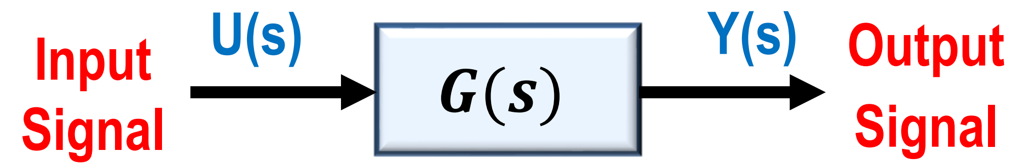
WE ARE HUMBER

LECTURE 8
System Analysis in Time Domain

- Response of First-Order Systems
- Response of Second-Order Systems
- Description and Specification of Step Response
- Stability of Systems

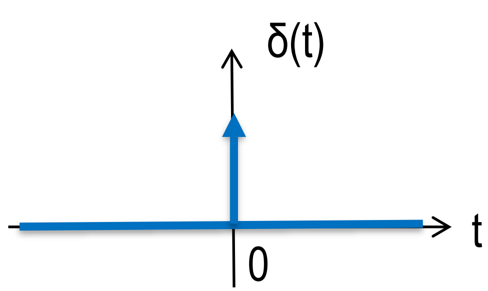
System Analysis in Time Domain

- Consider the following linear system with transfer function of $G(s)$
- We can use transfer functions to determine how the output of a system will change with <u>time</u> for a particular types of input.

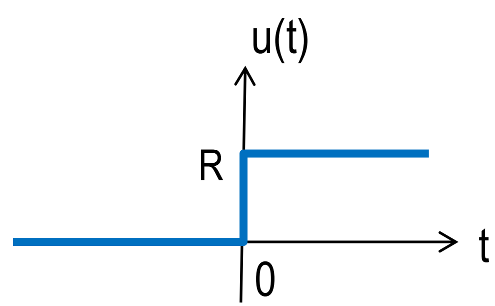


$$G(s) = \frac{Y(s)}{U(s)} \quad \rightarrow \quad Y(s) = G(s)U(s)$$

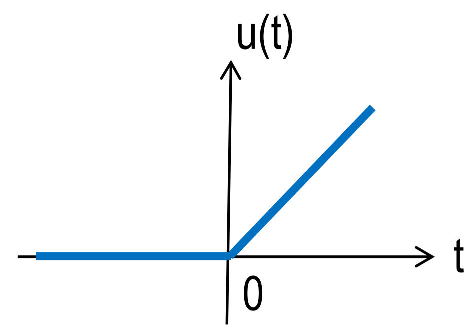
- In time response analysis of dynamic systems, the goal is:
  - To analyze and characterize input-output behavior of the system
  - To know how the system is performing
- We apply different test signals as an input $u(t)$ to study the system's time response $y(t)$.
  - **Impulse Function: $\delta(t)$**
  - **Step Function: $u(t) = R,\ t \geq 0$**
  - **Ramp Function: $u(t) = Rt,\ t \geq 0$**
  - **Parabolic Function: $u(t) = Rt^2,\ t \geq 0$**



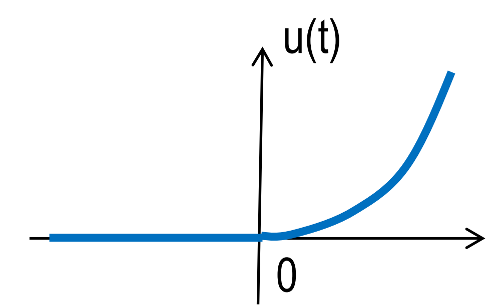
Effect of sudden large impact on the system



System's ability to track sudden input changes



System's ability to track varying input changes by constant rate



System's ability to track varying input faster than ramp function

First Order Systems

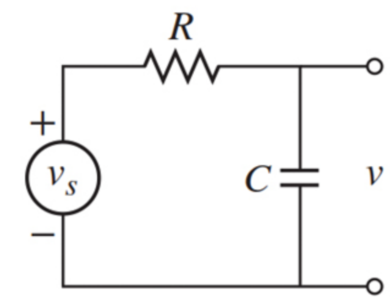
❑ Transfer Function Model

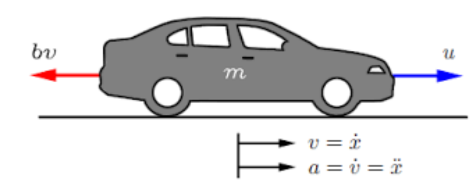
- First-order systems are systems whose input-output relationship is a first-order differential equation.

Differential Equation →
$$\boxed{\tau y'(t) + y(t) = K u(t)}$$

- First-order systems have a single energy-storage element.

- RC and RL Electric Circuits
- Single-Tank Liquid Level System
- Thermal Heating System
- Speed Control System



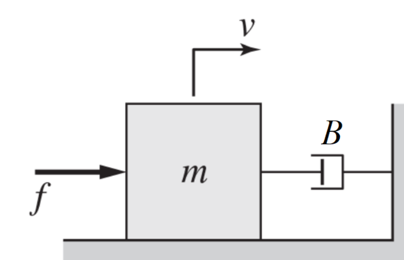


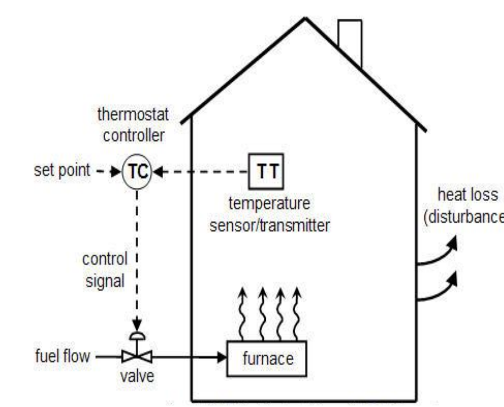
- **Standard Form of a First-Order Transfer Function**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

Steady-state gain (K)

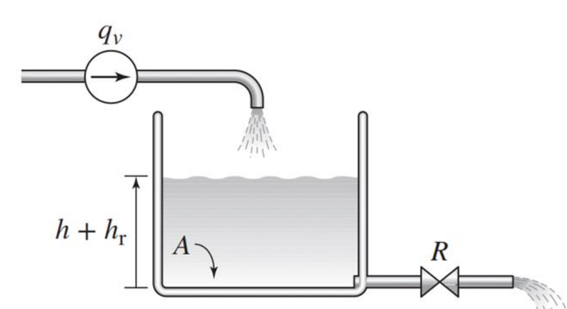
Time constant (τ)





Home-heating System

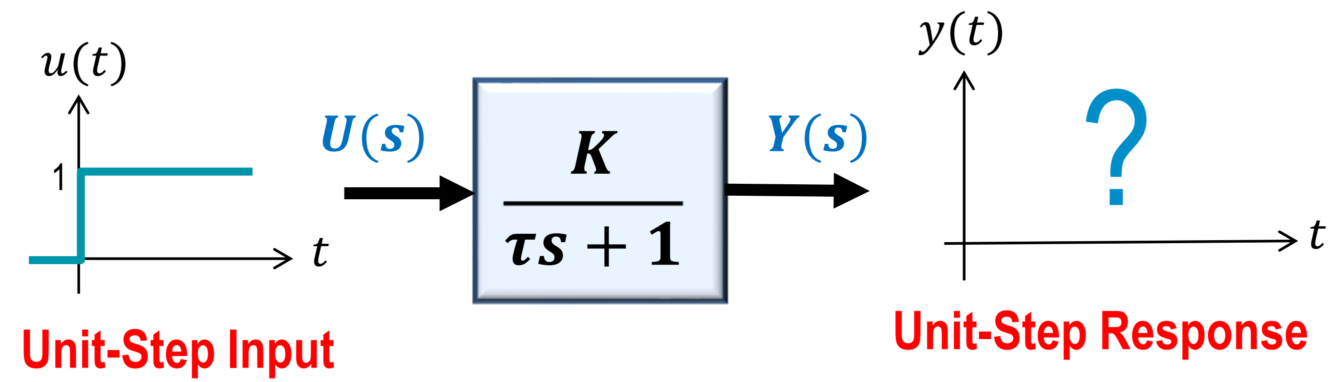
- The system order can also be defined as the order of the denominator polynomial of the transfer function, which is called Characteristic Equation.



First Order Systems

❑ Step Response

- Assume the standard first-order system:
- Unit-step response of a first-order system is determined as,



$$Y(s) = G(s)U(s)$$

$$Y(s) = \left(\frac{K}{\tau s+1}\right)\left(\frac{1}{s}\right) = \frac{K}{s(\tau s+1)} = \frac{K}{s} + \frac{-\tau K}{\tau s+1}$$

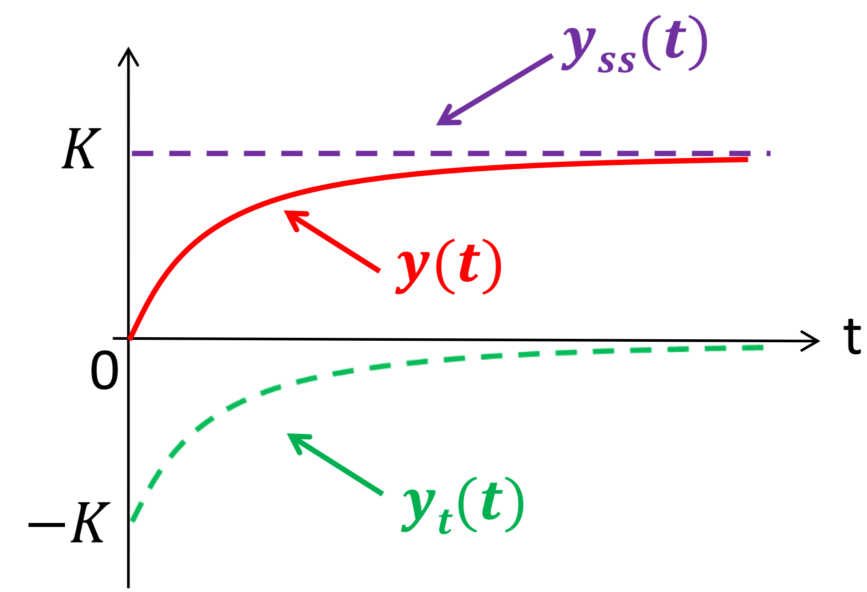
$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{K}{s}\right] + \mathcal{L}^{-1}\left[\frac{-\tau K}{\tau s+1}\right]$$

$$\boxed{\boldsymbol{y(t) = K - Ke^{-t/\tau}, \qquad t \geq 0}}$$

Unit-Step Response

Steady-State Response $y_{ss}(t)$

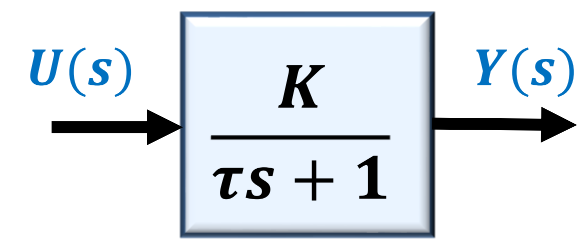
Transient Response $y_t(t)$



- **Steady-state Response:** Approaches to a <u>constant value</u> as $t \to \infty$
- **Transient Response:** Approaches to <u>zero</u> as $t \to \infty$

First Order Systems Parameters

❑ Time-Constant



- The step-response of a first-order system is an exponential curve.

$$\boxed{y(t) = K - Ke^{-t/\tau}, \qquad t \geq 0}$$ **Unit-step Response**

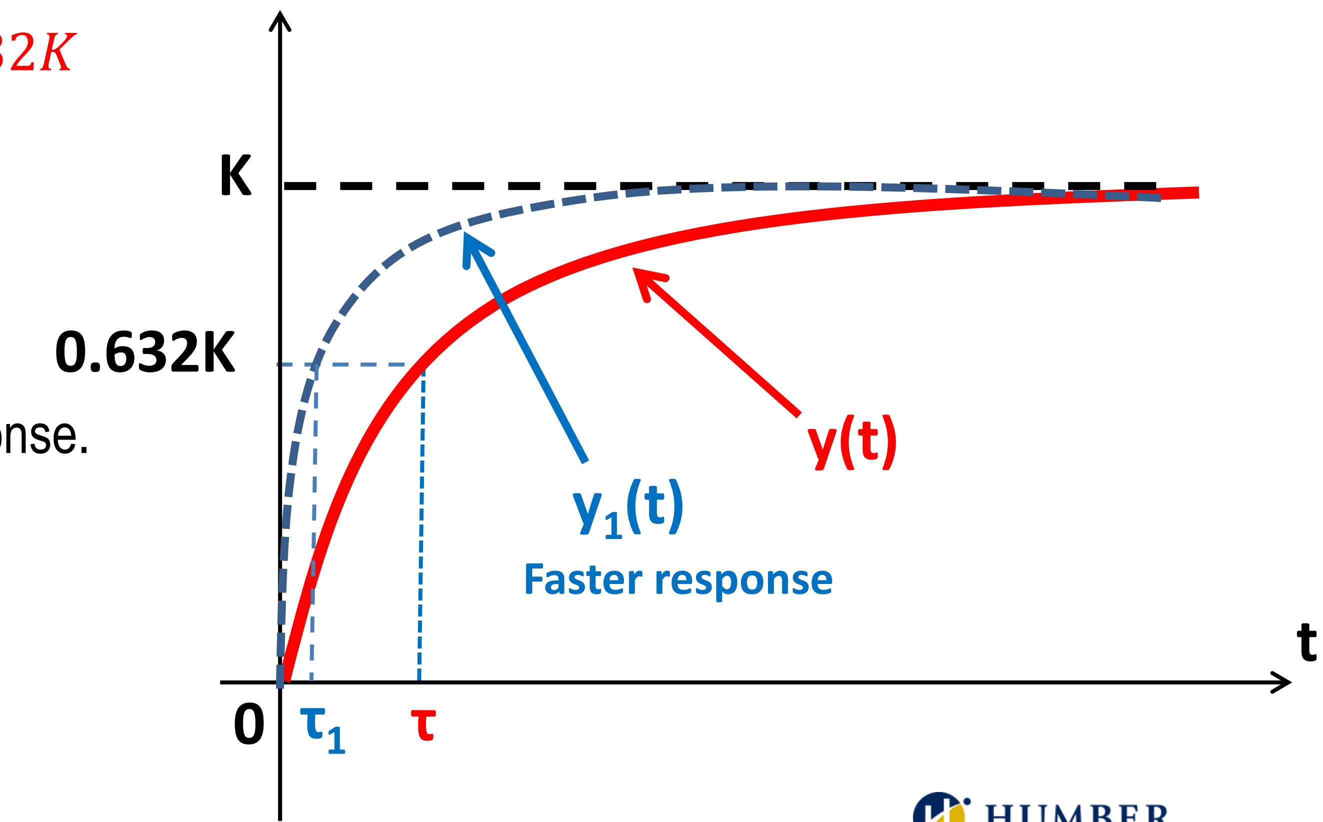
- **Time-Constant** is defined as the time when the unit-step response $y(t)$ has reached $\mathbf{63.2\%}$ of its total change from its initial value to the steady-state value.

At $t = \tau \quad \rightarrow \quad y(\tau) = K(1 - e^{-\frac{\tau}{\tau}}) \; = K(1 - e^{-1}) = 0.632K$

- Time-Constant shows <u>how fast</u> a first-order system responds to the input.

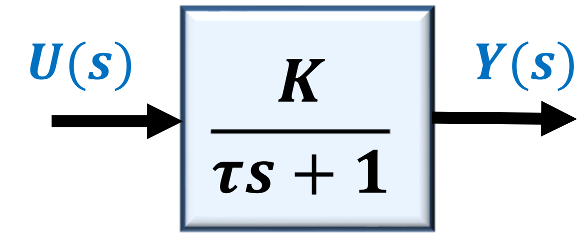
- The smaller the <u>time constant</u>, the faster the system response.

$$\boxed{\tau_1 < \tau}$$



First Order Systems Parameters

❑ Steady-State Gain (DC-Gain)



- The step-response of a first-order system is an exponential curve.

$$\boxed{y(t) = K - Ke^{-t/\tau}, \qquad t \geq 0}$$

Unit-step Response

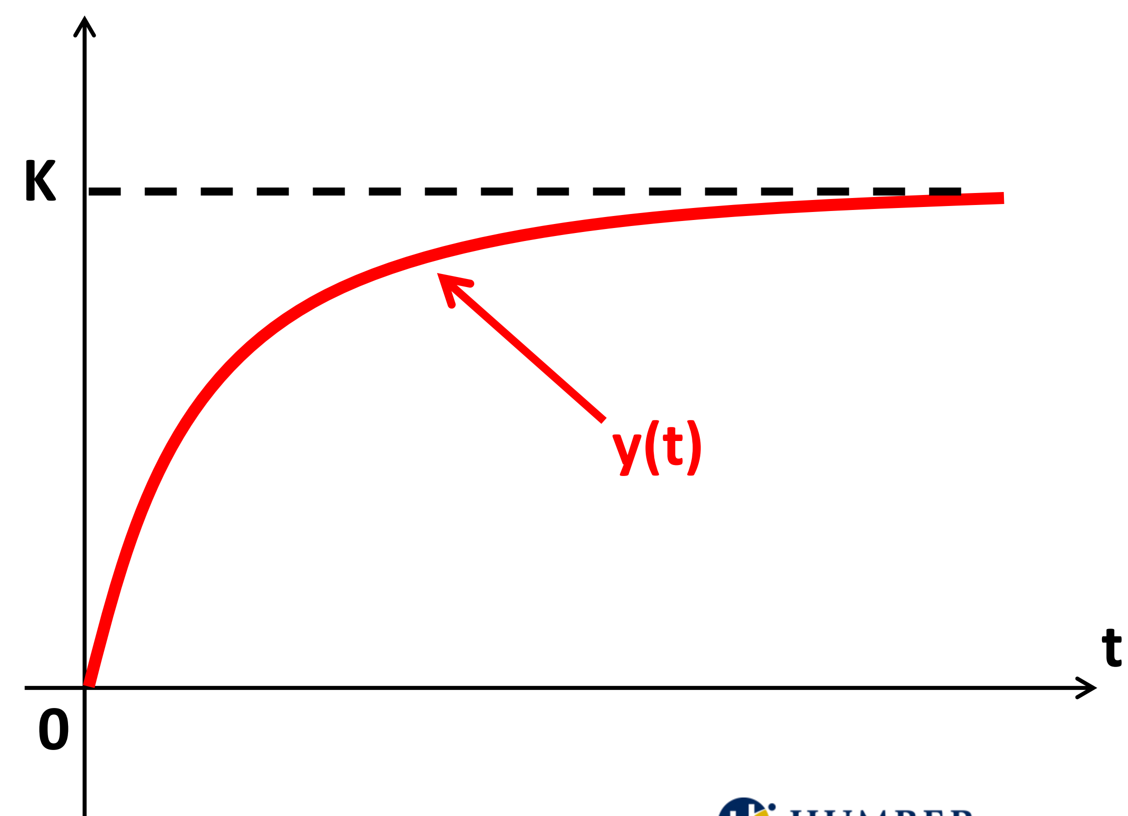
- **DC-gain** or **Steady-state gain** shows final value of the unit-step response in a stable system .

- **DC-gain** is also determined from **Final-Value Theorem**:

$$\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)}$$

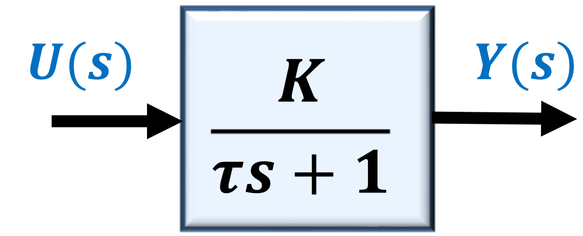
$$y(\infty) = \lim_{s\to 0} sG(s)U(s) = \lim_{s\to 0} sG(s)\left(\frac{1}{s}\right) = \lim_{s\to 0} G(s)$$

$$\boxed{\text{DC\_Gain} = \lim_{s\to 0} G(s)}$$



First Order Systems Parameters

❑ Settling Time



- **Settling Time (t_s)** is the required time for the step response to reach and stay within the specified percentage of its final value (usually 5%, 2% or 1% criteria)

At $t = 3\tau \rightarrow y(3\tau) = K(1 - e^{-3}) = 0.95K$

At $t = 4\tau \rightarrow y(4\tau) = K(1 - e^{-4}) = 0.982K$

At $t = 5\tau \rightarrow y(5\tau) = K(1 - e^{-5}) = 0.993K$

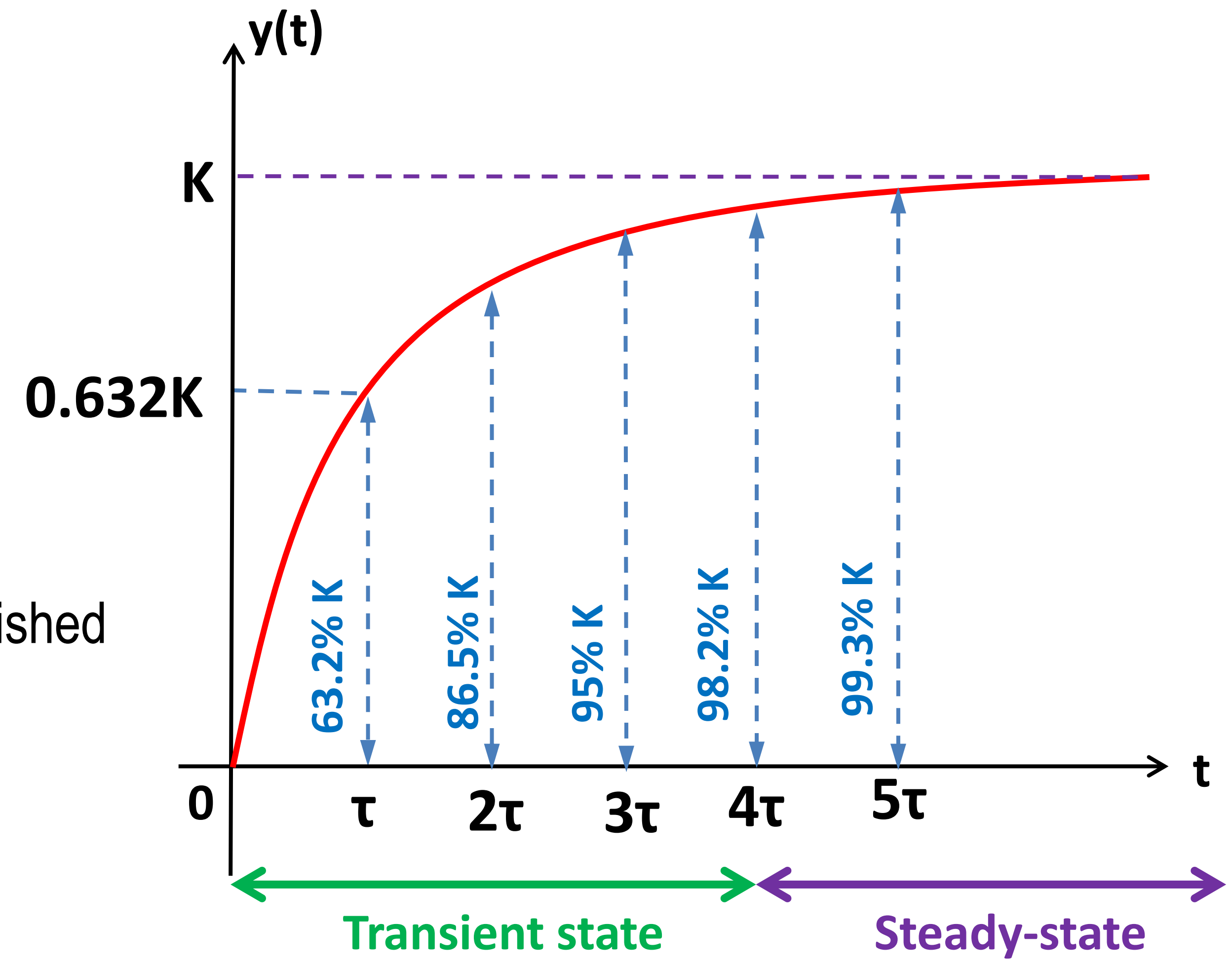
  - 1% criteria $\rightarrow \boldsymbol{t_s = 5\tau}$
  - 2% criteria $\rightarrow \boldsymbol{t_s = 4\tau}$
  - 5% criteria $\rightarrow \boldsymbol{t_s = 3\tau}$

- The transient state and the steady-state state are distinguished by the Settling Time.

$$\boxed{y(t) = K - Ke^{-\frac{t}{\tau}}, \qquad t \geq 0}$$

Steady-State Response

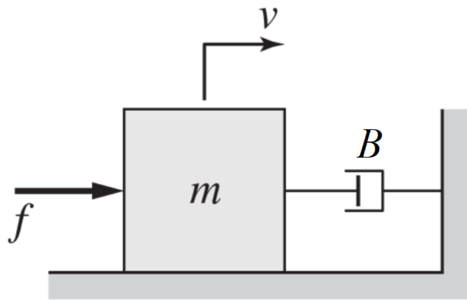
Transient Response



First-Order Systems – Example

$$G(s) = \frac{K}{\tau s + 1}$$

- Example of some systems that can be modeled as a first-order system.



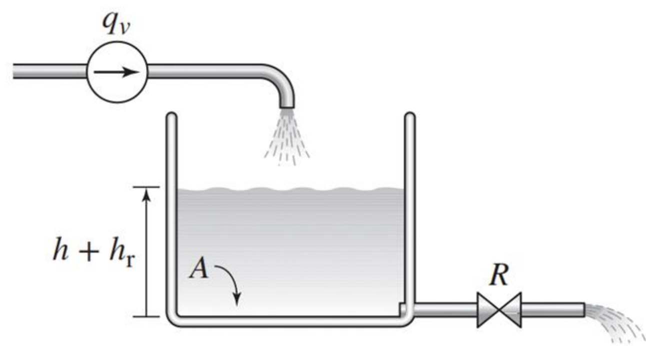
$$m\dot{v}(t) + bv(t) = f(t)$$

$$F(s) \longrightarrow \boxed{G(s)} \longrightarrow V(s)$$

$$G(s) = \frac{V(s)}{F(s)} = \frac{1}{ms + b} = \frac{\frac{1}{b}}{\frac{m}{b}s + 1}$$

- Time constant & DC-gain

$$\boxed{\tau = \frac{m}{b}, \qquad K = \frac{1}{b}}$$



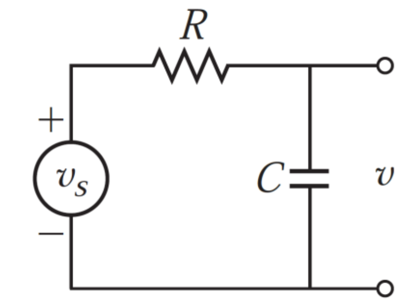
$$AR\frac{dh(t)}{dt} + gh(t) = Rq_v(t)$$

$$Q_v(s) \longrightarrow \boxed{G(s)} \longrightarrow H(s)$$

$$G(s) = \frac{H(s)}{Q_v(s)} = \frac{R}{ARs + g} = \frac{\frac{R}{g}}{\frac{AR}{g}s + 1}$$

- Time constant & DC-gain

$$\boxed{\tau = \frac{AR}{g}, \qquad K = \frac{R}{g}}$$



$$RC\frac{dv(t)}{dt} + v(t) = v_s(\mathrm{t})$$

$$V_s(s) \longrightarrow \boxed{G(s)} \longrightarrow V(s)$$

$$G(s) = \frac{V(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

- Time constant & DC-gain

$$\boxed{\tau = RC, \qquad K = 1}$$

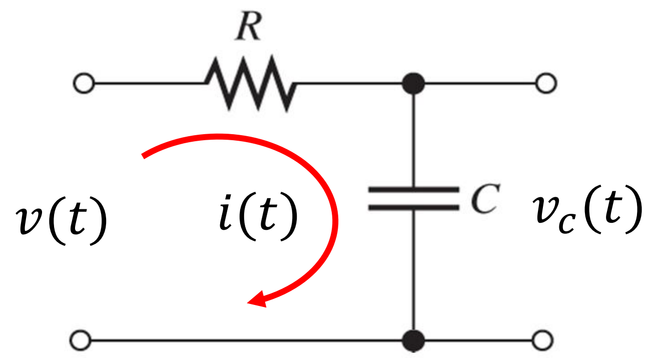
First-Order Systems – Example

Example 1

Consider the following electrical network with input $v(t)$ and output $v_c(t)$

a) Determine the differential equation and transfer function representing the dynamic model of the system.

$$\boldsymbol{G(s) = \frac{V_c(s)}{V(s)}}$$



Applying the Kirchhoff's Voltage Law (KVL) we have

$$v(t) = v_R(t) + v_c(t) \quad \rightarrow \quad v(t) = Ri(t) + v_c(t)$$

The differential equation relating $v(t)$ to $v_c(t)$ is determined as

$$\xrightarrow{i(t)=C\frac{dv_c(t)}{dt}} \quad \boxed{v(t) = RC\frac{dv_c(t)}{dt} + v_c(t)}$$

First-order differential equation

Taking the Laplace transform (**zero initial conditions,** $v_c(0) = 0$)

$$V(s) = RCsV_c(s) + V_c(s) \quad \longrightarrow \quad \boxed{G(s) = \frac{V_c(s)}{V(s)} = \frac{1}{RCs + 1}}$$

First order transfer function

b) Determine the transfer function, the steady-state gain and the time constant for $R = 100\Omega$, $C = 0.05F$.

$$\boldsymbol{G(s) = \frac{1}{5s + 1}} \quad \longrightarrow \quad \begin{cases} \text{Steady-state gain} \rightarrow K = 1 \\ \text{Time constant} \rightarrow \tau = RC = 5 \text{ sec} \end{cases}$$

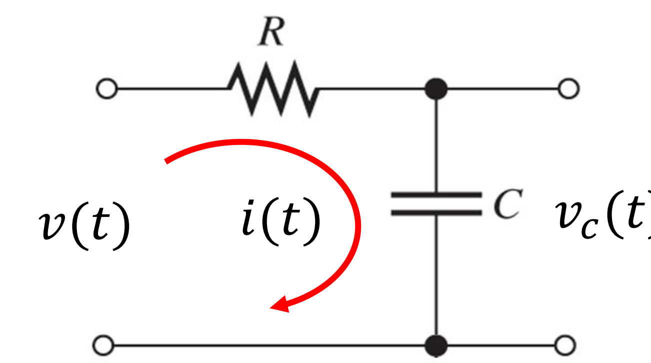
$$\boxed{\boldsymbol{G(s) = \frac{K}{\tau s + 1}}}$$

First-Order Systems – Example

Example 1

Consider the following electrical network with input $v(t)$ and output $v_c(t)$

c) If the applied voltage is v(t) = 6 V, determine and plot the response of the system if the initial voltage of the capacitor is zero, $v_c(0) = 0$.

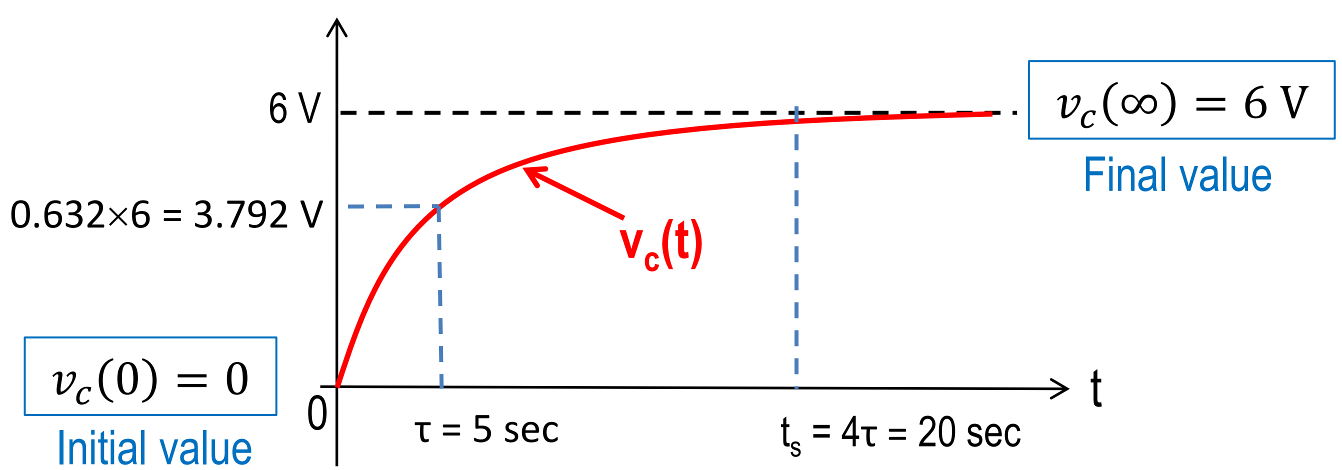


Find the output response by applying the partial fraction decomposition and taking the inverse Laplace transform:

$$\boldsymbol{G(s)} = \frac{\boldsymbol{V_c(s)}}{\boldsymbol{V(s)}} = \frac{1}{\boldsymbol{5s+1}} \quad \longrightarrow \quad V_c(s) = G(s)V(s) = \left(\frac{1}{5s+1}\right)\left(\frac{6}{s}\right)$$

$$V_c(s) = \frac{6}{s} + \frac{-30}{5s+1} = \frac{6}{s} - \frac{6}{s+1/5} \quad \longrightarrow \quad \boxed{v_c(t) = 6 - 6e^{-\frac{t}{5}}, \qquad t \geq 0}$$

System Response



Initial value $\rightarrow \lim_{t\to 0} v_c(t) = 0$

Final value $\rightarrow \lim_{t\to \infty} v_c(t) = 6$

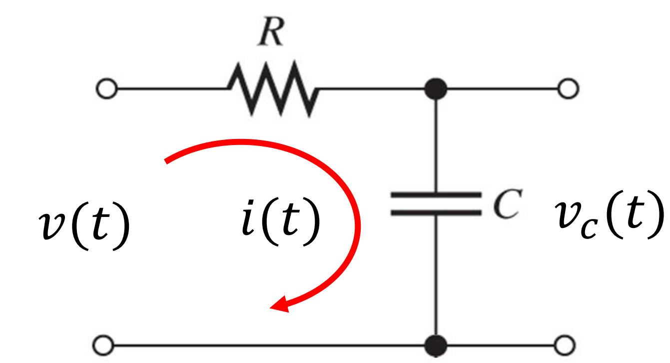
Settling time $\rightarrow t_s = 4\tau = 4RC = 20\ sec$

First-Order Systems – Example

Example 1

Consider the following electrical network with input $v(t)$ and output $v_c(t)$

d) If the applied voltage is v(t) = 1 V, determine and plot the response of the system if the initial voltage of the capacitor is zero, $v_c(0) = 0.1\ V$.



Having the differential equation model from Part (a): $v(t) = RC\dfrac{dv_c(t)}{dt} + v_c(t)$

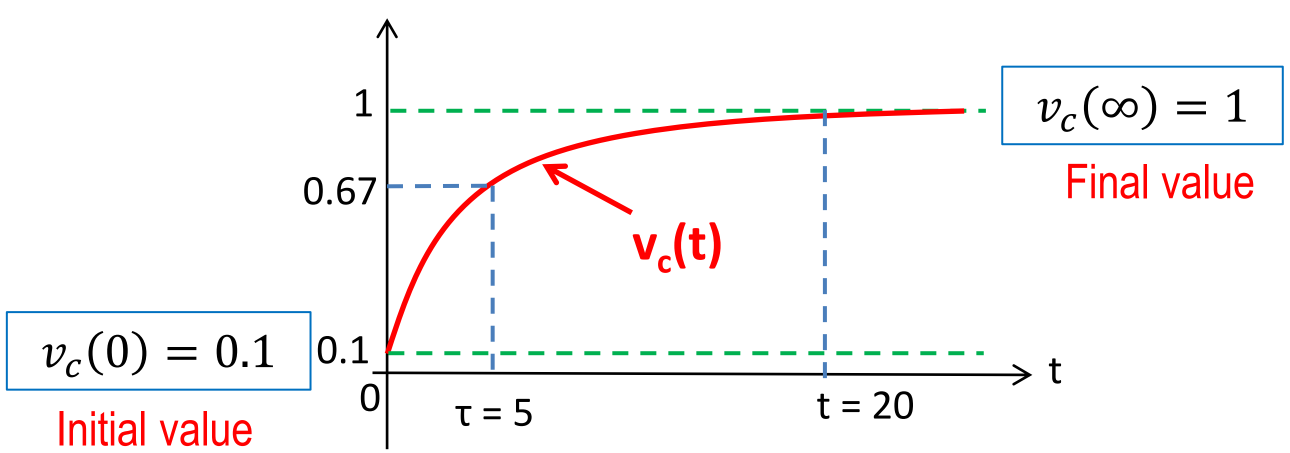
Taking Laplace transform by considering the initial conditions:

$$V(s) = RC(sV_c(s) - v_c(0)) + V_c(s) \quad \rightarrow \quad V(s) = (RCs + 1)V_c(s) - RCv_c(0)$$

$$V_c(s) = \underbrace{\frac{1}{RCs+1}V(s)}_{\text{Forced Response}} + \underbrace{\frac{RC}{RCs+1}v_c(0)}_{\text{Free Response}} \quad \rightarrow \quad V_c(s) = \left[\frac{1}{s(5s+1)}\right] + \left[\frac{0.5}{5s+1}\right] = \left[\frac{1}{s} + \frac{-5}{5s+1}\right] + \left[\frac{0.5}{5s+1}\right]$$

$$v_c(t) = \left(1 - e^{\frac{-t}{5}}\right) + \left(0.1e^{\frac{-t}{5}}\right) \quad \Rightarrow \quad \boxed{v_c(t) = 1 - 0.9e^{-\frac{t}{5}}, \qquad t \geq 0}$$

Unit-step Response



Initial value $\rightarrow$ $\lim_{t\to 0} v_c(t) = 0.1$

Final value $\rightarrow$ $\lim_{t\to\infty} v_c(t) = 1$

Settling time $\rightarrow$ $t_s = 4\tau = 4RC = 20\ sec$

Quick Review

1. Find the **time-constant** and **DC-gain** of a system represented by the following transfer function.

$$G(s) = \frac{2}{s+10}$$

a) 2 and 10 b) 0.5 and 5

c) 10 and 2 d) 0.1 and 0.2

2. Find the **DC-gain** of the following transfer function:

$$G(s) = \frac{50s+1}{(s+10)(s+2)(s+0.5)}$$

a) 50 b) 10

c) 0.1 d) 1

3. Which of the following systems has a <u>faster</u> step-response?

a) $G(s) = \frac{5}{s+10}$

b) $G(s) = \frac{7.5}{2s+1}$

c) $G(s) = \frac{3}{4s+50}$

d) $G(s) = \frac{5}{0.2s+1}$

Quick Review

1. Determine the unit-step response of the following first-order system.

$$\boldsymbol{G(s) = \frac{Y(s)}{U(s)} = \frac{5}{s+10}}$$

   a) $y(t) = (1 - 0.5e^{-0.1t})$

   b) $y(t) = 0.5(1 - e^{-10t})$

   c) $y(t) = 5(1 - 10e^{-10t})$

   d) $y(t) = 0.5(1 - e^{-0.1t})$

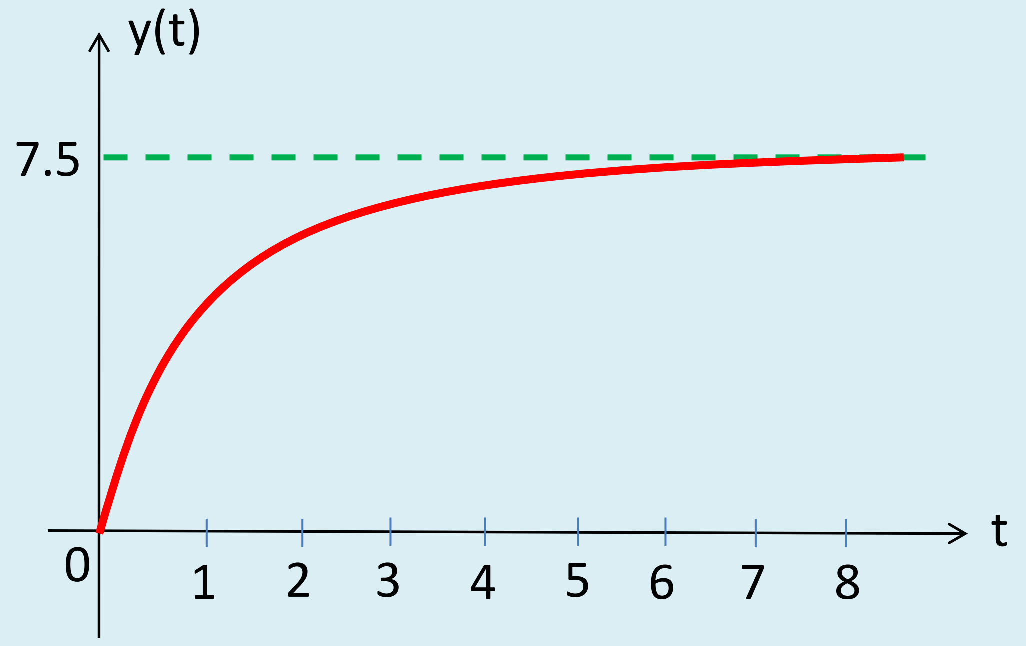
2. Given the unit-step response of a system, determine the transfer function model of the system.

   a) $G(s) = \frac{7.5}{s+1}$

   b) $G(s) = \frac{7.5}{2s+1}$

   c) $G(s) = \frac{5.3}{s+1}$

   d) $G(s) = \frac{5.3}{2s+1}$



Second Order Systems

- Second-order systems are systems whose input-output relationship is a second-order differential equation.
- Second-order systems have two energy-storage elements.

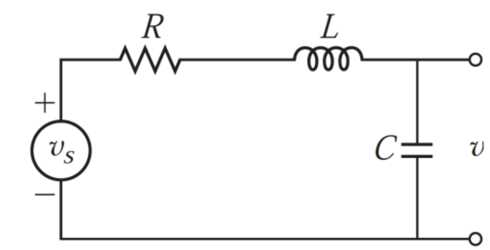
- RLC Electric Circuits
- Two-Tank Liquid Level System
- DC Motor Speed Model
- Mass-Spring-Damper System

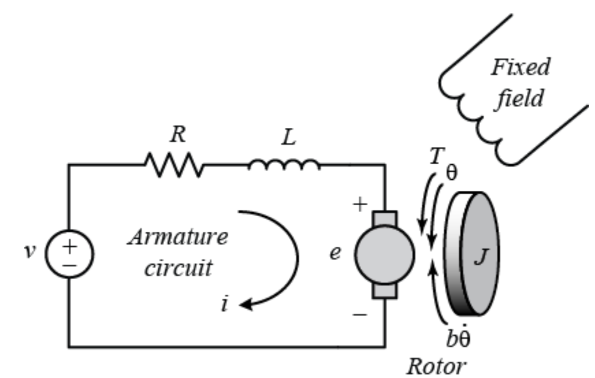
❑ Standard Form of Transfer Function

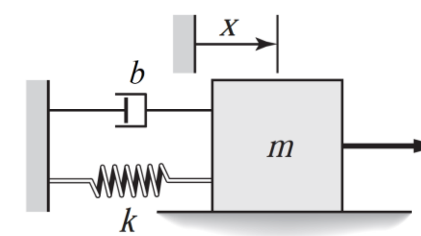
$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

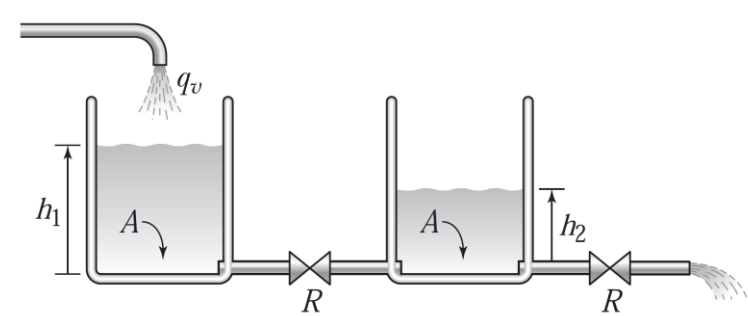
- ω_n is called Natural Undamped Frequency
- ζ is called Damping Ratio
- K is the steady-state gain

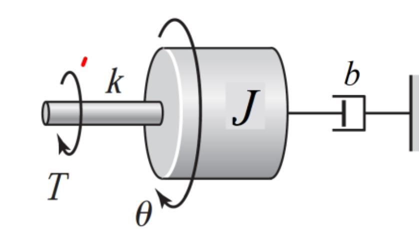
- Stability and dynamic behavior of the second-order system can be described in terms of the damping ratio ζ and the natural frequency ω_n.







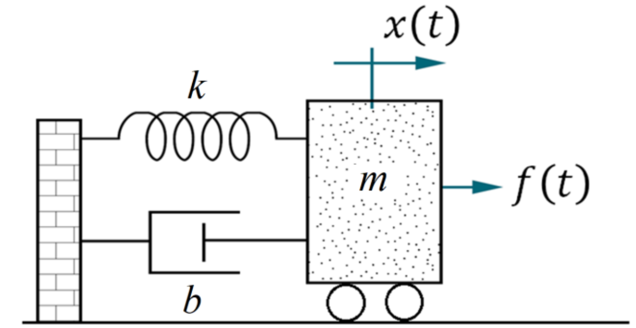




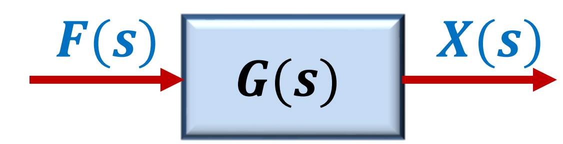
Second Order Systems – Example

- Example of systems can be modeled as a second-order system.

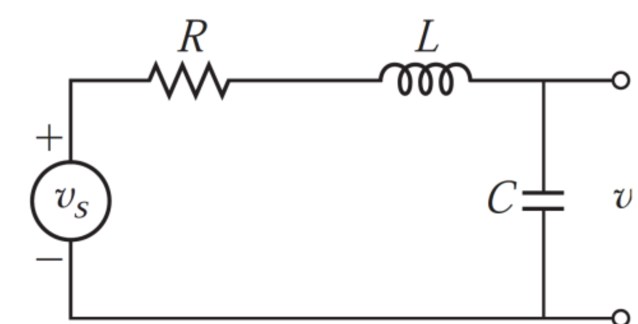
$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t) \longrightarrow G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$



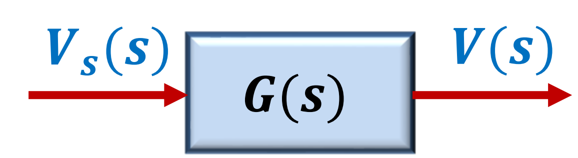
$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \rightarrow \quad \omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{b}{2\sqrt{mk}}, \quad K = \frac{1}{k}$$



$$LC\ddot{v}(t) + RC\dot{v}(t) + v(t) = v_s(t) \longrightarrow G(s) = \frac{V(s)}{V_s(s)} = \frac{1}{LCs^2 + RCs + 1}$$



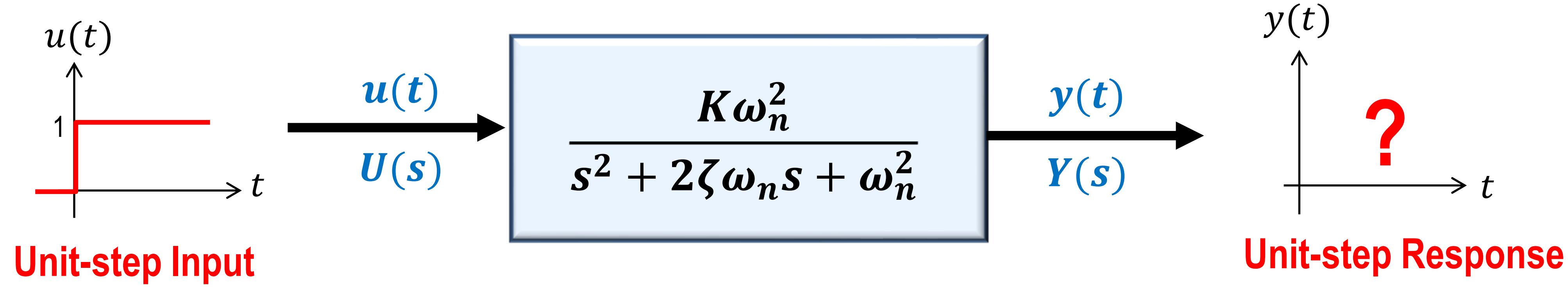
$$G(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \rightarrow \quad \omega_n = \frac{1}{\sqrt{LC}}, \quad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}, \quad K = 1$$



$$\boxed{G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

Second Order Systems

❑ Step Response



- Step response of a second order system is determined as below.

$$Y(s) = G(s)U(s) \quad \rightarrow \quad Y(s) = \left(\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{K\omega_n^2}{s(s-s_1)(s-s_2)} \quad \rightarrow \quad Y(s) = \frac{A}{s} + \frac{B}{s-s_1} + \frac{C}{s-s_2} \quad \rightarrow \quad y(t) = \mathcal{L}^{-1}[Y(s)]$$

where A, B and C are determined from partial fraction expansions method.
The values depend on the **pole** locations s_1 and s_2.

$$\boxed{s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}}$$

Second Order Systems

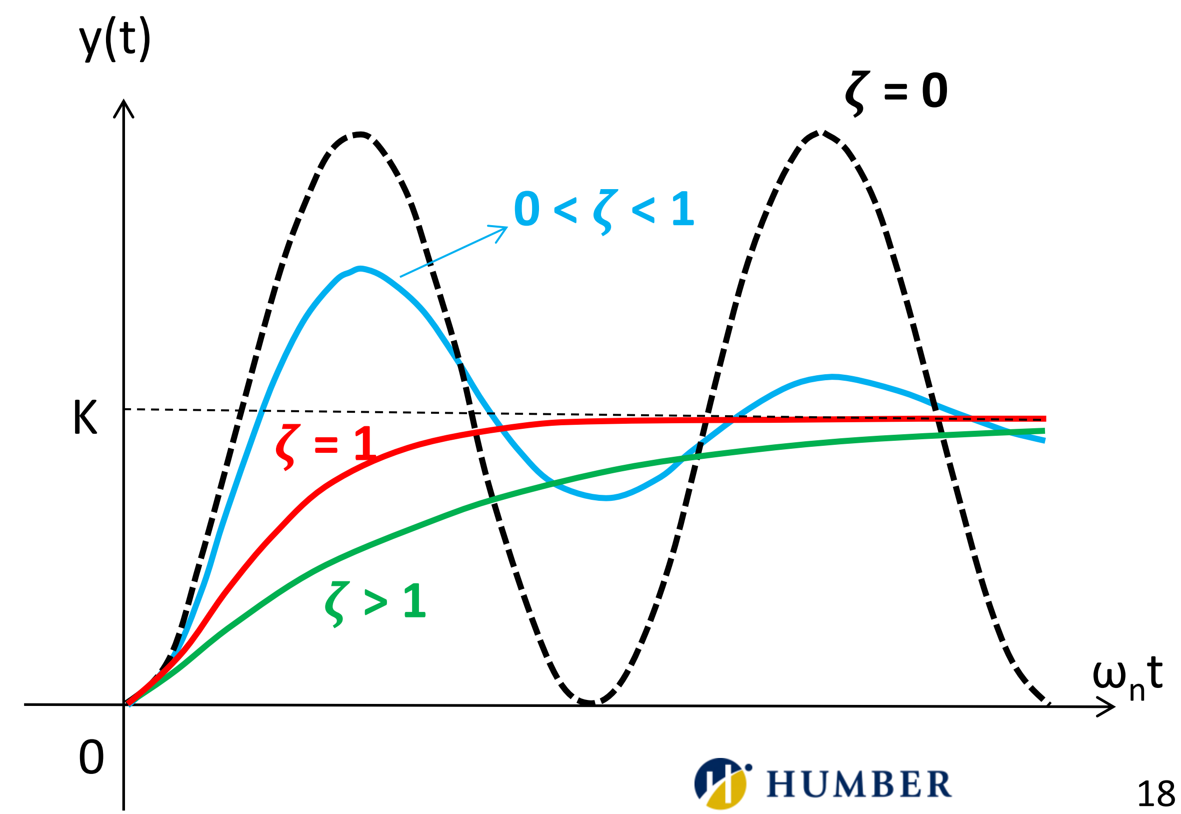
❑ Step Response

- The pole locations and the step response $y(t)$ depend on the natural frequency ω_n and the damping ratio ζ.
  - $\zeta = 1 \rightarrow$ The poles are real and equal $\rightarrow$ $s_1 = s_2 = -\omega_n$
  - $\zeta > 1 \rightarrow$ The poless are real but not equal $\rightarrow$ $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
  - $0 < \zeta < 1 \rightarrow$ The poles are complex conjugate $\rightarrow$ $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
  - $\zeta = 0 \rightarrow$ The poles are imaginary. $\rightarrow$ $s_{1,2} = \pm j\omega_n$

$$\boxed{\begin{aligned} s_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\ s_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \end{aligned}}$$

- Step response of the second-order systems can be classified based on the damping ratio ζ
  - **Critically-damped Systems:** $\zeta = 1$
  - **Over-damped Systems:** $\zeta > 1$
  - **Under-damped Systems:** $0 < \zeta < 1$
  - **Undamped Systems:** $\zeta = 0$

- Note that negative damping ratio $\zeta < 0$ means growing magnitude of oscillations, which is called unstable system.

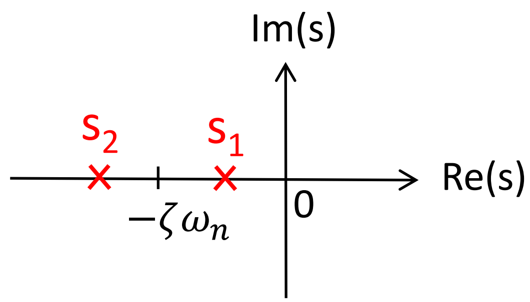


Second Order Systems

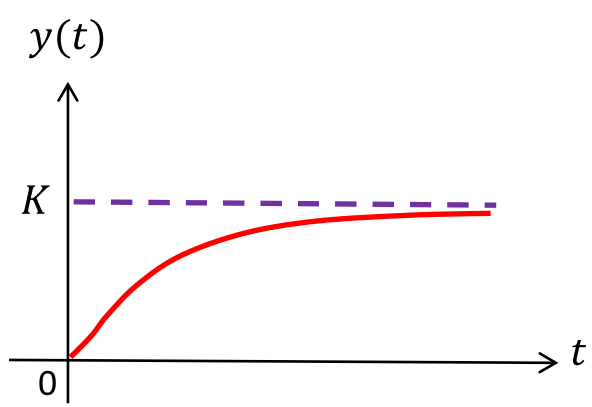
Over-damped System $\zeta > 1$

- System has two distinct real negative poles
- Output response is slow and is not oscillate
- Output response becomes slower by increasing the damping ratio ζ

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



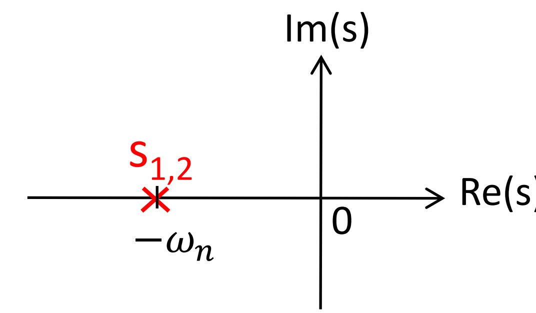
$$y(t) = K + C_1 e^{s_1 t} + C_2 e^{s_2 t}, \qquad t \geq 0$$



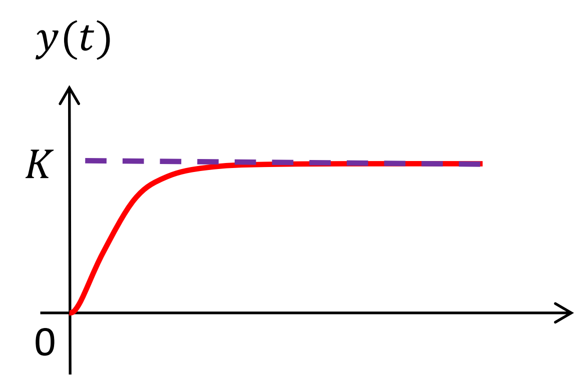
Critically-damped System $\zeta = 1$

- System has two repeated real negative poles
- Output response is not oscillated
- Fastest response without oscillation and overshoot

$$s_1 = s_2 = -\omega_n$$



$$y(t) = K - K e^{-\omega_n t}(1 + \omega_n t), \qquad t \geq 0$$

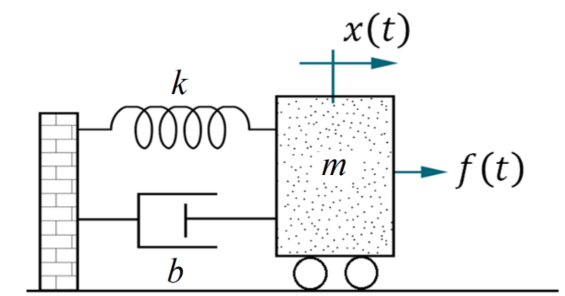


Second Order Systems

Example 2

Recall the transfer function model of a mass-spring-damper system.

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$



Assume $m = 1$ kg, $b = 8$ Ns/m and $k = 4$ N/m.

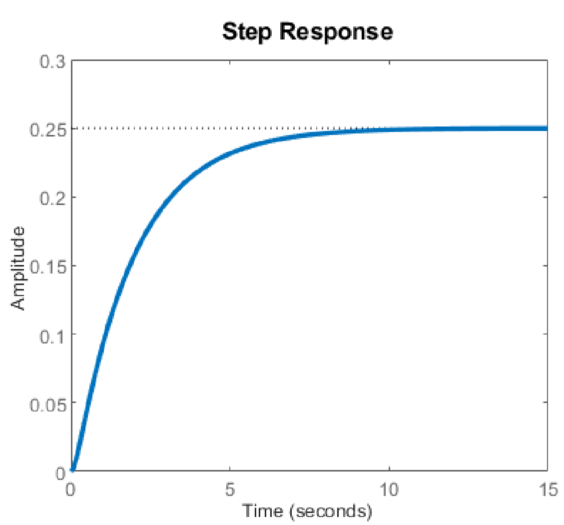
Transfer function $\rightarrow$ $G(s) = \dfrac{1}{s^2 + 8s + 4}$

Natural Frequency $\rightarrow$ $\omega_n^2 = 4$ $\rightarrow$ $\omega_n = 2$ rad/s

Damping Ratio $\rightarrow$ $2\zeta\omega_n = 8$ $\rightarrow$ $\zeta = 2$

Poles: $s^2 + 8s + 4 = 0$ $\rightarrow$ $s_1 = -0.54,\ s_2 = -7.46$

Unit-step response: $y(t) = 0.25 - 0.285e^{-0.54t} + 0.019e^{-7.46t}$



Over-damped System

Assume $m = 1$ kg, $b = 4$ Ns/m and $k = 4$ N/m.

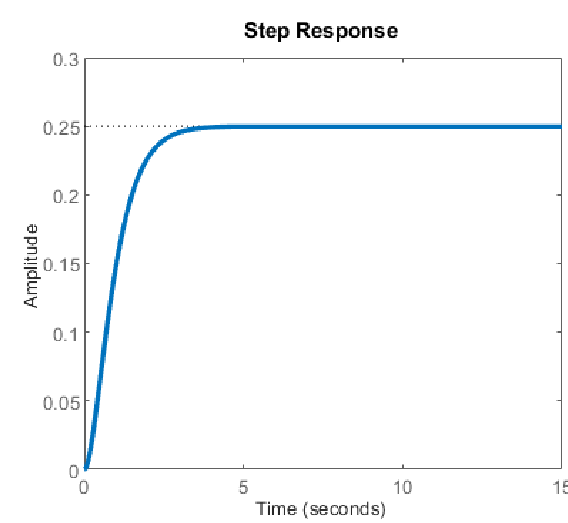
Transfer function $\rightarrow$ $G(s) = \dfrac{1}{s^2 + 4s + 4}$

Natural Frequency $\rightarrow$ $\omega_n^2 = 4$ $\rightarrow$ $\omega_n = 2$ rad/s

Damping Ratio $\rightarrow$ $2\zeta\omega_n = 4$ $\rightarrow$ $\zeta = 1$

Poles: $s^2 + 4s + 4 = 0$ $\rightarrow$ $s_1 = s_2 = -2$

Unit-step response: $y(t) = 0.25(1 - e^{-2t}(1 + 2t))$



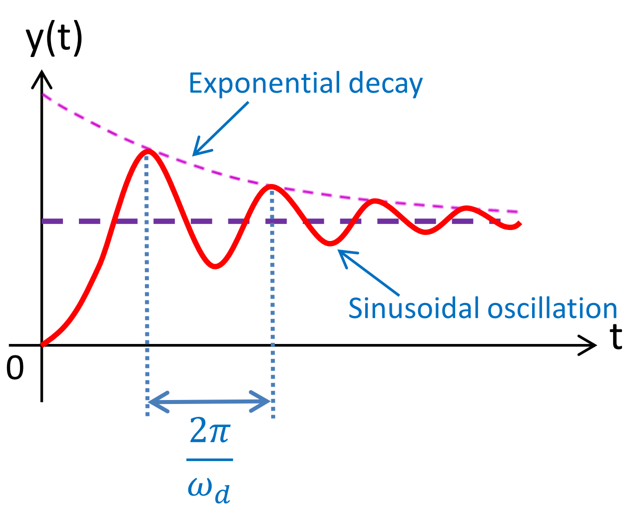
Critically-damped System

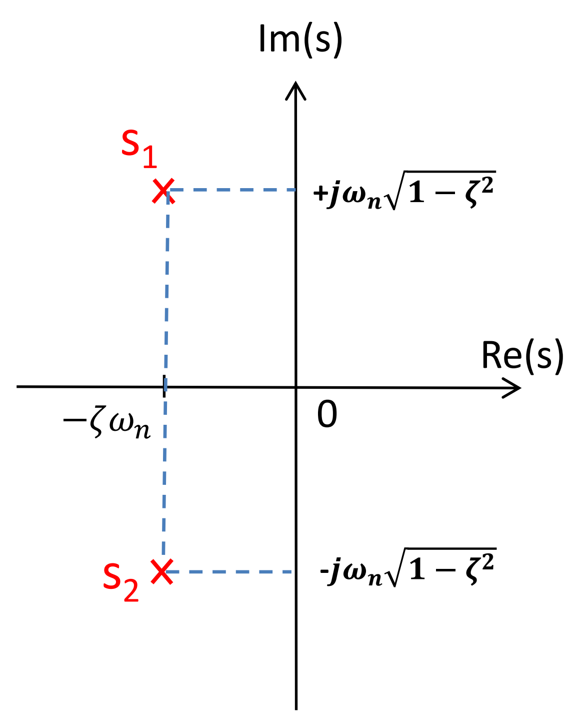
Second Order Systems

Underdamped System $0 < \zeta < 1$

- System has one pair of complex conjugated poles
- Transient response of the system would oscillate, and it becomes more oscillatory with larger overshoot by decreasing the ζ
- Frequency of oscillations is $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- ω_d is called damped natural frequency

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



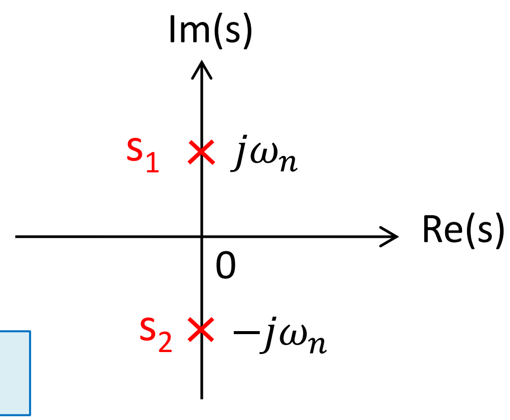


$$y(t) = K - \frac{Ke^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(sin(\omega_d t + cos^{-1}\zeta)), \qquad t \geq 0$$

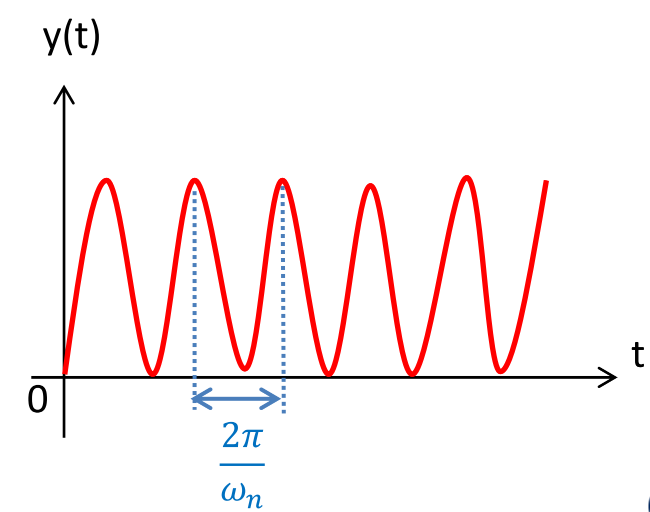
Undamped System $\zeta = 0$

- System has one pair of complex conjugate poles on the imaginary axes.
- The response has sustained oscillation with frequency of ω_n
- This is called marginally stable system.

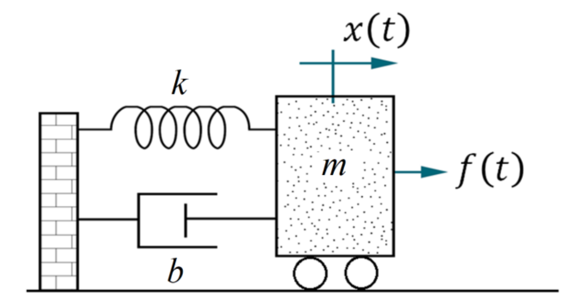
$$s_{1,2} = \pm j\omega_n$$



$$y(t) = K - Kcos(\omega_n t), \qquad t \geq 0$$



Second Order Systems



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Assume $m = 1$ kg, $b = 2$ Ns/m and $k = 4$ N/m.

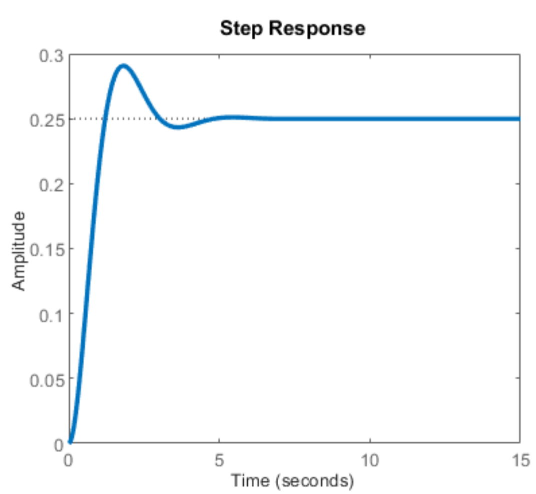
Transfer Function $\rightarrow$ $G(s) = \dfrac{1}{s^2 + 2s + 4}$

Natural Frequency $\rightarrow$ $\omega_n^2 = 4$ $\rightarrow$ $\omega_n = 2$ rad/s

Damping Ratio $\rightarrow$ $2\zeta\omega_n = 2$ $\rightarrow$ $\zeta = 0.5$

Poles: $s^2 + 2s + 4 = 0$ $\rightarrow$ $s_{1,2} = -1 \pm j\sqrt{3}$

Unit-step response: $y(t) = 0.25(1 - 1.15e^{-t}(\sin(1.73t + 60°)))$



Underdamped System

Assume $m = 1$ kg, $b = 0$ and $k = 4$ N/m.

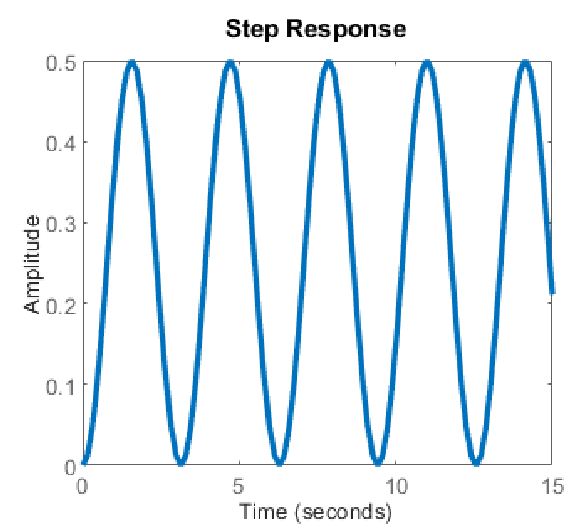
Transfer function is $\rightarrow$ $G(s) = \dfrac{1}{s^2 + 4}$

Natural Frequency $\rightarrow$ $\omega_n^2 = 4$ $\rightarrow$ $\omega_n = 2$ rad/s

Damping Ratio $\rightarrow$ $2\zeta\omega_n = 0$ $\rightarrow$ $\zeta = 0$ No damping

Poles: $s^2 + 4 = 0$ $\rightarrow$ $s_{1,2} = \pm j2$

Unit-step response: $y(t) = 0.25(1 - \cos(2t))$



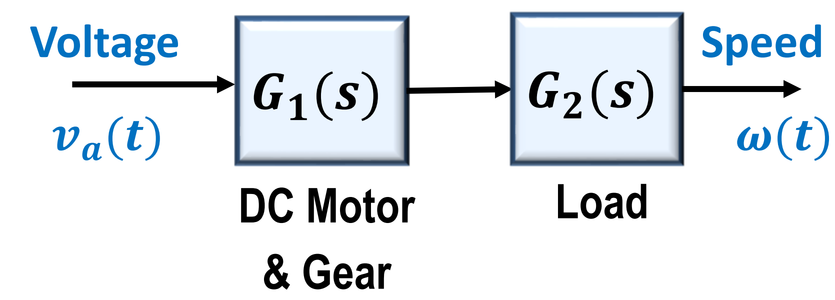
Undamped System

Second Order Systems – Example

Example 3

A DC motor drive system has a transfer function of $G_1(s)$ and drives a load which has a transfer function of $G_2(s)$.

$$G_1(s) = \frac{10}{s+5} \quad \text{and} \quad G_2(s) = \frac{0.5}{s+1}$$



a) Find overall transfer function of the system.

The overall transfer function:

$$G(s) = G_1(s)G_2(s) = \left(\frac{10}{s+5}\right)\left(\frac{0.5}{s+1}\right) = \frac{5}{(s+5)(s+1)} = \frac{5}{s^2+6s+5}$$

b) Determine damping ratio, undamped natural frequency and roots of the system? If the system is underdamped, critically-damped or overdamped?

Compare the characteristic polynomial with the standard form of a second order system $\rightarrow$ $s^2+6s+5 = s^2+2\zeta\omega_n s+\omega_n^2$

$$\omega_n^2 = 5 \quad \rightarrow \quad \omega_n = \sqrt{5} \ \ rad/sec$$

$$2\zeta\omega_n = 6 \quad \rightarrow \quad \zeta = \frac{6}{2\sqrt{5}} = 1.34$$

Here, $\zeta > 1$ so, the system is an over-damped system.

Find roots of the denominator:

$$s^2+6s+5 = 0 \quad \rightarrow \quad (s+1)(s+5) = 0 \quad \rightarrow \quad s_1 = -1, s_2 = -5$$

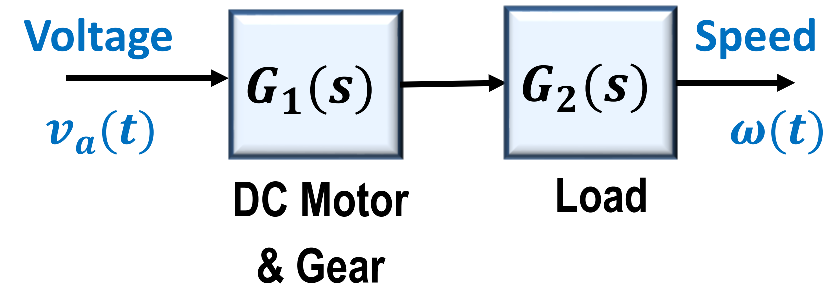
The system has two real distinct roots, so it is an over-damped system.

Second Order Systems – Example

Example 3

A DC motor drive system has a transfer function of $G_1(s)$ and drives a load which has a transfer function of $G_2(s)$.

$$G_1(s) = \frac{10}{s+5} \quad \text{and} \quad G_2(s) = \frac{0.5}{s+1}$$



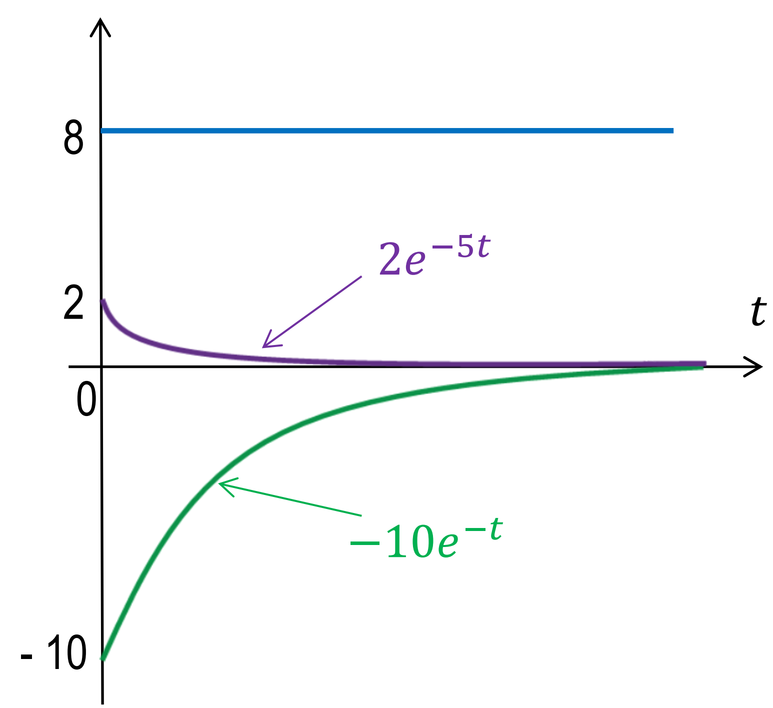
c) What will be the output of the system when the motor input is, $v_a(t) = 8V$.

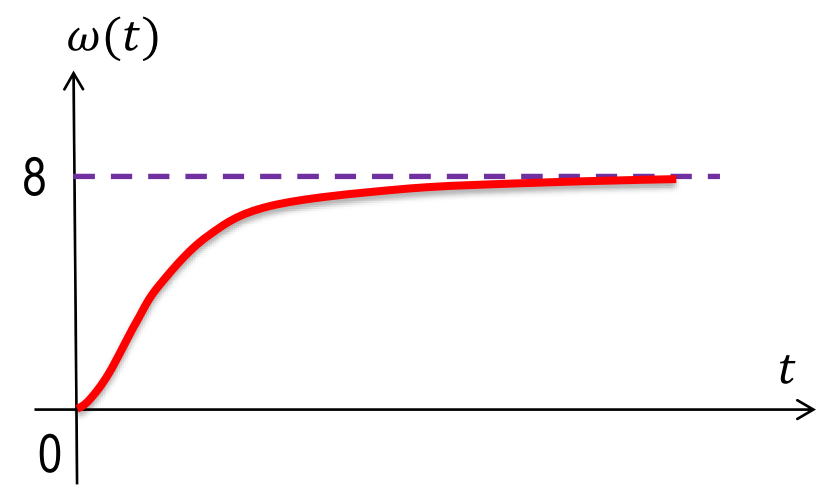
First, find the system output $\Omega(s)$ in Laplace-domain:

$$\Omega(s) = G(s)V_a(s) \quad \rightarrow \quad \Omega(s) = \left(\frac{5}{(s+5)(s+1)}\right)\left(\frac{8}{s}\right) = \frac{40}{s(s+5)(s+1)} = \frac{8}{s} + \frac{2}{s+5} + \frac{-10}{s+1}$$

We can find the output function $\omega(t)$ in time-domain using the Laplace transform table:

$$\boxed{\omega(t) = 8 + 2e^{-5t} - 10e^{-t}, \qquad t \geq 0}$$





Over-damped system

The initial-value and the final-value of $\omega(t)$:

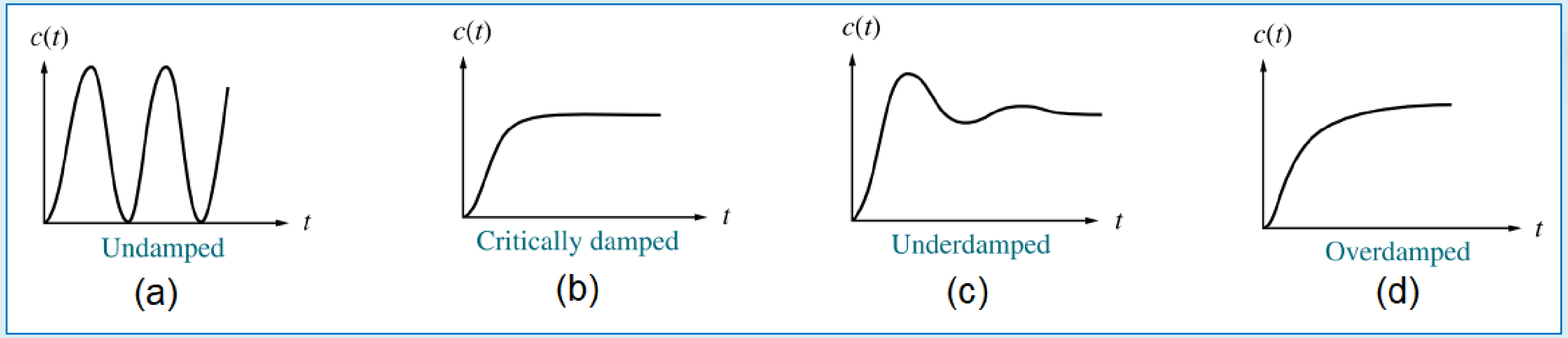
$$\lim_{t\to 0} \omega(t) = \lim_{s\to\infty} s\Omega(s) = 0$$

$$\lim_{t\to\infty} \omega(t) = \lim_{s\to 0} s\Omega(s) = 8 \ rad/sec$$

Quick Review

1. Match the unit-step response of second-order systems with the correct damping ratio:

   a) $\zeta = 1$
   b) $\zeta > 1$
   c) $\zeta = 0$
   d) $0 < \zeta < 1$



2. Consider the following transfer function. Choose which characteristic matches to its step response

   a) Over-damped Response
   b) Under-damped Response
   c) Critically-damped Response
   d) Undamped Response

$$\boldsymbol{G(s) = \frac{10}{s^2 + 30s + 200}}$$

3. A system has the following transfer function. What will be the natural frequency and the damping ratio?

   a) $\omega_n = 100, \quad \zeta = 1$
   b) $\omega_n = 10, \quad \zeta = 0.02$
   c) $\omega_n = 100, \quad \zeta = 0.01$
   d) $\omega_n = 10, \quad \zeta = 0.5$

$$\boldsymbol{G(s) = \frac{100}{s^2 + 10s + 100}}$$

$$\boxed{G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

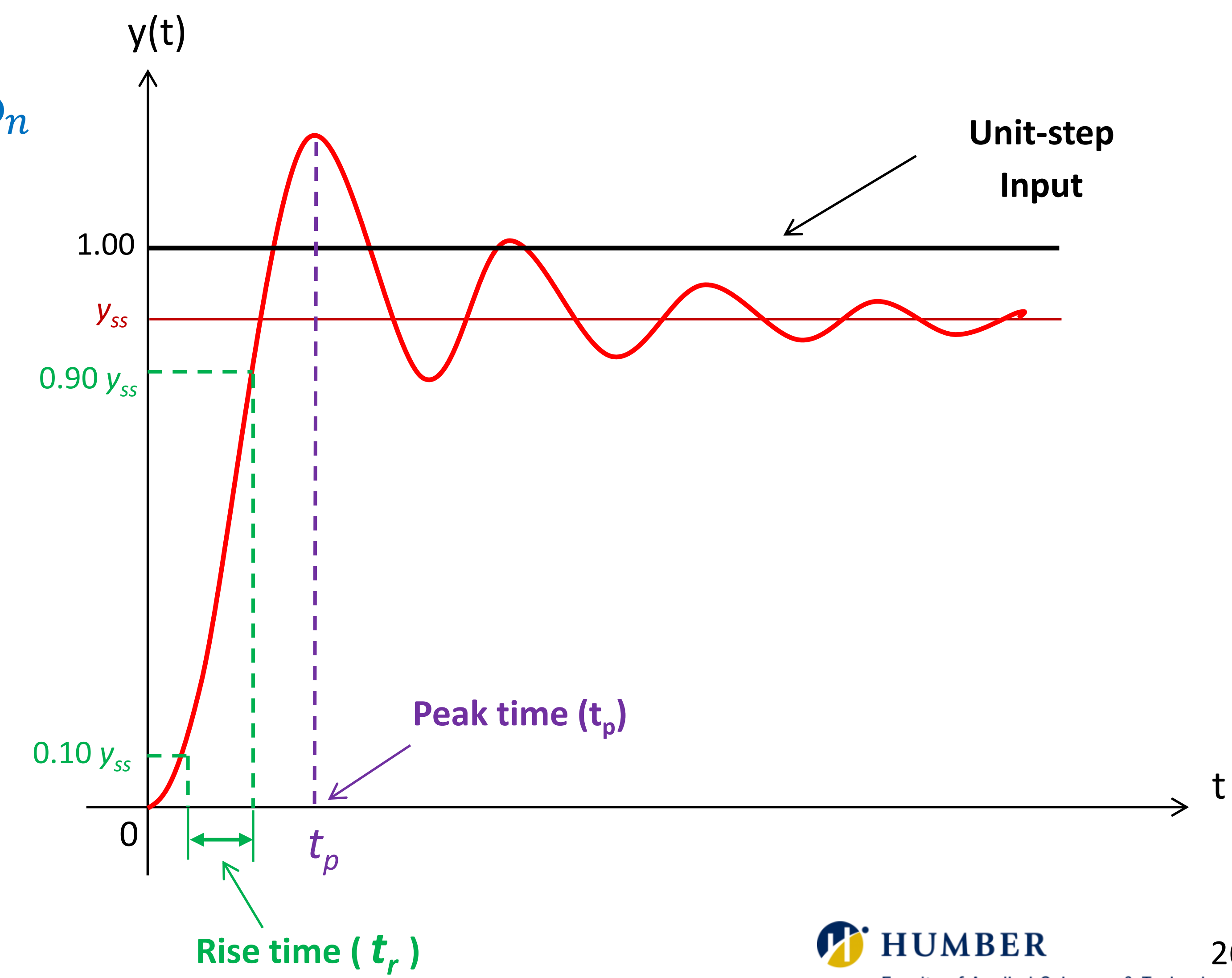
Time Response Specification of Underdamped Systems

Rise time (t_r): The time required for the step response to rise from 10% to 90% of its final value.

$$t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n}$$

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Rise-time shows how fast a system responds to an input.
- Rise-time is proportional to ζ and inversely proportional to ω_n



Peak time (t_p): The time required for the step response to reach the first peak of the overshoot.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

- Peak-time is inversely proportional to ω_n, increasing the ω_n will reduce the peak-time.

Time Response Specification of Underdamped Systems

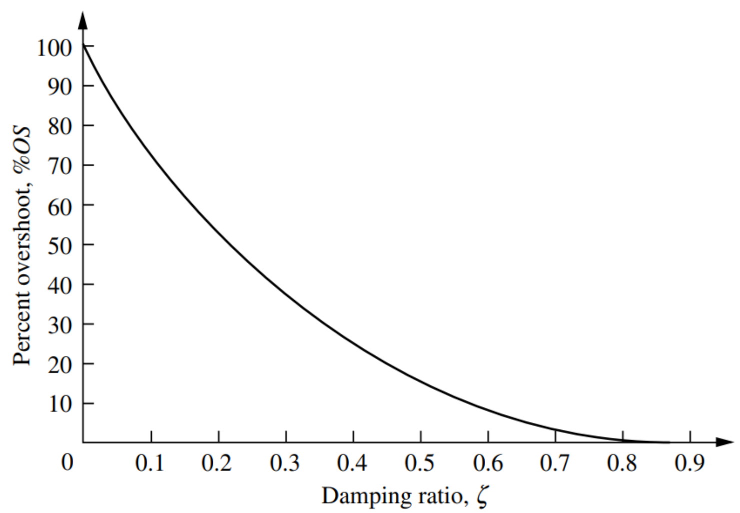
Maximum overshoot (M_p): The maximum peak value of the step response measured from the final value of the response.

$$M_p = y(t_p) - y_{ss} = y_{ss} e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\%\mathrm{O.S.} = \frac{M_p}{y_{ss}} \times 100\%$$

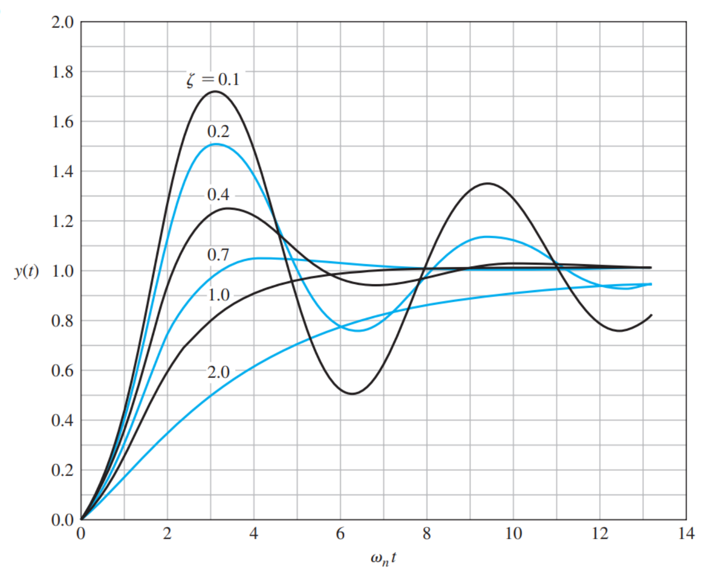
$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

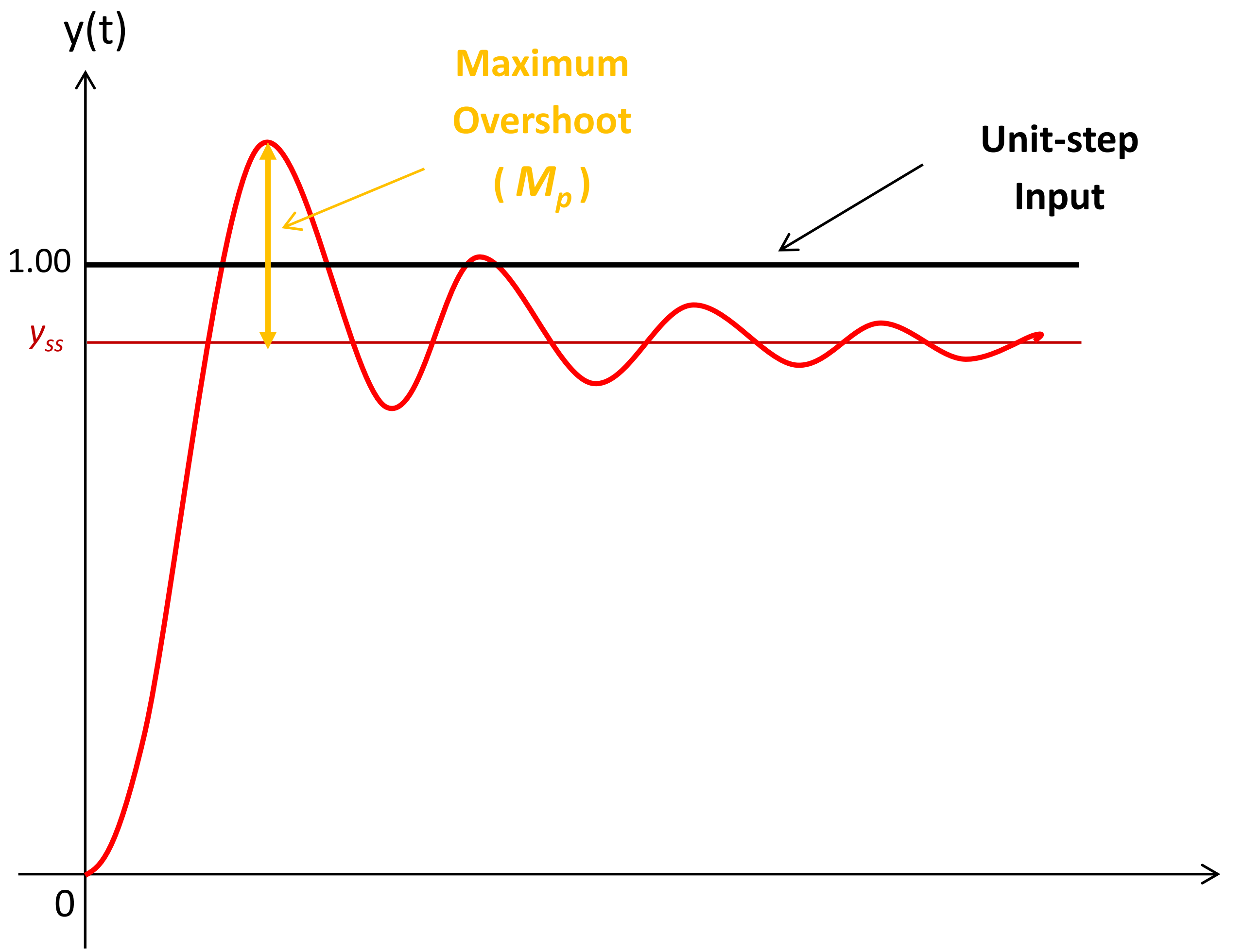
- Decreasing the damping ratio ζ will increase the overshoot.



ζ	%O.S.
0.690	5%
0.591	10%
0.517	15%
0.456	20%

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}}$$





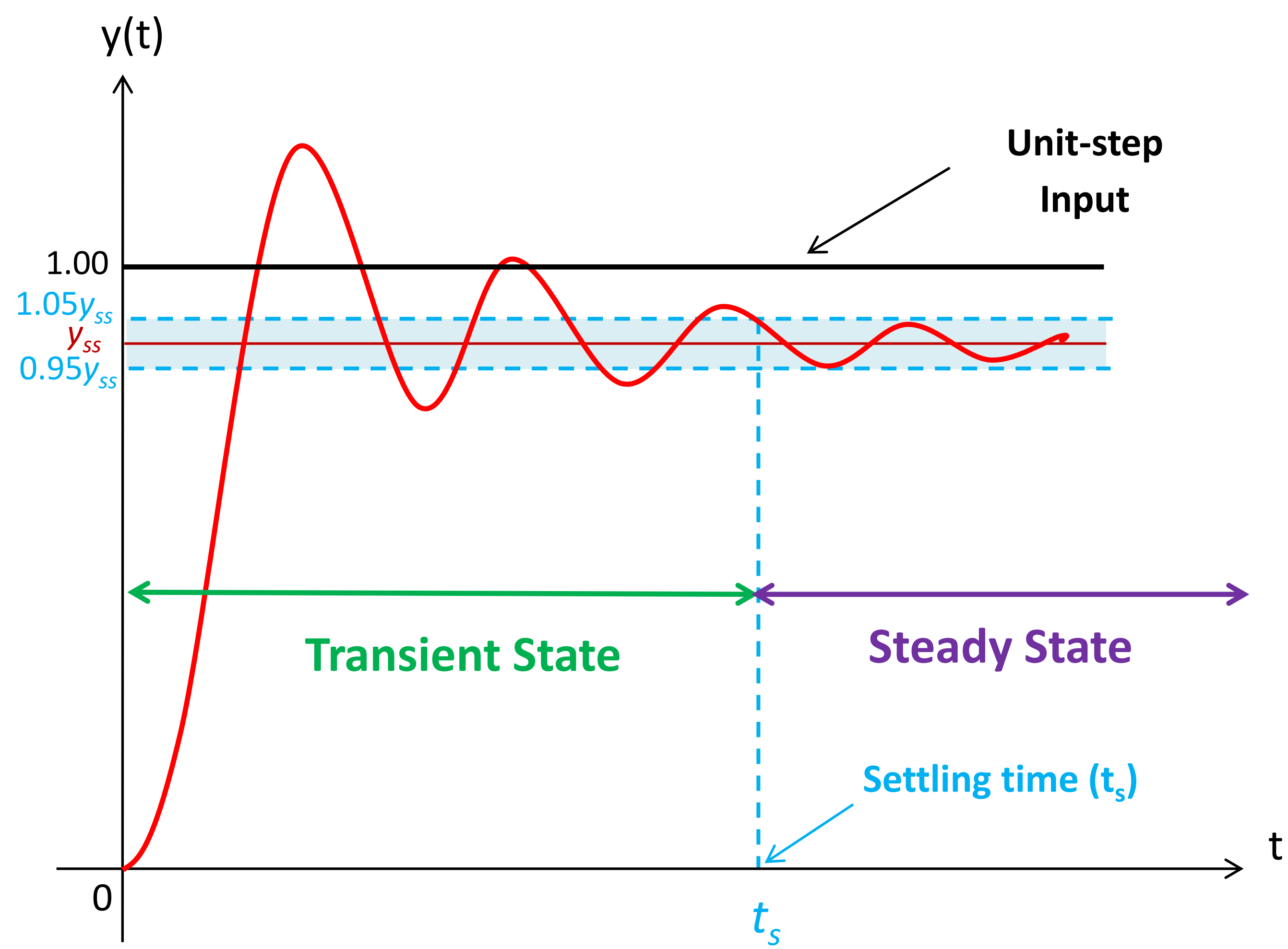
Time Response Specification of Underdamped Systems

Settling time (t_s): The time required for the step response to reach and stay within the specified percentage of its final value (usually 2% or 5%)

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$2\%\ \text{criteria} \rightarrow \quad t_s \approx \frac{4}{\zeta\omega_n}\ , \qquad 0 < \zeta < 0.9$$

$$5\%\ \text{criteria} \rightarrow \begin{cases} t_s \approx \dfrac{3.2}{\zeta\omega_n}\ , & 0 < \zeta < 0.69 \\[2ex] t_s \approx \dfrac{4.5\zeta}{\omega_n}\ , & \zeta > 0.69 \end{cases}$$

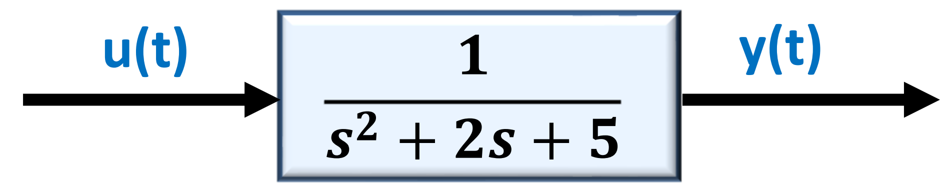


- Settling-time shows how fast the step response settles to its final value.
- The number of oscillations before settling time is calculated as:

$$\text{Number of oscillations} = \frac{\text{Settling Time}}{\text{Periodic Time}} = \frac{t_s}{2\pi/\omega_d}$$

Specification of Step Response in MATLAB

- We can obtain a list of all the step-response specifications with **`stepinfo`** function in MATLAB.

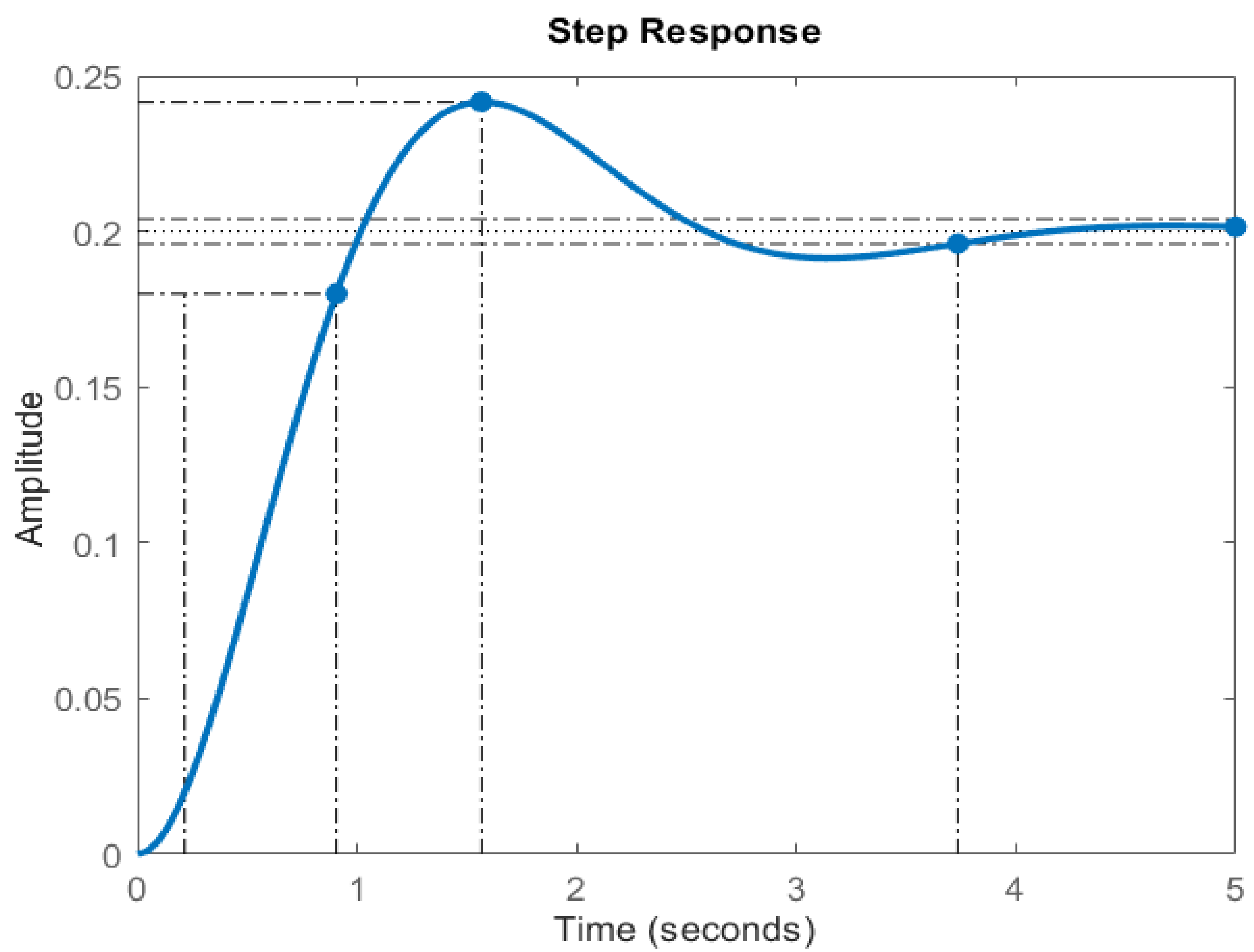



```
num = [1];
den = [1 2 5];
sys = tf(num,den);

step(sys)

stepinfo(sys)

ans =
          RiseTime: 0.6903
     TransientTime: 3.7352
      SettlingTime: 3.7352
       SettlingMin: 0.1830
       SettlingMax: 0.2416
         Overshoot: 20.7866
        Undershoot: 0
              Peak: 0.2416
          PeakTime: 1.5658
```



Second Order Systems – Example

Example 4

A second-order system has a natural frequency of 2.0 rad/s and a damped frequency of 1.8 rad/s. What are its damping ratio, rise-time, percentage overshoot, peak-time and 2% settling time?

Damping ratio is:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \rightarrow \quad \zeta = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1-\left(\frac{1.8}{2.0}\right)^2} = 0.436$$

Rise-time is:

$$t_r \cong \frac{0.8+2.5\zeta}{\omega_n} \quad \rightarrow \quad t_r \cong \frac{0.8+2.5(0.436)}{2.0} = 0.945\ sec$$

Percentage overshoot is:

$$\%O.S. = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\% \quad \rightarrow \quad \%O.S. = e^{-(0.436)\pi/\sqrt{1-0.436^2}} \times 100\% = 21.8\%$$

Peak-time is:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad \rightarrow \quad t_p = \frac{\pi}{(2.0)\sqrt{1-0.436^2}} = 1.74\ sec$$

Settling-time (2%) is:

$$t_s \approx \frac{4}{\zeta\omega_n} \quad \rightarrow \quad t_s = \frac{4}{(0.436)(2.0)} = 4.58\ sec$$

Quick Review

1. Consider the following second-order system, $G(s)$. Determine the settling time (2% criteria) and the peak time of the unit-step response for this system.

$$\boldsymbol{G(s) = \frac{10}{s^2 + 5s + 25}}$$

a) $t_s = 1.2\ sec$, $t_p = 0.72\ sec$

b) $t_s = 1.6\ sec$, $t_p = 0.72\ sec$

c) $t_s = 1.2\ sec$, $t_p = 0.92\ sec$

d) $t_s = 1.6\ sec$, $t_p = 0.92\ sec$

$$t_s \approx \frac{4}{\zeta\omega_n}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

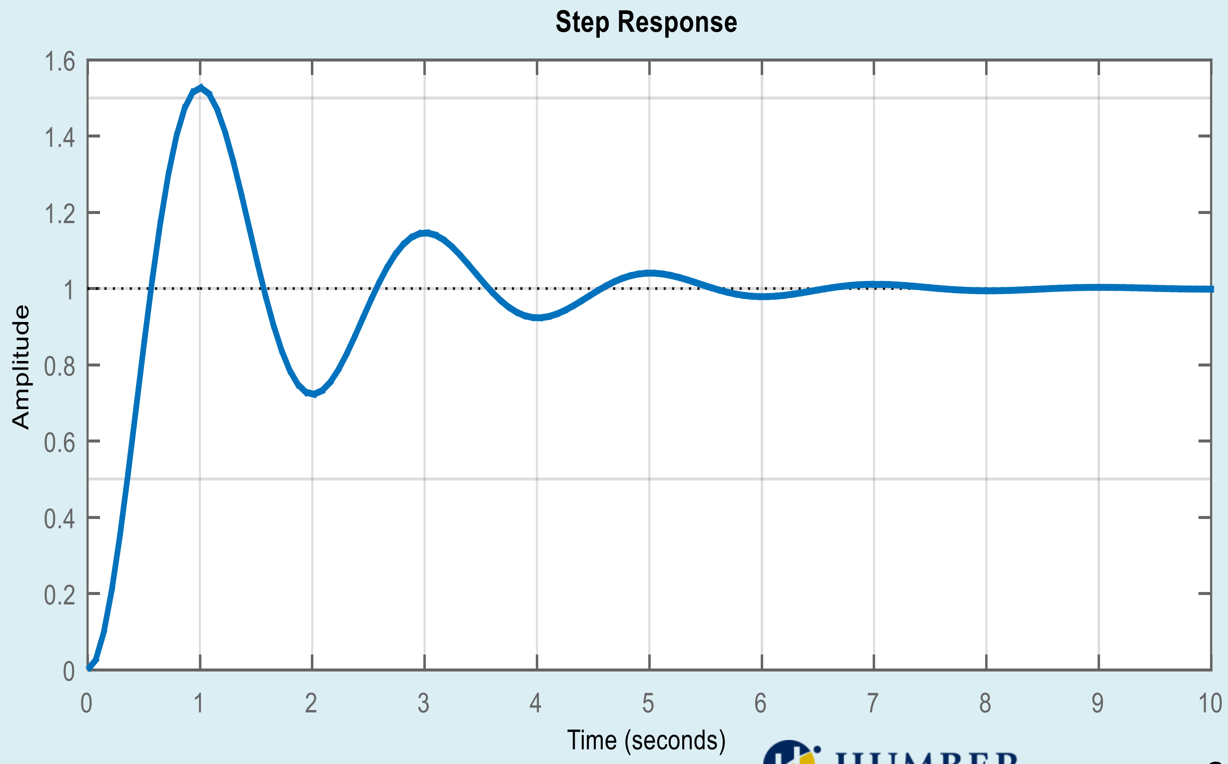
2. Consider the following unit-step response of a system. Which answer is the best estimation of the damping ratio, ζ, and the undamped natural frequency, ω_n , of this system?

a) $\zeta = 0.5$, $\omega_n = 3.2\ rad/sec$

b) $\zeta = 0.2$, $\omega_n = 3.2\ rad/sec$

c) $\zeta = 0.5$, $\omega_n = 1.2\ rad/sec$

d) $\zeta = 0.2$, $\omega_n = 1.2\ rad/sec$

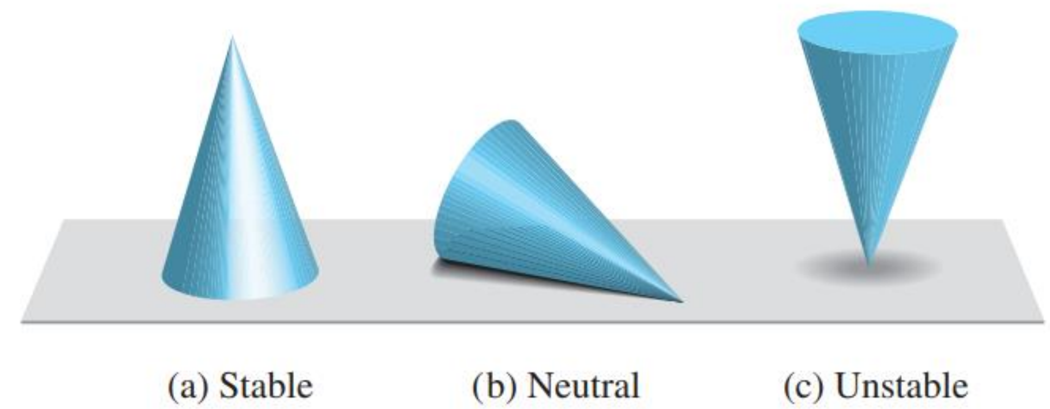


3. Find a second-order transfer function model for the system in question 2.

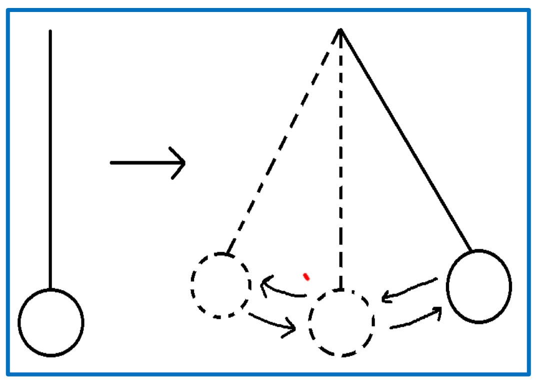
$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Stability of Systems

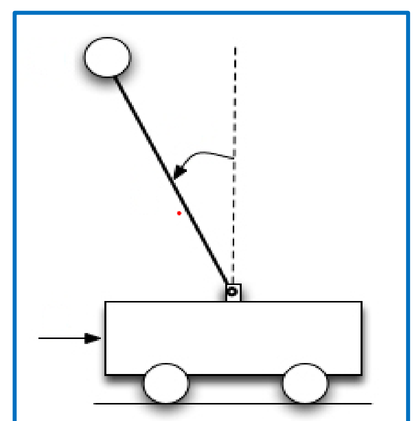
- In general, a stable system is the one that will
  - Remain at rest unless excited by an external source
  - Return to rest if all excitations are removed
- **Stability** is the most important specification of a control system.
- An unstable control system is generally useless.



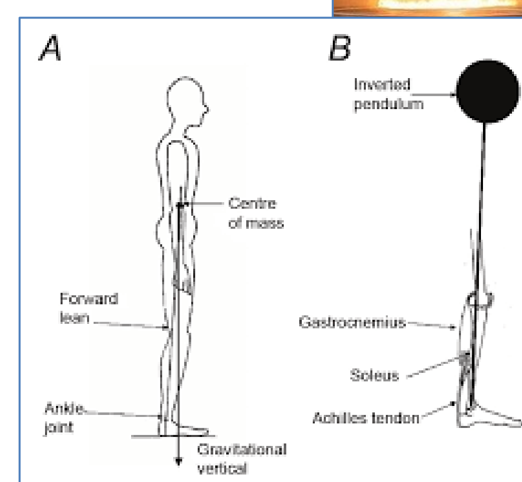
- Simple pendulum hanging from the ceiling is a stable system, but an inverted pendulum is an unstable system.



Simple Pendulum



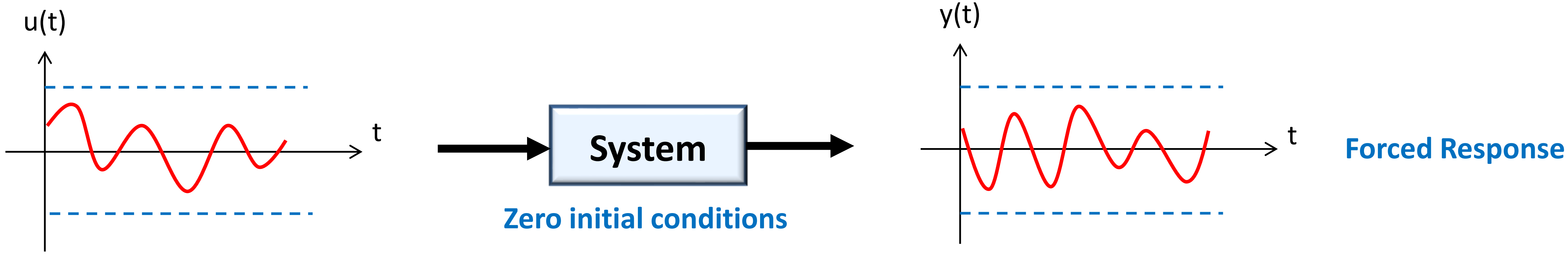
Inverted Pendulum



Stability of Systems

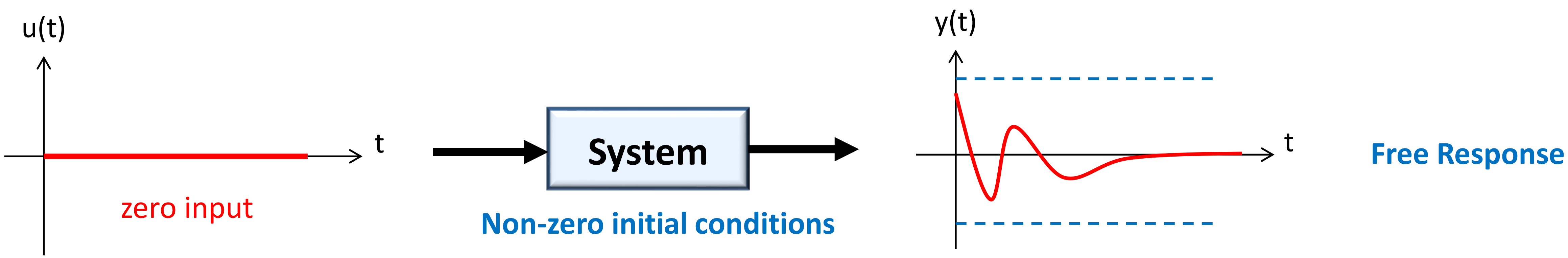
❑ Bounded-Input, Bounded-Output Stability (BIBO Stability)

- A system with zero initial conditions is **BIBO stable** if every bounded-input results in a bounded-output.



❑ Asymptotic Stability (Zero-Input Stability)

- A system with zero input is **asymptotically stable** if following conditions are satisfied:

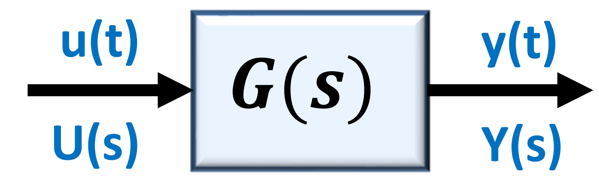


- In **LTI systems** the asymptotic stability is equivalent to BIBO stability, but in general they are different.

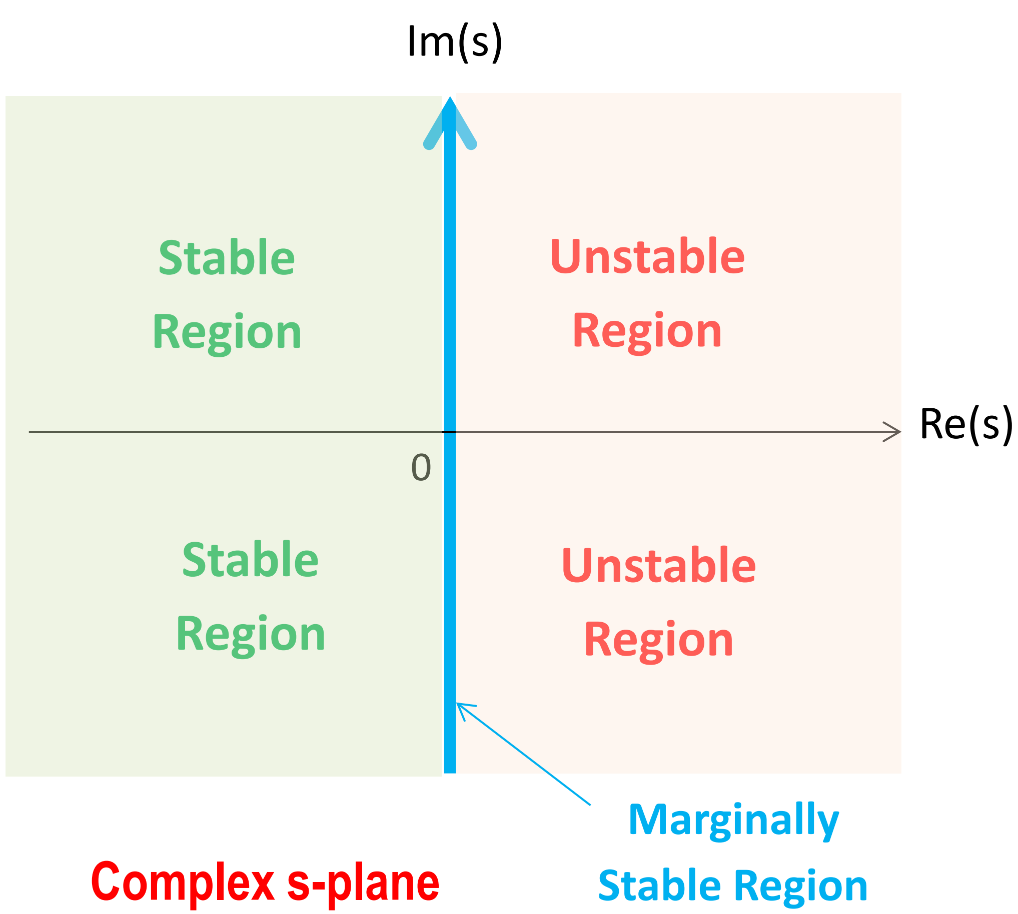
Stability of Linear Systems

❑ Stability of Linear Dynamic Systems

- Consider the a LTI dynamic system with input u(t), output y(t) and transfer function model of G(s)



A linear dynamic system is stable if and only if all poles of G(s) are in the left-half s-plane



Complex s-plane

Example 5

$$G_1(s) = \frac{s+1}{(s+2)(s^2+5)} \qquad poles \;\rightarrow \begin{cases} s_1 = -2 \\ s_{2,3} = \pm j\sqrt{5} \end{cases}$$

Marginally Stable

$$G_2(s) = \frac{2(s+2)}{(s+10)(s+3)} \qquad poles \;\rightarrow \begin{cases} s_1 = -10 \\ s_2 = -3 \end{cases}$$

Stable

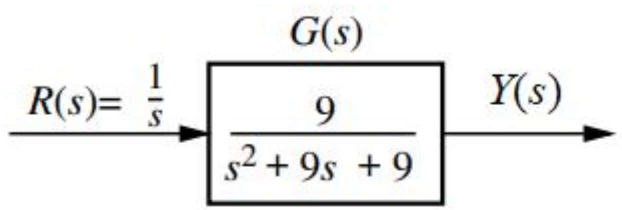
$$G_3(s) = \frac{10}{(s-10)(s^2+4)} \qquad poles \;\rightarrow \begin{cases} s_1 = 10 \\ s_2 = \pm j2 \end{cases}$$

Unstable

Step Response, Pole Locations & Stability

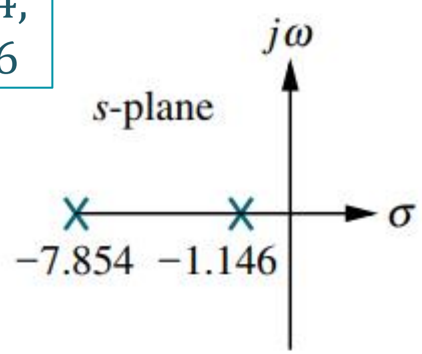
Example 6

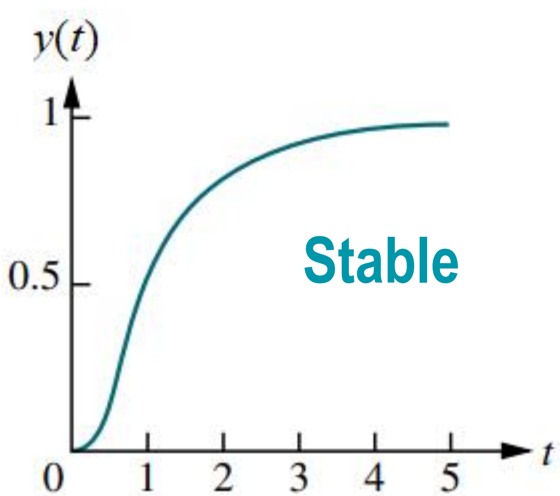
This example shows relationship of the step response, pole locations and stability in a second-order system.

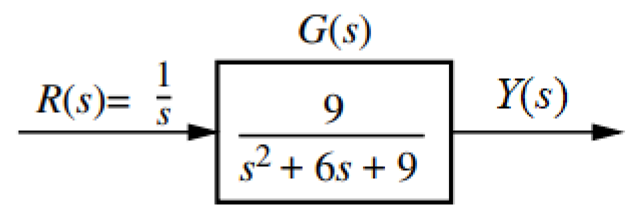


Overdamped

$\zeta = 1.5$
$s_1 = -7.854,$
$s_2 = -1.146$

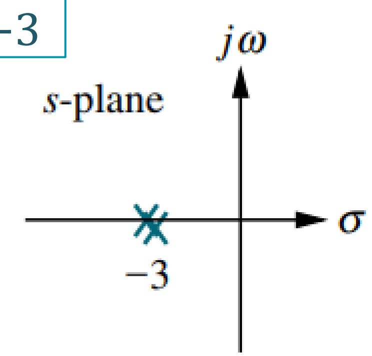


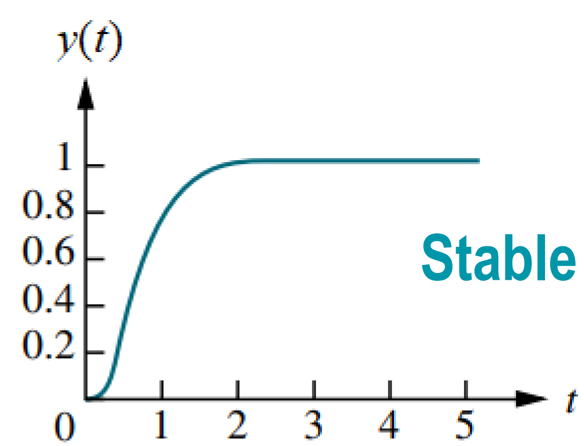


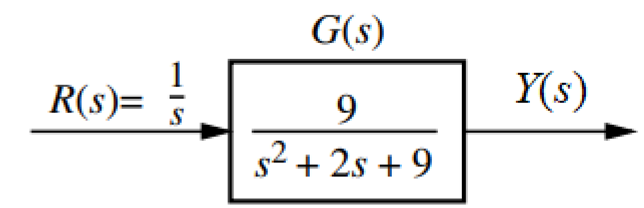


Critically damped

$\zeta = 1$
$s_1 = s_2 = -3$

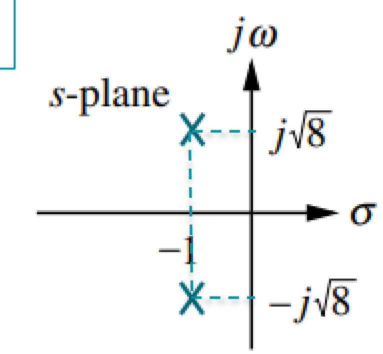


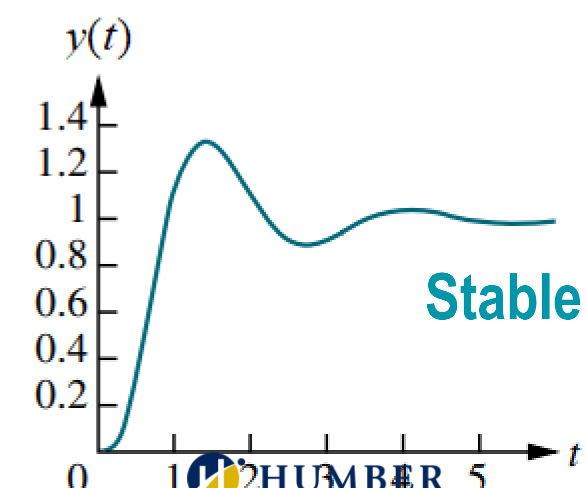




Underdamped

$\zeta = 0.33$
$s_{1,2} = -1 \pm j\sqrt{8}$





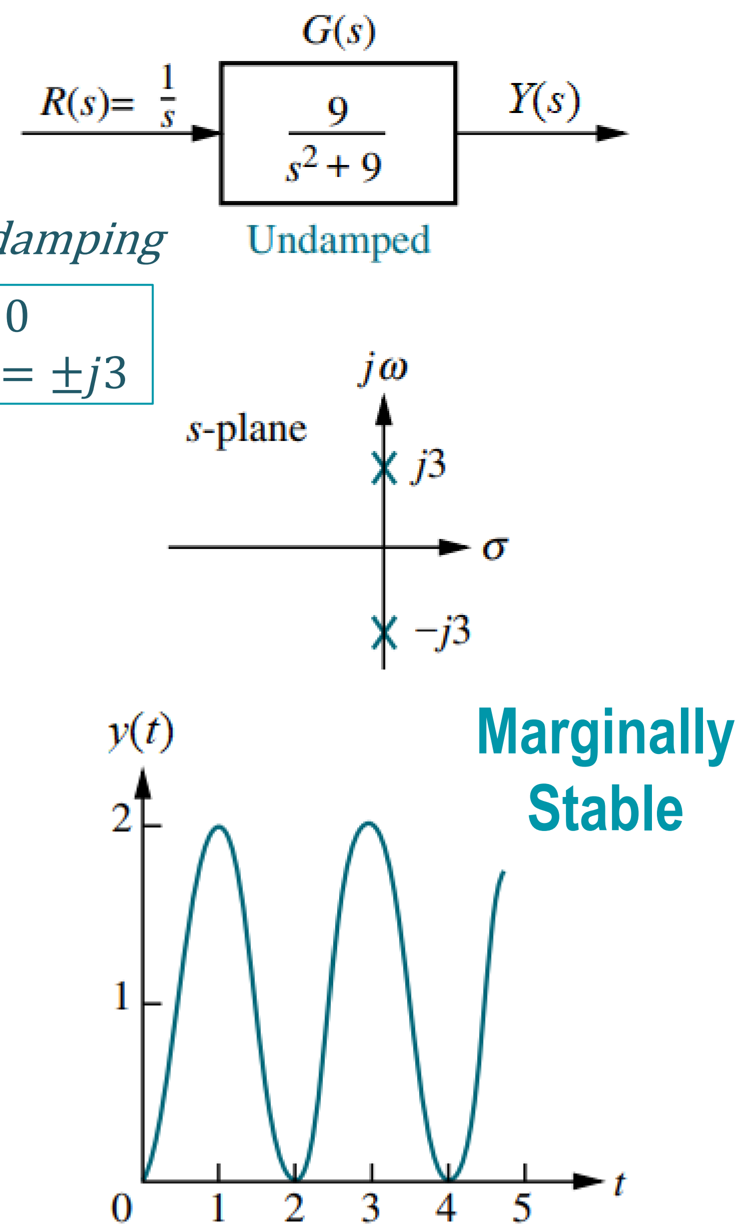
Step Response, Pole Locations & Stability

Example 6

This example shows relationship of the <u>step response</u>, <u>pole locations</u> and <u>stability</u> in a second-order system.

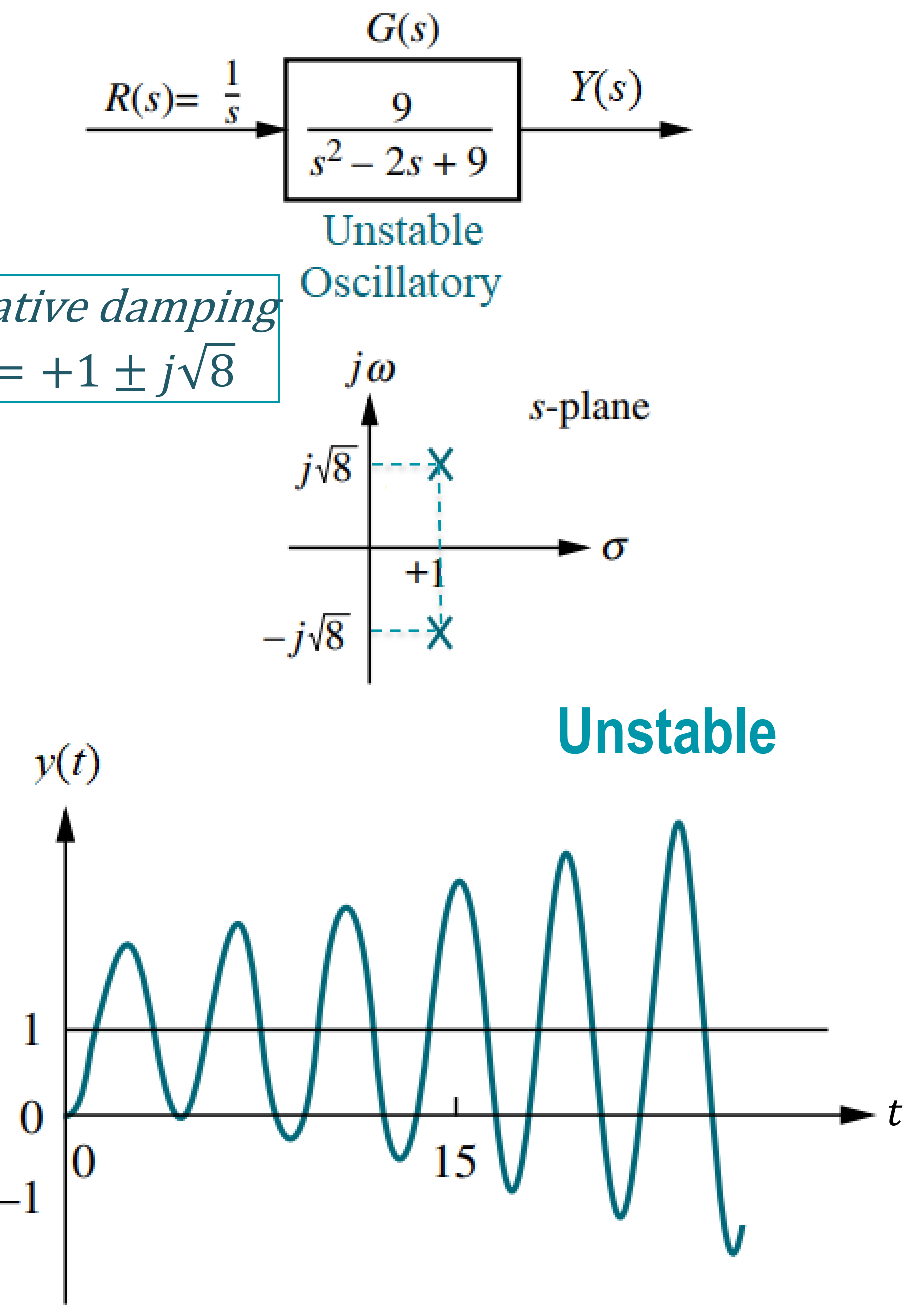
No damping

$\zeta = 0$
$s_{1,2} = \pm j3$



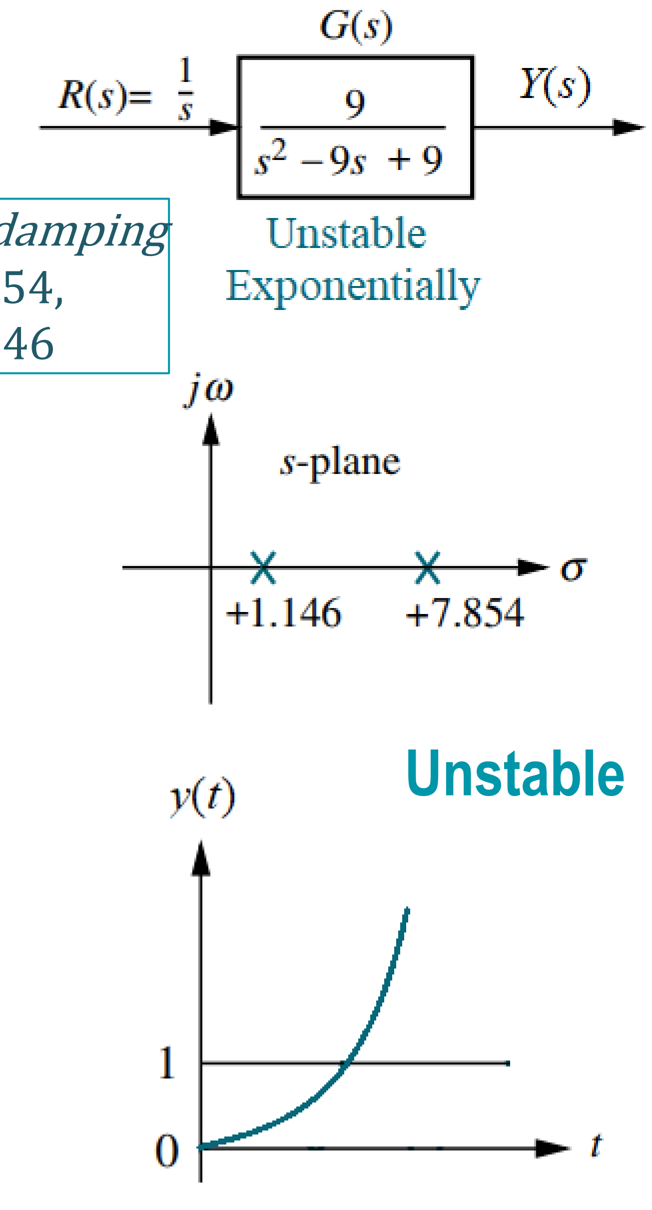
Negative damping

$s_{1,2} = +1 \pm j\sqrt{8}$



Negative damping

$s_1 = +7.854,$
$s_2 = +1.146$



Review of Complex s-Plane Characteristics

- Following graph shows relation between the pole locations in the s-plane with time response and stability of the systems.

❑ Poles on the Real axis

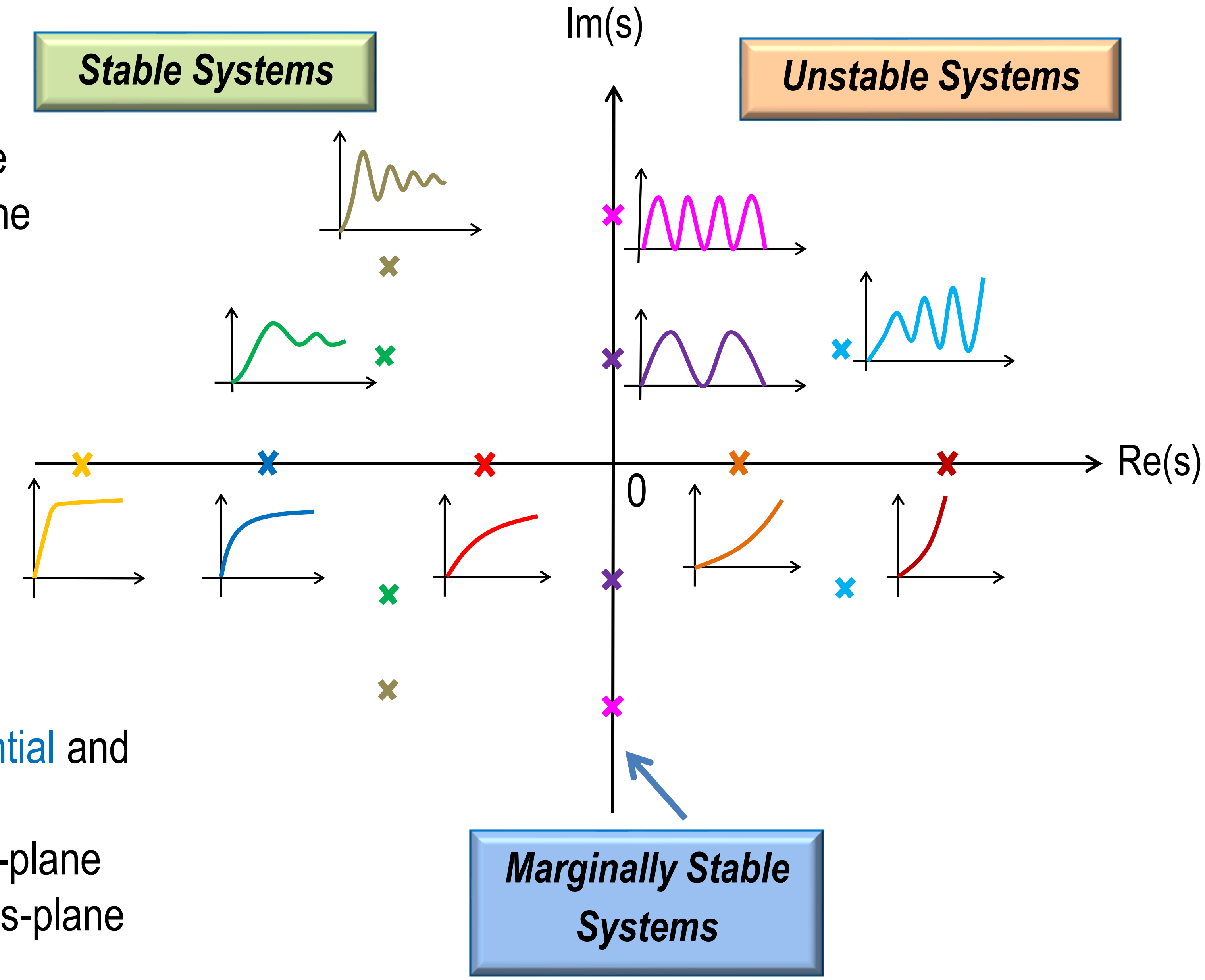
- Have an exponential response
- Exponential decay if the pole is in left-half s-plane
- Exponential grows if the pole is in right-half s-plane

❑ Poles on the Imaginary axis

- Always come in pair as a complex conjugated
- Have sinusoidal oscillations in the time domain

❑ Complex Conjugate Poles

- Have responses with mix of the two both exponential and sinusoidal motions
- Time response decay if the poles are in left-half s-plane
- Time response grows if the poles are in right-half s-plane



Pole Locations & Stability

Example 7

Assume the following dynamic system transfer function,

$$G(s) = \frac{3}{s-4}$$

a) Determine if the system is stable or not.

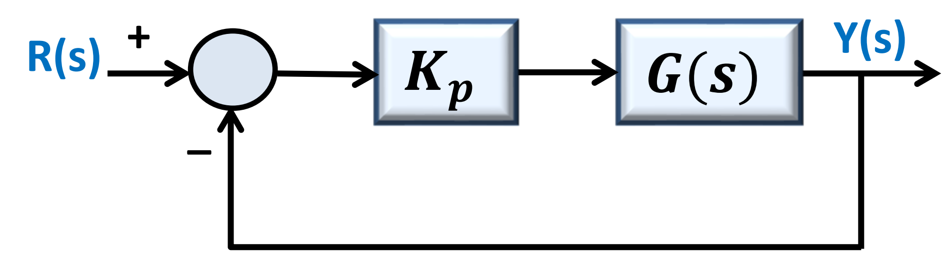
Check the pole location:

$$s - 4 = 0 \quad \rightarrow \quad s = 4$$

Since the pole is in the right-half side of the s-plane, the system is unstable.

b) Determine the required range of the proportional controller gain K_p to have a stable closed-loop system.

First find the closed-loop transfer function with controller gain of K_p:



$$\frac{Y(s)}{R(s)} = \frac{K_p G(s)}{1 + K_p G(s)} = \frac{\frac{3K_p}{s-4}}{1 + \frac{3K_p}{s-4}} = \frac{3K_p}{s - 4 + 3K_p}$$

Next determine the closed-loop poles in terms of gain K_p:

$$s - 4 + 3K_p = 0 \quad \rightarrow \quad s = 4 - 3K_p$$

To have a stable closed-loop system, poles must be in the left-half side of the s-plane:

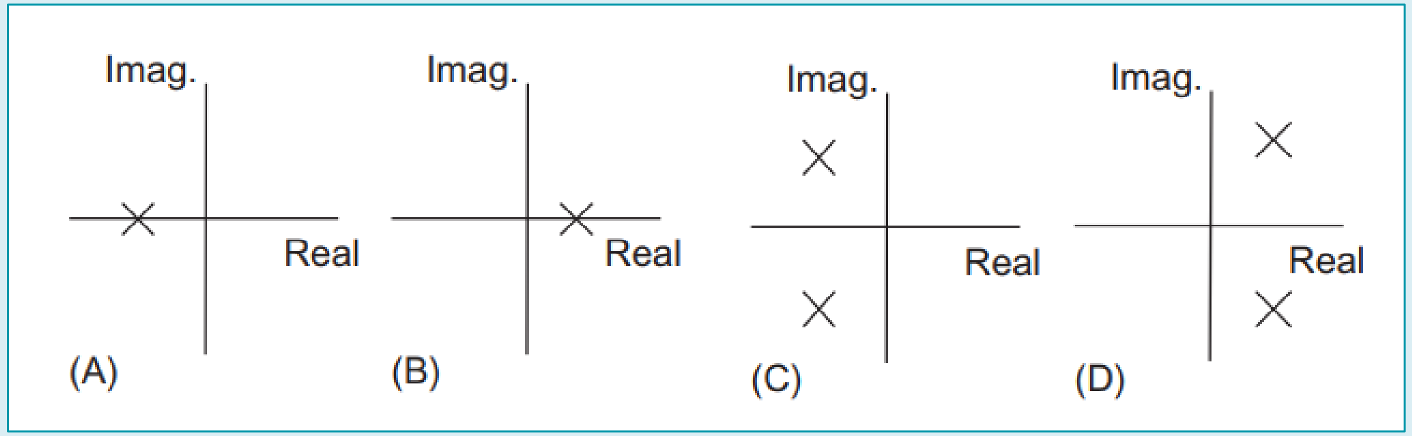
$$4 - 3K_p < 0 \quad \rightarrow \quad 4 < 3K_p \quad \rightarrow \quad \boxed{\frac{4}{3} < K_p}$$

For example, at $K_p = 2$ the closed-loop system is stable:

$$\frac{Y(s)}{R(s)} = \frac{6}{s+2}$$

Quick Review

1. For the poles shown on the s-planes, which will give rise to stable transients an which to unstable transients?



2. Which of the following transfer functions are stable?
   a) G_1 and G_2
   b) G_1 and G_3
   c) G_2 and G_3
   d) None of above

$$G_1(s) = \frac{1}{s^2 + 2s + 1} \qquad G_2(s) = \frac{1}{s^2 - 2s + 10} \qquad G_3(s) = \frac{1}{(s+1)(s+3)}$$

3. Which system is marginally stable?
   a) $G(s) = \frac{20}{(s+2)(s+5)}$
   b) $G_c(s) = \frac{0.5}{s^2+4}$
   c) $G_c(s) = \frac{5}{s^2-9}$
   d) $G_c(s) = \frac{10}{(s+3)(s-9)}$

THANK YOU

HUMBER

WE ARE HUMBER