Module 3 – Linear Systems of Equations – Part I

Lesson goals

1. Understanding matrix notation.
2. Being able to identify the following types of matrices: identity, diagonal, symmetric, triangular, and tridiagonal.
3. Knowing how to perform matrix multiplication and being able to assess when it is feasible.
4. Knowing how to represent a system of linear algebraic equations in matrix form.

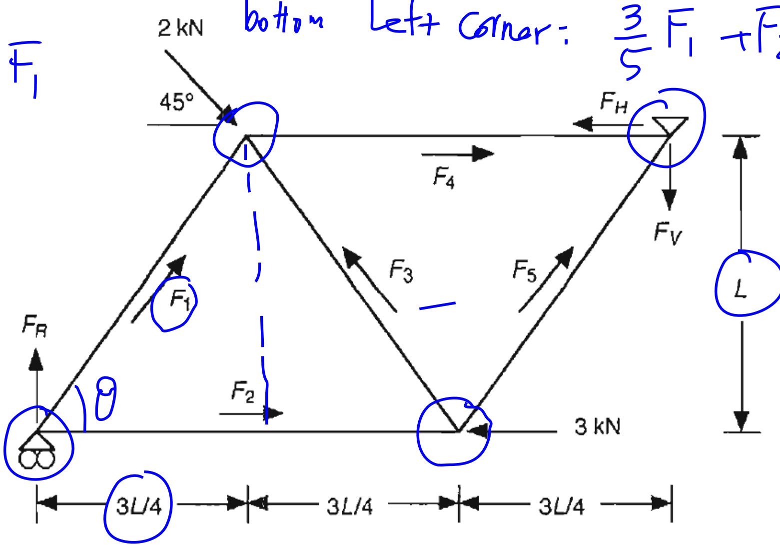
Introduction

Application 1. Consider the statically determinate plane truss shown in the following figure. The structure is pinned to a stationary support at the upper right and supported by a roller at the lower left. Furthermore, the structure is subjected to a 3-kN horizontal force at the lower right joint and to a 2-kN force acting at a 45° angle to the horizontal at the upper left joint. The objective is to determine the resulting forces within the members and the reaction forces at the stationary support and the roller.

bottom Left corner: $\frac{3}{5}F_1 + F_2 = 0$

horizontal component of F_1 $= F_1 \cos\theta = \frac{3}{5}F_1$

$\frac{4}{5}F_1 + F_R = 0$



$\tan\theta = \frac{L}{3L/4} = \frac{4}{3}$

$1 + \tan^2\theta = \frac{1}{\cos^2\theta} \to \frac{25}{9} = \frac{1}{\cos^2\theta}$

$\to \cos\theta = 3/5$

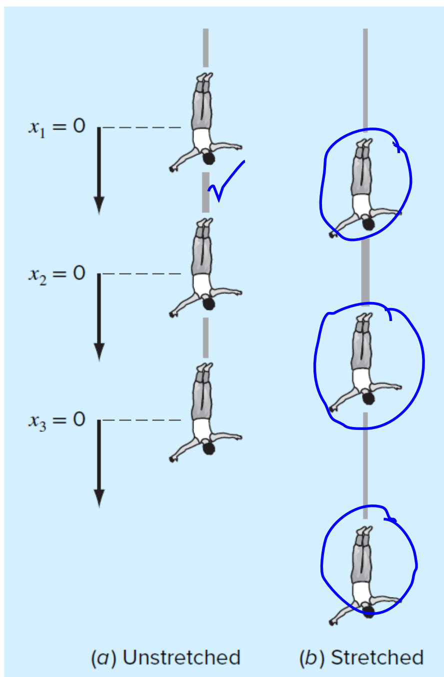
Let each force act in the direction indicated in the diagram. Then, balancing the horizontal and vertical components at each joint provides eight simultaneous linear equations for the eight unknowns. Here are the equations:

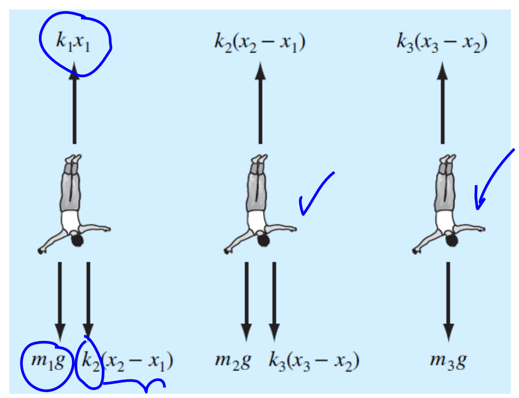
Joint	Horizontal component	Vertical component
Lower left	$\frac{3}{5}F_1 + F_2 = 0$ ✓	$\frac{4}{5}F_1 + F_R = 0$ ✓
Lower right	$F_2 - \frac{3}{5}F_3 + \frac{3}{5}F_5 - 3 = 0$ ✓	$\frac{4}{5}F_3 + \frac{4}{5}F_5 = 0$ ✓
Upper left	$\frac{3}{5}F_1 - \frac{3}{5}F_3 + F_4 + \sqrt{2} = 0$	$\frac{4}{5}F_1 + \frac{4}{5}F_3 - \sqrt{2} = 0$
Upper right	$F_4 + \frac{3}{5}F_5 - F_H = 0$	$\frac{4}{5}F_5 - F_V = 0$

System of 8 equations and 8 unknowns

Application 2. Suppose that three jumpers are connected by bungee cords, and they are being held in place vertically so that each cord is fully extended but unstretched. We can define three distances, x_1, x_2, and x_3, as measured downward from each of their unstretched positions (see below figure). After they are released, gravity takes hold, and the jumpers will eventually come to the equilibrium positions. Suppose that you are asked to compute the displacement of each of the jumpers. If we assume that each cord behaves as a linear spring and follows Hooke's law, free-body diagrams can be developed for each jumper as depicted in the following figures.

Using Newton's second law and considering the steady-state solution where the second derivatives can be set to zero, the problem reduces to solving the following system of three simultaneous equations for the three unknown displacements.





$$\begin{cases} (k_1 + k_2)x_1 - k_2x_2 = m_1g \\ -k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = m_2g \\ -k_3x_2 + k_3x_3 = m_3g \end{cases} \qquad (*)$$

where m_i = the mass of jumper i (kg), t = time (s), k_j = the spring constant for cord j (N/m), x_i = the displacement of jumper i measured downward from the equilibrium position (m), and g = gravitational acceleration (9.81 m/s^2).

Matrix Notations.

- A matrix is a rectangular collection of numbers arranged in rows and columns.
- A matrix is denoted by either A or $[A]$ and its element located in the ith row and jth column is denoted by a_{ij}. For example, a_{23} is in row 2 and column 3.

a_{rs} Column number

row number

$$\begin{array}{ccc} & \begin{array}{ccccc} \phantom{a_{11}} & \phantom{a_{12}} & \text{Column 3} & & \phantom{a_{1n}} \\ & & \downarrow & & \end{array} & \\ A = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{c} \\ \leftarrow \text{Row 2} \\ \\ \\ \\ \\ \end{array} \end{array}$$

$A_{m \times n}$

m rows
n columns

- A horizontal set of elements is called a row and a vertical set is called a column.
- A matrix with m rows and n columns is said to be of dimension m by n (or $m \times n$).
- A row vector is a matrix with row dimension $m = 1$, and is denoted by lowercase letter, such as

$$b = [b_1\ b_2\ \ldots\ b_n]$$

- A column vector is a matrix with column dimension $n = 1$, such as trace(A) = Sum of elements on principal diagonal.

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

- Matrices where $m = n$ are called square matrices. Square matrices are particularly important when solving sets of simultaneous linear equations.

a_{ij}

principal diagonal (main)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

Square

- The diagonal consisting of the elements a_{11}, a_{22}, and a_{33} is termed the principal or main diagonal of the matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 5 \\ 1 & 1 & 8 \end{bmatrix}$$

- A symmetric matrix is one where the rows equal the columns–that is, $a_{ij} = a_{ji}$, for all i's and j's. For example,

$$A^T = A$$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ -1 & 5 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 5 & -2 \\ 4 & -2 & \sqrt{3} \end{bmatrix}$$

Skew or asymmetric

- A diagonal matrix is a square matrix where all elements off the main diagonal are equal to zero.

$$A^T = -A$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

- An identity matrix is a diagonal matrix where all elements on the main diagonal are equal to 1.

I_n

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

- An upper triangular matrix is one where all the elements below the main diagonal are zero.

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

- A lower triangular matrix is one where all the elements above the main diagonal are zero.

$$L = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A banded matrix has all elements equal to zero, with the exception of a band centered on the main diagonal:

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

The preceding matrix has a bandwidth of 3 and is given a special name–the tridiagonal matrix.

- The transpose of a matrix involves transforming its rows into columns and its columns into rows. The transpose of A is denoted by A^T. For example,

$$\begin{array}{cccc} & C_1 & C_2 & C_3 \\ I_3 = & \multicolumn{3}{c}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \end{array}$$

$$A = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 5 & -2 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} -1 & 0 \\ 0 & 5 \\ 4 & -2 \end{bmatrix}$$

- A permutation matrix (also called a transposition matrix) is an identity matrix with rows and columns interchanged. For example,

$$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I_n: $n! = \#$ of permutation matrices

$$\begin{array}{cccc} & C_1 & C_3 & C_2 \\ P_1 = & \multicolumn{3}{c}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \end{array}$$

$$\begin{array}{cccc} & C_2 & C_3 & C_1 \\ P_2 = & \multicolumn{3}{c}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \end{array}$$

$3! = \#$ of permutation matrices.

- The final matrix manipulation that will have utility in our discussion is augmentation. A matrix is augmented by the addition of a column (or columns) to the original matrix. For example,

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -5 & 2 & 1 & 0 & 0 \\ 4 & 3 & -6 & 0 & 1 & 0 \\ -1 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$$

Operations with matrices

- Two m by n matrices are equal if, and only if, every element in the first is equal to every element in the second–that is, $A = B$ if $a_{ij} = b_{ij}$ for all i and j.

Ex. Find x and y so that the following matrices are equal.

$$A = \begin{bmatrix} 2 & -1 \\ x+3y & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2x+y \\ 0 & 5 \end{bmatrix}$$

$$-3\times \begin{cases} 2x+y = -1 \\ x+3y = 0 \end{cases} \rightarrow \begin{cases} -6x-3y = 3 \\ x+3y = 0 \end{cases}$$

Add

$$-5x = 3$$

$$\rightarrow x = -0.6$$

$A_{m\times n} + B_{m\times n} = C_{m\times n}$

$2(-0.6) + y = -1 \mapsto \boxed{y = 0.2}$

- Addition of two m by n matrices A and B is a matrix C whose elements are computed as

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1,2,\dots,m; \; j = 1,2,\dots,n$$

- Similarly, the subtraction of two matrices can be defined as

$$C = A - B = A + (-B)$$

- The multiplication of a matrix A by a scalar g is obtained by multiplying every element of A by g.

Row matrix $= A_{1\times n} = [a_1 \; a_2 \dots a_n]$

Column matrix $B_{n\times 1} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$$gA = \begin{bmatrix} ga_{11} & ga_{12} & ga_{13} \\ ga_{21} & ga_{22} & ga_{23} \\ ga_{31} & ga_{32} & ga_{33} \end{bmatrix}$$

$[2 \; -1 \; 3]\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} = (2)(4) + (-1)(5) + (3)(-1) = 0$

- The product of two matrices is represented as $C = AB$, whose elements are defined as

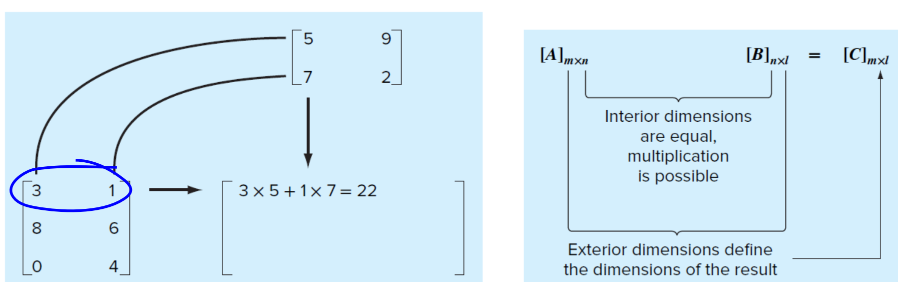
$A_{1\times n} \times B_{n\times 1} = C_{1\times 1} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

$A_{m\times p} \times B_{p\times n} = C_{m\times n}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

c_{ij} = product of i^{th} row of A by j^{th} Column of B

where n = the column dimension of A and the row dimension of B. See the following figure



Note. Addition of matrices is commutative and associative; however, multiplication of matrices is neither commutative nor associative in general.

Example. Determine $A + B$ and μA when

$$A = \begin{bmatrix} -1 & -1 & 4 \\ 4 & 3 & -6 \\ 2 & -5 & 2 \end{bmatrix}_{3\times 3}, \quad B = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 5 & -2 \\ 4 & -2 & 3 \end{bmatrix}_{3\times 3}, \quad \mu = -2$$

$$A + B = \begin{bmatrix} -2 & -1 & 8 \\ 4 & 8 & -8 \\ 6 & -7 & 5 \end{bmatrix}$$

$$B - A = B + (-A) = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 5 & -2 \\ 4 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -4 \\ -4 & -3 & 6 \\ -2 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$$

$$ uA = -2A = \begin{bmatrix} 2 & 2 & -8 \\ -8 & -6 & 12 \\ -4 & 10 & -4 \end{bmatrix} $$

$$ AB = \underset{\underline{3\times 2}}{\begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{bmatrix}} \underset{\underline{2\times 3}}{\begin{bmatrix} -1 & 0 & 4 \\ 3 & 5 & -2 \end{bmatrix}} = \begin{bmatrix} c_{11}=3 & c_{12}=10 & c_{13}=8 \\ c_{21}=4 & c_{22}=5 & c_{23}=-6 \\ c_{31}=11 & c_{32}=20 & c_{33}=-4 \end{bmatrix}_{3\times 3} $$

Example. Determine all possible products of the following matrices

$$ A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 4 \\ 3 & 5 & -2 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 & 2 & 4 \\ 0 & 5 & -1 & -2 \\ 3 & -1 & 4 & 0 \\ 4 & -2 & 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} $$

$$ \underset{\underline{2\times 3}}{B} \; \underset{\underline{3\times 2}}{A} = \begin{bmatrix} -1 & 0 & 4 \\ 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} c_{11}=1 & c_{12}=14 \\ c_{21}=2 & c_{22}=3 \end{bmatrix}_{2\times 2} $$

$AC, CA, BC, CB, BD, DA, DC, CD$ are not possible

$$ AD = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} c_{11}=6 & c_{12}=7 \\ c_{21}=-7 & c_{22}=1 \\ c_{31}=-8 & c_{32}=9 \end{bmatrix}_{3\times 2} $$

$DB =$

$$I_3 = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \begin{array}{ccc} c_3 & c_2 & c_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Note. Left multiplying a matrix A by a permutation matrix P will switch the *corresponding rows of* A. Right multiplying by P will switch the *corresponding columns*.

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 4 \\ 4 & 3 & -6 \\ 2 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 2 \\ 4 & 3 & -6 \\ -1 & -1 & 4 \end{bmatrix} \checkmark$$

$$AP = \begin{bmatrix} -1 & -1 & 4 \\ 4 & 3 & -6 \\ 2 & -5 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ -6 & 3 & 4 \\ 2 & -5 & 2 \end{bmatrix}$$

Inverse matrices. An $n \times n$ matrix A is said to be nonsingular (or invertible) if an $n \times n$ matrix A^{-1} exists with $\boxed{AA^{-1} = A^{-1}A = I.}$ The matrix A^{-1} is called the inverse of A. A matrix without an inverse is called singular (or noninvertible).

Note. For any nonsingular $n \times n$ matrix A:

$a \neq 0$ $\qquad a^{-1} = \frac{1}{a}$

A is invertible
$\to$ A is nonsingular.

a) A^{-1} is unique.

b) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

c) If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.

Note. The inverse of a 2×2 matrix can be represented simply by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{(given that } ad - bc \neq 0\text{)}$$

Similar formulas for higher-dimensional matrices are much more involved. One way is to use elementary row operations over the augmented matrix.

Elementary row operations:

1. Interchange two rows.
2. Multiply/divide a row by a nonzero scalar.
3. Add a multiple of a row to another row.

Example. Find the inverse matrix for the matrix

$$D = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\det(D) = (4)(2) - (-3)(1) = 11$$

$$D^{-1} = \frac{1}{11}\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2/11 & -1/11 \\ 3/11 & 4/11 \end{bmatrix}$$

Example. Find the inverse matrix for the matrix

$$A = \begin{bmatrix} 2 & -5 & 2 \\ 4 & 3 & -6 \\ -1 & -1 & 4 \end{bmatrix}$$

$$\text{Aug. matrix} = [A \mid I] = \left[\begin{array}{ccc|ccc} 2 & -5 & 2 & 1 & 0 & 0 \\ 4 & 3 & -6 & 0 & 1 & 0 \\ -1 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_1 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|ccc} -1 & -1 & 4 & 0 & 0 & 1 \\ 4 & 3 & -6 & 0 & 1 & 0 \\ 2 & -5 & 2 & 1 & 0 & 0 \end{array}\right] \overset{\text{new } R_1 \leftarrow (-1)R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & -1 \\ 4 & 3 & -6 & 0 & 1 & 0 \\ 2 & -5 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\underset{\text{new } R_3 \leftarrow -2R_1+R_3}{\overset{\text{new } R_2 \leftarrow -4R_1+R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & -1 \\ 0 & -1 & 10 & 0 & 1 & 4 \\ 0 & -7 & 10 & 1 & 0 & 2 \end{array}\right] \overset{\text{new } R_2 \leftarrow -R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & -1 \\ 0 & 1 & -10 & 0 & -1 & -4 \\ 0 & -7 & 10 & 1 & 0 & 2 \end{array}\right]$$

$$\underset{\text{new } R_3 \leftarrow 7R_2+R_3}{\overset{\text{new } R_1 \leftarrow -R_2+R_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 0 & 1 & 3 \\ 0 & 1 & -10 & 0 & -1 & -4 \\ 0 & 0 & -60 & 1 & -7 & -26 \end{array}\right] \overset{\text{new } R_3 \leftarrow \frac{R_3}{-60}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 0 & 1 & 3 \\ 0 & 1 & -10 & 0 & -1 & -4 \\ 0 & 0 & 1 & -1/60 & 7/60 & 26/60 \end{array}\right]$$

new $R_1 \leftarrow -6R_3 + R_1$

new $R_2 \leftarrow 10R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/10 & 3/10 & +2/5 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/3 \\ 0 & 0 & 1 & -1/60 & 7/60 & 26/60 \end{array}\right]$$

$$\underbrace{\quad\quad}_{I} \qquad \underbrace{\quad\quad\quad\quad}_{A^{-1}}$$

Representing Linear Algebraic Equations in Matrix Form

Consider the following (simultaneous) linear systems of equations:

$$
A = \begin{array}{c|c} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{array}{c} \text{row 1} \\ \text{row 2} \\ \vdots \\ \text{row } n \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n} \end{array}
$$

$$
\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}\right\}
\qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \underline{\underline{\text{RHS}}}
$$

The introduction of the matrix-vector product permits us to view the above linear system as the matrix equation

$$Ax = b \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

However, we may have a system of linear equation with more equations (rows) than unknowns (columns), the so-called ***overdetermined system*** $(m > n)$, or more unknowns than equations, the so-called ***underdetermined system*** $(m < n)$. A typical example for overdetermined system is least-squares regression where an equation with n coefficients is fit to m data (x, y).

Example. For the linear systems of equations

$$
ax = b \Rightarrow \begin{cases} a \neq 0 \mapsto x = a^{-1}b \\ a = 0 \mapsto \begin{cases} b = 0 \Rightarrow \text{infinitely many} \\ b \neq 0 \rightarrow \text{No Solution} \end{cases} \end{cases}
$$

$$
\begin{aligned}
4x_1 - 3x_2 + x_3 &= -2 \\
-3x_2 + 5x_3 &= 0 \\
x_1 + 2x_2 &= -4
\end{aligned}
\iff Ax = b
$$

identify the coefficient matrix A, the unknown vector x and the right-hand side vector b

$$
A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 4 & -3 & 1 \\ 0 & -3 & 5 \\ 1 & 2 & 0 \end{bmatrix} \end{array} \qquad b = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

Remark. A formal way to obtain a solution using matrix algebra when A is *nonsingular (invertible)* is to multiply each side of the equation by the inverse of A to yield

$$Ax=b \xRightarrow{A \text{ is nonsingular}} \underbrace{A^{-1}(Ax)}= A^{-1}b \Rightarrow \boxed{x = A^{-1}b} \checkmark$$

But this requires some knowledge from linear algebra on how to compute the inverse of a nonsingular matrix.

For a very simple case, let us solve the following linear systems of equations:

$$\begin{cases} 4x_1 - 3x_2 = -2 \\ -3x_1 + 5x_2 = 3 \end{cases}$$

$$A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

For this system, we have

$$A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{20-9}\begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{3}{11} \\ \frac{3}{11} & \frac{4}{11} \end{bmatrix}$$

Thus,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{A^{-1}b} = \begin{bmatrix} \frac{5}{11} & \frac{3}{11} \\ \frac{3}{11} & \frac{4}{11} \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} \\ \frac{6}{11} \end{bmatrix} \longrightarrow \begin{cases} x_1 = -\frac{1}{11} \\ x_2 = \frac{6}{11} \end{cases}$$

Solving linear algebraic equations with MATLAB. There are two direct ways in MATLAB to solve systems of linear algebraic equations. The most efficient way is to employ the backslash, or "left-division" operator as in

```
>> x = A\b
```

A\b

The second is to use matrix inversion:

```
>> x = inv(A)*b
```

Example. Use MATLAB to solve the bungee jumper problem described at the Application 2. The parameters for the problem are

Jumper	Mass (kg)	Spring Constant (N/m)	Unstretched Cord Length (m)
Top (1)	60	50	20
Middle (2)	70	100	20
Bottom (3)	80	50	20

Solution. Substituting these parameters into equation (*) in Application 2, we have:

$$A = \begin{bmatrix} 150 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 50 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 588.6 \\ 686.7 \\ 784.8 \end{bmatrix}$$

Start up MATLAB and enter the coefficient matrix and the right-hand-side vector:

```
>> A = [150 -100 0;-100 150 -50;0 -50 50]

>> b = [588.6; 686.7; 784.8]

>> x = A\b
```

```
>> x = inv(A)*b
```

Because the jumpers were connected by 20-m cords, their initial positions relative to the platform is

```
>> x0 = [20;40;60];
```

Thus, their final positions can be calculated as

```
>> xf = x+x0
```

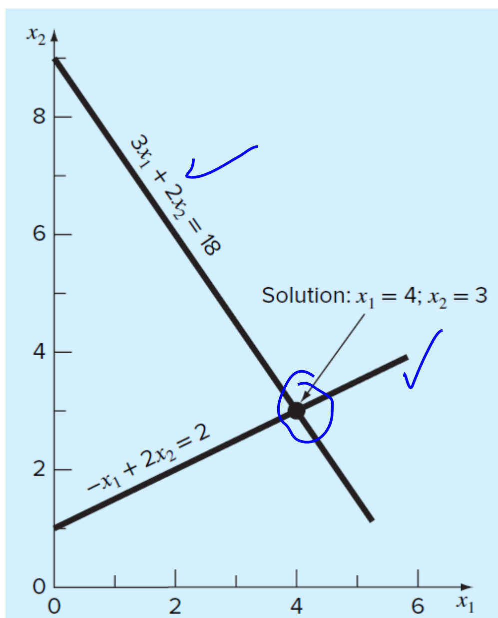
An Introduction to Gaussian Elimination

Overview (Graphical method for solving small numbers of equations)

A graphical solution can be obtained for two linear equations by plotting them on Cartesian coordinates with one axis corresponding to x_1 and the other to x_2.

Example. Consider the following linear systems of equations.

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$



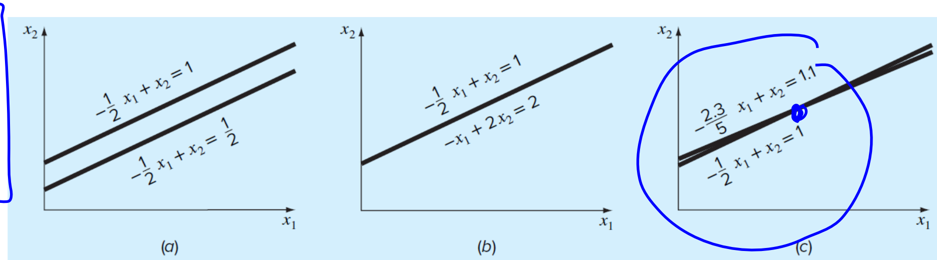
$$\begin{cases} -0.50001\, x_1 + x_2 = 1.1 \\ -0.5\, x_1 + x_2 = 1 \end{cases} \longrightarrow \text{Unique soluti}$$

$$\det(A) = (-0.50001) - (-0.5) = -0.0001 \neq 0$$

$$\tilde{A} = \begin{bmatrix} -0.5 & 1 \\ -0.5 & 1 \end{bmatrix}$$

$$\det(\tilde{A}) = 0$$

In general, check out all possible cases in the following graphs.



For the case system has no solution (Figure a) or infinitely many solutions (Figure b), then the system is said to be ***singular***. Systems that are very close to being singular (such as Figure c) are said to be ***ill-conditioned or ill-posed***.

Determinant of a square matrix. Suppose that A is a square matrix. The determinant of A is denoted by det (A).

(i) If $A = [a]$ is a 1×1 matrix, then $\boxed{\det(A) = a.}$

(ii) If A is an $n \times n$ matrix, with $n > 1$ the ***minor*** M_{ij} is the determinant of the $(n -$

element: a_{ij} $\longrightarrow$ minor $M_{ij} = \det(A_{ij})$

$A_{ij} =$ [ ] j^{th} col i^{th} row

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 5 & 0 \\ 8 & 1 & -3 \end{bmatrix} \qquad M_{21} = \det\begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} = -2$$

$$C_{21} = -M_{21} = 2$$

1) $\times$ $(n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

(iii) The ***cofactor*** A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

(iv) The determinant is given by

$$\det(A) = (8)C_{31} + (1)C_{32} + (-3)C_{33}$$

$$\det(A) = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij}$$

$$= 8M_{31} - (1)M_{32} + (-3)M_{33}$$

Note. There are simple ways to get the determinant of a 2×2 and 3×3 matrices.

$$= (8)(5) - (1)(3) + (-3)(7) = 16$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - afh - bdi \qquad \text{(Sarrus's rule)}$$

Example. Compute the determinant of the following matrices.

a) $A = \begin{bmatrix} 2 & -1 & 5 \\ -4 & 2 & 3 \\ -5 & 4 & 7 \end{bmatrix} \begin{matrix} 2 & -1 \\ -4 & 2 \\ -5 & 4 \end{matrix}$

$$= (2)(2)(7) + (-1)(3)(-5) + (5)(-4)(4) - (5)(2)(-5) - (2)(3)(4) - (-1)(-4)(7)$$

$$= 28 + 15 - 80 + 50 - 24 - 28 = -39$$

b) $B = \begin{bmatrix} \frac{2}{5} & -\frac{1}{3} \\ -\frac{4}{5} & -1 \end{bmatrix}$

$\det(B) = \frac{2}{5} \times (-1) - \left(-\frac{4}{5}\right)\left(-\frac{1}{3}\right) = -\frac{2}{5} - \frac{4}{15} = \frac{-2}{3}$

c) $A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & -2 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A) = (2)(-2)(1) = -4$

Some properties of determinant.

- If any row or column of A has only zero entries, then $\det(A) = 0$.

- If A has two rows or two columns the same, then $\det(A) = 0$.
- If $\tilde{A}$ is obtained from A by the interchanging two rows or two columns, then $\det(\tilde{A}) = -\det(A)$.
- If $\tilde{A}$ is obtained from A by the multiplying its one row or its one column by λ, then ~~det~~ $\det(\tilde{A}) = \lambda \det(A)$.
- If $\tilde{A}$ is obtained from A by adding a multiple of one row (column) to another row (column), then $\det(\tilde{A}) = \det(A)$.
- If B is also an $n \times n$ matrix, then $\det(AB) = \det(A)\det(B)$.
- $\det(A^T) = \det(A)$.
- When A^{-1} exists, $\det(A^{-1}) = \frac{1}{\det(A)}$.
- If A is an *upper triangular, lower triangular,* or *diagonal matrix*, then its determinant is the product of the elements on its main diagonal.

$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad A_1 = \begin{bmatrix} b & a_2 & a_3 & \cdots & a_n \end{bmatrix}$$

Cramer's rule. This rule states that each unknown in a nonsingular simultaneous linear system of equations $Ax = b$ can be obtained by a fraction of two determinants; the denominator is the determinant of the coefficient matrix A and the numerator is obtained from A by replacing its columns by the right-hand side vector b.

$$x_2 = \frac{\det(A_2)}{\det(A)} \qquad A_2 = \begin{bmatrix} a_1 & b & a_3 & \cdots & a_n \end{bmatrix}$$

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1,2,\ldots,n$$

b

A_i = the matrix obtained from A by replacing its ith column by the vector b

Example. Solve the following system by Cramer's rule

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

One flop: $\underbrace{a + bc}$ ✓ $\quad 10^{-10}$ seconds

Naïve elimination method (a system of two equations with two unknowns)

It is used to solve a pair of simultaneous equations:

1. The equations are manipulated to eliminate one of the unknowns from the equations by adding them. As a result, we get one equation with one unknown.

2. We solve this equation to get the unknown and substitute the result into one of the original equations to get the remaining unknown.

Example. Solve the following systems of equations by naïve elimination method.

$$(-1)\times\begin{cases} 3x_1 + 2x_2 = 18 \\ -x_1 + 2x_2 = 2 \end{cases} \longmapsto \begin{cases} 3x_1 + 2x_2 = 18 \\ x_1 - 2x_2 = -2 \end{cases} \quad \text{add}$$

$$4x_1 = 16 \longmapsto \boxed{x_1 = 4}$$

2^{nd} eqn. $\quad -4 + 2x_2 = 2 \longmapsto 2x_2 = 6 \longmapsto \boxed{x_2 = 3}$

Gaussian elimination

This method is a generalization of naïve elimination method for a simultaneous system of n equations:

$[A|b]$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \qquad (*)$$

It consists of **two main parts**: *Forward elimination* and *Backward substitution*.

$$\left.\begin{array}{c} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \\ \Big\downarrow \\ \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right] \end{array}\right\} \text{(a) Forward elimination}$$

$$\left.\begin{array}{l} x_3 = b''_3/a''_{33} \\ x_2 = (b'_2 - a'_{23}x_3)/a'_{22} \\ x_1 = (b_1 - a_{13}x_3 - a_{12}x_2)/a_{11} \end{array}\right\} \text{(b) Back substitution}$$

Forward elimination phase. It is designed to reduce the set of equations to an upper triangular system. We first remove x_1 from the second to the nth equation of (*): to do so, multiply the first equation (pivot equation) by $\frac{a_{21}}{a_{11}}$, $a_{11} \neq 0$ is called pivot element, and subtract it from the second equation. Note that division by a_{11} is called normalization. This leads to the following equation:

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or

$$a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2.$$

The procedure is then repeated for the remaining equations. This results in the following modified system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + \cdots + a'_{3n}x_n &= b'_3 \qquad (**) \\
&\vdots \\
a'_{n2}x_2 + \cdots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

pivot

$a'_{22} \neq 0$

The next step is to eliminate x_2 from the third to the nth equation of (**). To do this, we multiply the second equation (pivot equation) of (**) by $\frac{a'_{32}}{a'_{22}}$, $a'_{22} \neq 0$ is the pivot element, and subtract the result from the third equation. Perform a similar elimination for the remaining equations to yield:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a''_{n3}x_3 + \cdots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

The procedure can be continued using the remaining pivot equations. The final manipulation in the sequence is to use the $(n-1)$th equation to eliminate the x_{n-1} term from the nth equation. At this point, the system will have been transformed to an upper triangular system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \qquad (***) \\
&\ddots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Backward substitution phase. From the last equation of (***), we have

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$i=1 \mapsto x_1 = \frac{b_1 - \sum_{j=2}^{3} a_{1j}x_j}{a_{11}} = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

This result can be back-substituted into the $(n-1)$th equation to solve for x_{n-1}. The procedure is repeated to evaluate the remaining x's by the following formula:

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}}, \quad \text{for } i = n-1, n-2, \ldots, 1$$

In 3x3 system: $i=2$; $x_2 = \frac{b'_2 - \sum_{j=3}^{3} a'_{2j}x_j}{a'_{22}} = \frac{b'_2 - a'_{23}x_3}{a'_{22}}$

Remark (Computing the determinant using forward phase) The forward-elimination step of Gauss elimination results in an upper triangular system (***). Because the value of the determinant is not changed by the forward-elimination process, the determinant of the coefficient matrix can be simply evaluated at the end of this step via

$$\det(A) = a_{11}a'_{22}a''_{33}\dots a_{nn}^{(n-1)}$$

Example. Use the Gaussian elimination to solve the following system.

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Pivot

$$[A\,|\,b] = \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow -\frac{0.1}{3}R_1 + R_2 \\ R_3 \leftarrow -\frac{0.3}{3}R_1 + R_3 \end{array} \quad \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right]$$

a'_{22} a'_{23}

$$R_3 \leftarrow \frac{0.19}{7.0033}R_2 + R_3 \quad \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & 0 & 10.0120 & 70.0843 \end{array}\right]$$

end of forward elimination phase

Back Subs phase

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \mapsto 3x_1 = 9 \mapsto x_1 = 3 \\ 7.0033x_2 - 0.2933x_3 = -19.5617 \mapsto 7.0033x_2 = -17.5086 \mapsto x_2 = -2.5 \\ 10.012x_3 = 70.0843 \mapsto x_3 = 7 \end{cases}$$

The flops count for Gaussian elimination. One floating addition and one multiplication together is counted as *one flop* in the execution time of an algorithm. When dealing with the algorithmic procedure of Gaussian elimination, it can se seen that the total number of flops for this algorithm is

$$\frac{2n^3}{3} + O(n^2)$$

where $O(n^2)$ means "terms of order n^2 and lower". The result is written in this way because as n gets large, the $O(n^2)$ and lower terms become negligible. As it is seen from the following table, *most of the effort is incurred in the elimination step*.

Number of flops for naive Gauss elimination.

n	Elimination	Back Substitution	Total Flops	$2n^3/3$	Percent Due to Elimination
10	705	100	805	667	87.58
100	671550	10000	681550	666667	98.53
1000	6.67×10^8	1×10^6	6.68×10^8	6.67×10^8	99.85

Pivoting

Consider the following systems of linear equations.

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 4x_1 + 6x_2 + 7x_3 &= -3 \\ 2x_1 - 3x_2 + 6x_3 &= 5 \end{aligned}$$

During both the elimination and the back-substitution phases, it is possible that a *division by zero* can occur. Problems may also arise when the pivot element is close to zero because if the magnitude of the pivot element is small compared to the other elements, then *round off errors* can be introduced.

- **Partial pivoting.** The rows can be switched so that the largest element is the pivot element becomes the new pivot element.

- **Complete pivoting.** If columns as well as rows are searched for the largest element and then switched, the procedure is called complete pivoting. Complete pivoting is rarely used because most of the improvement comes from partial pivoting.

Example. Use Gaussian elimination with partial pivoting to solve the following systems of linear equations.

$$0.0003x_1 + 3.0000x_2 = 2.0001$$
$$1.0000x_1 + 1.0000x_2 = 1.0000$$

$$[A|b] = \left[\begin{array}{cc|c} 0.0003 & 3 & 2.0001 \\ 1 & 1 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0.0003 & 3 & 2.0001 \end{array}\right]$$

$$R_2 \leftarrow -0.0003R_1 + R_2 \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2.9997 & 1.9998 \end{array}\right] \rightarrow \begin{cases} x_1 + x_2 = 1 \\ 2.9997x_2 = 1.9998 \end{cases}$$

$$\longrightarrow x_2 = 0.6667$$

$$x_1 = 1 - x_2 = 1 - 0.6667 = 0.3333$$

References

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