Calc 1500, Week 5

Chain Rule

Assume $f(x,y)$, $x(s,t)$, and $y(s,t)$ are of class $\mathcal{C}^1$.

$$f\big(x(s,t), y(s,t)\big)$$

is also of class $\mathcal{C}^1$ and

$$\frac{\partial}{\partial s} f\big(x(s,t), y(s,t)\big) = \frac{\partial f}{\partial x}(x,y) \cdot \frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x,y) \cdot \frac{\partial y}{\partial s}(s,t),$$

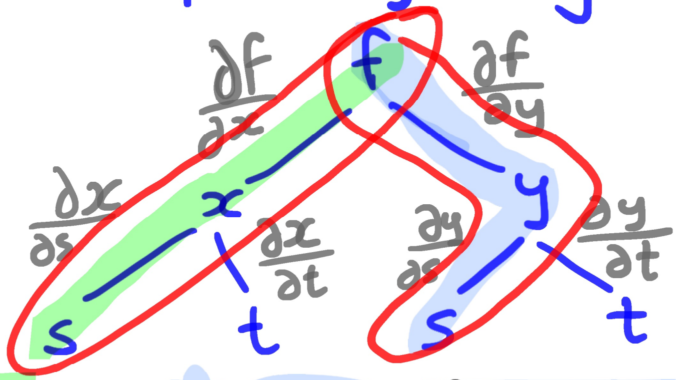
$$\frac{\partial}{\partial t} f\big(x(s,t), y(s,t)\big) = \frac{\partial f}{\partial x}(x,y) \cdot \frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}(x,y) \cdot \frac{\partial y}{\partial t}(s,t).$$

$f = f(x,y)$

$x = x(s,t)$

$y = y(s,t)$

Dependency diagram



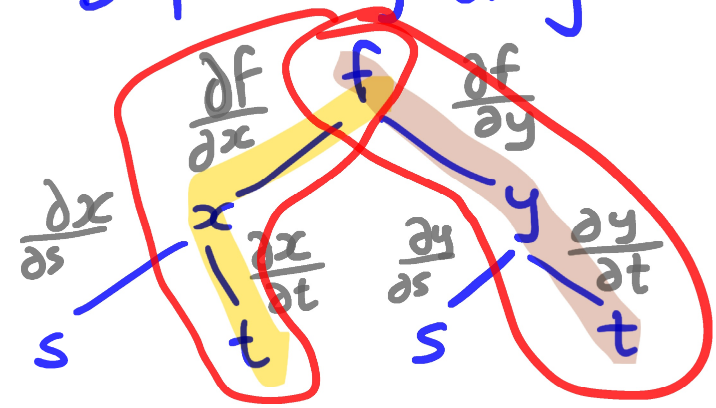
$$\frac{\partial f}{\partial s}(s,t) = \frac{\partial f}{\partial x}(x,y)\cdot\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x,y)\cdot\frac{\partial y}{\partial s}(s,t)$$

We also write

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

or $f_s = f_x x_s + f_y y_s$

Dependency diagram



$$\frac{\partial f}{\partial t}(s,t) = \frac{\partial f}{\partial x}(x,y) \cdot \frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}(x,y) \cdot \frac{\partial y}{\partial t}(s,t)$$

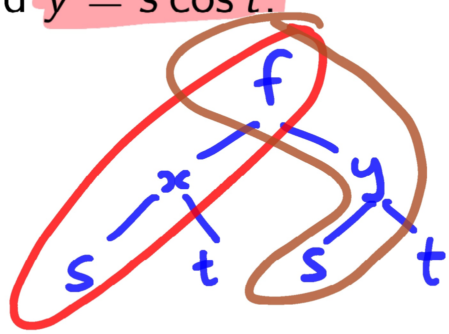
Example: Chain Rule

Take the partial derivative wrt s

of this function

Find $\dfrac{\partial}{\partial s} f\left(x(s,t), y(s,t)\right)$

for $f(x,y) = x^2 + y^2$, $x = st$, and $y = s\cos t$.



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= (2x)(t) + (2y)(\cos t)$$

$$= 2xt + 2y\cos t$$

$$\frac{\partial}{\partial s} f\big(x(s,t),\, y(s,t)\big)$$

let $F(s,t) = f\big(x(s,t),\, y(s,t)\big)$

$$\begin{tikzcd}
\mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R} \\
\begin{bmatrix} s \\ t \end{bmatrix} \arrow[r, mapsto] \arrow[rr, bend left] & \begin{bmatrix} x(s,t) \\ y(s,t) \end{bmatrix} \arrow[r, mapsto] & \begin{bmatrix} f(x,y) \end{bmatrix}
\end{tikzcd}$$

$x = st$

$y = s\cos t$

$f = x^2 + y^2$

$$\frac{\partial}{\partial s} f\big(x(s,t),\, y(s,t)\big) = \frac{\partial F}{\partial s}(s,t)$$

$L : \mathbb{R}^n \to \mathbb{R}^m$ $[L]$ is an $m \times n$ matrix

$$\begin{bmatrix} \frac{\partial F}{\partial s} & \frac{\partial F}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

Notation for Chain Rule

Assume $f(x, y)$, $x(s, t)$, and $y(s, t)$ are of class $\mathcal{C}^1$. Instead of writing

$$\frac{\partial}{\partial s} f\bigl(x(s,t), y(s,t)\bigr) = \frac{\partial f}{\partial x}(x, y) \cdot \frac{\partial x}{\partial s}(s, t) + \frac{\partial f}{\partial y}(x, y) \cdot \frac{\partial y}{\partial s}(s, t),$$

we may write

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s},$$

or

$$F_s = f_x x_s + f_y y_s,$$

where F is the composite function $F(s, t) = f\bigl(x(s, t), y(s, t)\bigr)$.

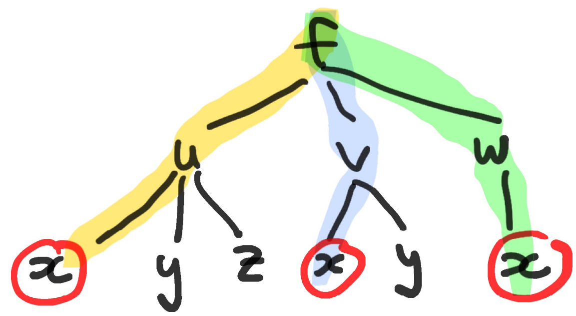
Example: Chain Rule (FRY Exercise III.2.4.5.1c)

composite function

Let $h(x, y, z) = \boxed{f\left(u(x, y, z), v(x, y), w(x)\right)}$. Find $\dfrac{\partial h}{\partial x}$.

(Show the dependency diagram.)

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x} f(u(x,y,z), v(x,y), w(x))$$



$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{dw}{dx}$$

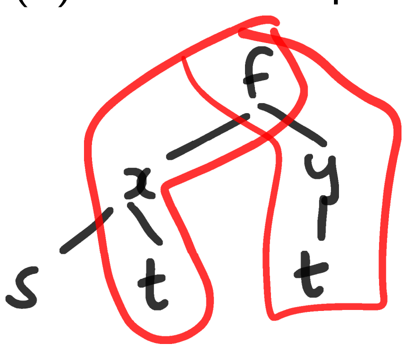
We use the ordinary (non-curly) d for the derivative of w with respect to x because w is a function of a single variable (and not a multivariable function)

Live Poll: Chain Rule

Let $f = f(x, y)$, $x = x(s, t)$, and $y = y(t)$.

(a) Draw a dependency diagram for the variables.

(b) Write an expression for $\dfrac{\partial}{\partial t} f(x(s,t), y(t))$.



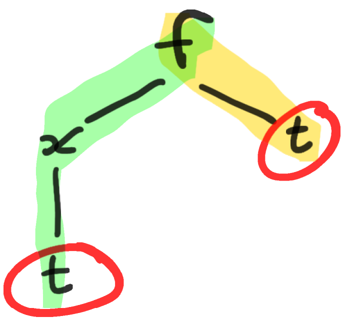
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Example: Chain Rule (Like FRY Exercise III.2.4.5.3)

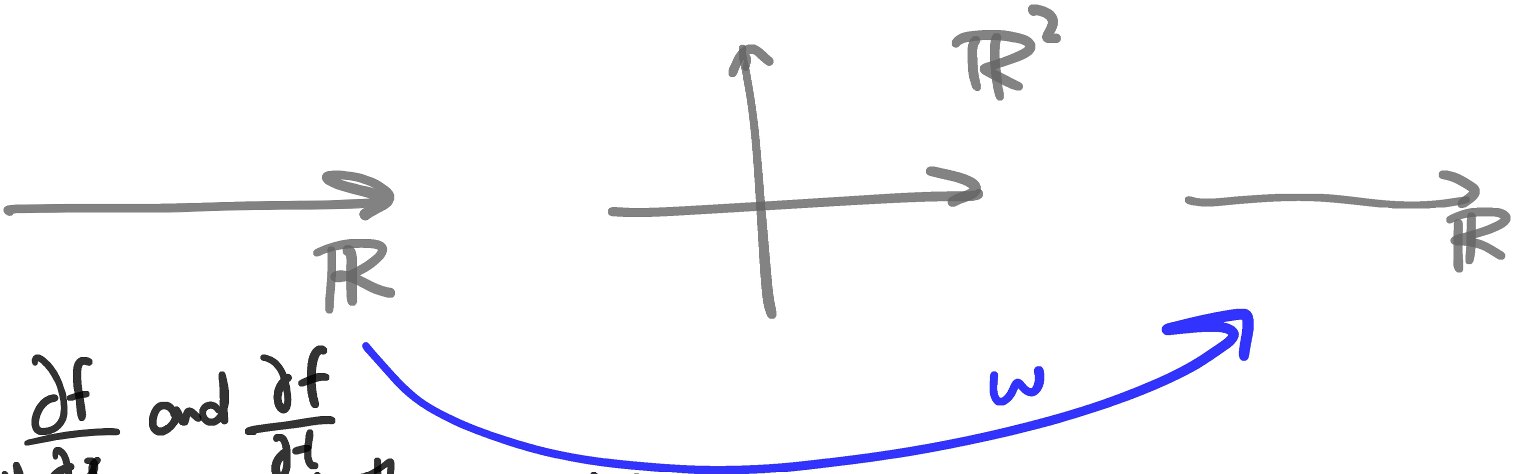
Let $w = f(x, t)$ where x depends on t. Suppose that at some point x_0 and at some time t_0,

- the partial derivatives f_x and f_t equal -2 and -3, respectively; and
- $\dfrac{dx}{dt} = 2$.

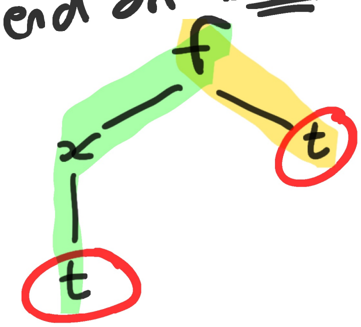
What is $\dfrac{dw}{dt}$?



$$\begin{tikzcd}
{[t]} \arrow[r, mapsto] \arrow[rr, bend right, "w"'] & \begin{bmatrix} x(t) \\ t \end{bmatrix} \arrow[r, mapsto] & {[f(x,t)]}
\end{tikzcd}$$



$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial t}$ may depend on <u>both</u> x and t



$$\frac{dw}{dt}(t) = \frac{d}{dt} f(x(t),t)$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial x}(x,t)\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial t}(x,t)$$

Given

- $f_x(x_0, t_0) = -2$
- $f_t(x_0, t_0) = -3$
- $\frac{dx}{dt}(t_0) = 2$

<u>Goal</u>: Find $\frac{dw}{dt}(t_0) = ?$

$$\frac{dw}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0,t_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial t}(x_0,t_0)$$

$$= (-2)(2) + (-3)$$

$$= -7$$

Example: Chain Rule (FRY Exercise III.2.4.5.21a)

Suppose that $f(x, y)$ is twice differentiable with $f_{xy} = f_{yx}$ and $x = r\cos\theta$ and $y = r\sin\theta$. Evaluate f_θ, f_r, and $f_{r\theta}$ in terms of r, θ, f_x and f_y.

Observe that

$$x_\theta = -r\sin\theta, \quad y_\theta = r\cos\theta, \quad x_r = \cos\theta, \quad y_r = \sin\theta.$$

and

$$f_\theta = f_x \cdot x_\theta + f_y \cdot y_\theta = -r\sin\theta\, f_x + r\cos\theta\, f_y$$

$$f_r = f_x \cdot x_r + f_y \cdot y_r = \cos\theta\, f_x + \sin\theta\, f_y$$

Example: Chain Rule (FRY Exercise III.2.4.5.21a)

$$
\begin{aligned}
f_{r\theta} = (f_r)_\theta &= \Big(\cos\theta\, f_x + \sin\theta\, f_y \Big)_\theta \\
&= -\sin\theta\, f_x + \cos\theta\,(f_x)_\theta + \cos\theta\ f_y + \sin\theta\,(f_y)_\theta \\
&= -\sin\theta\, f_x + \cos\theta\,(f_{xx}\, x_\theta + f_{xy}\, y_\theta) + \cos\theta\ f_y + \sin\theta\,(f_{yx}\, x_\theta + f_{yy}\, y_\theta) \\
&= -\sin\theta\, f_x + \cos\theta\Big(f_{xx}(-r\sin\theta) + f_{xy}\, r\cos\theta \Big) \\
&\qquad + \cos\theta\ f_y + \sin\theta\Big(f_{yx}\,(-r\sin\theta) + f_{yy}\, r\cos\theta \Big) \\
&= -\sin\theta\, f_x - r\sin\theta\cos\theta\, f_{xx} + r\cos^2\theta\, f_{xy} \\
&\qquad + \cos\theta\, f_y - r\sin^2\theta\, f_{yx} + r\sin\theta\cos\theta f_{yy} \\
&= -\sin\theta\, f_x - \frac{1}{2} r\sin(2\theta) f_{xx} + r\left(\cos^2\theta - \sin^2\theta\right) f_{xy} \\
&\qquad + \cos\theta\, f_y + \frac{1}{2} r\sin(2\theta) f_{yy} \\
&= -\sin\theta\, f_x - \frac{1}{2} r\sin(2\theta) f_{xx} + r\cos(2\theta) f_{xy} + \cos\theta\, f_y + \frac{1}{2} r\sin(2\theta) f_{yy}
\end{aligned}
$$

Tangent Plane and Normal Line to Surface $G(x, y, z) = K$ at the Point (x_0, y_0, z_0)

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be path on $G(x, y, z) = K$ that passes through (x_0, y_0, z_0) at $t = 0$. Assume $x = x(t)$, $y = y(t)$, $z = z(t)$ are differentiable at $t = 0$.

Since $G\big(\mathbf{r}(t)\big) = G\big(x(t), y(t), z(t)\big) = K$, differentiating both sides wrt t gives:

$$\frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} + \frac{\partial G}{\partial z}\frac{dz}{dt} = 0.$$

At $t = 0$, we get

$$\frac{\partial G}{\partial x}(x_0, y_0, z_0)\frac{dx}{dt}(0) + \frac{\partial G}{\partial y}(x_0, y_0, z_0)\frac{dy}{dt}(0) + \frac{\partial G}{\partial z}(x_0, y_0, z_0)\frac{dz}{dt}(0) = 0.$$

Tangent Plane and Normal Line to Surface $G(x, y, z) = K$ at the Point (x_0, y_0, z_0)

Written as a dot product:

$$\left\langle \frac{\partial G}{\partial x}(x_0, y_0, z_0), \frac{\partial G}{\partial y}(x_0, y_0, z_0), \frac{\partial G}{\partial z}(x_0, y_0, z_0) \right\rangle \cdot \left\langle \frac{dx}{dt}(0), \frac{dy}{dt}(0), \frac{dz}{dt}(0) \right\rangle = 0.$$

The left-hand side is such an important vector that we give it a special name.

The **gradient** of a function $f : \mathbb{R}^3 \to \mathbb{R}$ at whose partial derivatives exist at a point $\mathbf{a} \in \mathbb{R}^3$ is the vector

$$\nabla f(\mathbf{a}) = \left\langle \frac{\partial f}{\partial x}(\mathbf{a}), \frac{\partial f}{\partial y}(\mathbf{a}), \frac{\partial f}{\partial z}(\mathbf{a}) \right\rangle .$$

Tangent Plane and Normal Line to Surface $G(x, y, z) = K$ at the Point (x_0, y_0, z_0)

Thus

$$\left\langle \frac{\partial G}{\partial x}(x_0, y_0, z_0), \frac{\partial G}{\partial y}(x_0, y_0, z_0), \frac{\partial G}{\partial z}(x_0, y_0, z_0) \right\rangle \cdot \left\langle \frac{dx}{dt}(0), \frac{dy}{dt}(0), \frac{dz}{dt}(0) \right\rangle = 0.$$

can be expressed using the gradient of G at the point (x_0, y_0, z_0):

$$\nabla G(x_0, y_0, z_0) \cdot \mathbf{r}'(t) = 0.$$

Tangent Plane and Normal Line to Surface $G(x, y, z) = K$ at the Point (x_0, y_0, z_0)

$$\nabla G(x_0, y_0, z_0) \cdot \mathbf{r}'(t) = 0.$$

Since our choice of path along the surface $G(x, y, z) = K$ passing through (x_0, y_0, z_0) was arbitrary, the gradient $\nabla G(x_0, y_0, z_0)$ is perpendicular to every path through (x_0, y_0, z_0).

Thus, *the gradient vector $\nabla G(x_0, y_0, z_0)$ is also a normal vector to a tangent plane to the surface $G(x, y, z) = K$ at the point (x_0, y_0, z_0).*

Tangent Plane and Normal Line to Surface $G(x, y, z) = K$ at the Point (x_0, y_0, z_0)

Let K be a constant and (x_0, y_0, z_0) be a point on the surface $G(x, y, z) = K$. Assume that the gradient

$$\nabla G(x_0, y_0, z_0) = \left\langle \frac{\partial G}{\partial x}(x_0, y_0, z_0), \frac{\partial G}{\partial y}(x_0, y_0, z_0), \frac{\partial G}{\partial y}(x_0, y_0, z_0) \right\rangle$$

of G at (x_0, y_0, z_0) is nonzero. Then,

1. The equation of the tangent plane to the surface $G(x, y, z) = K$ at (x_0, y_0, z_0) is

$$\nabla G(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

2. The vector equation of the normal line to the surface $G(x, y, z) = K$ at (x_0, y_0, z_0) is

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \nabla G(x_0, y_0, z_0), \quad t \in \mathbb{R}.$$

Example: Equation of Tangent Plane (Exercise III.2.5.3.8)

Find the equation of the tangent plane and normal line (in vector, parametric, and symmetric forms) to the surface given by

$$xyz^2 + y^2z^3 = 3 + x^2$$

at the point $(-1, 1, 2)$.

$xyz^2 + y^2z^3 - x^2 = 3$

Let $G(x,y,z) = xyz^2 + y^2z^3 - x^2$. The surface has the form $G(x,y,z) = K$, where $K=3$

eqn of tangent plane: $\nabla G(-1,1,2) \cdot \langle x-(-1), y-1, z-2 \rangle = 0$

$$\nabla G(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{\begin{aligned}\nabla G(x,y,z) &= \left\langle \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right\rangle \\ &= \langle yz^2 - 2x,\ xz^2 + 2yz^3,\ 2xyz + 3y^2z^2 \rangle\end{aligned}}$$

$$G(x,y,z) = xyz^2 + y^2z^3 - x^2$$

So $\nabla G(-1,1,2) = \langle (1)(2)^2 - 2(-1),\ (-1)(2)^2 + 2(1)(2)^3,\ 2(-1)(1)(2) + 3(1)^2(2)^2 \rangle$

$$= \langle 6, 12, 8 \rangle$$

Eqn of tangent plane

$$\nabla G(-1,1,2) \cdot \langle x-(-1), y-1, z-2 \rangle = 0$$

$$\langle 6,12,8\rangle \cdot \langle x+1,\ y-1,\ z-2\rangle = 0$$

$$6x + 6 + 12y - 12 + 8z - 16 = 0$$

$$6x + 12y + 8z = 22$$

$$\boxed{3x + 6y + 4z = 11}$$

Equation of Tangent Plane to the given surface at $(-1,1,2)$.

Vector equation of normal line at $(-1,1,2)$

$$\langle x,y,z\rangle = \langle -1,1,2\rangle + t\langle 6,12,8\rangle,\quad t \in \mathbb{R}$$

direction vector

is $\nabla G(-1,1,2)$

Parametric Eqn of Normal Line

$$x = -1 + 6t$$
$$y = 1 + 12t \qquad , t \in \mathbb{R}$$
$$z = 2 + 8t$$

Isolate for t

Symmetric Eqn of Normal Line

$$\frac{x+1}{6} = \frac{y-1}{12} = \frac{z-2}{8}$$

Remark: $\nabla G(-1,1,2) = \langle 6, 12, 8 \rangle$

Any nonzero multiple of $\nabla G(-1,1,2)$ is a "good" normal vector. For example,

$$\vec{n} = \frac{1}{\sqrt{6^2+12^2+8^2}} \langle 6, 12, 8 \rangle$$

also is a normal vector.