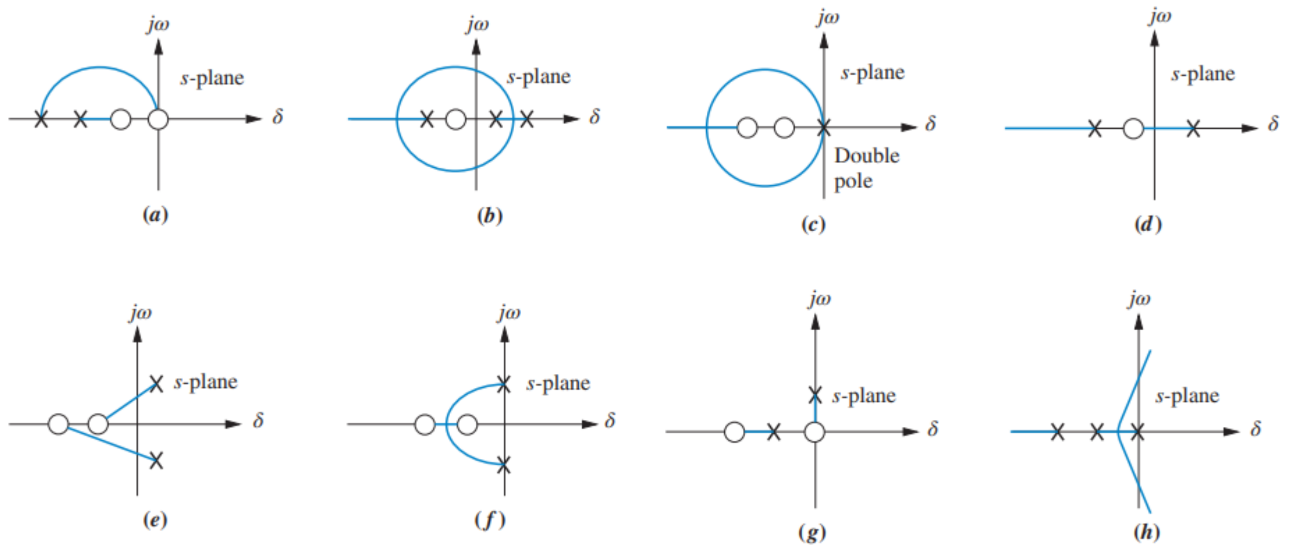
Worksheet 5 - Solution

1) For each of the root loci shown below, tell whether or not the sketch can be a root locus. If the sketch cannot be a root locus, explain why. Give all the reasons.



a. No: Not symmetric; On real axis to left of an even number of poles and zeros

b. No: On real axis to left of an even number of poles and zeros

c. No: On real axis to left of an even number of poles and zeros

d. Yes

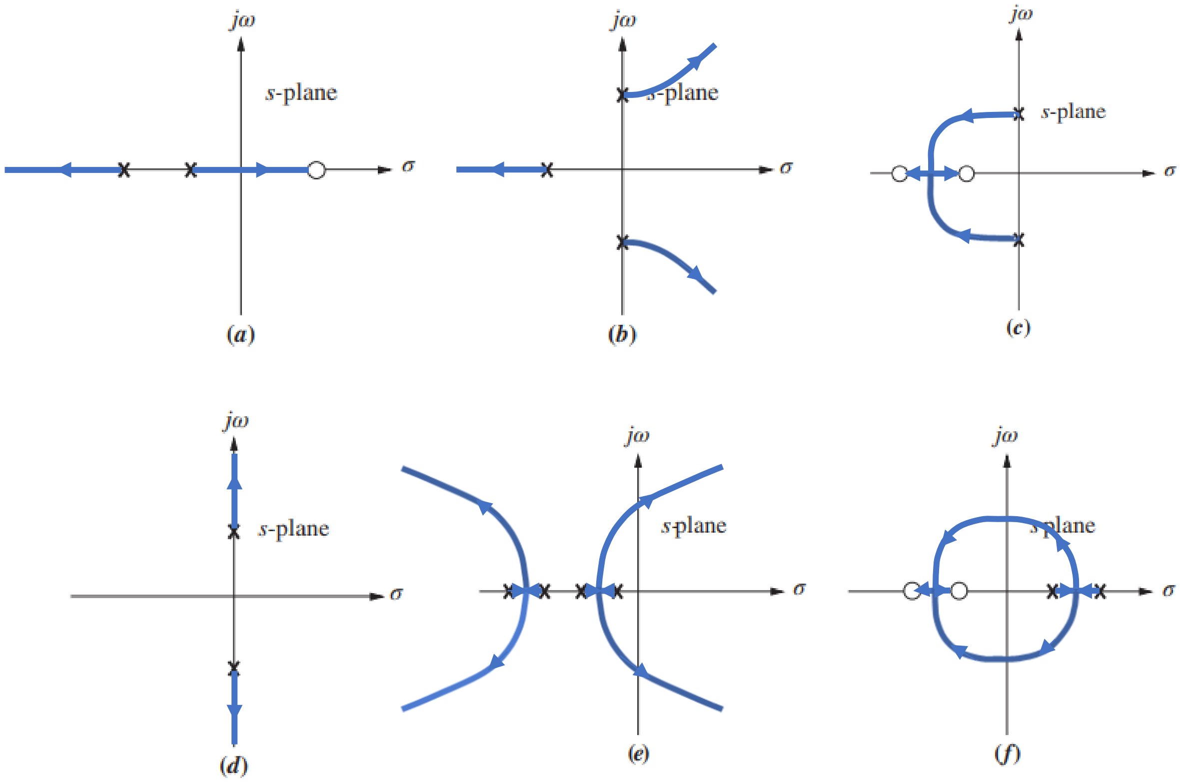
e. No: Not symmetric; Not on real axis to left of odd number of poles and/or zeros

f. Yes

g. No: Not symmetric; real axis segment is not to the left of an odd number of poles

h. Yes

2) Sketch the general shape of the root locus for each of the open-loop pole-zero plots shown below:



3) Sketch the root locus for the unity-feedback system shown below for the following transfer functions:

a) $G(s) = \frac{K(s+2)(s+6)}{s^2+8s+25}$

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)H(s)} = \frac{K(s^2+8s+12)}{(1+K)s^2+8(1+K)s+25+12K}$$

The closed-loop characteristic equation is: $\mathbf{(1+K)s^2+8(1+K)s+25+12K=0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_{1,2} = -4 \pm j3$

Zeros: $s_1 = -2, \;\; s_2 = -6$

Step 2: Draw the root-locus on the real axis

The segment between -2 to -6 is on the root-locus.

Step 3: Draw the asymptote lines for large K values

Number of asymptote lines: $n - m = 2 - 2 = 0$

There are **no asymptote lines**. The root-locus will **not** approach infinity for large K values.

Step 4: Intersection of root-locus with imaginary axis

$(1 + K)s^2 + 8(1 + K)s + 25 + 12K = 0$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(1 + K)(j\omega)^2 + 8(1 + K)(j\omega) + 25 + 12K = -(1 + K)\omega^2 + j8(1 + K)\omega + 25 + 12K = 0$

$(25 + 12K - (1 + K)\omega^2) + j(8(1 + K)\omega) = 0$

From the imaginary part:

$$8(1 + K)\omega = 0 \quad \rightarrow \quad \begin{cases} \omega = 0 \\ 1 + K = 0 \quad \rightarrow \quad K = -1 < 0 \quad \text{Not acceptable} \end{cases}$$

From the real part:

$$\text{For } \omega = 0 \quad \rightarrow \quad 25 + 12K - (1 + K)\omega^2 = 25 + 12K = 0 \quad \rightarrow \quad K = -2.083 < 0 \quad \text{Not acceptable}$$

Therefore, the root-locus will not cross the imaginary axis.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $\mathbf{(1 + K)s^2 + 8(1 + K)s + 25 + 12K = 0}$

Find the K from the characteristic equation

$$K = \frac{-s^2 - 8s - 25}{s^2 + 8s + 12}$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad \frac{(-2s - 8)(s^2 + 8s + 12) - (2s + 8)(-s^2 - 8s - 25)}{(s^2 + 8s + 12)^2} = 0 \quad \rightarrow \quad 26s + 104 = 0$$

The root is:

$\mathbf{s = -4}$ $\rightarrow$ On the root-locus (Break-in point)

The associate gain is:

$$K = \frac{-(-4)^2 - 8(-4) - 25}{(-4)^2 + 8(-4) + 12} = \frac{-9}{-4} = 2.25$$

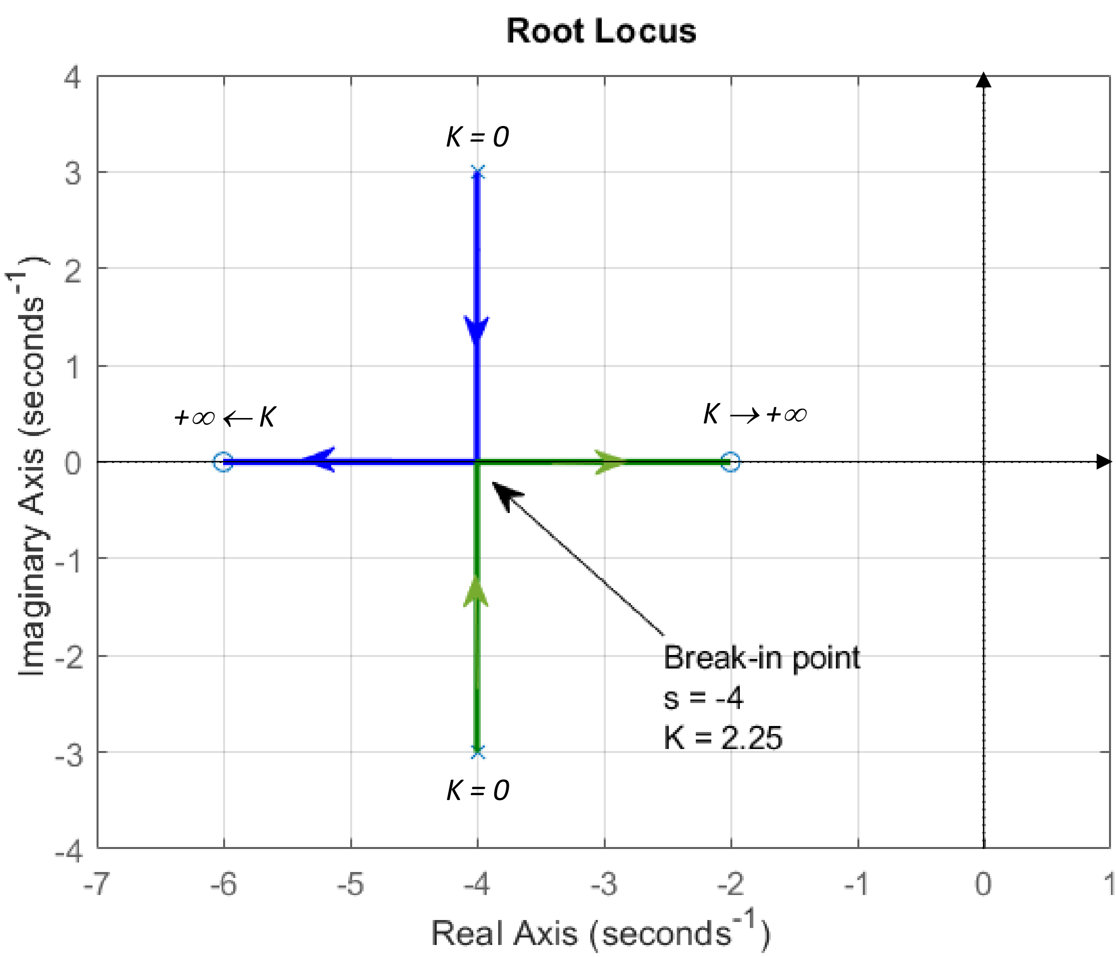
Step 6: Calculate angle of departure from the complex poles

The angle of departure from the complex pole at $s = -4 + j3$ is:

$$\boldsymbol{\phi_p = 180^{\circ} - \sum_i \angle p_i + \sum_j \angle z_j = 180^{\circ} - (\theta_1) + (\varphi_1 + \varphi_2) = 180^{\circ} - (90^{\circ}) + (124^{\circ} + 56^{\circ}) = 270^{\circ}}$$

Since the root-locus is symmetrical with respect to the real axis, the angle of departure for the complex pole $s = -4 - j3$ is $-270°$.

Step 7: Complete the root-locus diagram



b) $\boldsymbol{G(s) = \frac{K(s^2+4)}{s^2+1}}$

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K(s^2 + 4)}{(1 + K)s^2 + 1 + 4K}$$

The closed-loop characteristic equation is: $\boldsymbol{(1 + K)s^2 + 1 + 4K = 0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_{1,2} = \pm j1$

Zeros: $s_{1,2} = \pm j2$

Step 2: Draw the root-locus on the real axis

None of the real axis is on the root-locus.

Step 3: Draw the asymptote lines for large $\boldsymbol{K}$ values

Number of asymptote lines: $n - m = 2 - 2 = 0$

There are **<u>no asymptote lines</u>**. The root-locus will **not** approach infinity for large K values.

Step 4: Intersection of root-locus with imaginary axis

$(1 + K)s^2 + 1 + 4K = 0$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(1 + K)(j\omega)^2 + 1 + 4K = -(1 + K)\omega^2 + 1 + 4K = 0$

From the real part:

$$-(1 + K)\omega^2 + 1 + 4K = 0 \quad \rightarrow \quad \omega^2 = \frac{1 + 4K}{1 + K}$$

The value is always positive for all $K \geq 0$.

Therefore, the root-locus will be on the imaginary axis for all $K \geq 0$.

For example:

$K = 0 \quad \rightarrow \quad \omega^2 = 1 \quad \rightarrow \quad \omega = \pm 1$

$K = 1 \quad \rightarrow \quad \omega^2 = 2.5 \quad \rightarrow \quad \omega = \pm 1.58$

$K \rightarrow +\infty \quad \rightarrow \quad \omega^2 \rightarrow 4 \quad \rightarrow \quad \omega \rightarrow \pm 2$

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $\mathbf{(1+K)s^2+1+4K=0}$

Find the K from the characteristic equation

$$K = \frac{-s^2-1}{s^2+4}$$

$$\frac{dK}{ds}=0 \quad \rightarrow \quad \frac{(-2s)(s^2+4)-(2s)(-s^2-1)}{(s^2+4)^2}=0 \quad \rightarrow \quad -6s=0$$

The root is:

$\mathbf{s=0}$ $\rightarrow$ Not on the root-locus

There is no break-away or break-in point.

Step 6: Calculate angle of departure from the complex poles and angle of arrival to complex zeros

The angle of departure from the complex pole at $s=+j1$ is:

$$\boldsymbol{\phi_p = 180^\circ - \sum_i \angle p_i + \sum_j \angle z_j = 180^\circ - (\theta_1) + (\varphi_1+\varphi_2) = 180^\circ - (90^\circ) + (90^\circ - 90^\circ) = 90^\circ}$$

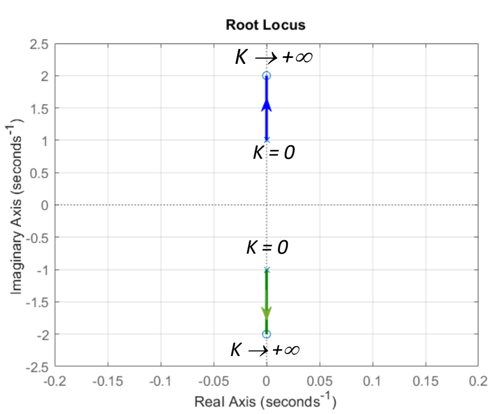
Since the root-locus is symmetrical with respect to the real axis, the angle of departure for the pole $s=-j1$ is $-90°$.

The angle of arrival to the complex zero at $s=+j2$ is:

$$\boldsymbol{\phi_p = 180^\circ - \sum_i \angle z_i + \sum_j \angle p_j = 180^\circ - (\varphi_1) + (\theta_1+\theta_2) = 180^\circ - (90^\circ) + (90^\circ + 90^\circ) = 270^\circ}$$

Since the root-locus is symmetrical with respect to the real axis, the angle of arrival to the zero at $s=-j2$ is $-270°$.

Step 7: Complete the root-locus diagram



c) $\boldsymbol{G(s) = \frac{K(s^2+1)}{s^2}}$

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K(s^2+1)}{(1+K)s^2 + K}$$

The closed-loop characteristic equation is: $\boldsymbol{(1+K)s^2 + K = 0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_1 = s_2 = 0$

Zeros: $s_{1,2} = \pm j1$

Step 2: Draw the root-locus on the real axis

The open-loop poles location at s = 0 is on the root locus.

Step 3: Draw the asymptote lines for large $\boldsymbol{K}$ values

Number of asymptote lines: $n - m = 2 - 2 = 0$

There are **no asymptote lines**. The root-locus will **not** approach infinity for large K values.

Step 4: Intersection of root-locus with imaginary axis

$(1+K)s^2 + K = 0$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(1+K)(j\omega)^2 + K = -(1+K)\omega^2 + K = 0$

From the real part:

$$-(1+K)\omega^2 + K = 0 \quad \to \quad \omega^2 = \frac{K}{1+K}$$

The value is always positive for all $K \geq 0$.

Therefore, the root-locus will be on the imaginary axis for all $K \geq 0$.

For example:

$K = 0 \quad \to \quad \omega^2 = 0 \quad \to \quad \omega = 0$

$K = 1 \quad \to \quad \omega^2 = 0.5 \quad \to \quad \omega = \pm 0.71$

$K \to +\infty \quad \to \quad \omega^2 \to 1 \quad \to \quad \omega \to \pm 1$

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $(\mathbf{1}+\boldsymbol{K})\boldsymbol{s}^{\mathbf{2}}+\boldsymbol{K}=\mathbf{0}$

Find the K from the characteristic equation

$$K=\frac{-s^2}{s^2+1}$$

$$\frac{dK}{ds}=0 \quad \rightarrow \quad \frac{(-2s)(s^2+1)-(2s)(-s^2)}{(s^2+1)^2}=0 \quad \rightarrow \quad -2s=0$$

The root is:

$\boldsymbol{s}=\mathbf{0}$ $\rightarrow$ On the root-locus

The $s = 0$ is a break-away point. The associate gain is $K=0$.

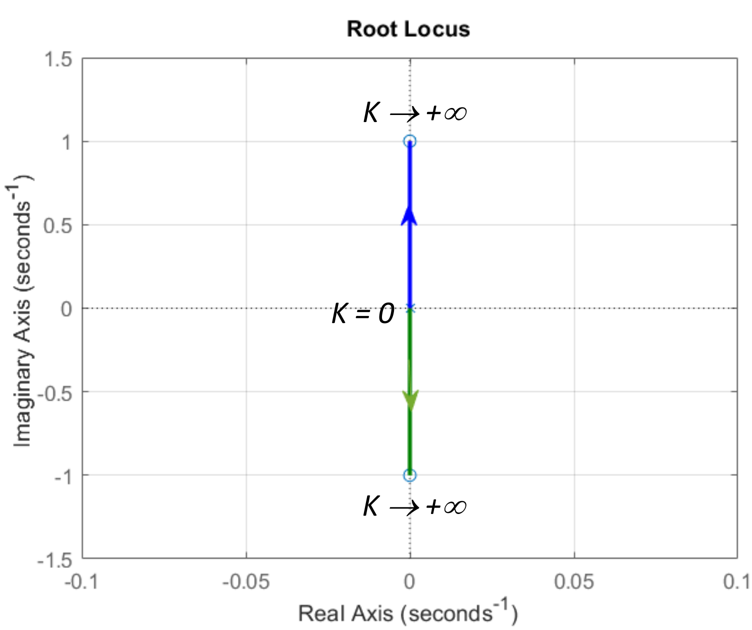
Step 6: Calculate angle of arrival to the complex zeros

The angle of arrival to the complex zero at $s=+j1$ is:

$$\boldsymbol{\phi_p}=\mathbf{180}^{\circ}-\sum_{i}\angle\boldsymbol{z_i}+\sum_{j}\angle\boldsymbol{p_j}=\mathbf{180}^{\circ}-(\boldsymbol{\varphi_1})+(\boldsymbol{\theta_1}+\boldsymbol{\theta_2})=\mathbf{180}^{\circ}-(\mathbf{90}^{\circ})+(\mathbf{90}^{\circ}+\mathbf{90}^{\circ})=\mathbf{270}^{\circ}$$

Since the root-locus is symmetrical with respect to the real axis, the angle of arrival to the zero at $s=-j1$ is $-270°$.

Step 7: Complete the root-locus diagram



d) $\boldsymbol{G(s) = \frac{K}{(s+1)^3(s+4)}}$

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K}{s^4 + 7s^3 + 15s^2 + 13s + 4 + K}$$

The closed-loop characteristic equation is: $\boldsymbol{s^4 + 7s^3 + 15s^2 + 13s + 4 + K = 0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_1 = s_2 = s_3 = -1, \quad s_4 = -4$

Zeros: No finite zeros, Four zeros at infinity

Step 2: Draw the root-locus on the real axis

The segment between s = -1 and s = -4 is on the root locus.

Step 3: Draw the asymptote lines for large K values

Number of asymptote lines: $n - m = 4 - 0 = 4$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m}} = \frac{[(-1) + (-1) + (-1) + (-4)]}{4 - 0} = -1.75$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i = \frac{180^\circ}{n - m}(2i + 1)} = \frac{180^\circ}{4 - 0}(2i + 1) = 45^\circ(2i + 1) \rightarrow \begin{cases} \boldsymbol{\varphi_0 = 45^\circ} \\ \boldsymbol{\varphi_1 = 135^\circ} \\ \boldsymbol{\varphi_2 = 225^\circ} \\ \boldsymbol{\varphi_3 = 315^\circ} \end{cases}$$

Step 4: Intersection of root-locus with imaginary axis

$s^4 + 7s^3 + 15s^2 + 13s + 4 + K = 0$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(j\omega)^4 + 7(j\omega)^3 + 15(j\omega)^2 + 13(j\omega) + 4 + K = \omega^4 - j7\omega^3 - 15\omega^2 + j13\omega + 4 + K = 0$

$(\omega^4 - 15\omega^2 + 4 + K) + j(-7\omega^3 + 13\omega) = 0$

From the imaginary part:

$$-7\omega^3 + 13\omega = 0 \quad \rightarrow \quad \rightarrow \quad \omega(-7\omega^2 + 13) = 0 \; \rightarrow \; \begin{cases} \omega = 0 \\ \omega^2 = \dfrac{13}{7} \rightarrow \quad \omega = \pm 1.86\, rad/s \end{cases}$$

From the real part:

For $\omega = 0 \quad \rightarrow \quad \omega^4 - 15\omega^2 + 4 + K = 0 - 15 \times 0 + 4 + K = 0 \; \rightarrow \; K = -4 < 0$ Not acceptable

For $\omega^2 = \dfrac{13}{7} \quad \rightarrow \quad \omega^4 - 15\omega^2 + 4 + K = \dfrac{169}{49} - \dfrac{195}{7} + 4 + K = 0 \quad \rightarrow \quad K = 20.41$

Therefore, the root-locus will cross the imaginary axis at $s = \pm j1.86$ for gain $K = 20.41$.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $s^4 + 7s^3 + 15s^2 + 13s + 4 + K = 0$

Find the K from the characteristic equation

$K = -s^4 - 7s^3 - 15s^2 - 13s - 4$

$\dfrac{dK}{ds} = 0 \quad \rightarrow \quad -4s^3 - 21s^2 - 30s - 13 = 0$

The roots are:

$\boldsymbol{s_1 = s_2 = -1} \; \rightarrow$ On the root-locus (Break-away point)

The associate gain is:

$K = -(-1)^4 - 7(-1)^3 - 15(-1)^2 - 13(-1) - 4 = 0$

$s = -3.25 \quad \rightarrow$ On the root-locus (Break-away point)

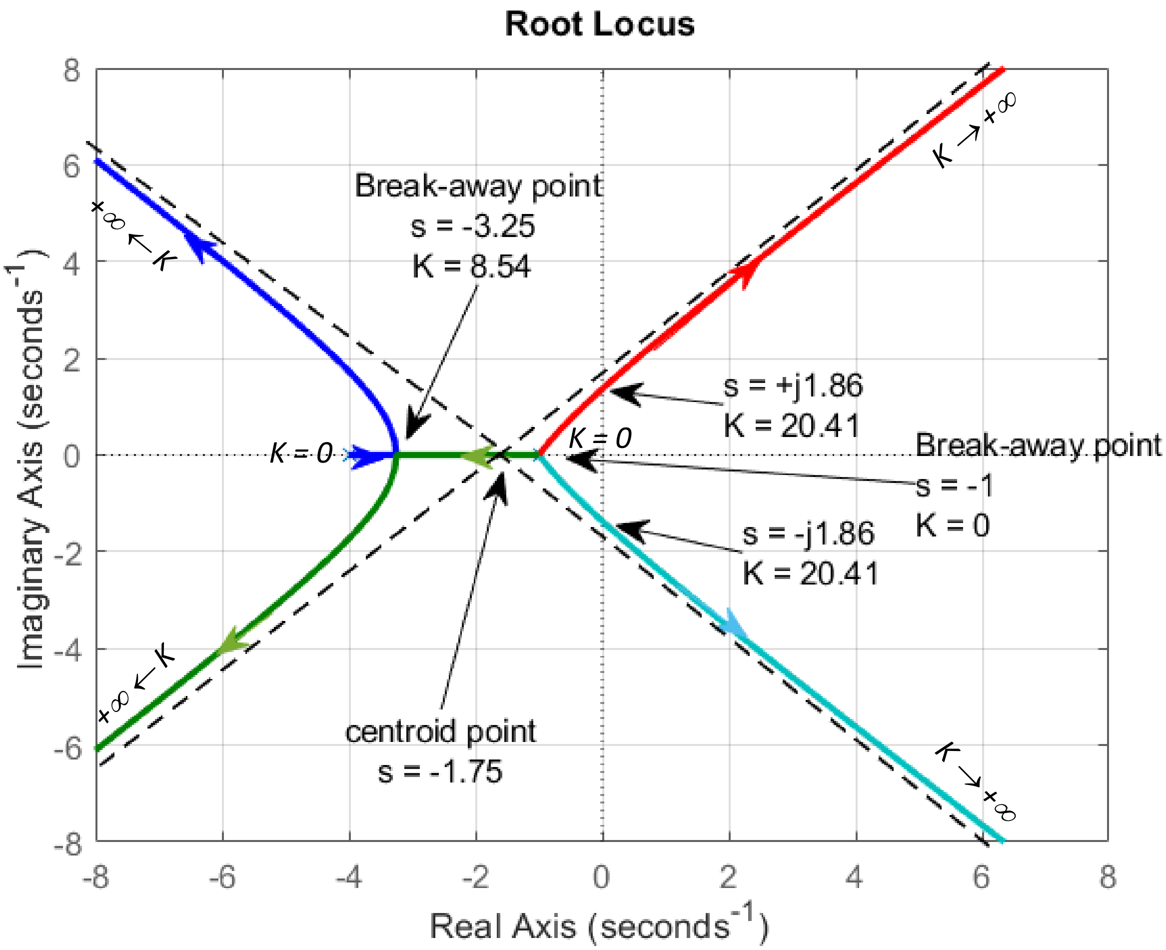
The associate gain is:

$K = -(-3.25)^4 - 7(-3.25)^3 - 15(-3.25)^2 - 13(-3.25) - 4 = 8.54$

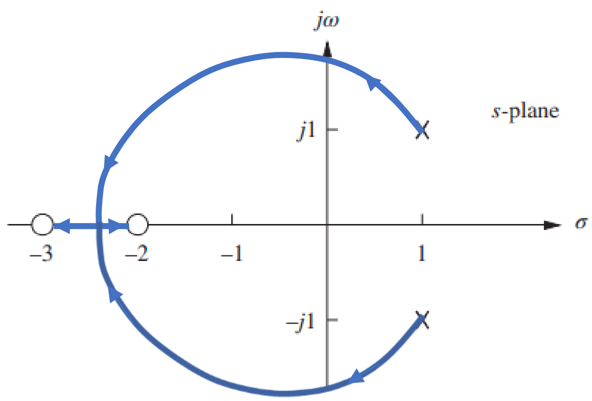
Step 6: Calculate angle departure from complex poles or angle of arrival to the complex zeros

Since there is no complex poles or zeros, we can skip this step.

Step 7: Complete the root-locus diagram



4) For the open-loop pole-zero plot shown below, sketch the root locus and find the break-in point.



We can find the pole-zero location of the open-loop system from the graph and form the open-loop system:

$$G(s)H(s) = \frac{(s+2)(s+3)}{(s-1-j)(s-1+j)} = \frac{s^2+5s+6}{s^2-2s+2}$$

Then determine the break-in point and its gain:

$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad 1 + \frac{K(s^2+5s+6)}{s^2-2s+2} = 0 \quad \rightarrow \quad K = \frac{-s^2+2s-2}{s^2+5s+6}$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad \frac{(-2s+2)(s^2+5s+6) - (2s+5)(-s^2+2s-2)}{(s^2+5s+6)^2} = 0 \quad \rightarrow \quad -7s^2 - 8s + 22 = 0$$

The roots are:

$\boldsymbol{s = -2.43}$ $\rightarrow$ On the root-locus (Break-in point)

$s = 1.29$ $\rightarrow$ Not on the root-locus

The associate gain is:

$$K = \frac{-(-2.43)^2 + 2(-2.43) - 2}{(-2.43)^2 + 5(-2.43) + 6} = \frac{-12.7649}{-0.2451} = 52.08$$

5) Sketch the root locus of the unity feedback system, where G(s) is given as below, and find the break-in and breakaway points. Find the range of K for which the system is closed-loop stable.

$$\boldsymbol{G(s) = \frac{K(s+1)(s+7)}{(s+3)(s-5)}}$$

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)H(s)} = \frac{K(s^2+8s+7)}{(1+K)s^2+2(-1+4K)s-15+7K}$$

The closed-loop characteristic equation is: $\boldsymbol{(1+K)s^2+2(-1+4K)s-15+7K=0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_1 = -3, \;\; s_2 = 5$

Zeros: $s_1 = -1, \;\; s_2 = -7$

Step 2: Draw the root-locus on the real axis

The segments between +5 to -1 and between -3 to -7 are on the root-locus.

Step 3: Draw the asymptote lines for large $\boldsymbol{K}$ values

Number of asymptote lines: $n - m = 2 - 2 = 0$

There are **<u>no asymptote lines</u>**. The root-locus will **not** approach infinity for large K values.

Step 4: Intersection of root-locus with imaginary axis

$(1+K)s^2+2(-1+4K)s-15+7K=0$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(1+K)(j\omega)^2+2(-1+4K)(j\omega)-15+7K = -(1+K)\omega^2 + j2(-1+4K)\omega - 15 + 7K = 0$

$(-15+7K-(1+K)\omega^2) + j(2(-1+4K)\omega) = 0$

From the imaginary part:

$$2(-1+4K)\omega = 0 \quad \rightarrow \quad \begin{cases} \omega = 0 \\ -1+4K = 0 \end{cases} \quad \rightarrow \quad K = 0.25$$

From the real part:

For $\omega = 0 \quad \rightarrow \quad -15+7K-(1+K)\omega^2 = -15+7K = 0 \quad \rightarrow \quad K = 2.14$

For $K = 0.25 \quad \rightarrow \quad -15+7K-(1+K)\omega^2 = -15+7(0.25)-(1.25)\omega^2 = 0 \quad \rightarrow \quad \omega^2 = -10.6 < 0 \quad \text{Not Acceptable}$

Therefore, the root-locus will cross the imaginary axis at $s = 0$ for gain of $K = 2.14$. The closed-loop system is stable for $K > 2.14$.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $\mathbf{(1+K)s^2+2(-1+4K)s-15+7K=0}$

Find the K from the characteristic equation

$$K=\frac{-s^2+2s+15}{s^2+8s+7}$$

$$\frac{dK}{ds}=0 \quad \rightarrow \quad \frac{(-2s+2)(s^2+8s+7)-(2s+8)(-s^2+2s+15)}{(s^2+8s+7)^2}=0 \quad \rightarrow \quad -10s^2-44s-106=0$$

The roots are:

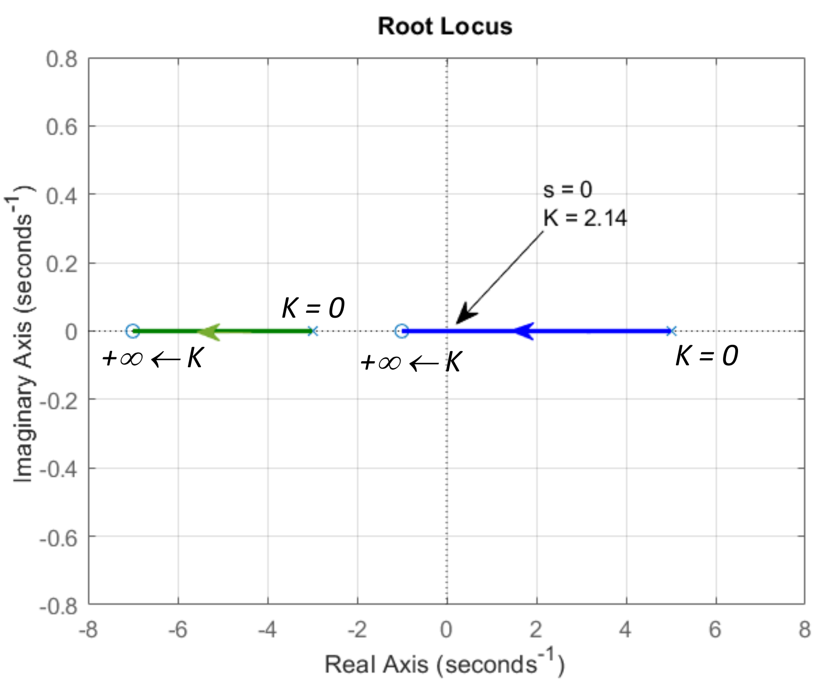
$s_{1,2}=-2.2\pm j2.4$ $\rightarrow$ Not on the root locus

There is no break-away or break-in point.

Step 6: Calculate angle departure from complex poles or angle of arrival to the complex zeros

Since there is no complex poles or zeros, we can skip this step.

Step 7: Complete the root-locus diagram



6) The characteristic polynomial of a feedback control system, which is the denominator of the closed-loop transfer function, is given below. Sketch the root locus for this system.

$$\boldsymbol{s^3 + 2s^2 + (20K + 7)s + 100K = 0}$$

First, find the open-loop system $G(s)H(s)$.

$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad G(s)H(s) = -\frac{1}{K}$$

$$s^3 + 2s^2 + (20K + 7)s + 100K = 0 \quad \rightarrow \quad (s^3 + 2s^2 + 7s) + (20s + 100)K = 0 \quad \rightarrow \quad K = \frac{-(s^3 + 2s^2 + 7s)}{20s + 100}$$

$$G(s)H(s) = \frac{20s + 100}{s^3 + 2s^2 + 7s}$$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_{1,2} = -1 \pm j2.45, \;\; s_3 = 0$

Zeros: $s_1 = -5$, Two zeros at infinity

Step 2: Draw the root-locus on the real axis

The segment between 0 to -5 is on the root-locus.

Step 3: Draw the asymptote lines for large K values

Number of asymptote lines: $n - m = 3 - 1 = 2$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha} = \frac{\sum_{i=1}^{n} \boldsymbol{p_i} - \sum_{i=1}^{m} \boldsymbol{z_i}}{\boldsymbol{n - m}} = \frac{[(-1 + j2.45) + (-1 - j2.45) + (0)] - [(-5)]}{3 - 1} = 1.5$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i} = \frac{\boldsymbol{180^\circ}}{\boldsymbol{n - m}}(\boldsymbol{2i + 1}) = \frac{180^\circ}{3 - 1}(2i + 1) = 90^\circ(2i + 1) \rightarrow \begin{cases} \boldsymbol{\varphi_0 = 90^\circ} \\ \boldsymbol{\varphi_1 = 270^\circ} \end{cases}$$

Step 4: Intersection of root-locus with imaginary axis

$$s^3 + 2s^2 + (20K + 7)s + 100K = 0$$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$$(j\omega)^3 + 2(j\omega)^2 + (20K + 7)(j\omega) + 100K = -j\omega^3 - 2\omega^2 + j(20K + 7)\omega + 100K = 0$$

$$(-2\omega^2 + 100K) + j(-\omega^3 + (20K + 7)\omega) = 0$$

From the imaginary part:

$$-\omega^3 + (20K + 7)\omega = \omega(-\omega^2 + 20K + 7) = 0 \quad \rightarrow \quad \begin{cases} \omega = 0 \\ \omega^2 = 7 + 20K \end{cases} \quad \rightarrow \quad \omega = \pm\sqrt{7 + 20K}$$

From the real part:

For $\omega = 0 \quad \rightarrow \quad -2\omega^2 + 100K = -2 \times 0 + 100K = 0 \quad \rightarrow \quad K = 0$

For $\omega^2 = 7 + 20K \quad \rightarrow \quad -2\omega^2 + 100K = -2(7 + 20K) + 100K = 0 \quad \rightarrow \quad K = 0.23$

Find the ω for $K = 0.23$,

$$\omega = \pm\sqrt{7 + 20K} = \pm\sqrt{7 + 20(0.23)} = \pm 3.41$$

Therefore, the root-locus will cross the imaginary axis at $s = \pm j3.41$ for gain of $K = 2.14$, and at $s = 0$ for gain of $K = 0$.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $s^3 + 2s^2 + (20K + 7)s + 100K = 0$

Find the K from the characteristic equation

$$K = \frac{-s^3 - 2s^2 - 7s}{20s + 100}$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad \frac{(-3s^2 - 4s - 7)(20s + 100) - (20)(-s^3 - 2s^2 - 7s)}{(20s + 100)^2} = 0 \quad \rightarrow -40s^3 - 340s^2 - 400s - 700 = 0$$

The roots are:

$s_{1,2} = -0.51 \pm j1.44 \quad \rightarrow$ Not on the root locus

$s_3 = -7.48 \quad \rightarrow$ Not on the root locus

There is no break-away or break-in point.

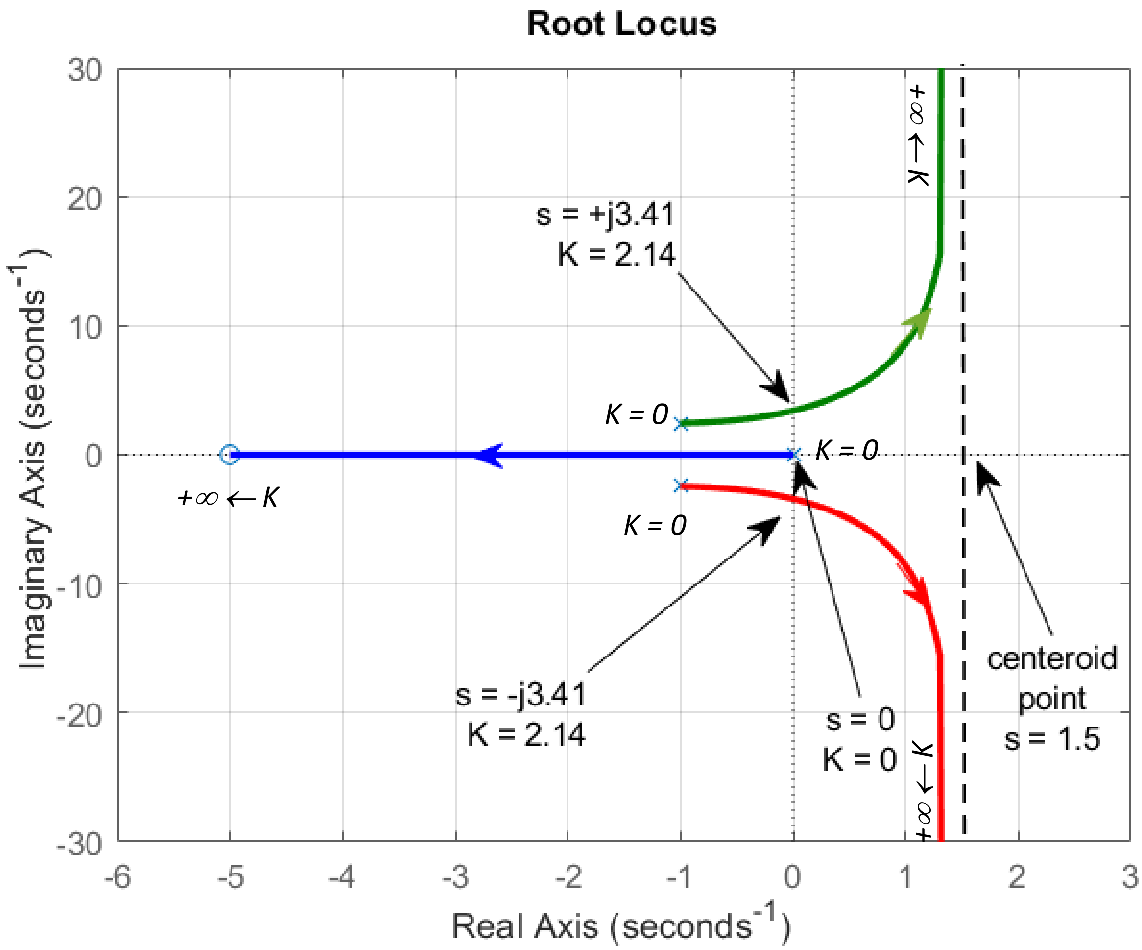
Step 6: Calculate angle of departure from the complex poles

The angle of departure from the complex pole at $s = -1 + j2.45$ is:

$$\boldsymbol{\phi_p = 180^{\circ} - \sum_i \angle p_i + \sum_j \angle z_j = 180^\circ - (\theta_1 + \theta_2) + (\varphi_1) = 180^\circ - (90^\circ + 112^\circ) + (30^\circ) = 8^\circ}$$

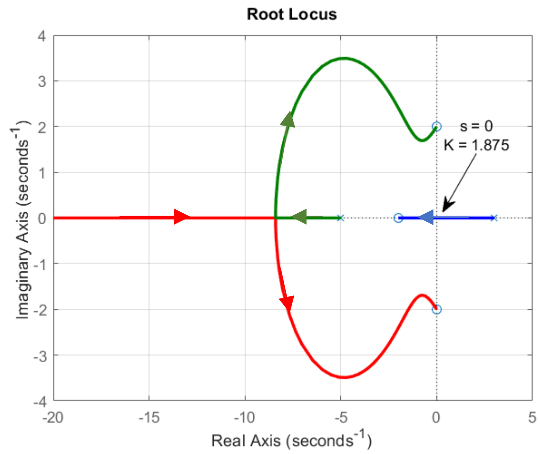
Since the root-locus is symmetrical with respect to the real axis, the angle of departure for the complex pole at $s = -1 - j2.45$ is -8°.

Step 7: Complete the root-locus diagram



7) Plot the root locus of the unity feedback system, where G(s) is given as below. For what range of K will the poles be in the right half-plane?

$$\boldsymbol{G(s) = \frac{K(s+2)(s^2+4)}{(s+5)(s-3)}}$$



We can plot the root locus using MATLAB:

$$G(s) = \frac{K(s^3+2s^2+4s+8)}{s^2+2s-15}$$

```
G = tf([1 2 4 8],[1 2 -15]);
rlocus(G)
```

We have to find the intersection of the root locus with imaginary axis.

The closed-loop system characteristic equation is:

$$Ks^3 + (1+2K)s^2 + (2+4K)s - 15 + 8K = 0$$

Create the Routh-Hurwitz table:

s^3	K	$2+4K$
s^2	$1+2K$	$-15+8K$
s^1	$\frac{2+23K}{1+2K}$	0
s^0	$-15+8K$	0

For stability,
$K > 0$

$1+2K > 0 \quad \rightarrow \quad K > -0.5$

$2+23K > 0 \quad \rightarrow \quad K > -0.087$

$-15+8K > 0 \quad \rightarrow \quad K > 1.875$

The stability range is: $K > 1.875$

The system is marginally stable for $K = 1.875$

For $0 < K < 1.875$ the poles will be in the right-half-plane.

8) Sketch the root locus for the shown unity feedback system, where G(s) is given as below. Give the values for all critical points of interest. Is the system ever unstable? If so, for what range of K?

$$\boldsymbol{G(s)=\frac{K(s^2+2)}{(s+3)(s+4)}}$$

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s)=\frac{Y(s)}{R(s)}=\frac{KG(s)}{1+KG(s)H(s)}=\frac{K(s^2+2)}{(1+K)s^2+7s+12+2K}$$

The closed-loop characteristic equation is: $\boldsymbol{(1+K)s^2+7s+12+2K=0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_1=-3,\;\; s_2=-4$

Zeros: $s_{1,2}=\pm j\sqrt{2}$

Step 2: Draw the root-locus on the real axis

The segment between -3 to -4 is on the root-locus.

Step 3: Draw the asymptote lines for large K values

Number of asymptote lines: $n-m=2-2=0$

There are **<u>no asymptote lines</u>**. The root-locus will **not** approach infinity for large K values.

Step 4: Intersection of root-locus with imaginary axis

$(1+K)s^2+7s+12+2K=0$

Set $s=j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(1+K)(j\omega)^2+2(j\omega)+12+2K=-(1+K)\omega^2+j2\omega+12+2K=0$

$(12+2K-(1+K)\omega^2)+j(2\omega)=0$

From the imaginary part:

$2\omega=0 \quad \rightarrow \quad \omega=0$

From the real part:

$\text{For} \quad \omega=0 \quad \rightarrow \quad 12+2K-(1+K)\omega^2=12+2K=0 \quad \rightarrow \quad K=-6 \quad \text{Not Acceptable}$

Therefore, the root-locus will not cross the imaginary axis.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $\boldsymbol{(1+K)s^2+7s+12+2K=0}$

Find the K from the characteristic equation

$$K=\frac{-s^2-7s-12}{s^2+2}$$

$$\frac{dK}{ds}=0 \quad \rightarrow \quad \frac{(-2s-7)(s^2+2)-(2s)(-s^2-7s-12)}{(s^2+2)^2}=0 \quad \rightarrow \quad 7s^2+20s-14=0$$

The roots are:

$s=0.5816$ → Not on the root locus

$s=-3.4387$ → On the root locus (Break-away point)

The associate gain for the break-away point:

$$K=\frac{-(-3.4387)^2-7(-3.4387)-12}{(-3.4387)^2+2}=0.0178$$

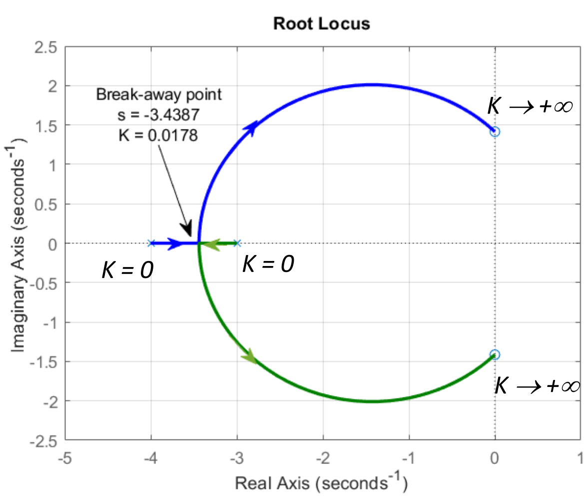
Step 6: Calculate angle of arrival to the complex zeros

The angle of arrival to the complex zero at $s=+j\sqrt{2}$ is:

$$\boldsymbol{\phi_p=180^{\circ}-\sum_i \angle z_i+\sum_j \angle p_j=180^\circ-(\varphi_1)+(\theta_1+\theta_2)=180^\circ-(90^\circ)+(34^\circ+27^\circ)=151^\circ}$$

Since the root-locus is symmetrical with respect to the real axis, the angle of arrival to the zero at $s=-j\sqrt{2}$ is $-151°$.

Step 7: Complete the root-locus diagram



9) Find the angles of the asymptotes and the intersect of the asymptotes of the root loci of the following equations when K varies from 0 to ∞.

a) $\boldsymbol{s^4+4s^3+4s^2+(K+8)s+K=0}$

Find the open-loop system $G(s)H(s)$.

$$1+KG(s)H(s)=0 \quad \rightarrow \quad G(s)H(s)=-\frac{1}{K}$$

$$(s^4+4s^3+4s^2+8s)+(s+1)K=0 \ \rightarrow \ K=\frac{-(s^4+4s^3+4s^2+8s)}{s+1}$$

$$G(s)H(s)=\frac{s+1}{s^4+4s^3+4s^2+8s}$$

Poles: $s_{1,2}=-0.25\pm j1.49,\ \ s_3=-3.51,\ \ s_4=0$

Zeros: $s_1=-1$, Three zeros at infinity

Number of asymptote lines:

$$n-m=4-1=3$$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha}=\frac{\sum_{i=1}^{n}\boldsymbol{p_i}-\sum_{i=1}^{m}\boldsymbol{z_i}}{\boldsymbol{n-m}}=\frac{[(-0.25+j1.49)+(-0.25-j1.49)+(-3.51)+(0)]-[(-1)]}{4-1}=-1.003$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i}=\frac{\mathbf{180^\circ}}{\boldsymbol{n-m}}(\mathbf{2}\boldsymbol{i}+\mathbf{1})=\frac{180^\circ}{4-1}(2i+1)=60^\circ(2i+1)\rightarrow\begin{cases}\boldsymbol{\varphi_0}=\mathbf{60^\circ}\\ \boldsymbol{\varphi_1}=\mathbf{180^\circ}\\ \boldsymbol{\varphi_2}=\mathbf{300^\circ}\end{cases}$$

b) $\boldsymbol{s^3 + 5s^2 + (K+1)s + K = 0}$

Find the open-loop system $G(s)H(s)$.

$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad G(s)H(s) = -\frac{1}{K}$$

$$(s^3 + 5s^2 + s) + (s+1)K = 0 \;\; \rightarrow \;\; K = \frac{-(s^3 + 5s^2 + s)}{s+1}$$

$$G(s)H(s) = \frac{s+1}{s^3 + 5s^2 + s}$$

Poles: $s_1 = -4.79, \;\; s_2 = -0.21, \;\; s_3 = 0$

Zeros: $s_1 = -1$, Two zeros at infinity

Number of asymptote lines:

$$n - m = 3 - 1 = 2$$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}}{\boldsymbol{n-m}} = \frac{[(-4.79) + (-0.21) + (0)] - [(-1)]}{3-1} = -2$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i} = \frac{\boldsymbol{180^\circ}}{\boldsymbol{n-m}}(\boldsymbol{2i+1}) = \frac{180^\circ}{3-1}(2i+1) = 90^\circ(2i+1) \rightarrow \begin{cases} \boldsymbol{\varphi_0 = 90^\circ} \\ \boldsymbol{\varphi_2 = 270^\circ} \end{cases}$$

c) $\boldsymbol{s^2 + K\left(s^3 + 3s^2 + 2s + 8\right) = 0}$

Find the open-loop system $G(s)H(s)$.

$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad G(s)H(s) = -\frac{1}{K}$$

$$s^2 + K(s^3 + 3s^2 + 2s + 8) = 0 \;\; \rightarrow \;\; K = \frac{-s^2}{s^3 + 3s^2 + 2s + 8}$$

$$G(s)H(s) = \frac{s^3 + 3s^2 + 2s + 8}{s^2}$$

Poles: $s_1 = s_2 = 0$, One pole at infinity

Zeros: $s_1 = -3.17, s_{2,3} = -0.083 \pm j1.59$

Number of asymptote lines:

$$m - n = 3 - 2 = 1$$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha} = \frac{\sum_{i=1}^{n} \boldsymbol{p_i} - \sum_{i=1}^{m} \boldsymbol{z_i}}{\boldsymbol{m - n}} = \frac{[(0) + (0)] - [(-3.17) + (-0.083 + j1.59) + (-0.083 - j1.59)]}{3 - 2} = -3.34$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i} = \frac{\mathbf{180^{\circ}}}{\boldsymbol{m - n}}(\boldsymbol{2i + 1}) = \frac{180^{\circ}}{3 - 2}(2i + 1) = 180^{\circ}(2i + 1) \quad \rightarrow \quad \boldsymbol{\varphi_0 = 180^{\circ}}$$

d) $\boldsymbol{s^3 + 2s^2 + 3s + K\left(s^2 - 1\right)(s + 3) = 0}$

Find the open-loop system $G(s)H(s)$.

$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad G(s)H(s) = -\frac{1}{K}$$

$$(s^3 + 2s^2 + 3s) + K(s^2 - 1)(s + 3) = 0 \;\; \rightarrow \;\; K = \frac{-(s^3 + 2s^2 + 3s)}{(s^2 - 1)(s + 3)}$$

$$G(s)H(s) = \frac{(s^2 - 1)(s + 3)}{s^3 + 2s^2 + 3s}$$

Poles: $s_{1,2} = -1 \pm j1.41, \quad s_3 = 0$

Zeros: $s_1 = -1, \;\; s_2 = +1, \;\; s_3 = -3$

Number of asymptote lines:

$$n - m = 3 - 3 = 0$$

There is no asymptote line. Since the number of poles and zeros are equal.

e) $\boldsymbol{s^5 + 2s^4 + 3s^3 + K(s^2 + 3s + 5) = 0}$

Find the open-loop system $G(s)H(s)$.

$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad G(s)H(s) = -\frac{1}{K}$$

$$s^5 + 2s^4 + 3s^3 + K(s^2 + 3s + 5) = 0 \ \rightarrow \ K = \frac{-(s^5 + 2s^4 + 3s^3)}{s^2 + 3s + 5}$$

$$G(s)H(s) = \frac{s^2 + 3s + 5}{s^5 + 2s^4 + 3s^3}$$

Poles: $s_1 = s_2 = s_3 = 0, \ \ s_{4,5} = -1 \pm j1.41$

Zeros: $s_{1,2} = -1.5 \pm j1.66$, Three zeros at infinity

Number of asymptote lines:

$$n - m = 5 - 2 = 3$$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha} = \frac{\sum_{i=1}^{n} \boldsymbol{p_i} - \sum_{i=1}^{m} \boldsymbol{z_i}}{\boldsymbol{n - m}} = \frac{[(0) + (0) + (0) + (-1 + j1.41) + (-1 - j1.41)] - [(-1.5 + j1.66) + (-1.5 - j1.66)]}{5 - 2} = -0.33$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i} = \frac{\mathbf{180^\circ}}{\boldsymbol{n - m}}(\mathbf{2}\boldsymbol{i} + \mathbf{1}) = \frac{180^\circ}{5 - 2}(2i + 1) = 60^\circ(2i + 1) \quad \rightarrow \quad \begin{cases} \boldsymbol{\varphi_0 = 60^\circ} \\ \boldsymbol{\varphi_1 = 180^\circ} \\ \boldsymbol{\varphi_2 = 300^\circ} \end{cases}$$

f) $\boldsymbol{s^4 + 2s^2 + 10 + K(s+5) = 0}$

Find the open-loop system $G(s)H(s)$.

$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad G(s)H(s) = -\frac{1}{K}$$

$$s^4 + 2s^2 + 10 + K(s+5) = 0 \;\; \rightarrow \;\; K = \frac{-(s^4 + 2s^2 + 10)}{s+5}$$

$$G(s)H(s) = \frac{s+5}{s^4 + 2s^2 + 10}$$

Poles: $s_{1,2} = -1.04 \pm j1.44, \;\; s_{3,4} = +1.04 \pm j1.44$

Zeros: $s_1 = -5$, Three zeros at infinity

Number of asymptote lines:

$$n - m = 4 - 1 = 3$$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha} = \frac{\sum_{i=1}^{n} \boldsymbol{p_i} - \sum_{i=1}^{m} \boldsymbol{z_i}}{\boldsymbol{n - m}} = \frac{[(-1.04 + j1.44) + (-1.04 - j1.44) + (+1.04 + j1.44) + (+1.04 - j1.44)] - [(-5)]}{4 - 1} = 1.67$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i} = \frac{\boldsymbol{180^\circ}}{\boldsymbol{n - m}}(\boldsymbol{2i + 1}) = \frac{180^\circ}{4 - 1}(2i + 1) = 60^\circ(2i + 1) \quad \rightarrow \quad \begin{cases} \boldsymbol{\varphi_0 = 60^\circ} \\ \boldsymbol{\varphi_1 = 180^\circ} \\ \boldsymbol{\varphi_2 = 300^\circ} \end{cases}$$

10) The feedforward transfer function of a unity-feedback system is

$$\boldsymbol{G(s)=\frac{K(s+2)^2}{(s^2+4)(s+5)^2}}$$

a) Construct the root loci for $K \geq 0$.

b) Find the range of K value for which the system is stable.

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s)=\frac{Y(s)}{R(s)}=\frac{KG(s)}{1+KG(s)H(s)}=\frac{K(s+2)^2}{s^4+10s^3+(29+K)s^2+(40+4K)s+100+4K}$$

The closed-loop characteristic equation is: $\boldsymbol{s^4+10s^3+(29+K)s^2+(40+4K)s+100+4K=0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_{1,2}=\pm j2,\;\; s_3=s_4=-5$

Zeros: $s_1=s_2=-2$

Step 2: Draw the root-locus on the real axis

None of the real axis is on the root-locus.

Step 3: Draw the asymptote lines for large K values

Number of asymptote lines: $n-m=4-2=2$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha=\frac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}}=\frac{[(+j2)+(-j2)+(-5)+(-5)]-[(-2)+(-2)]}{4-2}=-3$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i=\frac{180^\circ}{n-m}(2i+1)}=\frac{180^\circ}{4-2}(2i+1)=90^\circ(2i+1)\rightarrow\begin{cases}\boldsymbol{\varphi_0=\;90^\circ}\\ \boldsymbol{\varphi_1=\;270^\circ}\end{cases}$$

Step 4: Intersection of root-locus with imaginary axis

$s^4+10s^3+(29+K)s^2+(40+4K)s+100+4K=0$

Set $s=j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(j\omega)^4+10(j\omega)^3+(29+K)(j\omega)^2+(40+4K)(j\omega)+100+4K=0$

$\omega^4-j10\omega^3-(29+K)\omega^2+j(40+4K)\omega+100+4K=0$

$$(\omega^4 - (29+K)\omega^2 + 100 + 4K) + j(-10\omega^3 + (40+4K)\omega) = 0$$

From the imaginary part:

$$-10\omega^3 + (40+4K)\omega = 0 \quad \rightarrow \quad \begin{cases} \omega = 0 \\ \omega^2 = 4 + 0.4K \end{cases} \quad \rightarrow \quad \omega = \pm\sqrt{4+0.4K}$$

From the real part:

$$\text{For} \quad \omega = 0 \quad \rightarrow \quad \omega^4 - (29+K)\omega^2 + 100 + 4K = 0 - (29+K)\times 0 + 100 + 4K = 0 \quad \rightarrow \quad 100 + 4K = 0$$

$$\rightarrow \quad K = -25 \quad \text{Not Acceptable}$$

$$\text{For} \quad \omega^2 = 4 + 0.4K \quad \rightarrow \quad \omega^4 - (29+K)\omega^2 + 100 + 4K = (4+0.4K)^2 - (29+K)(4+0.4K) + 100 + 4K = 0$$

$$\rightarrow \quad -0.2K^2 - 8.4K = 0 \quad \rightarrow \quad \begin{cases} K = 0 & \text{Open} - \text{loop poles} \\ K = -35 & \text{Not Acceptable} \end{cases}$$

Therefore, the root-locus will cross the imaginary axis on at the open-loop locations at $s = \pm j2$.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $\boldsymbol{s^4 + 10s^3 + (29+K)s^2 + (40+4K)s + 100 + 4K = 0}$

Find the K from the characteristic equation

$$K = \frac{-s^4 - 10s^3 - 29s^2 - 40s - 100}{s^2 + 4s + 4}$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad \frac{(-4s^3 - 30s^2 - 58s - 40)(s^2 + 4s + 4) - (2s+4)(-s^4 - 10s^3 - 29s^2 - 40s - 100)}{(s^2+4s+4)^2} = 0$$

$$\rightarrow \; -2s^4 - 18s^3 - 60s^2 - 76s + 120 = 0$$

The roots are:

$s = 0.85$ → Not on the root locus

$s = -2.42 \pm j2.87$ → Not the real axis

$s = -5$ → On the root locus (Break-away point)

This is the open-loop pole location.

The associate gain for the break-away point:

$$K = \frac{-(-5)^4 - 10(-5)^3 - 29(-5)^2 - 40(-5) - 100}{(-5)^2 + 4(-5) + 4} = 0$$

Step 6: Calculate angle of departure from the complex poles

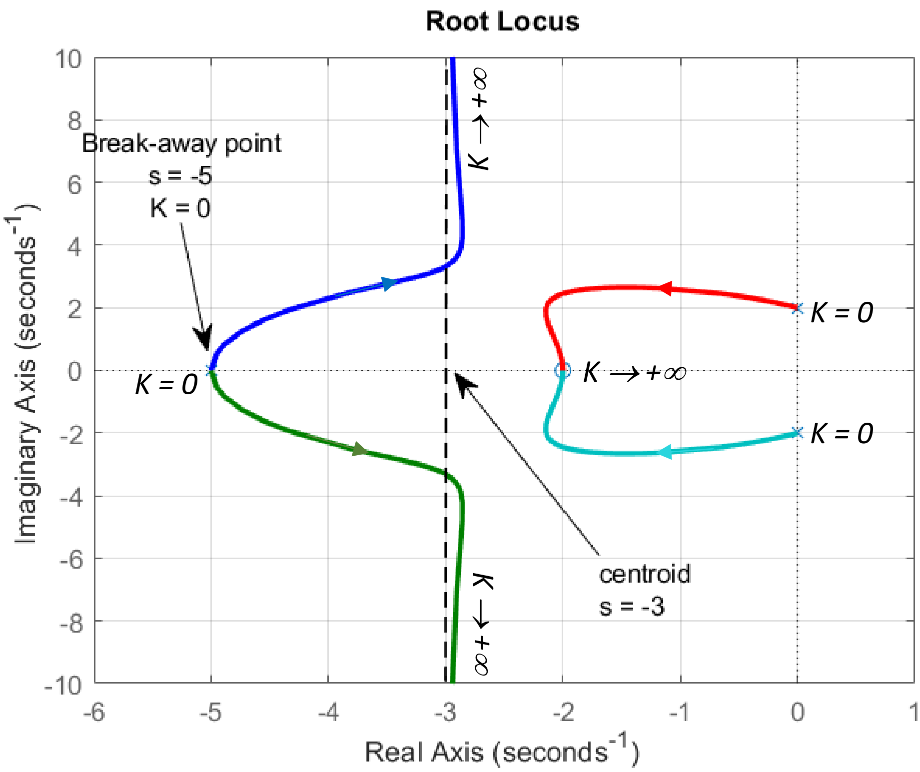
The angle of departure from the complex pole at $s = +j2$ is:

$$\phi_p = 180^{\circ} - \sum_i \angle p_i + \sum_j \angle z_j = 180° - (\theta_1 + \theta_2 + \theta_3) + (\varphi_1 + \varphi_2)$$

$$= 180° - (90° + 22° + 22°) + (45° + 45°) = 136°$$

Since the root-locus is symmetrical with respect to the real axis, the angle of departure for the complex pole $s = -j2$ is $-136°$.

Step 7: Complete the root-locus diagram



There is no closed loop pole on the right half s-plane; therefore, the system is stable for all $K > 0$

11) Given a unity-feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{s^2 - 4s + 13}$$

do the following:

a) Sketch the root locus.

b) Find the imaginary-axis crossing.

c) Find the gain K at the jω-axis crossing.

d) Find the break-in point.

e) Find the angle of departure from the complex poles.

Find the closed-loop transfer function and closed-loop characteristic equation:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K(s+2)}{s^2 + (-4+K)s + 13 + 2K}$$

The closed-loop characteristic equation is: $\boldsymbol{s^2 + (-4+K)s + 13 + 2K = 0}$

Step 1: Draw the axes of s-plane and mark the open-loop system poles and zeros on it.

Poles: $s_{1,2} = +2 \pm j3$

Zeros: $s_1 = -2$

Step 2: Draw the root-locus on the real axis

The segment between -2 to -∞ is on the root-locus.

Step 3: Draw the asymptote lines for large K values

Number of asymptote lines:

$n - m = 2 - 1 = 1$

Intersection of asymptotes on the real axis:

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m} = \frac{[(+2+j3) + (+2-j3)] - [(-2)]}{2-1} = 6$$

Angle of asymptote lines with real axis:

$$\varphi_i = \frac{180^\circ}{n-m}(2i+1) = \frac{180^\circ}{2-1}(2i+1) = 180^\circ(2i+1) \;\rightarrow\; \varphi_0 = 180^\circ$$

Step 4: Intersection of root-locus with imaginary axis

$s^2 + (-4 + K)s + 13 + 2K = 0$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$(j\omega)^2 + (-4 + K)(j\omega) + 13 + 2K = 0 \quad \rightarrow \quad -\omega^2 + j(-4 + K)\omega + 13 + 2K = 0$

$(-\omega^2 + 13 + 2K) + j\big((-4 + K)\omega\big) = 0$

From the imaginary part:

$$(-4 + K)\omega = 0 \quad \rightarrow \quad \begin{cases} \omega = 0 \\ -4 + K = 0 \quad \rightarrow \quad K = 4 \end{cases}$$

From the real part:

$\text{For} \quad \omega = 0 \quad \rightarrow \quad -\omega^2 + 13 + 2K = -0 + 13 + 2K = 0 \quad \rightarrow \quad 13 + 2K = 0 \quad \rightarrow \quad K = -6.5 \quad \text{Not Acceptable}$

$\text{For} \quad K = 4 \quad \rightarrow \quad -\omega^2 + 13 + 2K = -\omega^2 + 13 + 2(4) = 0 \quad \rightarrow \quad \omega^2 = 21 \quad \omega = \pm\sqrt{21} = \pm 4.58$

Therefore, the root-locus will cross the imaginary axis at $s = \pm j4.58$ for gain $K = 4$.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $\boldsymbol{s^2 + (-4 + K)s + 13 + 2K = 0}$

Find the K from the characteristic equation

$$K = \frac{-s^2 + 4s - 13}{s + 2}$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad \frac{(-2s + 4)(s + 2) - (1)(-s^2 + 4s - 13)}{(s + 2)^2} = 0 \quad \rightarrow -s^2 - 4s + 21 = 0$$

The roots are:

$s = 3 \ \rightarrow$ Not on the root locus

$s = -7 \ \rightarrow$ On the root locus (Break-in point)

The associate gain for the break-in point:

$$K = \frac{-(-7)^2 + 4(-7) - 13}{(-7) + 2} = 18$$

Step 6: Calculate angle of departure from the complex poles

The angle of departure from the complex pole at $s = +2 + j3$ is:

$$\boldsymbol{\phi_p = 180^{\circ} - \sum_i \angle p_i + \sum_j \angle z_j = 180^{\circ} - (\theta_1) + (\varphi_1)}$$

$$\boldsymbol{= 180^{\circ} - (90^{\circ}) + (37^{\circ}) = 127^{\circ}}$$

Since the root-locus is symmetrical with respect to the real axis, the angle of departure for the complex pole $s = +2 - j3$ is $-127°$.

Step 7: Complete the root-locus diagram