Lecture 1

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Chapter 4

$\mathbb{R}^n$

4.1 Vectors in $\mathbb{R}^n$

Outcomes

A. *Find the position vector of a point in* $\mathbb{R}^n$.

The notation $\mathbb{R}^n$ refers to the collection of ordered lists of n real numbers, that is

$$\mathbb{R}^n = \left\{ (x_1 \cdots x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, n \right\}$$

In this chapter, we take a closer look at vectors in $\mathbb{R}^n$. First, we will consider what $\mathbb{R}^n$ looks like in more detail. Recall that the point given by $0 = (0, \cdots, 0)$ is called the **origin**.

Now, consider the case of $\mathbb{R}^n$ for $n = 1$. Then from the definition we can identify $\mathbb{R}$ with points in $\mathbb{R}^1$ as follows:

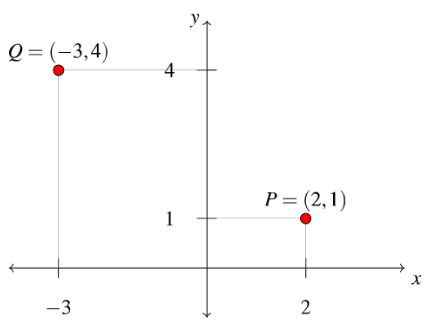
$$\mathbb{R} = \mathbb{R}^1 = \{(x_1) : x_1 \in \mathbb{R}\}$$

Hence, $\mathbb{R}$ is defined as the set of all real numbers and geometrically, we can describe this as all the points on a line.

Now suppose $n = 2$. Then, from the definition,

$$\mathbb{R}^2 = \left\{ (x_1, x_2) : x_j \in \mathbb{R} \text{ for } j = 1, 2 \right\}$$

Consider the familiar coordinate plane, with an x axis and a y axis. Any point within this coordinate plane is identified by where it is located along the x axis, and also where it is located along the y axis. Consider as an example the following diagram.



Hence, every element in $\mathbb{R}^2$ is identified by two components, x and y, in the usual manner. The coordinates x, y (or x_1,x_2) uniquely determine a point in the plan. Note that while the definition uses x_1 and x_2 to label the coordinates and you may be used to x and y, these notations are equivalent.

Now suppose $n = 3$. You may have previously encountered the 3-dimensional coordinate system, given by

$$\mathbb{R}^3 = \left\{ (x_1, x_2, x_3) : x_j \in \mathbb{R} \text{ for } j = 1, 2, 3 \right\}$$

Points in $\mathbb{R}^3$ will be determined by three coordinates, often written (x, y, z) which correspond to the x, y, and z axes. We can think as above that the first two coordinates determine a point in a plane. The third component determines the height above or below the plane, depending on whether this number is positive component determines the height above or below the plane, depending on whether this number is positive or negative, and all together this determines a point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

The idea behind the more general $\mathbb{R}^n$ is that we can extend these ideas beyond $n = 3$. This discussion regarding points in $\mathbb{R}^n$ leads into a study of vectors in $\mathbb{R}^n$. While we consider $\mathbb{R}^n$ for all n, we will largely focus on $n = 2, 3$ in this section.

Consider the following definition.

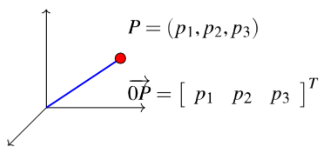
Definition 4.1: The Position Vector

Let $P = (p_1, \cdots, p_n)$ be the coordinates of a point in $\mathbb{R}^n$. Then the vector $\overrightarrow{0P}$ with its tail at $0 = (0, \cdots, 0)$ and its tip at P is called the **position vector** *of the point P. We write*

$$\overrightarrow{0P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

For this reason we may write both $P = (p_1, \cdots, p_n) \in \mathbb{R}^n$ and $\overrightarrow{0P} = [p_1 \cdots p_n]^T \in \mathbb{R}^n$.

This definition is illustrated in the following picture for the special case of $\mathbb{R}^3$.

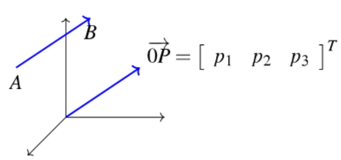


Thus every point P in $\mathbb{R}^n$ determines its position vector $\overrightarrow{0P}$. Conversely, every such position vector $\overrightarrow{0P}$ which has its tail at 0 and point at P determines the point P of $\mathbb{R}^n$.

Now suppose we are given two points, P, Q whose coordinates are $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ respectively. We can also determine the **position vector from P to Q** (also called the **vector from P to Q**) defined as follows.

$$\overrightarrow{PQ} = \begin{bmatrix} q_1 - p_1 \\ \vdots \\ q_n - p_n \end{bmatrix} = \overrightarrow{0Q} - \overrightarrow{0P}$$

Now, imagine taking a vector in $\mathbb{R}^n$ and moving it around, always keeping it pointing in the same direction as shown in the following picture.



After moving it around, it is regarded as the same vector. Each vector, $\overrightarrow{0P}$ and $\overrightarrow{AB}$ has the same length (or magnitude) and direction. Therefore, they are equal.

Consider now the general definition for a vector in $\mathbb{R}^n$.

Definition 4.2: Vectors in $\mathbb{R}^n$

Let $\mathbb{R}^n = \left\{ (x_1, \cdots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, n \right\}$. Then,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **vector**. *Vectors have both size (magnitude) and direction. The numbers x_j are called the* ***components*** *of $\vec{x}$.*

Using this notation, we may use $\vec{p}$ to denote the position vector of point P. Notice that in this context, $\vec{p} = \overrightarrow{0P}$. These notations may be used interchangeably.

You can think of the components of a vector as directions for obtaining the vector. Consider $n = 3$. Draw a vector with its tail at the point $(0, 0, 0)$ and its tip at the point (a, b, c). This vector it is obtained by starting at $(0,0,0)$, moving parallel to the x axis to $(a,0,0)$ and then from here, moving parallel to the y axis to $(a,b,0)$ and finally parallel to the z axis to (a,b,c). Observe that the same vector would result if you began at the point (d,e,f), moved parallel to the x axis to $(d+a,e,f)$, then parallel to the y axis to $(d+a,e+b,f)$, and finally parallel to the z axis to $(d+a,e+b,f+c)$. Here, the vector would have its tail sitting at the point determined by $A=(d,e,f)$ and its point at $B=(d+a,e+b,f+c)$. It is the **same vector** because it will point in the same direction and have the same length. It is like you took an actual arrow, and moved it from one location to another keeping it pointing the same direction.

We conclude this section with a brief discussion regarding notation. In previous sections, we have written vectors as columns, or $n\times 1$ matrices. For convenience in this chapter we may write vectors as the transpose of row vectors, or $1\times n$ matrices. These are of course equivalent and we may move between both notations. Therefore, recognize that

$$\left[\begin{array}{c} 2 \\ 3 \end{array}\right] = \left[\begin{array}{cc} 2 & 3 \end{array}\right]^T$$

Notice that two vectors $\vec{u}=[u_1\cdots u_n]^T$ and $\vec{v}=[v_1\cdots v_n]^T$ are equal if and only if all corresponding

components are equal. Precisely,

$$\begin{gathered}\vec{u}=\vec{v} \text{ if and only if} \\ u_j=v_j \text{ for all } j=1,\cdots,n\end{gathered}$$

Thus $\left[\begin{array}{ccc} 1 & 2 & 4 \end{array}\right]^T \in \mathbb{R}^3$ and $\left[\begin{array}{ccc} 2 & 1 & 4 \end{array}\right]^T \in \mathbb{R}^3$ but $\left[\begin{array}{ccc} 1 & 2 & 4 \end{array}\right]^T \neq \left[\begin{array}{ccc} 2 & 1 & 4 \end{array}\right]^T$ because, even though the same numbers are involved, the order of the numbers is different.

For the specific case of $\mathbb{R}^3$, there are three special vectors which we often use. They are given by

$$\begin{aligned}\vec{i} &= \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]^T \\ \vec{j} &= \left[\begin{array}{ccc} 0 & 1 & 0 \end{array}\right]^T \\ \vec{k} &= \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]^T\end{aligned}$$

We can write any vector $\vec{u}=\left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]^T$ as a linear combination of these vectors, written as $\vec{u}=u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$. This notation will be used throughout this chapter.

4.2 Algebra in $\mathbb{R}^n$

Outcomes

A. *Understand vector addition and scalar multiplication, algebraically.*

B. *Introduce the notion of linear combination of vectors.*

Addition and scalar multiplication are two important algebraic operations done with vectors. Notice that these operations apply to vectors in $\mathbb{R}^n$, for any value of n. We will explore these operations in more detail in the following sections.

4.2.1 Addition of Vectors in $\mathbb{R}^n$

Addition of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 4.3: Addition of Vectors in $\mathbb{R}^n$

If $\vec{u}=\left[\begin{array}{c} u_1 \\ \vdots \\ u_n \end{array}\right], \vec{v}=\left[\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right] \in \mathbb{R}^n$ *then* $\vec{u}+\vec{v}\in\mathbb{R}^n$ *and is defined by*

$$\begin{aligned}\vec{u}+\vec{v} &= \left[\begin{array}{c} u_1 \\ \vdots \\ u_n \end{array}\right]+\left[\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right] \\ &= \left[\begin{array}{c} u_1+v_1 \\ \vdots \\ u_n+v_n \end{array}\right]\end{aligned}$$

To add vectors, we simply add corresponding components. Therefore, in order to add vectors, they must be the same size.

Addition of vectors satisfies some important properties which are outlined in the following theorem.

Theorem 4.4: Properties of Vector Addition

The following properties hold for vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$.

- *The Commutative Law of Addition*

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

- *The Associative Law of Addition*

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

- *The Existence of an Additive Identity*

$$\vec{u} + \vec{0} = \vec{u} \tag{4.1}$$

- *The Existence of an Additive Inverse*

$$\vec{u} + (-\vec{u}) = \vec{0}$$

The additive identity shown in equation 4.1 is also called the **zero vector**, the $n \times 1$ vector in which all components are equal to 0. Further, $-\vec{u}$ is simply the vector with all components having same value as those of $\vec{u}$ but opposite sign; this is just $(-1)\vec{u}$. This will be made more explicit in the next section when we explore scalar multiplication of vectors. Note that subtraction is defined as $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

4.2.2 Scalar Multiplication of Vectors in $\mathbb{R}^n$

Scalar multiplication of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 4.5: Scalar Multiplication of Vectors in $\mathbb{R}^n$

If $\vec{u} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k\vec{u} \in \mathbb{R}^n$ is defined by

$$k\vec{u} = k \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$$

Just as with addition, scalar multiplication of vectors satisfies several important properties. These are outlined in the following theorem.

Theorem 4.6: Properties of Scalar Multiplication

The following properties hold for vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and k, p scalars.

- *The Distributive Law over Vector Addition*

$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$$

- *The Distributive Law over Scalar Addition*

$$(k + p)\vec{u} = k\vec{u} + p\vec{u}$$

- *The Associative Law for Scalar Multiplication*

$$k(p\vec{u}) = (kp)\vec{u}$$

- *Rule for Multiplication by 1*

$$1\vec{u} = \vec{u}$$

Proof. We will show the proof of:

$$k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}$$

Note that:

$$\begin{aligned} k(\vec{u}+\vec{v}) &= k\left[u_1+v_1 \cdots u_n+v_n\right]^T \\ &= \left[k(u_1+v_1)\cdots k(u_n+v_n)\right]^T \\ &= \left[ku_1+kv_1 \cdots ku_n+kv_n\right]^T \\ &= \left[ku_1 \cdots ku_n\right]^T + \left[kv_1 \cdots kv_n\right]^T \\ &= k\vec{u}+k\vec{v} \end{aligned}$$

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We now present a useful notion you may have seen earlier combining vector addition and scalar multiplication

Definition 4.7: Linear Combination

A vector $\vec{v}$ is said to be a ***linear combination*** *of the vectors $\vec{u}_1,\cdots,\vec{u}_n$ if there exist scalars, $a_1,\cdots,a_n$ such that*

$$\vec{v} = a_1\vec{u}_1+\cdots+a_n\vec{u}_n$$

For example,

$$3\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}.$$

Thus we can say that

$$\vec{v} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Exercises

Exercise 4.2.1 *Find* $-3\begin{bmatrix} 5 \\ -1 \\ 2 \\ -3 \end{bmatrix} + 5\begin{bmatrix} -8 \\ 2 \\ -3 \\ 6 \end{bmatrix}$. $\vec{V} = \begin{bmatrix} -55 \\ 13 \\ -21 \\ 39 \end{bmatrix}$

Exercise 4.2.2 *Find* $-7\begin{bmatrix} 6 \\ 0 \\ 4 \\ -1 \end{bmatrix} + 6\begin{bmatrix} -13 \\ -1 \\ 1 \\ 6 \end{bmatrix}$. $\vec{V} = \begin{bmatrix} -120 \\ -6 \\ -22 \\ 43 \end{bmatrix}$

Exercise 4.2.3 *Decide whether*

$$\vec{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \textit{ and } \vec{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Exercise 4.2.4 *Decide whether*

$$\vec{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 3 \end{bmatrix} \qquad \begin{bmatrix} 2 \end{bmatrix}$$

$$\vec{u}_1 = \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \text{ and } \vec{u}_2 = \left[\begin{array}{r} -2 \\ 1 \end{array}\right].$$

4.3 Geometric Meaning of Vector Addition

Outcomes

A. *Understand vector addition, geometrically.*

Recall that an element of $\mathbb{R}^n$ is an ordered list of numbers. For the specific case of $n = 2, 3$ this can be used to determine a point in two or three dimensional space. This point is specified relative to some coordinate axes.

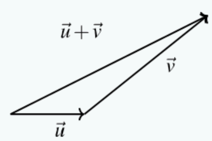
Consider the case $n = 3$. Recall that taking a vector and moving it around without changing its length or direction does not change the vector. This is important in the geometric representation of vector addition.

Suppose we have two vectors, $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$. Each of these can be drawn geometrically by placing the tail of each vector at 0 and its point at (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. Suppose we slide the vector $\vec{v}$ so that its tail sits at the point of $\vec{u}$. We know that this does not change the vector $\vec{v}$. Now, draw a new vector from the tail of $\vec{u}$ to the point of $\vec{v}$. This vector is $\vec{u} + \vec{v}$.

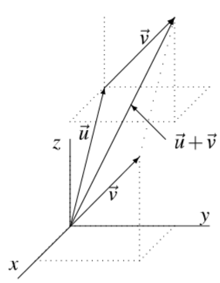
The geometric significance of vector addition in $\mathbb{R}^n$ for any n is given in the following definition.

Definition 4.8: Geometry of Vector Addition

Let $\vec{u}$ and $\vec{v}$ be two vectors. Slide $\vec{v}$ so that the tail of $\vec{v}$ is on the point of $\vec{u}$. Then draw the arrow which goes from the tail of $\vec{u}$ to the point of $\vec{v}$. This arrow represents the vector $\vec{u} + \vec{v}$.



This definition is illustrated in the following picture in which $\vec{u} + \vec{v}$ is shown for the special case $n = 3$.

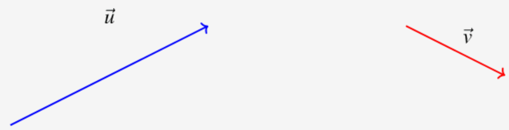


Notice the parallelogram created by $\vec{u}$ and $\vec{v}$ in the above diagram. Then $\vec{u} + \vec{v}$ is the directed diagonal of the parallelogram determined by the two vectors $\vec{u}$ and $\vec{v}$.

When you have a vector $\vec{v}$, its additive inverse $-\vec{v}$ will be the vector which has the same magnitude as $\vec{v}$ but the opposite direction. When one writes $\vec{u}-\vec{v}$, the meaning is $\vec{u}+(-\vec{v})$ as with real numbers. The following example illustrates these definitions and conventions.

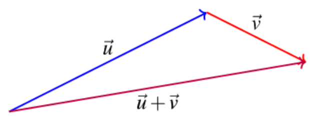
Example 4.9: Graphing Vector Addition

Consider the following picture of vectors $\vec{u}$ and $\vec{v}$.

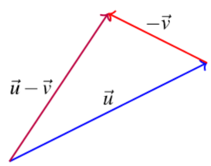


Sketch a picture of $\vec{u}+\vec{v}, \vec{u}-\vec{v}$.

Solution. We will first sketch $\vec{u}+\vec{v}$. Begin by drawing $\vec{u}$ and then at the point of $\vec{u}$, place the tail of $\vec{v}$ as shown. Then $\vec{u}+\vec{v}$ is the vector which results from drawing a vector from the tail of $\vec{u}$ to the tip of $\vec{v}$.



Next consider $\vec{u}-\vec{v}$. This means $\vec{u}+(-\vec{v})$. From the above geometric description of vector addition, $-\vec{v}$ is the vector which has the same length but which points in the opposite direction to $\vec{v}$. Here is a picture.



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4.4 Length of a Vector

Outcomes

A. *Find the length of a vector and the distance between two points in $\mathbb{R}^n$.*

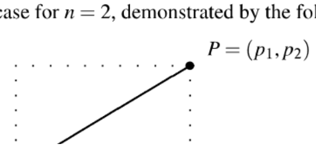
B. *Find the corresponding unit vector to a vector in $\mathbb{R}^n$.*

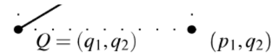
In this section, we explore what is meant by the length of a vector in $\mathbb{R}^n$. We develop this concept by first looking at the distance between two points in $\mathbb{R}^n$.

First, we will consider the concept of distance for $\mathbb{R}$, that is, for points in $\mathbb{R}^1$. Here, the distance between two points P and Q is given by the absolute value of their difference. We denote the distance between P and Q by $d(P,Q)$ which is defined as

$$d(P,Q)=\sqrt{(P-Q)^2} \tag{4.2}$$

Consider now the case for $n=2$, demonstrated by the following picture.

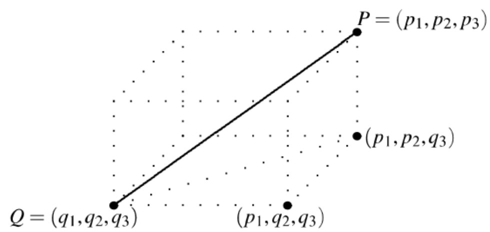




There are two points $P=(p_1,p_2)$ and $Q=(q_1,q_2)$ in the plane. The distance between these points is shown in the picture as a solid line. Notice that this line is the hypotenuse of a right triangle which is half of the rectangle shown in dotted lines. We want to find the length of this hypotenuse which will give the distance between the two points. Note the lengths of the sides of this triangle are $|p_1-q_1|$ and $|p_2-q_2|$, the absolute value of the difference in these values. Therefore, the Pythagorean Theorem implies the length of the hypotenuse (and thus the distance between P and Q) equals

$$\left(|p_1-q_1|^2+|p_2-q_2|^2\right)^{1/2}=\left((p_1-q_1)^2+(p_2-q_2)^2\right)^{1/2} \tag{4.3}$$

Now suppose $n=3$ and let $P=(p_1,p_2,p_3)$ and $Q=(q_1,q_2,q_3)$ be two points in $\mathbb{R}^3$. Consider the following picture in which the solid line joins the two points and a dotted line joins the points (q_1,q_2,q_3) and (p_1,p_2,q_3).



Here, we need to use Pythagorean Theorem twice in order to find the length of the solid line. First, by the Pythagorean Theorem, the length of the dotted line joining (q_1,q_2,q_3) and (p_1,p_2,q_3) equals

$$\left((p_1-q_1)^2+(p_2-q_2)^2\right)^{1/2}$$

while the length of the line joining (p_1,p_2,q_3) to (p_1,p_2,p_3) is just $|p_3-q_3|$. Therefore, by the Pythagorean Theorem again, the length of the line joining the points $P=(p_1,p_2,p_3)$ and $Q=(q_1,q_2,q_3)$ equals

$$\begin{aligned}&\left(\left(\left((p_1-q_1)^2+(p_2-q_2)^2\right)^{1/2}\right)^2+(p_3-q_3)^2\right)^{1/2}\\ &=\left((p_1-q_1)^2+(p_2-q_2)^2+(p_3-q_3)^2\right)^{1/2}\end{aligned} \tag{4.4}$$

This discussion motivates the following definition for the distance between points in $\mathbb{R}^n$.

Definition 4.10: Distance Between Points

Let $P=(p_1,\cdots,p_n)$ and $Q=(q_1,\cdots,q_n)$ be two points in $\mathbb{R}^n$. Then the distance between these points is defined as

$$\text{distance between } P \text{ and } Q=d(P,Q)=\left(\sum_{k=1}^{n}|p_k-q_k|^2\right)^{1/2}$$

*This is called the **distance formula**. We may also write $|P-Q|$ as the distance between P and Q.*

From the above discussion, you can see that Definition 4.10 holds for the special cases $n=1,2,3$, as in Equations 4.2, 4.3, 4.4. In the following example, we use Definition 4.10 to find the distance between two points in $\mathbb{R}^4$.

Example 4.11: Distance Between Points

Find the distance between the points P and Q in $\mathbb{R}^4$, where P and Q are given by

$$P=(1,2,-4,6)$$

and

$$Q = (2,3,-1,0)$$

Solution. We will use the formula given in Definition 4.10 to find the distance between P and Q. Use the distance formula and write

$$d(P,Q) = \left((1-2)^2 + (2-3)^2 + (-4-(-1))^2 + (6-0)^2\right)^{\frac{1}{2}} = (47)^{\frac{1}{2}}$$

Therefore, $d(P,Q) = \sqrt{47}$.

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There are certain properties of the distance between points which are important in our study. These are outlined in the following theorem.

Theorem 4.12: Properties of Distance

Let P and Q be points in $\mathbb{R}^n$, and let the distance between them, $d(P,Q)$, be given as in Definition 4.10. Then, the following properties hold .

- $d(P,Q) = d(Q,P)$
- $d(P,Q) \geq 0$, *and equals 0 exactly when* $P = Q$.

There are many applications of the concept of distance. For instance, given two points, we can ask what collection of points are all the same distance between the given points. This is explored in the following example.

Example 4.13: The Plane Between Two Points

Describe the points in $\mathbb{R}^3$ which are at the same distance between $(1,2,3)$ and $(0,1,2)$.

Solution. Let $P = (p_1, p_2, p_3)$ be such a point. Therefore, P is the same distance from $(1,2,3)$ and $(0,1,2)$. Then by Definition 4.10,

$$\sqrt{(p_1-1)^2 + (p_2-2)^2 + (p_3-3)^2} = \sqrt{(p_1-0)^2 + (p_2-1)^2 + (p_3-2)^2}$$

Squaring both sides we obtain

$$(p_1-1)^2 + (p_2-2)^2 + (p_3-3)^2 = p_1^2 + (p_2-1)^2 + (p_3-2)^2$$

and so

$$p_1^2 - 2p_1 + 14 + p_2^2 - 4p_2 + p_3^2 - 6p_3 = p_1^2 + p_2^2 - 2p_2 + 5 + p_3^2 - 4p_3$$

Simplifying, this becomes

$$-2p_1 + 14 - 4p_2 - 6p_3 = -2p_2 + 5 - 4p_3$$

which can be written as

$$2p_1 + 2p_2 + 2p_3 = 9 \tag{4.5}$$

Therefore, the points $P = (p_1, p_2, p_3)$ which are the same distance from each of the given points form a plane whose equation is given by 4.5. ♠

We can now use our understanding of the distance between two points to define what is meant by the length of a vector. Consider the following definition.

Definition 4.14: Length of a Vector

Let $\vec{u} = [u_1 \cdots u_n]^T$ be a vector in $\mathbb{R}^n$. Then, the length of $\vec{u}$, written $\|\vec{u}\|$ is given by

$$\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

This definition corresponds to Definition 4.10, if you consider the vector $\vec{u}$ to have its tail at the point $0 = (0, \cdots, 0)$ and its tip at the point $U = (u_1, \cdots, u_n)$. Then the length of $\vec{u}$ is equal to the distance between 0 and U, $d(0,U)$. In general, $d(P,Q) = \|\overrightarrow{PQ}\|$.

Consider Example 4.11. By Definition 4.14, we could also find the distance between P and Q as the length of the vector connecting them. Hence, if we were to draw a vector $\overrightarrow{PQ}$ with its tail at P and its point at Q, this vector would have length equal to $\sqrt{47}$.

We conclude this section with a new definition for the special case of vectors of length 1.

Definition 4.15: Unit Vector

Let $\vec{u}$ be a vector in $\mathbb{R}^n$. Then, we call $\vec{u}$ a ***unit vector*** *if it has length 1, that is if*

$$\|\vec{u}\| = 1$$

Let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then, the vector $\vec{u}$ which has the same direction as $\vec{v}$ but length equal to 1 is the corresponding unit vector of $\vec{v}$. This vector is given by

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

We often use the term **normalize** to refer to this process. When we **normalize** a vector, we find the corresponding unit vector of length 1. Consider the following example.

Example 4.16: Finding a Unit Vector

Let $\vec{v}$ be given by

$$\vec{v} = \left[\begin{array}{ccc} 1 & -3 & 4 \end{array}\right]^T$$

Find the unit vector $\vec{u}$ which has the same direction as $\vec{v}$.

Solution. We will use Definition 4.15 to solve this. Therefore, we need to find the length of $\vec{v}$ which, by Definition 4.14 is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Using the corresponding values we find that

$$\begin{aligned} \|\vec{v}\| &= \sqrt{1^2 + (-3)^2 + 4^2} \end{aligned}$$

$$\begin{aligned} &= \sqrt{1+9+16} \\ &= \sqrt{26} \end{aligned}$$

In order to find $\vec{u}$, we divide $\vec{v}$ by $\sqrt{26}$. The result is

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{v}\|}\vec{v} \\ &= \frac{1}{\sqrt{26}}\left[\begin{array}{ccc} 1 & -3 & 4 \end{array}\right]^T \\ &= \left[\begin{array}{ccc} \frac{1}{\sqrt{26}} & -\frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \end{array}\right]^T \end{aligned}$$

You can verify using the Definition 4.14 that $\|\vec{u}\| = 1$. ♠

4.5 Geometric Meaning of Scalar Multiplication

Outcomes

A. *Understand scalar multiplication, geometrically.*

Recall that the point $P = (p_1, p_2, p_3)$ determines a vector $\vec{p}$ from 0 to P. The length of $\vec{p}$, denoted $\|\vec{p}\|$, is equal to $\sqrt{p_1^2 + p_2^2 + p_3^2}$ by Definition 4.10.

Now suppose we have a vector $\vec{u} = \left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]^T$ and we multiply $\vec{u}$ by a scalar k. By Definition 4.5, $k\vec{u} = \left[\begin{array}{ccc} ku_1 & ku_2 & ku_3 \end{array}\right]^T$. Then, by using Definition 4.10, the length of this vector is given by

$$\sqrt{\left((ku_1)^2 + (ku_2)^2 + (ku_3)^2\right)} = |k|\sqrt{u_1^2 + u_2^2 + u_3^2}$$

Thus the following holds.

$$\|k\vec{u}\| = |k| \, \|\vec{u}\|$$

In other words, multiplication by a scalar magnifies or shrinks the length of the vector by a factor of $|k|$. If $|k| > 1$, the length of the resulting vector will be magnified. If $|k| < 1$, the length of the resulting vector will shrink. Remember that by the definition of the absolute value, $|k| \geq 0$.

What about the direction? Draw a picture of $\vec{u}$ and $k\vec{u}$ where k is negative. Notice that this causes the resulting vector to point in the opposite direction while if $k > 0$ it preserves the direction the vector points. Therefore the direction can either reverse, if $k < 0$, or remain preserved, if $k > 0$.

Consider the following example.

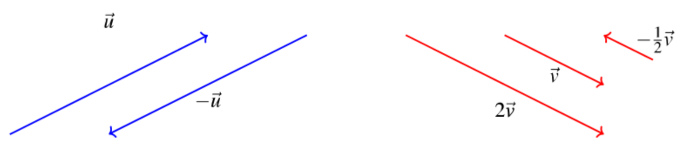
Example 4.17: Graphing Scalar Multiplication

Consider the vectors $\vec{u}$ and $\vec{v}$ drawn below.

Draw $-\vec{u}$, $2\vec{v}$, and $-\frac{1}{2}\vec{v}$.

Solution.

In order to find $-\vec{u}$, we preserve the length of $\vec{u}$ and simply reverse the direction. For $2\vec{v}$, we double the length of $\vec{v}$, while preserving the direction. Finally $-\frac{1}{2}\vec{v}$ is found by taking half the length of $\vec{v}$ and reversing the direction. These vectors are shown in the following diagram.



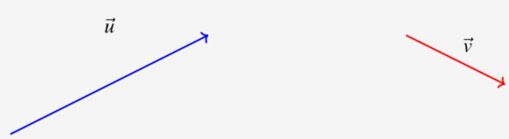
♠

Now that we have studied both vector addition and scalar multiplication, we can combine the two actions. Recall Definition 9.12 of linear combinations of column matrices. We can apply this definition to vectors in $\mathbb{R}^n$. A linear combination of vectors in $\mathbb{R}^n$ is a sum of vectors multiplied by scalars.

In the following example, we examine the geometric meaning of this concept.

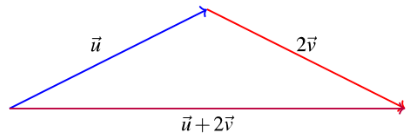
Example 4.18: Graphing a Linear Combination of Vectors

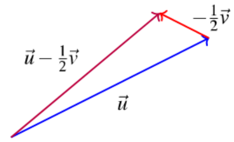
Consider the following picture of the vectors $\vec{u}$ and $\vec{v}$



Sketch a picture of $\vec{u} + 2\vec{v}, \vec{u} - \frac{1}{2}\vec{v}$.

Solution. The two vectors are shown below.





♠

$- \vec{p}_0)$ to $\vec{n}$ where t is some scalar. You can … by doing so, we

$$\vec{a} = \ldots$$