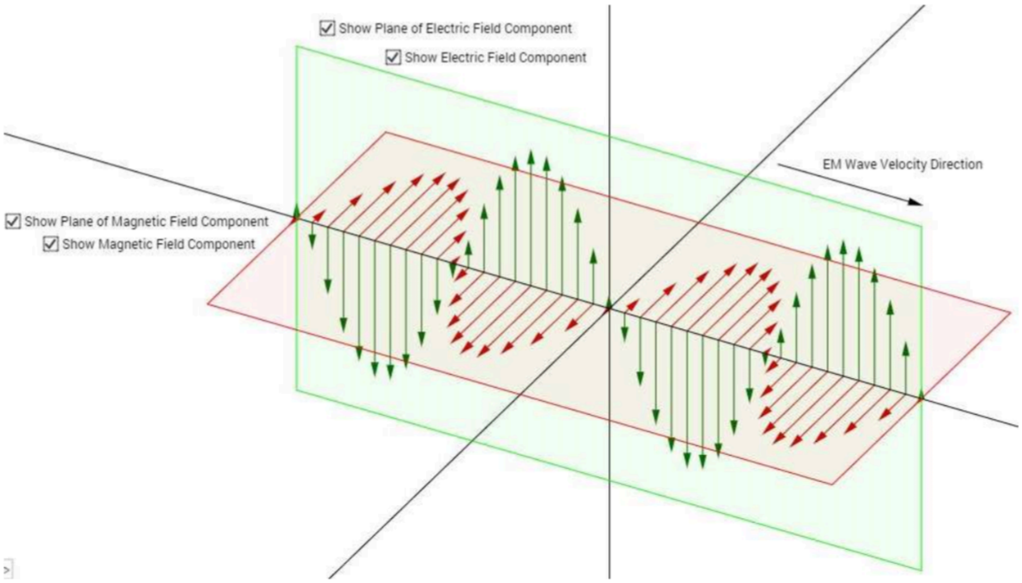
Class Note 2.1

September 14, 2022 10:49 AM



Module 2 Derivatives and Rules of Differentiation

2.1 Introduction. Definition and Interpretations.

2.2 Rules or Differentiation

- 2.2.1 Basic rules.
- 2.2.2 Product and quotient rules; chain rule;
- 2.2.3 Derivatives of elementary functions: exponential, trigonometric, and logarithmic.

2.3 Implicit Differentiation. Logarithmic Differentiation.

2.4 Derivatives of inverse trigonometric functions

2.5 Higher-order derivatives.

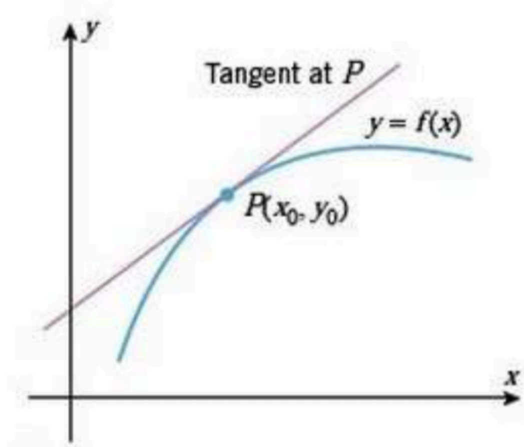
MODULE 2.1 - 2.2.1

MODULE 2.1 , 2.2.1
THE DERIVATIVE AND THE BASIC RULES OF DIFFERENTIATION

The Derivative

The TANGENT LINE problem

- Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at point P



The Derivative

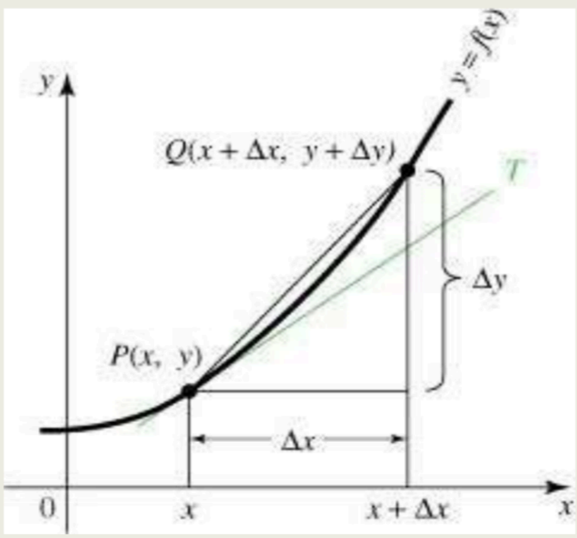
- **Derivative** is the mathematical tool for studying the rate at which one quantity changes relative to another quantity changes relative to another.

- The study of the rates of change is closely related to a concept of a tangent line to a curve.

- **Differentiation** is the process of finding the derivative to a function at a given point.

Aka "rate of change" and "gradient"

(PRE) DEFINITION



https://www.geogebra.org/m/CxxpzcbH
https://www.geogebra.org/m/qHekNnSS

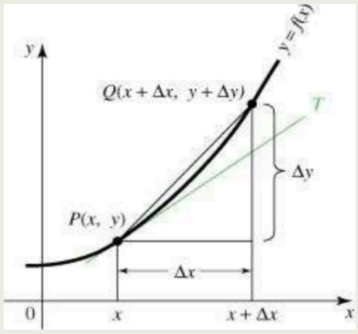
Slope of the secant line PQ:

- $m_{PQ} = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$
- $m_{PQ} = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$
- Measures the average rate of change in $f(x)$ over the interval Δx

As $\Delta x \to 0$, point Q approaches point P as it moves along the line, and the secant line PQ approaches the tangent Line T.

DEFINITION

DEFINITION



The **derivative** of function y with respect to x:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

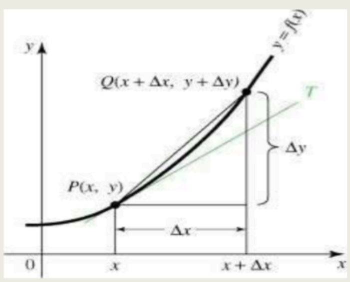
As $\Delta x \to 0$, point Q approaches point P as it moves along the line, and the secant line PQ approaches the tangent Line T.

The tangent line T is the *limiting* position of the secant line PQ as point Q

approaches point P ($\Delta x \to 0$)

The slope of the tangent line T: $\boldsymbol{m_T @ P(x,y) = \lim_{\Delta x \to 0} m_{PQ} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$

Applications of the Derivative.



The derivative of function y with respect to x:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- The derivative of function at a point gives the **slope of the tangent line** to the curve at this point

  Applications: plenty; any instantaneous rate of change of a quantity; measures the angle at which two curves intersect.

- Equation of the tangent line T to the curve at the point $\boldsymbol{P}(x_0, f(x_0))$ in the *point –slope form*:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

USEFUL

Applications of the Derivative.

The derivative of function $\boldsymbol{y}$ with respect to $\boldsymbol{x}$:

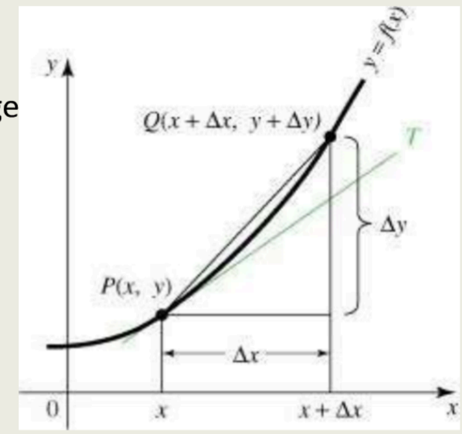
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The derivative of a function at a point represents the **instantaneous rate of change of** the **function** at this point.

Q&A

- True/False. The reading of a car speedometer shows the instantaneous speed of a moving vehicle.
- True/False. Most devices that measure current and voltage in circuits provide the accurate instantaneous value.

AVERAGE CHANGE vs INSTANTANEOUS CHANGE



Notations for the Derivative for the functions of a single variable

The derivative of function $y = u(x)$ may be written in any of the following ways:

OPERATOR

$$y', \quad u'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[u(x)] \quad D_x[u(x)]$$

A variable other than x may be used as *independent* variable.

If $g = g(t)$, then the derivative of g with respect to t:

$$g', \quad g'(t), \quad \frac{dg}{dt}, \quad \frac{d}{dt}[g(t)] \quad D_t[g(t)]$$

Computing Derivatives by the Definition

Let's Talk about Functions

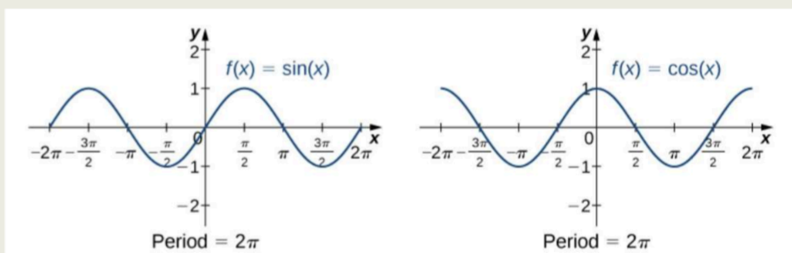
- Functions are elementary building blocks in modeling applications.
- Elementary functions are divided into classes or groups that share properties.
  - E.g. polynomial, rational, trigonometric, exponential, logarithmic.
- Compound functions are made up of simpler functions.
- New functions are made up of the known functions by arithmetic operations, by composition,

by differentiation and integration…

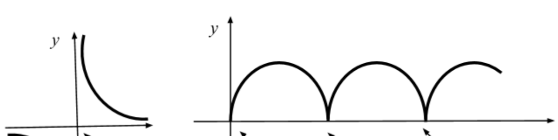
CONTINUITY AND DIFFERENTIABILITY

Differentiability – Existence of a Derivative

- A function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists. It is differentiable on open interval (a, b) if it is differentiable at every number in the interval.
- Functions that have the derivative at every point are continuous and are called **smooth functions** i.e. polynomial, exponential, power

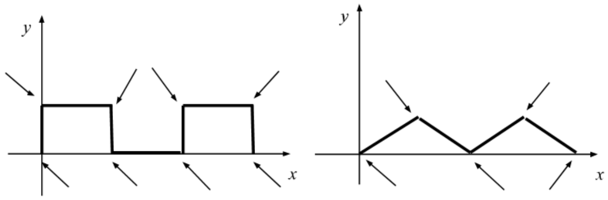


- If a function f is differentiable at a point, then it is continuous at this point.



The arrows show the points at which

the derivative does NOT exist.



The derivative DNE (does not exist)

- At the points of discontinuity of a function
- where the **curve has a jump, corner, cusp, or any other feature at which it is not possible to draw a tangent line**. At such points, we say that the function is *not differentiable*.

2.2.1 Basic Rules of Differentiation

1. **Constant Rule**: the derivative of a constant value is 0.
   For any constant value C: $\frac{d}{dx}[C] = 0$

EXAMPLE 1

EXAMPLE 2. (Self-check)

a)	$y = 15$, then $y' =$	d)	$u(x) = 10^{-5}, \quad u' = \frac{du}{dx} =$
b)	$y = -\sqrt{12}$, then $y' =$	e)	$s(t) = V \sin\frac{\pi}{6}\omega,$ Assume that V, ω are constants. $s' = \frac{ds}{dt} =$
c)	$\theta(t) = \frac{\pi}{2};$ then $\theta'(t) = \frac{d\theta}{dt} =$		

Rules of Differentiation

2. Power Rule: for any real number r,

$$\frac{d}{dx}[x^r] = rx^{r-1}.$$

Note. This is a specific function rule. It applies to the **power function**

$$y = x^r, r \in \mathbb{R}$$

$$output = (input)^r.$$

EXAMPLE 3
$y = x^2$, then $y' = 2x^{2-1} = 2x^1 = 2x$
$y = x^{-\frac{1}{2}}$, $\therefore y' = -\frac{1}{2}x^{-\frac{1}{2}-1} =$

Rules of Differentiation

3. Constant Times a Function: the derivative of a constant times a function equals the constant times the derivative of the function.
If C is a constant, then $\frac{d}{dx}[Cu(x)] = C\frac{d}{dx}[u(x)]$

EXAMPLE 4
$y = 4x^3$, then $y' = 4\frac{d}{dx}[x^3] = 4(3x^2) = 12x^2$

EXAMPLE 5. (Self-check)

a)	$y = 6x^{\frac{1}{3}}$, then $y' =$	e)	$y = 0.025t^{-0.4}$, then $y' = \frac{dy}{dt} =$
b)	$y = 5\sqrt{x}$, $y' =$	f)	$u = 5x$, then $u' =$ $5(1) = 5$
c)	$y = \frac{1}{2x^2}$, then $y' =$		
d)	$y = \frac{1}{-3\sqrt{}}$; then y'=		

	$7\sqrt{x}$		

Rules of Differentiation

4. Sum and Difference Rules: If u & v are functions of x, then

$$\frac{d}{dx}[u \pm v] = \frac{d}{dx}[u] \pm \frac{d}{dx}[v]$$

or $[u \pm v]' = u' \pm v'$

EXAMPLE 6
Differentiate the polynomial function: $y = 5x^3 - 4x^2 + 12x - 8$ $y' = 5(3)x^2 - 4(2)x + 12$

Rules of Differentiation

EXAMPLE 7. (Self-check) Combine the rules to find the derivatives.

a) Find $\frac{d}{dx}[x^{-2} + 6x^3 + 7] =$

b) Find $\frac{d}{dx}[2x^5 - 3x^{-7} + x + 2\sqrt{x}] =$ $\frac{d}{dx}\left[2x^5 - 3x^{-7} + x + 2x^{\frac{1}{2}}\right]$

$= 10x^4 + 21x^{-8} + 1 + 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$

$= 10x^4$

EXAMPLE 8. Calculate derivatives.

a) $y = \frac{x^3+1}{x}$

b) $f(x) = \frac{x^4+3\sqrt{x}}{x}$

Hint. It is often useful to manipulate an expression algebraically prior to differentiation. In this case, divide the numerator through by the denominator and reduce the resulting fractions prior to differentiation.

a) $y = \frac{x^3}{x} + \frac{1}{x} = x^2 + x^{-1} \quad \rightarrow \quad y' = 2x - x^{-2}$

b) $f(x) = \frac{x^4+3\sqrt{x}}{x} = \frac{x^4}{x} + \frac{3\sqrt{x}}{x} = x^3 + 3x^{-\frac{1}{2}}$

Then, $f'(x) = 3x^2 - \frac{3}{2}x^{-\frac{3}{2}}$

Evaluating Derivatives at the Given Values

EXAMPLE 9. If $f(x) = 7 - 4x^2$, find $f'(1)$ and $f'(-3)$

Solution:

Find the derivative first: $f'(x) = -8x$.

Treat the derivative as a function of x to be evaluated for the given values

For $x = 1$ $\quad f'(1) = -8(1) = -8$; and for $x = -3$ $\quad f'(-3) = -8(-3) = 24$

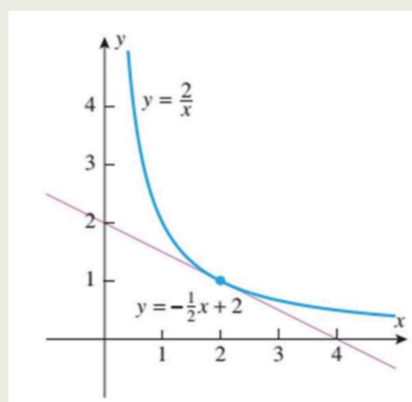
Note,
The derivative of a function is a function. Differentiation results in a function.
Alternative notations may be used in applications: for any $x = x_0$: $\quad f'(x_0) = \left.\frac{df}{dx}\right|_{x=x_0}$

Tangent Line to a Curve

EXAMPLE 10. Find an equation to the tangent line to the curve $y = \frac{2}{x}$ at the point (2,1) on this curve.

Hint. Equation of the tangent line T to the curve at the point $\boldsymbol{P}\big(x_0, f(x_0)\big)$ in the *point –slope form*:

$$y = f(x_0) + f'(x_0)(x - x_0)$$



https://openstax.org/books/calculus-volume-1/pages/4-2-linear-approximations-and-differentials