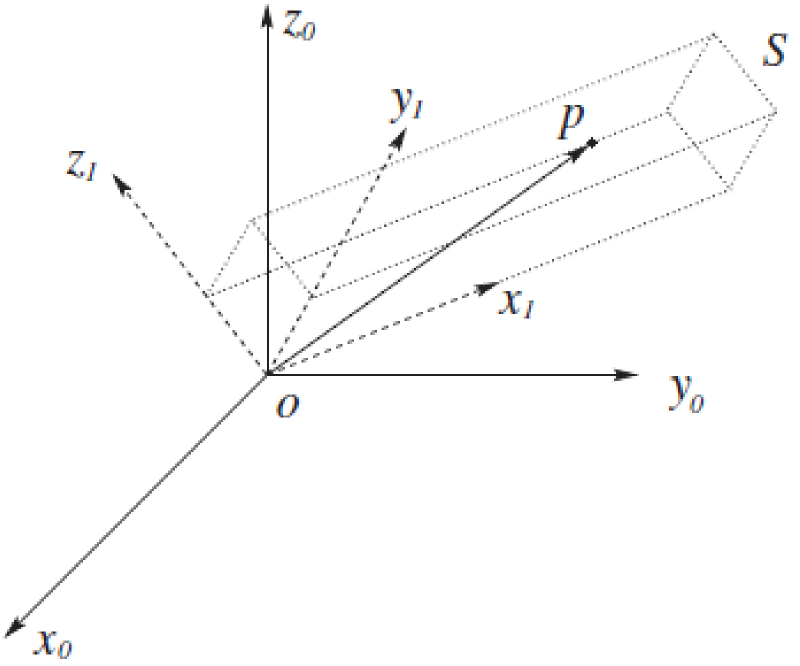
Kinematics and Dynamics of Robots

Module 3

Rotational Transformation

- consider a rigid object S to which a coordinate frame $o_1x_1y_1z_1$ is attached. Given the coordinates p^1 of the point p (in other words, given the coordinates of p with respect to the frame $o_1x_1y_1z_1$), we wish to determine the coordinates of p relative to a fixed reference frame $o_0x_0y_0z_0$. The coordinates $p^1 = (u, v, w)$ satisfy the equation

$$p^1 = ux_1 + vy_1 + wz_1$$



- In a similar way, we can obtain an expression for the coordinates p^0 by projecting the point p onto the coordinate axes of the frame $o_0x_0y_0z_0$, giving

$$p^0 = \begin{bmatrix} p \cdot x_0 \\ p \cdot y_0 \\ p \cdot z_0 \end{bmatrix}$$

HUMBER

Combining these two equations we obtain

$$
\begin{aligned}
p^0 &= \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_0 \\ (ux_1 + vy_1 + wz_1) \cdot y_0 \\ (ux_1 + vy_1 + wz_1) \cdot z_0 \end{bmatrix} \\
&= \begin{bmatrix} ux_1 \cdot x_0 + vy_1 \cdot x_0 + wz_1 \cdot x_0 \\ ux_1 \cdot y_0 + vy_1 \cdot y_0 + wz_1 \cdot y_0 \\ ux_1 \cdot z_0 + vy_1 \cdot z_0 + wz_1 \cdot z_0 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}
\end{aligned}
$$

- But the matrix in this final equation is merely the rotation matrix R_1^0, which leads to

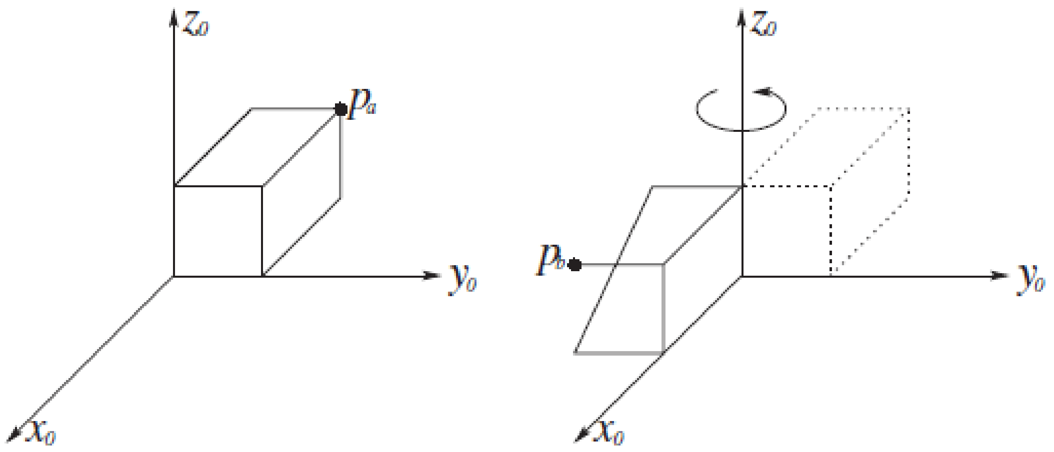
$$p^0 = R_1^0 p^1$$

- Thus, the rotation matrix R_1^0 can be used not only to represent the orientation of coordinate frame $o_1x_1y_1z_1$ with respect to frame $o_0x_0y_0z_0$, but also to transform the coordinates of a point from one frame to another. If a given point is expressed relative to $o_1x_1y_1z_1$ by coordinates p^1, then $R_1^0 p^1$ represents the **same point** expressed relative to the frame $o_0x_0y_0z_0$.

Pure Rotation

- We can also use rotation matrices to represent rigid motions that correspond to pure rotation. For example, in Figure (a) one corner of the block is located at the point p_a in space. Figure (b) shows the same block after it has been rotated about z_0 by the angle π. The same corner of the block is now located at point p_b in space. It is possible to derive the coordinates for p_b given only the coordinates for p_a and the rotation matrix that corresponds to the rotation about z_0. To see how this can be accomplished, imagine that a coordinate frame is rigidly attached to the block in Figure (a), such that it is coincident with the frame $o_0x_0y_0z_0$. After the rotation by π, the block's coordinate frame, which is rigidly attached to the block, is also rotated by π. If we denote this rotated frame by $o_1x_1y_1z_1$, we obtain

$$R_1^0 = R_{z,\pi} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



(a) (b)

Pure Rotation

- In the local coordinate frame o1x1y1z1, the point p_b has the coordinate representation p_b^1 To obtain its coordinates with respect to frame o0x0y0z0, we merely apply the coordinate transformation, giving

$$p_b^0 = R_{z,\pi} p_b^1$$

- It is important to notice that the local coordinates p_b^1 of the corner of the block do not change as the block rotates, since they are defined in terms of the block's own coordinate frame. Therefore, when the block's frame is aligned with the reference frame $o_0x_0y_0z_0$ (that is, before the rotation is performed), the coordinates p_b^1 equals p_a^0 , since before the rotation is performed, the point p_a is coincident with the corner of the block. Therefore, we can substitute p_a^0 into the previous equation to obtain

$$p_b^0 = R_{z,\pi} p_a^0$$

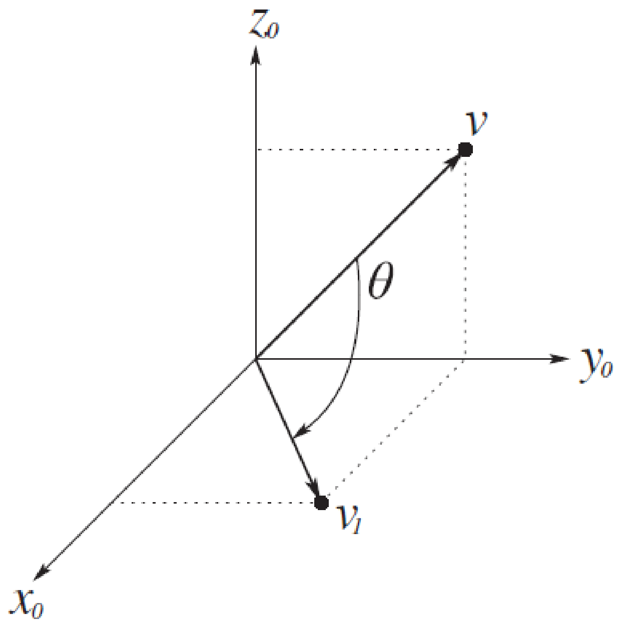
- This equation shows how to use a rotation matrix to represent a rotational motion. In particular, if the point p_b is obtained by rotating the point p_a as defined by the rotation matrix R, then the coordinates of p_b with respect to the reference frame are given by

$$p_b^0 = R p_a^0$$

- This same approach can be used to rotate vectors with respect to a coordinate frame.

Pure Rotation

- *The vector v with coordinates* $v^0 = (0, 1, 1)$ *is rotated about* y_0 *by* $\pi/2$ *as shown here. The resulting vector* v_1 *is given by*



Rotating a vector about axis y0.

Pure Rotation

- Thus, a third interpretation of a rotation matrix R is as an operator acting on vectors in a fixed frame. In other words, instead of relating the coordinates of a fixed vector with respect to two different coordinate frames, we can represent the coordinates in $o_0x_0y_0z_0$ of a vector v_1 that is obtained from a vector v by a given rotation.
- As we have seen, rotation matrices can serve several roles. A rotation matrix, either $R \in SO(3)$ or $R \in SO(2)$, can be interpreted in three distinct ways:

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.
2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.
3. It is an operator taking a vector and rotating it to give a new vector in the same coordinate frame.

Similarity Transformations

- A coordinate frame is defined by a set of **basis vectors**, for example, unit vectors along the three coordinate axes. This means that a rotation matrix, as a coordinate transformation, can also be viewed as defining a change of basis from one frame to another. The matrix representation of a general linear transformation is transformed from one frame to another using a so-called **similarity transformation**. For example, if A is the matrix representation of a given linear transformation in $o_0x_0y_0z_0$ and B is the representation of the same linear transformation in $o_1x_1y_1z_1$ then A and B are related as

$$B = (R_1^0)^{-1} A R_1^0$$

- where R_1^0 is the coordinate transformation between frames $o_1x_1y_1z_1$ and $o_0x_0y_0z_0$. In particular, if A itself is a rotation, then so is B, and thus the use of similarity transformations allows us to express the same rotation easily with respect to different frames.

- **Note:** *we use the shorthand notation* $c_\theta = \cos\theta$, $s_\theta = \sin\theta$ *for trigonometric functions, afterwards.*

Similarity Transformations

- *Suppose frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the rotation*

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

- *If $A = R_{z,\theta}$ relative to the frame $o_0x_0y_0z_0$, then, relative to frame $o_1x_1y_1z_1$ we have*

$$B \quad = \quad (R_1^0)^{-1} A R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

- *In other words,* ***B is a rotation about the z_0-axis but expressed relative to the frame $o_1x_1y_1z_1$***.

Composition of Rotations

- Recall that the matrix $p^0 = R_1^0 p^1$ represents a rotational transformation between the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$. Suppose we now add a third coordinate frame $o_2x_2y_2z_2$ related to the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ by rotational transformations. A given point p can then be represented by coordinates specified with respect to any of these three frames: p^0, p^1, and p^2. The relationship among these representations of p is

$$\begin{aligned} p^0 &= R_1^0 p^1 \\ p^1 &= R_2^1 p^2 \\ p^0 &= R_2^0 p^2 \end{aligned}$$

- where each R_j^i is a rotation matrix. Substituting Equation gives

$$p^0 = R_1^0 R_2^1 p^2$$

- Note that R_1^0 and R_2^0 represent rotations relative to the frame $o_0x_0y_0z_0$while R_2^1 represents a rotation relative to the frame $o_1x_1y_1z_1$. Comparing Equations we can immediately infer

$$R_2^0 = R_1^0 R_2^1$$

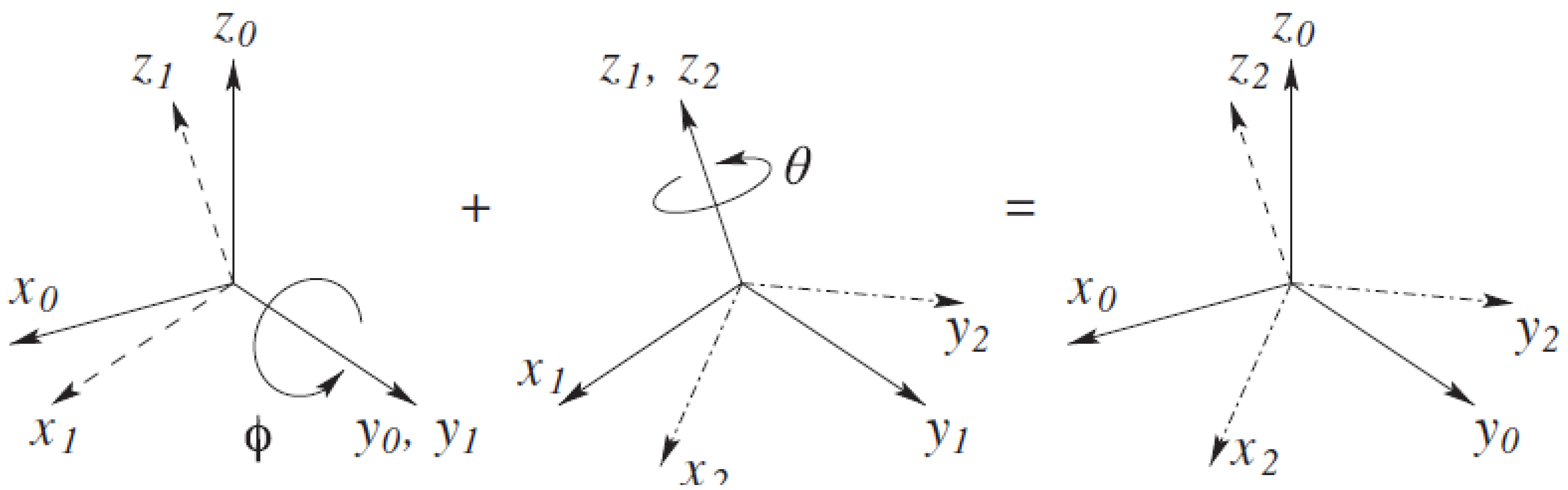
- This Equation is the **composition law** for rotational transformations

Composition of Rotations

- The composition law for rotational transformations states that, in order to transform the coordinates of a point p from its representation p^2 in the frame $o_2x_2y_2z_2$ to its representation p^0 in the frame $o_0x_0y_0z_0$, we may first transform to its coordinates p^1 in the frame $o_1x_1y_1z_1$ using R_2^1 and thentransform p^1 to p^0 using R_1^0 .
- Suppose that initially all three of the coordinate frames coincide. We first rotate the frame $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ according to the transformation R_1^0 . Then, with the frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ coincident, we rotate $o_2x_2y_2z_2$ relative to $o_1x_1y_1z_1$ according to the transformation R_2^1 . The resulting frame, $o_2x_2y_2z_2$ has orientation with respect to $o_0x_0y_0z_0$ given by $R_1^0R_2^1$. We call the frame relative to which the rotation occurs the **current frame**.

Composition of Rotations

- *Suppose a rotation matrix R represents a rotation of angle ϕ about the current y-axis followed by a rotation of angle θ about the current z-axis as shown here. Then the matrix R is given by*

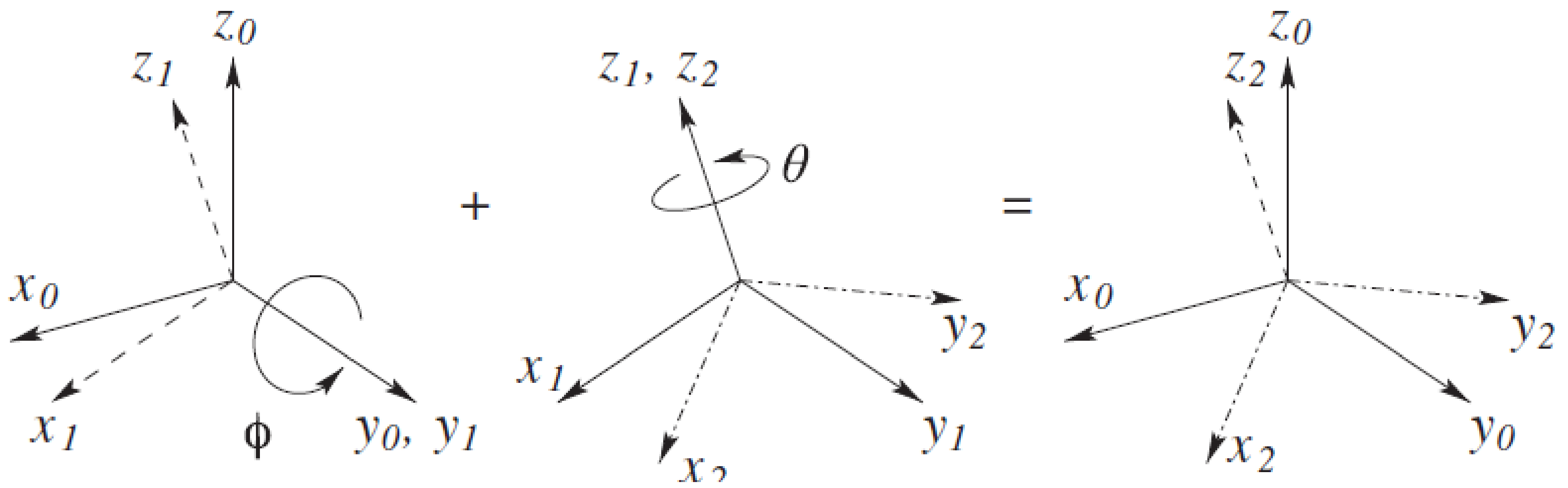


Composition of Rotations

- It is important to remember that the order in which a sequence of rotations is performed, and consequently the order in which the rotation matrices are multiplied together, is crucial. The reason is that rotation, unlike position, is not a vector quantity and so rotational transformations do not commute in general.

Composition of Rotations

- *Suppose that the above rotations are performed in the reverse order, that is, first a rotation about the current z-axis followed by a rotation about the current y-axis. Then the resulting rotation matrix is given by*



Questions?