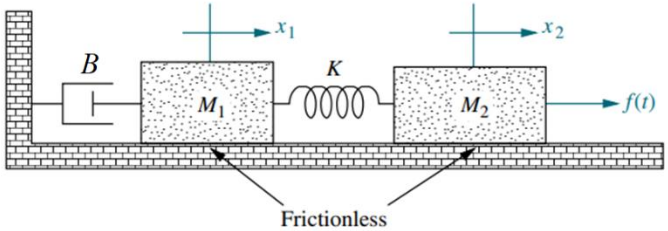
Worksheet 2 - Solution

PART 1: State-Space Modelling

1) Find the state-variale equations and output equations for the following translational mechanical systems.

a) Input is applied force $f(t)$ and the output is the relative displacement of masses M_1 and M_2.



First, write the equations of motion for mass M_1 and M_2.

$$Mass \quad M_1 \quad \rightarrow \quad -B\dot{x}_1 - K(x_1 - x_2) = M_1\ddot{x}_1$$

$$Mass \quad M_2 \quad \rightarrow \quad f(t) - K(x_2 - x_1) = M_2\ddot{x}_2$$

Define the state variables as the velocity of the mass M_1 and M_2, and displacement of spring K:

$$q_1 = \dot{x}_1$$

$$q_2 = \dot{x}_2$$

$$q_3 = x_2 - x_1$$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{q}_1 = \ddot{x}_1 \quad \rightarrow \quad \dot{q}_1 = \frac{1}{M_1}\left(-B\dot{x}_1 - K(x_1 - x_2)\right) = \frac{1}{M_1}(-Bq_1 + Kq_3)$$

$$\dot{q}_2 = \ddot{x}_2 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{M_2}\left(f(t) - K(x_2 - x_1)\right) = \frac{1}{M_2}(f(t) - Kq_3)$$

$$\dot{q}_3 = \dot{x}_2 - \dot{x}_1 \quad \rightarrow \quad \dot{q}_3 = q_2 - q_1$$

Find the output in terms of the state variables and the input.

$$y = x_2 - x_1 \quad \rightarrow \quad y = q_3$$

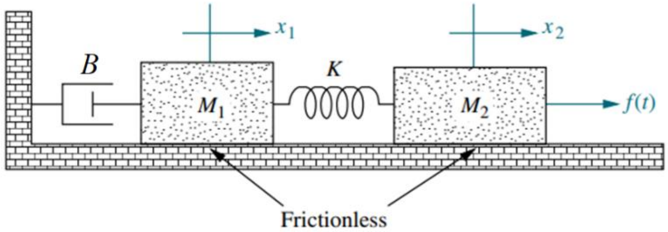
The system model has 3 state variables, 1 input, and 1 output.

Form the <u>state equation</u> and the <u>output equation</u> in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} -B/M_1 & 0 & K/M_1 \\ 0 & 0 & -K/M_2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M_2 \\ 0 \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + [0]f(t)$$

b) Input is applied force $f(t)$ and the outputs are the displacement of masses M_1 and M_2.



First, write the equations of motion for mass M_1 and M_2.

$$Mass\ \ M_1 \quad \rightarrow \quad -B\dot{x}_1 - K(x_1 - x_2) = M_1\ddot{x}_1$$

$$Mass\ \ M_2 \quad \rightarrow \quad f(t) - K(x_2 - x_1) = M_2\ddot{x}_2$$

NOTE: Since the displacement of the masses are defined as the output variables, we have to define the x_1 and x_2 separately as the state variables.

Define the state variables as the velocity of the mass M_1 and M_2, and displacement of each end of spring K:

$$q_1 = \dot{x}_1$$

$$q_2 = \dot{x}_2$$

$$q_3 = x_1$$

$$q_4 = x_2$$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{q}_1 = \ddot{x}_1 \quad \rightarrow \quad \dot{q}_1 = \frac{1}{M_1}\big(-B\dot{x}_1 - K(x_1 - x_2)\big) = \frac{1}{M_1}(-Bq_1 - Kq_3 + Kq_4)$$

$$\dot{q}_2 = \ddot{x}_2 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{M_2}\big(f(t) - K(x_2 - x_1)\big) = \frac{1}{M_2}(f(t) - Kq_4 + Kq_3)$$

$$\dot{q}_3 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_3 = q_1$$

$$\dot{q}_4 = \dot{x}_2 \quad \rightarrow \quad \dot{q}_4 = q_2$$

Find the outputs in terms of the state variables and the input.

$$y_1 = x_1 \quad \rightarrow \quad y_1 = q_3$$

$$y_2 = x_2 \quad \rightarrow \quad y_2 = q_4$$

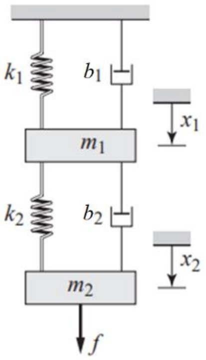
The system model has 4 state variables, 1 input, and 2 outputs.

Form the <u>state equation</u> and the <u>output equation</u> in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \end{bmatrix} = \begin{bmatrix} -B/M_1 & 0 & -K/M_1 & K/M_1 \\ 0 & 0 & K/M_2 & -K/M_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M_2 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} f(t)$$

c) Input is applied force $f(t)$ and the output is the displacement of masses m_1 and m_2.



First, write the equations of motion for mass m_1 and m_2.

$$Mass\ \ m_1 \quad \rightarrow \quad -k_1x_1 - k_2(x_1 - x_2) - b_1\dot{x}_1 - b_2(\dot{x}_1 - \dot{x}_2) = m_1\ddot{x}_1$$

$$Mass\ \ m_2 \quad \rightarrow \quad f(t) - k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) = m_2\ddot{x}_2$$

Define the state variables as the velocity of the mass m_1 and m_2, and displacement of springs k_1 and k_2:

$$q_1 = \dot{x}_1$$

$$q_2 = \dot{x}_2$$

$$q_3 = x_1$$

$$q_4 = x_2 - x_1$$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{q}_1 = \ddot{x}_1 \quad \rightarrow \quad \dot{q}_1 = \frac{1}{m_1}(-k_1x_1 - k_2(x_1 - x_2) - b_1\dot{x}_1 - b_2(\dot{x}_1 - \dot{x}_2))$$

$$\dot{q}_1 = \frac{1}{m_1}(-k_1q_3 + k_2q_4 - (b_1 + b_2)q_1 + b_2q_2)$$

$$\dot{q}_2 = \ddot{x}_2 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{m_2}\big(f(t) - k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1)\big) = \frac{1}{m_2}(f(t) - k_2q_4 - b_2q_2 + b_2q_1)$$

$$\dot{q}_3 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_3 = q_1$$

$$\dot{q}_4 = \dot{x}_2 - \dot{x}_1 \quad \rightarrow \quad \dot{q}_4 = q_2 - q_1$$

Find the outputs in terms of the state variables and the input.

$$y_1 = x_1 \quad \rightarrow \quad y_1 = q_3$$

$$y_2 = x_2 \quad \rightarrow \quad y_2 = q_4 - q_3$$

The system model has 4 state variables, 1 input, and 2 outputs.

Form the state equation and the output equation in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \end{bmatrix} = \begin{bmatrix} -(b_1+b_2)/m_1 & b_2/m_1 & -k_1/m_1 & k_2/m_1 \\ b_2/m_2 & -b_2/m_2 & 0 & -k_2/m_2 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m_2 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} f(t)$$

2) Obtain the state-space model for the following differential equation models:

a) $6\ddot{y}(t) + 4\dot{y}(t) + 9y(t) = 7f(t)$. The input is $f(t)$ and the output is $y(t)$.

The state variales are $q_1(t) = y(t)$ and $q_2(t) = \dot{y}(t)$

Given the state variables:

$q_1(t) = y(t)$

$q_2(t) = \dot{y}(t)$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$\dot{q}_1(t) = \dot{y}(t) \quad \rightarrow \quad \dot{q}_1(t) = q_2(t)$

$\dot{q}_2(t) = \ddot{y}(t) \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{6}(7f(t) - 4\dot{y}(t) - 9y(t)) = \frac{7}{6}f(t) - \frac{2}{3}q_2(t) - \frac{3}{2}q_1(t)$

Find the output in terms of the state variables and the input.

$y(t) = q_1(t)$

The system model has 2 state variables, 1 input, and 1 output.

Form the state equation and the output equation in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3}{2} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{7}{6} \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0]f(t)$$

b) $3\ddot{y}(t) + 5\dot{y}(t) + 2y(t) = 7f(t)$. The input is $f(t)$ and the output is $y(t)$.

The state variales are $q_1(t) = y(t)$ and $q_2(t) = \dot{y}(t)$

Given the state variables:

$q_1(t) = y(t)$

$q_2(t) = \dot{y}(t)$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$\dot{q}_1(t) = \dot{y}(t) \quad \rightarrow \quad \dot{q}_1(t) = q_2(t)$

$\dot{q}_2(t) = \ddot{y}(t) \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{3}(7f(t) - 5\dot{y}(t) - 2y(t)) = \frac{7}{3}f(t) - \frac{5}{3}q_2(t) - \frac{2}{3}q_1(t)$

Find the output in terms of the state variables and the input.

$y(t) = q_1(t)$

The system model has 2 state variables, 1 input, and 1 output.

Form the <u>state equation</u> and the <u>output equation</u> in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{3} & -\frac{5}{3} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{7}{3} \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0]f(t)$$

c) $4\ddot{z}(t) + 2\dot{z}(t) + 9z(t) = f(t) + 7g(t)$. The inputs are $f(t)$ and $g(t)$ and the output is $\dot{z}(t)$.

The state variales are $q_1(t) = z(t)$ and $q_2(t) = \dot{z}(t)$

Given the state variables:

$q_1(t) = z(t)$

$q_2(t) = \dot{z}(t)$

Find the first derivative of the state variable and rewrite them in terms of the state variables and the input.

$\dot{q}_1(t) = \dot{z}(t) \quad \rightarrow \quad \dot{q}_1(t) = q_2(t)$

$\dot{q}_2(t) = \ddot{z}(t) \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{4}(f(t) + 7g(t) - 2\dot{z}(t) - 9z(t)) = \frac{1}{4}f(t) + \frac{7}{4}g(t) - \frac{1}{2}q_2(t) - \frac{9}{4}q_1(t)$

Find the output in terms of the state variables and the input.

$y(t) = \dot{z}(t) = q_2(t)$

The system model has 2 state variables, 2 inputs, and 1 output.

Form the state equation and the output equation in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{9}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

d) $4\frac{d^3z(t)}{dt^3} + 6\frac{d^2z(t)}{dt^2} + 3\frac{dz(t)}{dt} + 8z(t) = 2f(t)$. The input is $f(t)$ and the output is $5z(t) - 2\dot{z}(t)$

The state variales are $q_1(t) = z(t),\ q_2(t) = \dot{z}(t)\ and\ q_3(t) = \ddot{z}(t)$

Given the state variables:

$q_1(t) = z(t)$

$q_2(t) = \dot{z}(t)$

$q_3(t) = \ddot{z}(t)$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$\dot{q}_1(t) = \dot{z}(t) \quad \rightarrow \quad \dot{q}_1(t) = q_2(t)$

$\dot{q}_2(t) = \ddot{z}(t) \quad \rightarrow \quad \dot{q}_2(t) = q_3(t)$

$\dot{q}_3(t) = \dddot{z}(t) \quad \rightarrow \quad \dot{q}_3(t) = \frac{1}{4}(2f(t) - 6\ddot{z}(t) - 3\dot{z}(t) - 8z(t)) = \frac{1}{2}f(t) - \frac{3}{2}q_3(t) - \frac{3}{4}q_2(t) - 2q_1(t)$

Find the output in terms of the state variables and the input.

$y(t) = 5z(t) - 2\dot{z}(t) = 5q_1(t) - 2q_2(t)$

The system model has 3 state variables, 1 input, and 1 output.

Form the state equation and the output equation in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -\frac{3}{4} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 5 & -2 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + [0]f(t)$$

3) Obtain the state-space model for the two-mass system whose equations of motion are:

$$\begin{cases} m_1\ddot{y}_1 + k_1(y_1 - y_2) = f(t) \\ m_2\ddot{y}_2 - k_1(y_1 - y_2) + k_2 y_2 = 0 \end{cases}$$

Having input $f(t)$ and output $y = y_1$. Define the state variables as $q_1 = y_1, q_2 = \dot{y}_1, q_3 = y_2,$ and $q_4 = \dot{y}_2$.

Define the state variables:

$q_1 = y_1$

$q_2 = \dot{y}_1$

$q_3 = y_2$

$q_4 = \dot{y}_2$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$$\dot{q}_1 = \dot{y}_1 \quad \rightarrow \quad \dot{q}_1 = q_2$$

$$\dot{q}_2 = \ddot{y}_1 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{m_1}(f(t) - k_1 y_1 + k_1 y_2) = \frac{1}{m_1}f(t) - \frac{k_1}{m_1}q_1 + \frac{k_1}{m_1}q_3$$

$$\dot{q}_3 = \dot{y}_2 \quad \rightarrow \quad \dot{q}_3 = q_4$$

$$\dot{q}_4 = \ddot{y}_2 \quad \rightarrow \quad \dot{q}_4 = \frac{1}{m_2}(k_1 y_1 - k_1 y_2 - k_2 y_2) = \frac{1}{m_2}(k_1 y_1 - (k_1 + k_2) y_2) = \frac{k_1}{m_2}q_1 - \left(\frac{k_1 + k_2}{m_2}\right)q_3$$

Find the output in terms of the state variables and the input.

$y = y_1 = q_1$

The system model has 4 state variables, 1 input, and 1 output.

Form the <u>state equation</u> and the <u>output equation</u> in the standard matrix-vector form.

$$\textit{State Equation} \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & 0 & \frac{k_1}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & 0 & -\left(\frac{k_1 + k_2}{m_2}\right) & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$\textit{Output Equation} \quad \rightarrow \quad y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + [0] f(t)$$

4) Obtain the state-space model for the two-mass system whose equation of motion for specific values of the mass, spring and damping conatans are:

$$\begin{cases} 15\ddot{y}_1 + 7\dot{y}_1 - 4\dot{y}_2 + 30y_1 - 15y_2 = 0 \\ 6\ddot{y}_2 - 15y_1 + 15y_2 - 4\dot{y}_1 + 4\dot{y}_2 = f(t) \end{cases}$$

Having input $f(t)$ and output $y = \dot{y}_1 - \dot{y}_2$. Define the state variables as $q_1 = y_1, q_2 = \dot{y}_1, q_3 = y_2,$ and $q_4 = \dot{y}_2$.

Define the state variables:

$q_1 = y_1$

$q_2 = \dot{y}_1$

$q_3 = y_2$

$q_4 = \dot{y}_2$

Find the first derivative of the state variables and rewrite them in terms of the state variables and the input.

$\dot{q}_1 = \dot{y}_1 \quad \rightarrow \quad \dot{q}_1 = q_2$

$\dot{q}_2 = \ddot{y}_1 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{15}(-7\dot{y}_1 + 4\dot{y}_2 - 30y_1 + 15y_2) = -\frac{7}{15}q_2 + \frac{4}{15}q_4 - 2q_1 + q_3$

$\dot{q}_3 = \dot{y}_2 \quad \rightarrow \quad \dot{q}_3 = q_4$

$\dot{q}_4 = \ddot{y}_2 \quad \rightarrow \quad \dot{q}_4 = \frac{1}{6}\left(f(t) + 15y_1 - 15y_2 + 4\dot{y}_1 - 4\dot{y}_2\right) = \frac{1}{6}f(t) + \frac{5}{2}q_1 - \frac{5}{2}q_3 + \frac{2}{3}q_2 - \frac{2}{3}q_4$

Find the output in terms of the state variables and the input.

$y = \dot{y}_1 - \dot{y}_2 = q_2 - q_4$

The system model has 4 state variables, 1 input, and 1 output.

Form the <u>state equation</u> and the <u>output equation</u> in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -\frac{7}{15} & 1 & \frac{4}{15} \\ 0 & 0 & 0 & 1 \\ \frac{5}{2} & \frac{2}{3} & -\frac{5}{2} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{6} \end{bmatrix} f(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + [0]f(t)$$

5) Given the state equations and the output equations, obtain the expressions for the matrices **A, B, C**, and **D**.

a) $\begin{cases} \dot{q}_1 = -6q_1 + 4q_2 + 7u_1 \\ \dot{q}_2 = -5q_2 + 9u_2 \end{cases}$

$\begin{cases} y_1 = q_1 + 4q_2 + 7u_1 \\ y_2 = q_2 \end{cases}$

The system model has 2 state variables, 2 inputs, and 2 outputs.

Form the state equation and the output equation in the standard matrix-vector form.

$$State\ Equation: \ \dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t) \ \rightarrow \ \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$Output\ Equation: \ \mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t) \ \rightarrow \ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The **A, B, C**, and **D** matrices are:

$$\mathbf{A} = \begin{bmatrix} -6 & 4 \\ 0 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$$

b) $\begin{cases} \dot{q}_1 = -7q_1 + 4q_2 \\ \dot{q}_2 = -3q_2 + 8u \end{cases}$

$\begin{cases} y_1 = q_1 \\ y_2 = q_2 \end{cases}$

The system model has 2 state variables, 1 input, and 2 outputs.

Form the state equation and the output equation in the standard matrix-vector form.

$$State\ Equation: \ \dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t) \ \rightarrow \ \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} u(t)$$

$$Output\ Equation: \ \mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t) \ \rightarrow \ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t)$$

The **A, B, C**, and **D** matrices are:

$$\mathbf{A} = \begin{bmatrix} -7 & 4 \\ 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

c) $\begin{cases} \dot{q}_1 = -7q_1 + 5q_2 + 3u_1 \\ \dot{q}_2 = -9q_2 + 2u_2 \end{cases}$

$\{y = q_1$

The system model has 2 state variables, 2 inputs, and 1 output.

Form the <u>state equations</u> and the <u>output equations</u> in the standard matrix-vector form.

$$State\ Equation{:}\quad \dot{\mathbf{q}}(t) = \mathbf{Aq}(t) + \mathbf{Bu}(t) \ \rightarrow\ \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -7 & 5 \\ 0 & -9 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$Output\ Equation{:}\quad \mathbf{y}(t) = \mathbf{Cq}(t) + \mathbf{Du}(t) \ \rightarrow\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The **A, B, C**, and **D** matrices are:

$$\mathbf{A} = \begin{bmatrix} -7 & 5 \\ 0 & -9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

a) $\begin{cases} \dot{q}_1 = -7q_1 + 9q_2 - 2q_3 + 3u_1 \\ \dot{q}_2 = -5q_2 + 4q_3 + 2u_2 \\ \dot{q}_3 = q_1 + 7q_2 \end{cases}$

$\begin{cases} y_1 = q_1 + 7q_2 + 4u_1 \\ y_2 = q_2 - 5q_3 \end{cases}$

The system model has 3 state variables, 2 inputs, and 2 outputs.

Form the <u>state equations</u> and the <u>output equations</u> in the standard matrix-vector form.

$$State\ Equation{:}\quad \dot{\mathbf{q}}(t) = \mathbf{Aq}(t) + \mathbf{Bu}(t) \ \rightarrow\ \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} -7 & 9 & -2 \\ 0 & -5 & 4 \\ 1 & 7 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$Output\ Equation{:}\quad \mathbf{y}(t) = \mathbf{Cq}(t) + \mathbf{Du}(t) \rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The **A, B, C**, and **D** matrices are:

$$\mathbf{A} = \begin{bmatrix} -7 & 9 & -2 \\ 0 & -5 & 4 \\ 1 & 7 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & -5 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

6) A linear dynamic system with input $u(t)$, output y, and state variables q_1 and q_2 is characterized by the equations:

$$\begin{cases} \dot{q}_1 - 2\dot{q}_2 = 3q_1 + 4q_2 - 5u(t) \\ \dot{q}_1 - \dot{q}_2 = 2q_1 + q_2 + u(t) \\ y = \dot{q}_1 + 2q_2 \end{cases}$$

Find the state equation and output equation in the standard form.

Find the first derivative of the state variables in terms of the state variables and the input.

$$Eqn.\ (1) \quad \rightarrow \quad \dot{q}_1 = 2\dot{q}_2 + 3q_1 + 4q_2 - 5u(t)$$

$$Eqn.\ (2) \quad \rightarrow \quad \dot{q}_2 = \dot{q}_1 - 2q_1 - q_2 - u(t)$$

Substitute $\dot{q}_2$ from Eqn. (2) into Eqn. (1), then find the $\dot{q}_1$:

$$\dot{q}_1 = 2\big(\dot{q}_1 - 2q_1 - q_2 - u(t)\big) + 3q_1 + 4q_2 - 5u(t) \quad \rightarrow \quad \dot{q}_1 = q_1 - 2q_2 + 7u(t)$$

Substitute $\dot{q}_1$ from Eqn. (1) into Eqn. (2), then find the $\dot{q}_2$:

$$\dot{q}_2 = 2\dot{q}_2 + 3q_1 + 4q_2 - 5u(t) - 2q_1 - q_2 - u(t) \quad \rightarrow \quad \dot{q}_2 = -q_1 - 3q_2 + 6u(t)$$

Find the output in terms of the state variables and the input.

$$y = \dot{q}_1 + 2q_2 = q_1 - 2q_2 + 7u(t) + 2q_2 = q_1 + 7u(t)$$

The system model has 2 state variables, 1 input, and 1 output.

Form the <u>state equations</u> and the <u>output equations</u> in the standard matrix-vector form.

$$State\ Equation \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 7 \\ 6 \end{bmatrix} u(t)$$

$$Output\ Equation \quad \rightarrow \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [7]u(t)$$

PART 2: Block Diagram Modelling

1) Draw block diagrams for each of the following input-output differential equation models, where $y(t)$ is the output, $f(t)$ is the input.

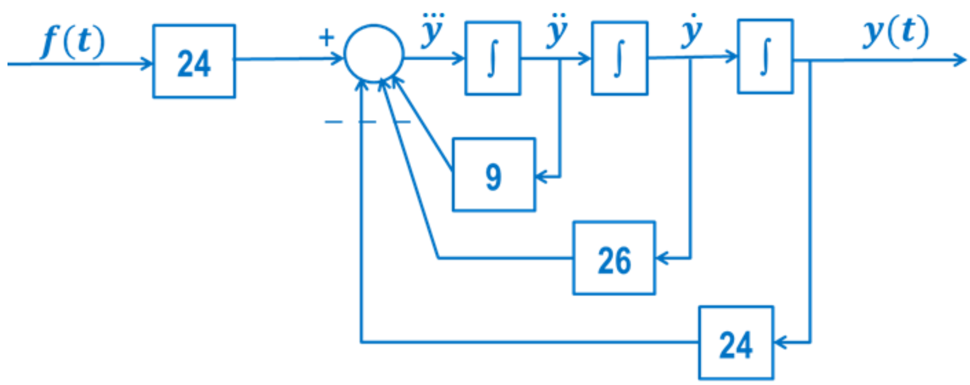
a) $\dddot{y}(t) + 9\ddot{y}(t) + 26\dot{y}(t) + 24y(t) = 24f(t)$

1 - The output variable is $y(t)$ and the input variable is $f(t)$.

2 - Solve the given equation for the highest derivative of the output variable.

$$\dddot{y}(t) = -9\ddot{y}(t) - 26\dot{y}(t) - 24y(t) + 24f(t)$$

3 - Create a sequence of output variables using integrator block and complete the diagram with addition/subtraction and gain blocks.



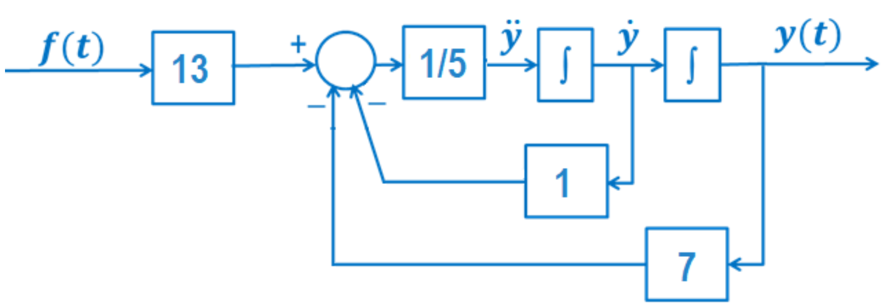
b) $5\ddot{y}(t) + \dot{y}(t) + 7y(t) = 13f(t)$

1 - The output variable is $y(t)$ and the input variable is $f(t)$.

2 - Solve the given equation for the highest derivative of the output variable.

$$\ddot{y}(t) = \frac{1}{5}\left(-\dot{y}(t) - 7y(t) + 13f(t)\right)$$

3 - Create a sequence of output variables using integrator block and complete the diagram with addition/subtraction and gain blocks.



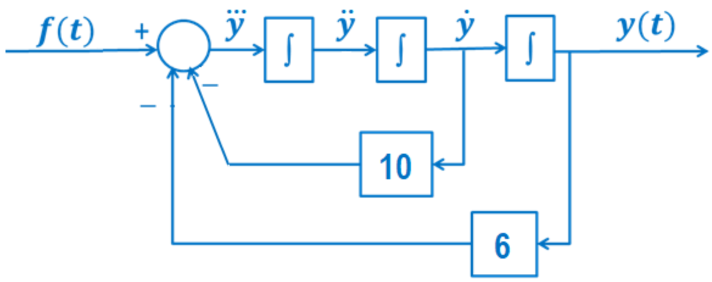
c) $\dddot{y}(t) + 10\dot{y}(t) + 6y(t) = f(t)$

1 - The output variable is $y(t)$ and the input variable is $f(t)$.

2 - Solve the given equation for the highest derivative of the output variable.

$$\dddot{y}(t) = -10\dot{y}(t) - 6y(t) + f(t)$$

3 - Create a sequence of output variables using integrator block and complete the diagram with addition/subtraction and gain blocks.



2) Draw block diagrams for each of the following sets of equations of motion, where $f(t)$ is the input.

a) $\begin{cases} M_1\ddot{x}_1 = -B\dot{x}_1 - K(x_1 - x_2) \\ M_2\ddot{x}_2 = f(t) - K(x_2 - x_1) \end{cases}$

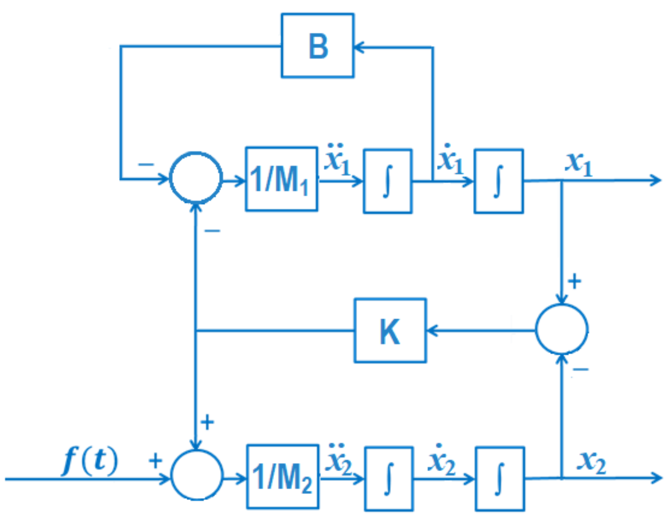
1 - The output variables are $x_1(t)$ and $x_2(t)$ and the input variable is $f(t)$.

2 - Solve the given equation for the highest derivative of the output variable.

$$\ddot{x}_1 = \frac{1}{M_1}\left(-B\dot{x}_1 - K(x_1 - x_2)\right)$$

$$\ddot{x}_2 = \frac{1}{M_2}\left(f(t) - K(x_2 - x_1)\right) = \frac{1}{M_2}\left(f(t) + K(x_1 - x_2)\right)$$

3 - Create a sequence of output variables using integrator block and complete the diagram with addition/subtraction and gain blocks.



b) $\begin{cases} M_1\ddot{x}_1 = -K_1x_1 - K_2(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2) \\ M_2\ddot{x}_2 = f(t) - K_2(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1) \end{cases}$

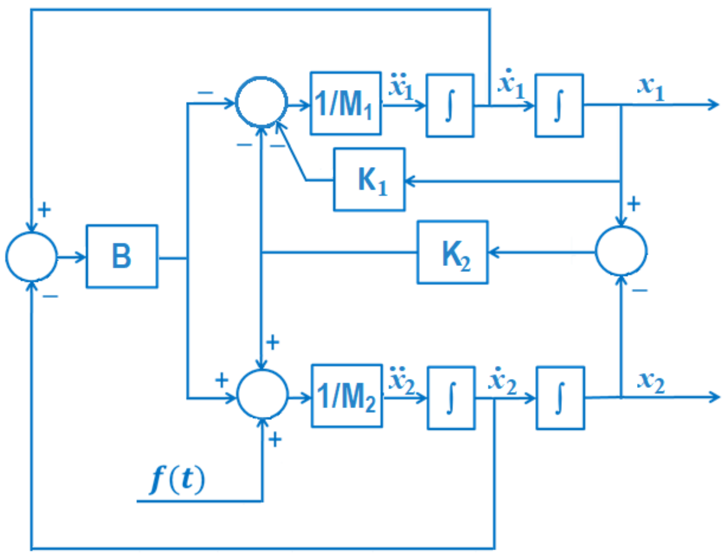
1 - The output variables are $x_1(t)$ and $x_2(t)$ and the input variable is $f(t)$.

2 - Solve the given equation for the highest derivative of the output variable.

$$\ddot{x}_1 = \frac{1}{M_1}\left(-K_1x_1 - K_2(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2)\right)$$

$$\ddot{x}_2 = \frac{1}{M_2}\left(f(t) - K_2(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1)\right) = \frac{1}{M_2}\left(f(t) + K_2(x_1 - x_2) + B(\dot{x}_1 - \dot{x}_2)\right)$$

3 - Create a sequence of output variables using integrator block and complete the diagram with addition/subtraction and gain blocks.

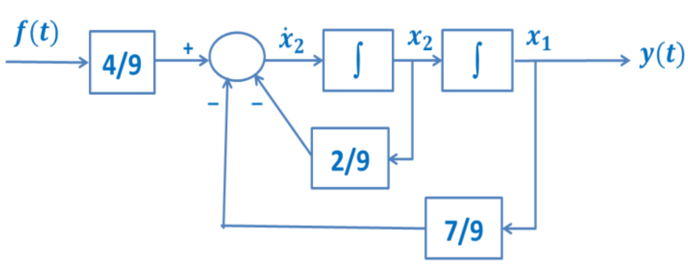


3) Draw block diagrams for each of the following sets of state-space equations, where $y(t)$ is the output, $f(t)$ is the input and x_1 *and* x_2 are the state variables.

a) $\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{4}{9}f(t) - \frac{2}{9}x_2(t) - \frac{7}{9}x_1(t) \\ y(t) = x_1(t) \end{cases}$

The system model has 2 state variables $x_1(t)$ and $x_2(t)$, 1 input $f(t)$, and 1 output $y(t)$.

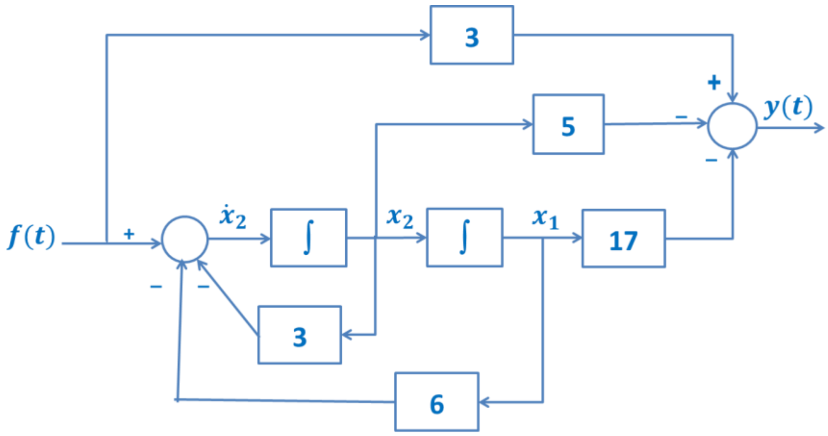
The block diagram to visualize the state variables, input, and output:



b) $\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -3x_2(t) - 6x_1(t) + f(t) \\ y(t) = -5x_2(t) - 17x_1(t) + 3f(t) \end{cases}$

The system model has 2 state variables $x_1(t)$ and $x_2(t)$, 1 input $f(t)$, and 1 output $y(t)$.

The block diagram to visualize the state variables, input, and output:



c) $\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -6x_2(t) - 5x_1(t) + f(t) \\ y(t) = 2x_2(t) + 9x_1(t) \end{cases}$

The system model has 2 state variables $x_1(t)$ and $x_2(t)$, 1 input $f(t)$, and 1 output $y(t)$.

The block diagram to visualize the state variables, input, and output: