Signal Processing (MENG3520)

Module 6

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MODULE 6

SYSTEM ANALYSIS USING Z TRANSFORM

Overview

- Every analysis method used in continuous time has a corresponding discrete time counterpart.
- The counterpart of the Laplace transform is the z-transform.
- The z-transform expresses DT signals as linear combination of DT complex exponential.
- The z-transform is critical in modern digital signal processing and system analysis because of the widely adaption of digital signals and systems.

Module Outline

- 6.1 Characterization of the DT LTI Systems using z-transform
- 6.2 System property analysis using z-transform
- 6.3 Block diagram of DT LTI systems

6.1

Characterization of LTI Systems Using z-Transform

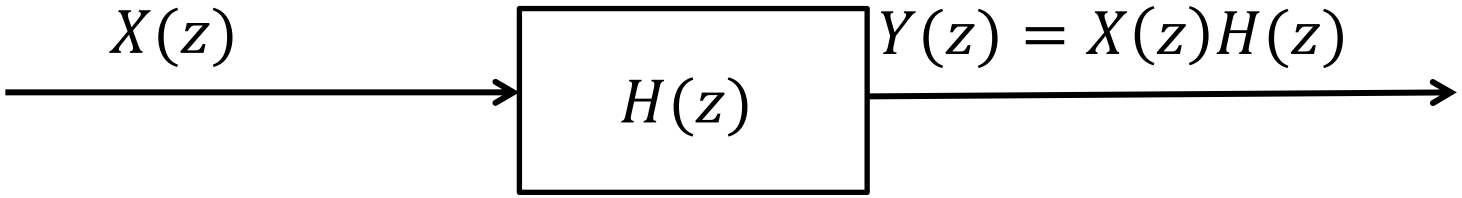
Transfer Function $\boldsymbol{H(z)}$: describes how the system "transfers" the excitation to the response.

Note: based on the convolution property of the z-transform, If the system is DT LTI, then through the z-transform, time-domain convolution becomes *z*-domain multiplication.

$$Y(z) = X(z)H(z)$$

Transfer function can be defined as:

$$H(z) = \frac{Y(z)}{X(z)}$$



The transfer function $\boldsymbol{H(z)}$ of a Nth-order DT LTI system defined by the following difference equation.

$$Q(E)y[n] = P(E)x[n] \qquad \textit{eq. 6.1}$$

Where operator notation E to represent operation for advancing a sequency by one time unit: $Ex[n] \equiv x[n+1];\ E^2x[n] \equiv x[n+2]; \cdots ; E^N x[n] \equiv x[n+N]$

The polynomials $Q(E)$ and $P(E)$ are:

$Q(E) = E^N + a_1\ E^{N-1} + \cdots + a_{N-1}E + a_N$

$P(E) =\ b_{N-M}\ E^M + b_{N-M+1}\ E^{M-1} + \cdots + b_{N-1}E + b_N \xRightarrow{let\ M=N} b_0\ E^N + b_1\ E^{N-1} + \cdots + b_{N-1}E + b_N$

Let’s derive the general expression for the zero-state response with all initial conditions to be zeros $y[-1] = y[-2] = \cdots = y[-N] = 0$. Also, excitation $x[n]$ is also causal.

Substitute $Q(E)$ and $P(E)$ back into equation 6.1:

$$(E^N + a_1\ E^{N-1} + \cdots + a_N)y[n] = (\,b_0\,E^N + b_1\ E^{N-1} + \cdots + b_N)x[n]$$

Apply E:

$$\begin{aligned} &y[n+N] + a_1 y[n+N-1] + \cdots + a_N y[n] \\ &= \ b_0\,x[n+N] + b_1 x[n+N-1] + \cdots + b_N x[n] \end{aligned}$$

Delay both sides by N:

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = \ b_0\,x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$

Recall that with all initial conditions to be zero: $y[-1] = y[-2] = \cdots = y[-N] = 0$.
Also, excitation $x[n]$ is also causal. Time–shift property of z-transform is:

- If: $x[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)$
- Then: $x[n-n_0] \overset{\mathcal{Z}}{\leftrightarrow} z^{-n_0} X(z)$

This means if we apply z-transform to this system:

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0\, x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$

$$Y(z) + a_1 z^{-1} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_N z^{-N} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_N}{z^N + a_1 z^{N-1} + \cdots + a_N} = \frac{P(z)}{Q(z)}$$

Conclusion: the transfer function of a system represented by a linear difference equation is rational.

Why do we use z-transform and the transfer function?

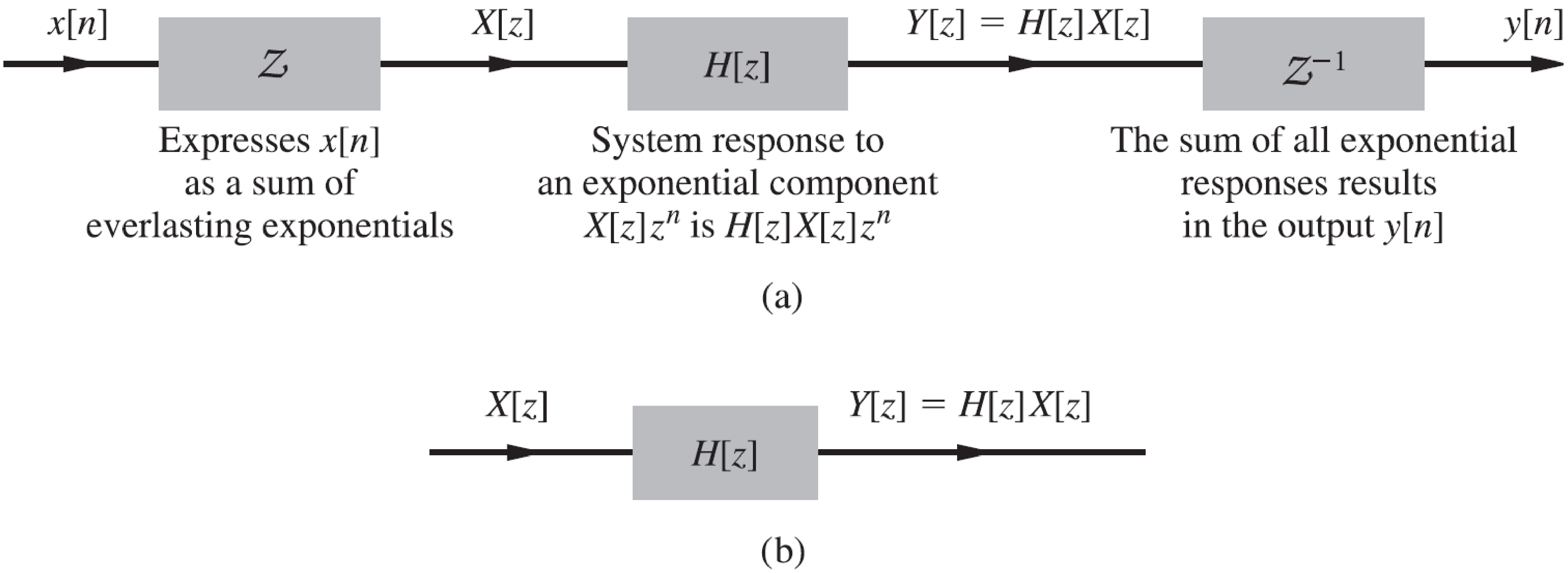


Figure 5.6 The transformed representation of an LTID system.

Textbook Example 5.6

Find the response y[n] described by the difference equation $y[n+2]+y[n+1]+0.16y[n]=x[n+1]+0.32x[n]$, here input $x[n]=(-2)^{-n}u[n]$. With all initial conditions zero.

Answer: from the differential equation, we have

$$H(z)=\frac{P(z)}{Q(z)}=\frac{(z+0.32)}{z^2+z+0.16}$$

For input $x[n]=(-2)^{-n}u[n]=(-0.5)^n u[n]$, its z-transform is:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}=\sum_{n=-\infty}^{\infty}(-0.5)^n u[n]z^{-n}=\sum_{n=0}^{\infty}(-0.5)^n z^{-n}=\frac{z}{z+0.5}$$

$$Y(z)=X(z)H(z)=\frac{z(z+0.32)}{(z^2+z+0.16)(z+0.5)}$$

$$\frac{Y(z)}{z}=\frac{(z+0.32)}{(z^2+z+0.16)(z+0.5)}=\frac{(z+0.32)}{(z+0.2)(z+0.8)(z+0.5)}=\frac{2/3}{\mathrm{z}+0.2}-\frac{8/3}{\mathrm{z}+0.8}+\frac{2}{\mathrm{z}+0.5}$$

$$Y(z)=\frac{2/3z}{\mathrm{z}+0.2}-\frac{8/3z}{\mathrm{z}+0.8}+\frac{2z}{\mathrm{z}+0.5}$$

$$y[n]=\frac{2}{3}(-0.2)^n-\frac{8}{3}(-0.8)^n+2(-0.5)^n$$

Frequency Response

Similar to its CT counterpart, we can derive frequency response through the transfer function sometimes. Recall the general format of DT complex exponential with z expressed in polar format:

$$z = re^{j\omega}$$

$$z^n = r^n e^{j\omega n} = r^n \cos(\omega n) + j r^n \sin(\omega n)$$

Thus, the impact of this transfer function at different frequencies is evaluated as:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = H(z)|_{r=1}$$

Frequency Response

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$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = H(z)|_{r=1}$$

Conclusion, the frequency response of a DT LTI system is the transfer function evaluated at **the unit circle where r=1**.

6.2

Systems Property Analysis Using z-Transform

System Properties

Previously, we have explored the properties of any systems, they are:

- Memory
- Causality
- Invertibility
- Stability
- Time invariance
- Linearity – additivity and homogeneity

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LTI System - Causality

LTI Causality Theorem 1. A DT LTI system with system function $H(z)$ is causal if and only if its impulse response $h[n] = 0$ for all $n < 0$.

LTI Causality Theorem 2: A DT LTI system with system function $H(z)$ is causal if and only if the ROC of $H(z)$ is:

The exterior of a circle, including $+\infty$.

LTI Causality Theorem 3: a DT LTI system with rational system function $H(z)$ is causal if and only if the order of the numerator of $H(z)$ is lower or equal to the order of the denominator and its ROC to be external to a circle.

LTI Causality Theorem 1. A DT LTI system with system function $H(z)$ is causal if and only if its impulse response $h[n] = 0$ for all $n < 0$.

Proof: $y[n]$ is causal if at any time n_0, $y[n_0]$ does not depend on its input at a time later than n_0. The value of output $y[n]$ at time n_0 is given by:

$$y[n_0] = x * h[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0 - k]$$

$$= \sum_{k=-\infty}^{n_0} x[k]h[n_0 - k] + \sum_{k=1+n_0}^{\infty} x[k]h[n_0 - k]$$

LTI Causality Theorem 1. A DT LTI system with system function $H(z)$ is causal if and only if its impulse response $h[n] = 0$ for all $n < 0$.

Proof: in order for

$$\sum_{k=1+n_0}^{\infty} x[k]h[n_0 - k]$$

to be independent of $x[k]$, $h[n_0 - k]$ has to be zero for all $k = 1 + n_0$ to $k = +\infty$, thus:

$$h[n] = 0, \text{for } n < 0$$

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LTI Causality Theorem 2: A DT LTI system with system function $H(z)$ is causal if and only if the ROC of $H(z)$ is:

The exterior of a circle, including $+\infty$.

Proof: a causal DT LTI system must satisfy

$$h[n] = 0, \qquad \text{for } n < 0$$

Also, because $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$, ROC will always include $+\infty$.

LTI Causality Theorem 3: a DT LTI system with rational system function $H(z)$ is causal if and only if the order of the numerator of $H(z)$ is lower or equal to the order of the denominator, and the ROC is the exterior of a circle.

Proof: a causal DT LTI system must satisfy

$$h[n] = 0, \qquad \text{for } n < 0$$

Also, because $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$, ROC must include $+\infty$. If $H(z)$ has higher order on the numerator compared to the denominator, then at $+\infty$, the $H(z)$ does not converge.

Activity: check the causality of the following DT LTI systems.

(a) $h[n] = \sin(\pi n/3)$

(b) $h[n] = \sin(\pi n/3)u[n]$

(c) $H(z) = \frac{2z^3}{(z-\frac{1}{2})(z-\frac{3}{4})}$ ROC $|z| < \frac{1}{2}$.

(d) $H(z) = \frac{2z^2}{(z-\frac{1}{2})(z-\frac{3}{4})}$ ROC $|z| < \frac{1}{2}$.

(e) $H(z) = \frac{2z^3}{(z-\frac{1}{2})(z-\frac{3}{4})}$ ROC $\frac{1}{2} < |z| < \frac{3}{4}$.

(f) $H(z) = \frac{2z^2}{(z-\frac{1}{2})(z-\frac{3}{4})}$ ROC $|z| > \frac{3}{4}$.

LTI Causality Theorem 1. A DT LTI system with system function $H(z)$ is causal if and only if its impulse response $h[n] = 0$ for all $n < 0$.

LTI Causality Theorem 2: A DT LTI system with system function $H(z)$ is causal if and only if the ROC of $H(z)$ is:
The exterior of a circle, including $+\infty$.

LTI Causality Theorem 3: a DT LTI system with rational system function $H(z)$ is causal if and only if the order of the numerator of $H(z)$ is lower or equal to the order of the denominator and its ROC to be external to a circle.

LTI System - Stability

LTI Stability Theorem 1: A DT LTI system is stable when its impulse response is absolutely summable.

LTI Stability Theorem 2: A DT LTI system with system function $H(z)$ is stable if and only if the ROC of $H(z)$ includes the unit circle, $|z| = 1$.

LTI Stability Theorem 3: a causal DT LTI system with rational system function $H(z)$ is stable if and only if all the poles of $H(z)$ lie inside the unit circle $|z| = 1$. i.e. all the poles have magnitude less than 1.

LTI Stability Theorem 1: A DT LTI system is stable when its impulse response is absolutely summable.

Proof: for a bounded input, the output is

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq \sum_{k=-\infty}^{\infty} B|h[k]|$$

LTI Stability Theorem 2: A DT LTI system with system function $H(z)$ is stable if and only if the ROC of $H(z)$ includes the unit circle, $|z| = 1$.

Proof:

If $\sum_{k=-\infty}^{+\infty} |h[n]| < \infty$,

$$H(z)|_{z=e^{j\Omega}} \leq \sum_{k=-\infty}^{+\infty} |h[n]z^{-n}||_{z=e^{j\Omega}} \leq \sum_{k=-\infty}^{+\infty} |h[n]||z^{-n}| \quad |_{z=e^{j\Omega}}$$

$$= \sum_{k=-\infty}^{+\infty} |h[n]| < \infty$$

LTI Stability Theorem 3: a causal DT LTI system with rational system function $H(z)$ is stable if and only if all the poles of $H(z)$ lie inside the unit circle, $|z| = 1$. i.e. they all have magnitude less than 1.

Proof: because unit circle is part of the ROC, and this system is also causal, thus, the ROC should be at least the exterior of the unit circle, which means all the poles need to be at least inside the unit circle.

Activity: consider the system with the transfer function.

$$H(z) = \frac{z}{\left(z - \frac{1}{5}\right)\left(z - \frac{3}{4}\right)}$$

If this system is BIBO stable, please determine the ROC.

6.3

Block Diagrams of DT LTI Systems

Similar to CT counterpart, many DT LTI systems of practical interest can be represented by N-th order linear difference equations :

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Observation: systems represented by a linear constant-coefficient difference equations are always rational.

Basic ways in which DT LTI systems can be inter-connected – series (cascade).

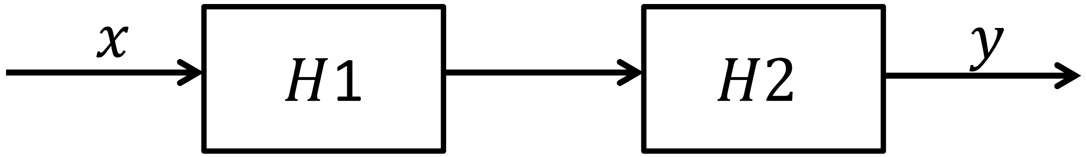
$$y[n] = H_2\{H_1\{x[n]\}\}$$

If the impulse response of the overall system is given by:

$$h[n] = h_1[n] * h_2[n]$$

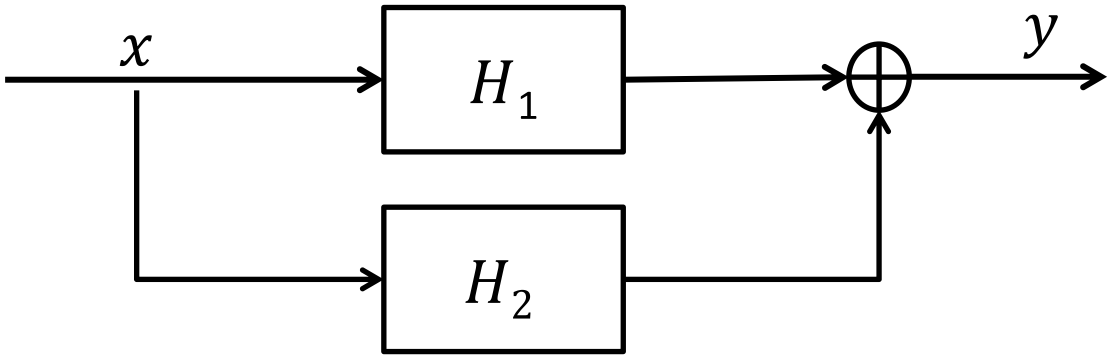
Then the transfer function is:

$$H(z) = H_1(z)H_2(z)$$



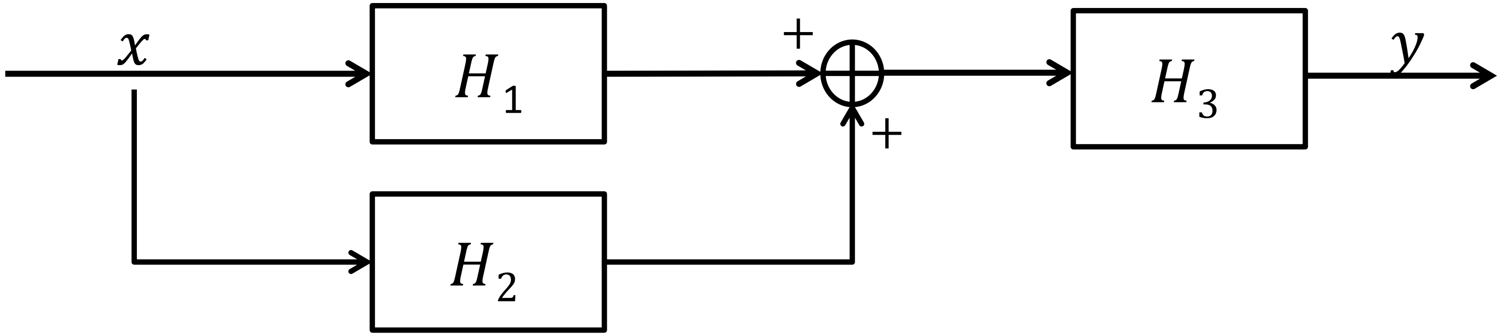
Basic ways in which DT LTI systems can be inter-connected – parallel.

$$y[n] = H_1\{x[n]\} + H_2\{x[n]\}$$



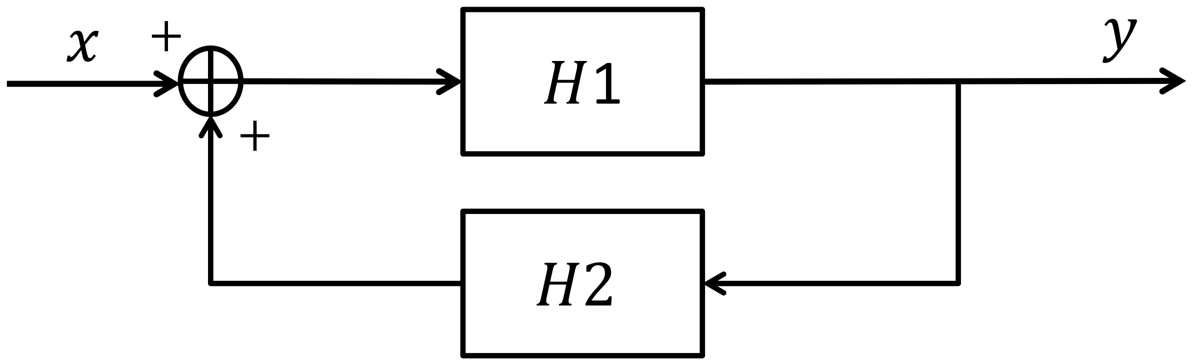
Basic ways in which DT LTI systems can be inter-connected are: series-parallel.

$$y = H_3\{H_1\{x[n]\} + H_2\{x[n]\}\}$$



Basic ways in which DT LTI systems can be inter-connected – feedback.

$$y = H_1\{x + H_2\{y\}\}$$



Activity: determine the rational transfer function and the block diagram for the causal LTI system described by the following difference equation:

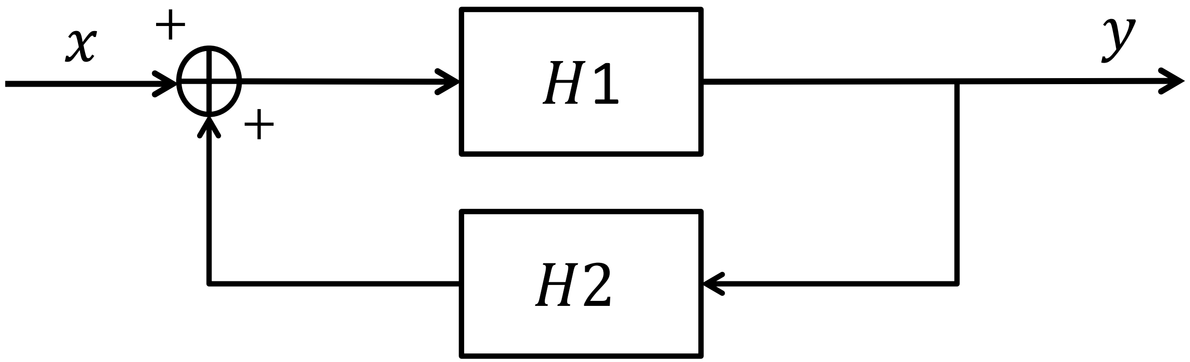
$$y[n] - \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = -x[n] + 2x[n-1]$$

Recall: Time shift in the z-transform

- If: $x[n] \overset{z}{\leftrightarrow} X(z)$, ROC=R
- Then: $x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}X(z)$, ROC=R, with possible addition of origin or $|z| = \infty$
- Multiplication of z^{-n_0} may:
  - introduce a pole at the origin if $n_0 > 0$
  - introduce a pole at the ∞ if $n_0 < 0$

Activity: consider a system below with the following two transfer function blocks. Decide its system function and if it is stable.

$$H_1(z) = \frac{\beta z}{z-1}, H_2(z) = 1$$



$$H(z) = \frac{H_1(z)}{1 - H_1(z)H_2(z)} = \frac{\frac{\beta z}{z-1}}{1 - \frac{\beta z}{z-1}} = \frac{\frac{\beta z}{(1-\beta)}}{z - \frac{1}{(1-\beta)}}$$

Since this system is causal, its ROC must include the unit circle. So:

$$\left|\frac{1}{(1-\beta)}\right| < 1$$

The system is BIBO stable when:

$$|1-\beta| > 1$$

Activity: consider a DT LTI system represented by the following transfer function. Plot the corresponding block diagram.

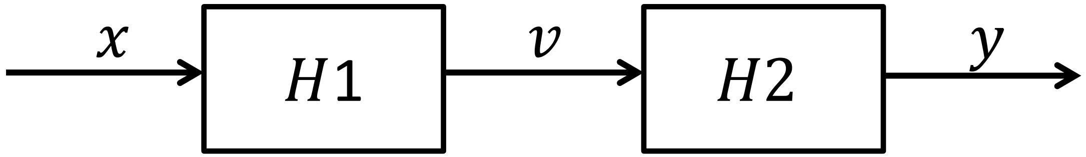
$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

Activity: consider a DT LTI system represented by the following transfer function. Plot the corresponding block diagram.

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{2}z^{-1}} = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right)(1 - 2z^{-1})$$

Activity: consider a DT LTI system represented by the following transfer function. Plot the corresponding block diagram.

$$H(z) = \frac{1-2z^{-1}}{1-\frac{1}{2}z^{-1}} = \underbrace{\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)}_{\text{H1}}\underbrace{\left(1-2z^{-1}\right)}_{\text{H2}}$$



Activity: consider a DT LTI system represented by the following transfer function. Plot the corresponding block diagram.

$$H1(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right) \leftrightarrow v[n] - \frac{1}{2}v[n-1] = x[n]$$

$$H2(z) = (1 - 2z^{-1}) \leftrightarrow y[n] = v[n] - 2v[n-1]$$

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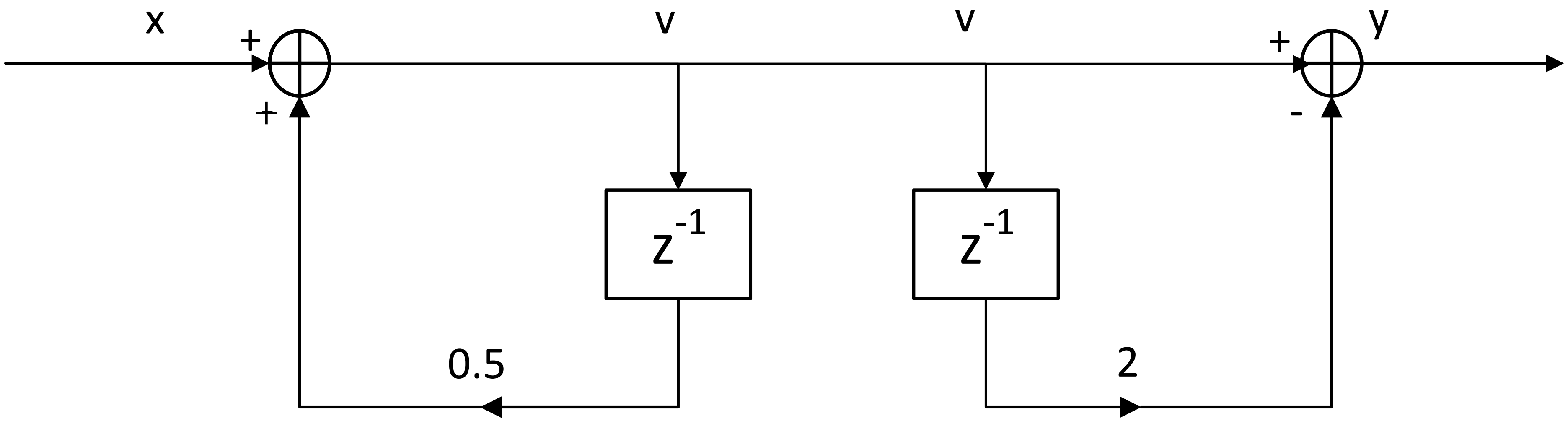

$$H1(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right) \leftrightarrow v[n] - \frac{1}{2}v[n-1] = x[n]$$

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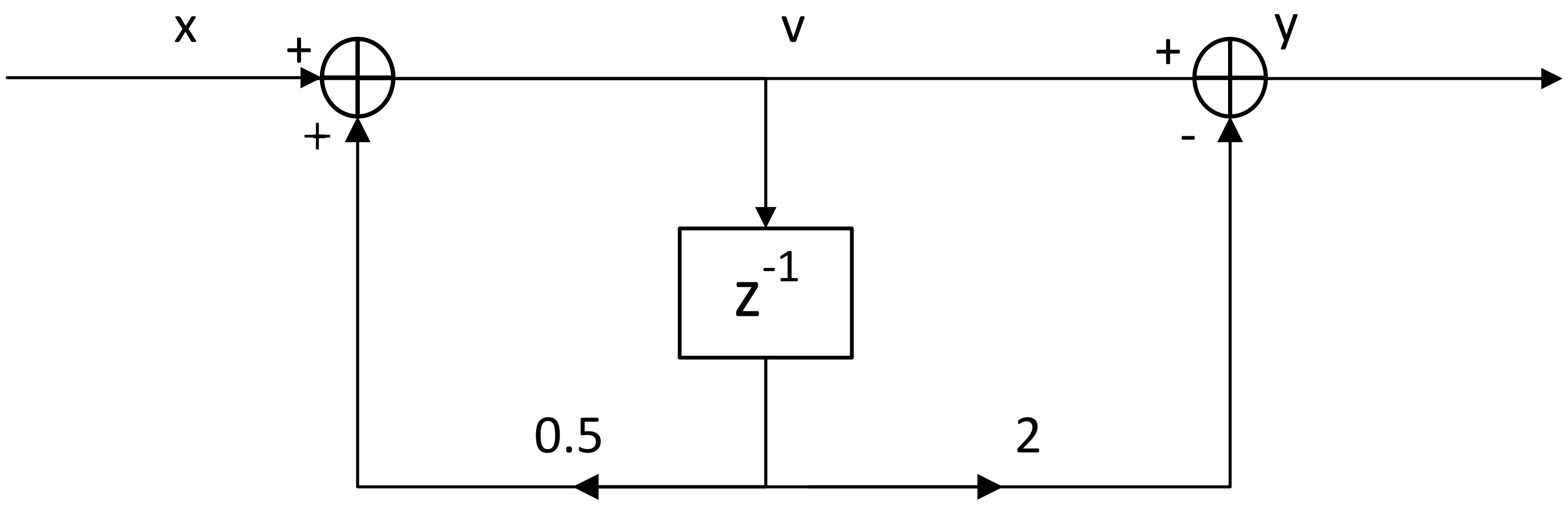

Activity: consider a DT LTI system represented by the following transfer function. Plot the corresponding block diagram using (a) direct form, (b) cascade form, (c) parallel form.

$$H(z) = \left(\frac{1}{1 + \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} \right)$$

Homework:

Review: in-class examples, textbook chapter 5. Section 5.3, 5.5.

Example: 5.10

Drill: 5.18

Problems: 5.3-6, 5.3-18, 5.5-8.