Module 7 – Numerical Integration

Lesson goals

1. Recognizing that Newton-Cotes integration formulas are based on the strategy of replacing a complicated function or tabulated data with a polynomial that is easy to integrate.

2. Knowing how to implement the following single application Newton-Cotes formulas:

    - Trapezoidal rule
    - Simpson's $\frac{1}{3}$ rule
    - Simpson's $\frac{3}{8}$ rule

3. Understanding how Gauss quadrature provides superior integral estimates by picking optimal abscissas at which to evaluate the function.

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Introduction

As you may know, the integration plays an important role in vast variety of applications. But, not every function is integrable in the sense of indefinite integral. As an example, the indefinite integral of

$$\int e^{x^2} dx$$

does not exist. Therefore the Fundamental Theorem of Calculus can not be applied to find the definite integral

$$\int_{2}^{5} e^{x^2} dx.$$

Numerical approaches are alternatives to get an estimation for these kinds of integrals.

Newton-Cotes Formulas

The Newton-Cotes formulas are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with a polynomial that is easy to integrate:

$$I = \int_{a}^{b} f(x)dx \approx \int_{a}^{b} P_n(x)dx$$

where

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

There are two types of Newton-Cotes formulas:

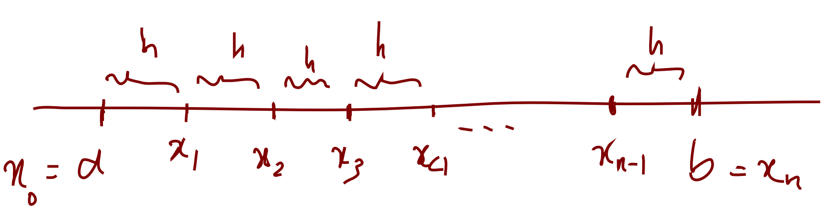
- *Closed forms:* the data points at the beginning and end of the limits of integration are known
- *Open forms:* the data points at the beginning and end of the limits of integration are not known

The Trapezoidal Rule

Assume that $f(x)$ is a continuous function in the interval $[a, b]$. To approximate the integral $\int_a^b f(x)dx$, we divide $[a, b]$ in to n subintervals of equal width with $n + 1$ equally spaced points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$. Let h be the width of each subinterval $[x_{i-1}, x_i]$. That is

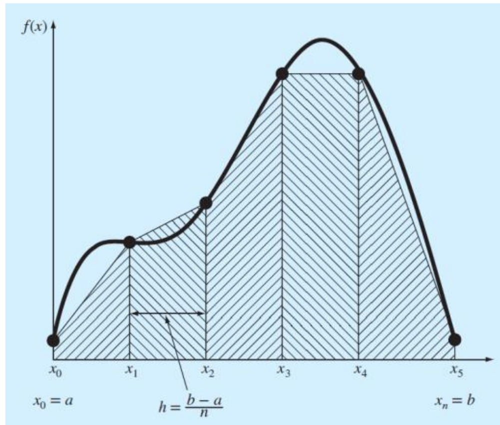
$$h = \frac{b - a}{n}$$

$$h = \frac{b-a}{n}$$



Using the properties of definite integral, we have:

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx \qquad (1)$$

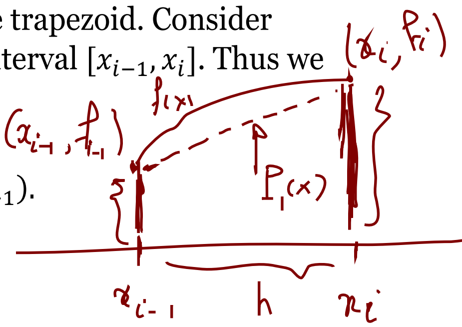


$$\text{Slope} = \frac{f_i - f_{i-1}}{x_i - x_{i-1}} = \frac{1}{h}(f_i - f_{i-1})$$

For each integral in the right hand side, we use the linear approximation of the function to estimate the area under the curve by the area of the trapezoid. Consider $P_1(x)$ as the linear approximation of the function $f(x)$ in the interval $[x_{i-1}, x_i]$. Thus we have:

$$\int_{x_{i-1}}^{x_i} f(x)dx \simeq \int_{x_{i-1}}^{x_i} P_1(x)dx$$

$$P_1(x) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1}).$$



By integrating this linear function (instead of f itself), we have

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx \int_{x_{i-1}}^{x_i} P_1(x)dx = (x_i - x_{i-1})\frac{f(x_i) + f(x_{i-1})}{2} = \frac{h}{2}[f(x_i) + f(x_{i-1})]$$

Now, by adding up the integrals in the right-hand side of (1), we get the composite trapezoidal rule as below:

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \simeq \frac{h}{2}[f_1 + f_0] + \frac{h}{2}[f_2 + f_1] + \frac{h}{2}[f_3 + f_2] + \cdots + \frac{h}{2}[f_{n-1} + f_{n-2}] + \frac{h}{2}[f_n + f_{n-1}]$$

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right].$$

Error of trapezoidal rule. Using the Remainder Theorem of the Taylor Polynomial approximation and by some calculus, it can be shown that the absolute error of this

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right].$$

$M = \max_{x\in[a,b]} |f''(x)|$

Error of trapezoidal rule. Using the Remainder Theorem of the Taylor Polynomial approximation and by some calculus, it can be shown that the absolute error of this rule is given by

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}'' \Rightarrow |E_a| < \frac{(b-a)^3}{12n^2}M = \frac{(b-a)h^2}{12}M$$

an upper bound for the truncate error

where $\bar{f}''$ is the average value of the second derivative f'' on $[a, b]$, and M is an upper bound for the second derivative f'' on $[a, b]$.

Note. For the function $y = g(x)$, the average value on the interval (a, b) is defined by

$$\bar{g} = \frac{1}{b-a}\int_a^b g(x)dx$$

Example. Use the *two-segment* trapezoidal rule to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Find an estimate for the error. Recall that the exact value of the integral is 1.640533.

$n = 2$

$[a,b] = [0, 0.8]$

$h = \frac{b-a}{2} = 0.4$

0 0.4 0.8

	x_0	x_1	x_2
x_i	0	0.4	0.8
f_i	0.2	2.456	0.232

$$\int_0^{0.8} f(x)\,dx \simeq \frac{h}{2}\left[f_0 + 2f_1 + f_2\right] = \frac{0.4}{2}\left[0.2 + 2 \times 2.456 + 0.232\right]$$

$$= 1.0688$$

An upper bound for the error:

$f'(x) = 25 - 400x + 2025x^2 - 3600x^3 + 2000x^4 \longmapsto f''(x) = -400 + 4050x - 10800x^2 + 8000x^3$

$M = \max_{x \in [0,0.8]} |f''(x)|$ =max(|-400|, 63.7)
=400

$$|E_T| \leq \frac{M(b-a)h^2}{12} = \frac{400 \times (0.8)(0.4)^2}{12} \simeq 4.27$$

MATLAB Instructions. Try these commands in MATLAB and check the output.

```
>> X = 0:pi/100:pi;
>> Y = sin(X);
>> Q = trapz(X,Y)
```

$y = \sin x$ $[0, \pi]$

$n = 100$

$$\int_0^{\pi} \sin x\,dx \simeq \text{trapz}(X,Y)$$

$$h = \frac{x_2 - x_0}{2}$$

$x_0 \quad x_1 \quad x_2$

The Simpson's 1/3 Rule

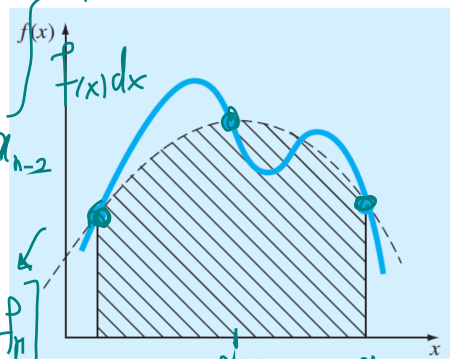
Consider three points (x_0, y_0), (x_1, y_1), and (x_2, y_2). Instead of using linear interpolation of the function in the subintervals $[x_0, x_1]$ and $[x_1, x_2]$, we use quadratic interpolation of the function at these points.

Requirement: n must be an even number.

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_4} f(x)dx$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \frac{h}{3}[f_4 + 4f_5 + f_6] + \cdots + \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n]$$

$$= \frac{h}{3}\left[f_0 + f_n + 4\sum_{i=0}^{\frac{n}{2}-1} f_{2i+1} + 2\sum_{i=1}^{\frac{n-2}{2}} f_{2i}\right]$$



Using Lagrange interpolation, we can get

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f_2\right] dx$$

Evaluating this integral along with the fact that $h = \frac{x_2 - x_0}{2}$ leads to

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}[f_0 + 4f_1 + f_2] \quad \checkmark$$

Using this procedure, we can get the composite 1/3 Simpson's rule as follows:

Assume that $[a, b]$ is divided into n subintervals, where n is an **even** number. In each two consecutive subintervals, we apply the above formula to get the composite one. In other words, consider the subintervals $[x_0, x_2], [x_2, x_4], [x_4, x_6], \ldots, [x_{n-2}, x_n]$ and assume that $h = \frac{b-a}{n}$. We have:

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$\approx \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \cdots + \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n]$$

$$\approx \frac{h}{3}[f_0 + 4f_1 + 2f_2 + +4f_3 + 2f_4 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n] = S_n$$

This is the composite Simpson's 1/3 rule.

Error analysis. Using the Intermediate Value Theorem, it can be shown that there is $\mu \in (a, b)$ so that the truncation error for Simpson's 1/3 rule satisfies the following

$$E_a = \int_a^b f(x)dx - s_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(\mu) = -\frac{(b-a)^5}{180n^4} \bar{f}^{(4)}$$

Where $\bar{f}^{(4)}$ is the average value of the fourth derivative of f in the interval (a, b). In case $|f^{(4)}(x)| \le M$ over (a, b), then an upper bound for the truncation error is:

$$|E_a| \le \frac{b-a}{180} h^4 M \qquad M = \max_{x \in [a,b]} |f^{(4)}(x)|$$

Example. Use Simpson's 1/3 rule with $n = 4$ to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Find an estimate for the error. Recall that the exact value of the integral is 1.640533.

$n = 4$, $h = \frac{b-a}{n} = 0.2$

0   0.2   0.4   0.6   0.8

	x_0	x_1	x_2	x_3	x_4
x_i	0	0.2	0.4	0.6	0.8
$f(x_i)$	0.2	1.288	2.456	3.464	0.232

$$\int_0^{0.8} f(x)\,dx \simeq \frac{h}{3}\left[f_0 + f_4 + 2f_2 + 4f_1 + 4f_3\right] = 1.6235$$

An upper bound for the truncation error:

$$f''(x) = -400 + 4050x - 10800x^2 + 8000x^3 \longrightarrow f''' = 4050 - 21600x + 24000x^2$$
$$\longrightarrow f^{(4)} = -21600 + 48000x$$

Note. There is no built-in function in MATLAB to approximate an integral using Simpson's rule.

$$M = \max_{x \in [0, 0.8]} |f^{(4)}(x)| = 21600 \longmapsto |E_T| \le \frac{M(b-a)h^4}{180} = 0.1536$$

The Simpson's 3/8 Rule

Consider four points (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3). Like before, a third order Lagrange polynomial can be fit to four points and integrated to yield

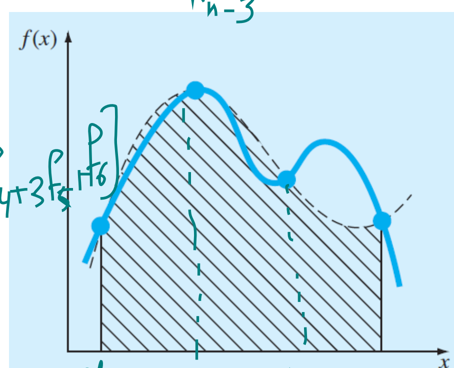
$$\int_{x_0}^{x_3} f(x)dx \approx \underbrace{\frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3]}$$

n must be multiple of 3

where $h = \frac{x_3 - x_0}{3}$.

$$\int_a^b f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)dx$$

$$= \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right] + \frac{3h}{8}\left[f_3 + 3f_4 + 3f_5 + f_6\right] + \cdots + \frac{3h}{8}\left[f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\right]$$



x_0 x_1 x_2 x_3

$$= \frac{3h}{8}\left[f_0 + f_n + 2\sum_{i=1}^{\frac{n}{3}-1} f_{3i} + 3\sum_{\substack{j=1\\ j\neq 3i}}^{\frac{n}{3}-1} f_j\right]$$

However, the truncation error for this version of the Simpson's rule has the same order as Simpson's 1/3 rule, that is, there is $\mu \in (x_0, x_3)$ so that

$$E_a = -\frac{(b-a)}{80} h^4 f^{(4)}(\mu)$$

$$|E_T| \le \frac{(b-a)\,h^4}{80} M$$

$$M = \max_{x\in[a,b]} |f^{(4)}(x)|$$

Remark. Simpson's 1/3 rule is usually the method of preference because it attains third-order accuracy with three points rather than the four points required for the 3/8 version. However, it can be combined with Simpson's 3/8 rule to handle the cases when the number of subintervals is odd.

$n = 7$

$[x_0, x_2]$ $\frac{1}{3}$ Simpson's rule

$[x_2, x_4]$ "

$[x_4, x_7]$ $\frac{3}{8}$ Simpson's rule

Example. Use Simpson's rules with $n = 5$ to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Use 1/3 rule for the first two subintervals and 3/8 rule for the last 3 subintervals.

$n = 5 \rightarrow h = \frac{b-a}{n} = \frac{0.8}{5}$

$0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

	x_0	x_1	x_2	x_3	x_4	x_5
x_i	0	h	2h	3h	4h	0.8
$f(x_i)$	0.2	1.2969	1.7434	3.186	3.1819	0.232

$$\int_0^{0.8} f(x)\,dx \simeq \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{3h}{8}\left[f_2 + 3f_3 + 3f_4 + f_5\right]$$

$$= \frac{0.8}{15}\left[0.2 + 4\times 1.2969 + 1.7434\right] + \frac{0.3}{5}\left[1.7434 + 3\times 3.186 + 3\times 3.1819 + 0.232\right]$$

$$= 1.645$$

MATLAB Functions for Multiple Integrals. The built-in functions `integral2` and `integral3` can be used for double and triple integrals in rectangular or rectangular cubic regions.

Example. We would like to integrate $f(x, y) = y\sin(x) + x\cos(y)$ over $x \in [\pi, 2\pi]$ and $y \in [0, \pi]$. Note that the true value of the integral is $-\pi^2$. Give a try to the following:

```
>> Q = integral2(@(x,y) y.*sin(x)+x.*cos(y),pi,2*pi,0,pi)
```

$$\int_{\pi}^{2\pi}\int_{0}^{\pi} \underbrace{(y \sin x + x \cos y)}_{\text{fun} = @(x,y)\ \ldots}\, dy\, dx = \text{integral2}(\text{fun}, \pi, 2\pi, 0, \pi)$$

Example: Integrate $\frac{1}{\sqrt{x+y}(1+x+y)^{\wedge}2}$ over the triangle $0 \le x \le 1$, and $0 \le y \le 1 - x$. Note that the integrand is infinite at $(0, 0)$, however, the true value of the integral is $\frac{\pi}{4} - \frac{1}{2}$.

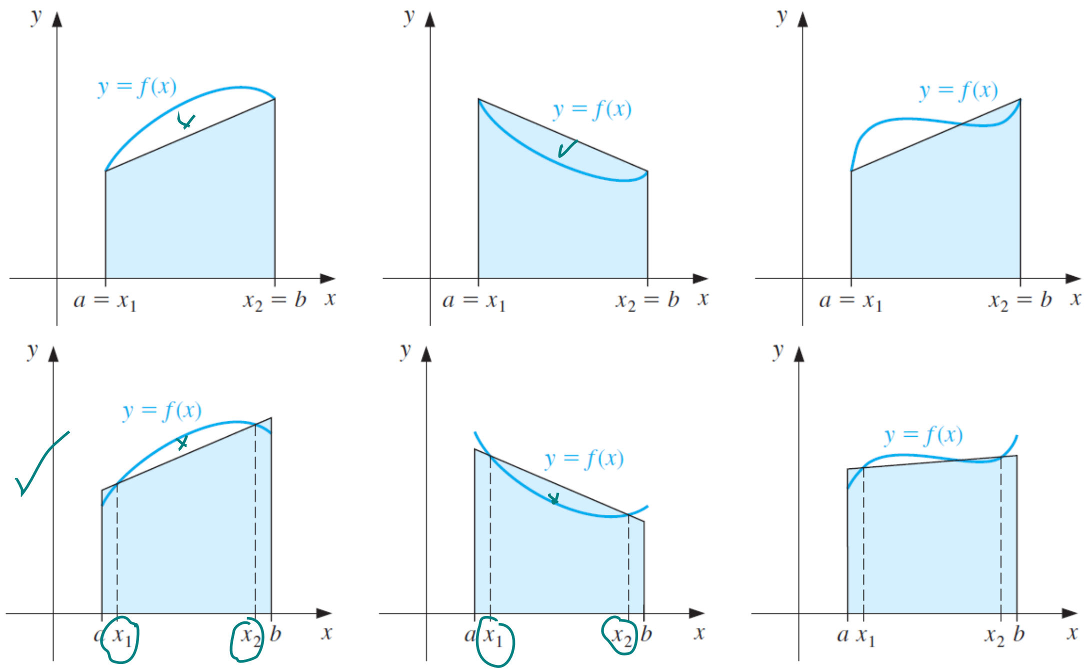
```
>> fun = @(x,y) 1./( sqrt(x + y) .* (1 + x + y).^2 )
>> ymax = @(x) 1 - x
>> Q = integral2(fun,0,1,0,ymax)
```

$$\int_0^1\int_0^{1-x} \frac{1}{\sqrt{x+y}\,(1+x+y)^2}\, dy\, dx$$

Remark. In command line of MATLAB, type `help integral3` to see the syntax format for the triple integral along with some examples.

Gaussian Quadrature

All the Newton–Cotes formulas use values of the function at equally spaced points. This restriction is convenient when the formulas are combined to form the composite rules, but it can significantly decrease the accuracy of the approximation (see the below figures).



Gaussian quadrature chooses the points for evaluation in an optimal, rather than equally spaced, way. The nodes $x_1, x_2, \dots, x_n$ in the interval $[a, b]$ and coefficients $c_1, c_2, \dots, c_n$ are chosen to minimize the expected error obtained in the approximation

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

2n variables, c_i's and x_i's

To measure this accuracy, assume that the best choice of these values produces the exact result for the largest class of polynomials, that is, the choice that gives the greatest degree of precision.

$\{1, x, x^2, x^3, x^4, \dots\}$

$p(x) = 3x^2 - 4x + 1 = 3(x^2) - 4(x) + 1(1)$

The coefficients $c_1, c_2, \dots, c_n$ in the approximation formula are arbitrary, and the nodes $x_1, x_2, \dots, x_n$ are restricted only by the fact that they must lie in $[a, b]$. This gives us $2n$ parameters to choose. If the coefficients of a polynomial are considered parameters, the class of polynomials of degree at most $2n - 1$ also contains $2n$ parameters. This is the largest class of polynomials for which it is reasonable to expect the formula to be exact.

$\int_{-1}^{1} f(x)dx \simeq c_1 f(x_1) + c_2 f(x_2)$

It is going to be exact for all polynomials of degree 3 or less.

Illustration of case $n = 2$. Consider $n = 2$ in the interval $[-1, 1]$. Suppose we want to determine c_1, c_2, x_1, and x_2 so that the integration formula

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

gives the exact result whenever $f(x)$ is a polynomial of degree $2n - 1 = 2(2) - 1 = 3$ or less, that is, when

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

for some collection of constants a_0, a_1, a_2, a_3. Instead of considering general case, we consider the basis functions $f(x) = 1, x, x^2, x^3$ and write the above equation in exact form. Thus

$$f(x) = 1 \Rightarrow \int_{-1}^{1} 1dx = c_1.1 + c_2.1 \rightarrow c_1 + c_2 = 2$$

$$f(x) = x \Rightarrow 0 = \int_{-1}^{1} xdx = c_1.x_1 + c_2.x_2 \rightarrow \boxed{c_1 x_1 + c_2 x_2 = 0}$$

$$f(x) = x^2 \Rightarrow \frac{2}{3} = \int_{-1}^{1} x^2dx = c_1.x_1^2 + c_2.x_2^2 \rightarrow \boxed{c_1 x_1^2 + c_2 x_2^2 = \frac{2}{3}}$$

$$f(x) = x^3 \Rightarrow 0 = \int_{-1}^{1} x^3dx = c_1 x_1^3 + c_2.x_2^3 \rightarrow \boxed{c_1 x_1^3 + c_2 x_2^3 = 0}$$

Solving the above system of equations for c_1, c_2, x_1, and x_2 leads to

$$c_1 = c_2 = 1, \qquad x_1 = -\frac{\sqrt{3}}{3}, \qquad x_2 = \frac{\sqrt{3}}{3}$$

$\int_{-1}^{1} f(x)dx \simeq f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3})$

degree of accuracy = 3

Thus,

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + c_2 f\left(\frac{\sqrt{3}}{3}\right)$$

This formula has degree of precision 3, that is, it produces the exact result for every polynomial of degree 3 or less.

$P_n(x) \begin{cases} \text{odd} & n \text{ is odd} \\ \text{even} & n \text{ is an even number.} \end{cases}$

$\int_{-1}^{1} f(x)dx \simeq c_1 f(x_1) + c_2 f(x_1) + c_3 f(x_3)$

Legendre Polynomials. The set of Legendre polynomials is a collection $\{P_0(x), P_1(x), \ldots, P_n(x), \ldots\}$ with properties:

1. For each n, $P_n(x)$ is a monic polynomial (leading coefficient is 1) of degree n.
2. $\int_{-1}^{1} P(x)P_n(x)dx = 0$ whenever $P(x)$ is a polynomial of degree less than n.

The first few Legendre polynomials are:

$P_2(x) = 0$

$x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$

$P_3(x) = 0$

$x = 0,\ x = \pm\sqrt{\frac{3}{5}}$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{3},$$

$$P_3(x) = x^3 - \frac{3}{5}x, \quad \text{and} \quad P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

Bonnet's recursion formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

$n = 4$

This relation, along with the first two polynomials P_0 and P_1, allows all the rest to be generated recursively.

The roots of these polynomials are distinct, lie in the interval $(-1, 1)$, have a symmetry with respect to the origin, and, most importantly, are the correct choice for determining the parameters that solve our problem. The nodes $x_1, x_2, \dots, x_n$ needed to produce an integral approximation formula that gives exact results for any polynomial of degree less than $2n$ are the roots of the nth-degree Legendre polynomial.

$n=1$

$\int_{-1}^{1} f(x) \cong c_0 f(x_0) = 2f(0)$

$n=2$

$\int_{-1}^{1} f(x)\,dx \cong c_0 f(x_0) + c_1 f(x_1)$

$n=3$

$\int_{-1}^{+1} f(x)\,dx \cong c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$

Weighting factors and function arguments used in Gauss-Legendre formulas.

Points	Weighting Factors	Function Arguments	Truncation Error
1	$c_0 = 2$	$x_0 = 0.0$	$\cong f^{(2)}(\xi)$
2	$c_0 = 1$ $c_1 = 1$	$x_0 = -1/\sqrt{3}$ $x_1 = 1/\sqrt{3}$	$\cong f^{(4)}(\xi)$
3	$c_0 = 5/9$ $c_1 = 8/9$ $c_2 = 5/9$	$x_0 = -\sqrt{3/5}$ $x_1 = 0.0$ $x_2 = \sqrt{3/5}$	$\cong f^{(6)}(\xi)$
4	$c_0 = (18 - \sqrt{30})/36$ $c_1 = (18 + \sqrt{30})/36$ $c_2 = (18 + \sqrt{30})/36$ $c_3 = (18 - \sqrt{30})/36$	$x_0 = -\sqrt{525 + 70\sqrt{30}}/35$ $x_1 = -\sqrt{525 - 70\sqrt{30}}/35$ $x_2 = \sqrt{525 - 70\sqrt{30}}/35$ $x_3 = \sqrt{525 + 70\sqrt{30}}/35$	$\cong f^{(8)}(\xi)$
5	$c_0 = (322 - 13\sqrt{70})/900$ $c_1 = (322 + 13\sqrt{70})/900$ $c_2 = 128/225$ $c_3 = (322 + 13\sqrt{70})/900$ $c_4 = (322 - 13\sqrt{70})/900$	$x_0 = -\sqrt{245 + 14\sqrt{70}}/21$ $x_1 = -\sqrt{245 - 14\sqrt{70}}/21$ $x_2 = 0.0$ $x_3 = \sqrt{245 - 14\sqrt{70}}/21$ $x_4 = \sqrt{245 + 14\sqrt{70}}/21$	$\cong f^{(10)}(\xi)$
6	$c_0 = 0.171324492379170$ $c_1 = 0.360761573048139$ $c_2 = 0.467913934572691$ $c_3 = 0.467913934572691$ $c_4 = 0.360761573048131$ $c_5 = 0.171324492379170$	$x_0 = -0.932469514203152$ $x_1 = -0.661209386466265$ $x_2 = -0.238619186083197$ $x_3 = 0.238619186083197$ $x_4 = 0.661209386466265$ $x_5 = 0.932469514203152$	$\cong f^{(12)}(\xi)$

Note. If the limits of the given integral are a, and b, we use the following substitution to convert the limits to $[-1, 1]$:

Ex. change the limits of the following integral to -1 and 1.

$$x = \frac{a+b}{2} + \frac{b-a}{2}u$$

$$\int_{-3}^{2} \cos x\, dx = \int_{-1}^{1} \cos\left(-\frac{1}{2} + \frac{5}{2}t\right)\left(\frac{5}{2}dt\right) = \frac{5}{2}\int_{-1}^{1} \cos\left(-\frac{1}{2} + \frac{5}{2}t\right) dt$$

$x = \frac{-1}{2} + \frac{5}{2}t \to dx = \frac{5}{2}dt$

Example. Use two-point Gaussian Quadrature to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$.

In order to use the weights and x_i values in the table, we have transform the interval $[0, 0.8]$ to $[-1, 1]$.

$$\int_0^{0.8} f(x)dx = \int_{-1}^{1} f(0.4+0.4t)(0.4dt) = (0.4)\int_{-1}^{1} \underbrace{f(0.4+0.4t)} dt$$

$x = 0.4 + 0.4t \Rightarrow dx = 0.4dt$

$$= (0.4)\int_{-1}^{1} g(t)\, dt$$

from table

$$= 0.4\left[c_0\, g(t_0) + c_1\, g(t_1)\right]$$

$$= 0.4\left[1 \times g\left(-\frac{1}{\sqrt{3}}\right) + 1 \times g\left(+\frac{1}{\sqrt{3}}\right)\right]$$

$$\int_0^{0.8} f(x)dx \simeq 0.4\, g\left(-\frac{1}{\sqrt{3}}\right) + 0.4\, g\left(\frac{1}{\sqrt{3}}\right)$$

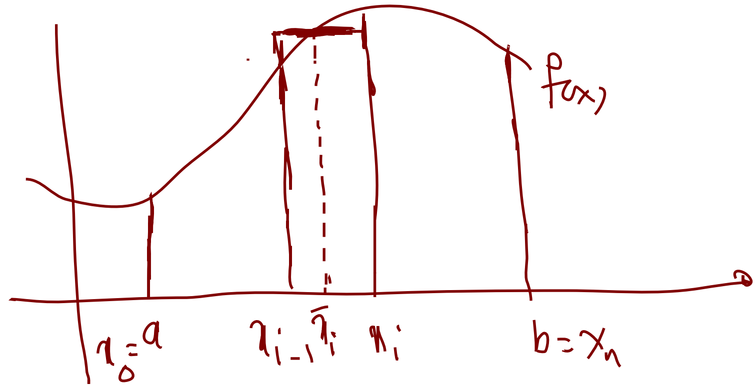
$$= (0.4)\, f\left(0.4 - \frac{0.4}{\sqrt{3}}\right) + (0.4)\, f\left(0.4 + \frac{0.4}{\sqrt{3}}\right)$$

$\underset{\text{use MATLAB}}{=}$ $\underline{\underline{1.8226}}$

References

1. Chapra, Steven C. (2018). *Numerical Methods with* MATLAB *for Engineers and Scientists*, 4th Ed. McGraw Hill.
2. Burden, Richard L., Faires, J. Douglas (2011). *Numerical Analysis*, 9th Ed. Brooks/Cole Cengage Learning

Midpoint rule :



$$A_i \simeq h f(\bar{x}_i)$$

$$\int_a^b f(x)\,dx \simeq h\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right] \quad \text{where } \bar{x}_i = \frac{x_i + x_{i-1}}{2}$$

$$|E_T| \leq \frac{(b-a)h^2}{24} M \quad \text{where } M = \max_{x\in[a,b]} |f''(x)|$$

Ex. $\int_0^{0.8}(0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)\,dx \simeq$?

with midpoint rule with $n=2$. $\quad \Big\}\ h = \frac{b-a}{n} = 0.4$

		$\bar{x}_1$		$\bar{x}_2$	
x_i	0	0.2	0.4	0.6	0.8
$f(x_i)$		1.288		3.464	

$$\int_0^{0.8} f(x)\,dx \simeq h\left[f(\bar{x}_1) + f(\bar{x}_2)\right]$$

$$= (0.4)(1.288 + 3.464)$$

$$= 1.9008$$