NUMM 2500 Humber College **Quiz 2 (5%)** **Quiz Period: 50 min.** **Feb. 7, 2024**

Solutions.

Student Name:

Student Number:

1. Consider the following equation:

$$x - \cos(x) = 0. \qquad (1)$$

a) [2 marks] Show this equation has a unique solution in the interval [0, 1].

b) [2 marks] Show that the solution of this equation is a fixed point of the function $g(x) = 1 - \frac{\sin^2 x}{x+1}$.

c) [5 marks] Use the *first 8 iterations* of the fixed-point algorithm with the function $g(x)$ given in **b)** to estimate the root of equation (1) starting from $x_0 = 1$. Create a table with three columns as below:

k	x_k	$\lvert\varepsilon_a\rvert$

Round your answers to 5-decimal digits.

a) $f(x) = x - \cos x$

$$\begin{cases} f(0) = 0 - \cos(0) = -1 \\ f(1) = 1 - \cos(1) = 0.45970 \end{cases} \mapsto$$

f(x) is a continuous function and f(0) f(1) < 0. So, by IVT, it has at least one solution on [0,1].

To show that the solution is unique, we consider the derivative of f(x).

$$f'(x) = 1 + \sin x \geq 0, \text{ for all } x.$$

So, f(x) is an increasing function on [0,1]. This shows that the root is unique.

b) We need to show that $x = g(x)$ will lead to $f(x) = 0$.

$$x = g(x) \Rightarrow x = 1 - \frac{\sin^2 x}{x+1} \Rightarrow x = \frac{x+1-\sin^2 x}{x+1}$$

$$\Rightarrow x(x+1) = x+1 - \sin^2 x \Rightarrow x^2 + \cancel{x} = 1 - \sin^2 x + \cancel{x}$$

$$\Rightarrow x^2 = \cos^2 x \Rightarrow x^2 - \cos^2 x = 0 \Rightarrow (x - \cos x)(x + \cos x) = 0$$

$$\Rightarrow \begin{cases} x - \cos x = 0 \Rightarrow \boxed{f(x) = 0} \checkmark \\ x + \cos x = 0 \end{cases}$$

K	x_K	$\lvert\varepsilon_a\rvert$
0	1	—
1	~~0.35404~~ 0.64596	0.35404
2	0.77985	0.20726
3	0.72220	0.07392
4	0.74627	0.03334
5	0.73606	0.01369
6	0.74037	0.00585
7	0.73854	0.00246
8	0.73931	0.00104

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2. [5 marks] The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has a zero in $[-1, 0]$. Use the modified secant method with $\delta = 10^{-2}$ and the midpoint of the interval as the initial approximation to estimate the root. Stop the iterative scheme if either $|\varepsilon_a| < 10^{-4}$ or the number of iterations exceeds 6. Provide a table as below:

k	x_k	$\|\varepsilon_a\|$

Round your answers to 5-decimal digits.

K	x_K	$\|\varepsilon_a\|$
0	−0.5	—
1	−0.15218	0.69564
2	−0.04190	0.72470
3	−0.04066	0.02951
4	−0.04066	0.00000208

STOP

$|\varepsilon_a| < 10^{-4}$

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3. a) [4 marks] Apply *5 iterations* of the bisection method on the interval $[2.5, 3.5]$ to find an approximation to $\sqrt[3]{25}$.

b) [2 marks] Find a bound for the number of iterations needed by the bisection method to achieve an approximation to $\sqrt[3]{25}$ in the above interval with accuracy 10^{-3}.

Round your answers to 4-decimal digits.

a) $f(x) = x^3 - 25$

$[a_0, b_0] = [a, b] = [2.5, 3.5]$

$f(a_0) = -9.375$

$f(b_0) = 17.875$

k	a_k	b_k	x_k	Sign $f(x_k)$
0	2.5	3.5	3.0000	+
1	2.5	3.0	2.75	−
2	2.75	3.0	2.875	−
3	2.875	3.0	2.9375	+
4	2.875	2.9375	2.9063	−
5	2.9063	2.9375	2.9219	−

b) Upper bound for the error $= \dfrac{b-a}{2^n} = \dfrac{1}{2^n}$

We want to have $\dfrac{1}{2^n} \leq 10^{-3}$

$\rightarrow 2^n \geq \dfrac{1}{10^{-3}} = 1000 \rightarrow n \geq \log_2 1000 \simeq 9.97$

So, $\boxed{n \geq 10}$