MENG 3510 – Quiz 2 Solution – Winter 2025

Question 1. [10 marks] Consider a dynamic system with the following state-space representation,

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

a) Find the <u>characteristic polynomial</u> and <u>eigenvalues</u> of the system matrix **A**. Determine the <u>stability</u> of the system? Show your work and justify your answer.

Find $\det(\lambda\mathbf{I} - \mathbf{A})$ to obtain the characteristics polynomial:

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda - 2 \end{bmatrix}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 2 \end{vmatrix} = \lambda^2 - 2\lambda - 1$$

Solve the characteristics polynomial to find the eigenvalues:

$$\lambda^2 - 2\lambda - 1 = 0 \quad \rightarrow \quad \lambda = 1 + \sqrt{2} = 2.4142 \text{ and } \lambda = 1 - \sqrt{2} = -0.4142$$

Since, one of the eigenvalues is at the right-half of the complex plane, the system is **unstable**.

b) Determine <u>transfer function model</u> of the open-loop system. Show your work.

$$G(s) = \frac{Y(s)}{U(s)}$$

Find the transfer function using the following equation:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{1}{s^2 - 2s - 1} \begin{bmatrix} s - 2 & 1 \\ 1 & s \end{bmatrix}$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{1}{s^2 - 2s - 1} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s - 2 & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 - 2s - 1} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = \frac{s + 1}{s^2 - 2s - 1}$$

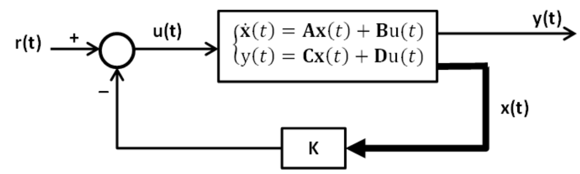
Question 2. [10 marks] Consider the given system in Question 1,

a) Determine the <u>controllability matrix</u> ($\boldsymbol{Q_c}$) of the open-loop system. Check the <u>controllability</u> of the open-loop system. Show your work and justify your answer.

$$Q_c = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad \rightarrow \quad \det(Q_c) = -1$$

Since determinant of the controllability matrix is non-zero, it is a full-rank matrix, and the open-loop system is **controllable**.

b) Consider the following closed-loop system with state feedback control,



Find the <u>state-feedback controller gain</u> **K** to locate the closed-loop eigenvalues at $s = -3$ and $s = -5$. Show your work.

$$u(t) = -\mathbf{Kx}(t) + r(t)$$

Step 1: Find the closed-loop system matrix,

$$\mathbf{A_{cl}} = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - k_1 & 2 - k_2 \end{bmatrix}$$

Step 2: Find the closed-loop system characteristic equation.

$$s\mathbf{I} - \mathbf{A_{cl}} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 - k_1 & 2 - k_2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -1 + k_1 & s - 2 + k_2 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A_{cl}}) = \begin{vmatrix} s & -1 \\ -1 + k_1 & s - 2 + k_2 \end{vmatrix} = s^2 + (k_2 - 2)s - 1 + k_1$$

Step 3: Find the desired closed-loop characteristic equation based on the desired pole locations at s = -3 and s = -5.

$$(s + 3)(s + 5) = s^2 + 8s + 15$$

Step 4: Match the desired characteristic equation with the characteristic equation of the closed-loop system to find the state-feedback controller gain:

$$s^2 + 8s + 15 = s^2 + (k_2 - 2)s - 1 + k_1$$

$$\begin{cases} k_2 - 2 = 8 \\ k_1 - 1 = 15 \end{cases} \quad \rightarrow \quad k_2 = 10, \quad k_1 = 16 \quad \rightarrow \quad \mathbf{K} = [16 \quad 10] \quad \text{State-feedback Gain}$$