Signal Processing (MENG3520)

Module 7

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MODULE 7

FOURIER METHODS – PART 1

Overview

- So far we have learned various types of transforms. It is time for us to reflect on what is a transform and why it is important.
- Transform: the conversion of a function from one domain to another with no loss of information.
- Transforms provide different perspectives to view the underlying information, in hope of clearer definition of the problem, easier analysis and more straightforward feature extraction.

What are the Relation Between The Laplace Transform, z-Transform and Fourier Transform

Fourier Transform

Laplace transform

z-Transform

Why Fourier Transform?

Human brain often find the frequency plane easier to visualize and more intuitive compared to complex planes such as the s domain and the z domain.

Wide range of applications in almost all areas of engineering and science.

- Circuit designers
- Material science and analysis
- Spectroscopy
- Signal processing
- Communication
- Imaging

FOURIER TRANSFORM FAMILY

The following four types are all part of the Fourier transform family:

- Continuous-time Fourier series (CTFS) – periodic continuous-time signals
- Continuous-time Fourier transform (CTFT) – aperiodic continuous-time signals
- Discrete-time Fourier transform (DTFT) – aperiodic discrete-time signals
- Discrete Fourier transform (DFT) – periodic discrete-time signals.

Characteristics of Different Fourier Transforms

To use Fourier methods to analyze a general system involving the mixing of CT, DT, aperiodic and periodic signals, we must:

1. Understand the four different Fourier methods;
2. Build bridges between different methods working with different classes of signals.

Let’s start with Continuous-time Fourier Series - CTFS.

Module Outline

- 7.1 Fourier Series Representation by Exponential Fourier Series
- 7.2 Fourier Series Representation by Trigonometric Fourier Series
- 7.3 Frequency Spectrum
- 7.4 Convergence of Fourier Series

7.1

Periodic Signal Representation by Exponential Fourier Series

Fourier Series: a type of mathematical technique that is used to represent a **periodic** signal as a sum of sinusoids.

Recall: **fundamental period**, the minimal positive, non-zero value of time interval T_0 such that $x(t) = x(t + T_0)$.

Fundamental frequency: the reciprocal of fundamental period T_0.

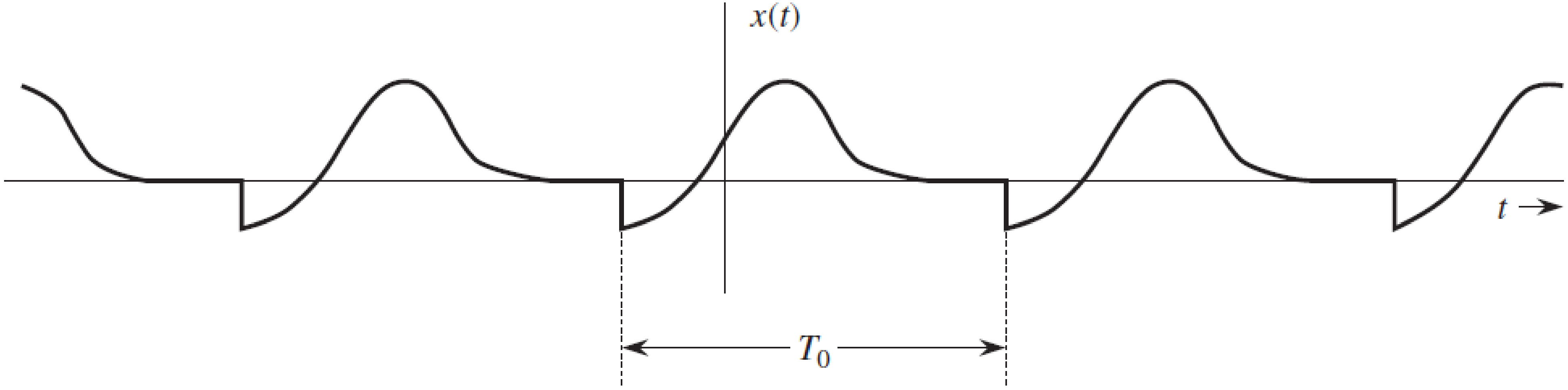


Figure 6.1 A periodic signal of period T_0.

Harmonically-related complex sinusoids: a group of complex sinusoids is harmonically related if there exists a constant ω_0 such that the **fundamental frequency** for each of these sinusoids is an integer multiple of ω_0.

Assume this set of harmonically-related complex sinusoids are:

$$\Phi_k(t) = e^{jk\omega_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, k = 0, \pm 1, \pm 2, \ldots$$

A linear combination of these sinusoids can be expressed as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$x(t)$ must be periodic with period T (why?).

Proof: since, $\omega_0 = \frac{2\pi}{T}$, and

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t+T)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)(t+T)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi t}{T} + j2k\pi} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi t}{T}} = x(t)$$

Thus $x(t)$ is periodic with period T.

Continuous-time Fourier Series (CTFS)

For a CT periodic signal $x(t)$ with **fundamental frequency** $\boldsymbol{\omega_0}$, it can be expressed as a sum of a series of harmonics:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$x(t) \xleftrightarrow{CTFS} c_k$$

When $k = K$ or $k = -K$, the term is called the Kth harmonic component with the fundamental frequency $K\omega_0$ and the corresponding Fourier series coefficient c_K.

CTFS Fourier Coefficients

The Fourier series coefficients c_k is defined as an integration of an arbitrary interval of length T:

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Here T is the fundamental period of signal $x(t)$, and:

$$T = \frac{1}{2\pi\omega_0}$$

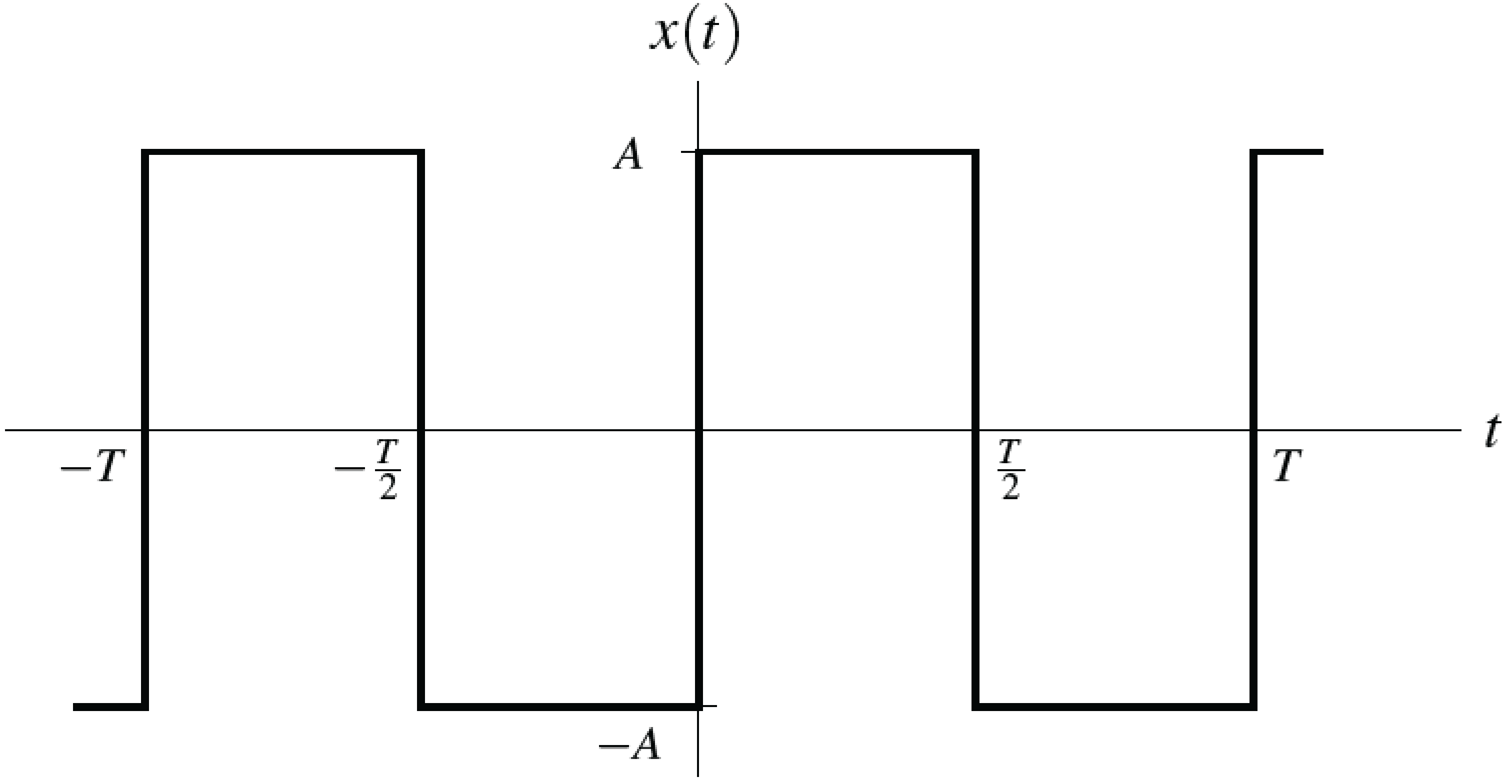
Exponential CTFS Example

Find the CT Fourier series coefficients c_k of a periodic square wave:

$T = T$

$f_0 = \frac{1}{T}$

$\omega_0 = \frac{2\pi}{T}$



Answer: This periodic square wave between $0 \le t < T$:

$$x(t) = \begin{cases} A, & 0 \le t < T/2 \\ -A, & T/2 \le t < T \end{cases}$$

$$c_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{0}^{T/2} Ae^{-jk\omega_0 t}dt + \frac{1}{T}\int_{T/2}^{T} -Ae^{-jk\omega_0 t}dt$$

(1) When $k = 0, c_0 = \frac{1}{T}\int_0^{T/2} Adt + \frac{1}{T}\int_{T/2}^{T} -Adt = \frac{1}{T}(At)\Big|_0^{T/2} + \frac{1}{T}(-At)\Big|_{T/2}^{T} = 0,$

(2) When $k \neq 0$, since $\omega_0 = 2\pi/T$, $c_k = \frac{1}{T}\frac{A}{(-jk\omega_0)}e^{-jk\omega_0 t}\Big|_0^{T/2} + \frac{1}{T}\frac{-A}{(-jk\omega_0)}e^{-jk\omega_0 t}\Big|_{T/2}^{T}$

$= \frac{jA}{(Tk\omega_0)}\left(e^{-jk\pi} - 1\right)$ - $\frac{jA}{(Tk\omega_0)}\left(e^{-2jk\pi} - e^{-jk\pi}\right)$

(2) When $k \neq 0$, and also since $\omega_0 = 2\pi/T$

$$c_k = \frac{1}{T}\frac{A}{(-jk\omega_0)}e^{-jk\omega_0 t}\Big|_0^{T/2} + \frac{1}{T}\frac{-A}{(-jk\omega_0)}e^{-jk\omega_0 t}\Big|_{T/2}^{T}$$

$$= \frac{jA}{(Tk\omega_0)}\left(e^{-jk\pi} - 1\right) - \frac{jA}{(Tk\omega_0)}\left(e^{-2jk\pi} - e^{-jk\pi}\right) = \frac{jA}{(Tk\omega_0)}\left(2e^{-jk\pi} - e^{-2jk\pi} - 1\right) =$$

$$\frac{jA}{(Tk\omega_0)}\left(2(-1)^k - 1 - 1\right) = \frac{jA}{(\pi k)}\left((-1)^k - 1\right) = \begin{cases} \frac{-j2A}{(\pi k)}, k \text{ is odd} \\ 0, k \text{ is even and } k \neq 0 \end{cases}$$

In conclusion, the Fourier series is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kt}, \text{ with } c_k = \begin{cases} \frac{-j2A}{(\pi k)}, k \text{ is odd} \\ 0, k \text{ is even} \end{cases}$$

7.2

Periodic Signal Representation by Trigonometric Fourier Series

Trigonometric Fourier Series

- The periodic signal can also be expressed as the sum of an infinite number of sine and/or cosine functions.

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

- Here ω_0 is the fundamental frequency of $x(t)$, and fundamental period $T = \frac{1}{2\pi\omega_0}$.
- $a_0 = \frac{1}{T} \int_T \, x(t)\, dt$
- $a_k = \frac{2}{T} \int_T \, x(t) \cos(k\omega_0 t)\, dt$
- $b_k = \frac{2}{T} \int_T \, x(t) \sin(k\omega_0 t)\, dt$

Exercise: Prove that the trigonometric Fourier series can be derived from the exponential Fourier series.

Proof: since $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, lets rewrite it into the positive and negative terms.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=1}^{\infty} c_{-k} e^{-jk\omega_0 t}$$

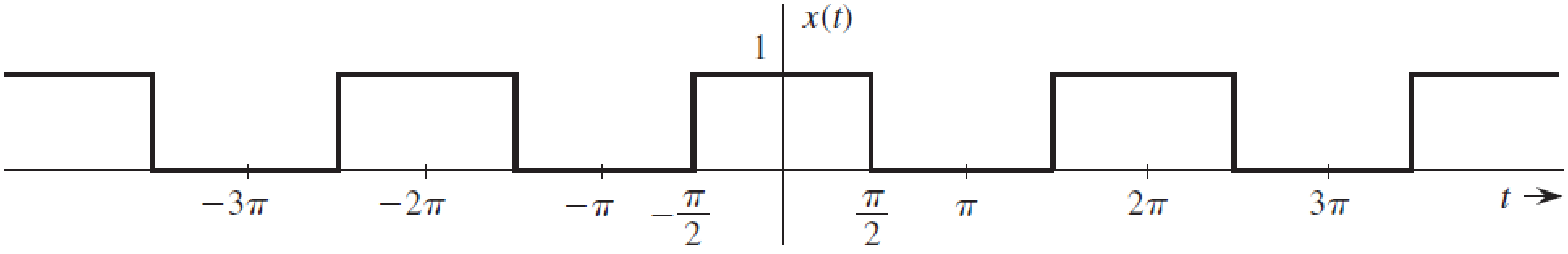
$$= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k (\cos(k\omega_0 t) + j\sin(k\omega_0 t)) + c_{-k} (\cos(k\omega_0 t) - j\sin(k\omega_0 t)) \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left((c_k + c_{-k}) \cos(k\omega_0 t) + j\left((c_k - c_{-k})\right) \sin(k\omega_0 t) \right)$$

- $a_0 = c_0 = \frac{1}{T} \int_T \ x(t)\, dt$
- $a_k = (c_k + c_{-k}) = \frac{1}{T} \int_T \ x(t) e^{-jk\omega_0 t} dt + \frac{1}{T} \int_T \ x(t) e^{jk\omega_0 t} dt = \frac{2}{T} \int_T \ x(t) \cos(k\omega_0 t)\, dt$
- $b_k = j(c_k - c_{-k}) = \frac{j}{T} \int_T \ x(t) e^{-jk\omega_0 t} dt - \frac{j}{T} \int_T \ x(t) e^{jk\omega_0 t} dt = \frac{2}{T} \int_T \ x(t) \sin(k\omega_0 t)\, dt$

Example: Rectangle Waveform



(a)

$T = 2\pi$

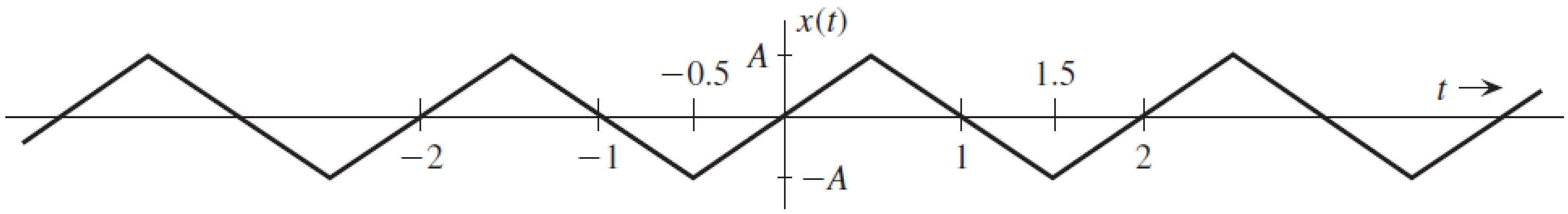
$f_0 = \frac{1}{2\pi}$

$\omega_0 = 1$

$$x(t) = \frac{1}{2} + 0.63\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

DC term | 1st harmonic | 3rd harmonic | 5th harmonic | 7th harmonic

Example: Triangle Waveform



(a)

$$T = 2$$
$$f_0 = \frac{1}{2}$$
$$\omega_0 = \pi$$

$$x(t) = 0.8A\left(\sin \pi t - \frac{1}{9}\sin 3\pi t + \frac{1}{25}\sin 5\pi t - \frac{1}{49}\cos 7\pi t + \cdots\right)$$

DC term=0

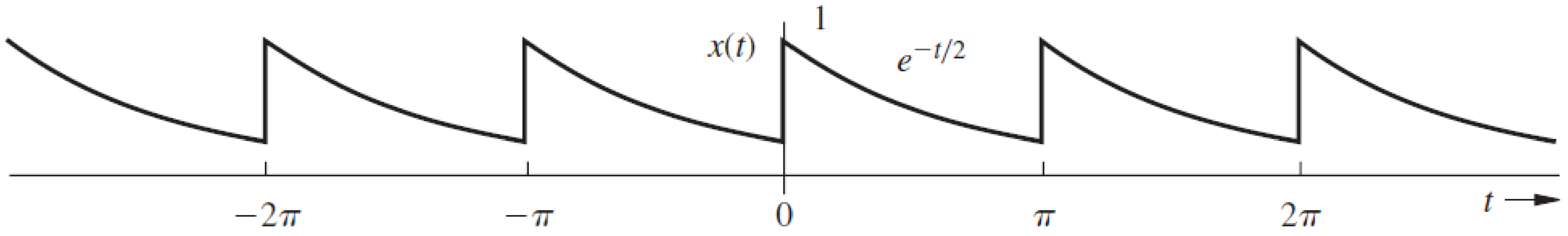
1st harmonic

3rd harmonic

5th harmonic

7th harmonic

Example: A Periodic Waveform Defined Below



(a)

$$T = \pi$$
$$f_0 = \frac{1}{\pi}$$
$$\omega_0 = 2$$

$$x(t) = 0.504(1 + (0.1\cos 2t + 0.47\sin 2t) + (0.03\cos 4t + 0.24\sin 4t) + (0.01\cos 6t + 0.16\sin 6t) + \cdots)$$

DC term

1st harmonic

2nd harmonic

3rd harmonic

Compact Trigonometric Fourier Series

- You can further express a periodic signal using the compact trigonometric Fourier series, which combines the sine and cosine terms into a single cosine term with a phase shift:

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) = c_0 + \sum_{k=1}^{\infty} d_k (\cos k\omega_0 t + \theta_k)$$

- Here d_k and θ_k are related to a_k and b_k.
- $c_0 = a_0 = \frac{1}{T}\int_T x(t)\,dt$
- $d_k = \sqrt{a_k^2 + b_k^2}$
- $\theta_k = \tan^{-1}\left(-\frac{b_n}{a_n}\right) = \sin^{-1}\left(\frac{-b_k}{d_k}\right) = \cos^{-1}\left(\frac{a_k}{d_k}\right)$

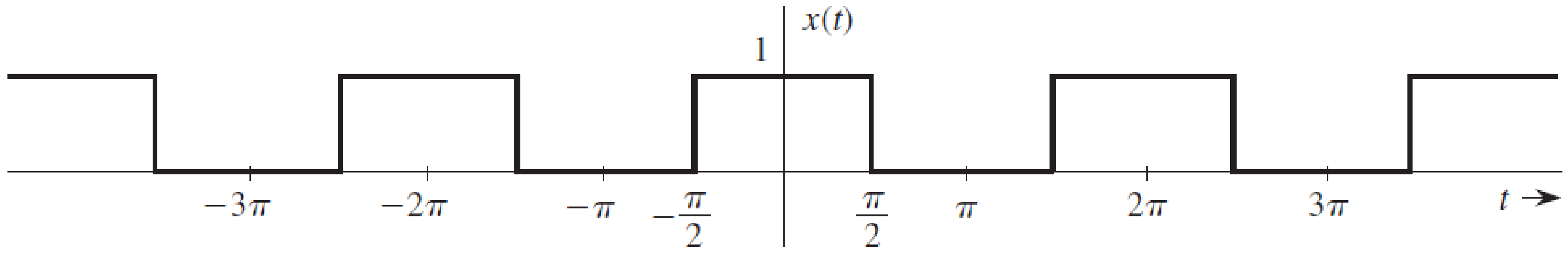
Example 6.4

Find the trigonometric Fourier Series for the square-pulse periodic signal.

$T = 2\pi$

$f_0 = \frac{1}{2\pi}$

$\omega_0 = 1$



(a)

- $a_0 = \frac{1}{T}\int_T \; x(t)\,dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dt = \frac{1}{2}$
- $a_k = \frac{2}{T}\int_T \; x(t)\cos(k\omega_0 t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(kt)\,dt = \frac{1}{k\pi}\left(\sin\frac{k\pi}{2} - \sin\frac{-k\pi}{2}\right) = \frac{2}{k\pi}\sin\frac{k\pi}{2}$
- $b_k = \frac{2}{T}\int_T \; x(t)\sin(k\omega_0 t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin(kt)\,dt = \frac{1}{k\pi}\left(-\cos\frac{k\pi}{2} + \cos\frac{-k\pi}{2}\right) = 0$
- $x(t) = \; a_0 + \sum_{k=1}^{\infty}(a_k\cos k\omega_0 t + b_k \sin k\omega_0 t) = \frac{1}{2} + \sum_{k=1}^{\infty}(\frac{2}{k\pi}\sin\frac{k\pi}{2}\cos kt) = \frac{1}{2} + 0.63\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$

Example 6.4

We can further express this signal using the compact trigonometric Fourier series:

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) = c_0 + \sum_{k=1}^{\infty} d_k(\cos k\omega_0 t + \theta_k)$$

- Since $x(t) = \frac{1}{2} + 0.63\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$
- Here d_k and θ_k are related to a_k and b_k.

$$c_0 = a_0 = 1/2$$

$$d_k = \sqrt{a_k^2 + b_k^2} = |a_k|$$

$$\theta_k = \cos^{-1}\left(\frac{a_k}{d_k}\right)$$

- $x(t) = \frac{1}{2} + 0.63\left(\cos t + \frac{1}{3}\cos(3t - \pi) + \frac{1}{5}\cos 5t + \frac{1}{7}\cos(7t - \pi) + \cdots\right)$

Trigonometric Fourier Series, Compact Trigonometric Fourier Series, and Exponential Fourier Series Comparison

- The **exponential Fourier Series** is the most compact and mathematically elegant representation. Unifies the representation of periodic and non-periodic signals (via the Fourier transform, to be covered later). However, it requires understanding of complex numbers and thus less intuitive for physical interpretation.
- The **trigonometric Fourier series** is intuitive and easy to understand/visualize. However, it requires two sets of coefficients (a_n and b_n to represent each harmonic), not as compact.
- The **compact trigonometric Fourier series** directly shows the amplitude and phase of each harmonic. However, it is more complex to compute.

7.3

FREQUENCY SPECTRUM

Frequency Spectrum

The frequency spectrum definition: it is a representation of the signal in the frequency domain, which shows how the signal's energy or amplitude is distributed across different frequencies.

The frequency spectrum is obtained by transforming the signal from the time domain to the frequency domain using mathematical tools such as:

- Continuous Time Fourier Series*: For CT periodic signals.
- Fourier Transform: For aperiodic signals.
- Discrete Fourier Transform (DFT): For DT periodic signals.

* Note: depending on which Fourier series and transform is used, the acquired spectra will vary accordingly.

Frequency Spectrum

Let's use Exponential Fourier Series as an example to extract Frequency spectrum of a signal:

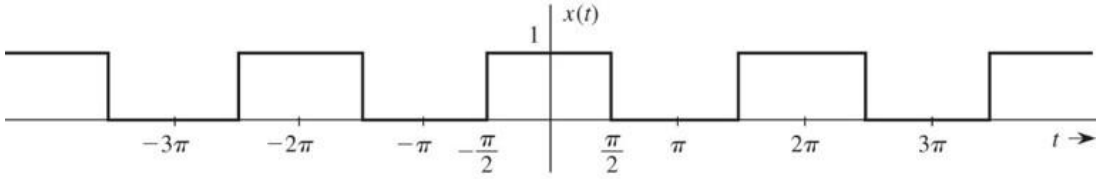
- The Fourier series coefficients c_k are referred to as the **frequency spectrum** of periodic signal $x(t)$.
- The magnitude of the Fourier series coefficients $|c_k|$ are referred to as the **magnitude spectrum** of the periodic signal $x(t)$.
- The phase of the Fourier series coefficients $\arg(c_k)$ are referred to as the **phase spectrum** of the periodic signal $x(t)$.
- The frequency spectrum is usually represented using two plots, the magnitude spectrum and the phase spectrum .

Frequency Spectrum

Let’s use Compact Trigonometric Fourier Series as an example to extract Frequency spectrum of a signal:

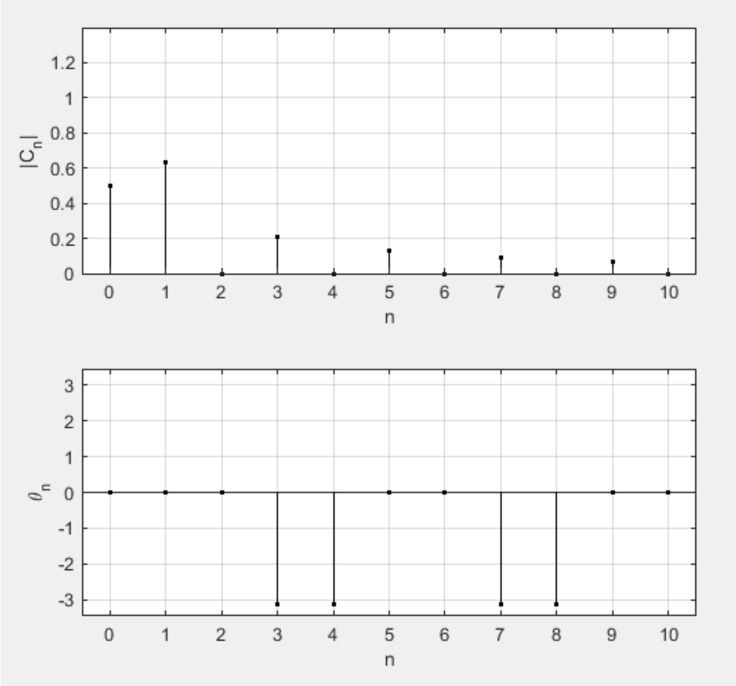
- The amplitude of harmonic d_k are referred to as the **magnitude spectrum** of the periodic signal $x(t)$.
- The phase of each harmonic θ_k are referred to as the **phase spectrum** of the periodic signal $x(t)$.
- The frequency spectrum is also represented using two plots, the magnitude spectrum and the phase spectrum .

Two Different Spectra to Represent a Periodic Signal

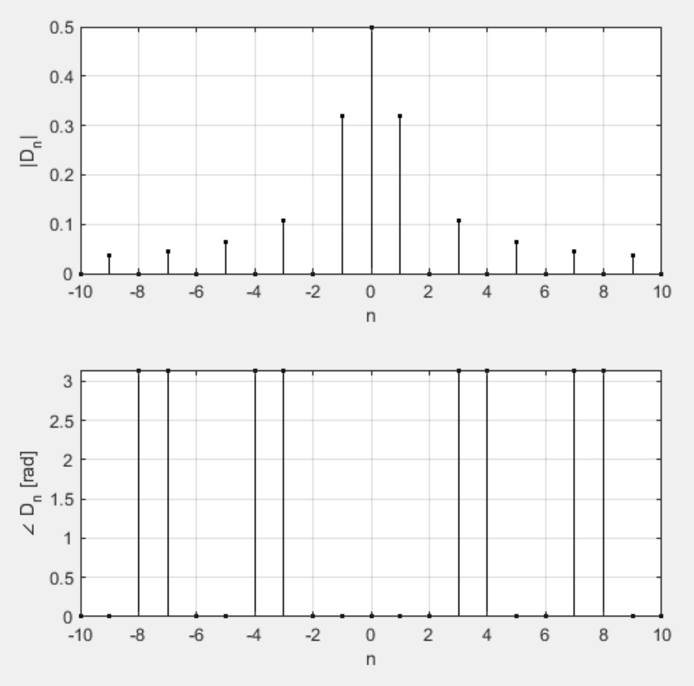


(a)

Compact Trigonometric FS



Exponential FS



7.4

Convergence of Fourier Series

Convergence of Fourier Series

Since the Fourier series have an infinite number of terms, there is need to discuss if the Fourier series is convergent.

Here we define a new type of convergence condition: MSE (mean-squared error) convergent.

Convergence of Fourier Series

MSE (mean-squared error) convergence. Truncate the Fourier series till the N^{th} harmonic components:

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}$$

The error in using $x_N(t)$ to approximate $x(t)$ is:

$$e_N(t) = x(t) - x_N(t)$$

Energy of the error is:

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2 dt$$

Convergence of Fourier Series

Definition of two types of convergence:

If $\lim_{N\to\infty} e_N(t) = 0$ for all t, the Fourier series is **pointwise convergent**;

If $\lim_{N\to\infty} E_N = 0$, the Fourier series is **MSE (mean squared error) convergent**.

Pointwise convergence is a much stronger condition than MSE convergence.

Convergence of Fourier Series

Convergence special case 1: If periodic signal $x(t)$ has finite energy in a single period:

$$\frac{1}{T}\int_{T} |x(t)|^2 dt < \infty$$

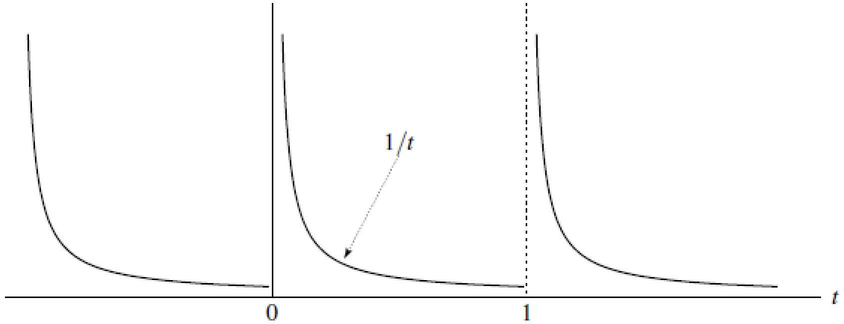
Then this signal is MSE convergent.

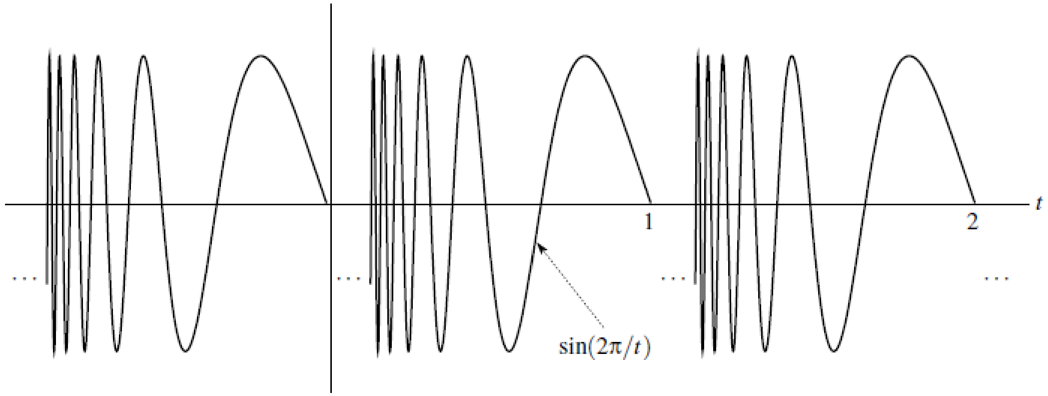
Convergence of Fourier Series

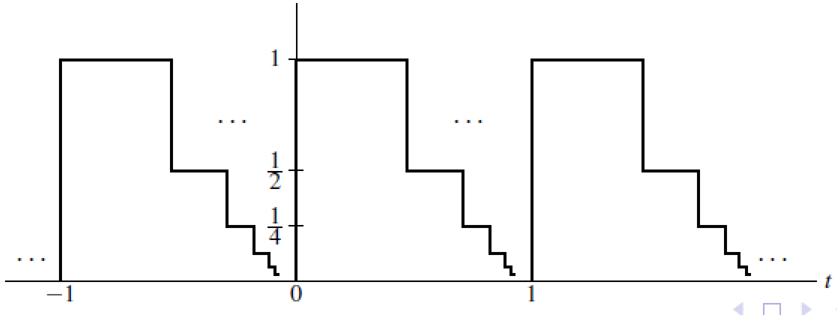
Convergence special case 2 (Dirichlet conditions): If periodic signal $x(t)$ satisfies:

- Absolutely integrable over a single period $\frac{1}{T}\int_T \; |x(t)|dt < \infty$;
- Has finite number of maxima and minima over a single period.
- Have finite number of discontinuities over an finite interval, each is finite.

Convergent or non-convergent?







Convergence of Fourier Series

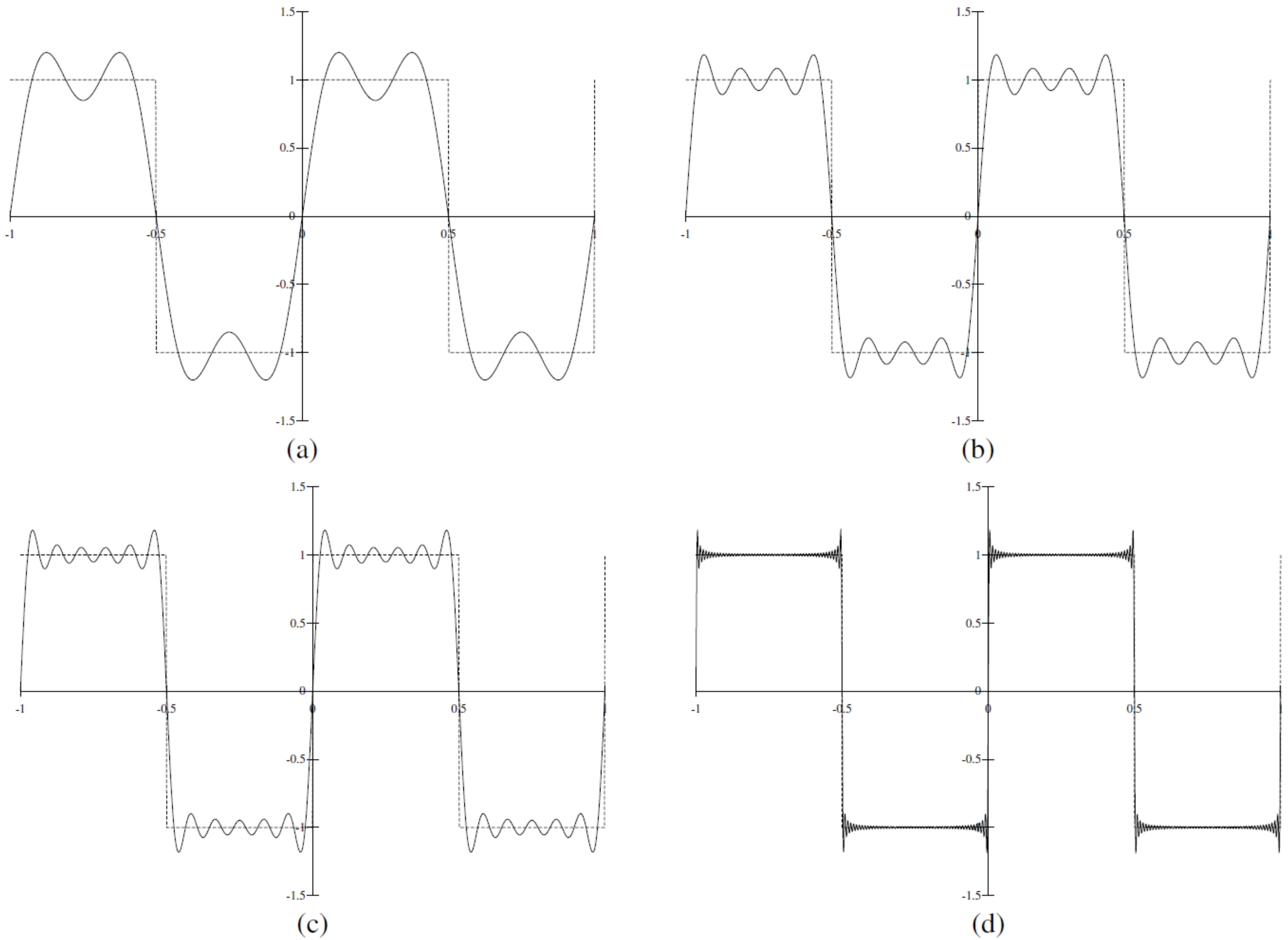
Convergence condition 2 (Dirichlet conditions): If periodic signal $x(t)$ satisfies these conditions:

- This signal is pointwise convergent everywhere except at the points of discontinuities of $x(t)$.
- Converges at the average of the left- and right-hand sides of the values of $x(t)$, at the points of discontinuities of $x(t)$.

Gibbs Phenomena

In practice, periodic signals with discontinuities are common.

- When a signal $x(t)$ has discontinuities, the Fourier series representation of $x(t)$ does not converge at the same rate everywhere.
- The rate of convergence is much slower at points close to a discontinuity relative to the rate of change.



Truncated Fourier series for the periodic square wave truncated after the Nth harmonics (a) N = 3, (b) N = 7, (c) N = 11, and (d) N = 101. (image by D. Michael Adams)