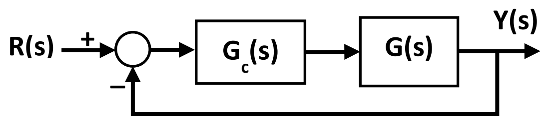
Worksheet 6 - Solution

1) Consider the following closed-loop system.



$$\boldsymbol{G(s) = \frac{1}{(s+10)(s+30)}}$$

$$\boldsymbol{G_c(s) = K}$$

a) Determine the K value so that the maximum overshoot of unit-step response is 5%.

First, calculate the desired damping ratio from the given desired maximum overshoot.

O.S. = 5% $\rightarrow \quad \zeta = \frac{-\boldsymbol{ln(O.S.)}}{\sqrt{\pi^2 + \boldsymbol{ln^2(O.S.)}}} \quad \rightarrow \quad \zeta = \frac{-ln(0.05)}{\sqrt{\pi^2 + ln^2(0.05)}} \quad \rightarrow \quad \zeta = 0.691$

Closed-loop transfer function

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{G_c(s)G(s)}}{\boldsymbol{1 + G_c(s)G(s)}} = \frac{K}{s^2 + 40s + K + 300}$$

Compare the characteristic equation with the standard second-order prototype system

$$\boldsymbol{s^2 + 2\zeta\omega_n s + \omega_n^2} = s^2 + 40s + K + 300$$

$2\zeta\omega_n = 40 \quad \rightarrow \quad 2 \times 0.6901\omega_n = 40 \quad \rightarrow \quad \boldsymbol{\omega_n = 28.981\ \mathrm{rad/sec}}$

$\omega_n^2 = K + 300 \quad \rightarrow \quad 28.981^2 = K + 300 \quad \rightarrow \quad \boldsymbol{K = 539.90}$ **Desired gain**

b) Determine the settling time (2% criteria), rise time and steady-state error of the unit-step response of the designed closed-loop system in Part (a).

Settling time:

$$\boldsymbol{t_s = \frac{4}{\zeta\omega_n}} \quad \rightarrow \quad t_s = \frac{4}{0.6901 \times 28.981} = \boldsymbol{0.2\ sec}$$

Rise time:

$$\boldsymbol{t_r \cong \frac{0.8 + 2.5\zeta}{\omega_n}} \quad \rightarrow \quad t_r = \frac{0.8 + 2.5 \times 0.6901}{28.981} = \boldsymbol{0.0871\ sec}$$

Steady-state error for unit-step response:

$$\boldsymbol{k_p = \lim_{s\to 0} G_c(s)G(s)} = \lim_{s\to 0} \frac{K}{(s+10)(s+30)} = \frac{K}{300} = \frac{539.90}{300} = 1.8$$

$$\boldsymbol{e_{ss} = \frac{R}{1 + k_p}} \quad \rightarrow \quad e_{ss} = \frac{1}{1 + 1.8} = 0.3571 \quad \rightarrow \quad \boldsymbol{e_{ss} = 35.7\%}$$

c) Find poles of the designed closed-loop system.

Transfer function of the designed closed-loop system for $K = 539.90$ is

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{G_c(s)G(s)}}{\boldsymbol{1 + G_c(s)G(s)}} = \frac{539.90}{s^2 + 40s + 839.90}$$

Closed-loop poles are obtained as

$$s^2 + 40s + 839.90 = 0 \qquad \rightarrow \qquad \boldsymbol{s = -20 \pm j20.97}$$

d) Design a lag compensator to achieve the steady-state error of 3% ($e_{ss} = 0.03$) for unit-step input without altering the closed-loop poles of the designed-system in Part (c).

$$\boldsymbol{G_c(s) = K_c\frac{s+z}{s+p}}$$

First, find the desired step-error constant k_p to achieve the desired steady-state error

$$\boldsymbol{e_{ss}} = \frac{\boldsymbol{1}}{\boldsymbol{1 + k_p}} = 0.03 \qquad \rightarrow \qquad k_p = 32.3$$

To not change the designed closed-loop poles with $K = 539.90$, the compensator's gain has to be selected equal to K

$$\boldsymbol{K_c = K = 539.90}$$

Step-error constant for compensated system is

$$\boldsymbol{k_p = \lim_{s \to 0} G_c(s)G(s)} = \lim_{s \to 0} K_c\frac{s+z}{s+p} \cdot \frac{1}{(s+10)(s+30)} = \frac{K_c z}{300p} \qquad \rightarrow \qquad 32.3 = \frac{539.9z}{300p} \qquad \rightarrow \qquad \boldsymbol{z \approx 18p}$$

Pole/zero of lag compensator must be selected far enough from the dominant closed-loop poles and close to the origin.

For example:

$$\text{If } \ z = 2 \quad \rightarrow \quad p = \frac{2}{18} = 0.11$$

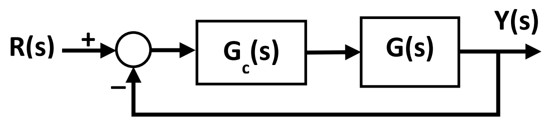
Lag Compensator

$$\boldsymbol{G_c(s) = K_c\frac{s+z}{s+p} = 539.90\frac{s+2}{s+0.11}}$$

2) Consider the following closed-loop system.

$$G(s) = \frac{s+2}{s(s+1)^2}$$

$$G_c(s) = K$$



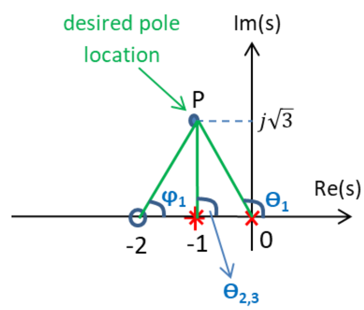
a) Determine whether it is possible to select a K value so that the dominant poles of the closed-loop system are located at $s_d = -1 \pm j\sqrt{3}$.

First, check the angle condition by calculating the angle of $G(s)$ at the desired closed-loop pole location

$$\angle G(s)|_{s=P} = \angle \frac{s+2}{s(s+1)^2}\bigg|_{s=-1+j\sqrt{3}}$$

$$= \angle(s+2) - \angle s - \angle(s+1) - \angle(s+1)|_{s=-1+j\sqrt{3}}$$

$$= \angle\varphi_1 - \angle\theta_1 - \angle\theta_2 - \angle\theta_3 = 60^\circ - 120^\circ - 90^\circ - 90^\circ = \mathbf{-240^\circ}$$



Angle condition is not satisfied

There is no K value to achieve the desired closed-loop poles

b) Design a lead compensator such that the compensated closed-loop system has dominant poles at $s_d = -1 \pm j\sqrt{3}$.

$$G_c(s) = K_c \frac{s+z}{s+p}$$

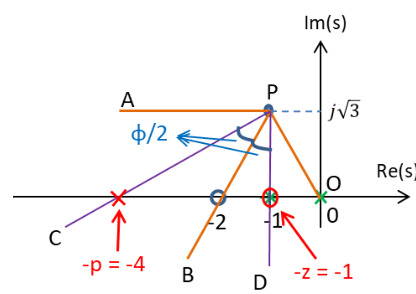
First, find the angle deficiency: $\phi = 240^\circ - 180^\circ = \mathbf{60^\circ}$

Next, design a **lead compensator** to contribute the angle of $\boldsymbol{\phi = 60^\circ}$ at the desired poles location.

Determine the <u>pole/zero locations</u> and the <u>gain</u> of the lead compensator.

- Draw lines PA and PO
- Draw bisector line PB so that $\angle APB = \angle BPO$
- Draw lines PC and PD so that $\angle CPB = \angle BPD = \frac{\phi}{2} = 30^\circ$
- Pole and zero are the intersections of PC and PD with real axis

$$\boldsymbol{z = 1,\ \ p = 4}$$



From the magnitude condition at the desired pole locations,

$$\left|K_c \frac{s+1}{s+4}\cdot\frac{s+2}{s(s+1)^2}\right|_{s=-1+j\sqrt{3}} = 1 \quad\rightarrow\quad |K_c| = \frac{|3+j\sqrt{3}||-1+j\sqrt{3}||j\sqrt{3}||j\sqrt{3}|}{|j\sqrt{3}||1+j\sqrt{3}|} \quad\rightarrow\quad \boldsymbol{K_c = 6}$$

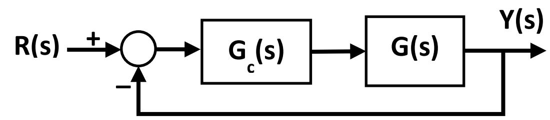
Therefore, the designed lead compensator is obtained as follows:

$$G_c(s) = K_c\frac{s+z}{s+p} = 6\frac{s+1}{s+4}$$

3) Consider the following closed-loop system.

$$\boldsymbol{G(s) = \frac{1}{s(s+2)}}$$

$$\boldsymbol{G_c(s) = K_p(1 + T_d s)}$$



a) Design a PD controller to achieve $O.S. = 5\%$ and $t_s = 1sec$.

First, find the desired poles based on the given specifications:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \rightarrow \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \rightarrow \boldsymbol{\zeta = 0.691}$$

$$t_s = \frac{4}{\zeta\omega_n} \rightarrow 1 = \frac{4}{0.7\omega_n} \rightarrow \boldsymbol{\omega_n = 5.7963}$$

The **desired closed-loop poles** location

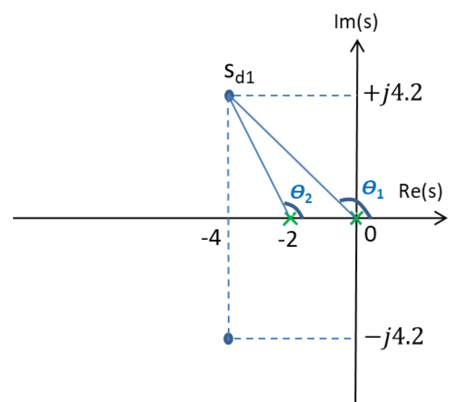
$$\boldsymbol{s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \rightarrow s_d = -4 \pm j4.2}$$

Check the angle condition at the desired pole locations:

$$\angle G(s)|_{s=s_{d1}} = \angle 1 - \angle(s) - \angle(s+2)|_{s=-4+j4.2}$$

$$= 0 - \angle\theta_1 - \angle\theta_2 = 0 - 134^\circ - 115.5^\circ = \boldsymbol{-249.5^\circ}$$

Angle condition is not satisfied



Calculate the angle deficiency: $\boldsymbol{\phi} = 249.5^\circ - 180^\circ = \boldsymbol{69.5^\circ}$

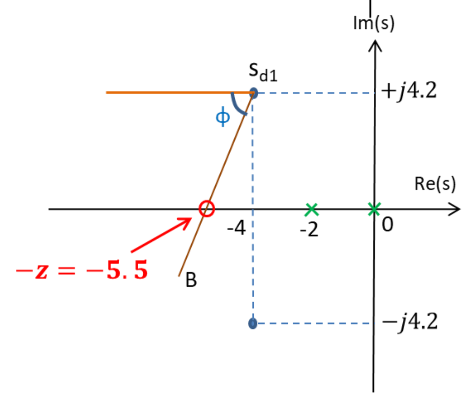
Determine the **zero** location and $\boldsymbol{T_d}$

- Draw line PA
- Draw line PB such that $\angle APB = \phi$
- The zero is located at the intersection of PB with real axis

$$\boldsymbol{z = 5.5}$$

- Determine the derivative time-constant $\boldsymbol{T_d}$

$$\boldsymbol{T_d = \frac{1}{z} = 0.18}$$



Determine the gain K_p from the magnitude condition:

$$\left|\frac{K_p(1+T_d s)}{s\,(s+2)}\right|_{s=-4+j4.2} = 1 \rightarrow |K_p| = \frac{|-4+j4.2||-2+j4.2|}{|0.28+j0.756|} \rightarrow K_p = 33.3$$

Therefore, the designed PD controller is obtained as:

$$\boldsymbol{G_c(s) = 33.3(1 + 0.18s)}$$

b) Determine steady-state error of the closed-loop system for unit-step and unit-ramp inputs.

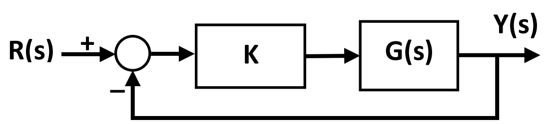
Steady-state error for unit-step input:

$$\boldsymbol{k_p} = \lim_{s\to 0} \boldsymbol{G(s)H(s)} = \lim_{s\to 0} \frac{33.65(1+0.18s)}{s(s+2)} = \infty \quad \rightarrow \quad \boldsymbol{e_{ss}} = \frac{\boldsymbol{R}}{\boldsymbol{1+k_p}} = 0$$

Steady-state error for unit-ramp input:

$$\boldsymbol{k_v} = \lim_{s\to 0} \boldsymbol{sG(s)H(s)} = \lim_{s\to 0} s\frac{33.3(1+0.18s)}{s(s+2)} = \lim_{s\to 0} \frac{33.3(1+0.18s)}{(s+2)} = 16.65 \quad \rightarrow \quad \boldsymbol{e_{ss}} = \frac{\boldsymbol{R}}{\boldsymbol{k_v}} = \frac{1}{16.65} = 0.06$$

4) Consider root-locus plot of the following system.



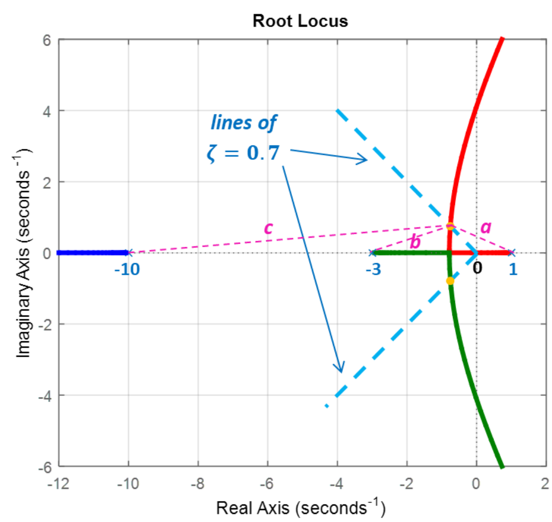
$$\boldsymbol{G(s)} = \frac{\boldsymbol{1}}{\boldsymbol{(s+3)(s+10)(s-1)}}$$

Determine value of *K* such that the dominant closed-loop poles have damping ratio of $\boldsymbol{\zeta = 0.7}$.

Plot the constant-damping-ratio-loci for $\zeta = 0.7$

$$\boldsymbol{\theta = \cos^{-1}(\zeta)} \quad \rightarrow \quad \theta = \cos^{-1}(0.7) \approx 45^\circ$$

Intersection points $\rightarrow$ $\boldsymbol{s = -0.75 \pm j0.75}$



Determine the magnitude:

Method 1: Calculation by evaluation at point A

$$|K| = |s-1||s+3||s+10|\Big|_{s=-0.75+j0.75}$$

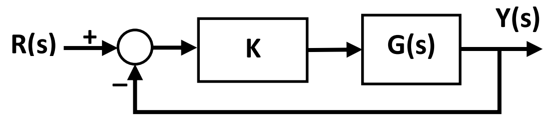
$$|K| = |-1.75+j0.75||2.25+j0.75||9.25+j0.75|$$

$$K = 1.90 \times 2.37 \times 9.28 = \mathbf{41.79}$$

Method 2: Geometrically by measuring the length of the vector

$$K = a \times b \times c = 1.9 \times 2.4 \times 9.3 = \mathbf{42.41}$$

5) Consider the following closed-loop system.



$$\boldsymbol{G(s)=\frac{s}{s^2+s+4.25}}$$

a) Sketch the root-locus for $K \in [0,+\infty)$ on the s-plane.

Step 1: Draw the axes of the s-plane and locate the open-loop poles/zeros

Poles $\rightarrow$ $p_1=-0.5+j2,\ \ p_2=-0.5-j2$

Zeros $\rightarrow$ $z_1=0$, one zero at infinity

Step 2: Draw the root-locus on the real axis.

The left side of 0 is on the root-locus.

Step 3: Draw asymptote lines for large *K* values

Number of asymptote lines: $n-m=2-1=1$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha=\frac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}}=\frac{[(-0.5+j2)+(-0.5-j2)]-[0]}{2-1}=\boldsymbol{-1}$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i=\frac{180^\circ}{n-m}(2i+1)}=\frac{180^\circ}{2-1}(2i+1)=180^\circ(2i+1)\ \rightarrow\ \boldsymbol{\varphi_0=\ 180^\circ}$$

The asymptote line lies on the real axis

Step 4: Intersection of root-locus with imaginary axis

$$s^2+(1+K)s+4.25=0$$

Set $s=j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$$(j\omega)^2+(1+K)(j\omega)+4.25=0\qquad\rightarrow\qquad-\omega^2+j\omega(1+K)+4.25=0$$

$$\underbrace{[-\omega^2+4.25]}_{\boldsymbol{real\ part}}+j\ \underbrace{[\omega(1+K)]}_{\boldsymbol{imaginary\ part}}\ =0$$

From the imaginary part:

$$\boldsymbol{-\omega^2+4.25=0}\quad\rightarrow\quad\omega^2=4.25\quad\rightarrow\quad\omega=\pm\sqrt{4.25}$$

From the real part:

For $\omega^2=4.25\quad\rightarrow\quad\omega(1+K)=\pm\sqrt{4.25}(1+K)=0\ \rightarrow\ \boldsymbol{K=-1<0}$ **Not acceptable**

Therefore, the root-locus will not cross the imaginary axis

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $s^2 + (1+K)s + 4.25 = 0$

Find the K from the characteristic equation:

$$K = \frac{-s^2 - s - 4.25}{s}$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad \frac{(-2s-1)(s) - (-s^2 - s - 4.25)}{(s)^2} = 0 \quad \rightarrow \quad -s^2 + 4.25 = 0$$

The roots are:

$s = +2.06$ → Not on the root locus

$s = -2.06$ → On the root locus (Break-in point)

The associate gain for the break-in point:

$$K = \frac{-(-2.06)^2 - (-2.06) - 4.25}{(-2.06)} = 3.12$$

Step 6: Calculate angle of departure from the complex poles

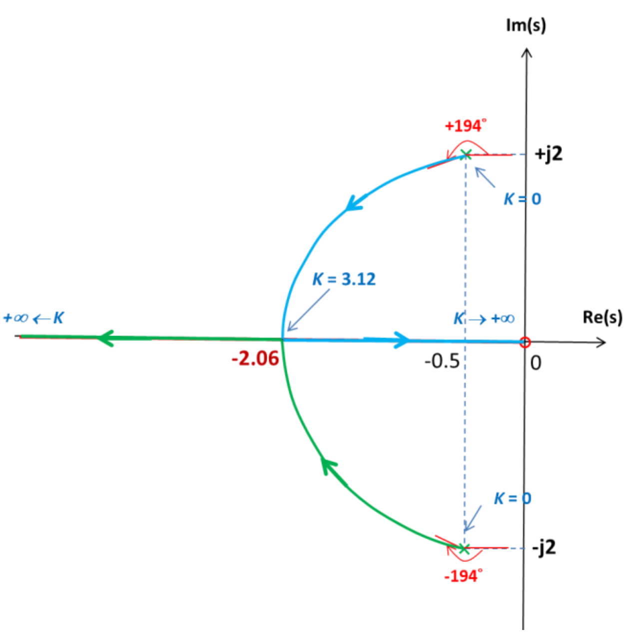
The angle of departure from the complex pole at $s = -0.5 + j2$ is:

$$\phi_p = 180^{\circ} - \sum_i \angle p_i + \sum_j \angle z_j = 180° - (\theta_1) + (\varphi_1)$$

$$= 180° - (90°) + (104°) = 194°$$

Since the root-locus is symmetrical with respect to the real axis, the angle of departure for the complex pole $s = -0.5 - j2$ is $-194°$.

Step 7: Complete the root-locus diagram

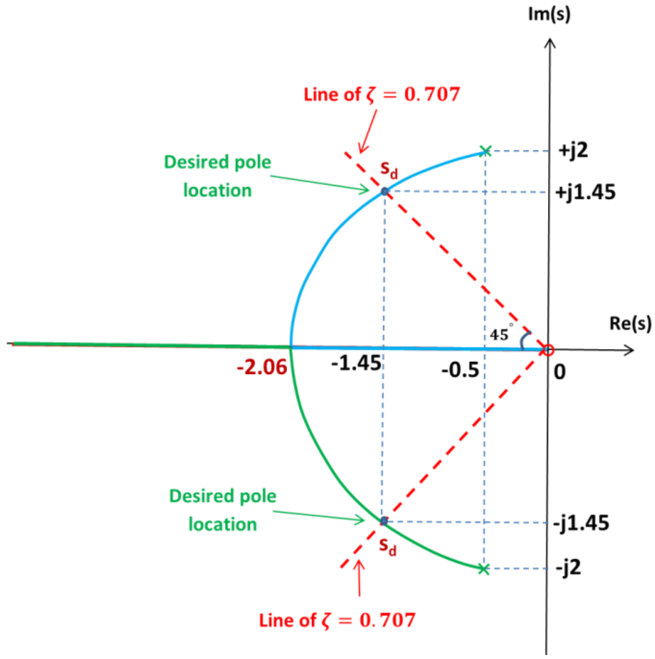


b) Determine the closed-loop poles with damping ratio of $\zeta = 0.707$ and the corresponding K value.

Plot the constant-damping-ratio-loci for $\zeta = 0.7$

$\boldsymbol{\theta = \cos^{-1}(\zeta)} \quad \rightarrow \quad \theta = \cos^{-1}(0.7) \approx 45^\circ$

Intersection points $\quad \rightarrow \quad \boldsymbol{s = -1.45 \pm j1.45}$



Determine the magnitude:

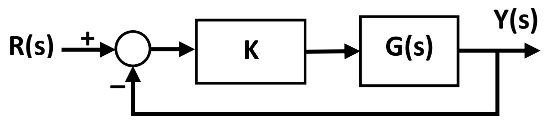
Calculation by evaluation at point A

$$|K| = \frac{|s + 0.5 + j2||s + 0.5 - j2|}{|s|}\Bigg|_{s=-1.45+j1.45}$$

$$|K| = \frac{|-0.95 + j3.45||-0.95 - j0.55|}{|-1.45 + j1.45|}$$

$K = \mathbf{1.91}$

6) Consider the following closed-loop system.



$$\boldsymbol{G(s) = \frac{s+2}{s^2 - 2s + 2}}$$

a) Find the closed-loop transfer function and the characteristic equation.

The closed-loop transfer function:

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{KG(s)}}{\boldsymbol{1 + KG(s)H(s)}} = \frac{K\dfrac{s+2}{s^2-2s+2}}{1 + K\dfrac{s+2}{s^2-2s+2}} = \frac{K(s+2)}{s^2 + (K-2)s + 2 + 2K}$$

The closed-loop characteristic equation:

$$\boldsymbol{1 + KG(s)H(s) = 0} \quad \rightarrow \quad s^2 + (K-2)s + 2 + 2K = 0$$

b) Using the Routh-Hurwitz table, find the range of K for which the closed-loop system is stable.

The characteristic equation is: $s^2 + (K-2)s + 2 + 2K = 0$

Create the Routh-Hurwitz table:

$\boldsymbol{s^2}$	1	$2 + 2K$
$\boldsymbol{s^1}$	$K - 2$	0
$\boldsymbol{s^0}$	$2 + 2K$	0

For stability all terms in the first column must be positive:

$K - 2 > 0 \quad \rightarrow \quad \boldsymbol{K} > 2$

$2 + 2K > 0 \quad \rightarrow \quad \boldsymbol{K > -1}$

$\boxed{\boldsymbol{K > 2} \qquad \textbf{Stability Condition}}$

- For $K = 2$ the closed-loop system is marginally stable.
- For $K < 2$ the closed-loop system is unstable.

c) Sketch the root-locus for $\boldsymbol{K} \in [\mathbf{0}, +\infty)$ on the s-plane.

Step 1: Draw the axes of the s-plane and locate the open-loop poles/zeros

Poles $\rightarrow \quad p_1 = 1 + j,\;\; p_2 = 1 - j$

Zeros $\rightarrow \quad z_1 = -2$, one zero at infinity

Step 2: Draw the root-locus on the real axis.

The left side of -2 is on the root-locus.

Step 3: Draw asymptote lines for large *K* values

Number of asymptote lines: $n - m = 2 - 1 = 1$

Intersection of asymptotes on the real axis:

$$\boldsymbol{\alpha} = \frac{\sum_{i=1}^{n} \boldsymbol{p_i} - \sum_{i=1}^{m} \boldsymbol{z_i}}{\boldsymbol{n} - \boldsymbol{m}} = \frac{[(1 + j1) + (1 - j1)] - [(-2)]}{2 - 1} = 4$$

Angle of asymptote lines with real axis:

$$\boldsymbol{\varphi_i} = \frac{\mathbf{180^\circ}}{\boldsymbol{n} - \boldsymbol{m}}(\mathbf{2}\boldsymbol{i} + \mathbf{1}) = \frac{180^\circ}{2 - 1}(2i + 1) = 180^\circ(2i + 1) \quad \rightarrow \quad \boldsymbol{\varphi_0} = \mathbf{180^\circ}$$

The asymptote line lies on the real axis

Step 4: Intersection of root-locus with imaginary axis

$$s^2 + (K - 2)s + 2 + 2K = 0$$

Set $s = j\omega$ in the closed-loop characteristic equation and solve for ω and K:

$$(j\omega)^2 + (K - 2)(j\omega) + 2 + 2K = 0 \quad \rightarrow \quad -\omega^2 + j(K - 2)\omega + 2 + 2K = 0$$

$$\underbrace{[-\omega^2 + 2 + 2K]}_{\boldsymbol{real\ part}} + j\ \underbrace{[K\omega - 2\omega]}_{\boldsymbol{imaginary\ part}} = 0$$

From the imaginary part:

$$\boldsymbol{K\omega - 2\omega = 0} \quad \rightarrow \quad \omega(K - 2) = 0 \quad \rightarrow \begin{cases} \boldsymbol{\omega = 0} \\ K - 2 = 0 \quad \rightarrow \quad \boldsymbol{K = 2} \end{cases}$$

From the real part:

For $\omega = 0 \quad \rightarrow \quad -\omega^2 + 2 + 2K = -0^2 + 2 + 2K = 0 \quad \rightarrow \quad \boldsymbol{K = -1 < 0}$ **Not acceptable**

For $K = 2 \quad \rightarrow \quad -\omega^2 + 2 + 2K = -\omega^2 + 2 + 2(2) = 0 \quad \rightarrow \quad \boldsymbol{\omega = \pm\sqrt{6} \pm 2.45}$

Therefore, the root-locus will cross the imaginary axis at $s = \pm j2.45$ for gain $K = 2$.

Step 5: Calculate the break-away/break-in points on real axis

The closed-loop characteristic equation is: $s^2 + (K-2)s + 2 + 2K = 0$

Find the K from the characteristic equation:

$$K = \frac{-s^2 + 2s - 2}{s+2}$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad \frac{(-2s+2)(s+2) - (-s^2+2s-2)}{(s+2)^2} = 0 \quad \rightarrow \quad -s^2 - 4s + 6 = 0$$

The roots are:

$s = 1.16$ $\rightarrow$ Not on the root locus

$s = -5.16$ $\rightarrow$ On the root locus (Break-in point)

The associate gain for the break-in point:

$$K = \frac{-(-5.16)^2 + 2(-5.16) - 2}{(-5.16) + 2} = 12.32$$

Step 6: Calculate angle of departure from the complex poles

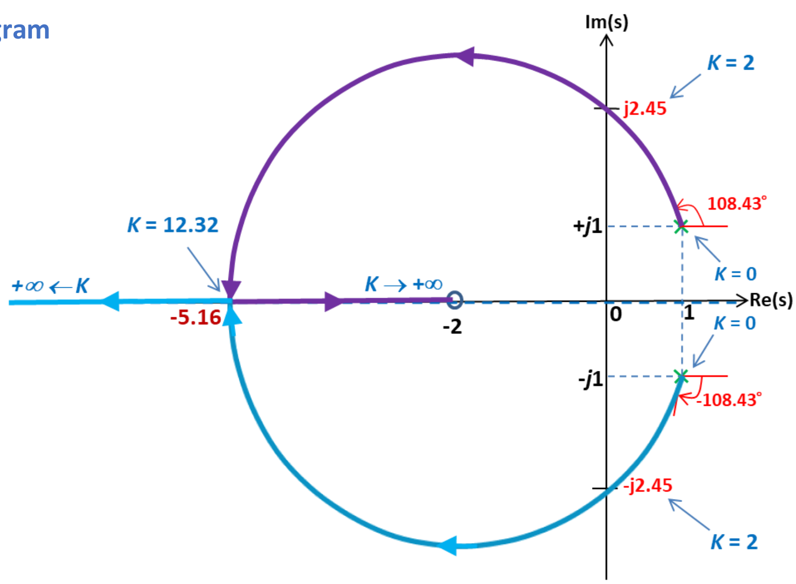
The angle of departure from the complex pole at $s = +1 + j1$ is:

$$\phi_p = 180^\circ - \sum_i \angle p_i + \sum_j \angle z_j = 180^\circ - (\theta_1) + (\varphi_1)$$

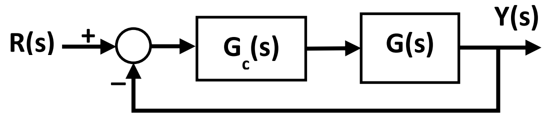
$$= 180^\circ - (90^\circ) + (18.43^\circ) = 108.43^\circ$$

Since the root-locus is symmetrical with respect to the real axis, the angle of departure for the complex pole $s = +1 - j1$ is $-108.43°$.

Step 7: Complete the root-locus diagram



7) Consider the following closed-loop system



$$\boldsymbol{G(s) = \frac{1}{s^3 + s^2 + 2s - 0.5}}$$

$$\boldsymbol{G_c(s) = 1 + \frac{K}{s}}$$

a) Determine if $G(s)$ is a stable system or not.

Characteristic equation: $\boldsymbol{s^3 + s^2 + 2s - 0.5 = 0}$

Since one of the coefficients is negative, the characteristic equation has pole at the right-half of s-plane.

Therefore, $G(s)$ is unstable.

b) Determine the closed-loop transfer function and closed-loop characteristic equation.

The closed-loop transfer function:

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{G_c(s)G(s)}}{\boldsymbol{1 + G_c(s)G(s)}} = \frac{(1 + \frac{K}{s})(\frac{1}{s^3 + s^2 + 2s - 0.5})}{1 + (1 + \frac{K}{s})(\frac{1}{s^3 + s^2 + 2s - 0.5})} = \frac{s + K}{s^4 + s^3 + 2s^2 + 0.5s + K}$$

The closed-loop characteristic equation:

$$\boldsymbol{1 + KG(s)H(s) = 0} \quad \rightarrow \quad \boldsymbol{s^4 + s^3 + 2s^2 + 0.5s + K = 0}$$

c) Determine the range of K such that the closed-loop system is stable.

Closed-loop system characteristic equation: $\boldsymbol{s^4 + s^3 + 2s^2 + 0.5s + K = 0}$

The Routh-Hurwitz table:

$\boldsymbol{s^3}$	1	2	K
$\boldsymbol{s^2}$	1	0.5	0
$\boldsymbol{s^2}$	1.5	K	0
$\boldsymbol{s^1}$	$\frac{0.75-K}{1.5}$	0	0
$\boldsymbol{s^0}$	K	0	0

For stability all terms in the first column must be positive:

$$\frac{0.75 - K}{1.5} > 0 \quad \rightarrow \quad 0.75 - K > 0 \quad \rightarrow \quad \boldsymbol{K < 0.75}$$

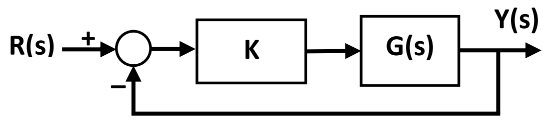
$$\boldsymbol{K > 0}$$

$\boxed{\boldsymbol{0 < K < 0.75}}$ **Stability Condition**

- For $K = 0$ and $K = 0.75$ the closed-loop system is marginally stable.
- For $K < 0$ and $K > 0.75$ the closed-loop system is unstable.

8) Consider the following closed-loop system.

$$G(s) = \frac{1}{s(s+2)}$$



a) Determine the range of K such that the closed-loop system is stable.

Closed-loop system transfer function

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)} = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{K}{s^2+2s+K}$$

Closed-loop system characteristic equation: $s^2 + 2s + K = 0$

The Routh-Hurwitz Table:

s^2	1	K
s^1	2	0
s^0	K	0

For stability all terms in the first column must be positive:

$\boxed{K > 0}$ **Stability Condition**

- For $K = 0$ the closed-loop system is marginally stable.
- For $K < 0$ the closed-loop system is unstable.

b) Determine the range of K such that the closed-loop system has over-damped, critically damped and under-damped dynamics.

Closed-loop system characteristic equation: $s^2 + 2s + K = 0$

Closed-loop system poles: $s_{1,2} = -1 \pm \sqrt{1-K}$

For over-damped system $\rightarrow \quad 1 - K > 0 \quad \rightarrow \quad 0 < K < 1$

For critically-damped system $\rightarrow \quad 1 - K = 0 \quad \rightarrow \quad K = 1$

For under-damped system $\rightarrow \quad 1 - K < 0 \quad \rightarrow \quad K > 1$