5.2 Trigonometric Substitutions

Intro

- The method of trigonometric substitution is a method of evaluating integrals containing radicals of the form

$$\sqrt{x^2 + a^2},\ \sqrt{x^2 - a^2},\ \sqrt{a^2 - x^2},\ \text{where } a \text{ is a positive constant (parameter).}$$

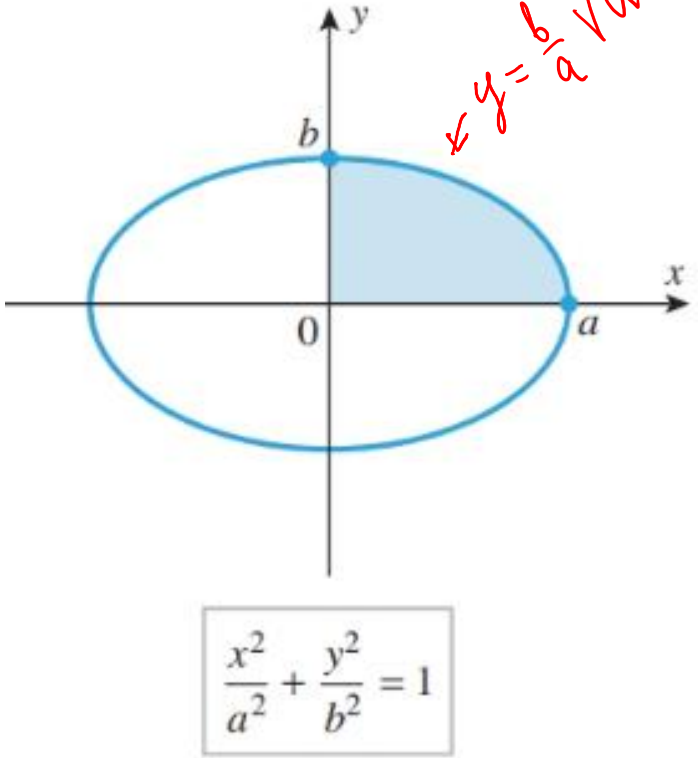
- The basic idea for evaluating such integrals is to make a substitution that eliminates the radical. This can be done by using the relations of the sides in the right triangle and trigonometric ratios associated with it.

Motivation

Example 1. Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\leftarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$



Solution. Since the ellipse is symmetric about both axis, its area A is four times the area in the first quadrant.

- The equation of the upper half of the ellipse is

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

- Hence, the area A is given

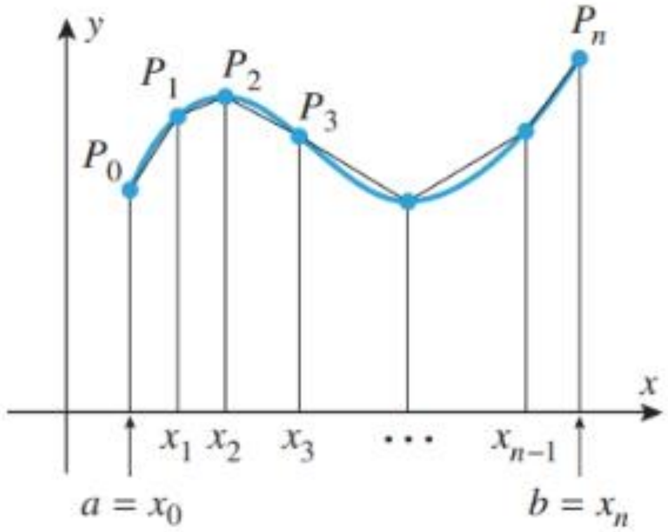
$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx$$

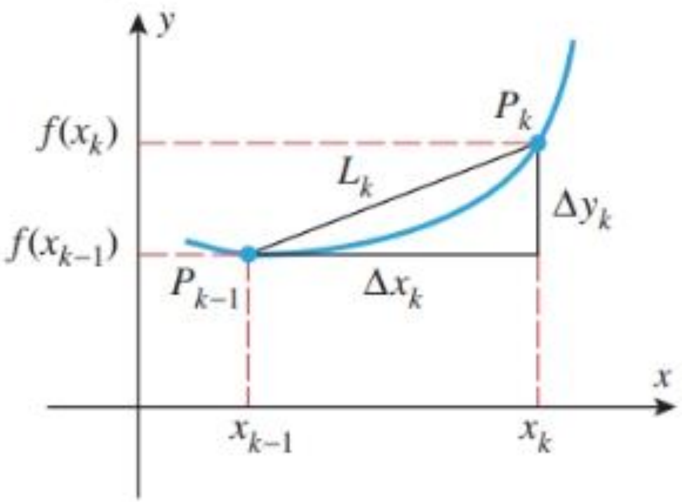
- Integration by trig.substitution yields the formula $A = \pi ab$

5.4.2 DEFINITION If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{3}$$

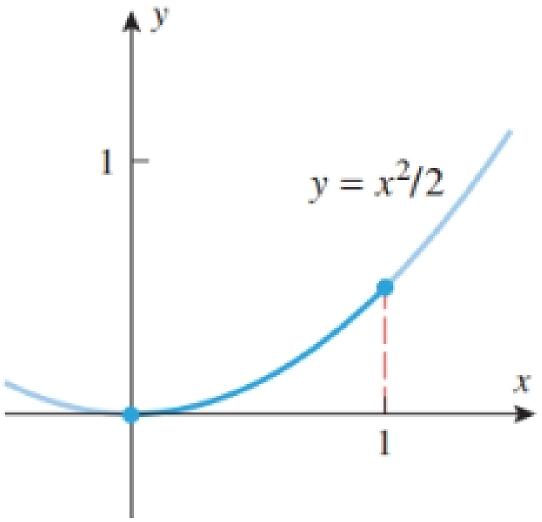
Shorter line segments provide a better approximation to the curve.





Example 2

Find the arc length of the curve $y = \frac{x^2}{2}$ from $x = 0$ to $x = 1$.



Solution. The arc length L of the curve is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + x^2}\, dx$$

- Integration by trig.substitution yields

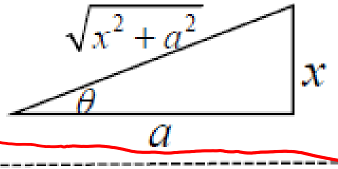
$$\frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2} + 1)\right] \approx 1.148$$

Trigonometric Substitutions

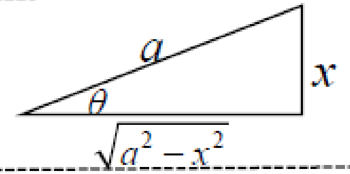
BKM

Expression	Substitution	Identity
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

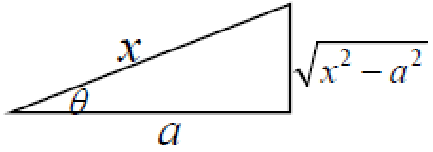
$\sqrt{x^2+a^2}$ Let $x = a\tan\theta$, $\boxed{\sqrt{x^2+a^2} = a\sec\theta}$



$\sqrt{a^2-x^2}$ Let $x = a\sin\theta$, $\sqrt{a^2-x^2} = a\cos\theta$



$\sqrt{x^2-a^2}$ Let $x = a\sec\theta$, $\sqrt{x^2-a^2} = a\tan\theta$



$\sqrt{a^2+x^2}$

$x = a\tan\theta \to$ sub

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$

$= \sqrt{a^2(1+\tan^2\theta)}$

$= a\sqrt{1+\tan^2\theta}$

$= a\sqrt{\sec^2\theta}$

$= a|\sec\theta|$

$= a\sec\theta$

Process Summary:

1. Match the radical to the substitution;
2. Adjust a, find dx, change the variable of integration from x to θ, simplify;
3. Integrate in terms of θ;
4. Return from θ to x for the final answer.
5. In case of definite integral, evaluate at the upper and lower limits of integration.

https://www.nasa.gov/press-release/nasa-reveals-webb-telescope-s-first-images-of-unseen-universe

https://apod.nasa.gov/apod/ap221121.html