Midterm Formula Sheet – MENG3520

Trigonometric Identities:

$e^{\pm jx} = \cos x \pm j\sin x$ $\quad$ $\cos x = \frac{1}{2}[e^{jx} + e^{-jx}]$ $\quad$ $\sin x = \frac{1}{2j}[e^{jx} - e^{-jx}]$

$\sin^2 x + \cos^2 x = 1$ $\quad$ $\cos^2 x - \sin^2 x = \cos 2x$ $\quad$ $2\sin x\cos x = \sin 2x$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$ $\quad$ $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos\left(x \pm \frac{\pi}{2}\right) = \mp\sin\ x$ $\quad$ $\sin\left(x \pm \frac{\pi}{2}\right) = \pm\cos\ x$

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$ $\quad$ $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

$\sin x\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$ $\quad$ $\cos x\cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

$\sin x\cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

Complex Numbers:

$e^{\pm j\pi/2} = \pm j$ $\quad$ $e^{\pm(2k+1)j\pi} = -1, k\ integer$ $\quad$ $e^{\pm 2kj\pi} = 1, k\ integer$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$ $\quad$ $a + jb = re^{j\theta}$, where $r = \sqrt{a^2+b^2}, \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\left(re^{j\theta}\right)^k = r^k e^{jk\theta}$ $\quad$ $\left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)}$

Common Derivative Formulas and Indefinite Integrals:

$$\frac{d}{dx}f(u) = \frac{d}{du}f(u)\frac{du}{dx}$$
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dx^n}{dx} = nx^{n-1}$$
$$\frac{d}{dx}\ln(ax) = \frac{1}{x}$$
$$\frac{d}{dx}\log(ax) = \frac{\log e}{x}$$
$$\frac{d}{dx}e^{bx} = be^{bx}$$
$$\frac{d}{dx}a^{bx} = b(\ln a)a^{bx}$$
$$\frac{d}{dx}\sin ax = a\cos ax$$
$$\frac{d}{dx}\cos ax = -a\sin ax$$
$$\frac{d}{dx}\tan ax = \frac{a}{\cos^2 ax}$$
$$\frac{d}{dx}(\sin^{-1} ax) = \frac{a}{\sqrt{1-a^2x^2}}$$
$$\frac{d}{dx}(\cos^{-1} ax) = \frac{-a}{\sqrt{1-a^2x^2}}$$
$$\frac{d}{dx}(\tan^{-1} ax) = \frac{a}{1+a^2x^2}$$

$$\int u\,dv = uv - \int v\,du$$
$$\int f(x)\dot{g}(x)\,dx = f(x)g(x) - \int \dot{f}(x)g(x)\,dx$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax$$
$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$
$$\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$$
$$\int x\cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax)$$
$$\int x^2\sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax)$$
$$\int x^2\cos ax\,dx = \frac{1}{a^3}(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$
$$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$
$$\int \sin ax\cos bx\,dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right] \qquad a^2 \neq b^2$$
$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$
$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$
$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

Basic Signals:

$\delta(t) = \frac{du(t)}{dt} \iff u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$

$\int_{-\infty}^{\infty} x(t)\delta(t - t0)dt = x(t0)$

$Even\{x(t)\} = \frac{1}{2}\big(x(t) + x(-t)\big)$

$Odd\{x(t)\} = \frac{1}{2}\big(x(t) - x(-t)\big)$

$Even\{x[n]\} = \frac{1}{2}(x[n] + x[-n])$

$Odd\{x[n]\} = \frac{1}{2}(x[n] - x[-n])$

$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} E_\infty$

$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$

$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

First derivative of a CT signal	**First difference of a DT signal**
$\frac{d}{dt} g(t) = \lim_{\Delta t\to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t}$ $\frac{d}{dt} g(t) = \lim_{\Delta t\to 0} \frac{g(t)-g(t\Delta t)}{\Delta t}$ $\frac{d}{dt} g(t) = \lim_{\Delta t\to 0} \frac{g(t+\Delta t)-g(t\Delta t)}{2\Delta t}$	Forward difference: $diff(g[n]) = g[n+1] - g[n]$ Backward difference: $diff(g[n]) = g[n] - g[n-1]$ Central difference: $diff(g[n]) = \frac{1}{2}\{g[n+1] - g[n-1]\}$

$\sum_{k=-\infty}^{+\infty} x[k]h[n-k] \triangleq x[n] * h[n]$

$\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \triangleq x(t) * h(t)$

$$sinc(x) \triangleq \frac{\sin(\pi x)}{\pi x} \qquad rect\left(\frac{t}{\tau}\right) = \begin{cases} 1, when -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0, otherwise \end{cases}$$

Properties of Convolution:

$x * h = h * x$ $\qquad x * (h_1 * h_2) = (x * h_1) * h_2$ $\qquad x * (h_1 + h_2) = (x * h_1) + (x * h_2)$

CT Transfer Function:

$$Y(s) = X(s)H(s) \qquad H(s) = \frac{Y(s)}{X(s)} = |H(s)|e^{j\Phi}, s = \sigma + j\omega$$

Laplace Transform and Inverse Laplace transform:

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt \qquad x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Laplace Transform Pairs:

Time domain signal	*S-domain transform*	*ROC*
$\delta(t)$	1	*All s*
$u(t)$	$1/s$	$Re\{s\} > 0$
$-u(-t)$	$1/s$	$Re\{s\} < 0$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$	$Re\{s\} > \lambda$
$-e^{\lambda t}u(-t)$	$\frac{1}{s-\lambda}$	$Re\{s\} < \lambda$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$	$Re\{s\} > \lambda$
$[\cos bt]u(t)$	$\frac{s}{s^2+b^2}$	$Re\{s\} > 0$
$[\sin bt]u(t)$	$\frac{b}{s^2+b^2}$	$Re\{s\} > 0$
$e^{-at}[\cos bt]u(t)$	$\frac{s-a}{(s-a)^2+b^2}$	$Re\{s\} > a$

$e^{-at}[\sin bt]u(t)$	$\dfrac{b}{(s-a)^2+b^2}$	$Re\{s\} > a$

Properties of Laplace Transform:

Property	*Signal*	*Laplace transform*	*ROC*
	$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	At least $R_1 \cap R_2$.
Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-domain	$e^{s_0t}x(t)$	$X(s-s_0)$	s is in ROC if $s-s_0$ is in R.
Time scaling	$x(at), a>0$	$\left\|\frac{1}{a}\right\| X\left(\frac{s}{a}\right)$	s is in ROC if s/a is in R.
Conjugation	$x^*(t)$	$X^*(s)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$.
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R.
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the time domain	$\int_{-\infty}^{\tau} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{Re\{s\} > 0\}$.

DT Transfer Function:

$$Y(z) = X(z)H(z) \qquad\qquad H(z) = \frac{Y(z)}{X(z)} = |H(z)|e^{j\Phi}$$

Z Transform and Inverse Z Transform:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]z^{-k}, \qquad x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz$$

Z Transform Pairs:

Time domain signal $x[n]$	*Z-domain transform* $X(z)$	*ROC*
$\delta[n]$	1	*All z*
$\delta[n-k]$	z^{-k}	All z, except for 0, if $k>0$, or ∞, if $k<0$.
$u[n]$	$\dfrac{z}{z-1}$	$\|z\| > 1$
$-u[-n-1]$	$\dfrac{z}{z-1}$	$\|z\| < 1$
$\gamma^{n-1}u[n-1]$	$\dfrac{1}{z-\gamma}$	$\|z\| > \|\gamma\|$
$\gamma^n u[n]$	$\dfrac{z}{z-\gamma}$	$\|z\| > \|\gamma\|$
$n\gamma^n u[n]$	$\dfrac{\gamma z}{(z-\gamma)^2}$	$\|z\| > \|\gamma\|$
$\|\gamma\|^n \cos(\beta n)\, u[n]$	$\dfrac{z(z-\|\gamma\|\cos\beta)}{z^2-(2\|\gamma\|\cos\beta)z+\|\gamma\|^2}$	$\|z\| > \|\gamma\|$
$\|\gamma\|^n \sin(\beta n)\, u[n]$	$\dfrac{z\|\gamma\|\sin\beta}{z^2-(2\|\gamma\|\cos\beta)z+\|\gamma\|^2}$	$\|z\| > \|\gamma\|$

$\cos(\beta n)\, u[n]$	$\frac{z(z-\cos\beta)}{z^2-2(\cos\beta)z+1}$	$\lvert z\rvert > 1$
$\sin(\beta n)\, u[n]$	$\frac{z\sin\beta}{z^2-2(\cos\beta)z+1}$	$\lvert z\rvert > 1$

Z Transform Properties:

Property	*Signal*	*Z transform*	*ROC*
	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	$R_1\cap R_2$
Time shifting	$x[n-n_0]u[n-n_0]$	$z^{-n_0}X(z)$	R except $z=0$ or $\lvert z\rvert=\infty$ in some cases.
Time shifting	$x[n-n_0]u[n]$	$z^{-n_0}X(z)$ $+z^{-n_0}\sum_{k=0}^{n_0-1}x[-k]z^k$	R except $z=0$ or $\lvert z\rvert=\infty$ in some cases.
Time reversal	$x[-n]$	$X(z^{-1})$	z is in ROC if z^{-1} is in R
Time expansion	$x_k[n]=\begin{cases} x\left[\frac{n}{k}\right], \frac{n}{k} \text{ is an integer} \\ 0, \text{otherwise}\end{cases}$	$X(z^k)$	z is in ROC if z^k is in R.
z-domain scaling	$a^n x[n]u[n]$	$X\left(\frac{z}{a}\right)$	z is in ROC if z/a is in R.
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least $R_1\cap R_2$.
First backward difference	$x[n]-x[n-1]$	$(1-z^{-1})X(z)$	At least $R\cap\{\lvert z\rvert>0\}$.
Differentiation in the z-domain	$-nx[n]$	$\frac{d}{dz}X(z)$	R
Accumulation in the time domain	$\sum_{k=-\infty}^{n}x[k]$	$\frac{z}{z-1}X(z)$	At least $R\cap\{\lvert z\rvert>1\}$.

CTFS and Inverse CTFS:

$$x(t)=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t},\qquad c_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt,\quad here\ T=\frac{1}{2\pi\omega_0}$$

$$x(t)=a_0+\sum_{k=1}^{\infty}(a_k\cos k\omega_0 t+b_k\sin k\omega_0 t),\qquad a_0=\frac{1}{T}\int_T x(t)\,dt, a_k=\frac{2}{T}\int_T x(t)\cos(k\omega_0 t)\,dt, b_k=\frac{2}{T}\int_T x(t)\sin(k\omega_0 t)\,dt$$

$$x(t)=a_0+\sum_{k=1}^{\infty}(a_k\cos k\omega_0 t+b_k\sin k\omega_0 t)=c_0+\sum_{k=1}^{\infty}d_k(\cos(k\omega_0 t+\theta_k)), c_0=a_0=\frac{1}{T}\int_T x(t)\,dt, d_k=\sqrt{a_k^2+b_k^2}, \theta_k=\tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

CTFT and Inverse CTFT:

$$X(\omega)\triangleq\mathcal{F}(x(t))=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt,\qquad x(t)\triangleq\mathcal{F}^{-1}(X(\omega))=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\omega)e^{j\omega t}d\omega$$

Fourier Transform Properties:

$x(t)\overset{CTFT}{\longleftrightarrow}X(\omega)\qquad y(t)\overset{CTFT}{\longleftrightarrow}Y(\omega)$

Property	*Time Domain*	*Fourier Domain*
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\omega) + \beta Y(\omega)$
Translation / time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Conjugation	$x^*(t)$	$X^*(\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Time and frequency scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
Differentiation in time	$\frac{d}{dt} x(t)$	$j\omega X(\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Differentiation in frequency	$tx(t)$	$j\frac{d}{d\omega} X(\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Even Symmetry	$x(t)$ real and even	$X(\omega)$ even and real
Conjugate symmetry	$x(t)$ real	$X(\omega) = X^*(-\omega)$

Discrete Fourier Transform

$$x[n] = \sum_{k=\langle N\rangle} X_k e^{jk\frac{2\pi}{N}n} = \sum_{k=\langle N\rangle} |X_k| e^{j(k\frac{2\pi}{N}n + \phi k)} \qquad X[k] = X_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n}$$