Class Note 2.2

September 16, 2022 9:26 AM

MODULE 2.2
FURTHER RULES OF DIFFERENTIATION

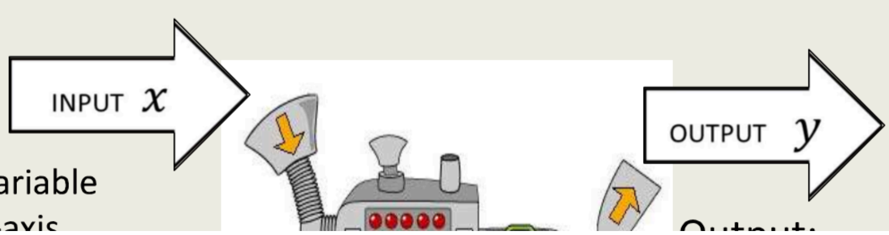
Function "machine" or a "computer program"

- a function is a rule that associates a unique **output** (y) with each **input** (x)

notation

placeholder

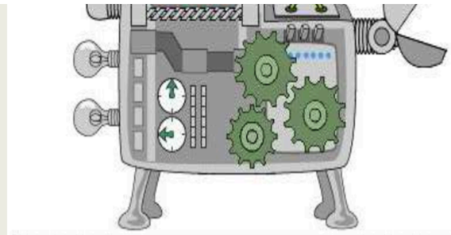
$$y = f(x)$$



Input:
- Independent variable
- Displayed on x-axis

Output:

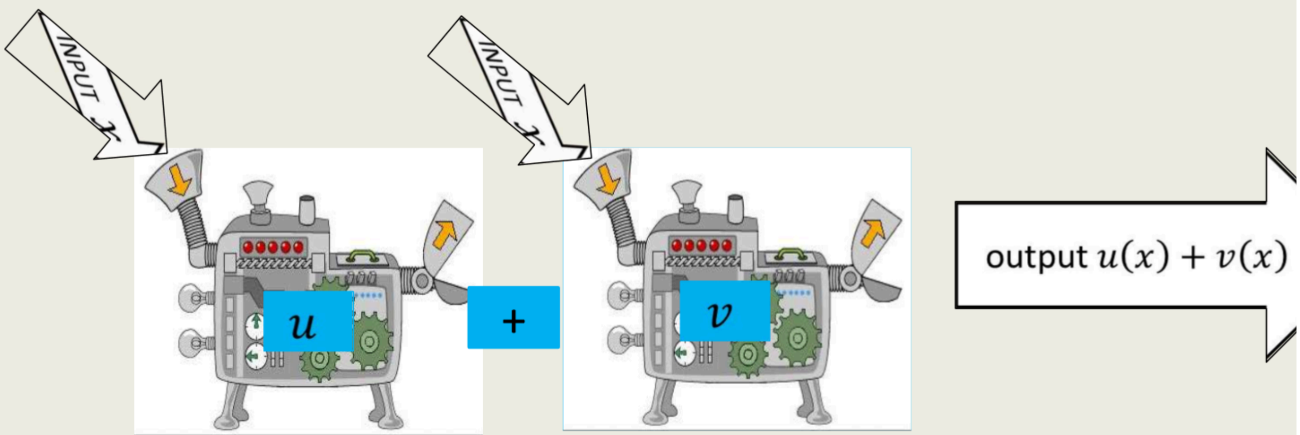
Displayed on x-axis
- Associated with DOMAIN



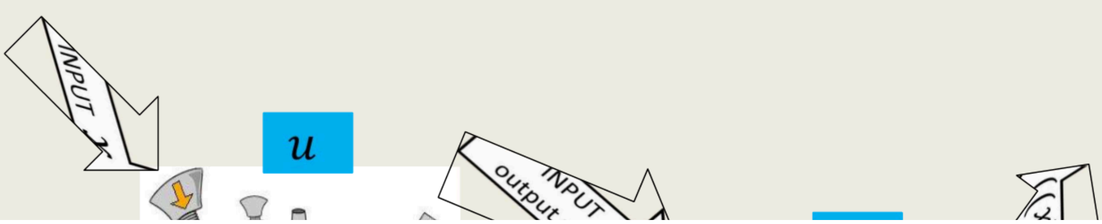
Output:
- Dependent variable, response
- Displayed on y-axis
- Associated with RANGE

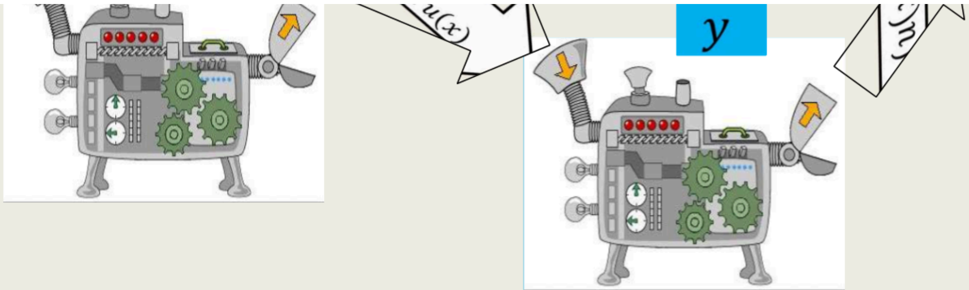
Operations with Functions.

- +/-, *, division, multiplication by a constant (Arithmetic)
- raising to a power, taking roots(Algebra)
- Differentiation, integration



Composition of Functions (Nested Functions)





$$y(x) = y(u(x))$$

The *output* of the first function becomes the *input* for the second function

Example of a Composite Function

How fast is the volume V decreases with time t?

$$V(t) = V[r(t)] = \frac{4}{3}\pi[r(t)]^3$$

$$= \frac{4}{3}\pi\left(6 - \frac{3}{17}\right.$$

- Candy. The volume of a "jawbreaker" for any given radius is given by the formula
- $V(r) = \frac{4}{3}\pi r^3$.
- Roger Guffey estimates the radius of a jawbreaker while in person's mouth to be
- $r(t) = 6 - \frac{3}{17}t$,

  where r in mm and t in minutes.

The Chain Rule

The Chain Rule

The chain rule allows us to find the derivative of a composite (nested) function.

How fast is the volume decreases with time?

$$V(t) = V[r(t)]$$

$$\frac{dV}{dt} = \frac{d}{dt}(V[r(t)])???$$

The Chain Rule

The chain rule allows us to find the derivative of a composite (nested) function.

- Suppose y is a function of u, that is
  $$y = y(u),$$
  And u is a function of x, that is
  $$u = u(x).$$
  Then y is a composite function of x, that is
  $$y(x) = y[u(x)].$$

Outside function

inside function

The Chain Rule

"Rates of change multiply"

$$y(x) = y[u(x)]$$

- $\frac{d}{dx}\left[y[u(x)]\right] = \frac{dy}{du} \times \frac{du}{dx}$

Derivative of outside function

Derivative of inside function

EXAMPLE 1 (chain rule and power rule)

- Use chain rule to find derivatives $y' = \frac{dy}{dx}$

$a.\ y = \left(5x^3 + 2\right)^{10}$

$b.\ y = -\frac{1}{2}\left(3x^4 - 5x\right)^{-6}$

EXAMPLE 1 (Solution)

- Use chain rule to find derivatives $y' = \frac{dy}{dx}$

a. $y = (5x^3 + 2)^{10}$

$x \to \underbrace{5x^3+2}_{\text{Inside}} \to (\quad)^{10}$ ← outside

$\frac{dy}{dx} = 10(5x^3+2)^9 \cdot \frac{d}{dx}[5x^3+2]$

$= 10[5x^3+2)^9 (15x^2)$

$= 150x^2(5x^3+2)^9$

$y' = 150x^2(5x^3 + 2$

EXAMPLE 1 (Solution)

- Use chain rule to find derivatives $y' = \frac{dy}{dx}$

b. $y = -\frac{1}{2}(3x^4 - 5x)^{-6}$

$x \to 3x^4 - 5x \to (\quad)^{-6}$

$y' = \frac{dy}{dx} = \left(-\frac{1}{2}\right)(-6)(3x^4 - 5x)^{-7} \cdot (12x^3 - 5)$

$y' = 3(12x^3 - 5)(3x^4 - 5x)$

$$y' = 3(12x^3 - 5)(3x^4 - 5x)$$

EXAMPLE 2 (Self-check)

Use chain rule to find the derivative of each function:

1. a. $y = (2x+3)^6$ b. $y = (2x^4 - x)^{-3}$ c. $y = 3(x^2 - 3x + 1)^{20}$
2. a. $y = \frac{-4}{(2x+3)^5}$ b. $y = \sqrt{x^3 - 3x + 2}$ c. $y = \sqrt{7x - 2x^2}$

Answers are on the next slide

EXAMPLE 2 (Answers)

Use chain rule to find the derivative of each function:

1. a. $y = (2x+3)^6$ b. $y = (2x^4 - x)^{-3}$ c. $y = 3(x^2 - 3x + 1)^2$
2. a. $y = \frac{-4}{(2x+3)^5}$ b. $y = \sqrt{x^3 - 3x + 2}$ c. $y = \sqrt{7x - 2x^2}$

1. a. $y' = 12(2x+5)^5$; b. $y' = -3(8x^3 - 1)(2x^4 - x)^{-4}$; c. $y' = 60(2x-3)(x^2 - 3x + 1$
2. a. $y' = 40(2x+3)^{-6}$; b. $y' = \frac{3}{2}\frac{(x^2-1)}{\sqrt{x^3-3x+2}}$; c. $y' = \frac{1}{2}\frac{(7-4x)}{\sqrt{7x-2x^2}}$.

The Product Rule

Given two functions of the same variable x:

$$u = u(x) \text{ and } v = v(x),$$

the derivative of their product $u \cdot v$ is computed by the formula:

$$\frac{d}{dx}[u \cdot v] = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

Or the short form:

$$[\boldsymbol{u} \cdot \boldsymbol{v}]' = \boldsymbol{u}'\boldsymbol{v} + \boldsymbol{u}\boldsymbol{v}'$$

EXAMPLE 3

Find the derivative $y' = \frac{dy}{dx}$. Do not simplify the result

$$y = \sqrt[3]{3x+5}\,(2x^2 - 3)$$

EXAMPLE 3 (Solution)

$$y = \sqrt[3]{3x+5}\,(2x^2 - 3) = \underbrace{(3x+5)^{\frac{1}{3}}}_{u}\underbrace{(2x^2-3)}_{v}$$

$[uv]' = u'v + uv'$

$u = (3x+5)^{\frac{1}{3}}$ $\qquad v = 2x^2 - 3$

$u' = \frac{1}{3}(3x+5)^{-\frac{2}{3}} \cdot (3) = (3x+5)^{-\frac{2}{3}}$ $\qquad v' = 4x$

$a^{2/3}\, a^{\frac{1}{3}} = a^{1} = a$

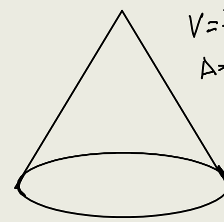
$$y' = \underset{u'}{(3x+5)^{-2/3}}\ \underset{v}{(2x^2-3)} + \underset{u}{(3x+5)^{1/3}}\ \underset{v'}{(4x)}$$

$$y' = \frac{2x^3-3}{(3x+5)^{2/3}} + \frac{4x(3x+5)^{1/3}}{1} = \frac{(2x^2-3) + 4x(3x+5)}{(3x+5)^{2/3}}$$

Answer: $y' = \frac{14x^2+20x-3}{(3x+5)^{2/3}}$

EXAMPLE 4 (video solution in Panopto)

Assume that $r = r(t)$ and $h = h(t)$. If $V = \frac{1}{3}\pi r^2 h$, find $\frac{dV}{dt}$.



$V = \frac{\pi r^2}{3}$

$A = \pi r^2$

$$\frac{dv}{dt} = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right]$$

$$= \frac{\pi}{3}\frac{d}{dt}\left[r^2(t)\,h(t)\right] = \frac{\pi}{3}\left[\left[2r\frac{dr}{dt}\right]h(t) + r^2\cdot\frac{dh}{dt}\right]$$

$f(x) = x^2$ $\quad$ $y = x^2$ $\quad$ $x = 2$

$f'(x) = 2x$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right]$$

$y = x^2 + 2x\,(x - 2)$

$y = x^2 + 2x^2 - 4x$

$y = 2x^2 - 4x$

$y' = 4x$

The Quotient Rule

Given two functions of the same variable x:

$$u = u(x) \text{ and } v = v(x),$$

the derivative of their quotient $\frac{u}{v}$ is computed by the formula:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{\frac{du}{dx}\cdot v - u\cdot\frac{dv}{dx}}{v^2}$$

Or the short form:

$$\left[\frac{u}{v}\right]' = \frac{u'v - uv'}{v^2}$$

EXAMPLE 5 (Self-check) Compute derivatives using the quotient rule

a. $y = \frac{x}{x^2+1}$ b. $y = \frac{x-1}{x+1}$ c. $s(t) = \frac{5t^2}{3-t}$

a. $y' = \frac{1-x^2}{(x^2+1)^2}$; b. $\frac{2}{(x+1)^2}$; c. $y' = \frac{5t(6-t)}{(3-t)^2}$.

EXAMPLE 6

Use the Quotient Rule to evaluate the derivative of $y = \frac{x}{\sqrt{1-4x}}$ for $x = -2$.

$$y' = \frac{1-2x}{(1-4x)^{3/2}}; y'(-2) = \frac{5}{27}$$

Rules of Differentiation (Summary)

1. Constant Rule: For any constant C: $\frac{d}{dx}[C] = 0$
2. Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$
3. Constant Times a Function: If C is a constant, then $\frac{d}{dx}[Cu(x)] = C\frac{d}{dx}[u(x)]$
4. Sum and Difference Rules: $[\boldsymbol{u} \pm \boldsymbol{v}]' = \boldsymbol{u}' \pm \boldsymbol{v}'$
5. Product Rule: $[\boldsymbol{u} \cdot \boldsymbol{v}]' = \boldsymbol{u}'\boldsymbol{v} + \boldsymbol{u}\boldsymbol{v}'$
6. Quotient Rule: $\left[\frac{u}{v}\right]' = \frac{u'v-uv'}{v^2}$

Chain Rule
The rates of change multiply for the neste functions

Differentiation is a LINEAR OPERATION