# HUMBER ENGINEERING

## MENG 3510 – Control Systems

## LECTURE 10

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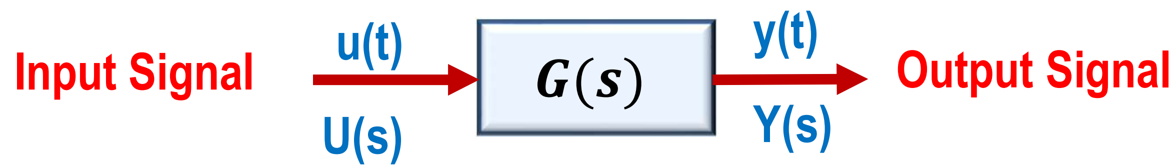
# LECTURE 10
# Stability Analysis via Frequency Response

- Frequency Response Specifications
  - System Identification
  - Type of System
  - Error Constants & Steady-State Error
- Stability Analysis
  - Gain Margin & Phase Margin
  - Nyquist Stability Criteria

# Frequency Response Specifications

- Consider the following stable LTI system with the transfer function model of G(s)




- Similar to the Time Response Analysis of control systems, in frequency domain we have the following Frequency Domain Specifications to identify the Stability and Performance of the system using the Bode Diagram.

| Time Domain | Frequency Domain |
|---|---|
| ○ **Rise time** | ○ **Bandwidth**<br>○ **Cutoff frequency** |
| ○ **Maximum overshoot**<br>○ **Peak time** | ○ **Resonant peak**<br>○ **Resonant frequency** |
| ○ **Steady-state error constants** | ○ **Steady-state error constants** |
| ○ **Routh-Hurwitz Criteria**<br>○ **Root Locus** | ○ **Gain margin & Phase margin**<br>○ **Nyquist Stability Criteria** |

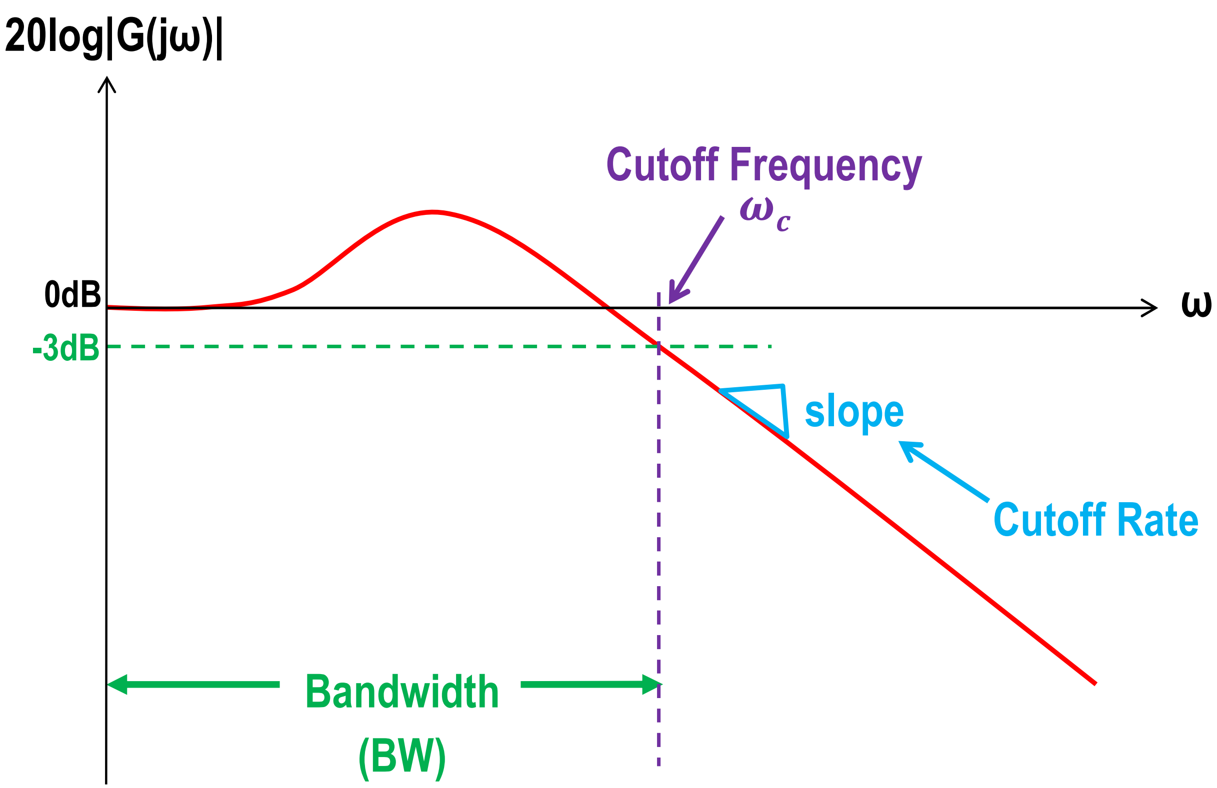
# Frequency Response Specifications

## ❑ Bandwidth ($BW$)

- The frequency at which $|G(j\omega)|$ drops 3dB down from its zero-frequency value. BW is also called **Cutoff Frequency, $\omega_c$**.

$$\boxed{20\log|G(j\omega)| = -3\text{dB} \;\rightarrow\; |G(j\omega)| = \frac{1}{\sqrt{2}} \cong 0.707}$$

- Bandwith of a control system gives indication on the transient response properties in time domain.
- Bandwith and rise-time are inversely proportional.
- Large bandwidth corresponds to a faster rise time.
- In second-order systems:
  - Increasing $\omega_n$, increases BW and decreases $t_r$
  - Increasing $\zeta$, decreases BW and increases $t_r$
- Bandwidth also indicates the noise-filtering characteristics and robustness of the system.
- The **cutoff rate (slope)** indicates the ability of a system to distinguish the signal from noise.



# Frequency Response Specifications

## ❑ Resonant Peak ($M_r$) & Resonant Frequency ($\omega_r$)

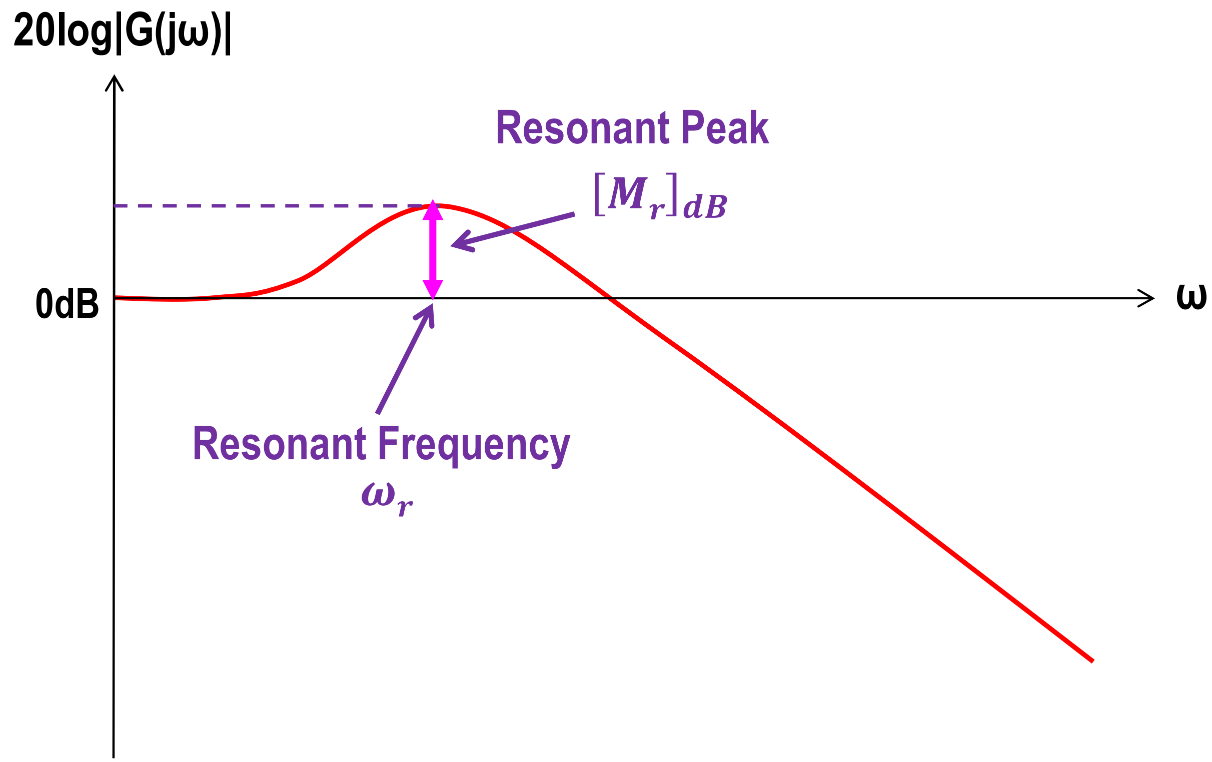
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

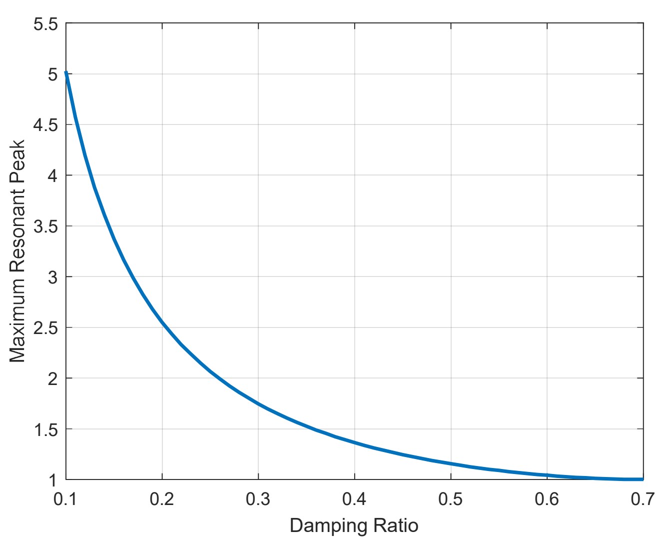
- Resonant peak is the maximum value of $|G(j\omega)|$.
- Resonant frequency is the frequency at which the resonant peak occurs.
- Resonant peak shows relative stability of a stable system.
- Large resonant peak corresponds to large maximum overshoot of the step response.
- Generally, the acceptable range is:

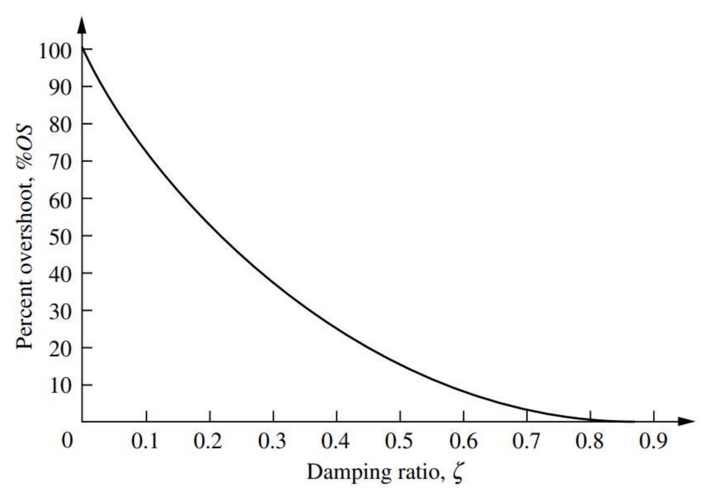
$$\mathbf{1.1 \cong 0.83dB \le M_r \le 1.5 \cong 3.5dB}$$

$$\boxed{M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}}$$

$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$



# Frequency Response Specifications

## Resonant Peak ($M_r$) & Resonant Frequency ($\omega_r$)

$$G(s) = \frac{K_{dc}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

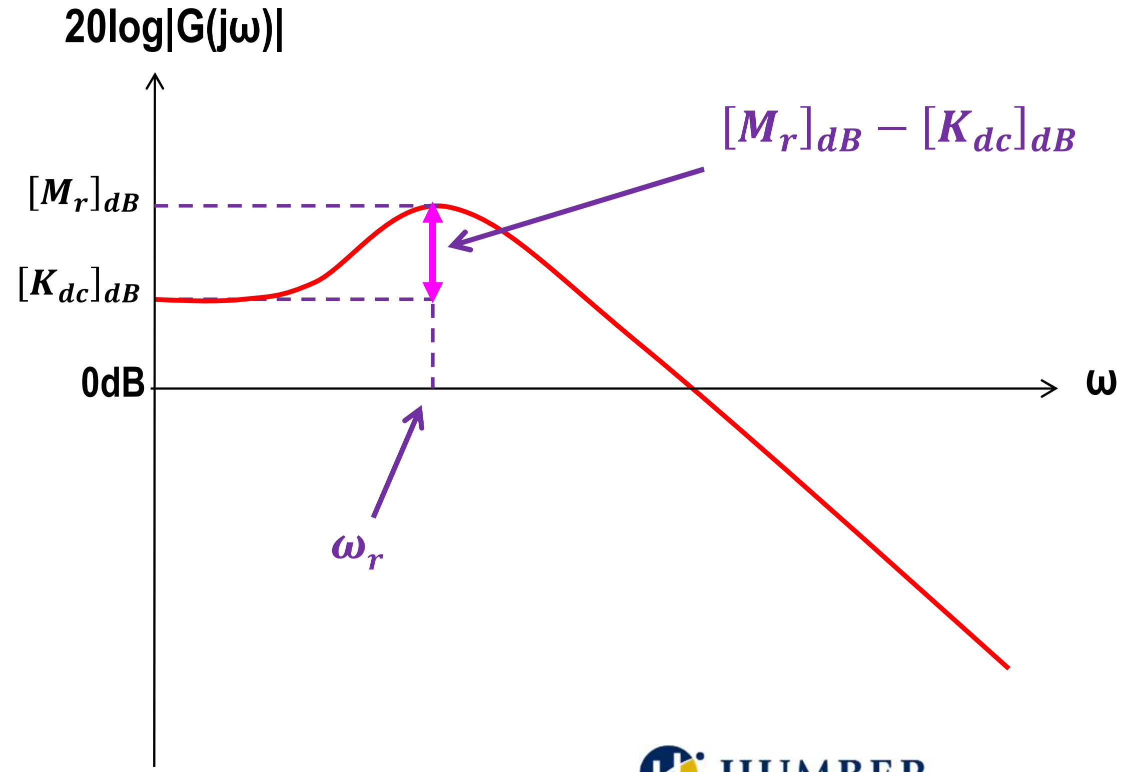
- In general form, if the transfer function has non-unit DC-gain:
  - The resonant frequency formula has no change.

$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

  - The resonance peak formula has to be modified as:

$$\boxed{M_r = K_{dc}\frac{1}{2\zeta\sqrt{1-\zeta^2}}}$$

$$\frac{M_r}{K_{dc}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \rightarrow \quad \left[\frac{M_r}{K_{dc}}\right]_{dB} = [M_r]_{dB} - [K_{dc}]_{dB}$$



# Frequency Response Specifications

**Example 1** Given Bode diagram of a dynamic system, estimate a second-order model for this system:

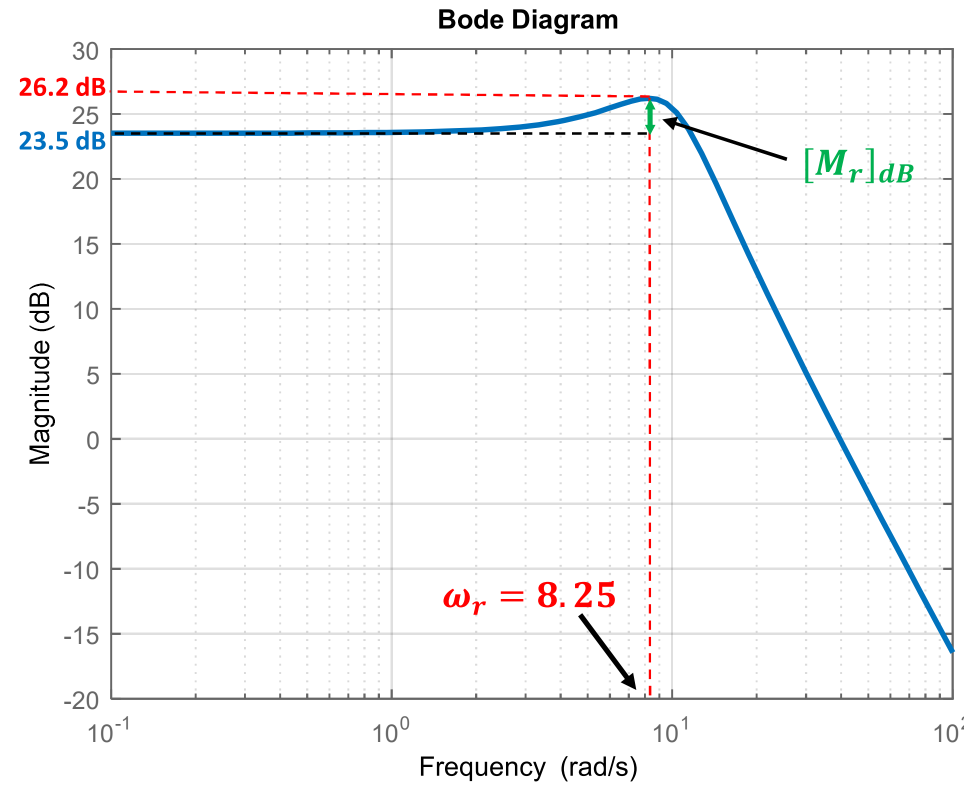
$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- The resonant frequency and resonant peak:

$\boxed{\omega_r = 8.25\ rad/sec}$ $\boxed{[M_r]_{\mathrm{dB}} = 26.2 - 23.5 = 2.7\ \mathrm{dB}}$

$$20\log M_r = 2.7\mathrm{dB} \;\rightarrow\; M_r = 10^{2.7/20} \;\rightarrow\; \boxed{M_r = 1.36}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \;\rightarrow\; \begin{cases} \boxed{\zeta = 0.4} \\ \zeta = 0.92 \;\longrightarrow\; \text{Not Acceptable} \end{cases}$$




- Determine the natural frequency, $\omega_n$:

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \;\rightarrow\; \omega_n = \frac{8.25}{\sqrt{1-2(0.4)^2}} \;\rightarrow\; \boxed{\omega_n = 10\ rad/s}$$

$$\boxed{G(s) = \frac{1500}{s^2 + 8s + 100}}$$

**Second-Order Model**

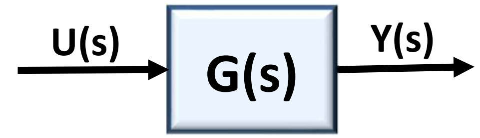
- Determine the DC-gain, $K$:

$$[K]_{\mathrm{dB}} = 23.5\ \mathrm{dB} \;\rightarrow\; 20\log_{10}(K) = 23.5\mathrm{dB} \;\rightarrow\; K = 10^{23.5/20} \;\rightarrow\; \boxed{K = 14.96 \approx 15}$$

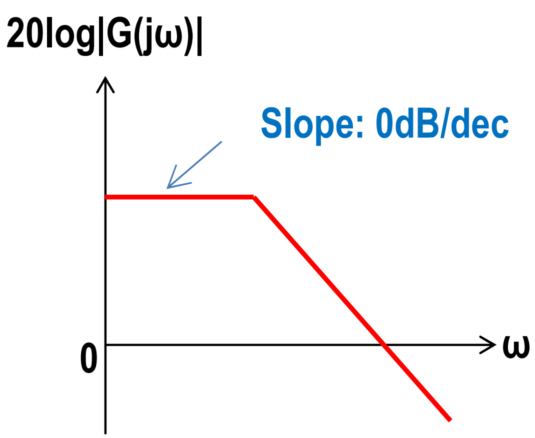
# Frequency Response Specifications

## ❑ Type of a System

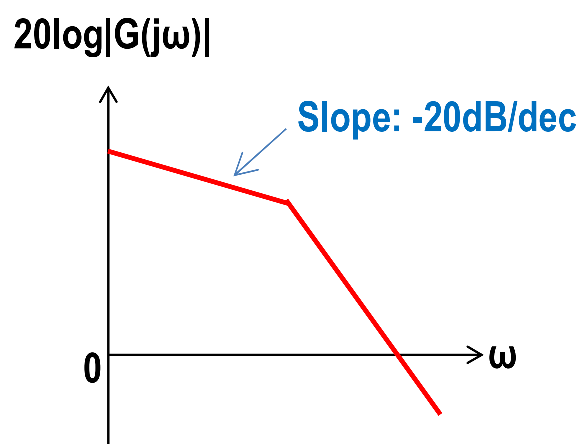
- Type of a system is determined by checking the **slope** of the log magnitude plot at **low frequencies**.
  - In Type 0 systems the low frequency asymptote is a Horizontal Line.
  - In Type 1 systems the low frequency asymptote is a line with slope of -20dB/dec .
  - In Type 2 systems the low frequency asymptote is a line with slope of -40dB/dec .



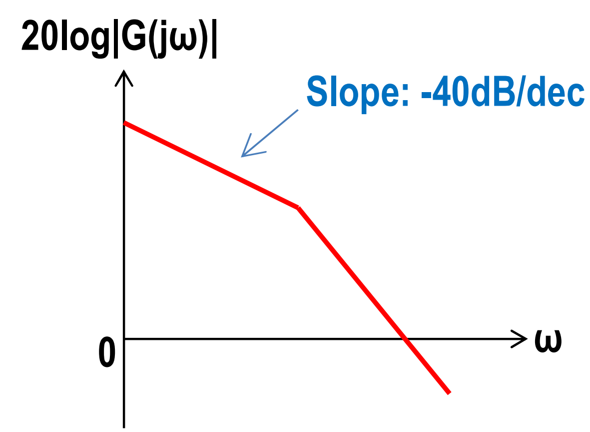

**Type 0 System**




**Type 1 System**

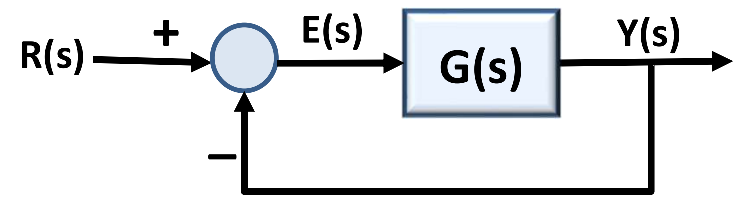



**Type 2 System**



# Frequency Response Specifications

## ❑ Steady-State Error Constants




- Consider the following unity-feedback closed-loop system, where $G(s)$ is stable.
- The steady-state error constants are determined from the Bode plot of the open-loop system $G(s)$.

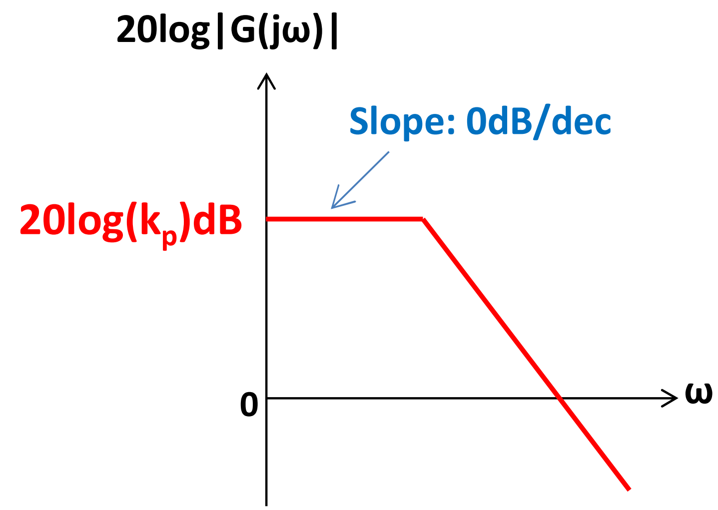
### ▪ Open-loop System is Type 0

- For Type 0 systems the step − error constant $\rightarrow$ $k_p = \lim_{s\to 0} G(s)$

- If $G(s)$ is **Type 0** the log magnitude plot of $G(j\omega)$ starts as a Horizontal Line at low frequencies with magnitude of $20\log(k_p)\text{dB}$

$$\boldsymbol{k_p = \lim_{s\to 0} G(s)} \quad \rightarrow \quad k_p = \lim_{\omega\to 0} G(j\omega)$$

$$k_p = G(j0) \quad \rightarrow \quad \boxed{\mathbf{20\log(k_p)dB = 20\log\big(G(j0)\big)}}$$

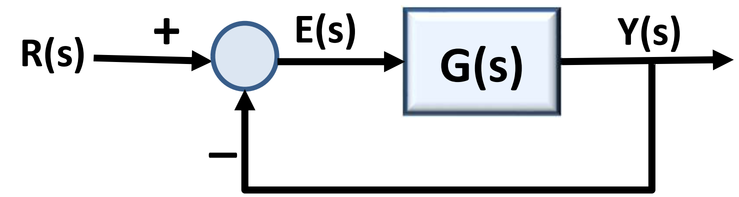


# Frequency Response Specifications

## Example 2

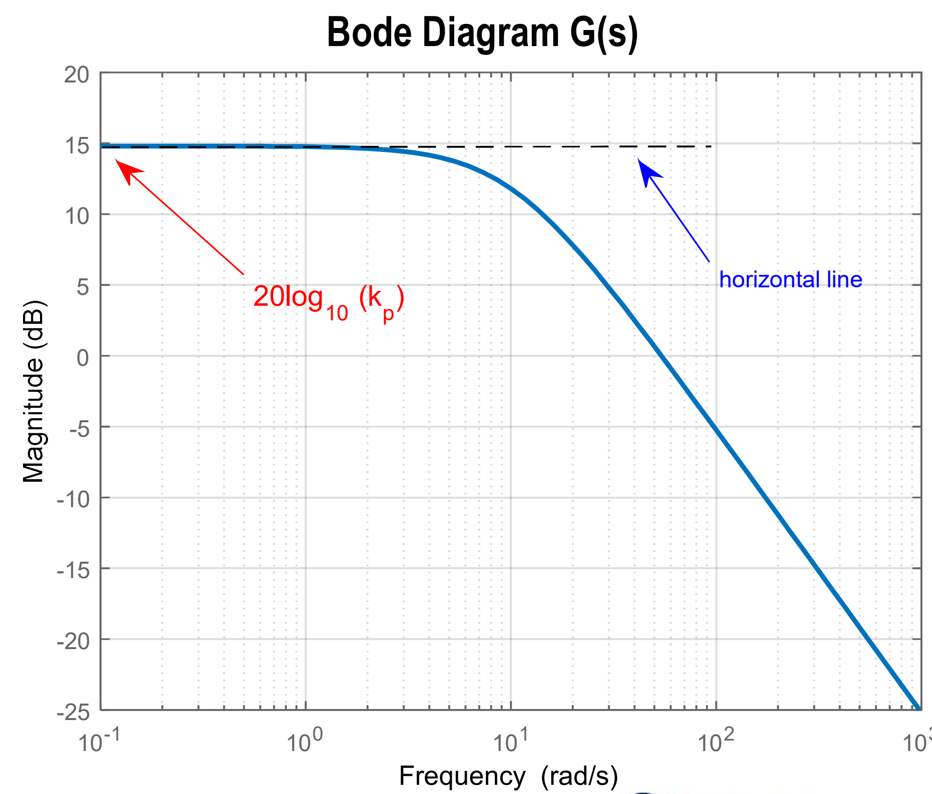
Consider a closed-loop system with the following Bode plot of the open-loop system G(s).

Determine Type of the open-loop system and corresponding steady-state error constant and the steady-state error of the closed-loop system.




The log magnitude plot starts at low frequencies with a horizontal line

**The system is Type 0**




- Determine the Step-error Constant:

$$\mathbf{20log(k_p)dB} = 15\text{dB}$$

$$k_p = 10^{15/20} \quad \rightarrow \quad \mathbf{k_p = 5.62}$$

$$\boldsymbol{e_{ss} = \frac{1}{1+k_p}} \quad \rightarrow \quad e_{ss} = \frac{1}{1+5.62} = 0.1511$$

$$\boxed{\boldsymbol{e_{ss} = 15.11\%}}$$

# Frequency Response Specifications

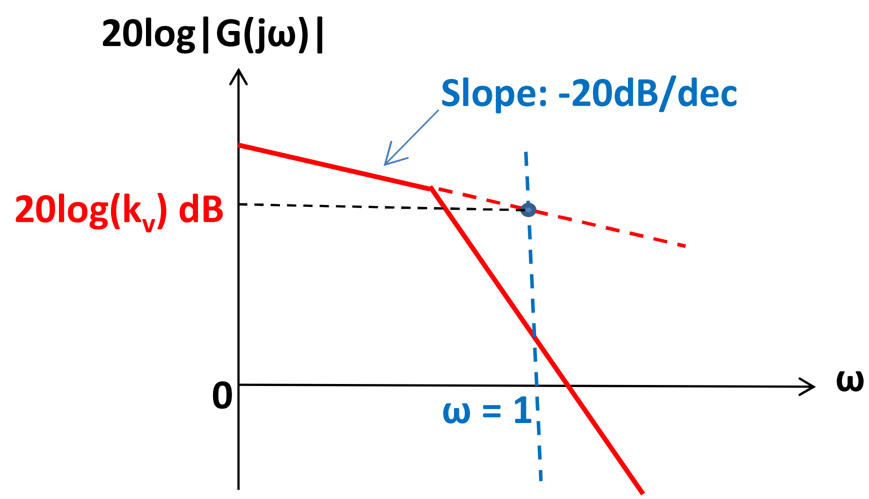
## ❑ Steady-State Error Constants & System Type

### ▪ Open-loop System is Type 1

- For Type 1 systems the ramp − error constant → $k_v = \lim_{s \to 0} sG(s)$

- If $G(s)$ is **Type 1** the log magnitude plot of $G(j\omega)$ starts as a line with the slope of -20 dB/dec.

- The intersection of the initial -20dB/dec segment (or its extension) with line ω = 1 has the magnitude of $20\log(k_v)$

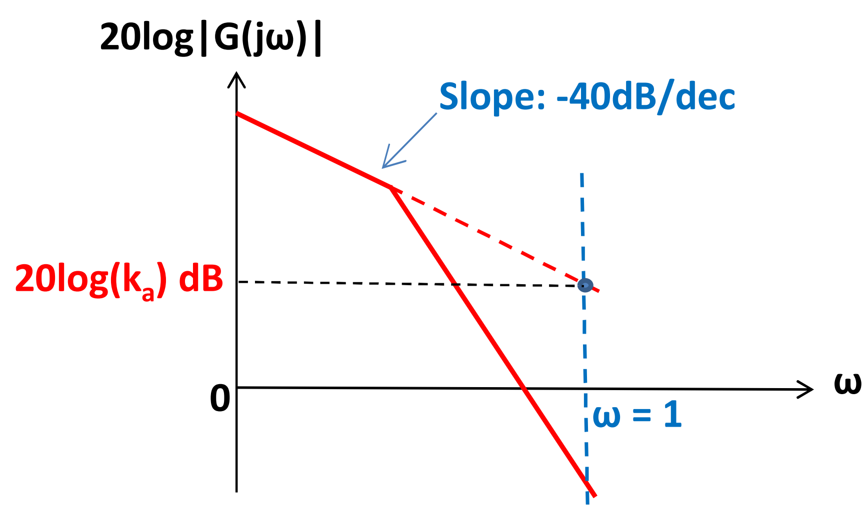



### ▪ Open-loop System is Type 2

- For Type 2 systems the parabolic − error constant → $k_a = \lim_{s \to 0} s^2 G(s)$

- If $G(s)$ is **Type 2** the log magnitude plot of $G(j\omega)$ starts as a line with the slope of -40 dB/dec.

- The intersection of the initial -40dB/dec segment (or its extension) with line ω = 1 has the magnitude of $20\log(k_a)$

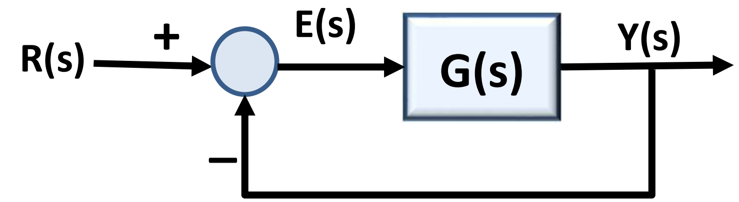


# Frequency Response Specifications

**Example 3**

Consider a closed-loop system with the following Bode plot of the open-loop system G(s).

Determine Type of the open-loop system and corresponding steady-state error constant and the steady-state error of the closed-loop system.




The log magnitude plot starts at low frequencies with the slope of -20 dB/dec

$$\boxed{\text{The system is Type 1}}$$

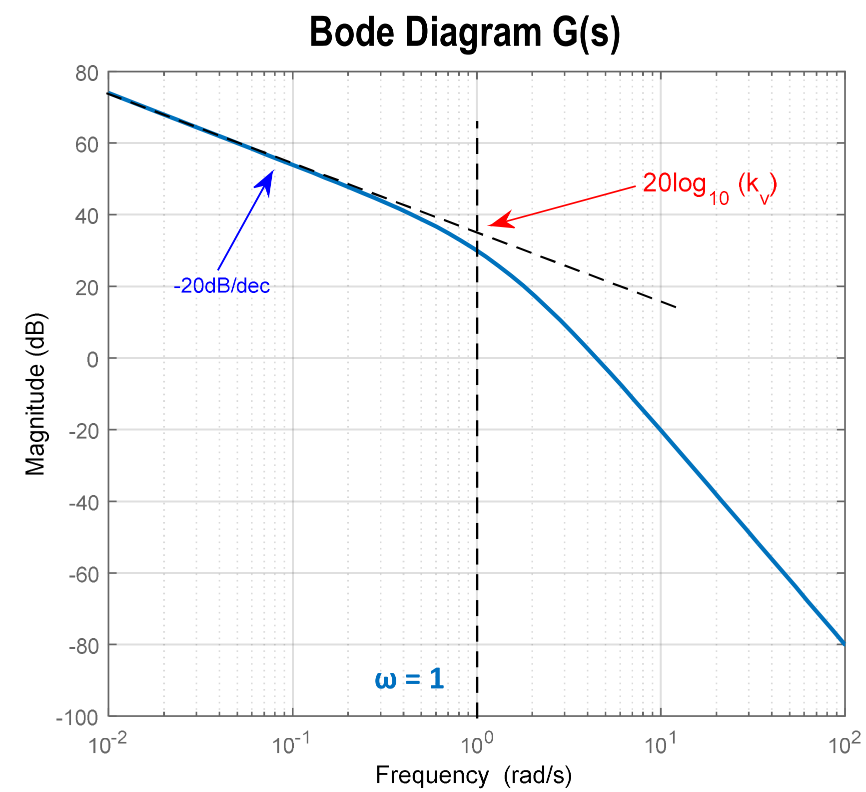
- Determine the Ramp-error Constant:

$$\mathbf{20log(}\boldsymbol{k_v}\mathbf{)dB} = 35\text{dB}$$

$$k_v = 10^{35/20} \quad \rightarrow \quad \boldsymbol{k_v = 56.23}$$

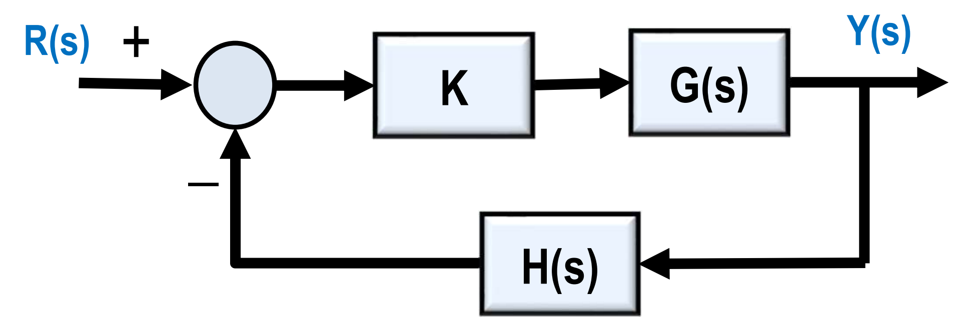
$$\boldsymbol{e_{ss} = \frac{1}{k_v}} \quad \rightarrow \quad e_{ss} = \frac{1}{56.23} = 0.0178$$

$$\boxed{\boldsymbol{e_{ss} = 1.78\%}}$$



# Stability Analysis

# Stability Analysis via Bode Diagram




- Consider the following closed-loop system with transfer function of $T(s)$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

- The closed-loop poles are determined from the characteristic equation

$$1 + KG(s)H(s) = 0$$

- Recall the root-locus, all closed-loop poles must satisfy the magnitude and phase conditions.

$$|KG(s)H(s)| = 1 \quad \text{and} \quad \angle\big(KG(s)H(s)\big) = -180^{\circ}$$

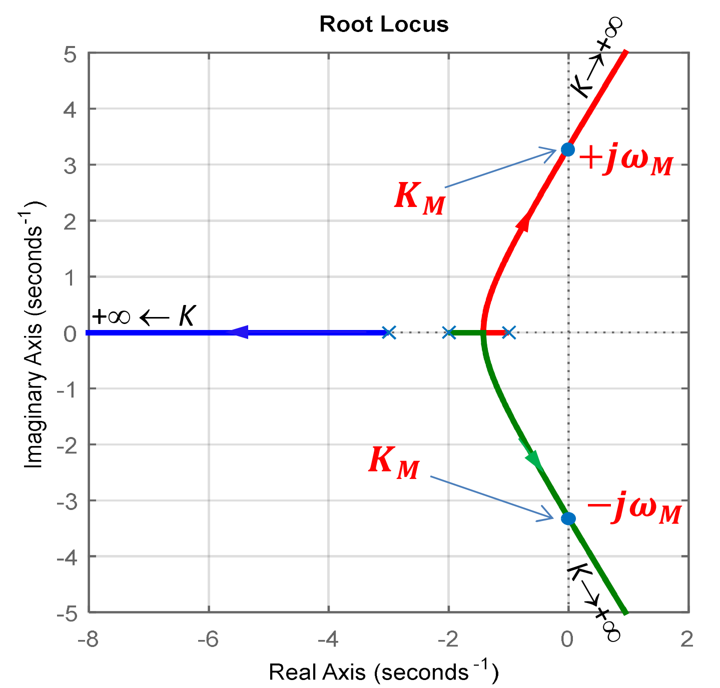
- The marginal stability gain, and points are determined as:

$$s = j\omega \quad \rightarrow \quad 1 + KG(j\omega)H(j\omega) = 0 \quad \Rightarrow \quad \boxed{K_M \;, \omega_M}$$

- If the following conditions are satisfied at the same time, the system will be at the marginal stability condition.

$$\boxed{|K_M G(j\omega_M)H(j\omega_M)| = 1 \quad \text{and} \quad \angle\big(K_M G(j\omega_M)H(j\omega_M)\big) = -180^{\circ}}$$

- Since we have access to $|KG(j\omega)H(j\omega)|$ and $\angle\big(KG(j\omega)H(j\omega)\big)$ from the Bode plot of the open-loop transfer function, we can determine the frequency $\omega$, which makes the closed-loop system marginally stable.



# Stability Analysis via Bode Diagram

- The magnitude and angle conditions can easily be identified from Bode diagram of the open-loop system, $\boldsymbol{KG(s)H(s)}$.

$$|KG(j\omega)H(j\omega)| = 1 \quad \text{and} \quad \angle\big(KG(j\omega)H(j\omega)\big) = -180^\circ$$

  - **Gain Crossover Frequency**

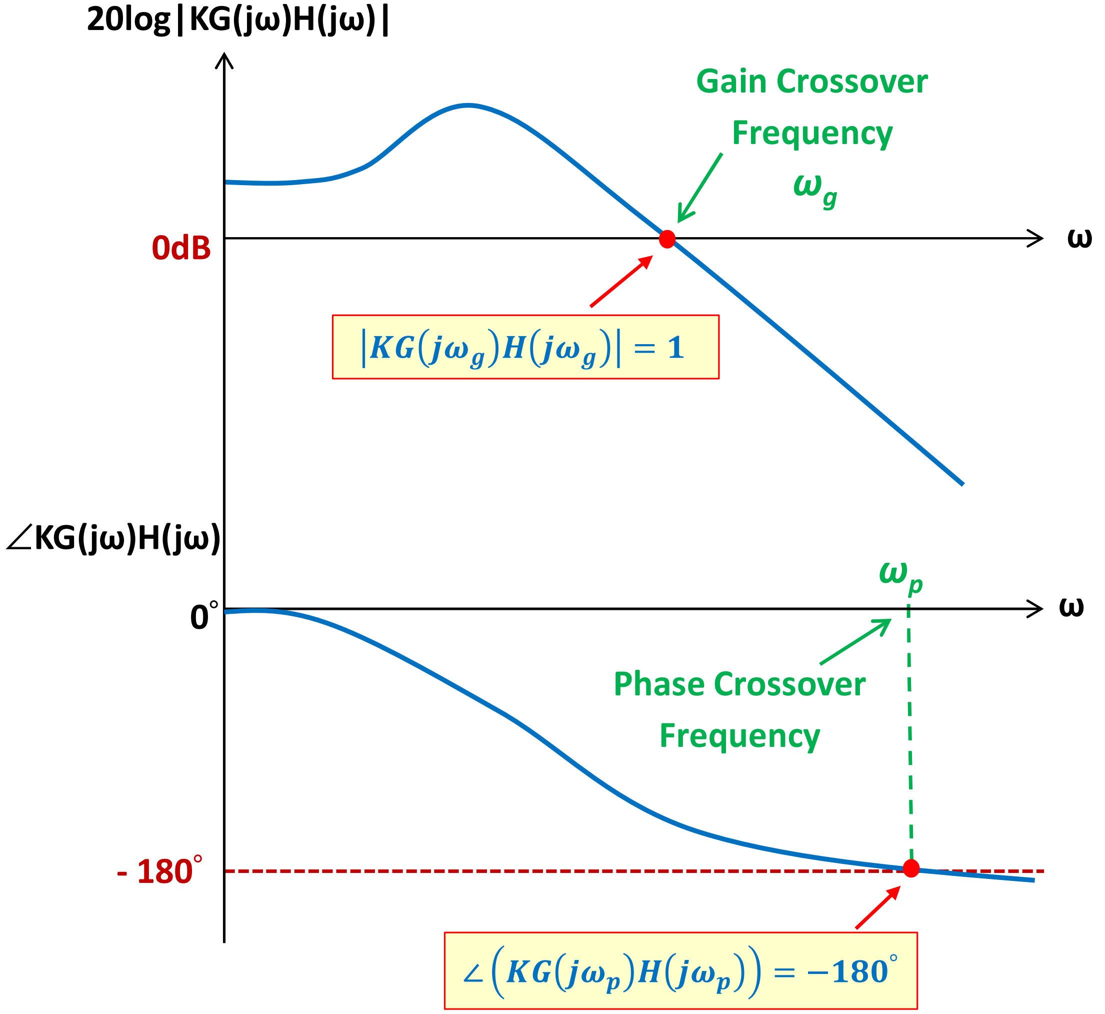
    The frequency, where the log magnitude plot crosses the **0dB** axis.

$$|KG(j\omega_g)H(j\omega_g)| = 1$$

  - **Phase Crossover Frequency**

    The frequency, where the phase plot crosses the $-180^\circ$ line.

$$\angle\big(KG(j\omega_p)H(j\omega_p)\big) = -180^\circ$$



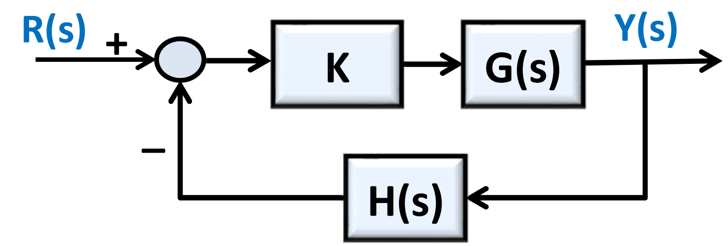

- If the gain crossover and the phase crossover happens at the same frequency the closed-loop system will be in the marginal stability condition.
- In marginal stability case, there is no margin to increase the gain $K$, because increasing the gain leads to instability.

# Stability Analysis via Bode Diagram

## Example 4

Consider the following unity feedback closed-loop system

$$G(s) = \frac{1}{s(s+1)^2}, \qquad H(s) = 1$$



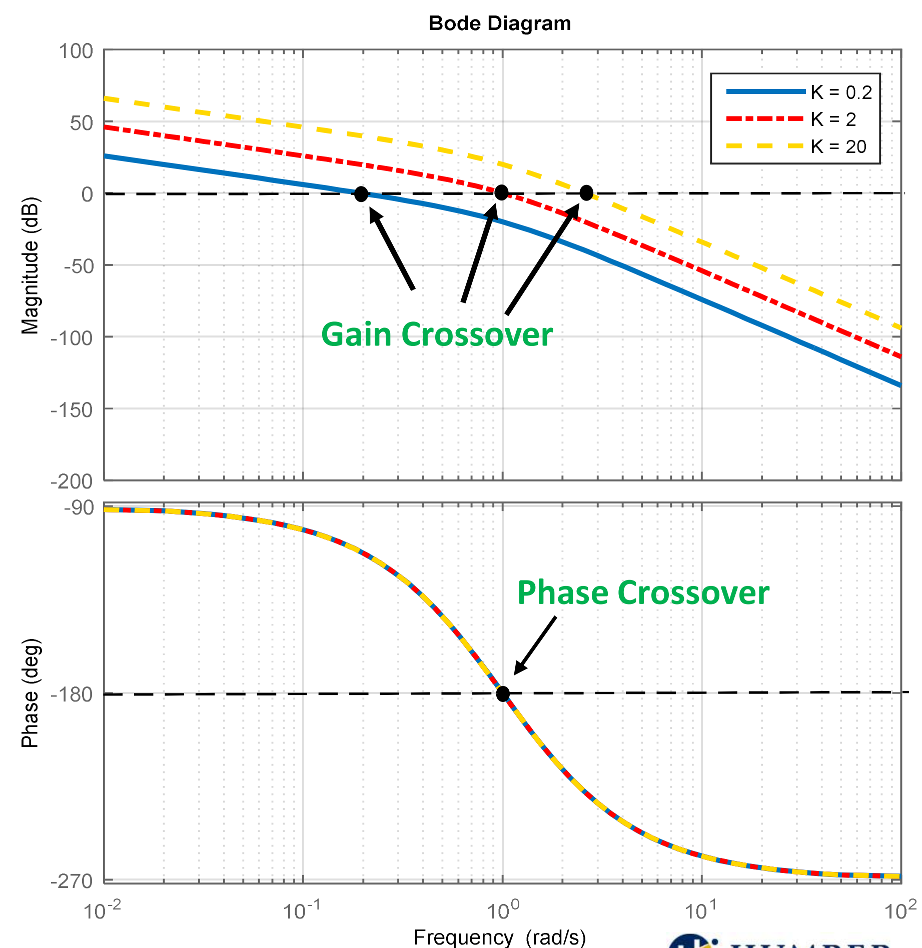

This example shows effect of increasing the open-loop gain $K$ on the Bode diagram of the open-loop system and stability of the closed-loop system.

The open-loop transfer function is:

$$KG(s)H(s) = \frac{K}{s(s+1)^2}$$

$$|KG(j\omega)H(j\omega)| = \frac{|K|}{|j\omega||j\omega+1||j\omega+1|}$$

$$\angle KG(j\omega)H(j\omega) = \angle K - \angle(j\omega) - \angle(1+j\omega) - \angle(1+j\omega)$$




Bode plot of the open-loop system has been plotted for $K = 0.2,\ 2 \text{ and } 20.$

Increasing the open-loop gain, $K$,

- Shifts the Bode magnitude plot up
- Shifts the gain crossover point to the right (higher frequencies),
- No effect on the phase crossover point.

For K = 2, the closed-loop system becomes marginally stable.
Therefore, for K = 0.2 the closed-loop system is stable, and for K = 20 the closed-loop system is unstable.

# Stability Analysis via Bode Diagram

- Bode diagram enables us to define stability margins, Gain margin and Phase margin, to check the relative stability of the closed-loop system, and to determine how far the system is from the marginal stability condition.

- **Gain Margin (GM)**

  The amount of gain in dB that can be added to the open-loop system before the closed-loop system becomes unstable.

$$\boxed{GM = 0\text{dB} - M_{\omega_p}}$$

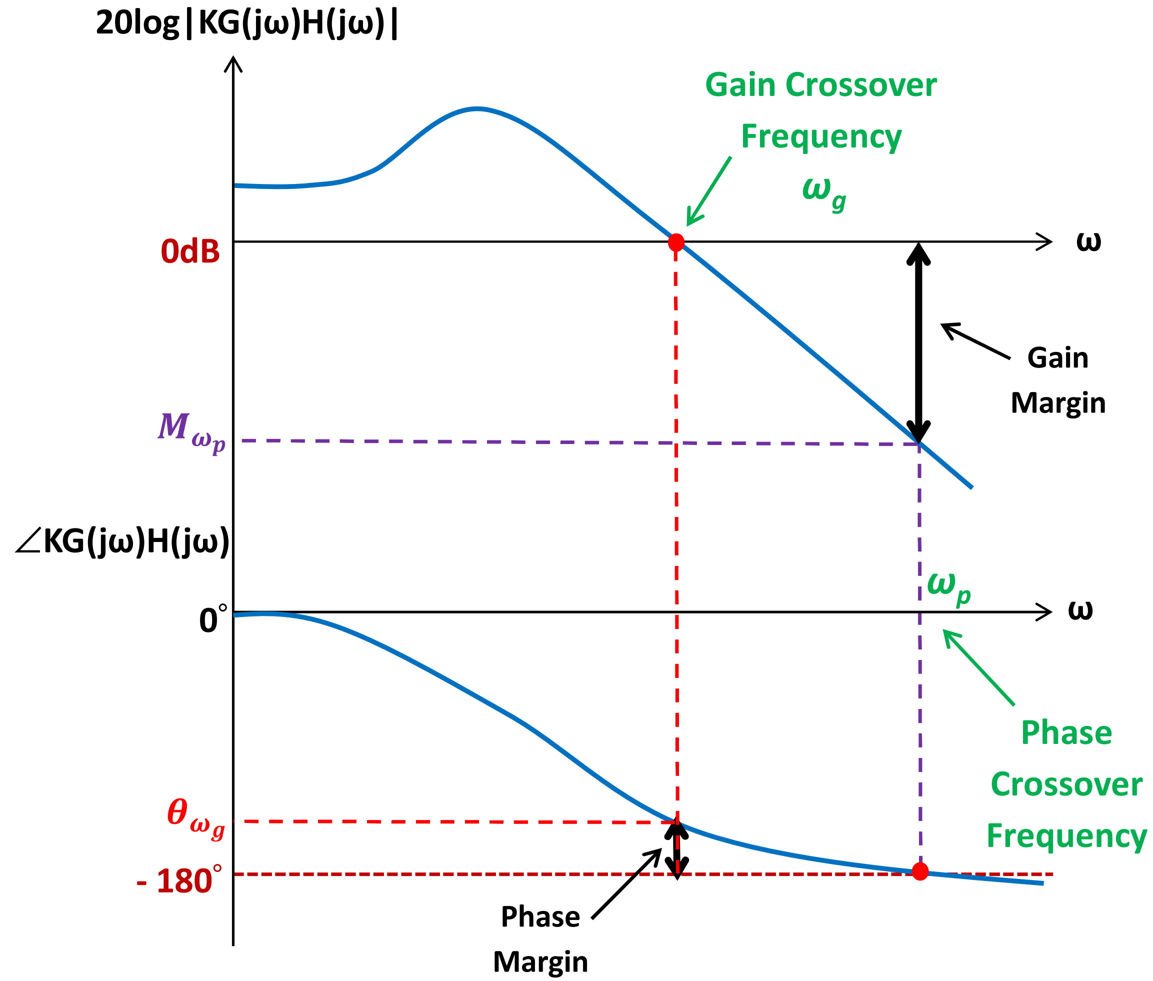
- **Phase Margin (PM)**

  The additional phase lag in degree can be added to the open-loop system before the closed-loop system becomes unstable.

$$\boxed{PM = 180^{\circ} + \theta_{\omega_g}}$$

- **Positive** Gain margin and Phase margin indicate the closed-loop stability.

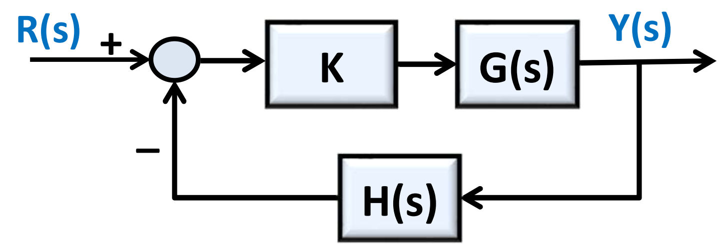
- Practically, for a satisfactory performance, a $45^{\circ} \le \text{PM}$ and a $\text{GM} > 6\text{dB}$ are required.



# Stability Analysis via Bode Diagram

## Example 5

Consider the following unity feedback closed-loop system




$$\boldsymbol{G(s) = \frac{1}{s(s+1)^2}}, \qquad \boldsymbol{H(s) = 1}$$

Determine relative stability of the closed-loop system by identifying the gain margin and phase margin of the open-loop system for $K = 0.2,\ 2\ \text{and}\ 20.$

$\boxed{K = 0.2}$

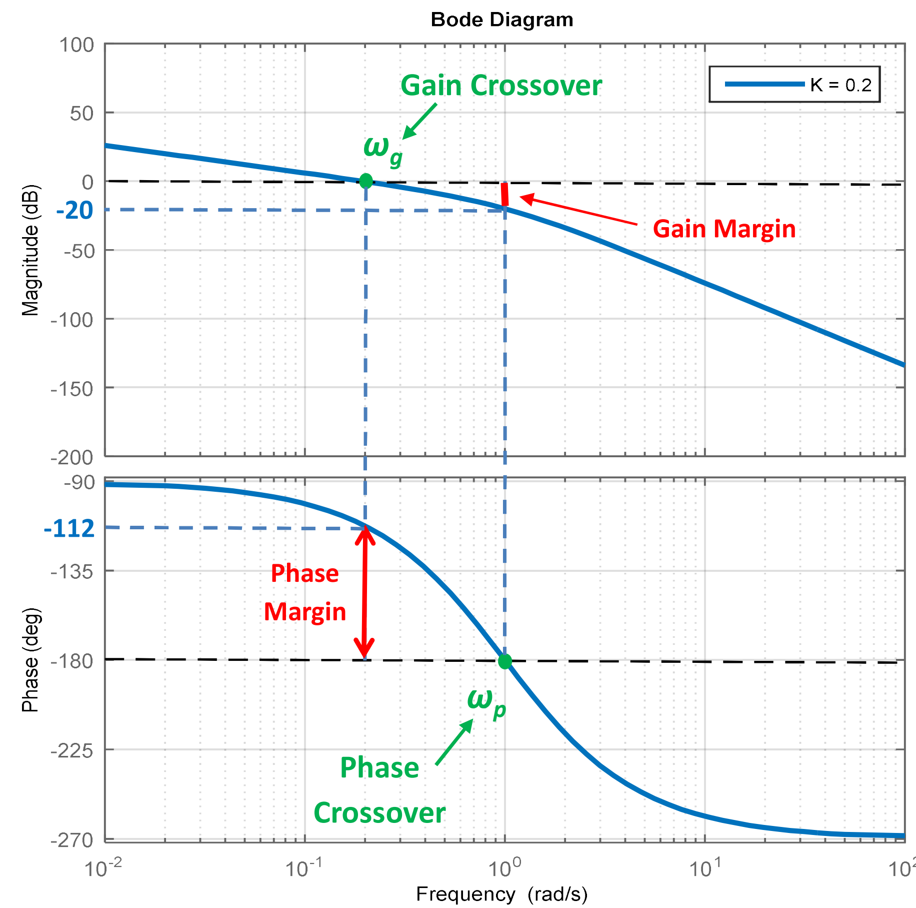
- Gain Crossover & Phase Crossover Frequencies

$$\boxed{\omega_p = 1\, rad/s} \qquad \boxed{\omega_g = 0.2\, rad/s}$$

- Gain Margin & Phase Margin

$$GM = 0\text{dB} - (-20\text{dB}) = 20\text{dB} \qquad \boxed{GM = 20\ dB > 0}$$

$$PM = 180^\circ + \left(-112^\circ\right) = 68^\circ \qquad \boxed{PM = 68^\circ\ > 0}$$

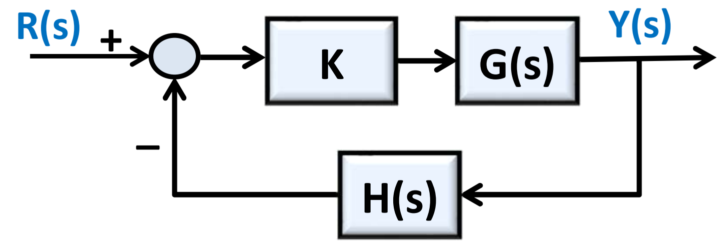



**For $K = 0.2$ the closed-loop system is stable.**

# Stability Analysis via Bode Diagram

**Example 5**

Consider the following unity feedback closed-loop system




$$G(s) = \frac{1}{s(s+1)^2}, \qquad H(s) = 1$$

Determine relative stability of the closed-loop system by identifying the gain margin and phase margin of the open-loop system for $K = 0.2,\ 2\ \text{and}\ 20.$

$\boxed{K = 2}$

- Gain Crossover & Phase Crossover Frequencies

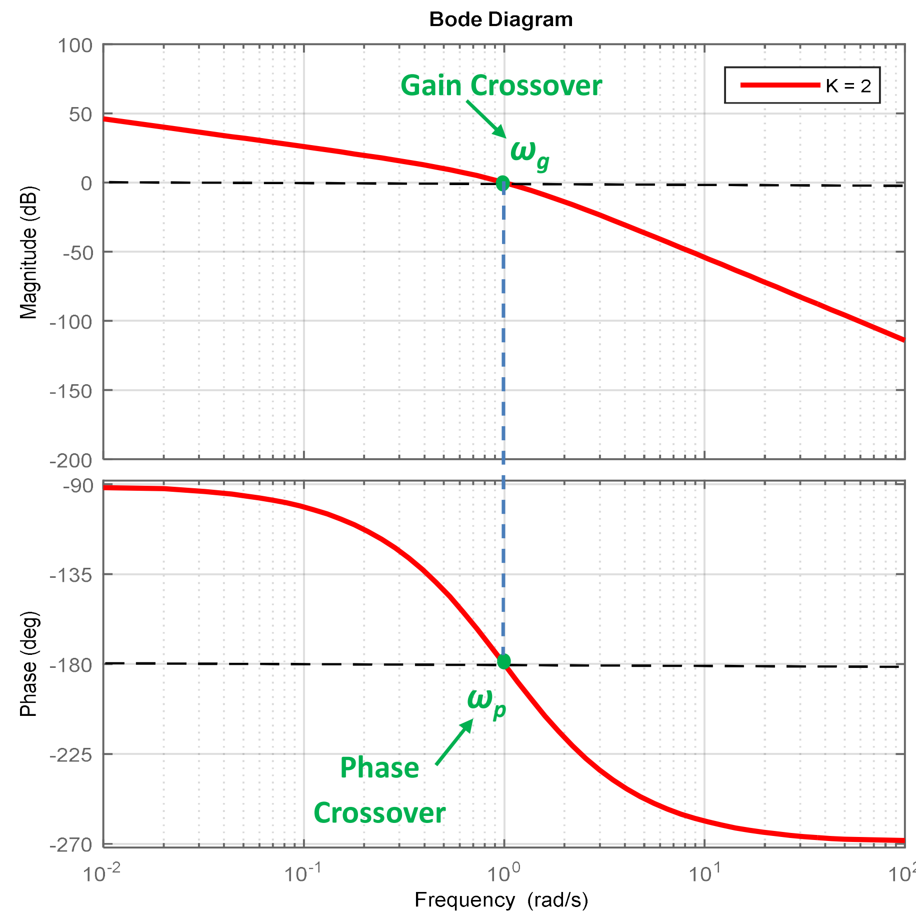
$\boxed{\omega_p = 1\, rad/s}$ $\boxed{\omega_g = 1\, rad/s}$

- Gain Margin & Phase Margin

$$GM = 0\text{dB} - (0\text{dB}) = 0\text{dB} \qquad \boxed{GM = 0dB}$$

$$PM = 180^\circ + \left(-180^\circ\right) = 0^\circ \qquad \boxed{PM = 0^\circ}$$

**For $K = 2$ the closed-loop system is marginally stable.**

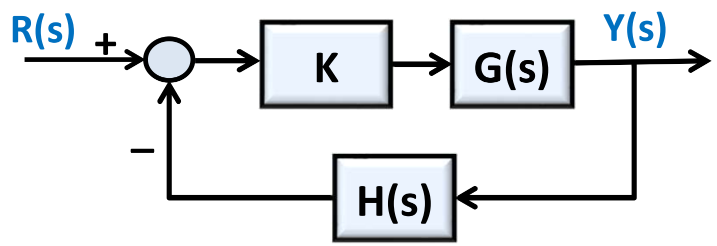


# Stability Analysis via Bode Diagram

**Example 5**

Consider the following unity feedback closed-loop system

$$G(s) = \frac{1}{s(s+1)^2}, \qquad H(s) = 1$$




Determine relative stability of the closed-loop system by identifying the gain margin and phase margin of the open-loop system for $K = 0.2,\ 2\ \text{and}\ 20.$

$K = 20$

- Gain Crossover & Phase Crossover Frequencies

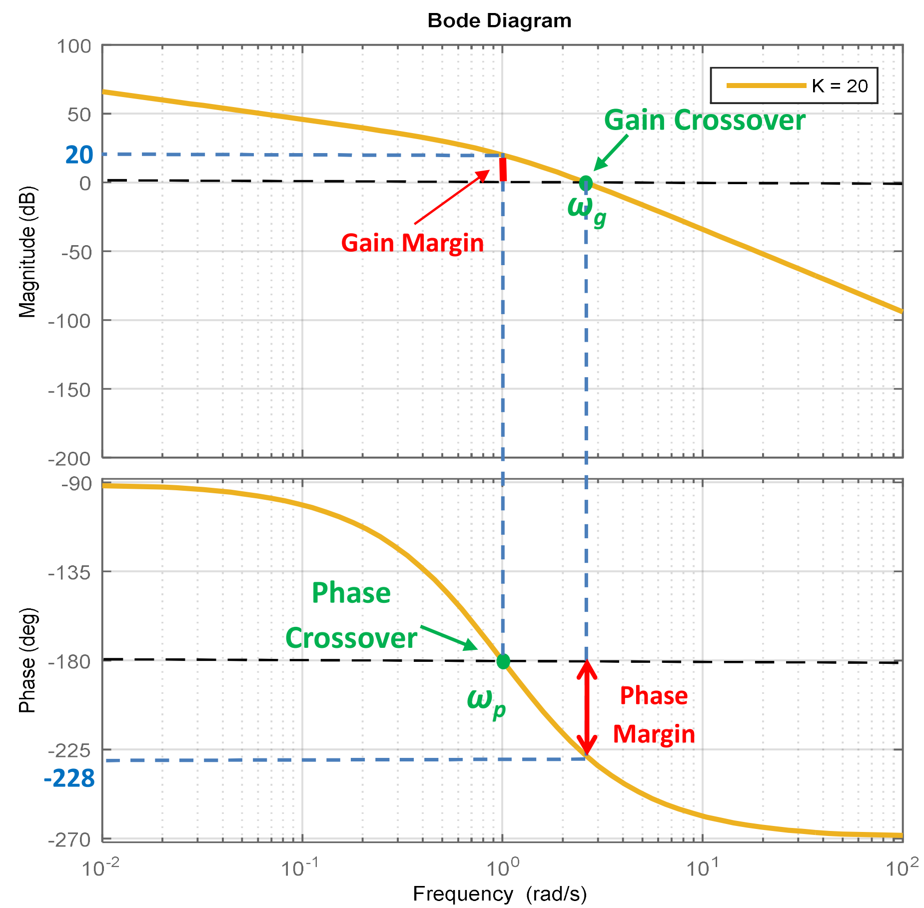
$\omega_p = 1\,rad/s$ $\qquad$ $\omega_g = 2.6\,rad/s$

- Gain Margin & Phase Margin

$GM = 0\text{dB} - (+20\text{dB}) = -20\text{dB}$ $\qquad$ $GM = -20\ dB < 0$

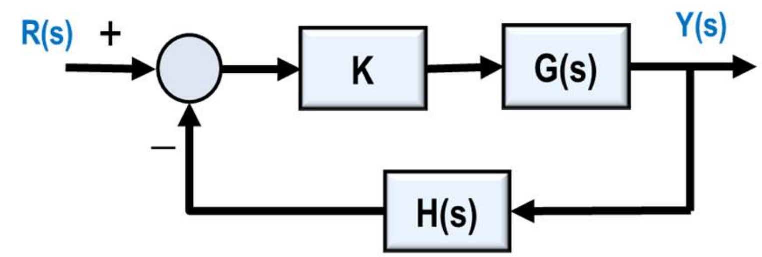
$PM = 180^\circ + (-228^\circ) = -48^\circ$ $\qquad$ $PM = -48^\circ < 0$

**For $K = 0.2$ the closed-loop system is unstable.**



# Stability Analysis via Nyquist Diagram

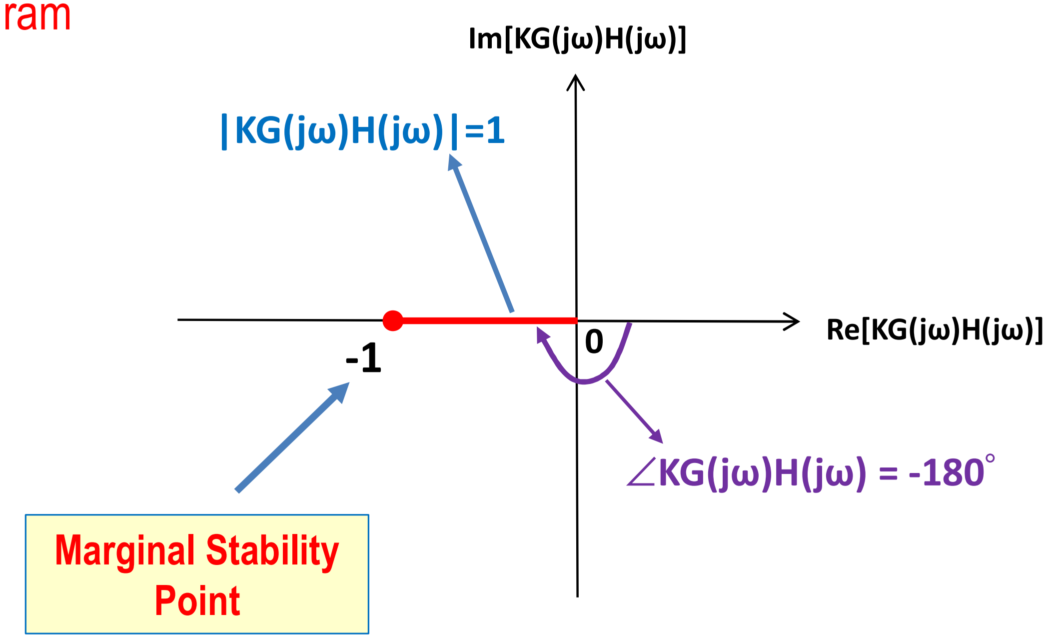
- The **marginal stability** condition can also be determined from the Nyquist Diagram of the open-loop system transfer function as below




$$|KG(j\omega)H(j\omega)| = 1 \quad \text{and} \quad \angle\big(KG(j\omega)H(j\omega)\big) = -180^{\circ}$$

- Therefore, the point (-1 , j0) is the **marginal stability point** on the Nyquist Diagram of the open-loop transfer function $KG(s)H(s)$.

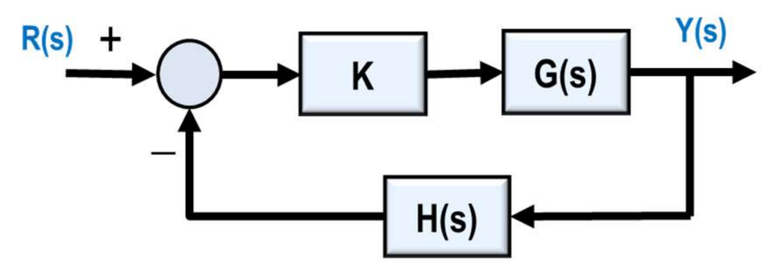
- If the Nyquist Diagram of the $KG(s)H(s)$ passes through the (-1 , j0) point, the closed-loop system is marginally stable.



# Stability Analysis via Nyquist Diagram

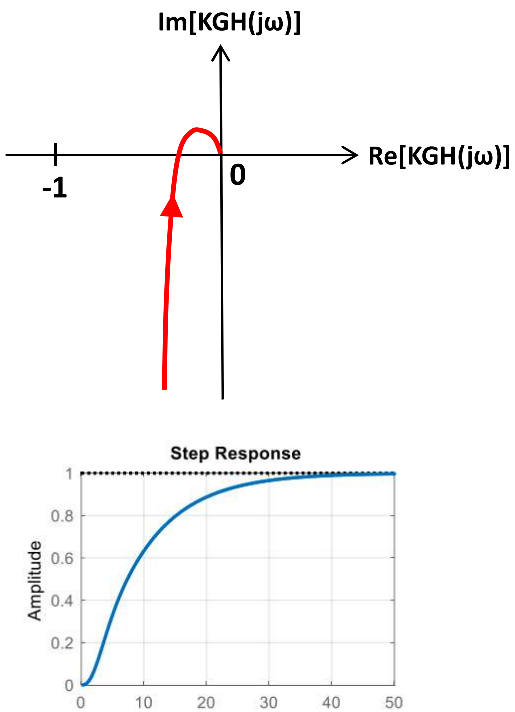
- The **marginal stability** condition can also be determined from the Nyquist Diagram of the open-loop system transfer function as below

$$|KG(j\omega)H(j\omega)| = 1 \quad \text{and} \quad \angle\big(KG(j\omega)H(j\omega)\big) = -180^\circ$$

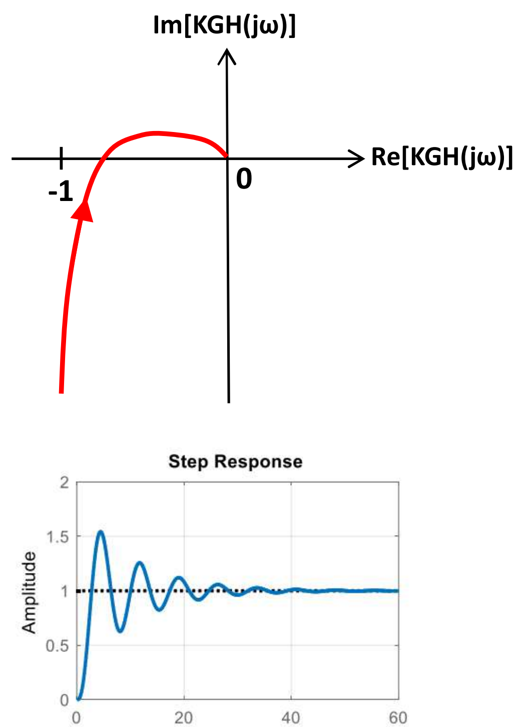



- For example, we can determine relative stability of the closed-loop system by checking the GM of each open-loop systems.

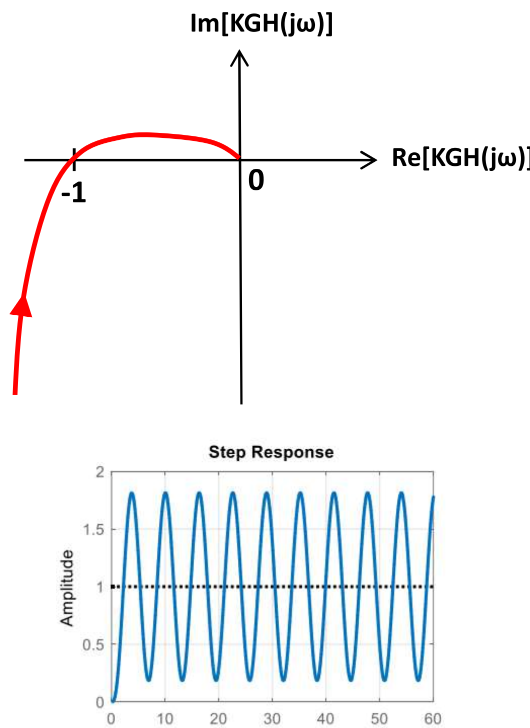
Stable and well damped system



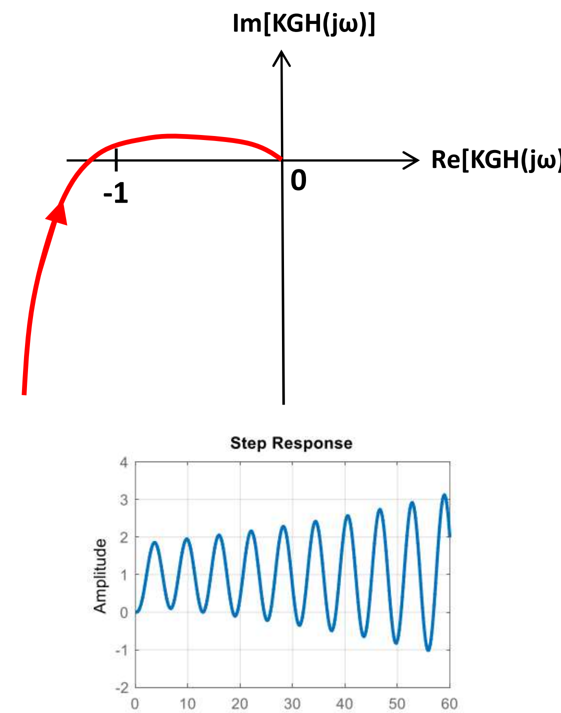

Stable but oscillatory system




Marginally stable system

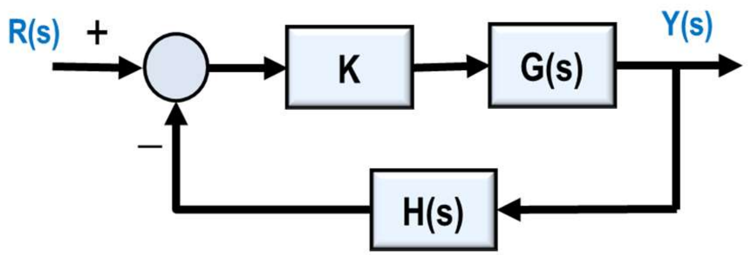



Unstable system



# Stability Analysis via Nyquist Diagram

- Similar to the Bode diagram the relative stability margins and the crossover frequencies can be determined from the Nyquist Diagram of the open-loop system transfer function $KG(s)H(s)$.




## ❑ Gain Margin

**At Phase Crossover Frequency** $\rightarrow \ \angle\left(KG(j\omega_p)H(j\omega_p)\right) = -180^\circ$

$$\boxed{GM = 0\text{dB} - 20\log\left|KG(j\omega_p)H(j\omega_p)\right|}$$

- If the phase crossover is between 0 and $-1$ point

  $0 < \left|KG(j\omega_p)H(j\omega_p)\right| < 1 \quad \rightarrow \quad GM > 0$

- If the phase crossover is at the $-1$ point

  $\left|KG(j\omega_p)H(j\omega_p)\right| = 1 \quad \rightarrow \quad GM = 0$

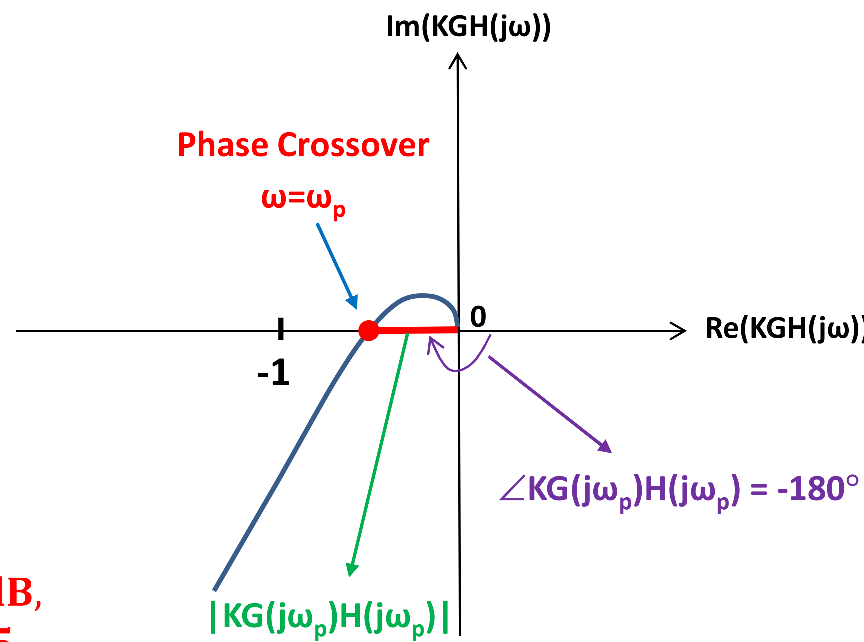
- If the phase crossover is at the left side of the $-1$ point

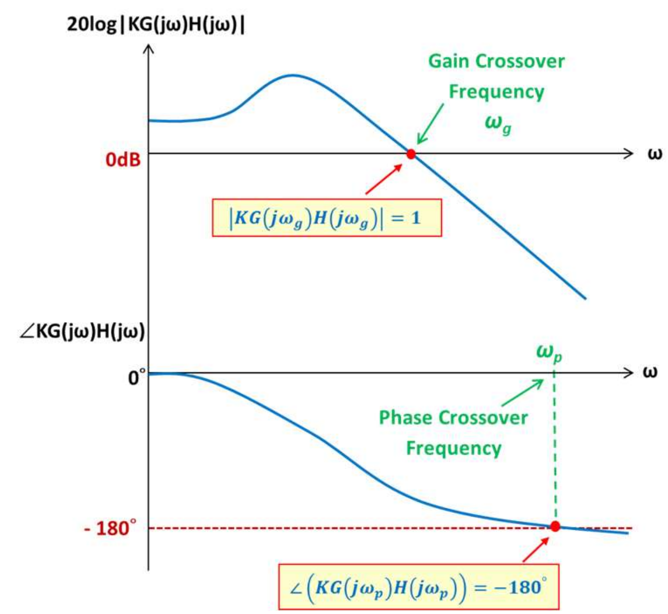
  $\left|KG(j\omega_p)H(j\omega_p)\right| > 1 \quad \rightarrow \quad GM < 0$

- If the Polar plot does not intersect the negative real axis

  $\left|KG(j\omega_p)H(j\omega_p)\right| = 0 \quad \rightarrow \quad GM = \infty$

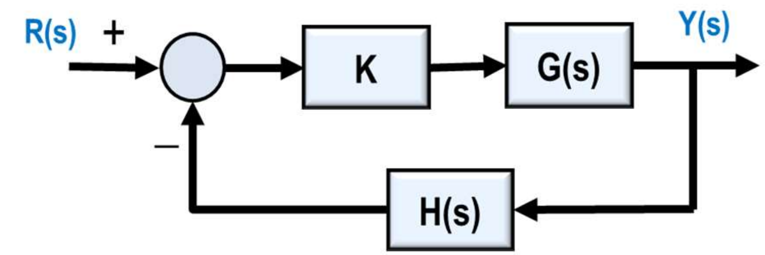
- Practically, a satisfactory performance yields if $\mathbf{GM > 6dB}$, means the phase crossover point is between $\mathbf{0}$ and $\mathbf{-0.5}$



# Stability Analysis via Nyquist Diagram

- Similar to the Bode diagram the relative stability margins and the crossover frequencies can be determined from the Nyquist Diagram of the open-loop system transfer function $KG(s)H(s)$.

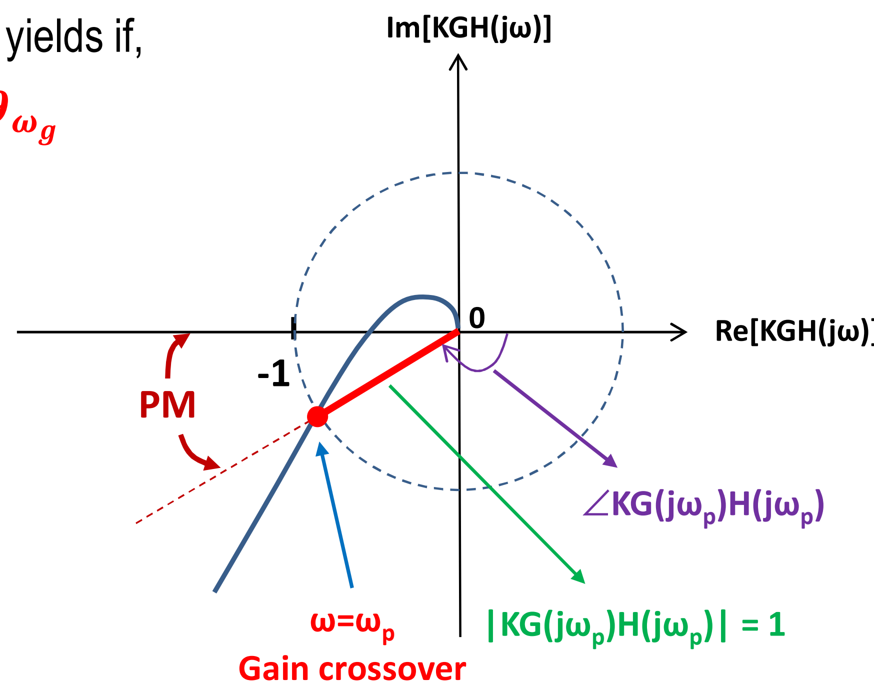


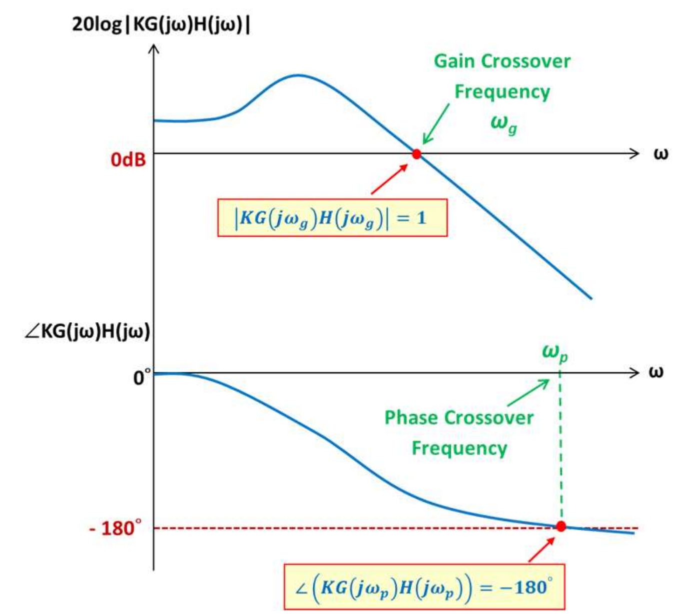

## ❑ Phase Margin

**At Gain Crossover Frequency** $\rightarrow \quad \left|KG(j\omega_g)H(j\omega_g)\right| = 1$

$$\boxed{PM = 180^{\circ} + \angle\left(KG(j\omega_g)H(j\omega_g)\right) = 180^{\circ} + \theta_{\omega_g}}$$

- Practically, for a satisfactory performance yields if, the $45^{\circ} \leq \mathrm{PM}$ that means $-135^{\circ} \leq \theta_{\omega_g}$



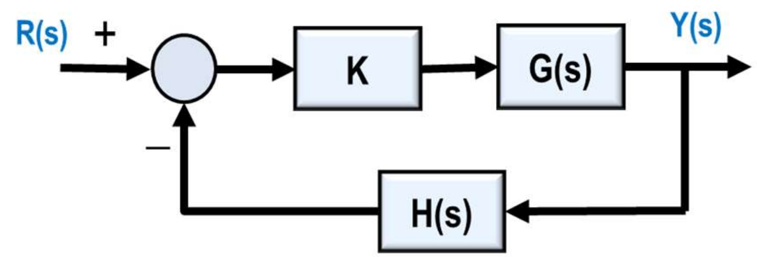



# Stability Analysis via Nyquist Diagram

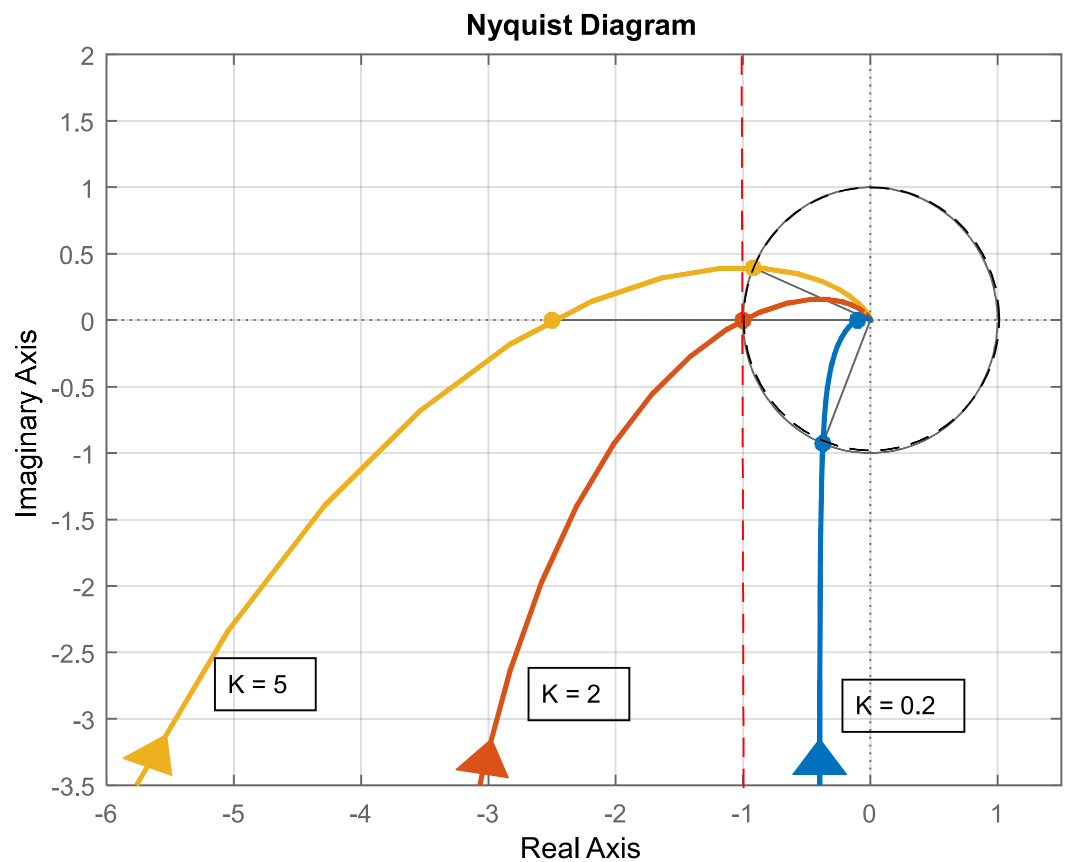
**Example 6**

Consider the following closed-loop system

$$G(s) = \frac{1}{s(s+1)^2}, \qquad H(s) = 1$$




Determine relative stability of the closed-loop system via the Polar plot and identify the gain margin and phase margin of the open-loop system for $K = 0.2,\ 2\ \text{and}\ 5.$

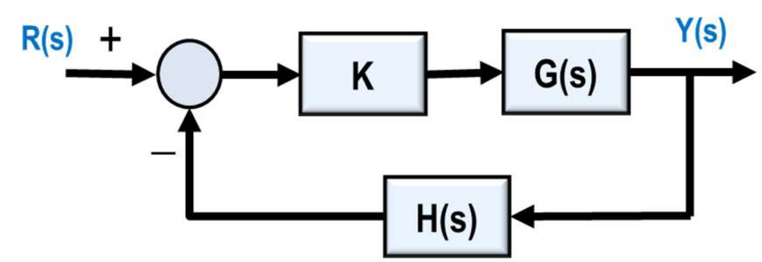


# Stability Analysis via Nyquist Diagram

## Example 6

Consider the following closed-loop system

$$G(s) = \frac{1}{s(s+1)^2}\ , \qquad H(s) = 1$$




Determine relative stability of the closed-loop system via the Polar plot and identify the gain margin and phase margin of the open-loop system for $K = 0.2,\ 2\ \text{and}\ 5.$

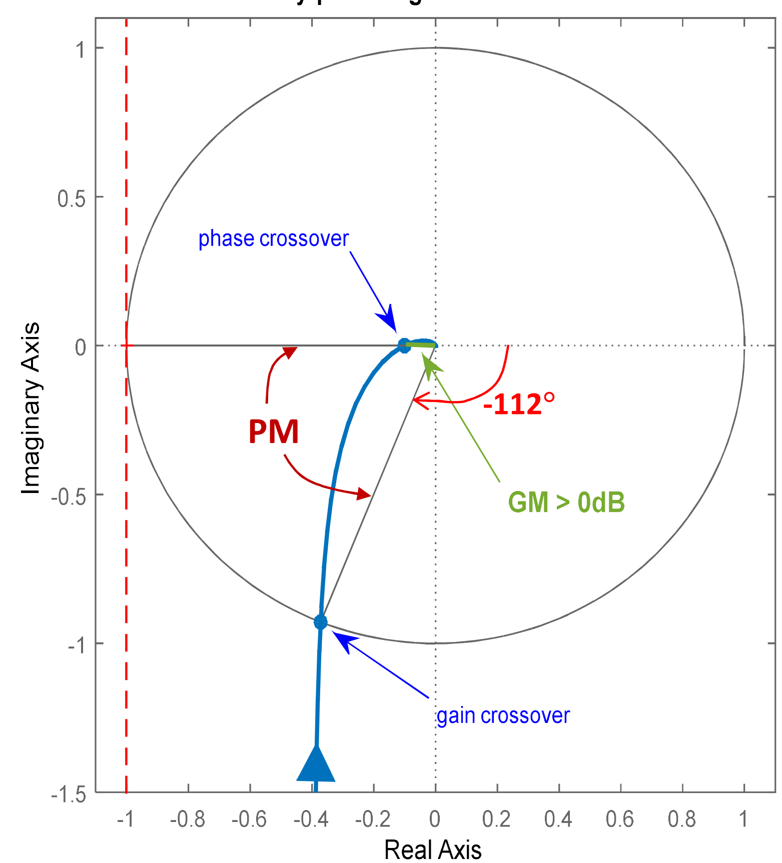
$\boxed{K = 0.2}$

**Phase crossover point at** → $-0.1$

$$GM = 0\text{dB} - 20\log|-0.1| = 20dB > 0$$

$$PM = 180^{\circ} + (-112^{\circ}) = 68^{\circ} > 0$$

Nyquist Diagram for K = 0.2




- The Nyquist plot is far enough from the (-1 , *j*0) point, and the phase crossover point is on the right-hand side of the (-1 , *j*0) point and between 0 and -0.5.

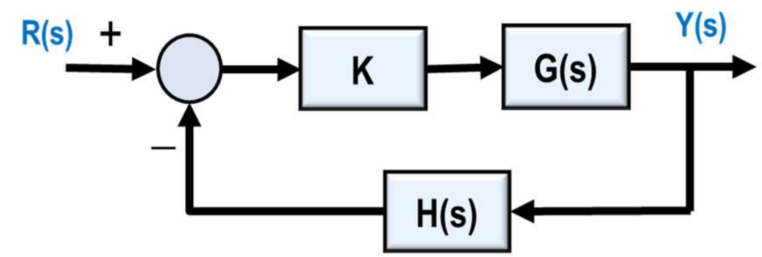
**For $K = 0.2$ the closed-loop system is stable**

# Stability Analysis via Nyquist Diagram

**Example 6**

Consider the following closed-loop system

$$G(s) = \frac{1}{s(s+1)^2}, \qquad H(s) = 1$$



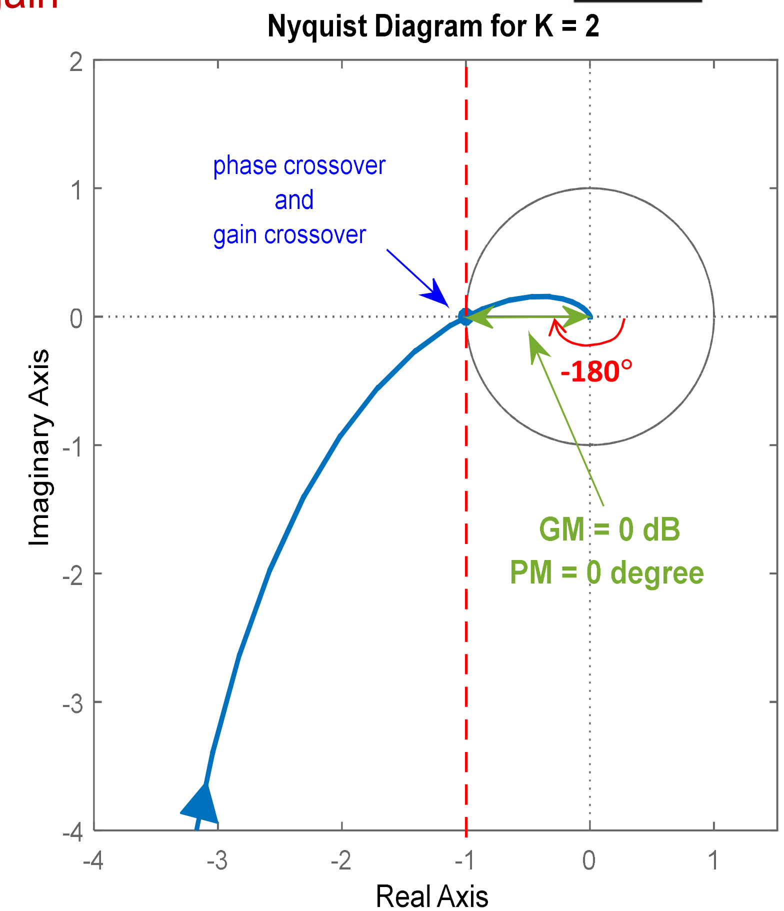

Determine relative stability of the closed-loop system via the Polar plot and identify the gain margin and phase margin of the open-loop system for $K = 0.2,\ 2 \text{ and } 5.$

$\boxed{K = 2}$

Phase crossover point at → $-1$

$$GM = 0\text{dB} - 20\log|-1| = 0\ dB$$

$$PM = 180^{\circ} + (-180^{\circ}) = 0^{\circ}$$




- The Nyquist plot crosses the negative real axis at the (-1 , *j*0) point, which is the marginal stability point

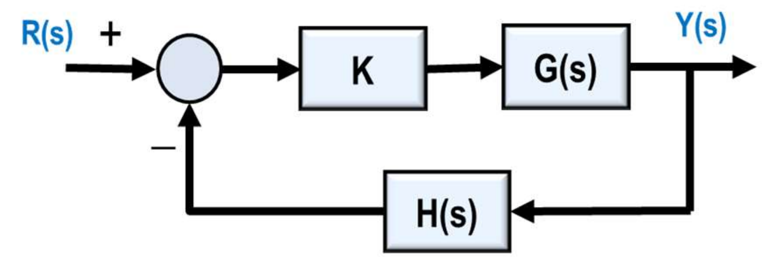
For $K = 2$ the closed-loop system is marginally stable

# Stability Analysis via Nyquist Diagram

**Example 6**

Consider the following closed-loop system

$$G(s) = \frac{1}{s(s+1)^2}, \qquad H(s) = 1$$



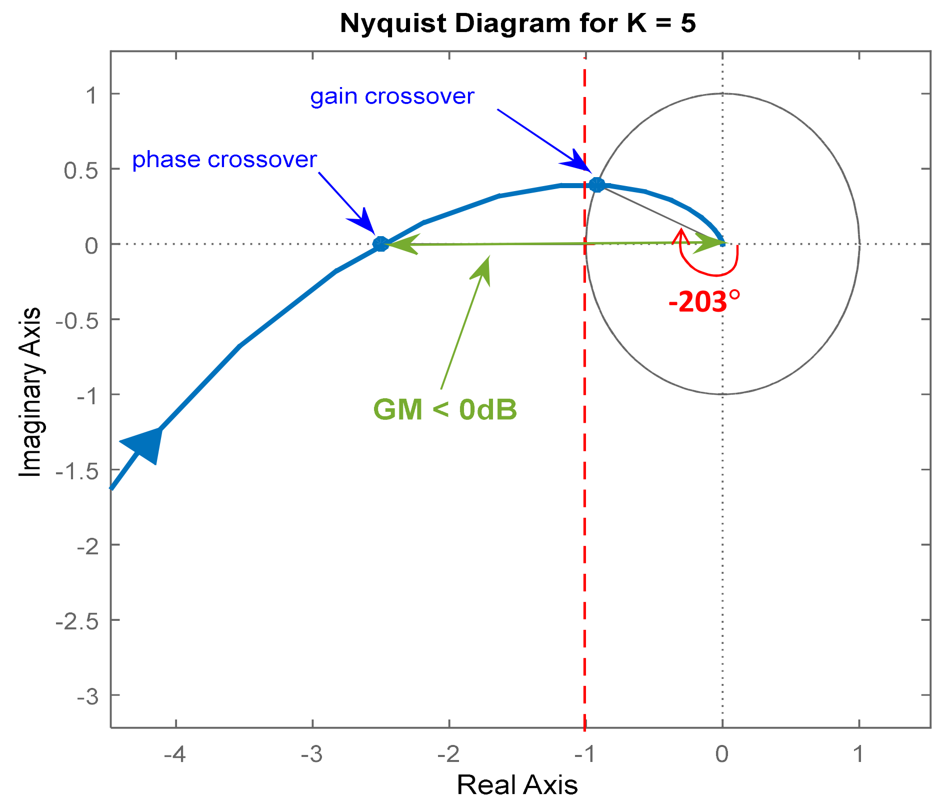

Determine relative stability of the closed-loop system via the Polar plot and identify the gain margin and phase margin of the open-loop system for $K = 0.2,\ 2\ \text{and}\ 5.$

$\boxed{K = 5}$

Phase crossover point at $\longrightarrow$ $-2.5$

$$GM = 0\text{dB} - 20\log|-2.5| = -8dB < 0$$

$$PM = 180^\circ + (-203^\circ) = -23^\circ < 0$$

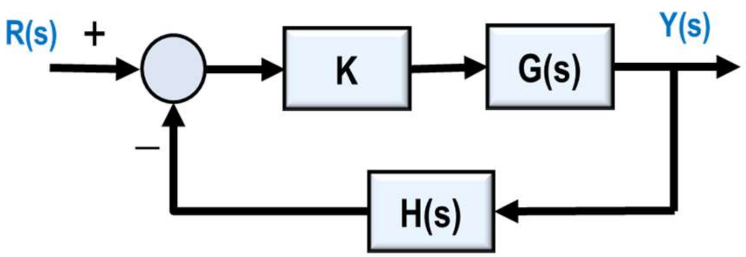



- The Nyquist plot intersects the negative real axis at the left-hand side of the (-1 , *j*0) point, the phase crossover point is at the left-hand side of the -1 point.

  **For $K = 5$ the closed-loop system is unstable**

# Nyquist Stability Criteria

- Consider the following closed-loop system with transfer function of $T(s)$




$$\boldsymbol{T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}}$$

- The Nyquist stability criterion is expressed as

$$\boxed{\boldsymbol{Z = N + P}}$$

$\boldsymbol{Z}$ → Number of closed-loop poles in the right-half s-plane

$\boldsymbol{N}$ → Number of clockwise encirclements of the (-1 , *j*0) point

$\boldsymbol{P}$ → Number of open-loop poles in the right-half s-plane

- To have a stable closed-loop system we must have $\boldsymbol{Z = 0}$. It means that $\boxed{\boldsymbol{N = -P}}$

- **Open-loop stable systems** → $\boxed{\boldsymbol{P = 0 \quad \rightarrow \quad Z = N}}$

  Therefore, for closed-loop stability ($Z = 0$) there must be no encirclement ($N = 0$) of the (-1 , *j*0) point.

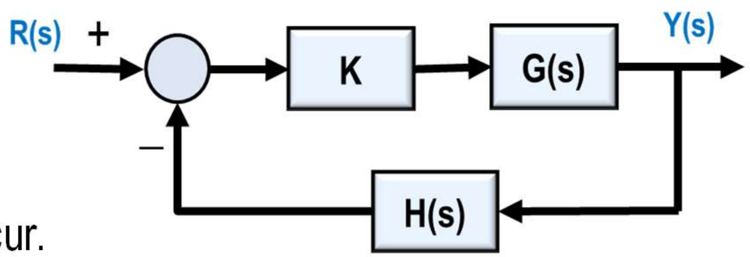
- **Open-loop unstable systems** → $\boxed{\boldsymbol{P \neq 0 \quad \rightarrow \quad Z = N + P}}$

  Therefore, for closed-loop stability ($Z = 0$) we have $N = -P$.

  It means that there must be $P$ counterclockwise (CCW) encirclements of the (-1 , *j*0) point.

# Nyquist Stability Criteria

- Consider the following closed-loop system with transfer function of $T(s)$




$$\boldsymbol{T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}}$$

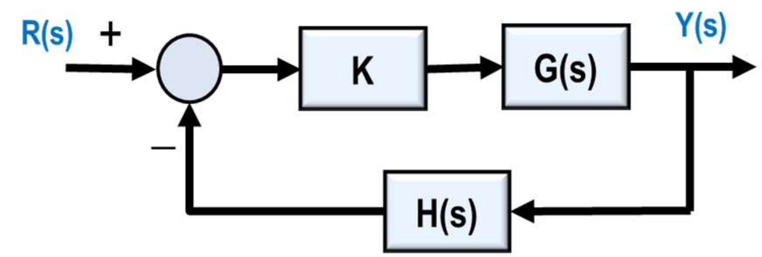
- In stability analysis of the control systems via the Nyquist criteria the following three possibilities can occur.
  - **There is no encirclement of the (-1 , *j*0) point.**
    - The closed-loop system is stable if the open-loop system is stable, ($KG(s)H(s)$ has no poles in the right-half s-plane). Otherwise, the closed-loop system is **unstable**.
  - **There are CCW encirclements of the (-1 , *j*0) point.**
    - The closed-loop system is stable if the number of CCW encirclements is the same as the number of unstable poles of the open-loop system, (number of poles of $KG(s)H(s)$ in the right-half s-plane). Otherwise, the system is **unstable**.
  - **There are CW encirclements of the (-1 , *j*0) point.**
    - The closed-loop system is unstable.
- Note that stability analysis of the closed-loop system via the Nyquist diagram is applicable for both open-loop stable and unstable systems. However, stability analysis of closed-loop system via the Bode diagram is only limited to the open-loop stable systems.
- Since, $G(-j\omega) = G^*(j\omega)$ the Polar plot for $-\infty < \omega < 0^-$ is mirror image of the Polar plot of $0^+ < \omega < +\infty$ with respect to the real axis.

# Nyquist Stability Criteria

**Example 7**

Consider the following closed-loop system

$$KG(s) = \frac{6}{(s+1)(s+2)(s+3)} \quad , \qquad H(s) = 1$$




Analyze stability of the closed-loop system using the Nyquist criterion.

The open-loop transfer function is

$$KG(s)H(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

The open-loop system is a stable: $\boxed{P = 0}$

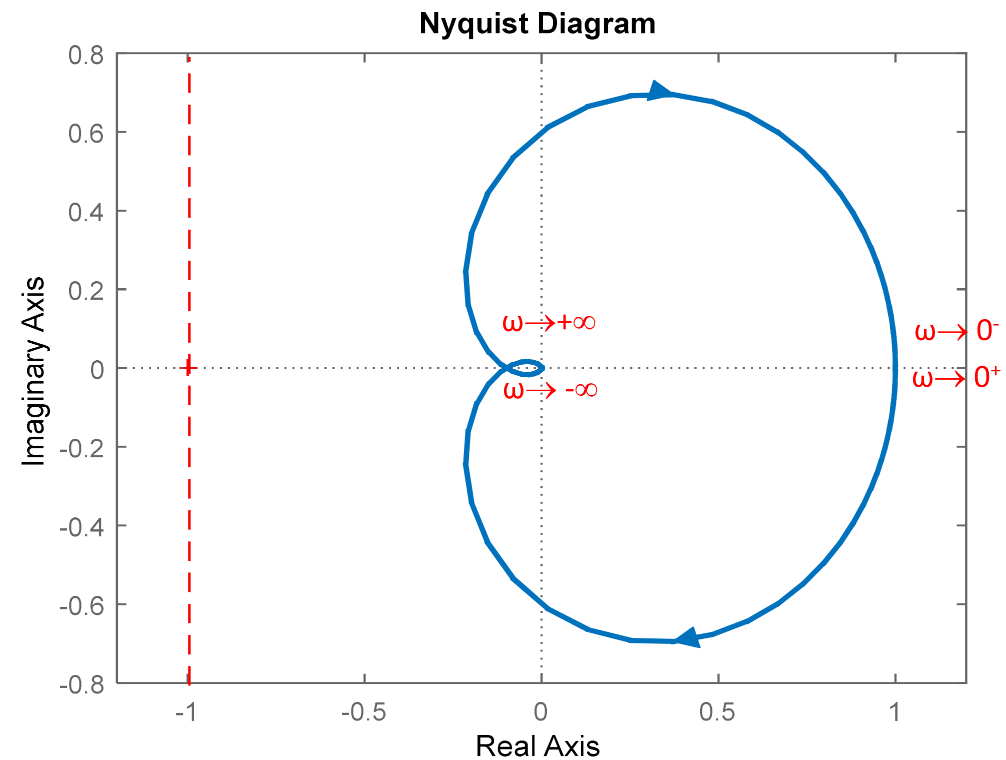
There is no encirclement of the (-1 , j0) point: $\boxed{N = 0}$

The Nyquist criterion:

$$\boxed{Z = N + P \quad \rightarrow \quad Z = 0 + 0 = 0}$$

No closed-loop poles in the right-half s-plane.

**The closed-loop system is stable**

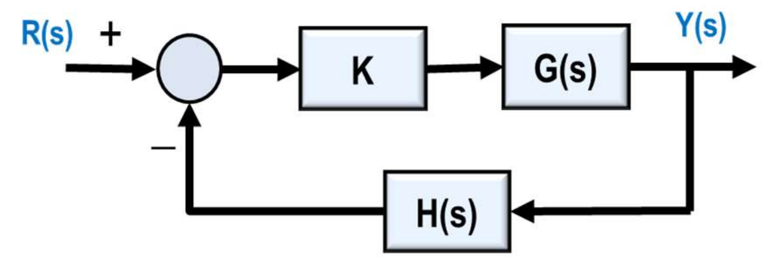


# Nyquist Stability Criteria

**Example 8**

Consider the following closed-loop system

$$KG(s) = \frac{20}{(s-1)(s+2)(s+3)}, \qquad H(s) = 1$$




Analyze stability of the closed-loop system using the Nyquist criterion.

The open-loop transfer function is

$$KG(s)H(s) = \frac{20}{(s-1)(s+2)(s+3)}$$

The open-loop system has one unstable pole: $\boxed{P = 1}$

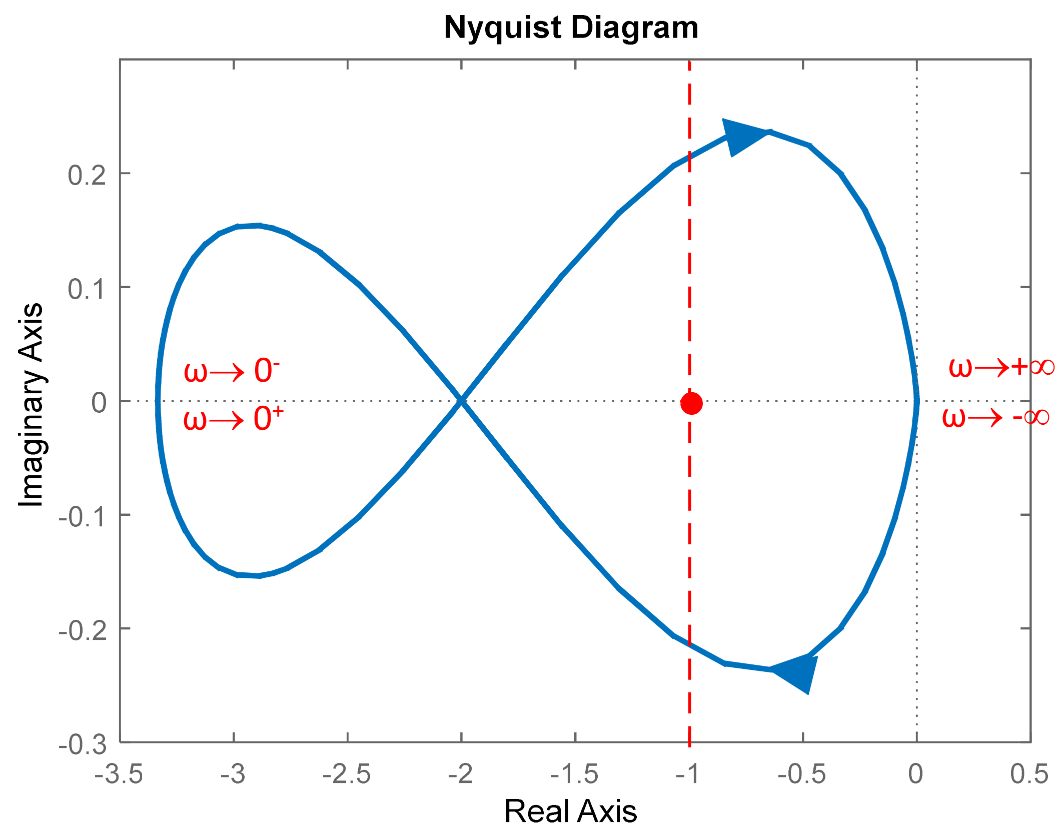
There is one CW encirclement of the (-1 , j0) point: $\boxed{N = 1}$

The Nyquist criterion:

$$\boxed{Z = N + P \quad \rightarrow \quad Z = 1 + 1 = 2}$$

Two closed-loop poles in the right-half s-plane.

**The closed-loop system is unstable**



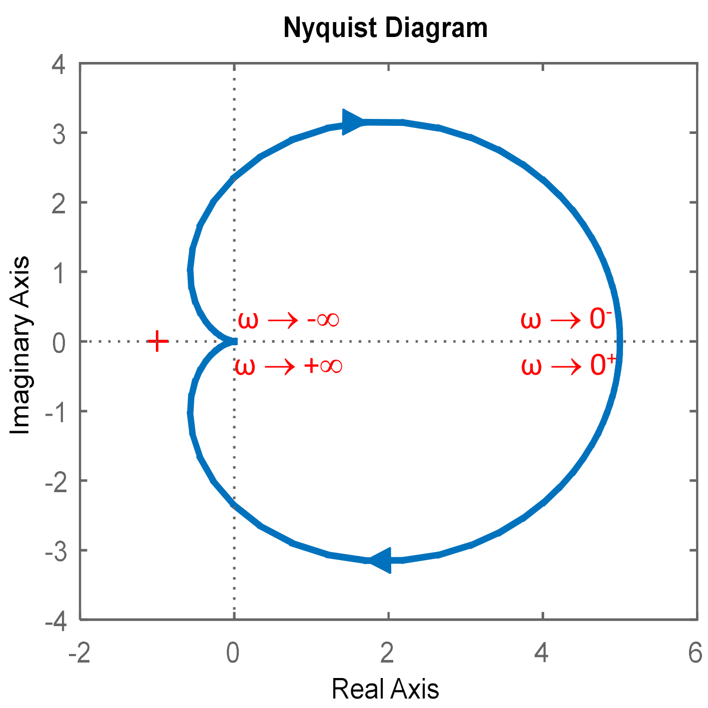
# Nyquist Stability Criteria

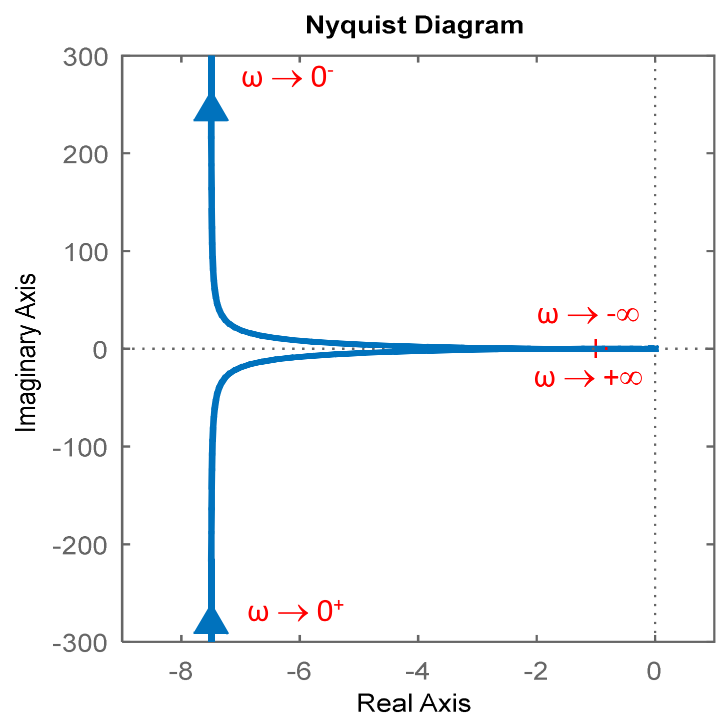
- In Type 0 open-loop systems, the Polar plot of the open-loop system for $-\infty < \omega < +\infty$ will be a closed-plot.
- However, in non-zero type open-loop systems, the Polar plot of the open-loop system for $-\infty < \omega < +\infty$ will not be a closed-plot.

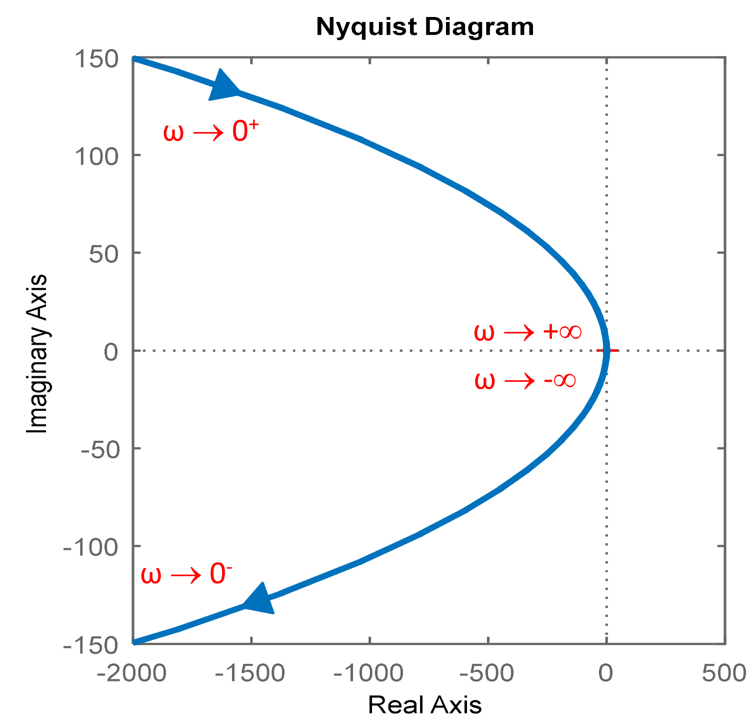
$$KG(s)H(s) = \frac{10}{(s+1)(s+2)}$$

$$KG(s)H(s) = \frac{10}{s(s+1)(s+2)}$$

$$G(s)H(s) = \frac{10}{s^2(s+1)(s+2)}$$



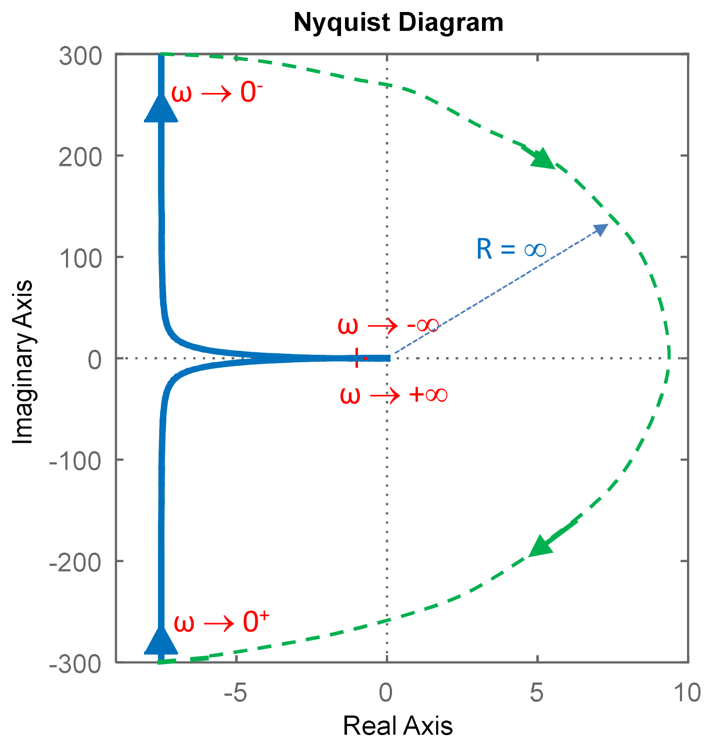



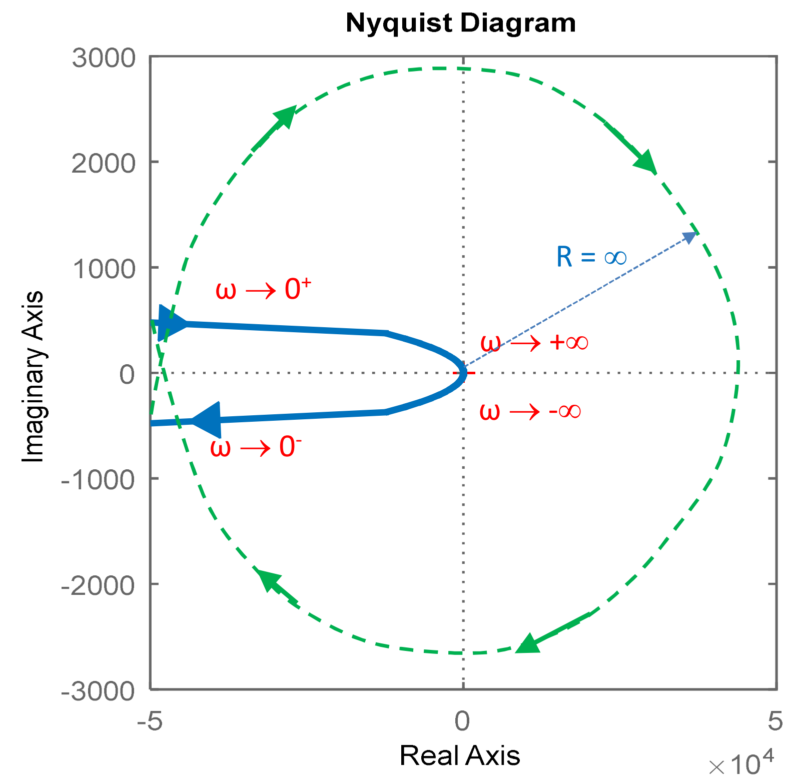
# Nyquist Stability Criteria

- To obtain a closed Polar plot, we have to modify the Polar plot by plotting $\beta$ clockwise semicircles of infinite radius from $0^-$ to $0^+$.

$$KG(s)H(s) = \frac{10}{s(s+1)(s+2)}$$




$$G(s)H(s) = \frac{10}{s^2(s+1)(s+2)}$$

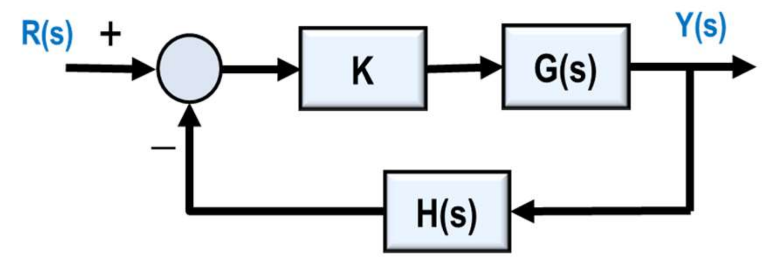


# Nyquist Stability Criteria

**Example 9**

Consider the following closed-loop system

$$\boldsymbol{KG(s) = \frac{6}{s(s+2)(s+3)}}, \qquad \boldsymbol{H(s) = 1}$$




Analyze stability of the closed-loop system using the Nyquist criterion.

The open-loop transfer function is

$$KG(s)H(s) = \frac{6}{s(s+2)(s+3)}$$

The open-loop system has no unstable poles: $\boxed{P = 0}$

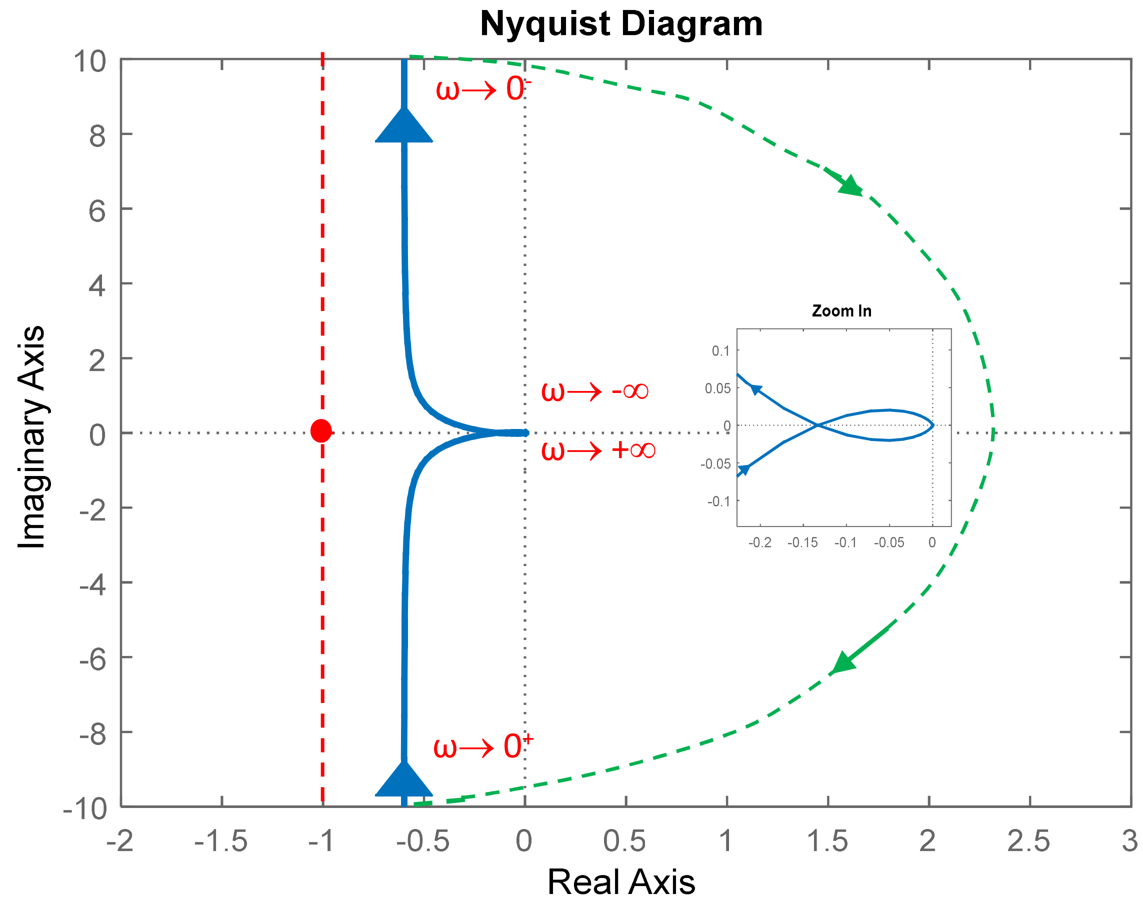
There is no encirclement of the (-1 , j0) point: $\boxed{N = 0}$

The Nyquist criterion:

$$\boxed{Z = N + P \quad \rightarrow \quad Z = 0 + 0 = 0}$$

No closed-loop poles in the right-half s-plane.

**The closed-loop system is stable**

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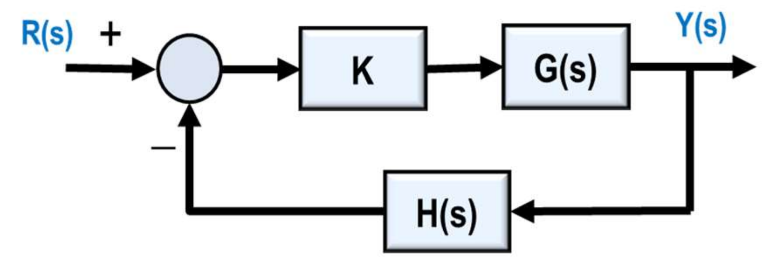
# Nyquist Stability Criteria

**Example 10**

Consider the following closed-loop system

$$KG(s) = \frac{100}{s(s-1)(s+5)}, \qquad H(s) = 1$$




Analyze stability of the closed-loop system using the Nyquist criterion.

The open-loop transfer function is

$$KG(s)H(s) = \frac{100}{s(s-1)(s+5)}$$

The open-loop system has one unstable poles: $\boxed{P = 1}$

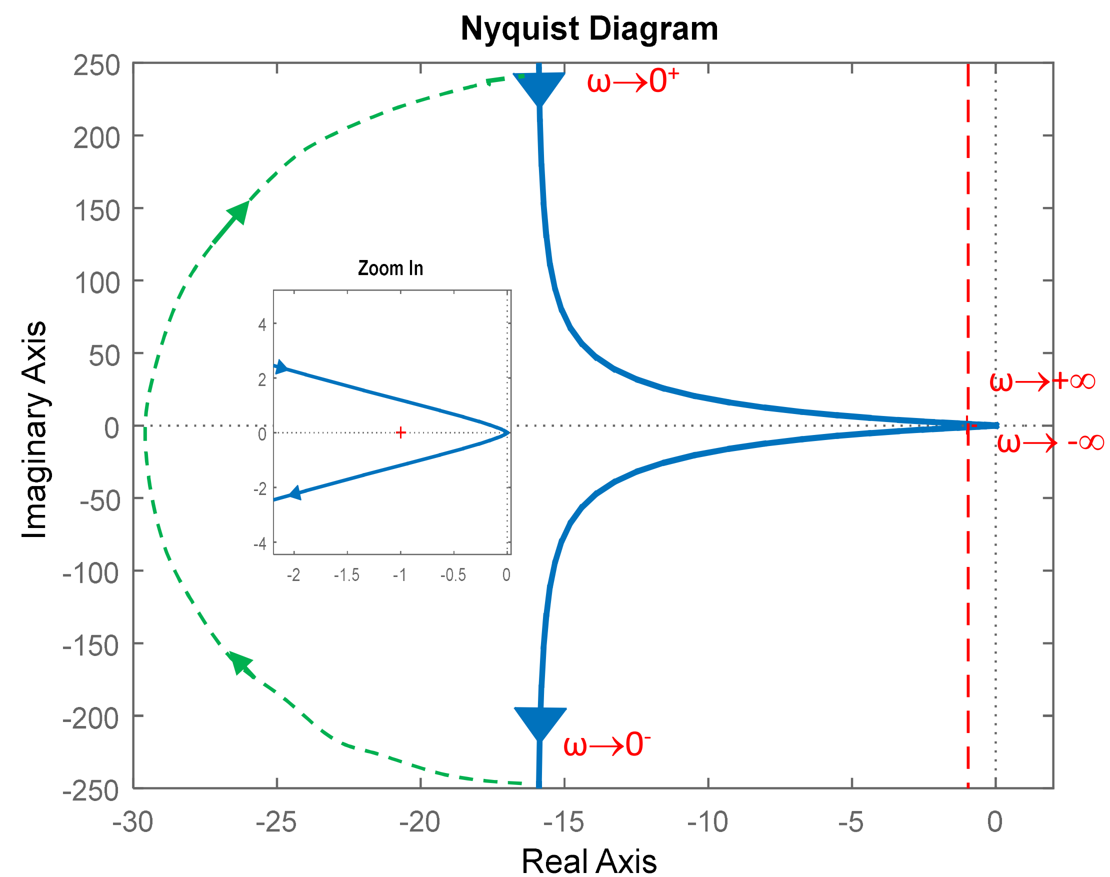
There is one CW encirclement of the (-1 , j0) point: $\boxed{N = 1}$

The Nyquist criterion:

$$\boxed{Z = N + P \quad \rightarrow \quad Z = 1 + 1 = 2}$$

Two closed-loop poles in the right-half s-plane.

**The closed-loop system is unstable**



# THANK YOU

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