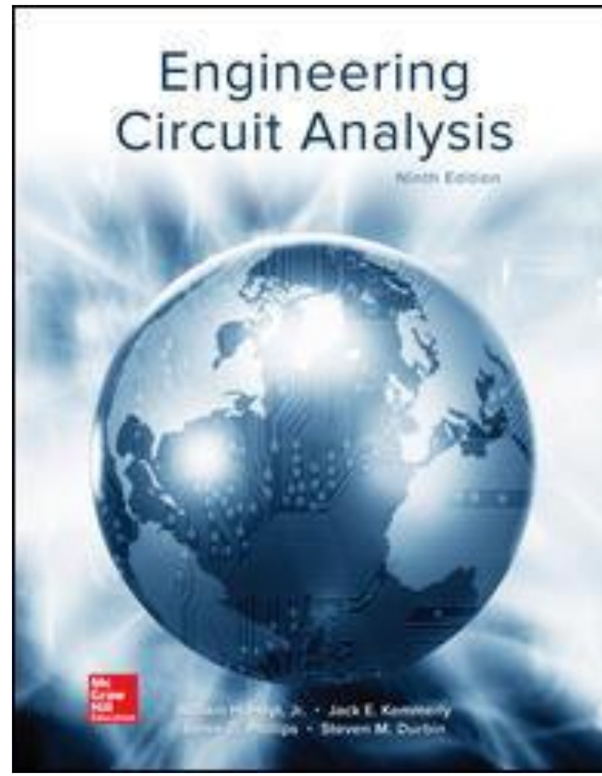


The *RLC* Circuit

The *RLC* Circuit

An *RLC* circuit has **both** an inductor and a capacitor

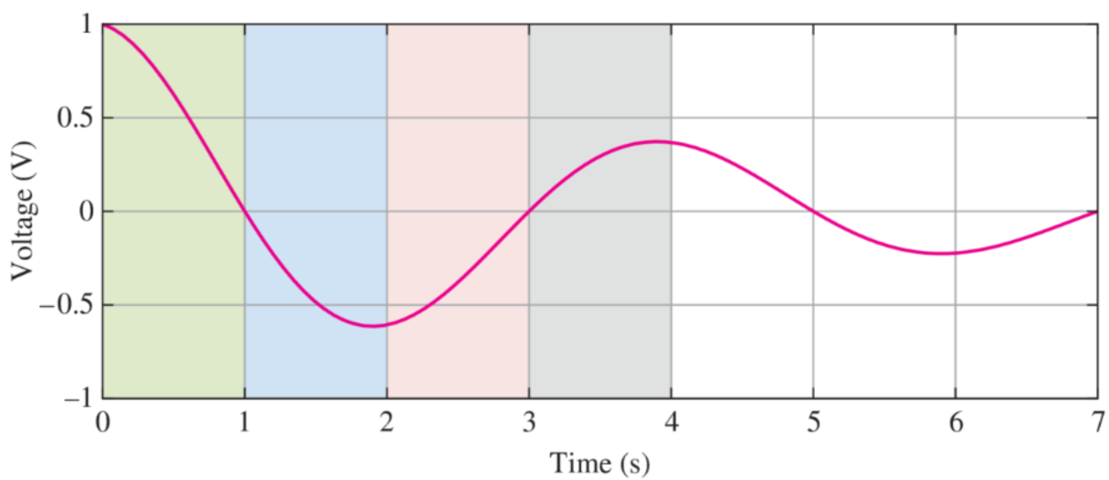
These circuits have a wide range of applications, including oscillators and frequency filters

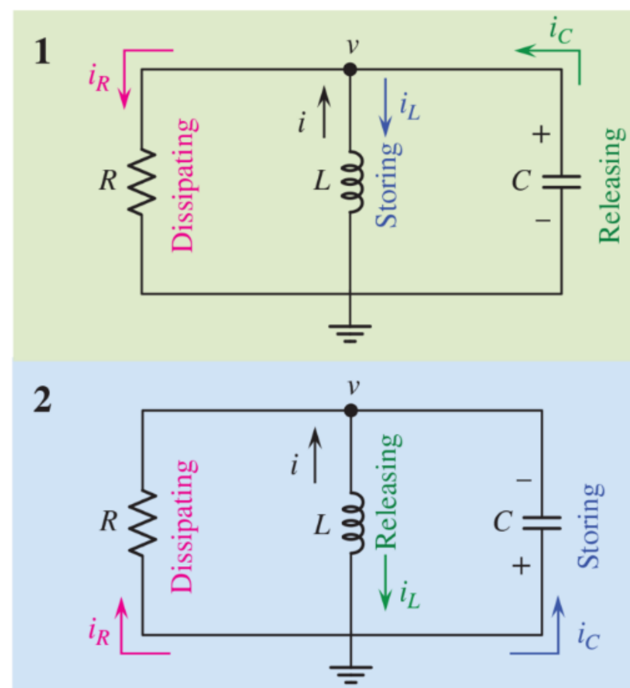
They also can model automobile suspension systems, temperature controllers, airplane responses, and more

Energy Transfer in *RLC* Circuits

Inductors and capacitors store and release energy with varying times, leading to *oscillation*

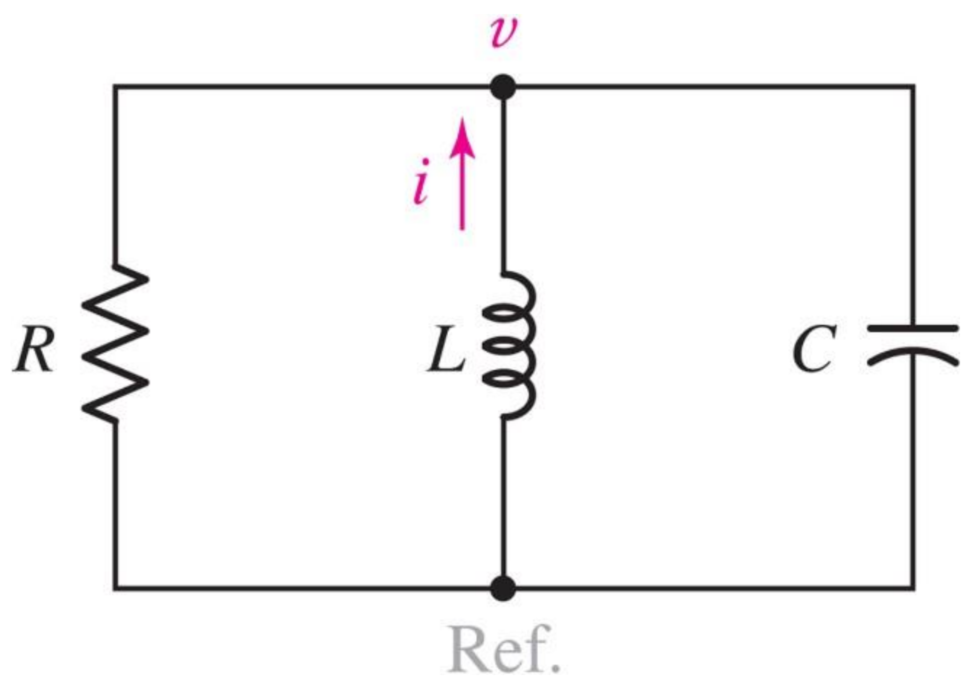
Resistors dissipate energy, leading to *damping*





The Source-Free Parallel Circuit

Apply KCL and differentiate:



$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v=0$$

Solving the Differential Equation

To solve, assume $v = Ae^{st}$.

The solution must then satisfy

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

which is called the *characteristic equation*.

If s_1 and s_2 are the solutions, then the natural response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Exploring the Solution

The solutions to the characteristic equation are

$$-\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Define ω_0 *the resonant frequency:* $\omega_0 = {}^{1}\!/\!_{\sqrt{LC}}$

and α *the damping coefficient:* $\alpha = \frac{1}{2RC}$

Exploring the Solution

With these definitions, the solutions can be expressed as:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The constants A_1 and A_2 are determined by the initial conditions.

Types of Responses

If $\alpha > \omega_0$ the solutions are real, unequal and the response is termed *overdamped*.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

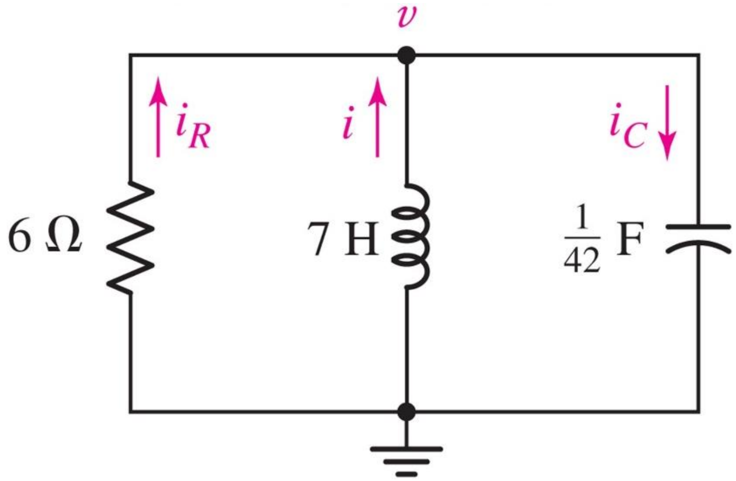
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

If $\alpha < \omega_0$ the solutions are complex conjugates and the response is termed *underdamped*.

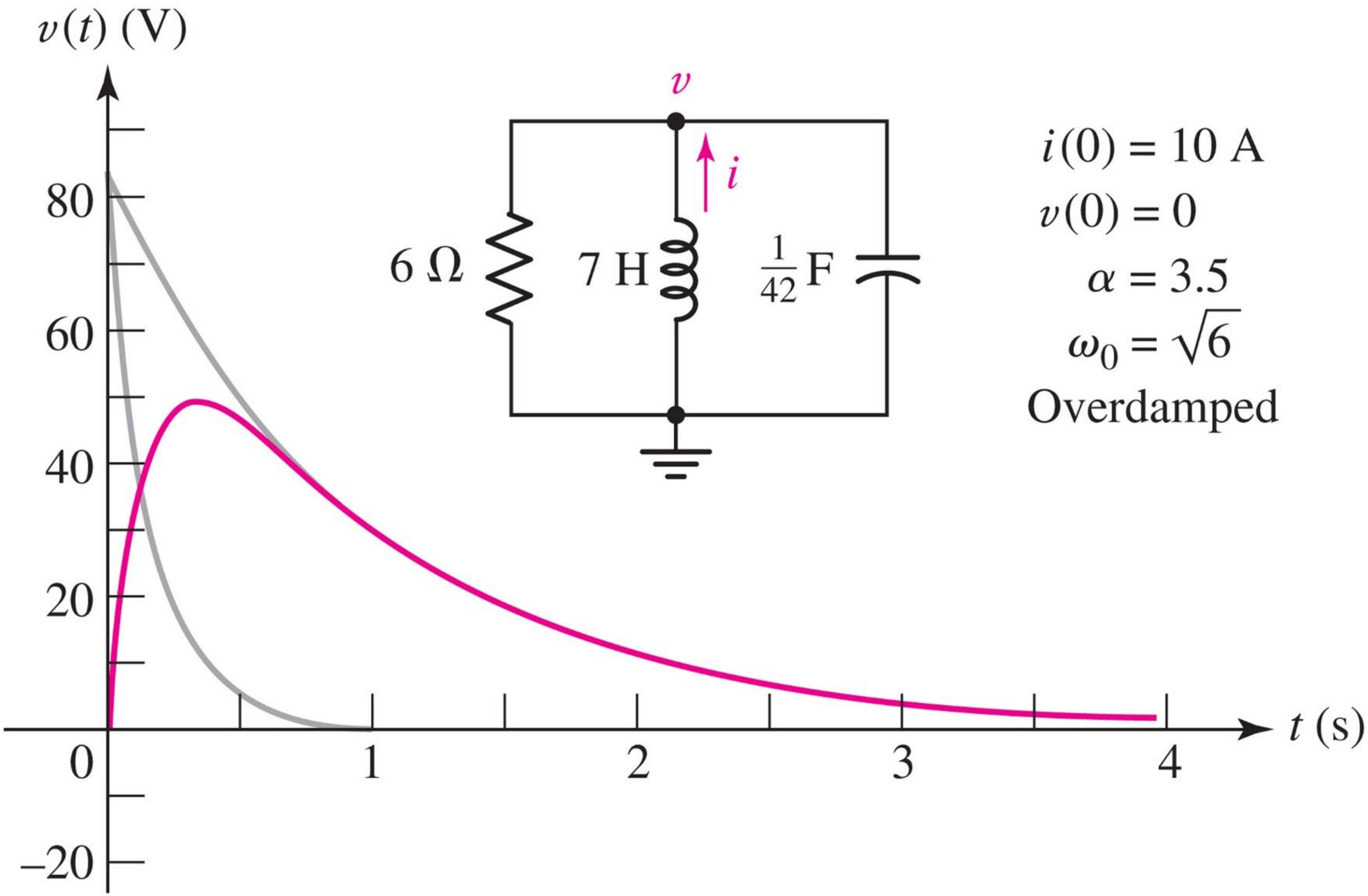
If $\alpha = \omega_0$ the solutions are real and equal and the response is termed *critically damped*.

Overdamped Parallel *RLC*

Show that $v(t) = 84(e^{-t} - e^{-6t})$ when $i(0^+)=10$ A and $v(0^+)=0$ V.



Graphing the Response



The Underdamped Response

If $\alpha < \omega_0$, define

$$\omega_d = \sqrt{\omega_0^{\;2} - \alpha^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

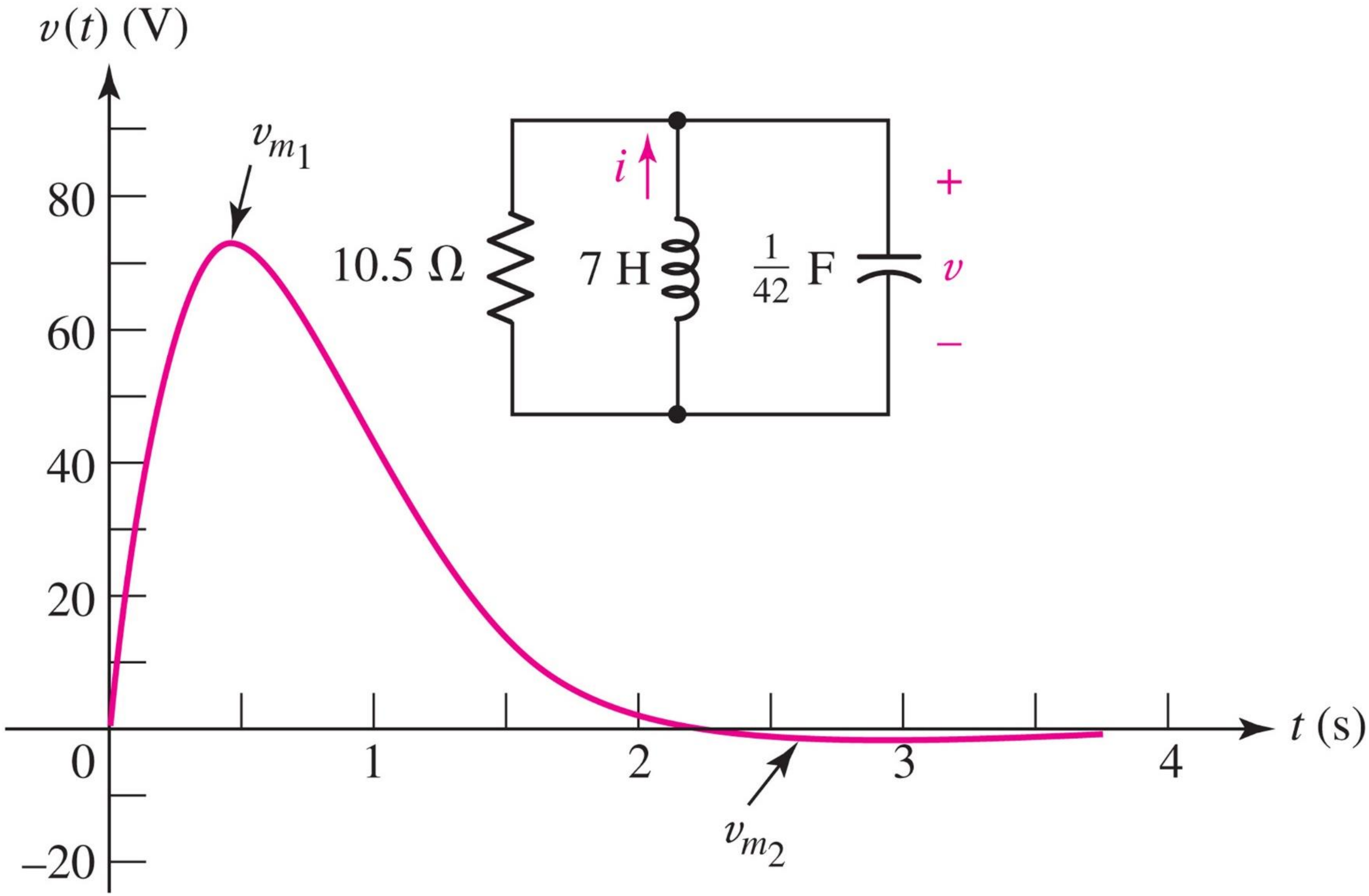
and the solution is

$$v(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

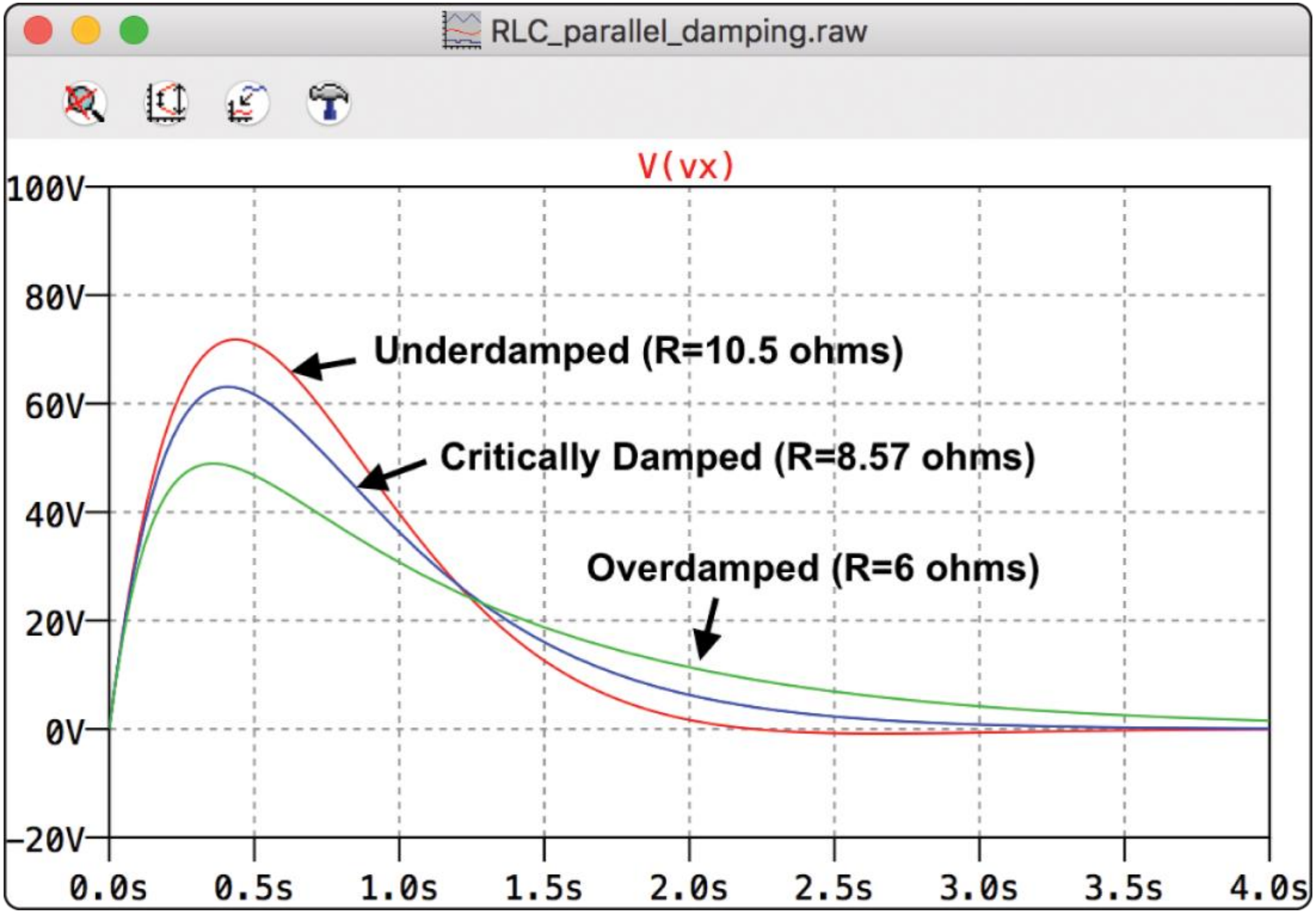
or equivalently

$$v(t) = e^{-\alpha t}\left(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right)$$

Example: Underdamped Response

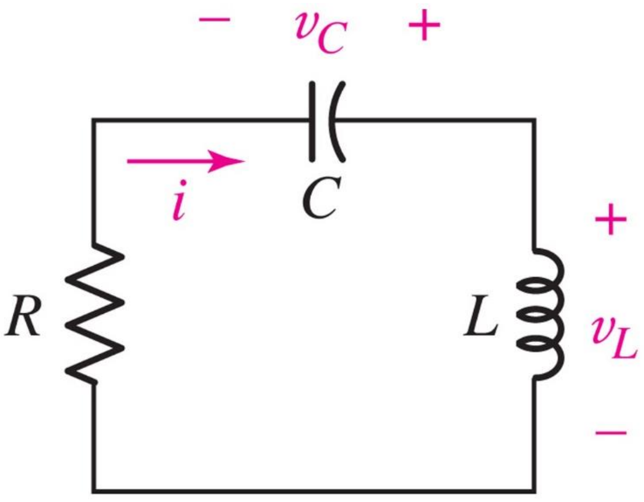


Comparing the Responses



Source-Free Series *RLC* Circuit

For the series RLC circuit,



$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

Series *RLC* Differential Equation

The characteristic equation is

$$Ls^2 + Rs + \frac{1}{C} = 0$$

and the solution is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Series *RLC* Circuit Solution

$$Define\ \omega_0 = {}^{1}\!\!\diagup\!{\sqrt{LC}}\ and\ \alpha = \frac{R}{2L}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Then if

$$\alpha > \omega 0 \left(overdamped \right): \quad v\left(t\right) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\alpha = \omega 0 \left(critically\ damped \right): \quad v\left(t\right) = e^{-\alpha t}\left(A_1 t + A_2\right)$$

$$\alpha < \omega 0 \left(underdamped \right): \quad v\left(t\right) = e^{-\alpha t}\left(B_1 \cos\left(\omega_d t\right) + B_2 \sin\left(\omega_d t\right)\right)$$

Summary for Source-Free *RLC*

Condition	Criteria	α	ω_0	Response
Overdamped	$\alpha > \omega_0$	$\frac{1}{2RC}(\text{parallel})$ $\frac{R}{2L}(\text{series})$	$\frac{1}{\sqrt{LC}}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$, Where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
Critically damped	$\alpha = \omega_0$	$\frac{1}{2RC}(\text{parallel})$ $\frac{R}{2L}(\text{series})$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(A_1 t + A_2)$
Underdamped	$\alpha < \omega_0$	$\frac{1}{2RC}(\text{parallel})$ $\frac{R}{2L}(\text{series})$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$, $\omega_\text{d} = \sqrt{\omega_0^2 - \alpha^2}$

The Complete Response

The response of *RLC* circuits with dc sources and switches will consist of the natural response and the forced response:

$$v(t) = v_f(t) + v_n(t)$$

The complete response must satisfy both the initial conditions and the “final conditions” or the forced response.

Summary of Procedure for Solving *RLC* Circuits

Determine initial conditions

Obtain a numerical value for the forced response

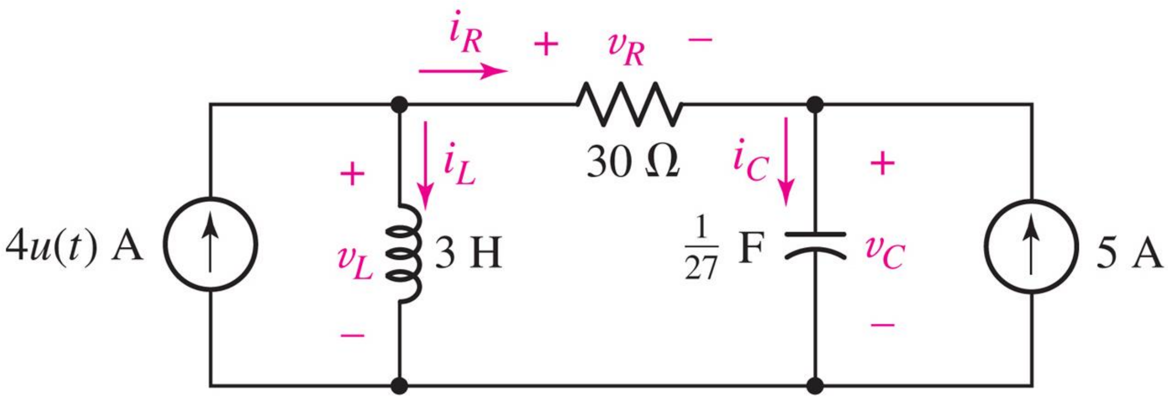
Write the appropriate form of the natural response with unknown constants. Calculate α and ω_0

Add forced and natural response to form complete response

Evaluate the response and its derivative at $t = 0$ and solve for unknown constants using initial conditions

Example: Initial Conditions

Find the labeled voltages and currents at $t = 0^{-}$ and $t = 0^{+}$.



Answer :

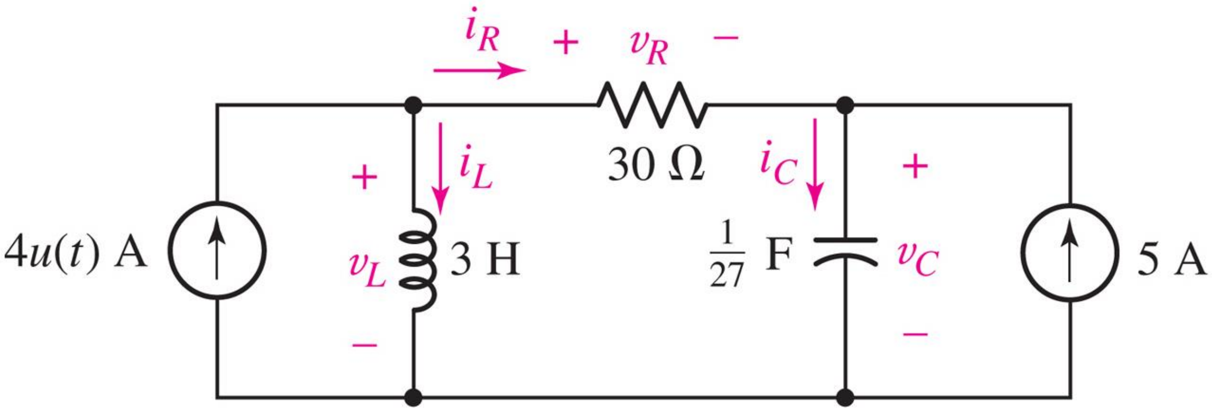
$$i_R(0^-) = -5\ A \quad v_R(0^-) = -150\ V \quad i_R(0^+) = -1\ A \quad v_R(0^+) = -30\ V$$

$$i_L(0^-) = 5\ A \quad v_L(0^-) = 0\ V \quad i_L(0^+) = 5\ A \quad v_L(0^+) = 120\ V$$

$$i_C(0^-) = 0\ A \quad v_C(0^-) = 150\ V \quad i_C(0^+) = 4\ A \quad v_C(0^+) = 150\ V$$

Example: Initial Slopes

Find the first derivatives of the labeled voltages and currents at $t = 0^+$.



Answer :

$$di_R / dt\left(0^+\right) = -40\ A/s \quad dvR / dt\left(0^+\right) = -1200\ V/s$$

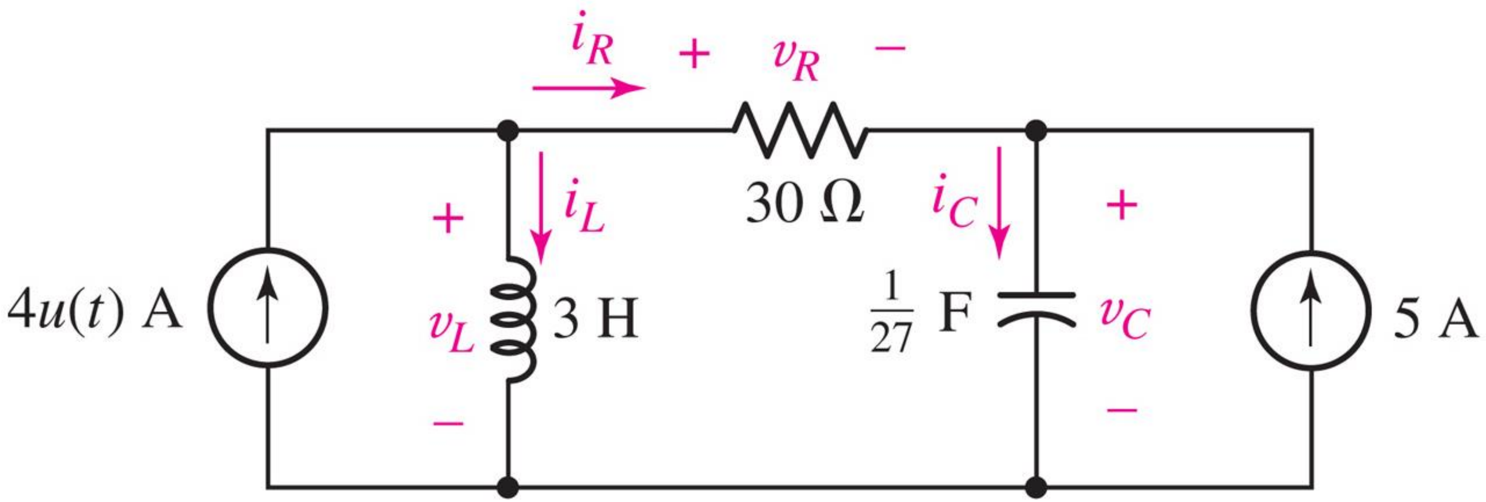
$$di_L / dt\left(0^+\right) = 40\ A/s \quad dvR / dt\left(0^+\right) = -1092\ V/s$$

$$di_C / dt\left(0^+\right) = -40\ A/s \quad dvR / dt\left(0^+\right) = 108\ V/s$$

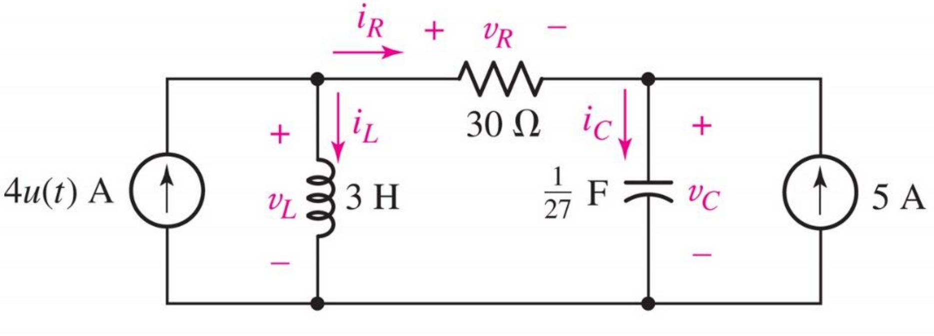
Example: Complete Response

Show that for $t > 0$

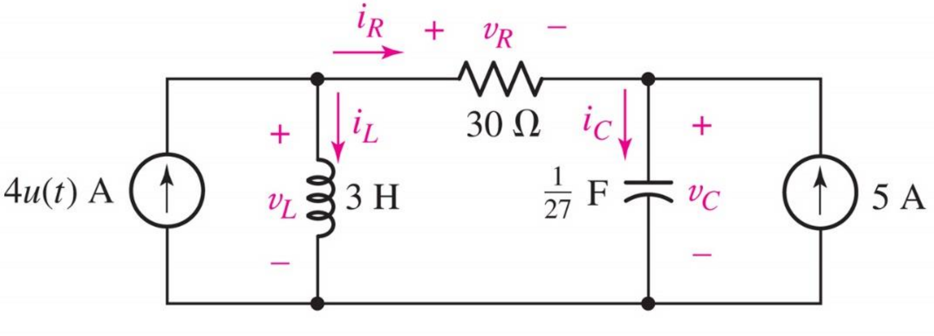
$$v_C(t) = 150 + 13.5\left(e^{-t} - e^{-9t}\right) \text{volts}$$



Example: Complete Response



Example: Complete Response



Example: Complete Response