Assignment 1

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Calculus Assignment 1

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1. Consider the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{5^n}$.

   (a) (1 point) Write the first three terms of the series.

   $-\frac{3}{5}, \frac{9}{25}, -\frac{27}{125}$

   $-0.6, 0.36, -0.216$

   (b) (1 point) Write the first three terms of the associated sequence of partial sums $\{S_n\}_{n=1}^{\infty}$.

   $S_1 = -\frac{3}{5} = \boxed{-0.6}$

   $S_2 = -\frac{3}{5} + \frac{9}{25} = \boxed{-\frac{6}{25} = -0.24}$

   $S_3 = -\frac{3}{5} + \frac{9}{25} + \left(-\frac{27}{125}\right) = \boxed{-\frac{57}{125} = -0.456}$

   After simplifying:

   $S_1 =$ $-\frac{3}{5} = -0.6$

   $S_2 =$ $-\frac{6}{25} = -0.24$

   $S_3 =$ $-\frac{57}{125} = -0.456$

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   (c) (1 point) Does the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$ satisfy the conditions necessary to apply the Alternating Series Test? Justify your answer.

   $\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n} = \frac{(-3)^1}{2^1} + \frac{(-3)^2}{2^2} + \frac{(-3)^3}{2^3}$

$n=1$

$= -\frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \frac{81}{16}$

We looked at the sequence $\{A_n\}_{n=1}^{\infty} = \left\{\frac{3^n}{2^n}\right\}_{n=1}^{\infty} = \left\{\left(\frac{3}{2}\right)^n\right\}_{n=1}^{\infty}$

We cannot apply the Alternating series

since $\left(\frac{3}{2}\right)^n \not\to 0$ as $n \to \infty$

Also $\left(\frac{3}{2}\right)^n$ is not decreasing sequence as $\left(\frac{3}{2}\right)^{n+1} > \left(\frac{3}{2}\right)^n$

(d) (1 point) Does the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{5^n}$ converge? If yes, explain why and write down the number that the series converges to. If not, why not?

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{(-3)}{5}\right)^n$$

$$= \frac{-\frac{1}{5}\left(1 - \left(\frac{-3n}{5}\right)\right)}{1 - \left(-\frac{3}{5}\right)} = -\frac{3}{8}$$

$$\lim_{n \to \infty} \left(\frac{-3n}{5}\right) = 0$$

$$\lim_{n \to \infty} \sqrt[n]{\left|\left(-\frac{3}{5}\right)^n\right|} = \frac{3}{5} < 1$$

So, $\sum_{n=1}^{\infty} \left(-\frac{3}{5}\right)^n$ absolutely converges

it converges since $\left(\frac{-3^n}{5^n}\right)$ is less than 1

2. (3 points) Determine whether or not the following series converges? Show the details of your work and justify your steps. (*Hint:* Use one of the comparison tests, comparing with an appropriate p-series.)

$$\sum_{n=2}^{\infty} \frac{n^2 + 7n + 5}{n^3 - 8}$$

Consider the comparison series $\sum_{n=2}^{\infty} \frac{1}{n}$

Since $\lim_{n \to \infty} \frac{\frac{n^2 + 7n + 5}{n^3 - 8}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2 + 7n + 5}{n^3 - 8} \cdot \frac{n}{1}$

$\lim_{n \to \infty} \frac{n(n^2 + 7n + 5)}{n^3 - 8}$

$= 1$

$\therefore$ Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is P-Series with $P = 1$, it is a diverging series

3. (3 points) Does the series $\sum_{k=1}^{\infty} \frac{k!}{(-5)^k}$ converge or diverge? Justify your answer.

$$\lim_{k\to\infty}\left|\frac{\frac{(k+1)!}{(-5)^{k+1}}}{\frac{k!}{(-5)^k}}\right| = \lim_{k\to\infty}\left|\frac{\overset{k+1}{\cancel{(k+1)!}}}{\underset{-5}{\cancel{(-5)^{k+1}}}}\cdot\frac{\overset{1}{\cancel{(-5)^k}}}{\underset{1}{\cancel{k!}}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{k+1}{-5}\right|$$

$$= \lim_{k\to\infty}\frac{k+1}{5}$$

$$= \infty$$

By the ratio test, $\sum_{k=1}^{\infty}\frac{k!}{(-5)^k}$ diverges

4. (4 points) Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} 3ne^{-n2}$ converges or not? Your work should also show that the conditions necessary to apply the Integral Test have been met.

Consider $f: [1:\infty] \to \mathbb{R}$ given by $f(x) = 3xe^{-x^2}$

(i) Since e^{-x^2} is positive forall $x \in \mathbb{R}$ & $3x > 0$ for $x \geq 1$, $f(x) = 3xe^{-x^2} > 0$ for all $x \in [1, \infty)$.

(ii) $f'(x) = 3x^{-x^2} - 6x^2e^{n^2}$

$= -(6x^2 - 3)e^{-x^2}$

< 0 Since $x \geq 1 \to x^2 \geq 1$

$\to 3x^2 \geq 3$

$\to 3x^2 > 1$ Since $3x^2 \geq 3$ & $3 > 1$

$\to 0 > 1 - 3x^2$ subtracting $3x^2$ from both sides

$\to 0 > 3e^{-x^2}(1-3x^2)$ Multiply both by positive quantity $3e^{-x^2}$

So for $x \geq 1$ is decreasing

(iii) For $n = 1, 2, 3$, $f(n) = 3ne^{-n^2}$ which is precisely the nth term at the series

Here we apply the Integral test.

$$\int_1^\infty 3xe^{-x^2}\,dx = \int_1^\infty e^{-u}\,du = \lim_{b\to\infty} \int_1^b e^{-u}\,du$$

let $u = x^2$
then $du = 2x\,dx$

$$= \lim_{b\to\infty} -e^{-u}\Big|_1^b$$

$$= \lim_{b\to\infty} (-e^{-b} - (-e^{-1}))$$

$$= \lim_{b\to\infty} \left(-e^{-b} + \frac{1}{e}\right)$$

$$= 0 + \frac{1}{e}, \text{ since as } b \to \infty, e^{-b} \to 0$$

$$= \frac{1}{e}$$

So, by integral test, since $\int^\infty 3xe^{-x^2}\,dx$ converges, the series $\sum_{n=1}^{\infty} 3ne^{-n^2}$ converges.

5. Express the taylor Series of the function $f(x) = \frac{5}{(2+x^3)}$ about $x=0$ in summation notation.

we know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So, $f(x) = \frac{5}{(2+x^3)}$

$$= \frac{5}{2} \cdot \frac{1}{1 + \frac{x^3}{2}}$$

$$= \frac{5}{2} \sum_{n=0}^{\infty} \left(-\frac{x^3}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5x^{3n}}{2^{n+1}}$$

$\therefore$ the summation notation is

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5x^{3n}}{2^{n+1}}$$