2.4 & 2.5

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2.4 Implicit and Logarithmic Differentiation

2.5 Higher Order Derivatives

Textbook ref: 2.11 and 2.14

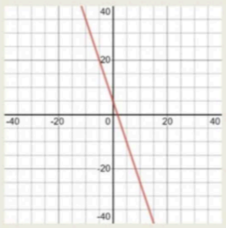
Explicit relation:

$y = -3x + 5$

Find $\frac{dy}{dx} = y'$

Treat

- x as independent variable
- y as specific function of x



Implicit relation:

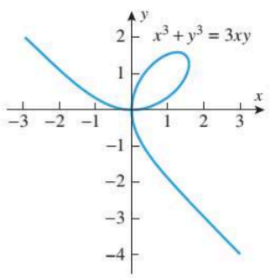
$y + 3x - 5 = 0$

Find $\frac{dy}{dx} = y'$

Treat

- x as an input
- y as **unspecified** function of x
- solve for $\frac{dy}{dx} = y'$

Example 1. The Folium of Descartes



This equation can be solved for $y = y(x)$, however the resulting expression is too complicated. (Ans. $y' = \frac{y-x^2}{y^2-x}$)

$x^3 + y^3 = 3xy$

$\frac{d}{dx}[x^3 + y^3] = 3\frac{d}{dx}[xy]$

$\frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] = 3\frac{d}{dx}[x \cdot y]$

$\frac{3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3y \cdot 3xy'}{3}$

$x^2 + y^2 y' = y + xy'$

$y^2 y' - xy' = y - x^2$

$y'(y^2 - x) = y - x^2$

$y' = \frac{y - x^2}{y^2 - x}$

Example 2. (video in Panopto)

Find the slope of the tangent to the curve $x^2 + y^2 = 25$ at point (3, 4). *Hint*: Use the point-slope form of the equation of a straight line $y - y_0 = m(x - x_0)$. (Ans. $3x + 4y = 25$)

Example 3

Use implicit differentiation to find the equation of the tangent line to the curve $xy^3 + xy = 20$ at the point (10,1) (Ans. $y = -0.05x + 1.5$)

Compute derivatives with respect to variables indicated.

1. a. Assume that $u = u(t)$. If $y = 3u^4$, find $\frac{dy}{dt}$.

   b. If $P = 2R^3 + 0.5t^2$ and $R = R(t)$, find $\frac{dP}{dt}$.

   c. Assume that $r = r(t)$. If $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dt}$

2. Find $\frac{dx}{dy}$, if $x = (y-3)^2$

3. Assume that $u = u(x)$, find

3. Assume that $y = y(x)$, find

   a. $\frac{d}{dx}[xy] =$

   b. $\frac{d}{dx}[x^2y^3] =$

4. $y^3 - 4x^2y^2 + y^4 = 9$, find $\frac{dy}{dx}$

5. Given $\cos(xy) = 1 + \sin y$, find $\frac{dy}{dx}$

6. Find the slope of the tangent line at the point $P(1,1)$ on the graph of $e^{x-y} = 2x^2 - y^2$.

7. Find the derivative $\frac{dy}{dx}$ by implicit differentiation $e^y = \sin(x+y)$

<u>Calculating the higher-order derivatives.</u>

Let $y = f(x)$. The 1st derivative of a function f is : $\frac{dy}{dx} = f'(x) = y' = D_x[y]$

The derivative of the 1st derivative is called the 2nd derivative:

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2} = y'' = f''(x) = D_x^2[y]$$

8. Find the first three derivatives of $y = 5x^3 - 2x, y', y'', y'''$

9. Find y'' for $y = \sqrt{5 - 4x^2}$

10. Find y' and y'' if $y = \sqrt{1 - \sec t}$

11. Evaluate y'' for $y = \frac{2}{1-x}$ for $x = -2$

12. Find y'' of $y = 6\tan 5x$

Additional Problems

13. Use implicit differentiation to find an equation of the tangent line to the curve $y\sin 2x = x\cos 2y$ at the point$(\pi/2, \pi/4)$.

14. The power P that a battery (source) supplies to a laptop (load) depends on the internal resistance of the battery. For a battery of voltage V and internal resistance R_S, the total power delivered to a laptop of resistance R_L is

$$P = \frac{V^2R_L}{(R_L + R_S)^2}$$

   a. Assume that V and R_S are constants, treat the power P as an unspecified function of R_L, such that $P = f(R_L)$, and find $\frac{dP}{dR_L}$.

   b. Determine the value(values) of the laptop resistance R_L for which the tangent line is horizontal? What does it mean in terms of the power?

14. Differentiate $y = x^{\sin x}$ using logarithmic differentiation.

15. Differentiate $f(x) = \frac{x^3 e^x}{(1+x)^4}$ using logarithmic differentiation

16. Use logarithmic differentiation to find the derivative of the function $y = (\sin x)^{\ln x}$

ANSWERS

1. a. $\frac{dy}{dt} = 12u^2\frac{du}{dt}$; b. $\frac{dP}{dt} = 6R^2\frac{dR}{dt} + t$; c. $y' = 4\pi r^2\frac{dr}{dt}$.
2. $\frac{dx}{dy} = 2(y-3)$
3. a. $\frac{d}{dx}[xy] = y + xy'$; b. $\frac{d}{dx}[x^2y^3] = 2xy^3 + 3x^2y^2y'$
4. $y' = \frac{8xy}{3y-8x^2+4y^2}$
5. $y' = -\frac{y\sin(xy)}{x\sin(xy)+\cos y}$
6. $m_T @ P(1,1) = 3$
7. $\frac{dy}{dx} = \frac{\cos(x+y)}{e^y-\cos(x+y)}$
8. $y' = 15x^2 - 2, y'' = 30x, y''' = 30$
9. $y' = -4x(5-4x^2)^{-\frac{1}{2}}; y'' = -20(5-4x^2)^{-\frac{3}{2}}$
10. $y' = -\frac{\sec t \tan t}{\sqrt{1-\sec t}}$, $y" = \text{good luck}$
11. $y' = 2(1-x)^{-2}; y'' = \frac{4}{(1-x)^3}; y''(-2) = \frac{4}{27}$
12. $y' = 30sec^2(5x); y" = 300sec^2(5x)\tan(5x)$
13. $\frac{dy}{dx} = \frac{\cos 2y - 2y\cos 2x}{\sin 2x + 2x\sin 2y}; y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right)$
14. a. $\frac{dP}{dR_L} = V^2\frac{R_S-R_L}{(R_L+R_S)^3}$; b. $R_L = R_S$
15. $y' = x^{sinx}\left(\cos x\ \ln x + \frac{sinx}{x}\right)$
16. $f'(x) = \frac{x^3\,e^x}{(1+x)^4}\left[1 + \frac{3}{x} - \frac{4}{1+x}\right]$
17. $y' = (\sin x)^{\ln x}\left[\frac{\ln(\sin x)}{x} + \ln x\ \cot x\right]$