Calc 1500

What is the Grad of a Scalar Function?

The gradient of a scalar function $f(x, y, z)$, denoted by

$$\nabla f \quad \text{or} \quad \text{grad } f,$$

is the vector field

$$\begin{aligned} \nabla f(x, y, z) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f(x, y, z) \\ &= \left\langle \frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right\rangle . \end{aligned}$$

Observe that the gradient of a scalar function is a vector field.

What is the Divergence of a Vector Field?

The divergence of a vector field $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$, denoted by

$$\nabla \cdot \mathbf{F} \quad \text{or} \quad \operatorname{div} \mathbf{F},$$

is the scalar-valued function

$$\begin{aligned} \nabla \cdot \mathbf{F}(x, y, z) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle \\ &= \frac{\partial F_1}{\partial x}(x, y, z) + \frac{\partial F_2}{\partial y}(x, y, z) + \frac{\partial F_3}{\partial z}(x, y, z). \end{aligned}$$

Observe that the divergence of a vector field is a scalar function. Also, div $\mathbf{F}$ is defined in a similar way for any vector field $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

Example: Find div **F**

Let $\mathbf{F}(x, y, z) = \langle \overbrace{xy}^{=F_1(x,y,z)}, \overbrace{y^2 + e^{xz}}^{=F_2(x,y,z)}, \overbrace{\sin(2y)}^{=F_3(x,y,z)} \rangle$.

Find $\nabla \cdot \mathbf{F}$, that is, div $\mathbf{F}$.

$$\operatorname{div} \vec{F}(x,y,z) = y + 2y + 0 = 3y$$

$$\nabla \cdot \vec{F}(x,y,z)$$

$$\frac{\partial(xy)}{\partial x} + \frac{\partial(y^2 + e^{xz})}{\partial y} + \frac{\partial(\sin(2y))}{\partial z}$$

What is the Curl of a 3D Vector Field?

The curl of a 3D vector field $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$, denoted by

$$\nabla \times \mathbf{F}(x, y, z) \quad \text{or} \quad \operatorname{curl} \mathbf{F},$$

is defined by

$$\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\boldsymbol{\imath}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\boldsymbol{k}} \\
&= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1(x, y, z) & F_2(x, y, z) & F_3(x, y, z) \end{vmatrix}.
\end{aligned}$$

Example: Find curl **F**

Let $\mathbf{F}(x, y, z) = \langle y + ze^x,\ x + e^y \sin z,\ z + e^x + e^y \cos z\rangle$.

Find $\nabla \times \mathbf{F}$, that is, curl $\mathbf{F}$.

$$\nabla \times \vec{F}(x,y,z) = \text{curl } \vec{F}(x,y,z)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+ze^x & x+e^y\sin z & z+e^x+e^y\cos z \end{vmatrix}$$

determinant

$$= \left\langle e^y \cos z - e^y \cos z \,,\ e^x - (0 + e^x + 0),\ 1 - (1 + 0) \right\rangle$$

$$\frac{\partial (z + e^x + e^y \cos z)}{\partial y} - \frac{\partial (x + e^y \sin z)}{\partial z}$$

$$\frac{\partial}{\partial z}(y + ze^x) - \frac{\partial}{\partial x}(z + e^x + e^y \cos z)$$

$$\frac{\partial}{\partial x}(x + e^y \sin z) - \frac{\partial}{\partial y}(y + ze^x)$$

$$= \langle 0, 0, 0 \rangle$$

If $\operatorname{curl} \vec{F} = \vec{0}$, we say that $\vec{F}$ is **"curl-free"** or **"irrotational"**

Live Poll

Find the divergence and $\boxed{\text{curl}}$ of a given vector field.

$$\vec{F}(x,y,z) = \langle \underbrace{4(y+z)}_{4y+4z}, 3x, -xyz \rangle$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4y+4z & 3x & -xyz \end{vmatrix}$$

$$= \left\langle \underbrace{-xz}_{\frac{\partial}{\partial y}(-xyz)} \underbrace{-}_{\text{minus}} \underbrace{0}_{\frac{\partial}{\partial z}(3x)} ,\ \underbrace{4}_{\frac{\partial}{\partial z}(4y+4z)} \underbrace{-}_{\text{minus}} \underbrace{(-yz)}_{\frac{\partial}{\partial x}(-xyz)} ,\ \underbrace{3}_{\frac{\partial}{\partial x}(3x)} \underbrace{-}_{\text{minus}} \underbrace{4}_{\frac{\partial}{\partial y}(4y+4z)} \right\rangle$$

$$= \langle -xz,\ 4+yz,\ -1 \rangle$$

Resume
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What is the "Curl of a 2D Vector Field"?

The "curl of a 2D vector field" $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$ is

$$\nabla \times \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ F_1(x, y) & F_2(x, y) \end{vmatrix}.$$

$$\text{div}(\text{grad } f) = \nabla \cdot (\nabla f) = \text{the Laplacian of } f$$

$$\text{curl}(\text{grad } f) = \nabla \times (\nabla f)$$

What is a Conservative Vector Field?

$$\mathbf{F} \text{ is conservative } \iff \mathbf{F} = \nabla f \text{ for some } f$$

We call f a potential for $\mathbf{F}$.

Live Poll

Is f a potential for $\mathbf{F}$?

Example: Is **F** a conservative vector field?

Is the vector field

$$\mathbf{F}(x, y, z) = \langle y + ze^x, x + e^y \sin z, z + e^x + e^y \cos z \rangle$$

a conservative vector field? If yes, then there exists $f(x,y,z)$ such that $\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

$$\frac{\partial f}{\partial x} = y + ze^x$$

constant wrt x

$$\boxed{f(x,y,z) = \int \frac{\partial f}{\partial x}\, dx = \int (y + ze^x)\, dx = xy + ze^x + g(y,z)}$$

$$\underbrace{x + e^y \sin z}_{F_2(x,y,z)} = \frac{\partial f}{\partial y} = x + 0 + \frac{\partial g}{\partial y}$$

$$\frac{\partial}{\partial y}\left(xy + ze^x + g(y,z)\right)$$

So $\frac{\partial g}{\partial y} = e^y \sin z$

Thus $g(y,z) = \int \frac{\partial g}{\partial y}\, dy = \int e^y \sin z \, dy = e^y \sin z + h(z)$

let's update f

So $\boxed{f(x,y,z) = \int \frac{\partial f}{\partial x}\, dx = \int (y + ze^x)\, dx = xy + ze^x + e^y \sin z + h(z)}$

g

$\partial\partial$

$$z + e^x + e^y \cos z = \frac{\partial f}{\partial z} = 0 + e^x + e^y \cos z + \underset{\uparrow \frac{dh}{dz}}{h'(z)}$$

So $h'(z) = z$ implying $h(z) = \frac{1}{2}z^2 + C$ ← numerical constant

Thus

$$\boxed{f(x,y,z) = xy + ze^x + e^y \sin z + \frac{1}{2}z^2 + C}$$

is a potential for the given vector field

$$\vec{F}(x,y,z) = \langle y + ze^x,\ x + e^y \sin z,\ z + e^x + e^y \cos z \rangle.$$

Example: Is $\mathbf{v}$ a conservative vector field?

Is the vector field

$$\mathbf{v}(x, y) = \langle -y, x\rangle$$

a conservative vector field?

A Test for Conservativeness

If

- $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ (or $\mathbb{R}^2$), *and*
- the components of $\mathbf{F}$ have continuous first-order partial derivatives,

then

$$\mathbf{F} \text{ is conservative } \iff \text{curl}\,\mathbf{F} = \mathbf{0}$$

Example: Is **F** a conservative vector field?

Is the vector field

$$\mathbf{F}(x, y, z) = \langle y + ze^x,\ x + e^y \sin z,\ z + e^x + e^y \cos z \rangle$$

a conservative vector field?

Example: Is $\mathbf{v}$ a conservative vector field?

Is the vector field

$$\mathbf{v}(x, y) = \langle -y, x \rangle$$

a conservative vector field?

Line Integrals of Conservative Vector Fields

Suppose **F** is a conservative vector field with potential function f, that is, $\mathbf{F} = \nabla f$.

Then, if $\mathcal{C}$ is any curve that starts at the point P_0 and ends at the point P_1,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(P_1) - f(P_0).$$

This is called the Fundamental Theorem of Line Integrals.

Example: Fundamental Theorem of Line Integrals

∇f = grad f

$\nabla \cdot \mathbf{F}$ = div $\mathbf{F}$

$\nabla \times \mathbf{F}$ = curl $\mathbf{F}$

Conservative Vector Fields

Surface Integrals of Scalar-Valued Functions

Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = \langle y + ze^x,\, x + e^y \sin z,\, z + e^x + e^y \cos z\rangle,$$

and $\mathcal{C}$ is the curve given by the parametrization

$$\mathbf{r}(t) = \langle t,\, e^t,\, \sin t\rangle,\quad 0 \le t \le \pi.$$

$$\int_{C} \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\underset{x}{\underset{\parallel}{t}}, \underset{y}{\underset{\parallel}{e^t}}, \underset{z}{\underset{\parallel}{\sin t}}) \cdot \langle 1, e^t, \cos t\rangle \, dt$$

$$= \int_0^{\pi} \langle e^t + \sin t\, e^t,\; t + e^{e^t} \sin(\sin t),\; \sin t + e^t + e^{e^t} \cos(\sin t)\rangle \cdot \langle 1, e^t, \cos t\rangle \, dt$$

We Know $\vec{F}$ is conservative with potential

$$\boxed{f(x,y,z) = xy + ze^{x} + e^{y}\sin z + \frac{1}{2}z^{2} + C}$$

By Fundamental Theorem of Line Integrals,

$$\int_{C} \vec{F} \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$$

$$\boxed{\begin{aligned} \vec{r}(t) &= \langle t, e^{t}, \sin t\rangle \\ \vec{r}(\pi) &= \langle \pi, e^{\pi}, 0\rangle \\ \vec{r}(0) &= \langle 0, 1, 0\rangle \end{aligned}}$$

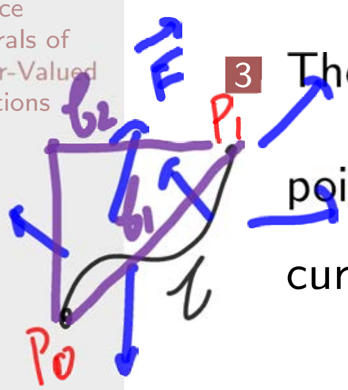
$$= f(\pi, e^{\pi}, 0) - f(0,1,0)$$

$$= (\pi e^{\pi} + C) - (0 + C) = \pi e^{\pi}$$

More Nice Properties of Conservative Vector Fields

Let $\mathbf{F}$ be a vector field that is defined and continuous on all of $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then, the following three statements are equivalent:

1. $\mathbf{F}$ is conservative, that is, there exists a function f such that $\mathbf{F} = \nabla f$.
2. The integral $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve $\mathcal{C}$.
3. The integral $\displaystyle\int \mathbf{F} \cdot d\mathbf{r}$ is path independent. That is, for any points P_0, P_1, we have that $\displaystyle\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$ for all curves $\mathcal{C}_1$, $\mathcal{C}_2$ that start at P_0 and end at P_1.



If any one of the three statements is true, then all three are true.