HUMBER ENGINEERING

MENG-3020
SYSTEMS MODELING & SIMULATION

LECTURE 6

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LECTURE 6
Rotational Mechanical Systems

- Modeling of Rotational Mechanical Systems
- Variables & Elements
- Element Laws
- Interconnection Laws
- Gear Systems
- Mathematical Modeling of Simple Mechanical Systems

Modeling of Mechanical Systems

- The motion of elements of mechanical systems can be described as:
  - **Translational Motion**
  - **Rotational Motion**
- The equations governing the motion of mechanical systems are called the ***equation of motion*** that often directly or indirectly formulated by applying **Newton's law of motion** to the **free-body diagram** (FBD).

Translational Motion $\boxed{\sum F_{ext} = Ma}$ $\boxed{\sum \tau_{ext} = J\alpha}$ **Rotational Motion**

- The number of equations of motion required is equal to the number of ***linearly independent*** motions or the number of ***degrees of freedom***.
  - **Step 1:** Identify reference point and positive direction of motion.
  - **Step 2**: Draw a free-body diagram for each inertia or junction with unknown motion.
  - **Step 3**: For each free-body diagram, find the torques acting on the body due only to its own motion and the torques create by the adjacent motion.
  - **Step 4**: Use Newton's law on each body to form the differential equation of motion.
  - **Step 5**: Represent the equation of motion in standard forms.

Rotational Mechanical Systems: Variables & Elements

- The rotational motion is defined as a motion that object **rotates** about an axis.
- The **variables** that are used to describe the rotational motion are:
  - $\tau(t)$: Torque (N.m)
  - $\theta(t)$: Angular Displacement (rad)
  - $\omega(t)$: Angular Velocity (rad/s)
  - $\alpha(t)$: Angular Acceleration (rad/s^2)

$$\omega(t) = \frac{d\theta(t)}{dt} = \theta'(t) = \dot{\theta}(t)$$

$$\alpha(t) = \frac{d^2\theta(t)}{dt^2} = \theta''(t) = \ddot{\theta}(t)$$

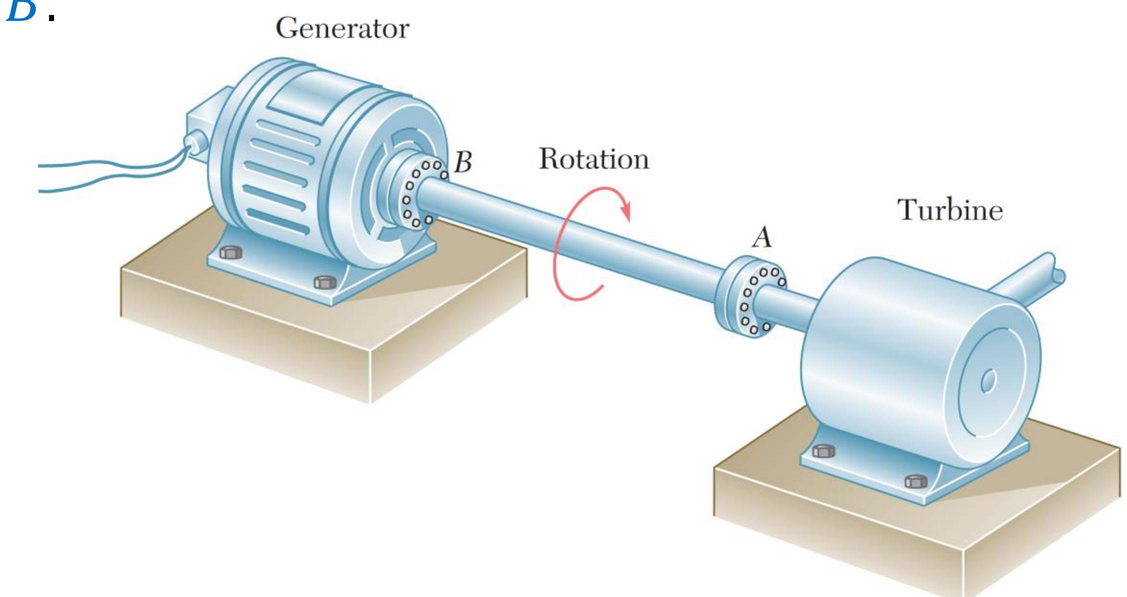
- All these variables are function of time.
- Angular Displacements are measured with respect to reference angle, which is the equilibrium orientation of the body.
- Velocities and accelerations are normally expressed as the derivatives of the corresponding displacement.
- **Torque** is defined as any causes that tends to produce a change in the rotational motion of a body on which it acts.
- Torque is the product of a force and the perpendicular distance from a point of rotation to the line of action of the force.
- The **elements** that we include in rotational systems are:
  - **Rotational Inertia Element** → **Moment of Inertia**
  - **Rotational Stiffness Element** → **Torsional Spring**
  - **Rotational Friction Element** → **Rotational Damper**

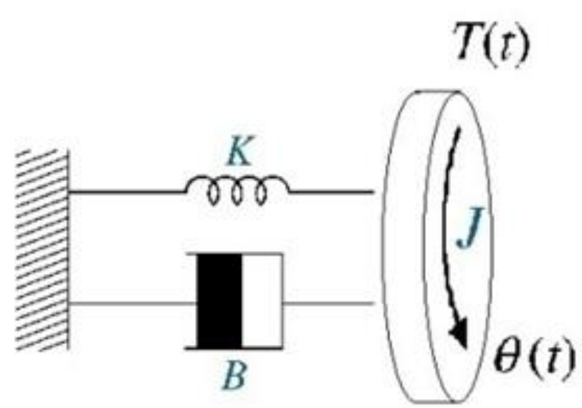
Rotational Mechanical Systems: Example

- Some real-word examples of rotational motion systems.

○ Transmission Shaft

- The transmission shaft is a common component of the rotational systems to transmit the power from one point to another. It is also a practical example of a rotational inertia-spring-damper system.
- The shaft (rod) under torsional load can be modeled as a rotational inertia-spring-damping system.
- The compliance and flexibility of a rod or a shaft when it is subject to an applied torque can be modeled by a torsional spring K.
- The internal energy loss in a rod is represented by viscus damping B.
- The inertia of the rod is represented by inertia disk J.

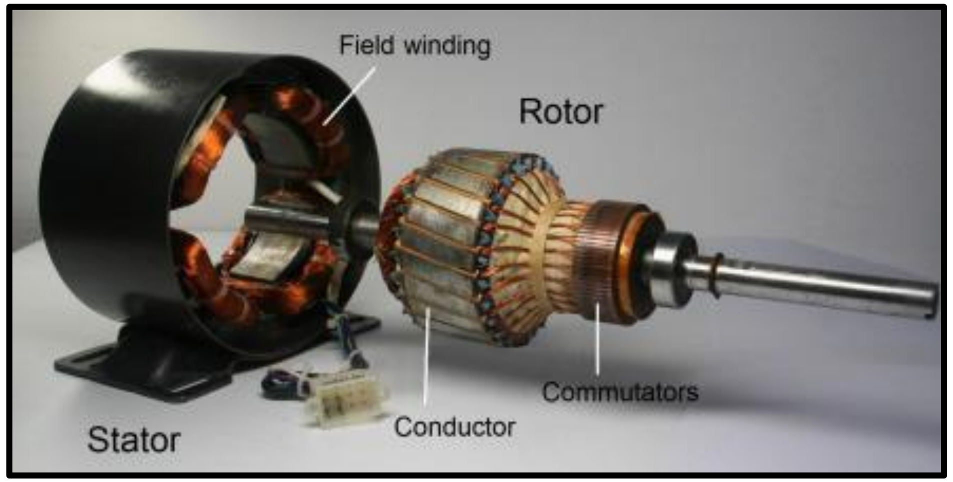




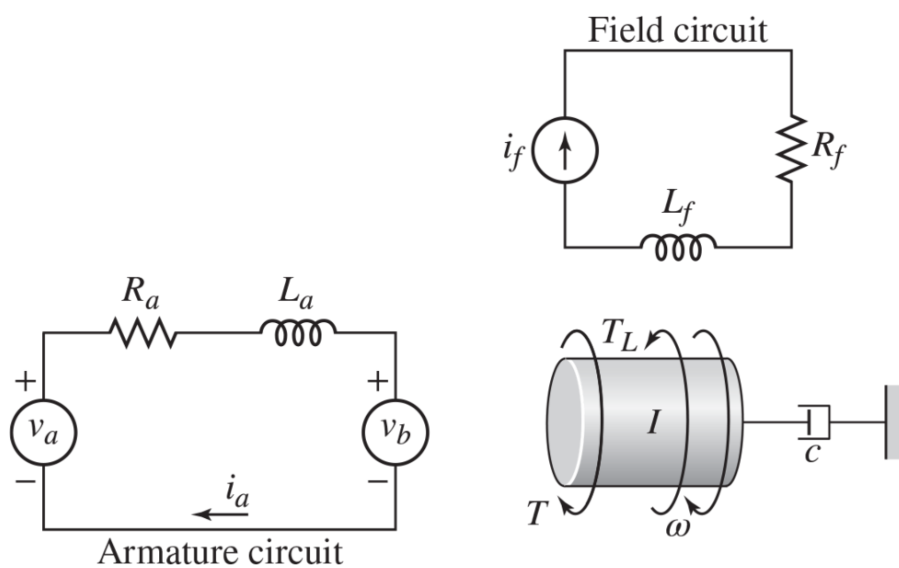
Rotational Mechanical Systems: Example

- Some real-word examples of rotational motion systems.

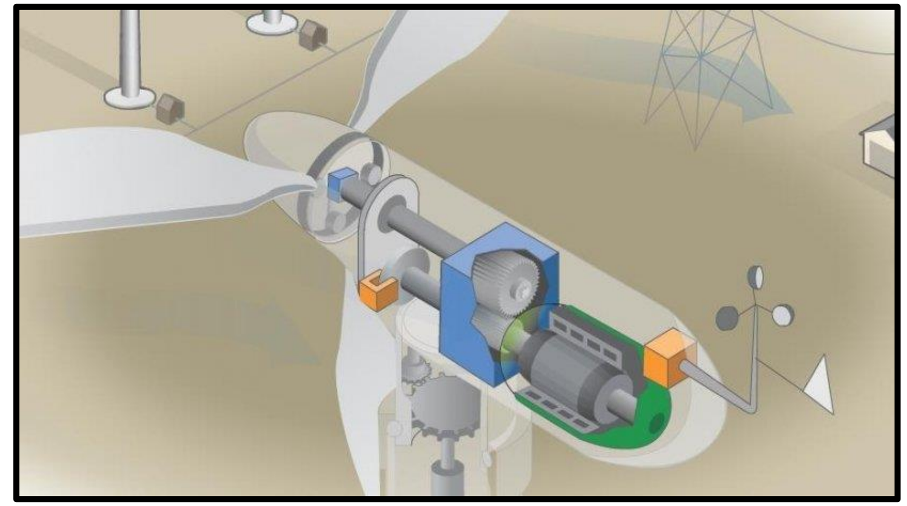
○ Armature Controlled DC Motor



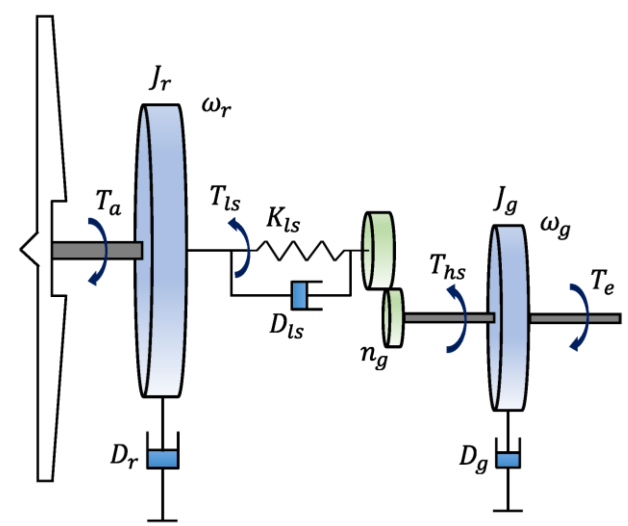
Input: Applied armature voltage
Output: Angular speed of rotor shaft

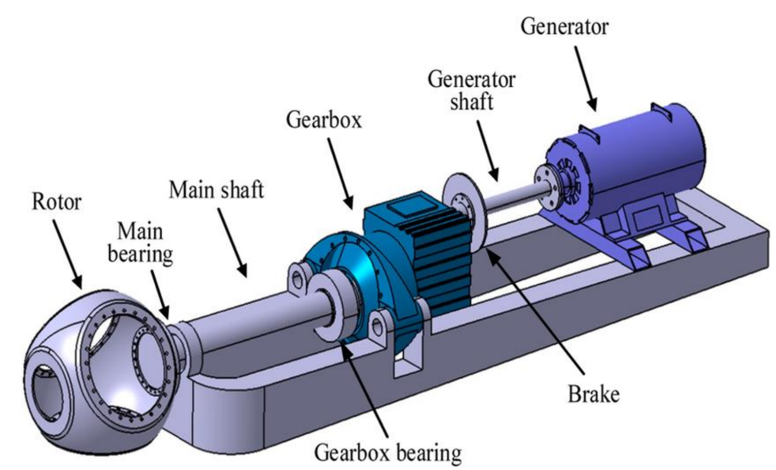


○ Wind Turbine Drivetrain



Input: Applied torque
Output: Speed of generator



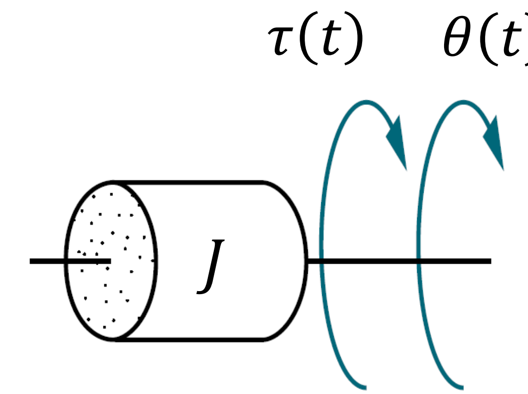


Rotational Mechanical Systems: Element Laws

❑ Inertia Element: Moment of Inertia

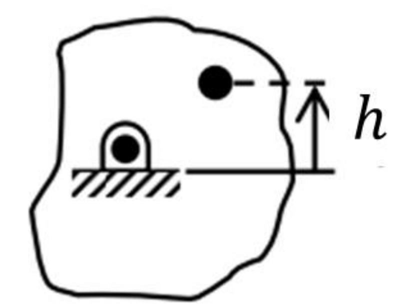
- If a torque is applied on a body having inertia, then it is opposed by an opposing torque due to the moment of inertia.
- Physically, the moment of inertia of a body is a measure of the resistance of the body to angular acceleration.
- From the Newton's second law, the applied torque is proportional to angular acceleration of the body.
- The J is the moment of inertia. The unit is (kg.m^2).

$$\boxed{\tau(t) = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}}$$



- Energy in a rotating body can store in both kinetic energy and potential energy forms.

$$\boxed{KE = \frac{1}{2}J\omega^2} \qquad \boxed{PE = Mgh}$$



where M is the mass, $g = 9.8\ m/s^2$ is gravitational acceleration and h is the height of the center of mass above its reference position.

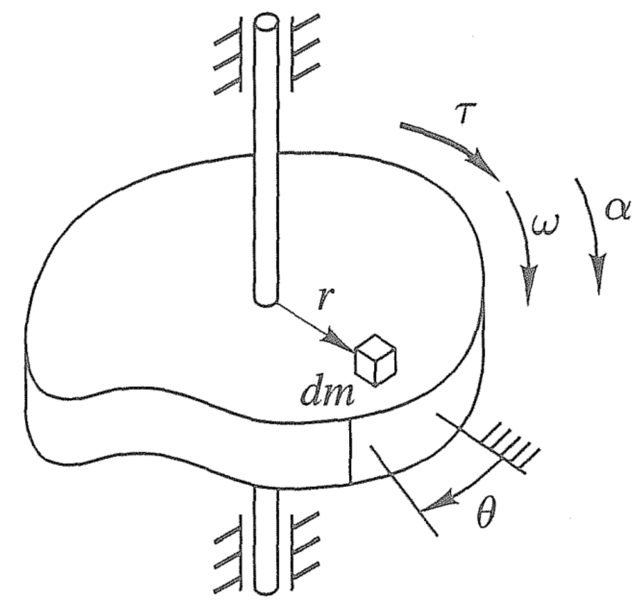
Rotational Mechanical Systems: Element Laws

❑ Moment of Inertia

- **Moment of Inertia J** of a rigid body about an axis is determined by:

$$J = \sum r^2 dm$$

  where dm is the element of mass, r is distance from the axis to dm and integration is performed over the body.

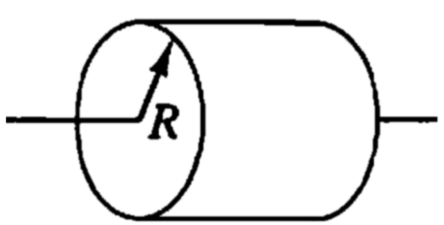


- The moment of inertia for a point mass M is $\boldsymbol{J = Mr^2}$

  where r is the distance from the point to the axis of rotation.

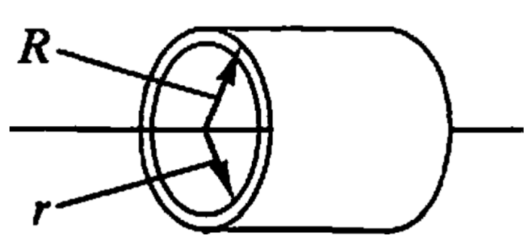
- Moment of inertia of rigid bodies with common shapes:

Solid Disk or Cylinder



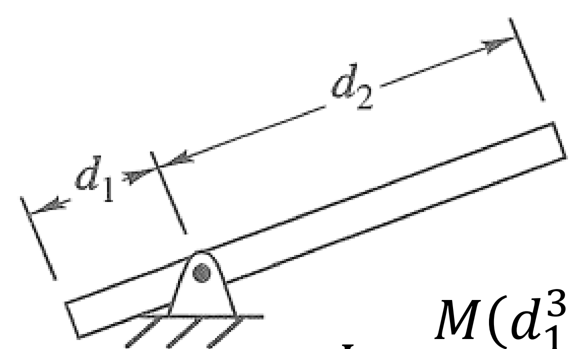
$$J = \frac{1}{2}MR^2$$

Hollow Cylinder



$$J = \frac{1}{2}M(R^2 + r^2)$$

Slender Bar



$$J = \frac{M(d_1^3 + d_2^3)}{3(d_1 + d_2)}$$

If $d_1 = d_2 = d$

$$J = \frac{1}{3}Md^2$$

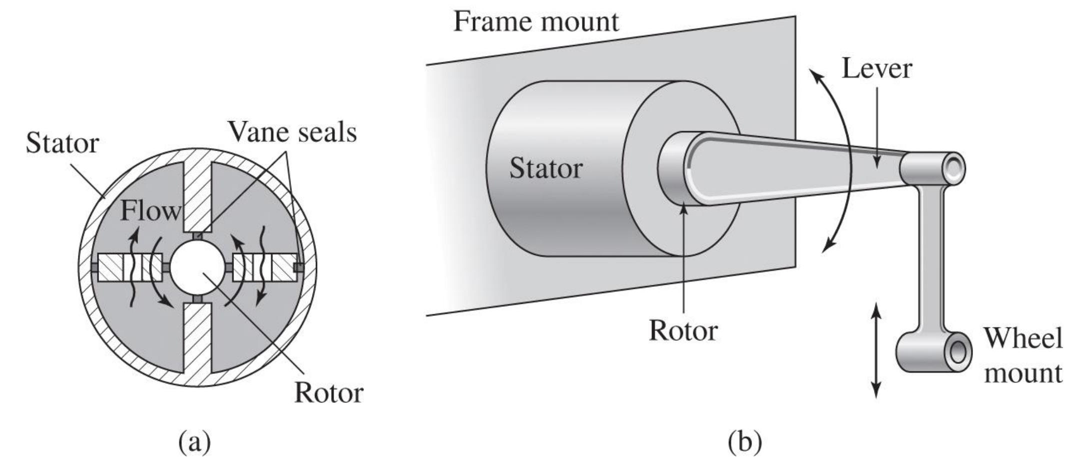
Rotational Mechanical Systems: Element Laws

❑ Rotational Friction Element

- Rotational viscous friction arises when two rotating bodies are separated by a film of oil.
- For example, the door closer rotary dampers or shock absorber dampers, ball bearings, clutches, torque limiters and brakes (mechanical friction).

Torque limiters & Brakes

Rotary damper



Ball bearings

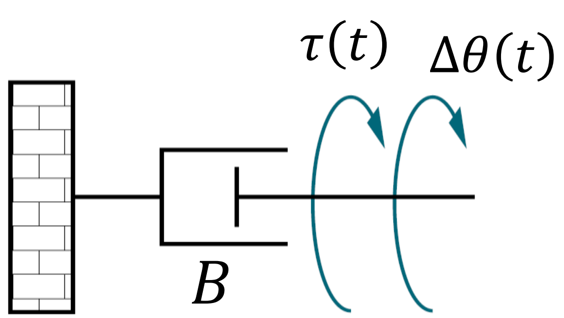
Rotational Mechanical Systems: Element Laws

❑ Rotational Friction Element

- Engineering systems can exhibit damping in bearings and other surfaces lubricated to prevent wear.
- Damping elements can be deliberately included as part of the design.
- Essentially, the damper absorbs energy, and the absorbed energy is dissipated as heat that flows away to the surroundings.

- The torque due to friction is proportional to the angular velocity of the motion, and its direction tends to reduce the relative angular velocity.
- The B is the viscus friction coefficient. The unit is (N.m.s/rad)

$$\boxed{\tau(t) = B\Delta\omega(t) = B\frac{d\Delta\theta(t)}{dt}}$$

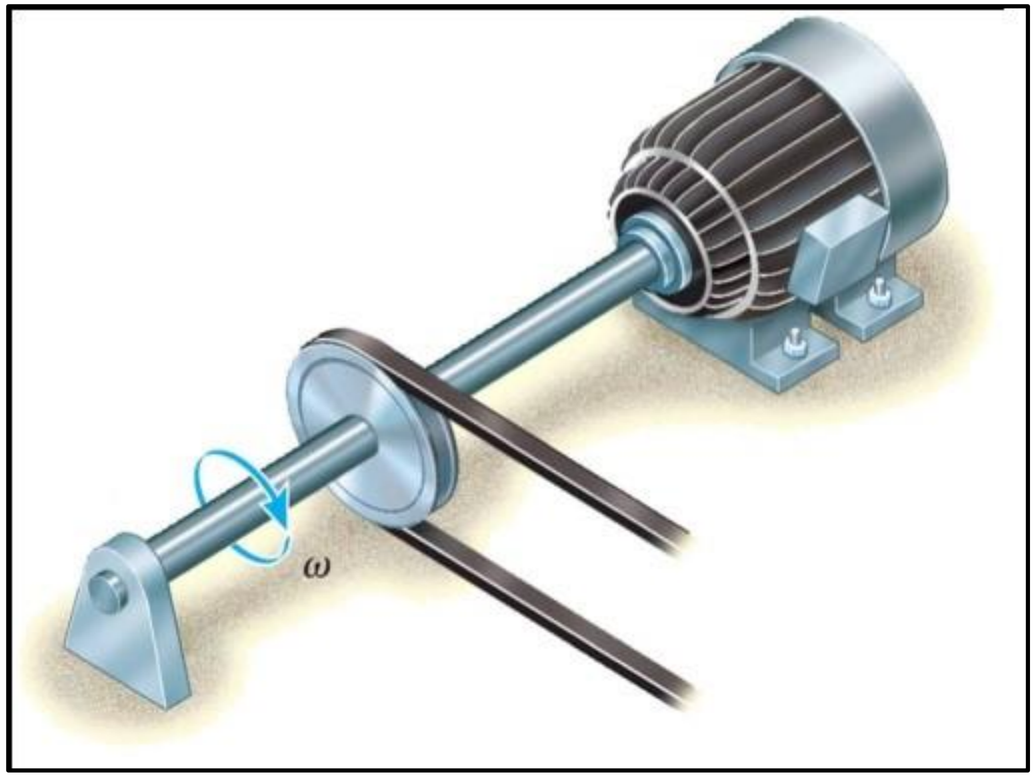


- All practical dampers produce inertia and spring effects.
  Here, we assume that these effects are negligible

Rotational Mechanical Systems: Element Laws

❑ Rotational Stiffness Element

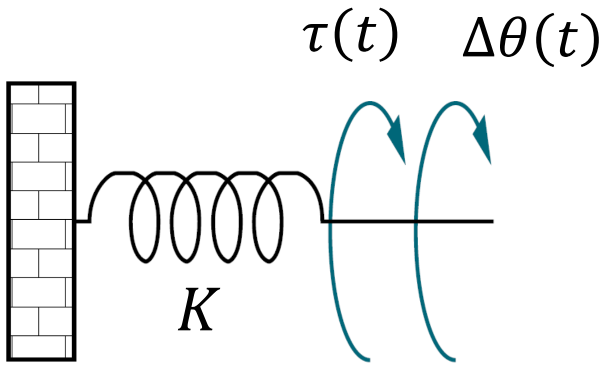
- Any mechanical element that undergoes a change in shape when subjected to a force or torque can be characterized by a **stiffness element**.
- The most common rotational stiffness elements are the **torsional springs** and a **rotational shafts.**



Rotational Mechanical Systems: Element Laws

❑ Rotational Stiffness Element

- If a torque is applied on a torsional spring or a shaft, then it is opposed by an opposing torque due to the elasticity of the element.
- The torque is proportional to the angular displacement of the torsional spring or the shaft.
- The K is the stiffness constant. The unit is (N.m/rad).



$$\boxed{\tau(t) = K\Delta\theta(t)}$$

- Potential energy is stored in a twisted stiffness element and can affect the response of the system at later time. For a linear torsional spring or shaft the potential energy is:

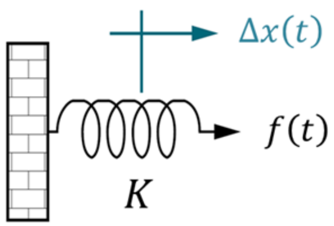
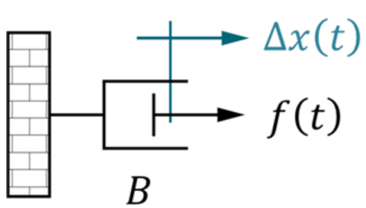
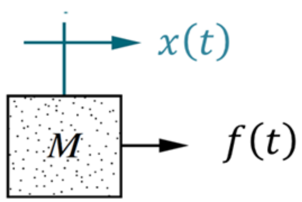
$$\boxed{PE = \frac{1}{2}K(\Delta\theta)^2}$$

- All practical stiffness elements have inertia and damping.
- Here we assume that the effect of the inertia and damping effects are negligibly small.

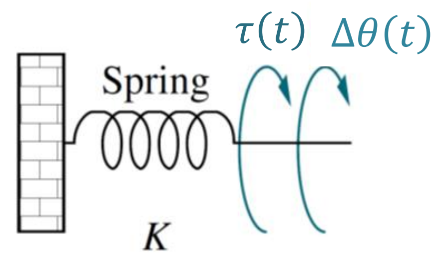
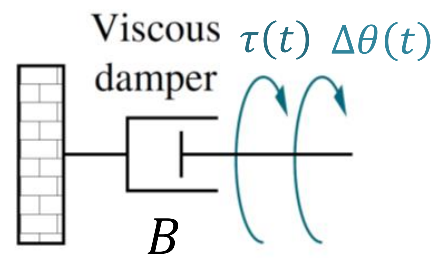
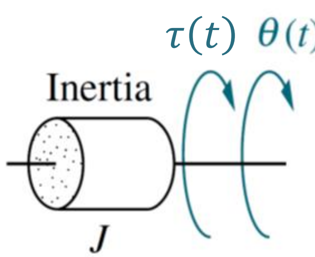
Mechanical Systems: Element Laws

❑ Summary

Translational Motion

Element	Force-velocity	Force-displacement
Spring 	$f(t) = K\int_0^t v(t)dt$	$f(t) = K\Delta x(t)$
Viscous damper 	$f(t) = B\Delta v(t)$	$f(t) = B\dfrac{d\Delta x(t)}{dt}$
Mass 	$f(t) = M\dfrac{dv(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$

Rotational Motion

Element	Torque-angular velocity	Torque-angular displacement
	$\tau(t) = K\int_0^t \omega(t)dt$	$\tau(t) = K\Delta\theta(t)$
	$\tau(t) = B\Delta\omega(t)$	$\tau(t) = B\dfrac{d\Delta\theta(t)}{dt}$
	$\tau(t) = J\dfrac{d\omega(t)}{dt}$	$\tau(t) = J\dfrac{d^2\theta(t)}{dt^2}$

Rotational Mechanical Systems: Interconnection Laws

❑ Newton's Second Law

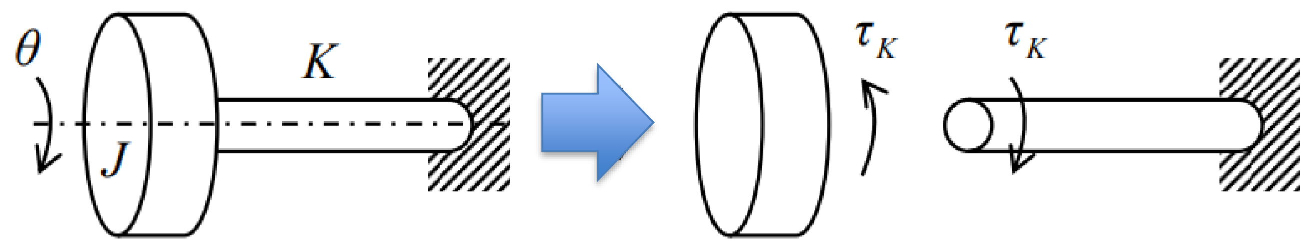
- For a rigid body in pure rotation about a fixed axis, if $\sum \tau_{ext}$ is the sum of all torques acting about a given axis and J is the moment of inertia of a body about that axis, then,

$$\sum \tau_{ext} = J\alpha(t)$$

where α is the angular acceleration of the body.

❑ Newton's Third Law: The Law of Reaction Torques

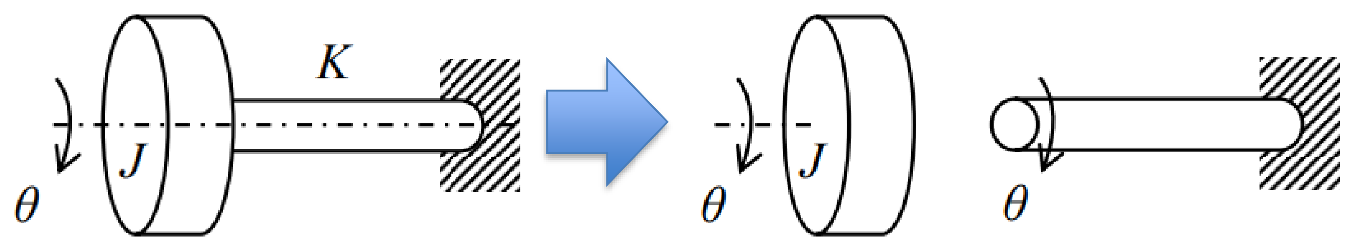
- For bodies that are rotating about the same axis, any torque exerted by one element on another is accompanied by a **reaction torque** of equal magnitude and opposite direction on the first element.



- Elements must be connected along the same axis. (Not applicable for gear meshing)

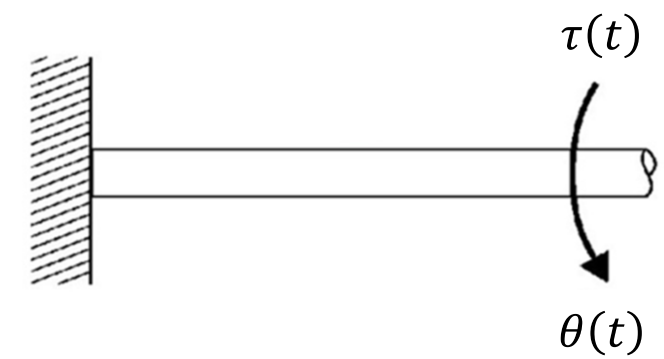
❑ The Law for Angular Displacement

- Two elements connected along the same axis have the same angular displacement. (Not applicable to gear meshing since the axes are not the same)

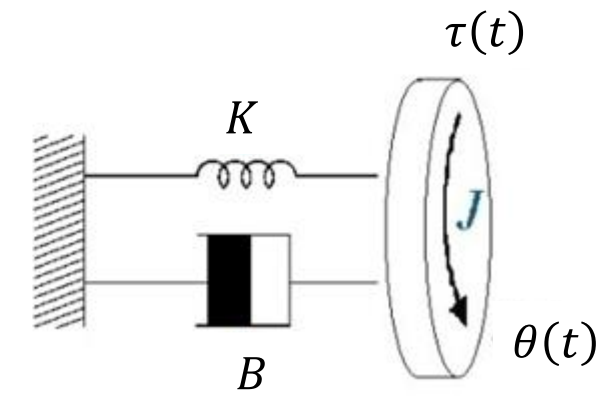


Modeling of Rotational Mechanical Systems

Example 1 Find the equation of motion and the transfer function model of $\theta(s)/T(s)$, for a rod under a torsional load system.



The rod under torsional load can be modeled as a rotational inertia-spring-damping system.

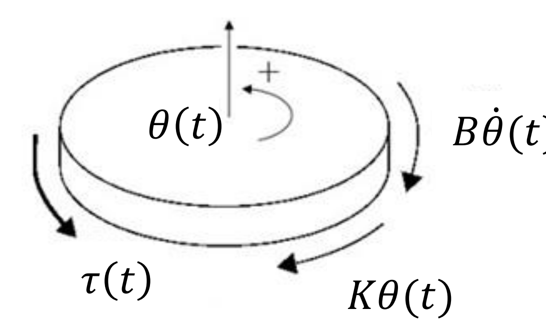


From the free-body diagram by considering the CCW as the positive direction, the differential equation model of system is obtained.

$$\sum \tau_{ext} = J\alpha(t) \quad \rightarrow \quad \tau(t) - \tau_K(t) - \tau_B(t) = J\alpha(t)$$

$$\tau(t) - K\theta(t) - B\dot{\theta}(t) = J\ddot{\theta}(t) \longrightarrow \boxed{\tau(t) = J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t)}$$

Equation of Motion



Taking the Laplace transform, assuming zero initial conditions, and solving for the transfer function

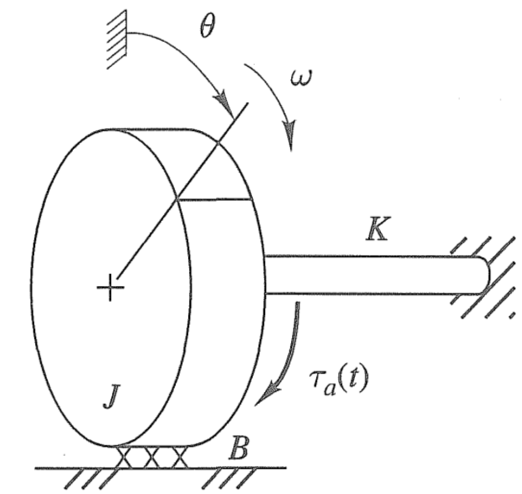
$$T(s) = Js^2\theta(s) + Bs\theta(s) + K\theta(s) \longrightarrow T(s) = (Js^2 + Bs + K)\theta(s)$$

$$\boxed{\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}}$$

Transfer Function Model

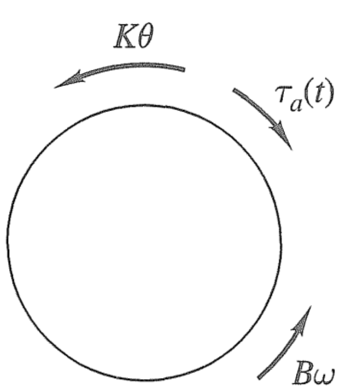
Modeling of Rotational Mechanical Systems

Example 2 Find the equation of motion and derive a state-space model for the following rotational system. Assume that the input is the applied torque τ_a and the outputs are the angular acceleration of the disk α and the torque exerted on the disk by the shaft τ_K.



From the free-body diagram by considering the CW as the positive direction, the equation of motion is obtained.

$$\tau_a(t) - K\theta(t) - B\omega(t) = J\alpha(t) \quad \rightarrow \quad \boxed{\tau_a(t) = J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t)}$$



Define the state variables q_1 and q_2 as the angular displacement of the shaft (spring K) and the angular velocity of the disk (inertia J).

$$q_1(t) = \theta(t) \quad \rightarrow \quad \dot{q}_1(t) = \dot{\theta}(t) \quad \rightarrow \quad \dot{q}_1(t) = q_2(t) \qquad Eqn.\ (1)$$

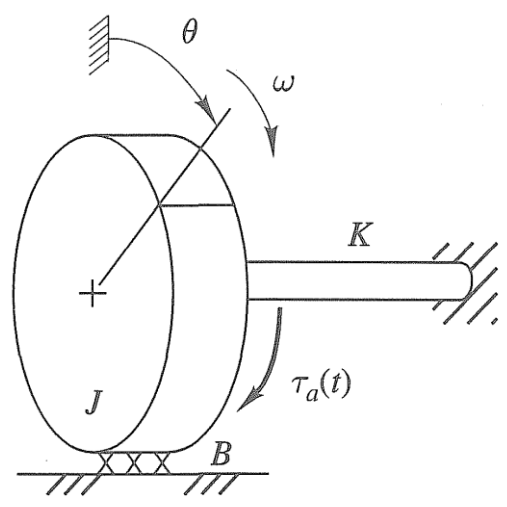
$$q_2(t) = \dot{\theta}(t) \quad \rightarrow \quad \dot{q}_2(t) = \ddot{\theta}(t) \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{J}\tau_a(t) - \frac{K}{J}\theta(t) - \frac{B}{J}\dot{\theta}(t)$$

$$\rightarrow \quad \dot{q}_2(t) = \frac{1}{J}\tau_a(t) - \frac{K}{J}q_1(t) - \frac{B}{J}q_2(t) \qquad Eqn.\ (2)$$

$$outputs \quad \rightarrow \quad \begin{cases} y_1(t) = \alpha(t) \quad \rightarrow \quad y_1(t) = \dfrac{1}{J}\tau_a(t) - \dfrac{K}{J}q_1(t) - \dfrac{B}{J}q_2(t) \\ y_2(t) = \tau_K(t) \quad \rightarrow \quad y_2(t) = K\theta(t) = Kq_1(t) \end{cases}$$

Modeling of Rotational Mechanical Systems

Example 2 Find the equation of motion and derive a state-space model for the following rotational system. Assume that the input is the applied torque τ_a and the outputs are the angular acceleration of the disk α and the torque exerted on the disk by the shaft τ_K.



From the state and output equations:

$$\begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = \dfrac{1}{J}\tau_a(t) - \dfrac{K}{J}q_1(t) - \dfrac{B}{J}q_2(t) \end{cases}$$

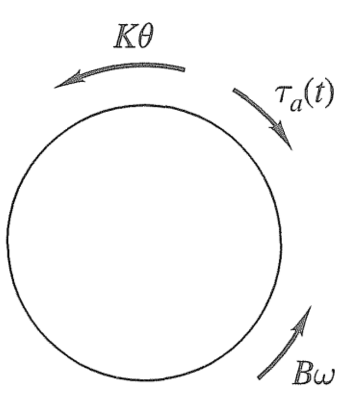
$$\begin{cases} y_1(t) = \dfrac{1}{J}\tau_a(t) - \dfrac{K}{J}q_1(t) - \dfrac{B}{J}q_2(t) \\ y_2(t) = K q_1(t) \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

$$\boxed{\dot{\boldsymbol{q}}(t) = \mathbf{A}\,\boldsymbol{q}(t) + \mathbf{B}\,\mathbf{u}(t)}$$

State Equation

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{J} & -\dfrac{B}{J} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{J} \end{bmatrix} \tau_a(t)$$



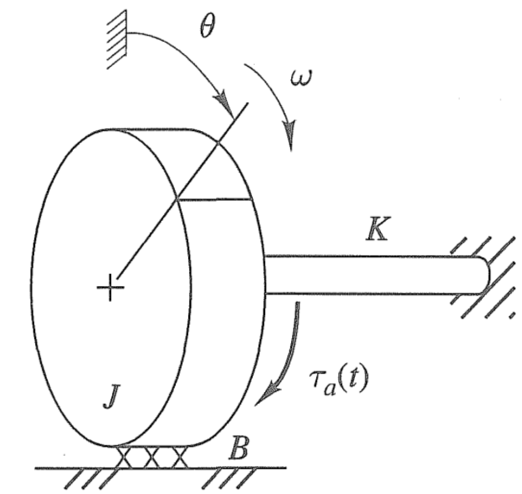
$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$$

Output Equation

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{K}{J} & -\dfrac{B}{J} \\ K & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{J} \\ 0 \end{bmatrix} \tau_a(t)$$

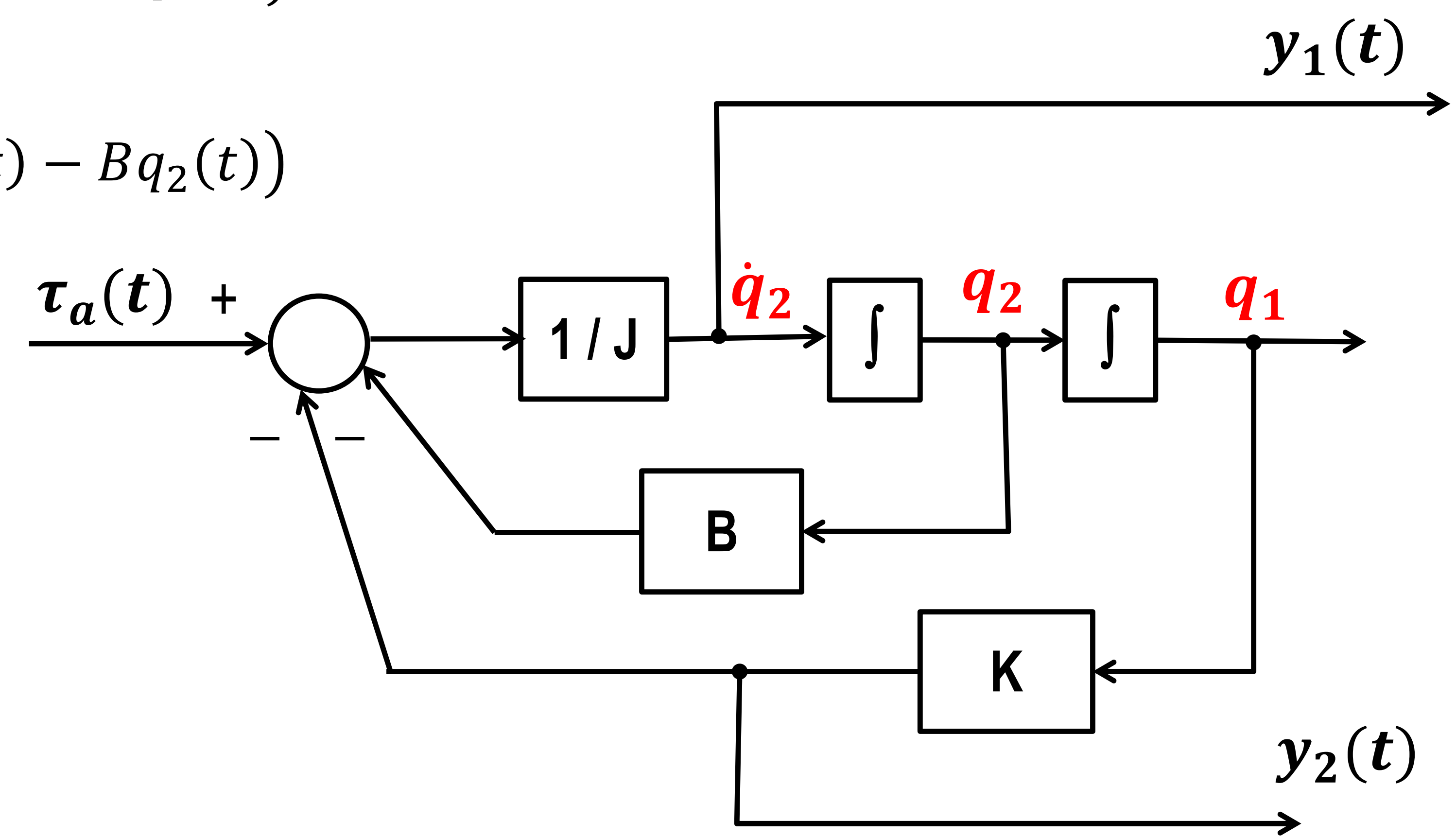
Modeling of Rotational Mechanical Systems

Example 2 Find the equation of motion and derive a state-space model for the following rotational system. Assume that the input is the applied torque τ_a and the outputs are the angular acceleration of the disk α and the torque exerted on the disk by the shaft τ_K.



Following block diagram visualizes the state variables and the system outputs.

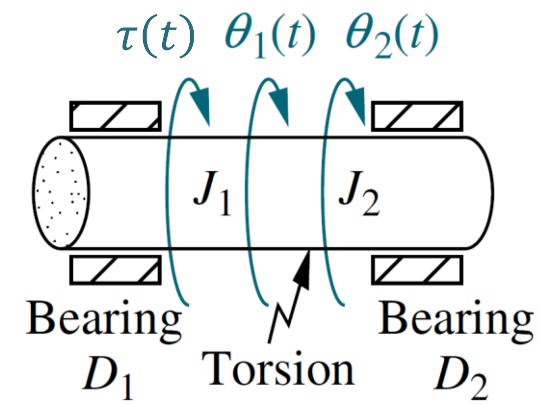
$$
\begin{cases}
\dot{q}_1(t) = q_2(t) \\[2ex]
\dot{q}_2(t) = \dfrac{1}{J}\big(\tau_a(t) - Kq_1(t) - Bq_2(t)\big) \\[2ex]
y_1(t) = \dfrac{1}{J}\big(\tau_a(t) - Kq_1(t) - Bq_2(t)\big) \\[2ex]
y_2(t) = Kq_1(t)
\end{cases}
$$



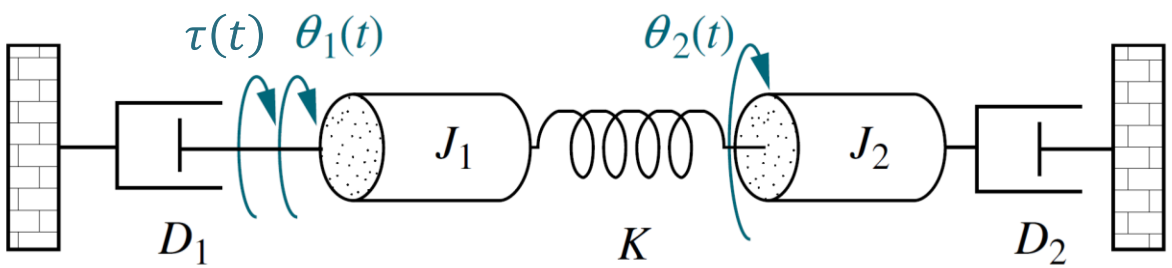
Modeling of Rotational Mechanical Systems

Example 3

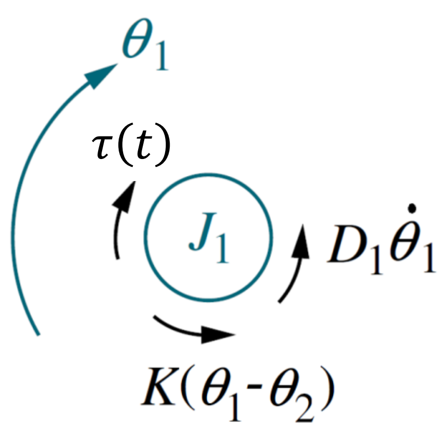
Find the equation of motion and a transfer function model of $\theta_2(s)/T(s)$, for the following rotational system. The rod is supported by bearings at either and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

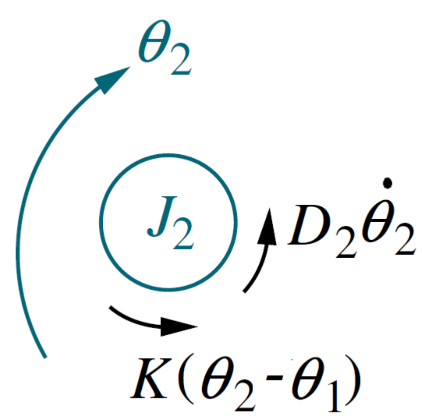


The system can be modeled as an inertia-spring-damper system as below



Draw the free-body diagram by considering the CW as the positive direction.





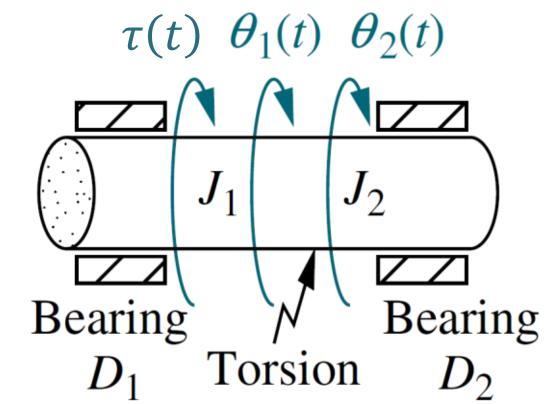
$$\tau(t) - K(\theta_1 - \theta_2) - D_1\dot{\theta}_1 = J_1\ddot{\theta}_1$$

$$-K(\theta_2 - \theta_1) - D_2\dot{\theta}_2 = J_2\ddot{\theta}_2$$

Modeling of Rotational Mechanical Systems

Example 3

Find the equation of motion and a transfer function model of $\theta_2(s)/T(s)$, for the following rotational system. The rod is supported by bearings at either and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.



Having the equation of motion for each rotational inertia:

$$\tau(t) - K(\theta_1 - \theta_2) - D_1\dot{\theta}_1 = J_1\ddot{\theta}_1$$

$$-K(\theta_2 - \theta_1) - D_2\dot{\theta}_2 = J_2\ddot{\theta}_2$$

Take the Laplace transform, assuming zero initial conditions, and solve for the transfer function

$$T(s) - K\theta_1(s) + K\theta_2(s) - D_1 s\theta_1(s) = J_1 s^2\theta_1(s)$$

$$-K\theta_2(s) + K\theta_1(s) - D_2 s\theta_2(s) = J_2 s^2\theta_2(s)$$

Simplify the equations:

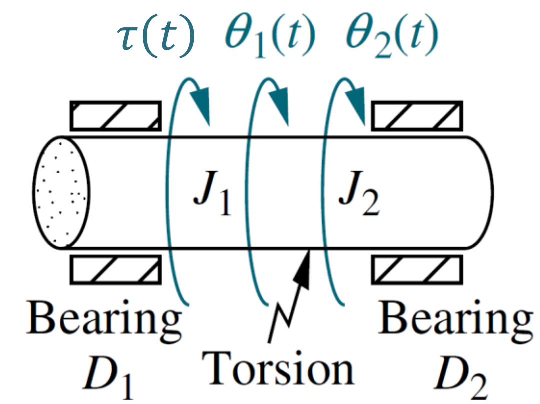
$$T(s) = (J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) \quad \rightarrow \quad Eqn.\,(1)$$

$$K\theta_1(s) - (J_2 s^2 + D_2 s + K)\theta_2(s) = 0 \quad \rightarrow \quad Eqn.\,(2)$$

Modeling of Rotational Mechanical Systems

Example 3 Find the equation of motion and a transfer function model of $\theta_2(s)/T(s)$, for the following rotational system. The rod is supported by bearings at either and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

Find $\theta_1(s)$ from Eqn. (2) in terms of $\theta_2(s)$ and substitute in Eqn. (1):



$$\text{From Eqn. (2)} \quad \rightarrow \quad \theta_1(s) = \frac{1}{K}(J_2 s^2 + D_2 s + K)\theta_2(s)$$

$$T(s) = (J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) \quad \rightarrow \quad \text{Eqn. (1)}$$

$$T(s) = (J_1 s^2 + D_1 s + K)\left(\frac{1}{K}(J_2 s^2 + D_2 s + K)\theta_2(s)\right) - K\theta_2(s)$$

$$KT(s) = (J_1 s^2 + D_1 s + K)(J_2 s^2 + D_2 s + K)\theta_2(s) - K^2\theta_2(s)$$

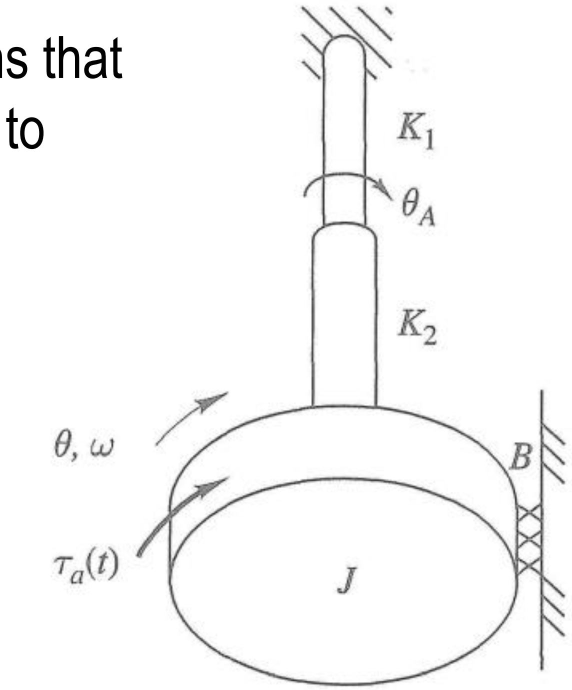
$$\boxed{\frac{\theta_2(s)}{T(s)} = \frac{K}{(J_1 s^2 + D_1 s + K)(J_2 s^2 + D_2 s + K) - K^2}}$$

Transfer Function Model

Modeling of Rotational Mechanical Systems

Example 4

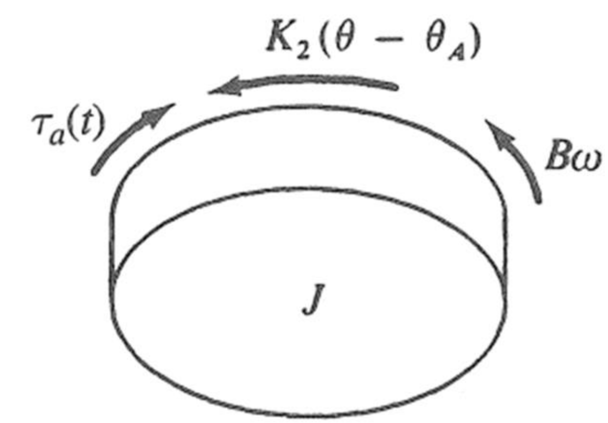
Consider the shaft supporting the disk system that is composed of two sections that have spring constants K_1 and K_2. Find the equation of motion and show how to replace the two sections by an equivalent stiffness element.



Draw the free-body diagrams by considering the CW as the positive direction.

Inertia J $\rightarrow$ $\tau_a(t) - K_2(\theta - \theta_A) - B\dot{\theta} = J\ddot{\theta}$

Massless Junction of two shafts A $\rightarrow$ $K_2(\theta_A - \theta) + K_1\theta_A = 0$



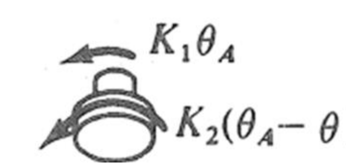
Solve the second equation for θ_A in terms of θ gives:

$$\theta_A = \left(\frac{K_2}{K_1 + K_2}\right)\theta$$

$\rightarrow$ two displacements are proportional to one another

Substituting the θ_A into the first equation we have:

$$\tau_a(t) - K_2\left(\theta - \frac{K_2}{K_1 + K_2}\theta\right) - B\dot{\theta} = J\ddot{\theta} \quad \rightarrow \quad \tau_a(t) = J\ddot{\theta} + B\dot{\theta} + \frac{K_1 K_2}{K_1 + K_2}\theta$$



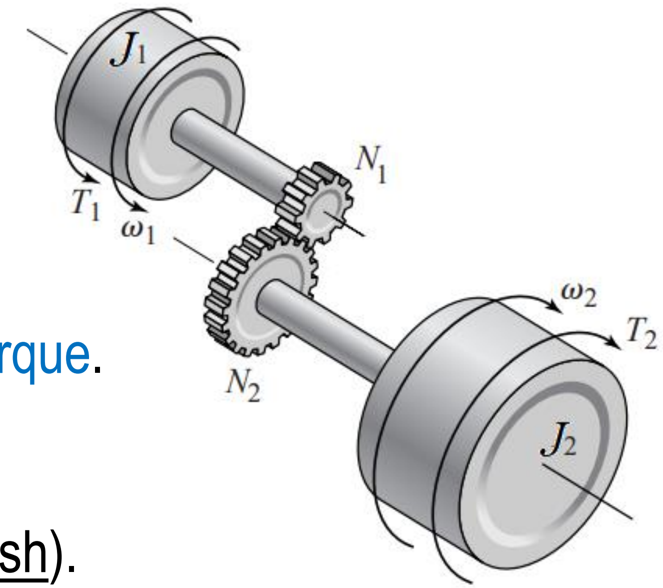
This equation describes the system when the two shafts are connected in series.

$$\boxed{K_{eq} = \frac{K_1 K_2}{K_1 + K_2}}$$

$$\boxed{\tau_a(t) = J\ddot{\theta} + B\dot{\theta} + K_{eq}\theta}$$

Modeling of Rotational Mechanical Systems

Rotational Transformer: Gears



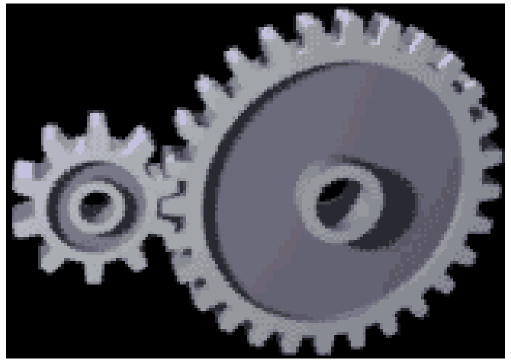
- The **gear** is a device that transmits energy from one part of the rotational system to another in such a way that force, torque, speed, and displacement may be altered.
- Gears allow us to match the drive system and the load—a trade-off between speed and torque.
- They also change direction of rotational motion.
- Ideal gears, have no moment of inertia, no friction and perfect meshing of teeth (no backlash).
- Actual gears have inertia, friction and backlash, but these can be represented by additional elements.
- The spacing between teeth must be equal for each gear in a pair, so the radii of the gears are proportional to the number of teeth.

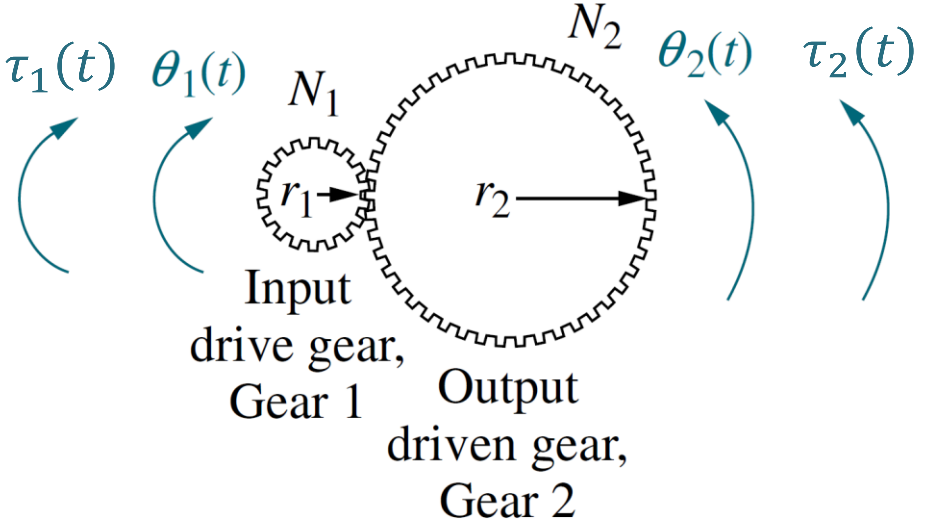
$$\boxed{\frac{N_2}{N_1} = \frac{r_2}{r_1}}$$

Teeth numbers

Radii of the gears

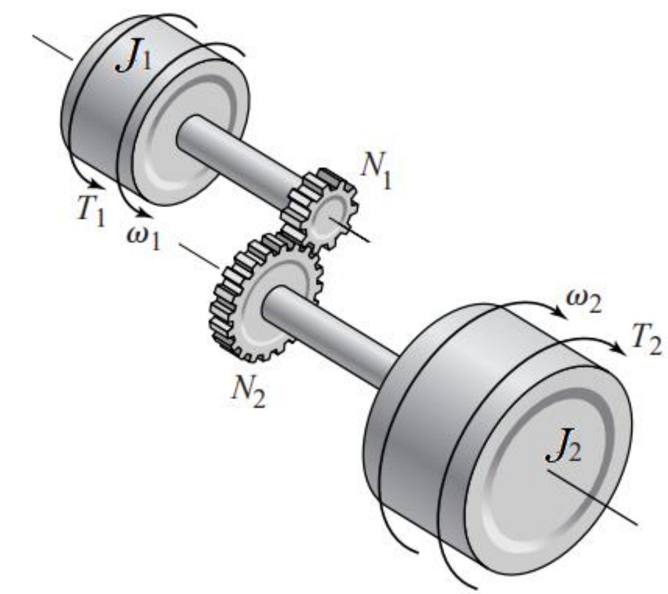
- The N_2/N_1 is called the **gear ratio**.





Modeling of Rotational Mechanical Systems

❑ Rotational Transformer: Gears

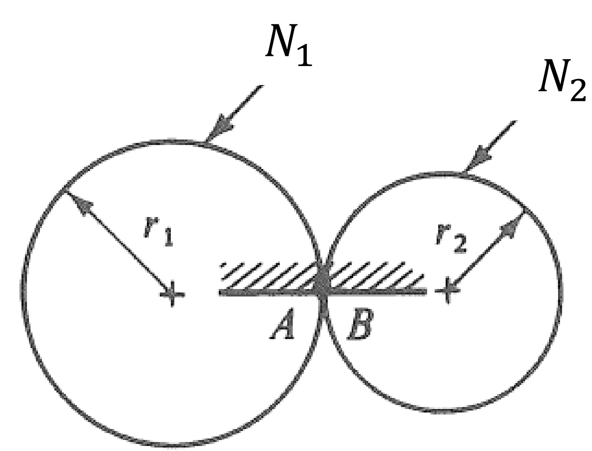


- Assume the ideal gears shown below, where points A and B denote points on the circles that are in contact with each other at some reference time.
- At some later time, A and B will have moved to the positions shown, where θ_1 and θ_2 denote the respective displacements from their original position.
- Since the arc PA and PB must be equal,

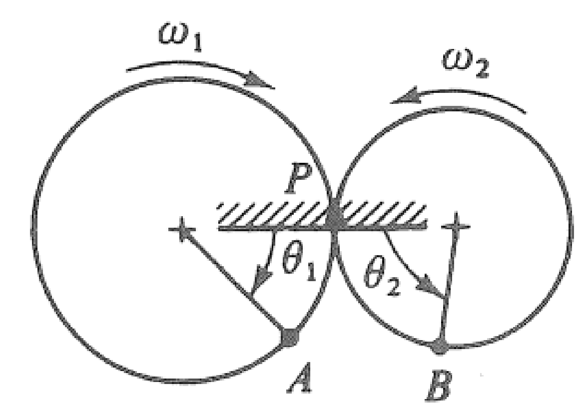
$$r_1\theta_1 = r_2\theta_2 \longrightarrow \boxed{\frac{\theta_1}{\theta_2} = \frac{r_2}{r_1}} \longrightarrow \boxed{\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}}$$

- Ideal gears are lossless. They do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear 2,

$$\tau_1\theta_1 = \tau_2\theta_2 \longrightarrow \boxed{\frac{\theta_1}{\theta_2} = \frac{\tau_2}{\tau_1}}$$



Reference Position



After Rotation

Modeling of Rotational Mechanical Systems

Example 5 The following system shows gears driving a rotational inertia, spring, and viscous damper. Find the transfer function, $\theta_1(s)/T_1(s)$ for the following system.

$$\frac{\tau_2}{\tau_1} = \frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} = \frac{r_2}{r_1}$$

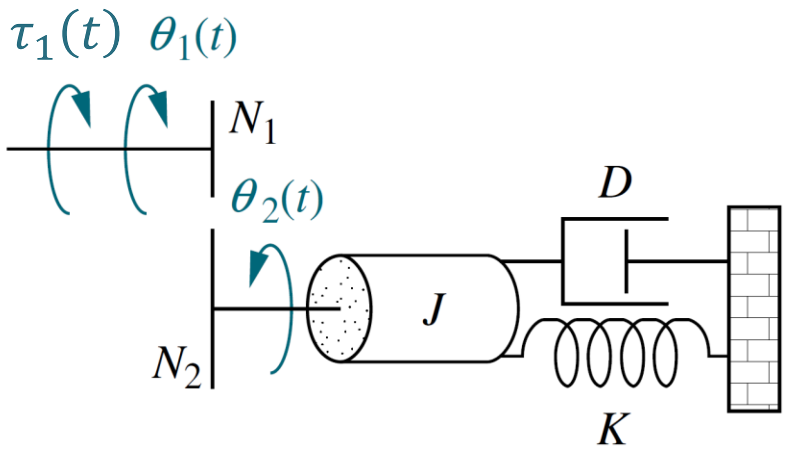
We want to represent the system as an equivalent system without the gears.

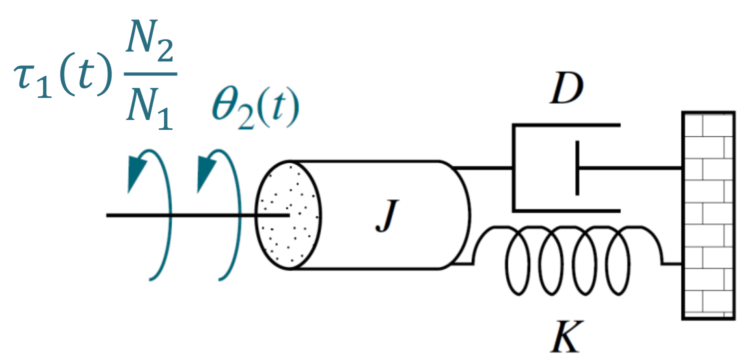
From the gear law, τ_1 can be reflected to the output by multiplying by N_2/N_1.

$$\tau_2 = \tau_1\left(\frac{N_2}{N_1}\right)$$

The equation of motion is:

$$\tau_1(t)\left(\frac{N_2}{N_1}\right) - K\theta_2(t) - D\dot{\theta}_2(t) = J\ddot{\theta}_2(t)$$





Now convert, θ_2 into an equivalent θ_1 by considering the gear ratio effect.

$$\theta_2 = \theta_1\left(\frac{N_1}{N_2}\right)$$

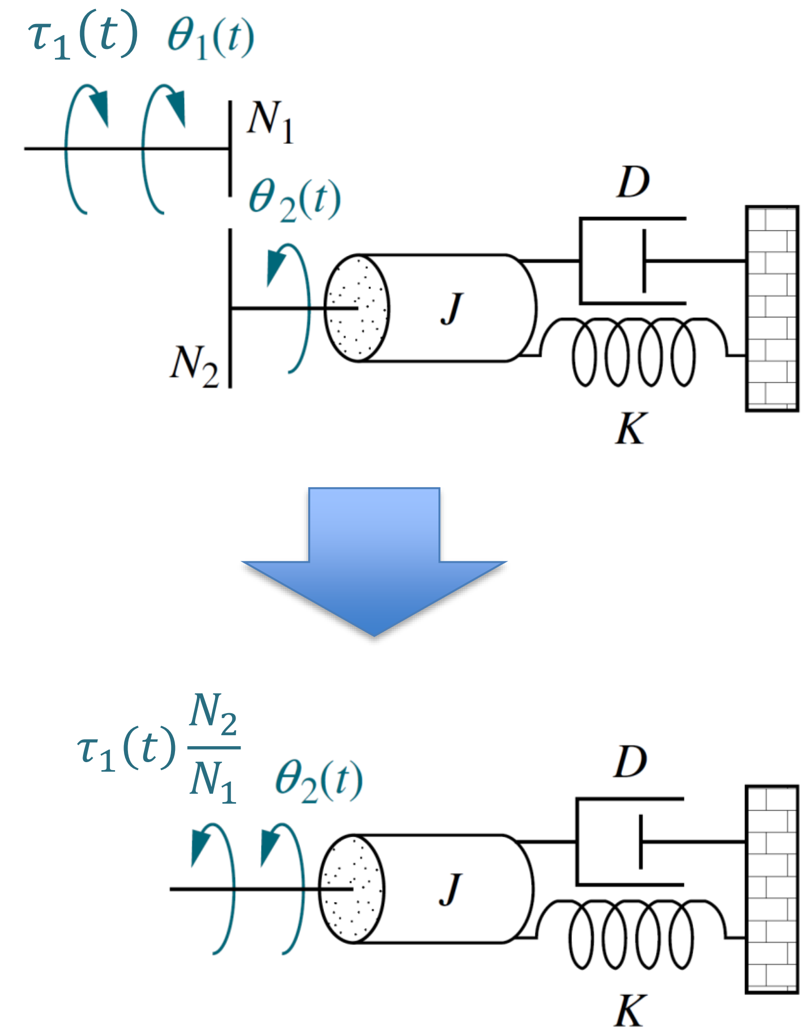
$$\tau_1(t)\left(\frac{N_2}{N_1}\right) = K\theta_1(t)\left(\frac{N_1}{N_2}\right) + D\dot{\theta}_1(t)\left(\frac{N_1}{N_2}\right) + J\ddot{\theta}_1(t)\left(\frac{N_1}{N_2}\right)$$

Modeling of Rotational Mechanical Systems

Example 5

The following system shows gears driving a rotational inertia, spring, and viscous damper. Find the transfer function, $\theta_1(s)/T_1(s)$ for the following system.

$$\boxed{\frac{\tau_2}{\tau_1}=\frac{\theta_1}{\theta_2}=\frac{\omega_1}{\omega_2}=\frac{N_2}{N_1}=\frac{r_2}{r_1}}$$



$$\tau_1(t)\left(\frac{N_2}{N_1}\right)=K\theta_1(t)\left(\frac{N_1}{N_2}\right)+D\dot{\theta}_1(t)\left(\frac{N_1}{N_2}\right)+J\ddot{\theta}_1(t)\left(\frac{N_1}{N_2}\right)$$

Take Laplace transform with <u>zero initial conditions</u> to find the transfer function:

$$T_1(s)\left(\frac{N_2}{N_1}\right)=K\left(\frac{N_1}{N_2}\right)\theta_1(s)+D\left(\frac{N_1}{N_2}\right)s\theta_1(s)+J\left(\frac{N_1}{N_2}\right)s^2\theta_1(s)$$

$$T_1(s)\left(\frac{N_2}{N_1}\right)=(K+Ds+Js^2)\left(\frac{N_1}{N_2}\right)\theta_1(s)$$

$$T_1(s)\left(\frac{N_2}{N_1}\right)^2=(K+Ds+Js^2)\theta_1(s)$$

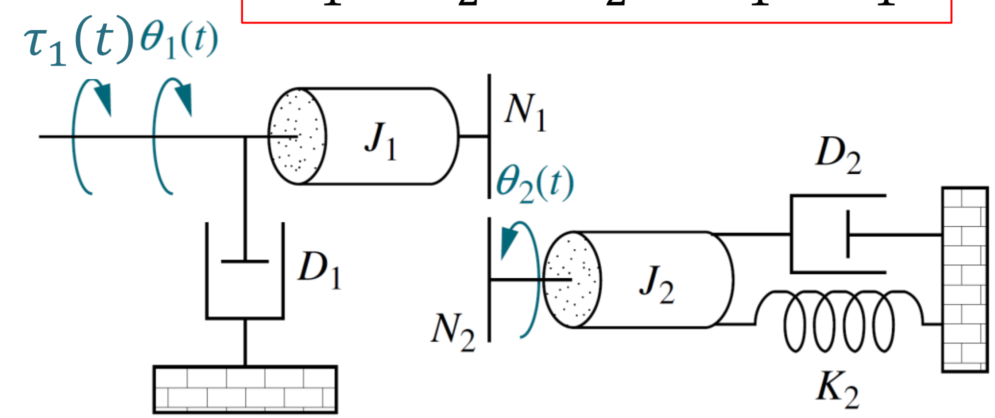
$$\boxed{\frac{\theta_1(s)}{T_1(s)}=\frac{\left(\frac{N_2}{N_1}\right)^2}{Js^2+Ds+K}}$$

Transfer Function Model

Modeling of Rotational Mechanical Systems

Example 6 The following figure shows gears driving a rotational inertia, spring, and viscous damper system. Find the transfer function, $\theta_2(s)/T_1(s)$ for the following system.

$$\frac{\tau_2}{\tau_1} = \frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} = \frac{r_2}{r_1}$$



We want to represent the system as an equivalent system without the gears.

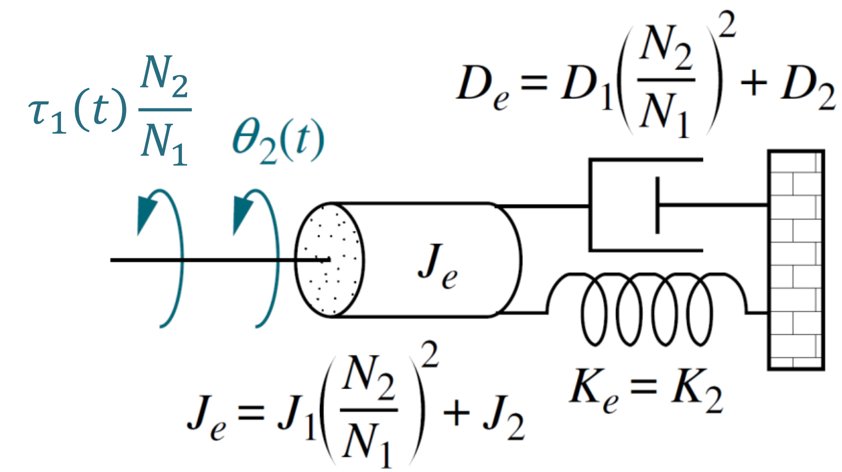
From the gear law:

τ_1 can be reflected to the output by multiplying by $\frac{N_2}{N_1}$.

$$\tau_2 = \tau_1\left(\frac{N_2}{N_1}\right)$$

J_1 and D_1 can be reflected to the output by multiplying by $\left(\frac{N_2}{N_1}\right)^2$

$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2 \qquad \text{and} \qquad D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2$$



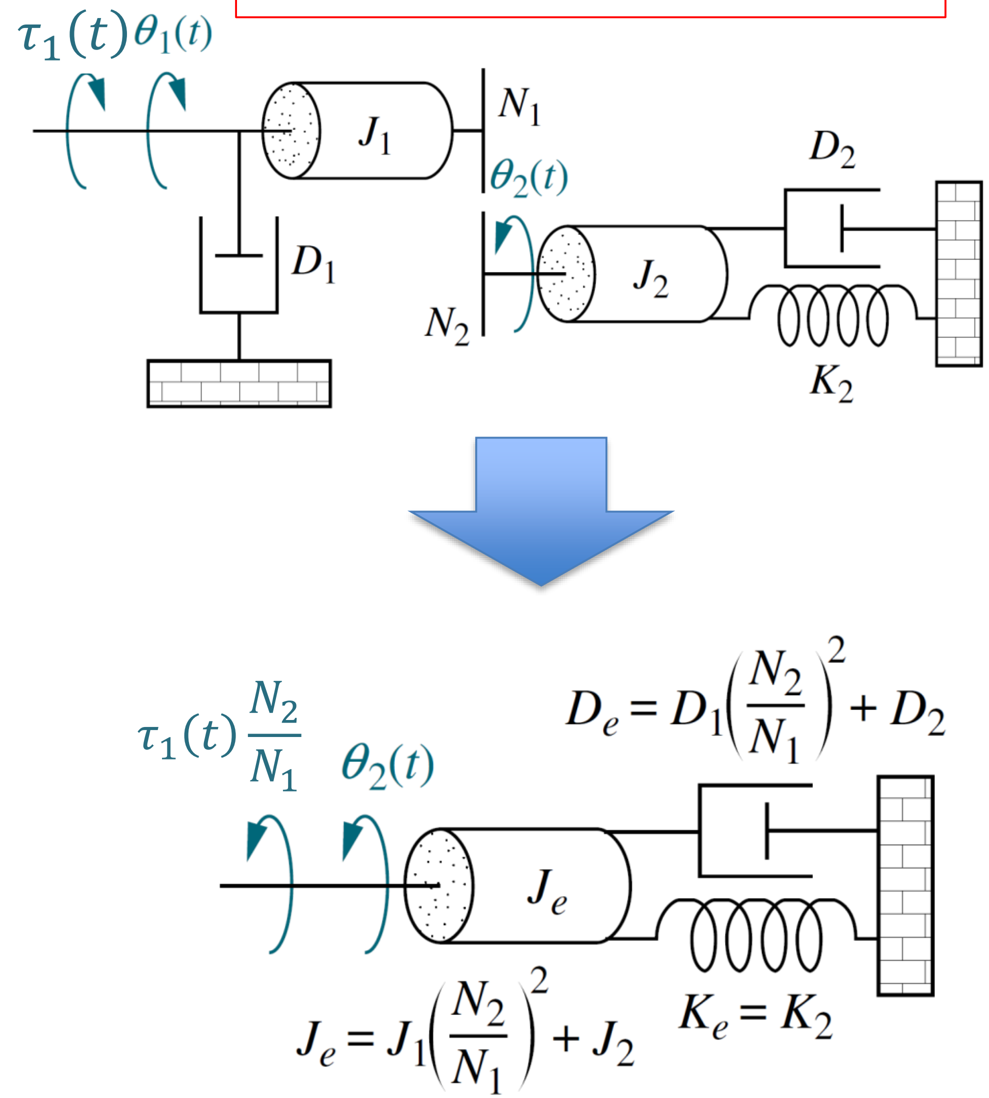
The equation of motion is:

$$\tau_1(t)\left(\frac{N_2}{N_1}\right) - K_2\theta_2(t) - D_e\dot{\theta}_2(t) = J_e\ddot{\theta}_2(t)$$

Modeling of Rotational Mechanical Systems

Example 6 The following figure shows gears driving a rotational inertia, spring, and viscous damper system. Find the transfer function, $\theta_2(s)/T_1(s)$ for the following system.

$$\boxed{\frac{\tau_2}{\tau_1} = \frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} = \frac{r_2}{r_1}}$$



The equation of motion is:

$$\tau_1(t)\left(\frac{N_2}{N_1}\right) - K_2\theta_2(t) - D_e\dot{\theta}_2(t) = J_e\ddot{\theta}_2(t)$$

Take Laplace transform with zero initial conditions to find the transfer function:

$$T_1(s)\left(\frac{N_2}{N_1}\right) = K_2\theta_2(s) + D_e s\theta_2(s) + J_e s^2\theta_2(s)$$

$$T_1(s)\left(\frac{N_2}{N_1}\right) = (K_2 + D_e s + J_e s^2)\theta_2(s)$$

$$\boxed{\frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_2}}$$

Transfer Function Model

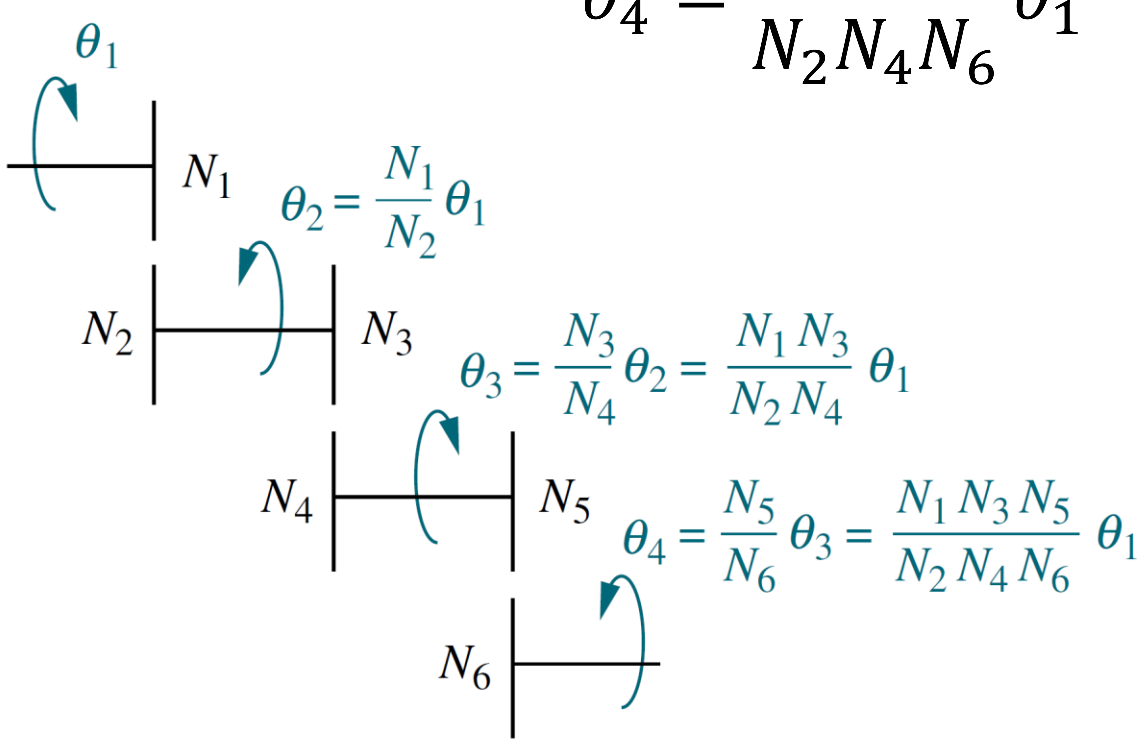
Modeling of Rotational Mechanical Systems

❑ Gear Trains

- In order to eliminate gears with large radii, a **gear train** is used to implement large gear rations by cascading smaller gear ratios.

- The equivalent gear ratio of this gear train is:

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1$$



$$\boxed{\frac{\tau_2}{\tau_1} = \frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} = \frac{r_2}{r_1}}$$

Modeling of Rotational Mechanical Systems

Example 7 The following system shows gears driving a rotational inertia, spring, and viscous damper system. Find the transfer function, $\theta_1(s)/T_1(s)$ for the following system.

$$\boxed{\frac{\tau_2}{\tau_1}=\frac{\theta_1}{\theta_2}=\frac{\omega_1}{\omega_2}=\frac{N_2}{N_1}=\frac{r_2}{r_1}}$$

We want to represent the system as an equivalent system without the gears.

In this system, all gears have inertia and for some shafts there is viscous friction.

We have to reflect the elements to the input shaft θ_1

$$J_e = J_1 + (J_2+J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4+J_5)\left(\frac{N_3}{N_4}\right)^2\left(\frac{N_1}{N_2}\right)^2$$

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

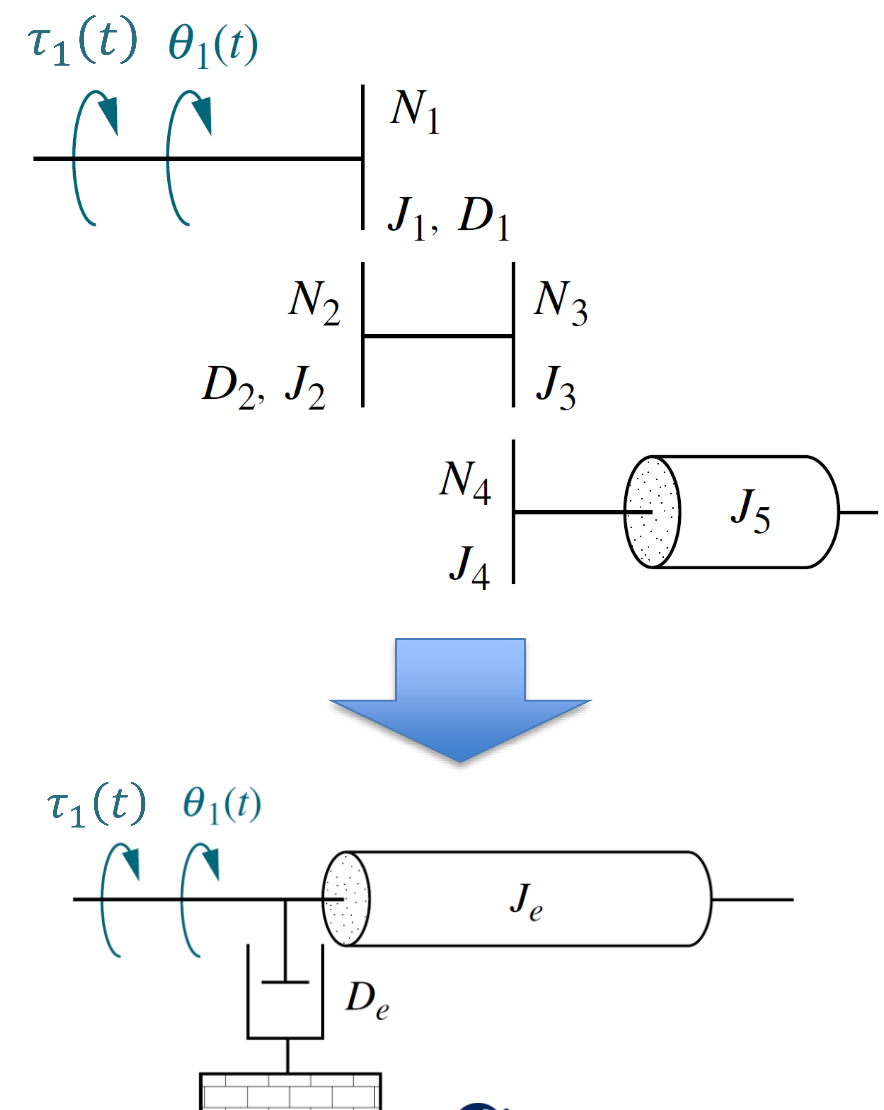
The equation of motion is: $\tau_1(t) - D_e\dot{\theta}_1(t) = J_e\ddot{\theta}_1(t)$

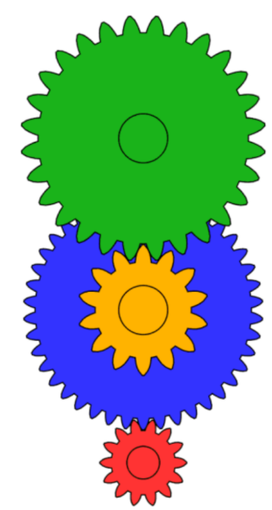
Take Laplace transform to find the transfer function:

$$T_1(s) = D_e s\theta_1(s) + J_e s^2\theta_1(s)$$

$$\boxed{\frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s}}$$

Transfer Function Model



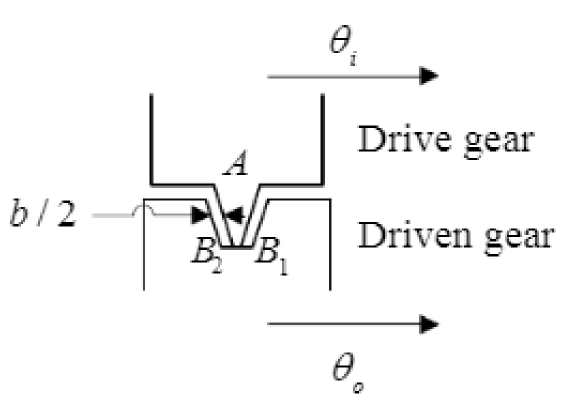


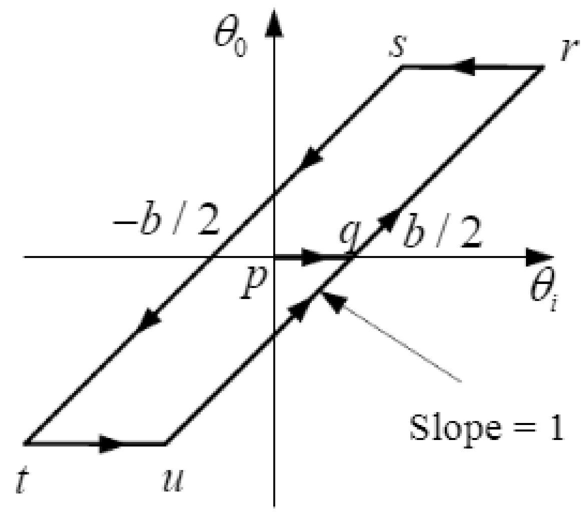
Modeling of Rotational Mechanical Systems

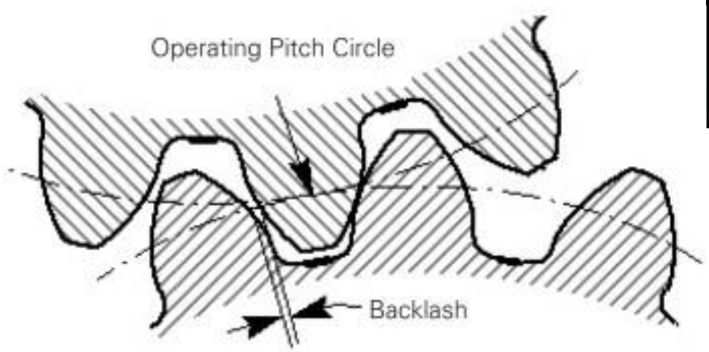
❑ Backlash in Gears (Nonlinear Characteristics)

- For many applications, gears exhibit **backlash**, which occurs because of the loose fit between two meshed gears.
- The drive gear rotates through a small angle before making contact with the meshed gear.
- The result is that the angular rotation of the output gear **does not occur** until a small angular rotation of the input gear has occurred.

- For example, as a motor reverses direction, the output shaft remains stationary while the motor begins to reverse.
- When the gears finally connect, the output shaft itself begins to turn in the reverse direction



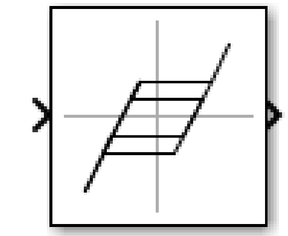




Modeling of Rotational Mechanical Systems

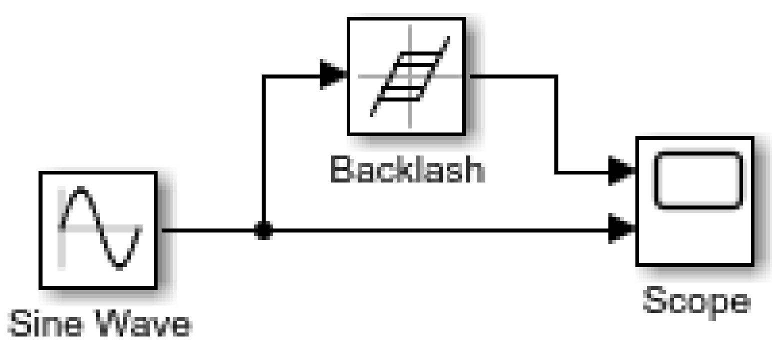
❑ Backlash in Gears (Nonlinear Characteristics)

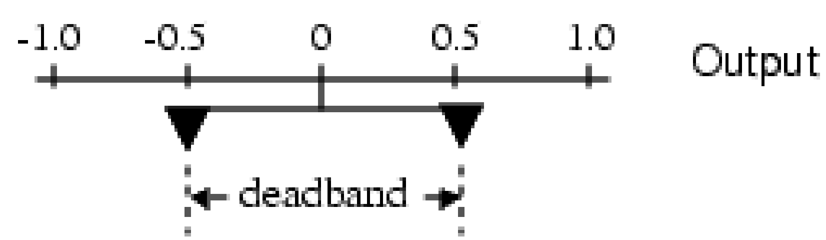
- The Backlash nonlinearity can be modeled in **Simulink** using the **Backlash** block.
- The Backlash block implements a system in which a change in input causes an equal change in output, except when the input changes direction.
- When the input changes direction, the initial change in input has **no effect** on the output.
- The amount of side-to-side play in the system is referred to as the **dead-band**.
- The dead-band is centered about the output.

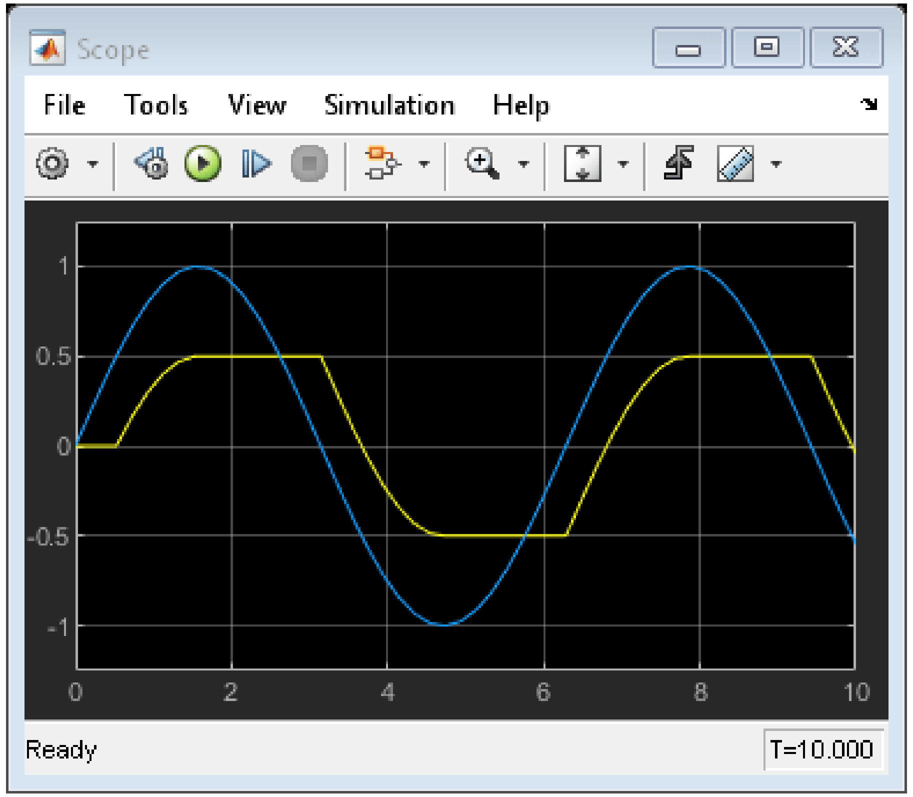


Backlash block

- This example shows the effect of the Backlash block on a sine wave.
- The initial **Dead-band width** is 1 and the **Initial output** is 0.







THANK YOU

HUMBER

WE ARE HUMBER