HUMBER ENGINEERING

MENG-3020
SYSTEMS MODELING & SIMULATION

LECTURE 3

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LECTURE 3
State-Space Approach to Modeling Dynamic Systems

- Standard Forms for System Models
- State Variable Equations
- Vector-Matrix Form of State-Variable Models
  - Standard Form of The State Equation
  - The Output Equation
- Block Diagram Models

Standard Forms for System Models

- In the previous lecture, we introduced the element and interconnection laws for translational mechanical systems and the procedure for drawing free-body diagrams and applying Newton's second law to find the equation of motion.
- These are essential steps regardless of the final form of the equation.
- However, the mathematical model should have a standard form
- **Why do we need consider standard forms?**
  - More convenient to find the solution of the equation
  - Easy to compare the model to another
  - Necessary for entry into a computer solver, such as MATLAB / Simulink
  - Different standard forms have different usages and purposes

- The two most common standard forms for dynamic system models:
  - **State-Space Model**
    - Provides a model in time domain, and internal description of the system via state variables
    - Applicable for time-varying, nonlinear and MIMO systems
  - **Transfer Function Model**
    - Provides a model in Laplace domain, and input-output description (External description)
    - Limited to LTI and SISO or two-input two-output systems

State-Space Representation of Dynamic Systems

- In State Space Representation, dynamic model of the system is described by a set of first-order differential equations in terms of the variables called the **state variables**.
- Formulating the model in state-space form is to begin by selecting a set of state variables.

❑ Definition of State Variables

- The **state variables** of a dynamic system are the minimum set of linearly independent variables that describes the effect of the history of the system (past inputs and dynamics) on its response in the future.
- The minimum number of required state variables equals the order of the differential equation describing the system.
- Note that there is **no unique** set of state variables that describe any given system; many different sets of state variables may be selected to obtain a complete system description.
- The state variables may be selected based on physical and measurable variables, or in terms of variables that are not directly measurable.
- In physical dynamic systems it is often convenient to associate the state variables with the energy storage elements in the system. Because any energy that is initially stored in these elements can affect the response of the system at a later time.

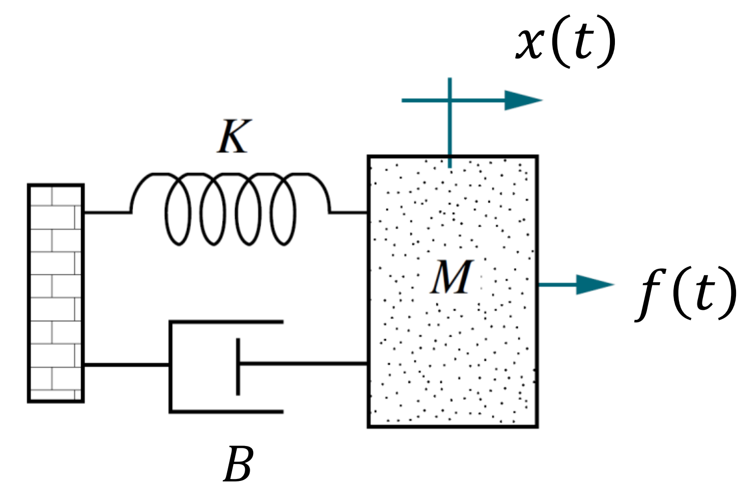
State Variable Selection

- Table of energy storage elements in physical dynamic systems:

System	Element	Energy	Physical Variable
Electrical Systems	Capacitor C	$\frac{1}{2}Cv^2$	Voltage $v(t)$
	Inductor L	$\frac{1}{2}Li^2$	Current $i(t)$
Translational Mechanical Systems	Mass M	$\frac{1}{2}Mv^2$	Translational Velocity $v(t)$
	Translational Spring K	$\frac{1}{2}Kx^2$	Translational Displacement $x(t)$
Rotational Mechanical Systems	Moment of Inertia J	$\frac{1}{2}J\omega^2$	Angular Velocity $\omega(t)$
	Tortional Spring K	$\frac{1}{2}K\theta^2$	Angular Displacement $\theta(t)$

State Variable Equations

Example 1 Find the state-variable equations for the given mass-spring-damper system.
Assume the applied force $f(t)$ as the input, and the displacement $x(t)$ is the output



Draw the free-body diagram of the system and write the equation of motion of the system.

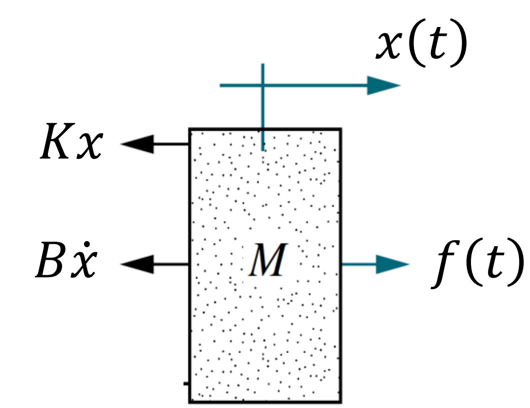
$$f(t) - Kx(t) - B\dot{x}(t) = M\ddot{x}(t)$$

The state variables q_1 and q_2 are selected as the displacement of the spring $x(t)$ and velocity of the mass $\dot{x}(t)$.

$$q_1(t) = x(t) \quad \rightarrow \quad \dot{q}_1(t) = \dot{x}(t) \quad \rightarrow \quad \dot{q}_1(t) = q_2(t) \qquad Eqn.\ (1)$$

$$q_2(t) = \dot{x}(t) \quad \rightarrow \quad \dot{q}_2(t) = \ddot{x}(t) \quad \rightarrow \quad \dot{q}_2(t) = \frac{1}{M}f(t) - \frac{K}{M}x(t) - \frac{B}{M}\dot{x}(t)$$

$$\rightarrow \quad \dot{q}_2(t) = \frac{1}{M}f(t) - \frac{K}{M}q_1(t) - \frac{B}{M}q_2(t) \qquad Eqn.\ (2)$$



Free-body Diagram

State-variable equations are obtained as:

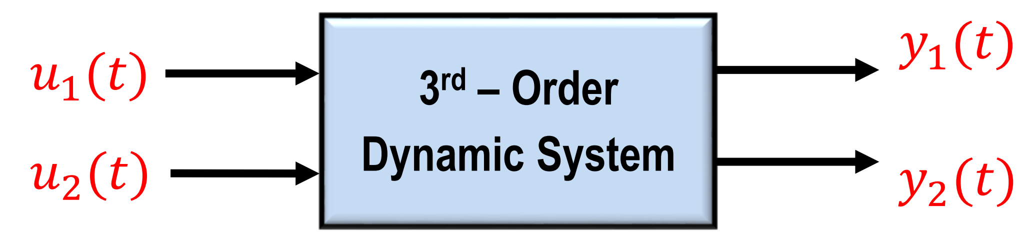
$$\begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = -\frac{K}{M}q_1(t) - \frac{B}{M}q_2(t) + \frac{1}{M}f(t) \end{cases}$$

The **output equation** is obtained as:

$$y(t) = q_1(t)$$

General Form of The State-Space Equations

- Consider a 3rd order LTI dynamic system with 2 inputs, 2 outputs and 3 state-variables.
  - Inputs are the external influences on the system (force, voltage, …)
  - Outputs are the variables of interest to be measured or calculated (position, velocity, current, …)
  - State variables represent the status or memory of the system



State variables: $q_1(t), q_2(t), q_3(t)$

- General form of a State Space Model for this system includes state equations and output equations:

State Equations

$$\begin{cases} \dot{q}_1 = a_{11}q_1 + a_{12}q_2 + a_{13}q_3 + b_{11}u_1 + b_{12}u_2 \\ \dot{q}_2 = a_{21}q_1 + a_{22}q_2 + a_{23}q_3 + b_{21}u_1 + b_{22}u_2 \\ \dot{q}_3 = a_{31}q_1 + a_{32}q_2 + a_{33}q_3 + b_{31}u_1 + b_{32}u_2 \end{cases}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\boxed{\dot{\mathbf{q}}(\boldsymbol{t}) = \mathbf{Aq}(\boldsymbol{t}) + \mathbf{Bu}(\boldsymbol{t})}$$

Output Equations

$$\begin{cases} y_1 = c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + d_{11}u_1 + d_{12}u_2 \\ y_2 = c_{21}q_1 + c_{22}q_2 + c_{23}q_3 + d_{21}u_1 + d_{22}u_2 \end{cases}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\boxed{\mathbf{y}(\boldsymbol{t}) = \mathbf{Cq}(\boldsymbol{t}) + \mathbf{Du}(\boldsymbol{t})}$$

Vector-Matrix Form of The State-Space Equations

- For **LTI systems**, the state and output equations can be written in **vector-matrix** form.
- General form of a State Space Representation for a nth order LTI system with m input and p output is:

$$\begin{cases} \dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t) & \longleftarrow \quad \textbf{state equation} \\ \mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t) & \longleftarrow \quad \textbf{output equation} \end{cases}$$

$\mathbf{A}_{n\times n}$: system matrix	$\mathbf{B}_{n\times m}$: input matrix	$\boldsymbol{q}(t)$: state vector	
$\mathbf{C}_{p\times n}$: output matrix	$\mathbf{D}_{p\times m}$: feed-forward matrix	$\mathbf{u}(t)$: input vector	$\mathbf{y}(t)$: output vector

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

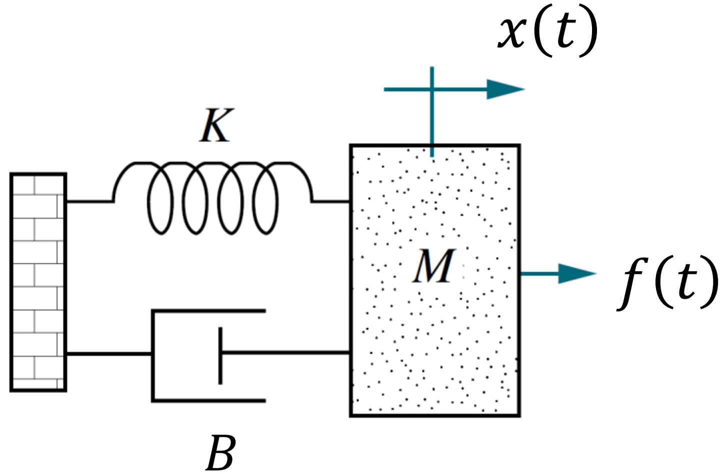
n×1 vector *m*×1 vector *p*×1 vector

- $\dot{\mathbf{q}}(t)$ is the derivative of the state vector, which shows how the state vector changes as a linear combination of the state vector and the input vector.
- In LTI systems $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and **D** are constant matrices.
- In time-varying linear systems some elements of the matrices would be functions of time.

Vector-Matrix Form of The State-Space Equations

Example 2 Find the vector-matrix form of the state-space equations for the given mass-spring-damper system. Assume the applied force $f(t)$ as the input, and the displacement $x(t)$ is the output

Recall the state equations and output equation from Example 1:



$$\begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = \dfrac{1}{M}f(t) - \dfrac{K}{M}q_1(t) - \dfrac{B}{M}q_2(t) \\ y(t) = q_1(t) \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

$$\boxed{\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)} \rightarrow \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{B}{M} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} f(t)$$

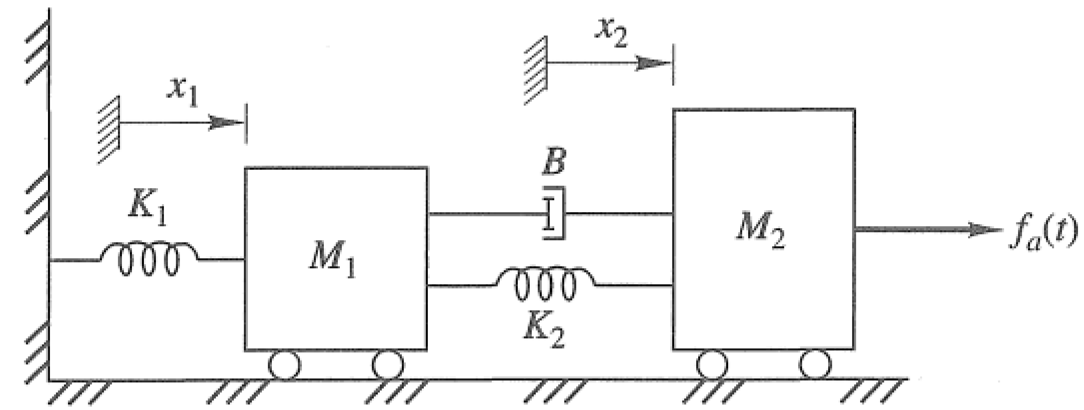
State Equation

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t)} \rightarrow y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0]f(t)$$

Output Equation

State-Space Representation

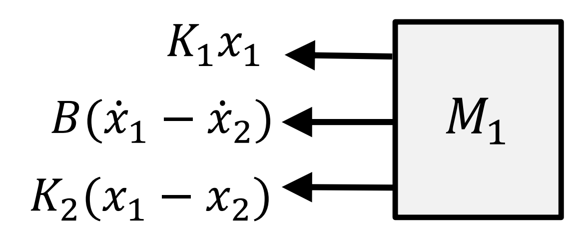
Example 3 Find the state-space model for the given mass-spring-damper system.
Assume the applied force $f_a(t)$ as the input, and the displacement of spring K_2 is the output.

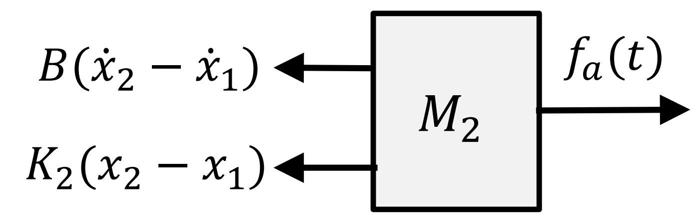


Draw the free-body diagram of the system and write the equation of motion of the system.

Mass M_1 $\rightarrow$ $\quad -K_1x_1 - K_2(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2) = M_1\ddot{x}_1$

Mass M_2 $\rightarrow$ $\quad f_a(t) - K_2(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1) = M_2\ddot{x}_2$





The state variables q_1, q_2, q_3 and q_4 are selected as the displacement of the springs and velocity of the masses.

$$q_1 = x_1 \quad \rightarrow \quad \dot{q}_1 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_1 = q_2 \qquad Eqn.\ (1)$$

$$q_2 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_2 = \ddot{x}_1 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{M_1}\left(-K_1x_1 - K_2(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2)\right)$$

$$\rightarrow \quad \dot{q}_2 = \frac{1}{M_1}\left(-K_1q_1 + K_2q_3 - B(q_2 - q_4)\right) \qquad Eqn.\ (2)$$

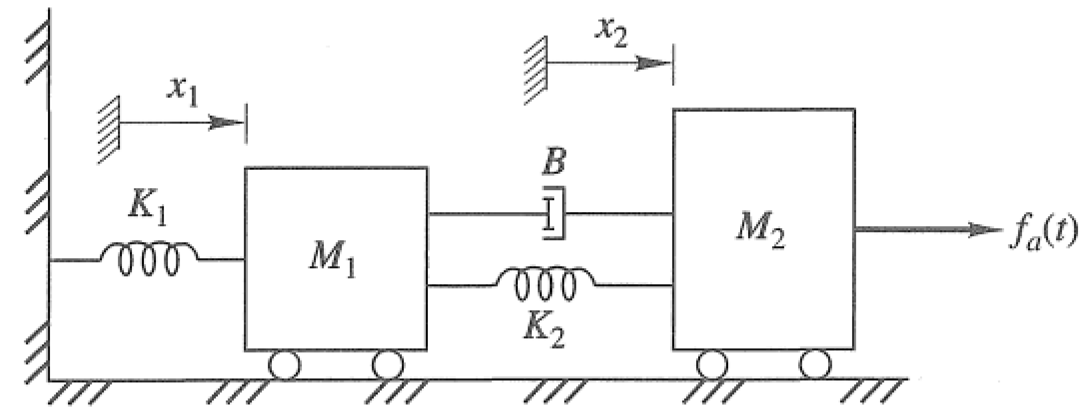
$$q_3 = x_2 - x_1 \quad \rightarrow \quad \dot{q}_3 = \dot{x}_2 - \dot{x}_1 \quad \rightarrow \quad \dot{q}_3 = q_4 - q_2 \qquad Eqn.\ (3)$$

$$q_4 = \dot{x}_2 \quad \rightarrow \quad \dot{q}_4 = \ddot{x}_2 \quad \rightarrow \quad \dot{q}_4 = \frac{1}{M_2}\left(f_a(t) - K_2(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1)\right)$$

$$\rightarrow \quad \dot{q}_4 = \frac{1}{M_2}\left(f_a(t) - K_2q_3 - B(q_4 - q_2)\right) \qquad Eqn.\ (4)$$

State-Space Representation

Example 3 Find the state-space model for the given mass-spring-damper system.
Assume the applied force $f_a(t)$ as the input, and the displacement of spring K_2 is the output.



The state-variable equations and output equation are obtained as:

$$\begin{cases} \dot{q}_1 = q_2 \\ \dot{q}_2 = \dfrac{1}{M_1}(-K_1 q_1 - B q_2 + K_2 q_3 + B q_4) \\ \dot{q}_3 = -q_2 + q_4 \\ \dot{q}_4 = \dfrac{1}{M_2}(B q_2 - K_2 q_3 - B q_4 + f_a(t)\,) \\ y = x_2 - x_1 \quad \rightarrow \quad y = q_3 \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

State Equation $\boxed{\dot{\mathbf{q}}(t) = \mathbf{A}\,\mathbf{q}(t) + \mathbf{B}\,\mathbf{u}(t)}$

$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t)}$ Output Equation

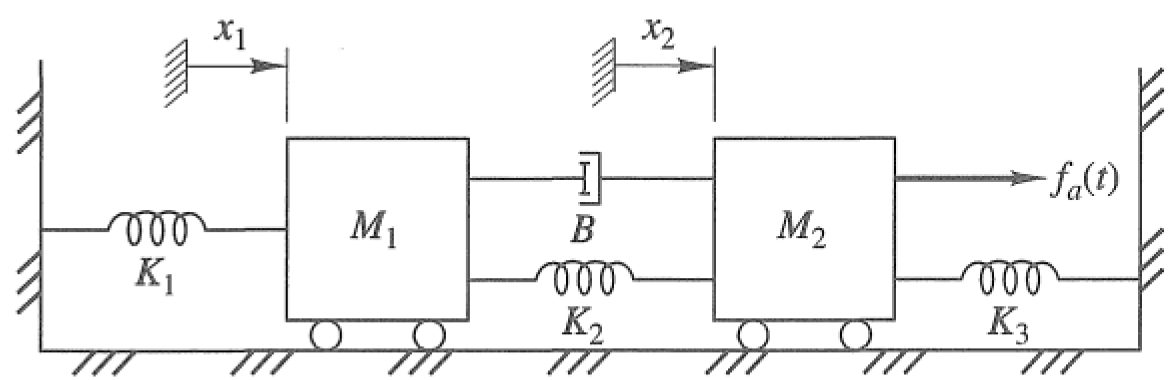
$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K_1/M_1 & -B/M_1 & K_2/M_1 & B/M_1 \\ 0 & -1 & 0 & 1 \\ 0 & B/M_2 & -K_2/M_2 & -B/M_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f_a(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + [0]u(t)$$

State-Space Representation

- The following example illustrates how the number of state variables can be less than the number of energy-storing elements.

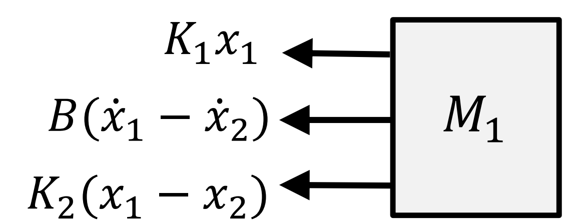
Example 4 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of spring K_2 is the output.



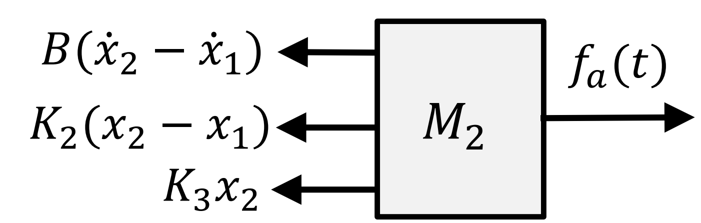
Draw the free-body diagram of the system and write the equation of motion of the system.

Mass M_1 $\rightarrow$ $$-K_1x_1 - K_2(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2) = M_1\ddot{x}_1$$

Mass M_2 $\rightarrow$ $$f_a(t) - K_2(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1) - K_3x_2 = M_2\ddot{x}_2$$



Although there are three springs, their elongations **are not all independent** and can be specified in terms of the two displacement variables x_1 and x_2.
The elongations of K_1, K_2, and K_3 are x_1 ,$x_2 - x_1$, and x_2, respectively.



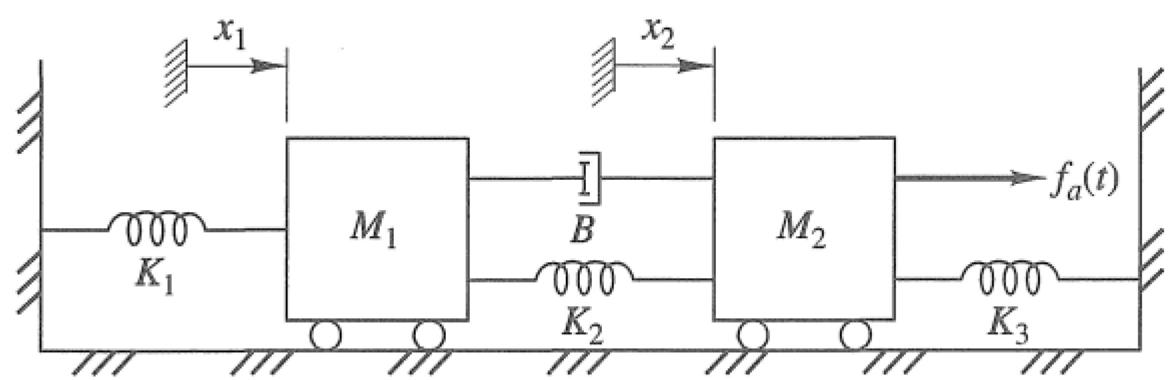
The state variables q_1, q_2, q_3 and q_4 are selected as the displacement of the springs K_1, K_3 and velocity of the masses M_1, M_2.

$$\boxed{q_1 = x_1 \qquad q_2 = \dot{x}_1 \qquad q_3 = x_2 \qquad q_4 = \dot{x}_2}$$

State-Space Representation

- The following example illustrates how the number of state variables can be less than the number of energy-storing elements.

Example 4 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of spring K_2 is the output.



Mass M_1 $\rightarrow$ $\quad -K_1x_1 - K_2(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2) = M_1\ddot{x}_1$

Mass M_2 $\rightarrow$ $\quad f_a(t) - K_2(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1) - K_3x_2 = M_2\ddot{x}_2$

The state-variable equations are derived as:

$$q_1 = x_1 \quad \rightarrow \quad \dot{q}_1 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_1 = q_2 \qquad Eqn.\ (1)$$

$$q_2 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_2 = \ddot{x}_1 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{M_1}\left(-K_1x_1 - K_2(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2)\right)$$

$$\rightarrow \quad \dot{q}_2 = \frac{1}{M_1}\left(-(K_1 + K_2)q_1 + K_2q_3 - B(q_2 - q_4)\right) \qquad Eqn.\ (2)$$

$$q_3 = x_2 \quad \rightarrow \quad \dot{q}_3 = \dot{x}_2 \quad \rightarrow \quad \dot{q}_3 = q_4 \qquad Eqn.\ (3)$$

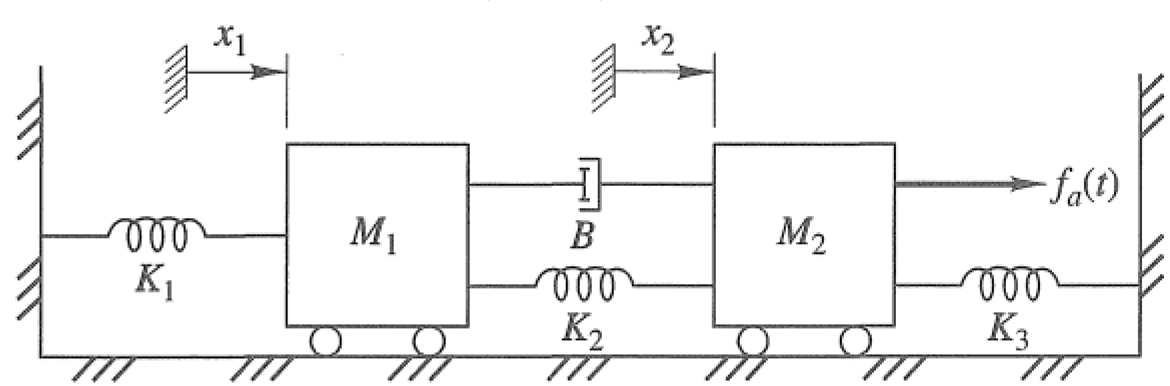
$$q_4 = \dot{x}_2 \quad \rightarrow \quad \dot{q}_4 = \ddot{x}_2 \quad \rightarrow \quad \dot{q}_4 = \frac{1}{M_2}\left(f_a(t) - K_2(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1) - K_3x_2\right)$$

$$\rightarrow \quad \dot{q}_4 = \frac{1}{M_2}\left(f_a(t) - K_2(q_3 - q_1) - B(q_4 - q_2) - K_3q_3\right) \qquad Eqn.\ (4)$$

State-Space Representation

- The following example illustrates how the number of state variables can be less than the number of energy-storing elements.

Example 4 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of spring K_2 is the output.



The state-variable equations and output equation are obtained as:

$$
\begin{cases}
\dot{q}_1 = q_2 \\
\dot{q}_2 = \dfrac{1}{M_1}\left(-(K_1+K_2)q_1 - Bq_2 + K_2 q_3 + Bq_4\right) \\
\dot{q}_3 = q_4 \\
\dot{q}_4 = \dfrac{1}{M_2}\left(K_2 q_1 + Bq_2 - (K_2+K_3)q_3 - Bq_4 + f_a(t)\right) \\
y = x_2 - x_1 \quad \rightarrow \quad y = q_3 - q_1
\end{cases}
$$

We can represent the state and output equations in the standard matrix-vector form as below:

State Equation $\boxed{\dot{\mathbf{q}}(t) = \mathbf{A}\,\mathbf{q}(t) + \mathbf{B}\,\mathbf{u}(t)}$

$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t)}$ Output Equation

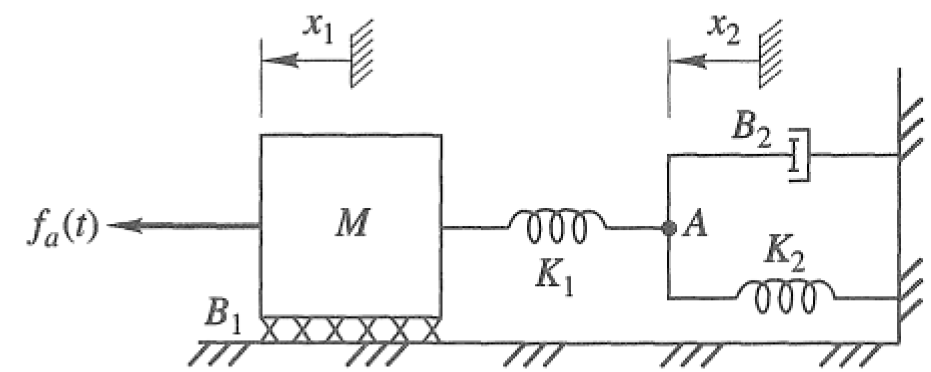
$$
\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(K_1+K_2)/M_1 & -B/M_1 & K_2/M_1 & B/M_1 \\ 0 & 0 & 0 & 1 \\ K_2/M_2 & B/M_2 & -(K_2+K_3)/M_2 & -B/M_2 \end{bmatrix} \begin{bmatrix} q_1\\ q_2\\ q_3\\ q_4\end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 0\\ 1/M_2\end{bmatrix} f_a(t)
$$

$$
y(t) = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1\\ q_2\\ q_3\\ q_4\end{bmatrix} + [0]u(t)
$$

State-Space Representation

- The following example, two springs and a damper are attached to a massless junction. The system contains three energy-storing elements and normally requires three state variables. However, when the damper is removed from the massless junction, the number of state variables is reduced.

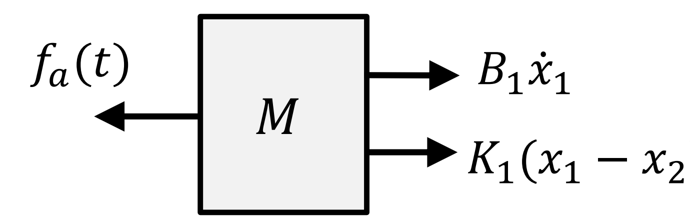
Example 6 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of massless junction A is the output.



Draw the free-body diagram of the system and write the equation of motion of the system.

Mass M $\rightarrow$ $\quad f_a(t) - K_1(x_1 - x_2) - B_1\dot{x}_1 = M\ddot{x}_1$

Junction A $\rightarrow$ $\quad K_1(x_1 - x_2) = K_2x_2 + B_2\dot{x}_2$

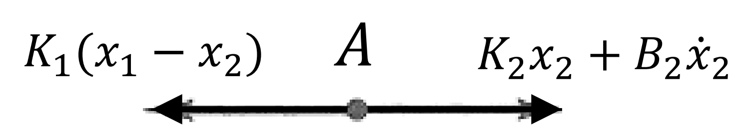


The state variables q_1, q_2 and q_3 are selected as the displacement of the springs K_1, K_2 and velocity of the mass M.

$$q_1 = x_1 - x_2 \quad \rightarrow \quad \dot{q}_1 = \dot{x}_1 - \dot{x}_2 \quad \rightarrow \quad \dot{q}_1 = q_3 - \frac{1}{B_2}(K_1q_1 - K_2q_2) \qquad Eqn.\ (1)$$

$$q_2 = x_2 \quad \rightarrow \quad \dot{q}_2 = \dot{x}_2 \quad \rightarrow \quad \dot{q}_2 = \frac{1}{B_2}(K_1(x_1 - x_2) - K_2x_2)$$

$$\rightarrow \quad \dot{q}_2 = \frac{1}{B_2}(K_1q_1 - K_2q_2) \qquad Eqn.\ (2)$$



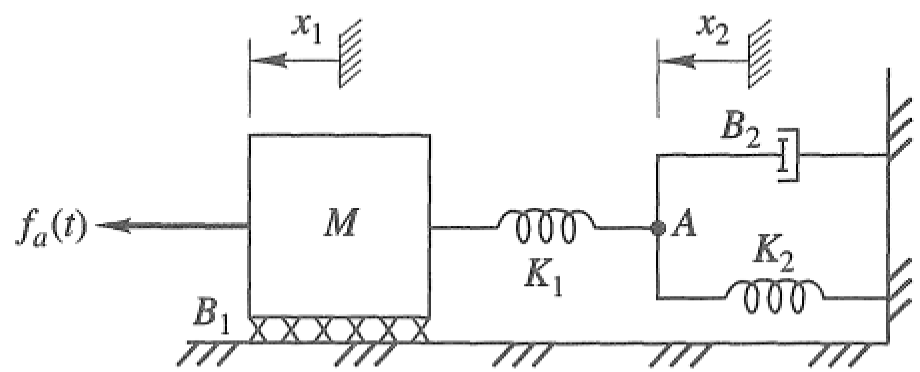
$$q_3 = \dot{x}_1 \quad \rightarrow \quad \dot{q}_3 = \ddot{x}_1 \quad \rightarrow \quad \dot{q}_3 = \frac{1}{M}(f_a(t) - K_1(x_1 - x_2) - B_1\dot{x}_1)$$

$$\rightarrow \quad \dot{q}_3 = \frac{1}{M}(f_a(t) - K_1q_1 - B_1q_3) \qquad Eqn.\ (3)$$

State-Space Representation

- The following example, two springs and a damper are attached to a massless junction. The system contains three energy-storing elements and normally requires three state variables. However, when the damper is removed from the massless junction, the number of state variables is reduced.

Example 6 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of massless junction A is the output.



The state-variable equations and output equation are obtained as:

$$\begin{cases} \dot{q}_1 = -\dfrac{1}{B_2}(K_1 q_1 - K_2 q_2) + q_3 \\ \dot{q}_2 = \dfrac{1}{B_2}(K_1 q_1 - K_2 q_2) \\ \dot{q}_3 = \dfrac{1}{M}(f_a(t) - K_1 q_1 - B_1 q_3) \\ y = x_2 \quad \rightarrow \quad y = q_2 \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

$$\boxed{\dot{\boldsymbol{q}}(t) = \mathbf{A}\,\mathbf{q}(t) + \mathbf{B}\,\mathbf{u}(t)}$$ State Equation

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -K_1/B_2 & K_2/B_2 & 1 \\ K_1/B_2 & -K_2/B_2 & 0 \\ -K_1/M & 0 & -B_1/M \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/M \end{bmatrix} f_a(t)$$

Output Equation

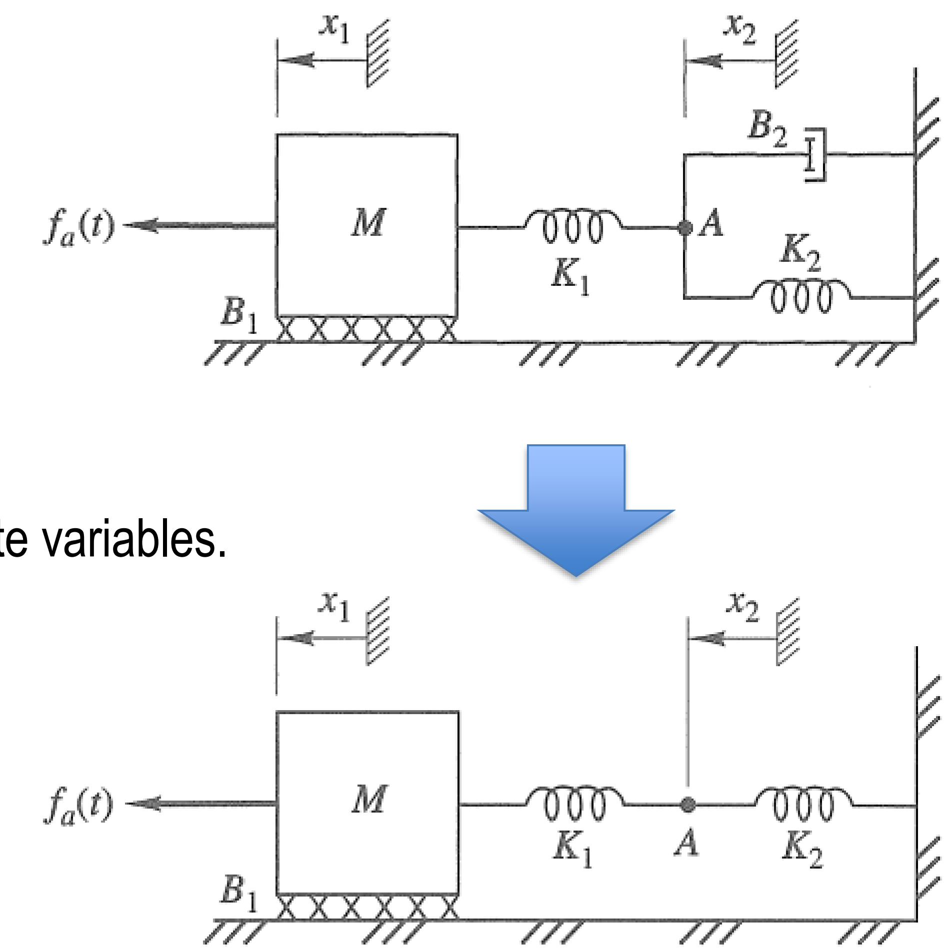
$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + [0]u(t)$$

State-Space Representation

- The following example, two springs and a damper are attached to a massless junction. The system contains three energy-storing elements and normally requires three state variables. However, when the damper is removed from the massless junction, the number of state variables is reduced.

Example 6 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of massless junction A is the output.



Removing the damper corresponds to setting $B_2 = 0$ in the equation of motion.

Mass M $\rightarrow$ $$f_a(t) - K_1(x_1 - x_2) - B_1\dot{x}_1 = M\ddot{x}_1$$

Junction A $\rightarrow$ $$K_1(x_1 - x_2) = K_2 x_2$$

From the second equation $\rightarrow$ $$x_2 = \left(\frac{K_1}{K_1 + K_2}\right)x_1$$

The displacements x_1 and x_2 are now proportional to each other, so they cannot both be state variables.

The two springs are now in series and can be replaced by a single equivalent spring,

$$K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$$

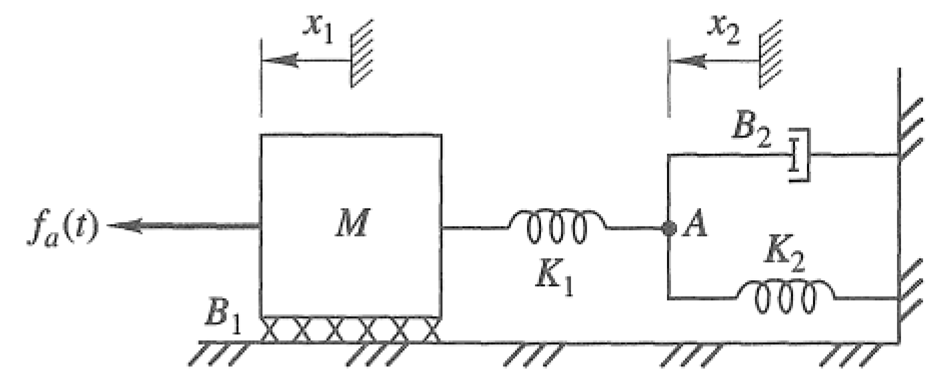
The equation of motion can be rewritten as,

Mass M $\rightarrow$ $$f_a(t) - K_{eq}x_1 - B_1\dot{x}_1 = M\ddot{x}_1$$

State-Space Representation

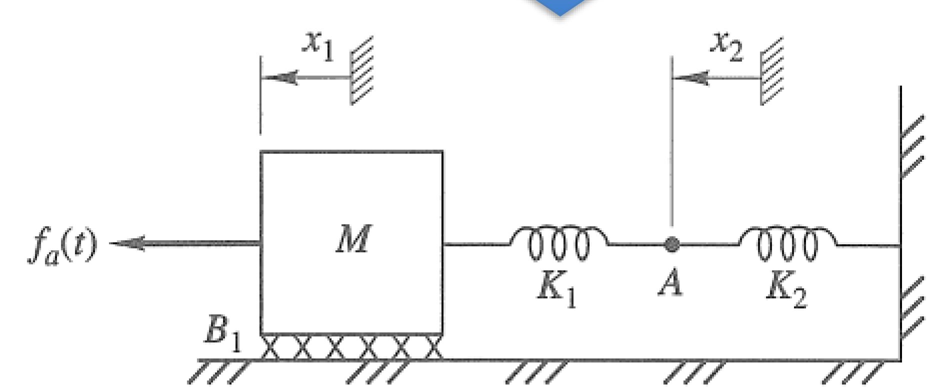
- The following example, two springs and a damper are attached to a massless junction. The system contains three energy-storing elements and normally requires three state variables. However, when the damper is removed from the massless junction, the number of state variables is reduced.

Example 6 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of massless junction A is the output.



Mass M $\rightarrow$ $f_a(t) - K_{eq}x_1 - B_1\dot{x}_1 = M\ddot{x}_1$

The state variables q_1 and q_2 are selected as the displacement of the equivalent spring K_{eq} and velocity of the mass M.



$$q_1 = x_1 \rightarrow \dot{q}_1 = \dot{x}_1 \rightarrow \dot{q}_1 = q_2 \quad Eqn.\ (1)$$

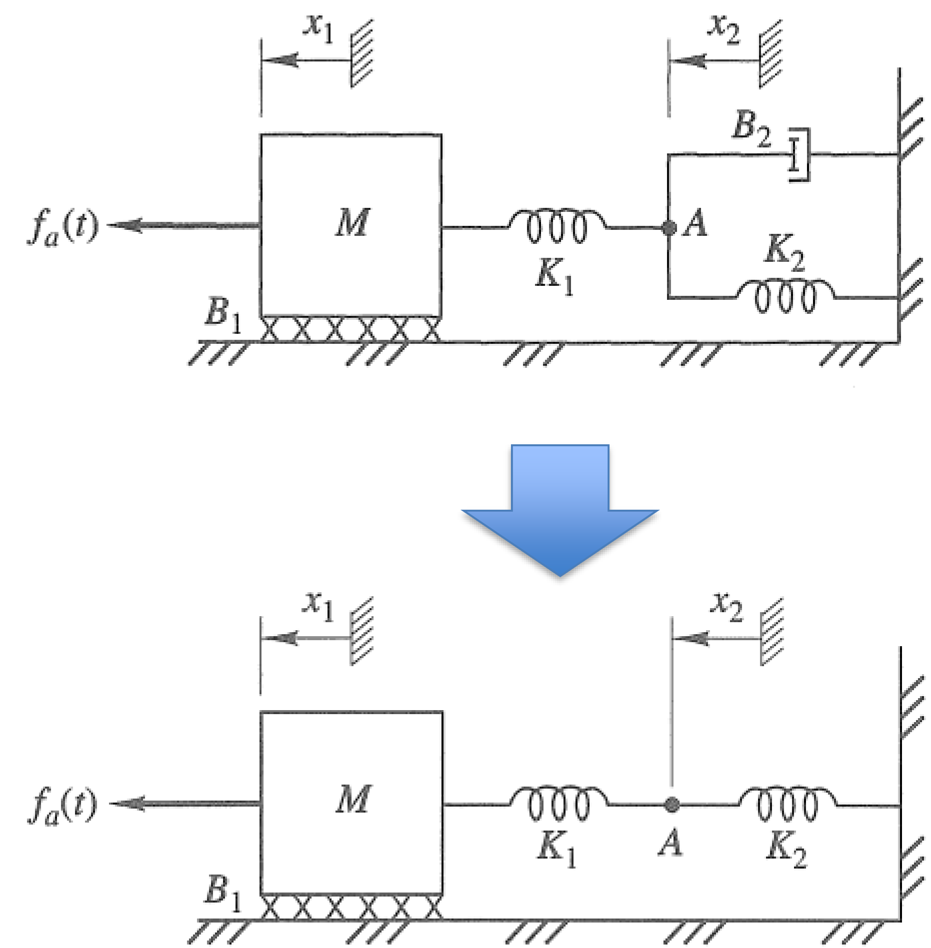
$$q_2 = \dot{x}_1 \rightarrow \dot{q}_2 = \ddot{x}_1 \rightarrow \dot{q}_2 = \frac{1}{M}\left(f_a(t) - K_{eq}x_1 - B_1\dot{x}_1\right)$$

$$\rightarrow \dot{q}_2 = \frac{1}{M}\left(f_a(t) - K_{eq}q_1 - B_1q_2\right) \quad Eqn.\ (2)$$

State-Space Representation

- The following example, two springs and a damper are attached to a massless junction. The system contains three energy-storing elements and normally requires three state variables. However, when the damper is removed from the massless junction, the number of state variables is reduced.

Example 6 Find the state-space model for the following LTI system.
Assume the applied force $f_a(t)$ as the input, and the displacement of massless junction A is the output.



The state-variable equations and output equation are obtained as:

$$\begin{cases} \dot{q}_1 = q_2 \\ \dot{q}_2 = \dfrac{1}{M}\left(f_a(t) - \dfrac{K_1K_2}{K_1+K_2}q_1 - B_1q_2\right) \\ y = x_2 \quad \rightarrow \quad y = \left(\dfrac{K_1}{K_1+K_2}\right)x_1 \quad \rightarrow \quad y = \left(\dfrac{K_1}{K_1+K_2}\right)q_1 \end{cases}$$

We can represent the state and output equations in the standard matrix-vector form as below:

State Equation
$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-K_1K_2}{M(K_1+K_2)} & -\dfrac{B_1}{M} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} f_a(t)$$

Output Equation
$$y(t) = \begin{bmatrix} \dfrac{K_1}{K_1+K_2} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + [0]f_a(t)$$

State-Space Representation with MATLAB

- We can create the State-Space Representation of a continuous-time system with **MATLAB** by using the following command:

```
sys = ss(A,B,C,D)
```

Output system state-space model (sys)

State-space model matrices (A,B,C,D)

- For example, create the given state-space model in MATLAB.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

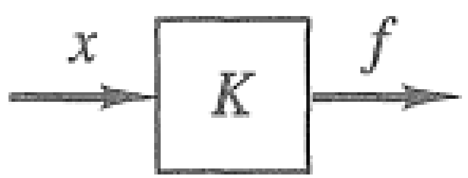
$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [0]u(t)$$

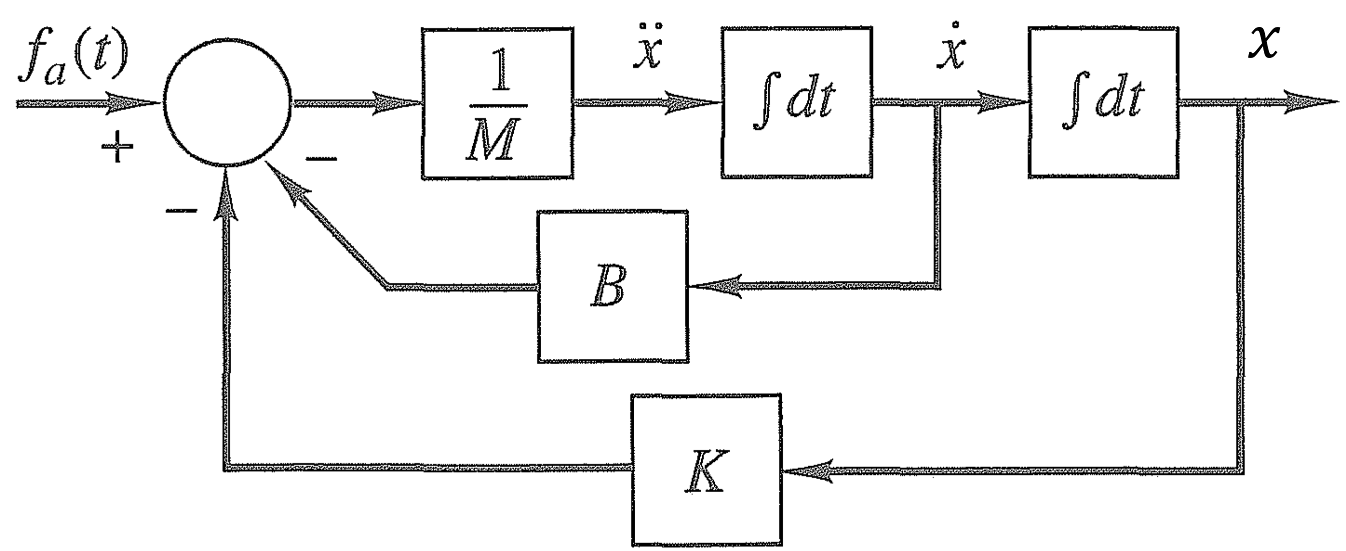
```
A = [0 1;-2 -3];
B = [0;1];
C = [1 0];
D = 0;
sys = ss(A,B,C,D)
sys =

  A =
         x1   x2
   x1     0    1
   x2    -2   -3

  B =
         u1
   x1     0
   x2     1

  C =
         x1   x2
   y1     1    0

  D =
         u1
   y1     0

Continuous-time state-space model.
```

Block Diagram Models

- A **block diagram model** is an interconnection of blocks representing basic mathematical operations in such a way that the overall diagram is **equivalent** to the system's mathematical model.

- In a block diagram model,
  - The arrows interconnecting the blocks represent the variables describing the system behavior and flow of the signals.
    - For example, inputs, outputs, state variables, ….
  - The blocks represents operations or functions that use one or more of these variables to calculate other variables.
    - For example, in equation $f = Kx$ the force can be calculated as,





❑ Basic Elements of Block Diagram:

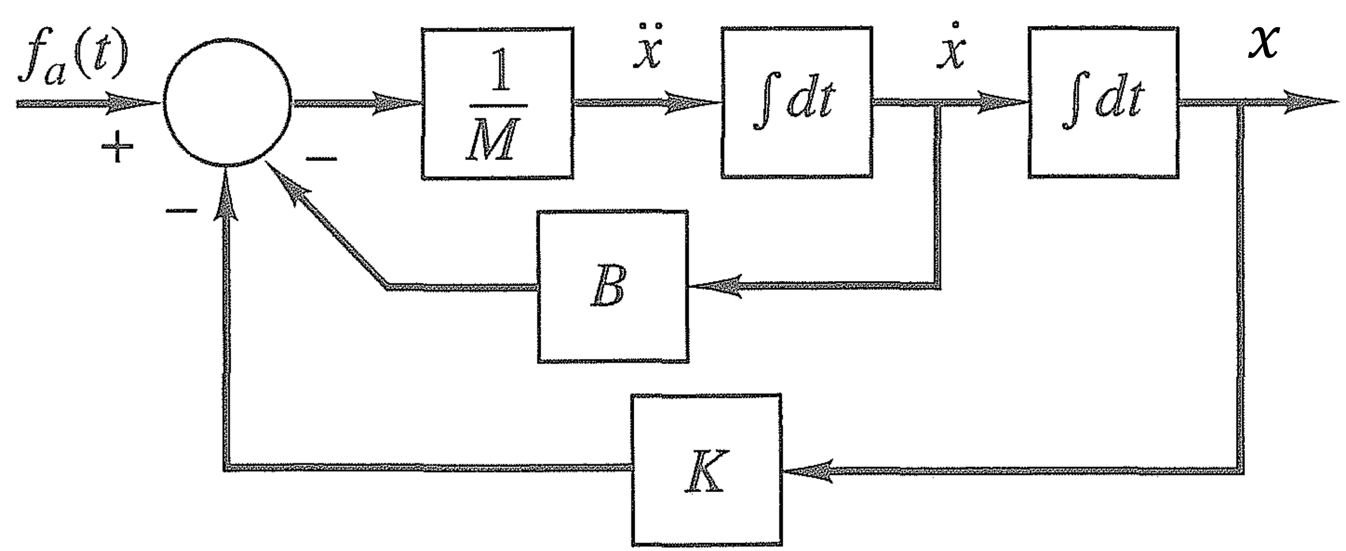
- Addition/Subtraction Block
- Gain Block
- Integrator Block
- Constant Block

Block Diagram Models for Input-Output Equations

- We will show that how to combine the block diagram elements, such as gains, summer, integrator, and constant value blocks, to represent and solve an input-output differential equation model of a system.

❑ Find the Block Diagrams Model for Input-Output Equations

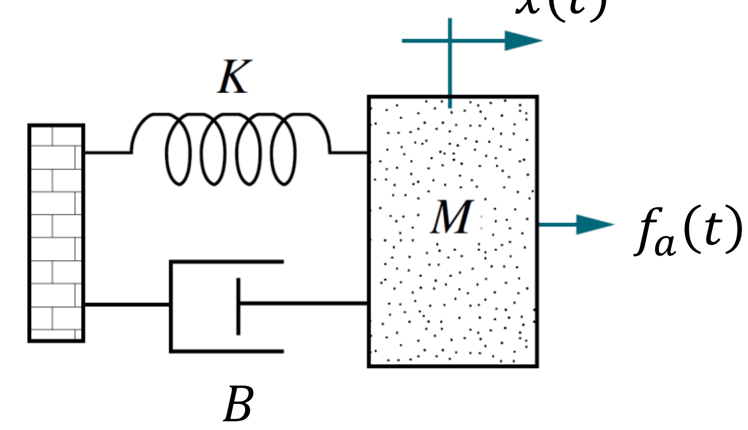
- Identify the output variables.
- Solve the given equation for the highest derivative of each output variables
- Create a sequence of output variables using integrator block to relate $\ddot{x}$, $\dot{x}$, and x to one another.
- Complete the diagram with addition/subtraction and gain blocks according to the model equation



Block Diagram Models for Input-Output Equations

Example 7 Construct a block diagram model for the following system, whose input-output model was given as below:

$$M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f_a(t)$$

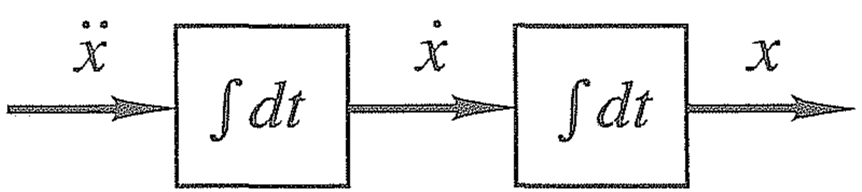


1) Identify the output variable $\rightarrow$ $x(t)$

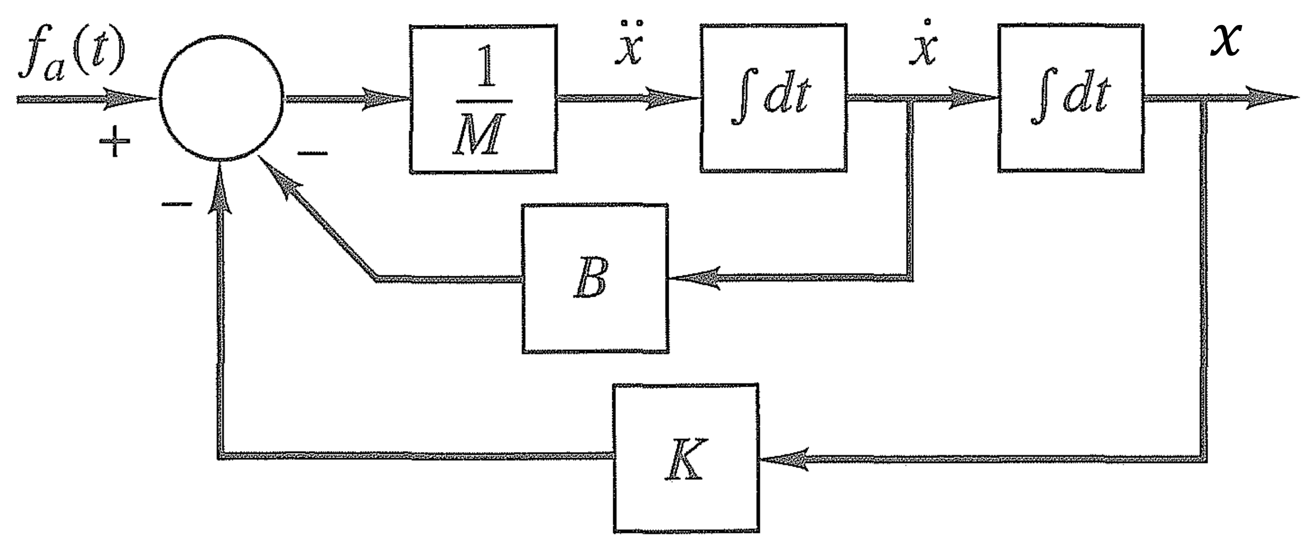
2) Solve for highest derivative of output variable

$$\ddot{x}(t) = \frac{1}{M}\left(-B\dot{x}(t) - Kx(t) + f_a(t)\right)$$

3) Create a sequence of output variables using integrator block to relate $\ddot{x}$, $\dot{x}$, and x to one another.

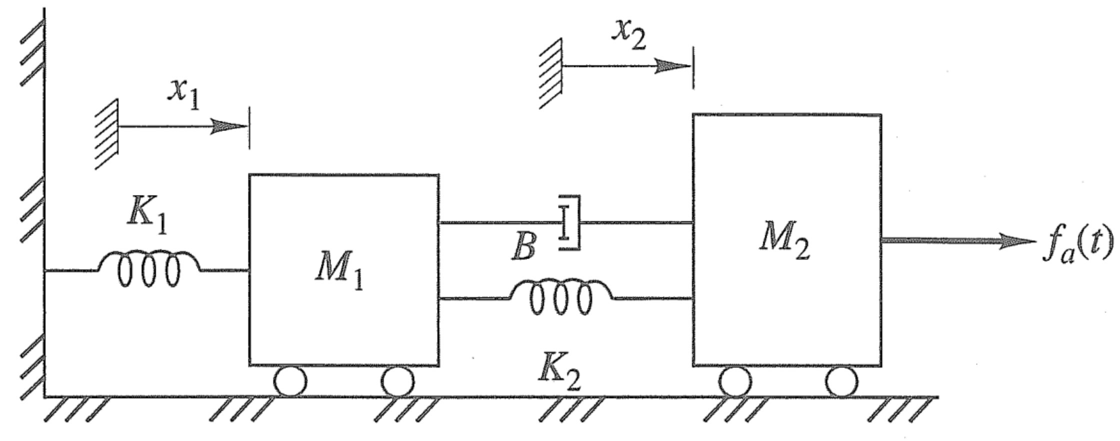


4) Complete the diagram with addition/subtraction and gain blocks according to the model equation



Block Diagram Models for Input-Output Equations

Example 8 Construct a block diagram model from the input-output equation for the following system. Assume $B = 0$ and take the output to be x_1.



Find the equation of motion for each mass:

$$\text{Mass } M_1 \rightarrow \quad -K_1 x_1 - B(\dot{x}_1 - \dot{x}_2) - K_2(x_1 - x_2) = M_1 \ddot{x}_1$$

$$\text{Mass } M_2 \rightarrow \quad f_a(t) - B(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) = M_2 \ddot{x}_2$$

Set $B = 0$ and rearrange the equations,

$$M_1 \ddot{x}_1 + (K_1 + K_2)x_1 - K_2 x_2 = 0 \qquad \text{Eqn. (1)} \rightarrow \quad x_2 = \frac{M_1 \ddot{x}_1 + (K_1 + K_2)x_1}{K_2}, \qquad \ddot{x}_2 = \frac{M_1 x_1^{(\mathrm{iv})} + (K_1 + K_2)\ddot{x}_1}{K_2}$$

$$M_2 \ddot{x}_2 - K_2 x_1 + K_2 x_2 = f_a(t) \qquad \text{Eqn. (2)}$$

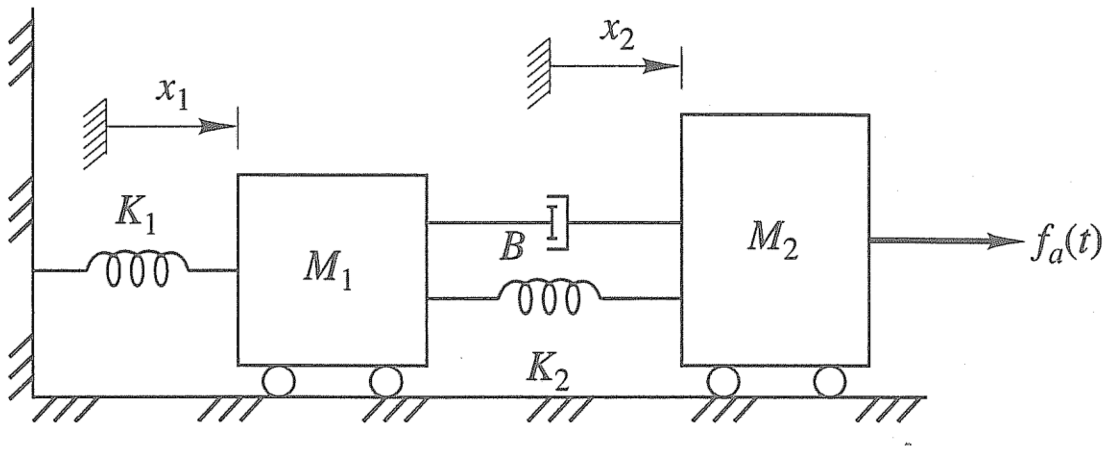
Find x_2 from equation (1) and substitute it in equation (2) and rearrange it:

$$M_1 M_2 x_1^{(\mathrm{iv})} + \big(M_1 K_2 + M_2(K_1 + K_2)\big)\ddot{x}_1 + K_1 K_2 x_1 = K_2 f_a(t)$$

$$x_1^{(\mathrm{iv})} = \frac{1}{M_1 M_2}\Big(-\big(M_1 K_2 + M_2(K_1 + K_2)\big)\ddot{x}_1 - K_1 K_2 x_1 + K_2 f_a(t)\Big)$$

Block Diagram Models for Input-Output Equations

Example 8 Construct a block diagram model from the input-output equation for the following system. Assume $B = 0$ and take the output to be x_1.



$$x_1^{(\mathrm{iv})} = \frac{1}{M_1 M_2}\Big(-\big(M_1K_2 + M_2(K_1+K_2)\big)\ddot{x}_1 - K_1K_2x_1 + K_2 f_a(t)\Big)$$

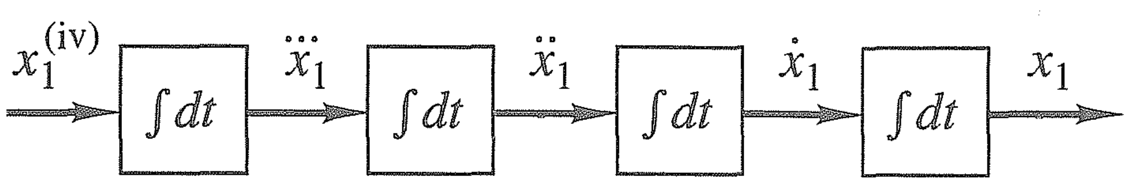
1) Identify the output variable $\rightarrow$ $x_1(t)$

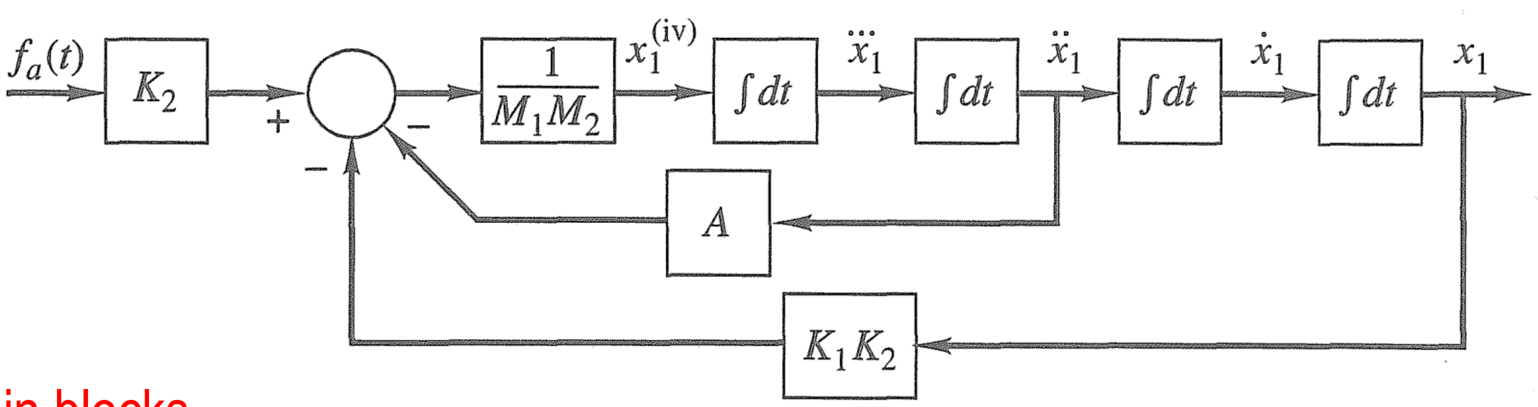
2) Solve for highest derivative of output variable. Define A as,

$$A = \big(M_1K_2 + M_2(K_1+K_2)\big)$$

$$x_1^{(\mathrm{iv})} = \frac{1}{M_1 M_2}\big(-A\ddot{x}_1 - K_1K_2x_1 + K_2 f_a(t)\big)$$

3) Create a sequence of output variables using integrator block.



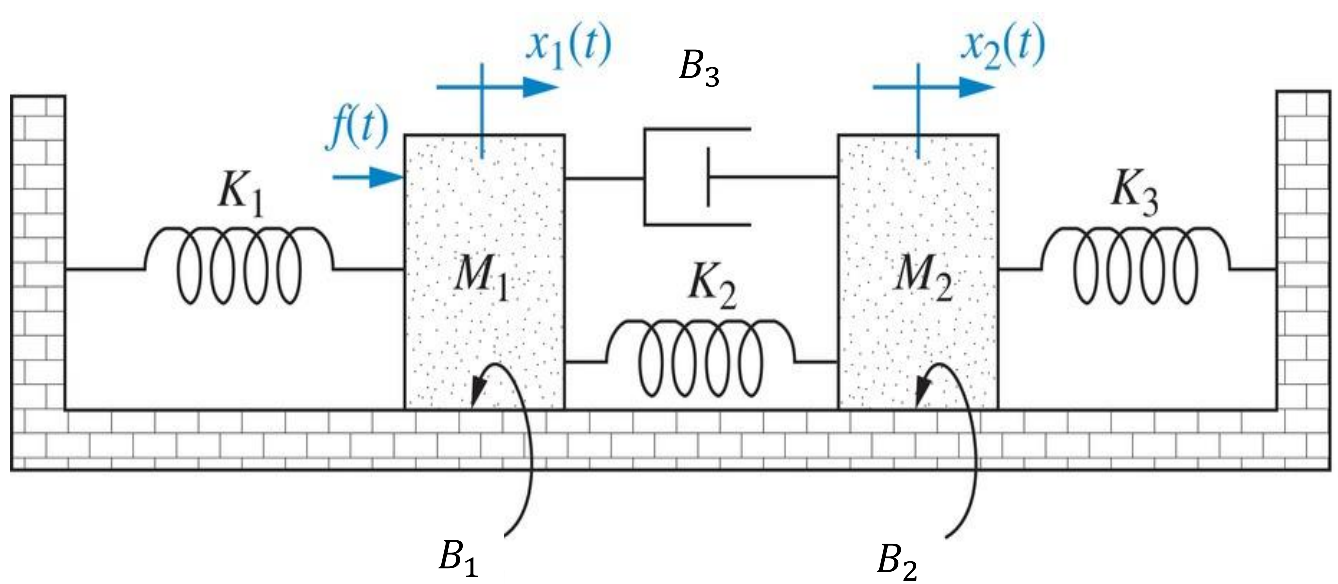


4) Complete the diagram with addition/subtraction and gain blocks.

Block Diagram Models for Original Model Equations

- If we have several free-body diagrams, then the overall model will probably consist of a set of coupled second-order equations.

- In this case, it is sensible to draw the block diagram directly from these coupled equations, rather than trying to first find an input-output model.

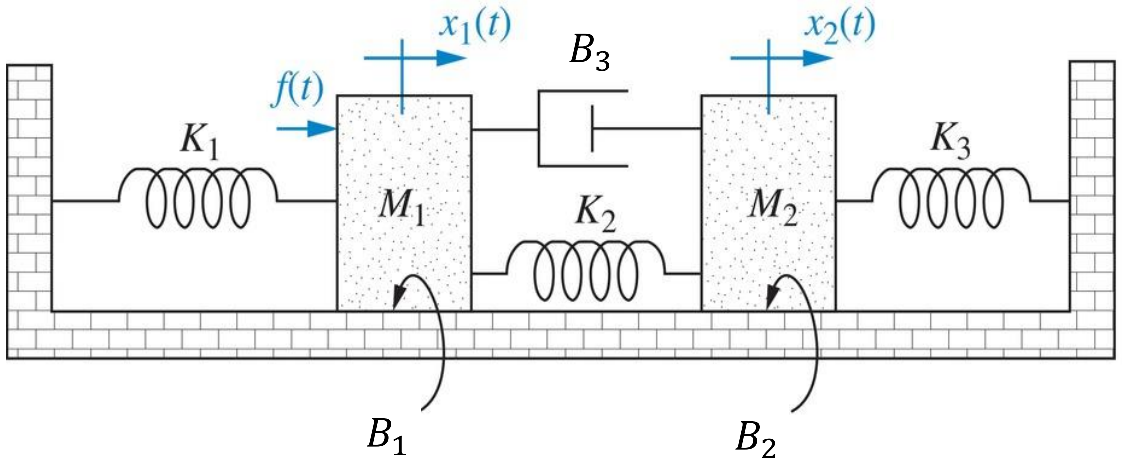


Block Diagram Models for Original Model Equations

Example 9 Construct a block diagram model for the following system, whose input-output model was given as below:

$$M_1\ddot{x}_1 + (B_1 + B_3)\dot{x}_1 + (K_1 + K_2)x_1 - B_3\dot{x}_2 - K_2x_2 = f(t)$$

$$M_2\ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + (K_2 + K_3)x_2 - B_3\dot{x}_1 - K_2x_1 = 0$$



1) Identify the output variables $\rightarrow$ $x_1(t)$ and $x_2(t)$

2) Solve for highest derivative of output variables

$$\ddot{x}_1 = \frac{1}{M_1}(f(t) - (B_1 + B_3)\dot{x}_1 - (K_1 + K_2)x_1 + B_3\dot{x}_2 + K_2x_2)$$

$$\ddot{x}_2 = \frac{1}{M_2}(-(B_2 + B_3)\dot{x}_2 - (K_2 + K_3)x_2 + B_3\dot{x}_1 + K_2x_1)$$

3) Create a sequence of output variables using integrator block to relate $\ddot{x}$, $\dot{x}$, and x to one another.

$$\ddot{x}_1 \rightarrow \boxed{\int dt} \xrightarrow{\dot{x}_1} \boxed{\int dt} \rightarrow x_1$$

$$\ddot{x}_2 \rightarrow \boxed{\int dt} \xrightarrow{\dot{x}_2} \boxed{\int dt} \rightarrow x_2$$

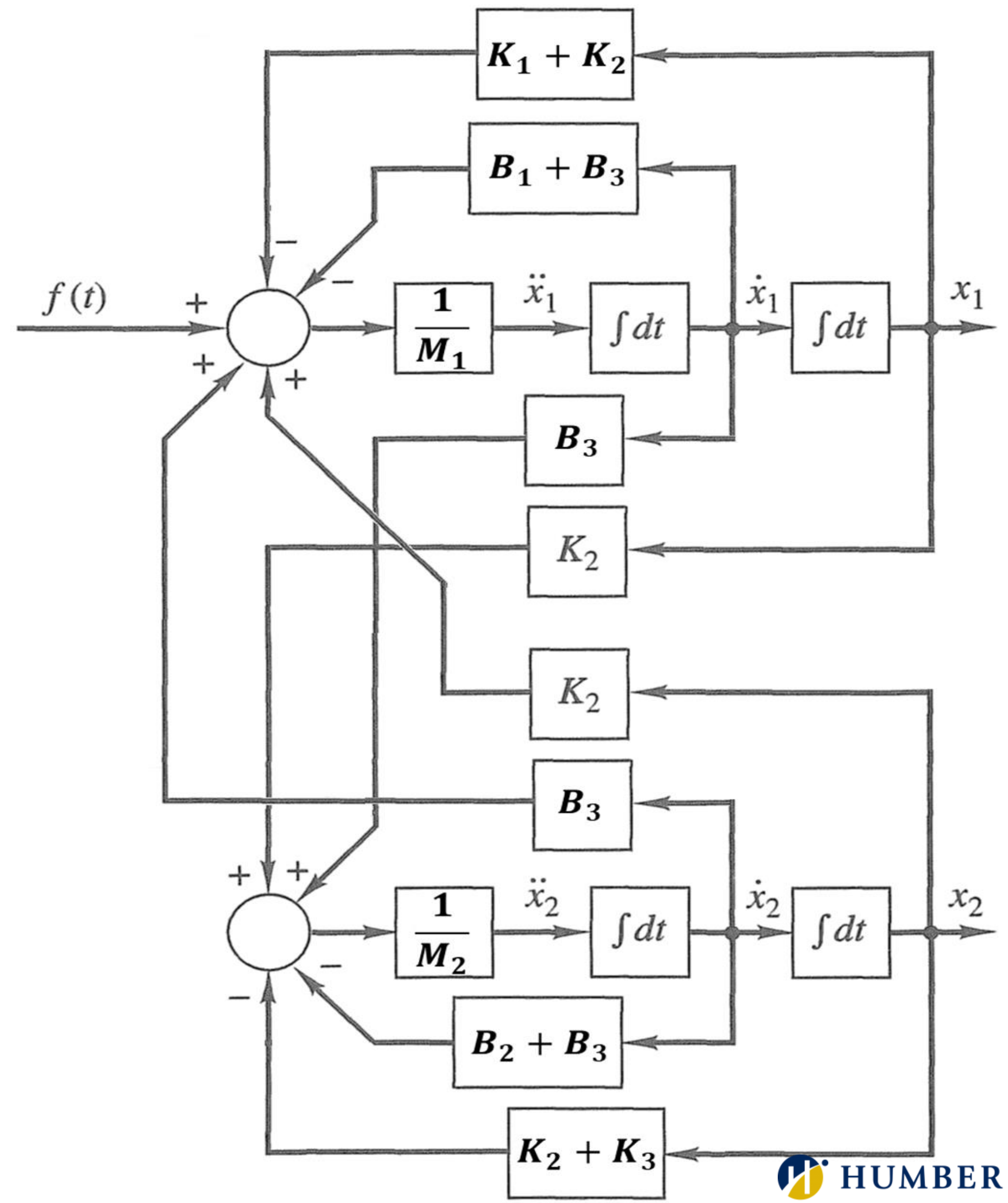
Block Diagram Models for Original Model Equations

Example 9 Construct a block diagram model for the following system, whose input-output model was given as below:

4) Complete the diagram with summer and gain blocks.

$$\ddot{x}_1 = \frac{1}{M_1}(f(t) - (B_1 + B_3)\dot{x}_1 - (K_1 + K_2)x_1 + B_3\dot{x}_2 + K_2 x_2)$$

$$\ddot{x}_2 = \frac{1}{M_2}(-(B_2 + B_3)\dot{x}_2 - (K_2 + K_3)x_2 + B_3\dot{x}_1 + K_2 x_1)$$



Block Diagram Models for Stace-Space Equations

- To construct a diagram for a state-variable model, we first draw an integrator for each state variable.
- The inputs to these integrators are the first derivatives of the corresponding state variables
- We then use the state-variable equations to form each of these derivatives in terms of state variables and inputs.

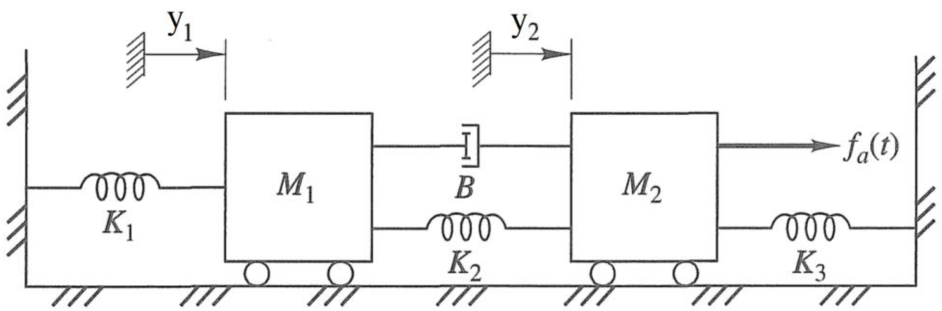
❑ Find the Block Diagrams Model for State-Variable Models

- Draw an integrator for each state variable.
- Assemble terms needed to create each state variable derivative. These will be functions of state variables and inputs.
- Assemble the blocks needed for any output equations, using the state variables and inputs required.

Block Diagram Models for Stace-Space Equations

Example 10 Consider the following translational mechanical system. The springs are undeflected when $y_1 = y_2 = 0$. The input is the applied force $f_a(t)$, and the system output is the velocity $v_1(t)$ of mass M_1.

Given the state-variables draw the block diagram model of the system.



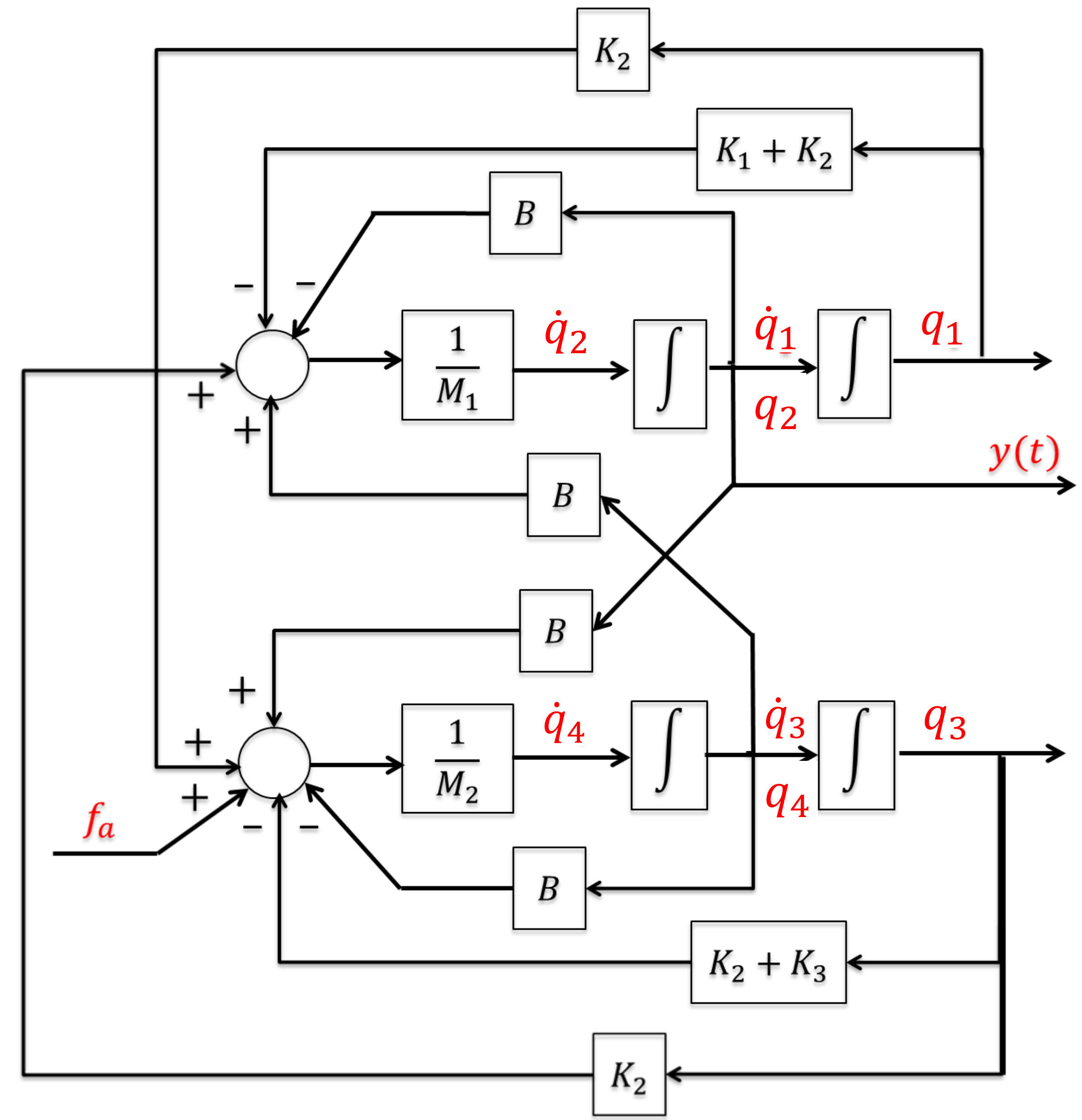
$$q_1 = y_1$$
$$q_2 = \dot{y}_1$$
$$q_3 = y_2$$
$$q_4 = \dot{y}_2$$

$$\begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = \dfrac{1}{M_1}\left(-Bq_2 - (K_1 + K_2)q_1 + Bq_4 + K_2 q_3\right) \\ \dot{q}_3(t) = q_4(t) \\ \dot{q}_4(t) = \dfrac{1}{M_2}\left(f_a(t) - Bq_4 - (K_2 + K_3)q_3 + Bq_2 + K_2 q_1\right) \\ y(t) = x_2(t) \end{cases}$$

1) Draw an integrator for each state variable

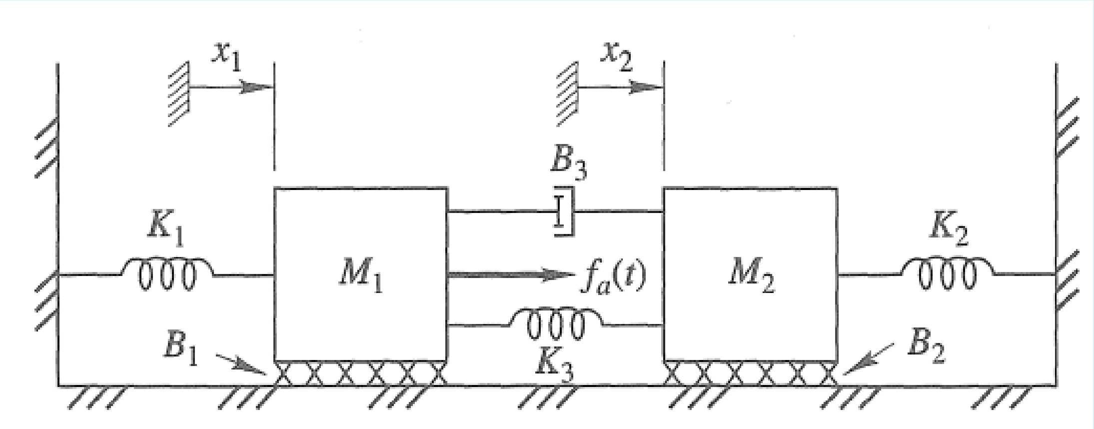
2) Assemble terms needed to create each state variable derivative

3) Assemble the blocks needed for any output equations



Quick Review

Draw a block diagram model to represent the state-variable of the system shown below from the given state-variable equations.



The state-variables are defined as:

$q_1 = x_1 \qquad q_2 = \dot{x}_1 \qquad q_3 = x_2 \qquad q_4 = \dot{x}_2$

The state-variable equations and output equation are obtained as:

$$
\begin{cases}
\dot{q}_1 = q_2 \\
\dot{q}_2 = \dfrac{1}{M_1}\left(f_a(t) - (K_1 + K_3)q_1 - (B_1 + B_3)q_2 + K_3 q_3 + B_3 q_4\right) \\
\dot{q}_3 = q_4 \\
\dot{q}_4 = \dfrac{1}{M_2}\left(K_3 q_1 + B_3 q_2 - (K_2 + K_3)q_3 - (B_2 + B_3)q_4\right) \\
y = x_2 \quad \rightarrow \quad y = q_3
\end{cases}
$$

THANK YOU

HUMBER

WE ARE HUMBER