HUMBER ENGINEERING

MENG 3510 – Control Systems

LECTURE 12

HUMBER

WE ARE HUMBER

LECTURE 12
Final Exam Review

- Plotting Bode Diagram & Nyquist Diagram
- Performance and Stability Analysis in Frequency Domain
- PI & PD Controller Design via Bode Diagram
- State-Space Analysis & State-Feedback Design
- Root Locus Plot & Gain Selection
- Lead & Lag Compensator Design via Root Locus
- PI & PD Controller Design from Time-Domain Specifications

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Example 1

Consider the following transfer function

$$G(s) = \frac{20(s+2)}{s(s+10)}$$

a) Determine the frequency response function $G(j\omega)$

The frequency response function is

$$G(s)\Big|_{s=j\omega} = G(j\omega) = \frac{20(j\omega+2)}{j\omega(j\omega+10)}$$

b) Sketch the Bode diagram of the $G(j\omega)$. (Determine the basic factors of $G(j\omega)$, find the corner frequencies (ω_c) and draw the asymptotic Bode diagram.)

First, convert the $G(j\omega)$ in the proper form

$$G(j\omega) = \frac{20(j\omega+2)}{j\omega(j\omega+10)} = \frac{40(j\frac{\omega}{2}+1)}{10j\omega(j\frac{\omega}{10}+1)} = \frac{4(j\frac{\omega}{2}+1)}{j\omega(j\frac{\omega}{10}+1)}$$

Next, find the basic factors of the $G(j\omega)$

$$\boldsymbol{G(j\omega) = (4)\left(j\frac{\omega}{2}+1\right)\left(\frac{1}{j\omega}\right)\left(\frac{1}{j\frac{\omega}{10}+1}\right)}$$

Constant Gain

Single Zero

First-order Integrator

Single Pole

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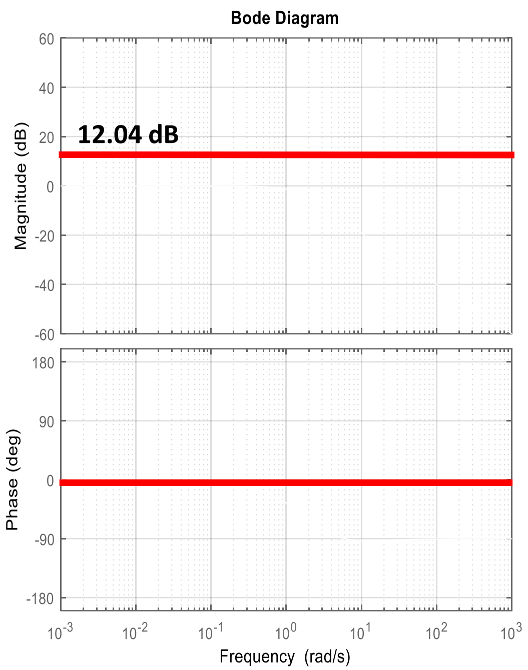
Example 1

$$|G(j\omega)|\text{dB} = 20\log(4) + 20\log\left(1 + j\frac{\omega}{2}\right) + 20\log\left(\frac{1}{j\omega}\right) + 20\log\left(\frac{1}{1 + j\frac{\omega}{10}}\right)$$

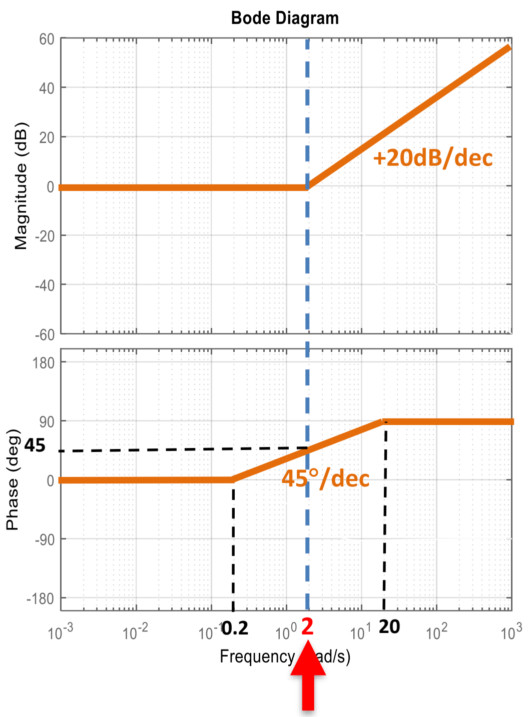
$$\angle G(j\omega) = \angle(4) + \angle\left(1 + j\frac{\omega}{2}\right) + \angle\left(\frac{1}{j\omega}\right) + \angle\left(\frac{1}{1 + j\frac{\omega}{10}}\right)$$

$$\boldsymbol{G(j\omega) = \frac{4(j\frac{\omega}{2} + 1)}{j\omega(j\frac{\omega}{10} + 1)}}$$

Constant Gain

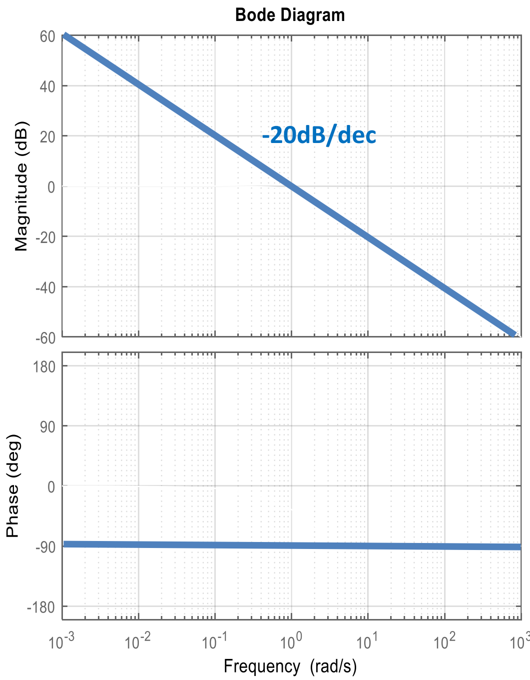


Single Zero

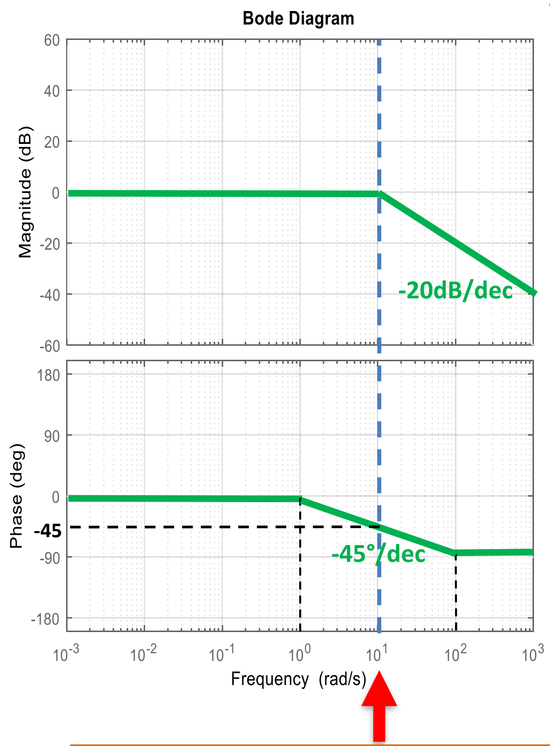


Corner frequency → ω = 2 rad/sec

First-order Integrator

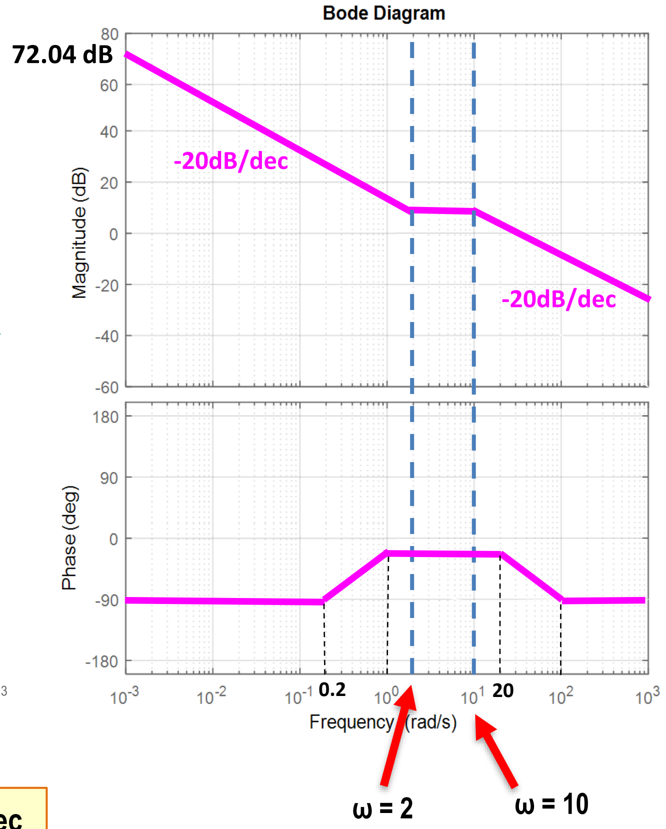


Single Pole



Corner frequency → ω = 10 rad/sec

The Overall Bode Plot



Starting Point: $20\log\left|\frac{K_B}{(j\omega)^\beta}\right| = 20\log\left|\frac{4}{(j0.001)^1}\right| = 20\log(4) - 20\log(0.001) = 12.04dB + 60\ dB = 72.04\ dB$

Starting Slope: $-20\beta\,\frac{dB}{dec} = -20(1)\ \frac{dB}{dec} = -20\ \ \frac{dB}{dec}$

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Example 1

Consider the following transfer function

$$G(s) = \frac{20(s+2)}{s(s+10)}$$

c) Sketch the Nyquist plot of the $G(j\omega)$ for both positive frequencies and negative frequencies.

The frequency response function is $\rightarrow$ $G(j\omega) = \dfrac{20(j\omega+2)}{j\omega(j\omega+10)}$

The magnitude and phase angle are obtained as below

$$\begin{cases} |G(j\omega)| = \left|\dfrac{20(j\omega+2)}{j\omega(j\omega+10)}\right| = \dfrac{20|j\omega+2|}{|j\omega||j\omega+10|} \\ \angle G(j\omega) = \angle\left(\dfrac{20(j\omega+2)}{j\omega(j\omega+10)}\right) = \tan^{-1}\left(\dfrac{\omega}{2}\right) - 90^\circ - \tan^{-1}\left(\dfrac{\omega}{10}\right) \end{cases}$$

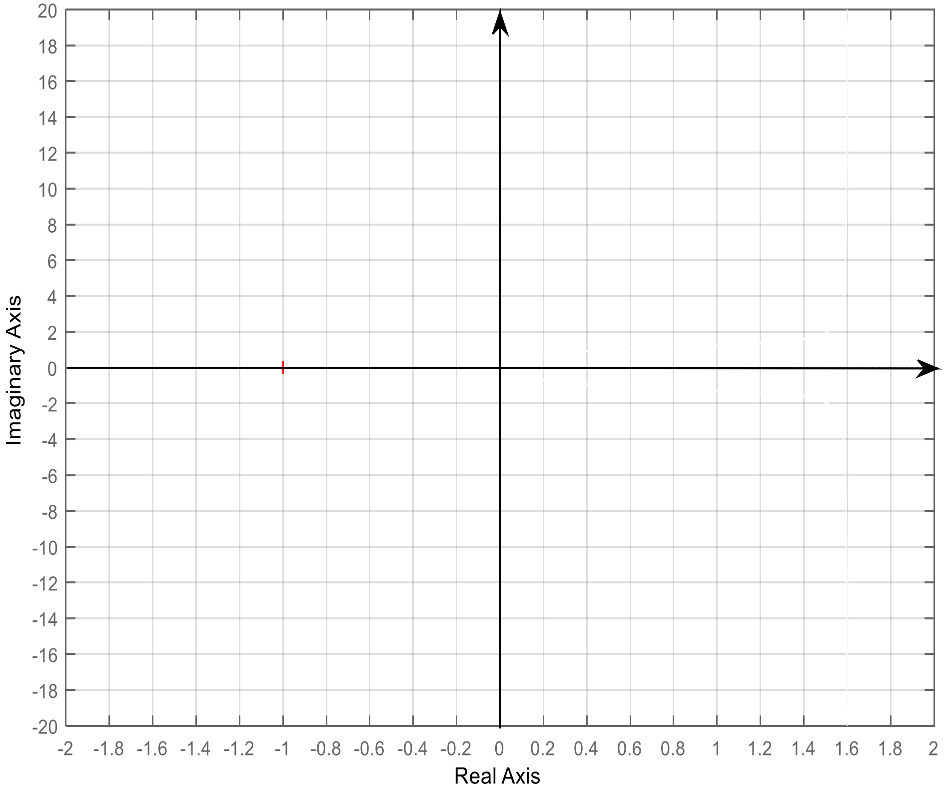
Determine starting point and ending point of the polar plot

Starting point $\rightarrow$ For $\omega \rightarrow 0^+ \Rightarrow G(j0) = \infty\angle -90^\circ$

Ending point $\rightarrow$ For $\omega \rightarrow +\infty \Rightarrow G(j\infty) = 0\angle -90^\circ$

For $\omega \rightarrow +\infty$ the graph is tangent to the negative imaginary axis.

Figure 1-2: Nyquist Diagram



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Example 1 Consider the following transfer function

$$G(s) = \frac{20(s+2)}{s(s+10)}$$

c) Sketch the Nyquist plot of the $G(j\omega)$ for both positive frequencies and negative frequencies.

The real part and the imaginary part of the $G(j\omega)$ are obtained as below

$$G(j\omega) = \frac{20(j\omega+2)}{j\omega(j\omega+10)} = \frac{20(j\omega+2)}{(-\omega^2+j10\omega)} \times \frac{(-\omega^2-j10\omega)}{(-\omega^2-j10\omega)} = \underbrace{\frac{\mathbf{160}}{\boldsymbol{\omega^2+100}}}_{\text{real part}} + j\underbrace{\frac{\mathbf{-20(20+\omega^2)}}{\boldsymbol{\omega(\omega^2+100)}}}_{\text{imaginary part}}$$

Find the intersection of the Polar plot with the real and imaginary axes

$$\mathrm{Re}[G(j\omega)] = 0 \quad \rightarrow \quad \frac{160}{\omega^2+100} = 0 \quad \rightarrow \quad \omega = \infty$$

$$\mathrm{Im}[G(j\omega)] = 0 \quad \rightarrow \quad \frac{-20(20+\omega^2)}{\omega(\omega^2+100)} = 0 \quad \rightarrow \quad \omega = \infty$$

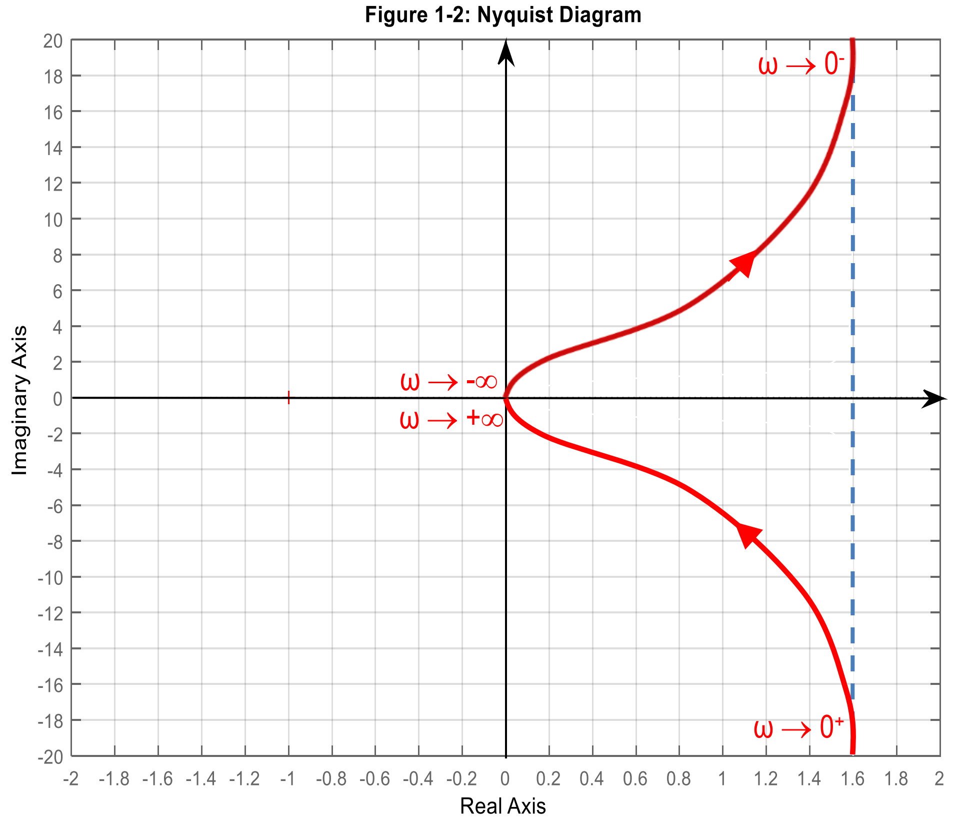
The Polar plot intersects the real axis and the imaginary axis only at the origin.

Intersection of the asymptote line with the real axis

$$\boxed{\boldsymbol{\alpha = \mathrm{Re}[G(j\omega)]\Big|_{\omega=0}}}$$

$$Re[G(j0^+)] = \frac{160}{(0)^2+100} = \mathbf{1.6}$$

For $\omega \rightarrow 0^+$ the graph is tangent to the line of $Re[G(j0^+)] = 1.6$



The Nyquist plot for negative frequency is mirror image of the positive frequency part with respect to the real axis.

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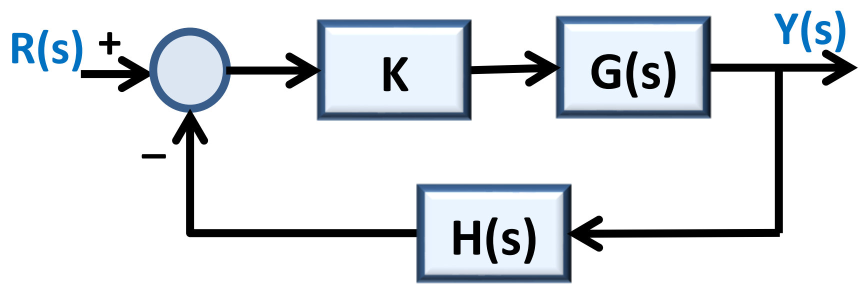
Example 2

Given the open-loop system, $KG(s)H(s)$, Bode diagram



a) Find the gain crossover frequency (ω_g), phase crossover frequency (ω_p), Gain margin (*GM*) and Phase margin (*PM*). Mark them on the Bode diagram

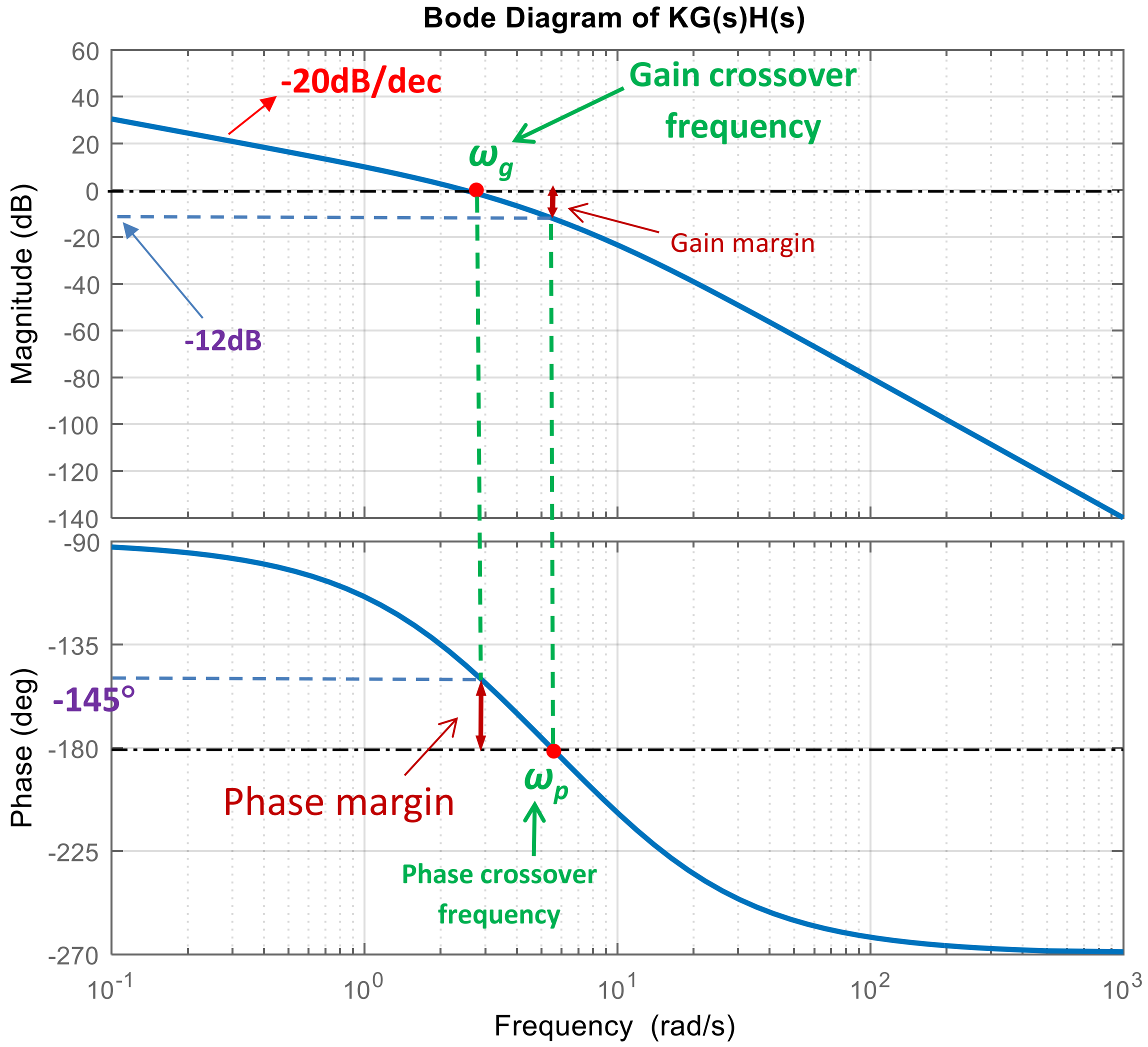
From the Bode plot the crossover frequencies can be determined as

$$\boxed{\omega_g \approx 2.5\,rad/s} \qquad \boxed{\omega_p \approx 5.5\,rad/s}$$

The gain margin and phase margin are obtained as

$$\mathbf{GM = 0dB - (-12dB) = 12dB}$$

$$\mathbf{PM = 180^{\circ} - 145^{\circ} = 35^{\circ}}$$



b) Determine stability of the closed-loop system based on the Gain margin and Phase margin values.

Since, PM > 0 and GM > 0, the closed-loop system is stable.

c) Determine Type of the open-loop system using the Bode plot.

Since, the slope of the log magnitude plot at low frequencies starts with -20dB/dec, the open-loop transfer function is **Type 1**.

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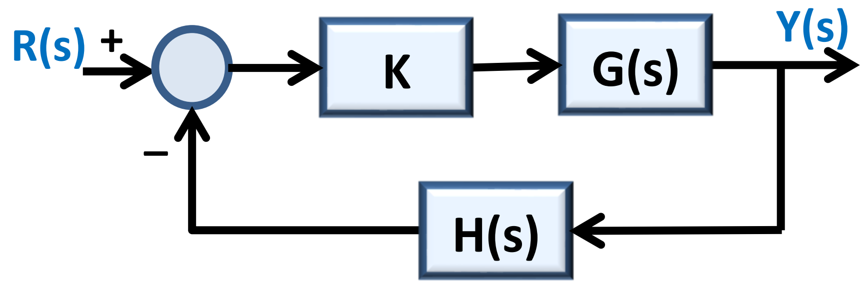
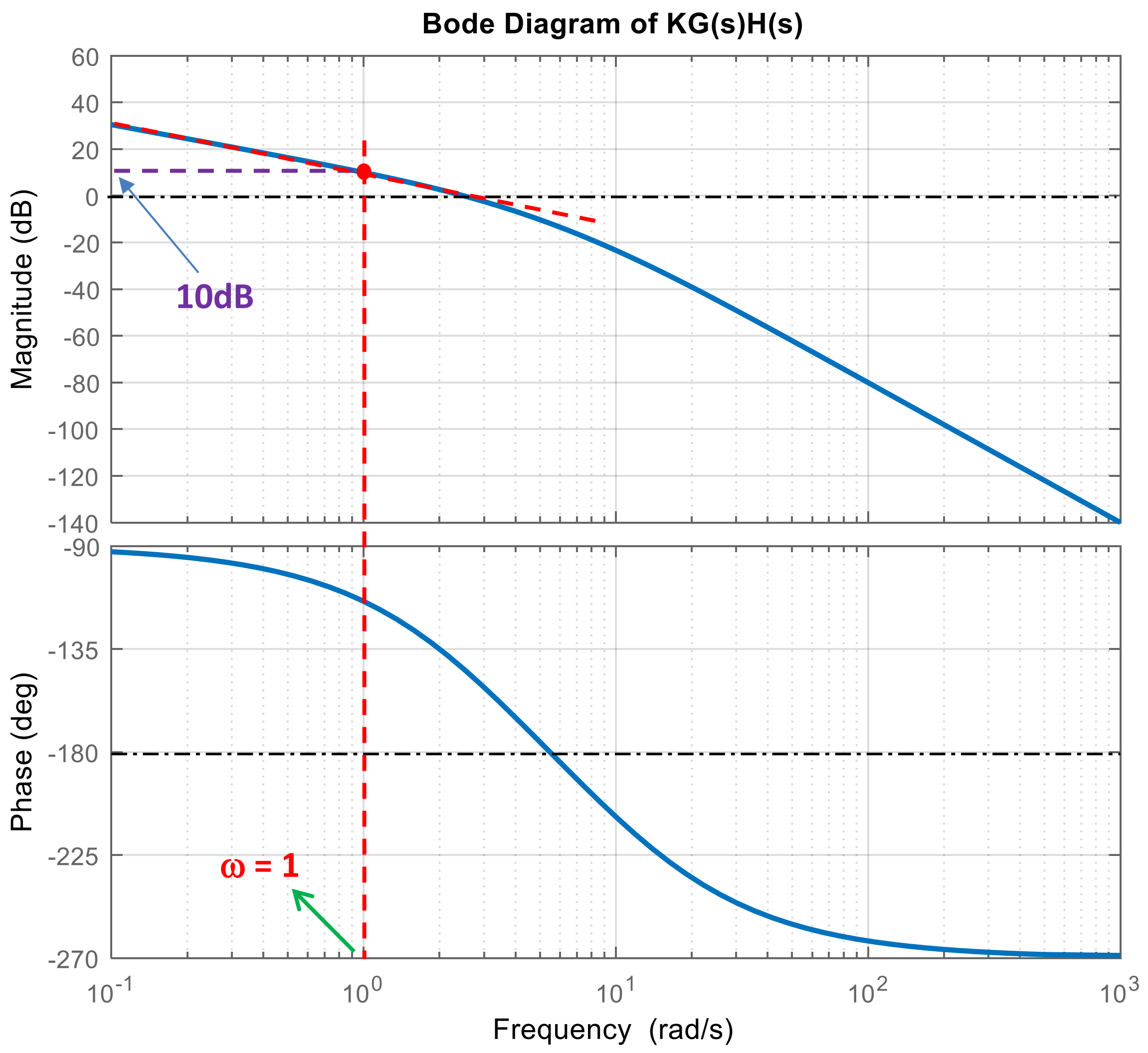
Example 2 Given the open-loop system, $KG(s)H(s)$, Bode diagram



d) Find the ramp-error constant (k_v) by using the Bode plot and calculate steady-state error (e_{ss}) of the closed-loop system for unit-ramp input.



Find the intersection of low frequency asymptote with line $\omega = 1$

$$\mathbf{20\log(k_v) = 10dB}$$

$$k_v = 10^{10/20} \rightarrow \quad \boldsymbol{k_v = 3.16}$$ Ramp-error constant

$$e_{ss} = \frac{1}{k_v} \quad \rightarrow \quad e_{ss} = \frac{1}{3.16} = 0.32$$

$$\boxed{\boldsymbol{e_{ss} = 0.32}}$$

Steady-state error

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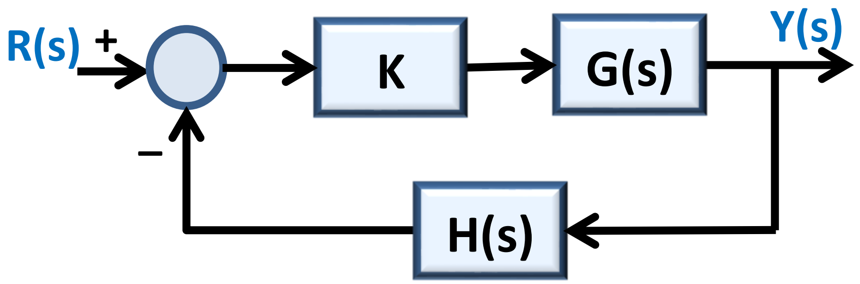
Example 3 Consider the following closed-loop system and the given open-loop Nyquist diagram



If the open-loop transfer function has TWO poles on the right-half s-plane. Determine stability of the closed-loop system and number of closed-loop poles on the right-half s-plane (if any) by using the Nyquist stability criteria.

$$\boxed{\boldsymbol{Z = N + P}}$$

The open-loop system has two unstable poles

$$\boxed{P = 2}$$

Two counterclockwise encirclement of the point -1

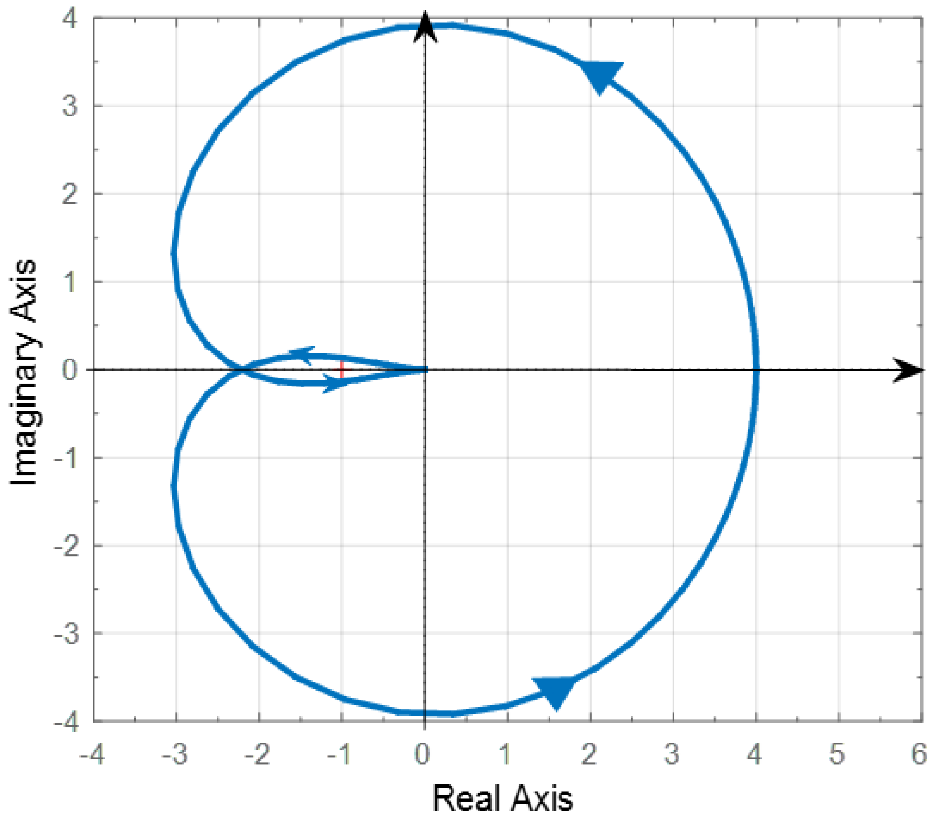
$$\boxed{N = -2}$$

From the Nyquist stability criteria

$$\boxed{Z = N + P = 0}$$

The closed-loop system is stable, and it has no poles on the right-half s-plane.

Figure 3: Nyquist Diagram



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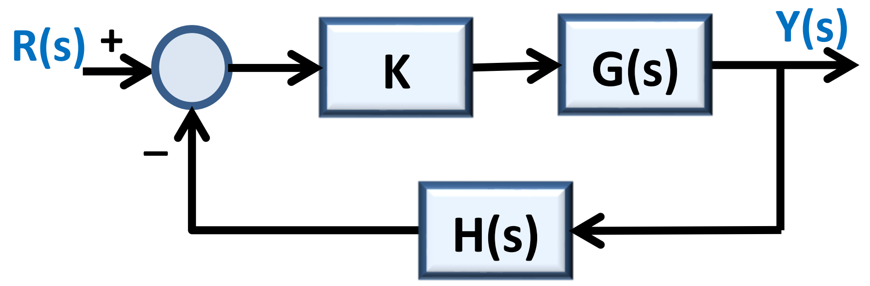
Example 4 Consider the following closed-loop system and the given open-loop Nyquist diagram



If the open-loop transfer function has NO poles on the right-half s-plane. Determine stability of the closed-loop system and number of closed-loop poles on the right-half s-plane (if any) by using the Nyquist stability criteria.

$$\boxed{Z = N + P}$$

First, close the Nyquist plot from $\omega = 0^-$ to $\omega = 0^+$ in clockwise direction.

No unstable poles for open-loop system

$$\boxed{P = 0}$$

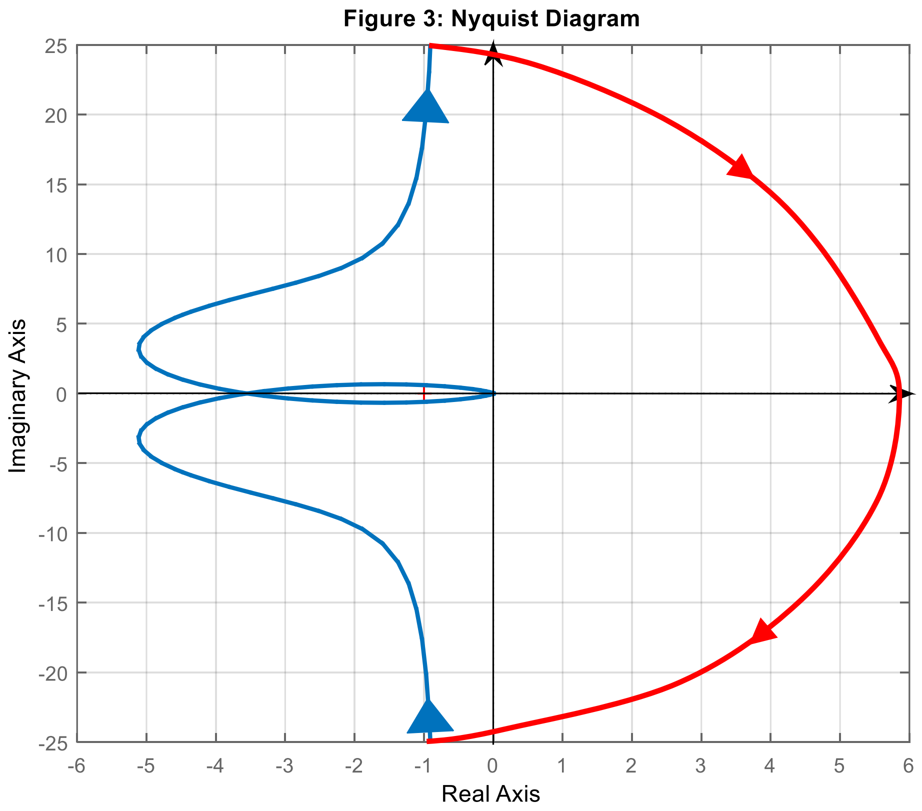
Two clockwise encirclement of the point -1

$$\boxed{N = 2}$$

From the Nyquist stability criteria

$$\boxed{Z = N + P = 2}$$

The closed-loop system is unstable, and it has two poles on the right-half s-plane.



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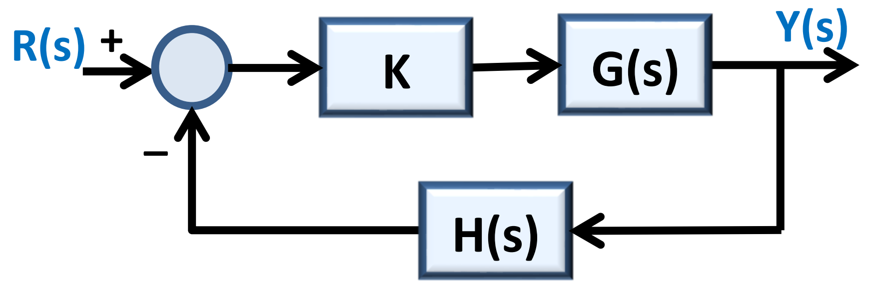
Example 5 Given the open-loop system, $KG(s)H(s)$, Bode diagram



a) Find the gain crossover frequency (ω_g), phase crossover frequency (ω_p), Gain margin (*GM*) and Phase margin (*PM*). Mark them on the Bode diagram

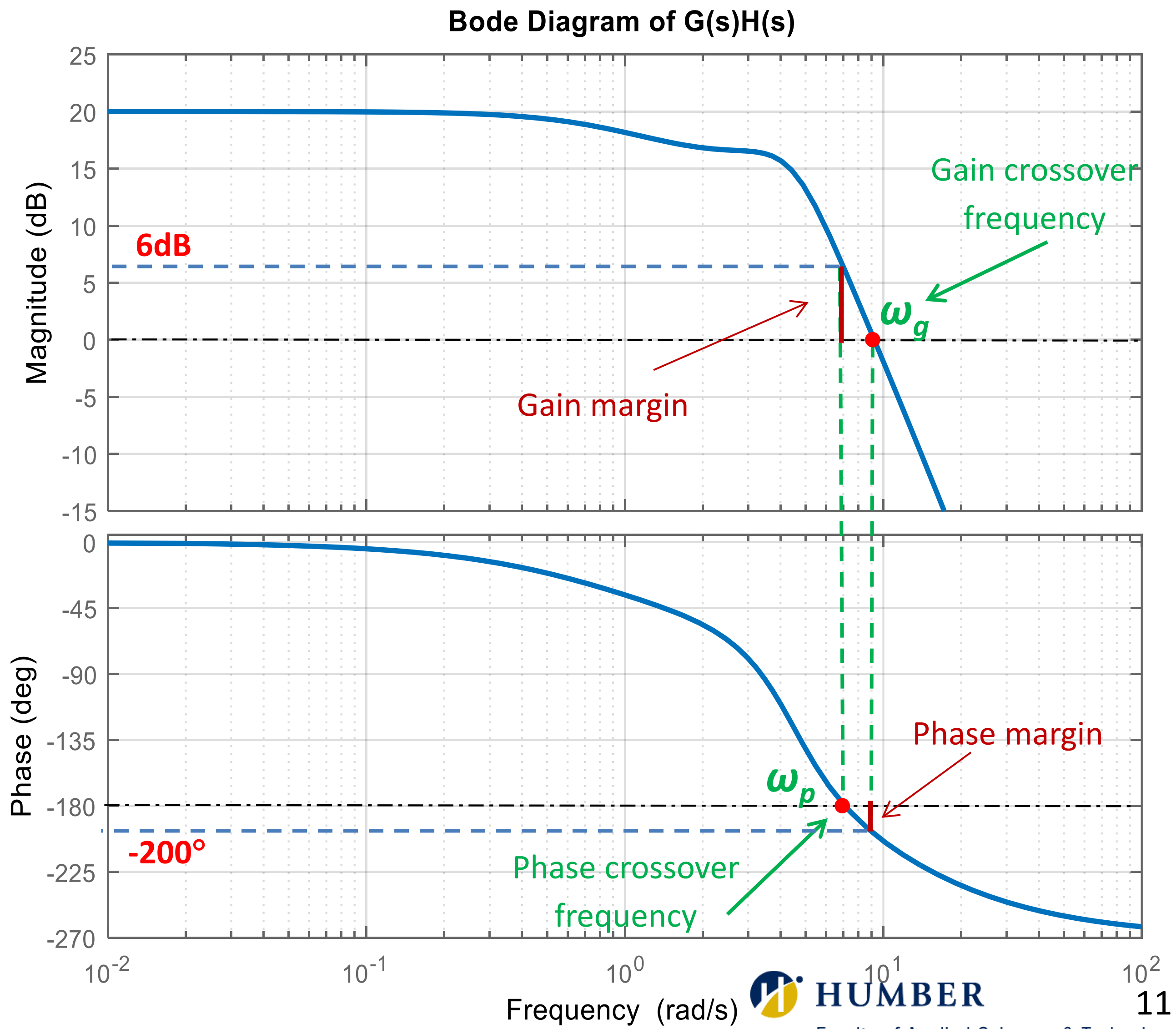
From the Bode plot the crossover frequencies can be determined as below

$$\boxed{\boldsymbol{\omega_g \approx 9\ \mathrm{rad/s}} \quad \text{and} \quad \boldsymbol{\omega_p \approx 7\ \mathrm{rad/s}}}$$

The gain margin and phase margin are obtained as follows

$$\mathrm{GM} \approx 0\mathrm{dB} - (+6\mathrm{dB}) = -6\mathrm{dB}$$

$$\mathrm{PM} \approx 180^\circ + (-200^\circ) = -20^\circ$$



b) Determine stability of the closed-loop system based on the Gain margin and Phase margin values.

Since, PM < 0 and GM < 0, the closed-loop system is unstable.

c) Determine type of the open-loop system using the Bode plot.

Since, the slope of the log magnitude plot at low frequencies starts with 0dB/dec, the open-loop transfer function is **Type 0**.

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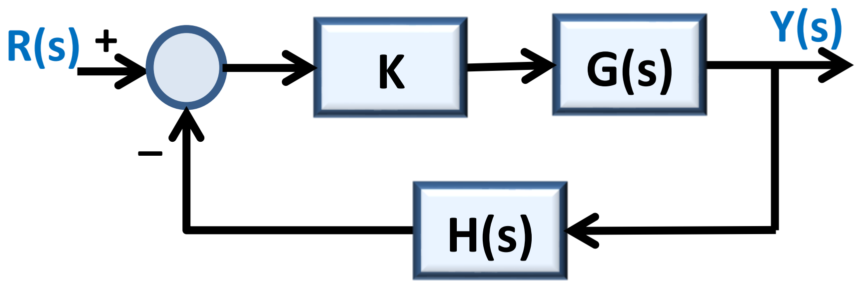
Example 5 Given the open-loop system, $KG(s)H(s)$, Bode diagram



d) Find the step-error constant (k_p) by using the Bode plot and calculate steady-state error (e_{ss}) of the closed-loop system for unit-step input.

Find the magnitude at low frequencies:

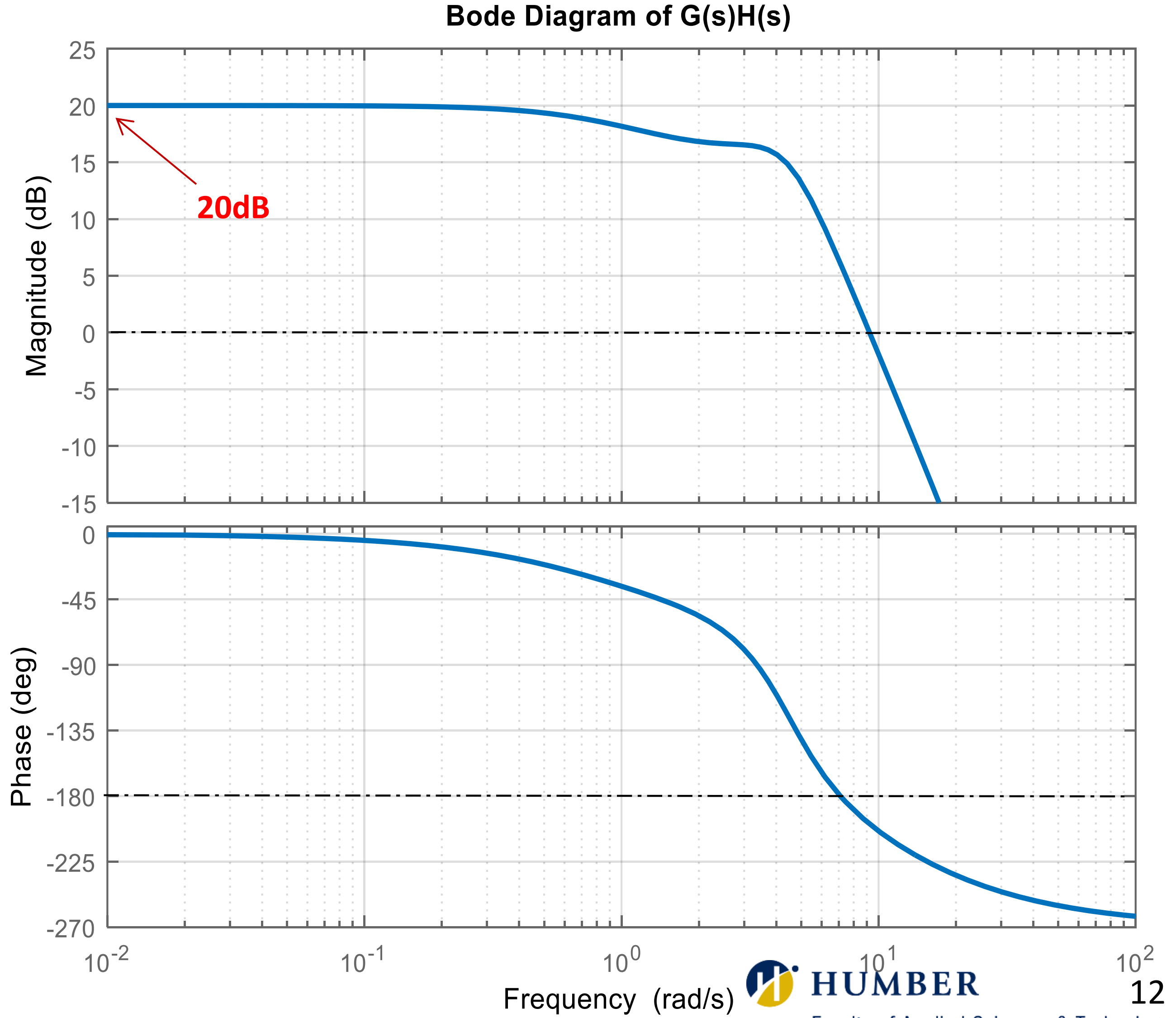
$$20\log(k_p) = 20\text{dB}$$

$k_p = 10^{20/20} \rightarrow$ $\boldsymbol{k_p = 10}$ Step-error constant

$$e_{ss} = \frac{1}{1+k_p} \rightarrow e_{ss} = \frac{1}{1+10} = 0.091$$

$$\boxed{e_{ss} = 9.1\%}$$

Steady-state error



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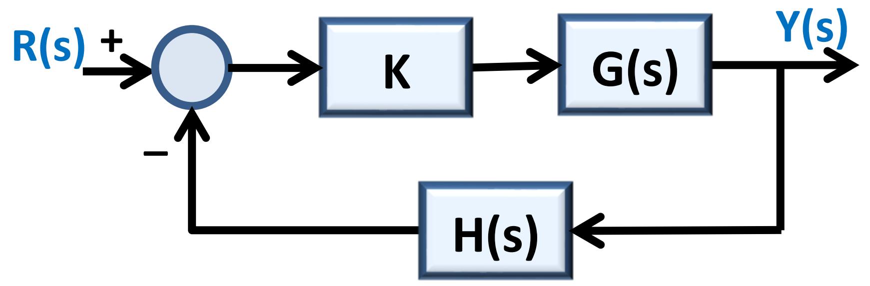
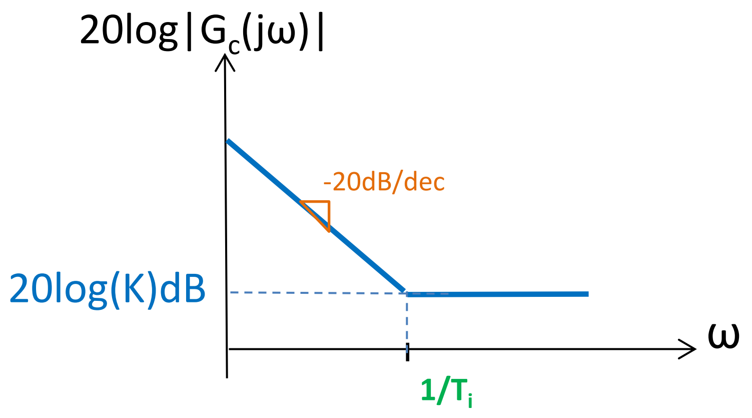
Example 5 Given the open-loop system, $KG(s)H(s)$, Bode diagram

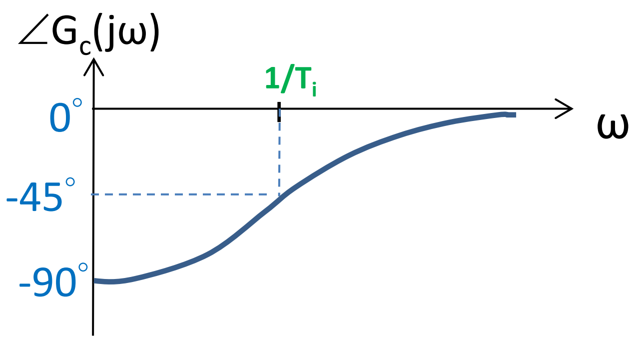


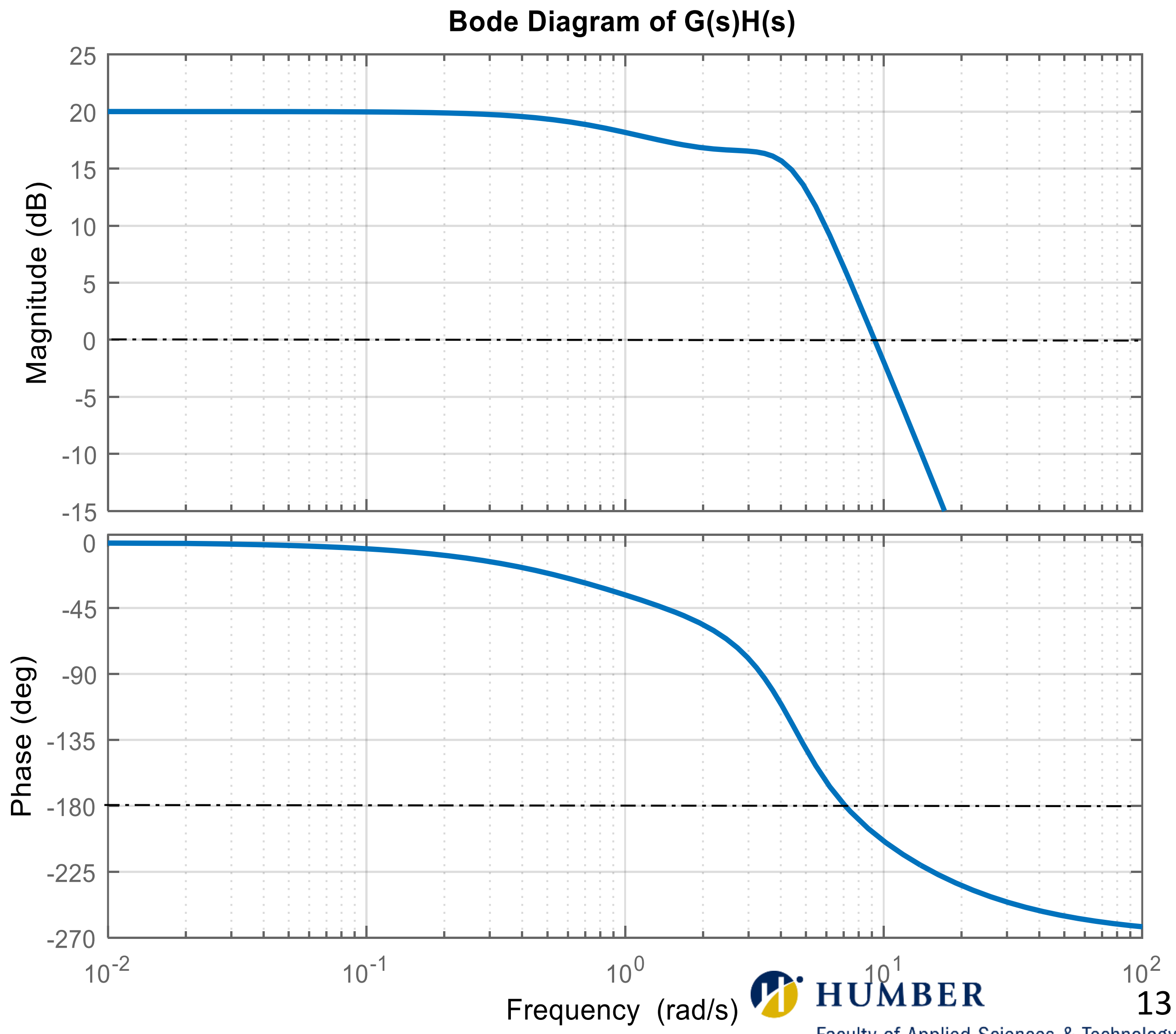
d) Design a PI controller to eliminate the steady state error of the closed-loop system ($e_{ss} = 0$), and to achieve the $PM = 60^\circ$ and $GM > 10dB$.

$$G_c(s) = K_P\left(1 + \frac{1}{T_i s}\right)$$



$$\boxed{\omega_z = \frac{1}{T_i}}$$





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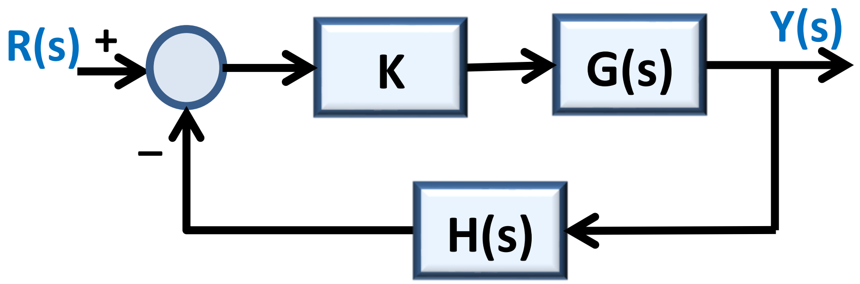
Example 5 Given the open-loop system, $KG(s)H(s)$, Bode diagram



d) Design a PI controller to eliminate the steady state error of the closed-loop system ($e_{ss} = 0$), and to achieve the $PM = 60^{\circ}$ and $GM > 10dB$.

$$G_c(s) = K_P\left(1 + \frac{1}{T_i s}\right)$$

Step 1: Plot Bode diagram of the open-loop system $KG(s)H(s)$, and find **PM** and **GM**

$\boxed{\mathrm{GM} = -6\mathrm{dB}}$ $\boxed{\mathrm{PM} = -20^{\circ}}$

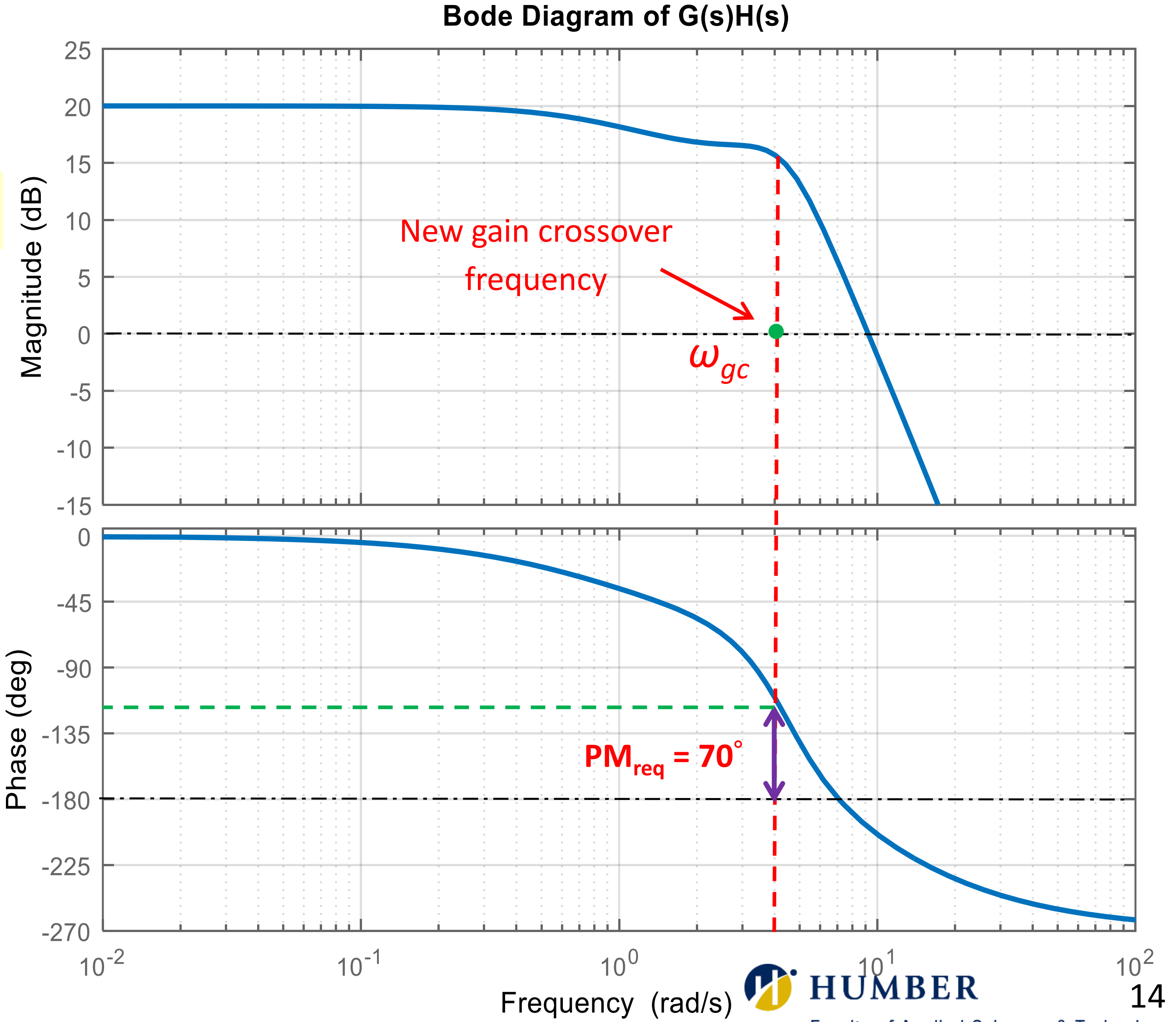
Step 2: Find the required phase margin, PM_{req}

$$PM_{req} = PM_d + 10^{\circ} \quad \rightarrow \quad PM_{req} = 60^{\circ} + 10^{\circ} = 70^{\circ}$$

Step 3: Determine the frequency on the Bode diagram to achieve the required phase margin PM_{req}. Select this frequency as the new gain crossover frequency, ω_{gc}

This frequency is selected as the new gain crossover frequency, ω_{gc} :

From the Bode diagram $\rightarrow$ $\boxed{\omega_{gc} \approx 4\ \mathrm{rad/s}}$



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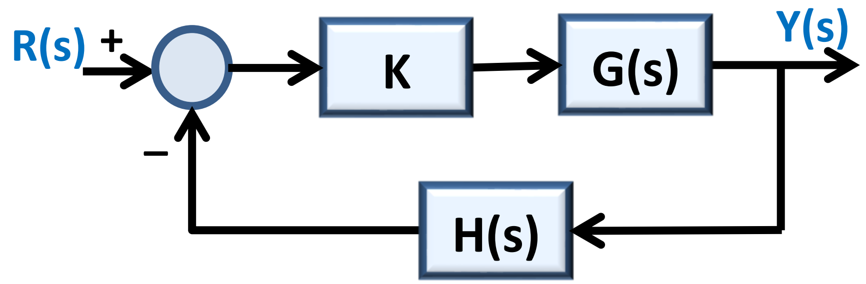
Example 5 Given the open-loop system, $KG(s)H(s)$, Bode diagram



d) Design a PI controller to eliminate the steady state error of the closed-loop system ($e_{ss} = 0$), and to achieve the $PM = 60^{\circ}$ and $GM > 10dB$.

$$G_c(s) = K_P\left(1 + \frac{1}{T_i s}\right)$$

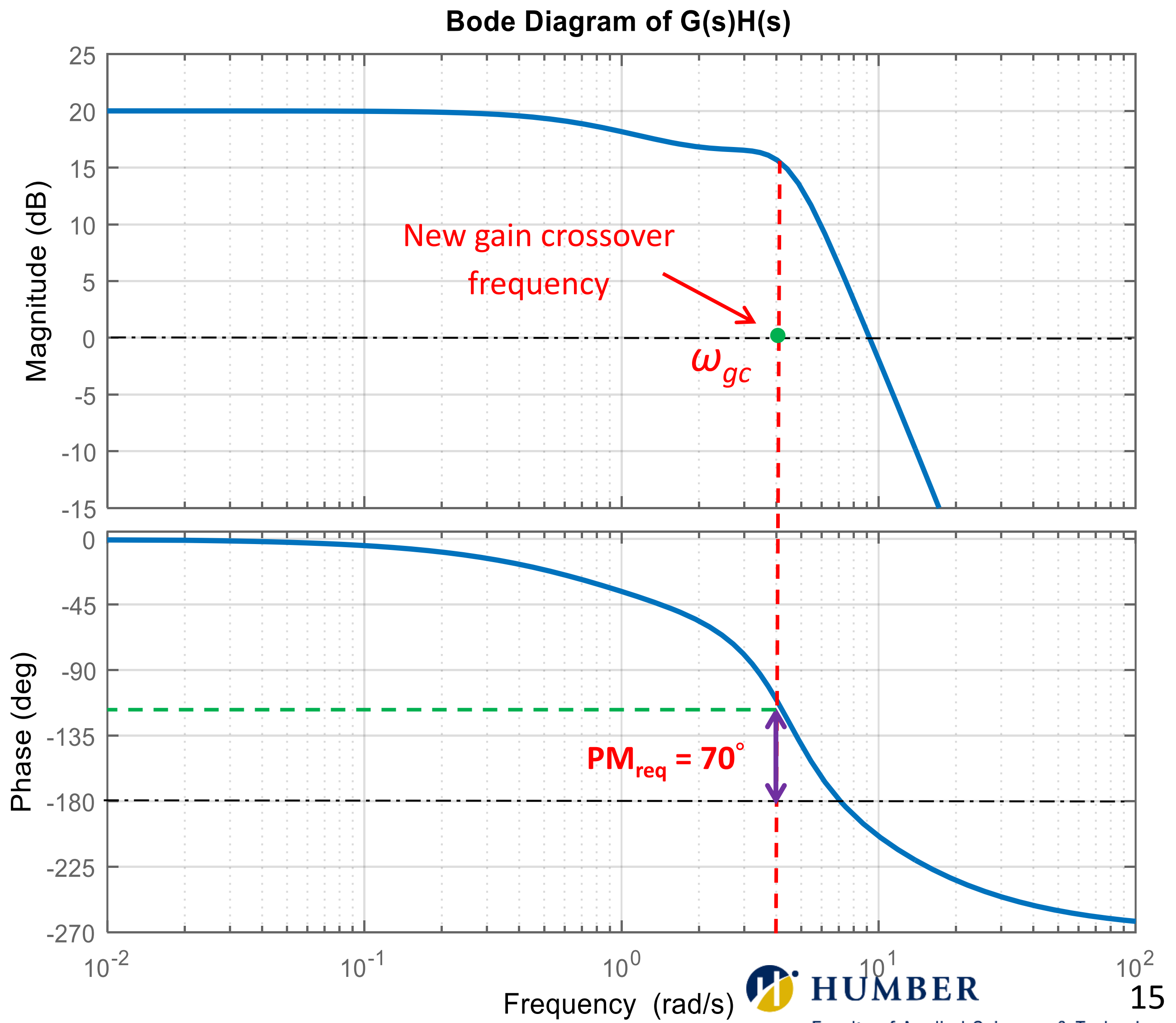
Step 4: Find the corner frequencies of zero for PI controller

- The corner frequency of the zero is selected one decade below the new gain crossover frequency, ω_{gc}.

$$\omega_z = 0.1\omega_{gc} = 0.1 \times 4 \rightarrow \boxed{\omega_z = 0.4 \text{ rad/s}}$$

Step 5: Select the integral time constant T_i

$$\omega_z = \frac{1}{T_i} \rightarrow \boxed{T_i = \frac{1}{0.4} = 2.5}$$



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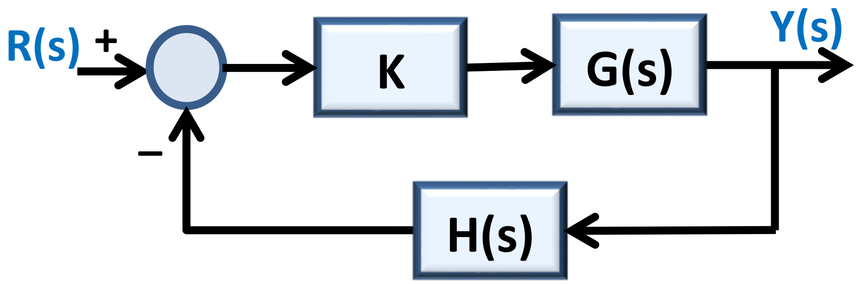
Example 5

Given the open-loop system, $KG(s)H(s)$, Bode diagram



d) Design a PI controller to eliminate the steady state error of the closed-loop system ($e_{ss} = 0$), and to achieve the $PM = 60^{\circ}$ and $GM > 10dB$.

$$G_c(s) = K_P\left(1 + \frac{1}{T_i s}\right)$$

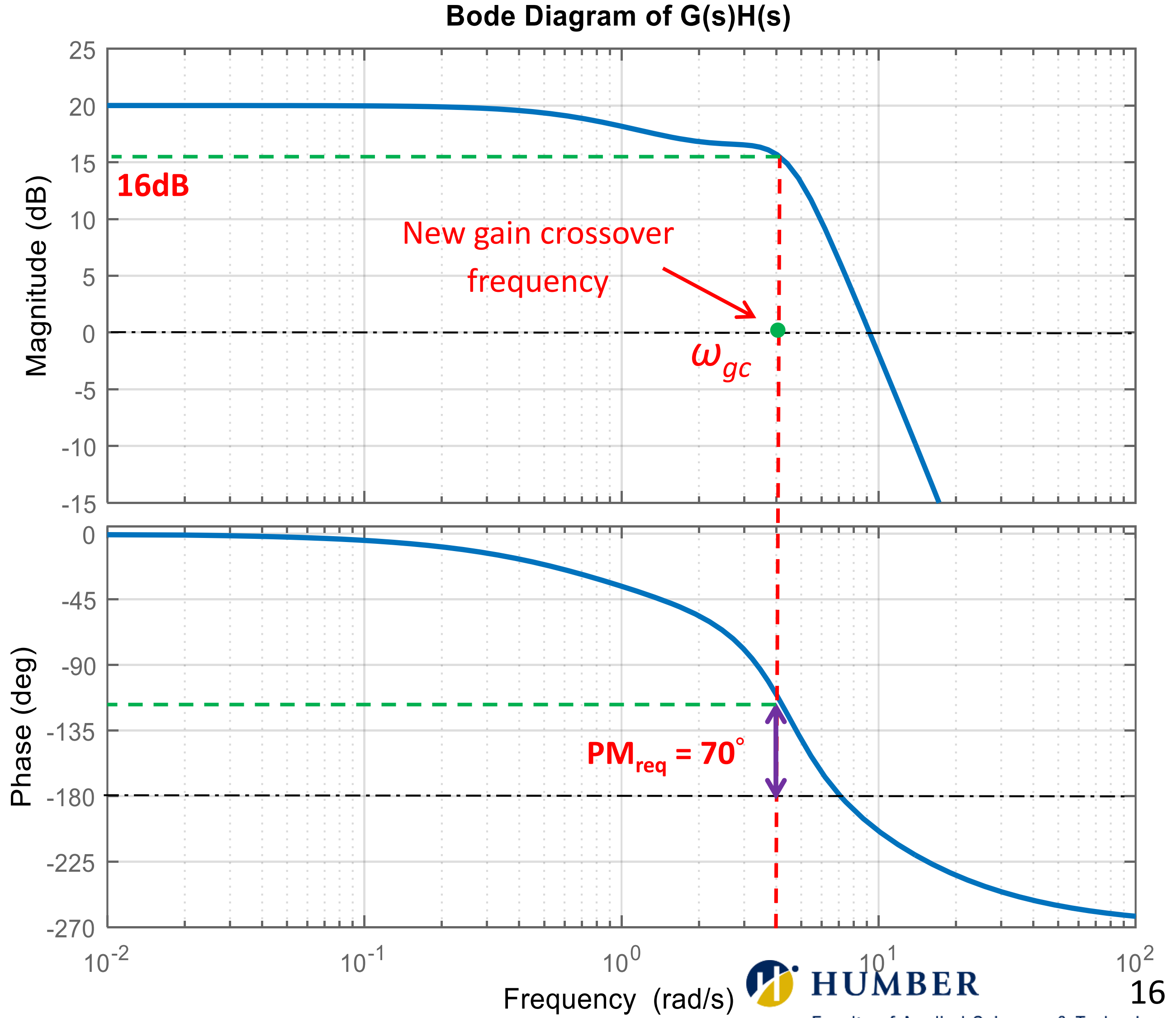
Step 6: Select the proportional gain K_p to bring down the magnitude plot to 0dB at the new crossover frequency ω_{gc}.

From the Bode plot the magnitude at the new gain crossover point, $\omega_{gc} = 4\ \text{rad/s}$, can be determine as 16dB

$$20\log_{10}(K_p) = -16\text{dB} \quad \rightarrow \quad K_p = 10^{-\frac{16}{20}} \quad \rightarrow \quad \boldsymbol{K_p = 0.16}$$

$$\boxed{G_c(s) = 0.16\left(1 + \frac{1}{2.5s}\right)}$$

PI Controller



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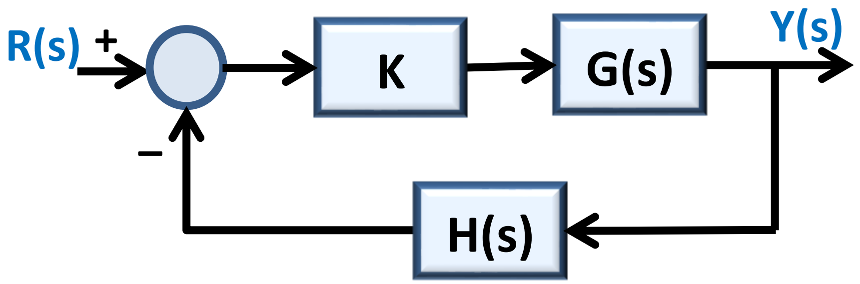
Example 6

Given the open-loop system, $KG(s)H(s)$, Bode diagram

$$K = 10, \qquad G(s) = \frac{1}{s(s+1)}\,, \qquad H(s) = 1$$



a) Given Bode diagram of the open-loop system, determine the Gain margin and the Phase margin of the open-loop system.

The gain crossover frequency ω_g is obtained from the Bode plot

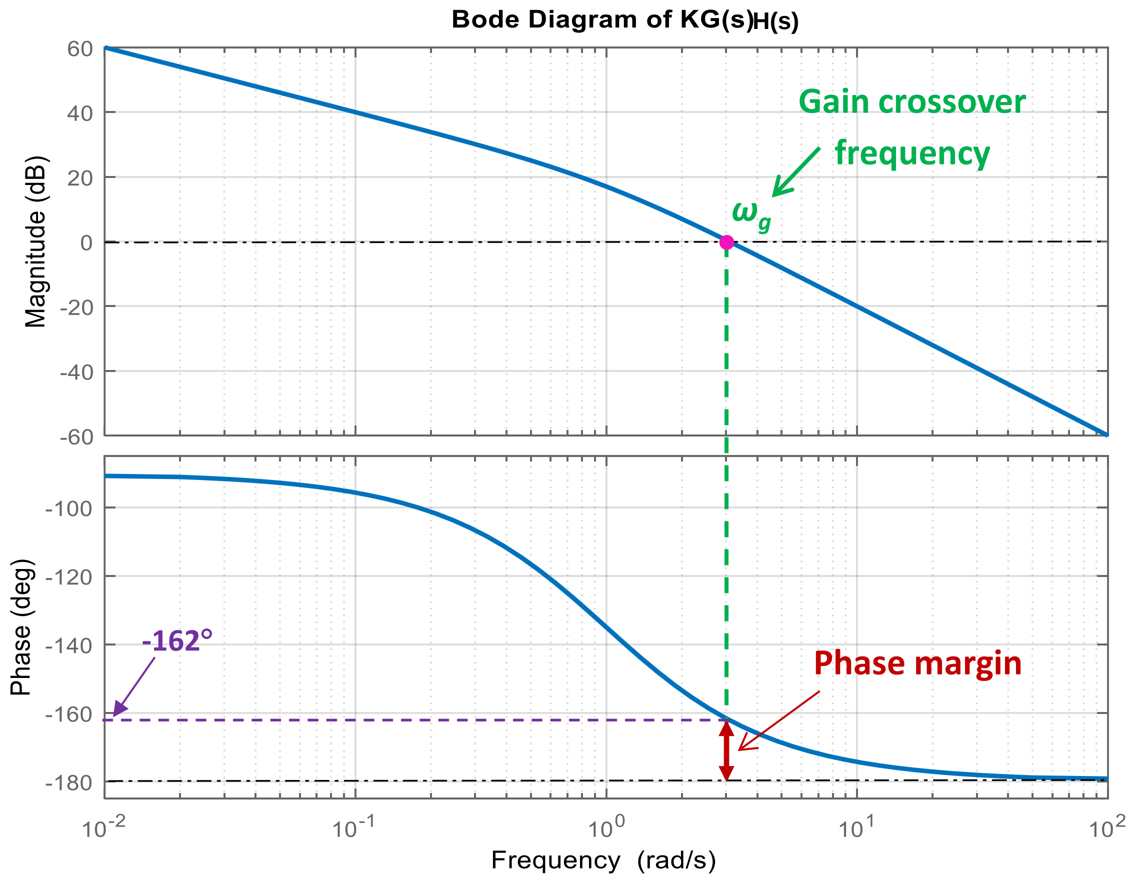
$$\boxed{\omega_g \approx 3\, rad/s}$$

The phase margin is determined as:

$$\boxed{\mathrm{PM} = 180^{\circ} - 162^{\circ} = 18^{\circ}}$$

Since, the phase plot approaches to -180° at $\omega \to \infty$, the gain margin will be

$$\boxed{\mathrm{GM} = +\infty}$$



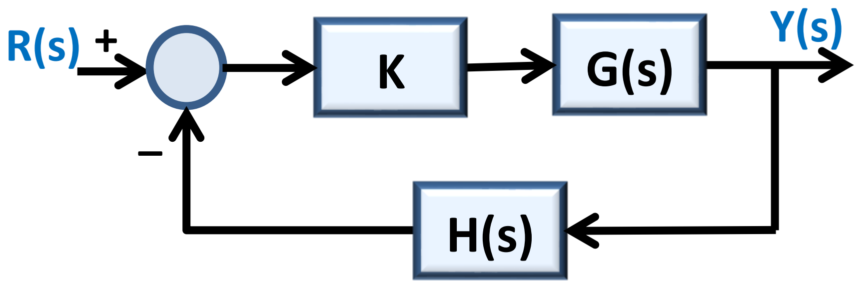
b) Determine stability of the closed-loop system based on the Gain margin and Phase margin values.

Since, PM > 0 and GM > 0, the closed-loop system is stable.

Final Exam Review

Example 6 Given the open-loop system, $KG(s)H(s)$, Bode diagram

$$K = 10, \qquad G(s) = \frac{1}{s(s+1)}, \qquad H(s) = 1$$



c) Determine type of the open-loop system using the Bode plot and the corresponding steady-state error of the closed-loop system.

Since, the slope of the log magnitude plot at low frequencies starts with -20dB/dec, the open-loop transfer function is **Type 1**.

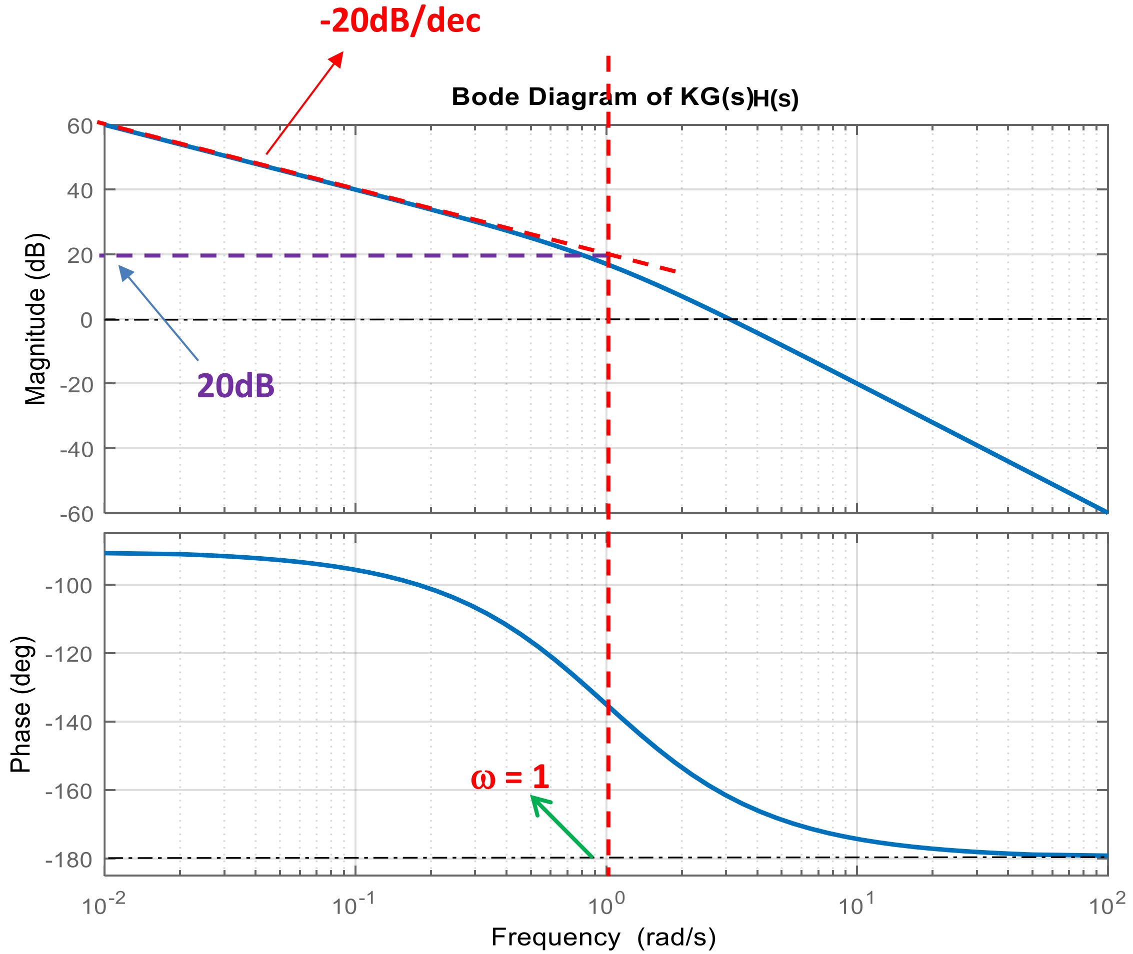
Since the open-loop system is Type 1, there is a constant ramp error.

Find the intersection of low frequency asymptote with line $\omega = 1$

$$\mathbf{20\log(k_v) = 20dB}$$

$$k_v = 10^{20/20} \rightarrow \quad \boldsymbol{k_v = 10}$$ Ramp-error constant

$$e_{ss} = \frac{1}{k_v} = \frac{1}{10} \rightarrow \quad \boldsymbol{e_{ss} = 0.1}$$ Steady-state error



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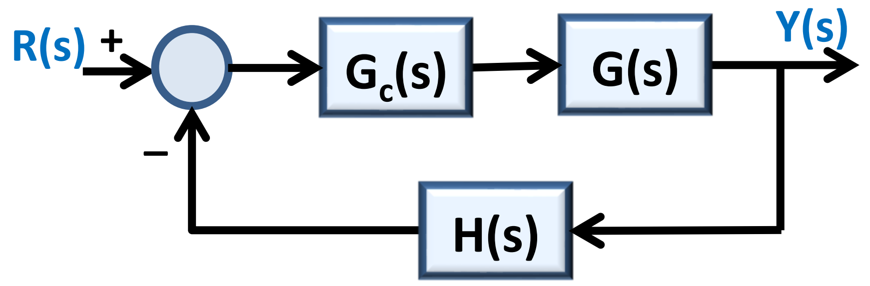
Example 6

Given the open-loop system, $KG(s)H(s)$, Bode diagram

$$K = 10, \qquad G(s) = \frac{1}{s(s+1)}, \qquad H(s) = 1$$



c) Design a PD controller to achieve the following performance characteristics without changing the unit-ramp steady-state error.

$$\boxed{PM > 70^\circ, \qquad GM > 15\text{dB}}$$

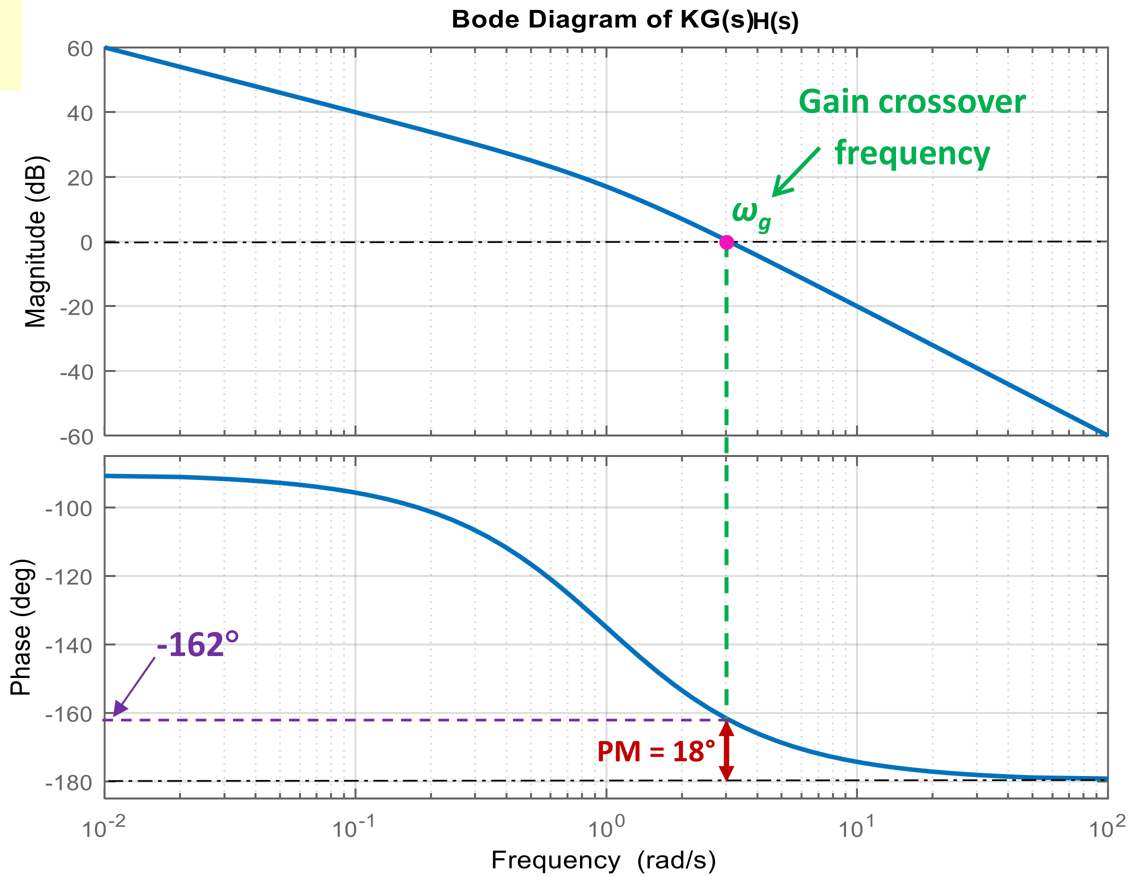
$$G_c(s) = K_P\left(1 + \frac{T_d s}{\frac{T_d}{\beta}s + 1}\right)$$

Step 1: Determine the proportional gain K_p to satisfy the desired steady-state error

Since, the ramp-error has not been changed,

$$\boxed{K_p = K = 10}$$ **Desired Proportional Gain**

Step 2: Plot Bode diagram of the open-loop system with proportional gain $K_pG(s)H(s)$, and find PM and GM



Open-loop system with desired proportional gain $$K_pG(s)H(s) = \frac{10}{s(s+1)}$$

From the Bode diagram the gain crossover frequency, the phase margin and gain margin of the system with $K_p = 10$

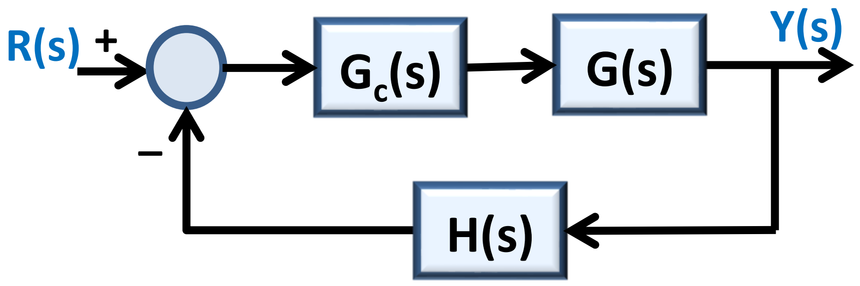
$$\boxed{\omega_g = 3.08\, rad/s} \qquad \boxed{PM = 18\,, GM = +\infty}$$

Final Exam Review

Example 6

Given the open-loop system, $KG(s)H(s)$, Bode diagram

$$K = 10, \qquad G(s) = \frac{1}{s(s+1)}, \qquad H(s) = 1$$



c) Design a PD controller to achieve the following performance characteristics without changing the unit-ramp steady-state error.

$$G_c(s) = K_P\left(1 + \frac{T_d s}{\frac{T_d}{\beta}s + 1}\right)$$

$$\boxed{PM > 70^\circ, \qquad \text{GM} > 15\text{dB}}$$

Step 3: Find the maximum phase angle, ϕ_m to be added to the system to achieve the desired PM criteria

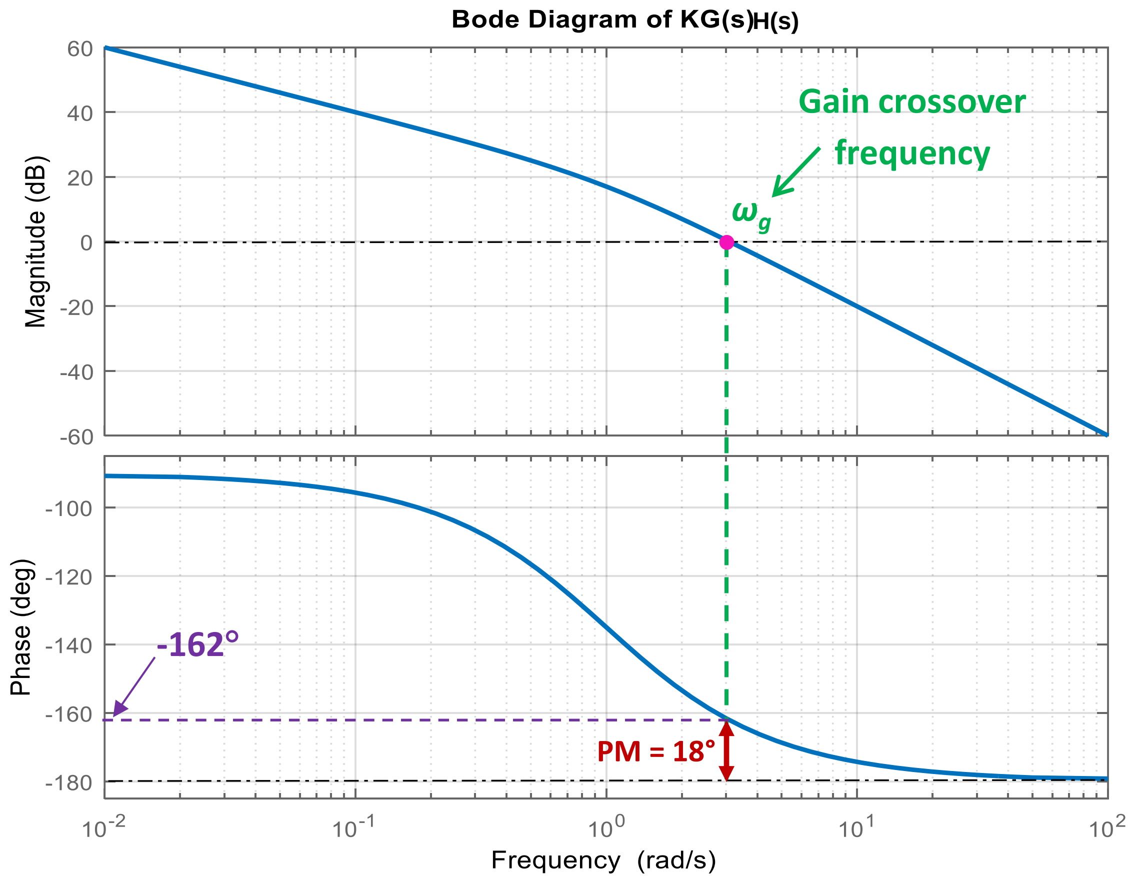
$$\phi_m = PM_d - PM + 10^\circ = 70^\circ - 18^\circ + 10^\circ = 62^\circ$$

$$\boxed{\phi_m > 62^\circ}$$

Step 4: Select the appropriate factor of β based on the ϕ_m value

$$\boxed{\beta = 20} \rightarrow \boxed{\phi_m = 65^\circ}$$

β	ϕ_m
10	55°
20	65°
30	70°
40	72°
50	74°
60	75°
70	76°
80	77°
90	78°
100	78.5°

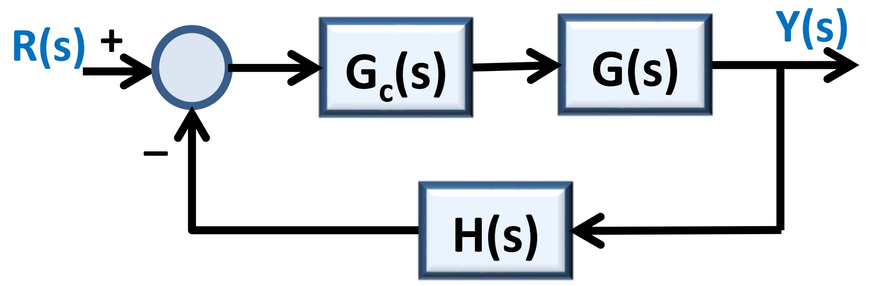


Final Exam Review

Example 6

Given the open-loop system, $KG(s)H(s)$, Bode diagram

$$K = 10, \qquad G(s) = \frac{1}{s(s+1)}, \qquad H(s) = 1$$



c) Design a PD controller to achieve the following performance characteristics without changing the unit-ramp steady-state error.

$$\boxed{PM > 70^\circ, \qquad GM > 15\text{dB}}$$

$$G_c(s) = K_P\left(1 + \frac{T_d s}{\frac{T_d}{\beta}s + 1}\right)$$

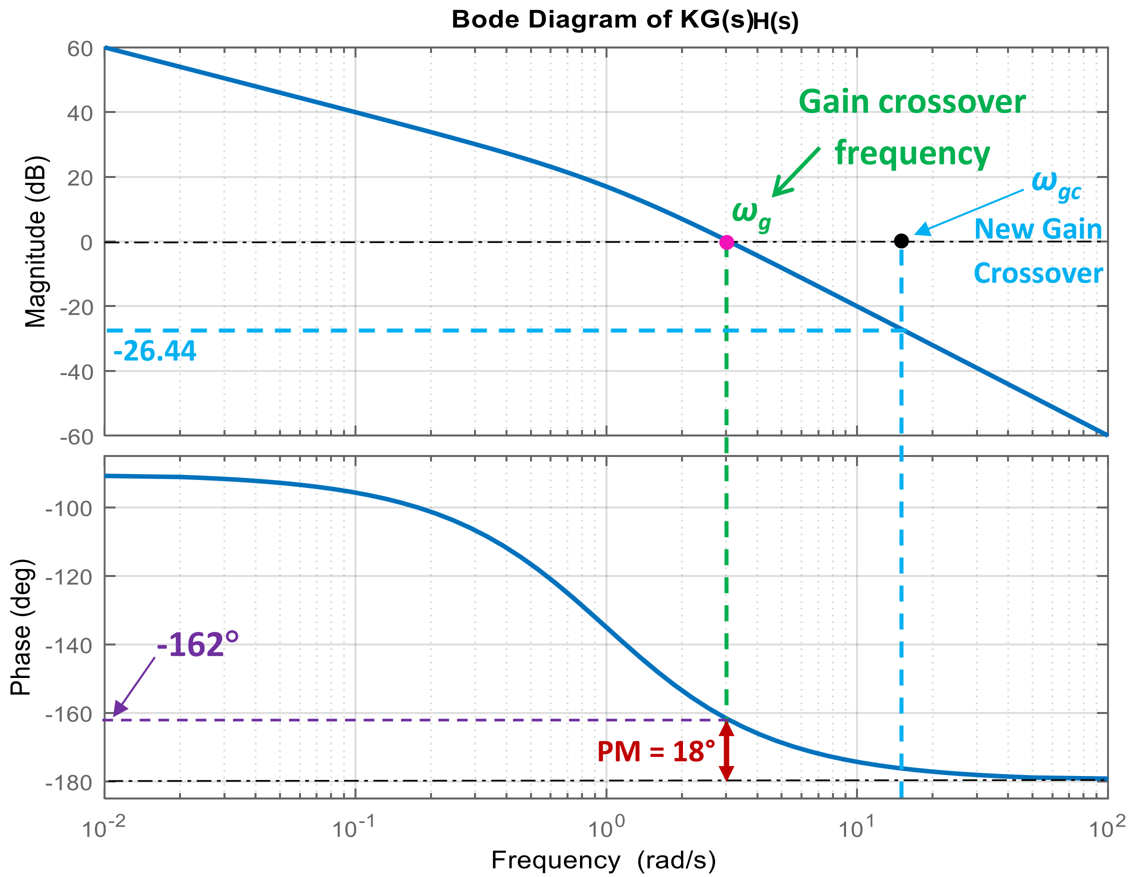
Step 5: **Find the new gain crossover frequency ω_{gc} where the magnitude is $-20\log\sqrt{1+\beta}$**

The new gain crossover frequency, ω_{gc}, can be determined from the Bode diagram at the magnitude of $-20\log\sqrt{1+\beta}$.

$$-20\log\sqrt{1+\beta} = -20\log\sqrt{21} = -26.44\text{ dB}$$

$$\boxed{\omega_{gc} = 14.4\text{ rad/sec}}$$

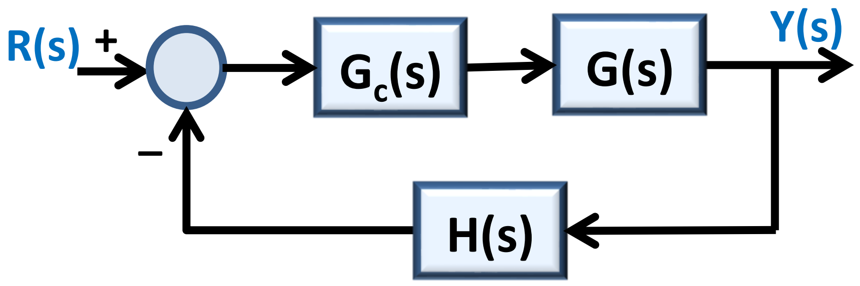
New Gain Crossover Frequency



Final Exam Review

Example 6 Given the open-loop system, $KG(s)H(s)$, Bode diagram

$$K = 10, \qquad G(s) = \frac{1}{s(s+1)}, \qquad H(s) = 1$$



c) Design a PD controller to achieve the following performance characteristics without changing the unit-ramp steady-state error.

$$\boxed{PM > 70^\circ, \qquad GM > 15\text{dB}}$$

$$G_c(s) = K_P\left(1 + \frac{T_d s}{\frac{T_d}{\beta}s + 1}\right)$$

Step 6: Assign the maximum phase frequency ω_m at the new gain crossover frequency ω_{gc} value

$$\boxed{\omega_m = \omega_{gc} = 14.4\ \text{rad/sec}}$$

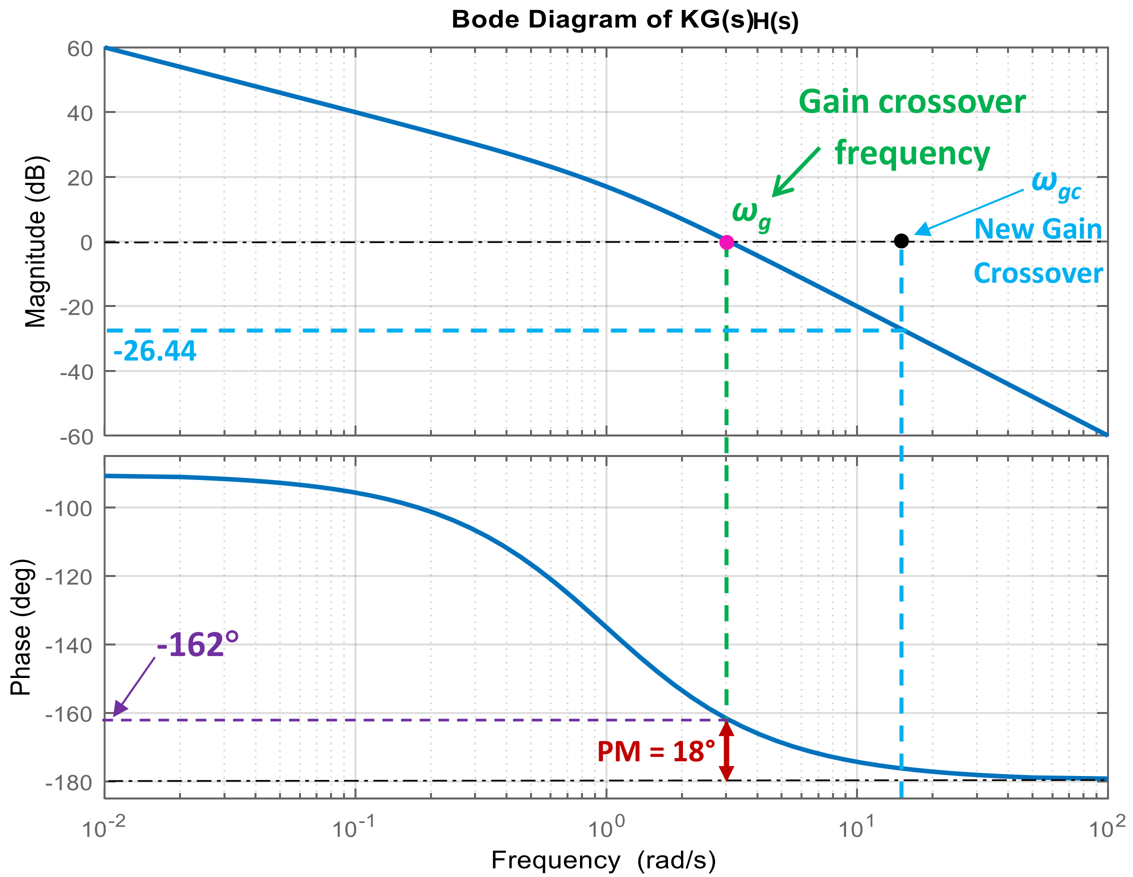
Step 7: Assign the derivative time constant T_d value

$$T_d = \frac{\sqrt{\beta}}{\omega_m} \quad \rightarrow \quad T_d = \frac{\sqrt{20}}{14.4} \quad \Rightarrow \quad \boxed{T_d = 0.31}$$

The designed PD controller is obtained as

$$\boxed{K_p = 10} \quad \boxed{\beta = 20} \quad \boxed{T_d = 0.31}$$

$$\Rightarrow \boxed{G_c(s) = 10\left(1 + \frac{0.31s}{0.0155s + 1}\right)}$$



Final Exam Review

Example 7

Consider a dynamic system with the following set of differential equations

$$\begin{cases} \dot{x}_1(t) + 2x_1(t) - 4x_2(t) = 4u(t) \\ \dot{x}_1(t) - \dot{x}_2(t) + 4x_1(t) + x_2(t) = 0 \\ y(t) = \dot{x}_1(t) - 2\dot{x}_2(t) \end{cases}$$

a) Determine state space representation of the system.

Rearrange the differential equations and find the state-space representation in the standard form,

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + 4x_2(t) + 4u(t) \\ \dot{x}_2(t) = 2x_1(t) + 5x_2(t) + 4u(t) \\ y(t) = -6x_1(t) - 6x_2(t) - 4u(t) \end{cases} \Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -6 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [-4]u(t) \end{cases}$$

b) Find the characteristic polynomial and eigenvalues of the system matrix **A**. Is the system stable?

The characteristic polynomial of the system matrix **A** is obtained as below

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda + 2 & -4 \\ -2 & \lambda - 5 \end{bmatrix} \longrightarrow \det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda + 2 & -4 \\ -2 & \lambda - 5 \end{vmatrix} = \boxed{\lambda^2 - 3\lambda - 18}$$

Characteristic Polynomial

Next, find the eigenvalues of the matrix **A**

$$\lambda^2 - 3\lambda - 18 = 0 \quad \rightarrow \quad \boxed{\boldsymbol{\lambda_1 = -3, \qquad \lambda_2 = 6}}$$

Eigenvalues

One of the eigenvalues is on the right-half of the s-plane, so the system is unstable.

Final Exam Review

Example 7 Consider a dynamic system with the following set of differential equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -6 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [-4] u(t)$$

c) Determine the transfer function model of the system.

The transfer function is determined by the following formula

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

First, find the $(s\mathbf{I} - \mathbf{A})^{-1}$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} s+2 & -4 \\ -2 & s-5 \end{bmatrix} \longrightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 - 3s - 18} \begin{bmatrix} s-5 & 4 \\ 2 & s+2 \end{bmatrix}$$

Substitute the $(s\mathbf{I} - \mathbf{A})^{-1}$, $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{D}$ in the transfer function formula

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 - 3s - 18} \begin{bmatrix} -6 & -6 \end{bmatrix} \begin{bmatrix} s-5 & 4 \\ 2 & s+2 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} + [-4] = \frac{1}{s^2 - 3s - 18} \begin{bmatrix} -6 & -6 \end{bmatrix} \begin{bmatrix} 4s-4 \\ 4s+16 \end{bmatrix} + [-4] = \frac{-48s - 72}{s^2 - 3s - 18} + [-4]$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{-4s^2 - 36s}{s^2 - 3s - 18}}$$

Transfer Function

Final Exam Review

Example 7 Consider a dynamic system with the following set of differential equations

$$\frac{Y(s)}{U(s)} = \frac{-4s^2 - 36s}{s^2 - 3s - 18}$$

d) Determine the Canonical Controllable model of the system from the transfer function .

First, find the associated differential equation

$$s^2Y(s) - 3sY(s) - 18Y(s) = -4s^2U(s) - 36sU(s) \longrightarrow \ddot{y}(t) - 3\dot{y}(t) - 18y(t) = -4\ddot{u}(t) - 36\dot{u}(t)$$

$$a_0 = -18, \quad a_1 = -3, \quad b_0 = 0 \quad b_1 = -36, \quad b_2 = -4$$

The state and output equations are obtained based on the general format

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 18 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

State Equation

$$y(t) = \begin{bmatrix} -72 & -48 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [-4]u(t)$$

Output Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 - b_m a_0 & b_1 - b_m a_1 & \cdots & b_{m-1} - b_m a_{m-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + [b_m]u(t)$$

Final Exam Review

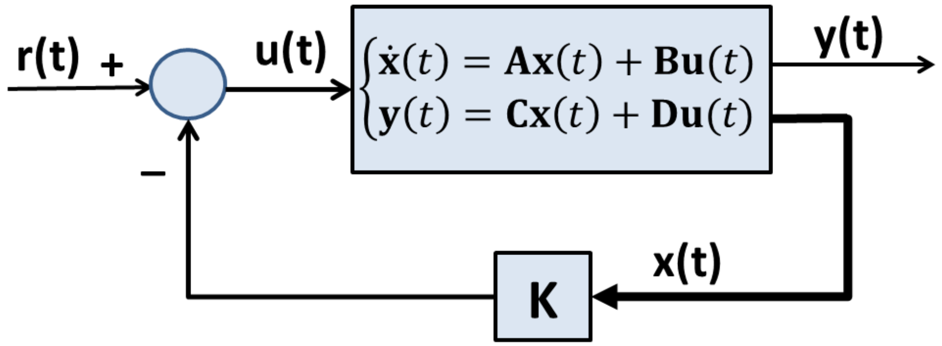
Example 7

Consider a dynamic system with the following set of differential equations

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 18 & 3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}-72 & -48\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [-4]u(t)$$

e) Design a state feedback controller of $u(t) = -\mathbf{Kx}(t) + r(t)$ for the system, such that the desired closed-loop eigenvalues should be located at $s_{1,2} = -3 \pm j6$.



Step 1: Check controllability of the open-loop system.

$$\mathbf{Q_c} = [\mathbf{B} \quad \mathbf{AB}] \rightarrow \mathrm{Q_c} = \begin{bmatrix}0 & 1\\ 1 & 3\end{bmatrix} \rightarrow \det(\mathrm{Q_c}) = -1$$

The controllability matrix is full rank, so the system is controllable.

Step 2: Determine the desired characteristic polynomial.

$$\boxed{(s + 3 + j6)(s + 3 - j6) = s^2 + 6s + 45}$$

Desired Characteristic Polynomial

Final Exam Review

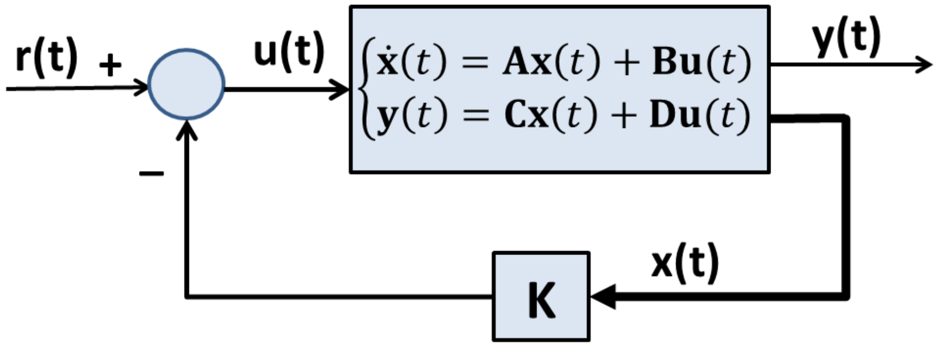
Example 7

Consider a dynamic system with the following set of differential equations

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ 18 & 3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}-72 & -48\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+[-4]u(t)$$

e) Design a state feedback controller of $u(t)=-\mathbf{K}\mathbf{x}(t)+r(t)$ for the system, such that the desired closed-loop eigenvalues should be located at $s_{1,2}=-3\pm j6$.



Step 3: Obtain the closed-loop system matrix and determine the characteristic polynomial

$$\boxed{\mathbf{K}=\begin{bmatrix}k_1 & k_2\end{bmatrix}}$$

$$\mathbf{A}_{cl}=\mathbf{A}-\mathbf{B}\mathbf{K}=\begin{bmatrix}0 & 1\\ 18 & 3\end{bmatrix}-\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 18 & 3\end{bmatrix}-\begin{bmatrix}0 & 0\\ k_1 & k_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 18-k_1 & 3-k_2\end{bmatrix}$$

$$s\mathbf{I}-\mathbf{A}_{cl}=\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1\\ 18-k_1 & 3-k_2\end{bmatrix}=\begin{bmatrix}s & -1\\ -18+k_1 & s-3+k_2\end{bmatrix}$$

$$\det(s\mathbf{I}-\mathbf{A}_{cl})=\begin{vmatrix}s & -1\\ -18+k_1 & s-3+k_2\end{vmatrix}=\boxed{s^2+(-3+k_2)s-18+k_1}$$

Closed-loop characteristic polynomial

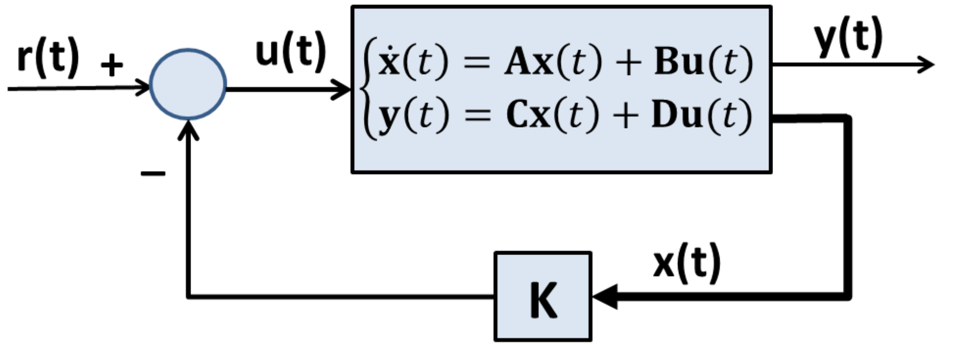
Final Exam Review

Example 7 Consider a dynamic system with the following set of differential equations

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 18 & 3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}-72 & -48\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [-4]u(t)$$

e) Design a state feedback controller of $u(t) = -\mathbf{K}\mathbf{x}(t) + r(t)$ for the system, such that the desired closed-loop eigenvalues should be located at $s_{1,2} = -3 \pm j6$.



Step 4: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the state feedback gain value $\mathbf{K}$.

Desired Characteristic Polynomial

$$\boxed{s^2 + 6s + 45}$$

Closed-loop System Characteristic Polynomial

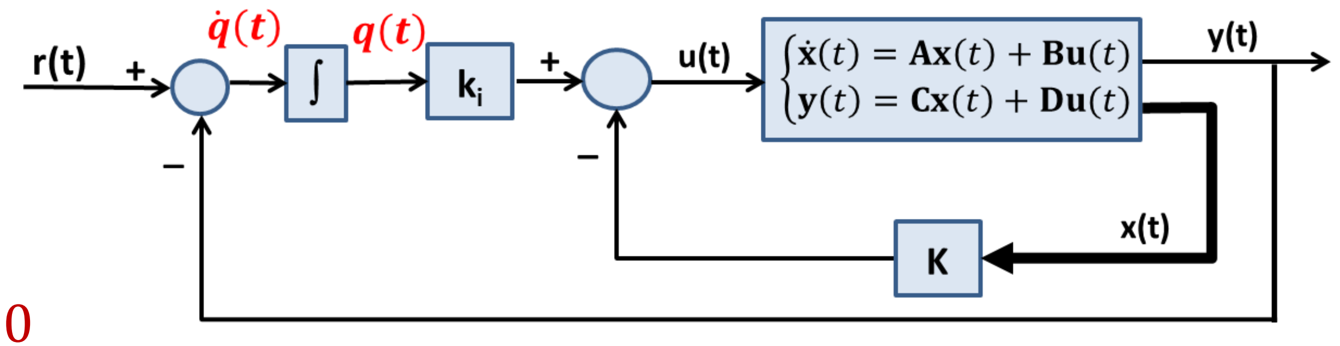
$$\boxed{s^2 + (-3 + k_2)s - 18 + k_1}$$

$$\begin{cases}-3 + \mathrm{k}_2 = 6\\ -18 + \mathrm{k}_1 = 45\end{cases} \rightarrow \begin{cases}k_2 = 9\\ k_1 = 63\end{cases} \Rightarrow \boxed{\mathbf{K} = \begin{bmatrix}63 & 9\end{bmatrix}}$$

State Feedback Gain

Final Example Review

Example 8 Consider the following control system of state feedback with integral control



Given A, B, C, D matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

design a state feedback with integral controller (find the K and k_i) such that the closed-loop poles are located at $s = -1, -2$

Step 1: Determine the augmented open-loop system.

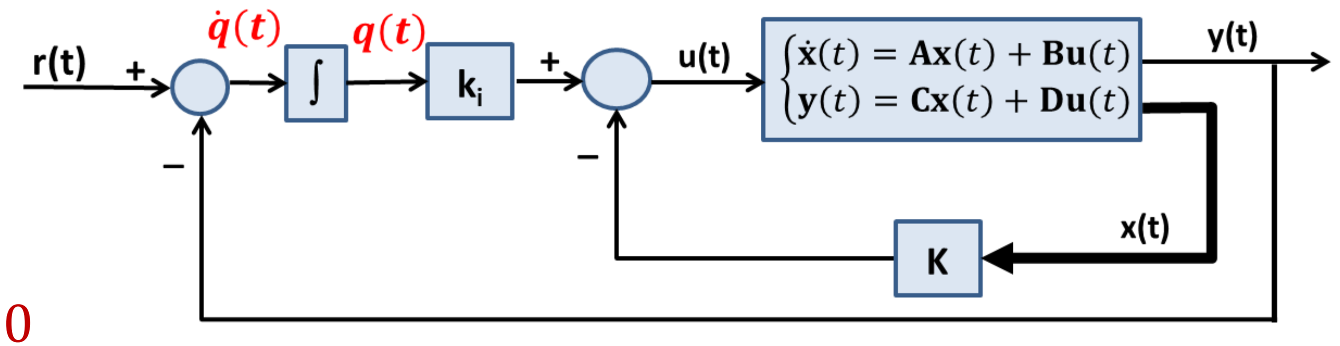
$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \\ y(t) = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \mathbf{D}u(t) \end{cases}$$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix} \end{cases}$$

Augmented open-loop system

Final Exam Review

Example 8 Consider the following control system of state feedback with integral control



Given A, B, C, D matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

design a state feedback with integral controller (find the K and k_i) such that the closed-loop poles are located at $s = -1, -2$

Step 2: Check controllability of the augmented open-loop system.

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \mathbf{Q_c} = [\bar{\mathbf{B}} \quad \bar{\mathbf{A}}\bar{\mathbf{B}} \quad \bar{\mathbf{A}}^2\bar{\mathbf{B}}] \rightarrow \mathbf{Q_c} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \boxed{\det(\mathbf{Q_c}) = 1}$$

Determinant is nonzero, so the controllability matrix is full rank, so the system is controllable.

Step 3: Determine the desired characteristic polynomial.

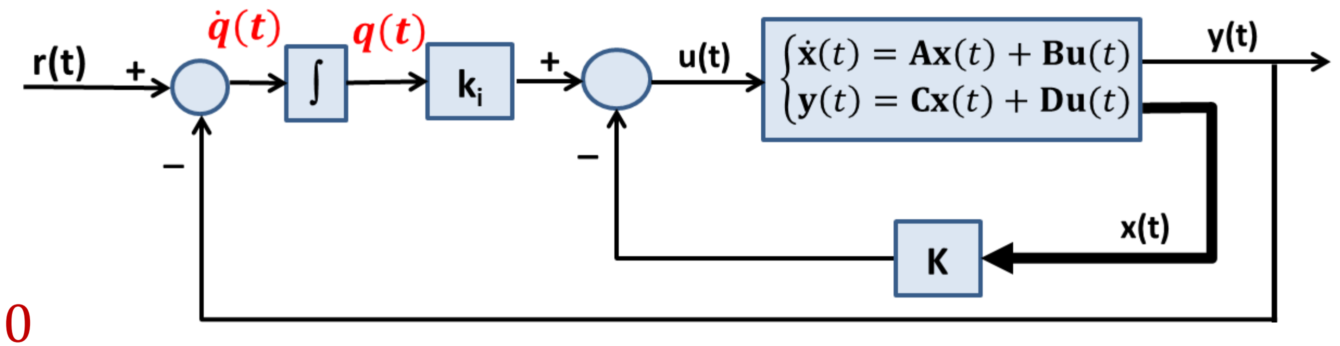
The desired characteristic equation is determined from the location of the desired closed-loop poles and considering the third pole more than ten times far from the desired poles

$$(s+1)(s+2)(s+20) = s^3 + 23s^2 + 62s + 40$$ Desired Characteristic Polynomial

Final Exam Review

Example 8

Consider the following control system of state feedback with integral control



Given A, B, C, D matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

design a state feedback with integral controller (find the K and k_i) such that the closed-loop poles are located at $s = -1, -2$

Step 4: Obtain the augmented closed-loop system matrix and determine the characteristic polynomial

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1-k_1 & -2-k_2 \end{bmatrix}$$

$$\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{B}k_i \\ -\mathbf{C} + \mathbf{DK} & -\mathbf{D}k_i \end{bmatrix} = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ -1-k_1 & -2-k_2 & k_i \\ \hdashline -1 & 0 & 0 \end{array}\right]$$

$$s\mathbf{I} - \mathbf{A}_{cl} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -1-k_1 & -2-k_2 & k_i \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 1+k_1 & s+2+k_2 & -k_i \\ -1 & 0 & s \end{bmatrix}$$

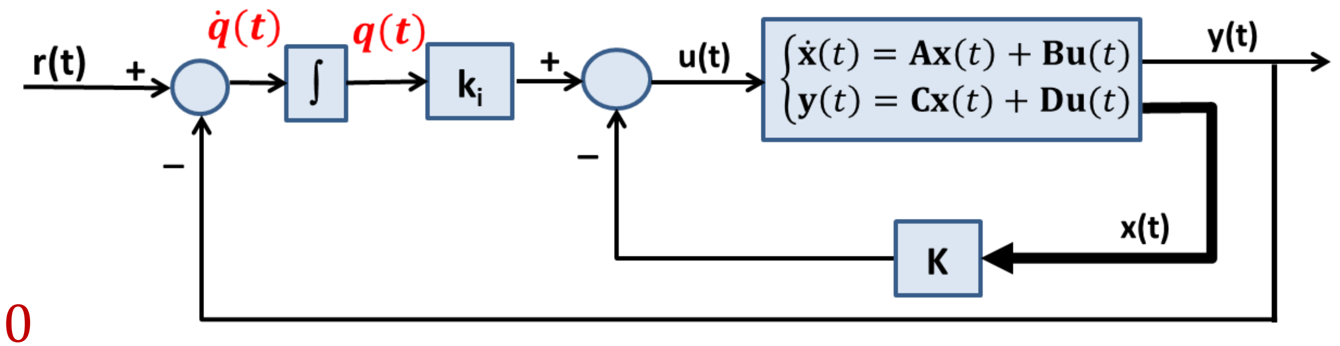
$$\det(s\mathbf{I} - \mathbf{A}_{cl}) = \begin{vmatrix} s & -1 & 0 \\ 1+k_1 & s+2+k_2 & -k_i \\ -1 & 0 & s \end{vmatrix} = s^3 + (2+k_2)s^2 + (1+k_1)s + k_i$$

Closed-loop characteristic polynomial

Final Exam Review

Example 8

Consider the following control system of state feedback with integral control



Given A, B, C, D matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

design a state feedback with integral controller (find the K and k_i) such that the closed-loop poles are located at $s = -1, -2$

Step 5: Compare the characteristic polynomial of the closed-loop system with the desired characteristic polynomial to determine the $\mathbf{K}$ and k_i values .

Desired Characteristic Polynomial

$$\boxed{s^3 + 23s^2 + 62s + 40}$$

Closed-loop Characteristic Polynomial

$$\boxed{s^3 + (2 + k_2)s^2 + (1 + k_1)s + k_i}$$

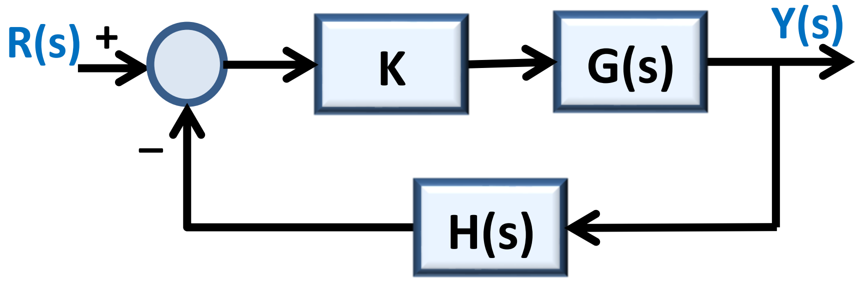
$$\begin{cases} 2 + k_2 = 23 \\ 1 + k_1 = 62 \\ k_i = 40 \end{cases} \rightarrow \begin{cases} k_2 = 21 \\ k_1 = 61 \\ k_i = 40 \end{cases} \Rightarrow \boxed{\mathbf{K} = [61 \quad 21]} \quad \boxed{k_i = 40}$$

State Feedback Gain **Integrator Gain**

Final Exam Review

Example 9

Consider the following closed-loop system



$$G(s) = \frac{s}{s^2 + s + 4.25}, \qquad H(s) = 1$$

a) Plot root-locus diagram of the system.

Step 1: Draw the axes of the s-plane

Mark poles $\times$ and zeros o of the open-loop system.

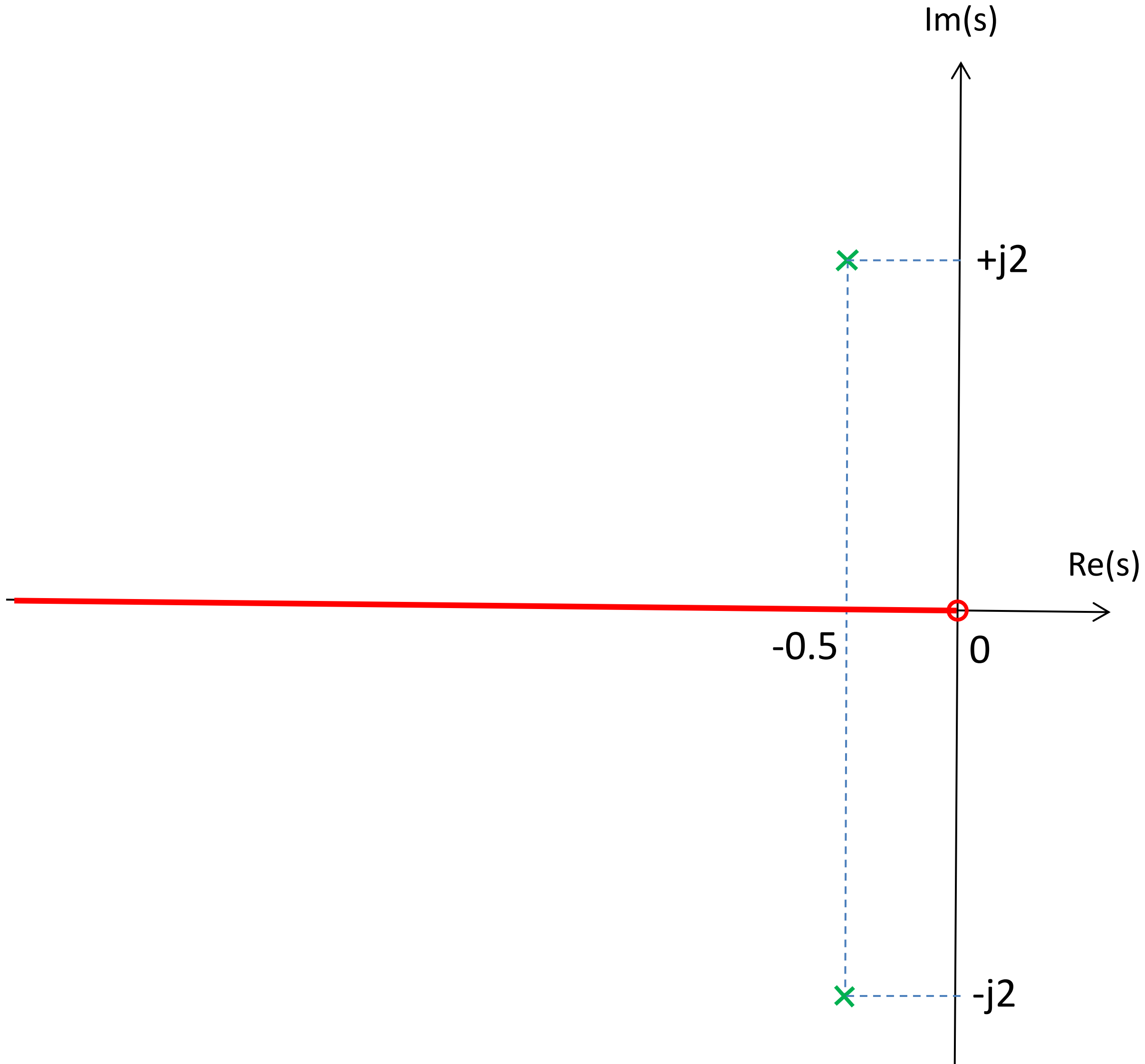
Poles: $p_1 = -0.5 + j2,\ p_2 = -0.5 - j2$

Zeros: $z_1 = 0$, One zero at infinity

Step 2: Draw the root-locus on the real axis

A point on the real axis is part of a locus if the number of poles and zeros to the right of that point is **ODD.**

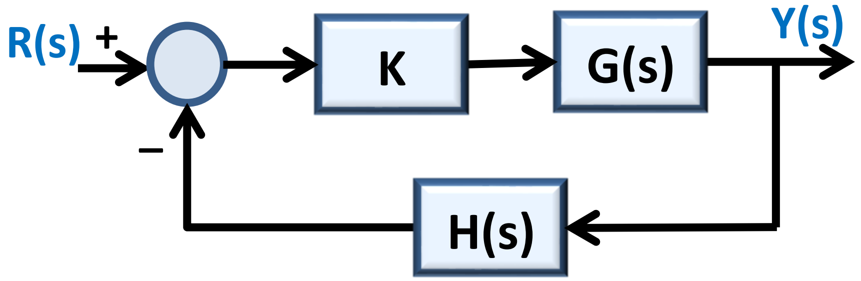
Here, zero is considered as an even number



Final Exam Review

Example 9

Consider the following closed-loop system



$$G(s) = \frac{s}{s^2 + s + 4.25}, \qquad H(s) = 1$$

a) Plot root-locus diagram of the system.

Step 3: Draw asymptote lines for large *K* values

Number of asymptotes → $n - m = 2 - 1 = 1$

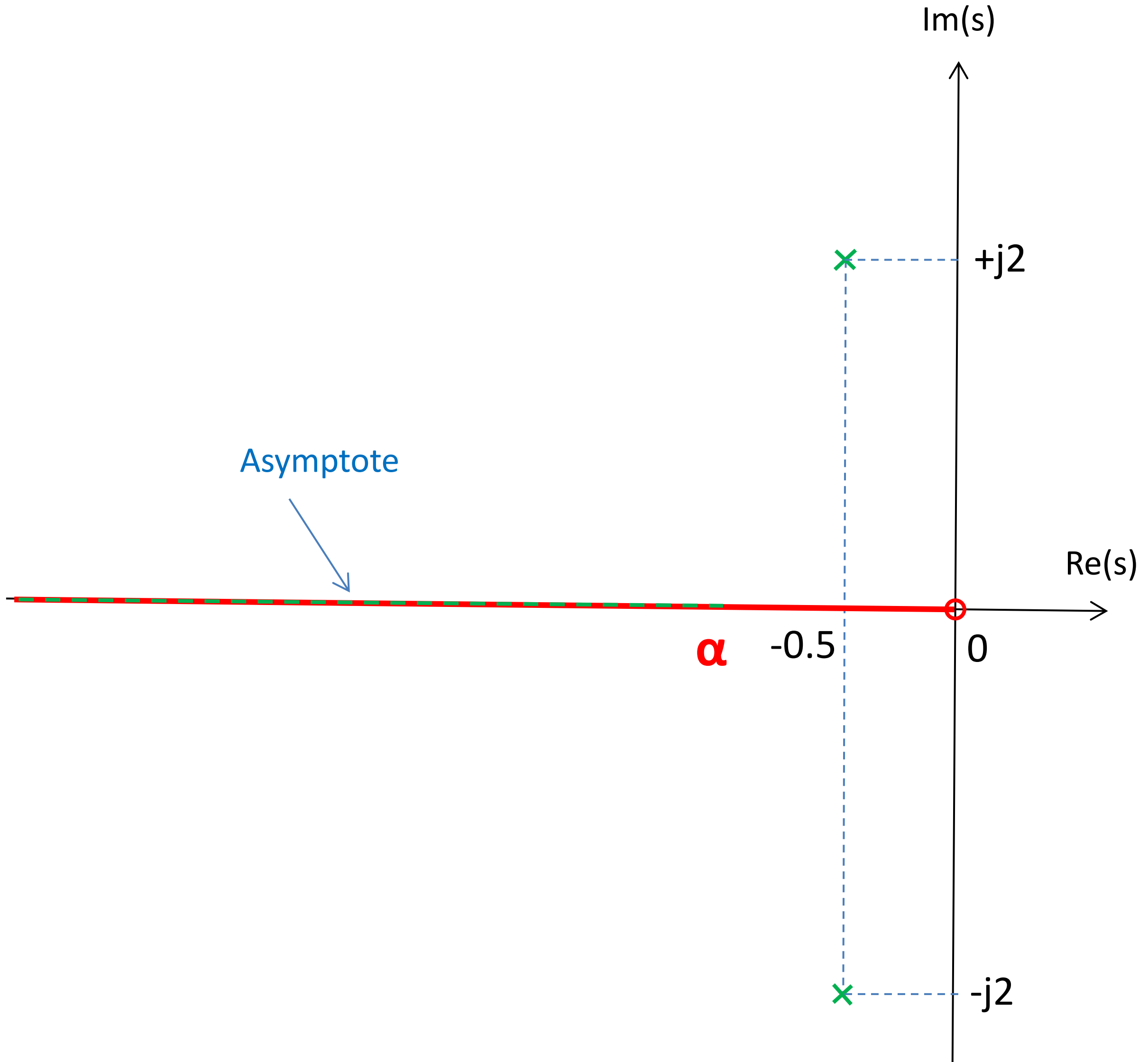
Intersection of asymptotes on the real axis

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{[(-0.5 + j2) + (-0.5 - j2)] - [0]}{2-1} = -1$$

Angle of asymptote lines with real axis

$$\varphi_i = \frac{180^\circ}{n-m}(2i+1) = \frac{180^\circ}{2-1}(2i+1) = 180^\circ(2i+1) \rightarrow \varphi_0 = 180^\circ$$

$$i = 0, 1, 2, \cdots$$

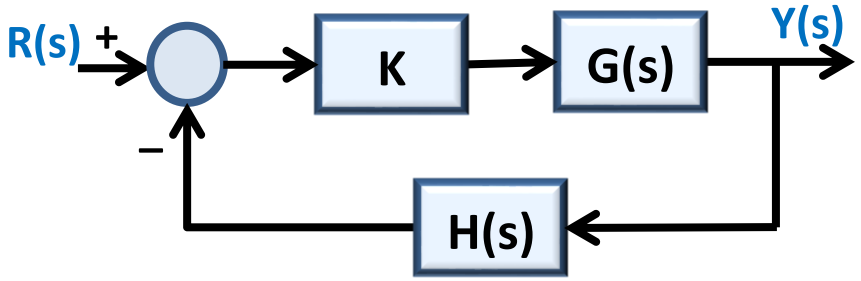


Final Exam Review

Example 9

Consider the following closed-loop system

$$G(s) = \frac{s}{s^2 + s + 4.25}, \qquad H(s) = 1$$



a) Plot root-locus diagram of the system.

Step 4: Intersection of root-locus with imaginary axis

$$\mathbf{1 + KG(s)H(s) = 0} \;\rightarrow\; s^2 + (1+K)s + 4.25 = 0$$

$$\boxed{s = j\omega} \longrightarrow (j\omega)^2 + (1+K)(j\omega) + 4.25 = -\omega^2 + j\omega(1+K) + 4.25 = 0$$

$$\underbrace{[-\omega^2 + 4.25]}_{real\ part} + j\underbrace{[\omega(1+K)]}_{imaginary\ part} = 0$$

From the real part:

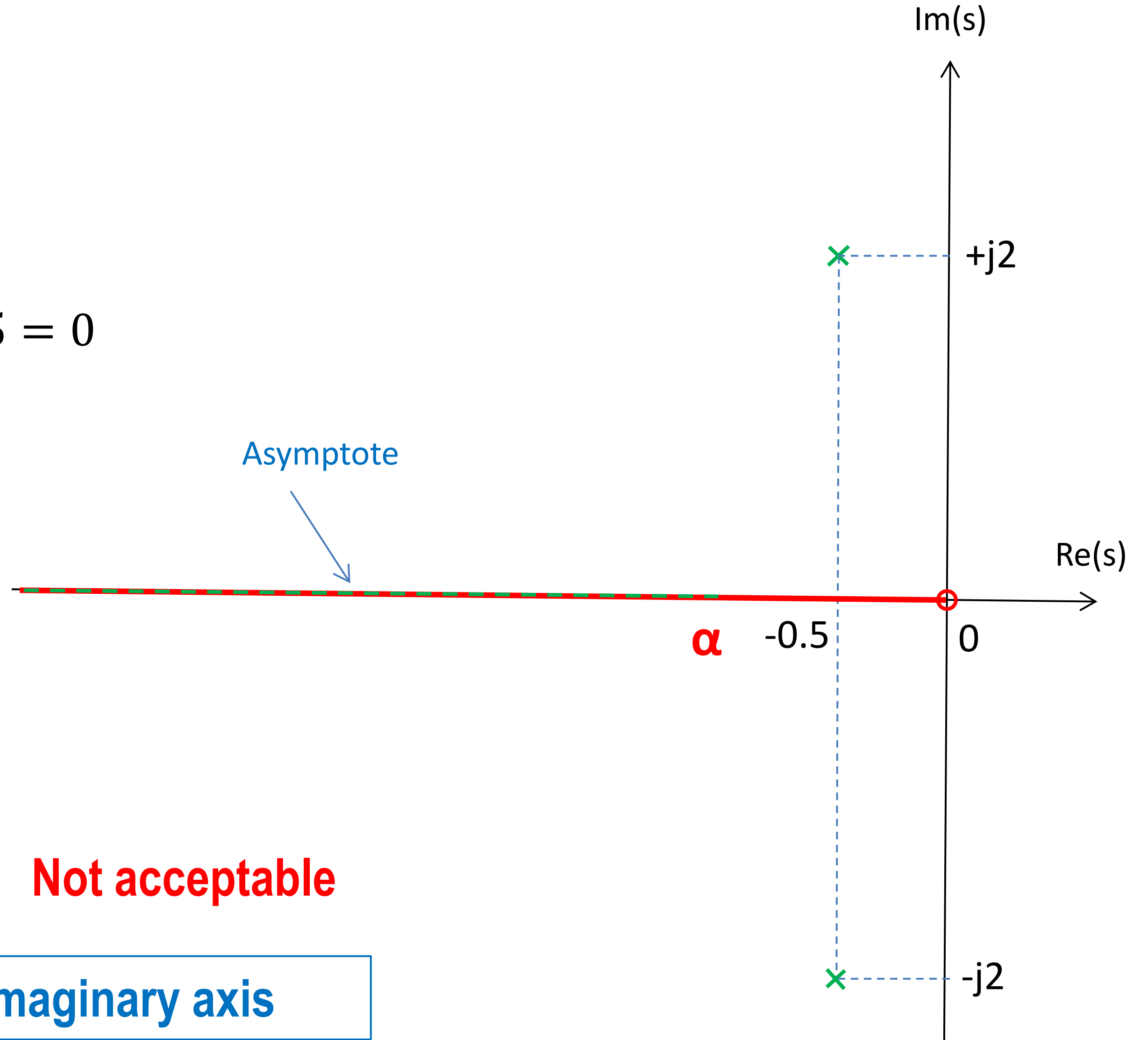
$$-\omega^2 + 4.25 = 0 \;\rightarrow\; \omega^2 = 4.25 \;\rightarrow\; \omega = \pm\sqrt{4.25}$$

From the imaginary part:

$$\omega^2 = 4.25 \longrightarrow \omega(1+K) = \pm\sqrt{4.25}(1+K) = 0 \;\rightarrow\; K = -1 < 0$$

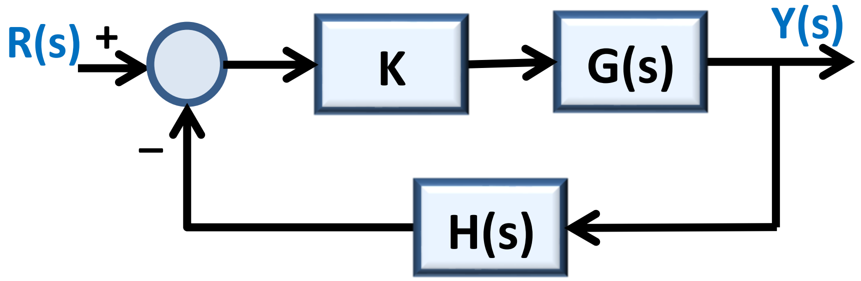
Not acceptable

The root-locus does not cross the imaginary axis



Final Exam Review

Example 9 Consider the following closed-loop system



$$G(s) = \frac{s}{s^2 + s + 4.25}, \qquad H(s) = 1$$

a) Plot root-locus diagram of the system.

Step 5: Calculate break-away/break-in points on real axis

$$\mathbf{1 + KG(s)H(s) = 0} \;\rightarrow\; s^2 + (1 + K)s + 4.25 = 0 \;\rightarrow\; K = \frac{-s^2 - s - 4.25}{s}$$

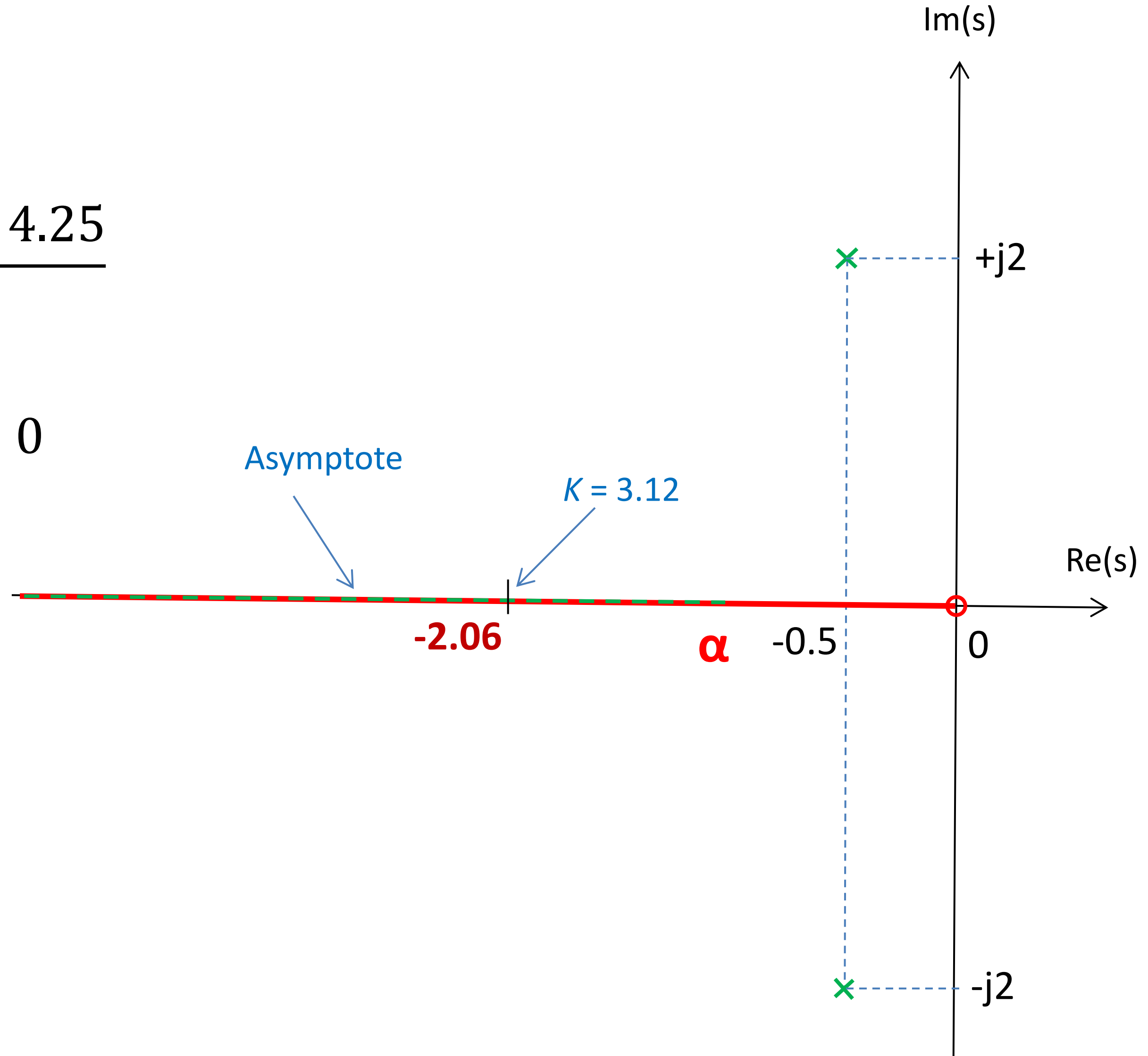
$$\frac{\boldsymbol{dK}}{\boldsymbol{ds}} = \mathbf{0} \;\rightarrow\; \frac{(-2s - 1)(s) - (-s^2 - s - 4.25)}{(s)^2} = 0 \;\rightarrow\; -s^2 + 4.25 = 0$$

$$\begin{cases} s = +2.06 & \rightarrow \; \textit{not on the root} - \textit{loci} \\ \boxed{\boldsymbol{s = -2.06}} & \rightarrow \; \textit{on the root} - \textit{loci} \end{cases}$$

Break-in Point

The associated gain at the break-in point:

$$K = \frac{-(-2.06)^2 - (-2.06) - 4.25}{-2.06} \;\longrightarrow\; \boxed{\boldsymbol{K = 3.12}}$$

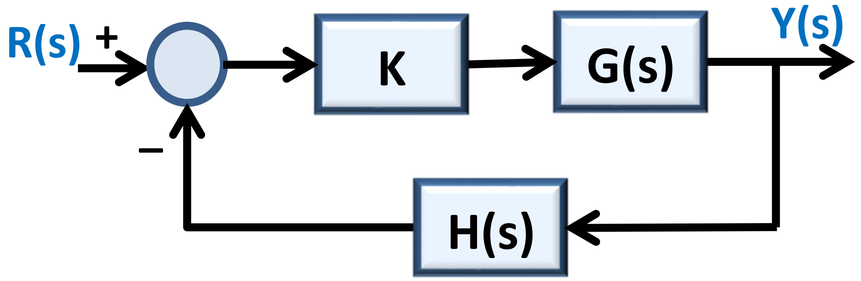


Final Exam Review

Example 9

Consider the following closed-loop system

$$G(s) = \frac{s}{s^2 + s + 4.25}, \quad H(s) = 1$$



a) Plot root-locus diagram of the system.

Step 6: Calculate angle of departure from the complex poles

$$\boxed{\phi_p = 180^\circ - \sum_i \angle p_i + \sum_j \angle z_j}$$

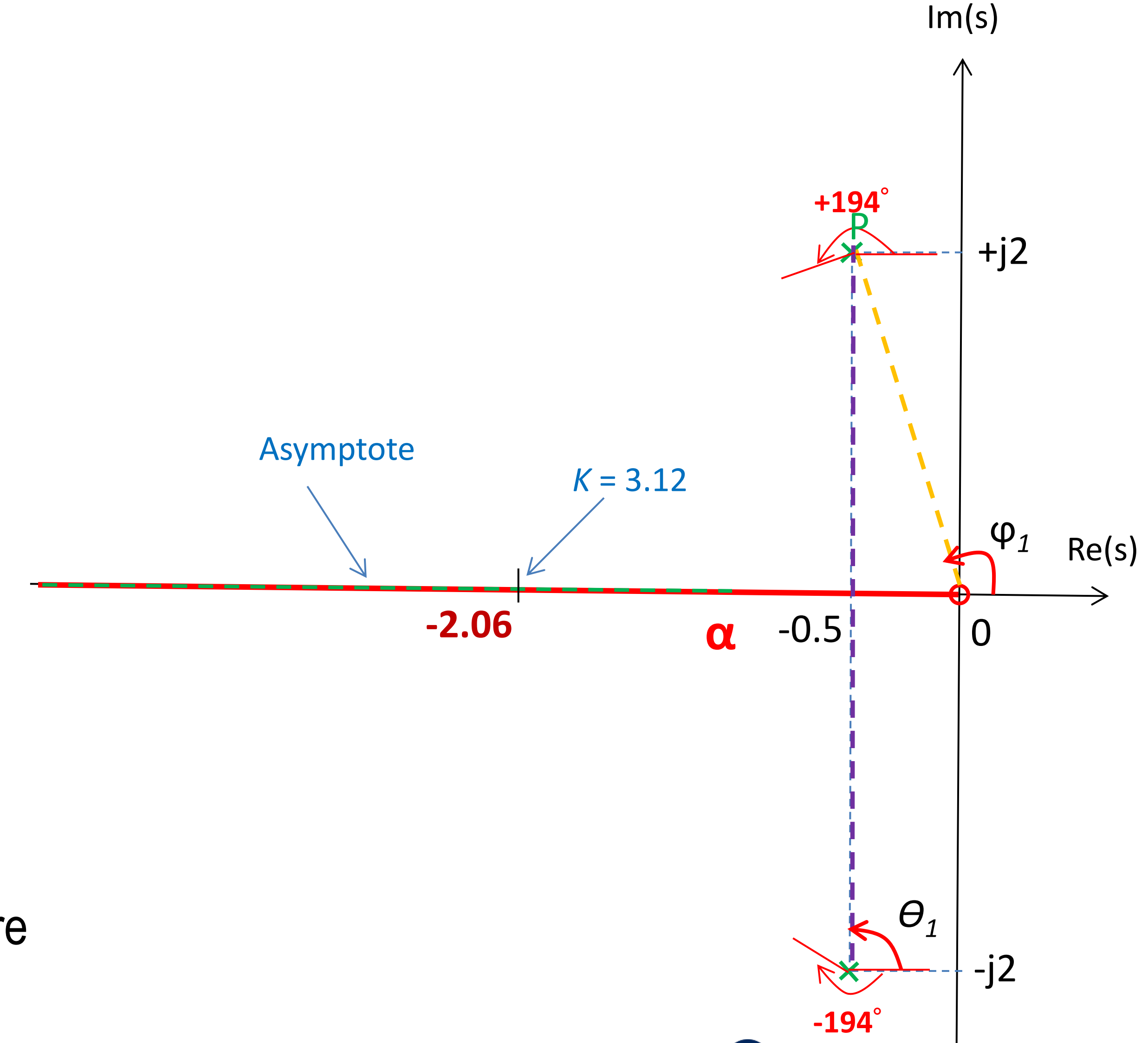
Angle of departure from the poles at $s = -0.5 \pm j2$

$$\phi_p = 180^\circ - (\theta_1) + (\varphi_1)$$

$$= 180^\circ - (90^\circ) + (tan^{-1}(0.25) + 90°)$$

$$= +194^\circ$$

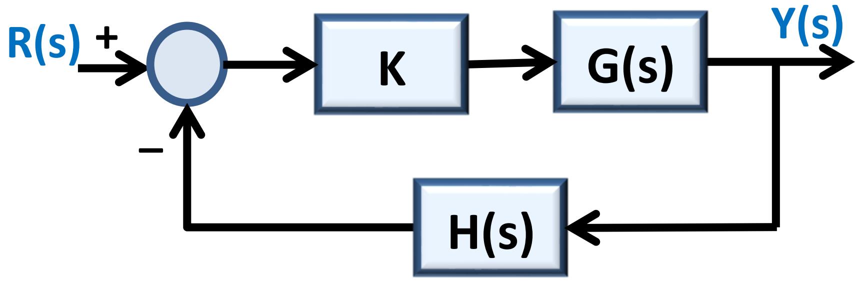
Angel of departure from the pole at $s = -0.5 + j2$



Since, root-locus is symmetric with respect to the real axis, the angle of departure for the pole at $s = -0.5 - j2$ will be $\phi_p = -194^\circ$

Final Exam Review

Example 9 Consider the following closed-loop system

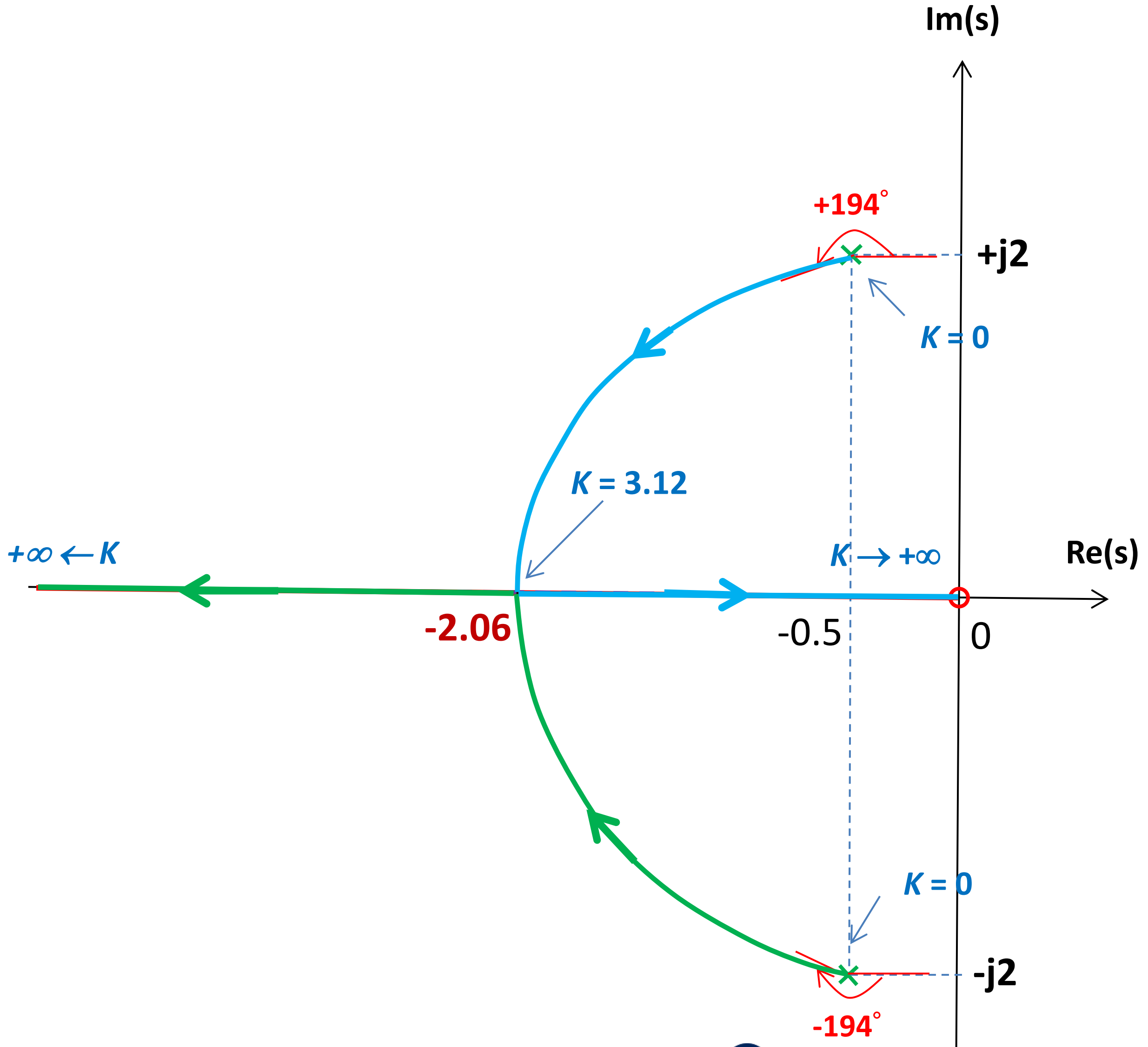


$$G(s) = \frac{s}{s^2 + s + 4.25}, \qquad H(s) = 1$$

a) Plot root-locus diagram of the system.

Step 7: Complete the root-locus diagram

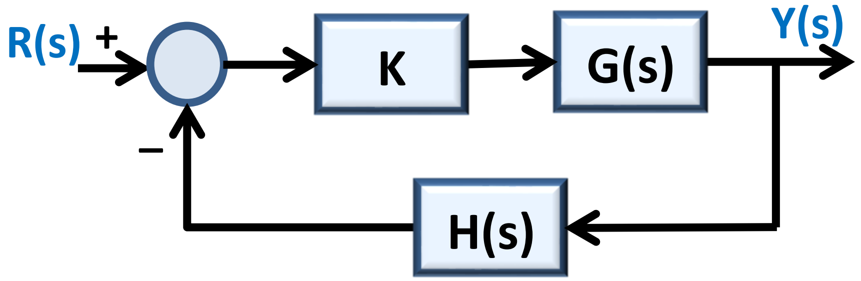
- Number of separate root-loci is equal to the order of open-loop transfer function, which is two here.
- For $K = 0$ the root-loci is at the open-loop poles including those at $s = \infty$.
- For $K = \infty$ the root-loci is at the open-loop zeros including those at $s = \infty$.
- Since open-loop transfer function has one finite zero at $s = 0$ one of the root-locus branches terminates at the zero and the other one goes to infinity approaching the asymptote line.
- Root-locus is symmetric with respect to the real axis.



Final Exam Review

Example 9 Consider the following closed-loop system

$$G(s) = \frac{s}{s^2 + s + 4.25}, \qquad H(s) = 1$$



b) Determine the location of the closed-loop poles with damping ratio of $\zeta = 0.707$ and the corresponding K value.

Draw the constant-damping-ratio lines of $\zeta = 0.707$

$$\boldsymbol{\zeta = 0.707} \quad \rightarrow \quad \theta = \cos^{-1}(\zeta) = 45^{\circ}$$

The intersection of the lines with root-locus will be the desired closed-loop pole locations.

$$\boxed{\boldsymbol{s_d = -1.45 \pm j1.45}}$$

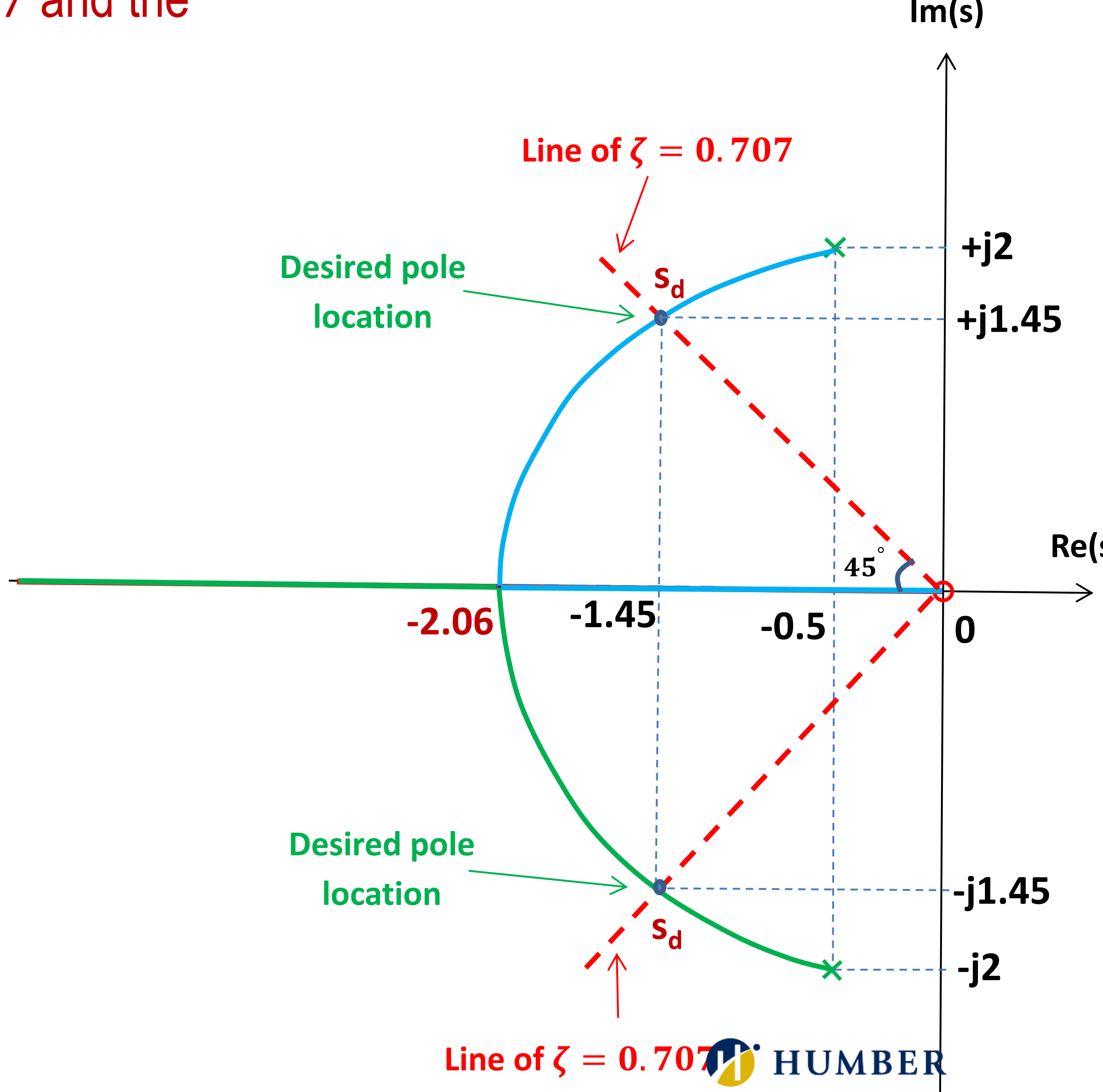
Find the gain K at the desired pole locations using the magnitude condition:

$$\boxed{|KG(s)H(s)|_{s=s_d} = 1 \quad \rightarrow \quad |K| = \frac{1}{|G(s_d)H(s_d)|}}$$

$$|K| = \left|\frac{s^2 + s + 4.25}{s}\right|_{s=s_d} = \left.\frac{|s + 0.5 + j2||s + 0.5 - j2|}{|s|}\right|_{s=-1.45+j1.45}$$

$$|K| = \frac{|-0.95 + j3.45||-0.95 - j0.55|}{|-1.45 + j1.45|} = 1.91$$

$$\boxed{\boldsymbol{K = 1.91}}$$

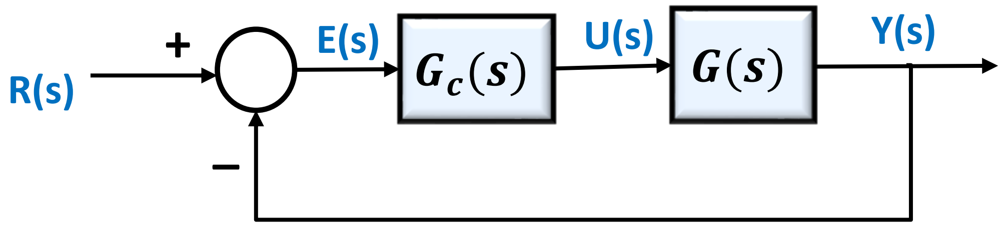


Final Exam Review

Example 10

Consider the following closed-loop system

$$G(s) = \frac{1}{(s+10)(s+30)} \qquad G_c(s) = K$$



a) Determine the K value so that the maximum overshoot of unit-step response is 5%.

First, calculate the desired damping ratio from the given desired maximum overshoot

$\boxed{\text{O.S.} = 5\%}$ → $$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \rightarrow \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \rightarrow \boxed{\zeta = 0.6901}$$ **Desired Damping Ratio**

Closed-loop transfer function → $$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K}{s^2 + 40s + K + 300}$$

Compare the characteristic equation with the standard second-order prototype system

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 40s + K + 300$$

$$\begin{cases} 2\zeta\omega_n = 40 \rightarrow 2 \times 0.6901\omega_n = 40 \rightarrow \boxed{\omega_n = 28.981\ \text{rad/sec}} \\ \omega_n^2 = K + 300 \rightarrow 28.981^2 = K + 300 \rightarrow \boxed{K = 539.90} \end{cases}$$

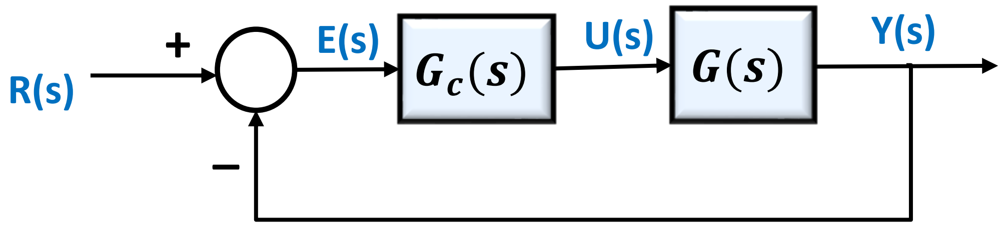
Desired Gain

Final Exam Review

Example 10

Consider the following closed-loop system

$$G(s) = \frac{1}{(s+10)(s+30)} \qquad G_c(s) = K_c \frac{s+z}{s+p}$$



b) Design a Lag Compensator to achieve the steady-state error of 3% ($e_{ss} = 0.03$) for unit-step input without altering the closed-loop poles of the designed-system in Part (b).

Step 1: Determine desired dominant closed-loop pole locations and the corresponding open-loop gain

$$\boxed{K = 539.90}$$

Transfer function of the designed closed-loop system with gain K is

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{539.90}{s^2 + 40s + 839.90}$$

Closed-loop poles are obtained as

$$s^2 + 40s + 839.90 = 0 \quad \rightarrow \quad \boxed{s = -20 \pm j20.97}$$ Closed-loop poles

Step 2: Calculate the desired error-constant, from the given e_{ss}.

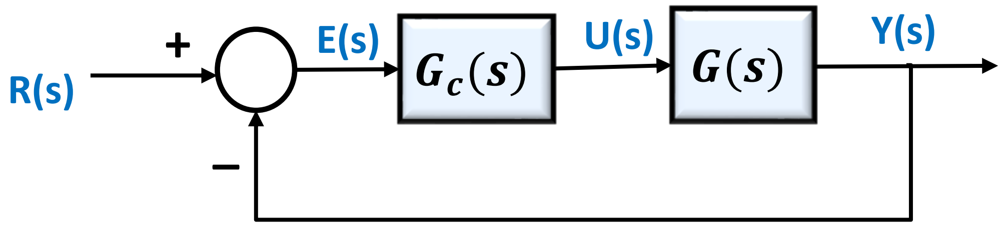
$$\boxed{e_{ss} = 0.03} \longrightarrow \quad e_{ss} = \frac{1}{1 + k_p} = 0.03 \quad \rightarrow \quad \boxed{k_p = 32.3}$$

Final Exam Review

Example 10

Consider the following closed-loop system

$$G(s) = \frac{1}{(s+10)(s+30)} \qquad G_c(s) = K_c \frac{s+z}{s+p}$$



b) Design a Lag Compensator to achieve the steady-state error of 3% ($e_{ss} = 0.03$) for unit-step input without altering the closed-loop poles of the designed-system in Part (b).

Step 3: Design a lag compensator to achieve the desired error value without altering the dominant poles

To not change the designed closed-loop poles with $K = 539.90$, the compensator's gain must be selected equal to K

$$\boxed{\boldsymbol{K_c = K = 539.90}}$$

Step-error constant for compensated system is

$$k_p = \lim_{s\to 0} G_c(s)G(s) = \lim_{s\to 0} K_c \frac{s+z}{s+p} \cdot \frac{1}{(s+10)(s+30)} = \frac{K_c z}{300p} \rightarrow 32.3 = \frac{539.9z}{300p} \rightarrow \boxed{z \approx 18p}$$

Pole/zero of the lag compensator must be selected far enough from the dominant closed-loop poles and close to the origin.

$$\boxed{\boldsymbol{s = -20 \pm j20.97}}$$

Closed-loop poles

$$z = 2 \;\rightarrow\; p = 0.11$$

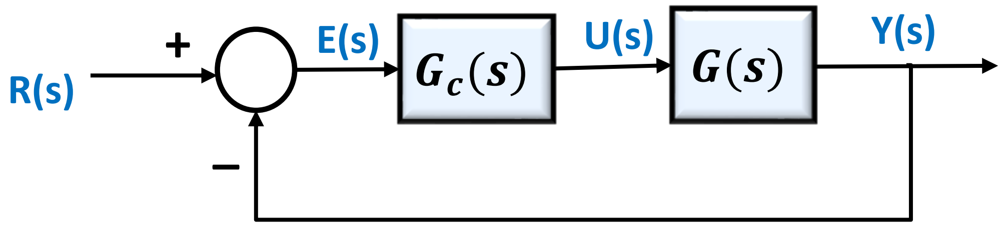
$$\boxed{G_c(s) = K_c \frac{s+z}{s+p} = 539.90 \frac{s+2}{s+0.11}}$$

Final Exam Review

Example 11

Consider the following closed-loop system

$$G(s) = \frac{s+2}{s(s+1)^2} \qquad G_c(s) = K$$



a) Determine whether or not it is possible to select a K value so that the dominant poles of the closed-loop system are located at $s_d = -1 \pm j\sqrt{3}$

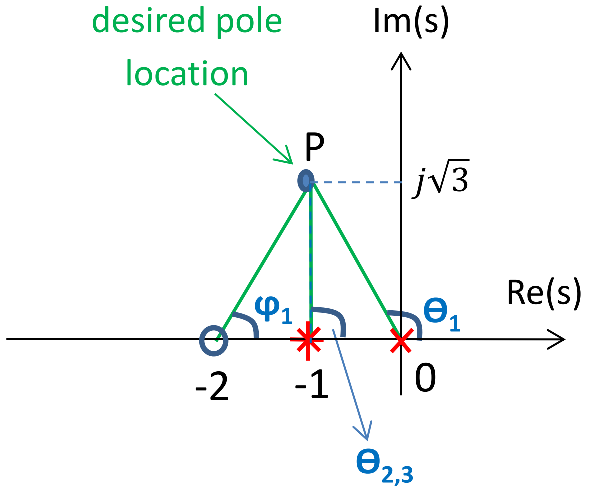
First, calculate the angle of $G(s)$ at the desired closed-loop pole location

$$\angle G(s)\Big|_{s=P} = \angle \frac{s+2}{s(s+1)^2}\Bigg|_{s=-1+j\sqrt{3}}$$

$$= \angle(s+2) - \angle s - \angle(s+1) - \angle(s+1)\Big|_{s=-1+j\sqrt{3}}$$

$$= \angle\varphi_1 - \angle\theta_1 - \angle\theta_2 - \angle\theta_3$$

$$= 60^\circ - 120^\circ - 90^\circ - 90^\circ = \mathbf{-240^\circ}$$



The angle condition is not satisfied

$$\boxed{240^\circ \neq (2i+1)180^\circ}$$

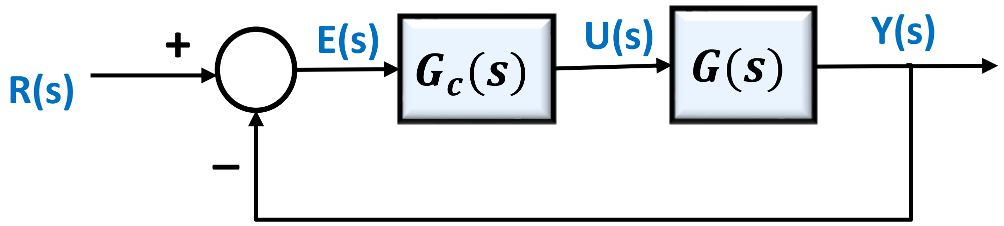
There is no K value to achieve the desired closed-loop poles

Final Exam Review

Example 11

Consider the following closed-loop system

$$G(s) = \frac{s+2}{s(s+1)^2} \qquad G_c(s) = K_c\frac{s+z}{s+p}$$



b) Design a lead compensator so that the compensated closed-loop system has dominant poles at $s_d = -1 \pm j\sqrt{3}$

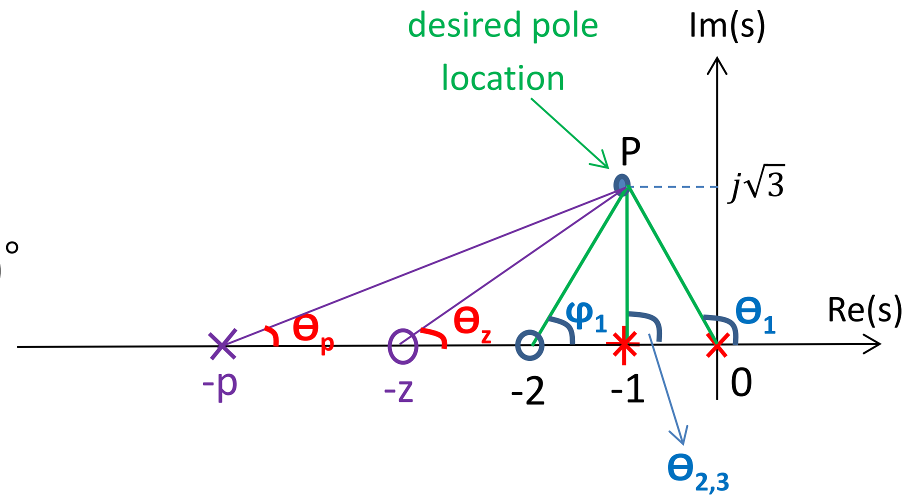
Step 4: Determine the required angle deficiency to satisfy the root-locus angle condition

First, find the angle deficiency → $\phi = 240^{\circ} - 180^{\circ} = \mathbf{60^{\circ}}$

Next, design a **lead compensator** to contribute the angle of $\boldsymbol{\phi} = \mathbf{60}^{\circ}$ at the desired poles location.

$$\angle G_c(s)G(s)\Big|_{s=-1+j\sqrt{3}} = \angle\frac{K_c(s+z)(s+2)}{s(s+p)(s+1)^2}\Bigg|_{s=-1+j\sqrt{3}}$$

$$= \underbrace{\angle K_c}_{0^{\circ}} + \underbrace{\angle\boldsymbol{\theta_z} - \angle\boldsymbol{\theta_p}}_{60^{\circ}} + \underbrace{\angle\varphi_1 - \angle\theta_1 - \angle\theta_2 - \angle\theta_3}_{-240^{\circ}} = -180^{\circ}$$

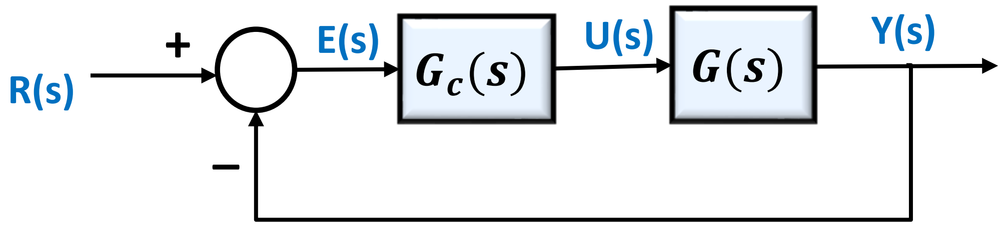


Final Exam Review

Example 11

Consider the following closed-loop system

$$G(s) = \frac{s+2}{s(s+1)^2} \qquad G_c(s) = K_c\frac{s+z}{s+p}$$



b) Design a lead compensator so that the compensated closed-loop system has dominant poles at $s_d = -1 \pm j\sqrt{3}$

Step 5: Determine pole/zero locations of the lead compensator to compensate the angle deficiency

Determine the pole/zero locations and the gain of the lead compensator.

- Draw lines PA and PO
- Draw bisector line PB so that

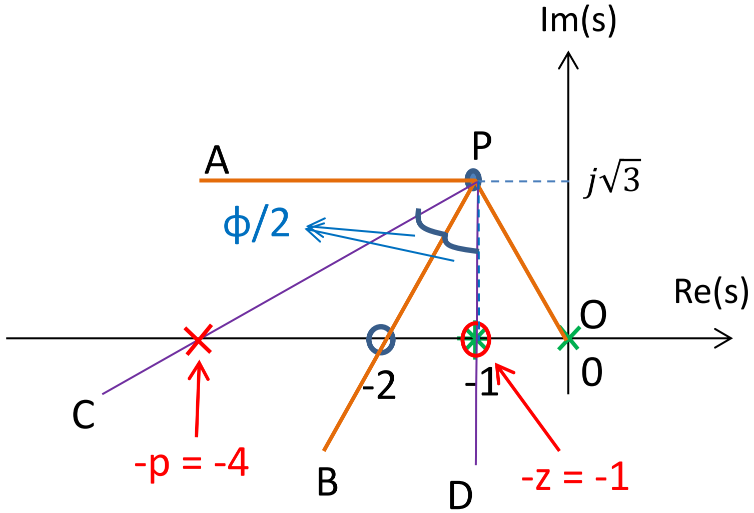
$$\angle APB = \angle BPO$$

- Draw lines PC and PD so that

$$\angle CPB = \angle BPD = \frac{\phi}{2} = 30^{\circ}$$

- Pole and zero are the intersections of PC and PD with real axis

$$\boxed{z = 1, \qquad p = 4}$$

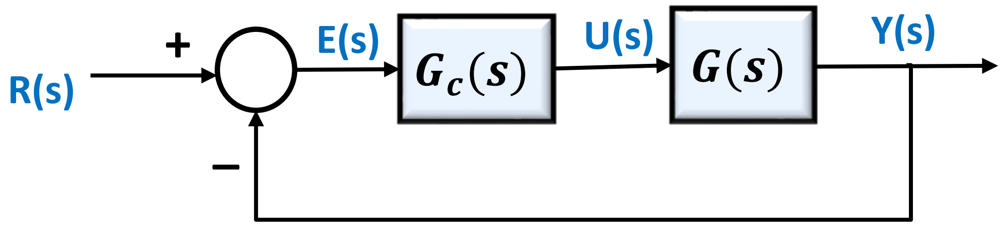


Final Exam Review

Example 11

Consider the following closed-loop system

$$G(s) = \frac{s+2}{s(s+1)^2} \qquad G_c(s) = K_c\frac{s+z}{s+p}$$



b) Design a lead compensator so that the compensated closed-loop system has dominant poles at $s_d = -1 \pm j\sqrt{3}$

Step 6: Determine gain of the lead compensator from magnitude condition

Magnitude condition at the desired pole locations

$$\left| K_c \frac{s+1}{s+4} \cdot \frac{s+2}{s(s+1)^2} \right|_{s=-1+j\sqrt{3}} = 1 \quad \rightarrow \quad |K_c| = \left. \frac{|s+4||s||s+1||s+1|}{|s+2||s+1|} \right|_{s=-1+j\sqrt{3}}$$

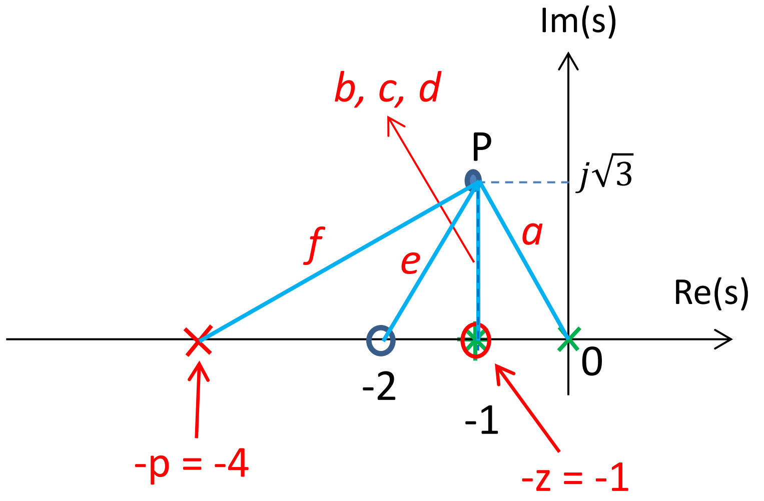
Method 1: Direct Calculation

$$|K_c| = \frac{|3+j\sqrt{3}||-1+j\sqrt{3}||j\sqrt{3}||j\sqrt{3}|}{|1+j\sqrt{3}||j\sqrt{3}|} = \frac{\sqrt{9+3}\,\sqrt{1+3}\,\sqrt{3}\,\sqrt{3}}{\sqrt{1+3}\,\sqrt{3}}$$

$\boldsymbol{K_c = 6}$

Method 2: Geometry

$$K_c = \frac{a \times b \times c \times f}{d \times e} = \frac{2 \times 1.7 \times 1.7 \times 3.5}{1.7 \times 2} = 5.95$$



$$\boxed{G_c(s) = K_c\frac{s+z}{s+p} = 6\frac{s+1}{s+4}}$$

Final Exam Review

Example 12 Consider the following transfer function model of a first-order system.

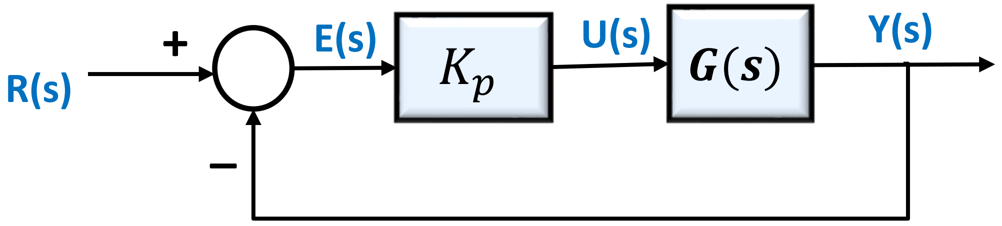
$$G(s) = \frac{Y(s)}{U(s)} = \frac{2.5}{35s + 1}$$

a) Determine the time-constant and steady-state gain of system.

Time-constant $\rightarrow$ $\tau = 35\ sec$, Steady-state gain $\rightarrow$ $K = 2.5$

b) The following closed-loop system with proportional control gain K_p has been developed to increase the speed of the system. Determine the required gain K_p to increase the speed 10 times faster than the current value.

First find the closed-loop transfer function:



$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_pG(s)}{1 + K_pG(s)H(s)} = \frac{\dfrac{2.5\ K_p}{35s + 1}}{1 + \dfrac{2.5\ K_p}{35s + 1}} = \frac{2.5\ K_p}{35s + 1 + 2.5\ K_p}$$

Find the time-constant of the closed-loop transfer function and make it equal to the desired time-constant, then find the required gain K_p.

Time-constant of the closed-loop system is: $\tau_{cl} = \frac{35}{1+2.5\ K_p}$

The desired time-constant is 35/10 = 3.5 sec. $\rightarrow$ $3.5 = \frac{35}{1+2.5\ K_p}$ $\rightarrow$ $3.5 + 8.75K_p = 35$ $\rightarrow$ $\boxed{K_p = 3.6}$

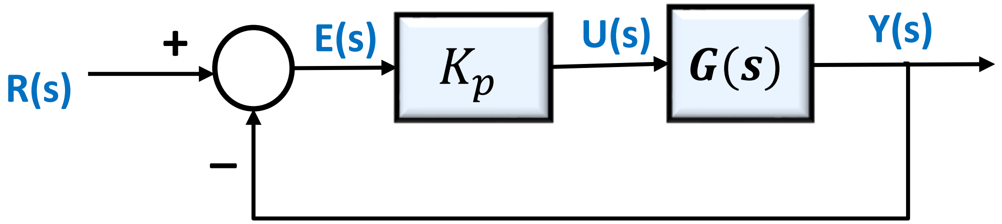
Final Exam Review

Example 12

Consider the following transfer function model of a first-order system.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2.5}{35s + 1}$$

c) The tracking error is defined as $E(s) = R(s) - Y(s)$. Determine the steady-state tracking error e_{ss} due to a unit-step response, $R(s) = 1/s$ for the obtained proportional gain K_p.



First, find the step-error constant

$$k_p = \lim_{s\to 0} K_p G(s) = \lim_{s\to 0}(3.6)\left(\frac{2.5}{35s+1}\right) = 9$$

The steady-state error for unit-step response is obtained as:

$$e_{ss} = \frac{1}{1+k_p} = \frac{1}{10} = 0.1 \quad \rightarrow \boxed{e_{ss} = 10\ \%}$$

Steady-state Error

Final Exam Review

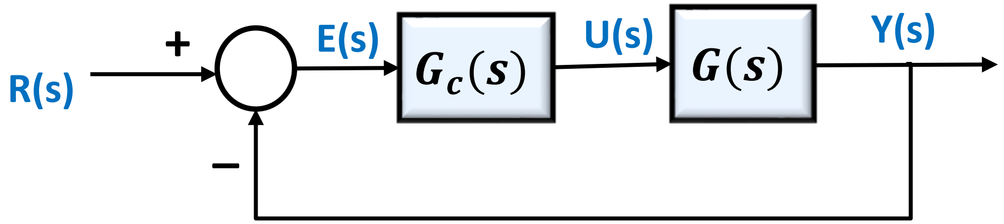
Example 12

Consider the following transfer function model of a first-order system.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2.5}{35s + 1}$$

d) Design a PI controller to achieve a zero steady-state error.

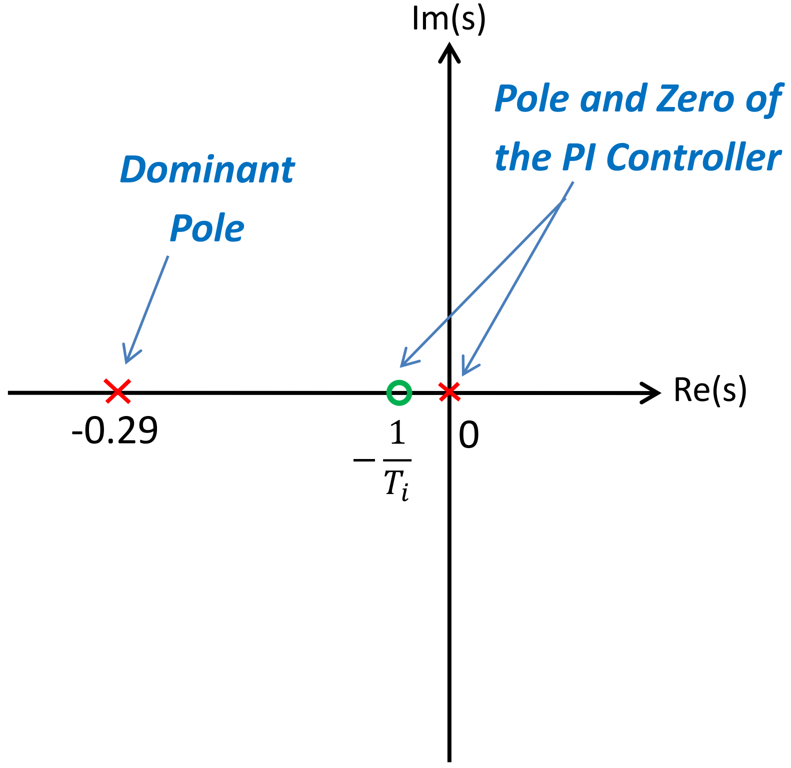
$$G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$



First, find the pole of the closed-loop transfer function for $K_p = 3.6$.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{9}{35s + 10} \quad \rightarrow \quad 35s + 10 = 0 \quad \rightarrow \quad s = -\frac{10}{35} = -0.29$$

The first-order closed-loop transfer function has one real stable pole.



The integral time constant T_i can be selected by the following stability consideration, where p_{cl} represent the closed-loop pole under the proportional control.

$$T_i \geq \frac{2}{|Re\{p_{cl}\}|} \quad \rightarrow \quad T_i = \frac{5}{0.29} = 17.2\ sec$$

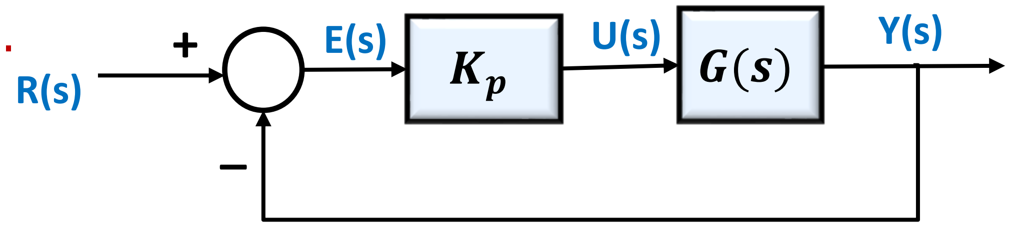
Therefore, the designed PI Controller is $\rightarrow$ $\boxed{G_c(s) = 3.6\left(1 + \frac{1}{17.2s}\right)}$

Final Exam Review

Example 13

Consider the transfer function model of a second-order dynamic system.

$$G(s) = \frac{1}{(s+1)(0.5s+1)}$$



a) Determine range of the proportional controller gain K_p to have the $\%O.S. \leq 5\%$

First find the closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_pG(s)}{1+K_pG(s)H(s)} = \frac{\dfrac{K_p}{(s+1)(0.5s+1)}}{1+\dfrac{K_p}{(s+1)(0.5s+1)}} = \frac{K_p}{0.5s^2+1.5s+1+K_p} = \frac{2K_p}{s^2+3s+2(1+K_p)}$$

Calculate the damping ratio from the required maximum overshoot value:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2+\ln^2(O.S.)}} \rightarrow \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2+\ln^2(0.05)}} \rightarrow \boxed{\zeta = 0.6901}$$

Desired Damping Ratio

Next, compare the characteristic equation with the standard second-order system to find the gain K_p.

$$s^2+2\zeta\omega_n s+\omega_n^2 = s^2+3s+2(1+K_p) \rightarrow \begin{cases} 2\zeta\omega_n = 3 & \rightarrow \quad 2(0.6901)\omega_n = 3 \quad \rightarrow \quad \omega_n = 2.17 \text{ rad/sec} \\ \omega_n^2 = 2(1+K_p) & \rightarrow \quad (2.17)^2 = 2+2K_p \quad \rightarrow \quad \boxed{K_p = 1.35} \end{cases}$$

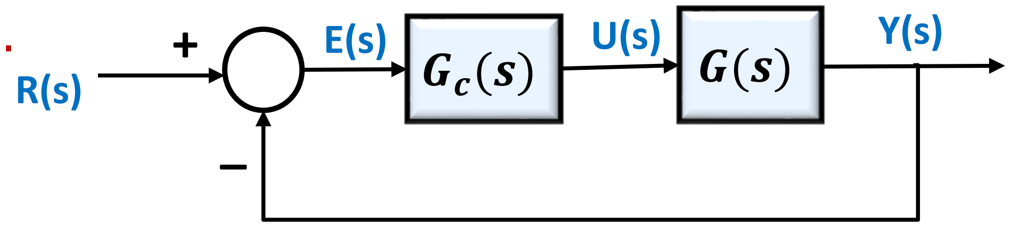
$$\boxed{\%O.S. \leq 5\% \quad \rightarrow \quad K_p < 1.36}$$

Final Exam Review

Example 13

Consider the transfer function model of a second-order dynamic system.

$$G(s) = \frac{1}{(s+1)(0.5s+1)}$$



b) Determine the steady-state tracking error e_{ss} due to a unit-step response, $R(s) = 1/s$ if the proportional gain is selected as $K_p = 1$.

First, find the step-error constant

$$k_p = \lim_{s \to 0} K_p G(s) = \lim_{s \to 0} (1)\left(\frac{1}{(s+1)(0.5s+1)}\right) = 1$$

The steady-state error for unit-step response is obtained as:

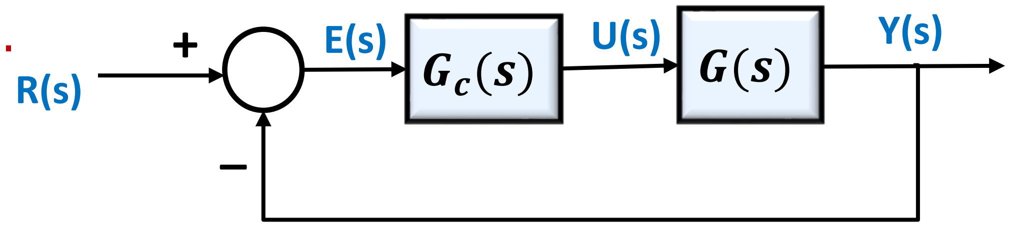
$$\boldsymbol{e_{ss} = \frac{1}{1+k_p}} = \frac{1}{2} \quad \rightarrow \quad \boxed{e_{ss} = 50\ \%}$$

Steady-state Error

Final Exam Review

Example 13 Consider the transfer function model of a second-order dynamic system.

$$G(s) = \frac{1}{(s+1)(0.5s+1)}$$



c) Design a PI controller to achieve a zero steady-state error.

$$\boldsymbol{G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)}$$

First, find the poles of the closed-loop transfer function for $K_p = 1$.

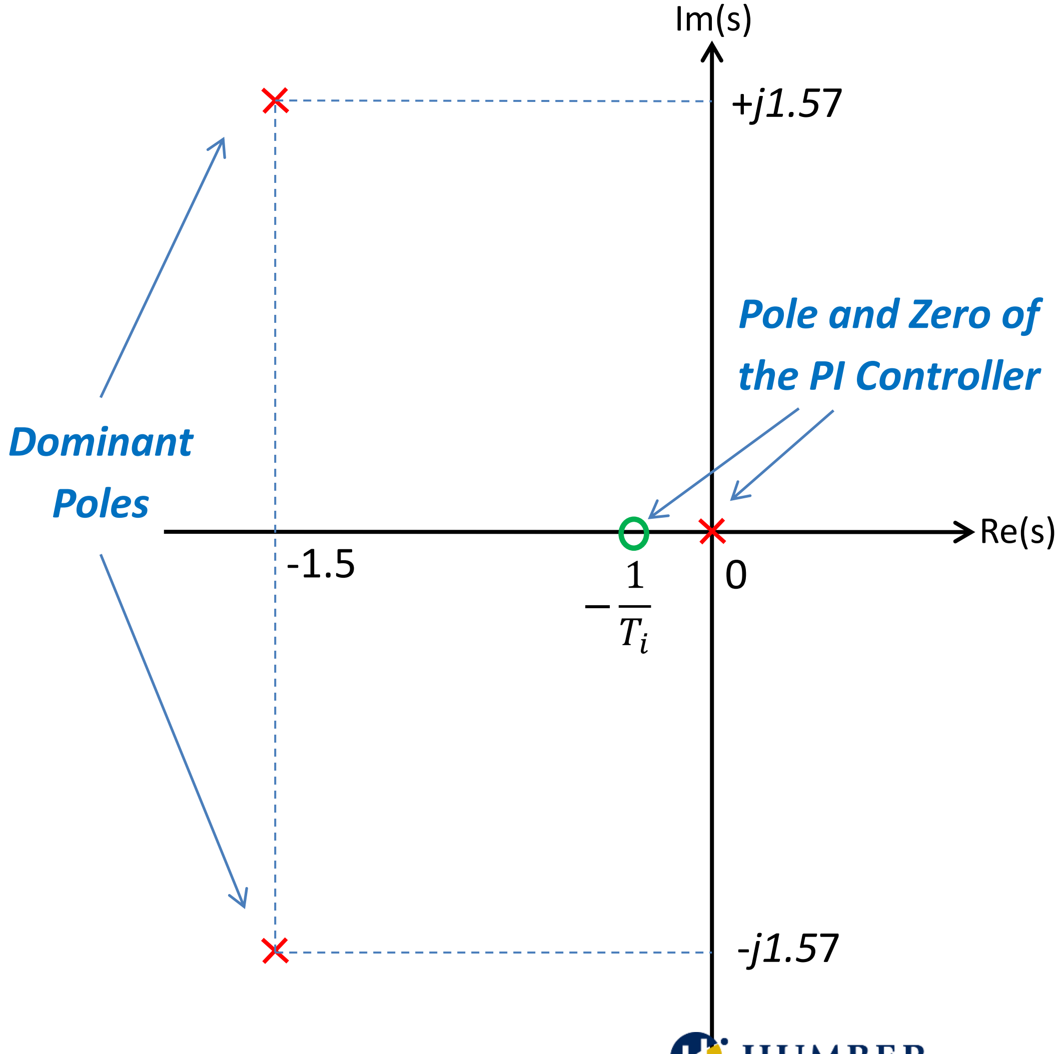
$$T(s) = \frac{Y(s)}{R(s)} = \frac{2}{s^2+3s+4} \rightarrow \quad s^2+3s+4=0 \quad \rightarrow \quad s = -1.5 \pm j1.32$$

The second-order closed-loop transfer function has one pair of complex-conjugate stable pole.

The integral time constant T_i can be selected by the following stability consideration, where p_{cl} represent the closed-loop pole under the proportional control.

$$\boldsymbol{T_i \geq \frac{2}{|Re\{p_{cl}\}|}} \quad \rightarrow \quad T_i = \frac{5}{1.5} = 3.33\ sec$$

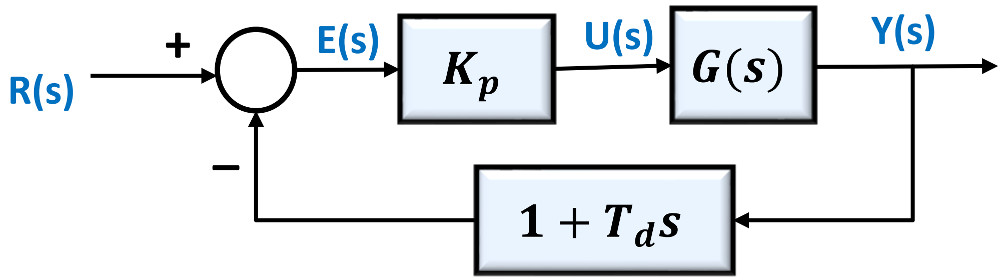
Therefore, the designed PI Controller is $\rightarrow$ $\boxed{\boldsymbol{G_c(s)} = 1 + \frac{1}{3.33s}}$



Final Exam Review

Example 14

Consider the following closed-loop system with proportional plus rate-feedback control



$$G(s) = \frac{1}{s(s+2)}$$

Determine the gains K_p and T_d so that the unit-step response has a maximum overshoot of 5% and the peak time of $t_p = 1sec.$

First, calculate the desired damping ratio from the given desired maximum overshoot

O.S. = 5% $\longrightarrow$ $\zeta = \dfrac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \rightarrow \zeta = \dfrac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \rightarrow \boxed{\zeta = 0.6901}$ **Desired Damping Ratio**

Then, calculate the undamped natural frequency from the given peak time value:

$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} \rightarrow 1 = \dfrac{\pi}{\omega_n\sqrt{1-(0.69)^2}} \rightarrow \boxed{\omega_n = 4.3409 \text{ rad/sec}}$ **Desired Natural Frequency**

Having the desired damping ratio and natural frequency, determine the desired characteristic equation for this closed-loop system

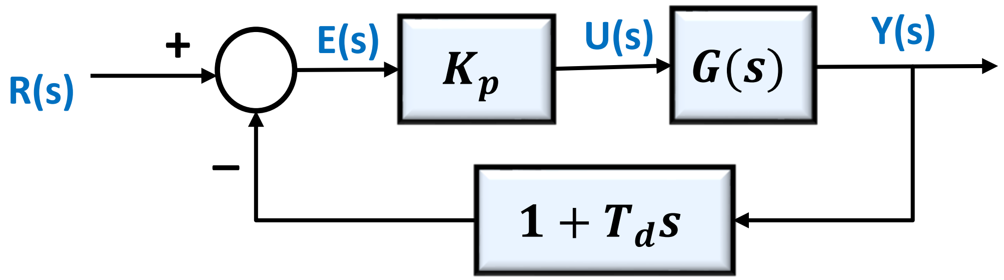
$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 5.99s + 18.84$ **Desired Characteristic Equation**

Final Exam Review

Example 14

Consider the following closed-loop system with proportional plus rate-feedback control

$$G(s) = \frac{1}{s(s+2)}$$



Determine the gains K_p and T_d so that the unit-step response has a maximum overshoot of 5% and the peak time of $t_p = 1sec$.

Find the transfer function of the closed-loop system

$$\frac{Y(s)}{R(s)} = \frac{G}{1+GH} = \frac{K_p \frac{1}{s(s+2)}}{1 + K_p \frac{1}{s(s+2)}(1+T_d s)} = \frac{\frac{K_p}{s(s+2)}}{\frac{s(s+2) + K_p(1+T_d s)}{s(s+2)}} = \frac{K_p}{s^2 + \left(2 + K_p T_d\right)s + K_p}$$

Compare the desired characteristic equation with the characteristic equation of the closed-loop system

$$s^2 + 5.99s + 18.84 = s^2 + \left(2 + K_\mathrm{p} T_\mathrm{d}\right)s + K_\mathrm{p}$$

$$\begin{cases} 2 + K_p T_d = 5.99 \\ K_p = 18.84 \end{cases} \rightarrow \boxed{K_p = 18.84}\,, \quad \boxed{T_d = 0.21}$$

THANK YOU

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