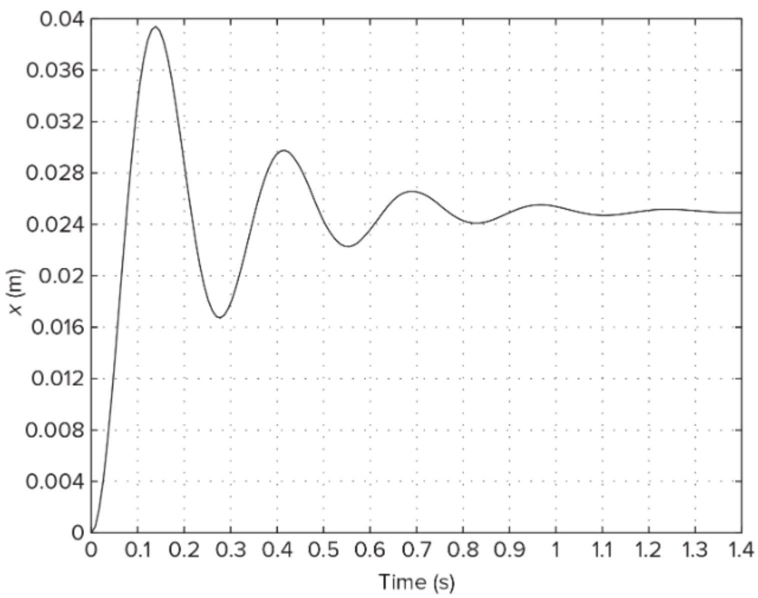
Worksheet 8 - Solution

1) The following figure shows the response of a system to a step input of magnitude 1000 N. The equation of motion is:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$$

Estimate the values of $m, b,$ and k.



From the equation of motion, the transfer function model of the system is obtained as:

$$ms^2X(s) + bsX(s) + kX(s) = F(s) \qquad \rightarrow \qquad G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Since the model is a second-order transfer function, we can find the model parameters in terms of the **DC-gain, damping ratio** and **undamped natural frequency** and determine those values from the step response and estimate the system parameters.

$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \qquad \text{and} \qquad \boldsymbol{G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

Comparing the transfer function with the standard form we have:

$$k = \frac{1}{K}, \qquad m = \frac{1}{K\omega_n^2}, \qquad b = \frac{2\zeta}{K\omega_n}$$

From the given step response we can find the steady-state value, peak-time and maximum deviation from the steady-state value:

$$x_{ss} = 0.025 \text{ m}, \;\; t_p = 0.14sec, \;\;\; M_p = 0.038 - 0.025 = 0.013 \text{ m}$$

The DC-gain of system is obtained as:

$$K = \frac{\Delta x}{\Delta f} = \frac{x_{ss} - x_0}{f_{ss} - f_0} = \frac{0.025 - 0}{1000 - 0} = 2.5 \times 10^{-5} m/N$$

The damping ratio is determined as:

$$O.S. = \frac{M_p}{x_{ss}} = \frac{0.013}{0.025} = 0.52$$

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} = \frac{-\ln(0.52)}{\sqrt{\pi^2 + \ln^2(0.52)}} = 0.204$$

The undamped natural frequency is calculated as:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad \rightarrow \quad 0.14 = \frac{\pi}{\omega_n\sqrt{1-(0.204)^2}} \quad \rightarrow \quad \omega_n = 22.92\, rad/s$$

The system parameters are estimated as:

$$k = \frac{1}{K} \quad \rightarrow \quad k = \frac{1}{2.5 \times 10^{-5}} = 40000\ \ N/m$$

$$m = \frac{1}{K\omega_n^2} \quad \rightarrow \quad m = \frac{1}{2.5 \times 10^{-5}(22.92)^2} = 76.14\, kg$$

$$b = \frac{2\zeta}{K\omega_n} \quad \rightarrow \quad b = \frac{2(0.204)}{2.5 \times 10^{-5}(22.92)} = 698.1\, N.s/m$$

2) For the following model, find the steady-state response and use the domiant pole approximation to find the dominat response (how long will take to reach steady-state? does it oscillate?). The initial conditions are zero. ($u_s(t)$ is the unit-step signal).

$$\dddot{x}(t) + 22\ddot{x}(t) + 131\dot{x}(t) + 110x(t) = u_s(t)$$

Obtain the exact solution for the response of the above model, and use it to check the prediction based on the dominant pole approximation.

We can find the transfer function and the chracteristics equation from the given differential equation:

$$s^3X(s) + 22s^2X(s) + 131sX(s) + 110X(s) = U_s(s)$$

$$G(s) = \frac{X(s)}{U_s(s)} = \frac{1}{s^3 + 22s^2 + 131s + 110}$$

Since, $u_s(t)$ is a unit-step input the steady-state response is:

$$x_{ss} = \lim_{s\to 0} sG(s) = s\left(\frac{1}{s^3 + 22s^2 + 131s + 110}\right)\left(\frac{1}{s}\right) = \frac{1}{110} = 0.0091$$

The characteristics equation and the poles are:

$$s^3 + 22s^2 + 131s + 110 = 0 \quad \rightarrow \quad s = -1, \quad s = -10, \quad s = -11$$

The dominant pole is $s = -1$ and the dominant time constant is 1sec.

Since all poles are real, there will be no oscillations in the response, and the steady-state response will be reached at approximately $t_s \approx 4(1\text{ sec}) = 4\,sec$

We can apply the partial fraction expansion to find the exact solution:

$$X(s) = \frac{1}{s^3 + 22s^2 + 131s + 110} U_s(s) = \frac{1}{(s+1)(s+10)(s+11)s} = \frac{A}{s+1} + \frac{B}{s+10} + \frac{C}{s+11} + \frac{D}{s}$$

$$A = \left[\frac{1}{(s+10)(s+11)s}\right]_{s=-1} = -0.011$$

$$B = \left[\frac{1}{(s+1)(s+11)s}\right]_{s=-10} = 0.011$$

$$C = \left[\frac{1}{(s+1)(s+10)s}\right]_{s=-11} = -0.0091$$

$$D = \left[\frac{1}{(s+1)(s+10)(s+11)}\right]_{s=0} = 0.0091$$

$$X(s) = -\frac{0.011}{s+1} + \frac{0.011}{s+10} - \frac{0.0091}{s+11} + \frac{0.0091}{s}$$

Taking the inverse Laplce transform the exact solution is:

$$x(t) = -0.011e^{-t} + 0.011e^{-10t} - 0.0091e^{-11t} + 0.0091$$

From the exact solution we can also find the staedy-state response as:

$$x_{ss} = \lim_{t\to\infty} x(t) = 0.0091$$

3) The following model has a dominant poles of $s = -3 \pm j5$ as long as the parameter μ is no less than 3. ($u_s(t)$ is the unit-step signal).

$$\dddot{y}(t) + (6+\mu)\ddot{y}(t) + (34+6\mu)\dot{y}(t) + 34\mu y(t) = u_s(t)$$

Investigate the accuracy of the estimate of the maximum overshoot, the peak time, the rise time, and the 2% settling time based on the dominant root, for the following cases: (a) $\mu = 30$, (b) $\mu = 6$, and $\mu = 3$. Discuss the effect of μ on these predictions.

The factor of the given dominant poles pair is:

$$(s+3+j5)(s+3-j5) = s^2 + 6s + 34$$

To find the third pole $s = -p$, we can expand the equation as follows:

$$(s^2 + 6s + 34)(s+p) = s^3 + (6+p)s^2 + (34+6p)s + 34p$$

Next, we can find the characteristic polynomial from the given differential equation and compare it with the obtained one:

$$s^3Y(s) + (6+\mu)s^2Y(s) + (34+6\mu)sY(s) + 34\mu Y(s) = U_s(s)$$

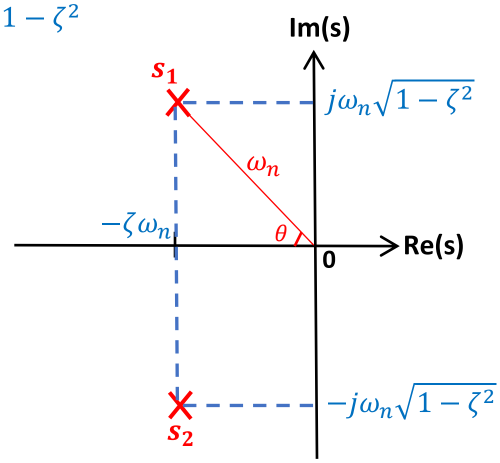
The characteristic polynomial is:

$$s^3 + (6+\mu)s^2 + (3+6\mu)s + 34 = 0$$

We can see that the third pole is $s = -\mu$. Thus, as long as $\mu > 3$, the poles at $s = -3 \pm j5$ will be closer to the imaginary axis, which means is dominant.

We can find the damping ratio and the undamped natural frequency from the dominant poles:

$$s = -3 \pm j5 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



The damping ratio is otained as:

$$\zeta = \cos\theta = \cos\left[\tan^{-1}\left(\frac{\mathrm{Im}[s]}{\mathrm{Re}[s]}\right)\right] = \cos\left[\tan^{-1}\left(\frac{5}{3}\right)\right] = 0.514$$

The undamped natural frequency is obtained as:

$$\omega_n = \sqrt{\mathrm{Re}[s]^2 + \mathrm{Im}[s]^2} = \sqrt{9+25} = \sqrt{34} = 5.83\ rad/s$$

The settling time from the dominant poles is about:

$$t_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{3} = 1.33\ sec$$

The maximum overshoot is estimated as:

$$O.S.\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% = 0.15 \times 100\% = 15\%$$

The peak-time and rise-time are estimated as:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{5.83\sqrt{1-(0.514)^2}} = 0.63\ sec$$

$$t_r = \frac{0.8+2.5\zeta}{\omega_n} = \frac{0.8+2.5(0.514)}{5.83} = 0.36\ sec$$

The above calculations are based on the dominant poles approximation, which is less accurate when the root separation factor is small. Thus the calculations for $\mu = 30$ have the most accuracy.

4) Estimate the maximum overshoot, the peak time, and the rise time of the unit-step response of the following model if $f(t) = 5000u_s(t)$ and the initial conditions are zero.

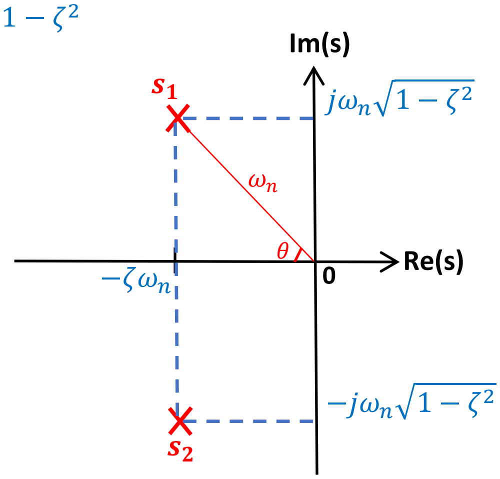
$$\frac{d^4y}{dt^4} + 26\frac{d^3y}{dt^3} + 269\frac{d^2y}{dt^2} + 1524\frac{dy}{dt} + 4680y(t) = f(t)$$

Its poles are $s = -3 \pm j6$ and $s = -10 \pm j2$.

The dominant pole pair is $s = -3 \pm j6$.

We can find the damping ration and the undamped natural frequency from the dominant poles:

$$s = -3 \pm j6 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



The damping ratio is otained as:

$$\zeta = \cos\theta = \cos\left[\tan^{-1}\left(\frac{\mathrm{Im}[s]}{\mathrm{Re}[s]}\right)\right] = \cos\left[\tan^{-1}\left(\frac{6}{3}\right)\right] = 0.447$$

The undamped natural frequency is obtained as:

$$\omega_n = \sqrt{\mathrm{Re}[s]^2 + \mathrm{Im}[s]^2} = \sqrt{9+36} = \sqrt{45} = 6.708\ rad/s$$

The settling time from the dominant poles is about:

$$t_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{3} = 1.33\ sec$$

The maximum overshoot is estimated as:

$$O.S.\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% = 0.21 \times 100\% = 21\%$$

The peak-time and rise-time are estimated as:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{6.708\sqrt{1-(0.447)^2}} = 0.524\ sec$$

$$t_r = \frac{0.8+2.5\zeta}{\omega_n} = \frac{0.8+2.5(0.447)}{6.708} = 0.286\ sec$$

5) What is the form of the unit-step response of the following model? Find the steady-state response. How long does the response take to reach steady-state?

$$2\frac{d^4y}{dt^4} + 52\frac{d^3y}{dt^3} + 6250\frac{d^2y}{dt^2} + 4108\frac{dy}{dt} + 1.1202 \times 10^4 y(t) = 5 \times 10^4 f(t)$$

First, find the characteristics equation and the poles:

$$2s^4Y(s) + 52s^3Y(s) + 6250s^2Y(s) + 4108sY(s) + 1.1202 \times 10^4 Y(s) = 5 \times 10^4 F(s)$$

Characteristic equation is:

$$2s^4 + 52s^3 + 6250s^2 + 4108s + 1.1202 \times 10^4 = 0$$

Poles are:

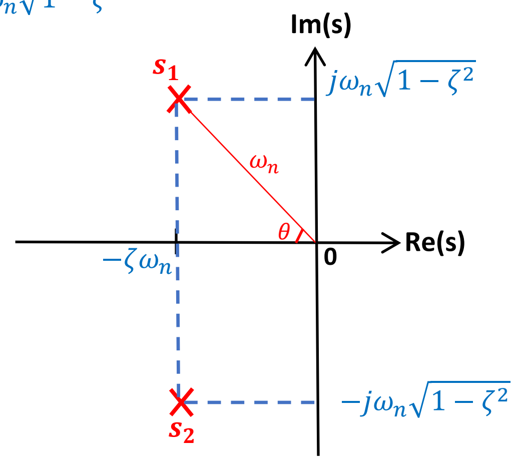
$$s = -12.68 \pm j5.28 \qquad and \qquad s = -0.323 \pm j1.3$$

The dominant pole pair is: $s = -0.323 \pm j1.3$

Since, dominant poles are complex pair, the unit-step response will be under-damped with oscillatory transient response. The overall step response can be approximated based on the dominant poles.

Thus, we can find the damping ration and the undamped natural frequency from the dominant poles:

$$s = -0.323 \pm j1.3 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



The damping ratio is otained as:

$$\zeta = \cos\theta = \cos\left[\tan^{-1}\left(\frac{\mathrm{Im}[s]}{\mathrm{Re}[s]}\right)\right] = \cos\left[\tan^{-1}\left(\frac{1.3}{0.323}\right)\right] = 0.241$$

The undamped natural frequency is obtained as:

$$\omega_n = \sqrt{\mathrm{Re}[s]^2 + \mathrm{Im}[s]^2} = \sqrt{0.323^2 + 1.3^2} = 1.34\, rad/s$$

The DC-gain (steady-state response value) of the system is:

$$K = \lim_{s\to 0} sG(s) = \frac{5 \times 10^4}{1.1202 \times 10^4} = 4.46$$

The damped natural frequency is:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.34\sqrt{1-0.241^2} = 1.3\, rad/s$$

The estimated step response based on the dominant poles will be:

$$y(t) = K - \frac{Ke^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(sin(\omega_d t + cos^{-1}\zeta)) \quad \rightarrow \quad y(t) = 4.46 - \frac{4.46e^{-0.323t}}{\sqrt{1-0.241^2}}(sin(1.3t + 76.05°))$$

$$y(t) = 4.46 - 4.6e^{-0.323t}(sin(1.3t + 76.05°))$$

The settling time from the dominant poles is:

$$t_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{0.323} = 12.38\, sec$$

which represent the required time for response to reach the steady-state.

6) Use a software package such as MATLAB to plot the step resposne of the folowing model for three cases: $a = 0.2$, $a = 1$ and $a = 10$. The step input has a magnitude of 2500.

$$\frac{d^4y}{dt^4} + 24\frac{d^3y}{dt^3} + 225\frac{d^2y}{dt^2} + 900\frac{dy}{dt} + 2500y(t) = f(t) + a\frac{df}{dt}$$

Compare the response to that predicted by the maximum overshoot, peak time, rise time, and 2% settling time calculated from the dominant poles.

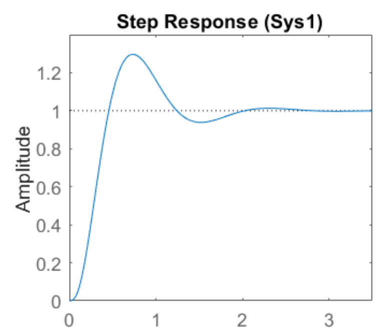
First, find the transfer function model of the system,

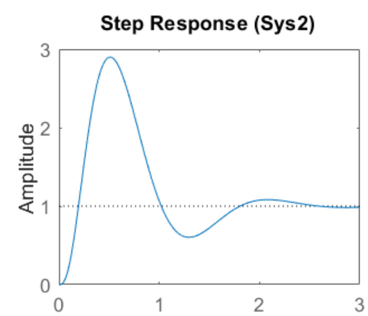
$$s^4Y(s) + 24s^3Y(s) + 225s^2Y(s) + 900sY(s) + 2500 = F(s) + asF(s)$$

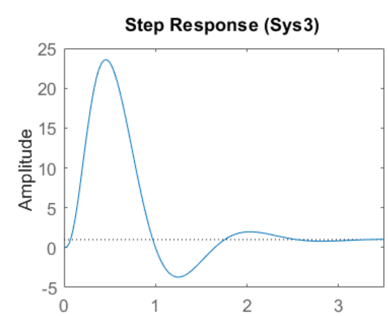
$$G(s) = \frac{Y(s)}{F(s)} = \frac{as + 1}{s^4 + 24s^3 + 225s^2 + 900s + 2500}$$

The MATLAB code to plot the step response and find the time response specifications is:

```
den = [1 24 225 900 2500];

num1 = 2500*[0.2 1];
sys1 = tf(num1,den);
figure; stepplot(sys1)
stepinfo(sys1)

num2 = 2500*[1 1];
sys2 = tf(num2,den);
figure; stepplot(sys2)
stepinfo(sys2)

num3 = 2500*[10 1];
sys3 = tf(num3,den);
figure; stepplot(sys3)
stepinfo(sys3)
```







The step response specifications are obtained as:

a	Overshoot	Peak time	Rise time	Settling time
0.2	30%	0.7368	0.2848	1.9
1	190%	0.5066	0.1130	2.9
10	2256%	0.4605	0.0385	2.3

We can estimate the system as a second-order system based on the dominant poles and compare the step response specifications with the obtained values for MATLAB simulation.

First, find the characteristics equation and the poles:

$$s^4Y(s) + 24s^3Y(s) + 225s^2Y(s) + 900sY(s) + 2500 = F(s) + asF(s)$$

Characteristic equation is:

$$s^4 + 24s^3 + 2250s^2 + 900s + 2500 = 0$$

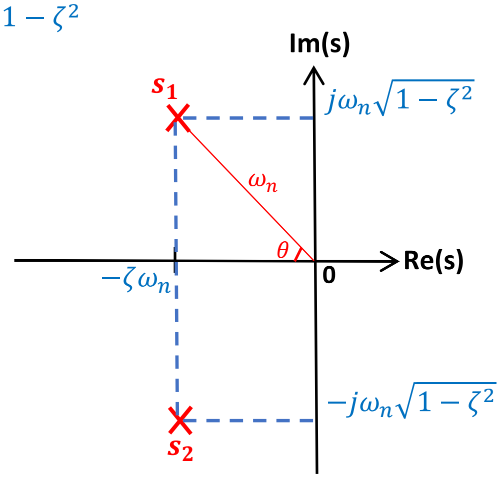
Poles are:

$$s = -10 \pm j5 \qquad and \qquad s = -2 \pm j4$$

The dominant pole pair is: $s = -2 \pm j4$

We can find the damping ration and the undamped natural frequency from the dominant poles:

$$s = -2 \pm j4 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



The damping ratio is otained as:

$$\zeta = \cos\theta = \cos\left[\tan^{-1}\left(\frac{\mathrm{Im}[s]}{\mathrm{Re}[s]}\right)\right] = \cos\left[\tan^{-1}\left(\frac{4}{2}\right)\right] = 0.447$$

The undamped natural frequency is obtained as:

$$\omega_n = \sqrt{\mathrm{Re}[s]^2 + \mathrm{Im}[s]^2} = \sqrt{4+16} = \sqrt{25} = 5\, rad/s$$

The settling time from the dominant poles is about:

$$t_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{2} = 2\, sec$$

The maximum overshoot is estimated as:

$$O.S.\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% = 0.21 \times 100\% = 21\%$$

The peak-time and rise-time are estimated as:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{5\sqrt{1-(0.447)^2}} = 0.702\, sec$$

$$t_r = \frac{0.8+2.5\zeta}{\omega_n} = \frac{0.8+2.5(0.447)}{5} = 0.384\, sec$$

Compare these values with the MATLAB simulation results, we can conclude that the dominant pole approimation is valid for small values of a, which means that the zero at $-1/a$ is far enough from the dominant poles.