HUMBER ENGINEERING

MENG 3510 – Control Systems
LECTURE 6

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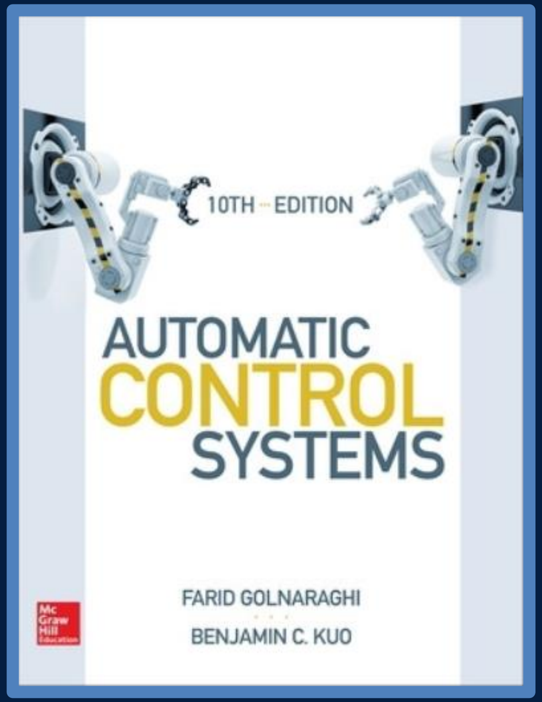
LECTURE 6
Root Locus Analysis

- Property of Transfer Functions
    - Proper, Improper and Strictly Proper Transfer Function
    - Poles & Zeros at Infinity
- Evaluation of Complex Functions
- Closed-loop Relation to Open-loop Gain
- Root Locus Method
    - Root Locus Plotting Rules for $K \geq 0$
    - Root Locus Examples



Chapter 9

Property of a Transfer Function

Proper, Improper and Strictly Proper Transfer Function

- Consider the following general form of a transfer function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

  - If $n = m$ the system is called proper
  - If $n > m$ the system is called strictly proper
  - If $n < m$ the system is called improper

Example 1

Determine property of the following transfer functions

$$G_1(s) = \frac{s^2 + 1}{s^3 + 2s + 1} \quad , \qquad G_2(s) = \frac{s^2 + 1}{s^2 + 5} \quad , \qquad G_3(s) = \frac{s^3 + 5s + 1}{s}$$

$G_1(s)$	$G_2(s)$	$G_3(s)$
$n = 3, m = 2$	$n = 2, m = 2$	$n = 1, m = 3$
$n > m$	$n = m$	$n < m$
Strictly proper system	**Proper system**	**Improper system**

Property of a Transfer Function

❑ Poles & Zeros at Infinity

- Consider the following general form of a transfer function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

- Recall the pole and zero definitions:
  - Poles are values of s that make the transfer function infinite.
  - Zeros are values of s that make the transfer function zero.

- Based on these definitions, theoretically (mathematically), we can also define poles and zeros at infinity

- In the improper transfer functions $(n < m)$ the following limit will be infinite, so we can assume that the transfer function has some **poles** at infinity:

$$\boxed{\lim_{s \to \infty} G(s) = \infty}$$

- In the strictly proper transfer functions $(n > m)$ the following limit will be zero, so we can assume that the transfer function has some **zeros** at infinity:

$$\boxed{\lim_{s \to \infty} G(s) = 0}$$

Property of a Transfer Function

Poles & Zeros at Infinity

- Consider the following general form of a transfer function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

  - **Proper transfer functions** $(n = m)$:

$$\lim_{s\to\infty} G(s) = \frac{b_m}{a_n} \Longrightarrow \text{No poles/zeros at infinity}$$

  - **Strictly proper transfer functions** $(n > m)$:

$$\lim_{s\to\infty} G(s) = 0 \Longrightarrow n - m \text{ zeros at infinity}$$

  - **Improper transfer functions** $(n < m)$:

$$\lim_{s\to\infty} G(s) = \infty \Longrightarrow m - n \text{ poles at infinity}$$

- Practically, we only consider the finite poles and zeros, and we analyze the control system based on the location of finite poles/zeros.
- **Considering the infinite poles/zeros, the total number of poles and zeros will always be equal.**

Property of a Transfer Function

Example 2 Find the poles and zeros of the following transfer functions (including ones at infinity, if any).

$$G(s) = \frac{2(s+4)}{s(s+6)(s-1)}$$

Strictly proper ($n > m$)

- Finite zeros: $s + 4 = 0 \quad \rightarrow \quad s = -4 \quad \rightarrow$ one finite zero
- Finite poles: $s(s+6)(s-1) = 0 \quad \rightarrow \quad s_1 = 0,\ s_2 = -6,\ s_3 = 1 \quad \rightarrow$ three finite poles
- Infinite poles/zeros: $\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{2(s+4)}{s(s+6)(s-1)} = 0 \quad \rightarrow \quad n - m = 2$ Two zeros at infinity

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$$G(s) = \frac{2(s+1)(s+4)}{(s+8)}$$

Improper ($n < m$)

- Finite zeros: $(s+1)(s+4) = 0 \quad \rightarrow \quad s_1 = -1,\ s_2 = -4 \quad \rightarrow$ two finite zeros
- Finite poles: $(s+8) = 0 \quad \rightarrow \quad s = -8 \quad \rightarrow$ one finite pole
- Infinite poles/zeros: $\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{2(s+1)(s+4)}{(s+8)} = \infty \quad \rightarrow \quad m - n = 1$ One pole at infinity

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$$G(s) = \frac{(s-2)(s+3)}{5(s^2+2s+2)}$$

Proper ($n = m$)

- Finite zeros: $(s-2)(s+3) = 0 \quad \rightarrow \quad s_1 = 2,\ s_2 = -3 \quad \rightarrow$ two finite zeros
- Finite poles: $s^2 + 2s + 2 = 0 \quad \rightarrow \quad s_1 = -1 + j1,\ s_2 = -1 - j1 \quad \rightarrow$ two finite poles
- Infinite poles/zeros: $\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{(s-2)(s+3)}{5(s^2+2s+2)} = \frac{1}{5} \quad \rightarrow \quad m - n = 0$

No poles and zeros at infinity

Evaluation of Complex Functions

- Consider the following general form of a transfer function in pole-zero form:

$$\boldsymbol{G(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}}$$

where z_i are the zeros and p_i are the poles of the transfer function.

- At any value of s, i.e. any point in the complex plane, $G(s)$ evaluates to a complex number in polar form with magnitude and phase.

$$\boxed{\boldsymbol{G(s) = |G(s)|\,\angle G(s)}}$$

where the magnitude and phase are determined as:

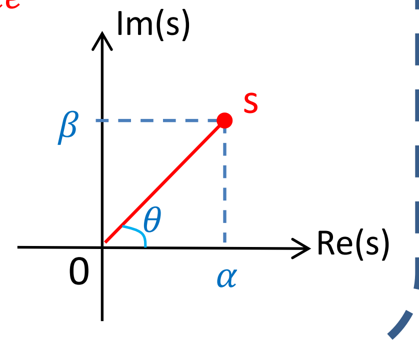
$$\boxed{\boldsymbol{|G(s)| = \frac{|K||s-z_1||s-z_2|\cdots|s-z_m|}{|s-p_1||s-p_2|\cdots|s-p_n|} = \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|}}}$$

$$\boxed{\boldsymbol{\angle G(s) = \angle K + \sum_{i=1}^{m}\angle(s-z_i) - \sum_{i=1}^{n}\angle(s-p_i)}}$$

$$s = \alpha + j\beta = \underbrace{|\alpha + j\beta|}_{magnitude}\ \underbrace{\angle(\alpha + j\beta)}_{angle}$$

$$|\alpha + j\beta| = \sqrt{\alpha^2 + \beta^2}$$

$$\angle(\alpha + j\beta) = \theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$$



- The magnitude and phase values can be determined either by calculation or a graphical method.
- The angles are measured from the positive extension of the real axis.

Evaluation of Complex Functions

Example 3 Evaluate the given transfer function at point $s = -3 + j4$

$$G(s) = \frac{s+1}{s(s+2)}$$

Method 1: Calculation of the gain by evaluation of $|G(s)|$ at point $s = -3 + j4$

$$|G(s)|\Big|_{s=-3+j4} = \frac{|s+1|}{|s||s+2|}\Bigg|_{s=-3+j4} = \frac{|-2+j4|}{|-3+j4||-1+j4|} = \frac{\sqrt{4+16}}{(\sqrt{9+16})(\sqrt{1+16})} = \frac{\sqrt{20}}{\sqrt{25 \times 17}} = 0.217$$

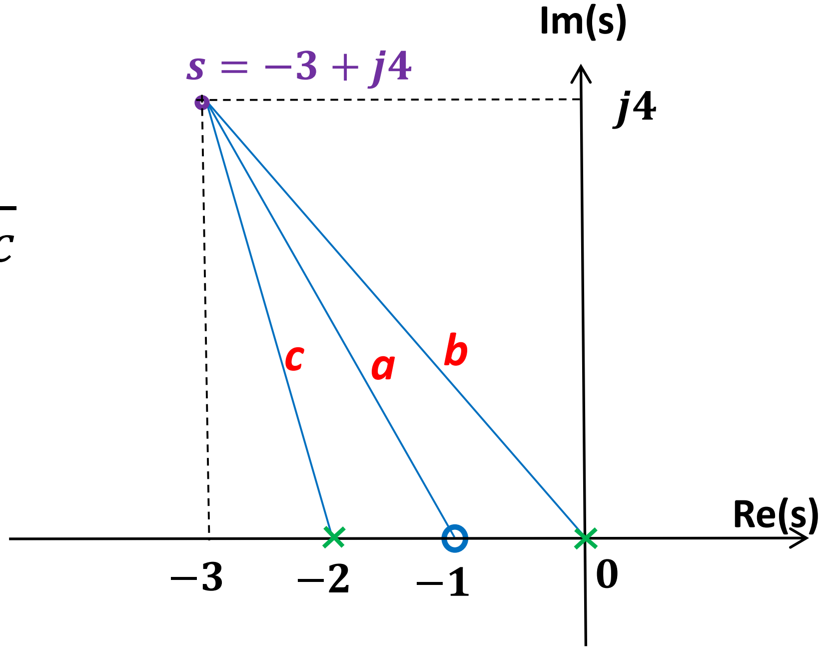
Method 2: Graphically measure the length of the vectors from point $s = -3 + j4$

$a = |-2 + j4|$

$b = |-3 + j4|$

$c = |-1 + j4|$

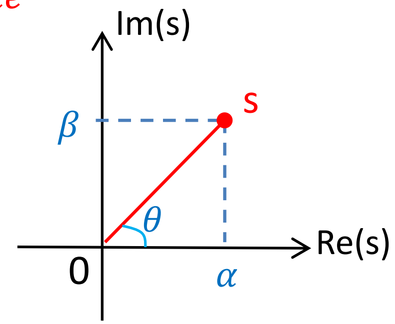
$$|G(s)|\Big|_{s=-3+j4} = \frac{|s+1|}{|s||s+2|}\Bigg|_{s=-3+j4} = \frac{a}{b \times c}$$



$$s = \alpha + j\beta = \underbrace{|\alpha + j\beta|}_{magnitude}\ \underbrace{\angle(\alpha + j\beta)}_{angle}$$

$$|\alpha + j\beta| = \sqrt{\alpha^2 + \beta^2}$$

$$\angle(\alpha + j\beta) = \theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$$



Evaluation of Complex Functions

Example 3 Evaluate the given transfer function at point $s = -3 + j4$

$$G(s) = \frac{s+1}{s(s+2)}$$

Method 1: Calculation of the angle by evaluation of $\angle G(s)$ at point $s = -3 + j4$

$$\angle G(s)\Big|_{s=-3+j4} = [\angle(s+1)] - [\angle s + \angle(s+2)]\Big|_{s=-3+j4} = \angle(-2+j4) - \angle(-3+j4) - \angle(-1+j4)$$

$$\angle G(s)\Big|_{s=-3+j4} = \tan^{-1}\left(\frac{4}{-2}\right) - \tan^{-1}\left(\frac{4}{-3}\right) - \tan^{-1}\left(\frac{4}{-1}\right) = 116.6° - 126.9° - 104.0° = -114.3°$$

Method 2: Graphically measure the angles

The angles are measured from the positive extension of the real axis.

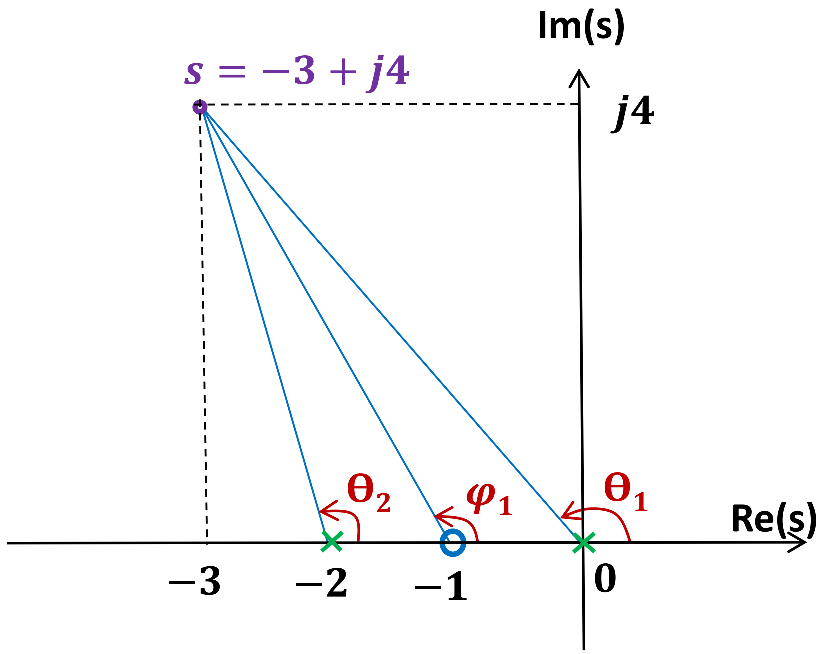
$$\varphi_1 = \angle(-2 + j4)$$

$$\theta_1 = \angle(-3 + j4)$$

$$\theta_2 = \angle(-1 + j4)$$

$$\angle G(s)\Big|_{s=-3+j4} = [\angle\varphi_1] - [\angle\theta_1 + \angle\theta_2]$$

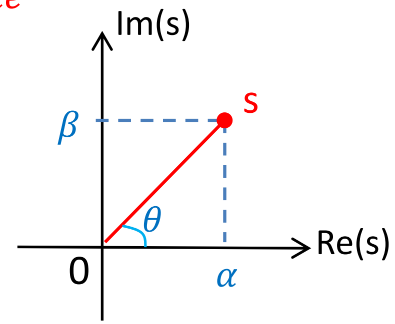
$$\boxed{G(s)\Big|_{s=-3+j4} = 0.217\angle -114.3°}$$



$$s = \alpha + j\beta = \underbrace{|\alpha + j\beta|}_{magnitude}\ \underbrace{\angle(\alpha + j\beta)}_{angle}$$

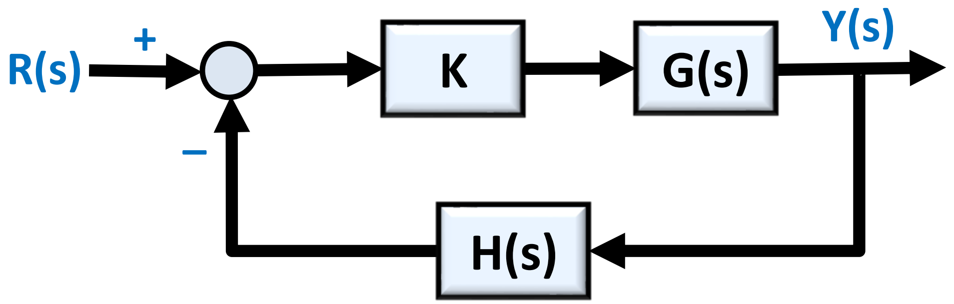
$$|\alpha + j\beta| = \sqrt{\alpha^2 + \beta^2}$$

$$\angle(\alpha + j\beta) = \theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$$



Introduction

❑ Closed-loop Poles Relation to Open-loop Gain



- Consider the following closed-loop system with a negative feedback

$$\boxed{T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}}$$

- The closed-loop characteristic equation is

$$\boxed{1 + KG(s)H(s) = 0} \rightarrow$$

Roots of this equation are poles of the closed-loop system

Open-Loop Gain

- Stability and performance of the closed-loop system depends on the closed-loop poles location.

- Closed-loop poles location in s-plane changes as the open-loop gain factor K is varied.

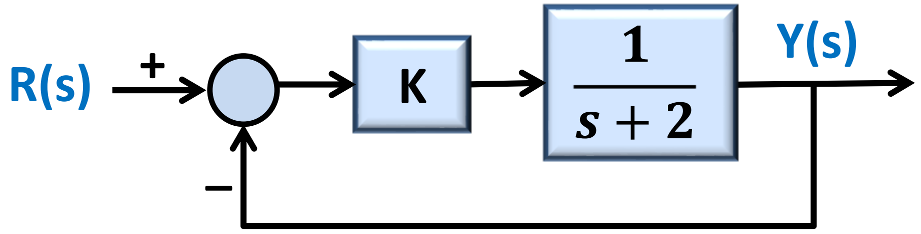
How variation of *K* affects the stability and performance?

Finding the *K* to achieve desire performance.

Closed-loop Poles Relation to Open-loop Gain

Example 4 Consider the following first-order system and the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K}{s+2+K}$$



Characteristic equation of the closed-loop system is

$$1 + KG(s)H(s) = 0 \;\rightarrow\; s + 2 + K = 0 \;\longrightarrow\; \boxed{s = -2 - K} \quad \text{Closed-loop Pole}$$

The closed-loop pole location depends on the parameter K

Plot the closed-loop poles in s-plane by varying K from zero to infinity, $K \in [0, +\infty)$.

$$K = 0 \quad \rightarrow \quad s = -2$$

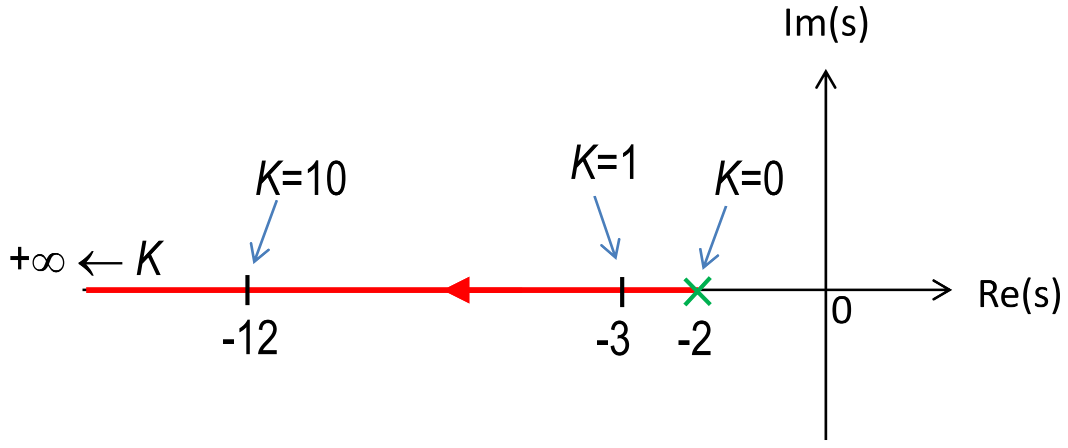
$$K = 1 \quad \rightarrow \quad s = -3$$

$$\vdots$$

$$K = 10 \quad \rightarrow \quad s = -12$$

$$\vdots$$

$$K \rightarrow +\infty \quad \rightarrow \quad s \rightarrow -\infty$$



Root-Locus Plot

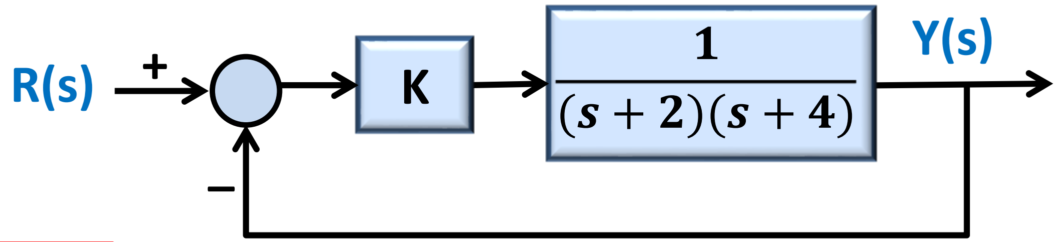
- Number of paths = Number of poles
- For $K = 0$ the root-loci is at the open-loop poles.
- For $K \rightarrow +\infty$ the root-loci goes to negative infinity.
- For all $K \in [0, +\infty)$ the closed-loop system is stable.

Closed-loop Poles Relation to Open-loop Gain

Example 5

Consider the following second-order system and the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K}{s^2 + 6s + 8 + K}$$



Characteristic equation of the closed-loop system is

$$1 + KG(s)H(s) = 0 \;\rightarrow s^2 + 6s + 8 + K = 0 \;\Longrightarrow\; \boxed{s_1 = -3 + \sqrt{1-K}} \quad \boxed{s_2 = -3 - \sqrt{1-K}}$$

Closed-loop Poles

Plot the closed-loop poles in s-plane by varying K from zero to infinity, $K \in [0, +\infty)$.

$$K = 0 \quad \rightarrow \quad s_1 = -2, \;\; s_2 = -4$$

$$K = 1 \quad \rightarrow \quad s_1 = s_2 = -3$$

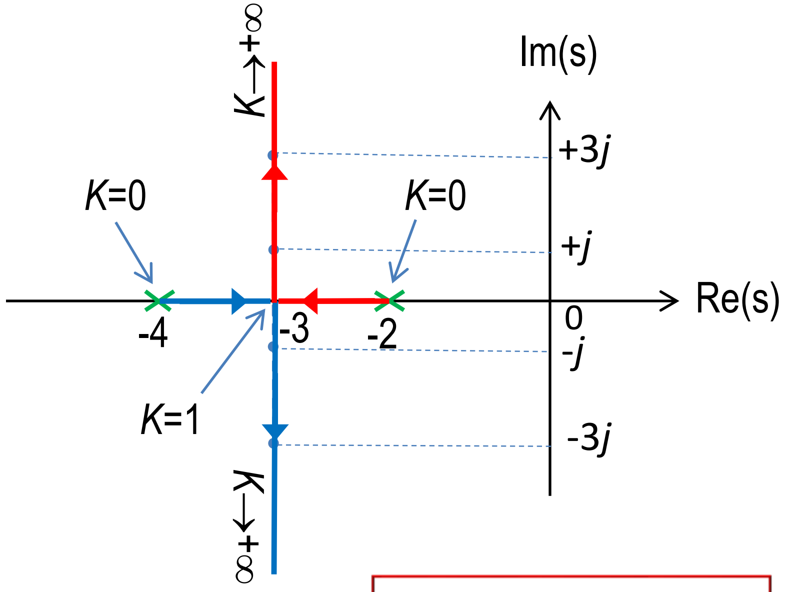
$$K = 2 \quad \rightarrow \quad s_{1,2} = -3 \pm j$$

$$K = 10 \quad \rightarrow \quad s_{1,2} = -3 \pm j3$$

$$\vdots$$

$$K \rightarrow +\infty \quad \rightarrow \quad \mathrm{Re}[s_1, s_2] = -3, \quad \mathrm{Im}[s_1, s_2] \rightarrow \pm\infty$$

- Number of paths = Number of poles
- For $K = 0$ the root-loci is at the open-loop poles.
- For $K \rightarrow \infty$ the root-loci goes to $\pm\infty$ parallel to the imaginary axis.
- For all $K \in [0, +\infty)$ the closed-loop system is stable.



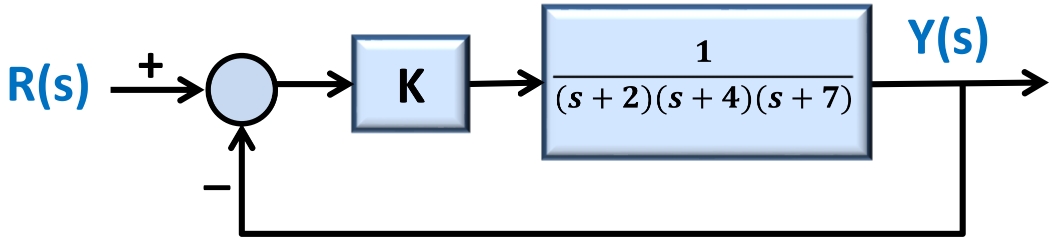
Root-Locus Plot

Closed-loop Poles Relation to Open-loop Gain

Example 6

Consider the following third-order system and the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K}{s^3 + 13s^2 + 50s + 56 + K}$$

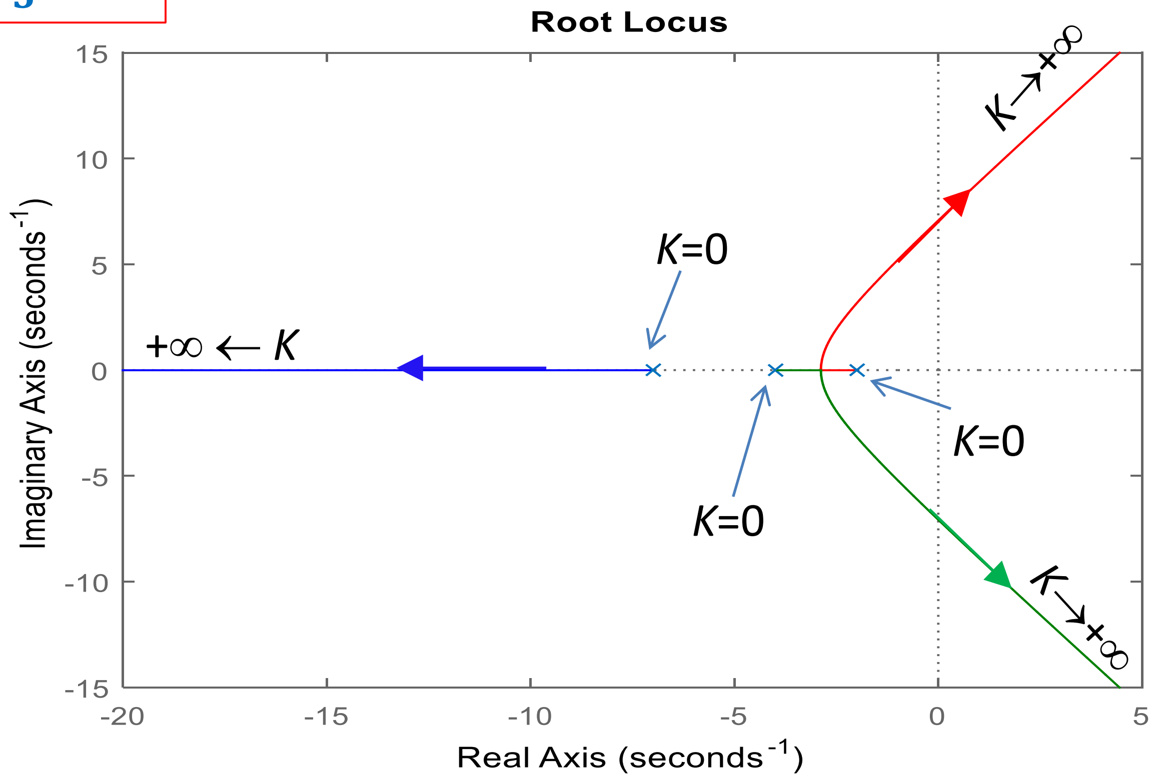


Characteristic equation of the closed-loop system is

$$1 + KG(s)H(s) = 0 \quad \rightarrow s^3 + 13s^2 + 50s + 56 + K = 0$$

$s_1 = ?$
$s_2 = ?$
$s_3 = ?$

- Number of paths = Number of poles
- For $K = 0$ the root-loci is at the open-loop poles.
- For $K \rightarrow \infty$ one of the root-loci paths goes to negative infinity on real axis, and the other two approaches to plus and minus infinity as well.
- The close-loop is not stable for all $K > 0$ values.
- For some of the $K > 0$ the closed-loop system is unstable.



Root-Locus Plot

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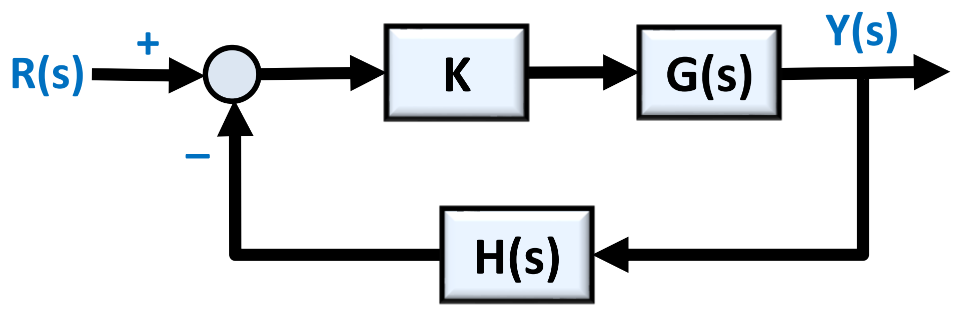
Root Locus Method

- **Root-locus** is a graphical technique to determine trajectories of the closed-loop poles by variation of a certain parameter, such as loop-gain *K* in the following closed-loop system:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$



A locus of closed-loop poles plotted in the s-plane as a function of open-loop gain factor *K* is called **Root-Locus**.

- **Root-locus** is a powerful method to analyze and design the control systems.
  - It provides a graphical representation of stability and transient response.
  - It relates the gain to the system dynamics, which can be used to find the specific gain to achieve a desire performance.
  - It provides a graphical method to design the controllers and compensators for high-order systems.
- The goal is finding a systematic way to draw the root-locus for any system by determining some critical points on the root-locus.
- A simple graphical method to plot the root-locus of closed-loop poles has been developed by Walter Evans, an American control theorist and the inventor of the root locus method.

Root Locus Method

❑ Root Locus Plotting Guidelines for $K \geq 0$

Step 1: Draw the axes of the s-plane, and mark open-loop poles and zeros.

Step 2: Draw the root-locus on the real axis.

Step 3: Draw the asymptote lines for large *K* values.

Step 4: Calculate the points where root-locus cross the imaginary axis.

Step 5: Calculate locations of breakaway (or break in) points on the real axis.

Step 6: Calculate angle of departure from complex poles or angle of arrival to complex zeros.

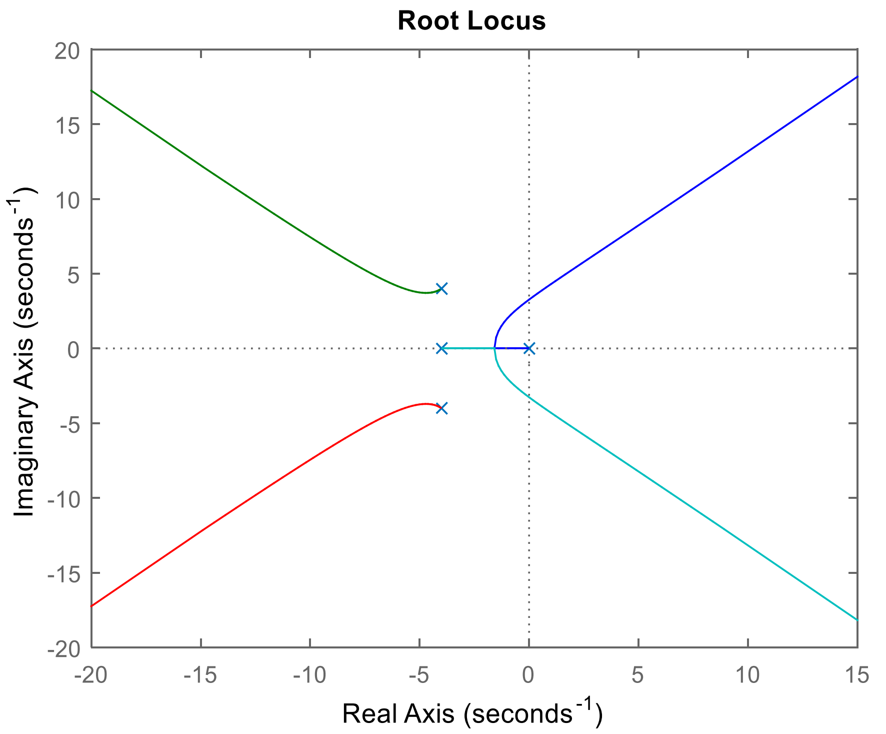
Step 7: Complete the plotting by considering the characteristics of the root-locus diagram.

Rule 1: Number of loci is equal to the order of the characteristic equation.

Rule 2: For $K \in [0, +\infty)$, each locus starts at an open-loop pole $(K = 0)$ and terminates at an open-loop zero, including the ones at infinity $(K \rightarrow +\infty)$

Rule 3: The root-locus diagram will be symmetrical about the real axis.

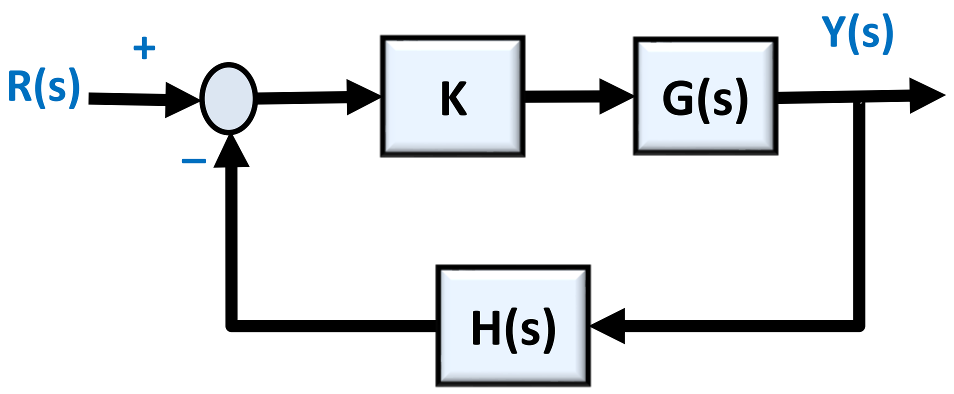
Rule 4: A locus will never cross over its own path.

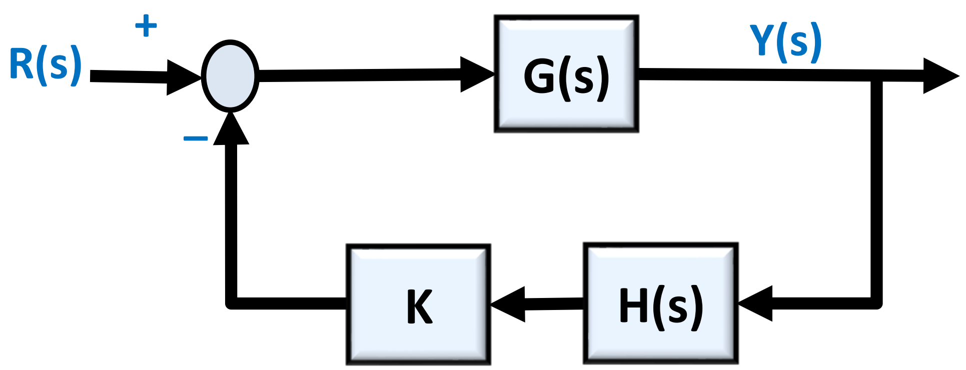


Root Locus Method

NOTE: The following two closed-loop systems have same closed-loop characteristic equations.

- Therefore, in terms of the stability both systems have same root-locus.





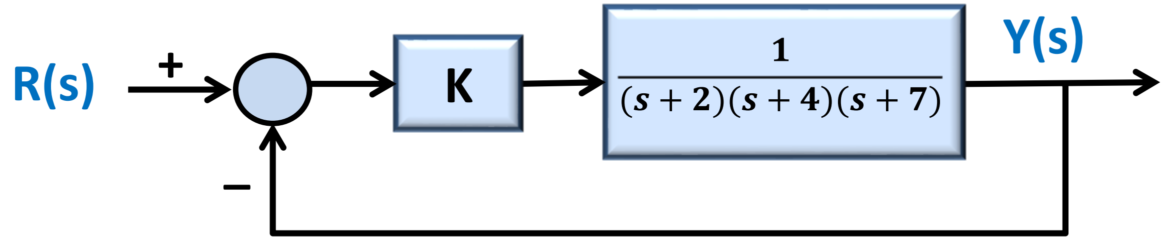
$$\frac{Y_1(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

$$\frac{Y_2(s)}{R(s)} = \frac{G(s)}{1 + KG(s)H(s)}$$

- Note that the systems behave differently to reference input $r(t)$, but it does not affect the root-locus.

Root Locus Method

Example 7 Plot the root-locus diagram for the following third-order system



First determine the closed-loop transfer function and the characteristics equation.

Closed-loop Transfer Function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K}{s^3 + 13s^2 + 50s + 56 + K}$$

Closed-loop Characteristic Equation

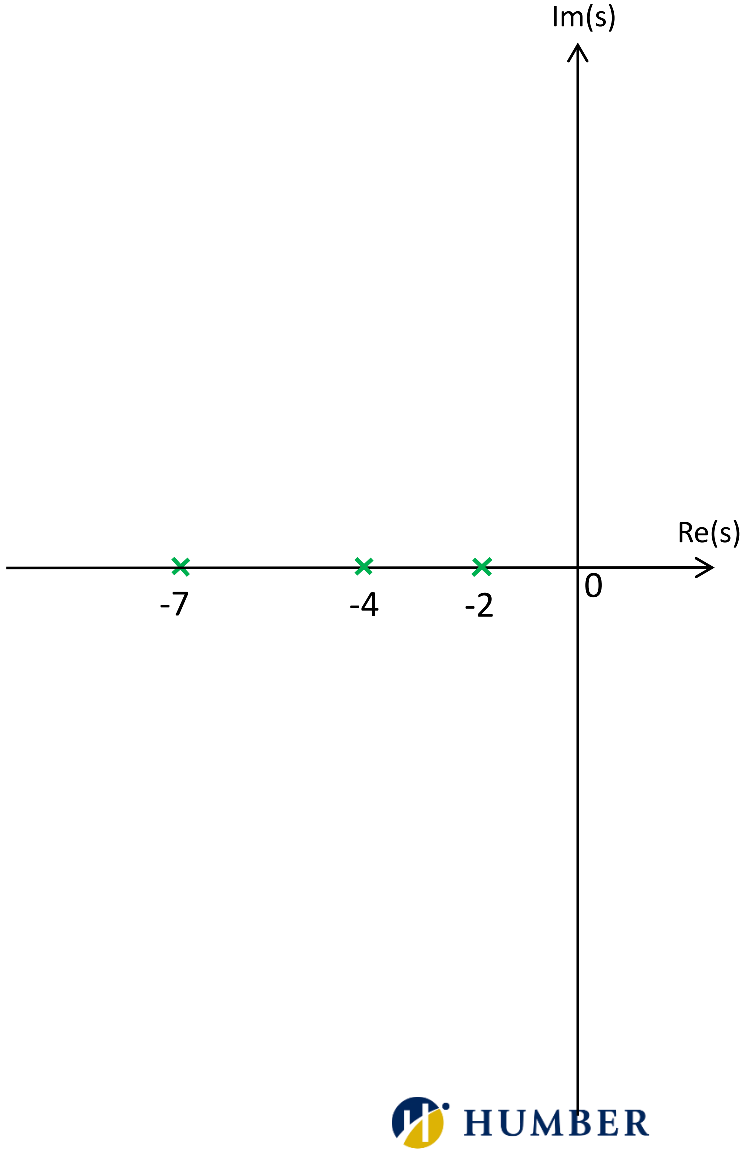
$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad s^3 + 13s^2 + 50s + 56 + K = 0$$

Step 1: Draw the axes of the s-plane

Mark poles × and zeros o of the open-loop system.

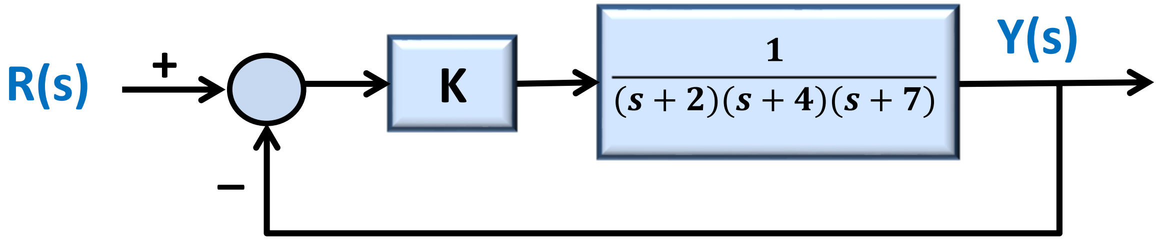
Poles: $p_1 = -2, \; p_2 = -4, \; p_3 = -7$

Zeros: No finite zeros, three zeros at infinity



Root Locus Method

Example 7 Plot the root-locus diagram for the following third-order system



Step 2: Draw the root-locus on the real axis

A point on the real axis is part of a locus if the number of poles and zeros to the right of that point is **ODD.**

Here, zero is considered as an even number

Step 3: Draw asymptote lines for large *K* values

Number of asymptotes $\rightarrow$ $\boldsymbol{n-m} = 3-0 = \mathbf{3}$

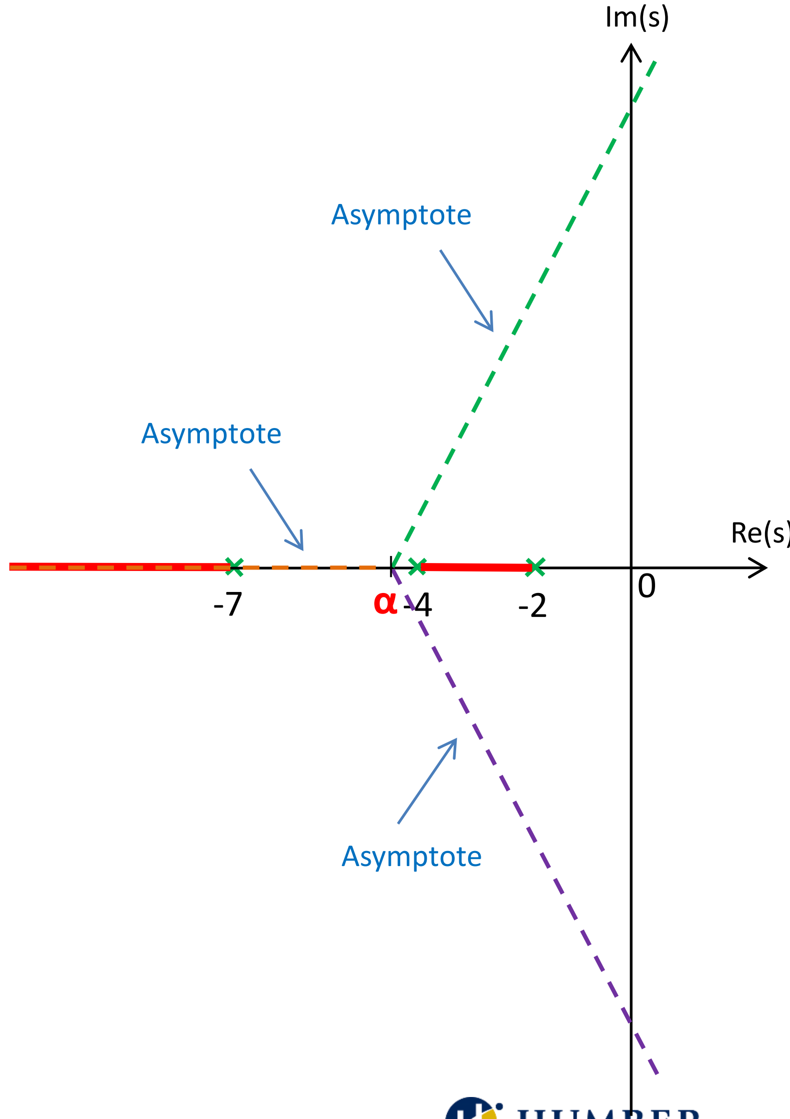
Intersection of asymptotes on the real axis

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{[(-2)+(-4)+(-7)]}{3-0} = -4.33$$

Angle of asymptote lines with real axis

$$\varphi_i = \frac{180^\circ}{n-m}(2i+1) = \frac{180^\circ}{3-0}(2i+1) = 60^\circ(2i+1) \rightarrow \begin{cases} \varphi_0 = 60^\circ \\ \varphi_1 = 180^\circ \\ \varphi_2 = 300^\circ \end{cases}$$

$i = 0, 1, 2, \cdots$



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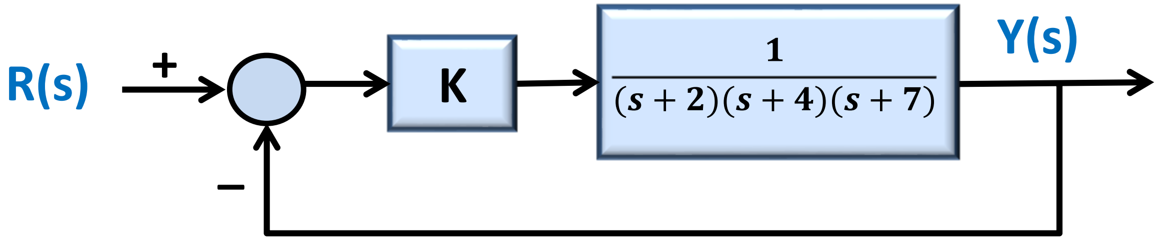
Root Locus Method

Example 7 Plot the root-locus diagram for the following third-order system

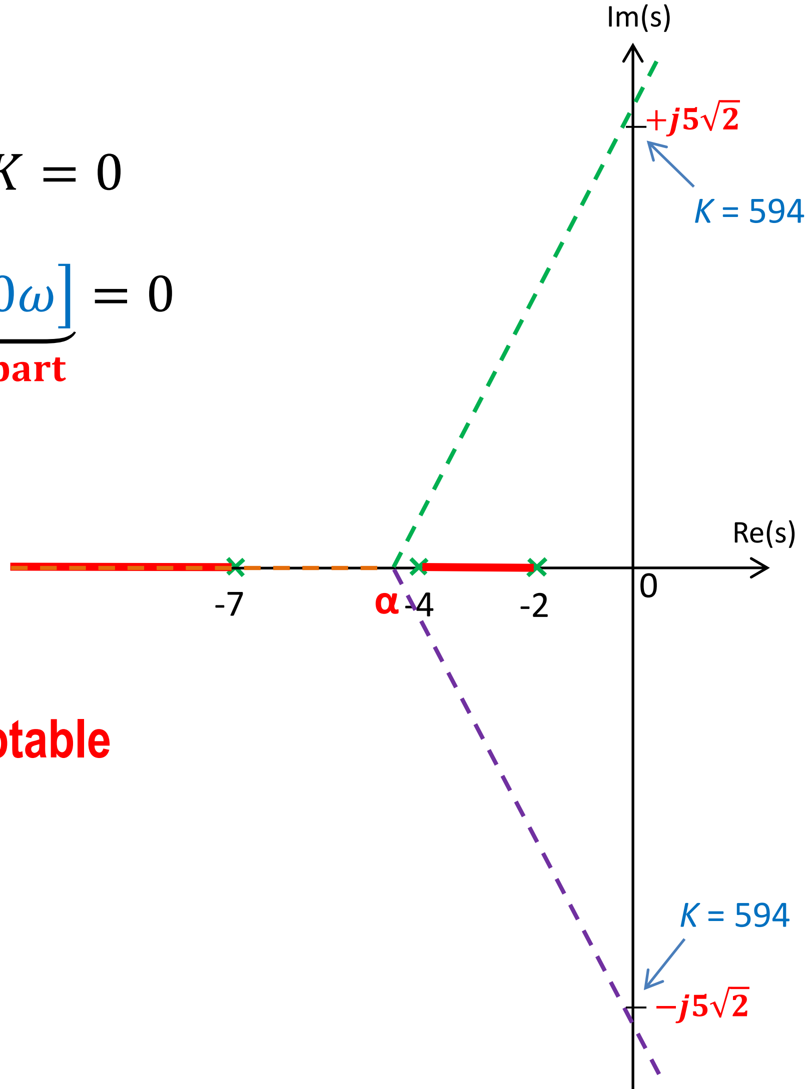


Step 4: Intersection of root-locus with imaginary axis

$$1 + KG(s)H(s) = 0 \;\rightarrow\; s^3 + 13s^2 + 50s + 56 + K = 0$$

$$\boxed{s = j\omega} \;\longrightarrow\; (j\omega)^3 + 13(j\omega)^2 + 50(j\omega) + 56 + K = -j\omega^3 - 13\omega^2 + j50\omega + 56 + K = 0$$

$$\underbrace{[-13\omega^2 + 56 + K]}_{\text{real part}} + j\,\underbrace{[-\omega^3 + 50\omega]}_{\text{imaginary part}} = 0$$



From the imaginary part:

$$-\omega^3 + 50\omega = 0 \;\rightarrow\; \omega(-\omega^2 + 50) = 0 \;\rightarrow\; \begin{cases} \omega = 0 \\ \omega^2 = 50 \rightarrow \boxed{\omega = \pm 5\sqrt{2}\ rad/s} \end{cases}$$

From the real part:

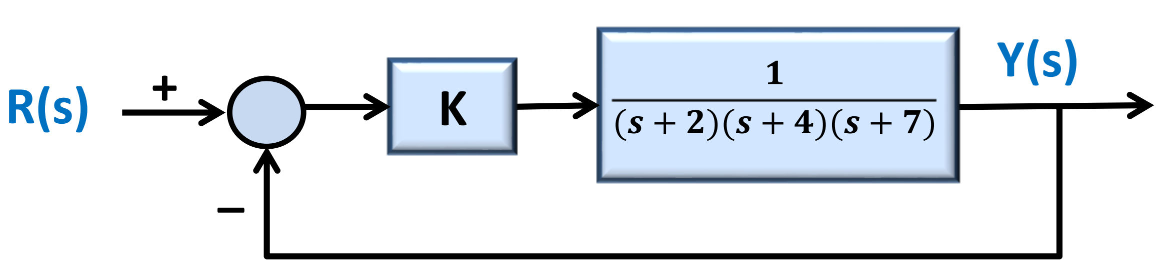
$$\omega = 0 \;\longrightarrow\; -13\omega^2 + 56 + K = -13 \times 0 + 56 + K = 0 \;\rightarrow\; K = -56 < 0 \quad \text{Not acceptable}$$

$$\omega^2 = 50 \;\longrightarrow\; -13\omega^2 + 56 + K = 0 \;\rightarrow\; -13 \times 50 + 56 + K = 0 \;\rightarrow\; \boxed{K = 594}$$

Therefore, the loci cross the imaginary axis at $s = j\omega \;\rightarrow\; \begin{cases} s = +j5\sqrt{2} \\ s = -j5\sqrt{2} \end{cases}$

Root Locus Method

Example 7 Plot the root-locus diagram for the following third-order system



Step 4: Intersection of root-locus with imaginary axis

NOTE: $\boldsymbol{K = 594}$ is the marginal-stability gain, which can also be determined by applying the Routh-Hurwitz Criterion on the characteristic equation.

$$\boxed{s^3 + 13s^2 + 50s + 56 + K = 0}$$

Routh-Hurwitz Table:

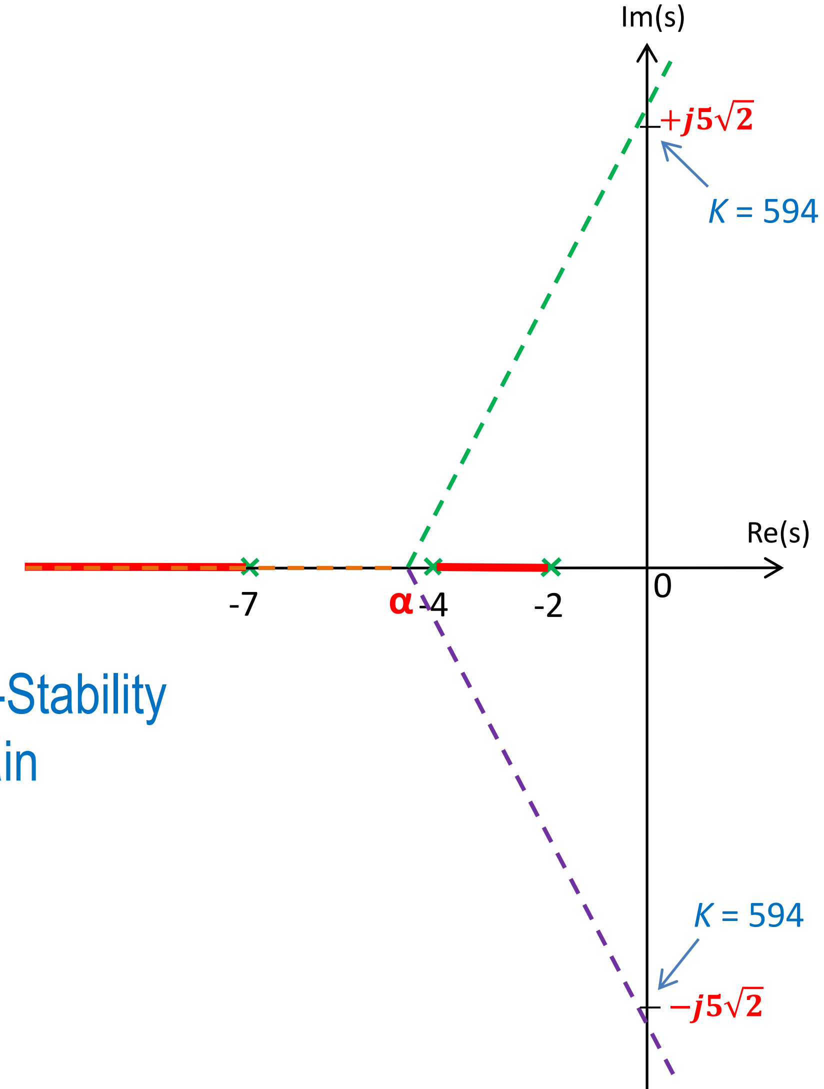
s^3	1	50
s^2	13	$56 + K$
s^1	$\frac{594-K}{13}$	0
s^0	$56 + K$	0

$$\frac{594 - K}{13} > 0 \quad \rightarrow \quad \boldsymbol{K < 594}$$

$$56 + K > 0 \quad \rightarrow \quad \boldsymbol{K > -56}$$

$$\boxed{\boldsymbol{-56 < K < 594}}$$

$$\boxed{\boldsymbol{K = 594}}$$ Marginal-Stability Gain

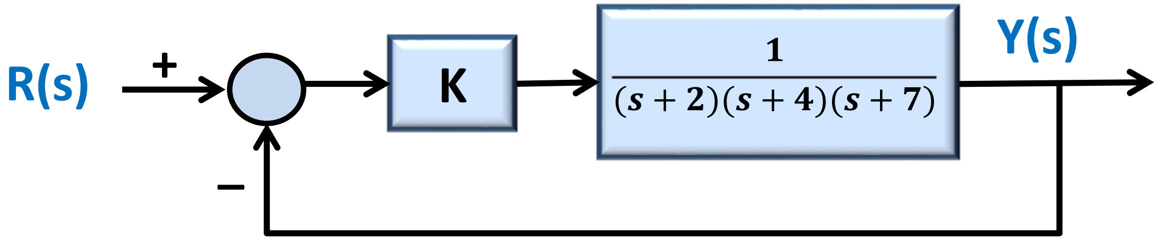


The intersection with imaginary axis is determined from the even polynomial by using s^2 row with $K = 594$

$$13s^2 + 56 + K = 0 \quad \rightarrow \quad 13s^2 + 650 = 0 \quad \rightarrow \quad \boldsymbol{s = \pm j5\sqrt{2}}$$

Root Locus Method

Example 7 Plot the root-locus diagram for the following third-order system

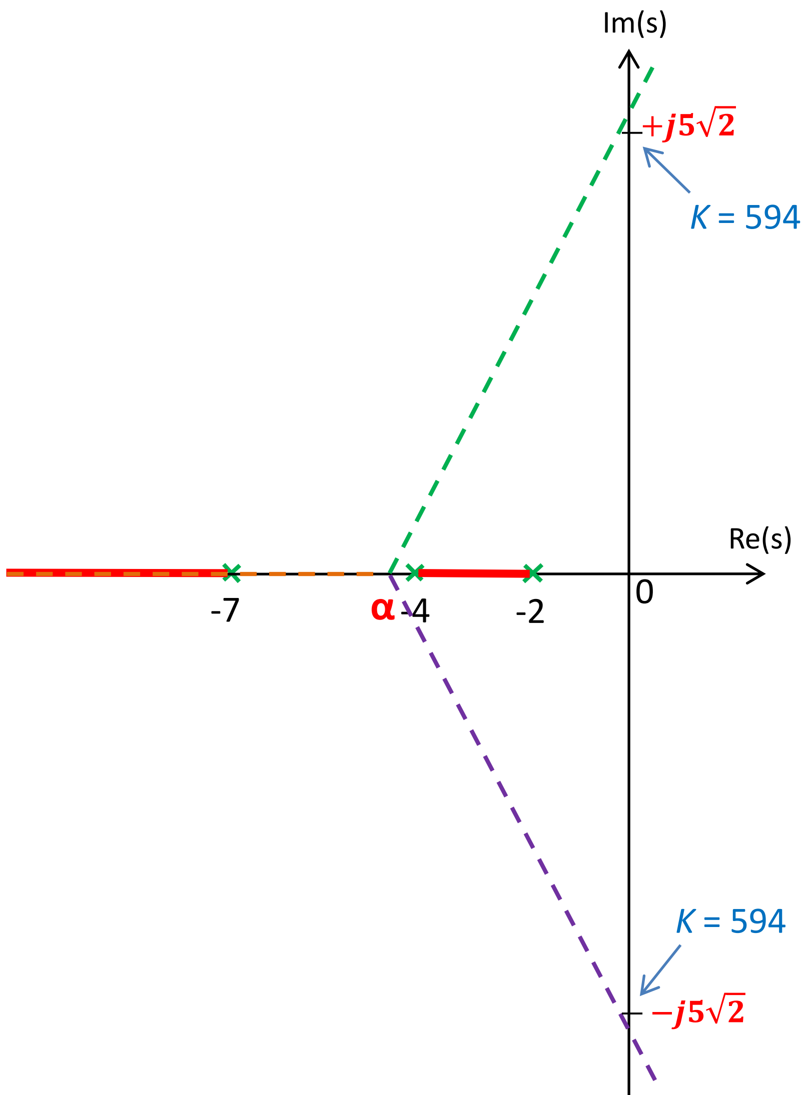


Step 5: Calculate break-away/break-in points on real axis

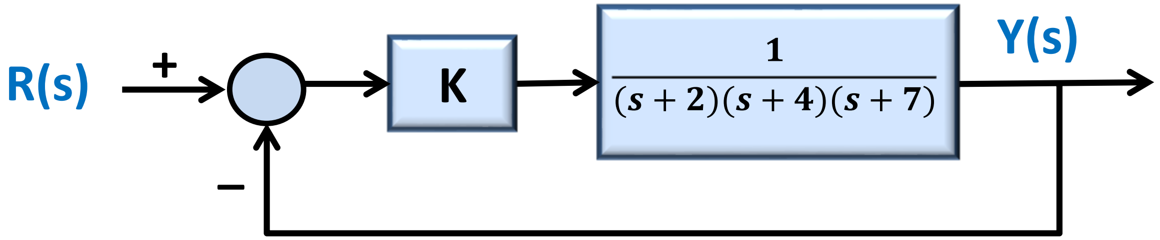
- The break-away (or break-in) point is where:
  - Two or more branches meet on the real axis
  - The closed-loop system has multiple roots
  - The parameter K is at maximum (or minimum) along the real axis

- The break-away (or break-in) point is determined as follows:

$$\boxed{1+KG(s)H(s)=0 \quad \rightarrow \quad K=\frac{-1}{G(s)H(s)} \quad \rightarrow \quad \frac{dK}{ds}=0}$$



Example 7 Plot the root-locus diagram for the following third-order system



Step 5: Calculate break-away/break-in points on real axis

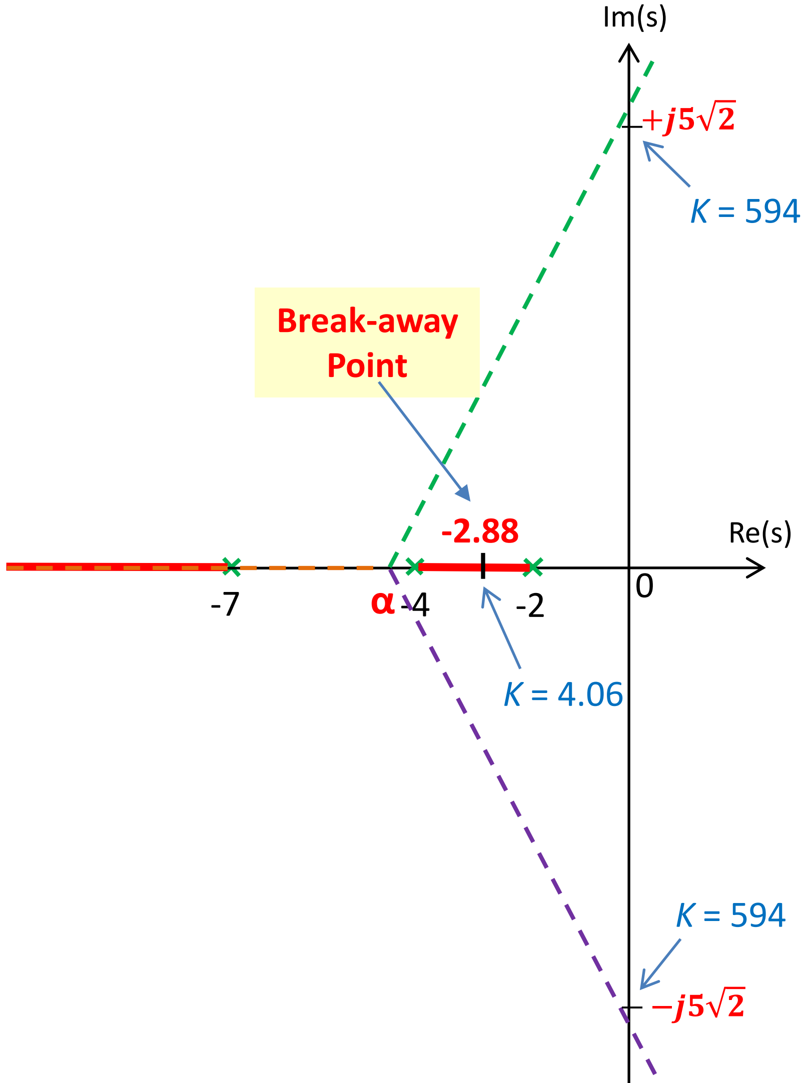
$$\mathbf{1 + KG(s)H(s) = 0} \quad \rightarrow \quad s^3 + 13s^2 + 50s + 56 + K = 0$$

$$K = -s^3 - 13s^2 - 50s - 56$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad -3s^2 - 26s - 50 = 0$$

$$\begin{cases} s = -5.79 & \rightarrow \quad \text{not on the root} - \text{loci} \\ \boxed{s = -2.88} & \rightarrow \quad \text{on the root} - \text{loci} \end{cases}$$

Break-away Point

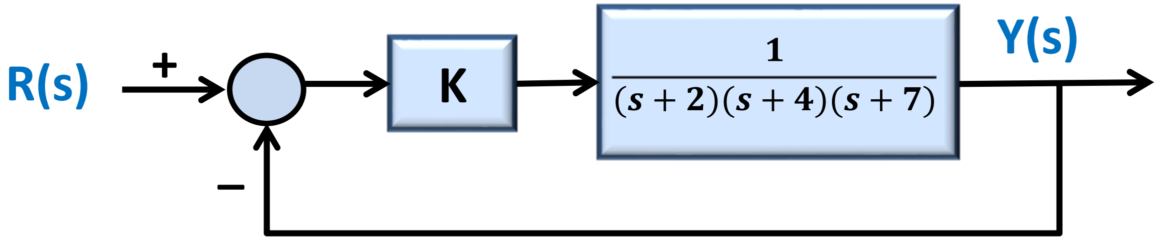


We can find the gain K at the break-away point:

$$K = -(-2.88)^3 - 13(-2.88)^2 - 50(-2.88) - 56 \longrightarrow \boxed{K = 4.06}$$

Root Locus Method

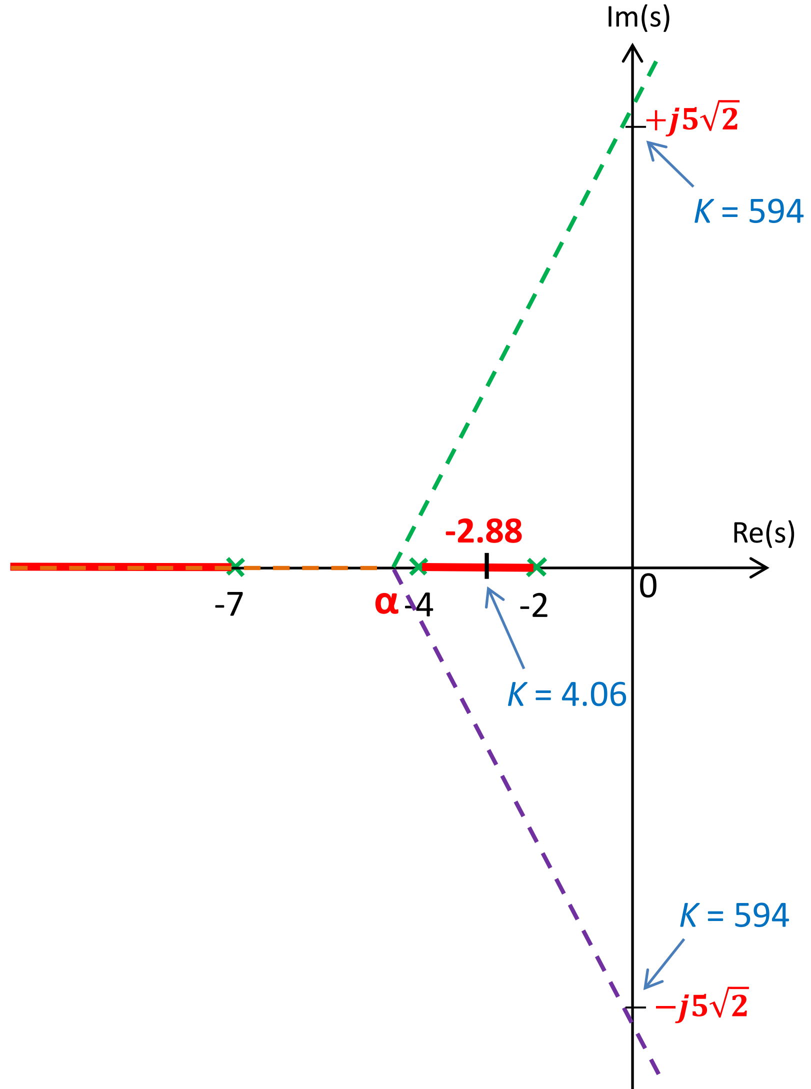
Example 7 Plot the root-locus diagram for the following third-order system



Step 6: Calculate angle of departure/angle of arrival

Calculate angle of departure from complex poles and/or angle of arrival to complex zeros of the root-loci.

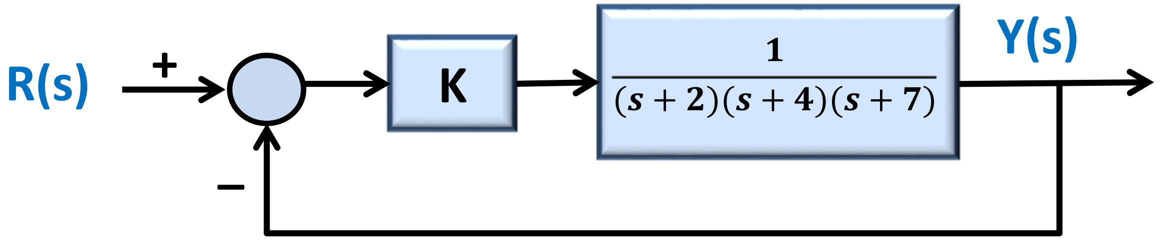
Skip this step, because the open-loop transfer function does not have any complex poles/zeros.



Root Locus Method

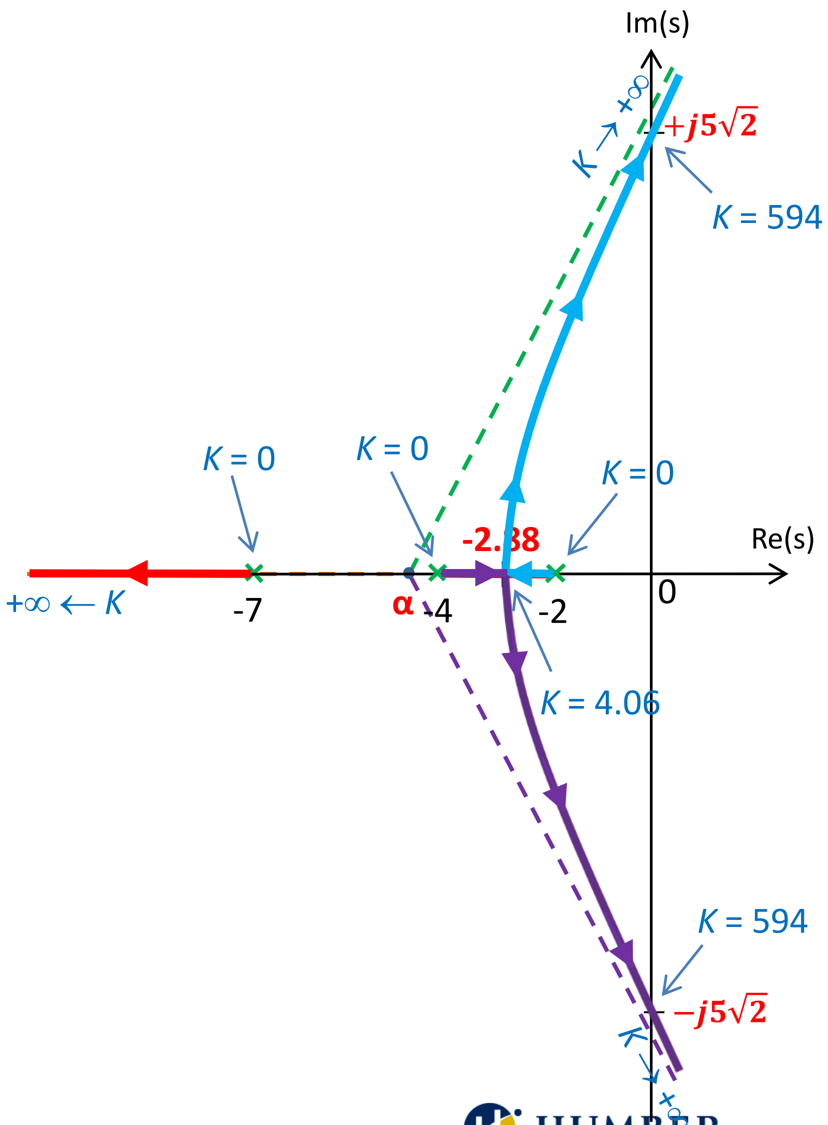
Example 7

Plot the root-locus diagram for the following third-order system



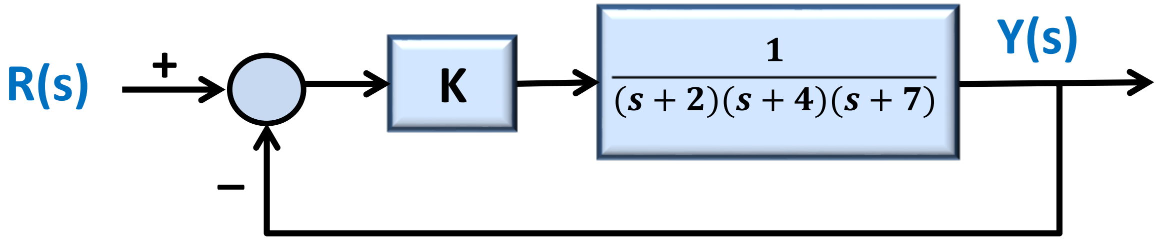
Step 7: Complete the root-locus diagram

- From the characteristic equation, number of separate root-loci is **three**.
- For $K = 0$ the root-loci is at the open-loop poles including those at $s = \infty$.
- For $K = \infty$ the root-loci is at the open-loop zeros including those at $s = \infty$.
- Since there is no finite zero the root-locus branches start from the poles of open-loop transfer function and go to infinity approaching the asymptote lines.
- Root-locus is symmetric with respect to the real axis.



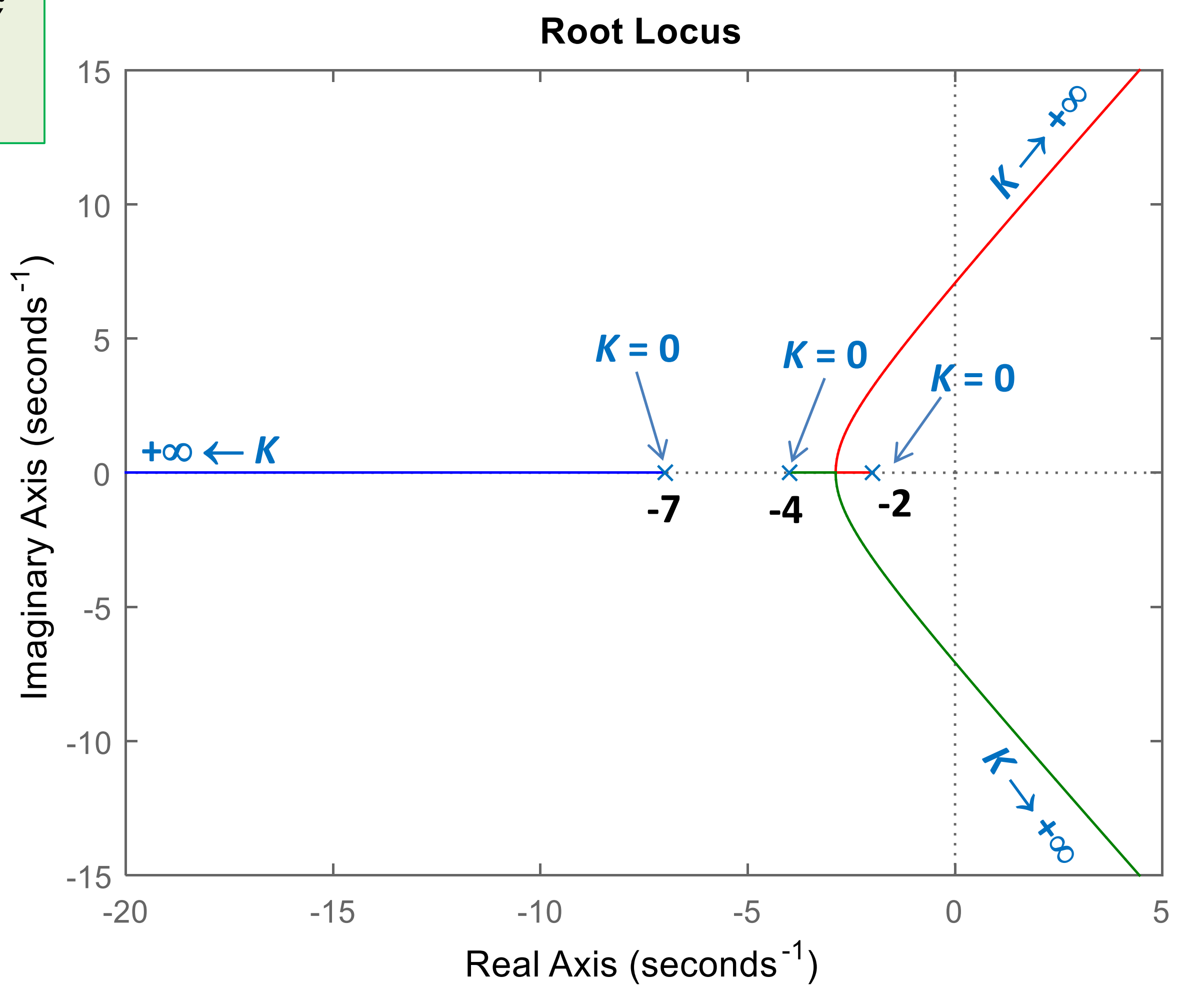
Root Locus Method

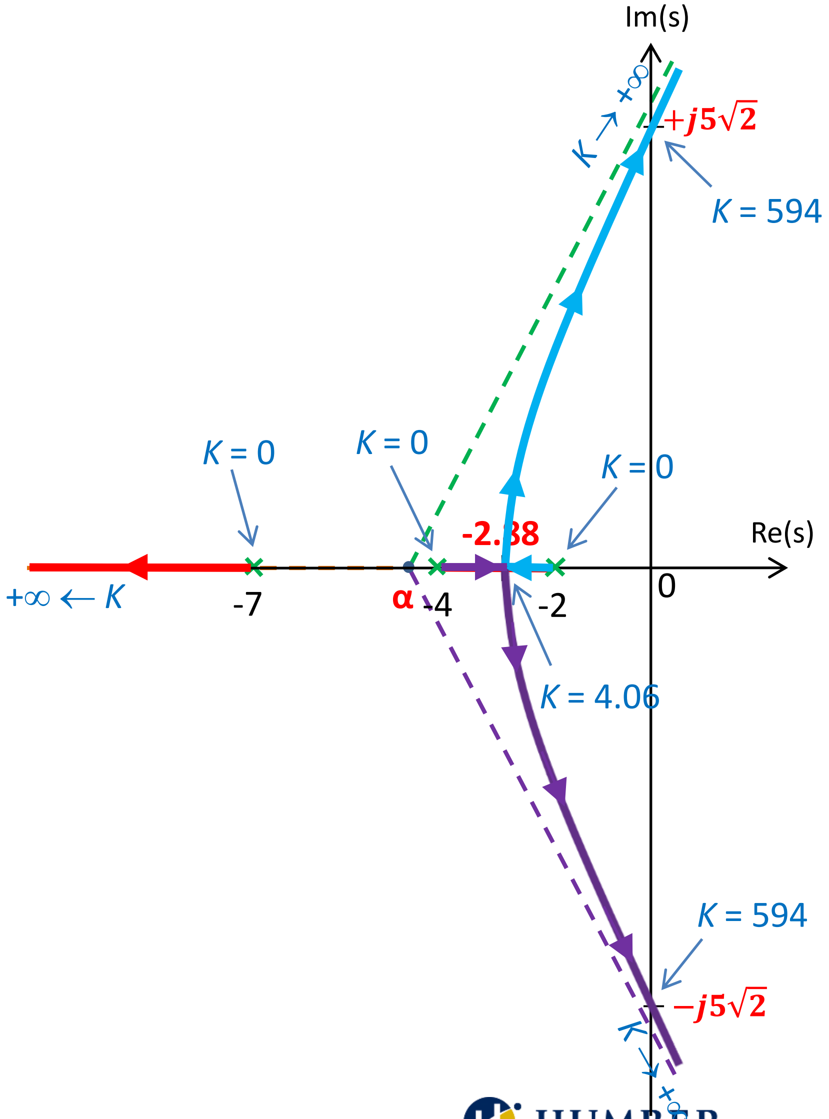
Example 7 Plot the root-locus diagram for the following third-order system



We can plot the root-locus by MATLAB to compare the results.

```
num = [1];
den = poly([-2 -4 -7]);
sys = tf(num,den);
rlocus(sys)
```

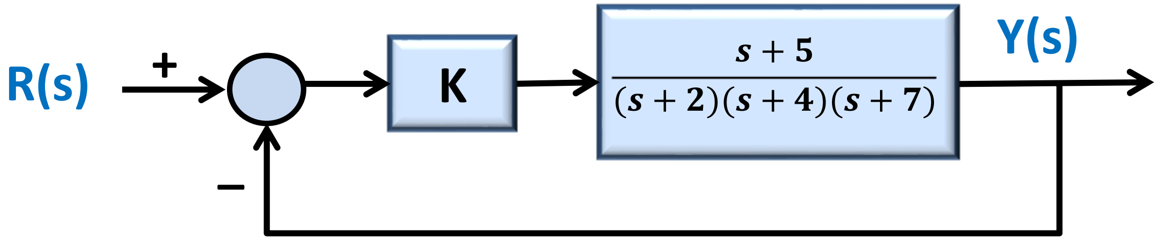




Root Locus Examples

Root Locus Method – Example

Example 8 Plot the root-locus diagram for the following third-order system



First determine the closed-loop transfer function and the characteristics equation.

Closed-loop Transfer Function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K(s+5)}{s^3 + 13s^2 + (50+K)s + 5K + 56}$$

Closed-loop Characteristic Equation

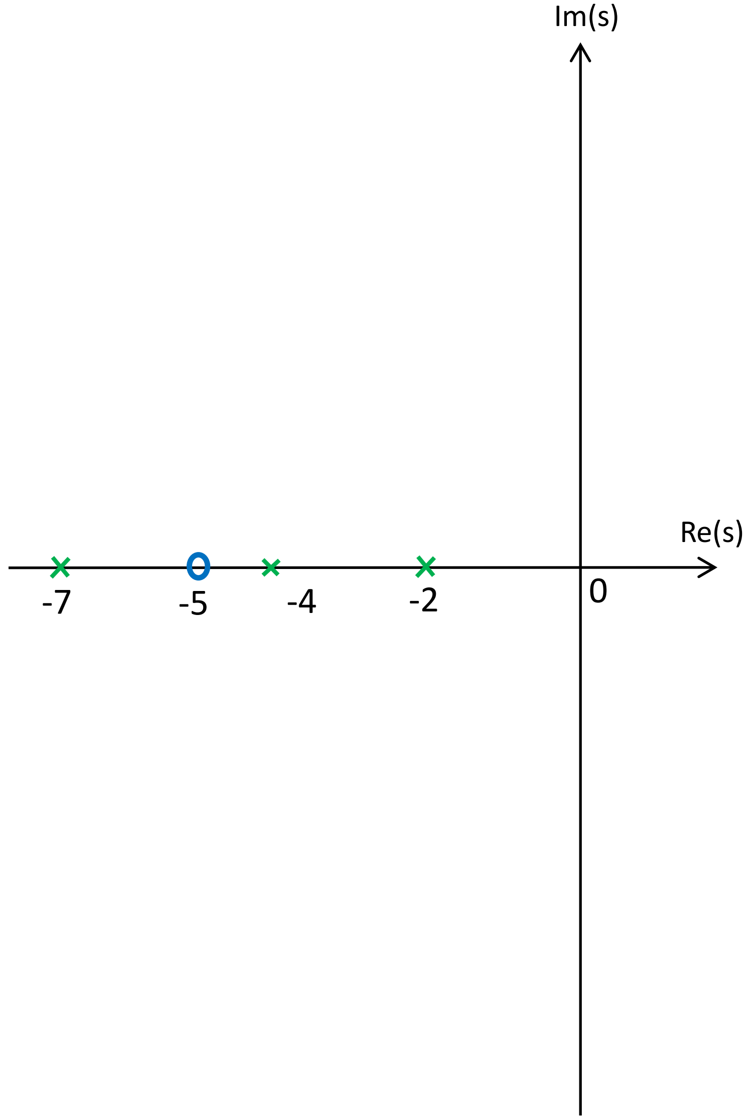
$$1 + KG(s)H(s) = 0 \quad \rightarrow s^3 + 13s^2 + (50+K)s + 5K + 56 = 0$$

Step 1: Draw the axes of the s-plane

Mark poles × and zeros o of the open-loop system.

Poles: $p_1 = -2,\ p_2 = -4,\ p_3 = -7$

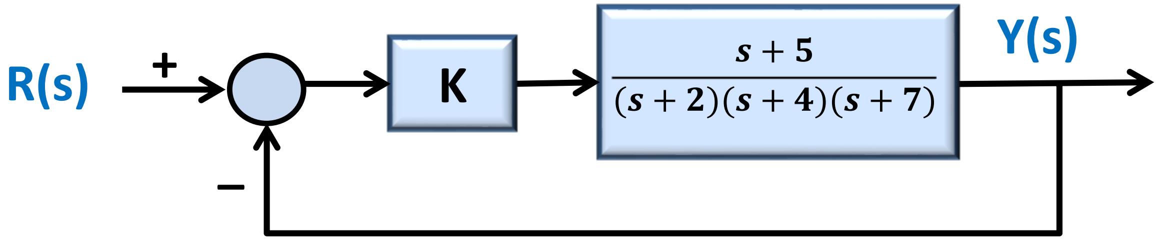
Zeros: $z_1 = -5,$ Two zeros at infinity



Root Locus Method – Example

Example 8

Plot the root-locus diagram for the following third-order system with a single zero.



Step 2: Draw the root-locus on the real axis

A point on the real axis is part of a locus if the number of poles and zeros to the right of that point is **ODD.**

Here, zero is considered as an even number

Step 3: Draw asymptote lines for large *K* values

Number of asymptotes → $n - m = 3 - 1 = 2$

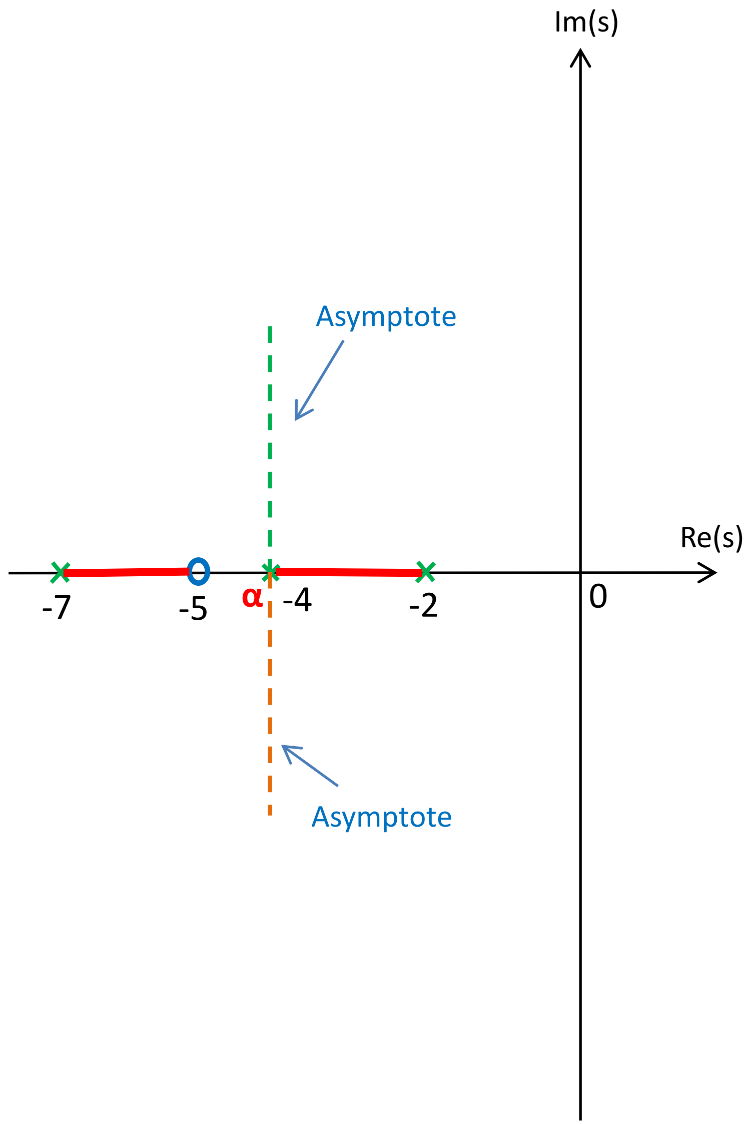
Intersection of asymptotes on the real axis

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{[(-2)+(-4)+(-7)] - [(-5)]}{3-1} = -4$$

Angle of asymptote lines with real axis

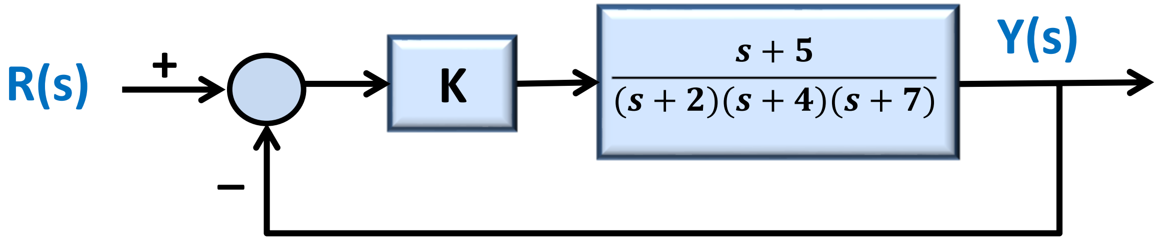
$$\varphi_i = \frac{180^\circ}{n-m}(2i+1) = \frac{180^\circ}{3-1}(2i+1) = 90^\circ(2i+1) \rightarrow \begin{cases} \varphi_0 = 90^\circ \\ \varphi_1 = 270^\circ \end{cases}$$

$i = 0, 1, 2, \cdots$



Root Locus Method – Example

Example 8 Plot the root-locus diagram for the following third-order system with a single zero.



Step 4: Intersection of root-locus with imaginary axis

$$\mathbf{1+KG(s)H(s)=0} \quad \rightarrow s^3+13s^2+(50+K)s+5K+56=0$$

$$\boxed{s=j\omega} \longrightarrow (j\omega)^3+13(j\omega)^2+(50+K)(j\omega)+5K+56=0$$

$$-j\omega^3-13\omega^2+j(50+K)\omega+5K+56=0$$

$$\underbrace{[-13\omega^2+56+5K]}_{\text{real part}}+j\underbrace{[-\omega^3+50\omega+K\omega]}_{\text{imaginary part}}=0$$

From the imaginary part:

$$-\omega^3+50\omega+K\omega=0 \rightarrow \omega(-\omega^2+50+K)=0 \rightarrow \begin{cases}\omega=0\\ \omega^2=50+K \rightarrow \omega=\pm\sqrt{50+K}\end{cases}$$

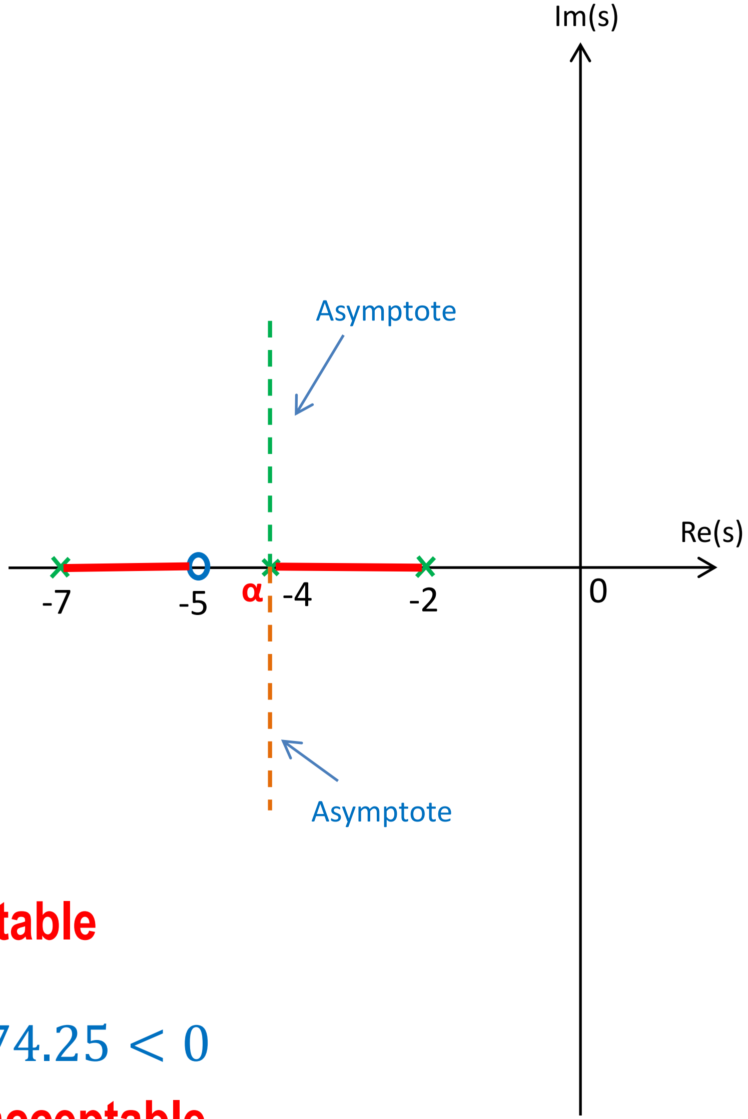
From the real part:

$$\omega=0 \rightarrow -13\omega^2+56+5K=-13\times 0+56+5K=0 \rightarrow K=-11.2<0$$ **Not acceptable**

$$\omega^2=50+K \rightarrow -13\omega^2+56+5K=0 \rightarrow -13(50+K)+56+5K=0 \rightarrow K=-74.25<0$$

Not acceptable

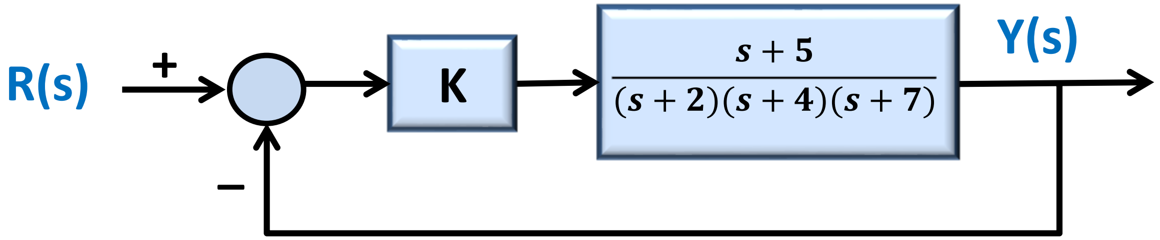
The root-locus does not cross the imaginary axis



Root Locus Method – Example

Example 8

Plot the root-locus diagram for the following third-order system with a single zero.



Step 5: Calculate break-away/break-in points on real axis

$$\mathbf{1 + KG(s)H(s) = 0} \quad \rightarrow \quad s^3 + 13s^2 + (50 + K)s + 5K + 56 = 0$$

$$K = \frac{-s^3 - 13s^2 - 50s - 56}{s + 5}$$

$$\frac{\boldsymbol{dK}}{\boldsymbol{ds}} = \mathbf{0} \quad \rightarrow \quad \frac{(-3s^2 - 26s - 50)(s + 5) - (-s^3 - 13s^2 - 50s - 56)}{(s + 5)^2} = 0$$

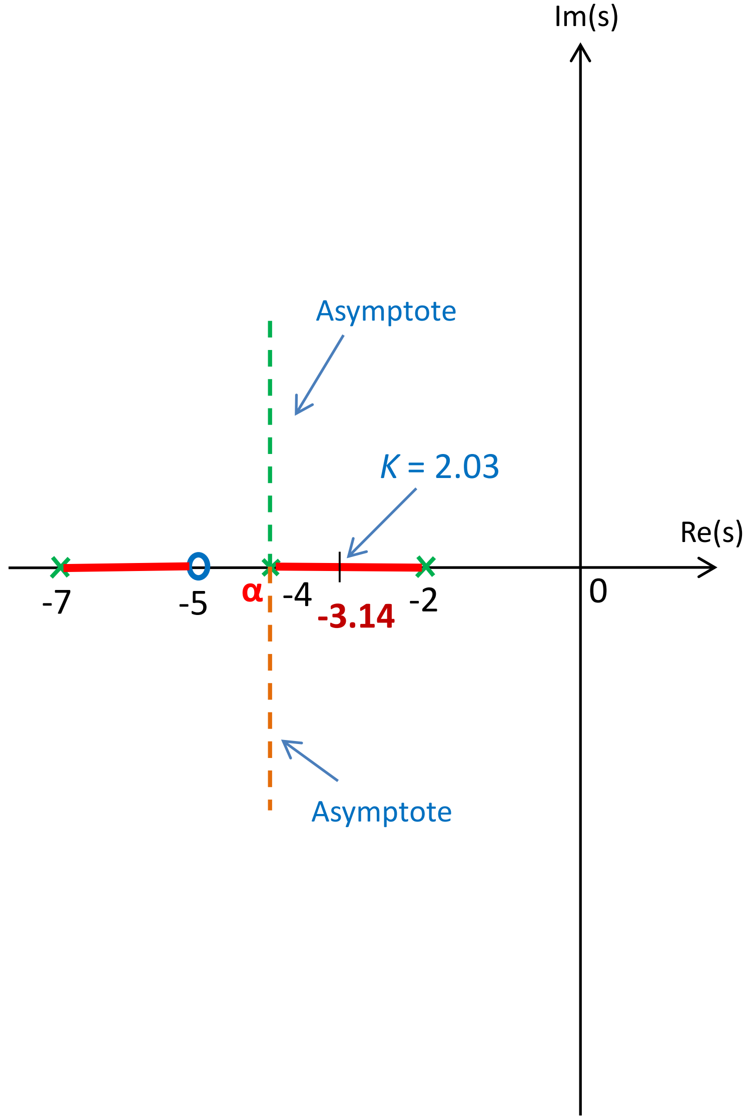
$$-2s^3 - 28s^2 - 130s - 194 = 0$$

$$\begin{cases} s = -5.43 \pm j1.19 & \rightarrow \quad \text{not on the root} - \text{loci} \\ \boxed{s = -3.14} & \rightarrow \quad \text{on the root} - \text{loci} \end{cases}$$

Break-away Point

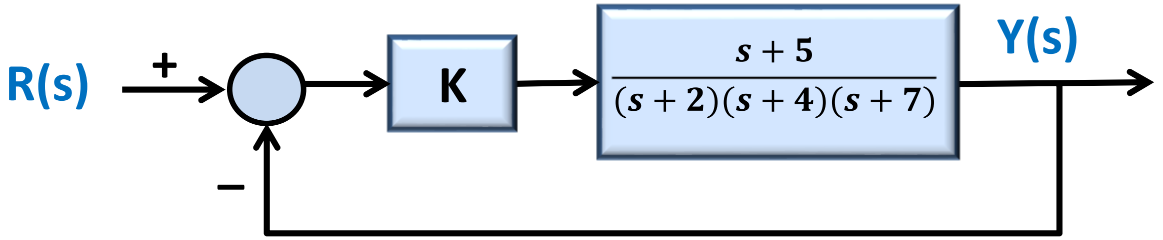
The gain K at the break-away point:

$$K = \frac{-(-3.14)^3 - 13(-3.14)^2 - 50(-3.14) - 56}{(-3.14) + 5} \quad \rightarrow \quad \boxed{\boldsymbol{K = 2.03}}$$



Root Locus Method – Example

Example 8 Plot the root-locus diagram for the following third-order system with a single zero.



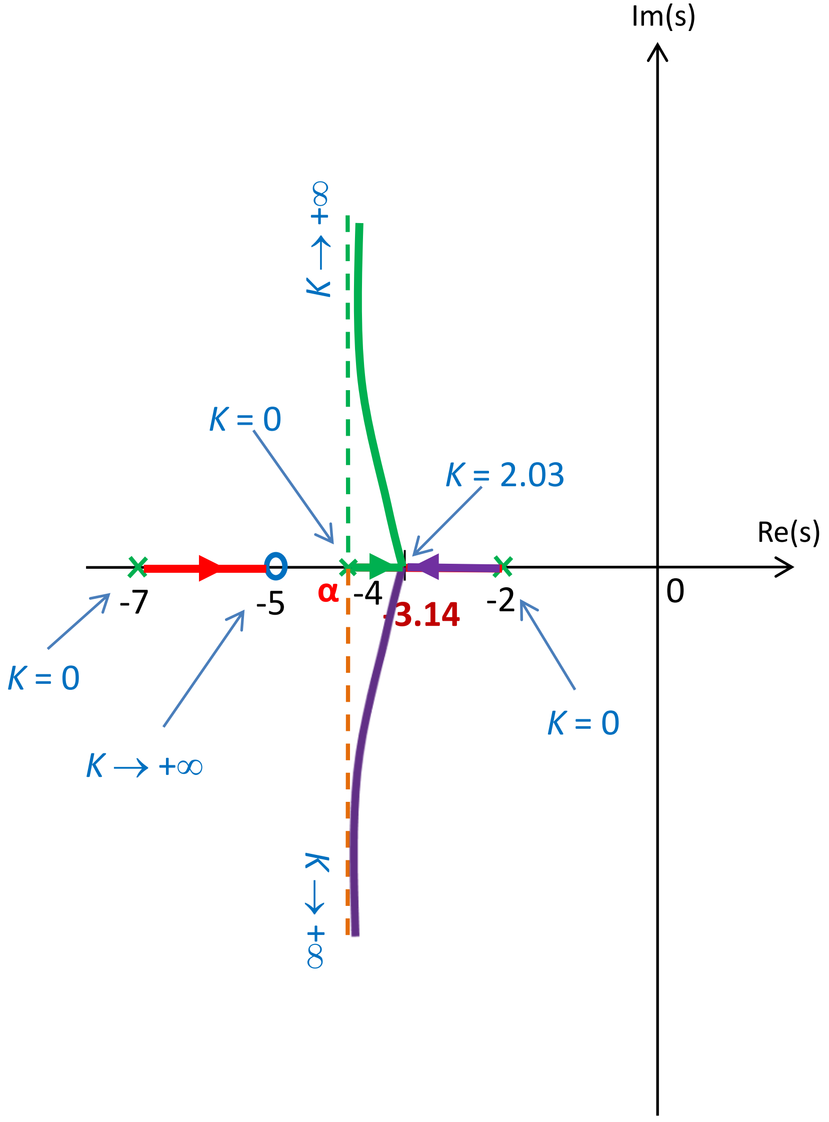
Step 6: Calculate angle of departure/angle of arrival

Calculate angle of departure from complex poles and/or angle of arrival to complex zeros of the root-loci.

Skip this step, because the open-loop transfer function does not have any complex poles/zeros.

Step 7: Complete the root-locus diagram

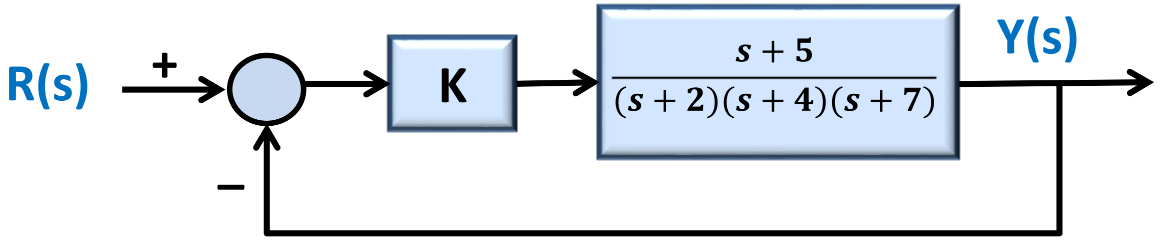
- Number of separate root-loci is equal to the order of open-loop transfer function, which is **three** here.
- For $K = 0$ the root-loci is at the open-loop poles including those at $s = \infty$.
- For $K = \infty$ the root-loci is at the open-loop zeros including those at $s = \infty$.
- Since open-loop transfer function has one finite zero at s = -5 one of the root-locus branches terminates at the zero of the open-loop transfer function and the others go to infinity approaching the asymptote lines.
- Root-locus is symmetric with respect to the real axis.



Root Locus Method – Example

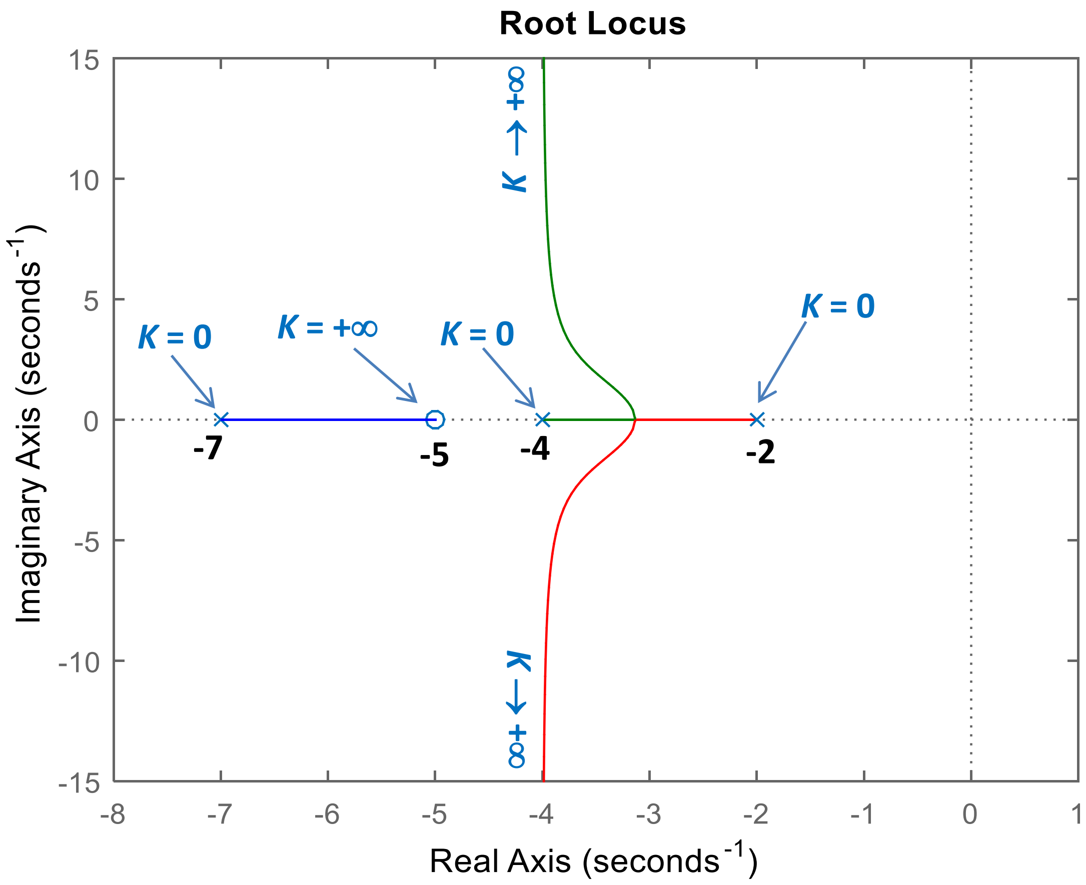
Example 8

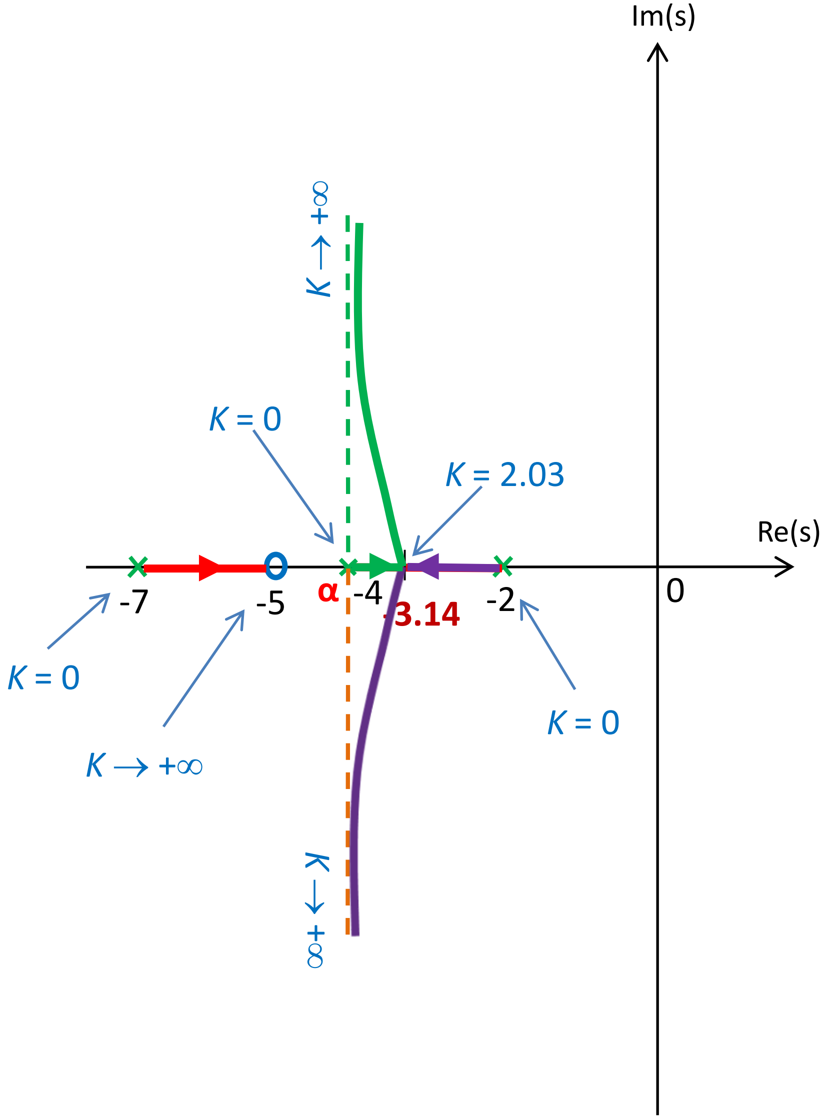
Plot the root-locus diagram for the following third-order system with a single zero.



We can plot the root-locus by MATLAB to compare the results.

```
num = [1 5];
den = poly([-2 -4 -7]);
sys = tf(num,den);
rlocus(sys)
```

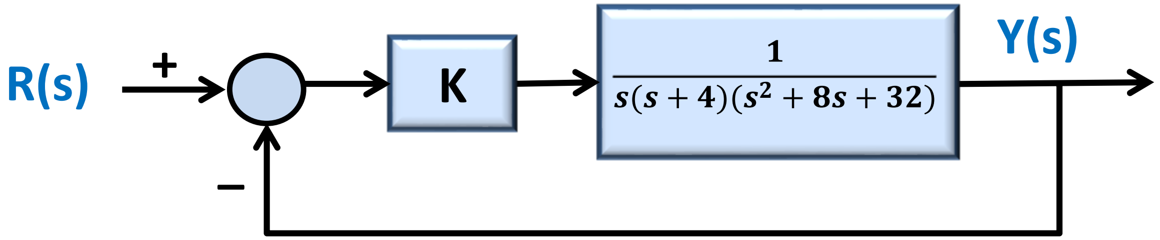




Root Locus Method – Example

Example 9

Plot the root-locus diagram for the following fourth-order system



First determine the closed-loop transfer function and the characteristics equation.

Closed-loop Transfer Function

$$\boldsymbol{T(s)=\frac{Y(s)}{R(s)}=\frac{KG(s)}{1+KG(s)H(s)}}=\frac{K}{s^4+12s^3+64s^2+128s+K}$$

Closed-loop Characteristic Equation

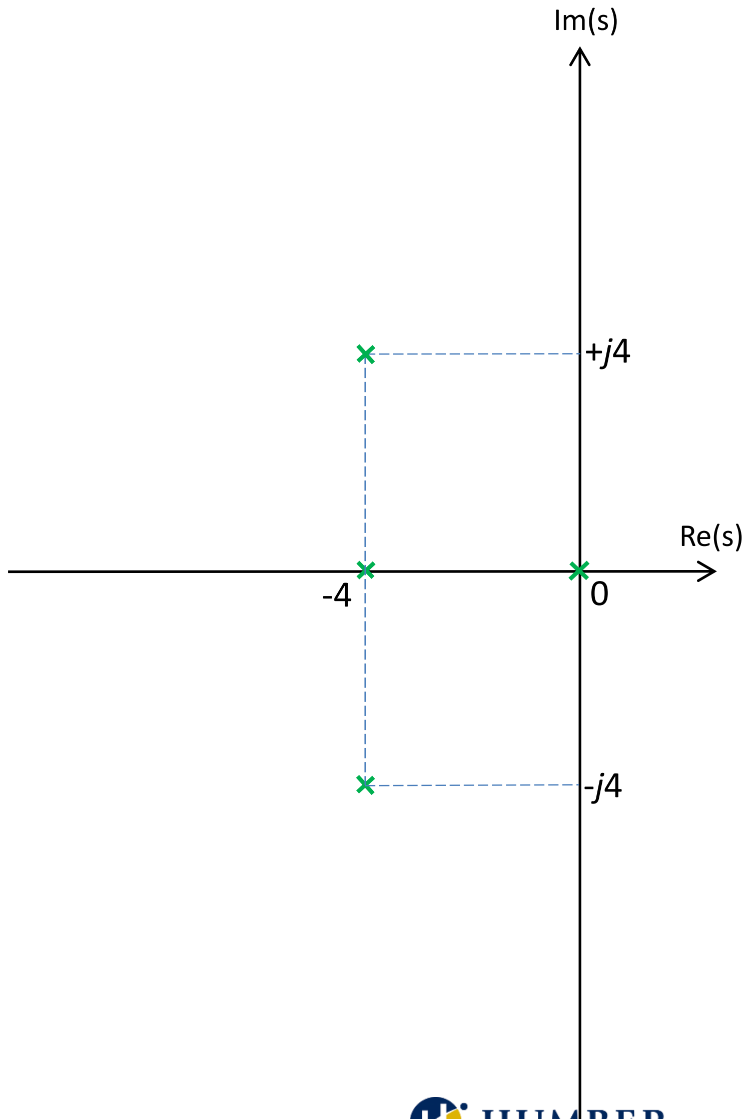
$$\boldsymbol{1+KG(s)H(s)=0} \quad \rightarrow \quad s^4+12s^3+64s^2+128s+K=0$$

Step 1: Draw the axes of the s-plane

Mark poles × and zeros o of the open-loop system.

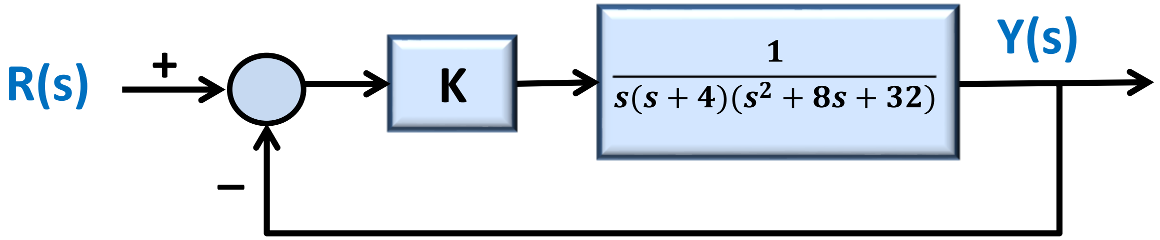
Poles: $p_1=0,\ p_2=-4,\ p_{3,4}=-4\pm j4$

Zeros: No finite zeros, Four zeros at infinity



Root Locus Method – Example

Example 9 Plot the root-locus diagram for the following fourth-order system



Step 2: Draw the root-locus on the real axis

A point on the real axis is part of a locus if the number of poles and zeros to the right of that point is **ODD.**

Here, zero is considered as an even number

Step 3: Draw asymptote lines for large *K* values

Number of asymptotes → $n - m = 4 - 0 = 4$

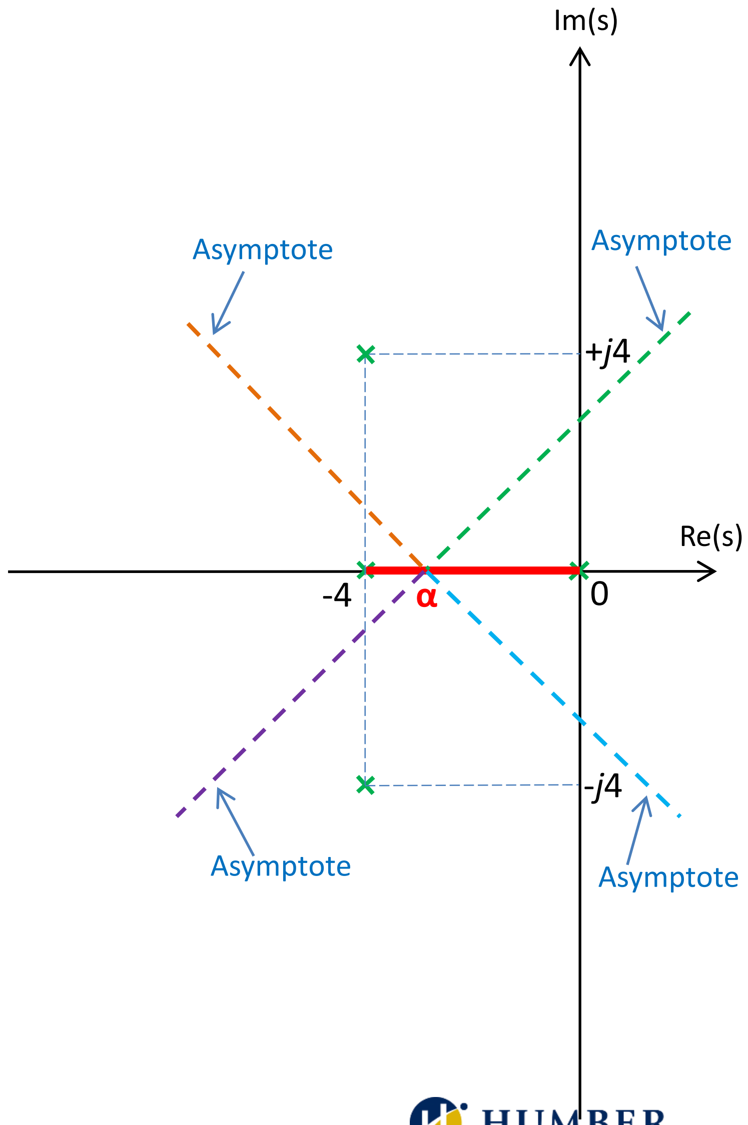
Intersection of asymptotes on the real axis

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{[(0)+(-4)+(-4+j4)+(-4-j4)]}{4-0} = -3$$

Angle of asymptote lines with real axis

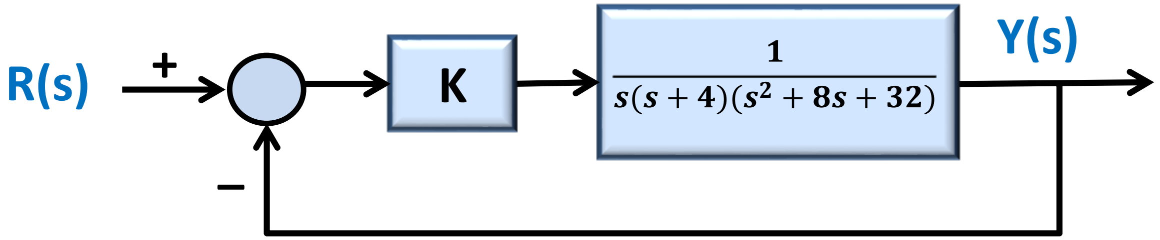
$$\varphi_i = \frac{180^\circ}{n-m}(2i+1) = \frac{180^\circ}{4-0}(2i+1) = 45^\circ(2i+1) \rightarrow \begin{cases} \varphi_0 = 45^\circ \\ \varphi_1 = 135^\circ \\ \varphi_2 = 225^\circ \\ \varphi_3 = 315^\circ \end{cases}$$

$i = 0, 1, 2, \cdots$



Root Locus Method – Example

Example 9 Plot the root-locus diagram for the following fourth-order system



Step 4: Intersection of root-locus with imaginary axis

$$\mathbf{1+KG(s)H(s)=0} \quad \rightarrow \quad s^4+12s^3+64s^2+128s+K=0$$

$$\boxed{s=j\omega} \quad \Rightarrow \quad (j\omega)^4+12(j\omega)^3+64(j\omega)^2+128(j\omega)+K=\omega^4-j12\omega^3-64\omega^2+j128\omega+K=0$$

$$\underbrace{[\omega^4-64\omega^2+K]}_{\text{real part}}+j\underbrace{[-12\omega^3+128\omega]}_{\text{imaginary part}}=0$$

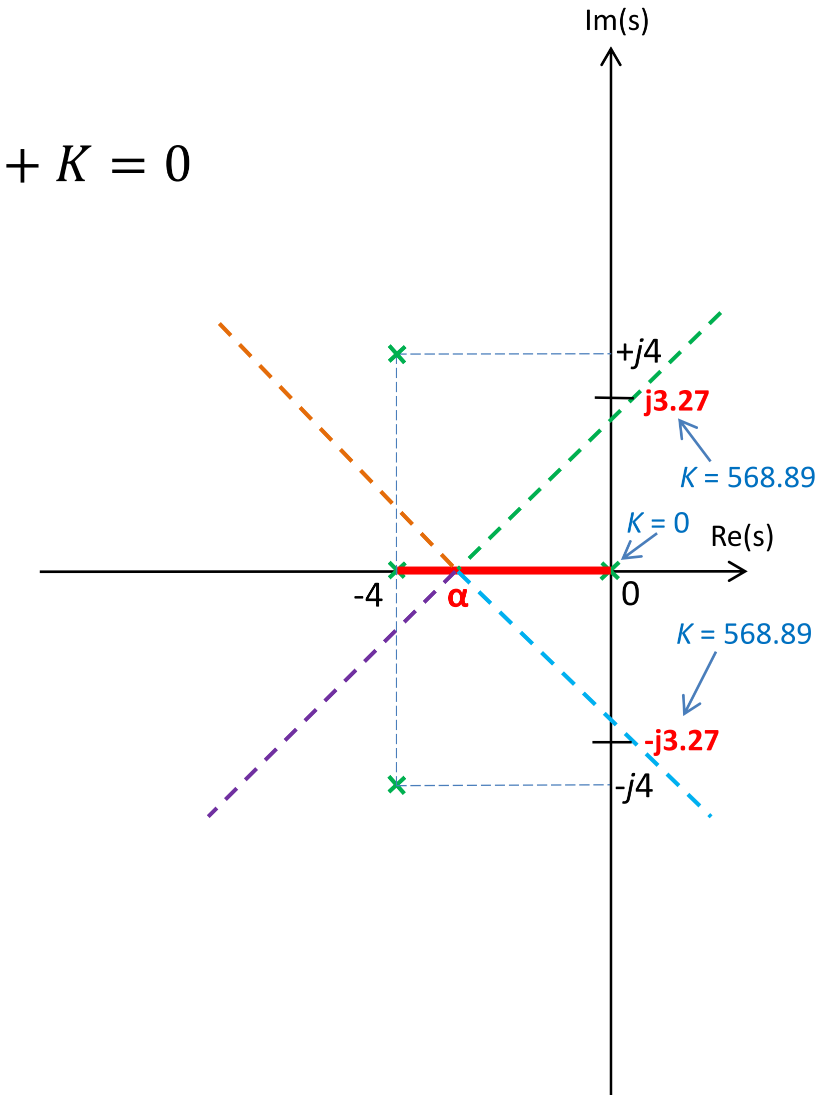
From the imaginary part:

$$-12\omega^3+128\omega=0 \quad \rightarrow \quad 4\omega(-3\omega^2+32)=0 \quad \rightarrow \begin{cases} \boxed{\omega=0} \\ \omega^2=\dfrac{32}{3} \rightarrow \boxed{\omega=\pm 3.27\ rad/s} \end{cases}$$

From the real part:

$$\omega=0 \quad \Rightarrow \quad \omega^4-64\omega^2+K=0-64\times 0+K=0 \rightarrow \boxed{K=0}$$

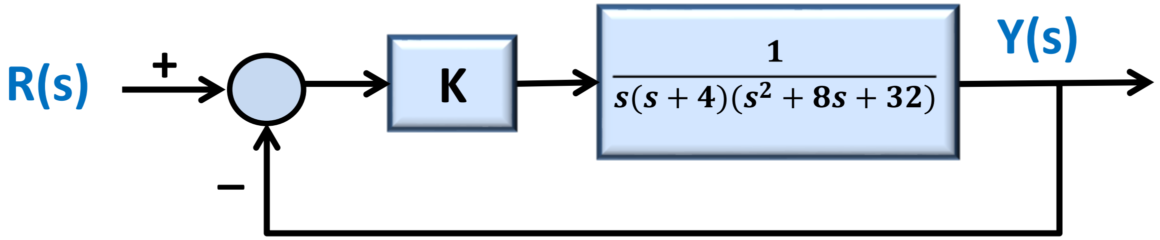
$$\omega^2=\frac{32}{3} \quad \Rightarrow \quad \omega^4-64\omega^2+K=\frac{1024}{9}-64\times\frac{32}{3}+K=0 \quad \rightarrow \boxed{K=\ 568.89}$$



Therefore, the loci cross the imaginary axis at $s=j\omega \quad \rightarrow \begin{cases} s=0 \\ s=\pm j3.27 \end{cases}$

Root Locus Method – Example

Example 9 Plot the root-locus diagram for the following fourth-order system



Step 4: Intersection of root-locus with imaginary axis

NOTE: $K = 0$ and $K = 568.89$ are the marginal-stability gains, which can also be determined by applying the Routh-Hurwitz Criterion on the characteristic equation.

$$\boxed{s^4 + 12s^3 + 64s^2 + 128s + K = 0}$$

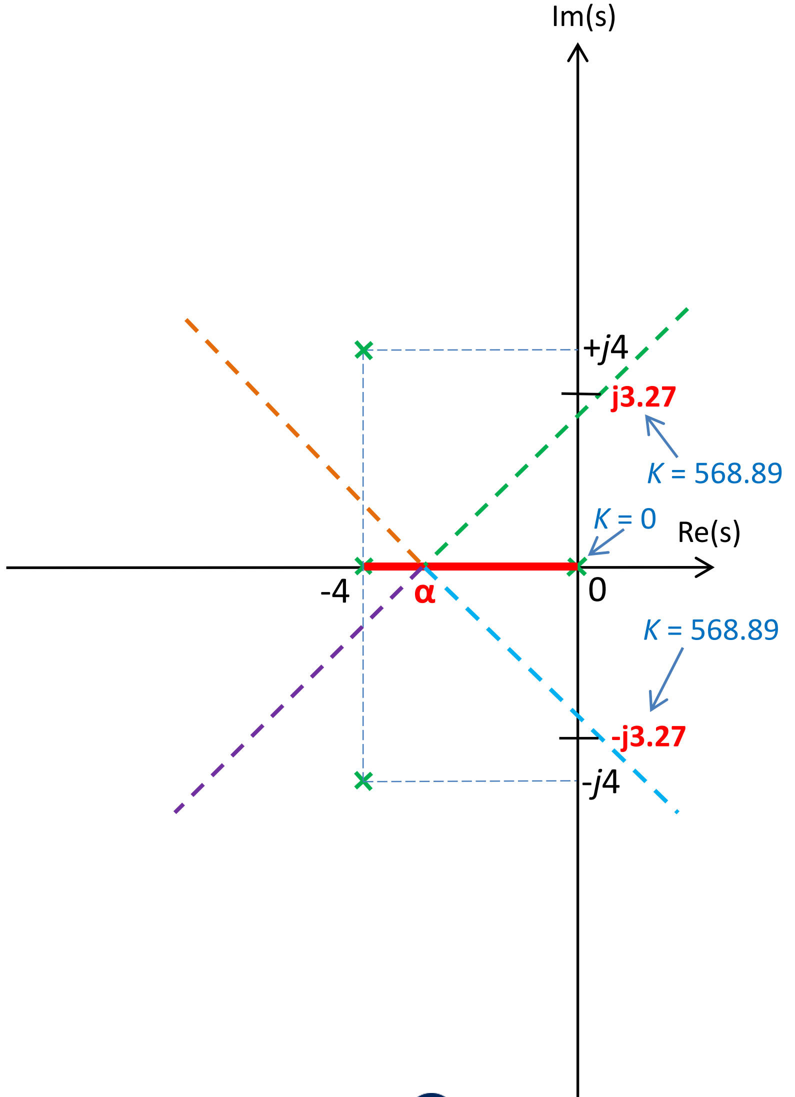
s^4	1	64	K
s^3	12	128	0
s^2	$\frac{160}{3}$	K	0
s^1	$\frac{20480-36K}{160}$	0	0
s^0	K	0	0

$$\frac{20480 - 36K}{160} > 0 \quad \rightarrow \quad K < 568.89$$

$$K > 0$$

$$\boxed{0 < K < 568.89}$$

$$\boxed{K = 0, \qquad K = 568.89}$$ Marginal-Stability Gains

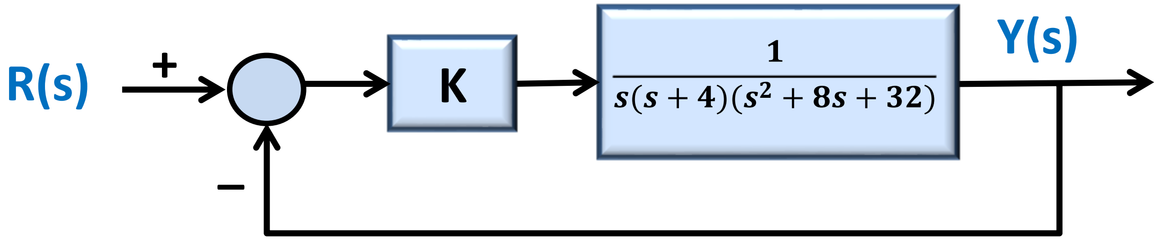


The intersection with imaginary axis is determined from the even polynomial by using s^2 row with $K = 0$, and $K = 568.89$

$$\frac{160}{3}s^2 + K = 0 \quad \rightarrow \begin{cases} \text{If} \quad K = 0 \quad \rightarrow \quad 53.33s^2 = 0 \quad \rightarrow \quad s = 0 \\ \text{If} \quad K = 568.89 \quad \rightarrow \quad 53.33s^2 + 568.89 = 0 \quad \rightarrow \quad s = \pm j3.27 \end{cases}$$

Root Locus Method – Example

Example 9 Plot the root-locus diagram for the following fourth-order system



Step 5: Calculate break-away/break-in points on real axis

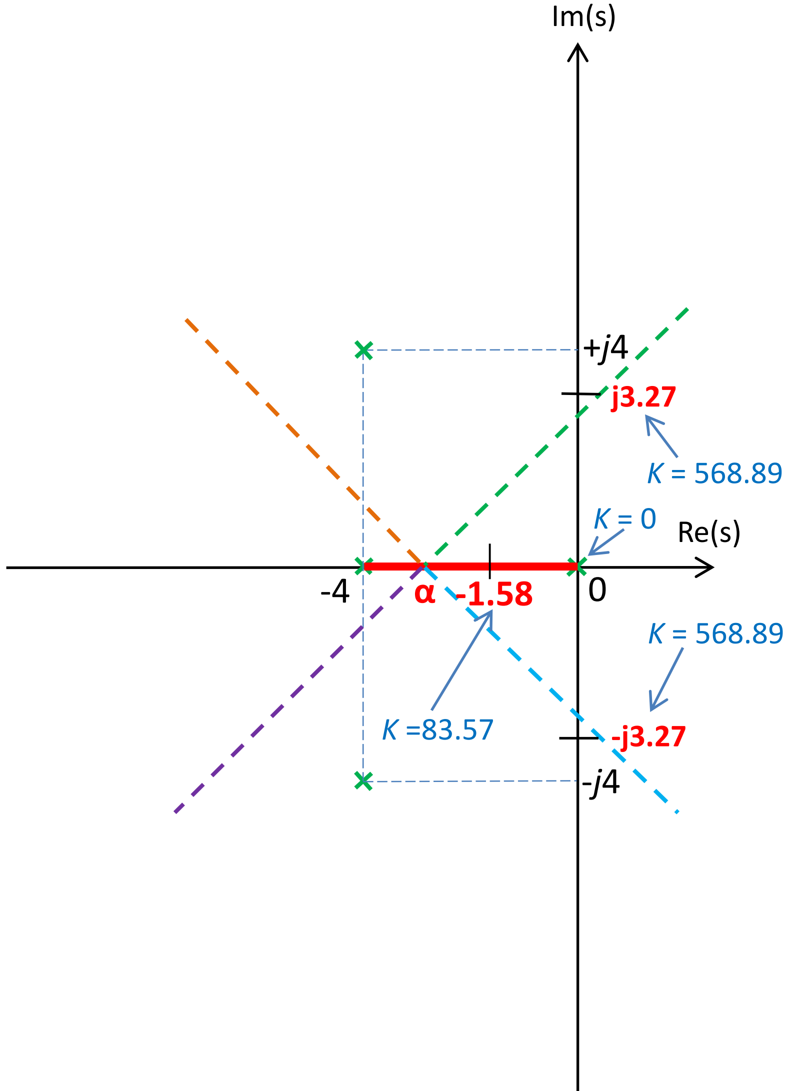
$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad s^4 + 12s^3 + 64s^2 + 128s + K = 0$$

$$K = -s^4 - 12s^3 - 64s^2 - 128s$$

$$\frac{dK}{ds} = 0 \quad \rightarrow \quad -4s^3 - 36s^2 - 128s - 128 = 0$$

$$\begin{cases} s = -3.7117 \pm j2.5533 \ \rightarrow \ \text{not on the root} - \text{loci} \\ \boxed{s = -1.58} \ \rightarrow \ \text{on the root} - \text{loci} \end{cases}$$

Break-away Point

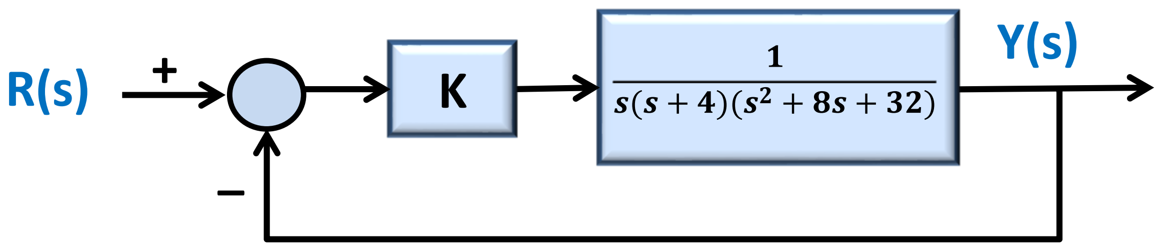


The gain K at the break-away point:

$$K = -(-1.58)^4 - 12(-1.58)^3 - 64(-1.58)^2 - 128(-1.58) \longrightarrow \boxed{K = 83.57}$$

Root Locus Method – Example

Example 9 Plot the root-locus diagram for the following fourth-order system



Step 6: Calculate angle of departure/angle of arrival

Angle of departure from the complex pole

= 180° - (*sum of the angles of vectors drawn to this pole from the other poles*)

+ (*sum of the angles of vectors drawn to this pole from zeros*)

$$\boxed{\boldsymbol{\phi_p = 180^\circ - \sum_i \angle p_i + \sum_j \angle z_j}}$$

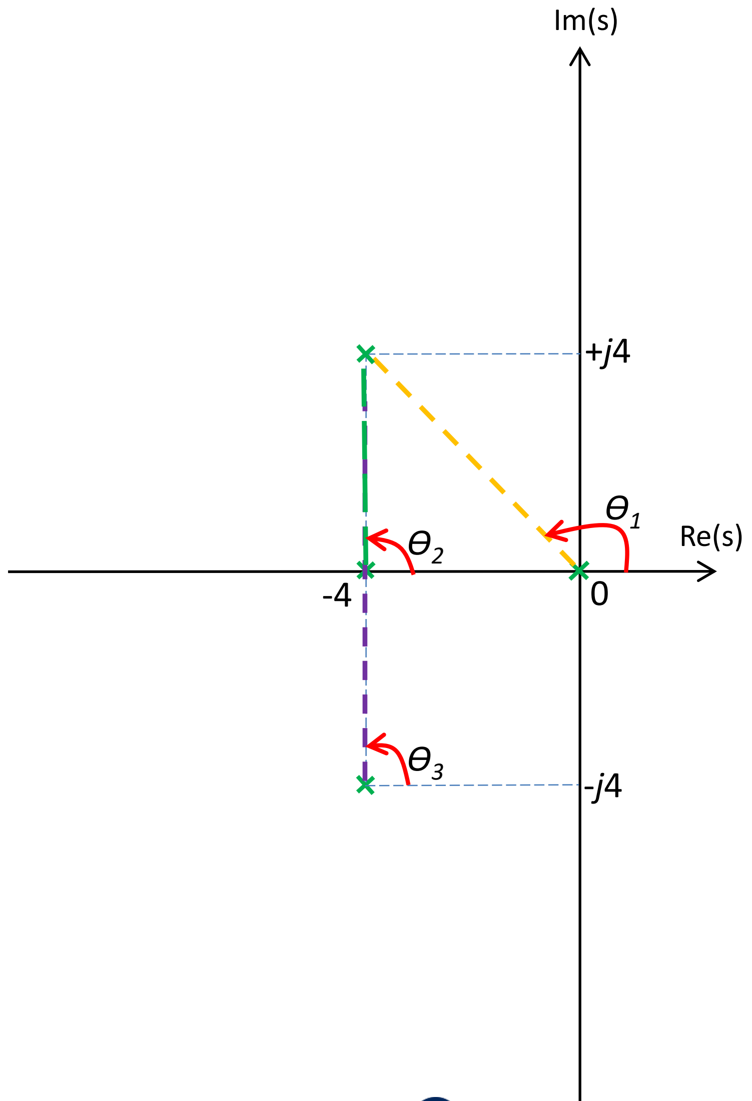
Angle of departure from the poles at $s = -4 \pm j4$

$$\boldsymbol{\phi_p = 180^\circ - (\theta_1 + \theta_2 + \theta_3)}$$

$$= 180^\circ - (135^\circ + 90^\circ + 90^\circ)$$

$$= \boldsymbol{-135^\circ}$$

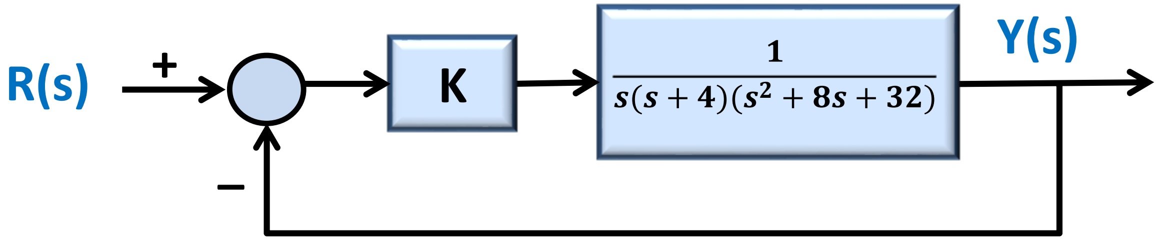
Angel of departure from the pole at $s = -4 + j4$



Since, root-locus is symmetric with respect to the real axis, the angle of departure for the pole at $s = -4 - j4$ will be $\boldsymbol{\phi_p = +135^\circ}$

Root Locus Method – Example

Example 9 Plot the root-locus diagram for the following fourth-order system



Step 6: Calculate angle of departure/angle of arrival

Angle of departure from the complex pole

= 180° - (*sum of the angles of vectors drawn to this pole from the other poles*)

+ (*sum of the angles of vectors drawn to this pole from zeros*)

$$\boldsymbol{\phi_p = 180^\circ - \sum_i \angle p_i + \sum_j \angle z_j}$$

Angle of departure from the poles at $s = -4 \pm j4$

$$\boldsymbol{\phi_p = 180^\circ - (\theta_1 + \theta_2 + \theta_3)}$$

$$= 180^\circ - (135^\circ + 90^\circ + 90^\circ)$$

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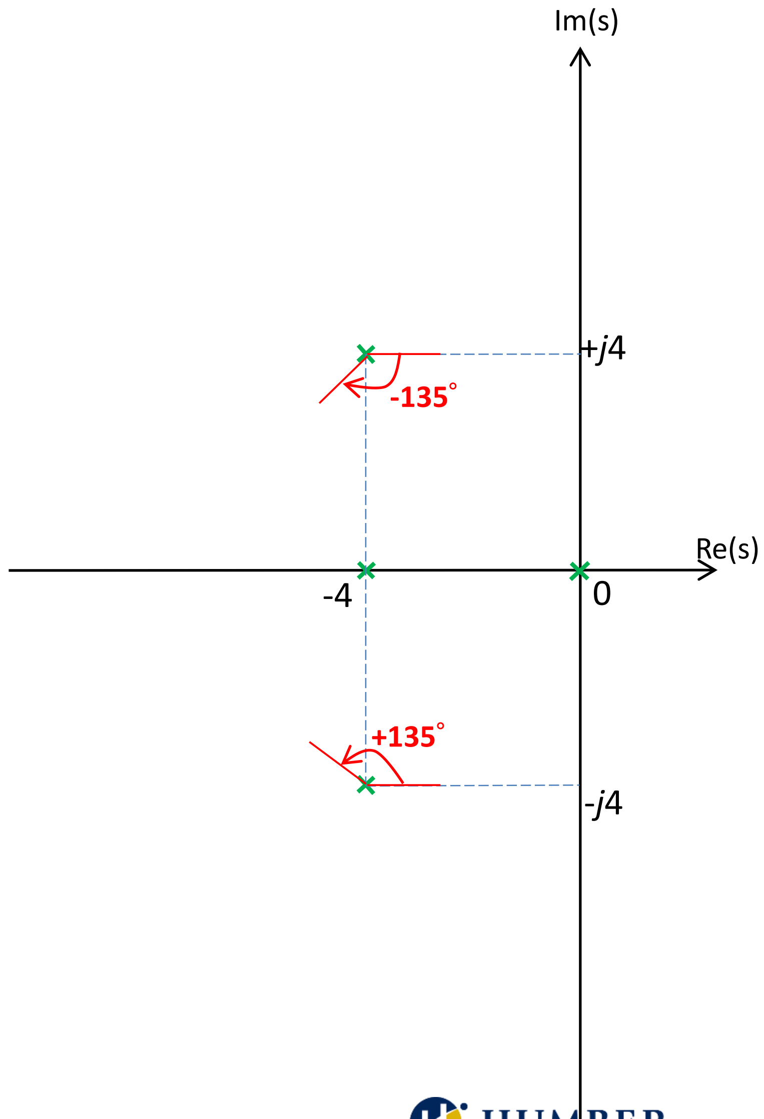
Angel of departure from the pole at $s = -4 + j4$



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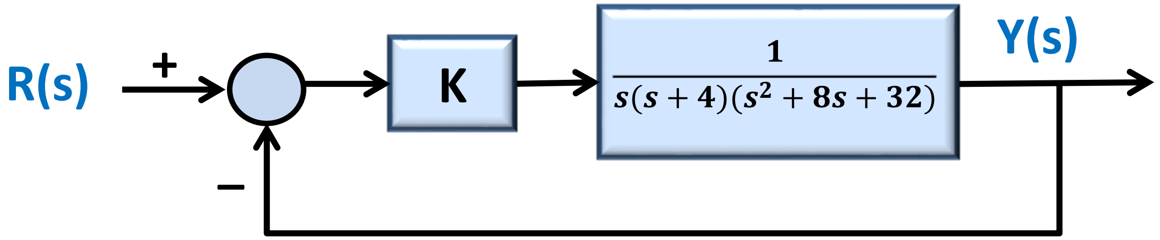
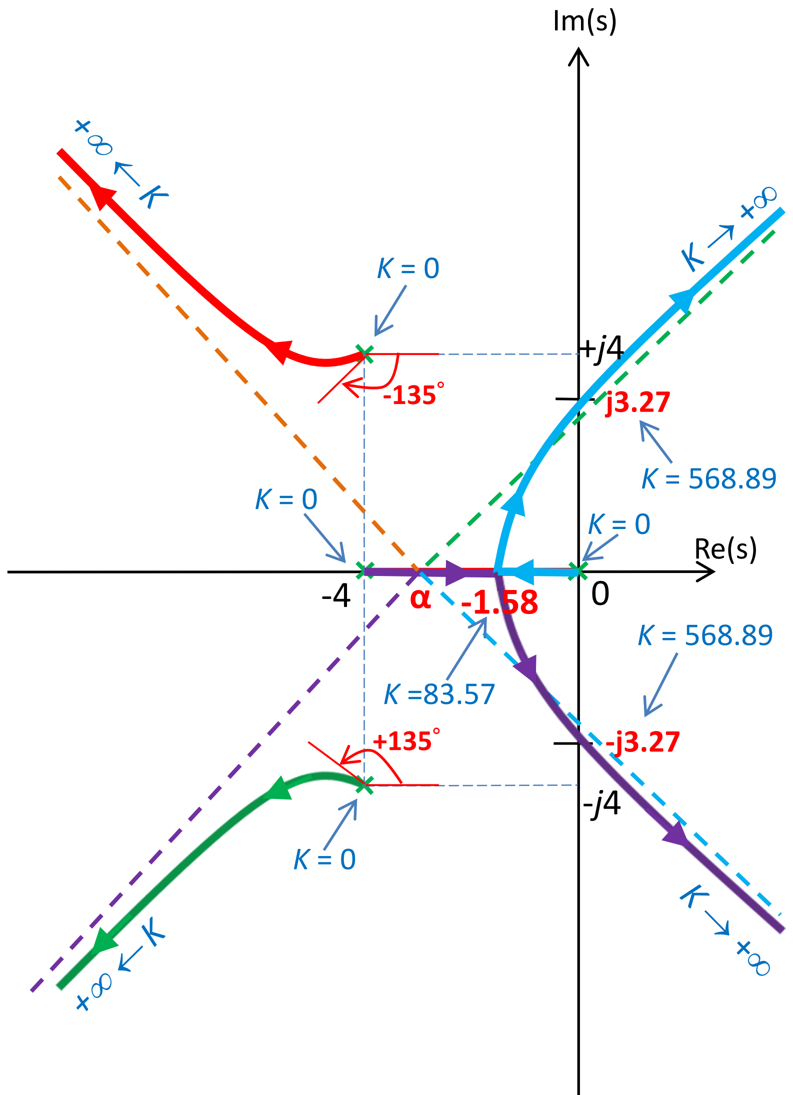
Root Locus Method – Example

Example 9 Plot the root-locus diagram for the following fourth-order system



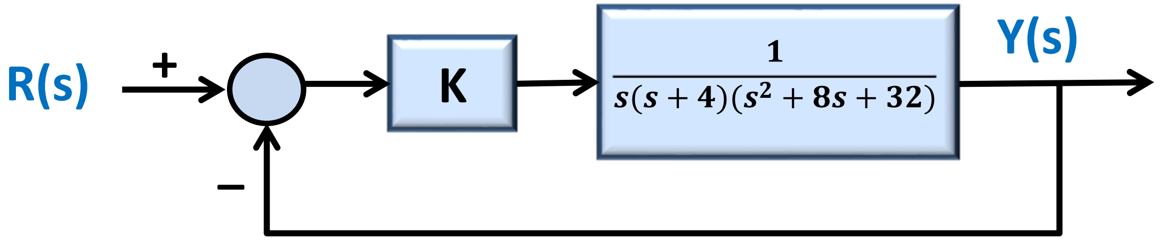
Step 7: Complete the root-locus diagram

- Number of separate root-loci is equal to the order of open-loop transfer function, which is four here.
- For $K = 0$ the root-loci is at the open-loop poles including those at $s = \infty$.
- For $K = \infty$ the root-loci is at the open-loop zeros including those at $s = \infty$.
- Since open-loop transfer function has no finite zero, so the root-locus branches start from open-loop poles and go to infinity approaching the asymptote lines.
- Root-locus is symmetric with respect to the real axis.



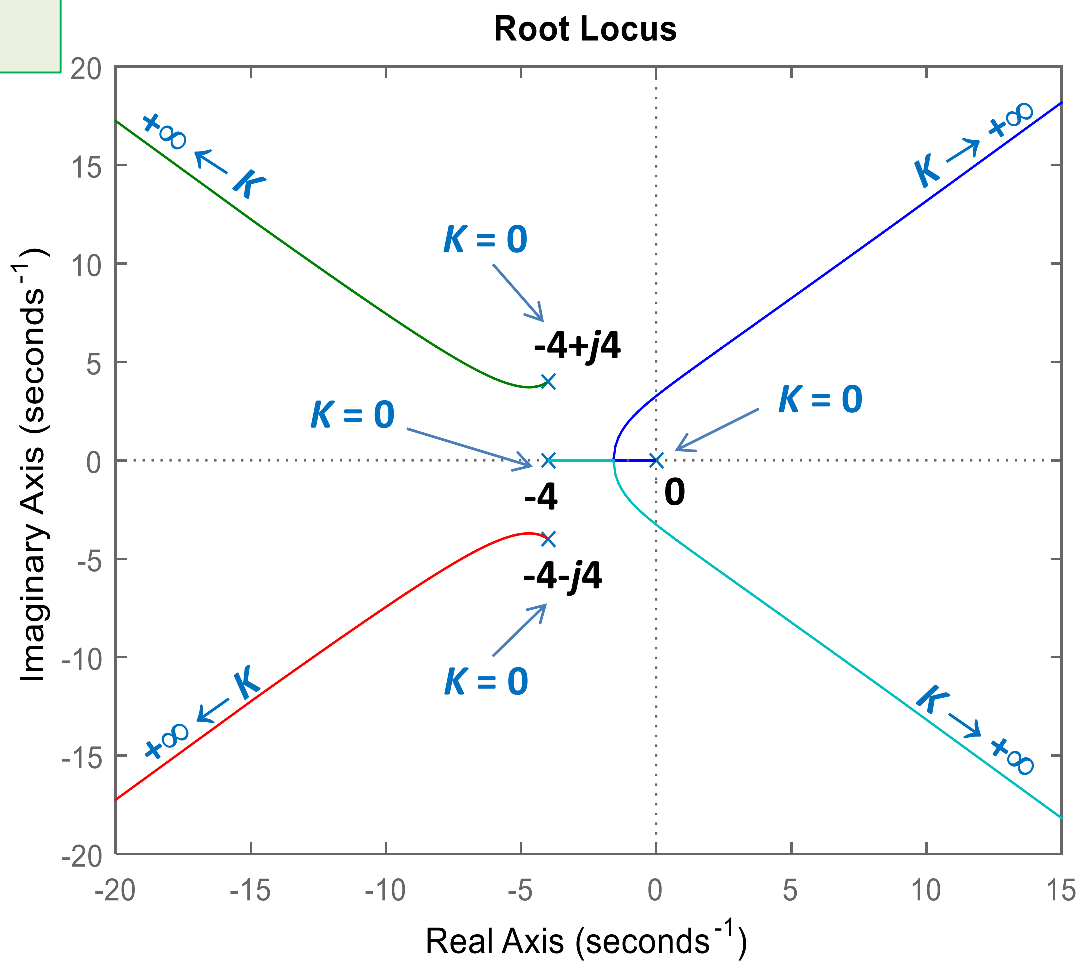
Root Locus Method – Example

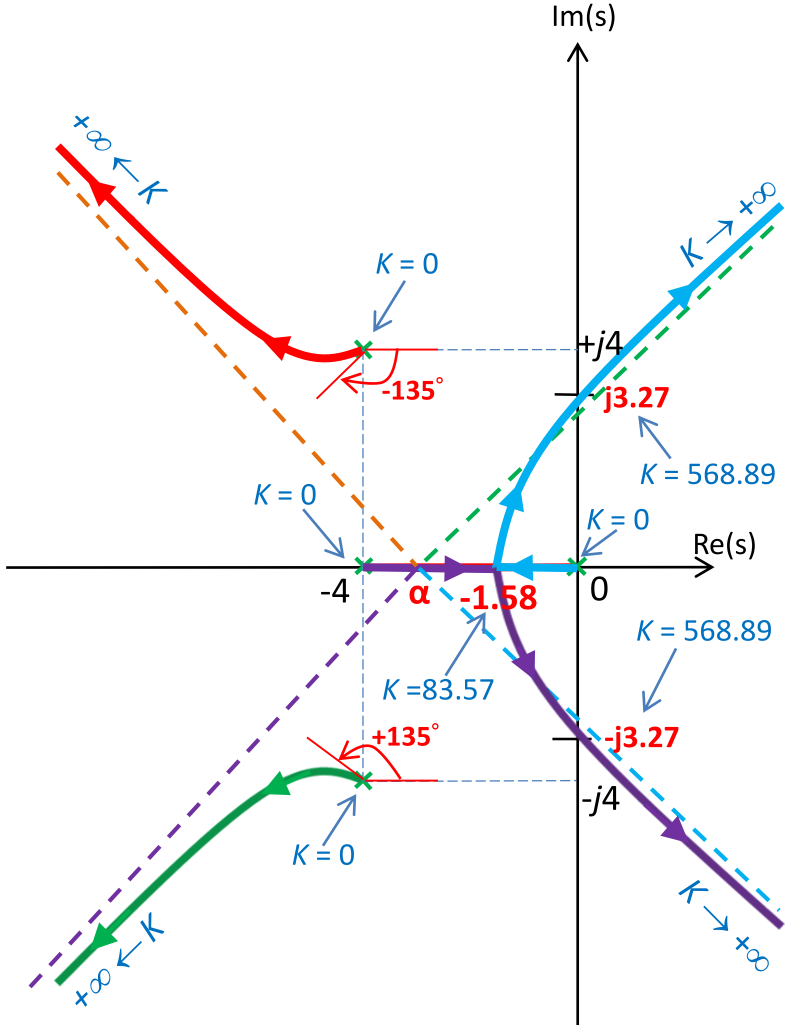
Example 9 Plot the root-locus diagram for the following fourth-order system



We can plot the root-locus by MATLAB to compare the results.

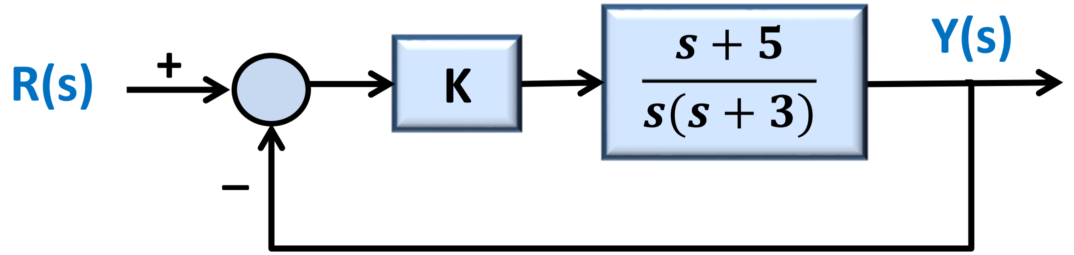
```
num = [1];
den = [1  12  64  128  0];
sys = tf(num,den);
rlocus(sys)
```





Root Locus Method – Example

Example 10 Plot the root-locus diagram for the following second-order system with a single zero.



First determine the closed-loop transfer function and the characteristics equation.

Closed-loop Transfer Function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{K(s+5)}{s^2 + (3+K)s + 5K}$$

Closed-loop Characteristic Equation

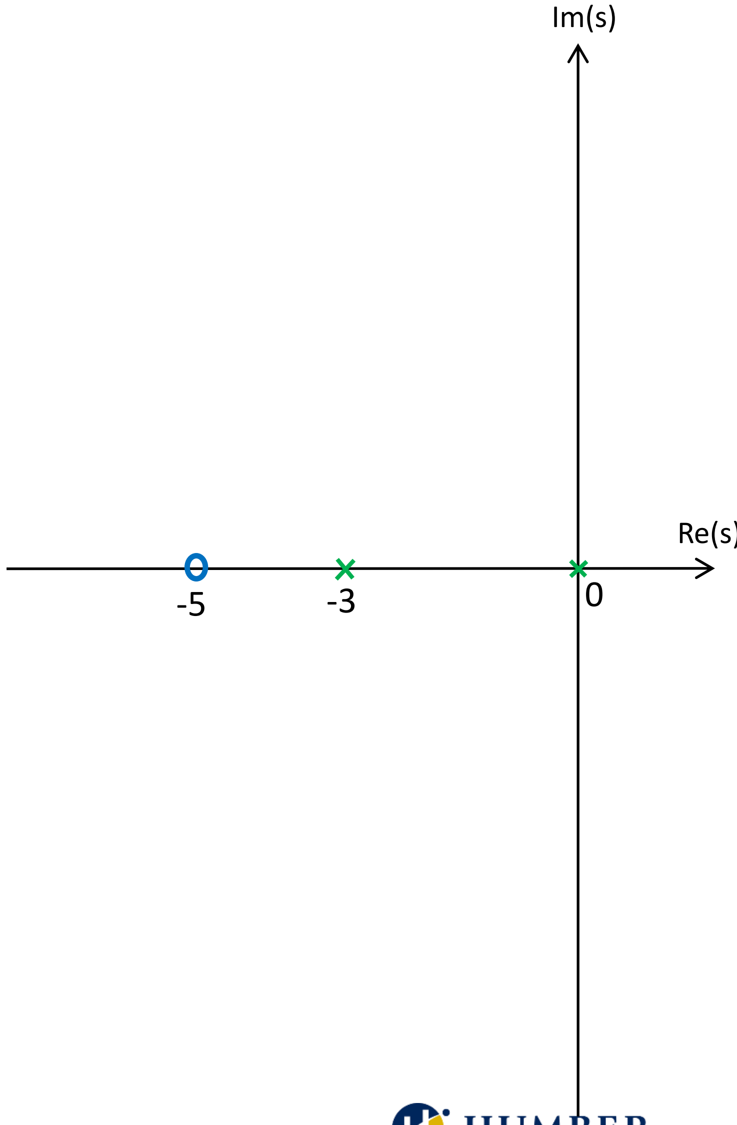
$$1 + KG(s)H(s) = 0 \quad \rightarrow \quad s^2 + (3+K)s + 5K = 0$$

Step 1: Draw the axes of the s-plane

Mark poles × and zeros o of the open-loop system.

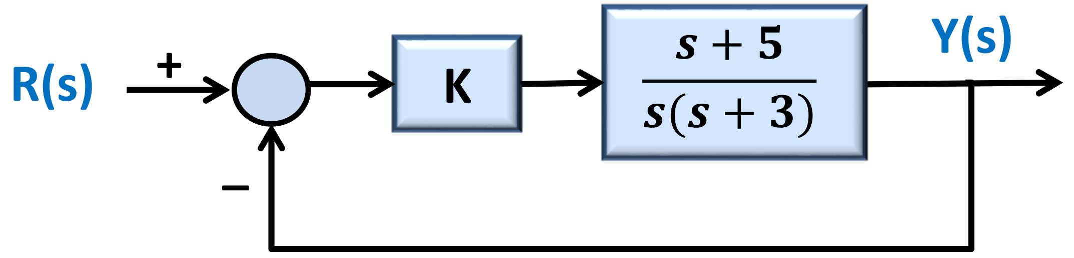
Poles: $p_1 = 0,\ p_2 = -3$

Zeros: $z_1 = -5$, one zero at infinity



Root Locus Method – Example

Example 10 Plot the root-locus diagram for the following second-order system with a single zero.



Step 2: Draw the root-locus on the real axis

A point on the real axis is part of a locus if the number of poles and zeros to the right of that point is **ODD.**

Here, zero is considered as an even number

Step 3: Draw asymptote lines for large *K* values

Number of asymptotes $\rightarrow$ $n - m = 2 - 1 = 1$

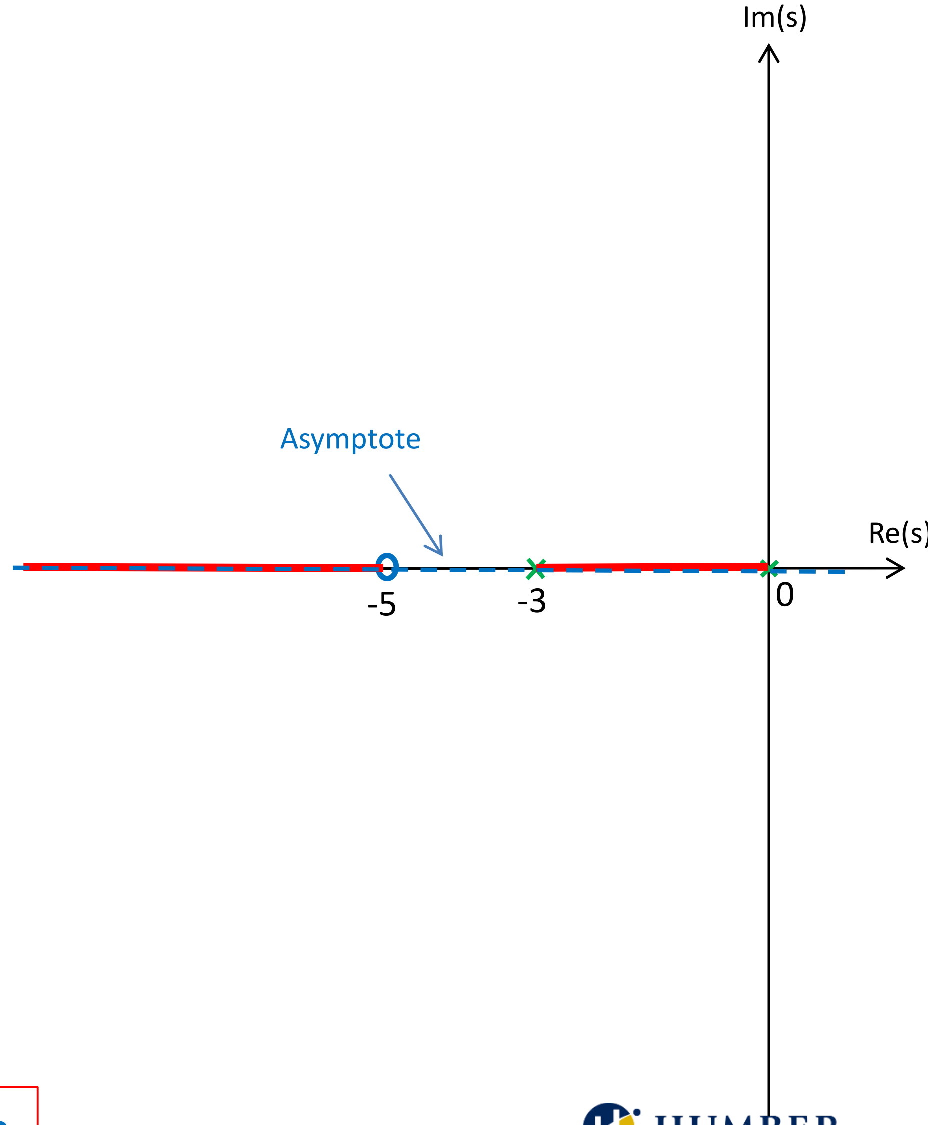
Intersection of asymptotes on the real axis

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{[(0)+(-3)]-[(-5)]}{2-1} = 2$$

Angle of asymptote lines with real axis

$$\varphi_i = \frac{180^\circ}{n-m}(2i+1) = \frac{180^\circ}{2-1}(2i+1) = 180^\circ(2i+1) \rightarrow \varphi_0 = 180^\circ$$

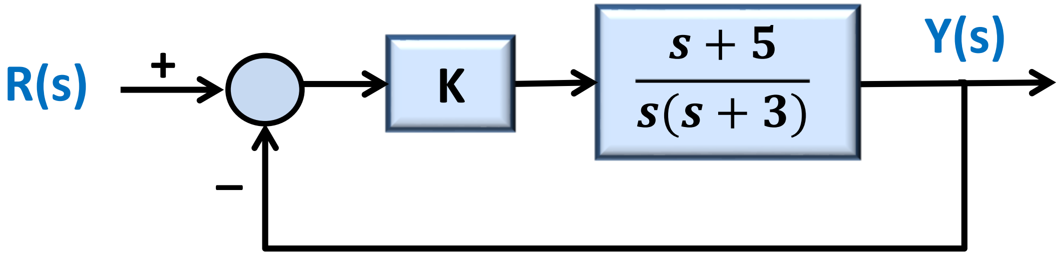
$i = 0, 1, 2, \cdots$



The asymptote line lies on the real axis

Root Locus Method – Example

Example 10 Plot the root-locus diagram for the following second-order system with a single zero.



Step 4: Intersection of root-locus with imaginary axis

$$\mathbf{1 + KG(s)H(s) = 0} \quad \rightarrow \quad s^2 + (3+K)s + 5K = 0$$

$$\boxed{\boldsymbol{s = j\omega}} \longrightarrow (j\omega)^2 + (3+K)(j\omega) + 5K = -\omega^2 + j(3+K)\omega + 5K = 0$$

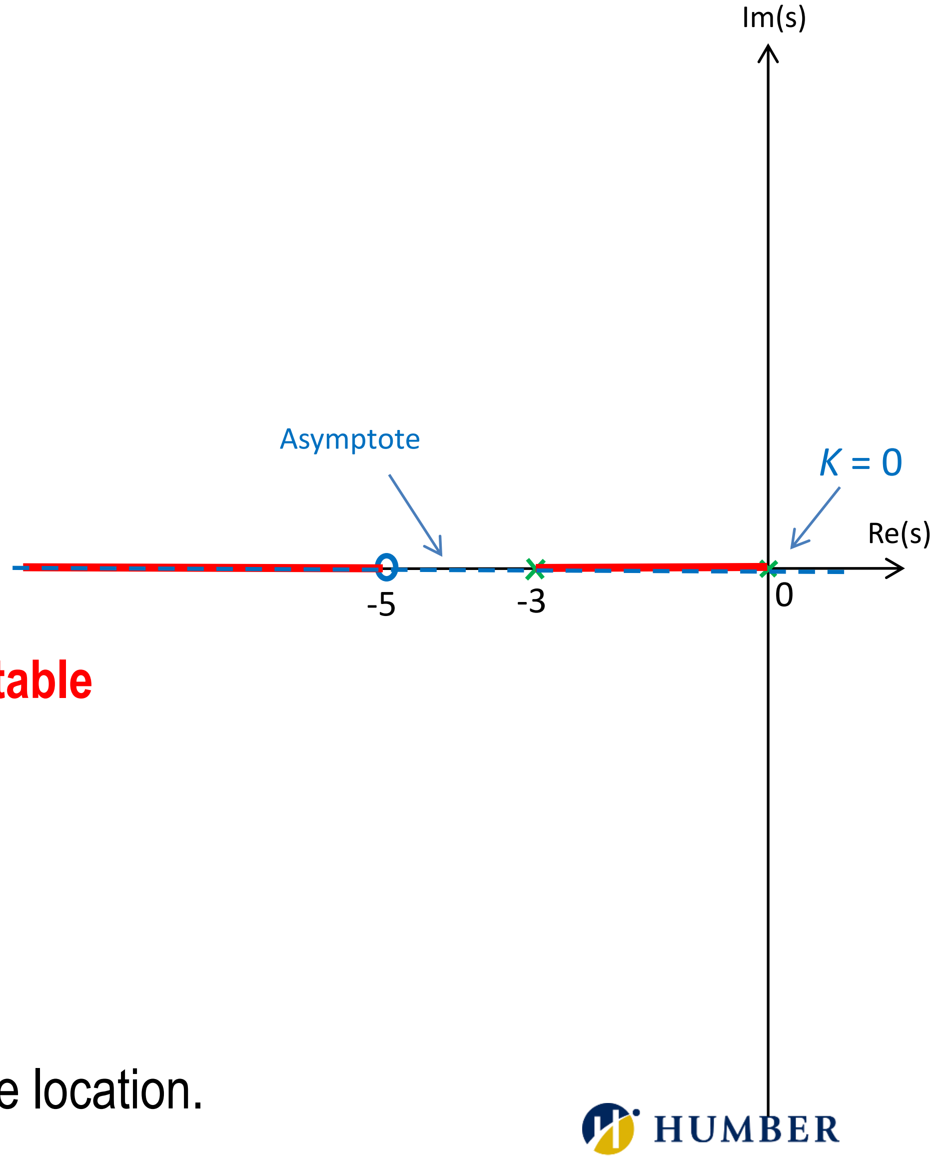
$$\underbrace{[-\omega^2 + 5K]}_{\textbf{real part}} + j\,\underbrace{[3\omega + K\omega]}_{\textbf{imaginary part}} = 0$$

From the imaginary part:

$$\boldsymbol{3\omega + K\omega = 0} \rightarrow \omega(3+K) = 0 \rightarrow \begin{cases} \boxed{\boldsymbol{\omega = 0}} \\ 3 + K = 0 \rightarrow K = -3 < 0 \quad \textbf{Not acceptable} \end{cases}$$

From the real part:

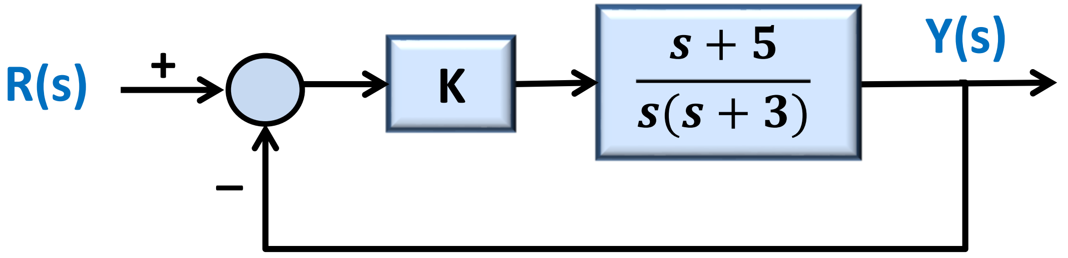
$$\boldsymbol{\omega = 0} \longrightarrow -\omega^2 + 5K = -0^2 + 5K = 0 \rightarrow \boxed{\boldsymbol{K = 0}}$$



Therefore, the loci do not cross the imaginary axis for $K \in [0, +\infty)$ other than open-loop pole location.

Root Locus Method – Example

Example 10 Plot the root-locus diagram for the following second-order system with a single zero.



Step 4: Intersection of root-locus with imaginary axis

NOTE: $\boldsymbol{K = 0}$ is the marginal-stability gain, which can also be determined by applying the Routh-Hurwitz Criterion on the characteristic equation.

$$\boxed{s^2 + (3+K)s + 5K = 0}$$

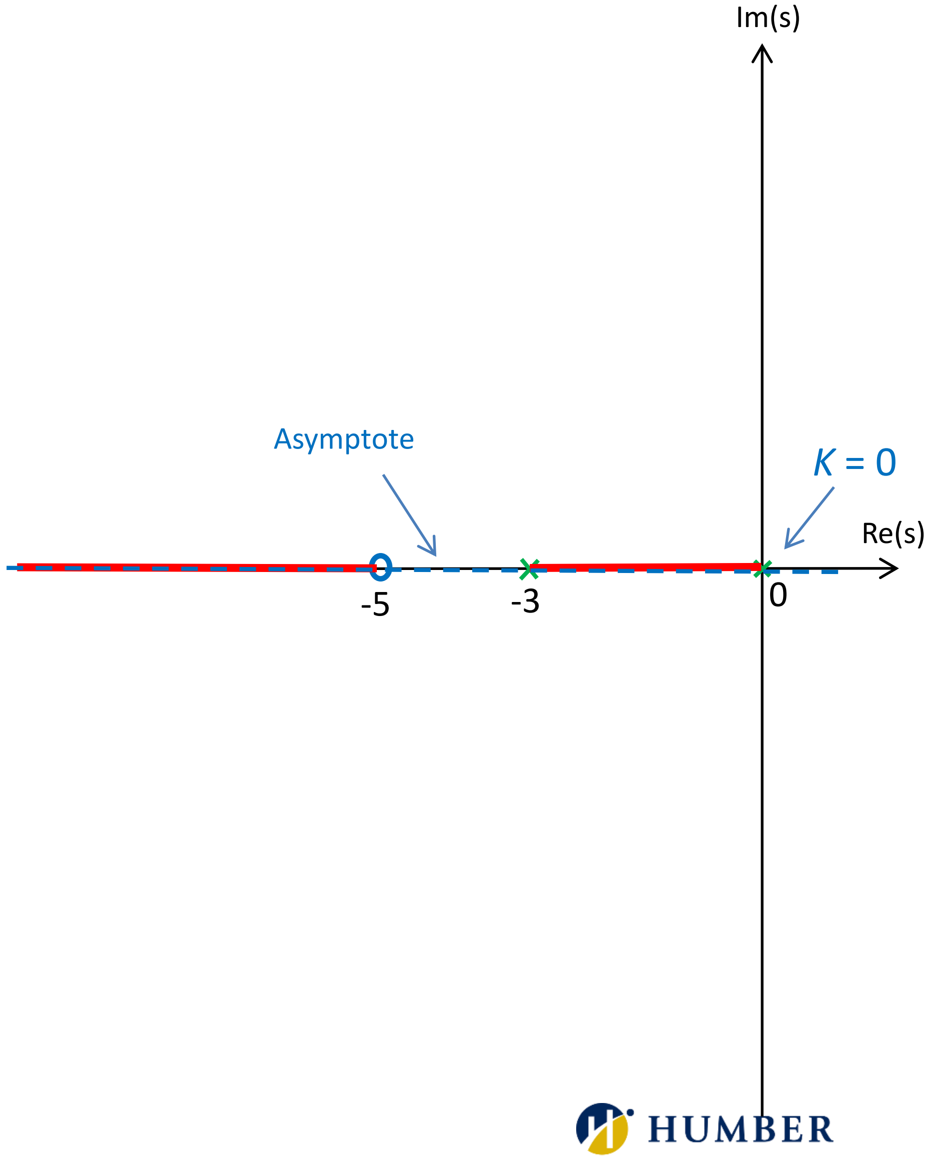
s^2	1	$5K$
s^1	$3+K$	0
s^0	$5K$	0

$3 + K > 0 \rightarrow \boldsymbol{K > -3}$

$5K > 0 \rightarrow \boldsymbol{K > 0}$

$\boxed{\boldsymbol{K > 0}}$

$\boxed{\boldsymbol{K = 0}}$ Marginal-Stability Gain

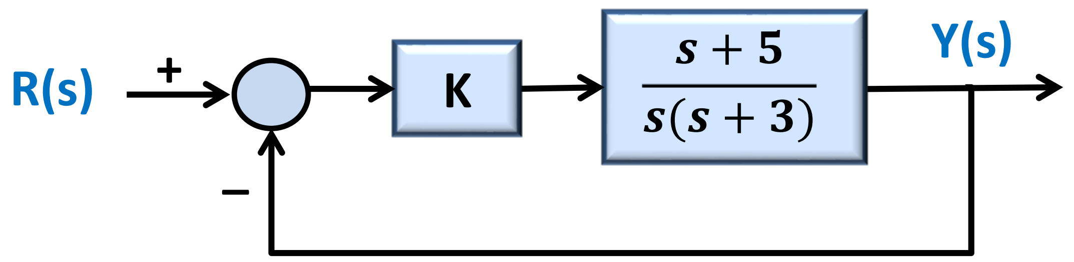


The intersection with imaginary axis is determined from the even polynomial by using s^2 row with $K = 0$.

$$s^2 + 5K = 0 \quad \rightarrow \quad s = 0$$

Root Locus Method – Example

Example 10 Plot the root-locus diagram for the following second-order system with a single zero.

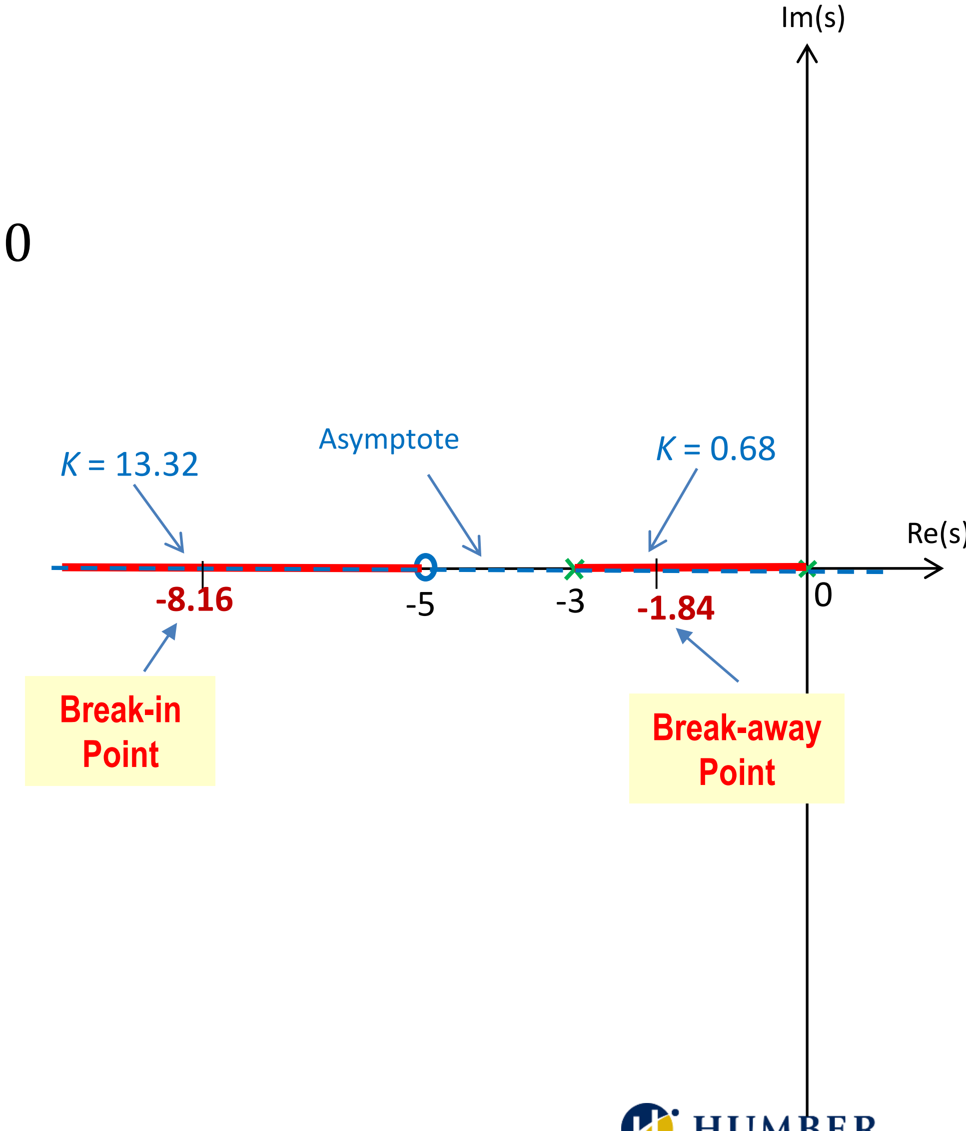


Step 5: Calculate break-away/break-in points on real axis

$$\mathbf{1 + KG(s)H(s) = 0} \rightarrow s^2 + (3+K)s + 5K = 0 \rightarrow K = \frac{-s^2 - 3s}{s+5}$$

$$\frac{dK}{ds} = 0 \rightarrow \frac{(-2s-3)(s+5) - (-s^2-3s)}{(s+5)^2} = 0 \rightarrow -s^2 - 10s - 15 = 0$$

$$\begin{cases} \boxed{s = -8.16} \rightarrow \text{on the root} - \text{loci} & \textbf{Break-in Point} \\ \boxed{s = -1.84} \rightarrow \text{on the root} - \text{loci} & \textbf{Break-away Point} \end{cases}$$

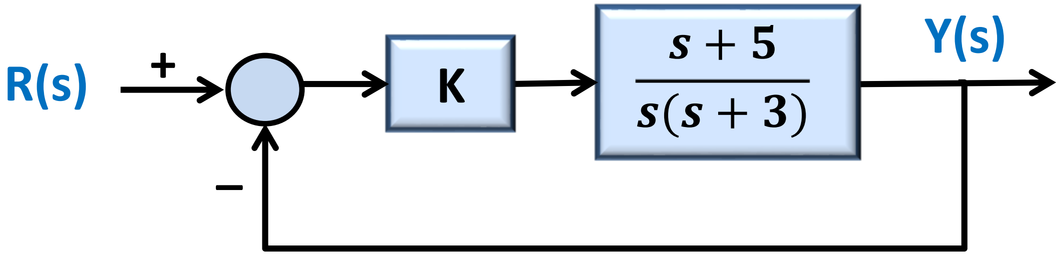


$$\boxed{s = -8.16} \rightarrow K = \frac{-(-8.16)^2 - 3(-8.16)}{(-8.16) + 5} \Rightarrow \boxed{K = 13.32}$$

$$\boxed{s = -1.84} \rightarrow K = \frac{-(-1.84)^2 - 3(-1.84)}{(-1.84) + 5} \Rightarrow \boxed{K = 0.68}$$

Root Locus Method – Example

Example 10 Plot the root-locus diagram for the following second-order system with a single zero.



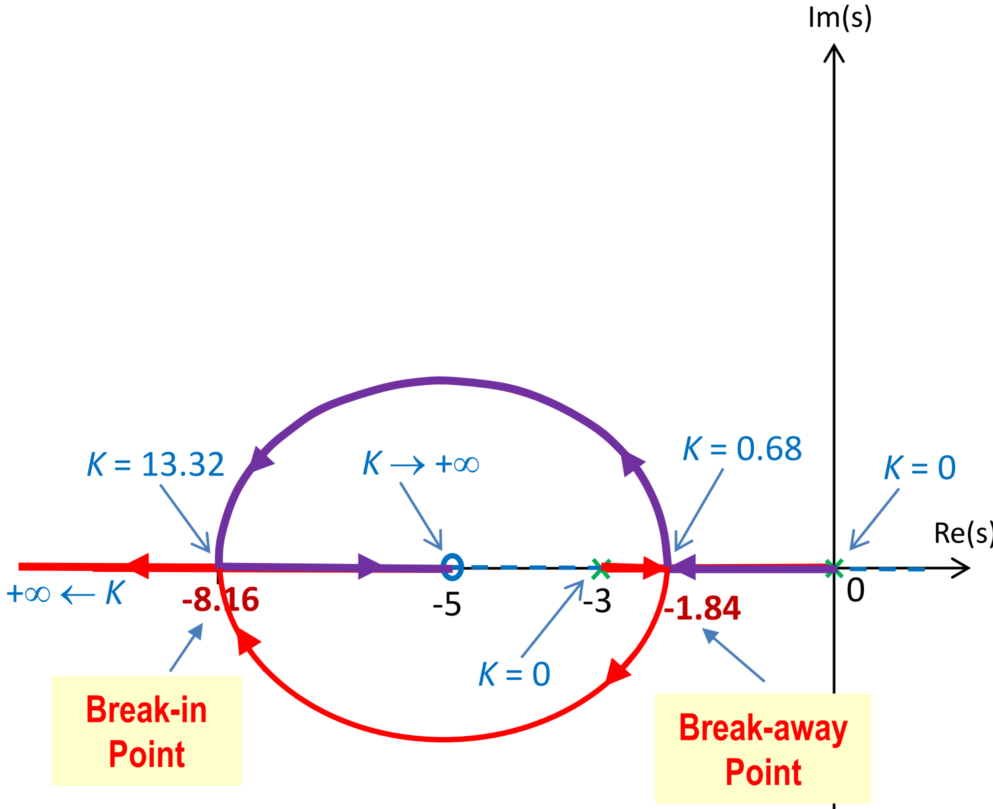
Step 6: Calculate angle of departure/angle of arrival

Calculate angle of departure from complex poles and/or angle of arrival to complex zeros of the root-loci.

Skip this step, because the open-loop transfer function does not have any complex poles/zeros.

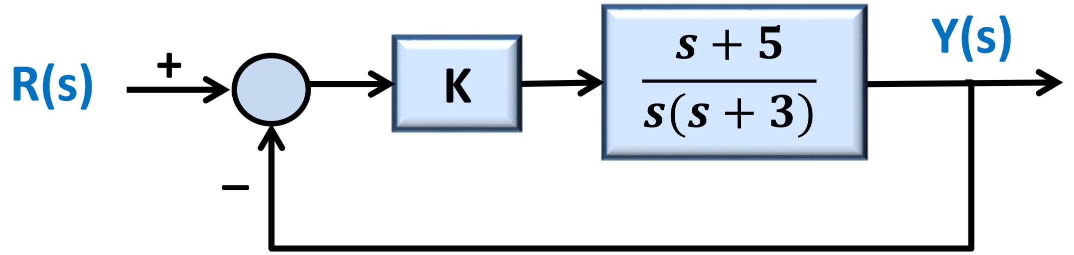
Step 7: Complete the root-locus diagram

- From the characteristic equation, number of separate root-loci is two.
- For $K = 0$ the root-loci is at the open-loop poles including those at $s = \infty$.
- For $K = \infty$ the root-loci is at the open-loop zeros including those at $s = \infty$.
- Since open-loop transfer function has one finite zero at $s = -5$ one of the root-locus branches terminates at the zero and the other one goes to infinity approaching the asymptote line.
- Root-locus is symmetric with respect to the real axis.



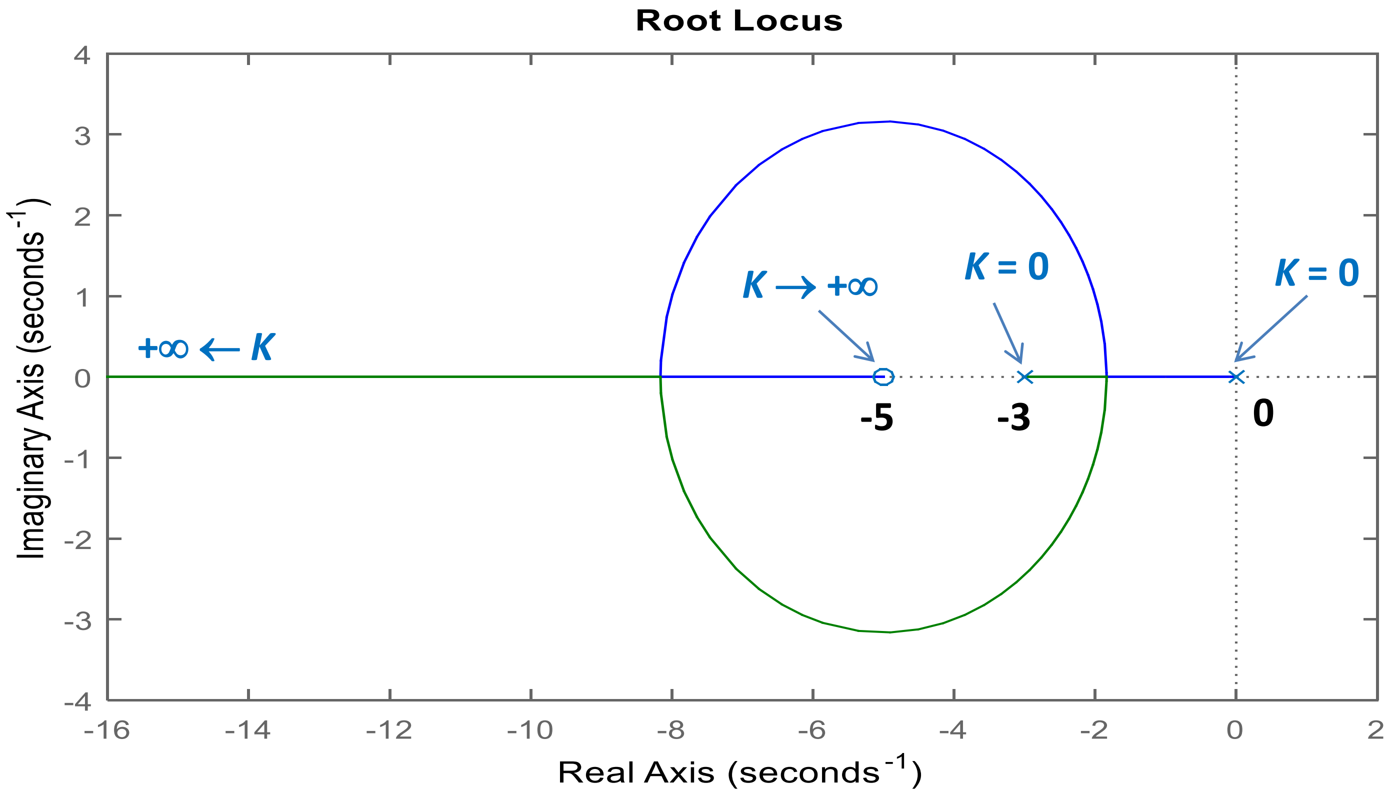
Root Locus Method – Example

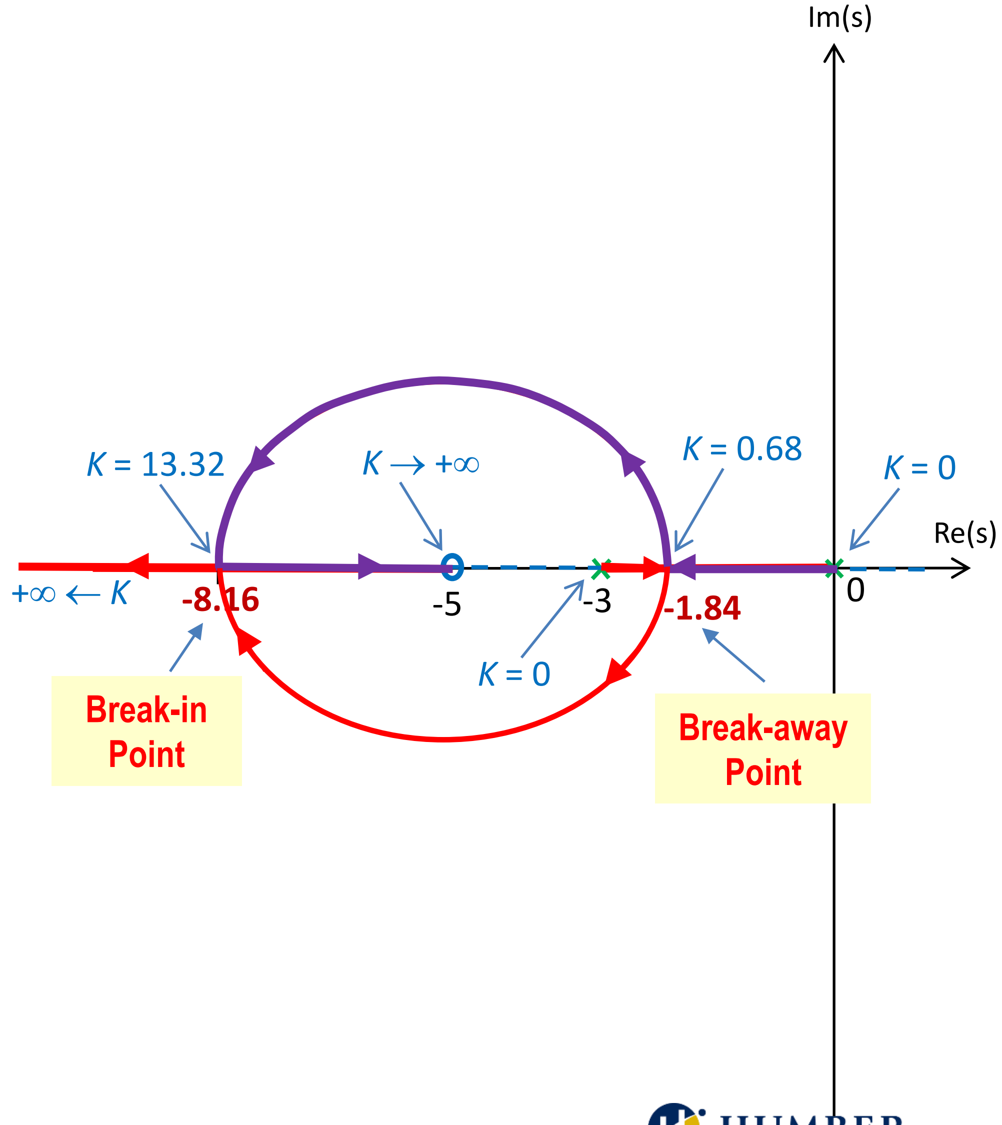
Example 10 Plot the root-locus diagram for the following second-order system with a single zero.



We can plot the root-locus by MATLAB to compare the results.

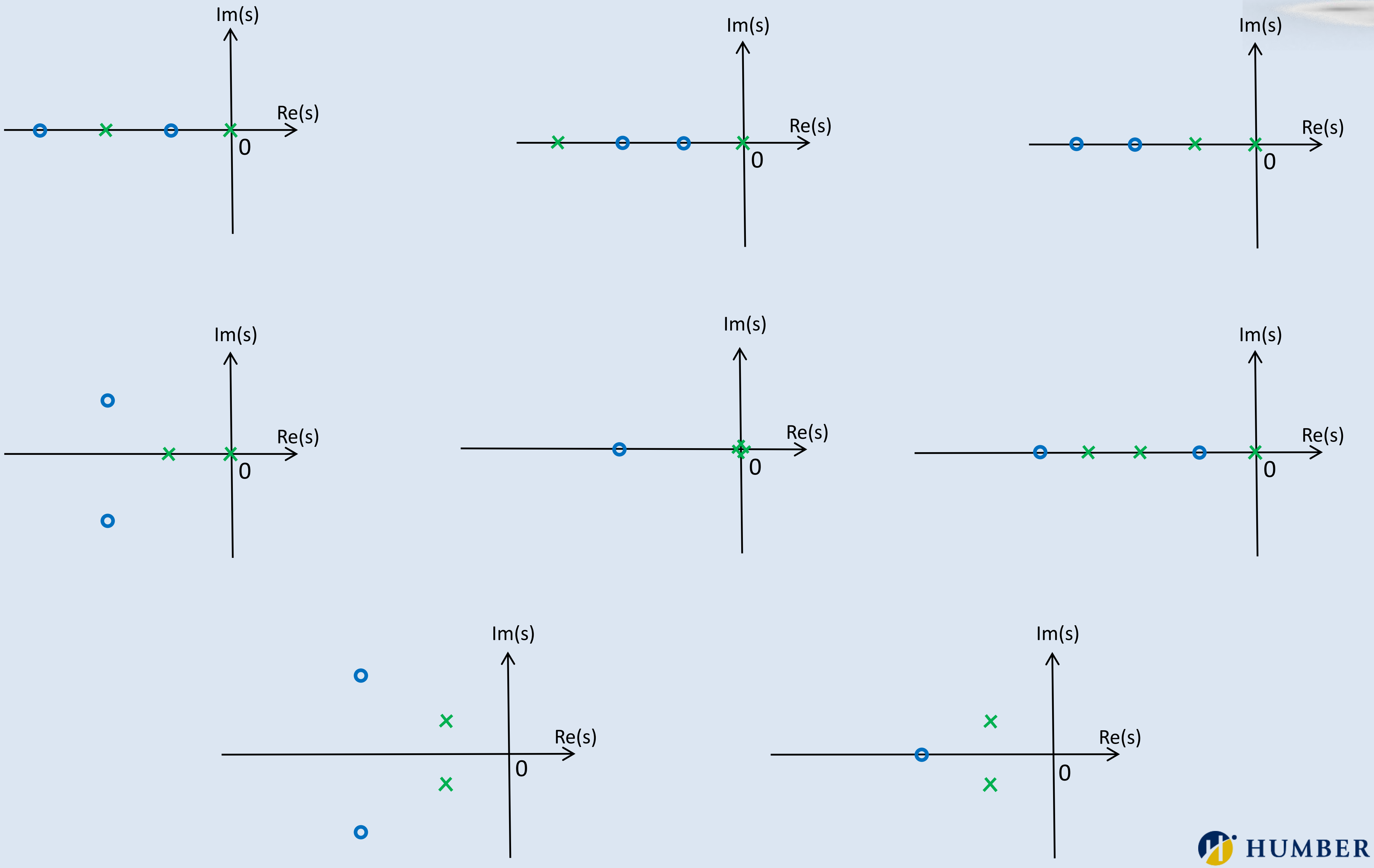
```
num = [1 5];
den = [1  3  0];
sys = tf(num,den);
rlocus(sys)
```





Quick Review

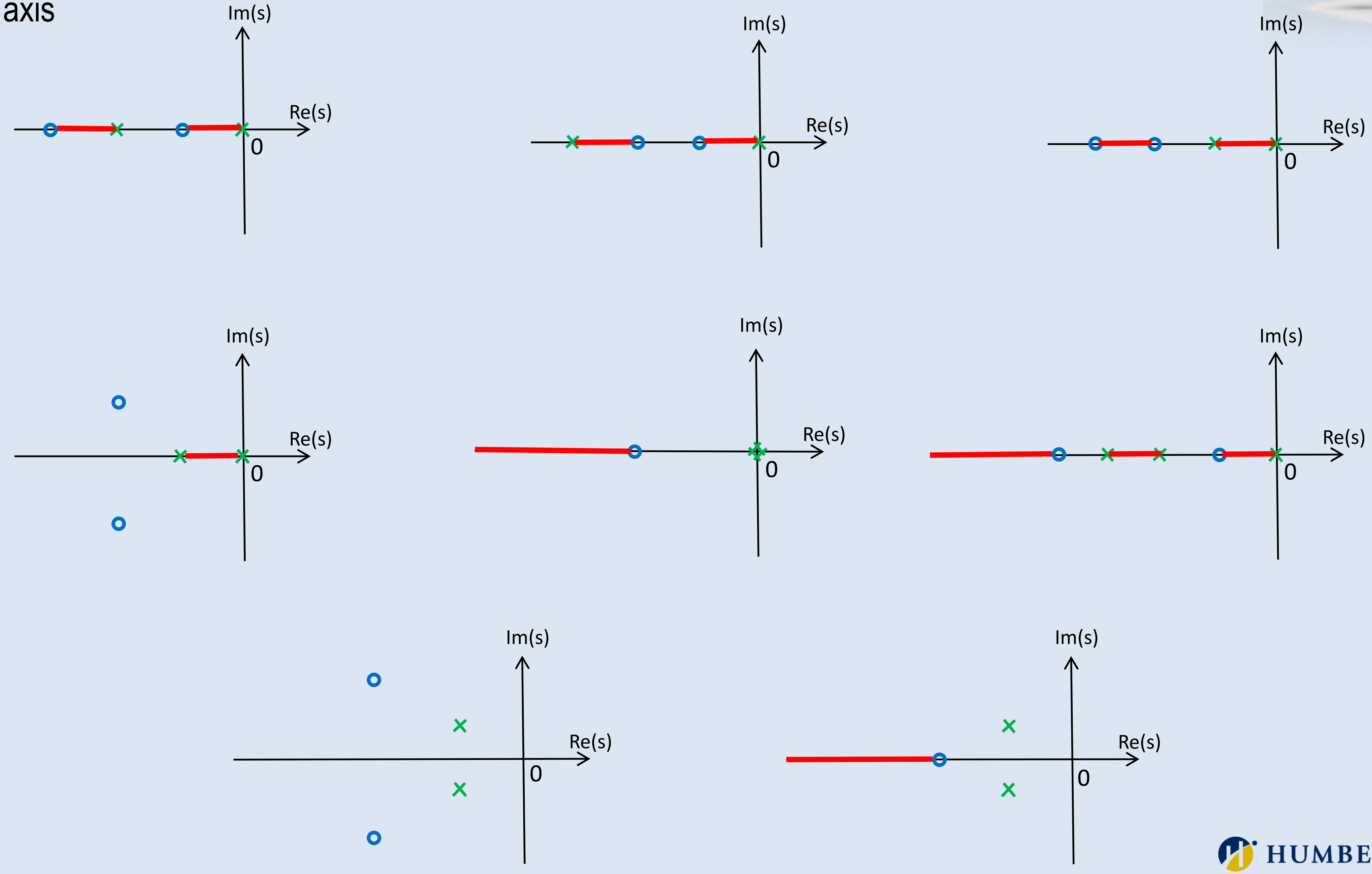
1) Roughly sketch the root-locus plots for the following open-loop pole/zero configurations. $\boldsymbol{K} \in [\mathbf{0}, +\infty)$



Quick Review

1) Roughly sketch the root-locus plots for the following open-loop pole/zero configurations. $\boldsymbol{K} \in [\mathbf{0}, +\infty)$

First, find the root-locus on the real axis

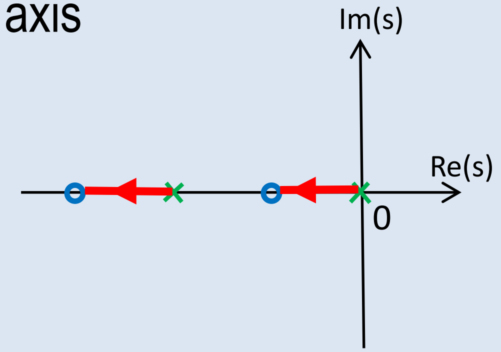


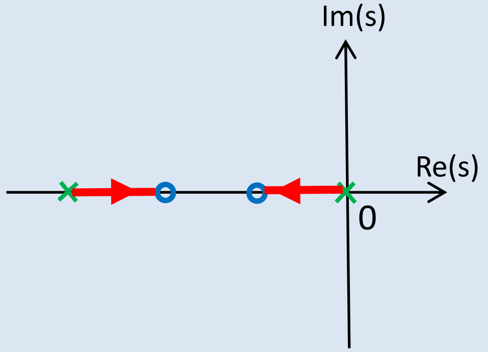
Quick Review

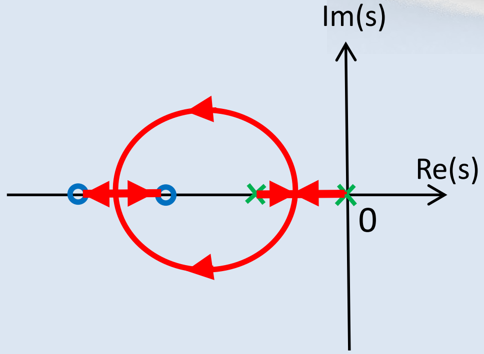
1) Roughly sketch the root-locus plots for the following open-loop pole/zero configurations. $\boldsymbol{K} \in [\mathbf{0}, +\infty)$

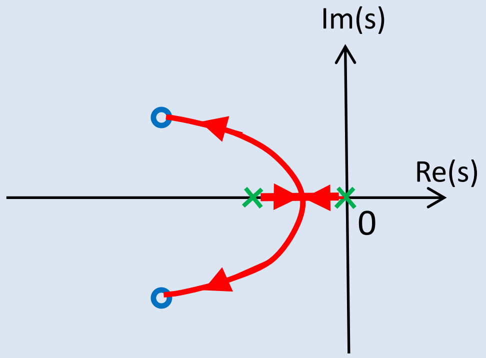
First, find the root-locus on the real axis

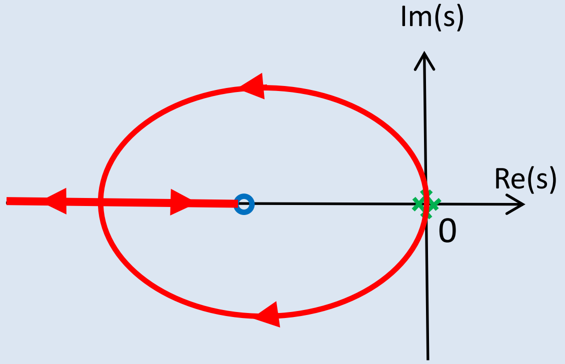
Next, draw the complete root-locus

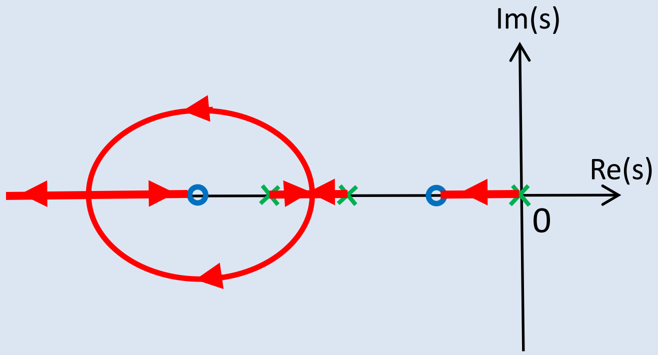


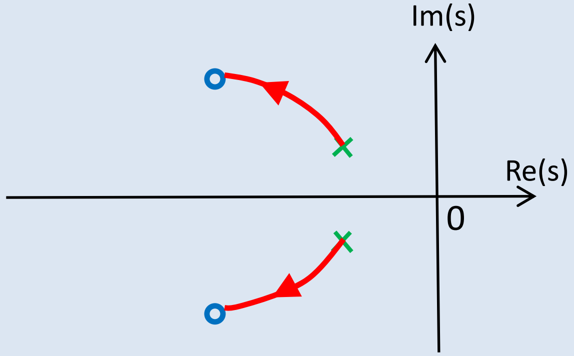


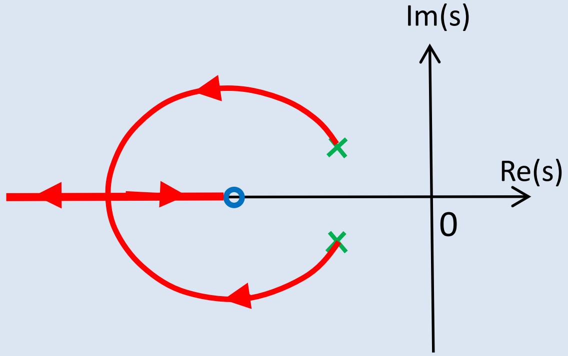












Root Locus for Negative Gains

$$\boldsymbol{K \leq 0}$$

Root Locus Method for $K \leq 0$

Root Locus Plotting Guidelines for $K \leq 0$

- In order to plot the root-locus for $K \in (-\infty, 0]$ some steps of the general plotting guideline must be modified in the following way:

Step 2: Draw the root-locus on the real axis

Step 2 is modified as follows:

- A point on the real axis is part of a locus if the number of poles and zeros to the right of that point is EVEN. (Here, 0 is considered as an even number)

Step 3: Draw asymptote lines for large *K* values

Step 3 is modified as follows:

- Angles of asymptote lines with real axis are determined as

$$\boxed{\varphi_i = \frac{360^\circ}{n-m} i \quad , \qquad i = 0, 1, 2, \cdots}$$

Root Locus Method for $K \leq 0$

❑ Root Locus Plotting Guidelines for $K \leq 0$

- In order to plot the root-locus for $K \in (-\infty, 0]$ some steps of the general plotting guideline must be modified in the following way:

Step 6: Calculate angle of departure/angle of arrival

Step 6 is modified as follows:

Angle of departure from the complex pole
= 0° - (*sum of the angles of vectors drawn to this pole from the other poles*)
+ (*sum of the angles of vectors drawn to this pole from zeros*)

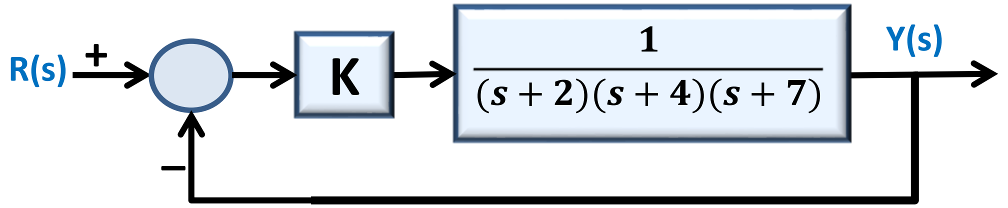
$$\boldsymbol{\phi_p = 0^\circ - \sum_i \angle p_i + \sum_j \angle z_j}$$

Angle of arrival to the complex zero
= 0° - (*sum of the angles of vectors drawn to this zero from the other zeros*)
+ (*sum of the angles of vectors drawn to this zero from poles*)

$$\boldsymbol{\phi_z = 0^\circ - \sum_j \angle z_j + \sum_i \angle p_i}$$

Root Locus Method for $K \leq 0$

Example 11 Compare the root-locus for both positive and negative values of K using MATLAB.

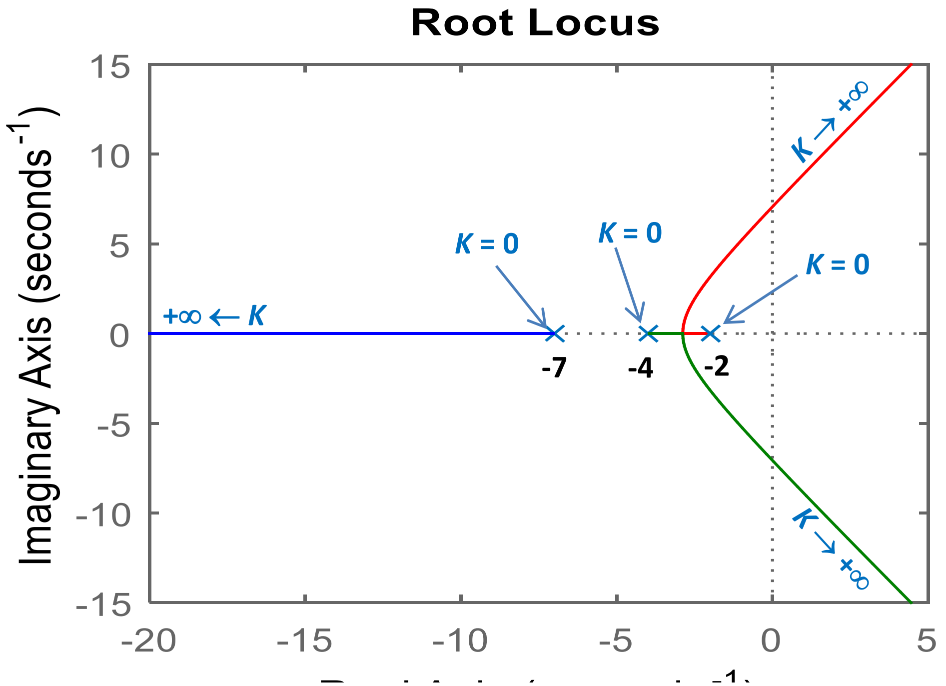


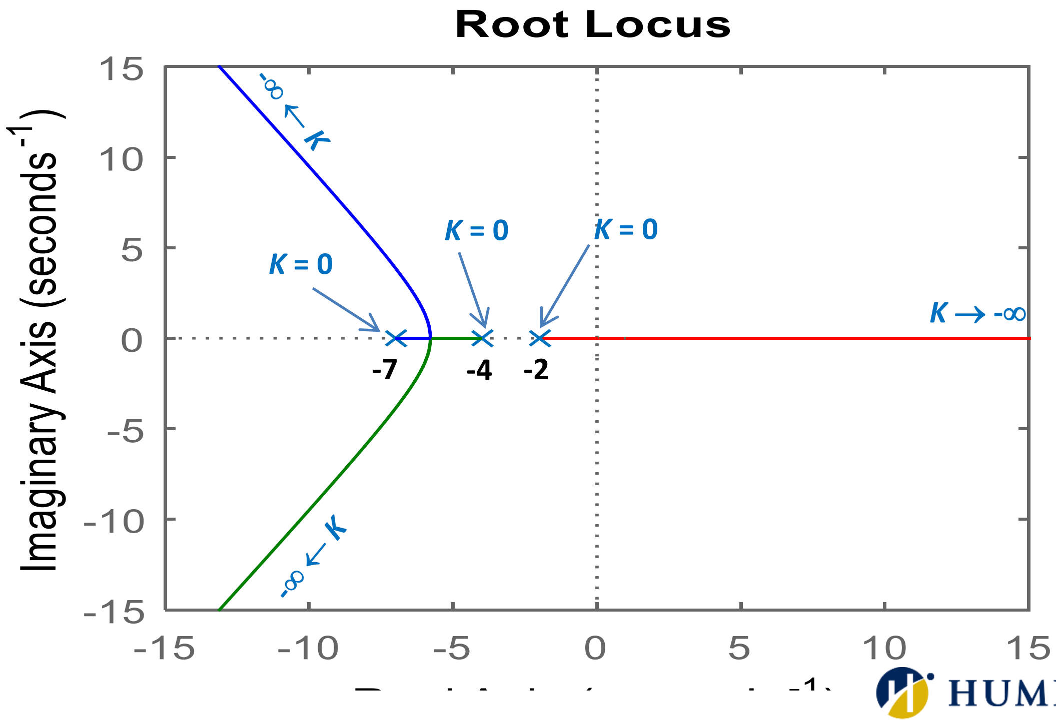

```
num = [1];
den = [1  13  50  56];
sys = tf(num,den);
rlocus(sys)
```

Positive K

Negative K

```
num = [-1];
den = [1  13  50  56];
sys = tf(num,den);
rlocus(sys)
```

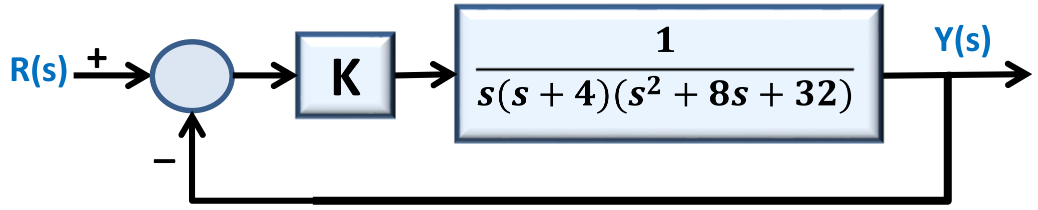




Root Locus Method for $K \leq 0$

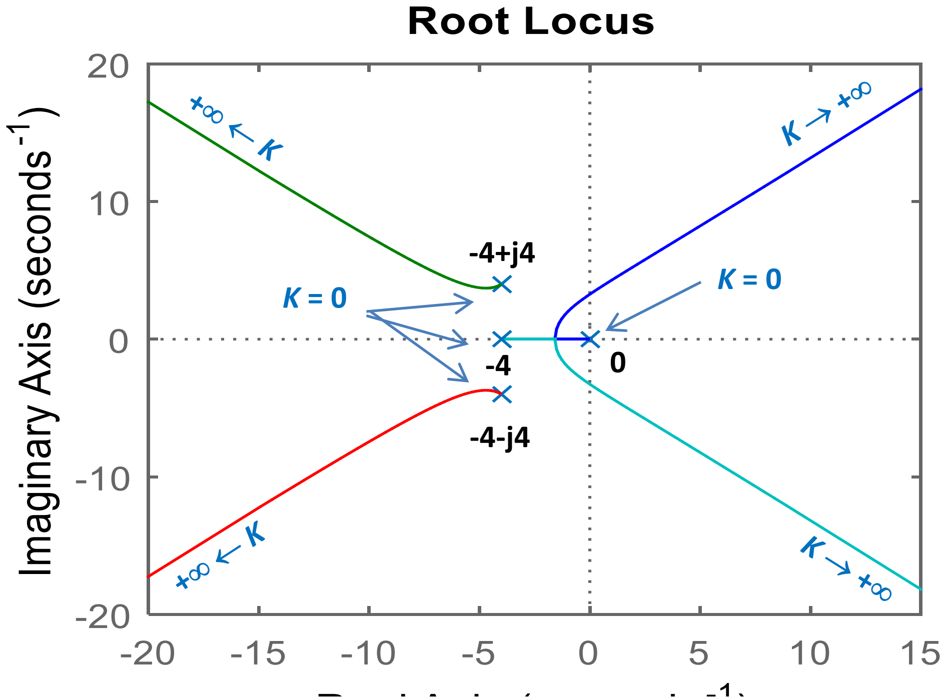
Example 11

Compare the root-locus for both positive and negative values of K using MATLAB.



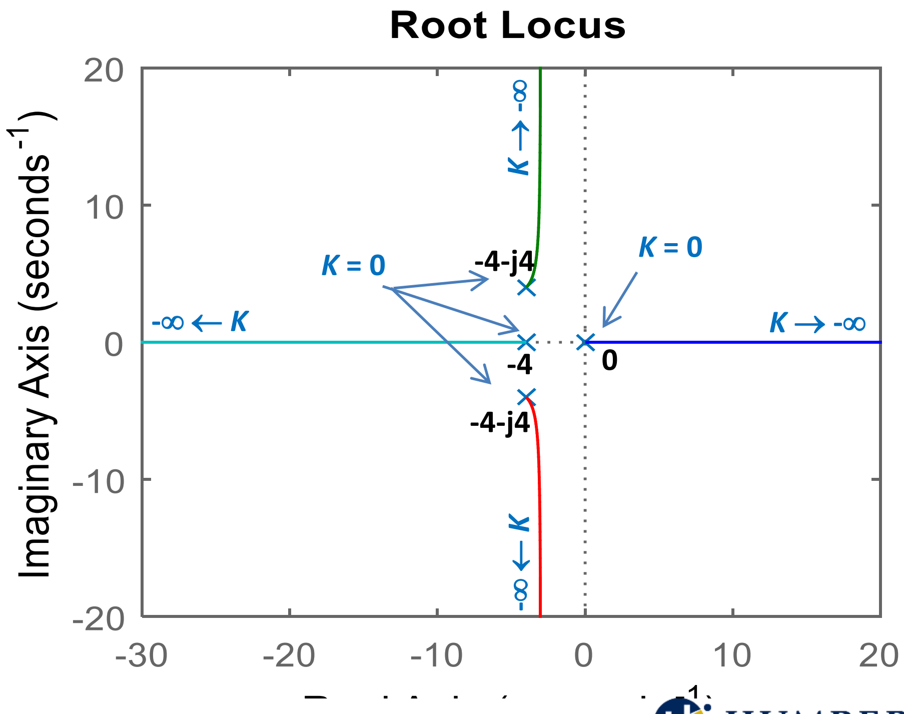

```
num = [1];
den = [1   12   64   128  0];
sys = tf(num,den);
rlocus(sys)
```

Positive K



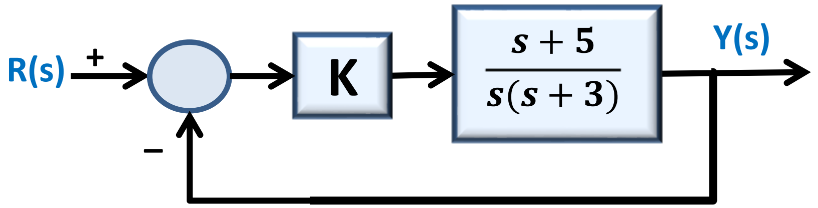
Negative K

```
num = [-1];
den = [1   12   64   128  0];
sys = tf(num,den);
rlocus(sys)
```



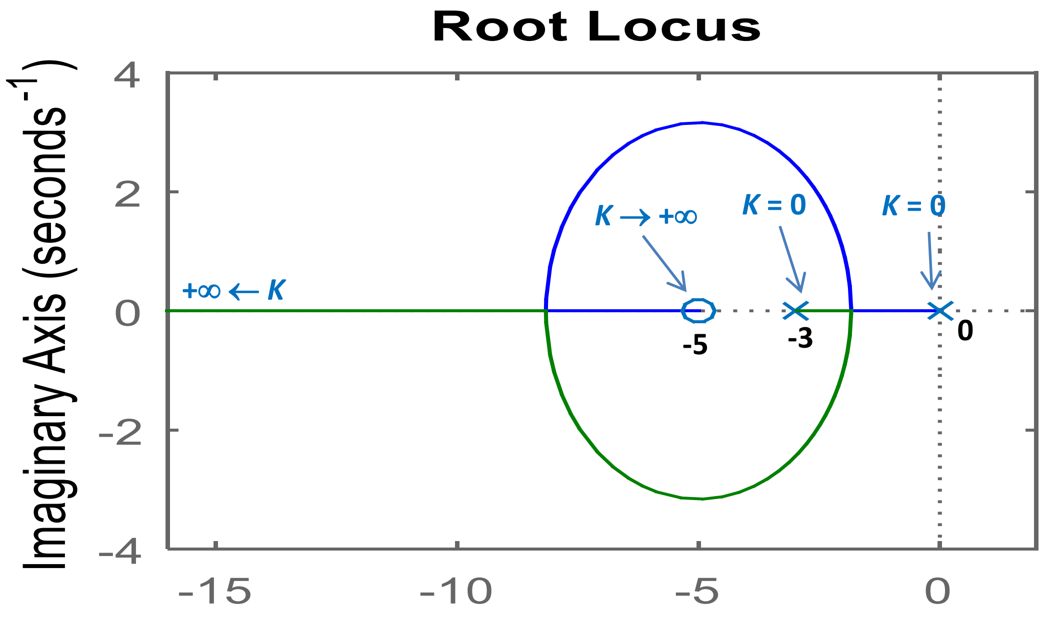
Root Locus Method for $K \leq 0$

Example 11 Compare the root-locus for both positive and negative values of K using MATLAB.



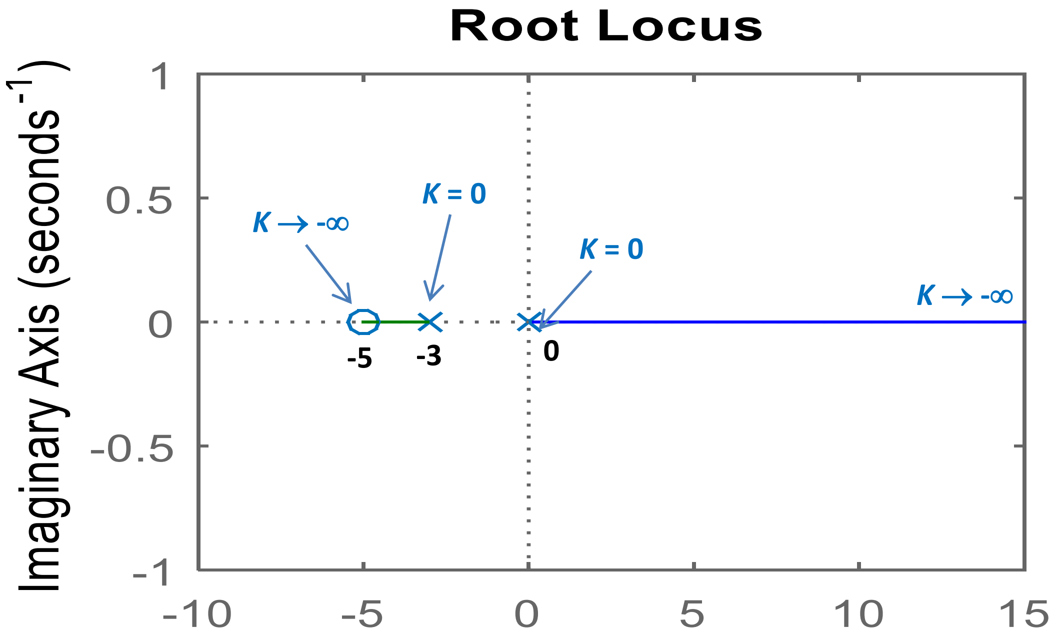

```
num = [1  5];
den = [1  3  0];
sys = tf(num,den);
rlocus(sys)
```

Positive K



Negative K

```
num = [-1 -5];
den = [1  3  0];
sys = tf(num,den);
rlocus(sys)
```



THANK YOU

WE ARE HUMBER

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