2.6 The Alternating Series Test

An *alternating series* is a series whose terms alternate in sign. For example, the series $1 - 1/2 + 1/3 - 1/4 + \cdots$ is an alternating series as is the series $\cos(\pi) + 2\cos(2\pi) + 3\cos(3\pi) + 4\cos(4\pi) + \cdots$. Notice that what makes the latter series an alternating series is that $\cos(n\pi)$ switches between -1 and 1 as n progresses through the values $1, 2, 3, \ldots$.

FRY Theorem 3.3.14, Alternating Series Test

Theorem 2.13. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of real numbers that are non-negative ($A_n \geq 0$ for all $n \geq 1$), nonincreasing ($A_{n+1} \leq A_n$ for all $n \geq 1$), and converging to 0 ($\lim_{n\to\infty} A_n = 0$). Then the alternating series with these terms,

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

converges. Moreover, if

$$\sum_{n=1}^{\infty} (-1)^{n-1} A_n = S,$$

then the difference between the limit S and the N^{th} partial sum S_N satisfies the inequality

$$|S - S_N| \leq A_{N+1},$$

where A_{N+1} is the first dropped term.

Example 2.14. (FRY Example II.3.3.15, convergence of the alternating harmonic series)
Show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{(-1)^{1-1}}{1} + \frac{(-1)^{2-1}}{2} + \frac{(-1)^{3-1}}{3} + \frac{(-1)^{4-1}}{4} + \cdots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \underset{\text{turns out to be}}{=} \ln 2$$

converges.

"conditionally convergent" series

Consider $\left\{A_n = \frac{1}{n}\right\}_{n=1}^{\infty}$.

(i) For all $n = 1, 2, 3, \ldots$, $\frac{1}{n} \geq 0$.

(ii) For all $n = 1, 2, 3, \ldots$, $\frac{1}{n+1} \leq \frac{1}{n}$ — the A_n are decreasing

(iii) $\lim_{n\to\infty} \frac{1}{n} = 0$

By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

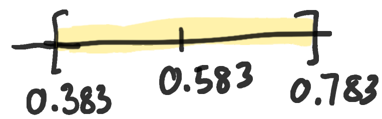
What does it converge to?

Suppose $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = S.$

$$\left| \underbrace{\sum_{n=1}^{4} \frac{(-1)^{n-1}}{n}}_{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}} - S \right| \le \frac{1}{5} \quad A_5$$

$$\left| \frac{7}{12} - S \right| \le 0.2$$

S lies in this interval



$$\left| \sum_{n=1}^{99} \frac{(-1)^{n}}{n} - S \right| \le \frac{1}{100} = 0.01$$

2.7 Ratio Test

The absolute value of the ratio of successive terms in a series can sometimes tell us about the convergence of a series.

FRY Theorem II.3.3.18, Ratio Test

Theorem 2.15. Let N be a positive integer such that $a_n \neq 0$ for all $n \geq N$. Then

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(c) If the ratio converges to 1, the Ratio Test is inconclusive.

Example 2.16. Examine the convergence (or lack of it) of the series $\cancel{\sum_{n=2}^{\infty} 5n\left(\frac{-3}{2}\right)^{n-1}}$. $\sum_{n=1}^{\infty} \frac{n}{2^n} = a_n$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}}{n/2^n}\right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}$$

$$= \lim_{n\to\infty}\frac{n+1}{2n}$$

$$= \lim_{n\to\infty}\frac{1+\cancel{1/n}^{\to 0}}{2}$$

$$= \frac{1+0}{2} = \frac{1}{2} < 1$$

By Ratio Test, $\sum_{n=1}^{\infty}\frac{n}{2^n}$ converges.

$$\lim_{n\to\infty}\left|\frac{5(n+1)\left(\frac{-3}{2}\right)^n}{5n\left(-3/2\right)^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{3}{2}\cdot\frac{n+1}{n}\right| = \lim_{n\to\infty}\frac{3}{2}\cdot\frac{n+1}{n}$$

Example 2.17. (Like FRY Exercise II.3.3.11.45a)

Determine whether $\cancel{\sum_{k=1}^{\infty} \frac{k^k}{8^k k!}}$ converges or diverges.

$\sum_{n=1}^{\infty} \frac{n!}{2^n} = a_n$ converges ?

defined
$0! \stackrel{\text{def}}{=} 1$
$1! = 1$
$2! = 2\cdot 1 = 2$
$3! = 3\cdot 2\cdot 1 = 6$
$4! = 4\cdot 3\cdot 2\cdot 1 = 24$
$\vdots$
$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

<u>Solution</u>: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} \right|$

$$= \lim_{n\to\infty} \frac{\cancelto{n+1}{(n+1)!}}{\cancelto{2}{2^{n+1}}} \cdot \frac{\cancelto{1}{2^n}}{\cancelto{1}{n!}}$$

$$\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{(n)}\cancel{(n-1)}\cdots\cancel{2\cdot 1}}{\cancel{n}\cancel{(n-1)}\cdots\cancel{2\cdot 1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{2}$$

$$\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{n!}}{\cancel{n!}} = n+1$$

$$= \infty$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ diverges

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$\sum_{k=1}^{\infty} \frac{k^k}{8^k k!}$ converges or not?

$$\lim_{k\to\infty} \left| \frac{\frac{(k+1)^{k+1}}{8^{k+1}\cdot(k+1)!}}{\frac{k^k}{8^k k!}} \right| = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{8^{k+1}\cdot(k+1)!} \cdot \frac{8^k\cdot k!}{k^k}$$

$$= \lim_{k\to\infty} \frac{\cancelto{1}{8^k}}{\cancelto{8}{8^{k+1}}} \cdot \frac{(k+1)^{\cancelto{k}{k+1}}}{k^k} \cdot \frac{\cancelto{1}{k!}}{\cancelto{1}{\cancelto{k+1}{(k+1)!}}}$$

$$\frac{(k+1)^{k+1}}{k+1} = \frac{\overbrace{\cancel{(k+1)}(k+1)\cdots(k+1)}^{k+1 \text{ times}}}{\cancel{k+1}} = (k+1)^k$$

$$= \lim_{k\to\infty} \frac{\cancel{8^k}^{\,1}}{\cancel{8^{k+1}}_{\,8}} \cdot \frac{(k+1)^{\cancel{k+1}\,k}}{k^k} \cdot \frac{\cancel{k!}^{\,1}}{\cancel{(k+1)!}_{\,\cancel{k+1}\,1}}$$

$$= \lim_{k\to\infty} \frac{1}{8} \cdot \left(\frac{k+1}{k}\right)^k \qquad \frac{k+1}{k} = \frac{k}{k} + \frac{1}{k} = 1 + \frac{1}{k}$$

$$= \lim_{k\to\infty} \frac{1}{8} \boxed{\left(1+\frac{1}{k}\right)^k}$$

$$= \frac{1}{8}\cdot e \qquad \text{, where } e \approx 2.718\ldots$$

$$\approx 0.3397$$

$$< 1$$

By the Ratio Test, $\sum_{k=1}^{\infty} \frac{k^k}{8^k k!}$ converges.

Example 2.18. (FRY Example II.3.3.22, series for which the Ratio Test is inconclusive)
What happens when we apply the Ratio Test to test the convergence of the following series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(i) $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

(ii) $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \underset{\text{dividing by } n^2}{=} \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = \frac{1}{1+0+0} = 1$

So, we have

- a divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ for which $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$; but also
- a convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ for which $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

This is why the Ratio Test is inconclusive when $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

2.8 Absolute and Conditional Convergence

Let $\sum a_n$ be a series.

- We say that $\sum a_n$ *converges absolutely* or is an *absolutely convergent* series if $\sum |a_n|$ converges.
- If $\sum a_n$ converges but $\sum |a_n|$ fails to converge, then we say that $\sum a_n$ *converges conditionally* or is a *conditionally convergent* series.

Given the above definitions, notice that a series converges absolutely, converges conditionally, or diverges.

FRY Theorem II.3.4.2

Theorem 2.19. Absolute convergence implies convergence.

Example 2.20. Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.

conditionally converged

(i) (FRY Example II.3.4.3) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges (Alternating Harmonic Series)

but $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges (Harmonic Series)

(ii) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$(2^2)^n = 2^{2n}$

(iii) (FRY Exercise II.3.4.3.5) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1+4^n}{3+2^{2n}}$

(ii) Let's see if $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely. That is, consider the series $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$.

Note that for every $n=1,2,3,\ldots$, $\left|\frac{\sin n}{n^2}\right| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$.

By the Comparison Test, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

so two does $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$.

Thus, $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely.

(Note that this absolute convergence implies that $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.)

(iii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1+4^n}{3-2^{2n}}$

(We'll show by the Divergence Test that this series diverges.)

$$\lim_{n\to\infty} (-1)^n\left(\frac{1+4^n}{3-2^{2n}}\right) = \lim_{n\to\infty} (-1)^n\left(\frac{1+(2^2)^n}{3-2^{2n}}\right)$$

$$= \lim_{n\to\infty} (-1)^n\left(\frac{1+2^{2n}}{3-2^{2n}}\right) \xrightarrow{\quad} -1$$

$$\lim_{n\to\infty} \frac{1+2^{2n}}{3-2^{2n}} = \lim_{n\to\infty} \frac{\frac{1}{2^{2n}}+1}{\frac{3}{2^{2n}}-1} = \frac{0+1}{0-1} = -1$$

So the limit does not exist, and it is certainly not zero.

So, by the Divergence Test, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1+4^n}{3-2^{2n}}$ diverges.

Since the statement of the Ratio Test involves examining the limit as $n \to \infty$ of $\left|\frac{a_{n+1}}{a_n}\right|$, we can strengthen its result as follows:

Ratio Test

Theorem 2.21. Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$. Then

(a) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges *absolutely.*

(b) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(c) If the ratio converges to 1, the Ratio Test is inconclusive.

2.9 References

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