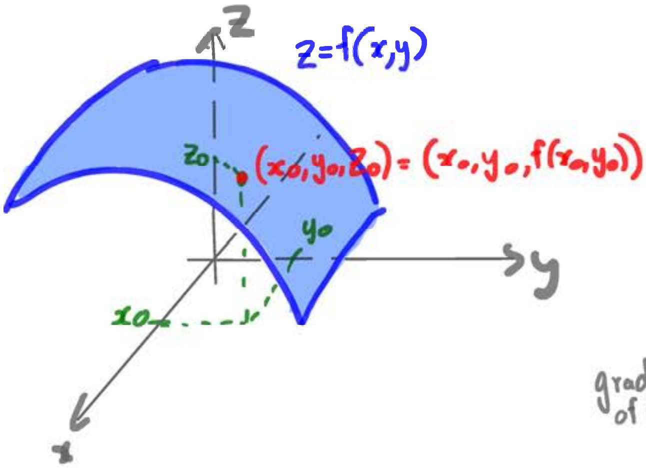


Surface: $z = f(x,y)$ $\boxed{z = e^{xy}}$

$\leftarrow$ graph of a function

$$\underbrace{z - f(x,y)}_{G(x,y,z)} = \underbrace{0}_{K} \qquad (x_0, y_0, z_0) = (x_0, y_0, f(x_0,y_0))$$

gradient of G $\rightarrow$ $\nabla G(x,y,z) = \left\langle -\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial y}(x,y), 1 \right\rangle$

Eqn of tangent plane:

$$\nabla G(x_0,y_0,z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\left\langle -\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1 \right\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$-\frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + z - z_0 = 0$$

$$z = z_0 + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$$

$$\boxed{z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)}$$

eqn of tangent plane to surface $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$

equation of normal line:

$$\langle x, y, z \rangle = \langle x_0, y_0, f(x_0,y_0) \rangle + t \left\langle -\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1 \right\rangle, \ t \in \mathbb{R}$$

FRY Thm III.2.5.1, Tangent Plane and Normal Line to surface $z = f(x, y)$

Corollary 5.9. Let (x_0, y_0, z_0) be a point on the surface $z = f(x, y)$. Then,

(i) The vector
$$-f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}$$
is normal to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$.

(ii) The equation of the tangent plane to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ may be written as
$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

(iii) The parametric equation of the normal line to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is
$$\langle x, y, z\rangle = \langle x_0, y_0, f(x_0, y_0)\rangle + t\,\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle\,,$$
where $t \in \mathbb{R}$. Written componentwise, the normal line is given by
$$x = x_0 - tf_x(x_0, y_0), \quad y = y_0 - tf_y(x_0, y_0), \quad z = f(x_0, y_0) + t, \quad t \in \mathbb{R}.$$

Example 5.10. (FRY Exercise III.2.5.3.7b)
Find the equations of the tangent plane and normal line to the surface given by
$$f(x, y) = e^{xy}$$
at the point $(2, 0)$.

Given: surface $z = f(x,y)$
$z = e^{xy}$

$z = e^{(2)(0)} = 1$

point on the surface $(2, 0, 1)$

Goal: Eqn of tangent plane = ?
Eqn of normal line = ?

$z = f(x,y)$
where $f(x,y) = e^{xy}$

$x_0 \; y_0$

Sol'n Remember $z = f(2,0) + \frac{\partial f}{\partial x}(2,0)(x-2) + \frac{\partial f}{\partial y}(2,0)(y-0)$

$\frac{\partial f}{\partial x}(x,y) = ye^{xy}$ $\qquad$ $\frac{\partial f}{\partial x}(2,0) = 0e^{(2)(0)} = 0 \cdot 1 = 0$

$\frac{\partial f}{\partial y}(x,y) = xe^{xy}$ $\qquad$ $\frac{\partial f}{\partial y}(2,0) = (2)e^{(2)(0)} = 2 \cdot 1 = 2$

$f(2,0) = e^{(2)(0)} = e^0 = 1$

eqn for tangent plane to graph $z=f(x,y)$ at $(2,0,1)$ is

gradient vector at $(2,0,1)$

$$z = 1 + 0(x-2) + 2(y-0) \qquad \langle 0,-2,1\rangle = \left\langle -\frac{\partial f}{\partial x}(2,0), -\frac{\partial f}{\partial y}(2,0), 1\right\rangle$$

$$\boxed{z = 1 + 2y}$$

$z = f(x,y)$

$\boxed{\underbrace{z - f(x,y)}_{G(x,y,z)} = 0}$

Equation for normal line to the graph $z=f(x,y)$ at $(2,0,1)$ is

vector: eqn $\quad \langle x,y,z\rangle = \langle 2,0,1\rangle + t\langle 0,-2,1\rangle, \; t \in \mathbb{R}$

$$\nabla G = \left\langle \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right\rangle$$

Parametric eqns: $\begin{cases} x = 2 + t(0) = 2 \\ y = 0 + t(-2) = -2t \\ z = 1 + t(1) = 1 + t \end{cases}, \; t \in \mathbb{R}$

$t = z - 1 \qquad t = \frac{y}{-2}$

Symmetric eqns: $\quad x = 2, \quad \frac{y}{-2} = z - 1$

$f(x,y) = 4x^2 + 25y^2 \qquad \left(\frac{1}{2}, \frac{1}{5}, 2\right)$

Find tangent plane to $z = f(x,y)$ at $\left(\frac{1}{2}, \frac{1}{5}, 2\right)$

$x_0 \quad y_0 \quad z_0 = f(x_0, y_0)$

$$f\left(\tfrac{1}{2}, \tfrac{1}{5}\right) = 4\left(\tfrac{1}{2}\right)^2 + 25\left(\tfrac{1}{5}\right)^2 = 1 + 1 = 2$$

$$\frac{\partial f}{\partial x} = 8x \qquad \frac{\partial f}{\partial x}\left(\tfrac{1}{2}, \tfrac{1}{5}\right) = 8\left(\tfrac{1}{2}\right) = 4$$

$$\frac{\partial f}{\partial y} = 50y \qquad \frac{\partial f}{\partial y}\left(\tfrac{1}{2}, \tfrac{1}{5}\right) = 50\left(\tfrac{1}{5}\right) = 10$$

$$z = f\left(\tfrac{1}{2}, \tfrac{1}{5}\right) + \frac{\partial f}{\partial x}\left(\tfrac{1}{2}, \tfrac{1}{5}\right)\left(x - \tfrac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\tfrac{1}{2}, \tfrac{1}{5}\right)\left(y - \tfrac{1}{5}\right)$$

$$z = 2 + 4\left(x - \tfrac{1}{2}\right) + 10\left(y - \tfrac{1}{5}\right)$$

$$z = 2 + 4x - 2 + 10y - 2$$

$$\boxed{z = 4x + 10y - 2}$$

5.4 Linear Approximations and Error

Linear Approximation to $f(x,y)$ near a point (x_0, y_0)

We may use the tangent plane to linearly approximate the value of a function f at a point (x,y) that is near a point (x_0, y_0) at which we know the function's value:

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

If we denote $x - x_0$ as Δx and $y - y_0$ as Δy, the linear approximation to $f(x,y)$ near the point (x_0, y_0) may be expressed as

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y.$$

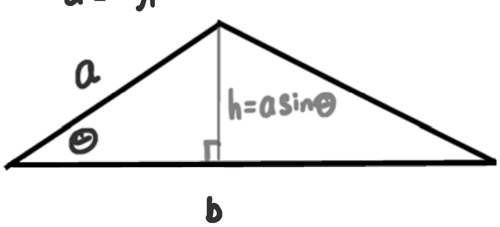
The same idea may be used to linearly approximate the value of a function of more variables near a point $(x_{0,1}, x_{0,2}, \ldots, x_{0,n})$:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) \approx f(x_{0,1}, x_{0,2}, \ldots, x_{0,n}) &+ \frac{\partial f}{\partial x}(x_{0,1}, x_{0,2}, \ldots, x_{0,n})\Delta x_1 \\ &+ \frac{\partial f}{\partial x}(x_{0,1}, x_{0,2}, \ldots, x_{0,n})\Delta x_2 + \cdots \frac{\partial f}{\partial x}(x_{0,1}, x_{0,2}, \ldots, x_{0,n})\Delta x_n, \end{aligned}$$

where $\Delta x_i = x_i - x_{0,i}$ for $i = 1, 2, \ldots, n$.

Example 5.11. (FRY Example III.2.6.9)
A triangle has sides $a = 10.1$ cm and $b = 19.8$ cm which include an angle $35°$. Approximate the area of the triangle.

$\sin\theta = \dfrac{h \leftarrow \text{opposite}}{a \leftarrow \text{hypotenuse}}$



Area $A = \frac{1}{2} b h = \frac{1}{2} b a \sin\theta$

$$\boxed{A(a,b,\theta) = \frac{1}{2} ab\sin\theta}$$

$(a_0, b_0, \theta_0) = (10, 20, \frac{\pi}{6})$ $\leftarrow 30°$

Goal: Linearly approximate

$A(10.1, 19.8, \frac{7\pi}{36})$
$\uparrow$ a $\uparrow$ b $\uparrow$ $35°$

$35° = \cancel{35°}^{7} \times \frac{\pi}{\cancel{180°}_{36}} = \frac{7\pi}{36}$

$$\boxed{f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}$$

$$A(a,b,\theta) \approx A(a_0,b_0,\theta_0) + \frac{\partial A}{\partial a}(a_0,b_0,\theta_0)(a-a_0) + \frac{\partial A}{\partial b}(a_0,b_0,\theta_0)(b-b_0) + \frac{\partial A}{\partial \theta}(a_0,b_0,\theta_0)(\theta-\theta_0)$$

$$\boxed{A(a,b,\theta) \approx 50 + 5(a-10) + \frac{5}{2}(b-20) + 50\sqrt{3}\left(\theta - \frac{\pi}{6}\right)}$$

$$A(a,b,\theta) = \frac{1}{2} b a \sin\theta$$

$$A(10,20,\pi/6) = \frac{1}{2}(10)(20)\sin(\pi/6) = 50$$

a_0, b_0, θ_0

$$\frac{\partial A}{\partial a} = \frac{1}{2} b \sin\theta. \quad \text{So } \frac{\partial A}{\partial a}(10,20,\tfrac{\pi}{6}) = \frac{1}{2}(20)\sin\left(\frac{\pi}{6}\right) = 5$$

$$\frac{\partial A}{\partial b} = \frac{1}{2} a \sin\theta. \quad \text{So } \frac{\partial A}{\partial b}(10,20,\tfrac{\pi}{6}) = \frac{1}{2}(10)\sin\left(\frac{\pi}{6}\right) = \frac{5}{2} = 2.5$$

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} b a \cos\theta. \quad \text{So } \frac{\partial A}{\partial \theta}(10,20,\tfrac{\pi}{6}) = \frac{1}{2}(10)(20)\cos\left(\frac{\pi}{6}\right) = 50\sqrt{3}$$

$\frac{\sqrt{3}}{2}$

$$\text{So } A\left(10.1, 19.8, \frac{7\pi}{36}\right) \approx 50 + 5(10.1-10) + \frac{5}{2}(19.8-20) + 50\sqrt{3}\left(\frac{7\pi}{36} - \frac{\pi}{6}\right)$$

$$\approx 57.56 \text{ cm}^2$$

a linear approximation

Live Poll Find a linear approximation for $f(x,y) = e^{x-2}\cos(y+\pi)$ centred at $(2,-\pi)$.

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$f(x,y) = e^{x-2}\cos(y+\pi)$$

$$\begin{aligned} f(2,-\pi) &= e^{2-2}\cos(-\pi+\pi) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

$$\frac{\partial f}{\partial x}(x,y) = e^{x-2}\cos(y+\pi), \quad \frac{\partial f}{\partial x}(2,-\pi) = 1$$

$$\frac{\partial f}{\partial y}(x,y) = -e^{x-2}\sin(y+\pi), \quad \frac{\partial f}{\partial y}(2,-\pi) = 0$$

$y-(-\pi) = y+\pi$

$$\begin{aligned} \text{So } f(x,y) &\approx f(2,-\pi) + \frac{\partial f}{\partial x}(2,-\pi)(x-2) + \frac{\partial f}{\partial y}(2,-\pi)(y+\pi) \\ &= 1 + 1(x-2) + 0(y+\pi) \\ &= 1 + x - 2 \\ &= x - 1 \end{aligned}$$

$$\boxed{f(x,y) \approx x-1}$$

$$\Delta f$$

$$\boxed{f(x,y)-f(x_0,y_0)}$$

We have seen that the linear approximation of $f(x, y)$ near the point (x_0, y_0) is given by

$$\boxed{f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y.}$$

If we subtract $f(x_0, y_0)$ from both sides and denote the change $f(x, y) - f(x_0, y_0)$ in the value of f by Δf, then the linear approximation to the *change in* f is given by

FRY Defn III.2.6.5, The linear approximation to the *change in f*

$$\Delta f \approx \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y.$$

When we want to emphasize that Δx, Δy, and Δf are very small ("infinitesimally" so), then we replace these with dx, dy, and df, respectively, and write

FRY Defn III.2.6.5, The linear approximation to the *change in f*

$$df \approx \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy.$$

Example 5.12. (FRY Exercise III.2.6.3.7)
If two resistors of resistance R_1 and R_2 are wired in parallel, the resulting resistance R satisfies the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Use a linear approximation to estimate the change in R if R_1 decreases from 2 to 1.9 ohms and R_2 increases from 8 to 8.1 ohms.

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}}$$

$R(R_1, R_2)$ ← R depends on (is a function of) R_1 and R_2

$$\Delta R \approx \frac{\partial R}{\partial R_1}(R_{1,0}, R_{2,0})\Delta R_1 + \frac{\partial R}{\partial R_2}(R_{1,0}, R_{2,0})\Delta R_2$$

$(R_{1,0}, R_{2,0}) = (2, 8)$

ohms

Since R_1 drops to $1.9\,\Omega$, $\Delta R_1 = -0.1\,\Omega$.

Since R_2 increases to $8.1\,\Omega$, $\Delta R_2 = +0.1\,\Omega$

We need to figure out $\frac{\partial R}{\partial R_1}(2,8)$ and $\frac{\partial R}{\partial R_2}(2,8)$.

Since $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$, implicitly differentiation

$R^{-1} = R_1^{-1} + R_2^{-1}$

9

$\frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1+R_2}{R_1R_2}$ So $\frac{1}{R} = \frac{R_1+R_2}{R_1R_2}$

and $\boxed{R = \frac{R_1R_2}{R_1+R_2}}$

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_1} = \frac{-1}{R_1^2}$$

$$\Longrightarrow \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2} = \frac{\left(\frac{R_1R_2}{R_1+R_2}\right)^2}{R_1^2} = \frac{\cancel{R_1^2}R_2^2}{(R_1+R_2)^2} \cdot \frac{1}{\cancel{R_1^2}} = \frac{R_2^2}{(R_1+R_2)^2}$$

Similarly, $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1+R_2)^2}$

So $\Delta R \approx \frac{\partial R}{\partial R_1}(2,8)\,\Delta R_1 + \frac{\partial R}{\partial R_2}(2,8)\,\Delta R_2$

$$= \frac{8^2}{(2+8)^2}(-0.1) + \frac{2^2}{(2+8)^2}(0.1)$$

$$= -0.064 + 0.004$$

$$= -0.060 \quad \text{ohms}$$

FRY Defn III.2.6.6, Absolute, relative, and percentage error

Definition 5.13. If the approximation of a quantity Q turns out to be $\underline{Q+\Delta Q}$, then

true

approximation

we are off the true value by ΔQ

- the absolute error in the approximation is $|\Delta Q|$;
- the relative error in the approximation is $\left|\frac{\Delta Q}{Q}\right|$; and
- the percentage error in the approximation is $100\left|\frac{\Delta Q}{Q}\right|$.

Example 5.14. (FRY Example III.2.6.9)
A triangle has sides $a = 10.1$ cm and $b = 19.8$ cm which include an angle 35°. What is the absolute error, the relative error, and the percentage error in the linear approximation of its area?

true value of area $A = \frac{1}{2}(10.1)(19.8)\sin(35°)$

≈ 57.35190787

≈ 57.35 cm^2

our $\boxed{\text{approxmn}}$ for area $A \overset{\text{was}}{=} 57.56$ cm^2 $= \underset{A}{57.35} + \underset{\Delta A}{0.21}$

absolute error $= |\Delta A| = |0.21| = 0.21$ cm^2

relative error $= \left|\frac{\Delta A}{A}\right| = \left|\frac{0.21}{57.35}\right| \approx 0.0037$

percentage error $= 100\left|\frac{\Delta A}{A}\right| = 100\left|\frac{0.21}{57.35}\right| \approx 0.37$

5.5 Degree Two Taylor Polynomials for a Function $f(x,y)$

Degree 2 Taylor polynomial for $f(x,y)$

Definition 5.15. Let U is an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ be of class $\mathcal{C}^2(U)$. Then the Taylor polynomial of degree 2 of f around $\mathbf{a} = (x_0, y_0)$ is

$$T^2_{f,(x_0,y_0)}(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2.$$

If we take x "close" to x_0 and y "close" to y_0, we may denote the differences $x - x_0$ and $y - y_0$ as Δx and Δy, respectively. We get the following expression:

Degree 2 Taylor polynomial for $f(x,y)$

Definition 5.16. Let U is an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ be of class $\mathcal{C}^2(U)$. Then the Taylor polynomial of degree 2 of f around $\mathbf{a} = (x_0, y_0)$ is

$$T^2_{f,(x_0,y_0)}(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(\Delta x)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\Delta x \Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)(\Delta y)^2.$$

Written using matrix notation, observe that

$$\begin{aligned} &T^2_{f,(x_0,y_0)}(x_0+\Delta x, y_0+\Delta y) \\ &= f(x_0,y_0) + \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x_0,y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) & \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \end{aligned}$$

The matrix involving the second-order derivatives above is given a special name.

Shifrin Defn 5.3.1, The Hessian matrix

Definition 5.17. Let U be an open subset of $\mathbb{R}^n$ and $\mathbf{a} \in U$. Let $f : U \to \mathbb{R}$ be of class $\mathcal{C}^2$ on U. Then the Hessian matrix of f at $\mathbf{a}$ is the symmetric matrix

$$\text{Hessian}(f)(\mathbf{a}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})\right] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{bmatrix}.$$

5.6 Quadratic Approximation

Just as we use the first-degree Taylor polynomial (the tangent plane) as a linear approximation to a function, we may use the second-degree Taylor polynomial to find a quadratic approximation to a function near the point at which the Taylor polynomial is constructed.

Example 5.18. Let $f(x, y) = e^{x-2}\cos(y + \pi)$. Estimate the value of $f(1.9, -3.1)$ using a quadratic approximation for f.

A quadratic approximation comes from a second-degree Taylor polynomial:

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2$$

$f(x,y) = e^{x-2}\cos(y+\pi)$ $\qquad$ $f(1.9,-3.1) \approx ?$

Let $x_0 = 2$ $\quad y_0 = -\pi$

$$f(2,-\pi) = e^{2-2}\cos(-\pi+\pi) = 1 \cdot 1 = 1$$

$\frac{\partial f}{\partial x}(x,y) = e^{x-2}\cos(y+\pi)$, $\quad \frac{\partial f}{\partial x}(2,-\pi) = 1$

$\frac{\partial f}{\partial y}(x,y) = -e^{x-2}\sin(y+\pi)$, $\quad \frac{\partial f}{\partial y}(2,-\pi) = 0$

$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{x-2}\cos(y+\pi)$, $\quad \frac{\partial^2 f}{\partial x^2}(2,-\pi) = 1$

$\frac{\partial^2 f}{\partial y \partial x}(x,y) = -e^{x-2}\sin(y+\pi)$, $\quad \frac{\partial^2 f}{\partial y \partial x}(2,-\pi) = 0$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -e^{x-2}\cos(y+\pi), \quad \frac{\partial^2 f}{\partial y \partial x}(2,-\pi) = -1$$

$y-(-\pi) = y+\pi$

$$\text{So } f(x,y) \approx f(2,-\pi) + \frac{\partial f}{\partial x}(2,-\pi)(x-2) + \frac{\partial f}{\partial y}(2,-\pi)(y+\pi)$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(2,-\pi)(x-2)^2 + \frac{\partial^2 f}{\partial x \partial y}(2,-\pi)(x-2)(y+\pi) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(2,-\pi)(y+\pi)^2$$

$$= 1 + 1(x-2) + 0(y+\pi) + \frac{1}{2}(1)(x-2)^2 + 0(x-2)(y+\pi) + \frac{1}{2}(-1)(y+\pi)^2$$

$$= \boxed{1 + x - 2 + \frac{1}{2}(x-2)^2 - \frac{1}{2}(y+\pi)^2}$$

2nd degree Taylor polynomial for $f(x,y)$ centred at (x_0, y_0)

$$\text{So } f(1.9,-3.1) \approx 1 + 1.9 - 2 + \frac{1}{2}(1.9-2)^2 - \frac{1}{2}(-3.1+\pi)^2$$

$$\approx 0.904\ 135\ 025\ 6$$

quadratic estimate of $f(1.9,-3.1)$

compare

True value of $f(1.9,-3.1) \approx 0.904\ 054\ 869\ 6$

linear approxim of $f(1.9,-3.1) = 1 - 0.1 = 0.9$

5.7 References

References:

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