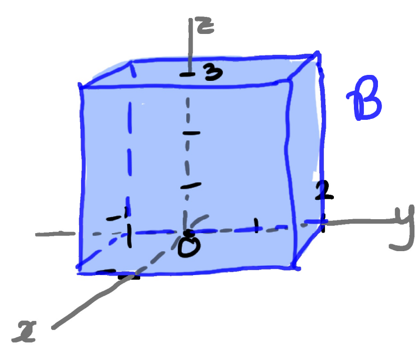
Chapter 10

Triple Integrals

10.1 Triple Integrals in Rectangular Coordinates

Example 10.1. (Stewart, Example 15.6.1)
Evaluate the triple integral $\iiint_B xyz^2 \, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) : \ 0 \le x \le 1, \ -1 \le y \le 2, \ 0 \le z \le 3\}.$$



$$\int_0^1 \int_{-1}^2 \int_0^3 xyz^2 \, dz \, dy \, dx$$

$$= \int_0^1 \int_{-1}^2 \frac{1}{3} xyz^3 \Big|_0^3 \, dy \, dx$$

$$= \int_0^1 \int_{-1}^2 9xy \, dy \, dx$$

$$= \int_0^1 \frac{9}{2} xy^2 \Big|_{-1}^2 \, dx = \int_0^1 \left(18x - \frac{9}{2}x \cdot 1\right) dx = \frac{27}{2} \int_0^1 x \, dx$$

$$18x - \frac{9}{2}x = \left(18 - \frac{9}{2}\right)x = \left(\frac{36}{2} - \frac{9}{2}\right) = \frac{27}{2}x$$

$$= \frac{27}{2} \cdot \frac{1}{2} x^2 \Big|_0^1 = \frac{27}{4}$$

Comment: $\int_0^1 \int_{-1}^2 \int_0^3 xyz^2 \, dz\,dy\,dx = \left(\int_0^1 x\,dx \int_{-1}^2 y\,dy\right)\left(\int_0^3 z^2\,dz\right) = \cdots = \frac{27}{4}$

In general, when the domain of integration in $\mathbb{R}^3$ is not a rectangular box, the bounds of integration will look like

$$\int_a^b \int_{u(x)}^{v(x)} \int_{f(x,y)}^{g(x,y)} f(x,y,z)\ dz\,dy\,dx \quad \text{or} \quad \int_c^d \int_{u(y)}^{v(y)} \int_{f(x,y)}^{g(x,y)} f(x,y,z)\ dz\,dx\,dy,$$

or

$$\int_{z_1}^{z_2} \int_{u(z)}^{v(z)} \int_{f(x,z)}^{g(x,z)} f(x,y,z)\ dy\,dx\,dz \quad \text{or} \quad \int_a^b \int_{u(x)}^{v(x)} \int_{f(x,z)}^{g(x,z)} f(x,y,z)\ dy\,dz\,dx,$$

or

$$\int_{z_1}^{z_2} \int_{u(z)}^{v(z)} \int_{f(y,z)}^{g(y,z)} f(x,y,z)\ dx\,dy\,dz \quad \text{or} \quad \int_c^d \int_{u(y)}^{v(y)} \int_{f(y,z)}^{g(y,z)} f(x,y,z)\ dx\,dz\,dy.$$

Double Integrals: $\text{area}(R) = \iint_R dA$

Triple Integrals: $\text{Vol}(B) = \iiint_B dV$

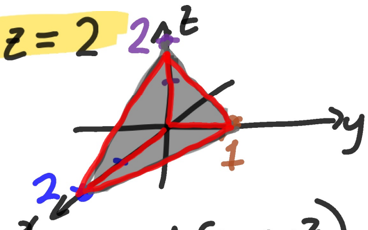
Example 10.2. (FRY Eg. III.3.5.19)
Let E be the solid within the tetrahedral region bounded by the coordinate planes and the plane $x+2y+z=2$. Suppose the density function for the solid is given by $\delta(x,y,z)=x$.

1. Find the total mass of the solid: $\text{Mass}(E)=\displaystyle\iiint_E \delta(x,y,z)\,dV$.
2. Rewrite the integral

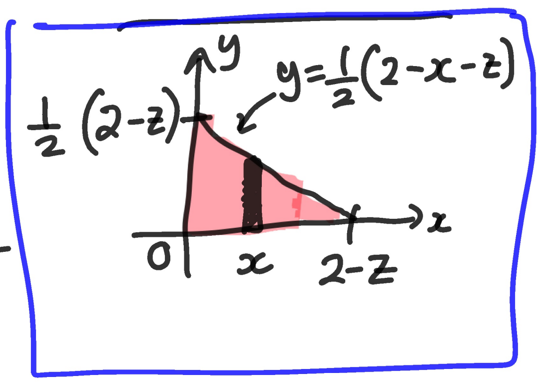
$$\int_{x=0}^{x=2}\int_{y=0}^{y=1-1/2x}\int_{z=0}^{z=2-x-2y} x\,dz\,dy\,dx$$

using a different order of integration. (In how many different ways can we write the above triple integral?)

$x+2y+z=2$



Fix z between 0 and 2



$$\int_{z=0}^{z=2}\int_{x=0}^{x=2-z}\int_{y=0}^{y=\frac{1}{2}(2-x-z)} \boxed{x}\,dy\,dx\,dz$$

density $\rho(x,y,z)$

Fix z:

$0+2y+z=2$

$2y=2-z$

$y=\frac{1}{2}(2-z)$

$x+2y+z=2$

$2y=2-x-z$

$y=\frac{1}{2}(2-x-z)$

$$=\int_{z=0}^{z=2}\int_{x=0}^{x=2-z} xy\Big|_{y=0}^{y=\frac{1}{2}(2-x-z)}\,dx\,dz$$

$$=\int_{z=0}^{z=2}\int_{x=0}^{x=2-z} \frac{1}{2}x(2-x-z)\,dx\,dz$$

$$= \int_{z=0}^{z=2} \int_{x=0}^{x=2-z} \left(-\frac{1}{2}x^2 + \frac{1}{2}x(2-z) \right) dx\, dz$$

$$= \int_{z=0}^{z=2} \left[\frac{-1}{6}x^3 + \frac{1}{4}x^2(2-z) \right]_0^{2-z} dz$$

$$= \int_{z=0}^{z=2} \left(-\frac{1}{6}(2-z)^3 + \frac{1}{4}(2-z)^2(2-z) \right) dz$$

$$\frac{-1}{6} + \frac{1}{4} = \frac{-2}{12} + \frac{3}{12} = \frac{1}{12}$$

$$= \int_{z=0}^{z=2} \frac{1}{12}(2-z)^3\, dz$$

$$= \frac{1}{12} \cdot \frac{-1}{4} (2-z)^4 \Big|_{z=0}^{z=2}$$

$$= \frac{-1}{48}(2-z)^4 \Big|_{z=0}^{z=2} = \cancelto{0}{\frac{-1}{48}(2-2)^4} - \left(\frac{-1}{48}(2-0)^4 \right)$$

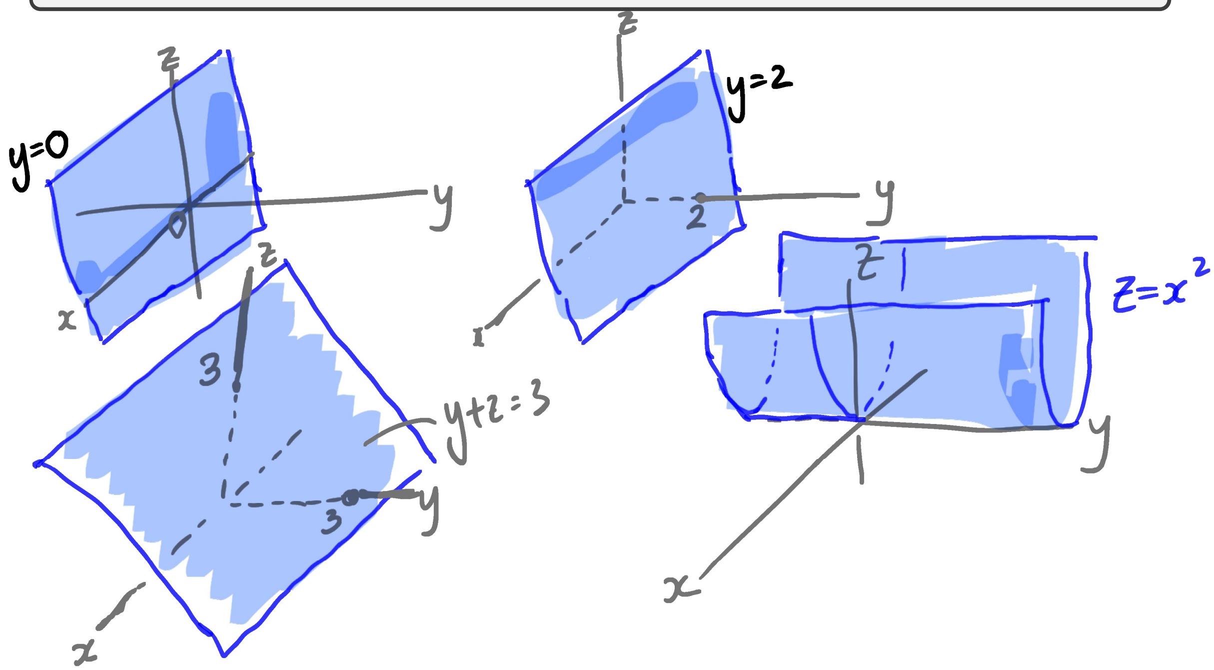
$$= \frac{1}{48} \cdot 2^4$$

times

$$= \frac{1}{3}$$

Example 10.3. (FRY Eg. III.3.5.19)
Let E be the region bounded by the planes $y = 0$, $y = 2$, $y + z = 3$, and the parabolic cylinder $z = x^2$.

1. Find the volume of the enclosed surface.
2. Rewrite the iterated integral corresponding to the $dx\,dy\,dz$ order of integration.



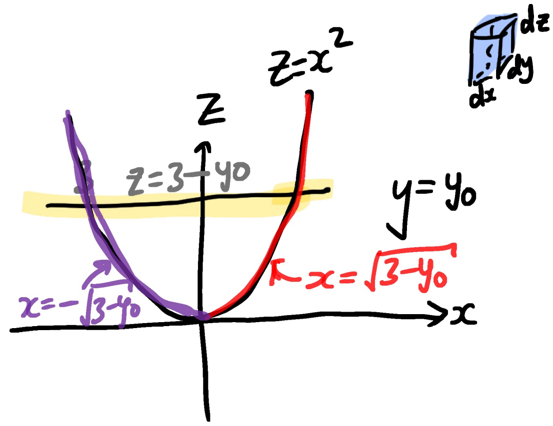
$$\boxed{\begin{aligned} &y=0,\\ &y=2,\\ &y+z=3,\\ &z=x^2 \end{aligned}}$$

If $y=y_0$ for some $y_0 \in [0,2]$,

$y_0+z=3 \Rightarrow z=3-y_0$

$x^2=z=3-y_0$

$x=\sqrt{3-y_0}$ or $x=-\sqrt{3-y_0}$

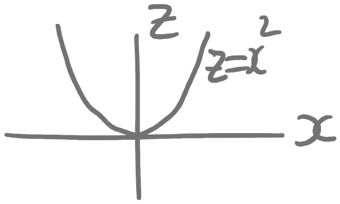


How would we set up the integral if we wanted z to be the outermost variable of integration?

$$\int_0^1 \int_0^2 \int_{-\sqrt{z}}^{\sqrt{z}} dx\,dy\,dz \quad + \quad \int_1^3 \int_0^{3-z} \int_{-\sqrt{z}}^{\sqrt{z}} dx\,dy\,dz$$

$z=x^2$

So $x=\sqrt{z}$ or $x=-\sqrt{z}$



$y+z=3$

$y=3-z$

10.2 References

References:

1. Butler, S., *Integration in cylindrical and spherical coordinates*, calc3.org.

2. Feldman J., Rechnitzer A., Yeager E., *CLP-3 Multivariable Calculus*, University of British Columbia, 2022.

3. Hubbard J.H., Hubbard B.B., *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach, 5th Edition*, Matrix Editions, 2009.

4. Maultsby, B. *Multivariable Calculus*, MA 242, NC State University, 2020.

5. Norman D., *Introduction to Linear Algebra for Science and Engineering*, Addison-Wesley, 1995.

6. Shifrin T., *Multivariable Mathematics: Linear Algebra, Multivariable Calculus, and Manifolds*, John Wiley & Sons, 2005.

7. Stewart J., *Multivariable Calculus, Eighth Edition*, Cengage, 2016.

Live Poll Problem

$$\int_0^1 \int_1^2 \int_{-3}^3 (y + x^2 z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_1^2 \left[xy + \frac{1}{3}x^3 z\right]_{-3}^3 dy\,dz$$

$$= \int_0^1 \int_1^2 \left([3y + 9z] - [-3y - 9z] \right) dy\,dz$$

$$= \int_0^1 \int_1^2 (6y + 18z)\,dy\,dz$$

$$= \int_0^1 \left[3y^2 + 18yz\right]_1^2 dz$$

$$= \int_0^1 \left([12 + 36z] - [3 + 18z] \right) dz$$

$$= \int_0^1 (9 + 18z)\,dz$$

$$= \left[9z + 9z^2\right]_0^1$$

$$= 18$$