Class Note 1.3

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$$\lim_{x\to 3}\left(-\sqrt[5]{x^7}+\frac{e^x}{1+\ln(x)}+\sin(x)\cos(x)\right)$$

$$\lim_{x\to 3}\left(-\sqrt[5]{3^7}+\frac{e^3}{1+\ln(3)}+\sin(3)\cos(3)\right)$$

$$\lim_{x\to 3}\left(8.185427274\right)$$

if we see anything in angles it needs to be in Radians

Module 1(continued)

Basic Limit Laws. Algebraic Properties of Limits.

Suppose that c is a constant and that the following limits exist:

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

Then

Elementary rules (building blocks) that allow us to manipulate expressions that are more complex. Functions can be combined by arithmetic operations.

1. $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)\cdot$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\cdot \quad if \ \lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer
6. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, where n is a positive integer

combined in a more general Rule 7

7. If p, q are integers with $q \neq 0$, then $\lim_{x\to c}[f(x)]^{p/q}$ exists and

$$\lim_{x\to c}[f(x)]^{p/q} = \left(\lim_{x\to c} f(x)\right)^{p/q}$$

8. For every continuous function, the limit of a function is the value of the function at the point:

$$\boldsymbol{\lim_{x\to a} f(x) = f(a)}\textbf{, if } \boldsymbol{f} \textbf{ is continuous at the point } \boldsymbol{x = a}\textbf{.}$$

$\Rightarrow$ hence the "direct substitution" works for all points of continuity.

Practice:

a. $\lim_{x\to 7} 5 = 5$

b. $\lim_{t\to 2} t^3 = 2^3 = 8$

c. $\lim_{x\to 2} 3x^2 = 3(2)^2 = 12$

d. $\lim_{\theta\to 7.56} \pi = \pi$

e. $\lim_{x\to 3}(2x^2 - 5x + 7) = 2(3)^2 - 5(3) + 7 = 15$ (poly is continuous on $\mathbb{R}$)

f. $\lim_{x\to 10.7} e^x = e^{10.7}$ (exp. function is continuous on $\mathbb{R}$)

g. $\lim_{x\to\frac{\pi}{2}} cosx = \cos\frac{\pi}{2} = 0$ (cos is continuous on $\mathbb{R}$)

h. Limit of a polynomial: $\lim_{x\to 1}(x^7 - 2x^5 + 1)^{35} = (1^7 - 2(1)^5 + 1)^{35} = 0$

i. Limit of a rational expression: $\lim_{x\to 2}\frac{5x^3+4}{x-3} = \{\text{sub } x=2\} = \frac{5(2)^3+4}{2-3} = -44$

Evaluating Limits Algebraically.

Find limits using the limit laws

a) $\lim_{x\to\pi} x = \pi$

b) $\lim_{x\to 2} x^4 = 2^4$

c) $\lim_{x\to 5}(2x^2 - 3x + 4) =$

> "Direct substitution" shortcut works whenever function is *continuous* at a point.
> Polynomial, power, exponential and some trigonometric (cosine, sine,tan) functions are continuous at all points in the domain.

{Find lim using Rules}

$$= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5}(4) = 2\lim_{x\to 5}(x^2) - 3\lim_{x\to 5}(x) + 4$$

$$= 2(5^2) - 3(5) + 4 = 39$$ "Direct sub" is a <u>shortcut</u>

d) $\lim_{x\to -2}\frac{x^3+2x^2-1}{5-x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-x)} = \frac{(-2)^3+2(-2)^2-1}{5-(-2)} = -\frac{1}{7}$

e) $\lim_{x\to -1}\frac{x^2+5x}{x^4+2} = \{\text{sub } x=-1\} = \frac{(-1)^2+5(-1)}{(-1)^4+2} = -\frac{4}{3}$

sub $x=3$

f) $\lim_{x\to 3}\sqrt{25-x^2} = \{\text{Rule } 6\} = \sqrt{\lim_{x\to 3}(25-x^2)} = \sqrt{25-3^2} = \sqrt{16} = 4$

g) $\lim_{x\to 2}(4x^2-3)^{1/3}$

$$= \left[\lim_{x\to 2}(4x^2-3)\right]^{1/3} = \left[4(2)^2-3\right]^{1/3} = [16-3]^{1/3} = 13^{1/3}$$

Finding a Limit of a Rational Function by Cancelling a Common Factor.

a) $\lim_{x\to 1}\frac{x-1}{x^2-1} = \{\text{sub } x=1\} = \frac{1-1}{1-1} = \boxed{\frac{0}{0}}$ ← Indeterminate Form $\frac{0}{0}$

Direct substitution results in "indetermanate form"

Apply algebraic simplification

$$\lim_{x\to1}\frac{\cancel{x-1}}{(\cancel{x-1})(x+1)} = \lim_{x\to1}\frac{1}{x+1} = \{\text{sub } x=1\} = \frac{1}{1+1} = \frac{1}{2}$$

Denominator is factored by "difference of squares" formula
$A^2-B^2=(A-B)(A+B)$

b) $\lim_{x\to3}\frac{x^2-9}{x-3} = \{\text{sub } x=3\} = \frac{3^2-9}{3-3} = \boxed{\frac{0}{0}}$ indeterminate form

$$\lim_{x\to3}\frac{(\cancel{x-3})(x+3)}{\cancel{x-3}} = \lim_{x\to3}\frac{x+3}{1} = \{\text{sub } x=3\} = 3+3 = 6$$

c) $\lim_{x\to1}\frac{x^2-1}{x^2-3x+2} = \{\text{sub } x=1\} = \frac{1^2-1}{1^2-3(1)+2} = \frac{0}{0}$ Hint: factor

$$\lim_{x\to1}\frac{(\cancel{x-1})(x+1)}{(\cancel{x-1})(x-2)} = \lim_{x\to1}\frac{x+1}{x-2} = \{\text{sub } x=1\} = \frac{1+1}{1-2} = \frac{2}{-1} = -2$$

Finding a Limit by Simplifying.

a) $\lim_{h\to0}\frac{(3+h)^2-9}{h} = \{\text{sub } h=0\} = \frac{3^2-9}{0} = \frac{0}{0}$ "IF"

$$\lim_{h\to0}\frac{(3^2+6h+h^2)-9}{h} = \lim_{h\to0}\frac{6h+h^2}{h} = \lim_{h\to0}\frac{\cancel{h}(6+h)}{\cancel{h}} = \lim_{h\to0}(6+h) = 6+0 = 6$$

b) $\lim_{x\to1}\frac{x-1}{\sqrt{x}-1} = \{\text{sub } x=1\} = \frac{0}{0}$

$$\lim_{x\to1}\frac{(\sqrt{x})^2-1^2}{\sqrt{x}-1} = \lim_{x\to1}\frac{(\cancel{\sqrt{x}-1})(\sqrt{x}+1)}{\cancel{\sqrt{x}-1}} = \lim_{x\to1}\sqrt{x}+1 = \sqrt{1}+1 = 2$$

Limits Involving Zero or Infinity and the End Behavior of a Function ($x\to\infty$)

Examples:

a) $\lim_{x\to+\infty}\frac{1}{x} = 0^+$ $\qquad \lim_{x\to-\infty}\frac{1}{x} = 0^-$ $\qquad \lim_{x\to\infty}\frac{8}{x} = 0^+$

b) $\lim_{x\to-\infty}\frac{100}{x^2} = 0^+$

c) $\lim_{x\to\infty}\left(\frac{5}{x}-4\right) = \lim_{x\to\infty}\left(\frac{5}{x}\right) - \lim_{x\to\infty}4 = 0-4 = -4$

d) $\lim_{x\to\infty}\left(50-\frac{12}{x^5}\right) = \lim_{x\to\infty}50 - \lim_{x\to\infty}\frac{12}{x^5} = 50-0 = 50$

Behaviours of the power function at infinity

Limits of Polynomials as $x\to\infty$

a) $\lim_{x\to+\infty} x = +\infty$ $\lim_{x\to-\infty} x = -\infty$

b) $\lim_{x\to+\infty} (3x-6) = +\infty$ $\lim_{x\to-\infty} (3x-6) = -\infty$

c) $\lim_{x\to+\infty} 2x^5 = +\infty$ $\lim_{x\to-\infty} 2x^5 = -\infty$

d) $\lim_{x\to+\infty} -7x^6 = -\infty$ $\lim_{x\to-\infty} -7x^6 = -\infty$ ← The power has <u>even</u> exponent

e) $\lim_{x\to+\infty} (5x^4 - 2x^3 + 7x - 59) = \lim_{x\to+\infty} (5x^4 - \dots) = +\infty$

"$5x^4$" is the leading (dominant) term that determines the behaviour of the poly function at infinity

tail is ignored

f) $\lim_{x\to-\infty} (5x^4 - 2x^3 + 7x - 59) = \lim_{x\to-\infty} (5x^4 - \boxed{\dots}) = +\infty$

Useful hint: The end behavior of a polynomial matches the end behavior of its leading term

Limits of Rational Functions as $x \to \infty$

Method: Divide each term in the Numerator and Denominator by the highest power of variable that occurs in the Denominator.

a) $\lim_{x\to+\infty} \frac{3x+5}{6x-8} = \frac{\infty}{\infty} = \lim_{x\to+\infty} \frac{3\cancel{x}/\cancel{x} + 5/x}{6\cancel{x}/\cancel{x} - 8/x} = \lim_{x\to+\infty} \frac{3 + 5/x}{6 - 8/x} = \lim_{x\to+\infty} \frac{3}{6} = \frac{1}{2}$

$\div x$

$\to 0$ $\to 0$

b) $\lim_{x\to\infty} \frac{x^2}{2-3x-x^2} =$

Ans: -1

c) $\lim_{x\to\infty} \frac{3x^3-5x^2}{x^3+6x-8} = \boxed{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{\frac{3x^3}{x^3} - \frac{5x^2}{x^3}}{\frac{x^3}{x^3} - \frac{6x}{x^3} - \frac{8}{x^3}} = \lim_{x\to\infty} \frac{3 - \frac{5}{x}}{1 - \frac{6}{x^2} - \frac{8}{x^3}} = \frac{3}{1} = 3$

hence $\div x^3$

Keep only the leading term for each polynomial

QUICK Method:

d) $\lim_{x\to\infty} \frac{3x^3-5x^2}{x^3+6x-8} = \lim_{x\to\infty} \frac{3x^3 - \dots}{x^3 + \dots} = \lim_{x\to\infty} \frac{3\cancel{x^3}}{\cancel{x^3}} = 3$

e) $\lim_{x\to\infty} \frac{3x^3+7x^4}{x^4-8x^5+x^6} = \lim_{x\to\infty} \frac{7x^4}{x^6} = \lim_{x\to\infty} \frac{7}{x^2} = 0$

f) $\lim\limits_{x \to +\infty} \frac{5x^3-2x^2+1}{1-3x} = \lim\limits_{x \to +\infty} \frac{5x^{\cancel{3}\,2}}{-3\cancel{x}} = \lim\limits_{x \to +\infty} -5x^2 = -\infty$