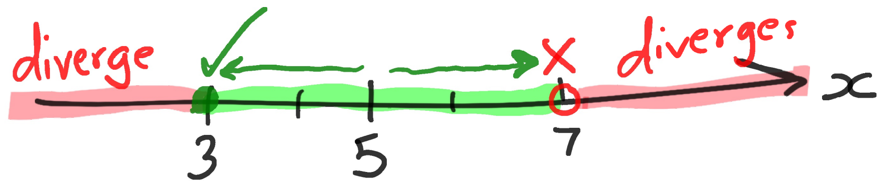
centre = 5, diverges at 7, converges at 3.

(i) radius of convergence $\boxed{R=2}$ (ii) interval of convergence



[3,7]

(3,7]

✓ [3,7)

(3,7)

none of the above

$3 \le x < 7$

[include the endpoint

) exclude the endpoint

3.5 Taylor Series

A smooth function

Definition 3.13. Let f be a function and a be an element in its domain. We say that f is smooth at a if $f^{(n)}(a)$, the n^{th} derivative of f at a, exists for all nonnegative integer values of n. We say that f is smooth function if it is smooth at every point in its domain.

FRY write, "Taylor polynomials provide a hierarchy of approximations to a smooth function $f(x)$ near a particular point a."

(i) The simplest approximation is the constant one: $f(x) \approx f(a)$.

(ii) The next best one is a linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$.

(iii) Then comes the quadratic approximation: $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

(iv) In general,

n^{th} degree Taylor polynomial

Definition 3.14. The Taylor polynomial of degree n for the function $f(x)$ about the expansion point a is the polynomial $T_n(x)$ given by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The n^{th} degree Taylor polynomial serves as an n^{th} degree approximation for $f(x)$ at $x = a$. We may also view the n^{th} Taylor polynomial T_n as being a member of a sequence of partial sums $\{T_n\}_{n=0}^{\infty}$ arising from an infinite series.

FRY Thm II.3.6.4, Ross Defn 31.2 Taylor & Maclaurin Series, Remainder/Error

Definition 3.15. Let f be a function defined on some open interval containing a. If f is smooth at a, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the Taylor series for f about a (or expanded around a). When $a = 0$, we call the Taylor series for f its Maclaurin series. For $n \geq 1$, the error E_n (or remainder R_n) is defined by

$$E_n(x) = f(x) - T_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

In general, the n^{th} remainder term $E_n(x)$ also depends on the function f and the point a at which the Taylor series is centred at, and we may write $E_n(x)$ as $E_n(f; a; x)$ to capture this dependence. Observe that

$$f(x) = T_n(x) + E_n(x),$$

that is, the value of a function at a point x is the sum of the Taylor polynomial of f evaluated at a plus the error (remainder) term. Also note that the value of the function f at x equals the value of the Taylor series at x if and only if the error (remainder) $E_n(x) \to 0$ as $n \to \infty$, that is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} E_n(x) = 0.$$

The next theorem gives us an expression for the error term if the $(n+1)^{\text{st}}$ derivative exists on an interval around the point that we're constructing the Taylor series.

Ross Thm 31.3 Taylor's Theorem

Theorem 3.16. Let f be defined on (α, β) where $\alpha < a < \beta$ (we allow $\alpha = -\infty$ or $\beta = +\infty$). Suppose the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}$ exists on (α, β). Then, for each $x \neq a$ in (α, β), there is some c strictly between a and x such that

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}.$$

Another version of Taylor's Theorem gives an expression for the error (remainder) using an integral.

Ross Thm 31.5 Taylor's Theorem with Integral Remainder

Theorem 3.17. Let f be defined on (α, β) where $\alpha < a < \beta$ (with possibly $\alpha = -\infty$ or $\beta = +\infty$). Suppose the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}$ exists and is continuous on (α, β). Then, for each $x \in (\alpha, \beta)$,

$$E_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\ dt.$$

If the error term does not converge to 0 as $n \to \infty$, then the value of the function f at x does not equal the value of the Taylor series at x. So, we should keep f and its Taylor series as distinct objects in our mind because the two need not agree. Functions that do match their Taylor series are given a special name:

An analytic function

Definition 3.18. A function f for which the Taylor series for f about a converges to f on an open interval around a is said to be analytic at a. If f is analytic at every point in an open interval, we say that f is analytic on that interval.

A function that is analytic at a point a is also smooth at that point because we can construct its Taylor series there, meaning that it had derivatives of every order at a. However, there exist functions that are smooth at a point a but fail to be analytic at a, that is, the value of f's Taylor series at a nearby point x does not match the value of the function at that nearby point.

Here is a condition that ensures that f will be analytic at a point:

Ross Corollary 31.4, A condition for analyticity

Theorem 3.19. Let f be defined on (α, β) where $\alpha < a < \beta$. If all the derivatives $f^{(n)}$ exist on (α, β) and are bounded by a single constant C, then $\lim_{n\to\infty} E_n(x) = 0$ for all $x \in (\alpha, \beta)$. Thus, f is analytic on (α, β).

Keeping the following in mind,

- The Taylor series for f about a is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-\overset{\text{centre}}{a})^n = f(a)+f^{(1)}(a)(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots,$$

$$= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1$$

n^{th} derivative of f

- The Maclaurin series for f is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0)+f^{(1)}(0)x+\frac{f^{(2)}(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots,$$

centre $a=0$

let's work through some examples.

Example 3.20. (FRY Example II.3.6.5)
Find the Maclaurin series for $f(x)=e^x$.

$f(x)=e^x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^x	$e^0=1$
1	e^x	$e^0=1$
2	e^x	$e^0=1$
$\vdots$	$\vdots$	$\vdots$
n	e^x	$e^0=1$
$\vdots$	$\vdots$	$\vdots$

Maclaurin Series for e^x $= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

$= \sum_{n=0}^{\infty}\frac{1}{n!}x^n$

$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

<u>Notes</u>: (i) Radius of convergence $= \infty$

(ii) Interval of convergence $= \mathbb{R} = (-\infty, \infty)$

(iii) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ e^x is an "analytic function"

really is an equality

Example 3.21. (FRY Example II.3.6.6)
Find the Maclaurin series for $\sin x$ and $\cos x$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1
6	$-\sin x$	0
7	$-\cos x$	-1
8	$\sin x$	0

Maclaurin Series for $\sin x$

$$= \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= \frac{0}{0!}x^0 \boxed{+\frac{1}{1!}x^1} + \frac{0}{2!}x^2 \boxed{-\frac{1}{3!}x^3} + \frac{0}{4!}x^4 \boxed{+\frac{1}{5!}x^5} + \frac{0}{6!}x^6 \boxed{-\frac{1}{7!}x^7} + \frac{0}{8!}x^8 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Notes: (i) radius of convergence $R = \infty$

(ii) interval of convergence $= \mathbb{R}$

(iii) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$\uparrow$ analytic function

Maclaurin series for $\cos x$

$$\cos x = (\sin x)' = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots \right)'$$

$$= 1 - \frac{3}{3!}x^2 + \frac{5}{5!}x^4 - \frac{7}{7!}x^6 + \cdots + \frac{(-1)^n(2n+1)}{(2n+1)!}x^{2n} + \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

Notes:

- radius of convergence $R = \infty$
- interval of convergence $= \mathbb{R}$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

$$\frac{3}{3!} = \frac{\cancel{3}}{\cancel{3}\cdot 2\cdot 1}$$

$$\frac{5}{5!} = \frac{\cancel{5}}{\cancel{5}\cdot 4\cdot 3\cdot 2\cdot 1}$$

Example 3.22. (FRY Exercise II.3.6.8.7, constructing a Taylor series using the definition)
Use the definition of a Taylor series to find the Taylor series for $g(x) = \frac{1}{x}$ about $x = 10$. What is the radius of convergence and the interval of convergence of the resulting series?

$g(x) = \frac{1}{x}$ centre $a = 10$

Goal: Taylor series for $g(x)$ about $x = 10$.

$$T(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(10)}{n!}(x-10)^n$$

$$= \frac{g^{(0)}(10)}{0!}(x-10)^0 + \frac{g^{(1)}(10)}{1!}(x-10)^1 + \frac{g^{(2)}(10)}{2!}(x-10)^2 + \cdots + \frac{g^{(n)}(10)}{n!}(x-10)^n + \cdots$$

$$= g(10) + g^{(1)}(10)(x-10) + \frac{g^{(2)}(10)}{2!}(x-10)^2 + \cdots + \frac{g^{(n)}(10)}{n!}(x-10)^n + \cdots$$

n	$g^{(n)}(x)$	$g^{(n)}(10)$
0	$\frac{1}{x} = x^{-1}$	$\frac{1}{10}$
1	$-x^{-2} = \frac{-1}{x^2}$	$\frac{-1}{10^2}$
2	$2x^{-3} = \frac{2}{x^3}$	$\frac{2}{10^3}$
3	$-3\cdot 2x^{-4} = \frac{-3!}{x^4}$	$\frac{-3!}{10^4}$
4	$4\cdot 3\cdot 2x^{-5} = \frac{4!}{x^5}$	$\frac{4!}{10^5}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(-1)^n n!}{x^{n+1}}$	$\frac{(-1)^n n!}{10^{n+1}}$
$\vdots$	$\vdots$	$\vdots$

$3\cdot 2 = 3\cdot 2\cdot 1 = 3!$

$4\cdot 3\cdot 2 = 4\cdot 3\cdot 2\cdot 1 = 4!$

$$\frac{2}{10^3} = \frac{2!}{10^3}$$

$$\boxed{T(x)} = g(10) + g^{(1)}(10)(x-10) + \frac{g^{(2)}(10)}{2!}(x-10)^2 + \cdots + \frac{g^{(n)}(10)}{n!}(x-10)^n + \cdots$$

$$= \frac{1}{10} - \frac{1}{10^2}(x-10) + \frac{\cancel{2!}/10^3}{\cancel{2!}}(x-10)^2 - \frac{\cancel{3!}/10^4}{\cancel{3!}}(x-10)^3 + \cdots + \frac{\frac{(-1)^n \cancel{n!}}{10^{n+1}}}{\cancel{n!}}(x-10)^n + \cdots$$

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(10)}{n!}(x-10)^n$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{n+1}}(x-10)^n}$$

Taylor series for $g(x) = \frac{1}{x}$ about $x=10$

Radius and Interval of Convergence = ?

$$\frac{1}{10^{n+1}} = \frac{1}{10 \cdot 10^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{10} \cdot \left[\frac{(-1)(x-10)}{10}\right]^n$$

$\sum_{n=0}^{\infty} ar^n$ converged if and only if $|r| < 1$

This series converges if and only if

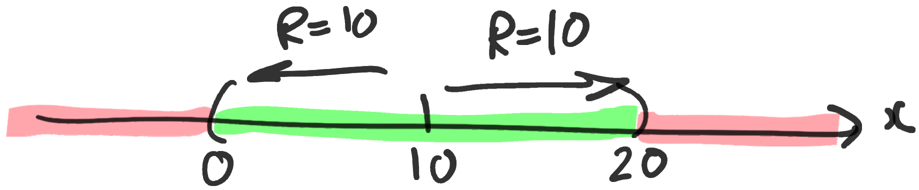
$$\left|\frac{(-1)(x-10)}{10}\right| < 1$$

$$\Longleftrightarrow \quad -1 < \frac{(-1)(x-10)}{10} < 1$$

$$\Longleftrightarrow \quad -10 < (-1)(x-10) < 10$$

$\Longleftrightarrow \quad 10 > x - 10 > -10$

$\Longleftrightarrow \quad 20 > x > 0$



radius of convergence $R = 10$

interval of convergence $= (0, 20)$

Example 3.23. (Taylor series can help us exactly evaluate some infinite series (of numbers))
Using the Alternating Series Test, we concluded that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converged. What number does the series converge to?

power series for $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

interval of convergence $= (-1, 1]$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(1+1) = \ln 2$

Example 3.24. (FRY Example II.3.6.22 Taylor series can help us evaluate limits.)
Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = \frac{1}{x}\sin x = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$

$$= 1 - 0 + 0 - 0 + \cdots$$

$$= 1$$

3.6 References

References:

1. Colemean R., *Calculus on Normed Vector Spaces*, Springer, 2012.

2. Feldman J., Rechnitzer A., Yeager E., *CLP-2 Integral Calculus*, University of British Columbia, 2020.

3. Ross K.A., *Elementary Analysis: The Theory of Calculus, Second Edition*, Springer, 2013.

4. Shifrin T., *Multivariable Mathematics: Linear Algebra, Multivariable Calculus, and Manifolds*, John Wiley & Sons, 2005.