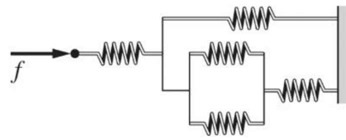
Worksheet 1 - Solution

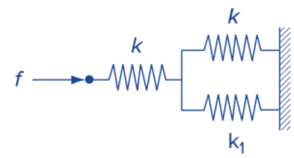
PART 1: Mass-Spring Systems

1) Determine the equivalent spring constant of the arrengement shown in the following figure. Asume all springs have the same spring constant k.

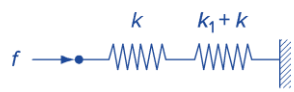


First reduce the system to the equivalent one shown in part (a) of the figure, where

$$\frac{1}{k_1} = \frac{1}{2k} + \frac{1}{k} = \frac{3}{2k} \quad \rightarrow \quad k_1 = \frac{2k}{3}$$



(a)

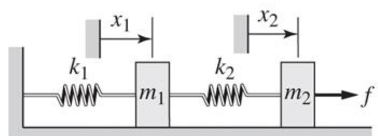


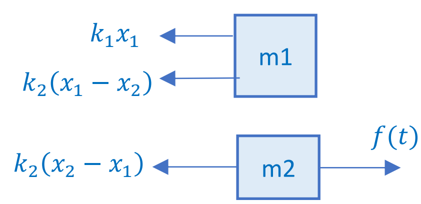
(b)

From part (b) of the figure,

$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{k_1 + k} = \frac{k_1 + 2k}{k(k_1 + k)} \quad \rightarrow \quad k_{eq} = \frac{k(k_1 + k)}{k_1 + 2k} = \frac{5k}{8}$$

2) For the mass-spring system shown below, the input is the force f and the outputs are the displacements x_1 and x_2 of the masses. The equilibrium possitions with $f = 0$ correspond to $x_1 = x_2 = 0$. Neglect any friction between the masses and the surface. Derive the equations of motion of the system.



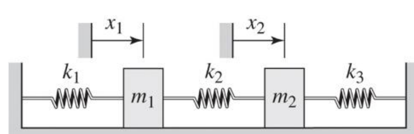


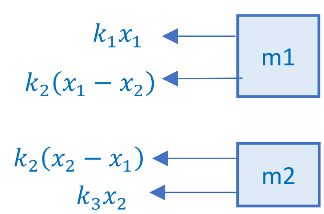
Direction of motion is to the right. The equation of motions are obtained as:

Mass m_1 $\rightarrow$ $-k_1x_1 - k_2(x_1 - x_2) = m_1\ddot{x}_1$ $\rightarrow$ $m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$

Mass m_2 $\rightarrow$ $f(t) - k_2(x_2 - x_1) = m_2\ddot{x}_2$ $\rightarrow$ $m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = f(t)$

3) In the following mass-spring system when $x_1 = x_2 = 0$, the springs are at their free lengths. Derive the equation of motion.



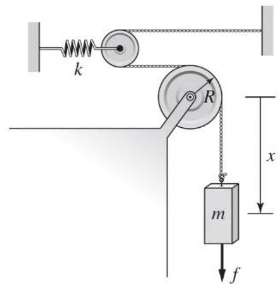


Direction of motion is to the right. The equation of motions are obtained as:

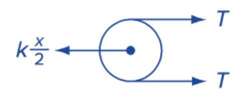
Mass m_1 $\rightarrow$ $-k_1x_1 - k_2(x_1 - x_2) = m_1\ddot{x}_1$ $\rightarrow$ $m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$

Mass m_2 $\rightarrow$ $-k_2(x_2 - x_1) - k_3x_2 = m_2\ddot{x}_2$ $\rightarrow$ $m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0$

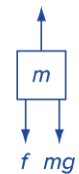
4) In the following mass-spring-pulley system, the input is the applied force f, and the output is the displacement x. Assume the pulley masses are negligible and derive the equation of motion.



Let T be the tension in the cable. Since the moving pulley is considered massless,



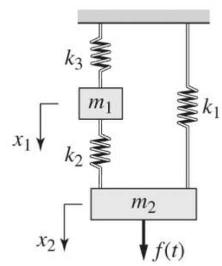
$$2T = \frac{kx}{2} \quad \rightarrow \quad T = \frac{kx}{4}$$

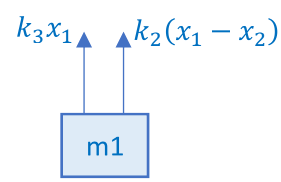


Direction of motion is downward.

Mass $m \quad \rightarrow \quad f(t) + mg - T - m\ddot{x} = 0 \quad \rightarrow \quad m\ddot{x} + \frac{k}{4}x = f(t) + mg$

5) For the system shown below, suppose that $k_1 = k$, $k_2 = k_3 = 4k$, and $m_1 = m_2 = m$. Obtain the equation of motion in terms of x_1 and x_2.





Direction of motion is downward.

Assume that $x_2 > x_1$. Then, the equation of motions are obtained as:

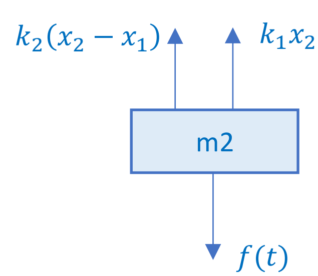
Mass $m_1 \quad \rightarrow \quad -k_3x_1 - k_2(x_1 - x_2) = m_1\ddot{x}_1 \quad \rightarrow \quad m_1\ddot{x}_1 + (k_2 + k_3)x_1 - k_2x_2 = 0$

Mass $m_2 \quad \rightarrow \quad f(t) - k_2(x_2 - x_1) - k_1x_2 = m_2\ddot{x}_2 \quad \rightarrow \quad m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_2x_1 = f(t)$

Using the given values,

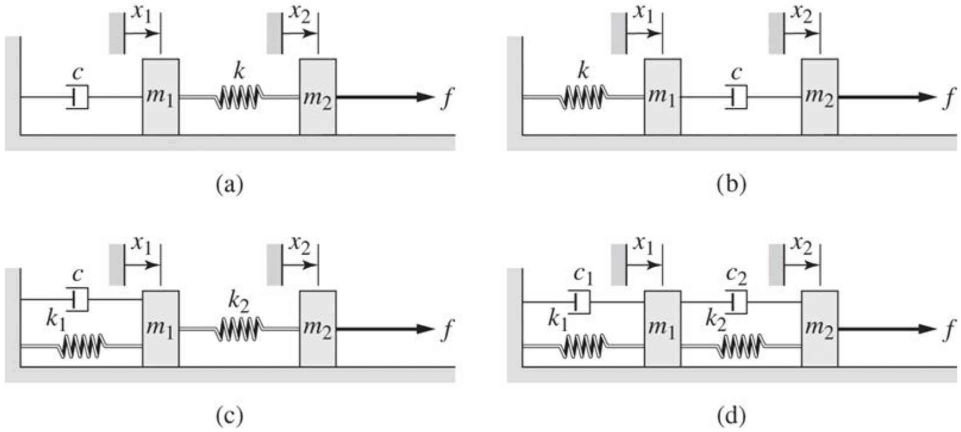
Mass $m_1 \quad \rightarrow \quad m\ddot{x}_1 + 8kx_1 - 4kx_2 = 0$

Mass $m_2 \quad \rightarrow \quad m\ddot{x}_2 + 5kx_2 - 4kx_1 = f(t)$



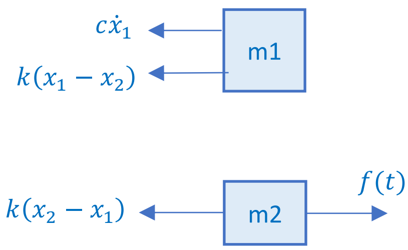
PART 2: Mass-Spring-Damper Systems

6) For each of the following mass-spring-damper systems, the input is the force f and the outputs are the displacements x_1 and x_2 of the masses. The equilibrium positions with $f = 0$ corresponds to $x_1 = x_2 = 0$. Neglect the friction between the masses and the surface. Derive the equations of motion of the systems.



Direction of motion is to the right for all cases.

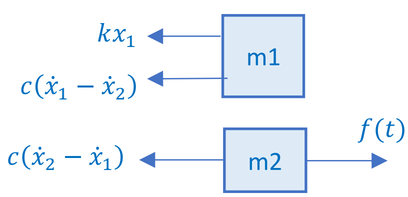
a) The equation of motions are obtained as:



Mass m_1 $\rightarrow$ $-c\dot{x}_1 - k(x_1 - x_2) = m_1\ddot{x}_1$ $\rightarrow$ $m_1\ddot{x}_1 + c\dot{x}_1 + kx_1 - kx_2 = 0$

Mass m_2 $\rightarrow$ $f(t) - k(x_2 - x_1) = m_2\ddot{x}_2$ $\rightarrow$ $m_2\ddot{x}_2 + kx_2 - kx_1 = f(t)$

b) The equation of motions are obtained as:



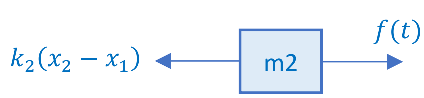
Mass m_1 $\rightarrow$ $-kx_1 - c(\dot{x}_1 - \dot{x}_2) = m_1\ddot{x}_1$ $\rightarrow$ $m_1\ddot{x}_1 + c\dot{x}_1 + kx_1 - c\dot{x}_2 = 0$

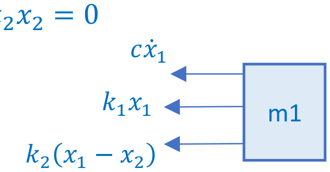
Mass m_2 $\rightarrow$ $f(t) - c(\dot{x}_2 - \dot{x}_1) = m_2\ddot{x}_2$ $\rightarrow$ $m_2\ddot{x}_2 + c\dot{x}_2 - c\dot{x}_1 = f(t)$

c) The equation of motions are obtained as:

Mass m_1 $\rightarrow$ $-c\dot{x}_1 - k_1x_1 - k_2(x_1 - x_2) = m_1\ddot{x}_1$ $\rightarrow$ $m_1\ddot{x}_1 + c\dot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$

Mass m_2 $\rightarrow$ $f(t) - k_2(x_2 - x_1) = m_2\ddot{x}_2$ $\rightarrow$ $m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = f(t)$

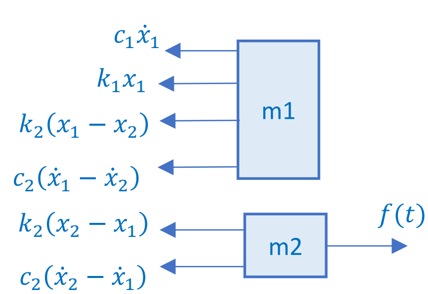




d) The equation of motions are obtained as:

Mass m_1 $\rightarrow$ $-c_1\dot{x}_1 - k_1x_1 - k_2(x_1 - x_2) - c_2(\dot{x}_1 - \dot{x}_2) = m_1\ddot{x}_1$

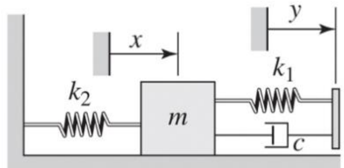
$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0$$



Mass m_2 $\rightarrow$ $f(t) - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = m_2\ddot{x}_2$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 = f(t)$$

7) In the following mass-spring-damper systems, the input is the displacement y and the output is the displacements x of the mass m. The equilibrium positions corresponds to $x = y = 0$. Neglect the friction between the mass and the surface. Derive the equations of motion and find the transfer function $X(s)/Y(s)$.



Direction of motion is to the right.

The equation of motion is obtained as:

Mass $m \;\rightarrow\; -k_2x - k_1(x-y) - c(\dot{x}-\dot{y}) = m\ddot{x}$

Rearrange the equation of motion:

$m\ddot{x} + c\dot{x} + (k_1+k_2)x = c\dot{y} + k_1y$

Take Laplace transform by considering the zero initial conditions:

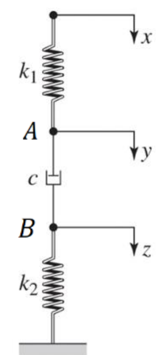
$ms^2X(s) + csX(s) + (k_1+k_2)X(s) = csY(s) + k_1Y(s)$

$\left(ms^2 + cs + (k_1+k_2)\right)X(s) = (cs+k_1)Y(s)$

The transfer function is:

$$\frac{X(s)}{Y(s)} = \frac{cs+k_1}{ms^2+cs+k_1+k_2}$$

8) Find the transfer function $Z(s)/X(s)$ for the system shown below.



Direction of motion is downward.

The equation of motions are obtained as:

Point $A \;\rightarrow\; -k_1(y-x) - c(\dot{y}-\dot{z}) = 0 \quad\rightarrow\quad c\dot{y} + k_1y - c\dot{z} - k_1x = 0$

Point $B \;\rightarrow\; -c(\dot{z}-\dot{y}) - k_2z = 0 \quad\rightarrow\quad c\dot{y} - c\dot{z} - k_2z = 0$

Take Laplace transform by considering the zero initial conditions:

$csY(s) + k_1Y(s) - csZ(s) - k_1X(s) = 0 \quad\rightarrow\quad (cs+k_1)Y(s) - csZ(s) - k_1X(s) = 0 \quad Eqn.(1)$

$csY(s) - csZ(s) - k_2Z(s) = 0 \quad\rightarrow\quad Y(s) = \frac{cs+k_2}{cs}Z(s) \quad Eqn.(2)$

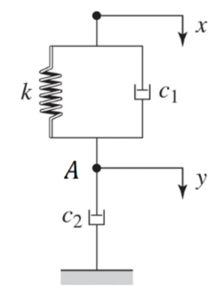
Subtitute Y(s) from Eqn. (2) into the Eqn. (1) and find the transfer function Z(s)/X(s):

$$(cs+k_1)\frac{cs+k_2}{cs}Z(s) - csZ(s) - k_1X(s) = 0$$

$$\left((cs+k_1)\frac{cs+k_2}{cs} - cs\right)Z(s) - k_1X(s) = 0$$

$$\rightarrow \quad \frac{Z(s)}{X(s)} = \frac{ck_1s}{(k_1+k_2)cs + k_1k_2}$$

9) Find the transfer function $Y(s)/X(s)$ for the system shown below.



Direction of motion is downward.

The equation of motion is obtained as:

Point A $\rightarrow$ $-k(y-x)-c_1(\dot{y}-\dot{x})-c_2\dot{y}=0$

$$(c_1+c_2)\dot{y}+ky-c_1\dot{x}-kx=0$$

Take Laplace transform by considering the zero initial conditions:

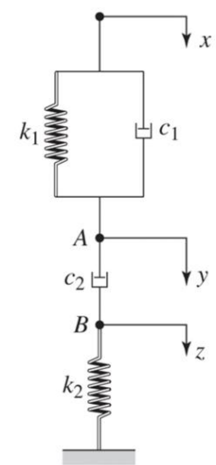
$$(c_1+c_2)sY(s)+kY(s)-c_1sX(s)-kX(s)=0$$

$$\big((c_1+c_2)s+k\big)Y(s)=(c_1s+k)X(s)$$

Find the transfer function Y(s)/X(s):

$$\frac{Y(s)}{X(s)}=\frac{c_1s+k}{(c_1+c_2)s+k}$$

10) Find the transfer function $Y(s)/X(s)$ for the system shown below.



Direction of motion is downward.

The equation of motions are obtained as:

Point A $\rightarrow$ $-k_1(y-x)-c_1(\dot{y}-\dot{x})-c_2(\dot{y}-\dot{z})=0$

$$(c_1+c_2)\dot{y}+k_1y-c_1\dot{x}-k_1x-c_2\dot{z}=0$$

Point B $\rightarrow$ $-c_2(\dot{z}-\dot{y})-k_2z=0$ $\rightarrow$ $c_2\dot{z}+k_2z-c_2\dot{y}=0$

Take Laplace transform by considering the zero initial conditions:

$$(c_1+c_2)sY(s)+k_1Y(s)-c_1sX(s)-k_1X(s)-c_2sZ(s)=0 \qquad Eqn.(1)$$

$$c_2sZ(s)+k_2Z(s)-c_2sY(s)=0 \quad\rightarrow\quad Z(s)=\frac{c_2s}{c_2s+k_2}Y(s) \qquad Eqn.(2)$$

Rearrange the Eqn. (1) and subtitute Z(s) from Eqn. (2) into the Eqn. (1):

$$\big((c_1+c_2)s+k_1\big)Y(s)-(c_1s+k_1)X(s)-c_2sZ(s)=0$$

$$\big((c_1+c_2)s+k_1\big)Y(s)-(c_1s+k_1)X(s)-c_2s\frac{c_2s}{c_2s+k_2}Y(s)=0$$

$$\left((c_1+c_2)s+k_1-c_2s\frac{c_2s}{c_2s+k_2}\right)Y(s)=(c_1s+k_1)X(s)$$

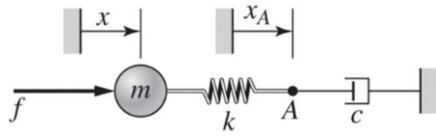
Find the transfer function Y(s)/X(s):

$$\frac{Y(s)}{X(s)}=\frac{(c_1s+k_1)(c_2s+k_2)}{c_1c_2s^2+\big((c_1+c_2)k_2+k_1c_2\big)s+k_1k_2}$$

11) In the system shown below, the input is the force f and the output is the displacement x_A of point A. When $x = x_A$, the spring is at its free length. Derive the equation of motion and transfer function $X_A(s)/F(s)$.

Direction of motion is to the right.

The equation of motion is obtained as:



Mass m $\rightarrow$ $f(t) - k(x - x_A) = m\ddot{x}$ $\rightarrow$ $f(t) = m\ddot{x} + kx - kx_A$

Point A $\rightarrow$ $-k(x_A - x) - c\dot{x}_A = 0$ $\rightarrow$ $c\dot{x}_A + kx_A = kx$

Take Laplace transform by considering the zero initial conditions:

$$F(s) = ms^2X(s) + kX(s) - kX_A(s) \qquad Eqn.(1)$$

$$csX_A(s) + kX_A(s) = kX(s) \quad \rightarrow \quad X(s) = \frac{cs + k}{k}X_A(s) \qquad Eqn.(2)$$

Substitute X(s) from Eqn. (2) into the Eqn. (1), and form the transfer function:

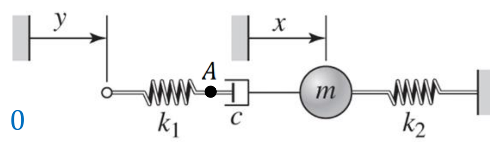
$$F(s) = (ms^2 + k)\frac{cs + k}{k}X_A(s) - kX_A(s) \quad \rightarrow \quad F(s) = \left((k + ms^2)\frac{cs + k}{k} - k\right)X_A(s)$$

$$\frac{X_A(s)}{F(s)} = \frac{k}{s(mcs^2 + mks + ck)}$$

12) In the system shown below, the input is the displacement y and the output is the displacement x. When $x = y = 0$, the springs are at their free length. Derive the equation of motion and transfer function $X(s)/Y(s)$.

Direction of motion is to the right.

The equation of motion is obtained as:



Mass m $\rightarrow$ $-c(\dot{x} - \dot{x}_A) - k_2x = m\ddot{x}$ $\rightarrow$ $m\ddot{x} + c\dot{x} + k_2x - c\dot{x}_A = 0$

Point A $\rightarrow$ $-k_1(x_A - y) - c(\dot{x}_A - \dot{x}) = 0$ $\rightarrow$ $c\dot{x}_A + k_1x_A = c\dot{x} + k_1y$

Take Laplace transform by considering the zero initial conditions:

$$ms^2X(s) + csX(s) + k_2X(s) - csX_A(s) = 0 \qquad Eqn.(1)$$

$$csX_A(s) + k_1X_A(s) = csX(s) + k_1Y(s) \quad \rightarrow \quad X_A(s) = \frac{cs}{cs + k_1}X(s) + \frac{k_1}{cs + k_1}Y(s) \qquad Eqn.(2)$$

Substitute $X_A(s)$ from Eqn. (2) into the Eqn. (1), and form the transfer function:

$$(ms^2 + cs + k_2)X(s) - csX_A(s) = 0 \quad \rightarrow \quad (ms^2 + cs + k_2)X(s) - \frac{ck_1s}{cs + k_1}Y(s) - \frac{c^2s^2}{cs + k_1}X(s) = 0$$

$$\frac{X(s)}{Y(s)} = \frac{ck_1s}{mcs^3 + mk_1s^2 + c(k_1 + k_2)s + k_1k_2}$$