Limits

Class Notes 1.2

Investigating Limits. Numerical Evaluation of Limits

Example 1: Evaluate function at the given points accurately to 5 decimal places.

x	$\boldsymbol{y(x) = \left(1+\frac{1}{x}\right)^x}$
1	2.00000
10	
100	
1000	
1000000	
1500000	
1550000	

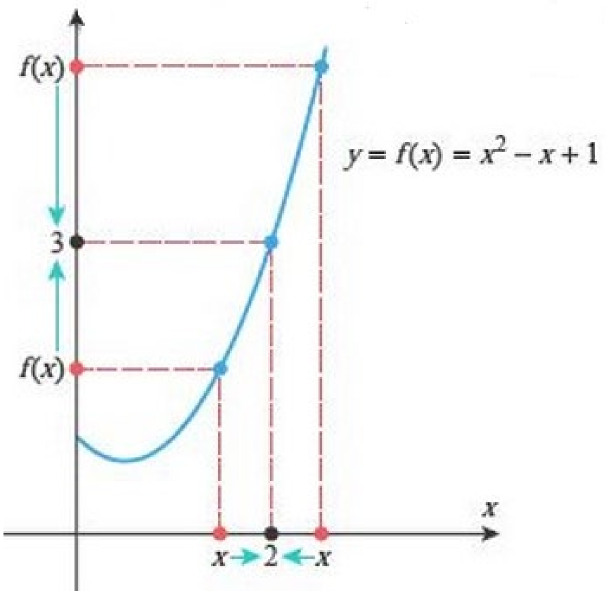
https://www.geogebra.org/m/Gc6YrBng

Symbolically, this can be expressed as $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$

The Limit of a Function in a Point of Continuity

Example 2:

Investigate numerically and graphically the behavior of the function *f* defined by $f(x)=x^2-x+1$ for values of *x* near 2 (finite input). The following tables A and B give values of $f(x)$ when *x* is close to 2 but not equal to 2.



x	$f(x)$
1.8	2.44000
1.9	2.71000
1.95	2.85250
1.99	2.97010
1.995	2.98503
1.999	2.99700

Table A

x	$f(x)$
2.2	3.64000
2.1	3.31000
2.05	3.15250
2.01	3.03010
2.005	3.01503
2.001	3.00300

Table B

vicinity of $x=2$:
$x \in (2-\delta, 2+\delta),\ \delta > 0$

Symbolically,

$$\left.\begin{array}{l}\lim\limits_{x\to 2^-} f(x) = 3 \\ \lim\limits_{x\to 2^+} f(x) = 3\end{array}\right\} \Rightarrow \lim_{x\to 2} f(x) = 3$$

or in words "as x approaches 2, the function $f(x)$ approaches 3.

From the tables and from the graph we can see that when *x* is close to 2 (on either side of 2), $f(x)$ is close to 3. In fact, we can make the values of $f(x)$ as close as we like to 3 by taking *x* sufficiently close to 2. This fact is expressed as:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} f(x^2 - x + 1) = 3$$

The Method of Direct Substitution

Observe, that the evaluation of the limit can be done by the **"method of direct substitution":** $\lim\limits_{x\to 2}(x^2-x+1) = f(2) = 2^2 - 2 + 1 = 3$. "Direct substitution" works for values at which the function is **continuous**.

Try it: Find $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5}(x^2 - x + 1) =$

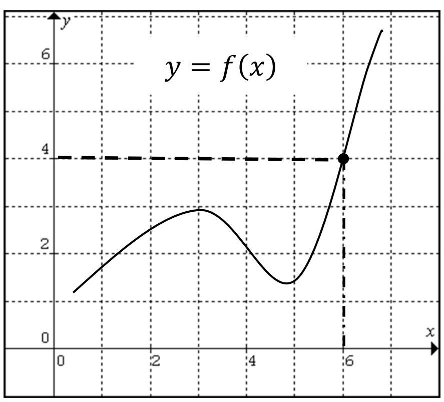
Definition and Denotations:

The limit of $f(x)$, as x approaches a, equals L if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a.	$\lim\limits_{x \to a} f(x) = L$

Note. $x \neq a$ in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined **near** a but not in a.

Example 3. For the function f graphed in the accompanying figure, find each quantity, if it exists. If it doesn't exist, explain why.

a)



(i) $\lim\limits_{x \to 6^-} f(x) =$ (ii) $\lim\limits_{x \to 6^+} f(x) =$

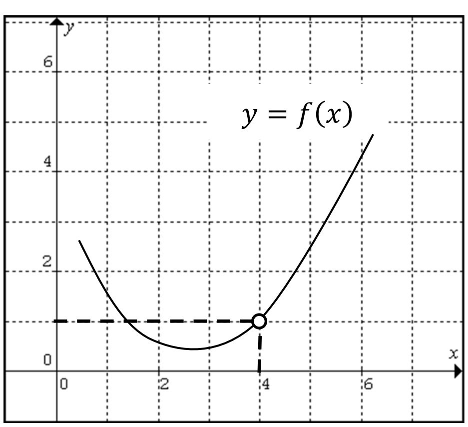
(iii) $\lim\limits_{x \to 6} f(x) =$

(iv) $f(6) =$

(v) f is ____________________ at $x = 6$

(continuous/discontinuous)

b)



(i) $\lim\limits_{x \to 4^-} f(x) =$ (ii) $\lim\limits_{x \to 4^+} f(x) =$

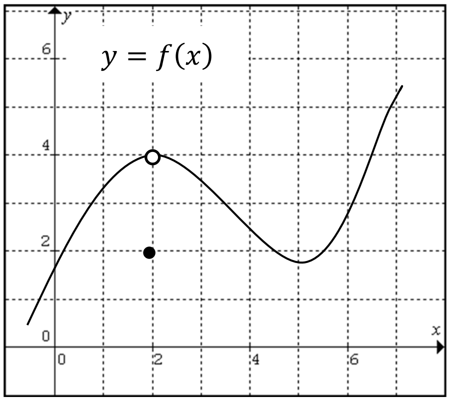
(iii) $\lim\limits_{x \to 4} f(x) =$

(iv) $f(4) =$

(v) f is ____________________ at $x = 4$

(continuous/discontinuous)

c)



(i) $\lim_{x \to 2^-} f(x) =$ (ii) $\lim_{x \to 2^+} f(x) =$

(iii) $\lim_{x \to 2} f(x) =$

(iv) $f(2) =$

(v) f is ____________________ at $x = 2$

One-Sided Limits more formally

One-sided limits are helpful in describing what may happen with a function at a point.

$\lim_{x \to a^+} f(x) = L$ is the right-hand limit $\lim_{x \to a^-} f(x) = L$ is the left- hand limit

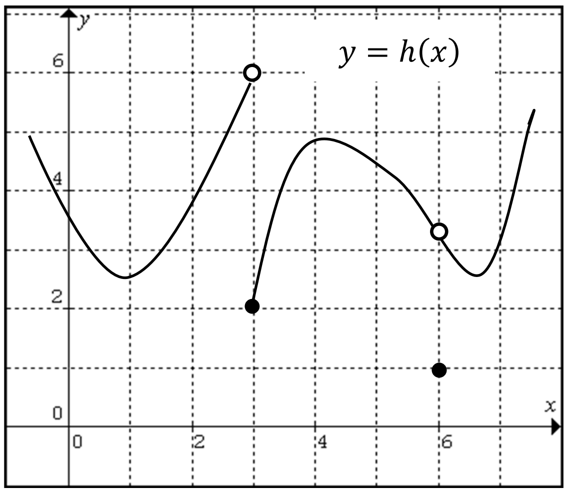
The symbol "$x \to a^+$" means that we consider only $x > a$, the symbol "$x \to a^-$" means that we consider only $x < a$.
The following is true.

$$\lim_{x \to a} f(x) = L, \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

Example 4. For the function h graphed in the accompanying figure, find each quantity, if it exists. If it doesn't exist, explain why.

a) $\lim_{x\to 3^-} h(x) =$ $\lim_{x\to 3^+} h(x) =$ $\lim_{x\to 3} h(x) =$

b) $\lim_{x\to 6^-} h(x) =$ $\lim_{x\to 6^+} h(x) =$ $\lim_{x\to 6} h(x) =$



c) $h(3) =$; $h(6) =$;

d) Complete the following statements by choosing the correct term.

(i) Function h is *continuous/discontinuous* at $x = 3$.

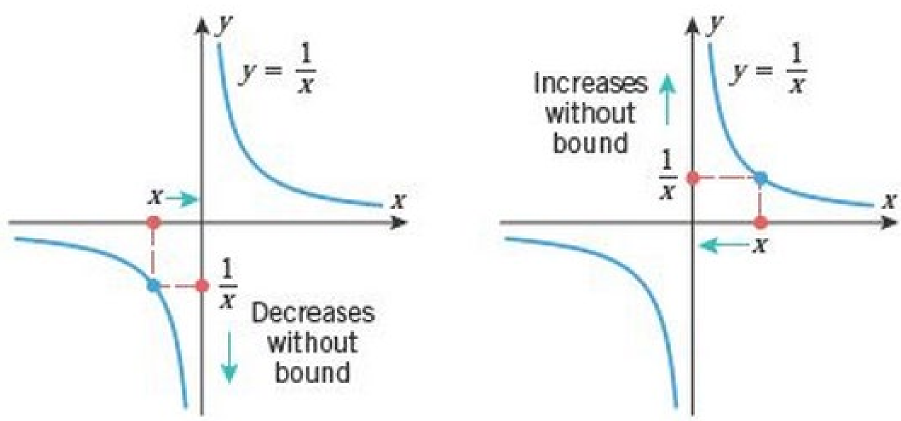
(ii) Function h is *defined/not defined* at $x = 3$

(iii) Function h is *continuous/discontinuous* at $x = 6$.

(iv) Function h is *defined/not defined* at $x = 6$

Limits Involving Infinity

Example 5. Evaluating limits graphically.



a) $\lim_{x\to 0^-} \frac{1}{x} =$

b) $\lim_{x\to 0^+} \frac{1}{x} =$

c) $\lim_{x\to 0} \frac{1}{x} =$

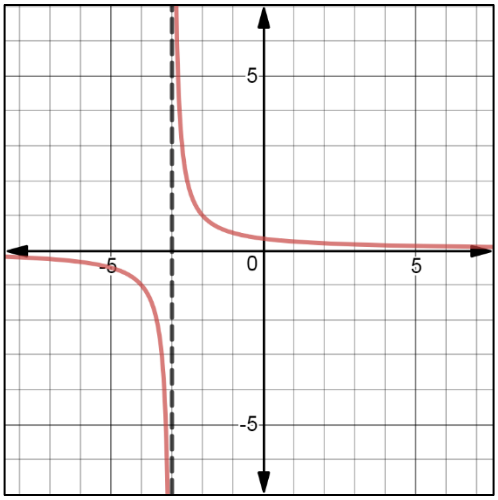
d) $\lim_{x\to -\infty} \frac{1}{x} =$

e) $\lim_{x\to +\infty} \frac{1}{x} =$

f) the equation of the vertical asymptote: __________.

g) the equation of the horizontal asymptote: _________.

Try it yourself. For the function $y = \frac{1}{x+3}$ graphed in the accompanying figure, find



a) $\lim_{x\to -3^-} \frac{1}{x+3} =$

b) $\lim_{x\to -3^+} \frac{1}{x+3} =$

c) $\lim_{x\to -3} \frac{1}{x+3} =$

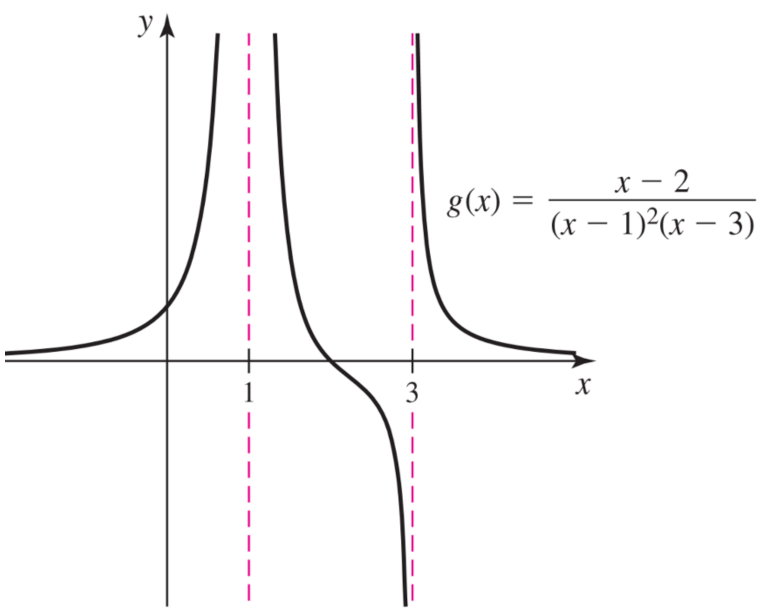
d) $\lim_{x\to -\infty} \frac{1}{x+3} =$

e) $\lim_{x\to +\infty} \frac{1}{x+3} =$

f) the equation of the vertical asymptote: ___________.

g) the equation of the horizontal asymptote: __________.

Example 6



a) $\lim_{x\to 1^-} g(x) =$

b) $\lim_{x\to 1^+} g(x) =$

c) $\lim_{x\to 3^-} g(x) =$

d) $\lim_{x\to 3^+} g(x) =$

e) $\lim_{x\to 3} g(x) =$

f) $\lim_{x\to -\infty} g(x) =$

g) $\lim_{x\to +\infty} g(x) =$

h)

i) the equation of the vertical asymptote(s): ___________.

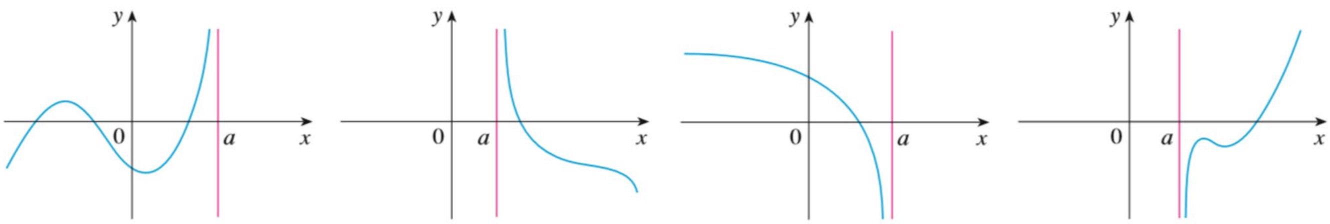
j) the equation of the horizontal asymptote: __________.

Vertical and Horizontal Asymptotes

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if

$$\lim_{x\to a} f(x) = \infty$$

Any variation of the statement applies, including one-sided limits. Some of the cases are illustrated below.

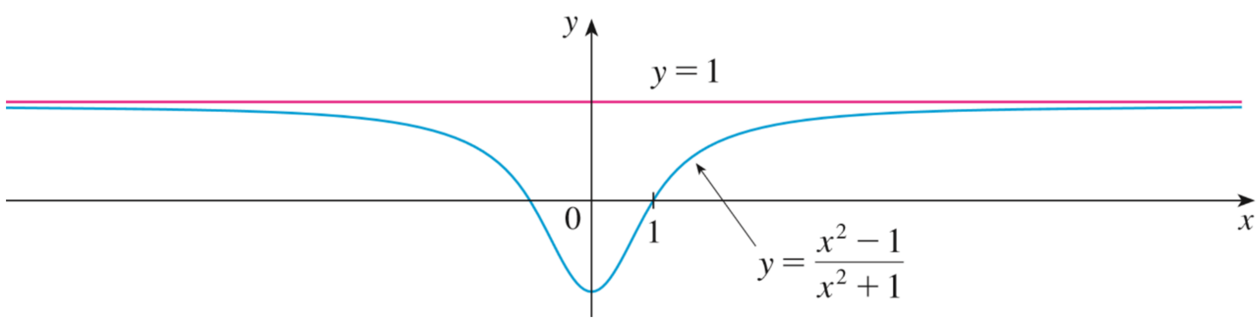


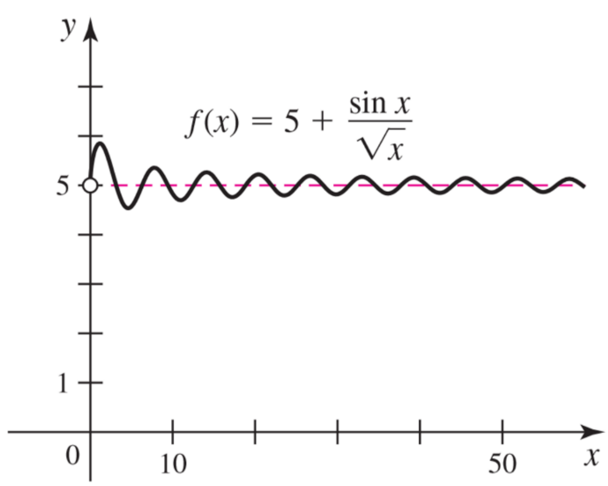
(a) $\lim_{x\to a^-} f(x) = \infty$ (b) $\lim_{x\to a^+} f(x) = \infty$ (c) $\lim_{x\to a^-} f(x) = -\infty$ (d) $\lim_{x\to a^+} f(x) = -\infty$

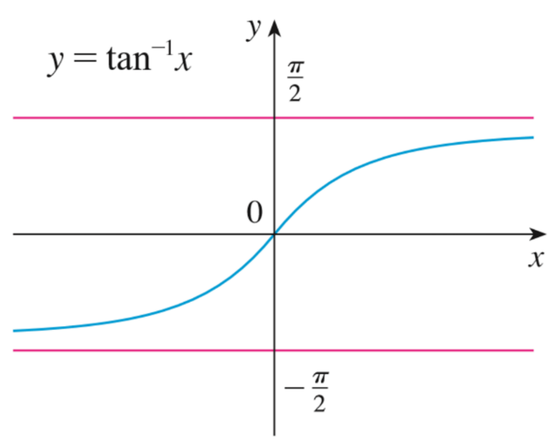
The line $y = L$ is called a **horizontal asymptote** if either

$$\lim_{x\to\infty} f(x) = L \; or \; \lim_{x\to -\infty} f(x) = L$$

Note: A function can have up to two different horizontal asymptotes, one corresponding to the limit at $+\infty$, and the other corresponding to the limit at $-\infty$.



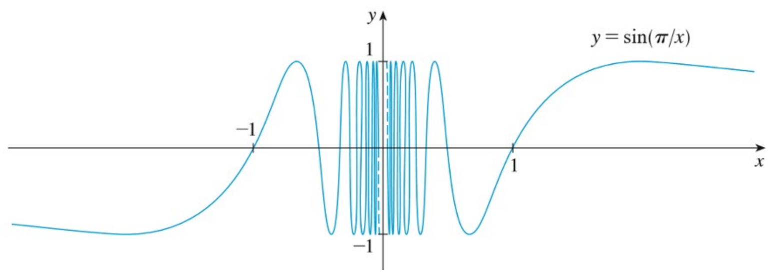




Examples of Limits Than Do Not Exist (DNE)

a. $\lim_{x\to\infty} \sin x$

b. $\lim_{x\to 0} \sin\frac{\pi}{x}$



c. Prove that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Absolute value function

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

or another useful representation:

$$|x| = \sqrt{x^2}$$

Since, LHS limit ≠ RHS limit, the limit at a point DNE: $\lim_{x\to 0}\frac{|x|}{x} = \text{DNE}$