Improper integrals

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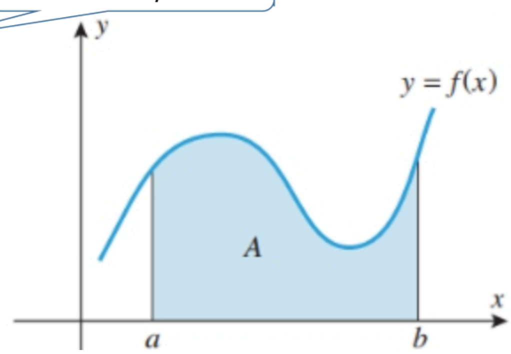
(BKM) The Definite Integral

With finite discontinuities/bounded

The **area** under the graph of a continuous positive function $f(x)$ over the finite closed interval $[a, b]$ is represented by the **definite integral**

$$\int_a^b f(x)dx = \mathrm{A}$$



Recall:

The FTC, part 2 provides the convenient computational method, if the **antiderivative** $F(x)$ exists, $F'(x) = f(x)$,

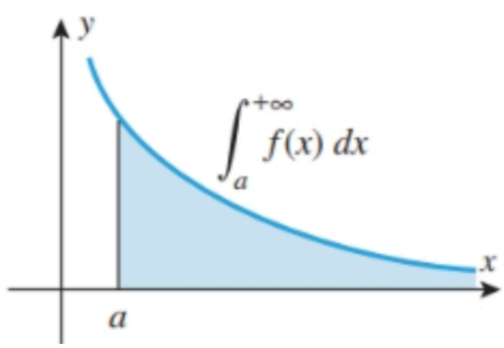
$$A = \int_a^b f(x)dx = \mathrm{F(x)}\Big|_{x\,=\,a}^{x\,=\,b} = F(b) - F(a)$$

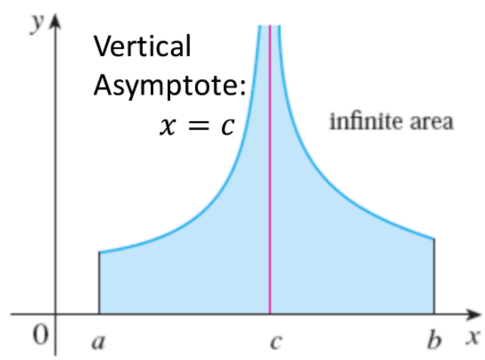
Finite value

Improper Integral: Type 1 and Type 2

Improper integral is an extension of the concept of a definite integral to allow for

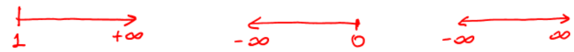
- Infinite intervals of integration → **Type 1**
- Integrands with *vertical asymptotes* within the interval of integration. The vertical asymptotes are called infinite discontinuities → **Type 2**
- There are improper integrals with Type 1 and Type 2 combined





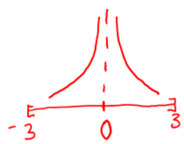
Improper Integral: Type 1 and Type 2

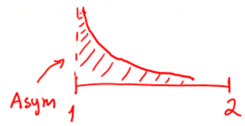
- **Type 1**: Infinite intervals of integration $\displaystyle\int_1^{+\infty} \frac{dx}{x^2},\quad \int_{-\infty}^{0} e^x\,dx,\quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

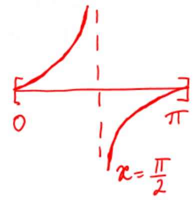


- **Type 2**: Integrands with *vertical asymptotes* within the interval of integration. The vertical asymptotes are called infinite discontinuities

$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_{1}^{2} \frac{dx}{x-1}, \quad \int_{0}^{\pi} \tan x \, dx$$







Improper Integral: Type 1 and Type 2

- **Type 1**: Infinite intervals of integration

$$\int_{1}^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x \, dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

- **Type 2**: Integrands with *vertical asymptotes* within the interval of integration. The vertical asymptotes are called infinite discontinuities

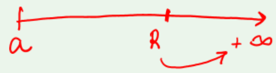
$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_{1}^{2} \frac{dx}{x-1}, \quad \int_{0}^{\pi} \tan x \, dx$$

- There are improper integrals with Type 1 and Type 2 combined

$$\int_{0}^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_{1}^{+\infty} \sec x \, dx$$

Type 1: Improper Integrals over Infinite Interval

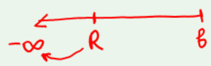
1. If the integral $\int_a^R f(x)\,dx$ exists for all $R > a$, then



$$\int_a^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$

when the limit exists (and is finite).

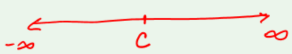
2. If the integral $\int_r^b f(x)\,dx$ exists for all $r < b$, then



$$\int_{-\infty}^{b} f(x)\,dx = \lim_{r\to-\infty} \int_r^b f(x)\,dx$$

when the limit exists (and is finite).

3. If the integral $\int_r^R f(x)\,dx$ exists for all $r < R$, then



$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to-\infty} \int_r^c f(x)\,dx + \lim_{R\to\infty} \int_c^R f(x)\,dx$$

when both limits exist (and are finite). Any c can be used.

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

Example 1

1. If the integral $\int_a^R f(x)\,dx$ exists for all $R > a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$

when the limit exists (and is finite).

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\lim_{R\to\infty} 1 - \lim_{R\to\infty} \frac{1}{R} = 1 - 0$$

$$\int_1^{\infty} \frac{\mathbf{1}}{\boldsymbol{x^2}}\,\boldsymbol{dx} = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \lim_{R\to\infty} \left[-\frac{1}{x}\right]_1^R = \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = 1$$

Finite value

In the case where the limit exists, the improper integral is said to **converge**, and the limit is defined to be the value of the integral.

In the case where the limit does not exist, the improper integral is said to **diverge**, and it is not assigned a value.

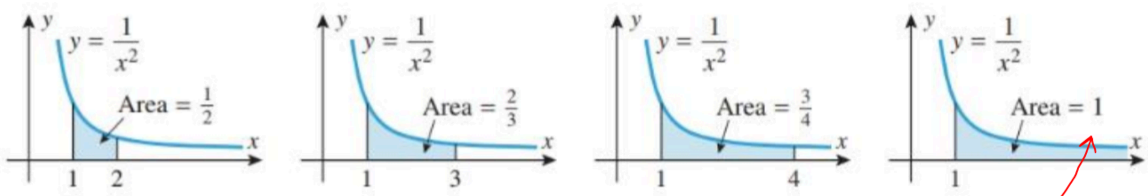
Example 1

Find $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

1. If the integral $\int_a^R f(x)\,dx$ exists for all $R > a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$

when the limit exists (and is finite).



$$\int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}, \qquad \int_1^3 \frac{1}{x^2}\,dx = \frac{2}{3} \qquad \int_1^4 \frac{1}{x^2}\,dx = \frac{3}{4}$$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots,$

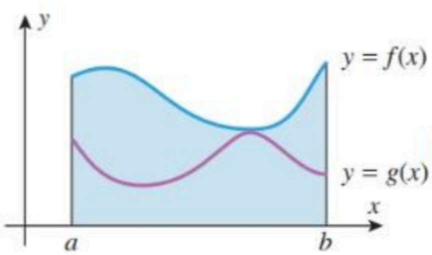
Important Improper Integral

For what values of p does the integral $\displaystyle\int_1^{+\infty} \frac{dx}{x^p}$ converge?

$$\int_1^{+\infty} \frac{dx}{x^p} = \left[0 - \frac{1}{1-p}\right] = \frac{1}{p-1} \quad (p > 1)$$

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & if \quad p > 1 \\ diverges & if \quad p \leq 1 \end{cases}$$

Comparing Functions and their Definite Integrals



Area under $f \geq$ area under g

If f and g are integrable on $[a, b]$, and

$$0 \leq g(x) \leq f(x)$$

then,

$$\mathbf{0} \leq \int_a^b \boldsymbol{g(x)dx} \leq \int_a^b \boldsymbol{f(x)dx}$$

A Comparison Test for Improper Integrals

- Does NOT help in finding the exact value of an improper integral

•DOES tell whether an improper integral is convergent of divergent

Comparison Theorem. Suppose that f and g are continuous on $[a, b]$, and for $x \geq a$

$$0 \leq g(x) \leq f(x)$$

then,

$\int_a^\infty f(x)dx$ is **convergent** $\Longrightarrow$ $\int_a^\infty g(x)dx$ is **convergent**

$\int_a^\infty g(x)dx$ is **divergent** $\Longrightarrow$ $\int_a^\infty f(x)dx$ is **divergent**

Dr. Trefor Bazett explains

Questions 9, 10 in OneNote

Improper Integral: Type 2

Type 2: Integrands with *vertical asymptotes* within the interval of integration.
The vertical asymptotes are called infinite discontinuities

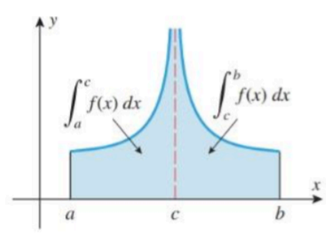
$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_{1}^{2} \frac{dx}{x-1}, \quad \int_{0}^{\pi} \tan x \, dx$$

If the integrant approaches infinity at either limit or at some point between the limits; $\int_a^b f(x)dx$

If $f(x)$ is discontinuous at a $\quad \int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$

If $f(x)$ is discontinuous at b $\quad \int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$

If $f(x)$ is discontinuous at c where $a < c < b$ $\quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$



$\int_a^b f(x)\,dx$ is improper.

Improper Integrals

Type 1
Infinite Intervals

Type 2
Discontinues Integrands

Example: $\displaystyle\int_0^4 \frac{1}{x^2}dx = \lim_{t\to 0^+}\int_t^4 \frac{1}{x^2}dx = \lim_{t\to 0^+}\left[-\frac{1}{x}\right]_t^4 = \lim_{t\to 0^+}\left(-\frac{1}{4}+\frac{1}{t}\right) = +\infty$

The limit does not exist $\Rightarrow$ The integral is *divergent*

Example: $\displaystyle\int_1^4 \frac{1}{1-x}dx = \lim_{t\to 1^+}\int_t^4 \frac{1}{1-x}dx = \lim_{t\to 1^+}[-\ln|1-x|\,]_t^4 = \lim_{t\to 1^+}(-\ln 3 + \ln|1-t|)$

$= -\infty$ The limit does not exist $\Rightarrow$ The integral is *divergent*

Example: $\displaystyle\int_0^3 \frac{1}{x^2-6x+5}dx = \int_0^3 \frac{1}{(x-1)(x-5)}dx$

$$= \int_0^1 \frac{1}{(x-1)(x-5)}dx + \int_1^3 \frac{1}{(x-1)(x-5)}dx$$

$$= \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)(x-5)}dx + \lim_{t\to 1^+}\int_t^3 \frac{1}{(x-1)(x-5)}dx$$