Faculty of Applied Science and Technology

Laboratory 3 Analysis of LTI Systems in the Time Domain

Student Name: Michael McCorkell Student Number: N01500049 Date: February 18th, 2025

Note 1: This is an individual lab. Please bring your laptop and install required software to complete this lab.
Note 2: For your lab report, please take necessary screenshots and proper captions and include them in the corresponding parts of the lab. Demonstrate your work during the lab time. Submit the report file and the related .m files on the course website.

1. Learning outcome:

1.1 Compute zero-input response of an LTI system.
1.2 Conduct impulse response analysis of an LTI system.
1.3 Compute zero-state response of an LTI system through convolution.
1.4 Differentiate between convolution sum and convolution integral.

2. Analysis of LTI Systems in the Time Domain:

In this lab, we are going to analyze and characterize CT and DT LTI systems in time domain. You will be solving system equations to find the zero-input response and the zero-state response of a system. Since convolution is a tool widely used in system analysis, we will explore the applications of convolution in the context of system analysis and characterization.

The total response to an LTI system consists of zero-input response and zero-state response.

2.1 CT LTI system total response

For CT LTI system, if the system equation is expressed as:

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

Using operator notation D to represent derivative $\frac{d}{dt}$, then:

$$(D^N + a_1\ D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_{N-M}\ D^M + b_{N-M+1}\ D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$
$$Q(D)\, y(t) \ =\ P(D) x(t)$$

Then the total response of the system is:

$$y_{total}(t) = \sum_{k=1}^{N} c_k e^{\lambda_k t} + x(t) * h(t)$$

Here, c_k is subject to the initial conditions. $h(t) =\ b_0\, \delta(t) + [P(D)\, y_n\ (t)]u(t)$ is the impulse response of the system. $x(t) * h(t) = \int_{\tau=-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$ is the convolution integral.

2.2 DT LTI system total response

For DT LTI system, if the system equation is expressed as:

$$y[n]+a_1y[n-1]+\cdots+a_{N-1}y[n-N+1]+a_Ny[n-N]=b_0x[n]+b_1x[n-1]+\cdots+b_Nx[n-N]$$

Using operator notation E to represent operation for advancing a sequency by one time unit, then:

$$(E^N+a_1\,E^{N-1}+\cdots+a_{N-1}E+a_N)h[n]=(b_0\,E^N+b_1\,E^{N-1}+\cdots+b_{N-1}E+b_N)\delta[n]$$
$$or,\;\;Q(E)h[n]\;=\;P(E)\delta[n]$$

Then the total response of the system is:

$$y_{total}[n]=\sum_{k=1}^{N}c_k\gamma_k^n+x[n]*h[n]$$

Here, c_k is subject to the initial conditions. $h[n]=\frac{b_N}{a_N}\delta[n]+y_c[n]u[n]$ is the impulse response of the system. $x[n]*h[n]=\sum_{m=-\infty}^{+\infty}x[m]h[n-m]$ is the convolution sum.

3. Procedures

3.1 Characteristic roots and zero-input response of a CT LTI system

Task 1.a (10%). Below is the system equation and corresponding initial conditions of a CT LTI systems:

- $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y(t)=\frac{d^2x}{dt^2}-5x(t)$ with $y(0)=2\;and\;\dot{y}(0)=0.$

Analytically, determine the constants $c_1, c_2, \lambda_1, \lambda_2$ for this second-order systems, which has zero-input response of the form $y_{ZIR}(t)=c_1e^{\lambda_1t}+c_2e^{\lambda_2t}$. Record your answers and the detailed steps below:

$$\lambda^2+2\lambda+5=0$$
$$\lambda=\frac{-2\pm\sqrt{2^2-4(1)(5)}}{2(1)}$$
$$\lambda=\frac{-2\pm\sqrt{4-20}}{2}$$
$$\lambda=\frac{-2\pm\sqrt{-16}}{2}$$
$$\lambda=\frac{-2\pm4j}{2}$$

$$y_{ZIR}(t)=c_1e^{\lambda_1t}+c_2e^{\lambda_2t}$$
$$y_{ZIR}(t)=c_1e^{(-1+2j)t}+c_2e^{(-1-2j)t}$$
$$y_{ZIR}(0)=c_1e^{(-1+2j)0}+c_2e^{(-1-2j)0}$$
$$2=c_1+c_2$$
$$\dot{y}_{ZIR}(0)=c_1(-1+2j)+c_2(-1-2j)$$
$$0=-c_1+2jc_1-c_2-2jc_2$$
$$0=-(c_1+c_2)+2j(c_1-c_2)$$
$$2=2j(c_1-c_2)$$
$$-j=c_1-c_2$$

$$c_1+c_2+c_1-c_2=2-j$$
$$2c_1=2-j$$
$$c_1=\frac{2-j}{2}\rightarrow1-0.5j$$

$$c_1+c_2-c_1+c_2=2+j$$
$$2c_2=2+j$$
$$c_2=\frac{2+j}{2}\rightarrow1+0.5j$$

$\lambda_1=-1+2j$ $\lambda_2=-1-2j$ $c_1=1-0.5j$ $c_2=1+0.5j$

Task 1.b For practice, please repeat task 1.a for the following systems, no need to record your steps or answers here:

- $\frac{d^2y}{dt^2}-5\frac{dy}{dt}+4y(t)=x(t)$ with $y(0)=4\;and\;\dot{y}(0)=5.$
- $\frac{d^2y}{dt^2}+4y(t)=x(t)$ with $y(0)=1\;and\;\dot{y}(0)=0.$
- $\frac{d^2y}{dt^2}+2\frac{dy}{dt}=x(t)$ with $y(0)=1\;and\;\dot{y}(0)=2.$

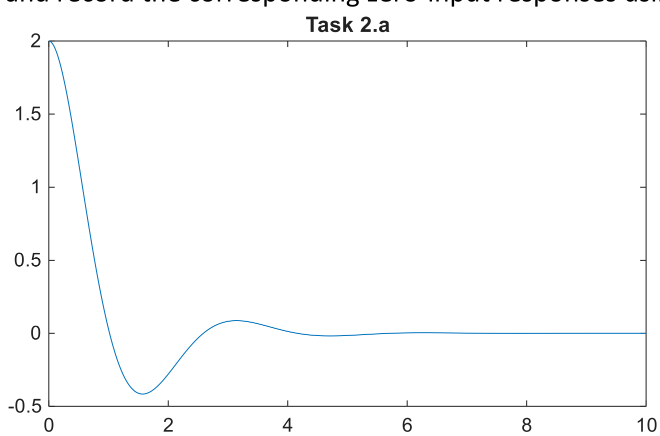
Task 2.a (10%). Use MATLAB script to validate your analytical results for the first system $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y(t) = \frac{d^2x}{dt^2} - 5x(t)$ with $y(0) = 2$ and $\dot{y}(0) = 0$. Use roots() and inv() functions in MATLAB. Plot its zero-input response using MATLAB over $0 \le t \le 10$. Record the entire MATLAB code below.

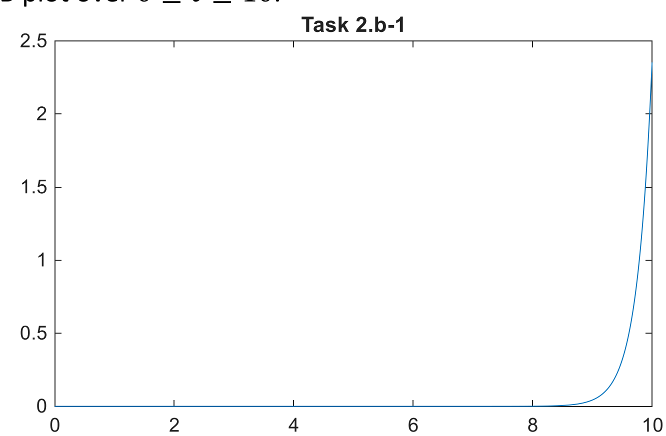
```
lamb = roots([1, 2, 5]);
lamba1 = lamb(1);
lamba2 = lamb(2);
disp("λ1 is " + lamba1 + " and " + "λ2 is " + lamba2);
cm = [1,1; lamba1, lamba2];
rhv = [2; 0];
c = inv(cm) * rhv;
c1 = c(1);
c2 = c(2);
disp("c1 is " + c1 + " and " + "c2 is " + c2);
t = 0:0.01:10;
y_ZIR = c1*exp(lamba1*t)+c2*exp(lamba2*t);
figure; plot(t,y_ZIR,'r','LineWidth',2);
title("Zero-Input Response of the System","Color",[0,0,0]);xlabel("Time (s)","Color",[0,0,0]);ylabel("Zero-Input Response","Color",[0,0,0]);
```

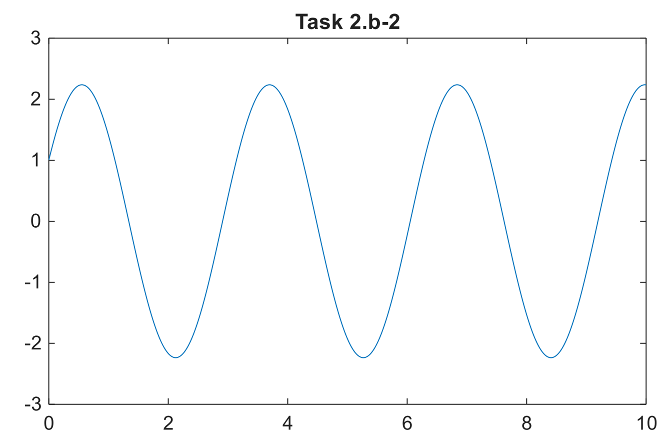
Task 2.b (10%). Please repeat task 2.a for the following systems,

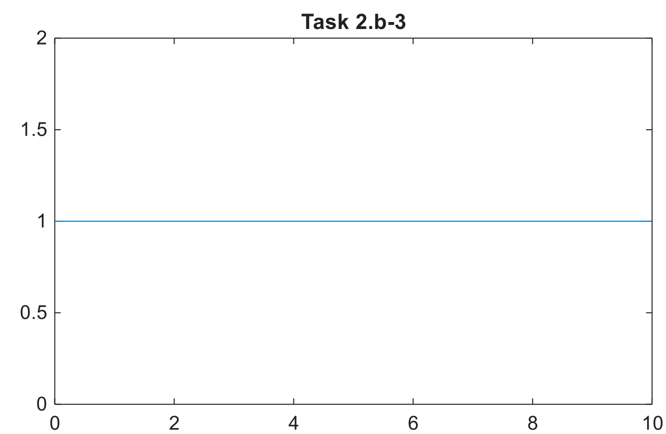
- 1. $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y(t) = x(t)$ with $y(0) = 4$ and $\dot{y}(0) = 5$.
- 2. $\frac{d^2y}{dt^2} + 4y(t) = x(t)$ with $y(0) = 1$ and $\dot{y}(0) = 0$.
- 3. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = x(t)$ with $y(0) = 1$ and $\dot{y}(0) = 2$.

and record the corresponding zero-input responses using MATLAB plot over $0 \le t \le 10$.









3.2 Unit impulse response $h(t)$ of a CT LTI system

In order to find the unit impulse response to a CT LTI system given by the following differential equation:

$$(D^N + a_1\ D^{N-1} + \cdots + a_{N-1}D + a_N)y(t) = (b_{N-M}\ D^M + b_{N-M+1}\ D^{M-1} + \cdots + b_{N-1}D + b_N)x(t)$$

In a practical system $M \leq N$. The unit impulse response is in the format of:

$$h(t) = \ b_0\ \delta(t) + [P(D)\ y_n\ (t)]u(t)$$

where $y_n(t)$ is a linear combination of the characteristic modes of the system subject to the following initial conditions:

$$y_n(0) = \dot{y}_n(0) = \ \ddot{y}_n(0) = \ \ \ldots = \ y_n^{(N-2)}(0) = 0 \qquad and\ \ y_n^{(N-1)}(0) = 1$$

Where $y_n^{(k)}(0)$ is the value of kth derivative of $y_n(t)$ at $t = 0$.

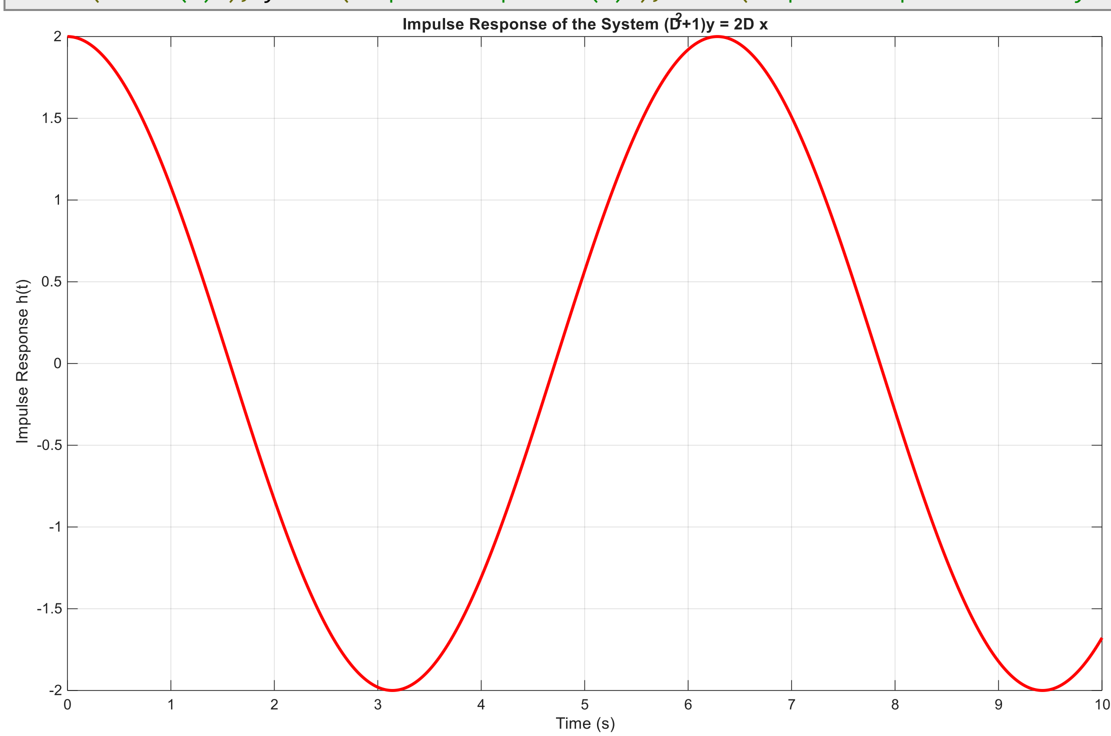
Task 3.a (10%). For each of the CT system below, please determine the characteristic equation, characteristic modes, and the impulse response $h(t)$:

• $(D^2 + 4D + 3)\{y(t)\} = (D + 5)\{x(t)\}$;	• $(D^2 + 2D + 1)\{y(t)\} = D\{x(t)\}$;
$D^2 + 4D + 3 = 0 \rightarrow (D + 3)(D + 1)$ $\lambda_1 = -3 \quad \lambda_2 = -1$ Characteristics are $\longrightarrow e^{-3t}, e^{-t}$ $h(t) = (e^{-t} - e^{-3t})u(t)$	$D^2 + 2D + 1 = 0 \rightarrow (D + 1)(D + 1)$ $\lambda_1 = -1 \quad \lambda_2 = -1$ Characteristics are $\longrightarrow e^{-t}, te^{-t}$ $h(t) = te^{-t}u(t)$

Task 3.b (10%). For the system below, please use MATLAB to determine the impulse response $h(t)$, plot the impulse response over $0 \leq t \leq 10$. Record your code and the screenshots below.

- $(D^2 + 1)\{y(t)\} = 2D\{x(t)\}$;

```
up = [2 0];
down = [1 0 1];
sys = tf(up,down);
[h, t]=impulse(sys, t);
figure; plot(t, h, 'r', 'LineWidth', 2); grid on;
xlabel('Time (s)'); ylabel('Impulse Response h(t)'); title('Impulse Response of the System (D^2+1)y = 2D x');
```



3.3 Zero-state response of an LTI system

Task 4 (10%). The unit impulse response of an LTI system is given below:

$h(t) = e^{-t}u(t)$

Use the convolution table in your textbook (table 2.1, pg. 176) or graphical method to find the zero-state response $y(t)$ of this system if the input $x(t)$ is:

a. $u(t)$
b. $e^{-t}u(t)$
c. $e^{-2t}u(t)$

a. $(1-e^{-t})u(t)$ b. $te^{-t}u(t)$ c. $(e^{-t}-e^{-2t})u(t)$

Note: if you use graphical method when solving for $y_{ZSR}(t)$, flip and shift $h(t)$ and explicitly show all integration steps.

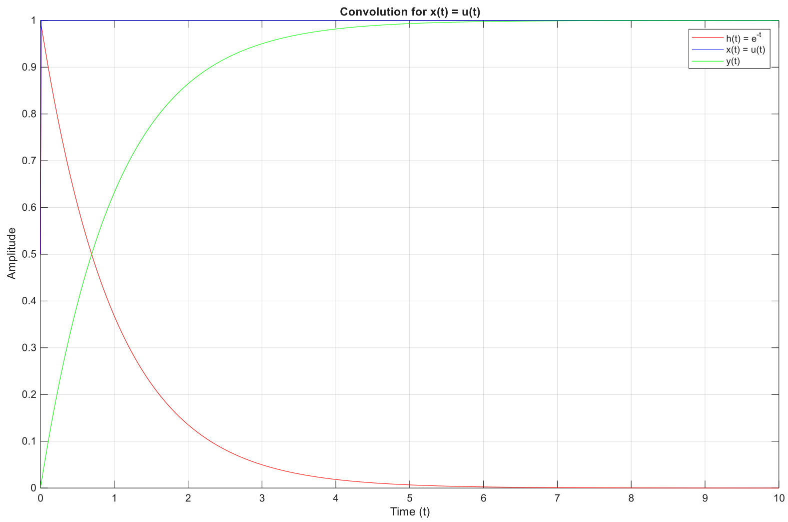
Task 5 (15%). In MATLAB, analytical integral can be implemented using int() from Symbolic Math Toolbox. Below is a code example of using symbolic integral function int() to compute the convolution of $x(t) = e^{-t}$ and $h(t) = \sin t$.

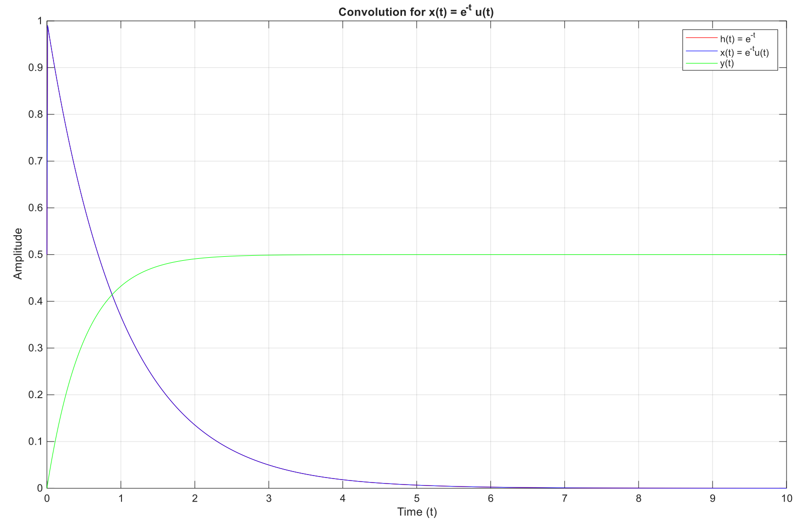
```
% Define the symbolic functions h(t) and x(t)
h = exp(-tau);          % Unit impulse response h(t) = e^(-t)
x = sin(t - tau);       % Input signal x(t) = sin(t - tau) for convolution

% Define the convolution integral for the zero-state response
y = int(h * x, tau, 0, t);  % Convolution: h(t) * x(t)
```

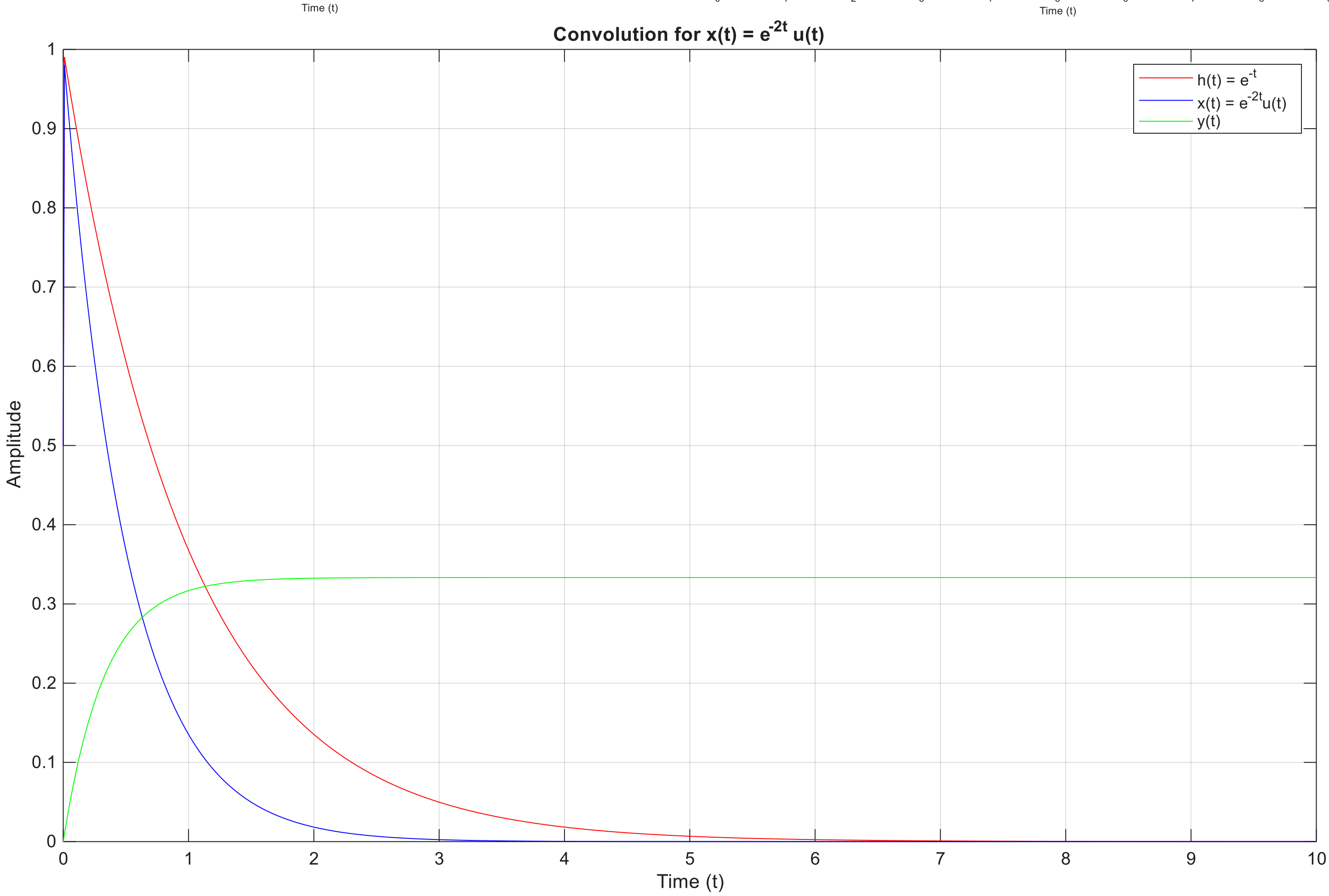
Based on the given information, please use MATLAB to calculate the convolution integral to find the zero-state response $y(t)$. Overlay the corresponding unit impulse response $h(t)$, the input signal $x(t)$ and the zero-state response $y(t)$ in one plot for each case. Record your code and the plots below.

```
syms t tau
t_vals = 0:1:100;
ui = exp(-tau);

c1 = heaviside(t-tau); %case 1
c2 = exp(-tau)*heaviside(t-tau); %case 2
c3 = exp(-2*tau)*heaviside(t-tau); %case 3
%integrals
int1 = int(ui * c1, tau, 0, t);
int2 = int(ui * c2, tau, 0, t);
int3 = int(ui * c3, tau, 0, t);

f_ui = matlabFunction(exp(-t).*heaviside(t));
f_c1 = matlabFunction(heaviside(t));
f_c2 = matlabFunction(exp(-t).*heaviside(t));
f_c3 = matlabFunction(exp(-2*t).*heaviside(t));
f_y1 = matlabFunction(int1);
f_y2 = matlabFunction(int2);
f_y3 = matlabFunction(int3);

ui_vals = f_ui(t_vals);
c1_vals = f_c1(t_vals);
c2_vals = f_c2(t_vals);
c3_vals = f_c3(t_vals);
y1_vals = f_y1(t_vals);
y2_vals = f_y2(t_vals);
y3_vals = f_y3(t_vals);


figure;
plot(t_vals,ui_vals,'r',t_vals,c1_vals,'b',t_vals,y1_vals,'g');
legend('h(t) = e^{-t}', 'x(t) = u(t)', 'y(t)');
title('Convolution for x(t) = u(t)'); xlabel('Time (t)'); ylabel('Amplitude');
```

```
grid on;
figure;
plot(t_vals,ui_vals,'r',t_vals,c2_vals,'b',t_vals,y2_vals,'g');
legend('h(t) = e^{-t}', 'x(t) = e^{-t}u(t)', 'y(t)');
title('Convolution for x(t) = e^{-t} u(t)'); xlabel('Time (t)'); ylabel('Amplitude');
grid on;
figure;
plot(t_vals,ui_vals,'r',t_vals,c3_vals,'b',t_vals,y3_vals,'g');
legend('h(t) = e^{-t}', 'x(t) = e^{-2t}u(t)', 'y(t)');
title('Convolution for x(t) = e^{-2t} u(t)'); xlabel('Time (t)'); ylabel('Amplitude');
grid on;
```







3.4 DT LTI system analysis

3.4.1 Zero-input response of a DT LTI system

Task 6.a (10%). Below is the system equation and corresponding initial conditions of a DT LTI systems:

- $y[n] - 0.6y[n-1] + 0.05y[n-2] = x[n-1] + x[n]$ with $y[-1] = 1 \; and \; y[-2] = 42.0$.

Analytically, determine the constants $c_1, c_2, \gamma_1, \gamma_2$ for this second-order systems, which has zero-input response of the form $y_{ZIR}[n] = c_1\gamma_1^n + c_2\gamma_2^n$. Record your answers and the detailed steps below.

$r^2 - 0.6r + 0.05 = 0 \quad r = \frac{0.6 \pm \sqrt{(0.6)^2 - 4(0.05)}}{2} \quad r = \frac{0.6 \pm \sqrt{0.16}}{2} \quad r = \frac{0.6 \pm 0.4}{2}$

$\gamma_1 = \dfrac{0.6 + 0.4}{2} \quad \gamma_2 = \dfrac{0.6 - 0.4}{2} \qquad \gamma_1 = 0.5 \quad \gamma_2 = 0.1$

$y_{ZIR}[n] = c_1(0.5)^n + c_2(0.1)^n$

<table>
<tr><td colspan="2">$1 = c_1(0.5)^{-1} + c_2(0.1)^{-1}$
$1 = 2c_1 + 10c_2 \;\rightarrow\; 2 = 4c_1 + 20c_2$

$2c_1 + 10(0.5) = 1 \rightarrow 2c_1 + 5 = 1$
$2c_1 = -4$
$c_1 = -2$</td><td colspan="2">$42 = c_1(0.5)^{-2} + c_2(0.1)^{-2}$
$42 = 4c_1 + 100c_2$

$42 - 2 = (4c_1 + 20c_2) - (4c_1 + 100c_2)$
$80c_2 = 40$
$c_2 = 0.5$</td></tr>
<tr><td>$\gamma_1 = 0.5$</td><td>$\gamma_2 = 0.1$</td><td>$c_2 = 0.5$</td><td>$c_1 = -2$</td></tr>
</table>

Task 6.b (5%). Please use MATLAB to confirm your answers in Task 6.a. Note, you can modify the MATLAB code for CT LTI zero-input response.

```
yn = [1, -0.6, 0.05];
roots_yn = roots(yn);
A = [2, 10; 4, 100];
B = [1; 42];
C = A \ B;

% Display results
disp('Characteristic Roots (γ1, γ2):');
disp(roots_yn);

disp('Constants (c1, c2):');
disp(C);
```

3.4.2 Impulse response of a DT LTI system

As is shown in the previous tasks, computing the zero-input response of a DT system is very similar to its CT counterpart. You can now review the course lectures to see how the computations of a zero-state response of a DT system is computed. It requires the following two steps:

Step 1. Find the unit impulse response to the system.

Step 2. Conduct convolution sum of the unit impulse response and the input signal.

For DT LTI system, MATLAB has a built-in function impz() to directly compute the DT impulse response. Below is an example of using impz():

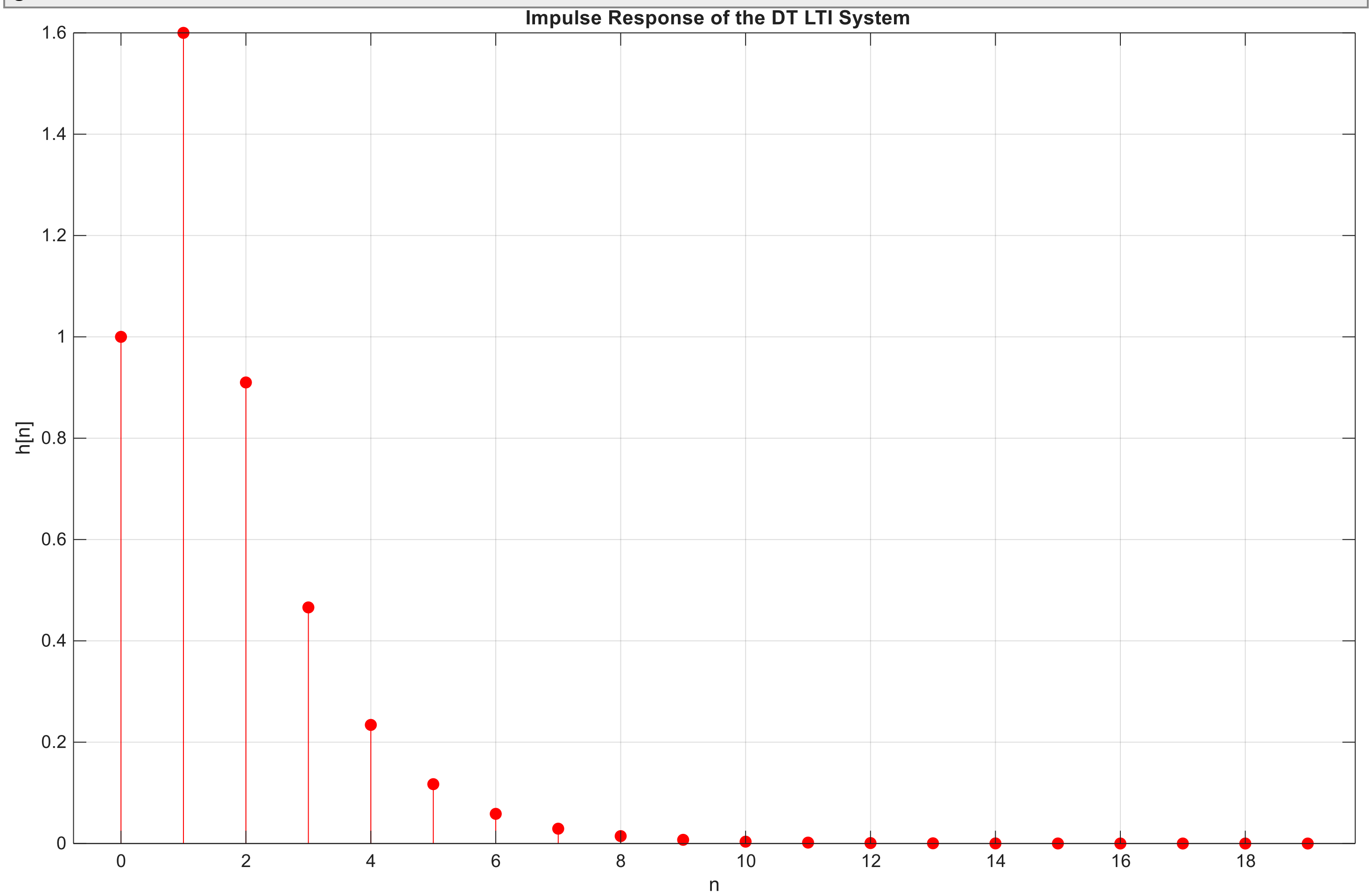
```
b = [1];              % input coefficients vector
a = [1 -0.5 -0.25]; % output coefficients (second-order)
N = 20;               % Number of impulse response samples
% Compute the impulse response using impz
h = impz(b, a, N);
% Plot the impulse response
stem(0:N-1, h, 'filled'); xlabel('n'); ylabel('h[n]'); grid on;
```

Task 7 (5%). Based on the information above, please use MATLAB to compute the impulse response of the system defined in section 3.4.1:

- $y[n] - 0.6y[n-1] + 0.05y[n-2] = x[n-1] + x[n]$ with $y[-1] = 1$ *and* $y[-2] = 42.0.$

Plot the first 20 samples of the impulse response, record your code and plot below.

```
yn = [1, -0.6, 0.05];
xn = [1 1];
N = 20;
h = impz(xn, yn, N);

% Plot the impulse response
figure;
stem(0:N-1, h, 'filled', 'r'); xlabel('n'); ylabel('h[n]');
title('Impulse Response of the DT LTI System');
grid on;
```



3.4.3 Zero-state response of a DT LTI system

Convolution sum calculation

When analysing DT LTI system, the convolution sum is a critical tool. Unlike its CT counterpart, convolution sum is MATLAB uses built-in function conv() to compute the convolution sum. The syntax is y = conv(x,h) where x and h are vectors representing the numerical values of the DT signals. y is the output vector containing the values of the convolution sum of x*h.

MATLAB is not able to compute an infinite sum (constraints imposed by the limited memory and computation capacity). In practice, MATLAB convolve time-limited signals. conv() use a parameter *shape* to determine whether only part of the convolution will be kept.

'full'	Full convolution (default).
'same'	Central part of the convolution of the same size as u.
'valid'	Only those parts of the convolution that are computed without the zero-padded edges. Using this option, length(w) is max(length(u)-length(v)+1,0), except when length(v) is zero. If length(v) = 0, then length(w) = length(u).

Table 1. Choice of Parameter 'shape'

Let $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$, and $h[n]$ to be the solution of section 3.4.2, here $\delta[n]$ is the unit impulse sequence.

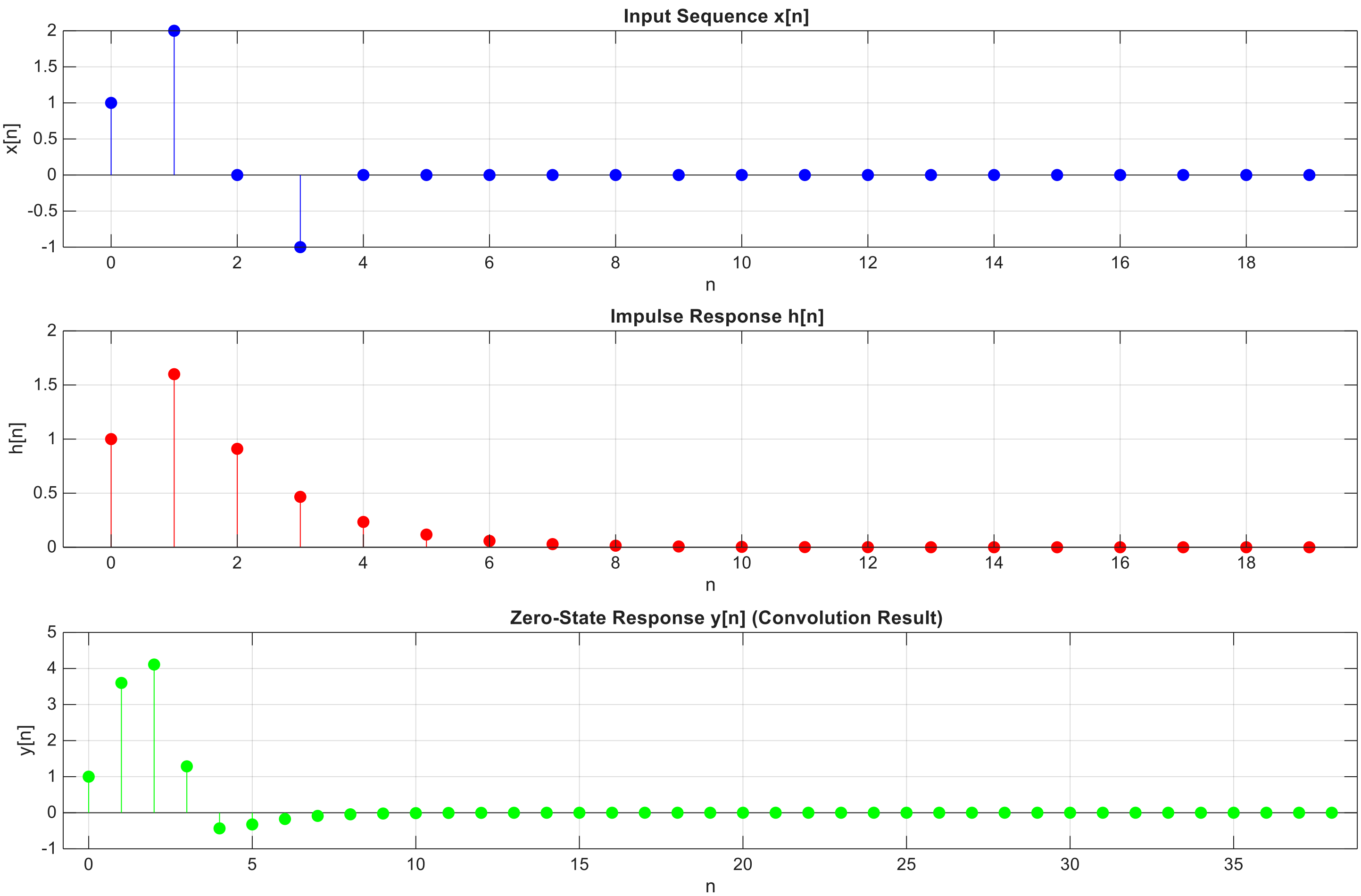
Task 8 (5%). Create input signal $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$ in MATLAB and plot the input sequence, impulse response sequence $h[n]$ and the output sequence $y[n]$. Use conv() to compute and plot the convolution of these two sequences. Record the code and the plotted convolution result in the space below:

Note 1: please note that unit impulse sequence $\delta[n]$ and unit impulse function $\delta(t)$ are defined differently in MATLAB.

Note 2: please use stem() to plot discrete time sequence for clarity.

Note 3: please don't overlay these plots, instead, use three subplot to create three panels for better visualization.

```
N = 20;
x = zeros(1, N);
x(1) = 1;  % δ[n]
x(2) = 2;  % 2δ[n-1]
x(4) = -1; % -δ[n-3]
xn = [1 1];
yn = [1 -0.6 0.05];
h = impz(xn, yn, N);
y = conv(x, h, 'full');
n_x = 0:N-1;
n_h = 0:N-1;
n_y = 0:length(y)-1;

figure;
subplot(3,1,1); stem(n_x, x, 'filled', 'b');
xlabel('n'); ylabel('x[n]'); title('Input Sequence x[n]');
grid on;

subplot(3,1,2); stem(n_h, h, 'filled', 'r');
xlabel('n'); ylabel('h[n]'); title('Impulse Response h[n]');
grid on;

subplot(3,1,3);
stem(n_y, y, 'filled', 'g'); xlabel('n'); ylabel('y[n]');
title('Zero-State Response y[n] (Convolution Result)');
grid on;
```


Input Sequence x[n]
x[n]
n
Impulse Response h[n]
h[n]
n
Zero-State Response y[n] (Convolution Result)
y[n]
n