# HUMBER ENGINEERING

## MENG-3020
## SYSTEMS MODELING & SIMULATION

## LECTURE 4

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# LECTURE 4
# Transfer Function Approach to Modeling Dynamic Systems

- Standard Forms of System Models
- Transfer Function Model
    - Transfer Function Properties
    - Block Diagram Transfer Function
- State-space Equations from Transfer Function

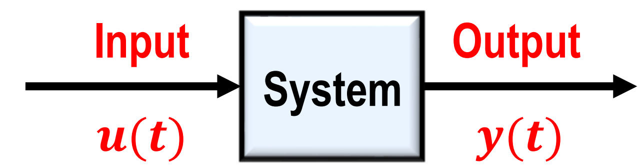
# Standard Forms of System Models

- In the previous lecture, we introduced the standard forms of dynamic system models
- The two most common standard forms for dynamic system models:
  - **State-Space Model**
    - Provides a model in time domain, and internal description of the system via state variables
    - Applicable for time-varying, nonlinear and MIMO systems
  - **Transfer Function Model**
    - Provides a model in Laplace domain, and input-output description (External description)
    - Limited to linear and time-invariant (LTI) and mostly applicable to SISO or two-input two-output systems

- **Linear Systems**
  - We will assume that systems that we are dealing with are **linear** systems.
  - **Property of Linear Systems (Superposition Principle):**
    - If a linear system for a particular input $x_1$ gives an output of $y_1$ and for another input $x_2$ gives an output of $y_2$ that the sum of those two inputs $ax_1 + bx_2$ would give an output of $ay_1 + by_2$.
  - In the real world, most physical systems can only be said to approximate to linear so assuming linearity means we are dealing with idealized systems, but to do so does make the calculation easier and give generally a good approximation to what the real-world answer would be.

# Laplace Transform Properties Review

- Laplace Transform converts a time-domain function to $s$-domain. $\boxed{\mathcal{L}\{f(t)\} = F(s)}$
- Input-output variable of physical dynamic systems are function of time and are showing with lower case letters. For example, input $u(t)$ and output $y(t)$
- Laplace transform of the variables are function of $s$ and are showing with upper case letters. For example, input $U(s)$ and output $Y(s)$

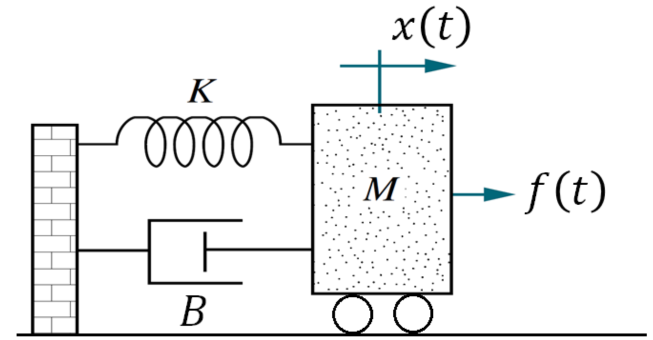



- **Linearity Property**
  - $f_1(t) \pm f_2(t) \quad \Leftrightarrow \quad F_1(s) \pm F_2(s)$
  - $kf(t) \quad \Leftrightarrow \quad kF(s)$
- For example:
  $2v(t) + 5u(t) \quad \Leftrightarrow \quad 2V(s) + 5U(s)$

- **Differentiation and Integration**
  - $\frac{df(t)}{dt}$ or $f'(t)$ or $\dot{f}(t) \quad \Leftrightarrow \quad sF(s) - f(0)$
  - $\frac{d^2f(t)}{dt^2}$ or $f''(t)$ or $\ddot{f}(t) \quad \Leftrightarrow \quad s^2F(s) - sf(0) - f'(0)$
  - $\int_0^t f(\mathrm{t})dt \quad \Leftrightarrow \quad \frac{1}{s}F(s)$
- For example:
  $2v'(t) - 3v(t) \quad \Leftrightarrow \quad 2\big(sV(s) - v(0)\big) - 3V(s)$

# Input-Output Relationship in Laplace Domain

- Consider a mass-spring-damper system that is represented by the following second-order differential equation.
- Assume that the applied force $f(t)$ is the input, and the displacement of the mass $x(t)$ is the output. $M = 50kg, B = 30\ N.s/m, K = 100\ N/m$




$$f(t) = 50x''(t) + 30x'(t) + 100x(t)$$

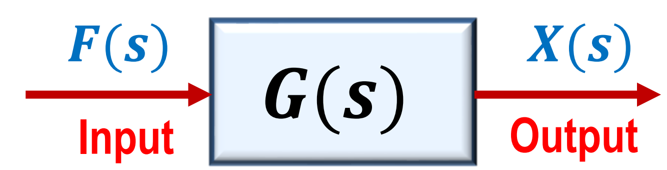
- We can find the input-output relationship in Laplace domain by taking Laplace transform of the differential equation and assuming zero initial conditions ($x(0) = 0, \quad x'(0) = 0$):

$$F(s) = 50s^2X(s) + 30sX(s) + 100X(s) \quad \rightarrow \quad F(s) = (50s^2 + 30s + 100)X(s)$$

$$\rightarrow \quad X(s) = \underbrace{\left(\frac{1}{50s^2 + 30s + 100}\right)}_{G(s)} F(s)$$

**Transfer Function**

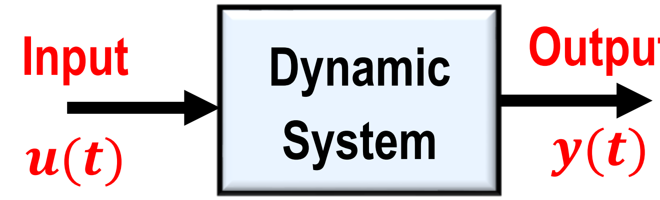
$$\boxed{X(s) = G(s)F(s)}$$




$$\boxed{\text{Output} = G(s) \times \text{Input}}$$

# Transfer Function Model

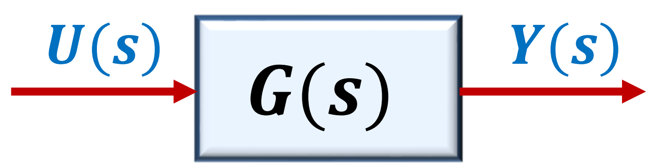
- Consider a dynamic system with input $u(t)$ and output $y(t)$
- In general, **linear** systems can be modeled by ordinary differential equations (ODE).




- Transfer function describes the **<u>input-output relationship</u>** of a linear system as a function of $s$ in Laplace domain when **all initial conditions before applying the input are zero.**

**Transfer Function** $\boxed{G(s) \triangleq \dfrac{Y(s)}{U(s)}}$

- A transfer function can be represented as a block diagram with input $U(s)$ and output $Y(s)$ and the transfer function $G(s)$ as the operator in the box that converts the input to the output.




- Having a transfer function model of a system, we can <u>find output of the system</u> for any input, <u>analyze performance</u> of the system and <u>design a controller</u>.

$$\boxed{Y(s) = G(s)U(s)}$$

# Transfer Function Properties

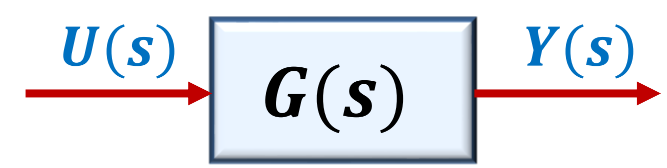
- ✓ The applicability of the concept of the transfer function is limited to **LTI (Ordinary Differential Equation)** systems.

- ✓ The transfer function provides an external description, input-output relation; however, it does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical – analogues systems)

- ✓ The transfer function is a **property** of a system itself, unrelated to the magnitude and nature of the input or driving function.

- ✓ If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.

$$\boxed{Y(s) = G(s)U(s)}$$

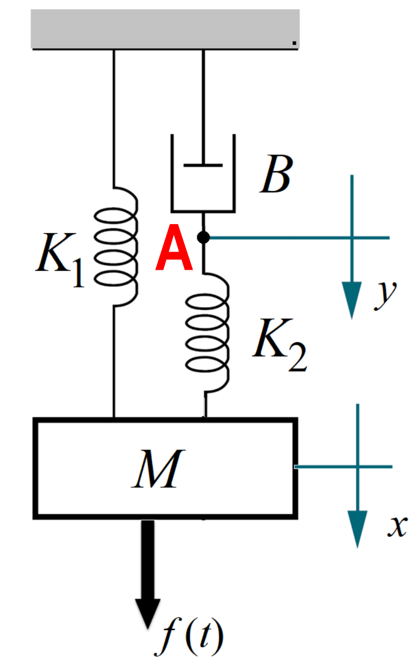



- ✓ If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system.

- ✓ If the transfer function of a system is known, it can be determined the differential equation model and the state-space model of the system.

# Transfer Function Model

**Example 1** Consider the following mechanical system. The system is at rest initially. The displacements $x$ and $y$ are measured from their respective equilibrium positions.




Assume that the applied force $f(t)$ is the input, and the displacement of the mass $x(t)$ and the displacement of junction A $y(t)$ are the outputs.

Determine the transfer function model of the system.

$$G_1(s) = \frac{X(s)}{F(s)} \qquad G_2(s) = \frac{Y(s)}{F(s)}$$

The equations of motion for the system are:

$$Mass\ M \ \rightarrow \ f(t) - K_1x - K_2(x - y) = M\ddot{x}$$

$$Junction\ A \ \rightarrow \ K_2(x - y) = B\dot{y}$$

Take Laplace transform of these two equations, assuming zero initial conditions.

$$F(s) - K_1X(s) - K_2\big(X(s) - Y(s)\big) = Ms^2X(s) \ \rightarrow \ F(s) = (Ms^2 + K_1 + K_2)X(s) - K_2Y(s) \qquad Eqn.1$$

$$K_2\big(X(s) - Y(s)\big) = BsY(s) \ \rightarrow \ K_2X(s) = (Bs + K_2)Y(s) \qquad Eqn.2$$

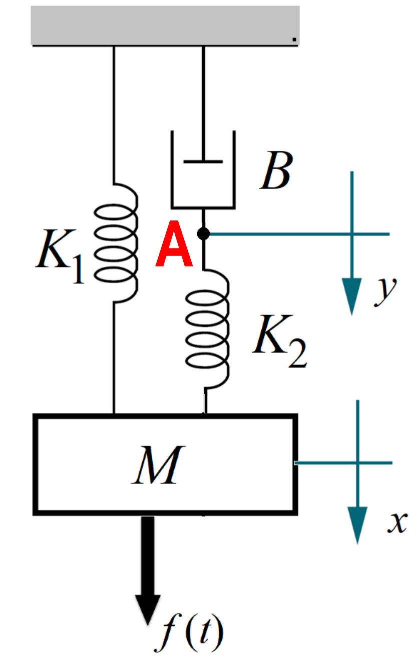
To obtain the transfer function $G_1(s)$, solve Eqn. 2 for $Y(s)$ and substitute the result into Eqn. 1.

From Eqn 2 $\rightarrow \ Y(s) = \dfrac{K_2}{Bs + K_2}X(s)$

Substitute into Eqn 1 $\rightarrow \ F(s) = (Ms^2 + K_1 + K_2)X(s) - \dfrac{K_2^2}{Bs + K_2}X(s)$

# Transfer Function Model

**Example 1** Consider the following mechanical system. The system is at rest initially. The displacements $x$ and $y$ are measured from their respective equilibrium positions.




Assume that the applied force $f(t)$ is the input, and the displacement of the mass $x(t)$ and the displacement of junction A $y(t)$ are the outputs.

Determine the transfer function model of the system. $G_1(s) = \frac{X(s)}{F(s)}$ $\quad G_2(s) = \frac{Y(s)}{F(s)}$

$$F(s) = (Ms^2 + K_1 + K_2)X(s) - \frac{K_2^2}{Bs + K_2}X(s)$$

$$(Bs + K_2)F(s) = (Bs + K_2)(Ms^2 + K_1 + K_2)X(s) - K_2^2 X(s)$$

$$(Bs + K_2)F(s) = (MBs^3 + MK_2s^2 + (K_1 + K_2)Bs + K_1K_2)X(s)$$

$$\boxed{\frac{X(s)}{F(s)} = \frac{Bs + K_2}{MBs^3 + MK_2s^2 + (K_1 + K_2)Bs + K_1K_2}}$$

**Transfer Function $G_1(s)$**

To obtain the transfer function $G_2(s)$, solve Eqn. 2 for $X(s)$ and substitute the transfer function $G_1(s)$.

From Eqn 2 $\rightarrow$ $X(s) = \frac{Bs + K_2}{K_2}Y(s)$

$$\left(\frac{Bs + K_2}{K_2}\right)\frac{Y(s)}{F(s)} = \frac{Bs + K_2}{MBs^3 + MK_2s^2 + (K_1 + K_2)Bs + K_1K_2}$$

$$\boxed{\frac{Y(s)}{F(s)} = \frac{K_2}{MBs^3 + MK_2s^2 + (K_1 + K_2)Bs + K_1K_2}}$$

**Transfer Function $G_2(s)$**

# Transfer Function & ODE Equivalence

- It is important to realize that the transfer function is equivalent to the ordinary differential equation (ODE).
  - For example, the following transfer function corresponds to the given equation:

$$\frac{X(s)}{F(s)} = \frac{0.4s + 4}{0.04s^3 + 0.4s^2 + 4s + 24} \quad \Longleftrightarrow \quad 0.04\dddot{x}(t) + 0.4\ddot{x}(t) + 4\dot{x}(t) + 24x(t) = 0.4\dot{f}(t) + 4f(t)$$

- Denominator of a transfer function is the characteristic polynomial.
- The highest power of the denominator polynomial is called the system order.
- The roots of the characteristic equation are the characteristic roots, which are also called **poles**.
- The roots of the numerator are called the **zeros**.
  - For example, in the previous transfer function, the characteristic polynomial is: $0.04s^3 + 0.4s^2 + 4s + 24$
  - The system is third order.
  - The poles are: $s_{1,2} = -1.3 \pm j8.9, \quad s_3 = -7.4$
  - The zero is: $s = -10$

# Transfer Function Model with MATLAB

- We can <u>create</u> the Transfer Function model of a continuous-time system with **MATLAB** by using the following command:

```
sys = tf(num,den)
```

**Transfer function model**

**Coefficients of the numerator and denominator polynomials in decreasing power of *s***

- We can determine the **poles** and **zeros** of a transfer function model:

```
p = pole(sys)
```

```
z = zero(sys)
```

- For example, create the given transfer function model in MATLAB and find the poles and zeros.

$$G(s) = \frac{s^2 + 5s + 6}{s^3 + s^2 + 4s + 4}$$

Poles $\rightarrow$ $s^3 + s^2 + 4s + 4 = 0 \;\; \rightarrow \;\; s_{1,2} = \pm 2j, \;\; s_3 = -1$

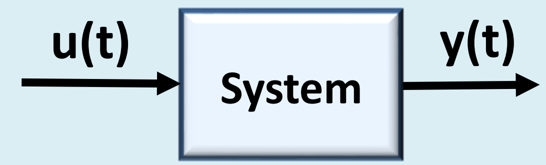
Zeros $\rightarrow$ $s^2 + 5s + 6 = 0 \;\; \rightarrow \;\; s_1 = -3, \; s_2 = -2$

```
num = [1 5 6];
den = [1 1 4 4];
G = tf(num,den)

G =
      s^2 + 5 s + 6
   -------------------
   s^3 + s^2 + 4 s + 4

p = pole(G)

p =
    0.0000 + 2.0000i
    0.0000 - 2.0000i
   -1.0000 + 0.0000i

z = zero(G)

z =
   -3.0000
   -2.0000
```

# Quick Review

1. Find the transfer function model of a linear system with the following differential equation?

$$\boldsymbol{y''(t) + 4y'(t) + 2y(t) = u'(t) + 3u(t)}$$



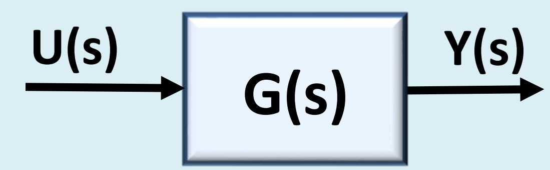

   a) $G(s) = \frac{s+3}{s^2+4s+2}$

   b) $G(s) = \frac{s^2+3}{s^2+2s+4}$

   c) $G(s) = \frac{s^2+4s+2}{s+3}$

   d) $G(s) = \frac{3s+1}{s^2+6s}$

2. Determine which system can be represented by the following transfer function model.




$$\boldsymbol{G(s) = \frac{Y(s)}{U(s)} = \frac{2s+5}{2s^3+5s+3}}$$

   a) $2y'(t) + 5y(t) + 3 = 2u(t) + 5$

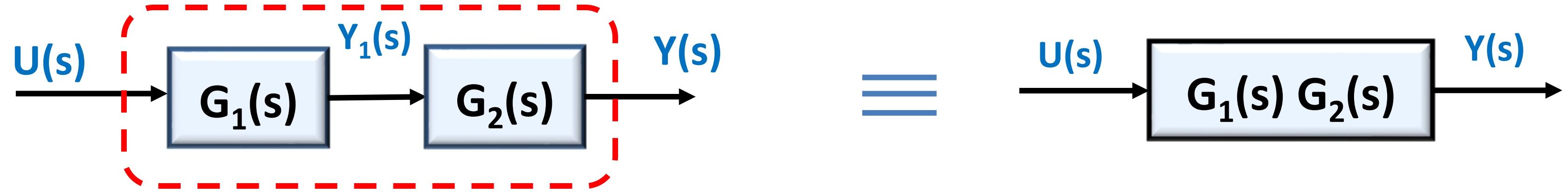
   b) $2y'''(t) + 5y'(t) + 3y(t) = 2u'(t) + 5u(t)$

   c) $2y''(t) + 5y(t) = 2u'(t) + 5u(t) + 3$

   d) $2y'(t) + 5y(t) = 2u''(t) + 5u'(t) + 3u(t)$

# Systems Transfer Function

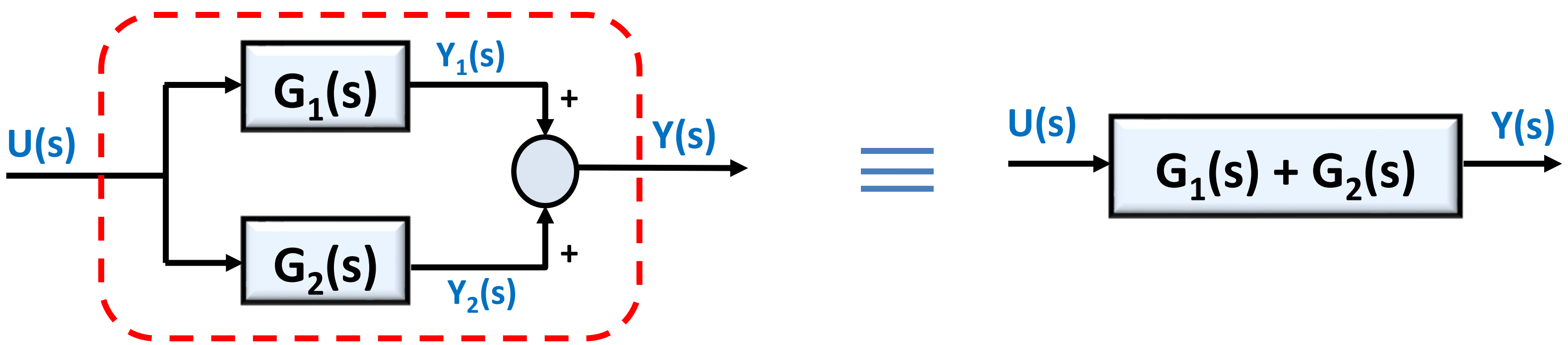
- **Series Connection:** The overall transfer function is equivalent to product of the individual systems transfer function.




$$\boxed{\frac{Y(s)}{U(s)} = G_1(s)G_2(s)}$$

$$Y(s) = G_2(s)Y_1(s) = G_2(s)[G_1(s)U(s)] = \underbrace{[G_1(s)G_2(s)]}_{overall\ TF} U(s)$$

- **Parallel Connection:** The overall transfer function is equivalent to summation of the individual systems transfer function.

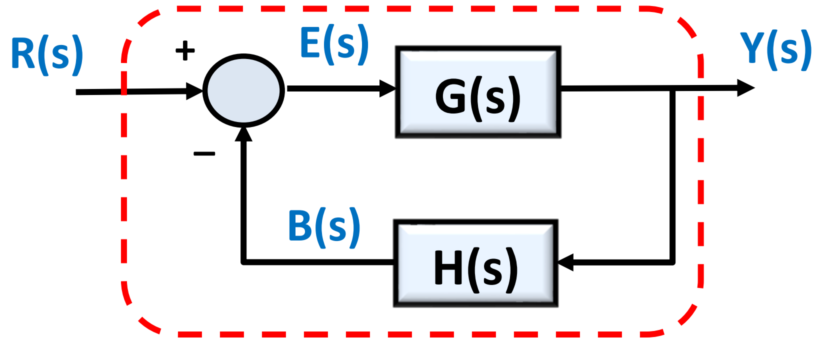


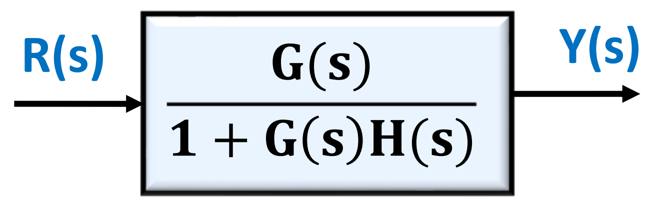

$$\boxed{\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)}$$

$$Y(s) = Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)U(s) = \underbrace{[G_1(s) + G_2(s)]}_{overall\ TF} U(s)$$

# Systems Transfer Function

- **Feedback Connection (Negative Feedback):** The overall transfer function is determined as below




$$Y(s) = G(s)E(s)$$

$$Y(s) = G(s)[R(s) - B(s)]$$

$$Y(s) = G(s)[R(s) - H(s)Y(s)]$$

$$Y(s) = G(s)R(s) - G(s)H(s)Y(s)$$

$$(1 + G(s)H(s))Y(s) = G(s)R(s)$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}}$$

Overall Closed-loop transfer function

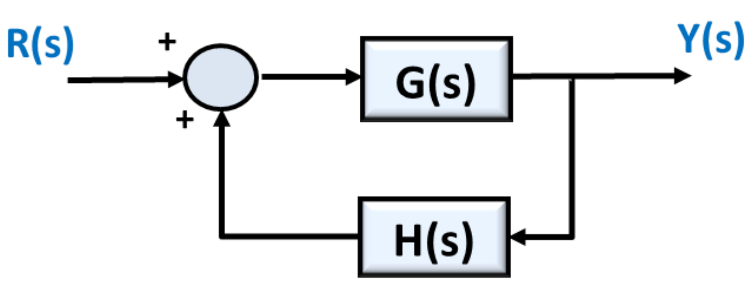
$R(s)$ : Input signal

$Y(s)$ : Output signal

$G(s)$ : Overall Forward-path transfer function

$H(s)$ : Overall Feedback transfer function

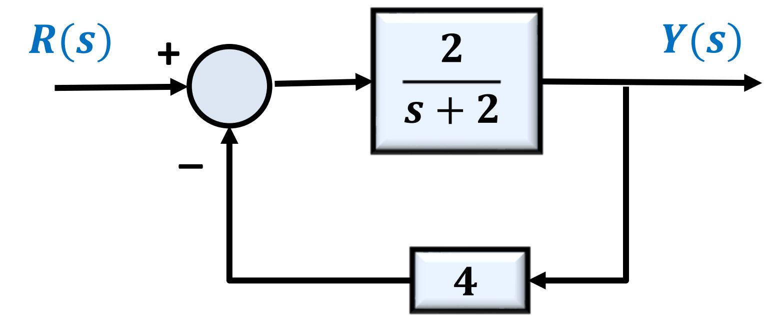
- **Feedback Connection (Positive Feedback):** The overall transfer function is determined as below



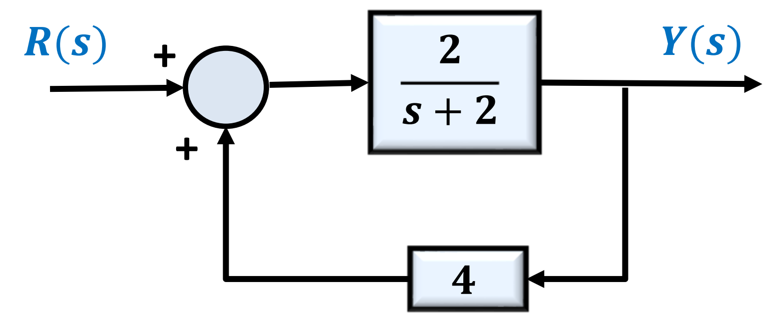

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}}$$

# Systems Transfer Function

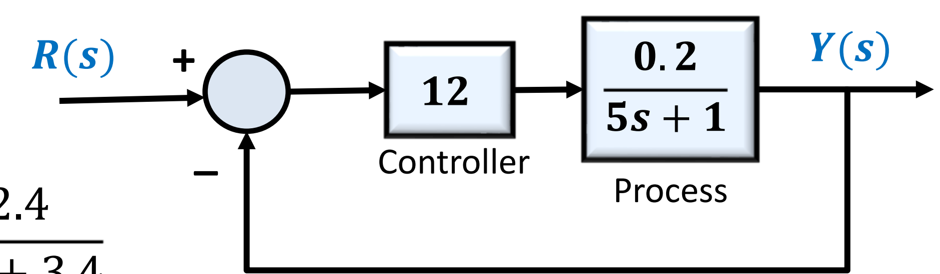
**Example 2** Determine the overall transfer function from Y(s) to R(s) for the control systems which have shown below.




$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{\frac{2}{s+2}}{1+\left(\frac{2}{s+2}\right)(4)}=\frac{\frac{2}{s+2}}{1+\frac{8}{s+2}}=\frac{\frac{2}{s+2}}{\frac{s+2+8}{s+2}}=\frac{2}{s+10}$$



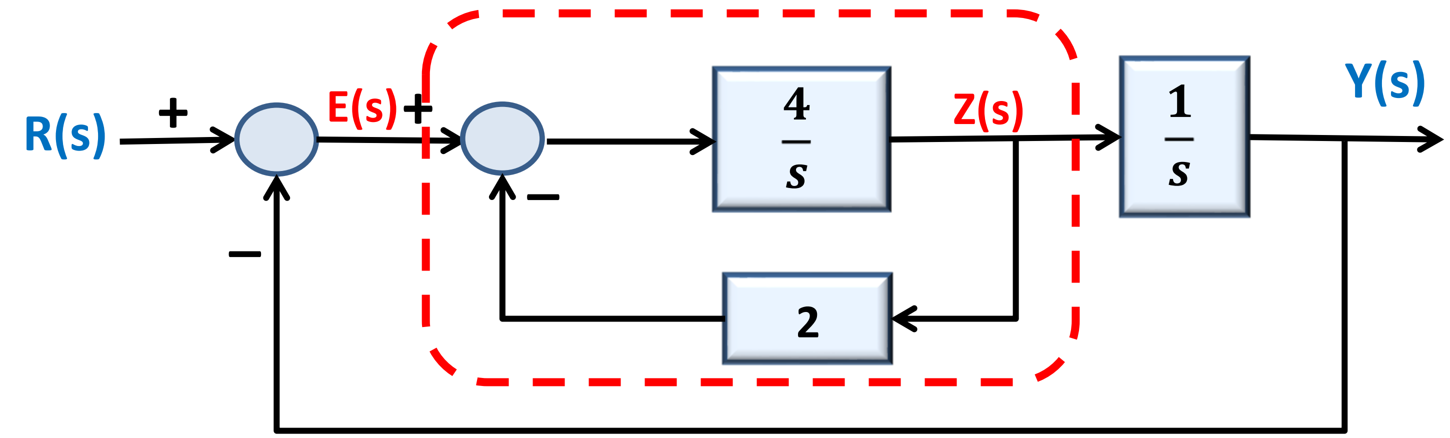

$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1-G(s)H(s)}=\frac{\frac{2}{s+2}}{1-\left(\frac{2}{s+2}\right)(4)}=\frac{\frac{2}{s+2}}{1-\frac{8}{s+2}}=\frac{\frac{2}{s+2}}{\frac{s+2-8}{s+2}}=\frac{2}{s-6}$$




$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{(12)\left(\frac{0.2}{5s+1}\right)}{1+(12)\left(\frac{0.2}{5s+1}\right)(1)}=\frac{\frac{2.4}{5s+1}}{1+\frac{2.4}{5s+1}}=\frac{\frac{2.4}{5s+1}}{\frac{5s+1+2.4}{5s+1}}=\frac{2.4}{5s+3.4}$$

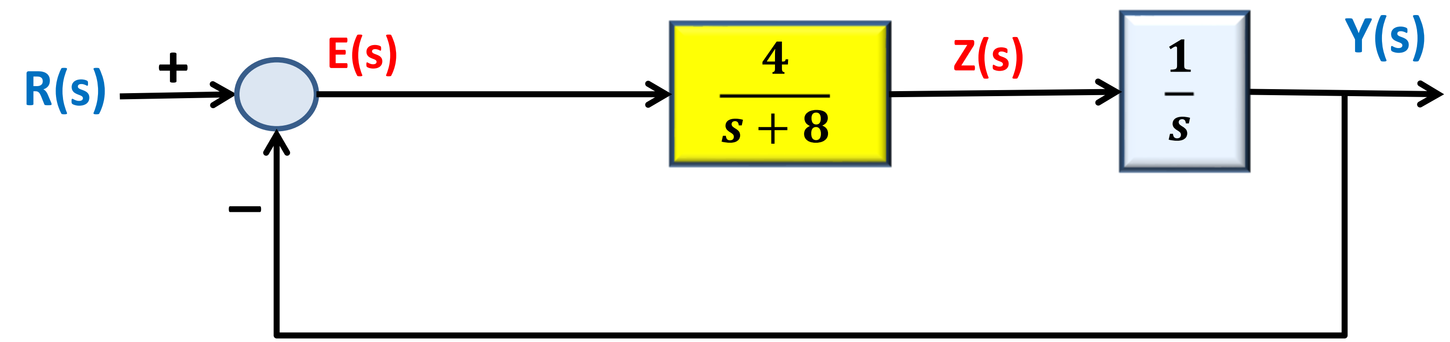
# Systems Transfer Function

**Example 3** Determine the overall transfer function Y(s) to R(s) for the following control system.




First determine the transfer function of internal feedback loop from Z(s) to E(s):

$$\frac{Z(s)}{E(s)} = \frac{G}{1+GH} = \frac{\frac{4}{s}}{1+\left(\frac{4}{s}\right)(2)} = \frac{\frac{4}{s}}{1+\frac{8}{s}} = \frac{\frac{4}{s}}{\frac{s+8}{s}} = \frac{4}{s+8}$$

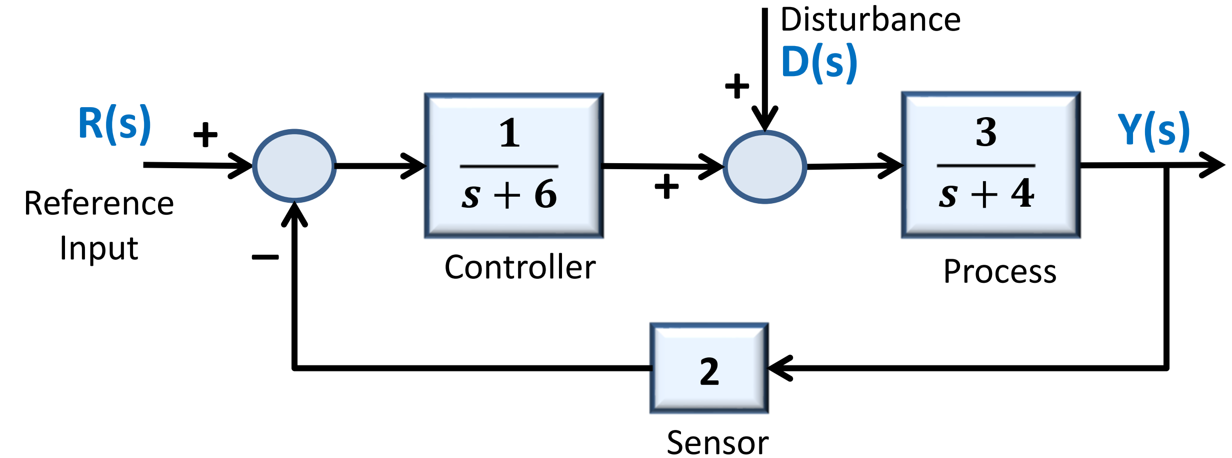



Thus, the overall transfer function from Y(s) to R(s) is:

$$\frac{Y(s)}{R(s)} = \frac{G}{1+GH} = \frac{\left(\frac{4}{s+8}\right)\left(\frac{1}{s}\right)}{1+\left(\frac{4}{s+8}\right)\left(\frac{1}{s}\right)(1)} = \frac{\frac{4}{s(s+8)}}{1+\frac{4}{s(s+8)}} = \frac{\frac{4}{s(s+8)}}{1+\frac{4}{s(s+8)}} = \frac{\frac{4}{s(s+8)}}{\frac{s(s+8)+4}{s(s+8)}} = \frac{4}{s^2+8s+4}$$

# Systems Transfer Function

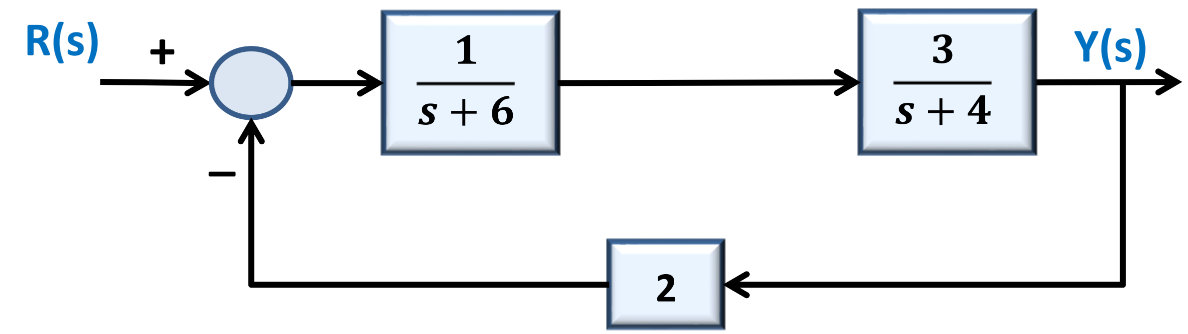
**Example 4** Assume the following control system with two inputs $R(s)$ and $D(s)$. Obtained the transfer functions $Y(s)/R(s)$ and $Y(s)/D(s)$.




When there is more than one input to a system, the superposition principle can be used.

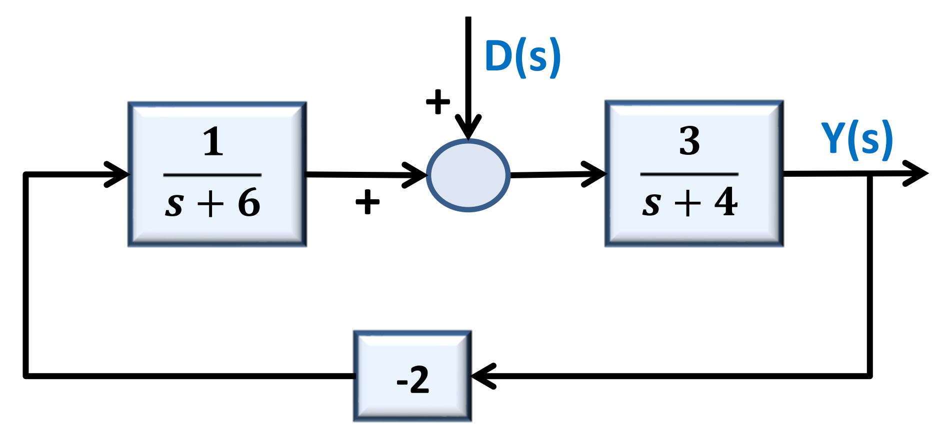
To obtain $Y(s)/R(s)$, set $D(s) = 0$ and redraw the diagram as shown

$$\frac{Y(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{3}{(s+6)(s+4)}}{1+\left(\frac{3}{(s+6)(s+4)}\right)(2)} = \frac{3}{(s+6)(s+4)+6} = \frac{3}{s^2+10s+30}$$




To obtain $Y(s)/D(s)$, set $R(s) = 0$ and redraw the diagram as shown

$$\frac{Y(s)}{D(s)} = \frac{G}{1-GH} = \frac{\frac{3}{s+4}}{1-\left(\frac{3}{s+4}\right)\left(\frac{-2}{s+6}\right)} = \frac{3(s+6)}{(s+4)(s+6)+6} = \frac{3(s+6)}{s^2+10s+30}$$

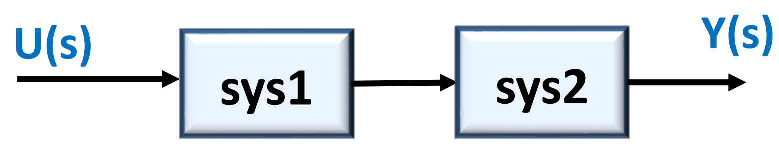



The output $Y(s)$ can be written in terms of both inputs:

$$Y(s) = \frac{3}{s^2+10s+30}R(s) + \frac{3(s+6)}{s^2+10s+30}D(s)$$

# Block Diagram Operations with MATLAB

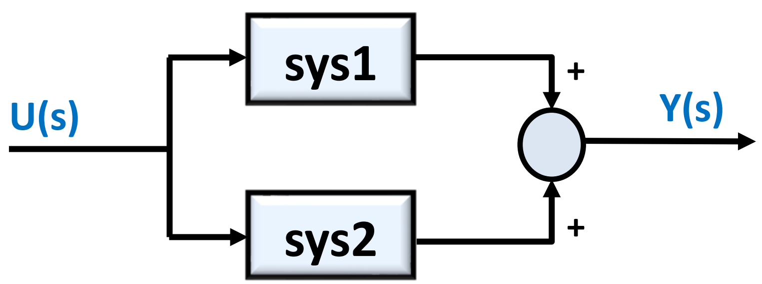
- We can determine series connection of **two TF models** in MATLAB using the following command:




```
sys = series(sys1,sys2)
```

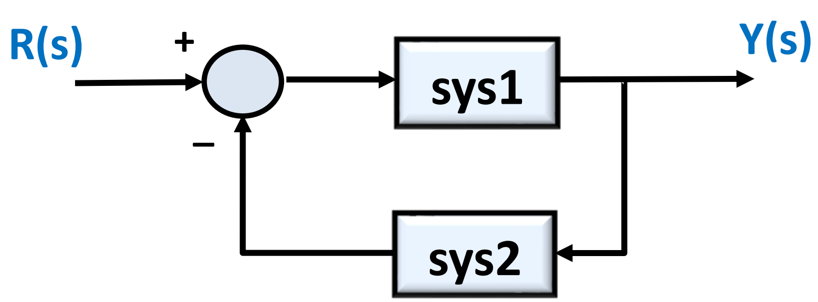
```
sys = sys1*sys2
```

- We can determine parallel connection of **two TF models** in MATLAB using the following command:




```
sys = parallel(sys1,sys2)
```

- We can determine overall negative feedback connection of multiple models in MATLAB using the following command:




```
sys = feedback(sys1,sys2)
```

Overall feedforward TF

Overall feedback TF

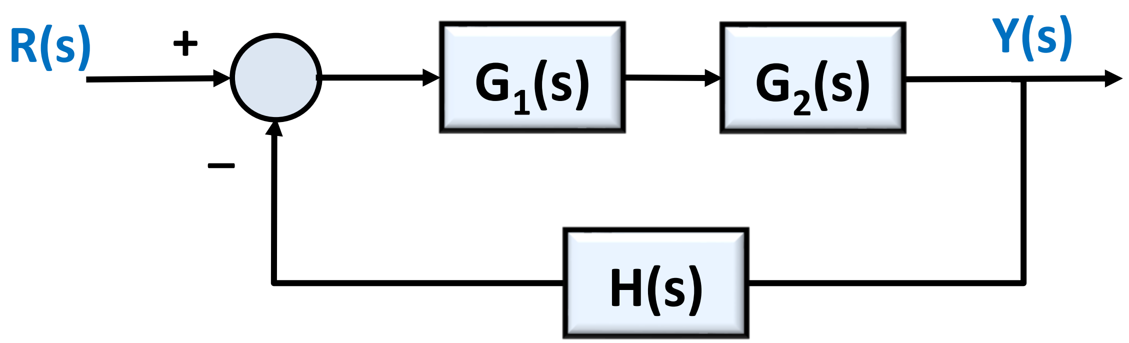
- For a positive feedback use the following syntax:

```
sys = feedback(sys1,sys2,+1)
```

# Block Diagram Operations with MATLAB

Given the block diagram, determine the transfer function $Y(s)/R(s)$ for the overall system.

We can find the overall transfer function with MATLAB:




$$G_1(s) = \frac{1}{s+6}$$

$$G_2(s) = \frac{1}{s+12}$$

$$H(s) = 8$$

```
num1 = [1];
den1 = [1 6];
G1 = tf(num1,den1)

num2 = [1];
den2 = [1 12];
G2 = tf(num2,den2)

H = 8;

sys = feedback(G1*G2,H)

sys =
         1
  ---------------
  s^2 + 18 s + 80
```

The overall feedback transfer function is

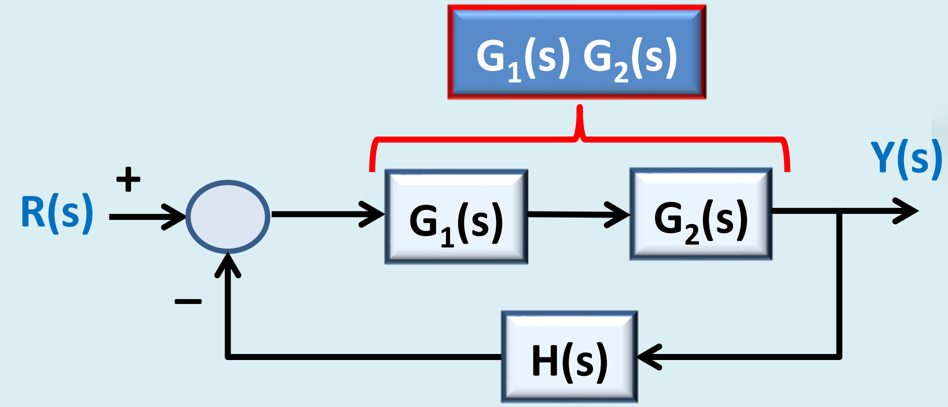
$$\frac{Y(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{1}{(s+6)(s+12)}}{1+\left(\frac{1}{(s+6)(s+12)}\right)(8)} = \frac{1}{(s+6)(s+12)+8} = \frac{1}{s^2+18s+80}$$

# Quick Review

1. What is the overall transfer function from Y(s) to R(s)?

a) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1}{1+G_1H}$  b) $\dfrac{Y(s)}{R(s)} = \dfrac{G_2}{1+G_2H}$

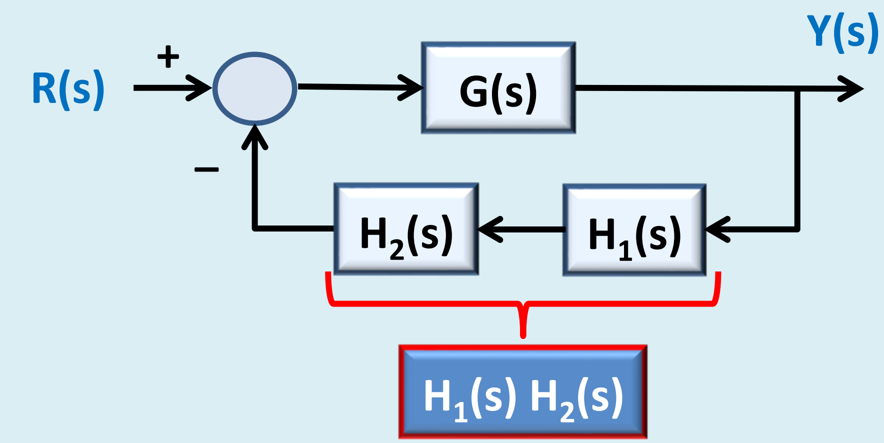
c) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1G_2}{1+G_1G_2H}$  d) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1G_2}{1+G_1G_2}$




2. What is the overall transfer function from Y(s) to R(s)?

a) $\dfrac{Y(s)}{R(s)} = \dfrac{G}{1+GH_1H_2}$  b) $\dfrac{Y(s)}{R(s)} = \dfrac{GH_1}{1+H_1H_2}$

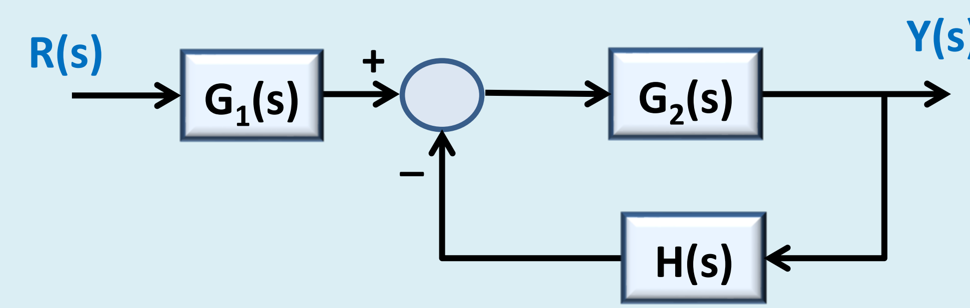
c) $\dfrac{Y(s)}{R(s)} = \dfrac{G}{1+H_1H_2}$  d) $\dfrac{Y(s)}{R(s)} = \dfrac{GH_2}{1+GH_1H_2}$




3. What is the overall transfer function from Y(s) to R(s)?

a) $\dfrac{Y(s)}{R(s)} = \dfrac{G_2}{1+G_1G_2H}$  b) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1G_2}{1+G_2H}$

c) $\dfrac{Y(s)}{R(s)} = \dfrac{G_2}{1+G_2(G_1-H)}$  d) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1G_2}{1+G_1G_2H}$



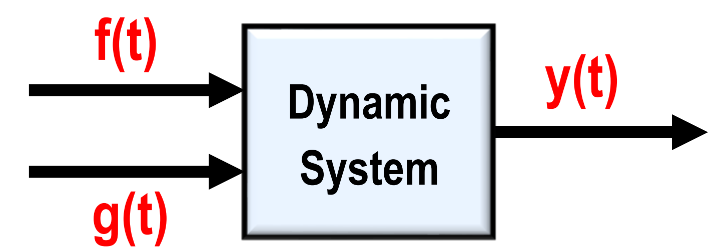
# Transfer Function of MIMO Systems

- In Multi-Input-Multi-Output (MIMO) models, a particular transfer function is the ratio of the output transform over the input transform, with all the remaining inputs ignored (set to zero temporarily – superposition property).

**Example 5**

Obtained the transfer functions $Y(s)/F(s)$ and $Y(s)/G(s)$ for the following two-inputs one-output model of a dynamic system.

$$5\ddot{y}(t) + 30\dot{y}(t) + 40y(t) = 6f(t) + 20g(t)$$




Taking Laplace transform of both side by assuming the **zero initial conditions**

$$5s^2Y(s) + 30sY(s) + 40Y(s) = 6F(s) + 20G(s) \quad \rightarrow \quad Y(s) = \frac{6}{5s^2 + 30s + 40}F(s) + \frac{20}{5s^2 + 30s + 40}G(s)$$

The transfer function for a specific input can be obtained by temporarily setting the other inputs equal to zero.

$$\boxed{\frac{Y(s)}{F(s)} = \frac{6}{5s^2 + 30s + 40}} \qquad \boxed{\frac{Y(s)}{G(s)} = \frac{20}{5s^2 + 30s + 40}}$$

**Transfer Function Models**

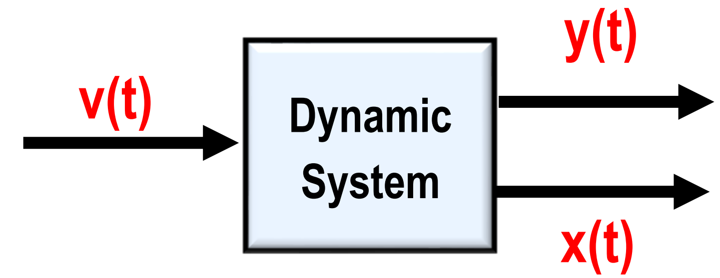
# Transfer Function of MIMO Systems

**Example 6**

Determine the transfer functions $X(s)/V(s)$ and $Y(s)/V(s)$ for the following system of equations:

$$\dot{x}(t) = -3x(t) + 2y(t)$$

$$\dot{y}(t) = -9y(t) - 4x(t) + 3v(t)$$




Here two outputs are specified, $x(t)$ and $y(t)$, with one input, $v(t)$.

Take Laplace transform of each equation, assuming **zero initial condition**.

$sX(s) = -3X(s) + 2Y(s)$   *Eqn. 1* $\rightarrow$ $Y(s) = \dfrac{s+3}{2}X(s)$ $\rightarrow$ Substitute into the Eqn. 2

$sY(s) = -9Y(s) - 4X(s) + 3V(s)$   *Eqn. 2* $\rightarrow$ $(s+9)\left(\dfrac{s+3}{2}\right)X(s) = -4X(s) + 3V(s)$

Solve for $X(s)/V(s)$:

$\rightarrow \quad (s+9)(s+3)X(s) = -8X(s) + 6V(s) \quad \rightarrow \quad (s^2 + 12s + 35)X(s) = 6V(s) \quad \rightarrow \quad \boxed{\dfrac{X(s)}{V(s)} = \dfrac{6}{s^2 + 12s + 35}}$

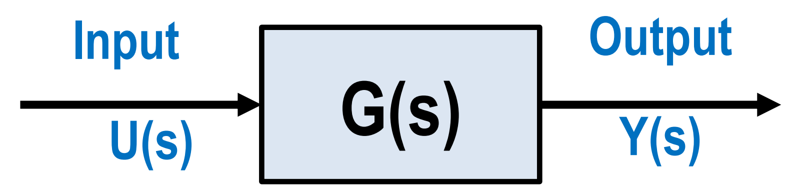
From Eqn. 1 $\rightarrow$ $X(s) = \dfrac{2}{s+3}Y(s)$ $\rightarrow$ Substitute into the $X(s)/V(s)$

$$\left(\frac{2}{s+3}\right)\frac{Y(s)}{V(s)} = \frac{6}{s^2 + 12s + 35} \quad \rightarrow \quad \boxed{\frac{Y(s)}{V(s)} = \frac{3(s+3)}{s^2 + 12s + 35}}$$

**Transfer Function Models**

# State-Space Equations From Transfer Function

- Consider an LTI, SISO system with the transfer function of G(s):




$$G(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

- Determining the state space representation from the transfer function is called **realization**.
- A transfer function is realizable if and only if the transfer function is proper or strictly proper.

**Strictly Proper Systems ($m < n$)**

- G(s) can be realized with minimum of *n* state variables as

$$\begin{cases} \dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) \end{cases}$$

In this case $\rightarrow$ $\mathbf{D} = \mathbf{0}$

**Proper Systems ($m = n$)**

- G(s) can be realized with minimum of *n* state variables as

$$\begin{cases} \dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

In this case $\rightarrow$ $\mathbf{D} = \lim_{s \to \infty} G(s)$

- General idea is deriving the differential equation from the given transfer function, and then realizing the state space equations from the differential equation.

# State-Space Equations From Transfer Function

**Example 7** Determine the state space representation of the following transfer function.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{12}{s^3 + 5s^2 + 11s + 8}$$

First, find the associated differential equation

$$s^3Y(s) + 5s^2Y(s) + 11sY(s) + 8Y(s) = 12U(s) \longrightarrow \dddot{y}(t) + 5\ddot{y}(t) + 11\dot{y}(t) + 8y(t) = 12u(t)$$

$$\begin{cases} q_1(t) = y(t) \\ q_2(t) = \dot{y}(t) \\ q_3(t) = \ddot{y}(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = \dot{y}(t) \\ \dot{q}_2(t) = \ddot{y}(t) \\ \dot{q}_3(t) = \dddot{y}(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = q_3(t) \\ \dot{q}_3(t) = -8q_1(t) - 11q_2(t) - 5q_3(t) + 12u(t) \\ \text{Output} \rightarrow \quad y(t) = q_1(t) \end{cases}$$

This state variables are called Phase Variables.

$$\boxed{\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)}$$

State Equation

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -11 & -5 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} u(t)$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$$

Output Equation

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + [0]u(t)$$

G(s) is a strictly proper transfer function, $\mathbf{D} = \mathbf{0}$

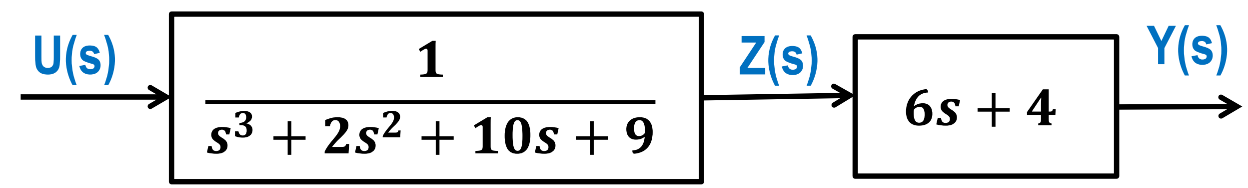
# State-Space Equations From Transfer Function

**Example 8** Determine the state space representation of the following transfer function.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{6s + 4}{s^3 + 2s^2 + 10s + 9}$$

Since, the numerator is a polynomial of *s*, we have to separate it into two parts as below

$$\frac{Y(s)}{U(s)} = \frac{Z(s)}{U(s)} \times \frac{Y(s)}{Z(s)} = \frac{1}{s^3 + 2s^2 + 10s + 9} \times (6s + 4)$$




First, find the state equation from the part with <u>denominator</u>

$$s^3Z(s) + 2s^2Z(s) + 10sZ(s) + 9Z(s) = U(s) \quad \rightarrow \quad \dddot{z}(t) + 2\ddot{z}(t) + 10\dot{z}(t) + 9z(t) = u(t)$$

$$\begin{cases} q_1(t) = z(t) \\ q_2(t) = \dot{z}(t) \\ q_3(t) = \ddot{z}(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = \dot{z}(t) \\ \dot{q}_2(t) = \ddot{z}(t) \\ \dot{q}_3(t) = \dddot{z}(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = q_3(t) \\ \dot{q}_3(t) = -9q_1(t) - 10q_2(t) - 2q_3(t) + u(t) \end{cases}$$

Next, find the output equation by considering the effect of the <u>numerator</u>:

$$Y(s) = (6s + 4)Z(s) \quad \rightarrow \quad Y(s) = 6sZ(s) + 4Z(s)$$

Take the inverse Laplace transform

$$y(t) = 6\dot{z}(t) + 4z(t) \quad \rightarrow \quad y(t) = 6q_2(t) + 4q_1(t)$$

# State-Space Equations From Transfer Function

**Example 8** Determine the state space representation of the following transfer function.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{6s+4}{s^3+2s^2+10s+9}$$

$$\begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = q_3(t) \\ \dot{q}_3(t) = -9q_1(t) - 10q_2(t) - 2q_3(t) + u(t) \\ y(t) = 6q_2(t) + 4q_1(t) \end{cases}$$

Therefore, the state and output equations are obtained as

$$\boxed{\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)}$$

State Equation

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -10 & -2 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{D}\mathbf{u}(t)}$$

Output Equation

$$y(t) = \begin{bmatrix} 4 & 6 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + [0]u(t)$$

G(s) is a strictly proper transfer function, $\mathbf{D} = \mathbf{0}$

# State-Space Equations From Transfer Function

**Example 9**

Determine the state space representation of the following transfer function.
Draw a block diagram to visualize the state variables.

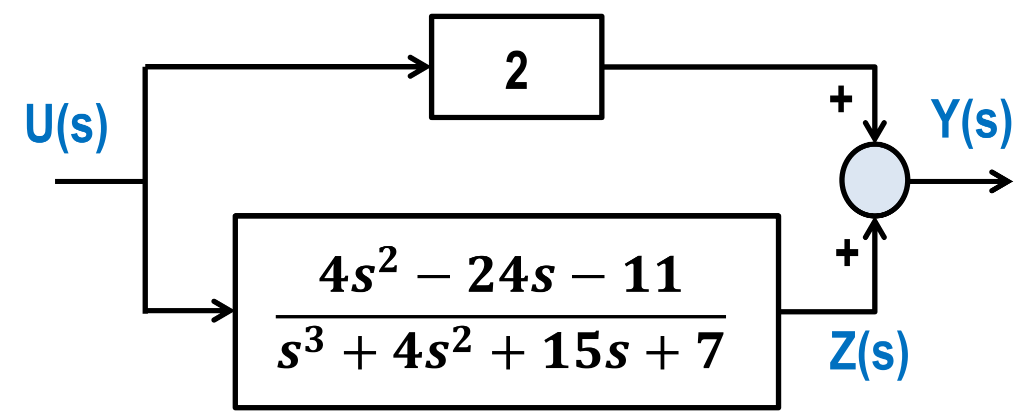
$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 12s^2 + 6s + 3}{s^3 + 4s^2 + 15s + 7}$$

Since, G(s) is a proper transfer function first, we have to rewrite it as a summation of a constant term and a strictly proper transfer function

$$G(s) = \lim_{s \to \infty} G(s) + G_1(s) \longrightarrow G(s) = \frac{Y(s)}{U(s)} = 2 + \frac{4s^2 - 24s - 11}{s^3 + 4s^2 + 15s + 7}$$

The feed-forward matrix **D** is obtained as

$$\boxed{\mathbf{D} = \lim_{s \to \infty} G(s) = 2}$$




Determine the state space matrices, **A, B** and **C** from the strictly proper transfer function by applying the same method in the previous example.

# State-Space Equations From Transfer Function

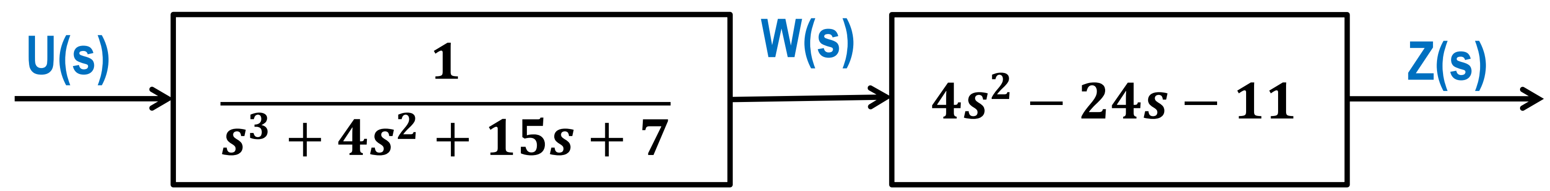
**Example 9**

Determine the state space representation of the following transfer function.
Draw a block diagram to visualize the state variables.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 12s^2 + 6s + 3}{s^3 + 4s^2 + 15s + 7}$$

Determine the state space matrices, **A, B** and **C** from the strictly proper transfer function.

$$\frac{Z(s)}{U(s)} = \frac{4s^2 - 24s - 11}{s^3 + 4s^2 + 15s + 7}$$




$$\frac{Z(s)}{U(s)} = \frac{W(s)}{U(s)} \times \frac{Z(s)}{W(s)} = \frac{1}{s^3 + 4s^2 + 15s + 7} \times (4s^2 - 24s - 11)$$

Find the state equation from the part with denominator

$$s^3W(s) + 4s^2W(s) + 15sW(s) + 7W(s) = U(s) \Rightarrow \dddot{w}(t) + 4\ddot{w}(t) + 15\dot{w}(t) + 7w(t) = u(t)$$

$$\begin{cases} q_1(t) = w(t) \\ q_2(t) = \dot{w}(t) \\ q_3(t) = \ddot{w}(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = \dot{w}(t) \\ \dot{q}_2(t) = \ddot{w}(t) \\ \dot{q}_3(t) = \dddot{w}(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = q_3(t) \\ \dot{q}_3(t) = -7q_1(t) - 15q_2(t) - 4q_3(t) + u(t) \end{cases}$$

# State-Space Equations From Transfer Function

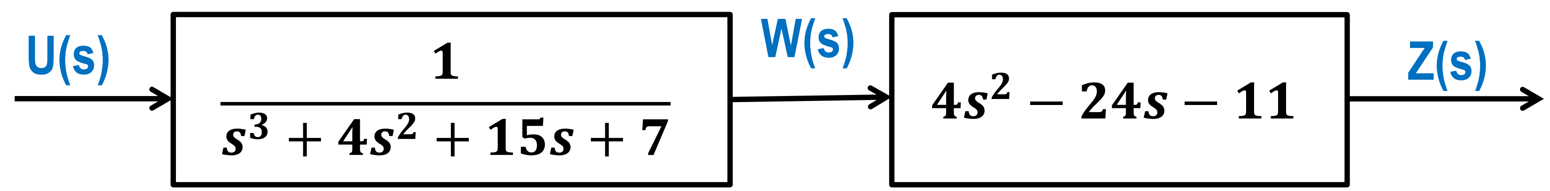
**Example 9**

Determine the state space representation of the following transfer function.
Draw a block diagram to visualize the state variables.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 12s^2 + 6s + 3}{s^3 + 4s^2 + 15s + 7}$$

Determine the state space matrices, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ from the strictly proper transfer function.

$$\frac{Z(s)}{U(s)} = \frac{4s^2 - 24s - 11}{s^3 + 4s^2 + 15s + 7}$$




$$\frac{Z(s)}{U(s)} = \frac{W(s)}{U(s)} \times \frac{Z(s)}{W(s)} = \frac{1}{s^3 + 4s^2 + 15s + 7} \times (4s^2 - 24s - 11)$$

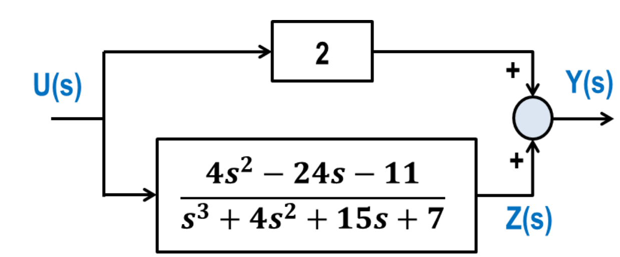
Next, find the output equation by considering the effect of the block with the numerator

$$Z(s) = (4s^2 - 24s - 11)W(s) \quad \rightarrow \quad Z(s) = 4s^2W(s) - 24sW(s) - 11W(s)$$

Take the inverse Laplace transform

$$z(t) = 4\ddot{w}(t) - 24\dot{w}(t) - 11w(t) \quad \rightarrow \quad z(t) = 4q_3(t) - 24q_2(t) - 11q_1(t)$$

The output is $\rightarrow y(t) = z(t) + 2u(t) \quad \rightarrow \quad y(t) = 4q_3(t) - 24q_2(t) - 11q_1(t) + 2\mathrm{u}(t)$



# State-Space Equations From Transfer Function

**Example 9** Determine the state space representation of the following transfer function. Draw a block diagram to visualize the state variables.

$$\boldsymbol{G(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 12s^2 + 6s + 3}{s^3 + 4s^2 + 15s + 7}}$$

$$G(s) = \frac{Y(s)}{U(s)} = 2 + \frac{4s^2 - 24s - 11}{s^3 + 4s^2 + 15s + 7}$$

$$\begin{cases} \dot{q}_1(t) = q_2(t) \\ \dot{q}_2(t) = q_3(t) \\ \dot{q}_3(t) = -7x_1(t) - 15q_2(t) - 4q_3(t) + u(t) \\ y(t) = 4q_3(t) - 24q_2(t) - 11q_1(t) + 2\text{u(t)} \end{cases}$$

Therefore, the state and output equations are obtained as

$$\boxed{\dot{\mathbf{q}}(t) = \mathbf{Aq}(t) + \mathbf{Bu}(t)}$$

State Equation

$$\longrightarrow \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -15 & -4 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\boldsymbol{q}(t) + \mathbf{Du}(t)}$$

Output Equation

$$\longrightarrow y(t) = \begin{bmatrix} -11 & -24 & 4 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + [2]u(t)$$

G(s) is a proper transfer function, $\mathbf{D} \neq \mathbf{0}$

# Models Conversion with MATLAB

- We can extract the state-space matrices (A, B, C, D) or the transfer function polynomials coefficients from an LTI object model in **MATLAB** by using the following commands:

```
[A,B,C,D] = ssdata(sys)
```

**State-space model matrices** — [A,B,C,D]

**Transfer function model** — sys

```
[num,den] = tfdata(sys)
```

**Transfer function polynomials coefficients** — [num,den]

**State-space model** — sys

- We can convert the state-space model to a transfer function model and vice-versa in **MATLAB** by using the following commands:

```
sys2 = ss(sys1)
```

**State-space model** — sys2

**Transfer function model** — sys1

```
sys1 = tf(sys2)
```

**Transfer function model** — sys1

**State-space model** — sys2

# Models Conversion with MATLAB

For example, given the transfer function model extract the state-space model in MATLAB.

$$G(s) = \frac{s^2 + 5s + 6}{s^3 + s^2 + 4s + 4}$$

```
num = [1 5 6];
den = [1 1 4 4];
G = tf(num,den);
sys = ss(G)
sys =
   A =
           x1   x2   x3
      x1   -1   -2   -2
      x2    2    0    0
      x3    0    1    0

   B =
           u1
      x1    2
      x2    0
      x3    0

   C =
              x1     x2     x3
      y1     0.5   1.25    1.5

   D =
           u1
      y1    0

Continuous-time state-space model.
```

```
num = [1 5 6];
den = [1 1 4 4];
G = tf(num,den);
[A,B,C,D] = ssdata(G)

A =

    -1     -2     -2
     2      0      0
     0      1      0

B =

     2
     0
     0

C =

    0.5000    1.2500    1.5000

D =

     0
```

# THANK YOU

HUMBER

WE ARE HUMBER