Signal Processing (MENG3520)

Module 10

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MODULE 10

SAMPLING AND INTERPOLATION

FOURIER TRANSFORM FAMILY

The following four types are all part of the Fourier transform family:

- ✓ Continuous-time Fourier series (CTFS) – periodic continuous-time signals
- ✓ Continuous-time Fourier transform (CTFT) – aperiodic continuous-time signals
- ❑ Discrete-time Fourier transform (DTFT) – aperiodic discrete-time signals
- ✓ Discrete Fourier transform (DFT) – periodic discrete-time signals.

Module Outline

- 10.1 Sampling
- 10.2 Interpolation
- 10.3 Downsampling and Decimation

10.1

SAMPLING

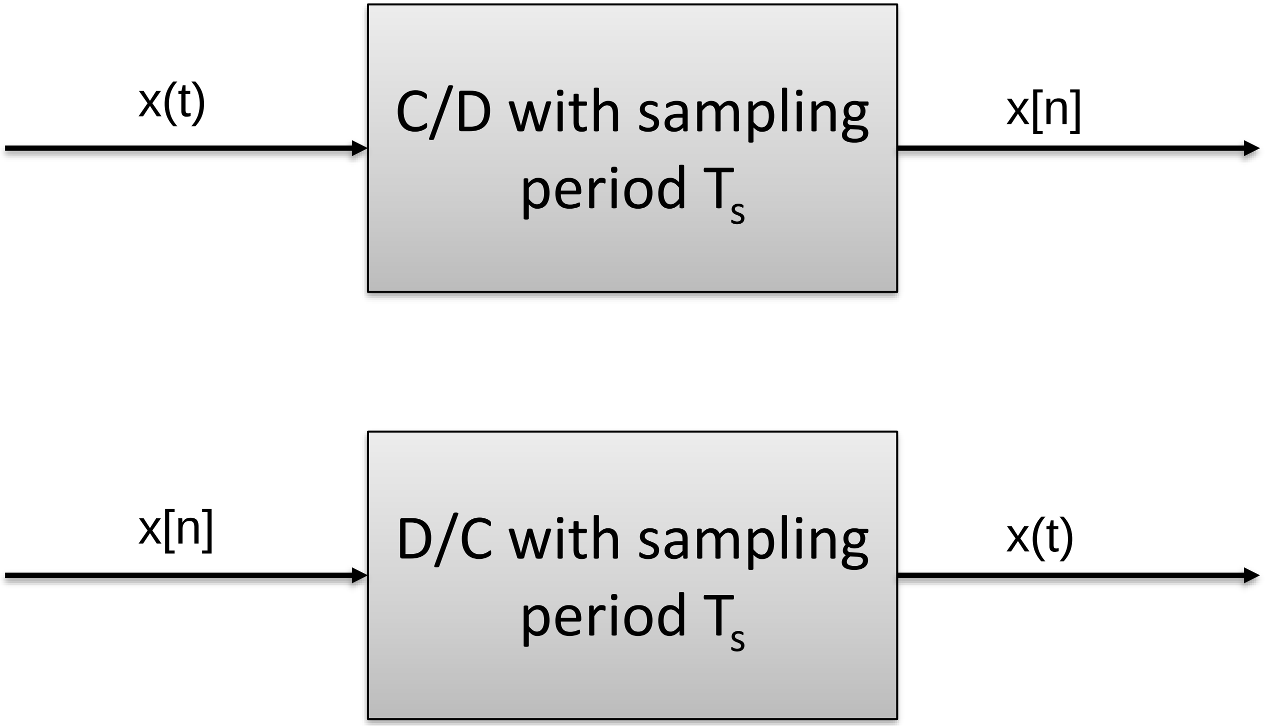
Recall

Sampling: obtaining the values of a signal at discrete points in time.

Sampling is performed by an ideal continuous-time to discrete-time (C/D) converter.

Interpolation: the opposite procedure to sampling, reconstructing the values of a signal in the continuous-time domain.

Interpolation is performed by an ideal discrete-time to continuous-time (D/C) converter.

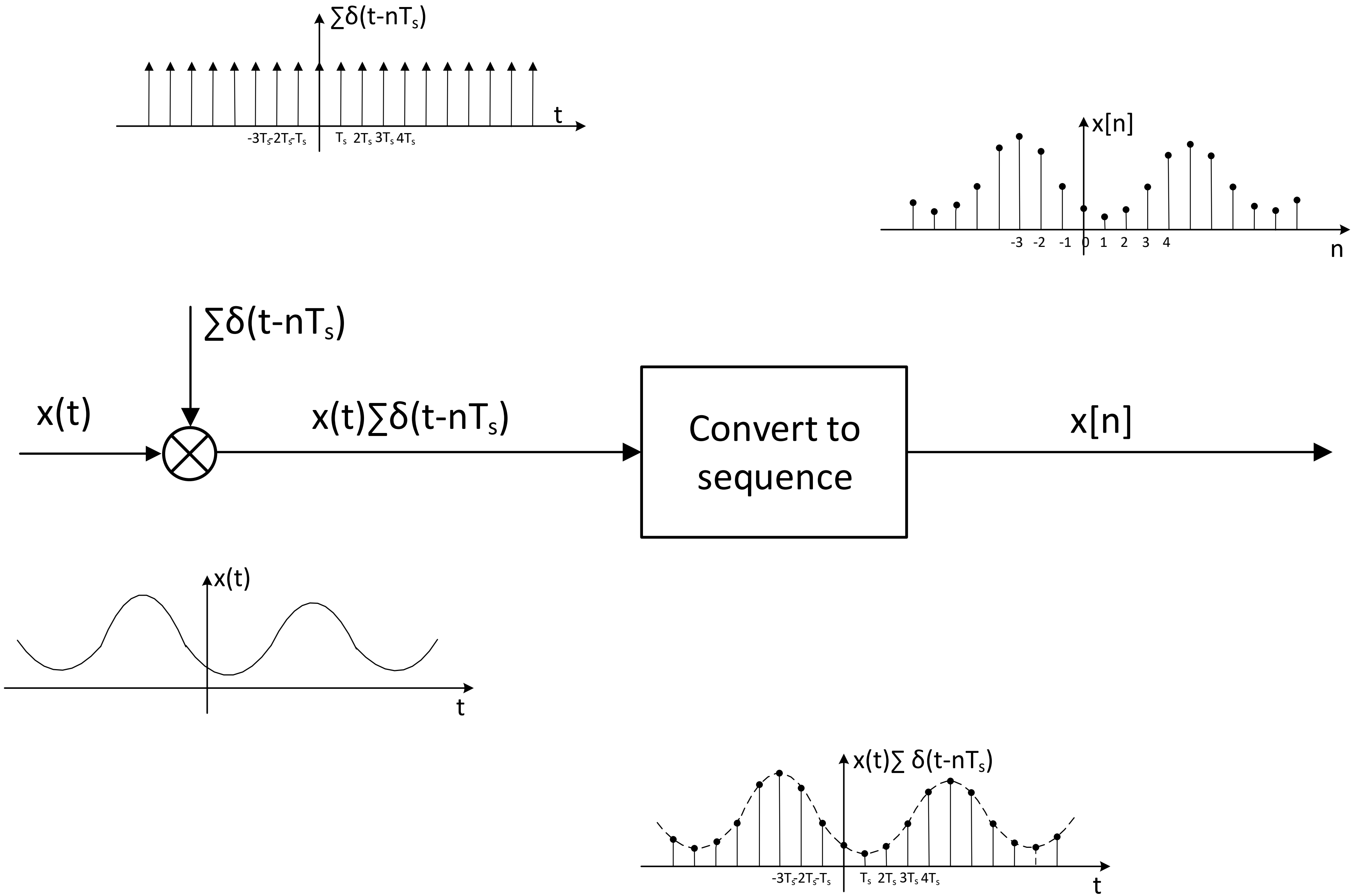

x(t)
C/D with sampling period T_s
x[n]
x[n]
D/C with sampling period T_s
x(t)

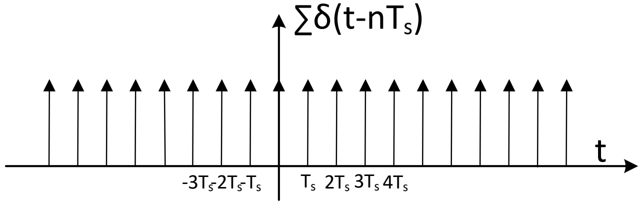
Mathematical Modeling of Sampling in the Time Domain

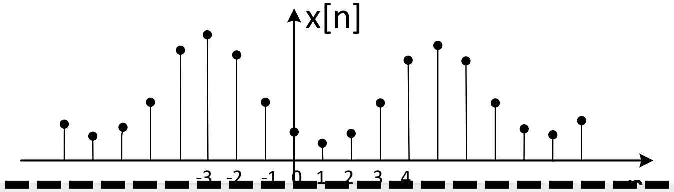
Sampling can be considered as the **multiplication** of a CT signal $x(t)$ with a sampling function.

The ideal sampling function is a periodic sequence of impulses of period T_s (referred to as impulse train $\delta_{T_s}(t)$):

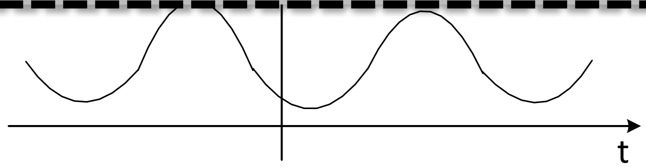
$$\delta_{T_s(t)} \triangleq \sum_n \delta(t - nT_s)$$

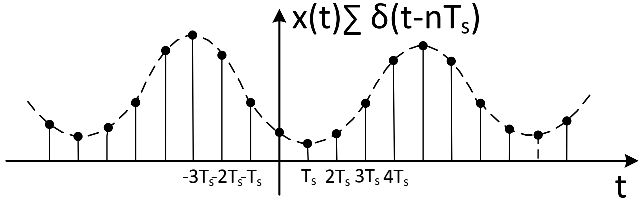

∑δ(t-nTs)
t
-3Ts -2Ts -Ts Ts 2Ts 3Ts 4Ts
x[n]
-3 -2 -1 0 1 2 3 4
n
∑δ(t-nTs)
x(t)
x(t)∑δ(t-nTs)
Convert to sequence
x[n]
x(t)
t
x(t)∑ δ(t-nTs)
-3Ts -2Ts -Ts Ts 2Ts 3Ts 4Ts
t





What is happening in the Frequency domain?





Mathematical Modeling of Sampling in the Frequency Domain

With ideal sampling, the impulse-sampled signal is given as:

$$v(t) = x(t)\delta_{T_s(t)} = x(t)\sum_n \delta(t - nT_s)$$

Use the Fourier multiplication property, sampled signal in the Fourier domain:

$$V(j\omega) = \mathcal{F}\{x(t)\delta_{T_s(t)}\} = X(j\omega) * \mathcal{F}\left\{\sum_n \delta(t - nT_s)\right\}$$

$$= X(j\omega) * \frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\sum_n \delta(t - nT_s)\, e^{-jk\omega_0 t} dt = X(j\omega) * \frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\sum_n e^{-jn\omega_0 nT_s}\, dt$$

$$= X(j\omega) * \frac{1}{T_s}\sum_n \delta\left(\omega - \frac{2\pi n}{T_s}\right) = f_s\sum_n X(j\omega) * \delta(\omega - n\omega_s) = f_s\sum_n X\big(j(\omega - n\omega_s)\big)$$

What does this mean?

Conclusion: discrete time signal can be considered as the sampled signal $v(t) = x(t)\delta_{T_s(t)}$ in the time domain, its representation in the Fourier domain is:

$$V(j\omega) = f_s \sum_{n} X\big(j(\omega - n\omega_s)\big)$$

This means:

1. The spectrum of the impulse-sampled signal $v(t)$ is a scaled sum of shifted copies of the spectrum of the original signal $x(t)$.
2. There are infinite numbers of copy of the scaled and shifted original spectrum $X(j\omega)$.
3. Each copy is separated by $f_s = \frac{1}{T_s}$ along the frequency axis.

Fourier Transform Family

The following four types are all part of the Fourier transform family:

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- ✓ Discrete Fourier transform (DFT) – periodic discrete-time signals.

Since DTFT is to convert discrete-time (sampled) signals in the Fourier domain. So, we have completed the DTFT forward transform already.

Fourier Transform Family

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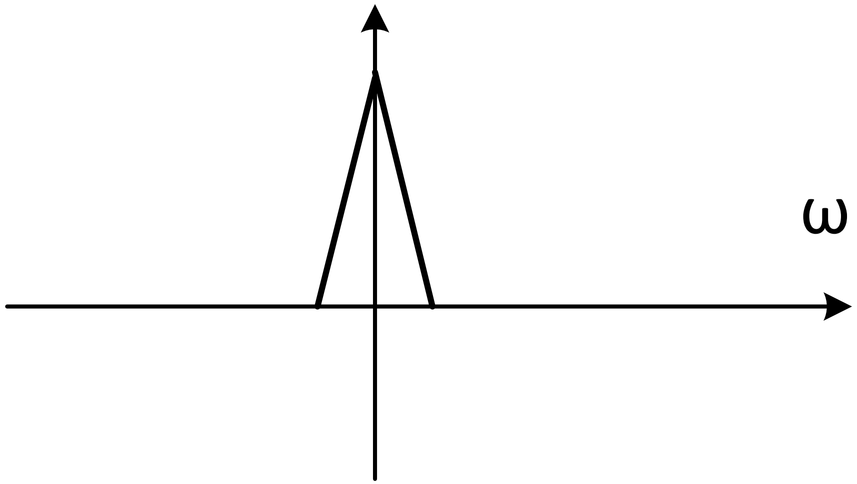
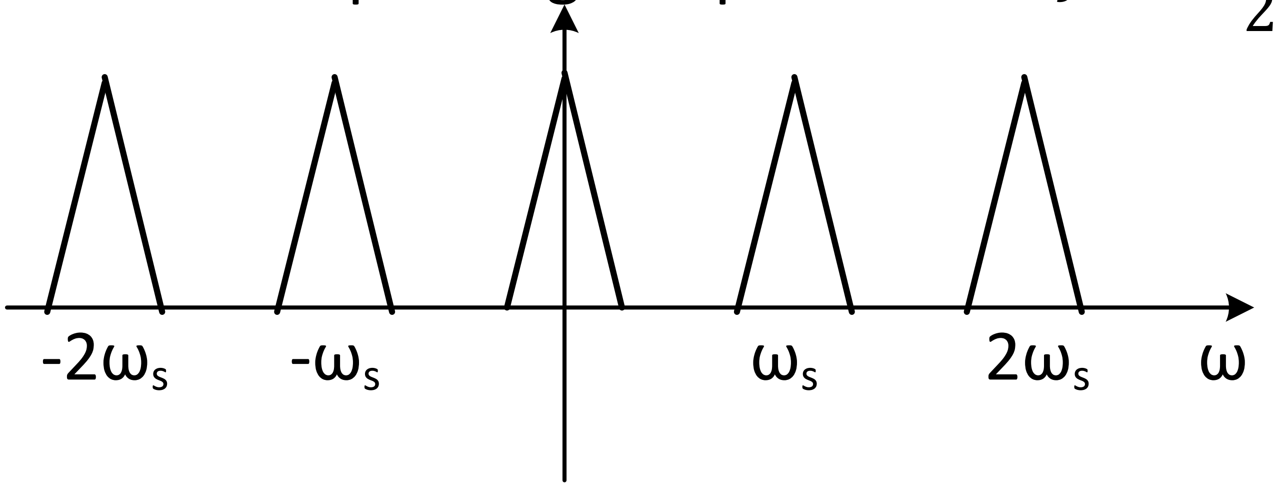
Let's review the relationship between these four methods again!

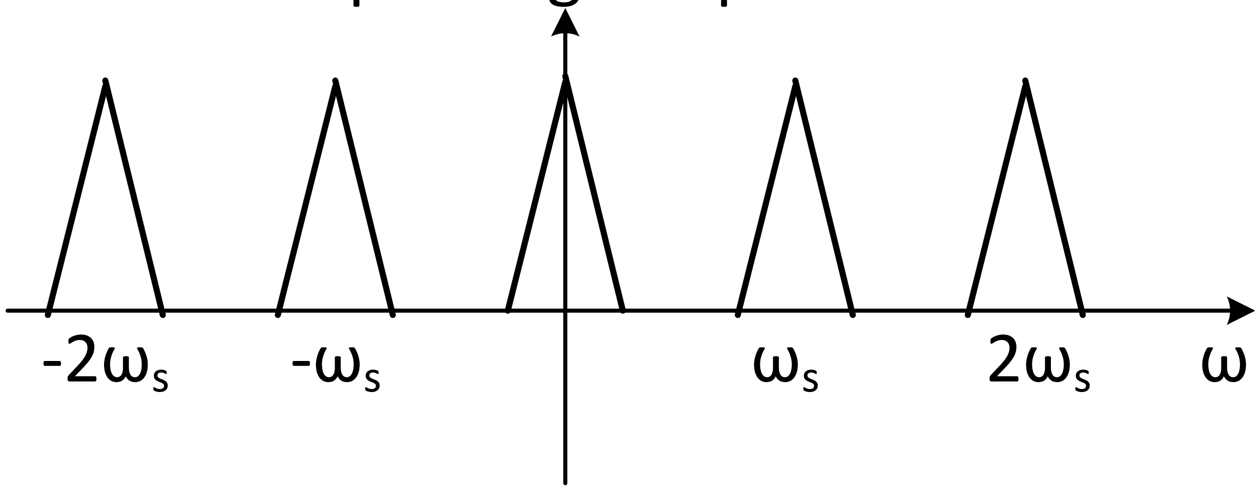
Original spectrum $X(j\omega)$

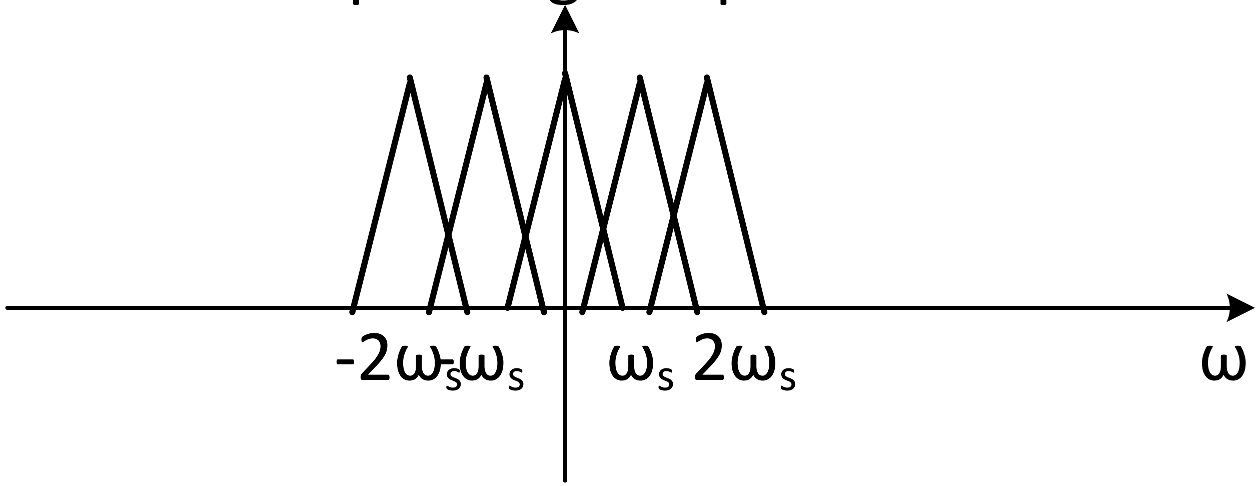


Sampled-signal spectrum $V(j\omega) = \dfrac{\omega_s}{2\pi} \sum_n X\big(j(\omega - n\omega_s)\big)$



Impact of T_s and ω_s





Depending on some factors, the frequency spectrum of the impulse-sampled signal is can be:

- Overlapping sum of copies of the original signal frequency spectrum.
- Non-overlapping sum of copies of the original signal frequency spectrum.

Wherever overlap occurs, the shifted copies of spectrum blended together so that the original spectrum is lost. It is known as **aliasing**.

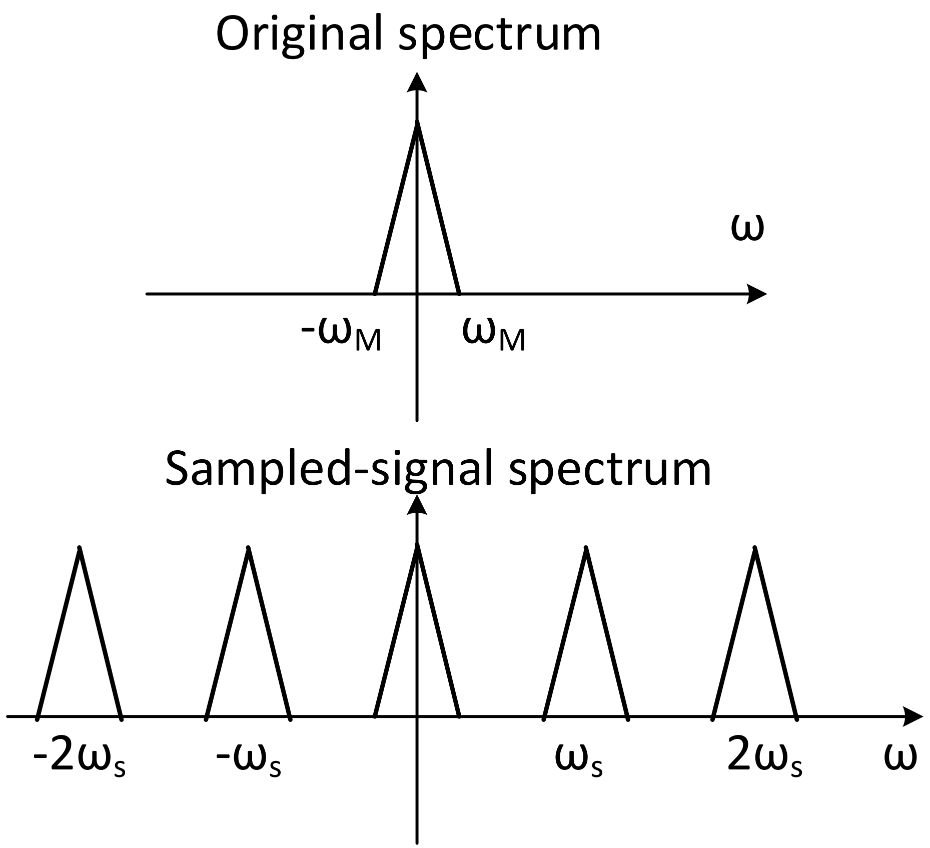
- When aliasing occurs, the original signal x cannot be recovered from its samples in v.

Aliasing is an unwanted effect of undersampling, i.e. when you are sampling a wide bandwidth signal too slowly.

Overlapping signals become indistinguishable from each other and thus create challenges in signal reconstruction.

Aliasing is a pervasive problem in any field where analog signals are sampled to create digital signals .

What is the requirement of sampling frequency ω_s and the signal bandwidth ω_M in order to avoid aliasing?



What is the requirement of sampling frequency ω_s and the signal bandwidth ω_M in order to avoid aliasing and completely determine $x(t)$?

$$\omega_s > 2\omega_M$$

Nyquist-Shannon sampling theorem: a system uniformly samples an analog signal at a rate that ω_s. If ω_s exceeds the signal's highest frequency ω_M by at least a factor of two, the original analog signal can be perfectly recovered from the discrete values produced by sampling.

- Nyquist rate: threshold $2\omega_M$.
- Nyquist frequency: a parameter of any sampler. 0.5 of the sampling frequency.
- Nyquist condition: $\omega_s > 2\omega_M$

Question: what if there is a limit to the sampling frequency thus this condition $\omega_s \geq 2\omega_M$ is impossible to meet?

10.2

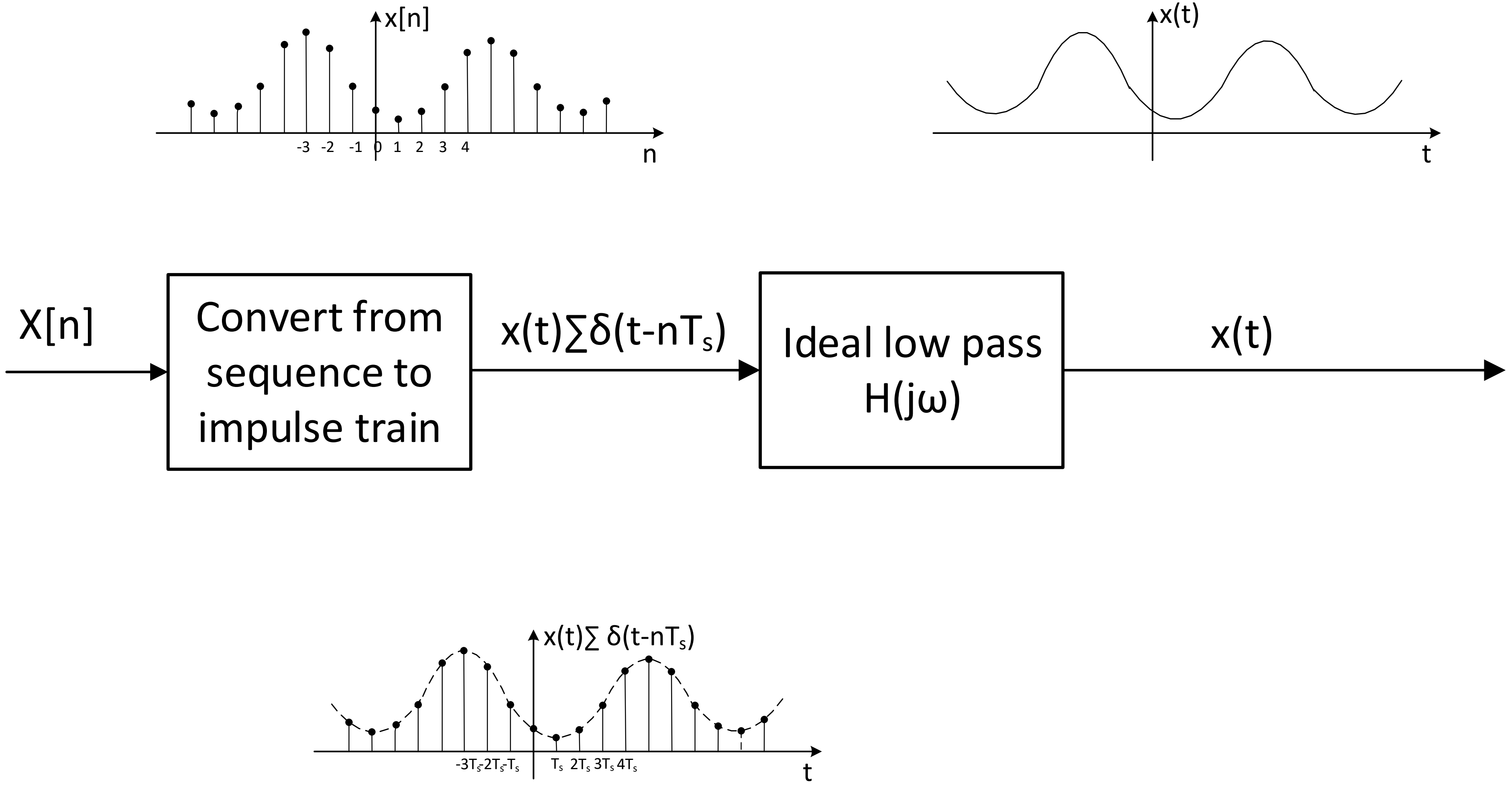
INTERPOLATION

Mathematical Modeling of Interpolation

Interpolation as the opposite of sampling, is a procedure to reconstruct the values of a signal in the continuous-time based on its discrete-time samples.

Methods to interpolate:

- Zero-hold
- Linear
- Polynomial
- Spline
- etc.


x[n]
-3 -2 -1 0 1 2 3 4
n
x(t)
t
X[n]
Convert from sequence to impulse train
$x(t)\sum\delta(t-nT_s)$
Ideal low pass H(jω)
x(t)
$x(t)\sum\delta(t-nT_s)$
$-3T_s$ $-2T_s$ $-T_s$ T_s $2T_s$ $3T_s$ $4T_s$
t

Mathematical Modeling of Interpolation

Interpolation can be considered as the **multiplication** of a signal spectrum with a low pass filter transfer function.

The ideal low pass transfer function is a rectangular function of bandwidth ω_s:

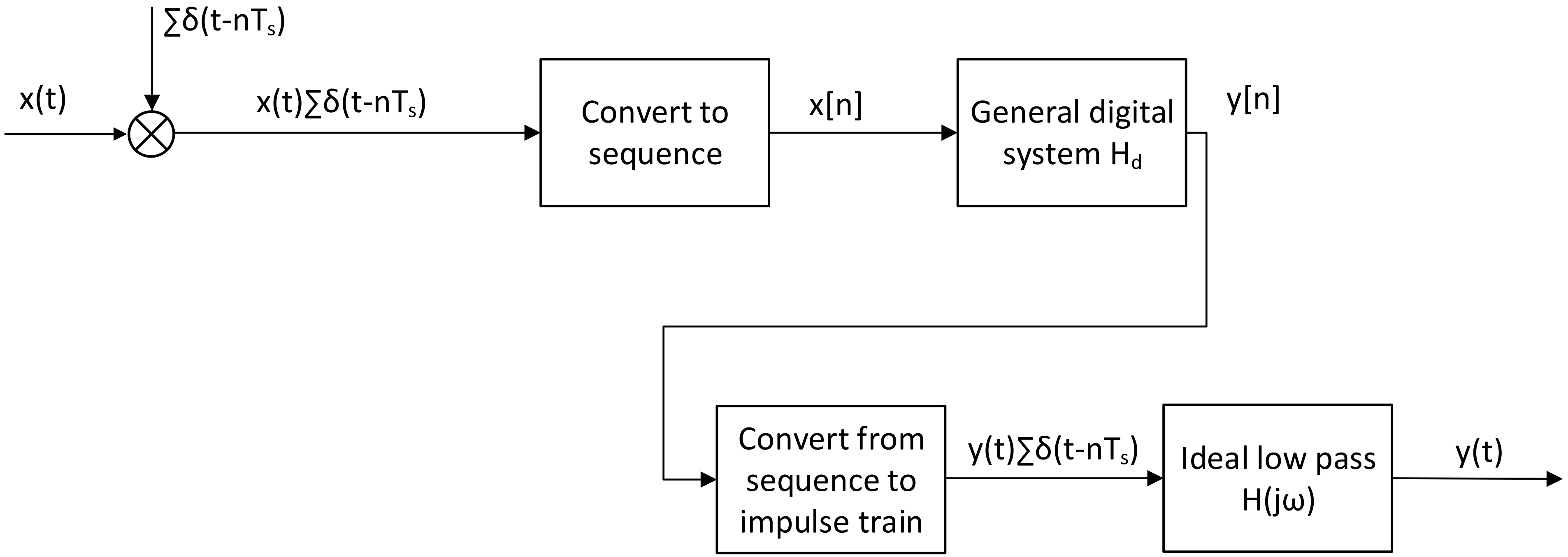
$$H(j\omega) \triangleq T\,\mathrm{rect}\left(\frac{\omega T_s}{2\pi}\right)$$

Mathematical Modeling of Interpolation

Question: what is the purpose of the ideal low-pass filter $H(j\omega)$ in this system?

Answer: to remove the extra copies of the original signal’s spectrum from the impulse-sampled signal spectrum.

Use of Digital System to Process Analog Signal



10.3

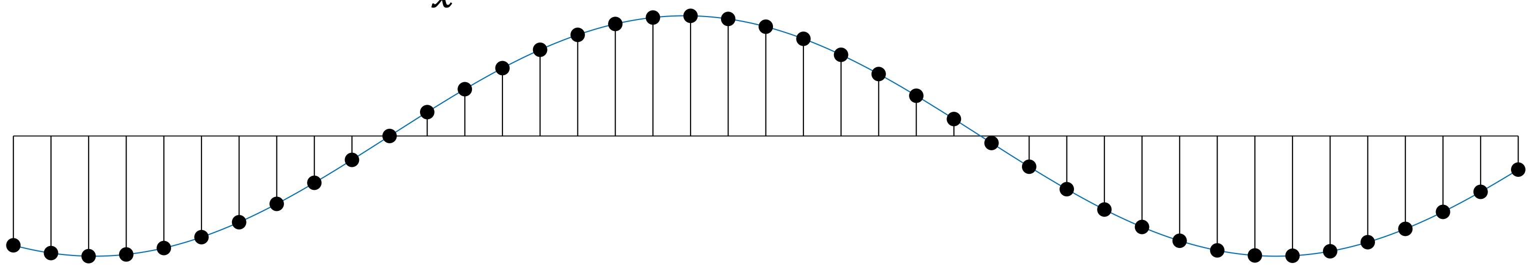
DOWNSAMPLING AND DECIMATION

Downsampling is a special case of sampling rate conversion, where $\frac{T_y}{T_x} = D$.

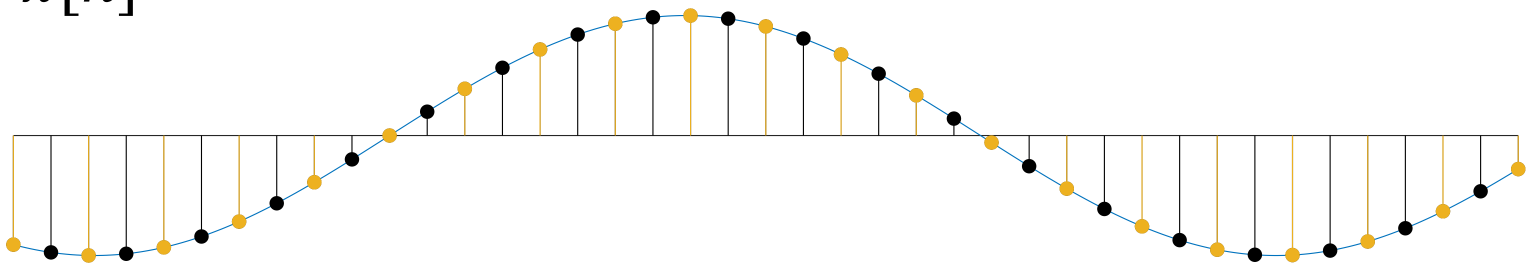
- T_y: The sampling period of the output signal $y[n]$.
- T_x: The sampling period of the input signal $x[n]$.
- D: An integer representing the ratio between the output and input sampling periods.

Downsampling

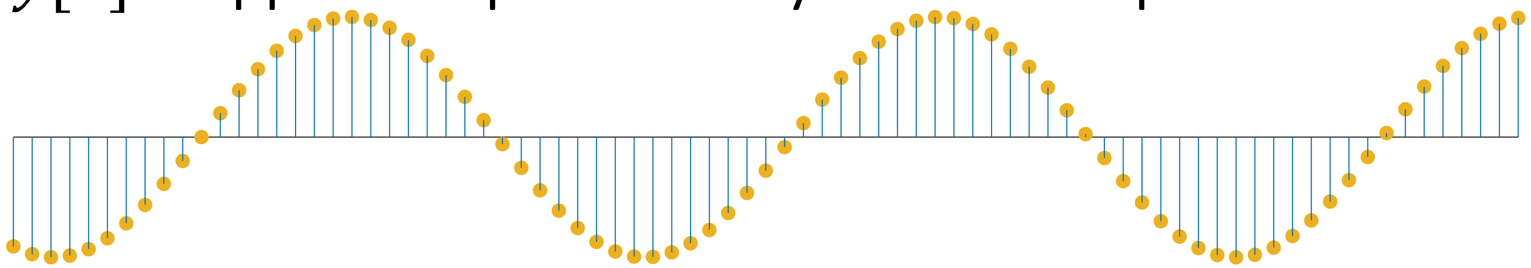
- Example: $D = \frac{T_y}{T_x} = 2.$



$x[n]$



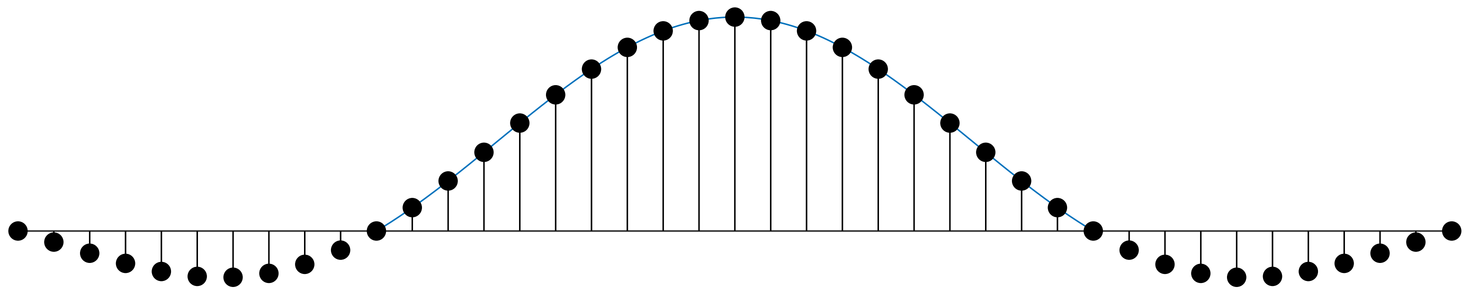
$y[n]$: skipped samples at every other x sample.



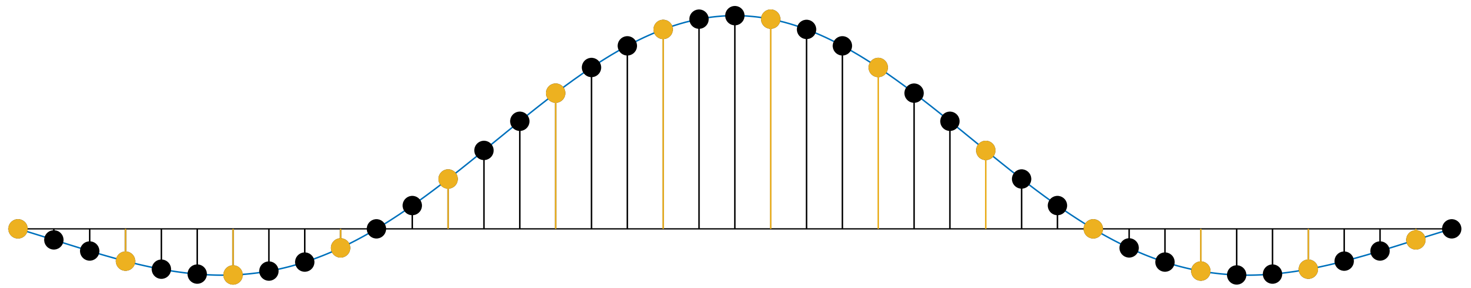
$y[n]$: compressed in the n axis compared to $x[n]$

Downsampling

- Example: $D = \frac{T_y}{T_x} = ?$



$x[n]$



$y[n]$: skipped samples at every other ? sample.

Downsampling

Question: what is the impact on the frequency spectrum if a signal x[n] is downsampled by $D = \frac{T_y}{T_x}$?

Answer: sampling frequency of the output signal is

$$T_y = DT_x, \qquad F_y = \frac{F_x}{D}$$

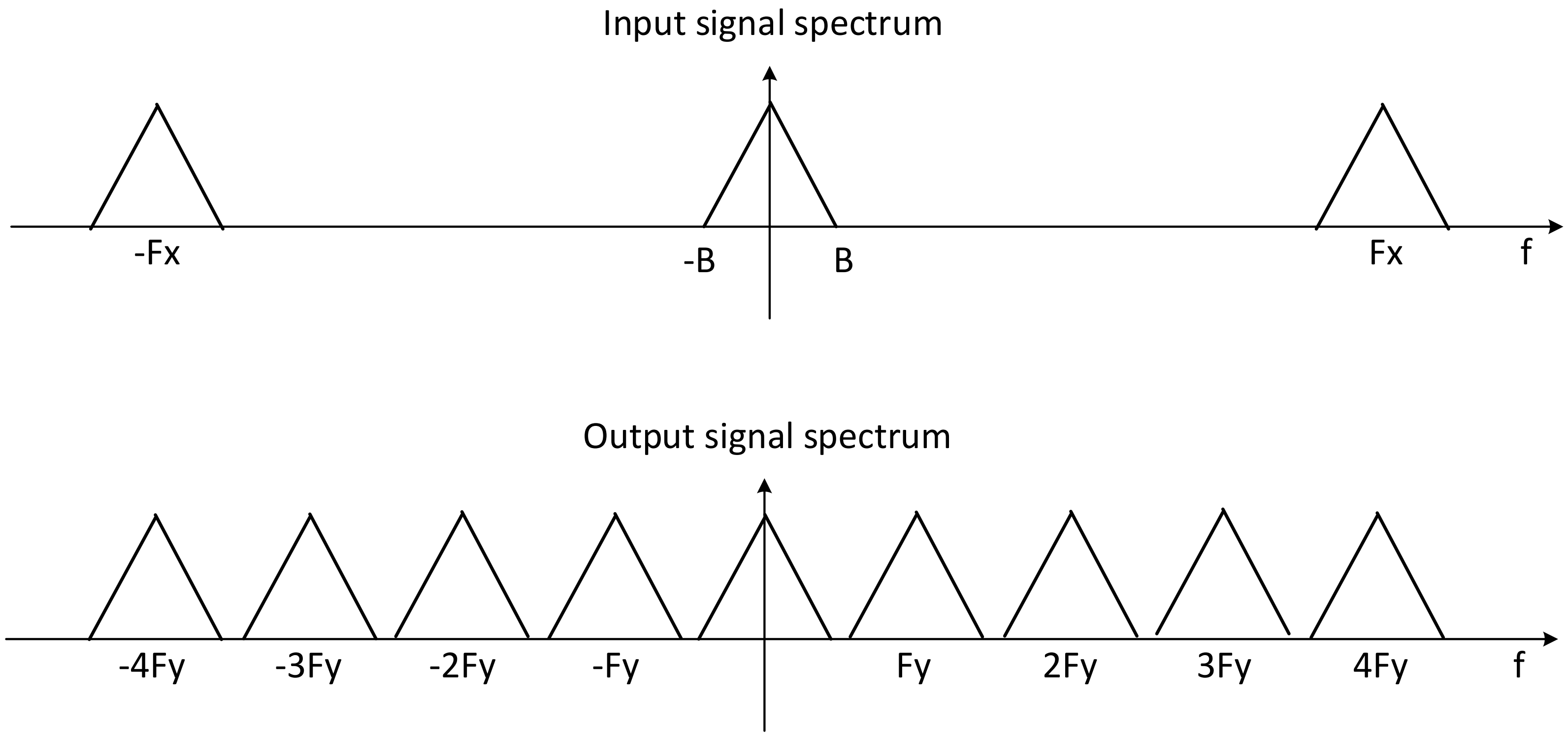
The sampling frequency of the output is $1/D$ of the sampling frequency of the input signal. From the input signal, keep one, discard $(D - 1)$ samples, keep one, discard $(D - 1)$ samples…

$$\tilde{x}[n] = \begin{cases} x[n], n = 0, \pm D, \pm 2D, \ldots \\ 0, otherwise \end{cases}$$

Downsampling

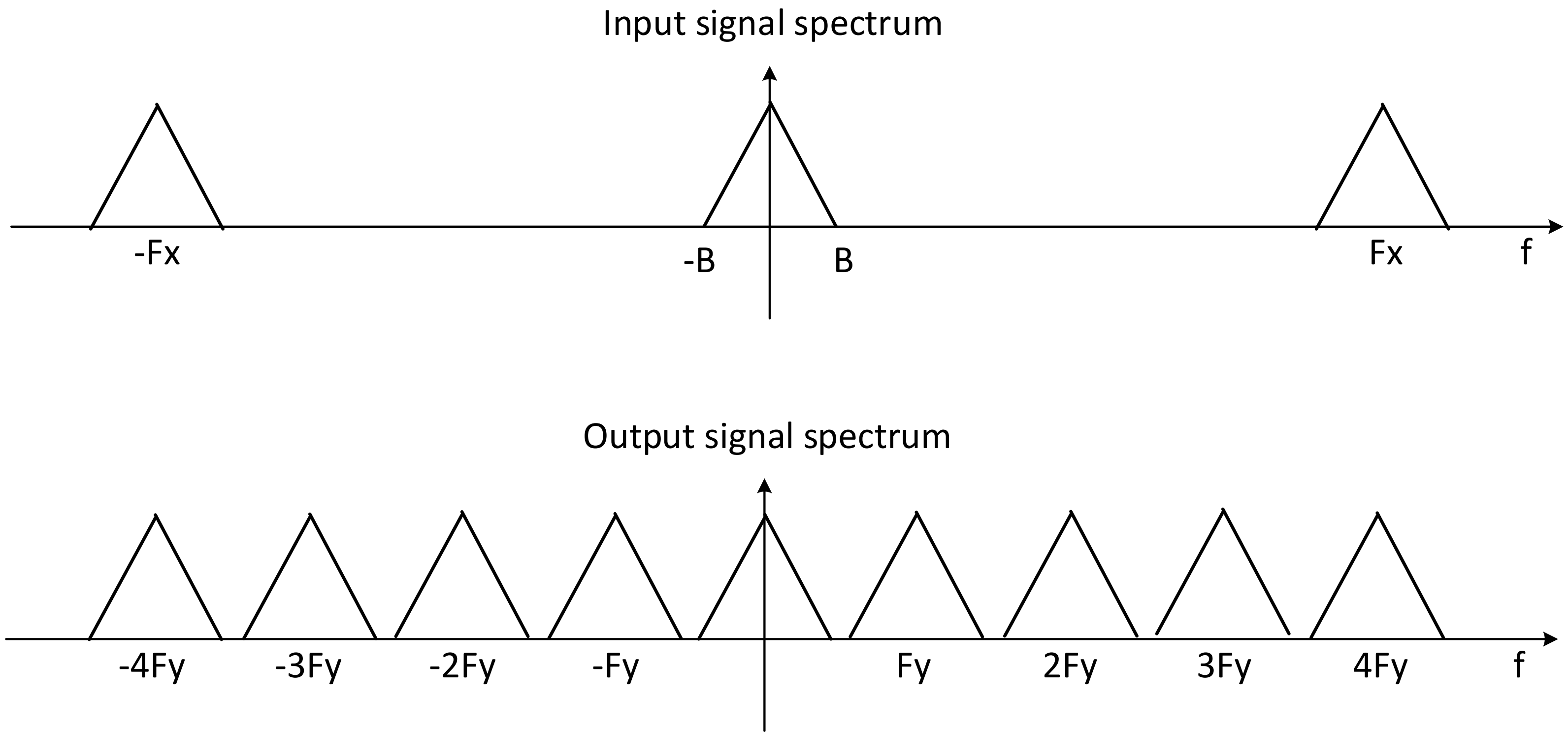
Question: what is the impact on the frequency spectrum if a signal $x[n]$ is downsampled by $D = \frac{T_y}{T_x}$?

Answer: in the frequency spectrum, assuming $D = 4$.



Three key observations:

1. $y[n]$ can be obtained directly from the continuous-time signal $x(t)$ at the new sampling rate.
2. There is a limit to the amount of downsampling: $Fy > 2B$ to avoid aliasing.



Must ensure Fy>2B to avoid aliasing

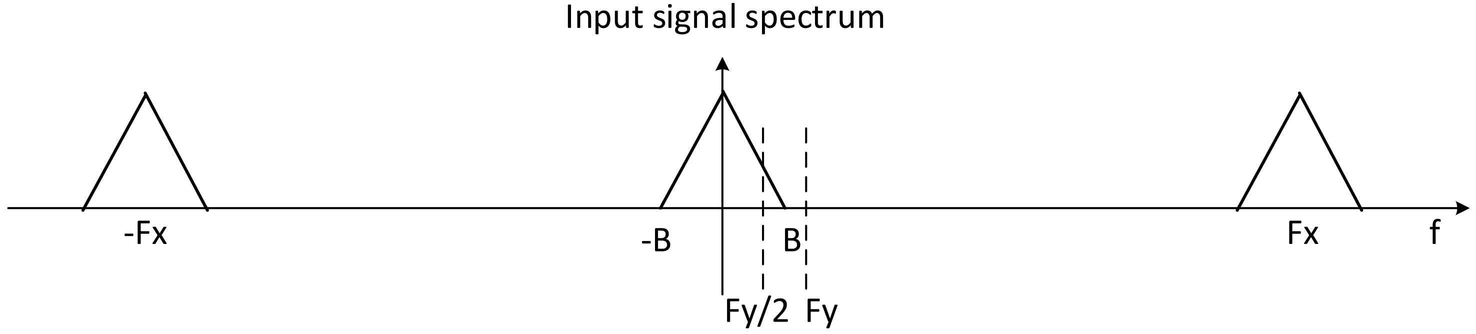
Decimation

If an application requires the downsampling to have a new sampling rate $Fy < 2B$, then the input signal needs to be filtered before the downsampling to avoid aliasing.

Discussion: what type of filter at which cutoff frequency should be used to implement decimation?

Decimation

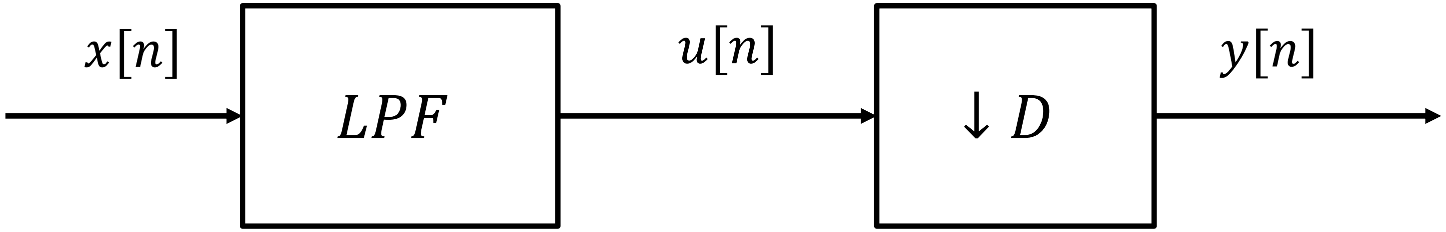
If an application requires the downsampling to have a new sampling rate $Fy < 2B$, then the input signal needs to be lowpass filtered before the downsampling to avoid aliasing.



The two-stage process of lowpass filter first and then downsample is referred to as **decimation**.

Decimation

A two-stage system that decimates an input signal by D is called a decimator.



Application of Decimation

Image sub-sampling:

1/2 1/4 (2x zoom) 1/16 (4x zoom)

Due to aliasing, simple downsampling leads to distortion in images (image by S. Seitz)

Application of Decimation

Image sub-sampling with Gaussian filtering:

1/2 1/4 (2x zoom) 1/16 (4x zoom)

Use a Gaussian filter as LPF and then downsample(image by S. Seitz)