Notes?

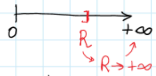
December 15, 2022 10:19 PM

Module 6 Introductory Examples

Wednesday, November 16, 2022 11:12 AM

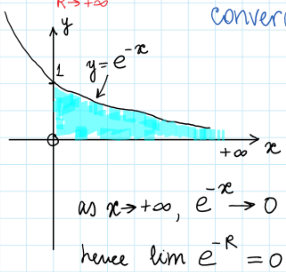
Improper Integrals

$$\boxtimes \quad \int_0^{\infty} e^{-x}\,dx = \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty}\left[1-e^{-R}\right] = 1 - \overbrace{\lim_{R\to\infty} e^{-R}}^{=0}$$



$$= 1$$

converges



as $x\to+\infty$, $e^{-x}\to 0$ asymptotically

hence $\lim_{R\to\infty} e^{-R} = 0$

Side Notes

$$\int_0^R e^{-x}\,dx = \left\{ \begin{matrix} u=-x \\ du=-dx \end{matrix} \right\}$$

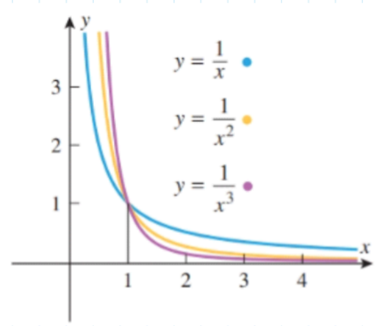
$$= \int_0^R e^{u}(-du) = -\int_0^R e^{u}\,du$$

$$= -\left[e^{u}\right]\Big|_0^R$$

$$= -e^{-x}\Big|_0^R = -\left[e^{R}-e^{0}\right]$$

$$= -\left[e^{R}-1\right] = 1-e^{R}$$

$\boxed{1}$ $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = 1$ $\quad\boxed{2}$ Show that $\displaystyle\int_1^{\infty}\frac{1}{x^3}\,dx = \frac{1}{2}$ (converges)



Soln. $$\int_1^{\infty}\frac{dx}{x^3} = \lim_{R\to\infty}\int_1^R x^{-3}\,dx = \lim_{R\to\infty}\left[\frac{x^{-2}}{-2}\right]\Big|_1^R$$

$$= \lim_{R\to\infty}\left[-\frac{1}{2x^2}\right]\Big|_1^R = \lim_{R\to\infty}\left[-\frac{1}{2R^2}-\left(-\frac{1}{2(1^2)}\right)\right]$$

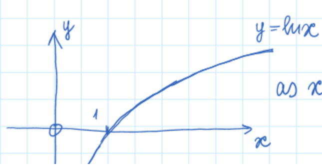
$$= \lim_{R\to\infty}\left[\frac{1}{2}-\frac{1}{2R^2}\right] = \frac{1}{2} - \underbrace{\lim_{R\to\infty}\frac{1}{2R^2}}_{=0} = \frac{1}{2}$$

converges

$\boxed{3}$ Show that $\displaystyle\int_1^{\infty}\frac{1}{x}\,dx$ diverges

$$\int_1^{\infty}\frac{dx}{x} = \lim_{R\to\infty}\int_1^R\frac{dx}{x} = \lim_{R\to\infty} \ln|R|\Big|_1^R = \lim_{R\to\infty}\left[\ln R - \overbrace{\ln 1}^{=0}\right] = \lim_{R\to\infty}\ln R$$

$$\int_1\frac{dx}{x} = \lim_{R\to\infty}\int_1\frac{dx}{x} = \lim_{R\to\infty} \ln|R|\Big|_1 = \lim_{R\to\infty}\left[\ln R - \ln 1\right] = \lim_{R\to\infty}\ln R$$



$$= +\infty$$

as $x\to+\infty$, $y=\ln x\to+\infty$

increases without bound

diverges

Q4

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Type 1

$$\frac{e^{2x}}{3} = \frac{(e^x)^2}{3} = \left(\frac{e^x}{\sqrt{3}}\right)^2$$

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx = \lim_{R\to\infty} \int_0^R \frac{e^x}{e^{2x}+3}dx \quad (*)$$

$$\int_0^R \frac{e^x}{e^{2x}+3}dx = \int_0^R \frac{1}{3}\frac{e^x}{\left(\frac{e^{2x}}{3}+1\right)}dx = \int_0^R \frac{1}{3}\frac{e^x}{\left(\frac{e^x}{\sqrt{3}}\right)^2+1}dx$$

U-substitution

$$u = \frac{e^x}{\sqrt{3}} \longrightarrow du = \frac{1}{\sqrt{3}}e^x dx \rightarrow \frac{\sqrt{3}\,du}{e^x} = dx$$

$$= \frac{1}{3}\int_0^R \frac{\cancel{e^x}}{u^2+1}\frac{\sqrt{3}}{\cancel{e^x}}du = \frac{\sqrt{3}}{3}\int_0^R \frac{du}{u^2+1} = \frac{\sqrt{3}}{3}\left[\tan^{-1}u\right]\Big|_0^R$$

$$= \frac{\sqrt{3}}{3}\left[\tan^{-1}\left(\frac{e^x}{\sqrt{3}}\right)\right]\Big|_0^R = \frac{\sqrt{3}}{3}\left\{\tan^{-1}\left(\frac{e^R}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{e^0}{\sqrt{3}}\right)\right\}$$

$$= \sqrt{3}\left\{\tan^{-1}\left(\frac{e^R}{\ }\right) - \tan^{-1}\left(\frac{1}{\ }\right)\right\} \qquad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$\frac{\pi}{6}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$= \frac{\sqrt{3}}{3}\left\{\tan^{-1}\left(\frac{e^R}{\sqrt{3}}\right) - \frac{\pi}{6}\right\}$$

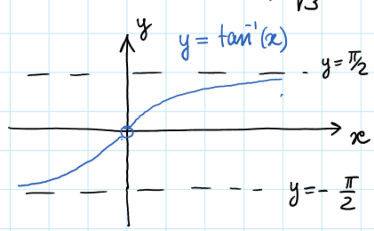
(*) $$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx = \lim_{R\to\infty} \frac{\sqrt{3}}{3}\left\{\tan^{-1}\left(\frac{e^R}{\sqrt{3}}\right) - \frac{\pi}{6}\right\}$$

$$= \frac{\sqrt{3}}{3}\left\{\lim_{R\to\infty}\left[\tan^{-1}\left(\frac{e^R}{\sqrt{3}}\right)\right] - \frac{\pi}{6}\right\} = \frac{\sqrt{3}}{3}\left\{\frac{\pi}{2} - \frac{\pi}{6}\right\} = \frac{\sqrt{3}}{3}\,\frac{\pi}{3}$$

$$= \frac{\sqrt{3}\,\pi}{9} = \frac{\pi}{3\sqrt{3}}$$

$$= \frac{\sqrt{3}\,\pi}{9} = \frac{\pi}{3\sqrt{3}}$$

as $R \to \infty$, $e^R \to \infty$, $\frac{e^R}{\sqrt{3}} \to \infty$, $\tan^{-1}\left(\frac{e^R}{\sqrt{3}}\right) = \frac{\pi}{2}$



Q5

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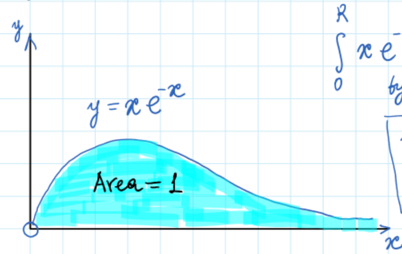
$$\int_0^\infty x e^{-x}\,dx = \lim_{R\to+\infty} \int_0^R x e^{-x}\,dx \quad (*)$$



$$\int_0^R x e^{-x}\,dx = uv - \int v\,du \Big]_0^R$$

by parts

$$\boxed{\begin{array}{l|l} u = x & dv = e^{-x} \\ du = dx & \hookrightarrow v = -e^{-x} \end{array}}$$

$$= -xe^{-x} - \int -e^{-x}\,dx$$

$$= -xe^{-x} + (-e^{-x}) = -e^{-x}(x+1)\Big|_0^R$$

$$= -e^{-R}(R+1) - \{-e^{0}(1)\}$$

$$= -e^{-R}(R+1) + 1 = 1 - \frac{R+1}{e^R}$$

$$(*)\quad \lim_{R\to+\infty}\left[1 - \frac{R+1}{e^R}\right] = 1 - \lim_{R\to+\infty}\frac{R+1}{e^R} = 1 - 0 = 1 \text{ Answer}$$

$$\lim_{R\to\infty}\frac{R+1}{e^R} = \frac{\infty}{\infty} = \{\text{L'H Rule}\} = \lim_{R\to\infty}\frac{1}{e^R} = 0$$

Q6

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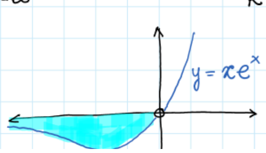
2. If the integral $\int_r^b f(x)\,dx$ exists for all $r < b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{r\to-\infty}\int_r^b f(x)\,dx$$

when the limit exists (and is finite).

$$\int_{-\infty}^0 x e^{x}\,dx = \lim_{R\to-\infty}\int_R^0 x e^{x}\,dx$$



$$\int_R^0 x e^x\,dx = xe^x - \int_R^0 e^x\,dx$$

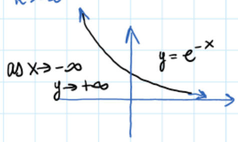
$$= xe^x - e^x\Big|_R^0 = e^x(x-1)\Big|_R^0$$

$$= e^{0}(0-1) - e^{R}(R-1)$$

$$= -1 - e^{R}(R-1)$$

$$\int_{-\infty}^{0} xe^{x}\,dx = \lim_{R\to-\infty}\left[-1 - e^{R}(R-1)\right] = -1 - \lim_{R\to-\infty} e^{R}(R-1)$$

$$\lim_{R\to-\infty} e^{R}(R-1) = 0\cdot\infty = \lim_{R\to-\infty}\frac{R-1}{e^{-R}} = \frac{\infty}{\infty} = \{\text{L'H Rule}\} = \lim_{R\to-\infty}\frac{1}{-e^{-R}} =$$



$$= -\frac{1}{\lim_{R\to-\infty} e^{-R}} = -\frac{1}{\infty} = 0$$

Q7

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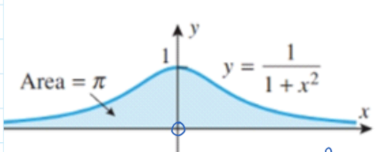
3. If the integral $\int_r^R f(x)\,dx$ exists for all $r < R$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to-\infty}\int_r^c f(x)\,dx + \lim_{R\to\infty}\int_c^R f(x)\,dx$$

when both limits exist (and are finite). Any c can be used.

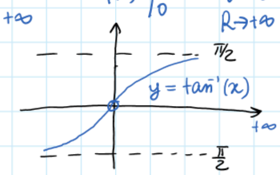


$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{+\infty}\frac{dx}{1+x^2}$$

$$\int_{0}^{+\infty}\frac{dx}{1+x^2} = \lim_{R\to+\infty}\int_{0}^{R}\frac{dx}{1+x^2} = \lim_{R\to+\infty}\tan^{-1}(x)\Big|_0^R = \lim_{R\to+\infty}\{\tan^{-1}(R) - \overbrace{\tan^{-1}(0)}^{=0}\}$$

$$= \lim_{R\to\infty}\tan^{-1}(R) = \frac{\pi}{2}$$

converges



$$\int_{-\infty}^{0}\frac{dx}{1+x^2} = \lim_{R\to-\infty}\int_{R}^{0}\frac{dx}{1+x^2} = \lim_{R\to-\infty}\tan^{-1}(x)\Big|_R^0 = \lim_{R\to-\infty}\{\tan^{-1}(0) - \tan^{-1}R\}$$

$$= \lim_{R\to\infty}\{0 - \tan^{-1}(R)\} = -\lim_{R\to-\infty}\tan^{-1}(R) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$ converges

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\to 2\int_0^{+\infty} \frac{dx}{1+x^2}$$ because $y = \frac{1}{1+x^2}$ is an "even" function

$f(x)$ is even if $f(-x) = f(x)$

i.e. $y = x^2$, $y = x^8$, $y = \cos x$

$f(x)$ is "odd" if $f(-x) = -f(x)$

$y = x^3$, $y = \sin x$, $\tan x$

Q8 The Comparison Theorem

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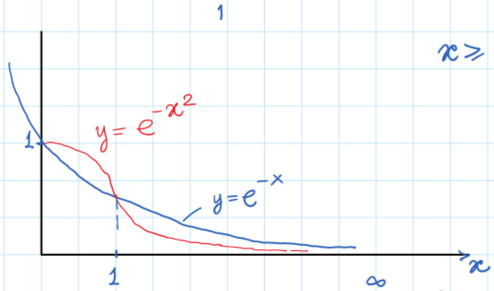
a) Show that $\int_1^{\infty} e^{-x^2} dx$ is convergent $e^{-x^2} \leq ???$



$x \geq 1 \Rightarrow x^2 \geq x \Rightarrow -x^2 \leq -x$

$e^{-x^2} \leq e^{-x}$

for all $x \geq 1$

$-9 < -3$

$e^{-9} < e^{-3}$

$$\int_1^{\infty} e^{-x} dx = \lim_{R\to+\infty} \int_1^R e^{-x} dx$$

$$= \lim_{R\to-\infty} -e^{-x}\Big|_1^R = \lim_{R\to-\infty} \{-e^{-R} - (-e^{-1})\}$$

$$= \lim_{R\to+\infty} \left[\frac{1}{e} - \frac{1}{e^R}\right] = \frac{1}{e} - \lim_{R\to+\infty} \frac{1}{e^R} = \frac{1}{e} - 0 = \frac{1}{e}$$

converges

Comparing integrands $e^{-x^2} \leq e^{-x}$

Conclusion: as $\int_1^{\infty} e^{-x} dx$ converges, then $\int_1^{\infty} e^{-x^2} dx$ converges

Q9 Show that $\int_1^{\infty} \frac{x-2}{x^3+1} dx$ converges

Comparing integrands $\frac{x-2}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}$

remove

remove

remove

Since $\int_1^\infty \frac{1}{x^2}\,dx$ - converges, and $\frac{x-2}{x^3+1} < \frac{1}{x^2}$, we conclude that $\int_1^\infty \frac{x-2}{x^3+1}\,dx$ converges by CT

Q10(from Assignment 4)

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$$\int_1^\infty \frac{\ln x}{x^5}\,dx = [\text{Type } 1] = \lim_{R\to+\infty} \int_1^R x^{-5}\ln x\,dx \quad (*)$$

by Parts $\int_1^R x^{-5}\ln x\,dx = -\frac{1}{4x^4}\ln x - \int -\frac{1}{4x^4}\,\frac{1}{x}\,dx\Big]_1^R \quad (*)$

$u = \ln x$ | $dv = x^{-5}\,dx$

$du = \frac{1}{x}\,dx$ | $\hookrightarrow v = \frac{x^{-4}}{-4} = -\frac{1}{4x^4}$

$$= -\frac{\ln x}{4x^4} + \frac{1}{4}\int x^{-5}\,dx\Big|_1^R = -\frac{\ln x}{4x^4} + \frac{1}{4}\left(-\frac{1}{4x^4}\right)\Big|_1^R = -\frac{\ln x}{4x^4} - \frac{1}{16x^4}\Big|_1^R$$

$$= -\frac{1}{16x^4}(4\ln x + 1)\Big|_1^R = -\frac{1}{16}\left\{\frac{1}{R^4}(4\ln R+1) - \frac{1}{1^4}(4\ln 1+1)\right\}$$

$$= -\frac{1}{16}\left\{\frac{1}{R^4}(4\ln R+1) - 1\right\} = \frac{1}{16} - \frac{1}{16}\left[\frac{1}{R^4}(4\ln R+1)\right]$$

$$\circledast \quad = \lim_{R\to+\infty}\left[\frac{1}{16} - \frac{1}{16}\left(\frac{1}{R^4}(4\ln 4+1)\right)\right] = \frac{1}{16} - \underbrace{\lim_{R\to+\infty}\frac{1}{16R^4}(4\ln R+1)}_{=0}$$

$$\lim_{R\to+\infty}\frac{4\ln R+1}{R^4} = \frac{\infty}{\infty} = \{\text{L'H Rule}\} = \lim_{R\to+\infty}\frac{4/R}{4R^3} = \lim_{R\to+\infty}\frac{1}{R^4} = 0$$

Module 6 (continued) Improper Integrals in Week 12B

Friday, December 2, 2022 8:57 AM

Warm Up

$\boxed{1}$ $$\int_e^{\infty}\frac{1}{x(\ln x)^3}\,dx = \lim_{R\to+\infty}\int_e^R \frac{1}{x(\ln x)^3}\,dx$$

Type I $\quad e \quad R \to +\infty$

$$\int\frac{1}{x(\ln x)^3}\,dx = \int\frac{1}{\cancel{x}u^3}\,\cancel{x}\,du = \int u^{-3}\,du = \frac{u^{-2}}{-2} = -\frac{1}{2(\ln x)^2} + C$$

u-sub: $\boxed{u = \ln x \quad du = \frac{1}{x}dx \quad x\,du = dx}$

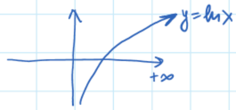
$\boxed{\ln e = 1}$

$$\lim_{R\to+\infty}\left[-\frac{1}{2\ln(x)^2}\right]\Big|_e^R = -\frac{1}{2}\left\{\lim_{R\to+\infty}\left[\frac{1}{(\ln(R))^2} - \underbrace{\frac{1}{(\ln e)^2}}_{=1}\right]\right.$$

$$= -\frac{1}{2}\left\{\lim_{R\to\infty}\frac{1}{(\ln R)^2} - 1\right\} = \frac{1}{2} - \frac{1}{2}\lim_{R\to\infty}\frac{1}{(\ln R)^2} = \frac{1}{2} \quad \text{converges}$$

as $R\to\infty$ then $\ln R \to +\infty$ then $(\ln R)^2 \to +\infty$ then $\frac{1}{\ldots} \to 0$

as $R \to +\infty$, then ... $(\ln R)^2$

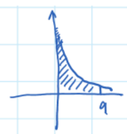


Type 2 Improper Integrals

Friday, December 2, 2022 9:50 AM

$\boxed{1}$ $\displaystyle\int_0^9 \frac{dx}{\sqrt{x}}$ ← improper

VA @ $x=0$



$$\int x^{-\frac{1}{2}}dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x} + C$$

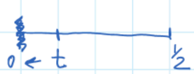
$$= \lim_{t\to 0^+} \int_t^9 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\left\{ 2\sqrt{x} \Big|_t^9 \right\} = \lim_{t\to 0^+}\left\{2\sqrt{9} - 2\sqrt{t}\right\}$$

$$= 2\sqrt{9} - 2\lim_{t\to 0^+}\sqrt{t} = 2\sqrt{9} - 2(0) = 2\sqrt{9} = 6$$

as $t \to 0^+$, $\sqrt{t} \to 0$

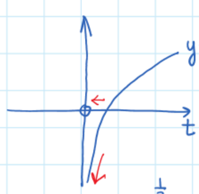
$\boxed{2}$ $$\int_0^{\frac{1}{2}} \frac{dx}{x} = \lim_{t\to 0^+}\int_t^{\frac{1}{2}} \frac{dx}{x} = \lim_{t\to 0^+}\Big[\ln|x|\Big]\Big|_t^{\frac{1}{2}} = \lim_{t\to 0^+}\left[\ln\tfrac{1}{2} - \ln t\right]$$

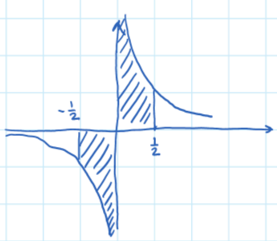
VA @ $x=0$ ← improper



$$= \ln\tfrac{1}{2} - \lim_{t\to 0^+}[\ln t] = \ln\tfrac{1}{2} - (-\infty) = \infty$$

diverges





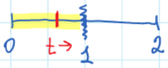
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{x} = \infty - \infty \quad \text{Indeterminate Form}$$

$f(x)$ is odd if $f(-x) = -f(x)$

2 1 2

$$\boxed{3}\quad \int_0^2 \frac{dx}{(x-1)^{2/3}} = \int_0^1 (x-1)^{-2/3}\,dx + \int_1^2 (x-1)^{-2/3}\,dx$$

VA @ $x=1$



$$\int_0^1 (x-1)^{-2/3}\,dx = \lim_{t\to 1^-} \int_0^t (x-1)^{-2/3}\,dx$$

Antiderivative

$$\int (x-1)^{-2/3}\,dx = \frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}} = 3\sqrt[3]{x-1}$$

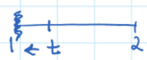
$$= \lim_{t\to 1^-} \left[3\sqrt[3]{x-1}\right]\Big|_0^t$$

$$= 3\lim_{t\to 1^-} \left[\sqrt[3]{t-1} - \sqrt[3]{-1}\right]$$

$$= 3\lim_{t\to 1^-} \left[\sqrt[3]{t-1} + 1\right] = 3 + 3\lim_{t\to 1^-} \sqrt[3]{t-1} = 3 + 0 = 3 \text{ converges}$$

"direct sub"

$$\int_1^2 (x-1)^{-2/3}\,dx = \lim_{t\to 1^+} \int_t^2 (x-1)^{-2/3}\,dx = \lim_{t\to 1^+} 3\sqrt[3]{x-1}\,\Big|_t^2 = 3\lim_{t\to 1^+}\left[\sqrt[3]{2-1} - \sqrt[3]{t-1}\right]$$



$$= 3\left(1 - \underbrace{\lim_{t\to 1^+} \sqrt[3]{t-1}}_{=0}\right) = 3 \text{ converges}$$

Ans $= 6$ and converges

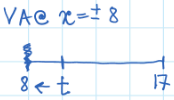
Assignment 4 Help

Friday, December 2, 2022 10:17 AM

$x^2 - 64 = (x-8)(x+8) = 0$

$$\int_8^{17} \frac{1}{\sqrt{x^2-64}}\,dx = \lim_{t\to 8^+} \int_t^{17} \frac{1}{\sqrt{x^2-64}}\,dx$$

VA @ $x = \pm 8$



$$\int \frac{1}{\sqrt{x^2-64}}\,dx = \int \frac{8\sec\theta\,\cancel{\tan\theta}\,d\theta}{8\,\cancel{\tan\theta}} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$$

$x^2 - a^2 \Rightarrow x = a\sec\theta$

$x^2 - 8^2 \Rightarrow x = 8\sec\theta \quad \sqrt{x^2-8^2} = \sqrt{8^2\sec^2\theta - 8^2} = 8\sqrt{\sec^2\theta - 1} = 8\tan\theta$

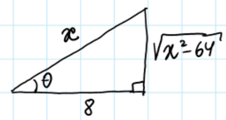
$a=8 \Rightarrow x = 8\sec\theta$ $\sqrt{x^2-8^2} = \sqrt{8^2\sec^2\theta - 8^2} = 8\sqrt{\sec^2\theta - 1} = 8\tan\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$dx = 8\, d(\sec\theta)\, d\theta = 8 \sec\theta \tan\theta\, d\theta$

$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$

Return from $\theta \to x$



$\sec\theta = \frac{1}{\cos\theta}$

$\cos\theta = \frac{adj}{hyp} = \frac{8}{x}$

$\sec\theta = \frac{x}{8}$

$\int \frac{1}{\sqrt{x^2-64}}\, dx = \ln|\sec\theta + \tan\theta| =$

$\tan\theta = \frac{opp}{adj} = \frac{\sqrt{x^2-64}}{8}$

$= \ln\left|\frac{x}{8} + \frac{\sqrt{x^2-64}}{8}\right| = \ln\left|\frac{x+\sqrt{x^2-64}}{8}\right|$

Going back

$\int_8^{17} \frac{1}{\sqrt{x^2-64}}\, dx = \lim_{t\to 8^+} \ln\left|\frac{x+\sqrt{x^2-64}}{8}\right|\Bigg|_t^{17} = \lim_{t\to 8^+}\left[\ln\frac{17+\sqrt{17^2-64}}{8} - \ln\left|\frac{t+\sqrt{t^2-64}}{8}\right|\right]$

$= \lim_{t\to 8^+}\left[\ln 4 - \ln\frac{t+\sqrt{t^2-64}}{8}\right] = \ln 4 - \underbrace{\lim_{t\to 8^+} \ln\frac{t+\sqrt{t^2-64}}{8}}_{=0}$

as $t\to 8^+$, then $\frac{t+\sqrt{t^2-64}}{8} \to 1$, then $\ln[\ldots] \to 0$

$= \ln 4$ converges