Calc 1500, Week 4

Multivariable and Vector-Valued Functions

A **multivariable function** is a function whose domain is a subset of $\mathbb{R}^n$ for some $n \geq 2$, that is, a function that depends on two or more variables.

Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x + y$.

A **vector-valued function** is a function whose codomain is a subset of $\mathbb{R}^n$ for some $n \geq 2$, that is, a function whose output consists of two or more components.

Example: Consider $\vec{f} : \mathbb{R} \to \mathbb{R}^2$ given by $\vec{f}(t) =< t, t^2 >$.

Limits tell us what *ought* to happen

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that

as (x, y) gets *close* to (a, b),

the values $f(x, y)$ of the function get *close* to L.

Limits behave well under arithmetic and composition

If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$, then

- $\lim_{(x,y)\to(a,b)} \Big(f(x,y) + g(x,y)\Big) = L + M$
- $\lim_{(x,y)\to(a,b)} f(x,y) \cdot g(x,y) = LM$
- $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ when $M \neq 0$

If h is continuous at L, then the composite function $h \circ f$ is continuous at (a,b) with $\lim_{(x,y)\to(a,b)} h\big(f(x,y)\big) = h(L)$.

Continuity is the agreement between what ought to happen and what actually happens

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in D$. If

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b),$$

then we say that f is **continuous at** (a, b).

If f is continuous at every point in its domain D, then we say that f **is continuous on** D.

Polynomials, rational functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions, and all the sums, products, quotients (where well-defined), and compositions of such functions are continuous.

Strategy 1: Continuous function, sensible answer

If the function is continuous and the answer is sensible when you plug in the point under consideration, then the limit exists and equals the value of the function at that point.

Example: Find

trigonometric exponential

$$\lim_{(x,y)\to(1,2)} f(x,y), \quad \text{where} \quad f(x,y) = \frac{\cos(x-1)\,e^{2-y}}{x^2+y^2}.$$

polynomial

$$= \frac{\cos(1-1)e^{2-2}}{1^2+2^2} = \frac{\cos(0)e^0}{1+4} = \frac{1\cdot 1}{5} = \frac{1}{5}$$

since $f(x,y)$ is continuous at $(1,2)$.

Live Poll. Strategy 1: Continuous function, sensible answer

$$\lim_{(x,y)\to(-1,1)} \frac{x^6y^3}{e^{x+y}} = \underbrace{\frac{(-1)^6(1)^3}{e^{-1+1}}}_{f(-1,1)} = \frac{(1)(1)}{e^0} = 1$$

$f(x,y) = \dfrac{x^6y^3}{e^{x+y}}$ is continuous because it is the quotient of a polynomial (x^6y^3) and an exponential function (e^{x+y})

Strategy 2: If the limit exists, it must be the same along *every* path

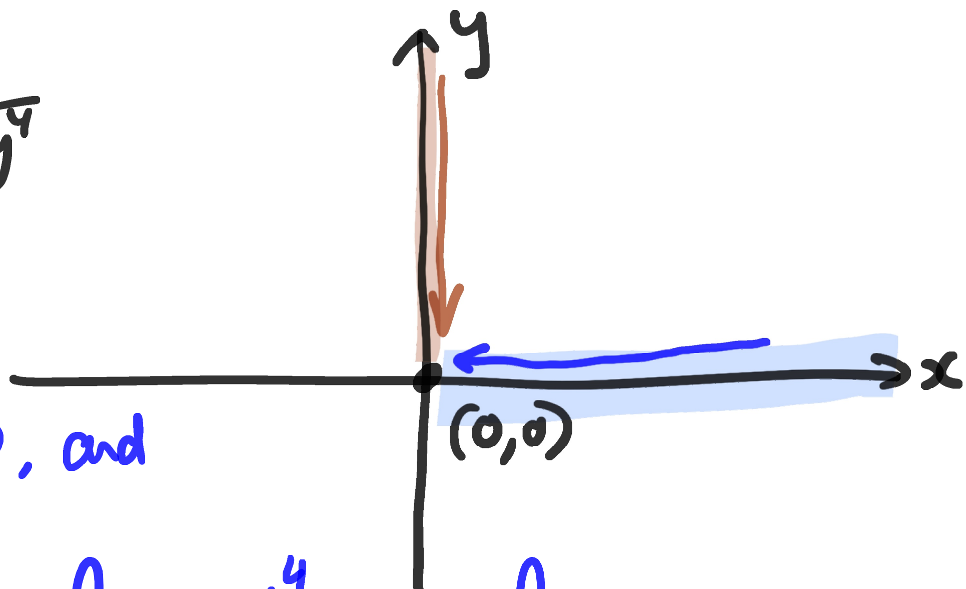
If the limit value differs depending on the path taken to the point, then the limit fails to exist.

Example: Determine whether the following limit exists? If it exists, what is the limiting value? If it does not, why not?

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^4}.$$

$\lim\limits_{x\to a} f(x)$ $\quad$ $\lim\limits_{x\to a^-} f(x)$ $\quad$ $\lim\limits_{x\to a^+} f(x)$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^4}$$



Let $x=t$, $y=0$, and let $t\to 0^+$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^4} = \lim_{t\to 0^+} \frac{t^4}{t^4+0^4} = \lim_{t\to 0^+} 1 = 1$$

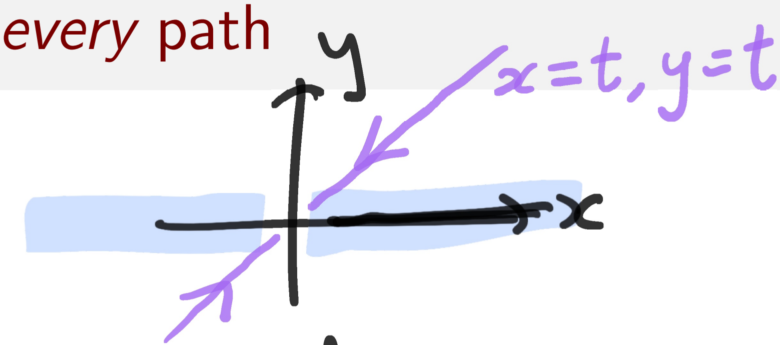
let $x=0$, $y=t$, and $t\to 0^+$.

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^4} = \lim_{t\to 0^+} \frac{0^4}{0^4+t^4} = \lim_{t\to 0^+} 0 = 0$$

Since the limit is different along different paths,

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^4}$$ does not exist.

Live Poll. Strategy 2: If the limit exists, it must be the same along *every* path



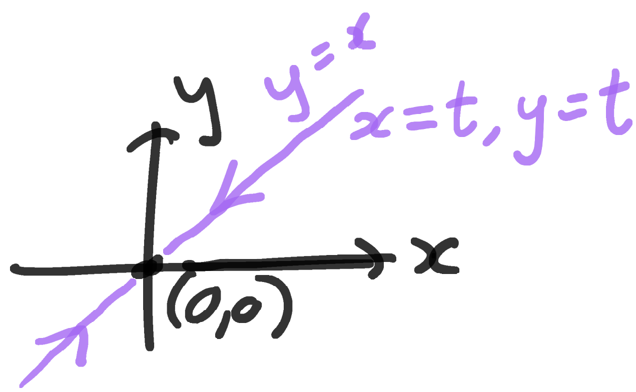
a) Let $x=t$, $y=0$, and $t \to 0$

$$f(x,y) = \boxed{f(t,0)} = \frac{t\cdot 0}{t^2+0^2} = \frac{0}{t^2} = \boxed{0}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} \boxed{f(t,0)} = \lim_{t\to 0} \boxed{0} = 0$$

b) We don't have enough information yet about whether the limit existed.

c)



$$f(t,t) = \frac{t \cdot t}{t^2+t^2} = \frac{t^2}{2t^2} = \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

d) Since limit is different along different paths, the limit does not exist.

Strategy 3: Squeeze Theorem for Limits

Bound the function in question between two functions that are easier to work with that have a common limiting value.

Example: Find

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4+y^4}.$$

Notice $0 \le \dfrac{x^4}{x^4+y^4} \le 1$

So $0 \le \left(\dfrac{x^4}{x^4+y^4}\right) y \le y$, if $y \ge 0$

[If $y<0$, then $0 \ge \dfrac{x^4}{x^4+y^4} \cdot y \ge y$]

Since $(x,y) \to (0,0)$, $y \to 0$.

Since $\lim_{(x,y)\to(0,0)} 0 = 0 = \lim_{(x,y)\to(0,0)} y$,

by the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4+y^4} = 0.$$

Strategy 4: Carry out a substitution that allows us to view the limit as that of a one-variable function

Example: Find

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}.$$

Let $u = x^2+y^2$. As $(x,y)\to(0,0)$, then $u = x^2+y^2 \to 0$. So

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

Strategy 5: Use Polar Coordinates

Suppose we wish to find $\lim_{(x,y)\to(0,0)} f(x,y)$. Let

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

If $f(r\cos\theta, r\sin\theta)$ ends up being

a function of r which goes to 0 as $r \to 0$

multiplied by

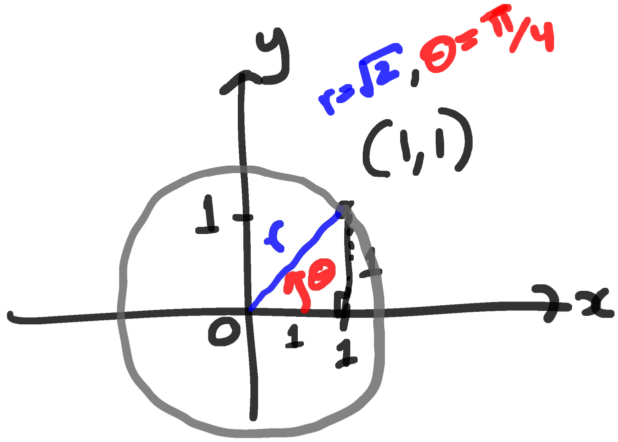
a bounded function of θ,

then $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Example: Find

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4 + y^4}.$$

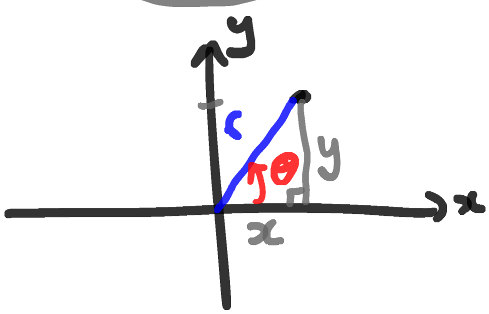


Polar coordinates

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 \cdot y}{x^4+y^4} = ?$$

Let $x = r\cos\theta$ and $y = r\sin\theta$.

As $(x,y)\to(0,0)$, $r\to 0$.

So
$$\lim_{(x,y)\to(0,0)} \frac{x^4 \cdot y}{x^4+y^4} = \lim_{r\to 0} \frac{(r\cos\theta)^4\, r\sin\theta}{(r\cos\theta)^4+(r\sin\theta)^4}$$

$$= \lim_{r\to 0} \frac{\cancel{r^5}^{\,r}\cos^4\theta\sin\theta}{\cancel{r^4}_{\,1}(\cos^4\theta+\sin^4\theta)}$$

$$= \lim_{r\to 0} r \frac{\cos^4\theta\sin\theta}{\cos^4\theta+\sin^4\theta}$$

is a bounded function of θ

$= 0$ since $r\to 0$

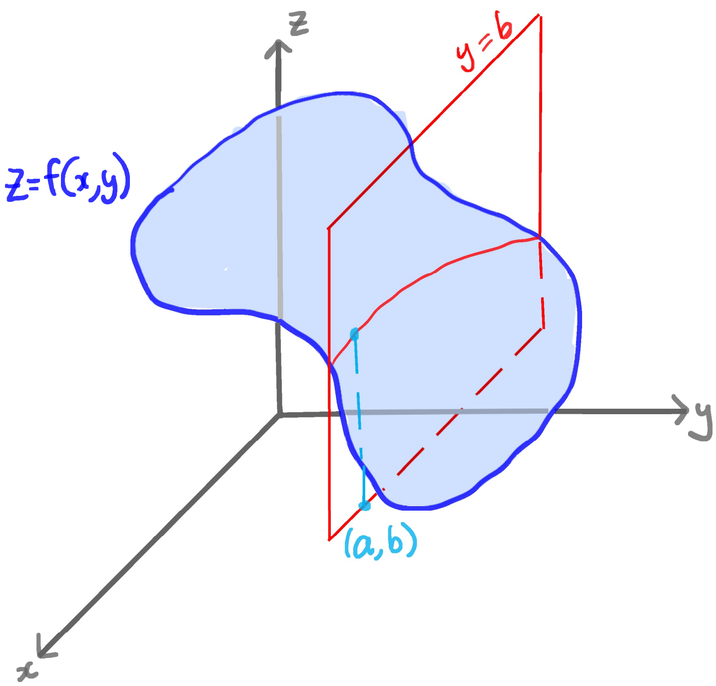
Comment:

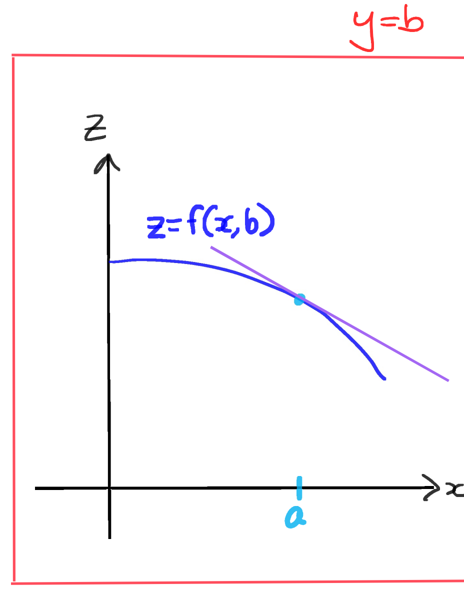
$\sin\theta \leq 1$

$$\frac{\cos^4\theta\sin\theta}{\cos^4\theta+\sin^4\theta} \leq \frac{\cos^4\theta}{\cos^4\theta+\sin^4\theta} \leq 1$$

What are partial derivatives?

$$\frac{\partial f}{\partial x}(a, b), \qquad \frac{\partial f}{\partial y}(a, b)$$

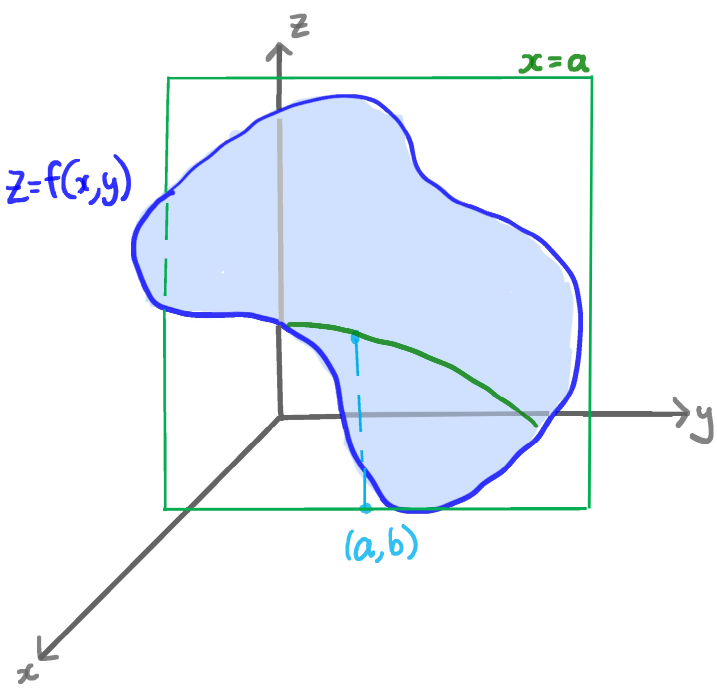


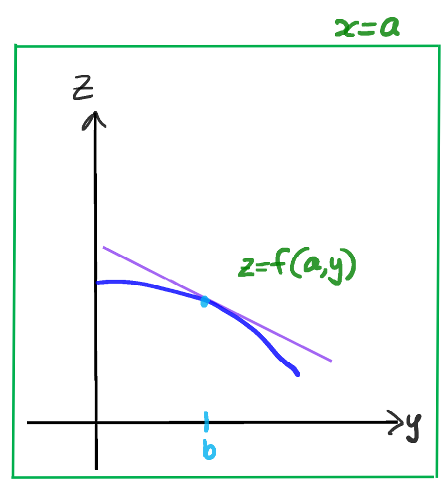


Slope of tangent line

$$= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$\overset{\text{definition}}{=} \frac{\partial f}{\partial x}(a,b)$$





Slope of tangent line

$$= \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

$$\overset{\text{definition}}{=} \frac{\partial f}{\partial y}(a,b)$$

Source: Shifrin's Multivariable Mathematics

Notation for partial derivatives

The partial derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}$ with respect to x can be denoted in several different ways:

$$\frac{\partial f}{\partial x}(x, y), \quad \frac{\partial f}{\partial x}, \quad f_x(x, y), \quad f_x, \quad D_x f(x, y), \quad D_x f, \quad D_1 f(x, y),$$

When we evaluate the partial derivative at a point (a, b), we usually write one of

$$\frac{\partial f}{\partial x}(a, b), \quad \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(a,b)}, \quad \left.\frac{\partial f}{\partial x}\right|_{x=a,y=b}.$$

Example: Calculating Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^2 y.$$

Find $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$. Evaluate these partial derivatives at the point $(-2, 1/2)$.

$\frac{\partial f}{\partial x}(x,y) = 2xy$, $\frac{\partial f}{\partial y}(x,y) = x^2$

$\frac{\partial f}{\partial x}(-2,\frac{1}{2}) = 2(-2)(\frac{1}{2})$, $\frac{\partial f}{\partial y}(-2,\frac{1}{2}) = (-2)^2 = 4$

$= -\frac{1}{2}$

Live Poll: Calculating Partial Derivatives

$$f(x,y) = x^2y^3$$

$$f_x(x,y) = 2xy^3$$

$$f_y(x,y) = 3x^2y^2$$

A more involved example in calculating partial derivatives

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be given by

$$f(x, y, z, t) = x \sin(y + 2z) + t^2 e^{3y} \ln(z).$$

Find all four partial derivatives and evaluate them at $(1, -2, 1, -1)$.

$$\frac{\partial f}{\partial z} = \underbrace{x\cos(y+2z)\cdot\frac{\partial}{\partial z}(y+2z)}_{\text{by Chain Rule}} + t^2e^{3y}\cdot\underbrace{\frac{1}{z}\cdot\frac{\partial}{\partial z}(z)}_{\text{Chain Rule}}$$

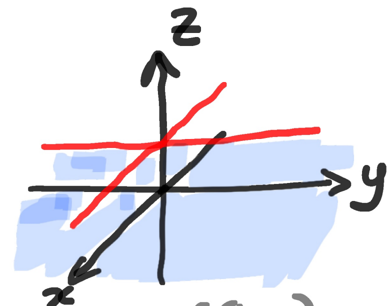
$$= x\cos(y+2z)\cdot(0+2) + t^2e^{3y}\cdot\frac{1}{z}\cdot 1$$

$$= 2x\cos(y+2z) + \frac{t^2e^{3y}}{z}$$

$$
\begin{aligned}
\frac{\partial f}{\partial z}(\underset{x}{1},\underset{y}{-2},\underset{z}{1},\underset{t}{-1}) &= 2(1)\cos(-2+2(1)) + \frac{(-1)^2 e^{3(-2)}}{1} \\
&= 2\cos(0) + e^{-6} \\
&= 2 + e^{-6}
\end{aligned}
$$

Use the definition of partial derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by



$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0. \end{cases}$$

1. Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.
2. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.
3. Is f continuous at $(0,0)$?

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\overbrace{f(0+h,0)}^{f(h,0)} - f(0,0)}{h}$$

fix y at 0

$$= \lim_{h\to 0} \frac{1-1}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

$$f(x,y)=\begin{cases}0, & \text{if } xy\neq 0\\ 1, & \text{if } xy=0\end{cases}$$

Fix x at 0

$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}$$

Since $0\cdot h=0$

Since $0\cdot 0=0$

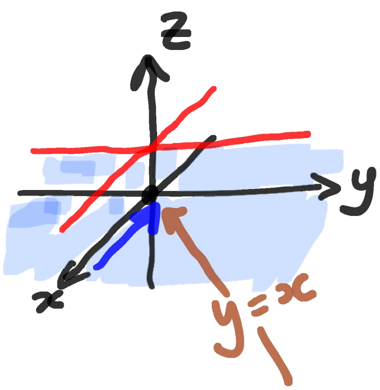
$$=\lim_{h\to 0}\frac{1-1}{h}$$

$$=\lim_{h\to 0}\frac{0}{h}$$

$$=0$$

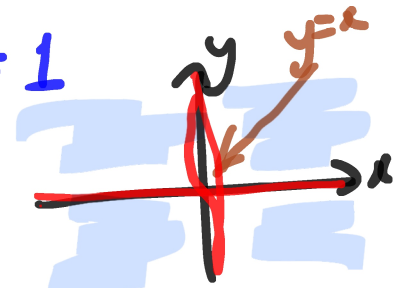
b) $$f(x,y)=\begin{cases}0, & \text{if } xy\neq 0\\ 1, & \text{if } xy=0\end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y)=?$$



Approach (0,0) along x-axis:

$$\lim_{t\to 0} f(t,0) = \lim_{t\to 0} 1 = 1$$



Approach (0,0) along line x=y:

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0} 0 = 0$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

and, thus, $f(x,y)$ is not continuous at (0,0).

For f to be continuous at (0,0), $\lim_{(x,y)\to(0,0)} f(x,y)$ must exist and must equal $f(0,0)$.

But this limit doesn't exist.