Worksheet 3 - Solution

1) Using the Routh-Hurwitz criterion, determine the stability of the closed-loop system that has the following characteristic equations. Determine the number of roots of each equation that are in the right-half s-plane and on the $j\omega$ axis.

a) $s^3 + 25s^2 + 10s + 450 = 0$

The Routh-Hurwitz table:

s^3	1	10
s^2	25	450
s^1	-8	0
s^0	450	0

Two sign changes in the first column. Two roots in right-half s-plane.

b) $s^3 + 25s^2 + 10s + 50 = 0$

The Routh-Hurwitz table:

s^3	1	10
s^2	25	50
s^1	8	0
s^0	50	0

No sign changes in the first column. No roots in right-half s-plane.

c) $s^3 + 25s^2 + 250s + 10 = 0$

The Routh-Hurwitz table:

s^3	1	250
s^2	25	10
s^1	2496	0
s^0	10	0

No sign changes in the first column. No roots in right-half s-plane.

d) $2s^4 + 10s^3 + 5.5s^2 + 5.5s + 10 = 0$

The Routh-Hurwitz table:

s^4	2	5.5	10
s^3	10	5.5	0
s^2	4.4	10	0
s^1	-758	0	0
s^0	10	0	0

Two sign changes in the first column. Two roots in right-half s-plane.

e) $s^6 + 2s^5 + 8s^4 + 15s^3 + 20s^2 + 16s + 16 = 0$

The Routh-Hurwitz table:

s^6	1	8	20	16
s^5	2	15	16	0
s^4	0.5	12	16	0
s^3	-33	-48	0	0
s^2	1127	16	0	0
s^1	-116	0	0	0
s^0	16	0	0	0

Four sign changes in the first column. Four roots in right-half s-plane.

f) $s^4 + 2s^3 + 10s^2 + 20s + 5 = 0$

The Routh-Hurwitz table:

s^4	1	10	5
s^3	2	20	0
s^2	0 ε	5	0
s^1	$\frac{20\varepsilon - 10}{\varepsilon} \cong -\frac{10}{\varepsilon}$	0	0
s^0	5	0	0

Replace 0 in the third row by a small positive number $\varepsilon > 0$ and continue to complete the table.

Two sign changes in the first column. Two roots in right-half s-plane.

g) $s^8 + 2s^7 + 8s^6 + 12s^5 + 20s^4 + 16s^3 + 16s^2 = 0$

The Routh-Hurwitz table:

s^8	1	8	20	16	0
s^7	2	12	16	0	0
s^6	2	12	16	0	0
s^5	0 12	0 48	0 32	0 0	0 0
s^4	2	$\frac{16}{3}$	0	0	0
s^3	16	32	0	0	0
s^2	1.33	0	0	0	0
s^1	32	0	0	0	0
s^0	0 32	0 0	0 0	0 0	0 0

There are all zero row in s^5. Form the auxiliary equation from row s^6

$A(s) = 2s^6 + 12s^4 + 16s^2$

Take derivative of the auxiliary equation with respect to s.

$$\frac{dA(s)}{ds} = 12s^5 + 48s^3 + 32s$$

Replace the coefficients in all zero row s^5

There are all zero row in s^0. Form the auxiliary equation from row s^1

$A(s) = 32s$

Take derivative of the auxiliary equation with respect to s.

$$\frac{dA(s)}{ds} = 32$$

Replace the coefficients in all zero row s^0

No sign changes in the first column. No roots in right-half s-plane.

2) For each of the characteristic equations of feedback control systems given, use Routh-Hurwitz criterion to determine the range of K so that the system is stable. Determine the value of K so that the system is marginally stable.

a) $s^4 + 25s^3 + 15s^2 + 20s + K = 0$

The Routh-Hurwitz table:

s^4	1	15	K
s^3	25	20	0
s^2	14.2	K	0
s^1	$20 - 1.76K$	0	0
s^0	K	0	0

For stability,
$20 - 1.76K > 0 \quad \rightarrow \quad K < 11.36$
$K > 0$

The stability range is: $0 < K < 11.36$

The system is marginally stable for $K = 11.36$

b) $s^4 + Ks^3 + 2s^2 + (K+1)s + 10 = 0$

The Routh-Hurwitz table:

s^4	1	2	10
s^3	K	$K+1$	0
s^2	$\frac{K-1}{K}$	10	0
s^1	$\frac{-9K^2-1}{K-1}$	0	0
s^0	10	0	0

For stability,
$K > 0$
$K - 1 > 0 \quad \rightarrow \quad K > 1$
$-9K2 - 1 > 0 \quad \rightarrow$ Since K^2 is always positive this condition cannot be met by any real value of K.

Thus, the system is unstable for all values of K.

c) $s^3 + (K+2)s^2 + 2Ks + 10 = 0$

The Routh-Hurwitz table:

s^3	1	$2K$
s^2	$K+2$	10
s^1	$\frac{2K^2+4K-10}{K+2}$	0
s^0	10	0

For stability,
$K + 2 > 0 \quad \rightarrow \quad K > -2$
$2K2 + 4K - 10 > 0 \quad \rightarrow \quad K > 1.4495 \quad or \quad K < -3.4495$

The stability range is: $K > 1.4495$

The system is marginally stable for $K = 1.4495$

d) $\boldsymbol{s^3 + 20s^2 + 5s + 10K = 0}$

The Routh-Hurwitz table:

s^3	1	5
s^2	20	$10K$
s^1	$\frac{100-10K}{20}$	0
s^0	$10K$	0

For stability,
$100 - 10K > 0 \quad \rightarrow \quad K < 10$
$10K > 0 \quad \rightarrow \quad K > 0$

The stability range is: $0 < K < 10$

The system is marginally stable for $K = 10$

e) $\boldsymbol{s^4 + Ks^3 + 5s^2 + 10s + 10K = 0}$

The Routh-Hurwitz table:

s^4	1	5	$10K$
s^3	K	10	0
s^2	$\frac{5K-10}{K}$	$10K$	0
s^1	$\frac{-10K^3+50K-100}{5K-10}$	0	0
s^0	$10K$	0	0

For stability,
$K > 0$
$5K - 10 > 0 \quad \rightarrow \quad K > 2$
$-10K3 + 50K - 100 > 0 \quad \rightarrow \quad K > 1.4233$
$10K > 0 \quad \rightarrow \quad K > 0$

The stability range is: $K > 2$

The system is marginally stable for $K = 2$

f) $s^4 + 12.5s^3 + s^2 + 5s + K = 0$

The Routh-Hurwitz table:

s^4	1	1	K
s^3	12.5	5	0
s^2	0.6	K	0
s^1	$\frac{3 - 12.5K}{0.6}$	0	0
s^0	K	0	0

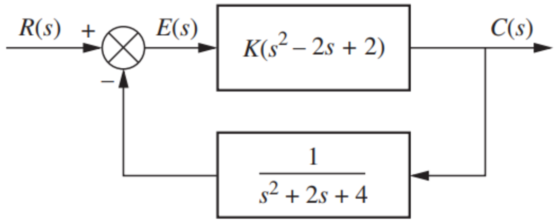
For stability,
$3 - 12.5K > 0 \quad \rightarrow \quad K < 0.24$
$K > 0$

The stability range is: $0 < K < 0.24$

The system is marginally stable for $K = 0.24$

3) Use the Routh-Hurwitz criterion to find the range of K for which the systems below are stable.

a)



The closed-loop transfer function is:

$$T(s) = \frac{Ks^4 + 2Ks^2 - 4Ks + 8K}{(K+1)s^2 + 2(1-K)s + (2K+4)}$$

Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

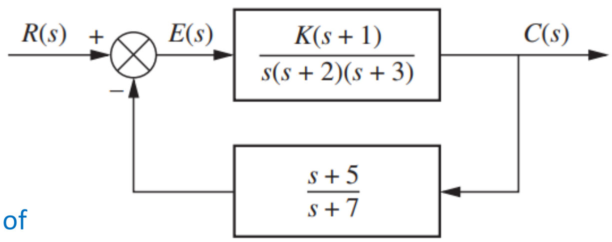
s^2	$K + 1$	$2K + 4$
s^1	$2 - 2K$	0
s^0	$2K + 4$	0

For stability,

$K + 1 > 0 \quad \rightarrow \quad K > -1$
$2 - 2K > 0 \quad \rightarrow \quad K < 1$
$2K + 4 > 0 \quad \rightarrow \quad K > -2$

The stability range is: $-1 < K < 1$

b)



The closed-loop system characteristic equation is:

$$s^4 + 12s^3 + (41 + K)s^2 + (42 + 6K)s + 5K = 0$$

Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

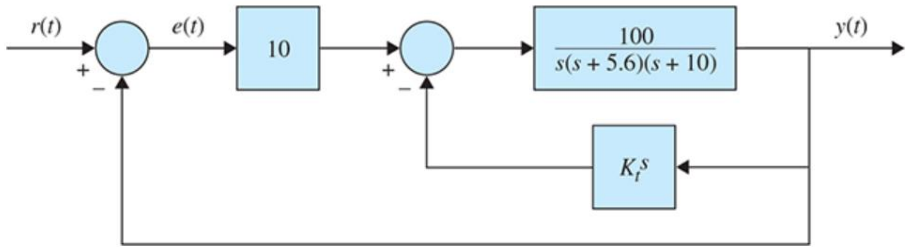
s^4	1	$41 + K$	$5K$
s^3	12	$42 + 6K$	0
s^2	$\frac{450 + 6K}{12}$	$5K$	0
s^1	$\frac{36(K + 10.12)(K + 51.88)}{450 + 6K}$	0	0
s^0	$5K$	0	0

For stability,

$450 + 6K > 0 \quad \rightarrow \quad K > -75$
$(K + 10.12)(K + 51.88) > 0 \quad \rightarrow \quad K > -10.12 \quad or \quad K < -51.88$
$5K > 0 \quad \rightarrow \quad K > 0$

The stability range is $K > 0$. The closed-loop system is stable for all positive K.

4) The block diagram of a motor-control system with tachometer feedback is shown below. Find the range of the tachometer constant K_t so that the system is asymptotically stable.



Using block diagram reduction, the overall transfer function of the system is:

$$\frac{Y(s)}{R(s)} = \frac{1000}{s^3 + 15.6s^2 + (56 + 100K_t)s + 1000}$$

The characteristic equation is:

$$s^3 + 15.6s^2 + (56 + 100K_t)s + 1000 = 0$$

Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

s^3	1	$56 + 100K_t$
s^2	15.6	1000
s^1	$\frac{1560K_t - 126.4}{15.6}$	0
s^0	1000	0

For stability,
$1560K_t - 126.4 > 0 \quad \rightarrow \quad K_t > 0.081$

The stability range is $K_t > 0.081$.

5) The following system in state space represents the forward path of a unity feedback system. Use the Routh-Hurwitz criterion to determine if the closed-loop system is stable.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ -\mathbf{5} & -\mathbf{4} & -\mathbf{3} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} u(t)$$

$$y(t) = [\mathbf{1} \quad \mathbf{0} \quad \mathbf{1}]\mathbf{x}(\mathrm{t})$$

The characteristic equation is:

$$|s\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} s & -1 & 0 \\ 1 & s-1 & -1 \\ 5 & 4 & s+3 \end{vmatrix} = s^3 + 2s^2 - 2s + 8 = 0$$

Create the Routh-Hurwitz table for the characteristic equation:

s^3	1	-2
s^2	2	8
s^1	-6	0
s^0	8	0

Since there are two sign changes in the first column the system is unstable.

6) Consider the system represented in state variable form

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\boldsymbol{k} & -\boldsymbol{k} & -\boldsymbol{k} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} u(t)$$

$$y(t) = [\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]\mathbf{x}(\mathrm{t}) + [\mathbf{0}]\mathrm{u}(\mathrm{t})$$

(a) What is the system transfer function?

The transfer function is:

$$G(s) = C(sI - A)^{-1}B + D = [1 \quad 0 \quad 0] \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k & k & s+k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^3 + ks^2 + ks + k} [1 \quad 0 \quad 0] \begin{bmatrix} s^2 + ks + k & s+k & 1 \\ -k & s^2 + ks & s \\ -ks & -ks - k & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^3 + ks^2 + ks + k} [1 \quad 0 \quad 0] \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{1}{s^3 + ks^2 + ks + k}$$

Thus, the transfer function is:

$$G(s) = \frac{1}{s^3 + ks^2 + ks + k}$$

(b) For what values of k is the system stable?

Create the Routh-Hurwitz table for the characteristic equation:

s^3	1	k
s^2	k	k
s^1	k – 1	0
s^0	k	0

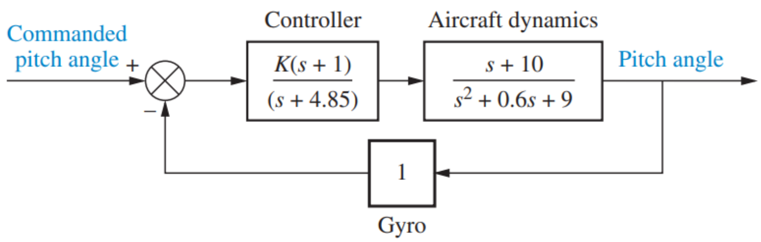
For stability:

$k > 0$

$k - 1 > 0 \quad \rightarrow \quad k > 1$

The stability range is: $k > 1$

7) A model for an airplane's pitch loop is shown below. Find the range of gain, K, that will keep the system stable. Can the system ever be unstable for positive values of K?



The closed-loop transfer function is:

$$T(s) = \frac{K(s+1)(s+10)}{s^3 + (5.45+K)s^2 + (11.91+11K)s + (43.65+10K)}$$

Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

s^3	1	11.91+11K
s^2	5.45+K	43.65+10K
s^1	$\frac{11K^2 + 61.86K + 21.26}{5.45 + K}$	0
s^0	43.65+10K	0

For stability,

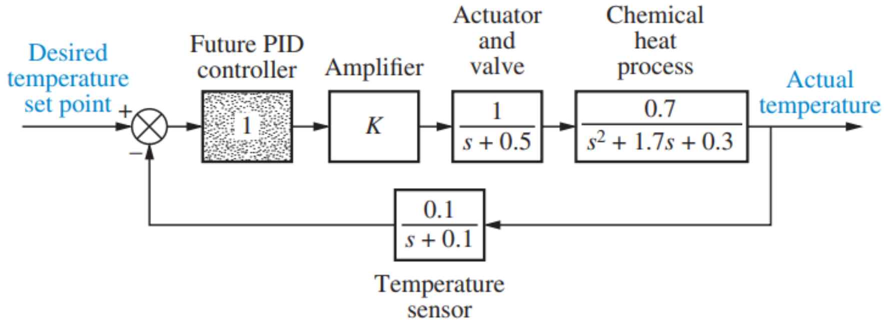
$5.45 + K > 0 \quad \rightarrow \quad K > -5.45$

$11K2 + 61.86K + 21.26 > 0 \quad \rightarrow \quad K > -0.3677 \quad or \quad K < -5.2559$

$43.65 + 10K > 0 \quad \rightarrow \quad K > -4.365$

The stability range is: $-0.3677 < K < \infty$. Stable for all positive K.

8) A common application of control systems is in regulating the temperature of a chemical process. The flow of a chemical reactant to a process is controlled by an actuator and valve. The reactant causes the temperature in the vat to change. This temperature is sensed and compared to a desired set-point temperature in a closed loop, where the flow of reactant is adjusted to yield the desired temperature. In the next lectures, we will learn how a PID controller is used to improve the performance of such process control systems. The figure below shows the control system prior to the addition of the PID controller. The PID controller is replaced by a shaded box with a gain of unity. For this system, prior to the design of the PID controller, find the range of amplifier gain, K, to keep the system stable.



The closed-loop transfer function is:

$$T(s)=\frac{0.7K(s+0.1)}{s^4+2.3s^3+1.37s^2+0.265s+(0.07K+0.015)}$$

Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

s^4	1	1.37	0.07K+0.015
s^3	2.3	0.265	0
s^2	1.2548	0.07K+0.015	0
s^1	0.23751 - 0.12831K	0	0
s^0	0.07K+0.015	0	0

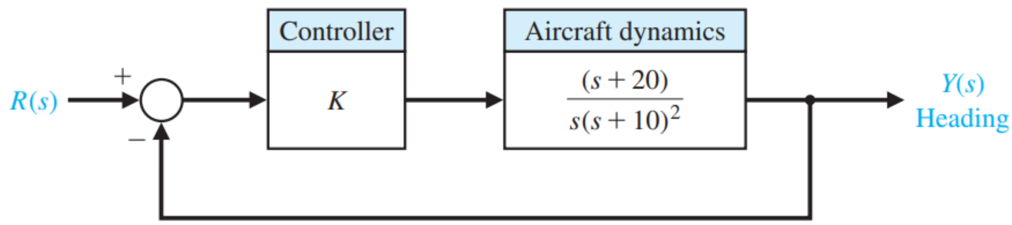
For stability,

$0.23751 - 0.12831K > 0 \quad \rightarrow \quad K < 1.85106$

$0.07K + 0.015 > 0 \quad \rightarrow \quad K > -0.21429$

The stability range is: $-\ 0.21429 < K < 1.85106$

9) Designers have developed small, fast, vertical- takeoff fighter aircraft that are invisible to radar (stealth aircraft). This aircraft concept uses quickly turning jet nozzles to steer the airplane. The control system for the heading or direction control is shown below. Determine the maximum gain of the system for stable operation.



The closed-loop transfer function is:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K(s+20)}{s^3 + 20s^2 + (100+K)s + 20K}$$

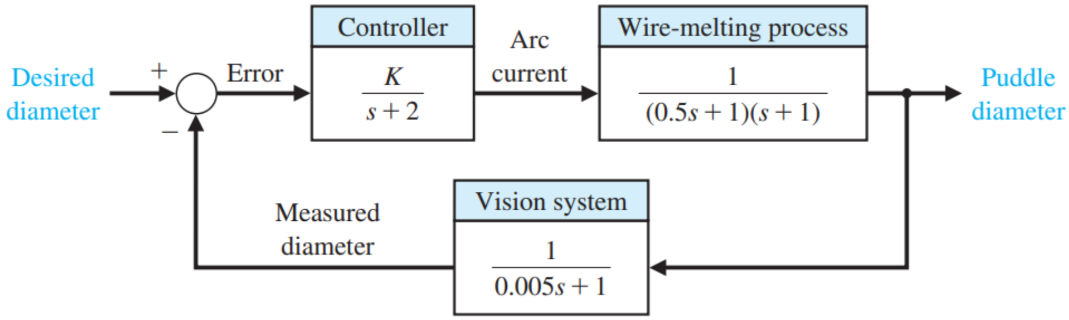
Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

s^3	1	100+K
s^2	20	20K
s^1	100	0
s^0	20K	0

For stability,
$20K > 0 \quad \rightarrow \quad K > 0$

Therefore, the system is stable for all $K > 0$.

10) Arc welding is one of the most important areas of application for industrial robots. In most manufacturing welding situations, uncertainties in dimensions of the part, geometry of the joint, and the welding process itself require the use of sensors for maintaining weld quality. Several systems use a vision system to measure the geometry of the puddle of melted metal, as shown below. This system uses a constant rate of feeding the wire to be melted.



(a) Calculate the maximum value for K for the system that will result in a stable system.

The closed-loop transfer function is:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K(0.005s+1)}{0.0025s^4 + 0.5125s^3 + 2.52s^2 + 4.01s + 2 + K}$$

Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

s^4	0.0025	2.52	2+K
s^3	0.5125	4.01	0
s^2	2.50	2+K	0
s^1	3.6 - 0.205K	0	0
s^0	2+K	0	0

For stability,

$3.6 - 0.205K > 0 \quad \rightarrow \quad K < 17.6$

$2 + K > 0 \quad \rightarrow \quad K > -2$

The required K range for stability is: $-2 < K < 17.6$

(b) For half of the maximum value of K found in part (a), determine the roots of the characteristic equation.

Using $K = 9$, the roots of the characteristic equation and its roots are:

$$0.0025s^4 + 0.5125s^3 + 2.52s^2 + 4.01s + 11 = 0$$

The roots are:

$$s_1 = -200, \quad s_{2,3} = -0.33 \pm j2.23, \quad and \quad s_4 = -4.35$$

(c) Estimate the overshoot of the system of part (b) when it is subjected to a step input.

Assuming the complex roots $-0.33 \pm j2.23$ are dominant, we compute the damping ratio:

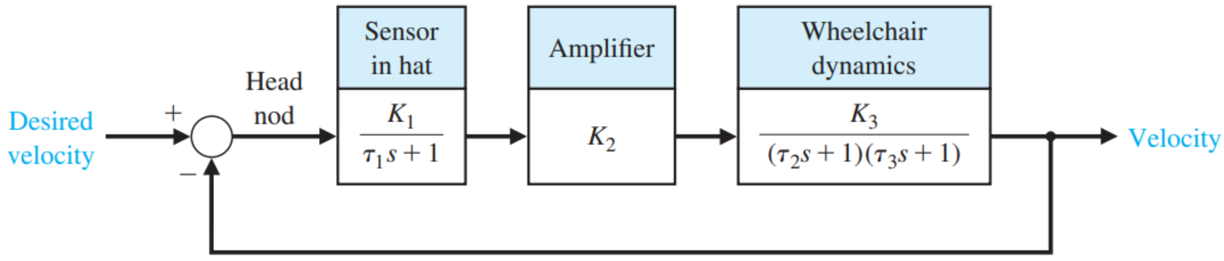
$$\zeta\omega_n = 0.33 \quad and \quad \omega_n\sqrt{1-\zeta^2} = 2.23$$

$$\frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{0.33}{2.23} \quad \rightarrow \quad \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{0.33}{2.23} \quad \rightarrow \quad \frac{\zeta^2}{1-\zeta^2} = \frac{0.1089}{4.9729} \quad \rightarrow \quad \zeta^2 = 0.0214 \quad \rightarrow \zeta = 0.15$$

Therefore, we estimate the percentage overshoot as

$$O.S.\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 100e^{-0.15\pi/\sqrt{1-(0.15)^2}} = 62\%$$

11) A very interesting and useful velocity control system has been designed for a wheelchair control system. A proposed system utilizing velocity sensors mounted in a headgear is shown below. The headgear sensor provides an output proportional to the magnitude of the head movement. There is a sensor mounted at 90° intervals so that forward, left, right, or reverse can be commanded. Typical values for the time constants are $\tau_1 = 0.5\ sec$, $\tau_2 = 1\ sec$, and $\tau_3 = 1/4\ sec$.



(a) Determine the limiting gain $K = K_1K_2K_3$ for a stable system.

The closed-loop transfer function is:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_1K_2K_3}{(0.5s+1)(s+1)(0.25s+1) + K_1K_2K_3} = \frac{8K}{s^3 + 7s + 14s + 8 + 8K}$$

Create the Routh-Hurwitz table for the characteristic equation of the closed-loop system:

s^3	1	14
s^2	7	8+8K
s^1	$\frac{90-8K}{7}$	0
s^0	8+8K	0

For stability,

$90 - 8K > 0 \quad \rightarrow \quad K < 11.25$

$8 + 8K > 0 \quad \rightarrow \quad K > -1$

Therefore, the system is stable for all $-1 < K < 11.25$.

(b) Determine the value of gain that results in a system with a settling time (2% criteria) of $t_s \leq 4\ sec$.

We want $t_s = 4\ sec$, so

$$t_s = \frac{4}{\zeta\omega_n} = 4 \quad \rightarrow \quad \zeta\omega_n = 1$$

Our desired characteristic polynomial is:

$$(s+b)(s^2 + 2\zeta\omega_n s + \omega_n^2) = s^3 + (2+b)s^2 + (\omega_n^2 + 2b)s + b\omega_n^2$$

where we have used the fact that $\zeta\omega_n = 1$ and ω_n and b are to be determined.

Our actual characteristic polynomial is:

$$s^3 + 7s + 14s + 8 + 8K$$

Comparing the coefficients of the actual and desired characteristic polynomials, we find the following relationships,

$2 + b = 7$

$\omega_n^2 + 2b = 14$

$b\omega_n^2 = 8 + 8K$

Solving these three equations yields:

$$b = 5, \ \ \omega_n = 2, \quad and \quad K = 1.5$$