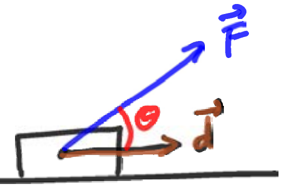
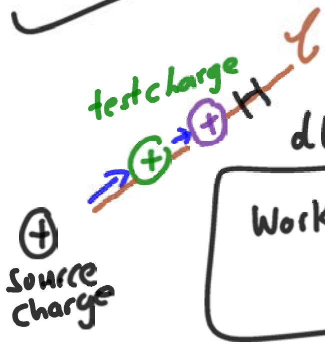
$W = Fd$ (force and displacement in same direction)

$$W = Fd\cos\theta = \|\vec{F}\|\,\|\vec{d}\|\cos\theta = \vec{F}\cdot\vec{d}$$





$$dW = \vec{F}\cdot d\vec{r}$$

$$\boxed{\text{Work} = \int_{\mathcal{C}} dW = \int_{\mathcal{C}} \vec{F}\cdot d\vec{r}}$$

11.5 Line Integrals of Vector Fields over Parametrized Curves

FRY Defn IV.2.4.1, Line integral of vector field $\mathbf{F}$ over path $\mathcal{C}$

Definition 11.21. Let $\mathbf{F}$ be a vector field and $\mathcal{C}$ be a parametric curve (path) with parametrization $\mathbf{r}(t)$ where $t_0 \le t \le t_1$.[a] Then the line integral[b] of the vector field $\mathbf{F}$ over the parametric curve $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

If $\mathcal{C}$ is a closed path[c], we also use the notation

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

[a] Assume that the vector field $\mathbf{F}$ is continuous (i.e., all of its component functions are continuous). Also assume that the component functions of $\mathbf{r}(t)$ have continuous derivatives. (We call such a curve or path a C^1 curve or path.)

[b] Line integrals are also called path integrals.

[c] A closed path is one in which the end point is the same as the start point.

If $\mathbf{F}$ is a vector field from $\mathbb{R}^2$ to $\mathbb{R}^2$, i.e., $\mathbf{F}(x, y) = \langle F_1(x, y),\ F_2(x, y)\rangle$ for some scalar-valued functions F_1 and F_2, then we sometimes write the line integral as

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} (\mathbf{F}_1 dx + \mathbf{F}_2 dy).$$

If $\mathbf{F}$ is a vector field from $\mathbb{R}^3$ to $\mathbb{R}^3$, i.e., $\mathbf{F}(x, y, z) = \langle F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)\rangle$ for some scalar-valued functions F_1, F_2, and F_3, then we sometimes write the line integral as

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} (\mathbf{F}_1 dx + \mathbf{F}_2 dy + \mathbf{F}_3 dz).$$

Since $\mathbf{r}'(t)$ is tangent to the curve and[1] $\int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \overbrace{\boxed{\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}}}^{\hat{T}(t)} \underbrace{\|\mathbf{r}'(t)\|\, dt}_{ds}$

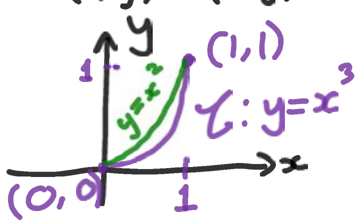
[1] Here, we are assuming that $\mathbf{r}'(t)$ is nonzero for all t in the interval that we are working over.

and $ds = \|\mathbf{r}'(t)\|\,dt$, the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ is also sometimes written as

$$\int_{\mathcal{C}} \mathbf{F}\cdot\hat{\mathbf{T}}\,ds.$$

Example 11.22. Let $\mathbf{F}(x,y) = -y\hat{\imath}+x\hat{\jmath} = \langle -y, x\rangle$. Let $\mathcal{C}$ be the curve $y=x^3$ where $0\le x\le 1$. Evaluate $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

Given $\vec{F}(x,y)=\langle -y,x\rangle$ Goal: Evaluate $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$



$\boxed{y=f(x)}$ $a\le x\le b$

$\vec{r}(x)=\langle x,f(x)\rangle,\ a\le x\le b$

Ans: (1) Parametrize $\mathcal{C}$

$\vec{r}(t)=\langle t,t^3\rangle$, where $0\le t\le 1$

$\vec{F}(x,y)=\langle -y,x\rangle$

$\vec{F}(t,t^3)$

(2) $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$$\int_0^1 \langle -t^2,t\rangle\cdot\underbrace{\langle 1,2t\rangle}_{\vec{r}\,'(t)}dt \qquad = \int_0^1 \langle -t^3,t\rangle\cdot\langle 1,3t^2\rangle\,dt$$

$$\int_0^1(-t^2+2t^2)\,dt \qquad = \int_0^1(-t^3+3t^3)\,dt$$

$$= \int_0^1 t^2\,dt = \frac{1}{3}t^3\Big|_0^1 = 1/3 \qquad = \int_0^1 2t^3\,dt = 2\cdot\frac{1}{4}t^4\Big|_0^1 = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}$$

Example 11.23. Let $\mathbf{F}(x,y,z) = (x-y^2)\hat{\imath} + (y-z^2)\hat{\jmath} + (z-x^2)\hat{k}$. Let $P_0 = (0,0,0)$ and $P_1 = (2,3,4)$ be two points in $\mathbb{R}^3$. Let $\mathcal{C}$ be the union of paths $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_1$ is the straight line path from P_1 to $(2,3,0)$ and $\mathcal{C}_2$ is the straight line path from $(2,3,0)$ to P_1. Evaluate the line integral:

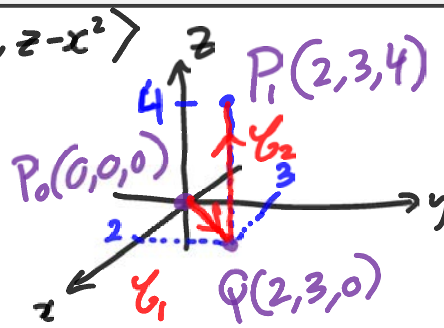
$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_1\cup\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}.$$

If $\mathcal{C}_3$ was the straight line segment from $P_0=(0,0,0)$ to $P_1=(2,3,4)$, would $\int_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{r}$ give the same result?

Given: $\vec{F}(x,y,z) = \langle x-y^2, y-z^2, z-x^2\rangle$

$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$

union



Goal $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r} + \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{r}$$

① Parametrize $\mathcal{C}_1$ and $\mathcal{C}_2$

$\mathcal{C}_1$: $\vec{r}_1(t) = \langle 0,0,0\rangle + t\langle 2,3,0\rangle$, $0\le t\le 1$

$= \langle 2t, 3t, 0\rangle$, $0\le t\le 1$

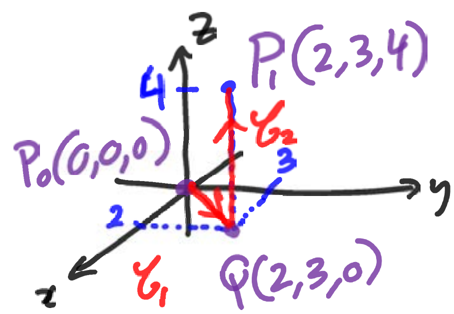
$\vec{r}_1{}'(t) = \langle 2,3,0\rangle$, $0\le t\le 1$

$\mathcal{C}_2$: $\vec{r}_2(t) = \langle 2,3,0\rangle + t\langle 0,0,4\rangle$, $0\le t\le 1$

$= \langle 2,3,4t\rangle$, $0\le t\le 1$

$\vec{r}_2{}'(t) = \langle 0,0,4\rangle$, $0\le t\le 1$

② $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r} + \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{r}$



$$= \int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1\,'(t)\,dt + \int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2\,'(t)\,dt$$

$$= \int_0^1 \langle 2t - 9t^2, 3t, -4t^2 \rangle \cdot \langle 2, 3, 0 \rangle\,dt + \int_0^1 \langle 2 - 9, 3 - 16t^2, 4t - 4 \rangle \cdot \langle 0, 0, 4 \rangle\,dt$$

$$= \int_0^1 (\underbrace{4t - 18t^2 + 9t}_{13t - 18t^2})\,dt + \int_0^1 (16t - 16)\,dt$$

$$= \left[\frac{13}{2}t^2 - 6t^3\right]_0^1 + \left[8t^2 - 16t\right]_0^1$$

$$= \frac{13}{2} - 6 + 8 - 16$$

$$= \frac{13}{2} - 14$$

$$= \frac{-15}{2}$$