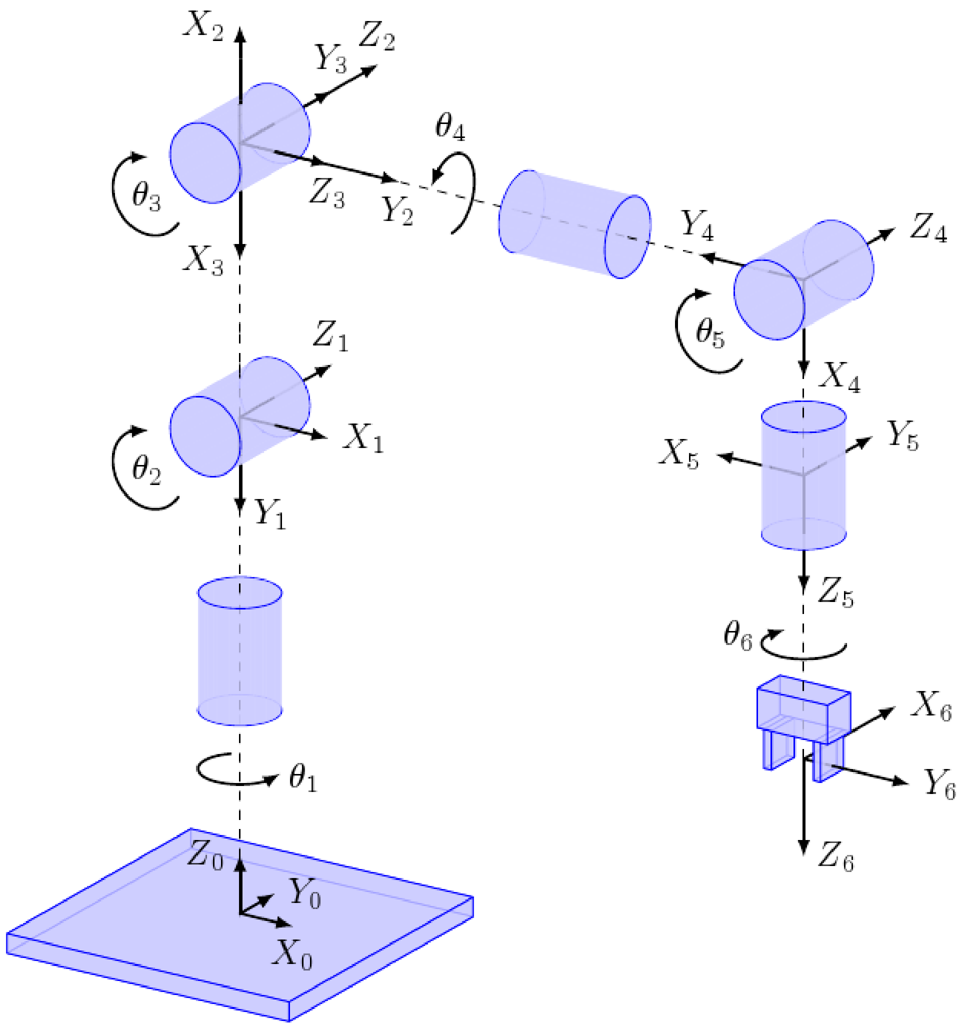
Kinematics and Dynamics of Robots

Module 9

Jacobian Matrix



Inputs: Joint Variables
output: End-Efector Position

$$\begin{bmatrix} P_{x_0} \\ P_y \\ P_z \\ 1 \end{bmatrix} = \begin{bmatrix} R_n^0 & d_n^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{x_n} \\ \vdots \\ \vdots \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{6\times1} = \underbrace{J}_{6\times n} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}}_{n\times1}$$

J: Jacobian Matrix

$\dot{q}_1$ → Joint Variables Velocities (it can be $\dot{\theta}$ (for revolute joint) or $\dot{d}$ (for prismatic))

n → number of joints

$\begin{bmatrix} \dot{x} & \cdots & \omega_z \end{bmatrix}^T$: The velocity of End Efector wrt World coord or Base frame

Jacobian Matrix

linear

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix}_{6\times1} = \begin{bmatrix}\square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\end{bmatrix}_{6\times3} \begin{bmatrix}\dot{\theta}_1 = \dot{q}_1\\ \dot{\theta}_2 \quad \dot{q}_2\\ \dot{d}_3 \quad \dot{q}_3\end{bmatrix}_{3\times1}$$

Rotation

$\dot{\theta}_1$ $\dot{\theta}_2$ $\dot{d}_3$

Joint Type →	Prismatic	Revolute
Linear	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d^0_n - d^0_{i-1})$ Cross Product
Rotational	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$

i = joint Number

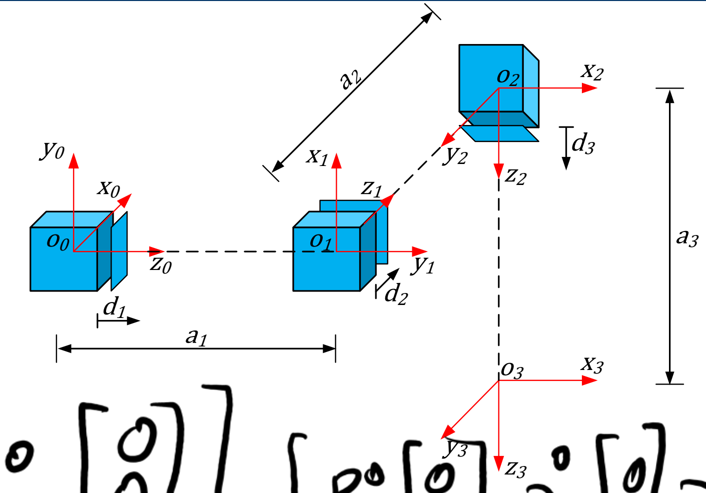
n = Total number of joints

$$H^0_1 = \begin{bmatrix} R^0_1 & d^0_1 \\ 0 & 1 \end{bmatrix}$$

Jacobian Matrix

HUMBER

Joint Type	Prim	Rev
linear	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times(d^0_n-d^0_{i-1})$
Rotational	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$



linear Part

$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\\\omega_x\\\omega_y\\\omega_z\end{bmatrix}=\begin{bmatrix}\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\end{bmatrix}\begin{bmatrix}\dot{d}_1\\\dot{d}_2\\\dot{d}_3\end{bmatrix}\Rightarrow J$$

Rotational Part

1st joint 2nd joint 3rd joint

$$\Rightarrow J=\begin{bmatrix}R^0_{\underbrace{1-1}_{0}}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{\underbrace{2-1}_{1}}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{\underbrace{3-1}_{2}}\begin{bmatrix}0\\0\\1\end{bmatrix}\\ \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix}\end{bmatrix}=\begin{bmatrix}R^0_0\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_2\begin{bmatrix}0\\0\\1\end{bmatrix}\\ \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix}\end{bmatrix}$$

$$R^0_0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

$$R^0_1=\begin{array}{c|ccc} & x_1 & y_1 & z_1\\ \hline x_0 & 0 & 0 & 1\\ y_0 & 1 & 0 & 0\\ z_0 & 0 & 1 & 0\end{array}$$

$$R^0_2=R^0_1R^1_2$$

$$R^1_2=\begin{array}{c|ccc} & x_2 & y_2 & z_2\\ \hline x_1 & 0 & 0 & -1\\ y_1 & 1 & 0 & 0\\ z_1 & 0 & -1 & 0\end{array}$$

$$R^0_2=\begin{bmatrix}0&-1&\\0&0&\\1&0&0\end{bmatrix}$$

Jacobian Matrix

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\end{bmatrix}\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix} \Rightarrow J$$

linear part, rotational part, 1st joint, 2nd joint, 3rd joint

$$\Rightarrow J = \begin{bmatrix} R^0_{1-1}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{2-1}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{3-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix} R^0_0\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_2 \dots\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$R^0_0 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

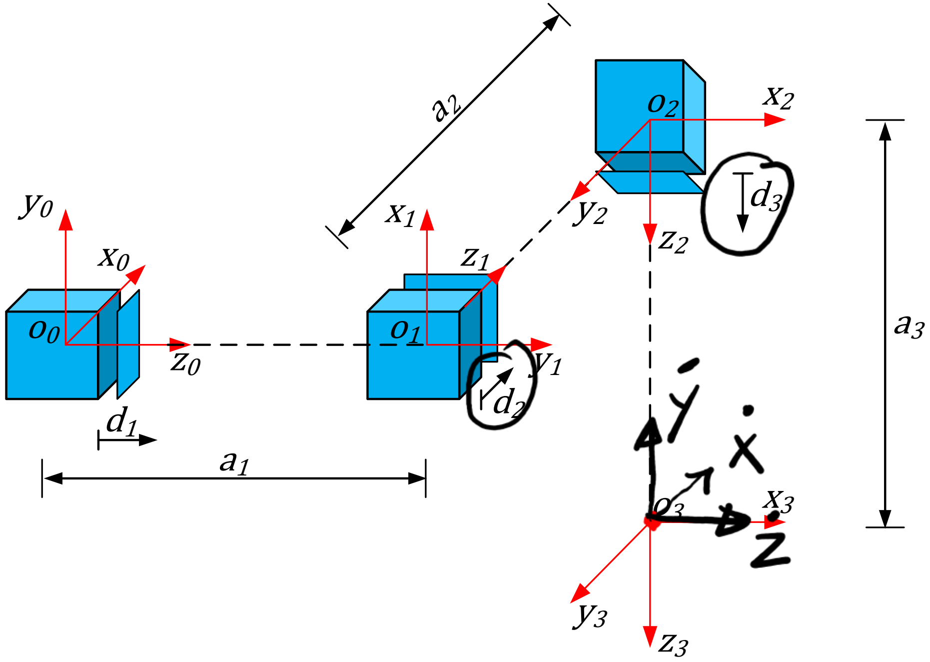
$$R^0_1 = \begin{array}{c|ccc} & x_1 & y_1 & z_1\\ \hline x_0 & 0 & 0 & 1\\ y_0 & 1 & 0 & 0\\ z_0 & 0 & 1 & 0\end{array}$$

$$R^0_2 = R^0_1 R^1_2$$

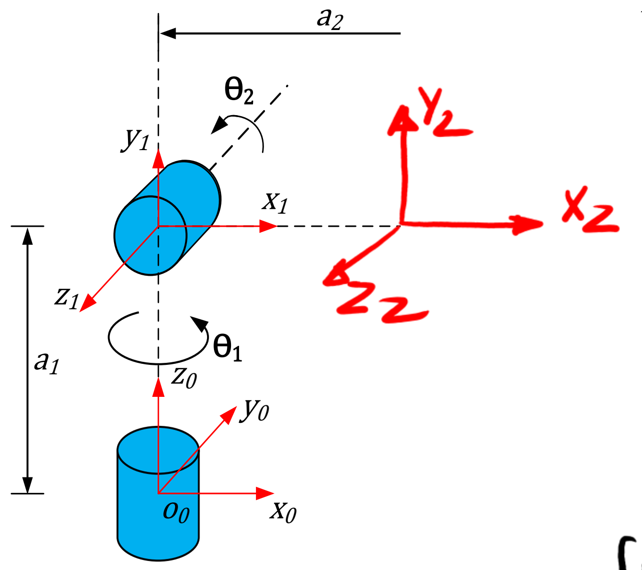
$$R^1_2 = \begin{array}{c|ccc} & x_2 & y_2 & z_2\\ \hline x_1 & 0 & 0 & -1\\ y_1 & 1 & 0 & 0\\ z_1 & 0 & -1 & 0\end{array}$$

$$R^0_2 = \begin{bmatrix}0 & -1 & \dots\\ 0 & 0 & \dots\\ 1 & 0 & \dots\end{bmatrix}$$

$$\dot{x} = \dot{d}_2$$
$$\dot{y} = -\dot{d}_3$$
$$\dot{z} = \dot{d}_1$$
$$\omega_x = 0$$
$$\omega_y = 0$$
$$\omega_z = 0$$



Joint Type	Prim	Rev
Linear	$R_{i-1}^{0}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R_{i-1}^{0}\begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(d_n^0-d_{i-1}^0\right)$
Rotation	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R_{i-1}^{0}\begin{bmatrix}0\\0\\1\end{bmatrix}$



$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\\\omega_x\\\omega_y\\\omega_z\end{bmatrix}=\begin{bmatrix}R_0^0\begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(d_2^0-d_0^0\right) & R_1^0\begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(d_2^0-d_1^0\right)\\ R_0^0\begin{bmatrix}0\\0\\1\end{bmatrix} & R_1^0\begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}$$

What is n? $=2\rightarrow\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\end{bmatrix}$

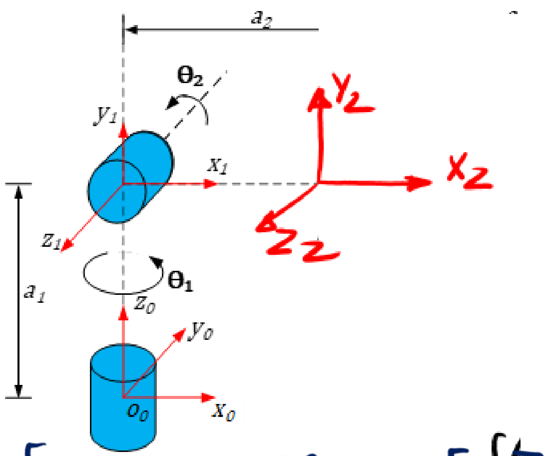
$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\\\omega_x\\\omega_y\\\omega_z\end{bmatrix}=\begin{bmatrix}\square&\square\\\square&\square\\\square&\square\\\square&\square\\\square&\square\\\square&\square\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\end{bmatrix}$$

$R_0^0 = I \qquad d_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1^0 \quad d_1^0 \leftarrow H_1^0$

$d_2^0 \leftarrow H_2^0 = H_1^0 H_2^1$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_0^0) & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_1^0) \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$



$$R_1^0 = R_{z_0,\theta_1} \begin{array}{c|ccc} & x_1 & y_1 & z_1 \\ \hline x_0 & 1 & 0 & 0 \\ y_0 & 0 & 0 & -1 \\ z_0 & 0 & 1 & 0 \end{array} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ s\theta_1 & 0 & -c\theta_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$d_1^0 = R_{z_0,\theta_1} \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix}$$

$$\Rightarrow H_1^0 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2^1 = R_{z_1,\theta_2} I = R_{z_1,\theta_2}$$

$$d_2^1 = R_{z_1,\theta_2} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\theta_2 a_2 \\ s\theta_1 a_2 \\ 0 \end{bmatrix} \Rightarrow H_2^1 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & c\theta_2 a_2 \\ s\theta_2 & c\theta_2 & 0 & s\theta_1 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 \cdot H_2^1$$

$$d_2^0 = \begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 + a_1 \end{bmatrix}$$

$R_0^0 = I \quad d_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1^0 \; d_1^0 \leftarrow H_1^0$

$H_2^0 \leftarrow H_2^0 = H_1^0 H_2^1$

$$R_1^0 = R_{z_0,\theta_1} \begin{array}{c|ccc} & x_1 & y_1 & z_1 \\ \hline x_0 & 1 & 0 & 0 \\ y_0 & 0 & 0 & -1 \\ z_0 & 0 & 1 & 0 \end{array} = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 \\ S\theta_1 & 0 & -C\theta_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$d_1^0 = R_{z_0,\theta_1} \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix}$$

$$\Rightarrow H_1^0 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2^1 = R_{z_1,\theta_2}\, I = R_{z_1,\theta_2}$$

$$d_2^1 = R_{z_1,\theta_2} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C\theta_2 a_2 \\ S\theta_2 a_2 \\ 0 \end{bmatrix} \Rightarrow H_2^1 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & C\theta_2 a_2 \\ S\theta_2 & C\theta_2 & 0 & S\theta_2 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 \cdot H_2^1$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - \underbrace{d_0^0}_{0}) & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_1^0) \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$d_2^0 = \begin{bmatrix} a_2 C\theta_1 C\theta_2 \\ a_2 S\theta_1 C\theta_2 \\ a_2 S\theta_2 + a_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 C\theta_1 C\theta_2 \\ a_2 S\theta_1 C\theta_2 \\ a_2 S\theta_2 + a_1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} a_2 C\theta_1 C\theta_2 \\ a_2 S\theta_1 C\theta_2 \\ a_2 S\theta_2 + a_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \right) \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} C\theta_1 & 0 & S\theta_1 \\ S\theta_1 & 0 & -C\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

Jacobian Matrix

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(-1)^{1+1}(a_2b_3 - a_3b_2) + j(-1)^{1+2}(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)$$

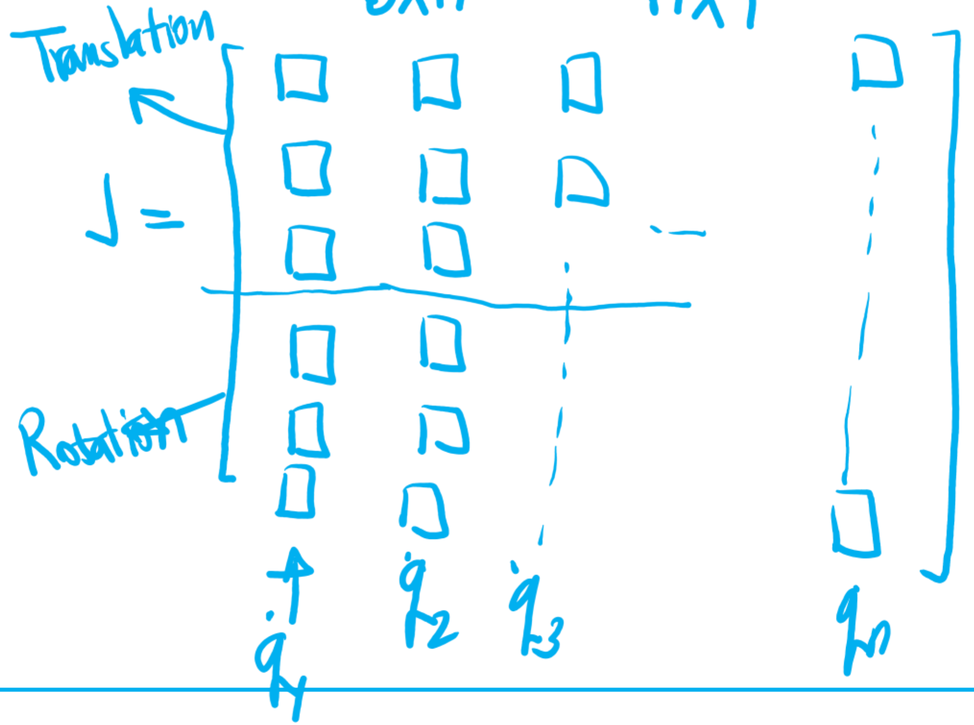
$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Jacobian Matrix

$$\underset{6\times 1}{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}} = \underset{6\times n}{J} \underset{n\times 1}{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}}$$

$q_1 \rightarrow \begin{cases} \theta \rightarrow \text{Revolute} \\ d \rightarrow \text{Prismatic} \end{cases}$

n = number of joints



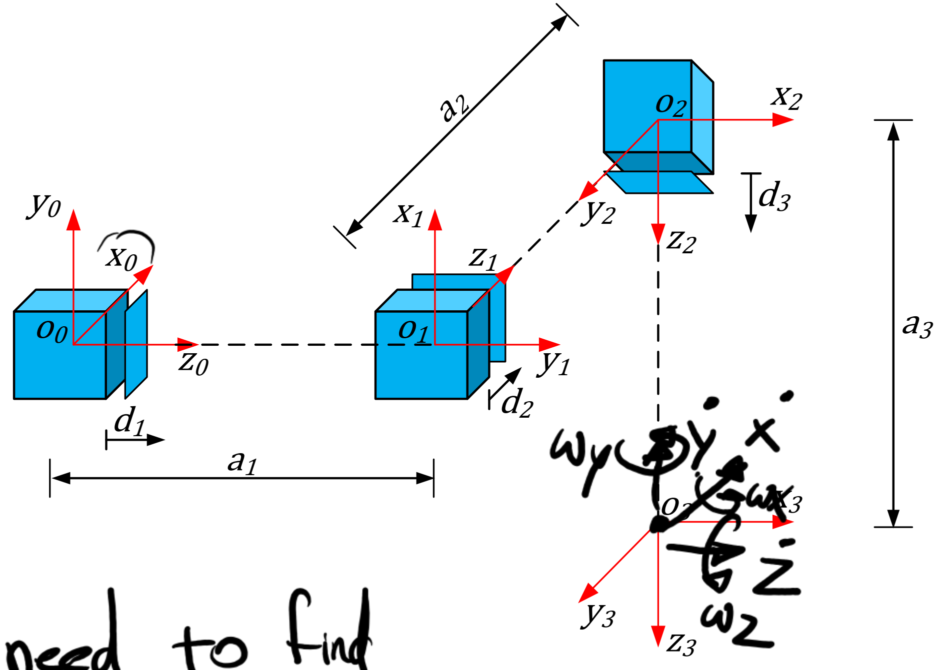
Joint Type	Prismatic	Revolute
Linear	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d^0_n - d^0_{i-1})$
Rotation	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$

i = joint number

n = number of joints

Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underbrace{\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}}_{6\times 3} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$



$$J = \begin{bmatrix} R^0_{1-1}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{2-1}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{3-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

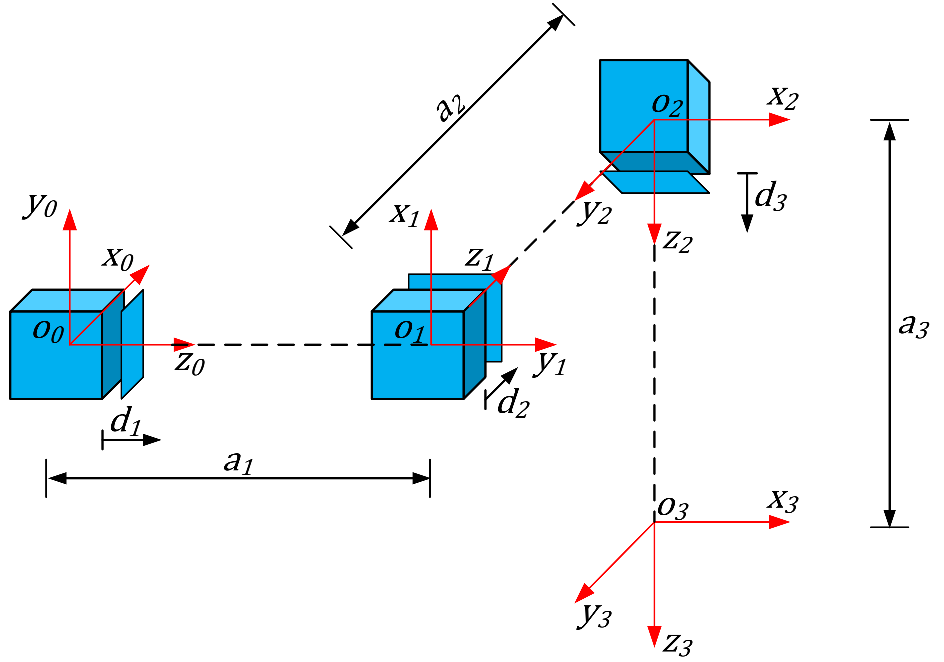
We need to find

R^0_1, $R^0_2 = R^0_1 R^1_2$

$$R_1^0 = \begin{array}{c|ccc} & x_1 & y_1 & z_1 \\ \hline x_0 & 0 & 0 & 1 \\ y_0 & 1 & 0 & 0 \\ z_0 & 0 & 1 & 0 \end{array}$$

$$R_2^1 = \begin{array}{c|ccc} & x_2 & y_2 & z_2 \\ \hline x_1 & 0 & 0 & -1 \\ y_1 & 1 & 0 & 0 \\ z_1 & 0 & -1 & 0 \end{array}$$

$$R_2^0 = R_1^0 R_2^1$$

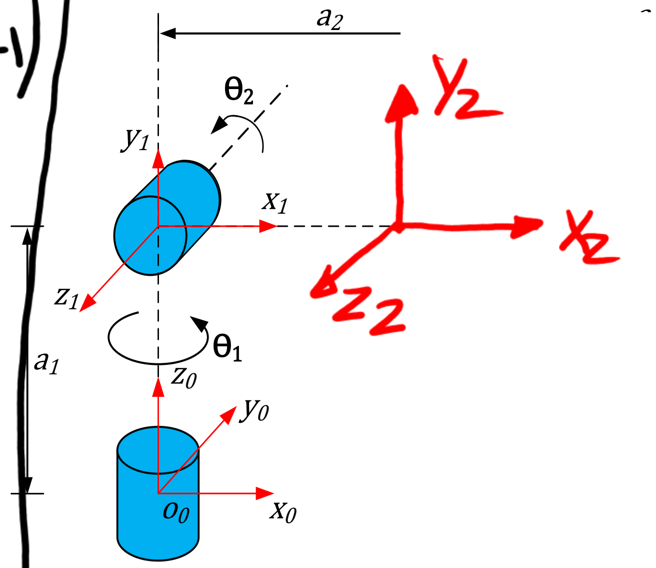


Jacobian Matrix

HUMBER

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix}_{6\times1} = \begin{bmatrix} R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(d^0_n-d^0_{i-1}\right) & R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(d^0_n-d^0_{i-1}\right) \\ R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}$$

$i=1$ $\quad$ $i=2$ $\quad$ $n=2$



$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} R^0_{1-1}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{2-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(d^0_2-d^0_{2-1}\right) \\ R^0_0\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}$$

We need to find

$R^0_1, d^0_1 \rightarrow H^0_1$

$d^0_2 \leftarrow H^0_2 = H^0_1 H^1_2$

H^0_1

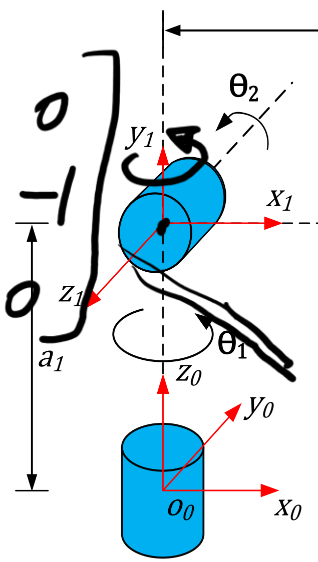
H^1_2

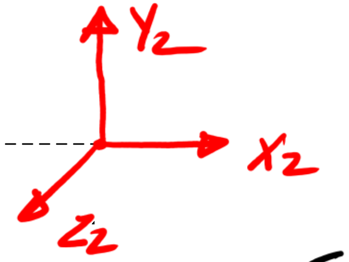
H_1^0

$$R_1^0 = R_{z_0,\theta_1}\cdot \begin{array}{c} \\ x_0 \\ y_0 \\ z_0 \end{array}\begin{array}{ccc} x_1 & y_1 & z_1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} = \begin{bmatrix} \cancel{c\theta_1} & \cancel{-s\theta_1} & 0 \\ \cancel{s\theta_1} & \cancel{c\theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_1^0 = \begin{array}{c} \\ x_0 \\ y_0 \\ z_0 \end{array}\begin{array}{ccc} x_1 & y_1 & z_1 \\ \hline - & - & - \\ - & - & - \\ - & - & - \end{array}$$

$$R_{y_1,\theta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ 0 & 1 & 0 \\ -s\theta_1 & 0 & c\theta_1 \end{bmatrix}$$





$$\begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ s\theta_1 & 0 & -c\theta_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\parallel$$

$$d_1^0 = R_{z_0,\theta_1}\begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix}$$

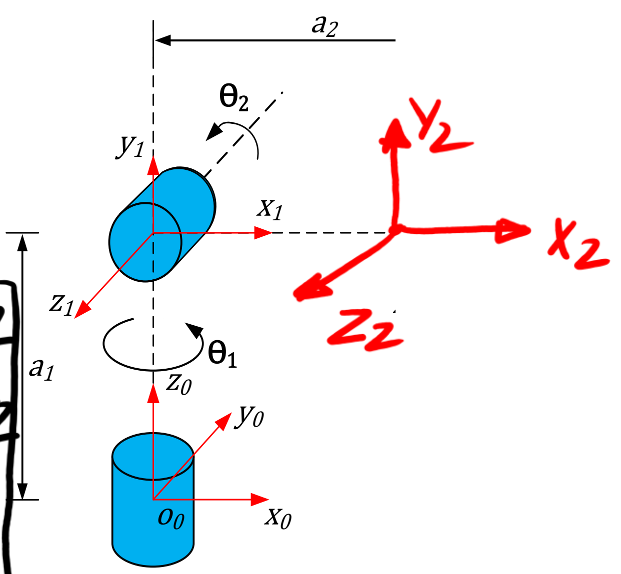
d_1^0

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 0 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^1_2 = R_{z_1,\theta_2}\quad \begin{array}{c|ccc} & x_2 & y_2 & z_2 \\ \hline x_1 & 1 & 0 & 0 \\ y_1 & 0 & 1 & 0 \\ z_1 & 0 & 0 & 1 \end{array} = R_{z_1,\theta_2} = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ S\theta_2 & C\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

H^1_2



$$d^1_2 = R_{z_1,\theta_2} \cdot \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ S\theta_2 & C\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_2 C\theta_2 \\ a_2 S\theta_2 \\ 0 \end{bmatrix}$$

$$H^1_2 = \begin{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ S\theta_2 & C\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} a_2 C\theta_2 \\ a_2 S\theta_2 \\ 0 \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} \begin{bmatrix} C\theta_1 & 0 & S\theta_1 \\ S\theta_1 & 0 & -C\theta_1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \qquad H_2^1 = \begin{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ S\theta_2 & C\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} a_2 C\theta_2 \\ a_2 S\theta_2 \\ 0 \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$d_2^0$$

$$H_2^0 = H_1^0 \, H_2^1 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2 C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta_1 C\theta_2 & -C\theta_1 S\theta_2 & S\theta_1 & a_2 C\theta_2 C\theta_1 \\ - & - & - & a_2 C\theta_2 S\theta_1 \\ - & - & - & a_1 \\ - & - & - & 1 \end{bmatrix}$$

$$d_2^0 = \begin{bmatrix} a_2 C\theta_2 C\theta_1 \\ a_2 C\theta_2 S\theta_1 \\ a_1 + a_2 S\theta_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \overset{i=1}{R_{1-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} & \overset{i=2}{R_{2-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_{2-1}^0)} \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} 0 & \begin{bmatrix} S\theta_1 \\ -C\theta_1 \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} a_2 C\theta_2 C\theta_1 \\ a_2 C\theta_2 S\theta_1 \\ a_1 + a_2 S\theta_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \right) \\ 0 & \\ 1 & \\ 0 & S\theta_1 \\ 0 & -C\theta_1 \\ 1 & 0 \end{bmatrix}$$

Jacobian Matrix

HUMBER

$$J = \begin{bmatrix} 0 & \begin{bmatrix} S\theta_1 \\ -C\theta_1 \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} a_2 C\theta_2 C\theta_1 \\ a_2 C\theta_2 S\theta_1 \\ a_1 + a_2 S\theta_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \right) \\ 0 & \\ 1 & \\ 0 & S\theta_1 \\ 0 & -C\theta_1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} S\theta_1 \\ -C\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_2 C\theta_2 C\theta_1 \\ a_2 C\theta_2 S\theta_1 \\ a_2 S\theta_2 \end{bmatrix} \\ 0 & \\ 1 & \\ 0 & S\theta_1 \\ 0 & -C\theta \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_2 C\theta_1 S\theta_2 \\ 0 & -a_2 S\theta_1 S\theta_2 \\ 1 & a_2 C\theta_2 \\ 0 & S\theta_1 \\ 0 & -C\theta_1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} S\theta_1 \\ -C\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_2 C\theta_2 C\theta_1 \\ a_2 C\theta_2 S\theta_1 \\ a_2 S\theta_2 \end{bmatrix} = \begin{vmatrix} i & j & k \\ S\theta_1 & -C\theta_1 & 0 \\ a_2 C\theta_2 C\theta_1 & a_2 C\theta_2 S\theta_1 & a_2 S\theta_2 \end{vmatrix} = i(-a_2 C\theta S\theta_2) - j\, S\theta\, a_2 S\theta_2 + k(a_2 C\theta_2 S\theta_1^2 + a_2 C\theta_2 C\theta_1^2)$$

$$i(-a_2 C\theta_1 S\theta_2) - j\, a_2 S\theta_1 S\theta_2 + k\, a_2 C\theta_2 (\underbrace{S\theta_1^2 + C\theta_1^2}_{1}) = \begin{bmatrix} -a_2 C\theta_1 S\theta_2 \\ -a_2 S\theta_1 S\theta_2 \\ a_2 C\theta_2 \end{bmatrix}$$