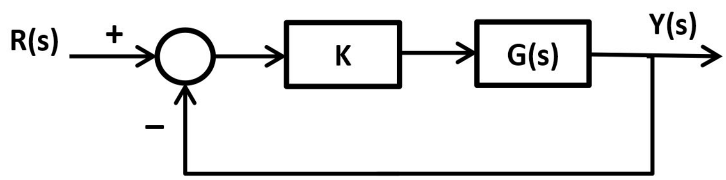
MENG 3510 Midterm Exam Solution – Winter 2025

Question 1: Consider the following closed-loop system



$$G(s) = \frac{1}{(s-1)(s+5)(s+8)}$$

Answer the following questions (use the blank figure on the next page for marking your answers on the s-plane.)

a) [1.5 marks] Find the open-loop poles and zeros (including ones at infinity). Mark them on the s-plane.

Poles: p1 = +1, p2 = -5 , p3 = -8

Zeros: No finite zeros, Three zeros at infinity

b) [1.5 mark] Find the segments of the real axis that belongs to the root-locus of G(s). Mark them on the s-plane.

The real axis segment between +1 and -5, and the segment to the left side of -8 are on the Root-locus.

c) [3 marks] Find the number of asymptote lines of the root-locus (if any), the point (if any) that the asymptotes intersect the real axis, and the angles of asymptote lines (if any) with the real axis. Mark them on the s-plane.

Number of asymptotes $\rightarrow$ $n - m = 3 - 0 = 3$

Intersection of asymptotes $\rightarrow$ $\alpha = \frac{(+1)+(-5)+(-8)}{3-0} = -4$

Angle of asymptote lines $\rightarrow$ $\varphi_i = \frac{180°}{3-0}(2i+1) = 60°(2i+1)$ $\rightarrow$ $\begin{cases} \varphi_0 = 60° \\ \varphi_1 = 180° \\ \varphi_2 = 300° \end{cases}$

d) [3 marks] Find the points (if any) where the root-locus crosses the imaginary axis, and the corresponding values of K. Mark these points on the s-plane.

Method 1:

Characteristics Equation $\rightarrow \quad 1 + KG(s)H(s) = 0 \quad \rightarrow \quad 1 + \frac{K}{(s-1)(s+5)(s+8)} = 0 \quad \rightarrow \quad s^3 + 12s^2 + 27s - 40 + K = 0$

For $s = j\omega \quad \rightarrow \quad (j\omega)^3 + 12(j\omega)^2 + 27(j\omega) - 40 + K = -j\omega^3 - 12\omega^2 + j27\omega - 40 + K = 0$

$$\underbrace{[-12\omega^2 - 40 + K]}_{\textbf{Real part}} + j\underbrace{[-\omega^3 + 27\omega]}_{\textbf{Imaginary part}} = 0$$

From the imaginary part:

$$-\omega^3 + 27\omega = 0 \quad \rightarrow \quad \omega(-\omega^2 + 27) = 0 \quad \rightarrow \quad \begin{cases} \omega = 0 \\ -\omega^2 + 27 = 0 \quad \rightarrow \quad \omega^2 = 27 \quad \rightarrow \quad \omega = \pm\sqrt{27} = \pm 5.2 \end{cases}$$

From the real part:

$$-12\omega^2 - 40 + K = 0 \quad \rightarrow \quad \begin{cases} \text{For} \quad \omega = 0 \quad \rightarrow \quad -12 \times 0 - 40 + K = 0 \quad \rightarrow \quad K = 40 \\ \text{For} \quad \omega^2 = 27 \quad \rightarrow \quad -12 \times 27 - 40 + K = 0 \quad \rightarrow \quad K = 364 \end{cases}$$

The Root-locus crosses the imaginary axis at $s = 0$ for $K = 40$, and at $s = \pm j5.2$ for $K = 364$.

Method 2:

Characteristics Equation $\rightarrow \quad 1 + KG(s)H(s) = 0 \quad \rightarrow \quad 1 + \frac{K}{(s-1)(s+5)(s+8)} = 0 \quad \rightarrow \quad s^3 + 12s^2 + 27s - 40 + K = 0$

Create the Routh-Hurwitz table for the closed-loop characteristics equation:

s^3	1	27
s^2	12	$-40 + K$
s^1	$\frac{364 - K}{12}$	0
s^0	$-40 + K$	0

All terms in the first column must be positive:

$$\frac{364 - K}{12} > 0 \quad \rightarrow \quad 364 - K > 0 \quad \rightarrow \quad 364 > K$$

$$-40 + K > 0 \quad \rightarrow \quad K > 40$$

The closed-loop system is marginally stable for $K = 40$ and $K = 364$. The intersection with the imaginary axis is determined from the even polynomial by using s^2 row with $K = 40$, and $K = 364$.

$$12s^2 - 40 + K = 0 \quad \rightarrow \quad \begin{cases} \text{If} \quad K = 40 \quad \rightarrow \quad 12s^2 = 0 \quad \rightarrow \quad s = 0 \\ \text{If} \quad K = 364 \quad \rightarrow \quad 12s^2 + 324 = 0 \quad \rightarrow \quad s = \pm j5.2 \end{cases}$$

The Root-locus crosses the imaginary axis at $s = 0$ for $K = 40$, and at $s = \pm j5.2$ for $K = 364$.

e) [3 marks] Find the break-away/break-in points (if any) of the root-locus and the corresponding value of *K*. Mark these points on the s-plane.

Characteristics Equation $\rightarrow \quad 1 + KG(s)H(s) = 0 \quad \rightarrow \quad s^3 + 12s^2 + 27s - 40 + K = 0$

$$K = -s^3 - 12s^2 - 27s + 40 \quad \rightarrow \quad \frac{dK}{ds} = -3s^2 - 24s - 27 = 0 \quad \rightarrow \quad \begin{cases} s = -6.65 & \rightarrow \ \text{not on the root} - \text{locus} \\ s = -1.35 & \rightarrow \ \text{on the root} - \text{locus} \end{cases}$$

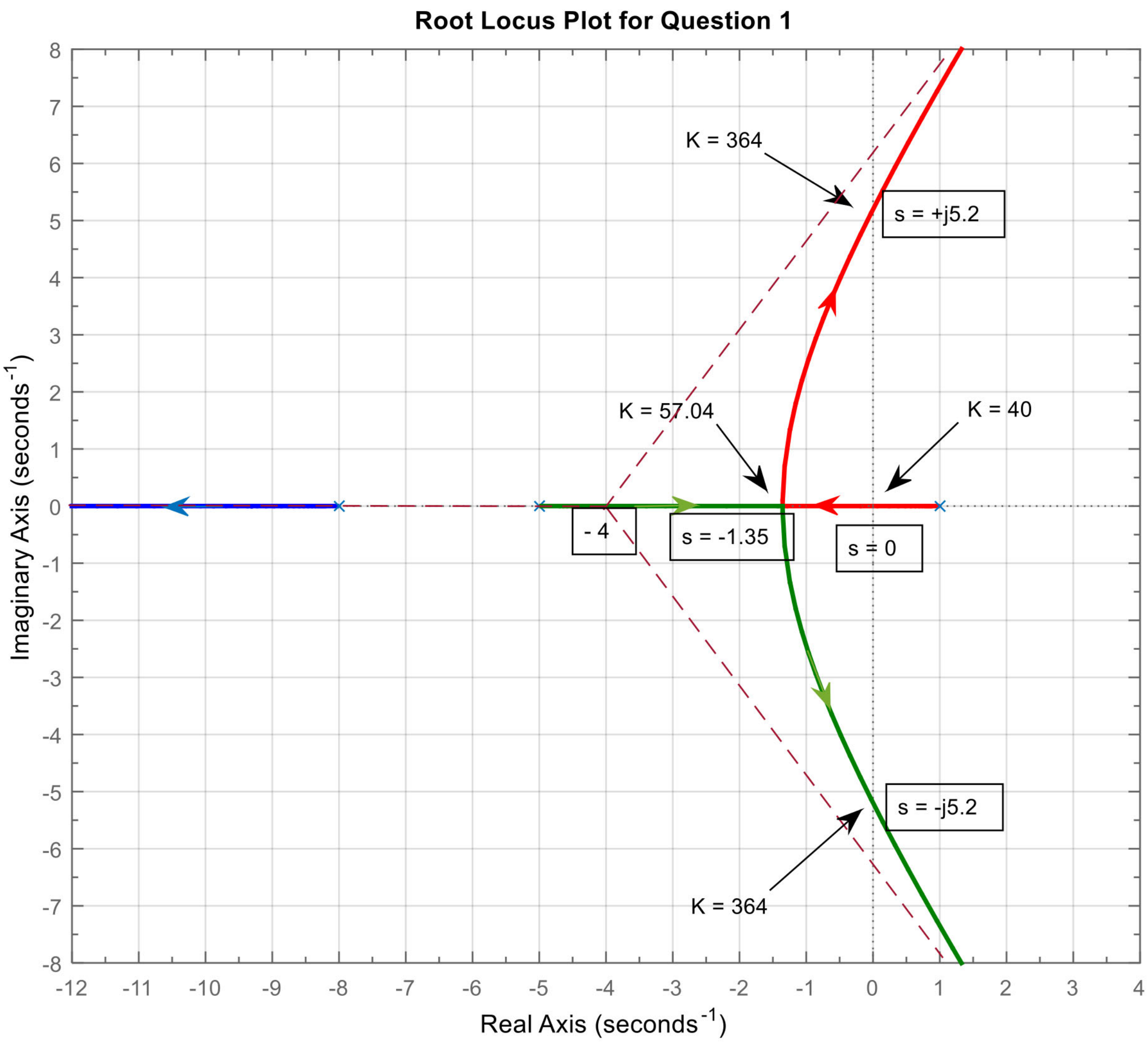
The multiple roots are located at $s = -1.35 \quad \rightarrow$ Break-away point

The corresponding K value is

$$K = -(-1.35)^3 - 12(-1.35)^2 - 27(-1.35) + 40 \quad \rightarrow \quad K = 57.04$$

f) [3 marks] Sketch the complete root-locus on the s-plane. Draw the root-locus paths. Show the directions, beginning and ending points of the paths.

See the following figure.



g) [5 mark] Determine the gain K from root-locus such that the closed-loop system has 5% overshoot. Show your work and the required drawings on the root-locus. Determine the dominant closed-loop poles from the root-locus. Show your work.

First, calculate the desired damping ratio from the desired maximum overshoot value,

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} = 0.691 \quad \rightarrow \quad \theta = \cos^{-1}(\zeta) = \cos^{-1}(0.691) = 46.3^\circ$$

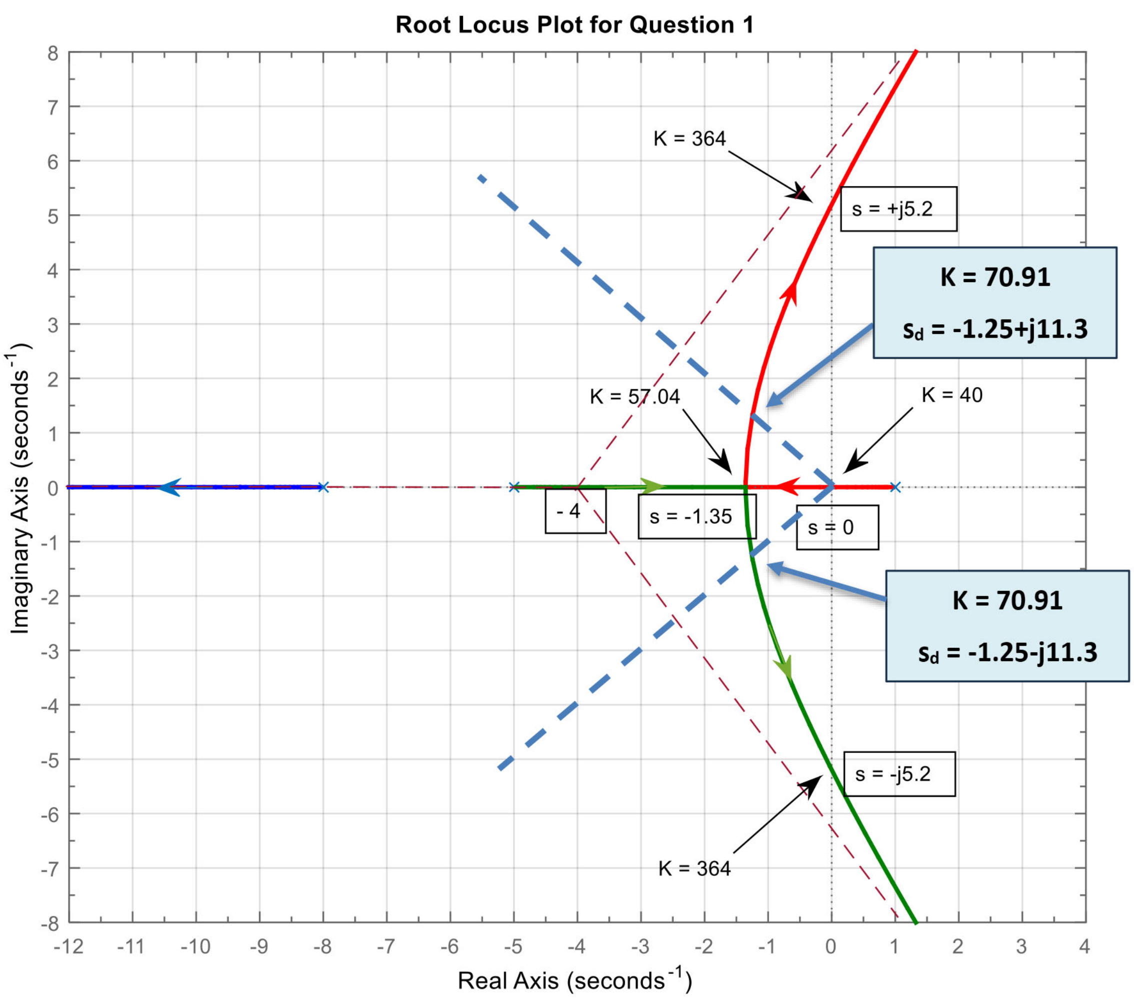
Next, sketch the constant-damping-ratio lines of $\zeta = 0.691$

The intersection of the lines with root-locus will be the desired pole locations. $s_d = -1.25 \pm j1.3$

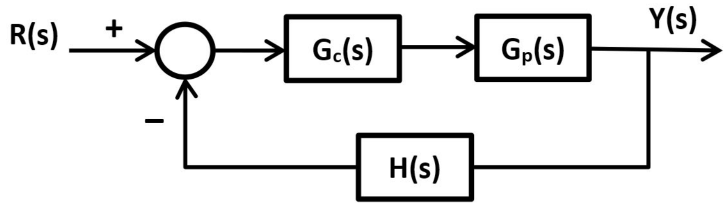
Find the gain K at the desired pole locations by using the magnitude condition:

$$\left|\frac{K}{(s-1)(s+5)(s+8)}\right| = 1 \quad \rightarrow \quad |K| = |(s-1)(s+5)(s+8)|_{s=s_d} \quad \rightarrow \quad |K| = |s_d - 1||s_d + 5||s_d + 8|$$

$$|K| = |-2.25 + j1.3||3.75 + j1.3||6.75 + j1.3| = 2.6 \times 3.97 \times 6.87 = 70.91$$



Question 2: **Consider the following closed-loop system with a proportional controller and a rate-feedback,**



$$G_c(s) = K_p, \qquad G_p(s) = \frac{10}{s(s-5)}, \qquad H(s) = 1 + T_d s$$

a) [1 mark] Determine if the system $G_p(s)$ is stable or not. Justify your answer.

$$G_p(s) = \frac{10}{s(s-5)}$$

Poles: $s = 0$ and $s = 5$

Since the system has pole at the right-half s-plane, the system is **unstable**.

b) [3 marks] Determine the transfer function and the characteristic equation of the closed-loop system in terms of the controller gains K_p and T_d. Show your work.

Closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} = \frac{K_p\left(\frac{10}{s(s-5)}\right)}{1 + K_p\left(\frac{10}{s(s-5)}\right)(1 + T_d s)} = \frac{10K_p}{s^2 + (10K_pT_d - 5)s + 10K_p}$$

Closed-loop characteristics equation:

$$s^2 + (10K_pT_d - 5)s + 10K_p = 0$$

c) [5 marks] Create the Routh-Hurwitz table and apply the Routh-Hurwitz stability criteria to determine the conditions on K_p and T_d to have a stable closed-loop system. Show your work and justify your answer.

Create the Routh-Hurwitz table for the closed-loop system characteristics equation:

s^2	1	$10K_p$
s^1	$10K_pT_d - 5$	0
s^0	$10K_p$	0

All terms in the first column must be positive:

$$10K_pT_d - 5 > 0 \quad \rightarrow \quad K_pT_d > 0.5 \quad \rightarrow \quad T_d > 0.5/K_p$$

$$10K_p > 0 \quad \rightarrow \quad K_p > 0$$

d) [6 marks] Determine the controller gains K_p and T_d so that the unit-step response has a maximum overshoot of 5% and the settling time of $t_s = 2\ sec$ (2% criteria). Show your work and justify your answer.

First, calculate the desired damping ratio from the given desired maximum overshoot:

$$\text{O.S.} = 5\% \rightarrow \zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2+\ln^2(O.S.)}} \rightarrow \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2+\ln^2(0.05)}} \rightarrow \zeta = 0.691$$

Next, determine the desired natural frequency from the given settling-time:

$$t_s = \frac{4}{\zeta\omega_n} \rightarrow 2 = \frac{4}{0.6901\omega_n} \rightarrow \omega_n = 2.89\ \ rad/s$$

Closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)} = \frac{10K_p}{s^2+(10K_pT_d-5)s+10K_p}$$

Compare the characteristic equation with the standard second-order prototype system

$$s^2+2\zeta\omega_n s+\omega_n^2 = s^2+(10K_pT_d-5)s+10K_p$$

$$\omega_n^2 = 10K_p \rightarrow (2.89)^2 = 10K_P \rightarrow K_p = \frac{8.3521}{10} = 0.835$$

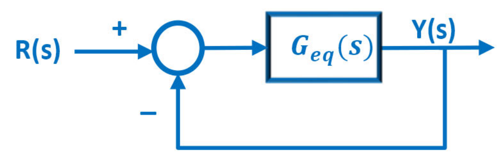
$$2\zeta\omega_n = 10K_pT_d - 5 \rightarrow 2(0.6901)(2.89) = 10(0.835)T_d - 5 \rightarrow T_d = \frac{8.99}{8.35} = 1.077$$

e) [5 marks] For $K_p = 5$ and $T_d = 1$ determine the error constants and the steady-state error of the closed-loop system for unit-step and unit-ramp inputs? Show your work. Hint: First, convert the non-unity feedback to a unity-feedback system.

Since the closed-loop system has non-unity feedback, first we have to convert it to unity-feedback configuration:

$$G_{eq}(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} \rightarrow G_{eq}(s) = \frac{\frac{50}{s(s-5)}}{1+\frac{50(1+s)}{s(s-5)}-\frac{50}{s(s-5)}} = \frac{50}{s^2+45s}$$

The step-error constant and steady-state error of step response:



$$k_p = \lim_{s\to 0} G_{eq}(s) = \lim_{s\to 0}\left(\frac{50}{s^2+45s}\right) = \infty$$

$$e_{ss} = \frac{1}{1+k_p} = \frac{1}{\infty} = 0$$

Since the open-loop system is **Type 1**, the steady-state error of step response will be **zero**.

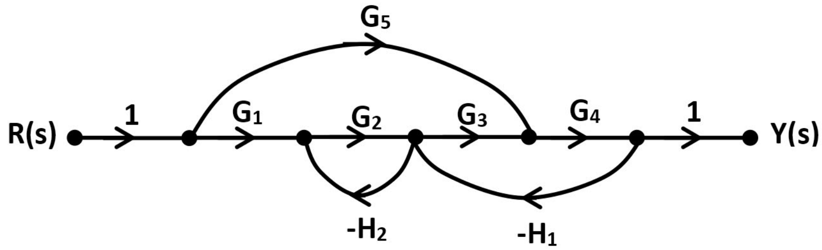
The ramp-error constant and steady-state error of ramp response:

$$k_v = \lim_{s\to 0} sG_{eq}(s) = \lim_{s\to 0} s\left(\frac{50}{s^2+45s}\right) = \frac{50}{45} = 1.11$$

$$e_{ss} = \frac{1}{k_v} = \frac{1}{1.11} = 0.91$$

Since the open-loop system is **Type 1**, the steady-state error of ramp response will be a **finite value**.

Question 3: Consider the following signal flow graph (SFG). Determine the transfer function of $\frac{Y(s)}{R(s)}$ using Mason's Gain Formula.



where, $G_1 = 5,\ G_2 = G_3 = G_4 = \frac{1}{s},\ G_5 = 4,\ H_1 = 1,\ H_2 = 2.$

Answer the following questions and determine the transfer function of $\frac{Y(s)}{R(s)}$ using Mason's Gain Formula.

a) [2 Marks] Find the number of forward paths from R(s) to Y(s) and determine the forward path gains.

Number of forward paths $\rightarrow$ N = 2

The forward paths' gains $\rightarrow$ $M_1 = G_1G_2G_3G_4$, $M_2 = G_5G_4$

b) [2 Marks] Find all loops of the SFG and determine the loop gains. Find the non-touching loop (if any).

The loop gains $\rightarrow$ $L_1 = -G_2H_2$, $L_2 = -G_3G_4H_1$

There are no non-touching loops.

c) [2 Marks] Determine the determinant of the SFG.

The SFG determinant $\rightarrow$ $\Delta = 1 - (L_1 + L_2) = 1 - (-G_2H_2 - G_3G_4H_1)$

d) [2 Marks] Determine the cofactors of each forward path.

The cofactors of each forward path $\rightarrow$ $\Delta_1 = 1$, $\Delta_2 = 1 + G_2H_2$

e) [2 Marks] Determine the overall transfer function of the SFG using Mason's Gain Formula

The overall transfer function of the SFG

$$\frac{Y(s)}{R(s)} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4 + G_5G_4(1 + G_2H_2)}{1 - (-G_2H_2 - G_3G_4H_1)} = \frac{G_1G_2G_3G_4 + G_5G_4 + G_5G_4G_2H_2}{1 + G_2H_2 + G_3G_4H_1}$$

Apply the numerical values:

$$\frac{Y(s)}{R(s)} = \frac{\frac{5}{s^3} + \frac{4}{s} + \frac{8}{s^2}}{1 + \frac{2}{s} + \frac{1}{s^2}} = \frac{\frac{5 + 4s^2 + 8s}{s^3}}{\frac{s^2 + 2s + 1}{s^2}} = \frac{4s^2 + 8s + 5}{s(s^2 + 2s + 1)}$$