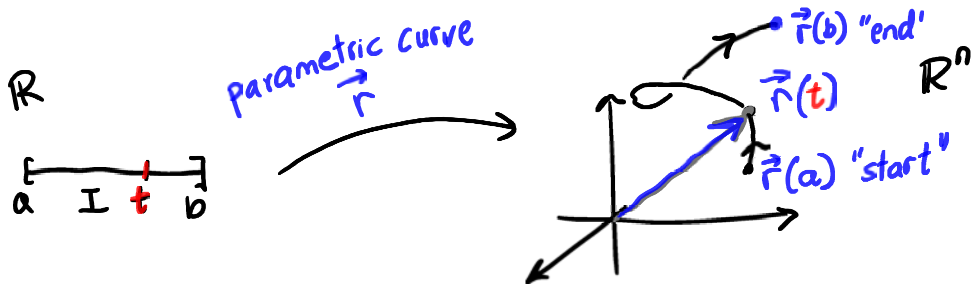


Chapter 11

Curves and Vector Fields

11.1 Curves

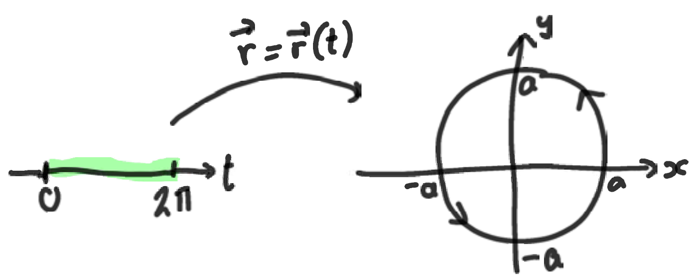
A parametric curve is a special case of a vector-valued function whose domains is an interval in $\mathbb{R}$.

Parametric Curve or Path

Definition 11.1. A *parametric curve* (*curve* or *path*) in $\mathbb{R}^n$ is a non-constant continuous map $\mathbf{r} : I \to \mathbb{R}^n$ where I is an interval in $\mathbb{R}$. If the domain of $\mathbf{r}$ is the interval $[a, b]$, we say that $\mathbf{r}$ is a parametric curve that "starts" at $\mathbf{r}(a)$, "ends" at $\mathbf{r}(b)$, and "runs" from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. If $\mathbf{r}(a) = \mathbf{r}(b)$, we say that the parametric curve is *closed*.

Often, we refer to the function $\mathbf{r}$ and its image in $\mathbb{R}^n$ both as "the parametric curve."

Example 11.2. (FRY Example IV.1.0.1)
Parametrize the circle $x^2 + y^2 = \cancel{r^2}\, a^2$



(i) Consider $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$
where $0 \le t \le 2\pi$

going around circle counterclockwise

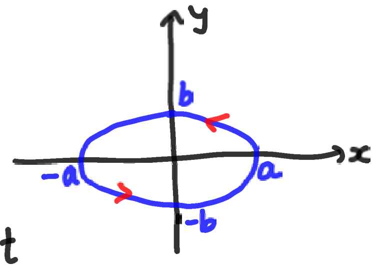
(ii) If we wanted to go clockwise,
$\vec{r}(t) = \langle a\cos t, -a\sin t\rangle$ where $0 \le t \le 2\pi$

(iii) Here's another $\vec{r}(t) = \langle a\cos(2t), a\sin(2t) \rangle$ where $0 \le t \le \pi$

Example 11.3. (FRY Example IV.1.0.3)
Parametrize the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. where $a, b > 0$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

$\cos^2 t + \sin^2 t = 1$

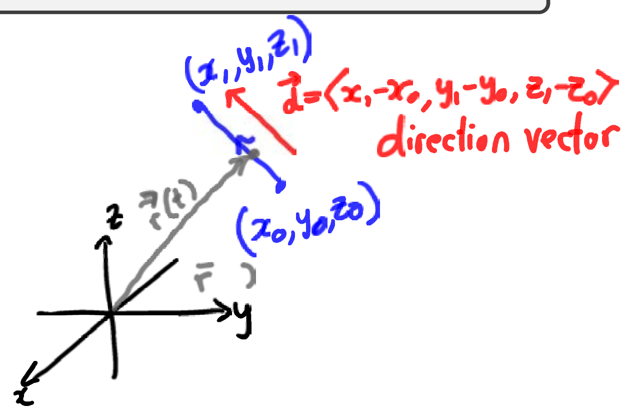


$\frac{x}{a} = \cos t \rightarrow x = a\cos t$

$\frac{y}{b} = \sin t \rightarrow y = b\sin t$

So, $\vec{r}(t) = \langle a\cos t, b\sin t \rangle$, where $0 \le t \le 2\pi$
parametrizes the ellipse.

Example 11.4. Parametrize the line segment that runs from the point (x_0, y_0, z_0) to the point (x_1, y_1, z_1).



$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\vec{d}$, where $0 \le t \le 1$

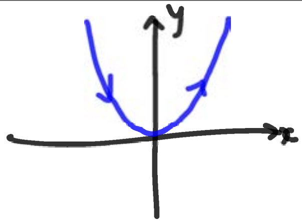
$= \langle x_0, y_0, z_0 \rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$, $0 \le t \le 1$

$= \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0) \rangle$, $0 \le t \le 1$

Example 11.5. Parametrize $y = x^2$

$\vec{r}(x) = \langle x, x^2 \rangle$

where $-\infty < x < \infty$



Example 11.6. Unparametrize[a] $\mathbf{r}(t) = \langle \underset{x}{t}, \underset{y}{t^2} \rangle$.

[a]To "unparametrize" means to "hide" the parametrization and relate the components to each other directly.

Observe $y = t^2 = (t)^2 = x^2$.

$\vec{r}(t) = \langle t, t^2 \rangle$ parametrizes $\boxed{y = x^2}$

11.2 Derivatives, Velocity, Etc.

The component-wise nature of the operations of addition and scalar multiplication on vectors allows us to calculate the derivative of a vector-valued function component by component.

FRY Defn IV.1.1.1 Derivative of $\mathbf{r}(t)$

Definition 11.7. Let $\mathbf{r} : I \to \mathbb{R}^n$ be a vector-valued function on an interval $I \subseteq \mathbb{R}$. Its derivative is defined to be

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

or, equivalently, as

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}(t) = \lim_{\Delta t\to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t},$$

when the limit exists. We say that $\mathbf{r}$ is differentiable at $t \in I$ when the derivative of $\mathbf{r}$ exists at t. We say that $\mathbf{r}$ is differentiable on I if it is differentiable at every point in I.

If $\mathbf{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t)\rangle$, then

$$\mathbf{r}'(t) = \left\langle x_1'(t), x_2'(t), \ldots, x_n'(t)\right\rangle,$$

that is, the derivative of $\mathbf{r}(t)$ can be found by differentiating each of its components.

For example, if $\mathbf{r}(t) = \langle t, \sin t, e^t\rangle$ where $t \in \mathbb{R}$, then $\mathbf{r}'(t) = \langle 1, \cos t, e^t\rangle$ for $t \in \mathbb{R}$.

FRY Defn IV.1.1.3 Arithmetic of differentiation

Theorem 11.8. Let I be an interval in $\mathbb{R}$ (possibly all of $\mathbb{R}$). Let $\mathbf{f}$ and $\mathbf{g}$ be differentiable vector-valued functions and $a(t)$ and $b(t)$ be real-valued differentiable functions on I. Let $c, d \in \mathbb{R}$ be constants. Then

(a) $\dfrac{d}{dt}\big[c\,\mathbf{f}(t) + d\,\mathbf{g}(t)\big] = c\,\mathbf{f}'(t) + d\,\mathbf{g}'(t)$

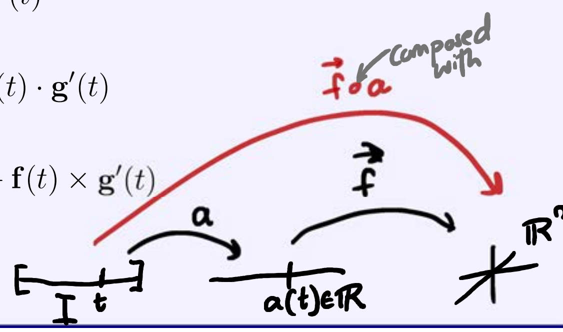
"Product Rules"

(b) $\dfrac{d}{dt}\big[a(t)\,\mathbf{f}(t)\big] = a'(t)\mathbf{f}(t) + a(t)\mathbf{f}'(t)$

(c) $\dfrac{d}{dt}\big[\mathbf{f}(t) \cdot \mathbf{g}(t)\big] = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)$

(d) $\dfrac{d}{dt}\big[\mathbf{f}(t) \times \mathbf{g}(t)\big] = \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t)$

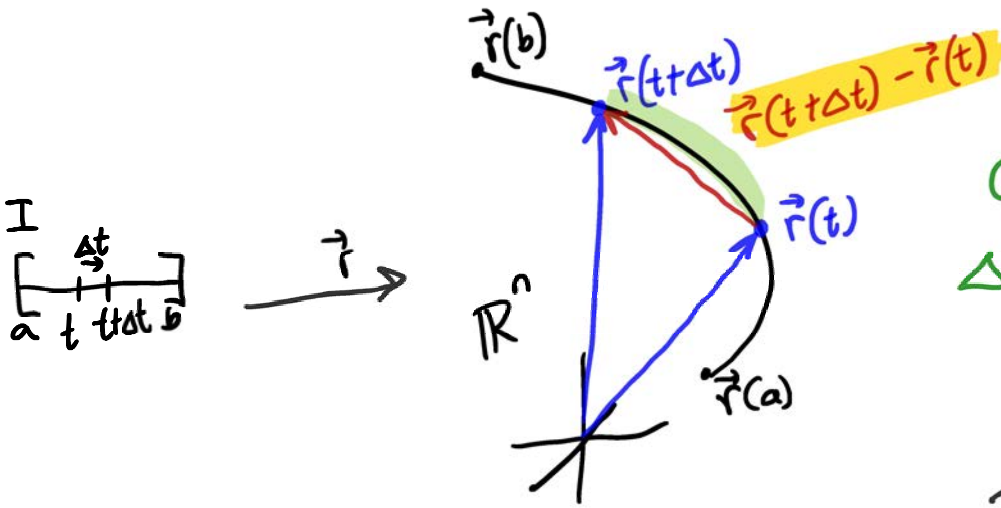
(e) $\dfrac{d}{dt}\big[\mathbf{f}\big(a(t)\big)\big] = \mathbf{f}'\big(a(t)\big)\,a'(t)$



Example 11.9. (FRY Example IV.1.1.2)
Let $\mathbf{f}(t) = \langle t^2, t^4, t^6 \rangle$, $\mathbf{g}(t) = \langle e^{-t}, e^{-3t}, e^{-5t} \rangle$, $a(t) = t^2$, and $b(t) = \sin(t)$. Use these functions to see some of the identities described in the theorem above in action.

Let $\vec{r}: \underset{=[a,b]}{I} \subseteq \mathbb{R} \to \mathbb{R}^n$. Suppose $\vec{r}$ is differentiable at $t \in I$.

That is, $\displaystyle\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right) = \vec{r}\,'(t)$



Observe that

$\Delta s =$ distance along the curve between $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$

for small Δt $\approx$ length of the vector $\vec{r}(t+\Delta t) - \vec{r}(t)$

$= \|\vec{r}(t+\Delta t) - \vec{r}(t)\|$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right) = \vec{r}\,'(t)$$

$\approx \|\vec{r}\,'(t)\,\Delta t\|$

$= \|\vec{r}\,'(t)\|\,\Delta t$

In the limit,

$$ds = \|\vec{r}\,'(t)\|\,dt$$

$$\underbrace{\int ds}_{\substack{\text{length of curve} \\ \text{between } \vec{r}(a) \text{ and } \vec{r}(b)}} = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

arclength

adding up all the distances along the curve

A consequence of the definition of the derivative being

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

is that when Δt is small,

- The vector $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$ has roughly the same direction as the tangent vector to the curve at $\mathbf{r}(t)$ that points in the direction of increasing t; and
- $\|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\|$ has roughly the same length as the portion of the curve between $\mathbf{r}(t)$ and $\mathbf{r}(t + \Delta t)$.

$\vec{r}: I \to \mathbb{R}^n$

FRY Lemma IV.1.1.4, The tangent vector, unit tangent vector, and arclength

Lemma 11.10. Let I be an interval in $\mathbb{R}$ (possibly all of $\mathbb{R}$). Let $\mathbf{r}$ be a differentiable vector-valued function with a nonzero continuous derivative[a] at every $t \in I$. Then

(a) $\mathbf{r}'(t)$ is a tangent vector to the curve at $\mathbf{r}(t)$ that points in the direction of increasing t.

(b) The unit tangent vector $\hat{\mathbf{T}}(t)$ to the curve at $\mathbf{r}(t)$ pointing in the direction of increasing t is given by

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

(c) The arclength (or length), $s(t)$, of the part of the curve between $\mathbf{r}(a)$, where $a \in I$, and $\mathbf{r}(t)$ is given by

$$\begin{aligned} \text{arclength function } s(t) &= \int_a^t \|\mathbf{r}'(\tau)\| \; d\tau = \int_a^t \left\| \frac{d\mathbf{r}}{d\tau}(\tau) \right\| \; d\tau \\ &= \int_a^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2} \; d\tau. \end{aligned}$$

(d) The arclength (or length) of the curve from $\mathbf{r}(a)$ to $\mathbf{r}(b)$, where $a, b \in I$ with $a \leq b$, equals $\displaystyle\int_a^b \|\mathbf{r}'(t)\| \; dt$.

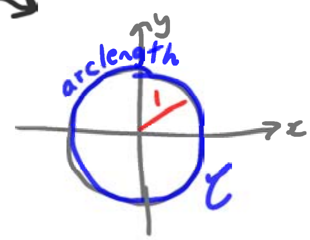
[a] A vector-valued function with a nonzero continuous derivative at every point in its domain is also called a *regular* function.

Example 11.11. (FRY Example IV.1.1.6)
Find the arclength of the parametric curve

$$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$$

from $t = 0$ to $t = 2\pi$.

This is the unit circle C



$$\text{arclength} = \int_C ds$$

$$= \int_0^{2\pi} \|\vec{r}\,'(t)\|\, dt$$

$$= \int_0^{2\pi} 1\, dt$$

$$= t \Big|_0^{2\pi}$$

$$= 2\pi - 0$$

$$= 2\pi$$

$\vec{r}(t) = \langle \cos t, \sin t \rangle, \; 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= \sqrt{1}$$

$$= 1$$

FRY Lemma IV.1.1.5

Lemma 11.12. Let I be an interval in $\mathbb{R}$ (possibly all of $\mathbb{R}$). Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a differentiable vector-valued function with nonzero continuous derivative at every $t \in I$. Suppose that $\mathbf{r}(t)$ describes the position of a particle at time t.

(a) The position of the particle at time t is given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

(b) The velocity of the particle at time t is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

(c) The speed of the particle at time t is given by

$$\frac{ds}{dt}(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$ [a]

(d) The acceleration of the particle at time t is given by

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle.$$

(e) The distance travelled between times t_0 and t is

$$\begin{aligned} s(t) - s(t_0) &= \int_{t_0}^{t} \|\mathbf{r}'(\tau)\|\, d\tau = \int_{t_0}^{t} \|\mathbf{v}(\tau)\|\, d\tau \\ &= \int_{t_0}^{t} \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\, d\tau. \end{aligned}$$

---

[a] Note that the velocity of the particle at time t may be written as its speed multiplied by the direction: $\mathbf{v}(t) = \dfrac{ds}{dt}\,\hat{\mathbf{T}}(t)$, where $\hat{\mathbf{T}}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ is the unit tangent vector.

$$x^2+y^2=[3\cos(2\pi t)]^2+[3\sin(2\pi t)]^2=9\cos^2(2\pi t)+9\sin^2(2\pi t)$$
$$=9\left[\cos^2(2\pi t)+\sin^2(2\pi t)\right]=9(1)=9$$

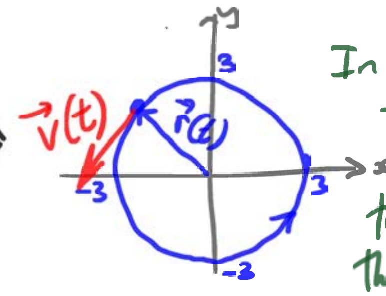
Example 11.13. At time t, the position of a particle is described by $\mathbf{r}(t) = \langle 3\cos(2\pi t), 3\sin(2\pi t)\rangle$. Determine its velocity, speed, and acceleration at time t. For this particular motion, describe any relations that you see in terms of the directions of the position, velocity, and acceleration vectors.

$$\vec{r}(t)=\langle \underbrace{3\cos(2\pi t)}_{x}, \underbrace{3\sin(2\pi t)}_{y}\rangle \quad 0\le t\le 1$$

- velocity $\vec{v}(t)=\vec{r}\,'(t)$

$$=\langle 3(-\sin(2\pi t)\cdot 2\pi, 3\cos(2\pi t)\cdot 2\pi\rangle$$
$$=\langle -6\pi\sin(2\pi t), 6\pi\cos(2\pi t)\rangle,$$

where $0\le t\le 1$



In general, $\vec{r}\,'(t)$ is tangent to the curve at $\vec{r}(t)$.

Note $\vec{v}(t)\cdot\vec{r}(t)$

$$=\langle -6\pi\sin(2\pi t), 6\pi\cos(2\pi t)\rangle\cdot\langle 3\cos(2\pi t), 3\sin(2\pi t)\rangle$$
$$=-18\pi\sin(2\pi t)\cos(2\pi t)+18\pi\sin(2\pi t)\cos(2\pi t)$$
$$=0$$

That is, $\vec{v}(t)\perp\vec{r}(t)$

↑ perpendicular.

For circular trajectories, the velocity vector is perpendicular to the position vector.

- speed $=\|\vec{v}(t)\|$

$$=\sqrt{(-6\pi\sin(2\pi t))^2+(6\pi\cos(2\pi t))^2}$$
$$=\sqrt{36\pi^2\sin^2(2\pi t)+36\pi^2\cos^2(2\pi t)}$$
$$=\sqrt{36\pi^2\underbrace{(\sin^2(2\pi t)+\cos^2(2\pi t))}_{=1}}=\sqrt{36\pi^2}=6\pi$$

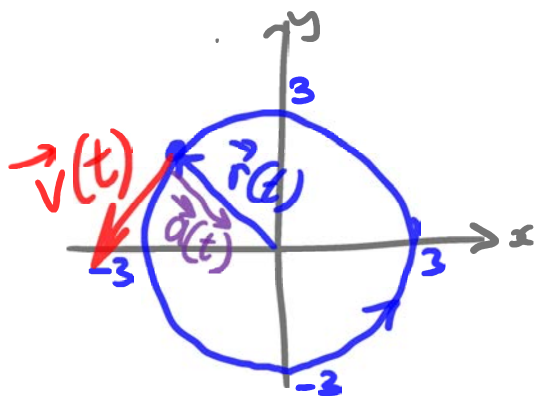
■ acceleration

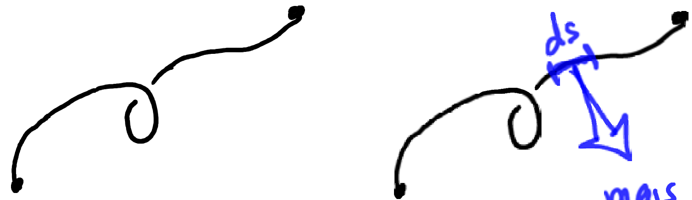
$\vec{v}(t) = \langle -6\pi \sin(2\pi t), 6\pi\cos(2\pi t)\rangle$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$
$$= \langle -12\pi^2 \cos(2\pi t), -12\pi^2 \sin(2\pi t)\rangle$$

Recall $\vec{r}(t) = \langle 3\cos(2\pi t), 3\sin(2\pi t)\rangle$. Notice that

$$\vec{a}(t) = -4\pi^2 \vec{r}(t)$$





mass $= \rho\, ds$ of this small segment

11.3 Integrating Along a Curve

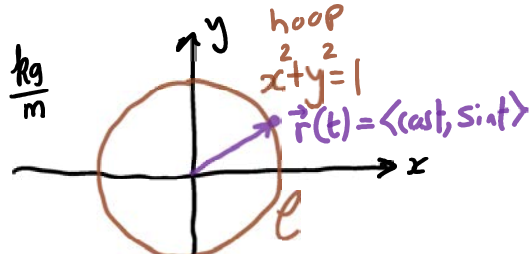
Let $\mathcal{C}$ be a (smooth, regular) curve with parametrization $\mathbf{r}(t)$ where $a \leq t \leq b$. If ρ is a continuous mass density function along the curve $\mathcal{C}$, then

$$\text{mass of } \mathcal{C} \;=\; \int_a^b \rho(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt = \int_{\mathcal{C}} \rho\, ds.$$

> **Example 11.14.** (FRY Exercise IV.1.6.1.4)
> A hoop traces out the curve $x^2+y^2=1$ where x and y are measured in metres. At a point (x,y), its density is x^2 kg per metre. What is the mass of the hoop?

$$\boxed{\int_{\mathcal{C}} ds \quad \text{arclength}}$$

$\rho(x,y) = x^2 \ \frac{\text{kg}}{\text{m}}$



$$\boxed{\text{mass} = \int_{\mathcal{C}} \rho\, ds}$$

$$\boxed{\text{Let } \vec{r}(t) = \langle \cos t, \sin t\rangle \text{ where } 0 \le t \le 2\pi}$$

$\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

$$= \int_0^{2\pi} \rho(\vec{r}(t))\, \underbrace{\|\vec{r}\,'(t)\|}_{=1}\, dt$$

$$= \int_0^{2\pi} \cos^2 t \cdot 1\, dt$$

$\rho(\cos t, \sin t)$

$$= \int_0^{2\pi} \tfrac{1}{2}(1+\cos(2t))\, dt$$

$$= \tfrac{1}{2}\int_0^{2\pi} 1\, dt + \tfrac{1}{2}\int_0^{2\pi} \cos(2t)\, dt$$

$$= \tfrac{1}{2} t \Big|_0^{2\pi} + \tfrac{1}{2}\cdot\tfrac{1}{2}\sin(2t)\Big|_0^{2\pi}$$

$$= \tfrac{1}{2}(2\pi) + \tfrac{1}{4}(0-0)$$

$$= \pi \text{ kg}$$

$\cos(2t) = \cos(t+t)$

$= \cos t\cos t - \sin t \sin t$

$= \cos^2 t - \sin^2 t$

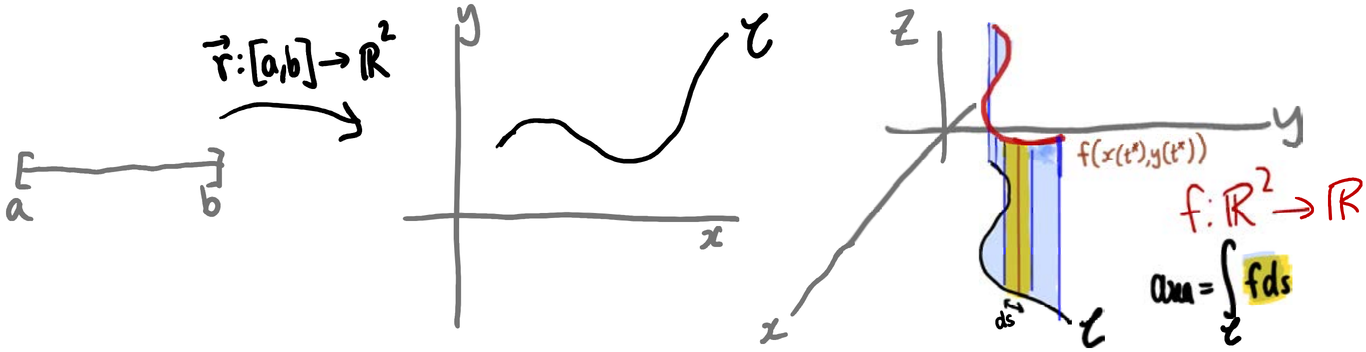
$= \cos^2 t - (1-\cos^2 t)$

So $\cos(2t) = 2\cos^2 t - 1$

and $\boxed{\cos^2 t = \tfrac{1}{2}(1+\cos(2t))}$.

$\int \cos(mt)\, dt = \frac{1}{m}\sin(mt) + C$, $m \neq 0$

$\sin^2 t = \frac{1}{2}(1-\cos(2t))$



$\vec{r}:[a,b]\to\mathbb{R}^2$
a
b
y
x
$\mathcal{C}$
z
y
x
$f(x(t^*),y(t^*))$
$f:\mathbb{R}^2\to\mathbb{R}$
ds
$\mathcal{C}$
area $=\int_{\mathcal{C}} f\,ds$

More generally,

FRY Defn IV.1.6.1 Integral of f over parametric curve

Definition 11.15. Let $\mathcal{C}$ be a curve parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, where $a \le t \le b$. Let f be a continuous function along $\mathbf{C}$. Then

$$\int_{\mathcal{C}} f\ ds = \int_a^b f\left(\mathbf{r}(t)\right) \|\mathbf{r}'(t)\|\ dt = \int_a^b f\left(x(t), y(t), z(t)\right) \|\mathbf{r}'(t)\|\ dt.$$

In the special case that the curve $\mathcal{C}$ lies in $\mathbb{R}^2$ and is described by the function $y = f(x)$, then for a continuous function $g(x, y)$,

$$\int_{\mathcal{C}} g\ ds = \int_a^b g(x, f(x))\sqrt{1 + (f'(x))^2}\ dx.$$

The reason this works is because we can use the parametrization $\mathbf{r}(x) = \langle x, f(x)\rangle$ in which case $g(\mathbf{r}(x)) = g(x, f(x))$ and $\|\mathbf{r}'(x)\| = \|\langle 1, f'(x)\rangle\| = \sqrt{1 + (f'(x))^2}$.

Example 11.16. (FRY Exercise IV.1.6.1.5)
Let $f(x, y, z) = xy + z$ and $\mathcal{C}$ be the straight line segment from $(1, 2, 3)$ to $(2, 4, 5)$. Evaluate $\displaystyle\int_{\mathcal{C}} f\ ds$.

$\boxed{f(x,y,z) = xy+z}$

$\mathcal{C}$: straight line segment $(1,2,3)$ to $(2,4,5)$.

Goal: Evaluate $\boxed{\displaystyle\int_{\mathcal{C}} f\,ds}$ scalar-valued function

line integral

Ans: $\displaystyle\int_{\mathcal{C}} f\,ds$

$\vec{r}(t) = \langle 1,2,3\rangle + t\langle 2-1, 4-2, 5-3\rangle$
$= \langle 1,2,3\rangle + t\langle 1,2,2\rangle$
$= \langle 1+t, 2+2t, 3+2t\rangle$ where $0 \le t \le 1$

(1) Parametrize $\mathcal{C}$, say $\vec{r}(t)$ where $\underset{0}{a} \le t \le \underset{1}{b}$

(2) $\boxed{\displaystyle\int_{\mathcal{C}} f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$

$f(x,y,z) = xy+z$

$$= \int_0^1 f(\overbrace{1+t}^{x}, \overbrace{2+2t}^{y}, \overbrace{3+2t}^{z}) \, \| \langle 1,2,2 \rangle \| \, dt$$

$$\sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$$

since $f(x,y,z) = xy + z$

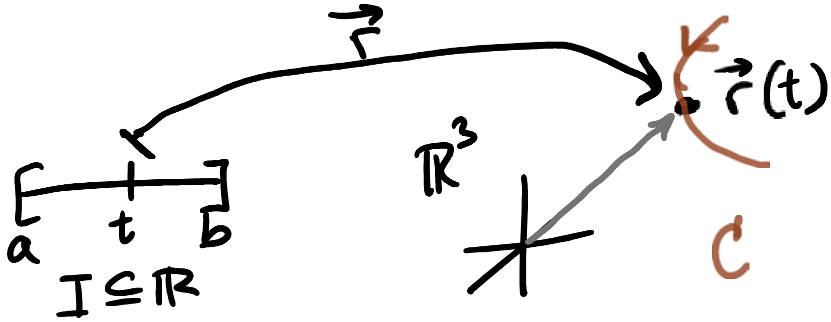
$$= \int_0^1 \left((1+t)(2+2t) + 3+2t \right) \cdot 3 \, dt$$

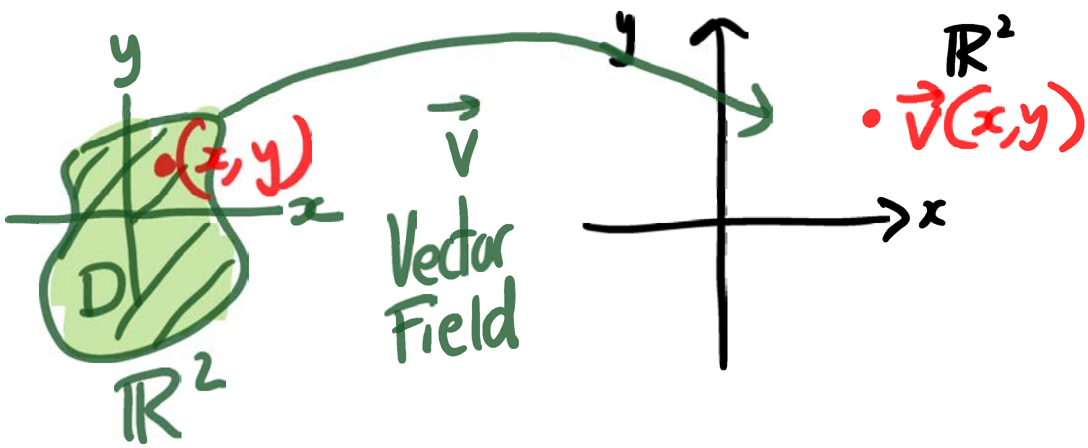
$$= 3\int_0^1 \left(2+2t+2t+2t^2+3+2t \right) dt$$

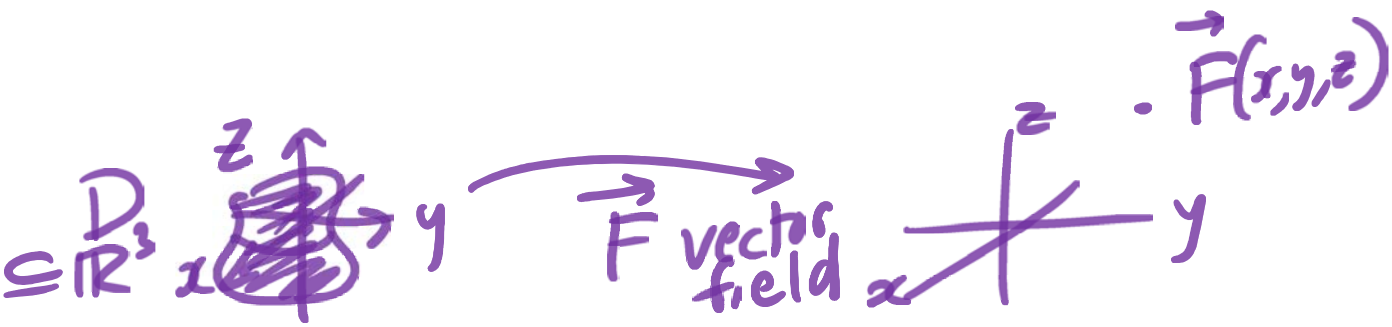
$$= 3 \int_0^1 (5+6t+2t^2)\, dt$$

$$= 3 \left[5t + 3t^2 + \frac{2}{3}t^3 \right]_0^1$$

$$= 3\left(5+3+\frac{2}{3}\right) = 3\left(\frac{26}{3}\right) = 26$$


$\vec{r}$
$\vec{r}(t)$
$\mathbb{R}^3$
C
a t b
$I \subseteq \mathbb{R}$


y
(x, y)
x
D
$\mathbb{R}^2$
$\vec{V}$
Vector
Field
y
$\mathbb{R}^2$
$\vec{V}(x, y)$
x


z
$D \subseteq \mathbb{R}^3$
x
y
$\vec{F}$ vector
field
z
$\vec{F}(x, y, z)$
y
x

11.4 Vector Fields: Definitions and First Examples

FRY Defn IV.2.1.1, What is a vector field?

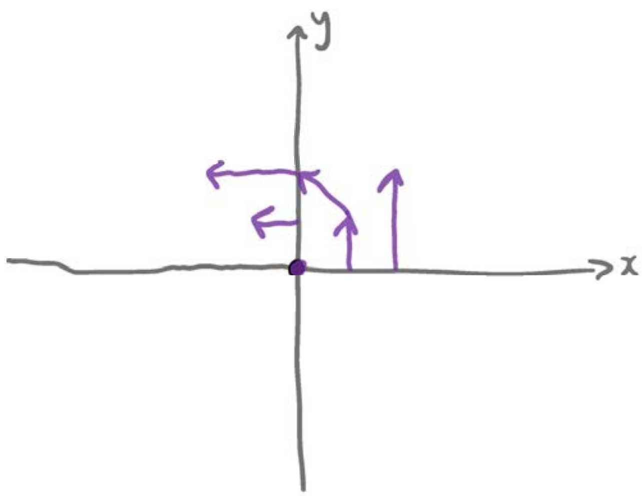
Definition 11.17. A vector field is a function that assigns to each point $(x_1, x_2, \ldots, x_n)$ in a subset D of $\mathbb{R}^n$ a vector $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ in $\mathbb{R}^n$.

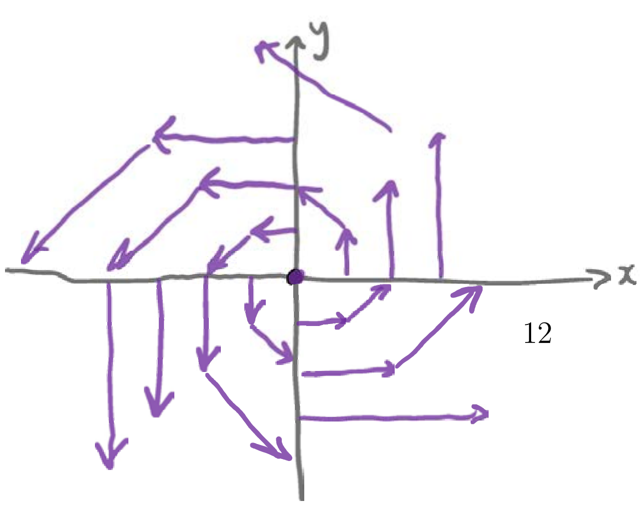
We will primarily work with vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 11.18. Sketch the vector field $\mathbf{v}(x, y) = \langle -y, x \rangle$.

$\vec{V}(x,y) = \langle -y, x \rangle$

Input	Output
$(0,0)$	$\langle -0, 0 \rangle = \langle 0,0 \rangle$
$(1,0)$	$\langle -0, 1 \rangle = \langle 0,1 \rangle$
$(2,0)$	$\langle -0, 2 \rangle = \langle 0,2 \rangle$
$(1,1)$	$\langle -1, 1 \rangle$
$(2,3)$	$\langle -3, 2 \rangle$
$(0,1)$	$\langle -1, 0 \rangle$
$\boxed{(0,2)}$	$\boxed{\langle -2, 0 \rangle}$
$(-1,1)$	$\langle -1, -1 \rangle$
$-1,0)$	$\langle -0, -1 \rangle = \langle 0,-1 \rangle$
$(-2,-2)$	$\langle -(-2), -2 \rangle = \langle 2,-2 \rangle$





If we use unit vectors ("directions") where we can, we get a direction field of a given vector field:

FRY Defn IV.2.1.8, What is a direction field?

Definition 11.19. Let $\mathbf{v}$ be a vector field. Then, the direction field of $\mathbf{v}$ is the vector field defined by

$$\mathbf{D}(x_1, x_2, \ldots, x_n) = \begin{cases} \dfrac{\mathbf{v}(x_1, x_2, \ldots, x_n)}{\|\mathbf{v}(x_1, x_2, \ldots, x_n)\|}, & \text{if } \mathbf{v}(x_1, x_2, \ldots, x_n) \neq \mathbf{0} \\ \mathbf{0}, & \text{if } \mathbf{v}(x_1, x_2, \ldots, x_n) = \mathbf{0}. \end{cases}$$

Example 11.20. Sketch the direction field for the vector field $\mathbf{v}(x, y) = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} = \langle x, y\rangle$.

11.5 References

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