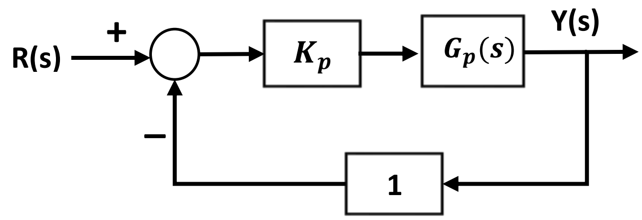
# Worksheet 4 - Solution

**1) Consider the mobile robot that uses a vision system as the measurement device. The system can be modeled as a second-order transfer function as below:**

$$G_p(s) = \frac{1}{(s+1)(0.5s+1)}$$




**a) Determine range of the proportional controller gain $K_p$ to have the percent of overshoot $\%O.S. \leq 5\%$**

First find the closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_p G_p(s)}{1 + K_p G_p(s) H(s)} = \frac{\frac{K_p}{(s+1)(0.5s+1)}}{1 + \frac{K_p}{(s+1)(0.5s+1)}} = \frac{K_p}{0.5s^2 + 1.5s + 1 + K_p}$$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2K_p}{s^2 + 3s + 2(1+K_p)}$$

Calculate the damping ratio from the required maximum overshoot value:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \quad \rightarrow \quad \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \quad \rightarrow \quad \zeta = 0.691$$

Next, compare the characteristic equation with the standard second order system to find the natural frequency $\omega_n$ and damping ratio $\zeta$ in terms of gain $K_p$.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 3s + 2(1+K_p) \quad \rightarrow \quad \begin{cases} 2\zeta\omega_n = 3 \\ \omega_n^2 = 2(1+K_p) \end{cases} \quad \rightarrow \quad \zeta = \frac{3}{2\omega_n} = \frac{3}{2\sqrt{2(1+K_p)}}$$

Calculate the required proportional gain $K$ to achieve the given maximum overshoot:

$$\zeta = \frac{3}{2\sqrt{2(1+K_p)}} \quad \rightarrow \quad 0.691 = \frac{3}{2\sqrt{2(1+K_p)}} \quad \rightarrow \quad K_p = 1.36$$

To achieve the maximum overshoot less than 5% the proportional gain must be selected less than 1.36.

$$\%O.S. \leq 5\% \quad \rightarrow \quad K_p < 1.36$$

**b) The tracking error is defined as $E(s) = R(s) - Y(s)$. Determine the steady-state tracking error $e_{ss}$ due to a unit-step response, $R(s) = 1/s$ if the proportional gain is selected as $K_p = 1$.**

The step-error-constant and the steady-state error are obtained as:

$$\boldsymbol{k_p} = \lim_{s\to 0} \boldsymbol{G(s)} = \lim_{s\to 0} K_p G_p(s) = \lim_{s\to 0} \frac{1}{(s+1)(0.5s+1)} = 1$$

$$\boldsymbol{e_{ss}} = \frac{\mathbf{1}}{\mathbf{1} + \boldsymbol{k_p}} = \frac{1}{1+1} = 0.5 \quad \rightarrow \quad e_{ss} = 50\ \%$$

**c) Determine the dominant poles of the closed-loop transfer function if $K_p = 1$ and design a PI controller as below to achieve a zero steady-state error.**

$$\boldsymbol{G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)}$$

First, find the dominant poles of the closed-loop transfer function for $K_p = 1$.

$$s^2 + 3s + 4 = 0 \quad \rightarrow \quad s = -\frac{3}{2} \pm j\frac{\sqrt{7}}{2} = -1.5 \pm j1.32$$

The second order closed-loop transfer function has one pair of complex-conjugated poles, which are the dominant poles.

The integral time constant $T_i$ can be selected by the following stability consideration:

$$\frac{2}{|Re\{-1.5 \pm j1.32\}|} \le T_i \quad \rightarrow \quad \frac{2}{1.5} \le T_i \quad \rightarrow \quad \boldsymbol{3.33 \le T_i}$$

For example, the following selection can be an appropriate PI controller.

$$K_p = 1, \quad T_i = 5 \quad \rightarrow \quad G_c(s) = 1 + \frac{1}{5s}$$

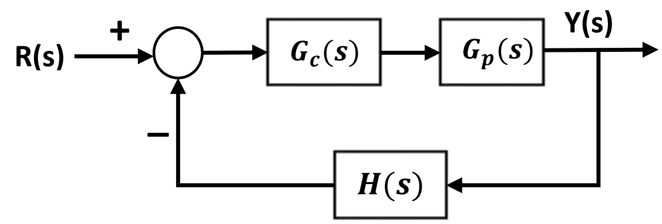
Note that we must check the step response and fine tune the integral time-constant if required.

**2) Consider the following closed-loop system with the following components.**

$$G_p(s) = \frac{100}{s+100}$$

$$G_c(s) = K_p$$

$$H(s) = \frac{5}{s+5}$$




**a) Compute the closed-loop transfer function**

$$T(s) = \frac{Y(s)}{R(s)}$$

The closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} = \frac{\dfrac{100\,K_p}{s+100}}{1 + \dfrac{500\,K_p}{(s+100)(s+5)}} = \frac{100(s+5)K_p}{s^2 + 105s + 500 + 500K_p}$$

**b) The tracking error is defined as $E(s) = R(s) - Y(s)$. Determine the steady-state tracking error due to a unit-step response, $R(s) = 1/s$ if $K_p = 10$.**

The step-error-constant and the steady-state error are obtained as:

$$k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} K_p G_p(s) = \lim_{s\to 0} \frac{1000}{s+100} = 10$$

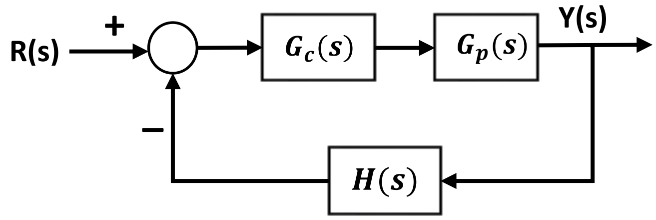
$$e_{ss} = \frac{1}{1+k_p} = \frac{1}{1+10} = 0.091 \quad \rightarrow \quad e_{ss} = 9.1\ \%$$

**3) Consider the unity feedback control system with the following components.**

$$G_p(s) = \frac{1}{s(s+5)}$$

$$G_c(s) = K_p$$

$$H(s) = 1$$




**Determine the proportional controller gain $K_p$ to achieve a peak-time less than 1.0 sec $(t_p \leq 1.0\ sec)$**

**What is the steady-state error of the closed-loop system to the unit-step input $R(s) = 1/s$ ?**

First find the closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} = \frac{\frac{K_p}{s(s+5)}}{1 + \frac{K_p}{s(s+5)}} = \frac{K_p}{s^2 + 5s + K_p}$$

Determine the damping ratio and the natural frequency from the characteristic equation of the closed-loop system:

$$s^2 + 2\zeta\omega_n + \omega_n^2 = s^2 + 5s + K_p \quad \rightarrow \quad \omega_n^2 = K_p \quad and \quad 2\zeta\omega_n = 5$$

Next calculate the damping ratio from the given maximum overshoot value:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \quad \rightarrow \quad \zeta = \frac{-\ln(0.1)}{\sqrt{\pi^2 + \ln^2(0.1)}} \quad \rightarrow \quad \zeta = 0.591$$

To achieve the $\%O.S. \leq 10\%$ the damping ratio must be $\zeta \geq 0.591$. Therefore,

$$2\zeta\omega_n = 5 \quad \rightarrow \quad \omega_n = \frac{5}{2\zeta} = \frac{5}{2(0.591)} = 4.23$$

$$\omega_n^2 = K_p \quad \rightarrow \quad \boldsymbol{K_p = 17.89}$$

To achieve the maximum overshoot less than $10\%$ the proportional gain must be selected less than 17.89.

$$\boldsymbol{\%O.S. \leq 10\% \quad \rightarrow \quad K_p \leq 17.89}$$

Since the open-loop system has an integrator, it is **Type 1** system. Therefore, the steady-state error of step response is **zero.**

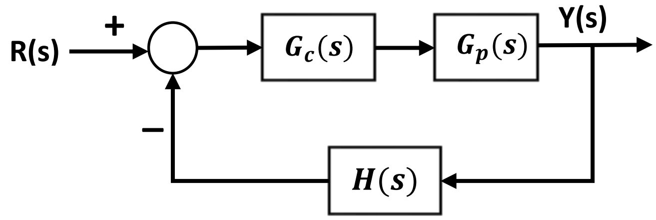
$$\boldsymbol{e_{ss}} = 0$$

**4) Consider the unity feedback control system with the following components.**

$$G_p(s) = \frac{1}{s+2}$$

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

$$H(s) = 1$$




**Note that the steady-state error of closed-loop system for a step input is zero. Select $T_i$ and $K_p$ so that percentage of the overshoot to a step is $O.S. \leq 5\%$ and the settling time (with 2% criterion) is $t_s \leq 1\ sec$.**

From the design specifications, we determine the desired damping ratio and the natural frequency:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \quad \rightarrow \quad \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \quad \rightarrow \quad \zeta = 0.691$$

$$t_s = \frac{4}{\zeta\omega_n} \quad \rightarrow \quad 1 = \frac{4}{(0.691)\omega_n} \quad \rightarrow \quad \omega_n = 5.79\ rad/sec$$

The closed-loop transfer function and the characteristic equation is:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{K_p\left(1 + \frac{1}{T_i s}\right)\left(\frac{1}{s+2}\right)}{1 + K_p\left(1 + \frac{1}{T_i s}\right)\left(\frac{1}{s+2}\right)(1)} = \frac{K_p(T_i s + 1)}{T_i s^2 + (2 + K_p)T_i s + K_p} =$$

The characteristic equation is $\quad \rightarrow \quad s^2 + (2 + K_p)s + \frac{K_p}{T_i} = 0$

The desired characteristic equation is obtained based on the desired $\zeta$ and $\omega_n$:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 8s + 33.6 = 0$$

Matching the characteristic equation of the closed-loop system with the desired one:

$$K_p + 2 = 8 \quad \rightarrow \quad K_p = 6$$

$$\frac{K_p}{T_i} = 33.6 \quad \rightarrow \quad T_i = \frac{K_p}{33.6} = \frac{6}{33.6} = 0.179$$

The designed PI controller is:

$$G_c(s) = 6\left(1 + \frac{1}{0.179s}\right)$$

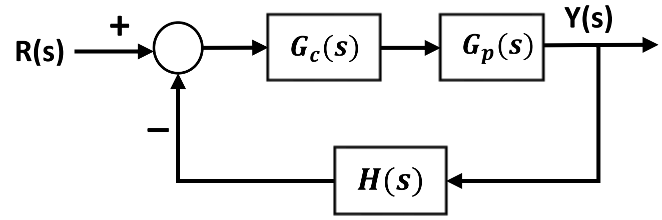
Note that we must check the step response and adjust the integral time-constant if required.

**5) Consider the feedback control system with the following components.**

$$G_p(s) = \frac{1}{s(s+2)}$$

$$G_c(s) = K_p$$

$$H(s) = 1 + 0.2s$$




**a) Determine the proportional gain value $K_p$ such that the following performance specifications can be achieved: The damping ration of the poles $\zeta \geq 0.707$ and the settling-time $t_s \leq 2\ sec$.**

The closed-loop transfer function and the characteristic equation is:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{K_p\left(\frac{1}{s(s+2)}\right)}{1 + K_p\left(\frac{1}{s(s+2)}\right)(1+0.2s)} = \frac{K_p}{s^2 + (2+0.2K_p)s + K_p}$$

The characteristic equation is $\rightarrow \quad s^2 + (2+0.2K_p)s + K_p = 0$

Comparing the characteristic equation with the second-order standard system:

$$2\zeta\omega_n = 2 + 0.2K_p \quad and \quad \omega_n = \sqrt{K_p}$$

To have a damping ratio of $\zeta = 0.707$

$$2(0.707)\sqrt{K_P} = 2 + 0.2K_p \quad \rightarrow \quad 2K_p = \left(2 + 0.2K_p\right)^2 \quad \rightarrow \quad 2K_p = 0.04K_p^2 + +0.8K_p + 4$$

Simplify and solve the equation for $K_p$:

$$K_p^2 - 30K_p + 100 = 0 \quad \rightarrow \quad K_p = 3.82 \quad \text{and} \quad K_P = 26.18$$

There are two solutions for this equation, we have to check which solution is acceptable:

For $K_p = 3.82$

$$\omega_n = \sqrt{3.82} = 1.95$$

$$2\zeta\omega_n = 2 + 0.2(3.82) = 2.76 \quad \rightarrow \quad \zeta = \frac{2.76}{2(1.95)} = 0.707$$

$$\%O.S. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 100e^{-\frac{\pi(0.707)}{\sqrt{1-(0.707)^2}}} = 4.3\%$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.707(1.95)} = 2.9\ sec$$

For $K_p = 26.18$

$$\omega_n = \sqrt{26.18} = 5.12$$

$$2\zeta\omega_n = 2 + 0.2(26.18) = 7.24 \quad \rightarrow \quad \zeta = \frac{7.24}{2(5.12)} = 0.707$$

$$\%O.S. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 100e^{-\frac{\pi(0.707)}{\sqrt{1-(0.707)^2}}} = 4.3\%$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.707(5.12)} = 1.11\ sec$$

Therefore, the gain $\boldsymbol{K_p = 26.18}$ will fulfill the required specifications.

**b) For the value of $K_p$ as found above, estimate the following closed-loop specifications: percent overshoot, rise-time and settling time (2% criteria).**

For $K_p = 26.18$

$$\omega_n = \sqrt{26.18} = 5.12$$

$$2\zeta\omega_n = 2 + 0.2(26.18) = 7.24 \quad \rightarrow \quad \zeta = \frac{7.24}{2(5.12)} = 0.707$$

$$\%O.S. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 100e^{-\frac{\pi(0.707)}{\sqrt{1-(0.707)^2}}} = 4.3\%$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.707(5.12)} = 1.11\ sec$$

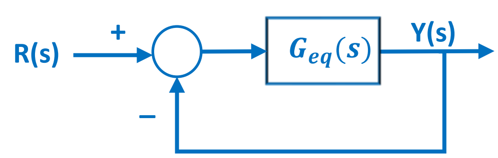
$$t_r = \frac{0.8 + 2.5\zeta}{\omega_n} = \frac{0.8 + 2.5(0.707)}{5.12} = 0.5\ sec$$

**c) Determine the steady-state error for unit-step and unit-ramp inputs.**

Since the closed-loop system has non-unity feedback, first we have to convert it to unity-feedback configuration:

$$\boldsymbol{G_{eq}(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}} \quad \rightarrow \quad G_{eq}(s) = \frac{\frac{K_p}{s(s+2)}}{1 + \frac{K_p(1+0.2s)}{s(s+2)} - \frac{K_p}{s(s+2)}} = \frac{K_p}{s^2 + (2 + 0.2K_p)s}$$

The step-error constant and steady-state error for a unit-step input.




$$k_p = \lim_{s\to 0} G_{eq}(s) = \lim_{s\to 0} \frac{K_p}{s^2 + (2 + 0.2K_p)s} = \infty$$

$$e_{ss} = \frac{R}{1 + k_p} = \frac{1}{\infty} = 0$$

The ramp-error constant and steady-state error for a unit-ramp input.

$$k_v = \lim_{s\to 0} sG_{eq}(s) = \lim_{s\to 0} \frac{K_p s}{s^2 + (2 + 0.2K_p)s} = \lim_{s\to 0} \frac{K_p}{s + (2 + 0.2K_p)} = \frac{K_p}{2 + 0.2K_p} = \frac{26.18}{2 + 0.2(26.18)} = 7.236$$

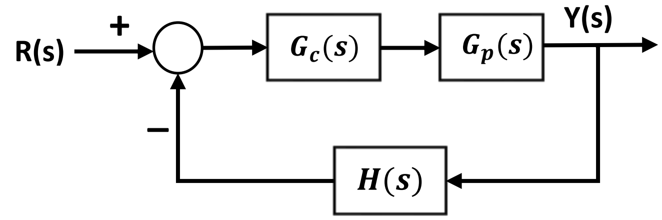
$$e_{ss} = \frac{R}{k_v} = \frac{1}{7.236} = 3.62$$

**6) Consider the feedback control system with the following components.**

$$G_p(s) = \frac{10}{(s+2)(s+4)}$$

$$G_c(s) = K_p$$

$$H(s) = 1 + K_D s$$




**a) Determine the controller gains $K_p$ and $K_D$ such that the closed-loop system will meet the following conditions:**

- **The closed-loop operation is stable.**
- **The steady-state error for unit-step is equal to 5%**
- **The closed-loop unit-step response has 5% overshoot.**

The closed-loop transfer function and the characteristic equation is:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{K_p\left(\frac{10}{(s+2)(s+4)}\right)}{1 + K_p\left(\frac{10}{(s+2)(s+4)}\right)(1 + K_D s)} = \frac{10K_p}{s^2 + (6 + 10K_pK_D)s + 8 + 10K_p}$$

The characteristic equation is $\rightarrow$ $s^2 + (6 + 10K_pK_D)s + 8 + 10K_p = 0$

Stability can be obtained from **Routh-Hurwitz** criteria.

| | | |
|---|---|---|
| $s^2$ | $1$ | $8 + 10K_p$ |
| $s^1$ | $6 + 10K_pK_D$ | $0$ |
| $s^0$ | $8 + 10K_p$ | $0$ |

For the stability of closed-loop system:

$$6 + 10K_pK_D > 0 \qquad K_pK_D > -0.9$$

$$8 + 10K_p > 0 \quad \rightarrow \quad K_p > -0.8$$

The system will be stable for all positive $K_p$ and $K_D$ values.

Given the steady-state error of 5% we can find the required $K_p$.

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\,T(s)R(s) = \lim_{s\to 0} s\,T(s)\left(\frac{1}{s}\right) = T(0) = \frac{10K_p}{8 + 10K_p}$$

$$e_{ss} = r - y_{ss} \quad \rightarrow \quad 0.05 = 1 - \frac{10K_p}{8 + 10K_p} \quad \rightarrow \quad \boldsymbol{K_p = 15.2}$$

Given the percent overshoot the desired damping ratio is found as:

$$\boldsymbol{\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}}} \quad \rightarrow \quad \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \quad \rightarrow \quad \zeta = 0.691$$

Comparing the characteristic equation with the second-order standard form we have:

$$2\zeta\omega_n = 6 + 10K_pK_D \quad \text{and} \quad \omega_n = \sqrt{8 + 10K_p}$$

Therefore,

$$\omega_n = \sqrt{8 + 10K_p} = \sqrt{160} = 12.65 \ rad/sec$$

$$2\zeta\omega_n = 6 + 10K_pK_D \quad \rightarrow \quad 2(0.691)(12.65) = 6 + 10(15.2)K_D \quad \rightarrow \quad \boldsymbol{K_D = 0.0754}$$

The closed-loop transfer function is:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{10K_p}{s^2 + (6 + 10K_pK_D)s + 8 + 10K_p} = \frac{152}{s^2 + 17.46s + 160}$$

**b) With the calculated controller gain values, what will be the step response settling-time (2% criteria)?**

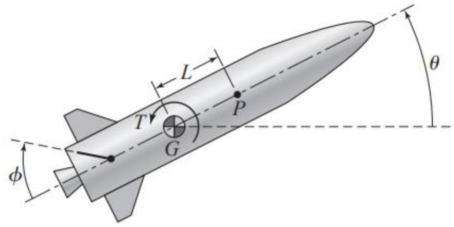
$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.691(12.65)} = 0.46 \ sec$$

**7) Control of the attitude θ of a missile by controlling the fin angle ϕ, as shown below, involves controlling an inherently unstable plant. Consider the specific plant transfer function,**

$$\boldsymbol{G_p(s) = \frac{\theta(s)}{\phi(s)} = \frac{1}{s^2 - 5}}$$

**Determine the PD control gains so that the closed-loop system is stable, the damping ratio is 0.707, and the closed-loop settling-time is 0.4.**

$$\boldsymbol{G_c(s) = K_p + K_D s}$$




**Find the steady-state error of closed-loop system for unit-step input.**

The closed-loop transfer function and the characteristic equation is:

$$\boldsymbol{T(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}} = \frac{(K_p + K_D s)\left(\frac{1}{s^2-5}\right)}{1 + (K_p + K_D s)\left(\frac{1}{s^2-5}\right)} = \frac{K_p + K_D s}{s^2 + K_D s + K_p - 5}$$

The characteristic equation is $\rightarrow \quad s^2 + K_D s + K_p - 5 = 0$

Stability can be obtained from **Routh-Hurwitz** criteria.

| | | |
|---|---|---|
| $s^2$ | $1$ | $K_p - 5$ |
| $s^1$ | $K_D$ | $0$ |
| $s^0$ | $K_p - 5$ | $0$ |

For the stability of closed-loop system:

$K_D > 0$

$K_p - 5 > 0 \quad \rightarrow \quad K_p > 5$

To have a settling-time of $t_s = 0.4\ sec$

$$t_s = \frac{4}{\zeta\omega_n} \quad \rightarrow \quad 0.4 = \frac{4}{\zeta\omega_n} \quad \rightarrow \quad \zeta\omega_n = 10$$

The undamped natural frequency $\omega_n$ is determined based on the desired damping ratio of $\zeta = 0.707$

$$\zeta\omega_n = 10 \quad \rightarrow \quad \omega_n = \frac{10}{0.707} = 14.14\ \ rad/s$$

Comparing the characteristic equation of the closed-loop system with the second-order standard form we have:

$2\zeta\omega_n = K_D \quad \rightarrow \quad K_D = 2(10) = 20$

$\omega_n^2 = K_p - 5 \quad \rightarrow \quad K_p = \ \omega_n^2 + 5 = (14.14)^2 + 5 = 204.94 \approx 205$

The steady-state error is of unit-step input obtained from the overall closed-loop transfer function:

$$\boldsymbol{y_{ss}} = \lim_{t\to\infty} \boldsymbol{y(t)} = \lim_{s\to 0} \boldsymbol{sY(s)} = \lim_{s\to 0} s\,T(s)R(s) = \lim_{s\to 0} s\,T(s)\left(\frac{1}{s}\right) = T(0) = \frac{K_p}{K_p - 5} = \frac{205}{205 - 5} = 1.025$$

$$\boldsymbol{e_{ss}} = \boldsymbol{r} - \boldsymbol{y_{ss}} \quad \rightarrow \quad e_{ss} = 1 - 1.025 = -0.025 \quad \rightarrow \quad e_{ss} = 2.5\%$$

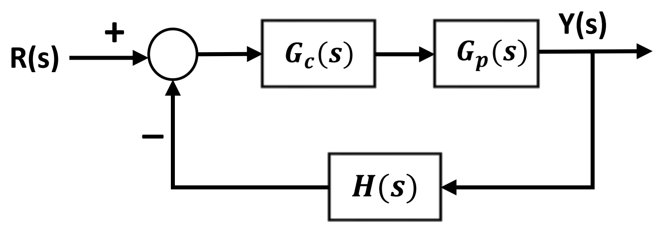
The negative sign represents the output is greater than the reference input.

**8) A control system with PD controller is shown below**

$$\boldsymbol{G_p(s) = \frac{1000}{s(s+10)}}$$

$$\boldsymbol{G_c(s) = K_p + K_D s}$$

$$\boldsymbol{H(s) = 1}$$




**a) Find the values of $K_p$ and $K_D$ so that the ramp-error-constant is $k_v = 1000$ and the damping ratio is 0.5.**

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{1000(K_p + K_D s)}{s(s+10)} = 100K_p \quad \rightarrow \quad 1000 = 100K_p \quad \rightarrow \quad K_p = 10$$

The closed-loop transfer function and the characteristic equation is:

$$\boldsymbol{T(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}} = \frac{(K_p + K_D s)\left(\frac{1000}{s(s+10)}\right)}{1 + (K_p + K_D s)\left(\frac{1000}{s(s+10)}\right)} = \frac{1000(K_p + K_D s)}{s^2 + (10 + 1000K_D)s + 1000K_p}$$

Comparing the characteristic equation of the closed-loop system with the second-order standard form we have:

$$\omega_n^2 = 1000K_p \quad \rightarrow \quad \omega_n^2 = 10000 \quad \rightarrow \quad \omega_n = 100$$

$$2\zeta\omega_n = 10 + 1000K_D \quad \rightarrow \quad K_D = \frac{2\zeta\omega_n - 10}{1000} = \frac{2(0.5)(100) - 10}{1000} = 0.09$$

**b) Find the values of $K_p$ and $K_D$ so that the ramp-error-constant is $k_v = 1000$ and the damping ratio is 0.707.**

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{1000(K_p + K_D s)}{s(s+10)} = 100K_p \quad\rightarrow\quad 1000 = 100K_p \quad\rightarrow\quad K_p = 10$$

The closed-loop transfer function and the characteristic equation is:

$$\boldsymbol{T(s)} = \frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{G_c(s)G(s)}}{\boldsymbol{1 + G_c(s)G(s)H(s)}} = \frac{(K_p + K_D s)\left(\frac{1000}{s(s+10)}\right)}{1 + (K_p + K_D s)\left(\frac{1000}{s(s+10)}\right)} = \frac{1000(K_p + K_D s)}{s^2 + (10 + 1000K_D)s + 1000K_p}$$

Comparing the characteristic equation of the closed-loop system with the second-order standard form we have:

$$\omega_n^2 = 1000K_p \quad\rightarrow\quad \omega_n^2 = 10000 \quad\rightarrow\quad \omega_n = 100$$

$$2\zeta\omega_n = 10 + 1000K_D \quad\rightarrow\quad K_D = \frac{2\zeta\omega_n - 10}{1000} = \frac{2(0.707)(100) - 10}{1000} = 0.1314$$

**c) Find the values of $K_p$ and $K_D$ so that the ramp-error-constant is $k_v = 1000$ and the damping ratio is 1.**

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{1000(K_p + K_D s)}{s(s+10)} = 100K_p \quad\rightarrow\quad 1000 = 100K_p \quad\rightarrow\quad K_p = 10$$

The closed-loop transfer function and the characteristic equation is:

$$\boldsymbol{T(s)} = \frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{G_c(s)G(s)}}{\boldsymbol{1 + G_c(s)G(s)H(s)}} = \frac{(K_p + K_D s)\left(\frac{1000}{s(s+10)}\right)}{1 + (K_p + K_D s)\left(\frac{1000}{s(s+10)}\right)} = \frac{1000(K_p + K_D s)}{s^2 + (10 + 1000K_D)s + 1000K_p}$$

Comparing the characteristic equation of the closed-loop system with the second-order standard form we have:

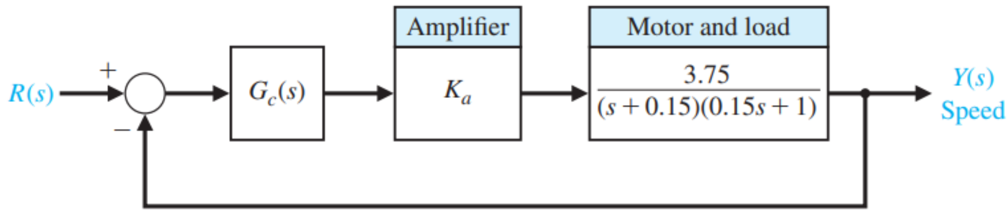
$$\omega_n^2 = 1000K_p \quad\rightarrow\quad \omega_n^2 = 10000 \quad\rightarrow\quad \omega_n = 100$$

$$2\zeta\omega_n = 10 + 1000K_D \quad\rightarrow\quad K_D = \frac{2\zeta\omega_n - 10}{1000} = \frac{2(1)(100) - 10}{1000} = 0.19$$

**9) A stabilized precision rate table uses a precision tachometer and a DC direct-drive torque motor, as shown below. We want to maintain a high steady-state accuracy for the speed control. To obtain a zero steady-state error for a step command design, select a proportional plus integral compensator.**

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

**Select the appropriate gain constants so that the system has a percentage overshoot of $10\%$, and a settling time (with a $2\%$ criterion) of $t_s \leq 1.5\ sec$. Assume that $K_a = 1$.**




First, design the proportional controller gain $K_p$ based on the desired percentage of overshoot 10%.

From the design specifications, we determine the desired damping ratio and the natural frequency:

$$\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}} \quad \rightarrow \quad \zeta = \frac{-\ln(0.1)}{\sqrt{\pi^2 + \ln^2(0.1)}} \quad \rightarrow \quad \zeta = 0.591$$

The closed-loop transfer function with only proportional controller is:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{\left(\frac{3.75K_p}{(s+0.15)(0.15s+1)}\right)}{1 + \left(\frac{3.75K_p}{(s+0.15)(0.15s+1)}\right)(1)} = \frac{3.75K_p}{0.15s^2 + 1.0225s + 0.15 + 3.75K_p}$$

$$\frac{Y(s)}{R(s)} = \frac{25K_p}{s^2 + 6.82s + 1 + 25K_p}$$

The characteristic equation is $\rightarrow$ $s^2 + 6.82s + 1 + 25K_p = 0$

Comparing the characteristic equation of the closed-loop system with the second-order standard form we have:

$$2\zeta\omega_n = 6.82 \quad \rightarrow \quad \omega_n = \frac{6.82}{2(0.591)} = 5.77\ rad/s$$

$$\omega_n^2 = 1 + 25K_p \quad \rightarrow \quad K_p = \frac{(5.77)^2 - 1}{25} \quad \rightarrow \quad K_p = 1.29$$

Next, find the appropriate range of the integral time-constant $T_i$ based on the dominant closed-loop poles.

First, find the dominant poles of the closed-loop transfer function for $K_p = 1.29$.

$$s^2 + 6.82s + 33.25 = 0 \quad \rightarrow \quad s = -3.41 \pm j4.65$$

The integral time constant $T_i$ can be selected by the following stability consideration:

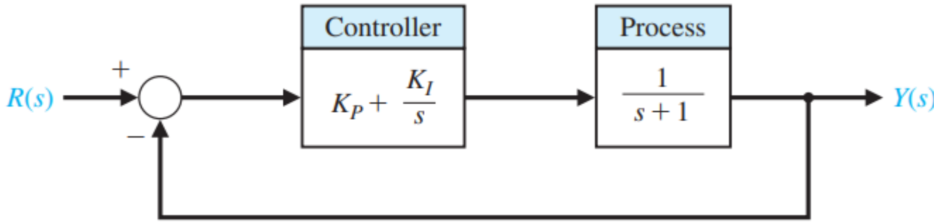
$$\frac{2}{|Re\{-3.41 \pm j4.65\}|} \leq T_i \quad \rightarrow \quad \frac{2}{3.41} \leq T_i \quad \rightarrow \quad \mathbf{0.59 \leq \boldsymbol{T_i}}$$

For example, the following selection can be an appropriate PI controller.

$$K_p = 1.29, \quad T_i = 2 \qquad \rightarrow \qquad G_c(s) = 1.29\left(1 + \frac{1}{2s}\right)$$

Note that we must check the step response and fine tune the integral time-constant if required.

**10) A control system with a controller is shown below. Select $K_p$ and $K_I$ so that the percentage overshoot to a step input is $5\%$, and the ramp-error constant $k_v$ is equal to $5$.**




The ramp-error constant is:

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{K_p + \frac{K_I}{s}}{s+1} = \lim_{s\to 0}\frac{K_p s + K_I}{s+1} = K_I \quad \to \quad 5 = K_I$$

Given the percent overshoot the desired damping ratio is found as:

$$\boldsymbol{\zeta = \frac{-\ln(O.S.)}{\sqrt{\pi^2 + \ln^2(O.S.)}}} \quad \to \quad \zeta = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \quad \to \quad \zeta = 0.691$$

The closed-loop transfer function and the characteristic equation is:

$$\boldsymbol{T(s) = \frac{Y(s)}{R(s)}} = \frac{\left(K_p + \frac{K_I}{s}\right)\left(\frac{1}{s+1}\right)}{1 + \left(K_p + \frac{K_I}{s}\right)\left(\frac{1}{s+1}\right)} = \frac{K_p s + K_I}{s^2 + \left(1 + K_p\right)s + K_I}$$

Comparing the characteristic equation of the closed-loop system with the second-order standard form we have:

$$\omega_n^2 = K_I \quad \to \quad \omega_n^2 = 5 \quad \to \quad \omega_n = \sqrt{5}$$

$$2\zeta\omega_n = 1 + K_p \quad \to \quad K_p = 2(0.691)\left(\sqrt{5}\right) - 1 = 2.09$$

The designed PI controller is:

$$\boldsymbol{G_c(s) = 2.09 + \frac{5}{s}}$$

Note that we must check the step response and adjust the integral time-constant if required.