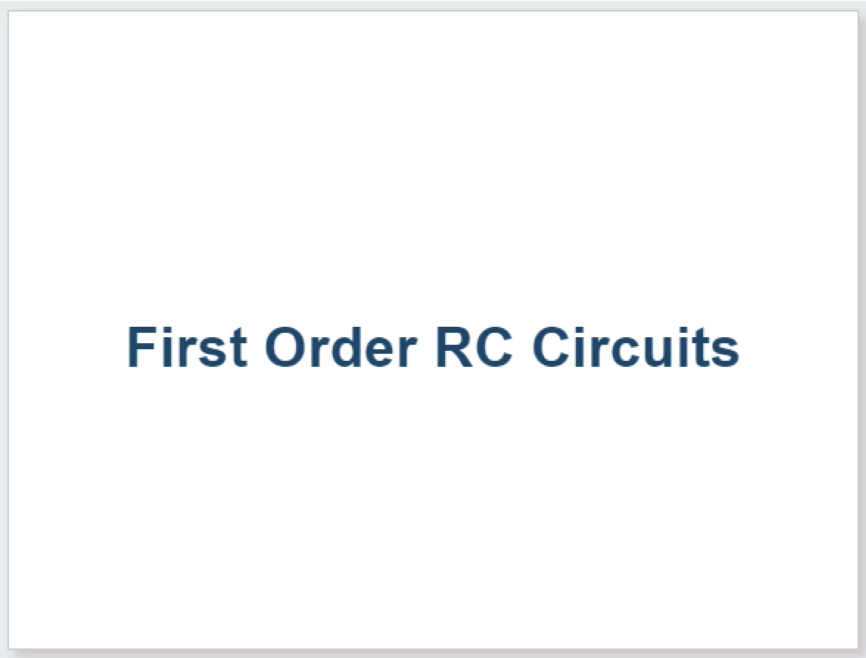
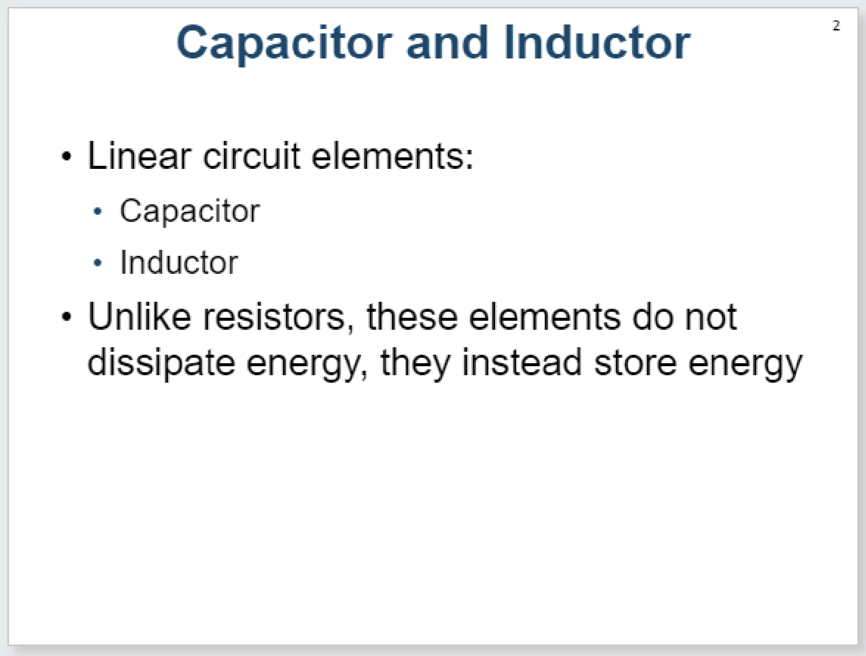
Lecture 4

November 9, 2023 8:30 PM

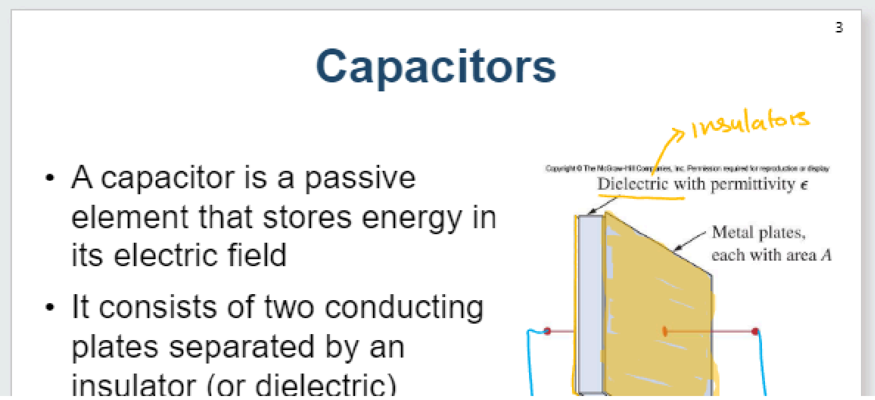


First Order RC Circuits



Capacitor and Inductor

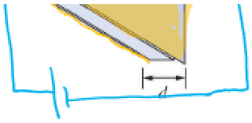
- Linear circuit elements:
  - Capacitor
  - Inductor
- Unlike resistors, these elements do not dissipate energy, they instead store energy



Capacitors

- A capacitor is a passive element that stores energy in its electric field
- It consists of two conducting plates separated by an insulator (or dielectric)

- The plates are typically aluminum foil
- The dielectric is often air, ceramic, paper, plastic, or mica

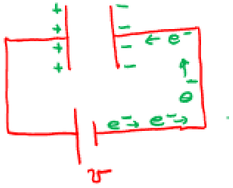


Capacitors

- When a voltage source *v* is connected to the capacitor, the source deposits a positive charge *q* on one plate and a negative charge –*q* on the other
- The charges will be equal in magnitude
- The amount of charge is proportional to the voltage: $q = 1C$

$q \propto v$

$$q = Cv$$



- Where C is the capacitance
- The unit of capacitance is the Farad (F)
- One Farad is 1 Coulomb/Volt

$\frac{q}{v} = c \Rightarrow 1 \frac{\text{Coulomb}}{\text{Volt}} = 1\ \text{Farad}$

Capacitors – Stored Charge

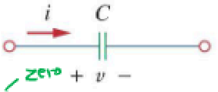
$dt \times \frac{1}{c} \times i = \cancel{c}\frac{dv}{\cancel{dt}} \times \frac{1}{\cancel{c}} \times d\cancel{t} \Rightarrow \int_{v(t_0)}^{v(t)} dv = \int_{t_0}^{t} \frac{1}{c}\, i\, dt$

- Similarly, the voltage current relationship is:

$v(t) - v(t_0) = \frac{1}{c}\int_{t_0}^{t} i\,dt$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt + v(t_0)$$

$v(t) = \frac{1}{c}\int_{t_0}^{t} i\,dt + v(t_0) \begin{cases} \text{zero} \\ \text{non-zero} \end{cases}$



- This shows the capacitor has a memory
- The instantaneous power delivered to the capacitor is:

$$p = vi = Cv\frac{dv}{dt}$$

$= v\left(c\frac{dv}{dt}\right) = cv\frac{dv}{dt}$

- The energy stored in a capacitor is:

$$\boxed{w = \frac{1}{2}Cv^2}$$

Joules

F

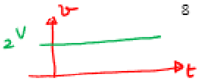
V

$dw = p\,dt = cv\frac{dv}{\cancel{dt}}\cancel{dt}$

$dw = cv\,dv$

$W = \int cv\,dv = c\frac{v^2}{2} = \frac{1}{2}cv^2$

Properties of Capacitors

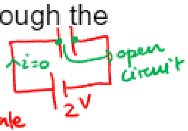


- Ideal capacitors all have these characteristics:
- When the voltage is not changing, the current through the capacitor is zero

$i = c\frac{0}{2} = 0\ A$

$$i = C\frac{dv}{dt}$$

@ steady state

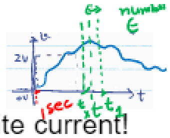


- ② This means that with DC applied to the terminals no current will flow A Fully charged capacitor → open circuit

very small

- ③ The voltage on the capacitor's plates can't change instantaneously

$i = C\frac{(2-0)}{(1-1)} = C\frac{2}{0} = \infty$

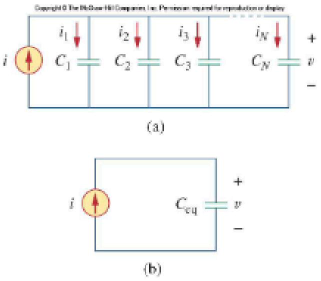


- An abrupt change in voltage would require an infinite current!

$v(t_x) = v(t) = v(t_1)$

- This means if the voltage on the cap does not equal the applied voltage, charge will flow and the voltage will finally reach the applied voltage
- ① An ideal Capacitor do not dissipate energy, stored energy can be retrieved later

Parallel Capacitors



- Starting with *N* parallel capacitors, one can note that the voltages on all the caps are the same
- Applying KCL:

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

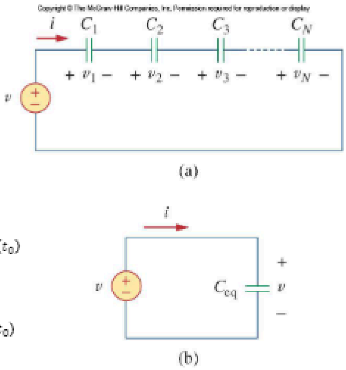
$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt}$$

$$= \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{eq}\frac{dv}{dt}$$

$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N$$

Parallel capacitors combine as the sum of all capacitance

Series Capacitors



- Each capacitor shares the same current.
- Applying KVL to the loop:

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

- Now apply the voltage current relationship

$$v = \frac{1}{C_1}\int_{t_0}^{t} i(t)dt + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(t)dt + v_2(t_0) + \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(t)dt + v_N(t_0)$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(t)dt + v_1(t_0) + v_2(t_0) + v_3(t_0) + \cdots + v_N(t_0)$$

$$= \frac{1}{C_{eq}}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$$

$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}}$

Series combination of capacitors resembles the parallel combination of resistors

Series and Parallel Capacitors

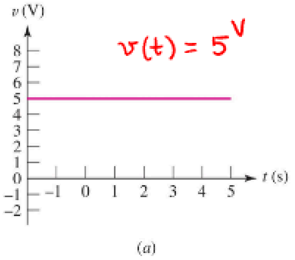
- Another way to think about the combinations of capacitors is this:

this:

- Combining capacitors in parallel is equivalent to increasing the surface area of the capacitors
- This would lead to an increased overall capacitance (as is observed)
- A series combination can be seen as increasing the total plate separation
- This would result in a decrease in capacitance (as is observed)

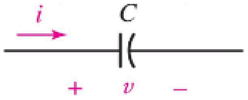
Example: Capacitor *i*–*v* Curves

Find *i*(*t*) for the voltages shown, if *C* = 2 F.

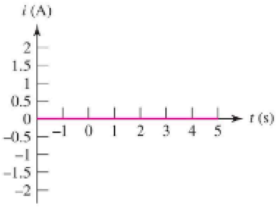


$v(t) = 5^V$

(a)



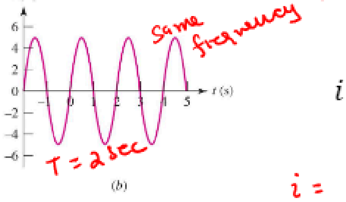
$$i = C\frac{dv}{dt} = 2\frac{d}{dt}(5) = 2\times 0 = 0^A$$



(a)

Example: Capacitor *i*–*v* Curves

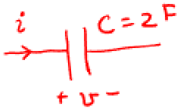
$v(t) = 5\sin(\omega t)$



same frequency

T = 2 sec

(b)



$$i = C\frac{dv}{dt}$$

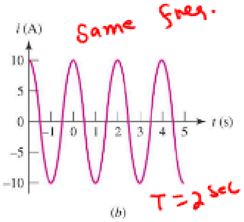
$$i = 2\frac{d}{dt}\left(5\sin\omega t\right)$$

$$= 2\times 5\frac{d}{dt}(\sin\omega t)$$

assume : ω = 1 rad/sec

$$i = 10\cos(\omega t)^A$$

Same freq.



(b)

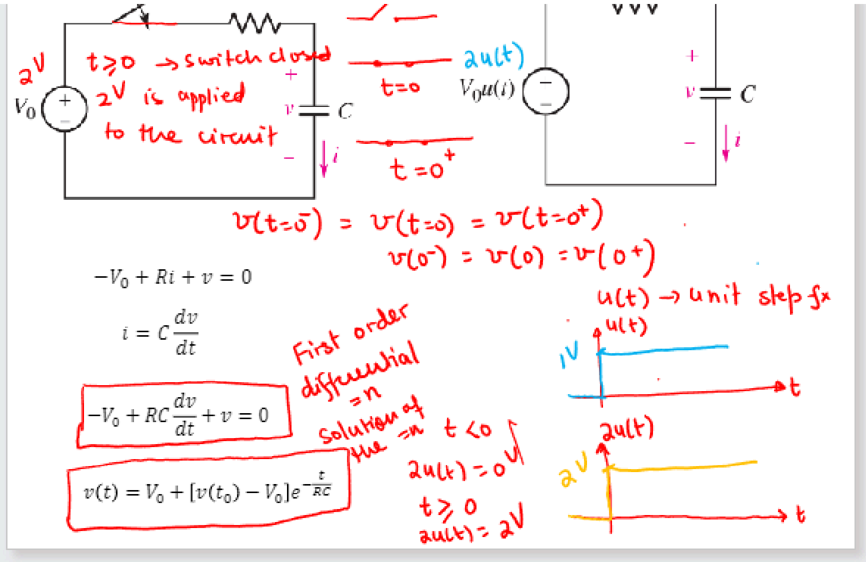
T = 2 sec

First order RC Circuit

t < 0 → switch open

0 voltage applied to the circuit

t = 0⁻



t ≥ 0 → switch closed 2V is applied to the circuit

t=0

$t=0^+$

$v(t=0^-) = v(t=0) = v(t=0^+)$

$v(0^-) = v(0) = v(0^+)$

$$-V_0 + Ri + v = 0$$

$$i = C\frac{dv}{dt}$$

$$\boxed{-V_0 + RC\frac{dv}{dt} + v = 0}$$

First order differential =n

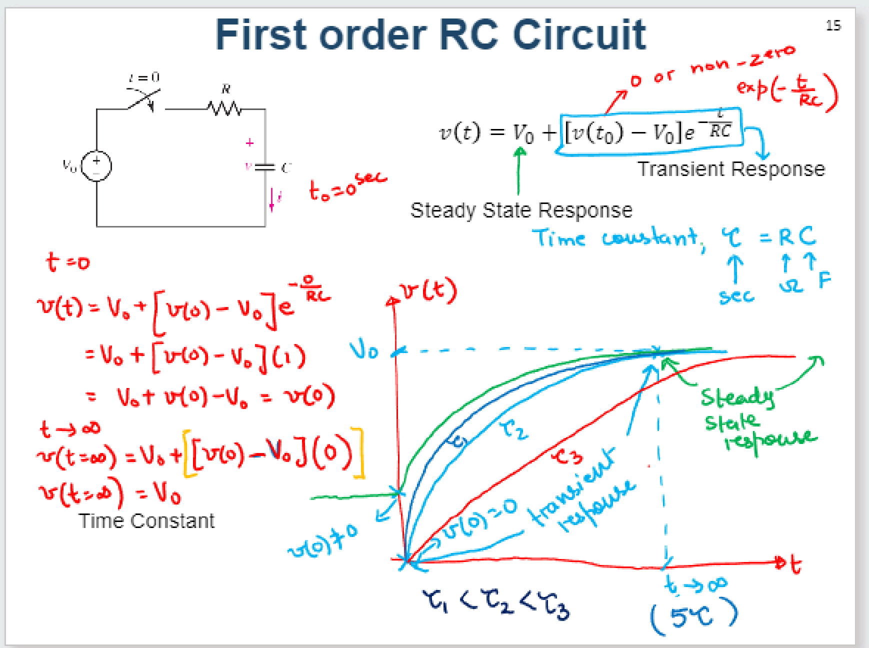
Solution of the =n

$$\boxed{v(t) = V_0 + [v(t_0) - V_0]e^{-\frac{t}{RC}}}$$

t < 0 $2u(t) = 0V$

t ≥ 0 $2u(t) = 2V$

u(t) → unit step fx



First order RC Circuit

$t_0 = 0$ sec

$$v(t) = V_0 + \boxed{[v(t_0) - V_0]e^{-\frac{t}{RC}}}$$

0 or non-zero

$\exp(-\frac{t}{RC})$

Transient Response

Steady State Response

Time constant, $\tau = RC$

sec Ω F

t = 0

$v(t) = V_0 + [v(0) - V_0]e^{-\frac{0}{RC}}$

$= V_0 + [v(0) - V_0](1)$

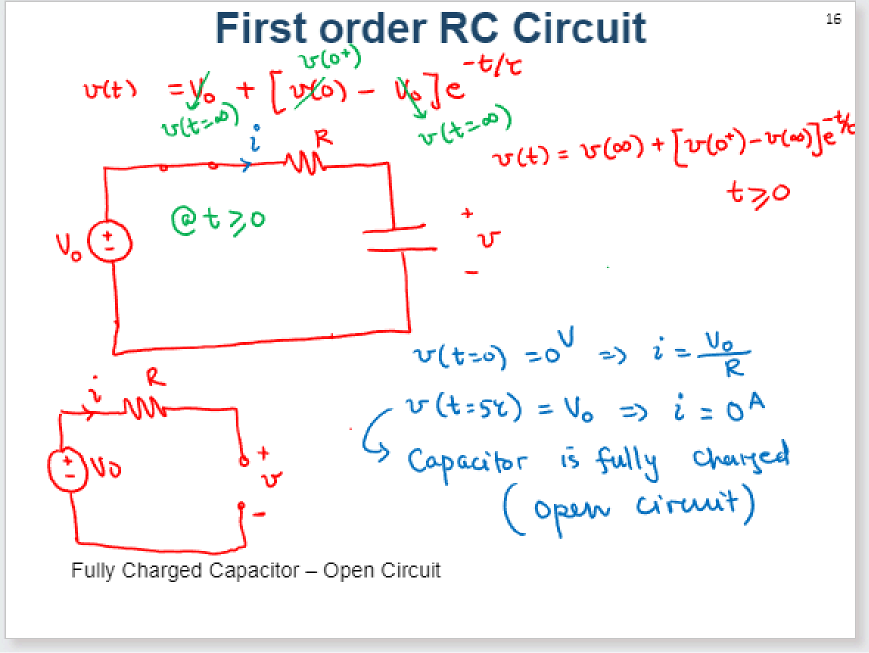
$= V_0 + v(0) - V_0 = v(0)$

$t \to \infty$

$v(t=\infty) = V_0 + [[v(0) - V_0](0)]$

$v(t=\infty) = V_0$

Time Constant



First order RC Circuit

$v(t) = \overset{v(t=\infty)}{\cancel{V_0}} + [\overset{v(0^+)}{\cancel{v(0)}} - \underset{v(t=\infty)}{\cancel{V_0}}]e^{-t/\tau}$

$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$

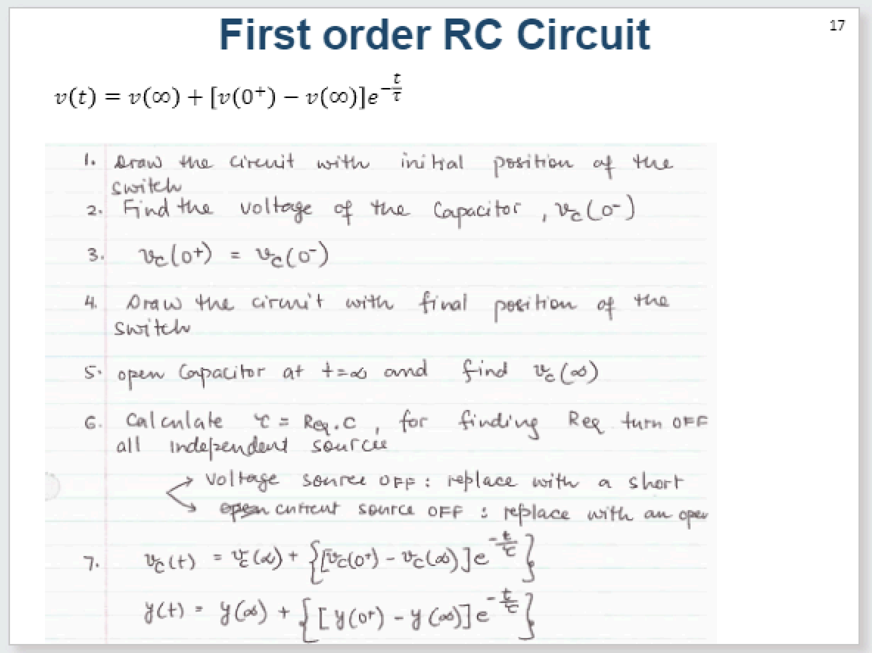
$t \geq 0$

$v(t=0) = 0^V \Rightarrow i = \frac{V_0}{R}$

$v(t=5\tau) = V_0 \Rightarrow i = 0^A$

Capacitor is fully charged (open circuit)

Fully Charged Capacitor – Open Circuit

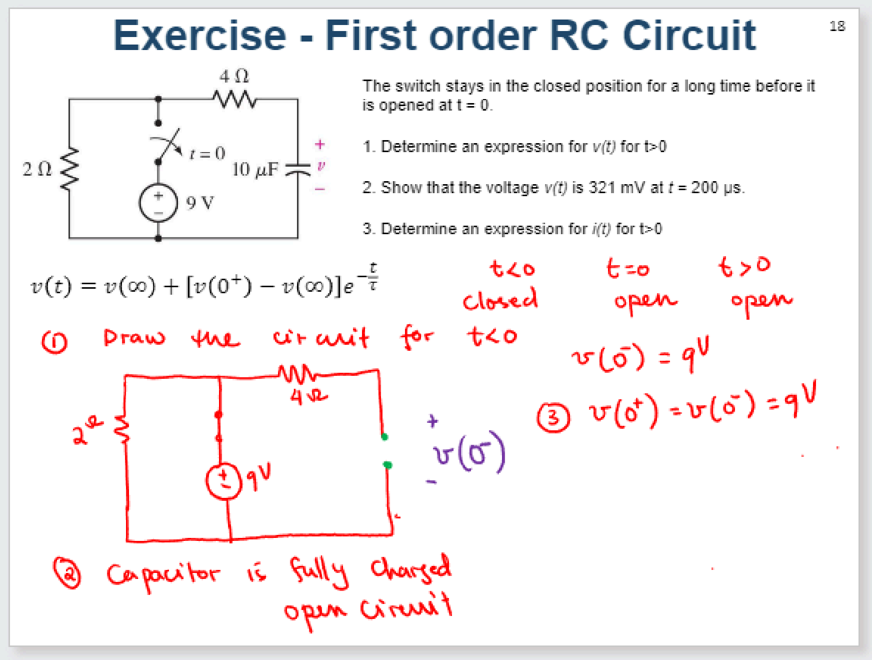


First order RC Circuit

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$

1. Draw the circuit with initial position of the switch
2. Find the voltage of the Capacitor , $v_c(0^-)$
3. $v_c(0^+) = v_c(0^-)$
4. Draw the circuit with final position of the switch
5. open Capacitor at $t=\infty$ and find $v_c(\infty)$
6. Calculate $\tau = R_{eq}.C$, for finding Req turn OFF all independent source
   - voltage source OFF : replace with a short
   - ~~open~~ current source OFF : replace with an open
7. $v_c(t) = v_c(\infty) + \{[v_c(0^+) - v_c(\infty)]e^{-\frac{t}{\tau}}\}$

   $y(t) = y(\infty) + \{[y(0^+) - y(\infty)]e^{-\frac{t}{\tau}}\}$



Exercise - First order RC Circuit

The switch stays in the closed position for a long time before it is opened at t = 0.

1. Determine an expression for *v(t)* for t>0
2. Show that the voltage *v(t)* is 321 mV at *t* = 200 μs.
3. Determine an expression for *i(t)* for t>0

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$

$t<0$ closed

$t=0$ open

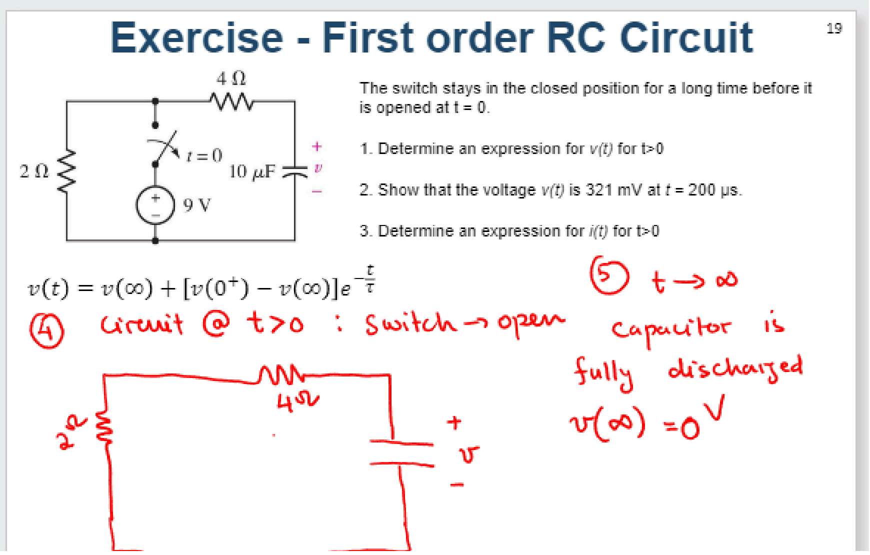
$t>0$ open

① Draw the circuit for $t<0$

$v(0^-) = 9V$

③ $v(0^+) = v(0^-) = 9V$

② Capacitor is fully charged open circuit



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$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$

④ Circuit @ $t>0$: switch → open

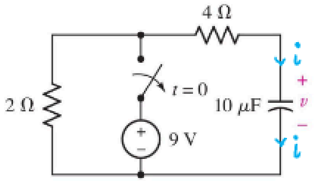
⑤ $t \to \infty$

capacitor is fully discharged

$v(\infty) = 0V$

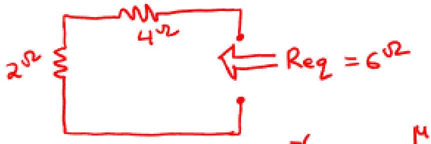
capacitor is discharging

Exercise - First order RC Circuit



⑥ circuit t>0

$\tau = RC$



$\tau = 6 \times 10 \times 10^{-6} = 60^{\mu sec}$

$\tau = 60 \times 10^{-6}$ sec

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$

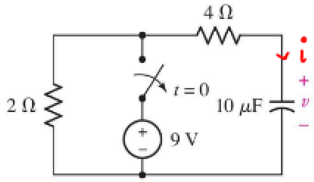
⑦ $v(t) = 0 + [9 - 0]e^{\frac{-t}{60\times10^{-6}}}$

$v(t) = 9e^{\frac{-t}{60\times10^{-6}}}$ $t \geq 0$ Volts

part 2 $v(200^{\mu sec}) = 9e^{\left(-\frac{200}{60}\right)} = 0.321^{V} = 321^{mV}$

$v(t) = 9e^{-t/60\times10^{-6}}$ V

Exercise - First order RC Circuit



$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$

part 3

$i = C\frac{dv}{dt}$

$= 10\times10^{-6}\frac{d}{dt}\left(9e^{\frac{-t}{(60\times10^{-6})}}\right)$

$= \cancel{10}\times\cancel{10^{-6}}\times9\times\left(-\frac{1}{\cancel{6}0\times\cancel{10^{-6}}}\right)e^{\frac{-t}{(60\times10^{-6})}}$

$i(t) = -1.5e^{\frac{-t}{(60\times10^{-6})}}$ A

$v(t) = 9e^{-t/60\times10^{-6}}$ V

Exercise - First order RC Circuit

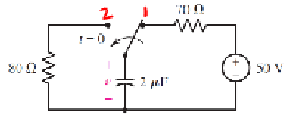


FIGURE 8.18

The switch stays in position '1' for a long time before it is moved to position '2' at t = 0.

1. Determine an expression for *v(t)* for t>0

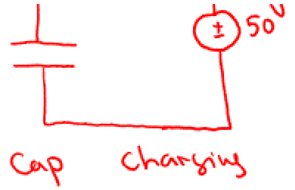
2. Calculate the value of *v(t)* at *t* = 0 sec and at *t* = *160 μs*.

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$

t<0	t=0	t>0
position 1	position 2	position 2

① circuit @ t<0

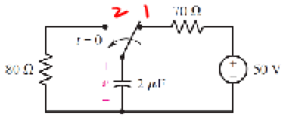
②



$v(0^-) = 50^V$

③ $v(0^+) = v(0^-) = 50^V$

Exercise - First order RC Circuit



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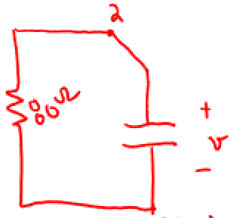
$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$

④ Circuit for $t>0$



Cap is discharging

⑤ Fully discharged

$v(\infty) = 0$

⑥ $\tau = 80 \times 2 \times 10^{-6} = 160^{\mu sec}$

$v(t) = 50\, e^{-\frac{t}{160\times10^{-6}}}$ Volts $t \geq 0$

part 2

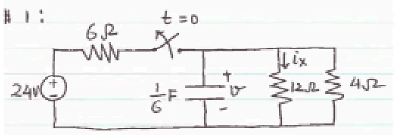
$v(t=0) = 50^V$

$v(t = 160^{\mu sec}) = 50\, e^{\frac{-160}{160}} = 50e^{-1}$

$= 18.39^V$

50 V; 18.39 V

Exercise - First order RC Circuit



The switch stays closed for a long time before it is opened at t = 0.

1. Determine an expression for *v(t)* for t>0
2. Determine an expression for $i_x(t)$ for t>0
3. Calculate the energy stored in the capacitor at *t = 0* *(5.33 J)*

$\frac{1}{2} C v^2$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-\frac{t}{\tau}}$$