MTH 320: HW 8

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Due: April 1, 2019

Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. **(Abbott 4.4.3)** Show that $f(x) = \dfrac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$ but not on $(0, 1]$.

2. **(Abbott 4.4.4)** Decide whether each of the following statements is true or false, justifying each conclusion.

   (a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$.

   (b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

   (c) If f is defined on $\mathbb{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbb{R}$.

3. **(Abbott 4.4.9)** A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \qquad \text{for all} \qquad x \neq y \text{ in } A$$

   Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

   (a) Show that if $f : A \to \mathbb{R}$ is Lipschitz then it is uniformly continuous on A.

   (b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz? (Hint: Consider $f(x) = \sqrt{x}$ on $[0, \infty)$.)

4. **(Abbott 4.4.12)** Determine which of the following statements are true or false. If true then provide a short argument why it is and if false then give a counterexample.

   Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Let $B \subset \mathbb{R}$ and recall that $f^{-1}(B) = \{x \in \mathbb{R} : f(x) \in B\}$.

   (a) $f^{-1}(B)$ is a finite set whenever B is a finite set.

   (b) $f^{-1}(B)$ is compact whenever B is compact.

   (c) $f^{-1}(B)$ is bounded whenever B is bounded.

   (d) $f^{-1}(B)$ is closed whenever B is closed. (Hint: Use that $f^{-1}(B^c) = [f^{-1}(B)]^c$.)

5. **(Abbott 4.5.2)** Provide an example of each of the following or explain why the request is impossible: $f : A \to \mathbb{R}$ is a continuous function on A where

   (a) A is an open set and $f(A)$ is a closed set.

   (b) A is a closed interval and $f(A)$ is an open interval.

   (c) A is an open interval and $f(A)$ is an unbounded closed set different from $\mathbb{R}$.

   (d) $A = \mathbb{R}$ and $f(A) = \mathbb{Q}$. (Hint: Use the Intermediate Value Theorem.)

6. **(Abbott 4.5.3)** Let $f : [a, b] \to \mathbb{R}$. Show that if f satisfies the intermediate value theorem and is increasing on $[a, b]$ then f is continuous on $[a, b]$.

7. **(Abbott 4.5.4)** Let I be an interval and $f : I \to \mathbb{R}$ be continuous on I. Define the set F to be the points at which the function fails to be one-to-one, that is,

$$F = \{x \in I : f(x) = f(y) \text{ for some } y \in I \setminus \{x\}\}.$$

Show that F is either empty or uncountable.

8. **(Abbott 4.5.7)** Let f be a continuous function on $[0, 1]$ and suppose $f([0, 1]) \subset [0, 1]$. Prove that f must have a fixed point, that is, there exists $c \in [0, 1]$ such that $f(c) = c$.

9. **(Abbott 4.6.5)** Let $= (a, b)$ be an open bounded interval and $f : I \to \mathbb{R}$ be a monotone increasing function. Show that if f has a discontinuity at $c \in I$ then c is a jump discontinuity.