MTH 320: HW 7

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Due: March 20, 2019

Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. **(Abbott 4.2.3)** Let $t : \mathbb{R} \to \mathbb{R}$ be the Thomae function defined by

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   (a) Construct three different sequence (x_n), (y_n) and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

   (b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$, and $\lim t(z_n)$.

   (c) Make an educated conjecture for $\lim_{x\to 1} t(x)$, and use Definition 4.2.1B to verify the claim. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbb{R} : t(x)\epsilon\}$. Argue that all the points in this set are isolated.)

2. **(Abbott 4.2.5)** Use Definition 4.2.1 to supply a proper proof for: $\lim_{x\to 2} x^2 + x - 1 = 5$.

3. **(Abbott 4.2.7)** Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$.

4. **(Abbott 4.3.1)** Let $g(x) = x^{1/3}$.

   (a) Prove that g is continuous at $c = 0$.

   (b) Prove that g is continuous at any point $c \in \mathbb{R}$. (The identity $a^3b^3 = (ab)(a^2 + ab + b^2)$ will be helpful.)

5. **(Abbott 4.3.3)** Supply a proof for Theorem 4.3.9 using the $\epsilon - \delta$ characterization of continuity.

6. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

7. **(Abbott 4.3.5)** Show using Definition 4.3.1 that if c is an isolated point of A, that is c is not a limit point of A, then $f : A \to \mathbb{R}$ is continuous at c.

8. **(Abbott 4.3.9)** Assume that $h : \mathbb{R} \to \mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.