Analyzing the Black-Litterman Model and its Applications

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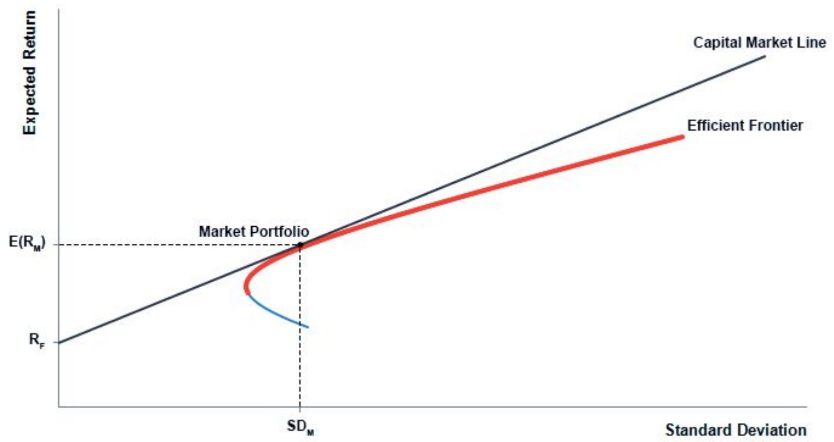
Presentation Agenda

- Overview of Traditional Mean-Variance Portfolio Method
- Overview of Black-Litterman Portfolio Method
- Data Collection
- Mean-Variance with Black-Litterman Scenarios
- Goldman Sachs Replication
- Goldman Sachs Black-Litterman Model Replication with more Complex Scenarios
- Enhancing the Black-Litterman Model with Investor Confidence Calibration

Traditional Mean-Variance Approach

- The mean-variance approach is a way for portfolio analysis invented by Markowitz in 1952
- Goal: Maximizing expected return and Minimizing risk(minimize standard deviation)
- Investors choose from efficient portfolios which are consistent with their risk tolerance

Figure 1: Efficient Frontier with and without a Risk-free asset



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Mean-Variance Approach with and without a Risk-free Asset

Without a Risk-free Asset

- $\min_{\boldsymbol{w}} \frac{1}{2} \boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w}$ (2)
  $s.t\ \mathbf{w}'\boldsymbol{\mu} = p\ and\ \mathbf{w}'\mathbf{1} = 1$
- n risky assets with the expected return vector R
- $R \sim MVN_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (1)
- p: The value that depends on the risk aversion level of investors
- w = (w1,...,wn)' as a vector portfolio weights.

With a Risk-free Asset

- $\min_{\boldsymbol{w}} \frac{1}{2} \boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w}$ (3)
  $s.t\ \ (1 - \sum_{i=1}^{n} w_i) r_f + \boldsymbol{w}'\boldsymbol{\mu} = p$

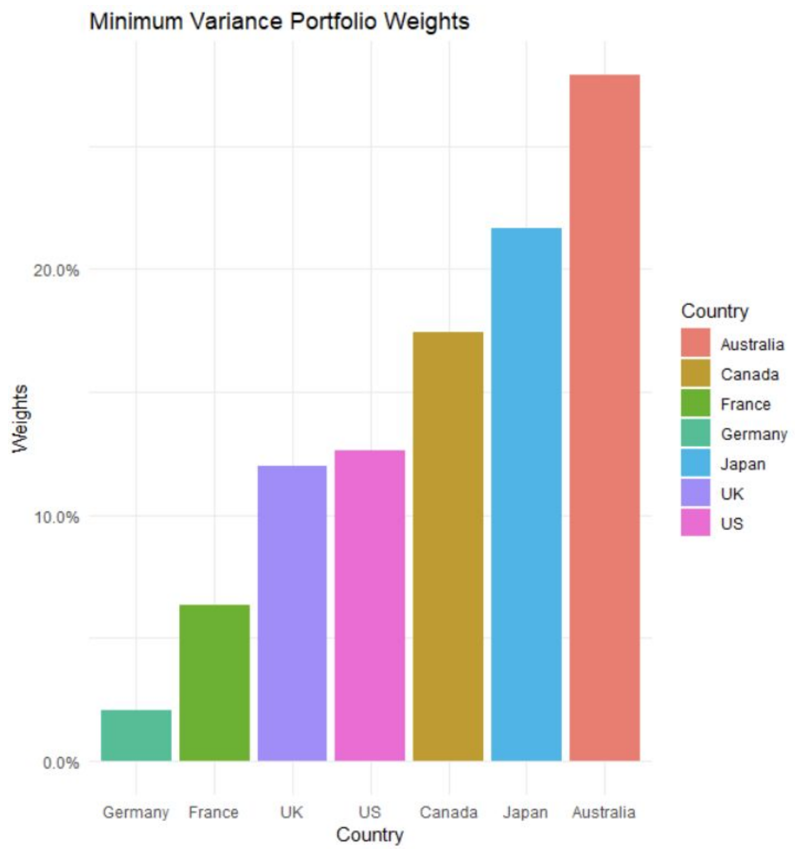
Drawback of Mean-Variance Analysis

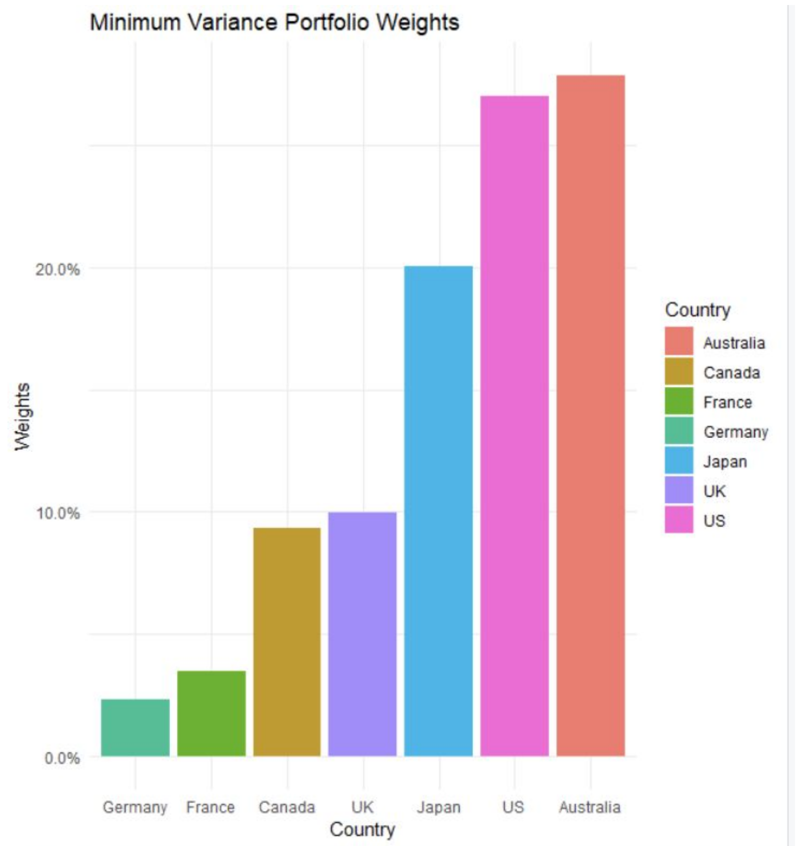
- Mean-Variance preferences the whole market M-V portfolio analysis
- Mean-Variance utility treats gains and losses symmetrically

  not able to capture the investor behaviors
- Assumes constant risk aversion
- Sensitive to expected return inputs

$$w = \xi \cdot \mathbf{C}^{-1}(\mathbf{R} - r\mathbf{1})$$

Unstable Behavior in Portfolio Weights using Optimizers





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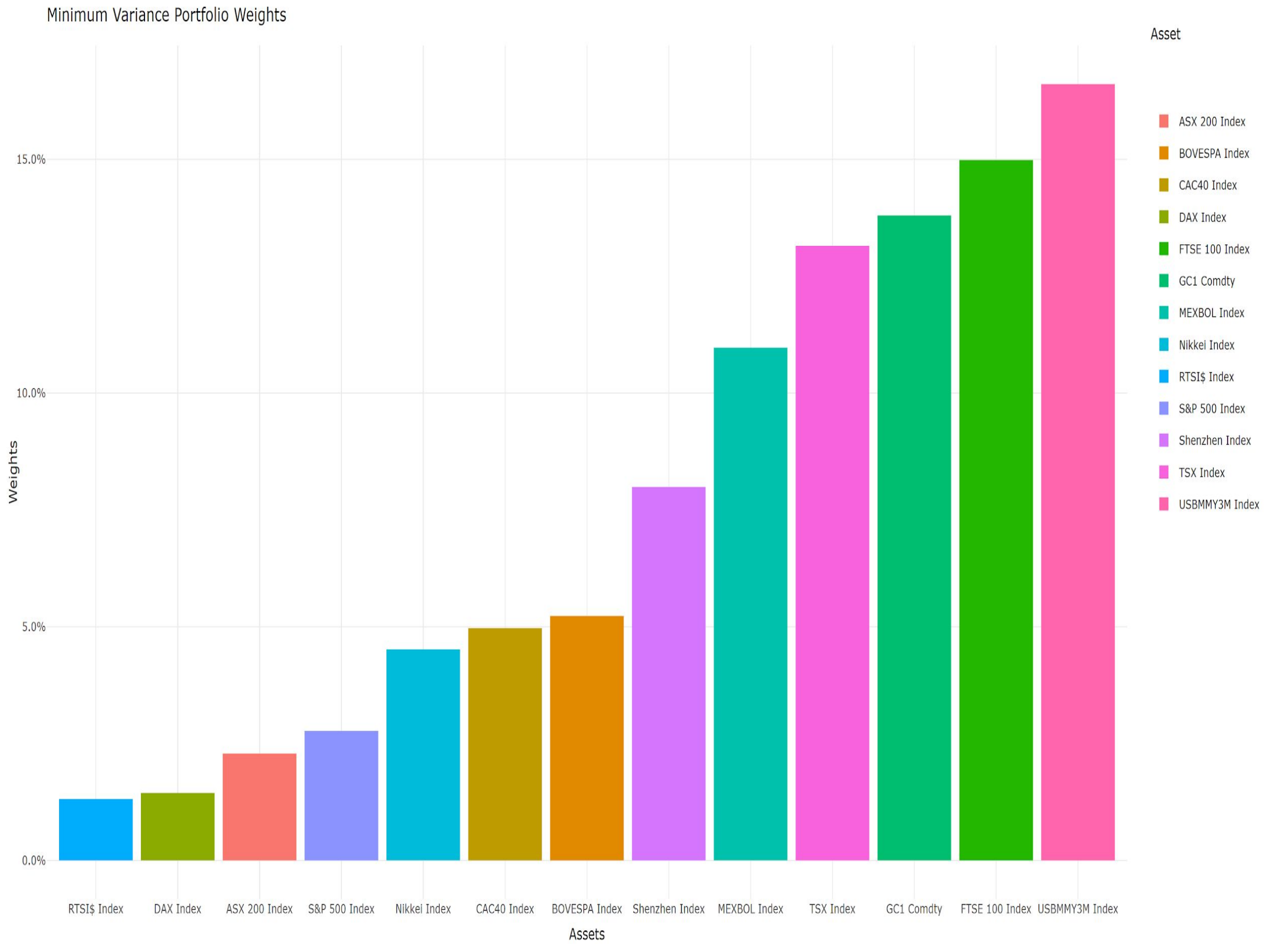
Minimum-Variance Portfolio (Real Data)

- Portfolio is heavily weighted in Cash and Gold
- Both of these assets have the lowest Standard Deviations
- Portfolio heavily invested in the least volatile assets



Black-Litterman Model

- **<u>Bayesian approach + subjective views</u>**

  Estimate of expected returns =

  An investor of expected returns of assets +

  Market equilibrium vector of expected returns

- It improves the Mean-Variance Model by one more step
  - Include an investor's personal view about the assets in asset allocations

- Assumption:
  - Known: Variance of the prior and the conditional distributions
  - Unknown: Actual mean of returns

Mean Variance vs. Black-Litterman

	Mean-Variance Optimization	**Black-Litterman**
Asset Mean	Mean of historical asset returns	Blended asset returns estimated from investor views and equilibrium returns
Asset Covariance	Covariance of historical returns	Covariance of historical asset returns + Estimation uncertainty of the blended asset returns

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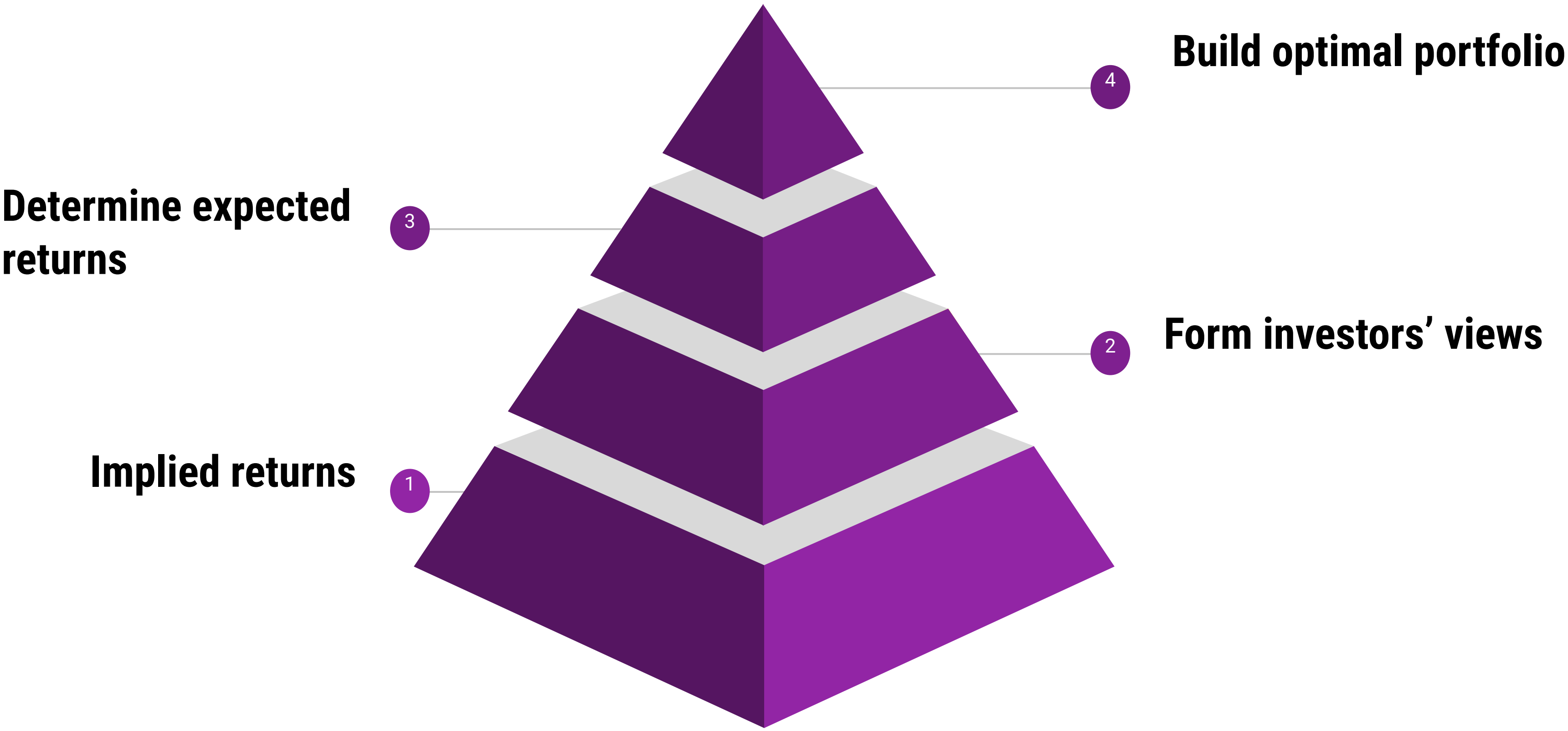
Process of Black-Litterman Model



Process of Black-Litterman Model

- k assets
- r: vector of asset returns (random variable)
  - $r \sim N(\mu, \Sigma)$
  - Σ: The covariance from historical asset returns
  - μ: Expected return (unknown model parameter)
- Assume the prior knowledge:
  - $\mu \sim N(\pi, C)$
    - Without views, π = equilibrium returns
    - C: The uncertainty in the prior
      - $C = \tau\Sigma$
        - τ: a small constant

Process of Black-Litterman Model

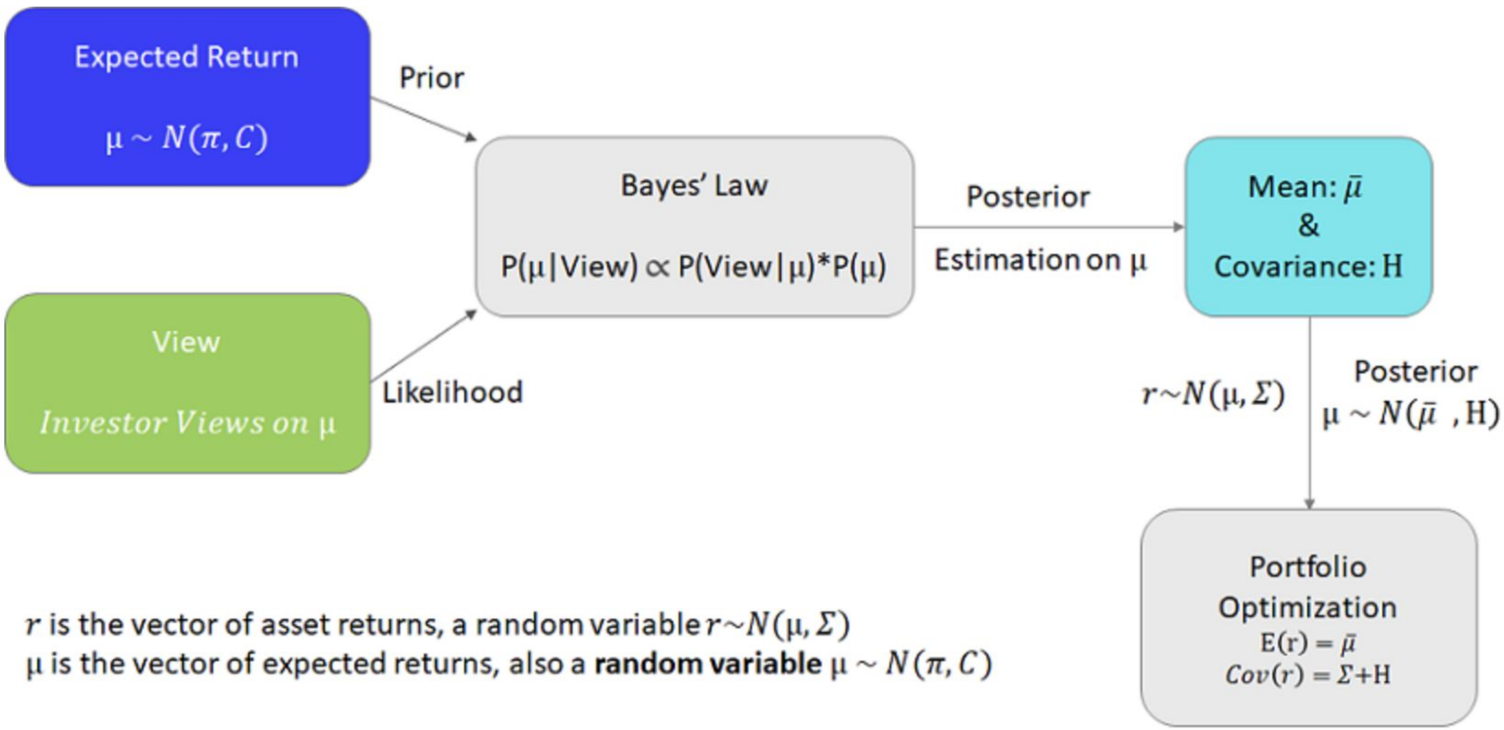


Figure 1. Black-Litterman Model

Bayesian Definition of the Black-Litterman Model

- Posterior ∝ likelihood * prior ⟹ $f(\mu|q) \propto f(q|\mu) * f(\mu)$ (1)
- Likelihood: how likely it is for the views to happen given μ

$$\Longrightarrow f(q|\mu) \propto \exp\left[-\frac{1}{2}(P\mu - q)'\Omega^{-1}(P\mu - q)\right]$$

- *Prior assumes* $\mu \sim N(\pi, C)$ $\Longrightarrow f(\mu) \propto \exp\left[-\frac{1}{2}(\mu - \pi)'C^{-1}(\mu - \pi)\right]$
-
- *Posterior: distribution of* μ *given view and*

$$\Longrightarrow f(\mu|q) \propto \exp\left[-\frac{1}{2}(P\mu - q)'\Omega^{-1}(P\mu - q) - \frac{1}{2}(\mu - \pi)'C^{-1}(\mu - \pi)\right]$$

Black-Litterman Model Views

Step 1

$$q_i = \mathrm{E}\left[\boldsymbol{p}_i * \boldsymbol{r} \mid \mu\right] + \varepsilon_i \ , \quad i = 1, 2, \ldots, v$$

Step 2

$$\boldsymbol{q} = \mathrm{E}[\boldsymbol{P} * \boldsymbol{r} \mid \mu] + \varepsilon, \ \ \varepsilon \sim N(0, \Omega)$$

Step 3

$$\boldsymbol{q} = \boldsymbol{P} * \mu + \varepsilon, \ \ \varepsilon \sim N(0, \Omega)$$

Blended expected return:

$$\bar{\mu} = [P^T \Omega^{-1} P + C^{-1}]^{-1} [P^T \Omega^{-1} q + C^{-1} \pi]$$

$$\Omega = \mathrm{diag}(\omega_1, \omega_2, \ldots \omega_v)$$

Estimation uncertainty:

$$\mathrm{cov}(\mu) = [P^T \Omega^{-1} P + C^{-1}]^{-1}$$

Drawback of BL Model

- The relative magnitude of the active views
- The level of confidence in each individual active view
- An overall level of confidence in the active views versus the equilibrium expected returns

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Data Collection - Real Prices

- Data set collected from Bloomberg
- Dates of Collection: 12/31/2010 - 3/31/2020
- Benchmark: SPX: US
- Other Assets included in Analysis:

AS51: Australia Index	SPTSX: Canada Index
CAC: France Index	DAX: Germany Index
NKY: Japan Index	UKX: UK Index
SHSZ300: China Index	IBOV: Brazil Index
MEXBOL: Mexico Index	RTSI$: Russia Index
GC1: Gold Commodity	Cash: 3-Month T-Bill

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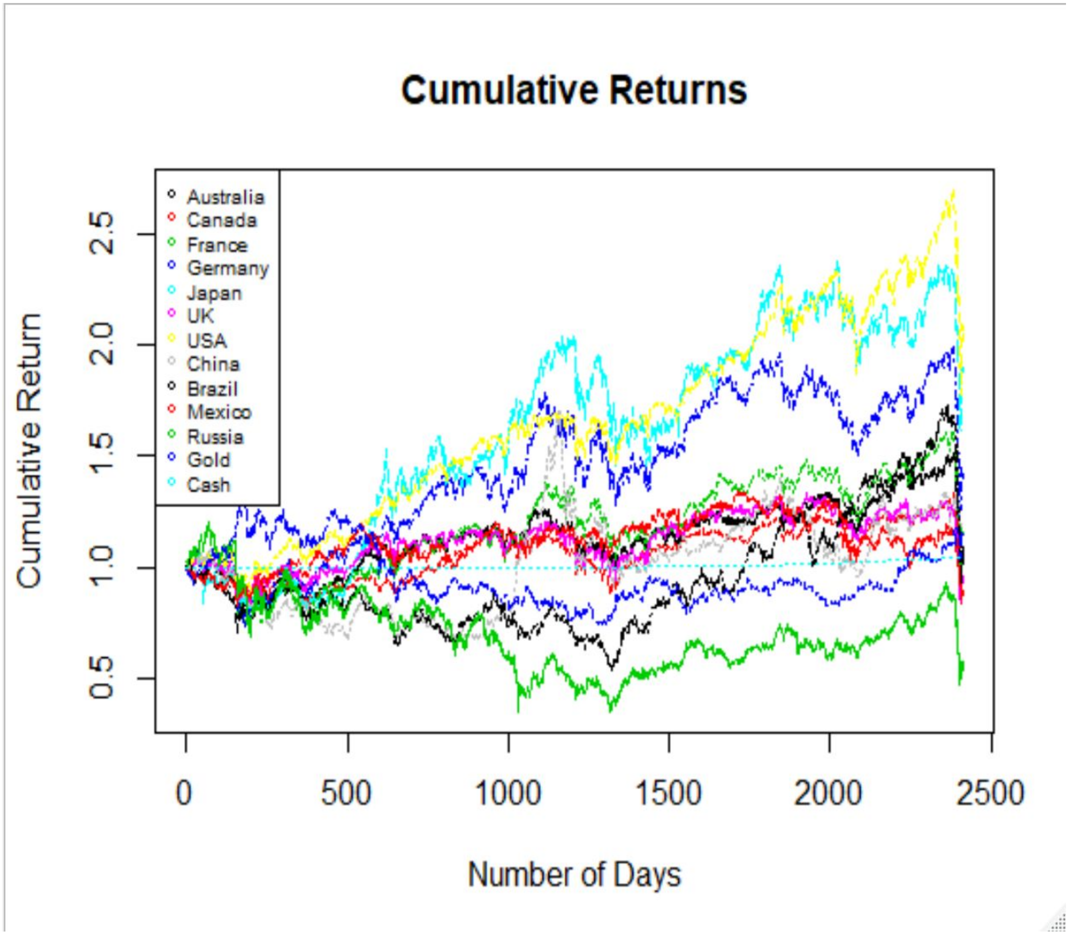
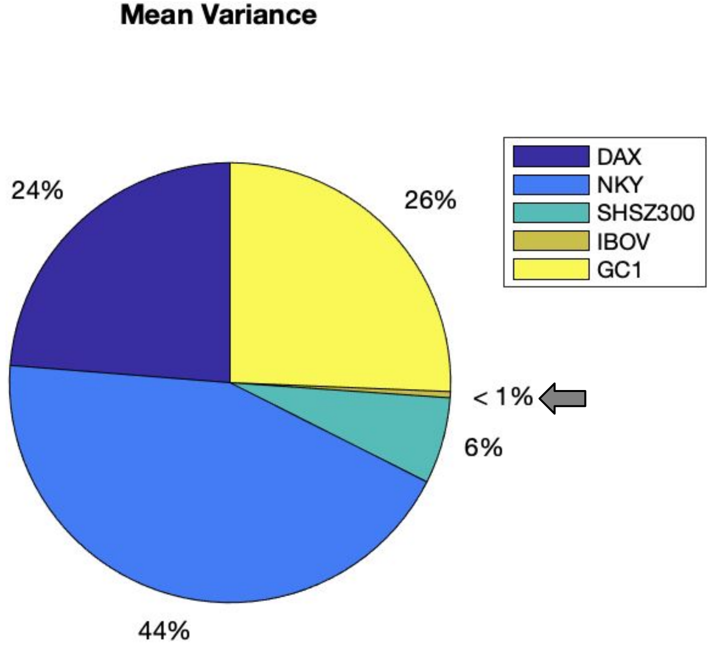
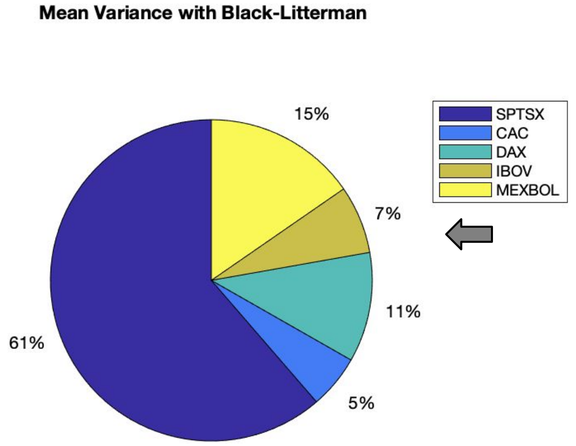
Asset Cumulative Returns from Data

- Cumulative Returns over the dataset period: 12/31/2010 - 3/31/2020
- Cash is the Benchmark Return set at 1 (Dotted Teal Line)
- Lines Below Cash are Underperformers during this period
- Above Cash are considered Outperformers

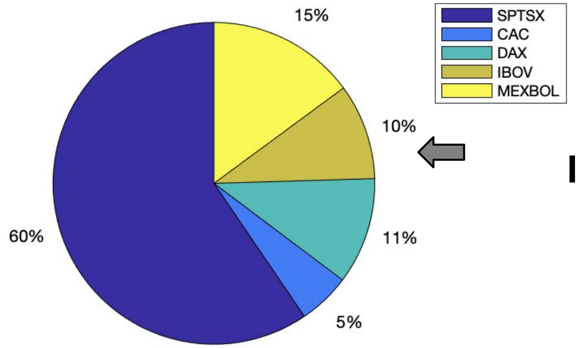


Influence of One Positive View on the Smallest Weight of MV To Black-Litterman Model

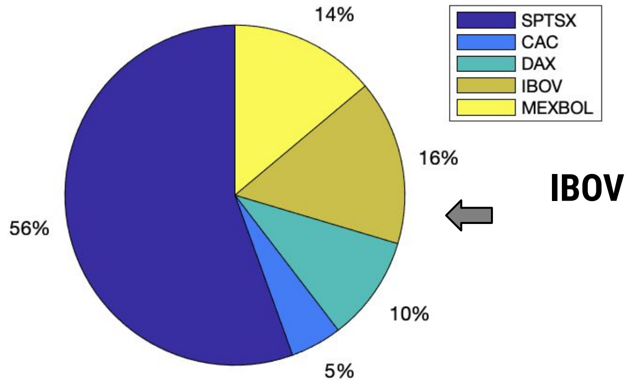




IBOV increase by 1%



IBOV increase by 30%

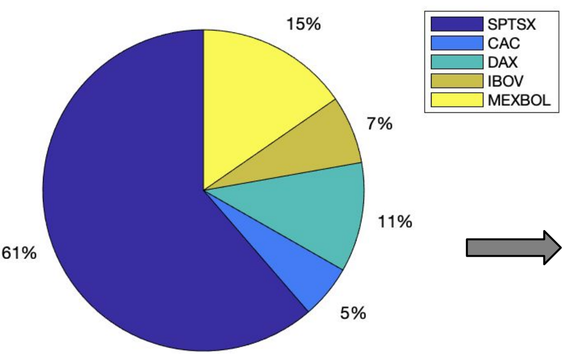


IBOV increase by 50%

IBOV increase by 1%

AS51	SPTSX	CAC	DAX	NKY	UKX	SHSZ300	IBOV	MEXBOL	RTSI	GC1	View_Return	View_Uncertainty
0	0	0	0	0	0	0	1	0	0	0	0.01	0.001

Mean Variance with Black-Litterman

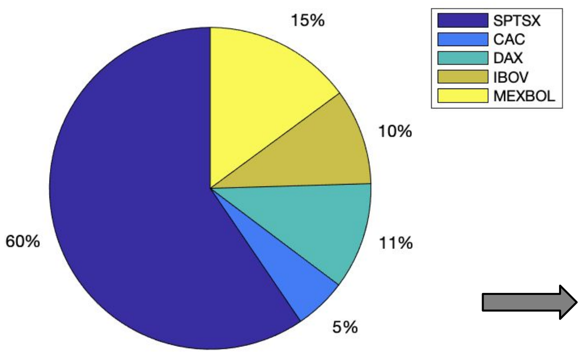


Asset_Name	Prior_Belief_of_Expected_Return	Black_Litterman_Blended_Expected_Return
"AS51"	0.036215	0.03581
"SPTSX"	0.075747	0.074979
"CAC"	0.081269	0.080493
"DAX"	0.079249	0.078532
"NKY"	0.026816	0.026589
"UKX"	0.064005	0.063357
"SHSZ300"	0.024187	0.023916
"IBOV"	0.095011	0.092901
"MEXBOL"	0.055156	0.054499
"RTSI"	0.079187	0.078271
"GC1"	0.0084967	0.0083985

AssetName	Mean_Variance	Mean_Variance_with_Black_Litterman
"AS51"	1.4847e-15	8.942e-07
"SPTSX"	1.3954e-15	0.61339
"CAC"	1.447e-15	0.054005
"DAX"	0.23747	0.11049
"NKY"	0.43805	1.0007e-06
"UKX"	8.5848e-16	1.5364e-06
"SHSZ300"	0.063502	9.7824e-07
"IBOV"	0.0045109	0.068511
"MEXBOL"	1.2013e-15	0.1536
"RTSI"	3.9587e-16	1.1402e-06
"GC1"	0.25646	8.9667e-07

IBOV increase by 30%

	AS51	SPTSX	CAC	DAX	NKY	UKX	SHSZ300	IBOV	MEXBOL	RTSI	GC1	View_Return	View_Uncertainty
Mean Variance with Black-Litterman	0	0	0	0	0	0	0	1	0	0	0	0.3	0.001



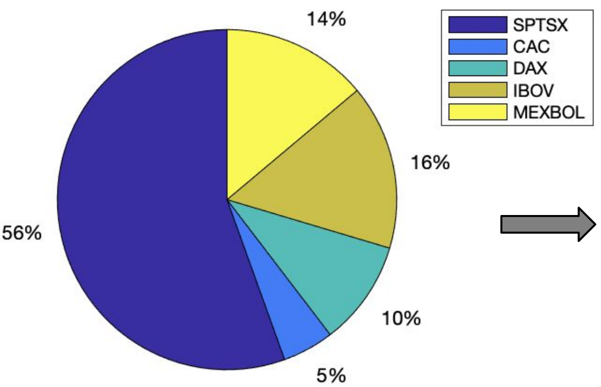
Asset_Name	Prior_Belief_of_Expected_Return	Black_Litterman_Blended_Expected_Return
"AS51"	0.036215	0.037192
"SPTSX"	0.075747	0.077598
"CAC"	0.081269	0.08314
"DAX"	0.079249	0.080979
"NKY"	0.026816	0.027361
"UKX"	0.064005	0.065569
"SHSZ300"	0.024187	0.024841
"IBOV"	0.095011	0.1001
"MEXBOL"	0.055156	0.056741
"RTSI"	0.079187	0.081398
"GC1"	0.0084967	0.0087334

AssetName	Mean_Variance	Mean_Variance_with_Black_Litterman
"AS51"	1.4847e-15	8.0654e-07
"SPTSX"	1.3954e-15	0.59509
"CAC"	1.447e-15	0.052394
"DAX"	0.23747	0.10719
"NKY"	0.43805	8.82e-07
"UKX"	8.5848e-16	1.3749e-06
"SHSZ300"	0.063502	8.6335e-07
"IBOV"	0.0045109	0.096304
"MEXBOL"	1.2013e-15	0.14902
"RTSI"	3.9587e-16	1.0455e-06
"GC1"	0.25646	7.9386e-07

IBOV increase by 50%

AS51	SPTSX	CAC	DAX	NKY	UKX	SHSZ300	IBOV	MEXBOL	RTSI	GC1	View_Return	View_Uncertainty
0	0	0	0	0	0	0	1	0	0	0	0.5	0.001

Mean Variance with Black-Litterman



Asset_Name	Prior_Belief_of_Expected_Return	Black_Litterman_Blended_Expected_Return
"AS51"	0.036215	0.038144
"SPTSX"	0.075747	0.079404
"CAC"	0.081269	0.084965
"DAX"	0.079249	0.082666
"NKY"	0.026816	0.027894
"UKX"	0.064005	0.067095
"SHSZ300"	0.024187	0.025478
"IBOV"	0.095011	0.10506
"MEXBOL"	0.055156	0.058287
"RTSI"	0.079187	0.083554
"GC1"	0.0084967	0.0089644

AssetName	Mean_Variance	Mean_Variance_with_Black_Litterman
"AS51"	1.4847e-15	7.9268e-07
"SPTSX"	1.3954e-15	0.58309
"CAC"	1.447e-15	0.051338
"DAX"	0.23747	0.10503
"NKY"	0.43805	8.3775e-07
"UKX"	8.5848e-16	1.2802e-06
"SHSZ300"	0.063502	8.2192e-07
"IBOV"	0.0045109	0.11453
"MEXBOL"	1.2013e-15	0.14601
"RTSI"	3.9587e-16	1.0009e-06
"GC1"	0.25646	8.126e-07

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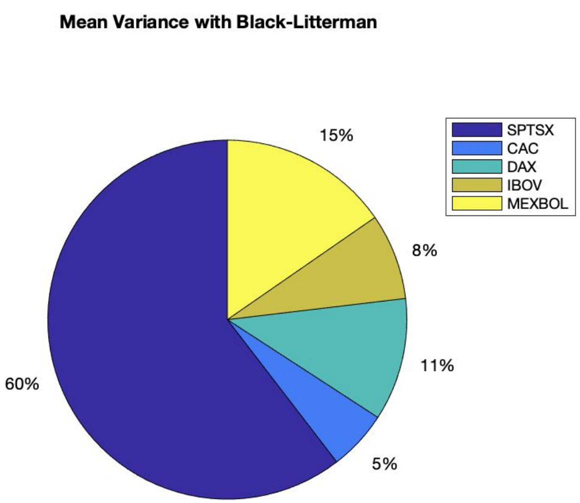
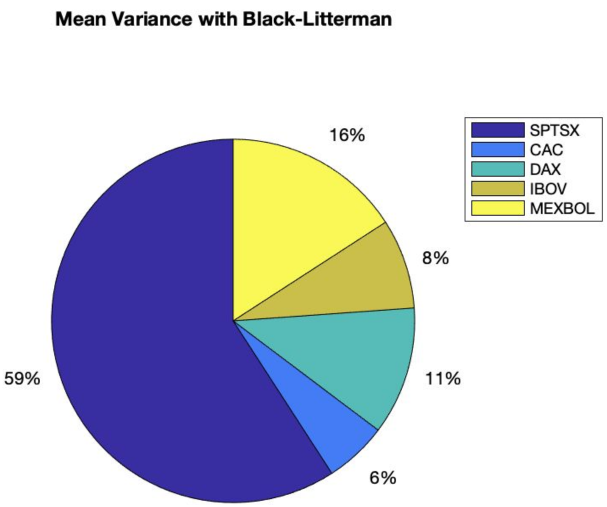
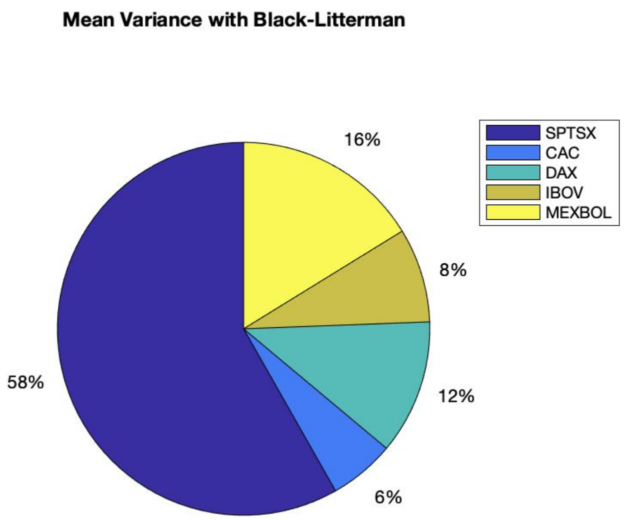
Influence of One Negative View on the Largest Weight of Black-Litterman Model



SPTSX decrease by 1%



SPTSX decrease by 30%



SPTSX decrease by 50%

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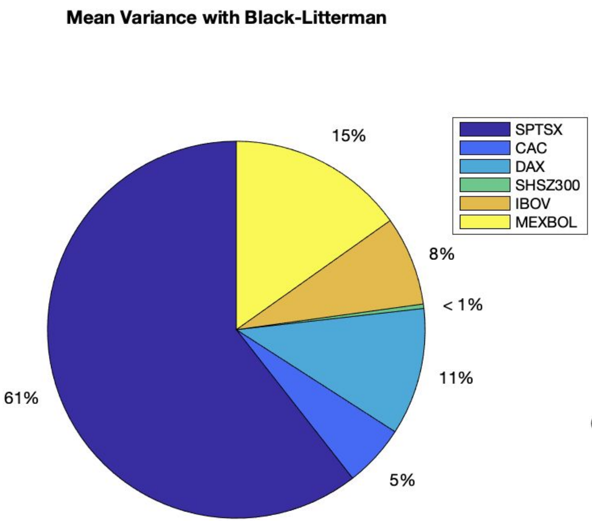
Black Litterman Blended Expected Return for SPTSX Decrease

Asset Name	1%	30%	50%
"AS51"	0.035877	0.034731	0.033941
"SPTSX"	0.074976	0.072367	0.070568
"CAC"	0.080631	0.078473	0.076985
"DAX"	0.078638	0.076571	0.075145
"NKY"	0.026595	0.02585	0.025337
"UKX"	0.063477	0.061689	0.060456
"SHSZ300"	0.023979	0.023274	0.022789
"IBOV"	0.094224	0.091562	0.089727
"MEXBOL"	0.05474	0.053333	0.052363
"RTSI"	0.078537	0.076336	0.074819
"GC1"	0.0083808	0.0079889	0.0077186

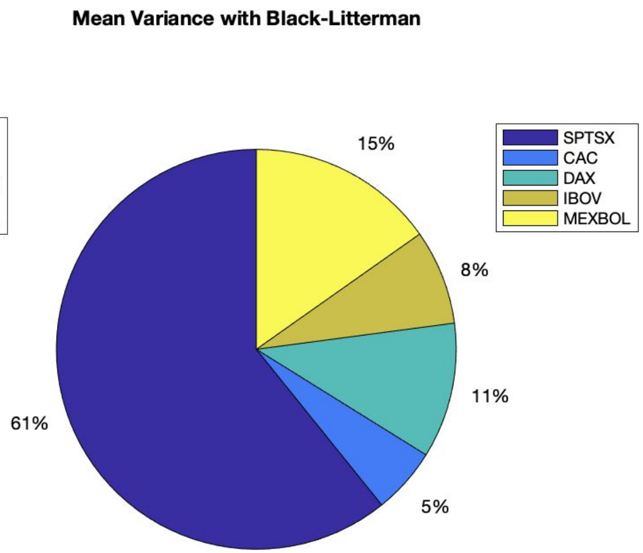
Mean Variance with Black Litterman for SPTSX Decrease

Asset Name	Mean_Variance	1%	30%	50%
"AS51"	1.4847e-15	8.5754e-07	8.5266e-07	8.406e-07
"SPTSX"	1.3954e-15	0.60429	0.59152	0.58222
"CAC"	1.447e-15	0.054018	0.055761	0.05703
"DAX"	0.23747	0.11051	0.11408	0.11667
"NKY"	0.43805	9.3705e-07	8.8505e-07	8.4721e-07
"UKX"	8.5848e-16	1.4618e-06	1.4364e-06	1.3992e-06
"SHSZ300"	0.063502	9.1703e-07	8.6848e-07	8.3242e-07
"IBOV"	0.0045109	0.07754	0.080042	0.081864
"MEXBOL"	1.2013e-15	0.15363	0.15859	0.1622
"RTSI"	3.9587e-16	1.0801e-06	1.0637e-06	1.0393e-06
"GC1"	0.25646	8.4291e-07	8.0856e-07	7.7878e-07

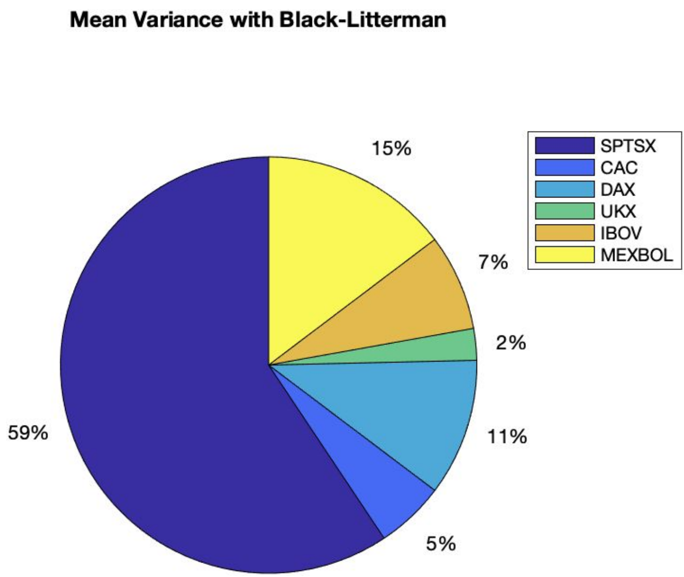
Influence of Number of Views



SHSZ30 increase by 6%



SHSZ30 increase by 6%
AS51 increase by 30%



SHSZ30 increase by 6%
AS51 increase by 30%
UKX larger than SHSZ30 by 5%

Goldman Sachs - “*The Intuition behind Black-Litterman Model Portfolios*”

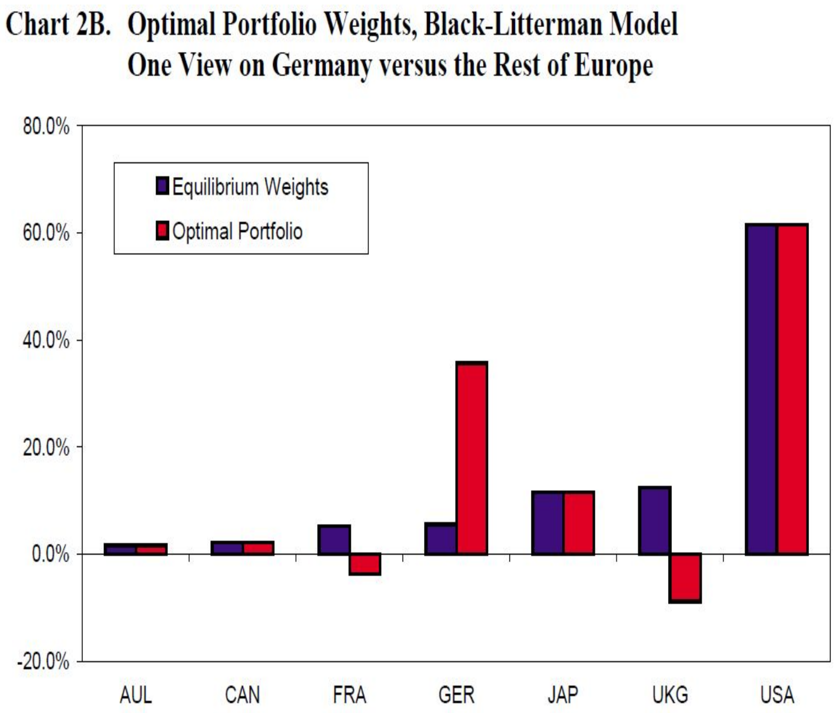
- Replicated Black-Litterman graph from the Goldman Sachs Black-Litterman research paper

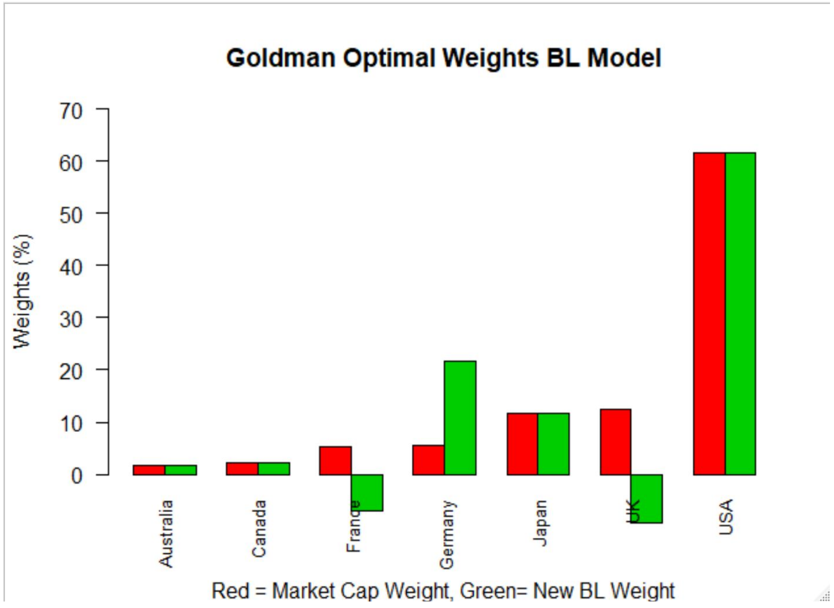
- The paper discusses the difference in optimal weights between the Black-Litterman Model and the Mean-Variance Optimization Process

- Group replicated the results from Goldman Sach’s One View Model using daily real-data to determine if the model represented in the paper was accurate and if the model could be updated
    - One View Scenario: Expect Germany to outperform UK and France y 5% for the year
    - We used this as our stepping stone to our next analysis

Black-Litterman Replication Graph

- Goldman Sachs Report Graph
- Replicating Graph using our Model





Source: Goldman Sach Report ("Intuition Behind Black-Litterman Model Portfolio")

Black-Litterman Model with Real Data

- Utilized replicated Black-Litterman Model with updated, real data
- Original Weights are normalized weights with respect to country GDP
  - Gold and Cash were weighted at approximately 5% and 1% of the portfolio, respectively
- Weighted-Average Portfolio Return = 5.77%
- Weighted-Average Standard Deviation = 19.17%

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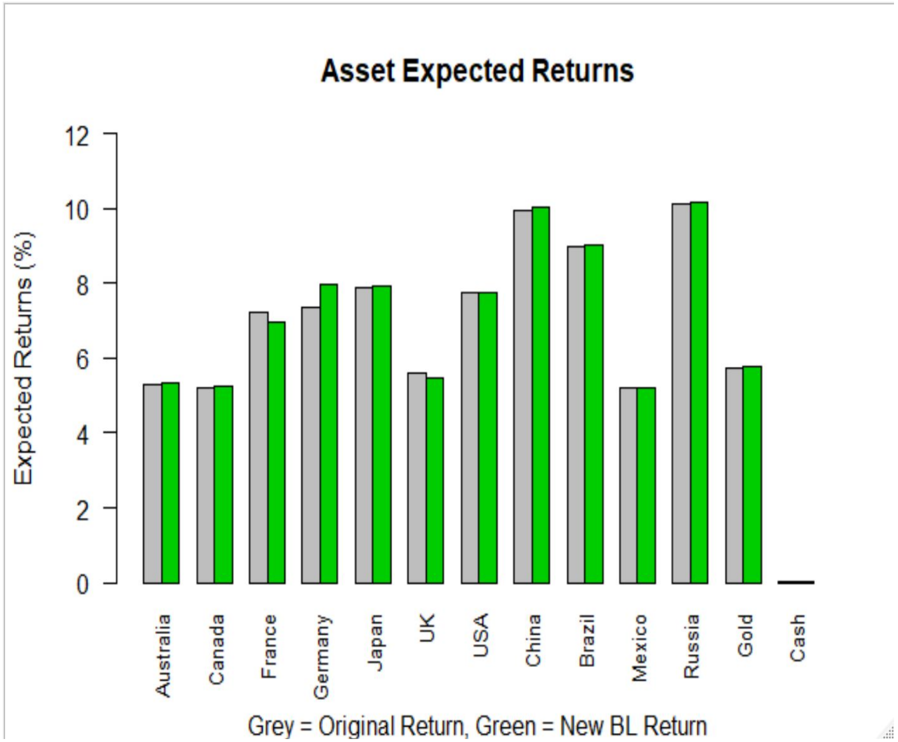
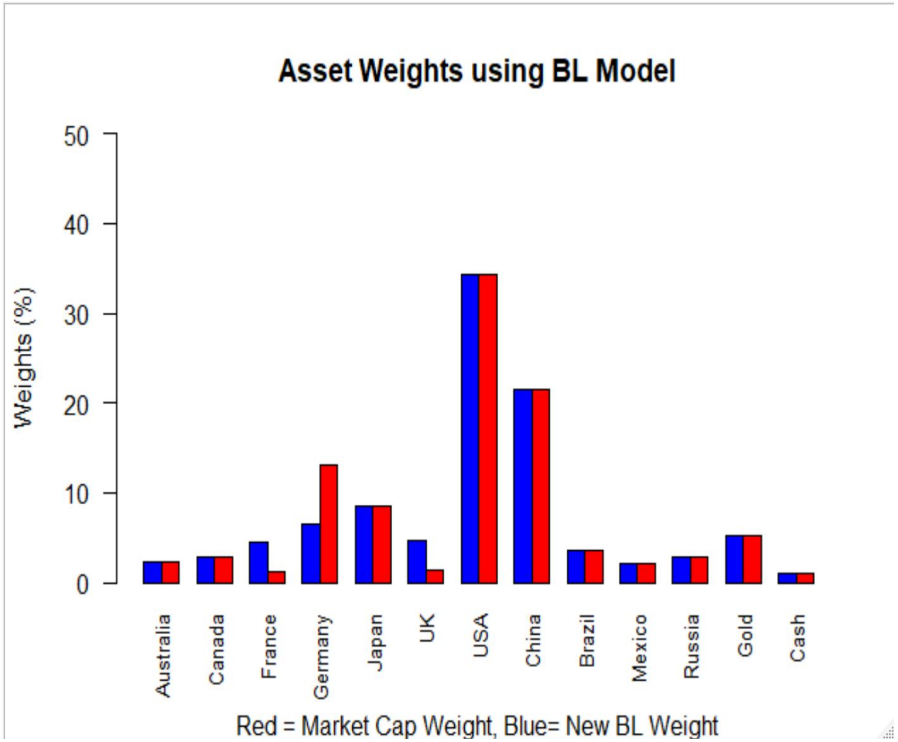
Adjusted BL Model with One View

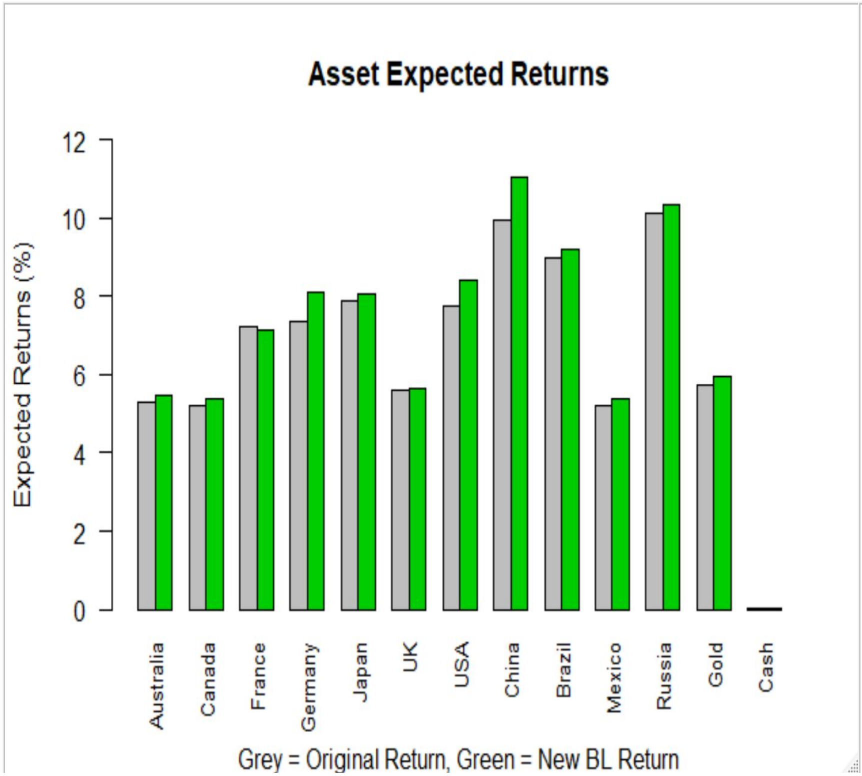
- Implemented Goldman report scenario: Germany Outperforms France and UK by 5%

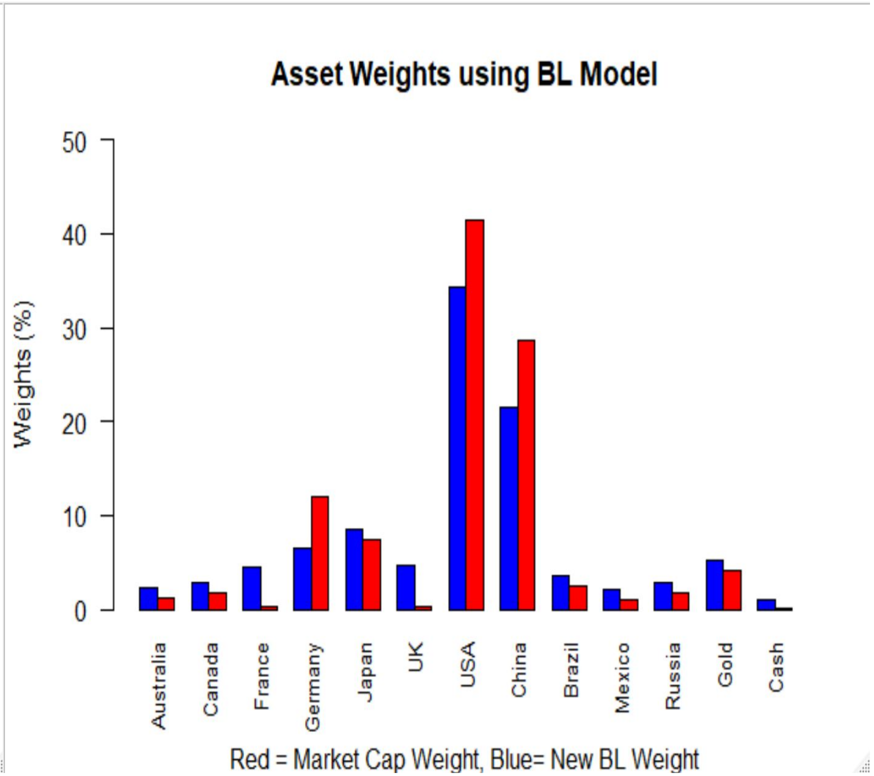




Adjusted BL Model with Two Views

- Implemented Goldman report scenario: Germany Outperforms France and UK by 5%
- Additional View: US and China expected to outperform all other assets





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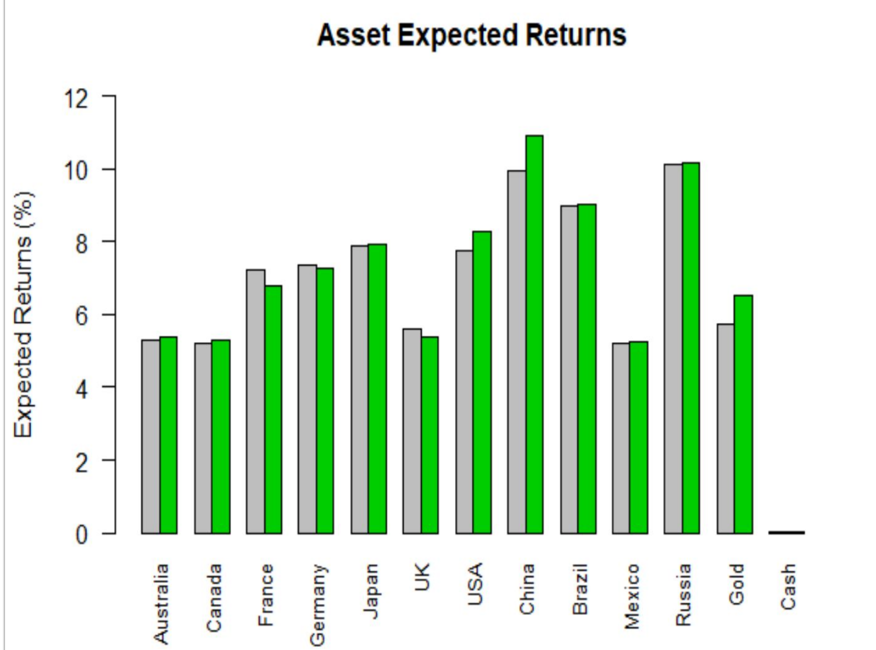
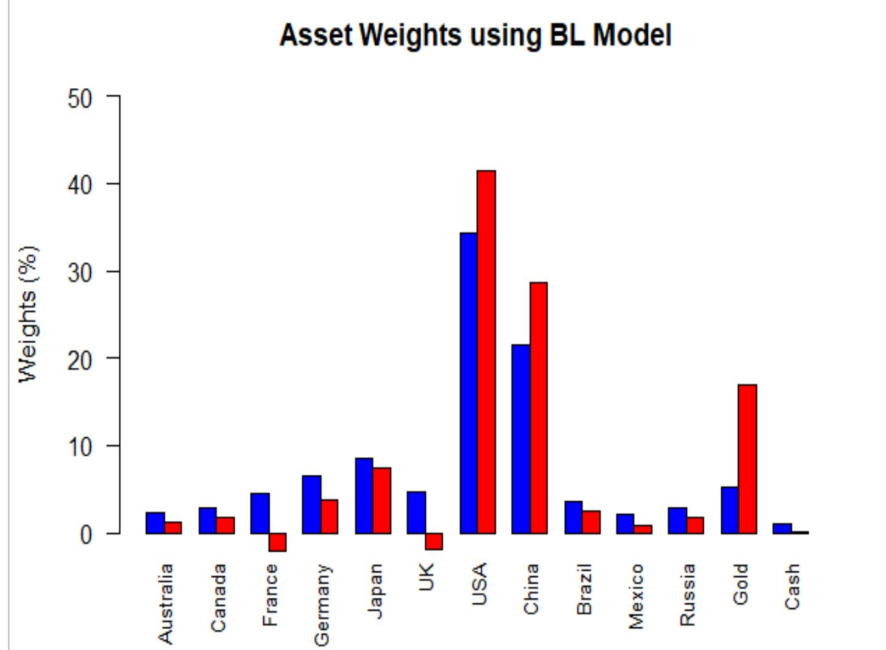
Adjusted BL Model with Three Views

- Implemented Goldman report scenario: Germany Outperforms France and UK by 5%
- Additional View: US and China expected to outperform all other assets
- Gold is relatively expected to outperform Germany



Grey = Original Return, Green = New BL Return



Red = Market Cap Weight, Blue= New BL Weight

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Enhance BL Model - Investor Confidence Calibration

- **Investor Confidence Calibration (ICC)** - used to enhance the posterior expectation by incorporating investor's confidence levels for individual securities' views by allowing the investor to place individual conviction levels on individual securities

- Tau allows investor to specify the confidence in the overall model, but we added an additional constraint in the model component which allows investors to adjust views on the individual security level, without adjusting Omega or Tau

- ICC adds another level of protection for the investor

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Investor Confidence Calibration
Weights Calculation

- Original Posterior Expectation:

$$E\left[\mathbf{R}|\mathbf{q}\right] = \boldsymbol{\pi} + \boldsymbol{\Sigma}\mathbf{P}^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\pi})$$

- Current Optimal Weights in BL Model:

$$w^* = \xi\bar{\mathbf{C}}^{-1}\bar{\mathbf{R}}$$

- Proposed Investor Calibration in BL Model:

$$\omega_k = [\lambda\Sigma]^{-1}[(\tau\Sigma)^{-1} + P^T_k\,\Omega_{k,k}^{-1}\,P_k]^{-1}\,[(\tau\Sigma)^{-1}\,\pi + P^T_k\,\Omega_{k,k}^{-1}\,Q_k]$$

Investor Confidence Calibration Method Part 1

- First Define Confidence Levels
  - Example: only 80% confidence in view Germany outperforms UK and France, not 100% convinced

- Calculate Black-Litterman Posterior Expectation

- Specify deviation from Black-Litterman Optimal allocation

$$\text{Deviation} = (W_{fc} - W_{mkt}) * C\%$$

- W_{fc} = Full Investor Conviction (100% belief in idea)
- W_{mkt} = Original BL Optimal Allocation
- C% = Confidence in individual security weight

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Investor Confidence Calibration Method
Part 2

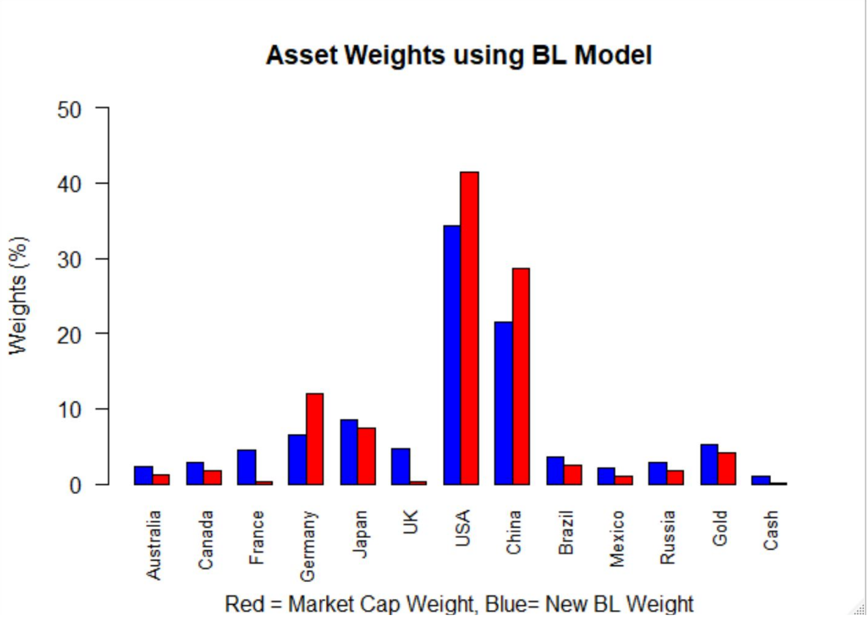
- Find new confidence weighting:
  - $W^* = W_{mkt} + \text{Deviation}$

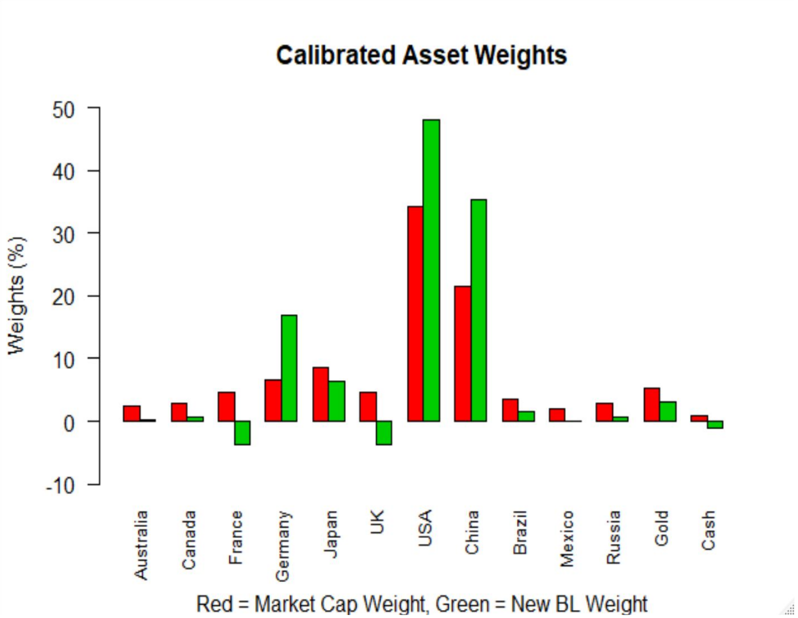
- Solve for $\Omega_{k,k}$, so that the squared difference between the Confidence weighting and the BL weights are minimized

BU

Refined BL Model with ICC (Refining Model with Two Views)

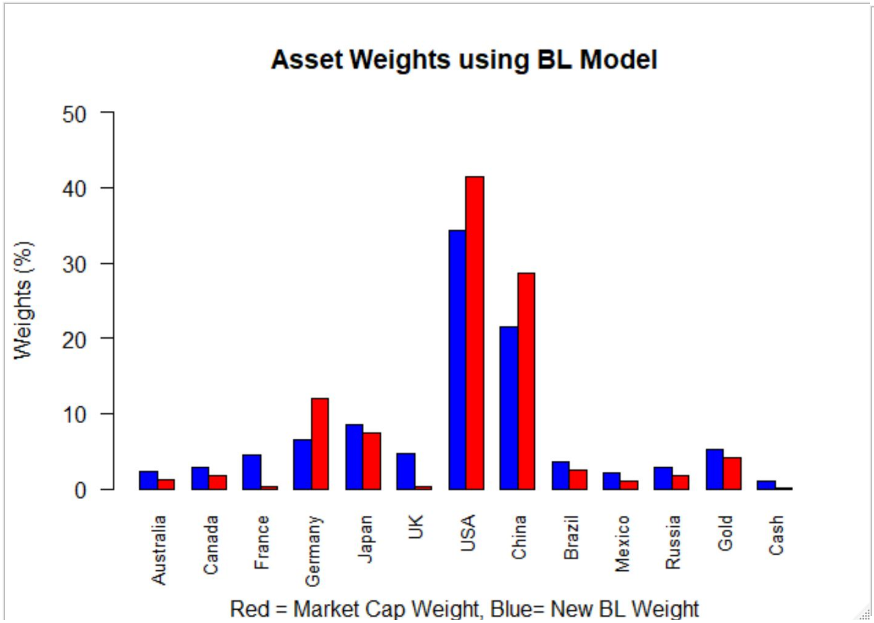
- 80% Confident in View One (Germany outperforms UK and France)
- 60% Confident in View Two (US and China outperform)

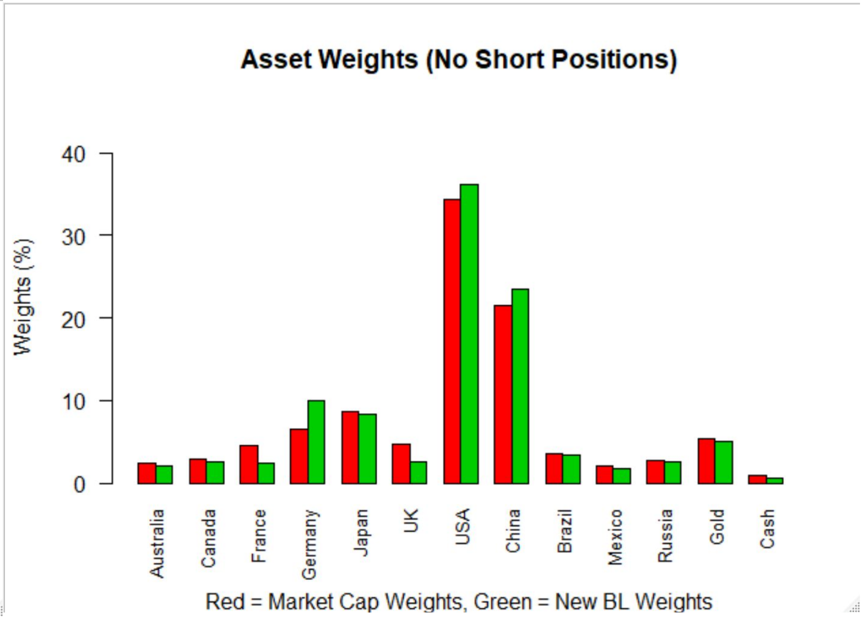




Refined BL Model with ICC - No Shorts (Refining Model with Two Views)

- 80% Confident in View One (Germany outperforms UK and France)
- 60% Confident in View Two (US and China outperform)





Final Weight Differences Between Models

Asset	Original Mkt Cap Weights	Minimum Variance Weights	Black-Litterman Weights	New Calibrated Weights	New Calibrated Weights (No Shorts)
Australia	2.33%	14.53%	1.21%	0.19%	1.57%
Canada	2.90%	1.38%	1.77%	0.74%	2.13%
France	4.55%	3.22%	0.21%	-3.67%	2.28%
Germany	6.50%	0.90%	11.63%	16.18%	8.64%
Japan	8.58%	0.98%	7.32%	6.19%	7.73%
UK	4.65%	12.98%	0.30%	-3.58%	2.37%
USA	34.31%	0.88%	40.42%	46.19%	38.53%
China	21.54%	9.48%	27.96%	33.95%	25.95%
Brazil	3.62%	1.50%	2.52%	1.43%	2.84%
Mexico	2.03%	10.95%	0.92%	-0.10%	1.27%
Russia	2.78%	3.30%	1.65%	0.62%	2.01%
Gold	5.28%	19.35%	4.10%	3.02%	4.48%
Cash	0.93%	20.53%	0.00%	-1.15%	0.19%

Thank You, Any Questions?

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