Light OJ Solution

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Number Theory

1278 - Sum of Consecutive Integers

Description Given an integer n, you have to find the number of ways you can express n as sum of consecutive integers. You have to use at least two integers. For example, $n = 15$ has three solutions,$(1+2+3+4+5), (4+5+6), (7+8)$.

My Solution
For $a, b \in \mathbb{N}, a \neq b$, let

$$\begin{aligned}
& a + (a+1) + (a+2) + \cdots + b = n \\
\Rightarrow\ & \sum_{k=a}^{b} k = n \\
\Rightarrow\ & \sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k = n \\
\Rightarrow\ & \frac{b(b+1)}{2} - \frac{(a-1)(a-1+1)}{2} = n \\
\Rightarrow\ & b^2 + b - (a^2 - a) = 2n \\
\Rightarrow\ & b^2 - a^2 + (b+a) = 2n \\
\Rightarrow\ & (b+a)(b-a) + (b+a) = 2n \\
\therefore\ & (b+a)(b-a+1) = 2n
\end{aligned}$$

The number of ways n be expressed as sum of consecutive natural numbers is the number of possible natural values of a, b.
Now as $a, b \in \mathbb{N}$ and $(b+a)(b-a+1) = 2n$,the values of $(b+a)$ and $(b-a+1)$ would be among the divisors of $2n$. If $pq = 2n$ then $(b+a) = p, (b-a+1) = q$ or $(b+a) = q, (b-a+1) = p$. But if $q < p$ the later assignment would result in negative value of a and b. So of any divisor pair the larger one is assigned to $b+a$. Let $p \geq q$ now,

$$\left.\begin{aligned} b + a &= p \\ b - a &= q - 1 \end{aligned}\right\} \begin{aligned} b &= \tfrac{p+q-1}{2} \\ a &= \tfrac{p-q+1}{2} \end{aligned}$$

But if p, q are both even or odd then $p+q-1, p-q+1$ are odd,so a, b would be fraction,which is not allowed. But in this case p, q both can't be odd,as $pq = 2n$. But both even can occur. So the way of choosing p, q is:
For a pair of divisor of $2n$ if any of them is odd(both can't be odd), make the bigger one p and smaller one q.

So the possible values of a, b are **number of odd divisors of** $2n$ with an exception $\{1, 2n\}$ pair. Because for $\{1, 2n\}$ pair, $p = 2n, q = 1$ gives $b = a = n$,but the problem does not allow to use only one integer.
So the **answer** is **one less than number of odd divisors of 2n**.
Finding number of odd divisor is to make $2n$ free from power of 2, as odd divisors of $2n$ has no 2.

1045 - Digits of Factorial

Description In this problem, you have to find the number of digit(s) of the factorial of an integer in a certain base.

Logarithm,Some awesome use! The basic definition and properties of logarithm are not stated.Here are some awesome use.
Let $n = \sum_{i=0}^{k} a_i \cdot b^i$ be the b base representation of n;where, a_i is the i^{th} digit,a_k is the most significant digit(left most) ,a_0 is the least significant digit(right most). Now n can also be written as,

$$n = (a_k.a_{k-1}a_{k-2}\cdots a_0)\cdot b^k; \text{like}, 123456 = 1.23456 \cdot 10^5$$

- **Finding Number of Digits :** the number of digits in base b is, $k+1$.

$$\begin{aligned} n &= (a_k.a_{k-1}a_{k-2}\cdots a_0)\cdot b^k \\ \log_b n &= \log_b\left[(a_k.a_{k-1}a_{k-2}\cdots a_0)\cdot b^k\right] \\ \log_b n &= \log_b(a_k.a_{k-1}a_{k-2}\cdots a_0) + \log_b b^k \end{aligned}$$

  as $a_k.a_{k-1}a_{k-2}\cdots a_0 < b$, $k = \lfloor \log_b n \rfloor$

$$\therefore \#\text{digits} = \lfloor \log_b n \rfloor + 1$$

  Its **NOT** $\lceil \log_b n \rceil$,because for $n = b^k$,$\lceil \log_b n \rceil$ would give k,but the answer is $k+1$.
  The most amazing fact is,this can be used to find the number of digits of huge number!!Like 112358^{314159} in base 10.It is huge!!

$$\lfloor \log_{10} 112358^{314159} \rfloor + 1 = \lfloor 314159 \cdot \log_{10} 112358 \rfloor + 1 = \lfloor 1586692.7020.. \rfloor + 1 = 1586693$$

  So,even if we can not store that huge number,the power rule of Logarithm gives us its number of digits!! Its magic ;).

- **Finding digits from left most sides :** finding right most digit is easy,just find $n \bmod b$. Finding last two digits,$n \bmod b^2$ and so on. Finding left most digits,first two digits,last three digits,...are a bit challenging.Specially if the number is too huge to find!! Lets see magic.

$$\begin{aligned} n &= (a_k.a_{k-1}a_{k-2}\cdots a_0)\cdot b^k \\ \log_b n &= \log_b(a_k.a_{k-1}a_{k-2}\cdots a_0) + \log_b b^k \\ a_k.a_{k-1}a_{k-2}\cdots a_0 &= b^{\log_b n - k} \\ a_k.a_{k-1}a_{k-2}\cdots a_0 &= b^{\log_b n - \lfloor \log_b n \rfloor} \end{aligned}$$

  Now that expression can give the first d digits,

$$\left\lfloor b^{\log_b n - \lfloor \log_b n \rfloor} \cdot b^{d-1} \right\rfloor$$

  Again lets try to find first 3 digits of a huge number,like 1505064^{314159} in base 10.

$$\log_{10} 1505064^{314159} = 314159 \cdot \log_{10} 1505064 = 1940734.491152316347124..$$
$$\left\lfloor 10^{1940734.491152316347124 - 1940734} \cdot 10^{3-1} \right\rfloor = 309$$

  You can verify this result in Wolfram Alpha.
  Its magic,isn't it?

My Solution

$$\text{\#digits in base } b = \lfloor \log_b n! \rfloor + 1$$

for large values of n its not possible to store $n!$. But properties of Logarithm to the rescue.

$$\begin{aligned} \lfloor \log_b n! \rfloor + 1 &= \lfloor \log_b(1\cdot 2\cdot 3\cdots n)\rfloor + 1 \\ &= \lfloor \log_b 1 + \log_b 2 + \log_b 3 \cdots + \log_b n \rfloor + 1 \\ &= \left\lfloor \frac{1}{\log b} \cdot \sum_{k=1}^{n} \log k \right\rfloor + 1 \end{aligned}$$

The solution is now simple. $\displaystyle\sum_{k=1}^{n} \log k$ is pre-calculated and for each values of b answer is given in $O(1)$.

1282 - Leading and Trailing

Description You are given two integers: n and k, your task is to find the most significant three digits, and least significant three digits of n^k.

My Solution

- **least significant three:** this is simply, $n^k \bmod 10^3$,which can be calculated by Modular Exponentiation (Big Mod) in $O(\log k)$.
- **most significant three:** Can be got directly from this formula in $O(1)$,

$$\left\lfloor 10^{\log_{10} n^k - \lfloor \log_{10} n^k \rfloor} \cdot 10^{3-1} \right\rfloor = \lfloor 10^{k \cdot \log_{10} n - \lfloor k \cdot \log_{10} n \rfloor} \cdot 100 \rfloor$$

1336 - Sigma Function

Description For some n the value of $\sigma(n)$ is odd and for others it is even. Given a value n, you will have to find how many integers from 1 to n have even value of σ.

$\tau(n)$ and $\sigma(n)$ Function
In number theory the,for any natural number n,

$$\tau(n) = \text{number of divisors of } n = \sum_{d|n} 1$$

$$\sigma(n) = \text{sum of all divisors of } n = \sum_{d|n} d$$

Its easy to find τ and σ for any prime p .As the only divisors of p are 1 and p itself. $\tau(p) = 2$ and $\sigma(p) = p + 1$.
Also for p^a,the divisors are only $\{1, p, p^2, \cdots, p^a\}$.
So $\tau(n) = a + 1$ and $\sigma(n) = 1 + p + p^2 + \cdots + p^a = \dfrac{p^{a+1} - 1}{p - 1}$
Also τ and σ are **Multiplicative Function**. A function f is multiplicative if $f(mn) = f(m)f(n); \gcd(m, n) = 1$.
So,if $n = \prod {p_i}^{a_i}$ is the prime factorization of n,by the multiplicative property,

$$\begin{aligned} \tau(n) &= \prod (a_i + 1) \\ \sigma(n) &= \prod \frac{{p_i}^{a_i+1} - 1}{p_i - 1} \end{aligned}$$

Now,$\tau(n)$ is **odd** iff all a_i are even. In other word iff the number is **perfect square**.
$\sigma(n)$ is **odd** iff n is **perfect square** or **a power of** 2. That is $n = 2^k N^2; k \in \mathbb{N}_0, 2 \nmid N$.

My Solution Its a direct result of condition for $\sigma(n)$ be odd.$\sigma(n)$ is odd iff $n = 2^k N^2; k \in \mathbb{N}_0, 2 \nmid N$.
From 1 to n,the number of square of perfect square are $\lfloor\sqrt{n}\rfloor$.
The number of power of 2 and perfect square its $\left\lfloor\sqrt{\lfloor n/2\rfloor}\right\rfloor$.
As the problem asks for even,

$$\text{ans} = n - \left(\lfloor\sqrt{n}\rfloor + \left\lfloor\sqrt{\lfloor n/2\rfloor}\right\rfloor\right).$$

1035 - Intelligent Factorial Factorization

Description Given an integer n, you have to prime factorize $n!$ (factorial n).

Legendre's Formula For any prime number p and any integer n, let $\nu_p(n!)$ be the exponent of the largest power of p that divides n (that is, the p-adic valuation of n).Then,

$$\nu_p(n!) = \sum_{i=1}^{\infty}\left\lfloor\frac{n}{p^i}\right\rfloor$$

where $\lfloor x\rfloor$ is the floor function. While the formula on the right side is an infinite sum, for any particular values of n and p it has only finitely many nonzero terms: for every i large enough that $p^i > n$, $\left\lfloor\frac{n}{p^i}\right\rfloor = 0$

$$\therefore \nu_p(n!) = \sum_{i=1}^{\lfloor\log_p n\rfloor}\left\lfloor\frac{n}{p^i}\right\rfloor$$

My Solution Using **Legendre's formula**, $n!$ can be prime factorized.

$$n! = \prod_{i=1}^{k} p_i^{\alpha_i}; \alpha_i = \nu_{p_i}(n!)$$

1090 - Trailing Zeroes (II)

Description Find the number of trailing zeroes for the following function:

$${}^nC_r \cdot p^q$$

Trailing Zeros of $n!$ in base b

Let,$b = \prod_{i=1}^{k} p_i^{\beta_i}$ be any base.Where right side is the prime factorization of b.
Using **Legendre's formula**, $n!$ can be prime factorized.Then let,

$$n! = \prod_{i=1}^{k} p_i^{\alpha_i} = k \cdot b^{\tau}; \alpha_i = \nu_{p_i}(n!); k, \tau \in \mathbb{N}_0$$

So this concludes $n!$ has τ trailing zeros in base b. τ is the maximum power of b is $n!$.This can be calculated by

$$\tau = \min \left\{ \left\lfloor \frac{\alpha_i}{\beta_i} \right\rfloor \right\}_{i=1}^{i=k}$$

My Solution In this problem the trailing zeros in base ten is wanted.As $b = 10 = 2 \cdot 5$.

$$^nC_r \cdot p^q = \frac{n!}{r!(n-r)!} \cdot p^q = 2^a \cdot 5^b \cdot k; a, b, k \in \mathbb{N}_0$$

Here number of trailing zeros would be $\min\{a, b\}$.
Power of 2 ans 5 in the factorials can be calculated using Legendre's Formula.

$$^nC_r = 2^{\nu_2(n!)-(\nu_2(r!)+\nu_2((n-r)!))} \cdot 5^{\nu_5(n!)-(\nu_5(r!)+\nu_5((n-r)!))} \cdot k_1; k_1 \in \mathbb{N}_0$$

Now finding 2 and 5's power in p^q is to be done by finding maximum power of 2 and 5 that divides p and multiplying it by q,as p is raised to the power q.

1138 - Trailing Zeroes (III)

Description You task is to find minimal natural number n, so that $n!$ contains exactly q zeroes on the trail in decimal notation. For example, $5! = 120$, 120 contains one zero on the trail.

My Solution In this problem the trailing zeros in base ten is wanted.
So,$b = 10 = 2 \cdot 5$.

$$\therefore \tau = \min\{\nu_2(n!), \nu_5(n!)\} = \nu_5(n!); \text{as there are more } 2'\text{s than } 5'\text{s}$$

Now in the problem τ is given. The minimum value of n is to be found that has exactly q trailing zeros.This can be done by Binary Search on b.But there is also a beautiful mathematical solution.

$$q = \tau = \nu_5(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor \le \sum_{i=1}^{\infty} \frac{n}{5^i} = n \sum_{i=1}^{\infty} \frac{1}{5^i} = n \cdot \frac{1/5}{1 - \frac{1}{5}} = \frac{n}{4}$$

$$\therefore q \le \frac{n}{4} \Rightarrow n \ge 4q$$

As n has to have exactly q trailing zeros,

$$4q \le n < 4(q+1)$$

So a linear search from $4q$ can give n within maximum 5 tries. Therefore time complexity is O(1) as time does not depend on n.
If no n is possible a linear search can detect that.

1340 - Story of Tomisu Ghost

Description $n!$ (factorial n) has at least t trailing zeroes in b based number system. Given the value of n and t, what is the maximum possible value of b?

My Solution As the problem asks for maximum value of b,then for each prime,p_i their power β_i have to be maximum. Also as $\tau \geq t$,

$$\frac{\alpha_i}{\beta_i} \geq t \Rightarrow \beta_i \leq \frac{\alpha_i}{t}, \forall i$$
$$\therefore (\beta_i)_{max} = \left\lfloor \frac{\alpha_i}{t} \right\rfloor$$

So the maximum value of b is,

$$b = \prod_{i=1}^{k} p_i^{\lfloor \alpha_i/t \rfloor}$$

This is the value of b. Now b can be very large so the problem asks for mod by 10000019.So **Modular Exponentiation**(Big Mod) can be used to find the mod in $O(\lg n)$. Primes are found in $O(n \lg \lg n)$ by **Sieve of Eratosthenes**.

Impossible Case

- if $b = 1$ its impossible.As 1 can not be a number base

Matrix Exponentiation

1052 - String Growth

Description Zibon just started his courses in Computer science. After having some lectures on programming courses he fell in love with strings. He started to play with strings and experiments on them. One day he started a string of arbitrary (of course positive) length consisting of only a, b. He considered it as 1st string and generated subsequent strings from it by replacing all the b's with ab and all the a's with b. For example, if he ith string is abab, (i+1)th string will be b(ab)b(ab) = babbab. He found that the Nth string has length X and M^{th} string has length Y. He wondered what will be length of the K^{th} string. Can you help him?

My Solution Let a_k, b_k, f_k be the number of a's,number of b's and length of k^{th} string respectively.
Now,as b becomes ab and a becomes b all b's come from a and b of the previous string.

$$\therefore b_k = a_{k-1} + b_{k-1} = \text{length of previous string} = f_{k-1}$$

As a comes from only by b becoming ab,

$$a_k = b_{k-1} = f_{k-2}$$

So, $f_k = a_k + b_k = f_{k-2} + f_{k-1}$

$$\begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{k-1} \\ f_{k-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-2} \cdot \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} ; \text{for } k > 2$$

f_1, f_2 are unknown.
f_n and f_m are given.
Let,

$$\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \cdot \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} f_2 \\ f_1 \end{bmatrix}$$
$$\begin{bmatrix} f_m \\ f_{m-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{m-2} \cdot \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdot \begin{bmatrix} f_2 \\ f_1 \end{bmatrix}$$

These implies,

$$a_1\ f_2 + b_1\ f_1 = f_n \tag{1}$$
$$a_2\ f_2 + b_2\ f_1 = f_m \tag{2}$$

Now solve Equation (1),(2) to get f_1, f_2.

Corner Cases

- If $n = 1$,Equation (1) becomes, $(0)\ f_2 + (1)\ f_1 = f_n$
- If $n = 2$,Equation (1) becomes, $(1)\ f_2 + (0)\ f_1 = f_n$
- If $m = 1$,Equation (2) becomes, $(0)\ f_2 + (1)\ f_1 = f_m$
- If $m = 2$,Equation (2) becomes, $(1)\ f_2 + (0)\ f_1 = f_m$

Impossible Cases

- any of f_1, f_2 are fraction,zero or negative.
- $\forall k, f_{k-1} \le f_k \le 2f_{k-1}$,So impossible if $f_2 < f_1$ or $f_2 > 2f_1$

1065 - Number Sequence

Description Let's define another number sequence, given by the following function:

$$\begin{aligned} f(0) &= a \\ f(1) &= b \\ f(n) &= f(n-1) + f(n-2),\ n > 1 \end{aligned}$$

When $a = 0$ and $b = 1$, this sequence gives the Fibonacci sequence. Changing the values of a and b, you can get many different sequences. Given the values of a, b, you have to find the last m digits of $f(n)$.

My Solution Its a straight forward Matrix Exponentiation problem. A Dynamic Programming solution would not work because of values of n.

$$\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \text{; for } n > 1$$

$f_0 = a, f_1 = b$ are given. Use Matrix Exponentiation to find f_n.
Finding last m digit means the number is to be $f_n \bmod 10^m$.

1070 - Algebraic Problem

Description Given the value of $a + b$ and ab you will have to find the value of $a^n + b^n$. a and b not necessarily have to be real numbers.

Linear Homogeneous Recurrence Relation Linear homogeneous recurrence relation of degree two,

$$aF_n + bF_{n-1} + cF_{n-2} = 0$$

The corresponding solution equation is,

$$ax^2 + bx + c = 0$$

α, β are two solutions of this equation.
If $\alpha \neq \beta$,general solution,

$$F_n = c_1\,\alpha^n + c_2\,\beta^n$$

If $\alpha = \beta$,general solution,

$$F_n = c_1\,\alpha^n + c_2\,n\,\alpha^n$$

To find particular solution use initial value F_0, F_1 to find c_1, c_2.

My Solution Let $a + b = p$ and $ab = q$.
Now a quadratic equation with root a, b is,

$$x^2 - px + q = 0$$

Now this equation corresponds to,

$$F_n - pF_{n-1} + qF_{n-2} = 0$$

Now if $a = b$,finding $a^n + b^n$ is easy. Its $a^n + b^n = 2(p/2)^n$.
For $a \neq b$, solution to the recurrence is,

$$F_n = c_1\,a^n + c_2\,b^n$$

So,if $c_1 = c_2 = 1$, F_n is the answer.
For that,

$$\begin{aligned} n = 0 &\Rightarrow F_0 = c_1 + c_2 = 1 + 1 = 2 \\ n = 1 &\Rightarrow F_1 = c_1\,a + c_2\,b = 1 \cdot a + 1 \cdot b = a + b = p \end{aligned}$$

So, $a^n + b^n$ is the n^{th} term of, $F_n = pF_{n-1} - qF_{n-2}; F_0 = 2, F_1 = p$.
Now,

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \text{(for } n > 1)$$

Now just Matrix Exponentiation is used to find the value.

1096 - nth Term

Description You have to find the n^{th} term of the following function:

$$\begin{aligned} f(n) &= a\,f(n-1) + b\,f(n-3) + c, \text{if } n > 2 \\ &= 0, \text{if } n \leq 2 \end{aligned}$$

My Solution Its a straight forward Matrix Exponentiation problem. A Dynamic Programming solution would not work because of values of n.

$$\begin{bmatrix} f_n \\ f_{n-1} \\ f_{n-2} \\ c \end{bmatrix} = \begin{bmatrix} a & 0 & b & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ c \end{bmatrix} = \begin{bmatrix} a & 0 & b & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{n-2} \cdot \begin{bmatrix} f_2 \\ f_1 \\ f_0 \\ c \end{bmatrix} \text{for } n > 2$$

$f_0 = f_1 = f_2 = 0$ as given $f_n = 0, \text{if } n \leq 2$.
Use Matrix Exponentiation to find f_n.

1096 - nth Term

Description Let,

$$f_n = a_1\,f_{n-1} + b_1\,f_{n-2} + c_1\,g_{n-3}$$

$$g_n = a_2\,g_{n-1} + b_2\,g_{n-2} + c_2\,f_{n-3}$$

Find $f_n\%M$ and $g_n\%M$. (% stands for the modulo operation.)

My Solution Its a straight forward Matrix Exponentiation problem. A Dynamic Programming solution would not work because of values of n.

$$\begin{bmatrix} f_n \\ f_{n-1} \\ f_{n-2} \\ g_n \\ g_{n-1} \\ g_{n-2} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & c_1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & a_2 & b_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ g_{n-1} \\ g_{n-2} \\ g_{n-3} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & c_1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & a_2 & b_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{n-2} \cdot \begin{bmatrix} f_2 \\ f_1 \\ f_0 \\ g_2 \\ g_1 \\ g_0 \end{bmatrix}, \text{for } n > 2$$

All initial values and M is given.
Use Matrix Exponentiation to find f_n.

1132 - Summing up Powers

Description Given n and k, you have to find $(1^k + 2^k + 3^k + ... + n^k)$ **mod** 2^{32}

My Solution Let

$$f(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

. Base case,$f(1) = 1$.

$$\begin{aligned} f(n+1) &= 1^k + 2^k + 3^k + \cdots + n^k + (n+1)^k \\ &= f(n) + (n+1)^k \end{aligned}$$

This could have been solved by DP. But the value of n is too big. This indicates the solution would be Matrix Exponentiation. Now the recurrence has to be converted into a matrix.

$$\begin{aligned} f(n+1) &= f(n) + (n+1)^k \\ &= f(n) + \binom{k}{0} n^k + \binom{k}{1} n^{k-1} + \binom{k}{2} n^{k-2} + \cdots + \binom{k}{k} n^0 \end{aligned}$$

$$\begin{bmatrix} f(n+1) \\ (n+1)^k \\ (n+1)^{k-1} \\ \vdots \\ (n+1)^1 \\ (n+1)^0 \end{bmatrix} = \begin{bmatrix} 1 & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-1} & \binom{k}{k} \\ 0 & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-1} & \binom{k}{k} \\ 0 & 0 & \binom{k-1}{0} & \binom{k-1}{1} & \cdots & \binom{k-1}{k-2} & \binom{k-1}{k-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{1}{0} & \binom{1}{1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \binom{0}{0} \end{bmatrix} \cdot \begin{bmatrix} f(n) \\ n^k \\ n^{k-1} \\ \vdots \\ n^1 \\ n^0 \end{bmatrix}$$

Solving the recurrence,

$$\begin{bmatrix} f(n+1) \\ (n+1)^k \\ (n+1)^{k-1} \\ \vdots \\ (n+1)^1 \\ (n+1)^0 \end{bmatrix} = \begin{bmatrix} 1 & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-1} & \binom{k}{k} \\ 0 & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-1} & \binom{k}{k} \\ 0 & 0 & \binom{k-1}{0} & \binom{k-1}{1} & \cdots & \binom{k-1}{k-2} & \binom{k-1}{k-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{1}{0} & \binom{1}{1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \binom{0}{0} \end{bmatrix}^n \cdot \begin{bmatrix} f(1) \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

The binomial coefficients are pre-calculated. The exponent is calculated in $O(\log n)$ and each step a multiplication of matrix of size k takes k^3 time.
So total complexity $O(k^3 \log n)$.

And about **mod** 2^{32}. Taking **mod** 2^{32} is the same as taking last 32 digits of the number in binary.So for language like C/C++ that has 32-bit integer,just take an `unsigned int` which is just considering the 32-bits of the result. That is the same as **mod** 2^{32}. So no need of wasting time doing a **mod** 2^{32}.

1142 - Summing up Powers (II)

Description Let A be an $n \times n$ matrix. We define $A^k = A * A * ... * A$ (k times). Here, $*$ denotes the usual matrix multiplication. You are to write a program that computes the matrix $A + A^2 + A^3 + ... + A^k$.

My Solution Let

$$S(k) = A + A^2 + A^3 + \cdots + A^k$$

. Base case,$S(0) = \mathbf{0}$.

$$\begin{aligned} S(k) &= A + A^2 + A^3 + \cdots + A^k \\ &= A + A * (A + A^2 + \cdots + A^{k-1}) \\ &= A + A * S(k-1) \end{aligned}$$

Now this can be formulated in Matrix form as follows,

$$\begin{bmatrix} S(k) \\ I \end{bmatrix} = \begin{bmatrix} A & A \\ \mathbf{0} & I \end{bmatrix} \cdot \begin{bmatrix} S(k-1) \\ I \end{bmatrix}$$

Where I is Identity matrix.

Solving the recurrence,

$$\begin{bmatrix} S(k) \\ I \end{bmatrix} = \begin{bmatrix} A & A \\ \mathbf{0} & I \end{bmatrix}^k \cdot \begin{bmatrix} S(0) \\ I \end{bmatrix}$$

As $A, I, \mathbf{0}$ are constant value matrix Matrix Exponentiation can be applied to calculate the matrix power k in $O(\lg k)$ time.

Also in every step of the exponentiation, there wound be a $O(n^3)$,($n =$ the size of matrix A),time to multiply matrices.

So final complexity, $O(n^3 \lg k)$