Strange Correlator for 1D Fermionic Symmetry-Protected Topological Phases

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I. INTRODUCTION

Symmetry-protected topological (SPT) phases, including the topological insulators and topological superconductors, are short-range-entangled topological phases beyond Landau's paradigm. An important feature of SPT phases is the bulk-boundary correspondence: when the bulk is topologically nontrivial, its boundary will exhibit anomalous behaviors, including symmetry-protected gaplessness. (The boundary can also exhibit symmetry breaking or realize a gapped topological order with an anomalous symmetry enrichment in 2+1 dimensions.) Therefore, a gapless surface is usually used as a signature to detect a topologically nontrivial bulk. However, this signature is sometimes inconvenient or impossible to use: In numerical simulations such as exact diagonalization and quantum Monte Carlo, the extra computational cost of simulating a boundary (instead of the common practice of using periodic boundary conditions) is often unacceptable; it is impossible to design fully symmetric boundaries for SPT phases protected by crystalline or average symmetries. Therefore, it is desirable to have bulk-detection tools for nontrivial SPT phases.

Unlike free-fermion topological states, which can be detected using topological invariant (such as the Chern number and the Fu-Kane invariant) computed from the bulk band structure, the bulk detection of interacting SPT states is more challenging. In principle, SPT phases can be detected by evaluating the partition function on nontrivial space-time manifold with symmetry fluxes. However, this tool could be inconvenient to use in practice, due to the difficulties of generalzing a lattice model to curved manifolds, and evaluating the partition function for a realistic model. In one dimension, SPT states can be detected using symmetry-transformation properties of the Schmidt eigenstates of the entanglement Hamiltonian, but this approach is difficult to generalize to higher dimensions. Another useful tool for bulk detection is the strange correlator first proposed by You *et al.* [1]. Defined as a correlation function evaluated between two different SPT wave functions, it has proven to be a powerful bulk diagnostic tool for identifying non-trivial topological properties of symmetry protected topological phases, because it uses the ground state wave function which is easily obtained in numerical simulations, and it can be generalized to higher dimensions. Hence, the strange correlator will be the main focus of this work.

The main property of the strange correlator is that [1], for short-range entangled topological phases of matter there exist some on-site observables $\hat{O}(r), \hat{O}(r')$, such that it displays a power law decay (or saturates to a finite value) as a function of distance when the state under detection $|\Psi\rangle$ is a non-trivial topological state. Conversely, if both the detected state $|\Psi\rangle$ and the reference state $|\Omega\rangle$ are trivial, all the strange correlators decay exponentially in $|r - r'|$.

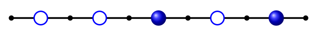
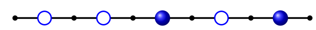
Indeed, previous research has primarily focused on the strange correlator of bosonic symmetry-protected topological (SPT) phases and its ability to detect such states [1–5]. The properties of the strange correlator in detecting SPT states have been extensively studied. However, there have been no investigation into the effectiveness of the strange correlator method in detecting fermionic symmetry-protected topological (FSPT) states. As a result, the availability and applicability of the strange correlator method in characterizing FSPT states remain unclear and require further exploration. Furthermore, it is worth noting that previous works on the strange correlator have often relied on specific examples to illustrate its properties in symmetry-protected topological states. However, a general proof or comprehensive theoretical framework for the strange correlator method is still lacking. While the strange correlator has shown promise in detecting certain features of symmetry-protected topological states, a more rigorous and systematic understanding of its underlying theory is needed. This would involve establishing a general framework and providing a theoretical foundation for the strange correlator method in characterizing a broader range of symmetry-protected topological states.

In this paper, we extensively investigate the strange correlator in 1D FSPT phases [6]. Our main contribution is a rigorous proof of its applicability to 1D FSPT phases using the transfer matrix method. We also propose a general method for constructing the FSPT strange correlator, considering the fermionic symmetry group's relationship to the bosonic symmetry group. Our work further supports the stability and validity of general SPT and FSPT models [6–20], highlighting the role of the strange correlator in detecting and characterizing FSPT states and enhancing our understanding of these theoretical frameworks.

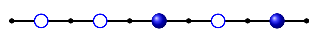
II. STRANGE CORRELATOR FOR 1D FSPT PHASES WITH $G_f = G_b \times \mathbb{Z}_2^f$

A. Fixed-Point Wavefunction for FSPT Phases in 1D

In this section, we briefly review the fixed-point wave function for a 1D FSPT state, which will be used later to compute the strange correlators. The general form of such wave function is given as a superposition over states labeled by group-element and complex-fermion decorations, [6]:

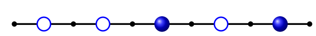
$|\Psi\rangle = \sum_{\text{all conf.}} \Psi\,($  $)\,|$  $\rangle. \qquad (1)$

Here, we first explain the set of states the wave function is built upon, then give the explicit expression of the superposition coefficients. In the 1D lattice, each vertex i (represented by a small black dot in the above equation) has a bosonic degree of freedom labeled by a group element $g_i \in G_b$. The center of each bond between nearest-neignbor vertices i and $i+1$ has a complex-fermion degree of freedom denoted by $c_{i,i+1}$ (represented by a big blue dot in the above equation). The Hilbert space we consider is a tensor product between the Fock space of these fermion modes, and the bosonic Hilbert space spanned by the basis states labeled by group-element configurations. In other words, the basis states are labeled by a fermion occupation number $n_{i,i+1}$ on each bond $(i, i+1)$, and a group element g_i on each vertex. Moreover, the phase factor in the superposition is also determined from the group-element decoration as the following,

$\Psi($  $) = \prod_i \nu_2(e, g_i, g_{i+1}), \qquad (2)$

where $\nu_2 \in H^2[G_b, \mathrm{U}(1)]$ is a 2-cocycle, and $e \in G_b$ is the identity element.

Therefore, a fixed-point wave function for a FSPT state labeled by n_1 and ν_2 can be expressed as:

$|\Psi\rangle = \sum_{g_k \in G_b} \prod_{\mathrm{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1})\,|$  $\rangle. \qquad (3)$

B. Strange Correlator for 1D FSPT Phases with $G_f = G_b \times \mathbb{Z}_2^f$

To detect the fermionic part FSPT phase, we use the complex-fermion annihilating operators $c_{i,i+1}$ as the local observables $O(r)$, and consider the following strange correlator,

$$C(i,j) = \frac{\langle \Omega | c_{i,i+1}, c_{j,j+1} | \Psi \rangle}{\langle \Omega | \Psi \rangle}, \tag{4}$$

where $|\Psi\rangle$, the wave function being tested, has the form of Eq. (41), and we choose $|\Phi\rangle$ to be a trivial FSPT wave function.

To compute this correlation function, we first expand the wave functions over the group-element-configuration basis,

$$C(i,j) = \frac{\sum_{\{g_i\}} \langle \{g_i\} | c_{i,i+1} c_{j,j+1} \nu_2(e, g_i, g_{i+1}) | \{n_1(g_i, g_{i+1})\}; \{g_i\} \rangle}{\sum_{\{g_i\}} \langle \{g_i\} | \nu_2(e, g_i, g_{i+1}) | \{n_1(g_i, g_{i+1})\}; \{g_i\} \rangle} \tag{5}$$

Because different Fock states are orthogonal, the overlap between two states is nonvanishing only when the fermion decoration in the bra and ket states is identical. Since the bra states $\langle \{g_i\}|$ has no fermion, the overlap in the numerator of (4) is nonvanishing if and only if $|\{n_1(g_i, g_{i+1})\}; \{g_i\}\rangle$ has exactly two fermions at bonds $(i, i+1)$ and $(j, j+1)$, and the overlap in the denominator of (4) is nonvanishing when $|\{n_1(g_i, g_{i+1})\}; \{g_i\}\rangle$ has no fermion. Therefore, we need to sum over subset of group elements such that $n_1(g_i, g_{i+1})$ is 0 and 1, respectively. In fact, when n_1 is a nontrival cocycle, G_b can be divided into two cosets, such that $n_1(g_i, g_{i+1}) = 0$ if g_i and g_{i+1} belong to the same coset, and $n_1(g_i, g_{i+1}) = 1$ if otherwise.

We notice that $n_1 \in H^1(G_b, \mathbb{Z}_2)$ can be interpreted as a group homomorphism $\rho : G_b \to \mathbb{Z}_2$, such that $n_1(g_i, g_{i+1}) = \rho(g_i^{-1} g_{i+1})$, or a $\mathbb{Z}_2$ grading of G_b. Therefore, the kernel of ρ forms a normal subgroup of G_b, which we denote by $G_b^0 = \rho^{-1}(0)$. When n_1 is trivial, ρ is also a trivial homomorphism that maps all elements to 0. Therefore, $G_b^0 = G_b$. In this case, it is straightforward to see that $C(i,j) = 0$, because there is no fermion in any configuration in $|\Psi\rangle$. When n_1 is nontrivial, ρ is surjective and $G_b/O = \mathbb{Z}_2$. Therefore, G_b can be divided into two cosets, one of which is G_b^0 and the other we denote by G_b^1. G_b^1 can be viewed as σG_b^0, where σ is an arbitrarily chosen element in G_b^1. Then, from the definition of ρ, it is straightforward to check that $n_1(g_i, g_{i+1}) = 0$ (or 1) if g_i and g_j belong to the same (different) cosets, respectively. Using this $\mathbb{Z}_2$ grading, the nonvanishing entries in the summation in Eq. (5) can be summarized as the following: In the denominator, all group elements must belong to the same coset; in the numerator, group elements on vertices in the interval between i and j must belong to one coset, and elements on vertices outside this interval must belong to the other coset.

Following Sec. ??, we can express the summation in Eq. (5) using the transfer matrix $T_{ij} = \nu_2(e, g_i, g_j)$. Since the summation involves the $\mathbb{Z}_2$ grading $G_b = G_b^0 \cup G_b^1$, we divide T into blocks: $T_{ij}^{ab} = \nu_2(e, \sigma^a g_i, \sigma^b g_j)$ (here $a, b = 0, 1$ denotes the grading, and $g_i, g_j \in G_b^0$). Using this notation, Eq. (5) is expressed as

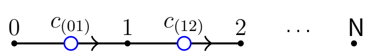
$$C(i,j) = \frac{\mathrm{tr}\left[T^{01}(T^{11})^{|i-j|}T^{10}(T^{00})^{N-|i-j|}\right]}{\mathrm{tr}\left[(T^{00})^N\right] + \mathrm{tr}\left[(T^{11})^N\right]} \tag{6}$$

Specifically, for all $k \neq i, j$, it requires $n_1(g_k, g_{k+1}) = 0$, while $n_1(g_i, g_{i+1}) = n_1(g_j, g_{j+1}) = 1$. As a result, the configurations g_k that contribute to the numerator of the strange correlator must satisfy the following conditions: $g_i^{-1}g_{i+1} \in O$, $g_j^{-1}g_{j+1} \in O$, and $g_k^{-1}g_{k+1} \in Z$ for all $k \neq i, j$. There are two cases to consider. First, if $g_0 \in Z$, then $g_{i+1}, ..., g_j \in O$ and $g_{j+1}, ..., g_i \in Z$. Alternatively, if $g_0 \in O$, then $g_{i+1}, ..., g_j \in Z$ and $g_{j+1}, ..., g_i \in O$.

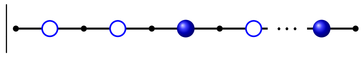
Note that $n_1(g_k, g_{k+1})$ is a Z_2-valued cochain with the property $n_1(g_k, g_{k+1}) = g \bullet n_1(g_k, g_{k+1}) = n_1(gg_k, gg_{k+1})$ [11]. Therefore, we can divide G_b into two non-intersecting sets $Z = \{g|n_1(e,g) = 0\}$ and $O = \{g|n_1(e,g) = 1\}$. Since $n_1(g_k, g_{k+1})$ is also a Z_2-valued cocycle with the property $\mathrm{d}n_1(g_0, g_1, g_2) = n_1(g_1, g_2) + n_1(g_0, g_2) + n_1(g_0, g_1) = 0 \pmod 2$, we can set $g_0 = e$ and obtain $\mathrm{d}n_1(e, g_1, g_2) = n_1(g_1, g_2) + n_1(e, g_2) + n_1(e, g_1) = 0 \pmod 2$. If g_1 and g_2 belong to subgroup Z, it follows that $g_1^{-1}g_2 \in Z$, $g_2^{-1}g_1 \in Z$, $g_1^{-1}g_1 = e \in Z$, $g_1^{-1}e = g_1^{-1} \in Z$, and $g_1 g_2 \in Z$. These properties establish that Z is a subgroup of G_b. Conversely, if g_1 and g_2 are elements of O, we observe that $g_1^{-1}g_2 \in Z$ and $n_1(g_1, g_2) = 0$. Furthermore, by examining the case where $g_1 \in O$ and $g_2 \in Z$, we find that $g_1 g_2 \in O$, implying that the cardinality of Z matches that of O, denoted as $|Z| = |O|$. Consequently, the sets Z and O partition G_b into two distinct cosets, with O being represented as σZ. (Our theory remains consistent with the fact that groups with odd order only have the trivial 1-cocycle n_1.).

According to periodic boundary conditions with $g_0 = g_N$, the numerator $\langle \Omega | c_{i,i+1} c_{j,j+1} | \Psi \rangle$ can be expressed as follows:

$$\sum_{g_k} \prod_{\mathrm{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \Big\langle$$



$$\Big| c_{i,i+1} c_{j,j+1} \Big|$$



$$\Big\rangle \tag{7}$$

$$= \sum_{g_{j+1},...,g_i \in Z} \sum_{g_{i+1},...,g_j \in O} \prod_{\mathrm{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) + \sum_{g_{j+1},...,g_i \in O} \sum_{g_{i+1},...,g_j \in Z} \prod_{\mathrm{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1})$$

To simplify the calculation, we can utilize the transfer matrix method. The numerator can be expressed as:

$$\langle \Omega | c_{i,i+1} c_{j,j+1} | \Psi \rangle = \mathrm{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \tag{8}$$

Here, T_{OZ}, T_Z, T_{ZO}, T_O are defined as follows: $T_Z(i,j) = \nu_2(e, g_i, g_j)$ for $g_i, g_j \in Z$, $T_O(i,j) = \nu_2(e, g_i, g_j)$ for $g_i, g_j \in O$, $T_{ZO}(i,j) = \nu_2(e, g_i, g_j)$ for $g_i \in Z, g_j \in O$, and $T_{OZ}(i,j) = \nu_2(e, g_i, g_j)$ for $g_i \in O, g_j \in Z$. By using the relationship $T_O = S^{-1} T_Z S$ (see Appendix B), where S is a diagonal matrix, we can rewrite the numerator as:

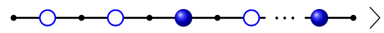
$$\langle \Omega | c_{i,i+1} c_{j,j+1} | \Psi \rangle = 2\nu_2(e, \sigma, e)\mathrm{tr}(\tilde{T} D^{|i-j|} \tilde{T} D^{N-|i-j|}) \tag{9}$$

Here, $D = A^{-1} T_Z A$ is the diagonalization of T_Z, and $A^{-1} T_{ZO} S^{-1} A = \nu_2(e, \sigma, e)^* A^{-1} S T_{OZ} A = \tilde{T}$. Furthermore, the denominator can be expressed as:

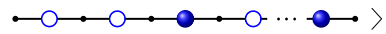
$$\langle \Omega | \Psi \rangle = \mathrm{tr}(T_Z^N) + \mathrm{tr}(T_O^N) \tag{10}$$

Under the assumption of periodic boundary conditions, we have $\langle \Omega | c_{i,i+1} | \Psi \rangle = 0$, implying that the correlation background is always zero. In Appendix B, it is demonstrated that in the thermodynamic limit ($N \longrightarrow \infty$), the strange correlator will converge to a constant, exhibit power-law decay, or oscillate. This observation aligns perfectly with the predictions of the strange correlation theory.

To detect the non-triviality of the bosonic part in a 1D FSPT state, we can choose a non-trivial reference state Eq. (11) with a trivial 2-cocycle $\nu_2(g_0, g_1, g_2) = 1$. However, since we have already determined the non-triviality of the fermionic part with $n_1(g_0, g_1)$, we can select $n_1(g_0, g_1)$ to be identical to that of the non-trivial state under consideration. This choice allows us to extract the non-trivial bosonic component of our 1D FSPT state.

$$|\Omega\rangle = \sum_{\text{all conf.}} |$$

$$\rangle \tag{11}$$

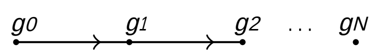
and the nontrivial FSPT state under measure is

$$|\Psi\rangle = \sum_{\text{all conf.}} \prod_{\text{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) |$$

$$\rangle. \tag{12}$$

By selecting appropriate detection operators $\hat{O}(r_i)$ and $\hat{O}(r_j)$ for the bosonic component, we can demonstrate that the peculiar strange correlator $C(r_i, r_j) = \frac{\langle\Omega|\hat{O}(r_i)\hat{O}(r_j)|\Psi\rangle}{\langle\Omega|\Psi\rangle}$ of our 1D FSPT state is exactly the traditional SPT strange correlator, which conforms to the general rules of strange correlation theory. Although the stability of SPT strange correlation theory has been frequently confirmed, in the next subsection, we will provide a rigorous proof based on the transfer matrix method. Consequently, we conclude that the strange correlator for 1D FSPT with trivial central extension $G_f = G_b \times \mathbb{Z}_2^f$ aligns with the general strange correlation theory.

C. Proof of Correctness of Strange Correlation Theory for 1D SPT Phases

As always, we assume periodic boundary conditions. The 1D symmetry-protected topological (SPT) state under an on-site symmetry G_b can be regarded as a 1D fermionic symmetry-protected topological (FSPT) state with $n_1(g_0, g_1) = 0$. Physically, this implies that there are no complex fermion decorations on each link $\langle i, j\rangle$. We can express the 1D SPT fixed-point state as follows:

$$|\Psi\rangle = \sum_{\text{all conf.}} \prod_{\text{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \Big|$$


$$\Big\rangle \tag{13}$$

With this peculiar form of the wave function, we can define a transfer matrix T such that $T_{ij} = \nu_2(e, g_i, g_j)$. This matrix is of size $|G_b| \times |G_b|$ and represents the peculiar SPT phase. Consequently, for the strange correlator, the denominator can be expressed as follows:

$$\langle\Omega|\Psi\rangle = \sum_{\text{all conf.}} \prod_{\text{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) = \mathrm{tr}(T^N) \tag{14}$$

Similarly, the numerator can be expressed as:

$$\langle\Omega|\hat{O}_i\hat{O}_j|\Psi\rangle = \mathrm{tr}(\hat{O}_i T^{|i-j|} \hat{O}_j T^{N-|i-j|}) \tag{15}$$

Here, $\hat{O}_i$ and $\hat{O}_j$ are operators acting on sites i and j respectively, and $|i - j|$ represents the distance between two sites.

In Appendix A, we will demonstrate that for any local observable $\hat{O}_i$, if the transfer matrix can be diagonalized, the strange correlator exhibits exponential decay only if there is a single dominant eigenvalue (where dominance is determined by the magnitude $|\lambda|$). Otherwise, the strange correlator will display power-law decay, oscillations, or gradually converge to a constant value.

On the other hand, we can prove that for a non-trivial symmetry-protected topological (SPT) state with a non-trivial 2-cocycle $\nu_2 \in H^2(G_b, U_T(1))$, the eigenvalues of the corresponding transfer matrix always exhibit degeneracy due to protected symmetry. Conversely, for a trivial SPT state with a trivial 2-cocycle $\nu_2 \in H^2(G_b, U_T(1))$, the dominant eigenvalue of the transfer matrix (except for certain extreme cases with accidental degeneracy) is always unique. Consequently, the strange correlator for a trivial SPT state will consistently display exponential decay, while for a non-trivial SPT state, the strange correlator may exhibit power-law decay, oscillations, or converge to a constant value. This observation perfectly aligns with the predictions of the strange correlation theory. Therefore, we have rigorously proven the correctness of the strange correlation theory for 1D FSPT states with $G_f = G_b \times \mathbb{Z}_2^f$.

III. STRANGE CORRELATOR FOR 1D FSPT PHASES WITH NON-TRIVIAL CENTRAL EXTENSION: $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$

A. Proof of Correctness of Strange Correlation Theory for 1D FSPT Phases with Non-trivial Central Extension

Now we consider the general case of 1D FSPT state with non-trivial central extension: $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$. According to Ref [6], 1D FSPT state $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$ is classified by the group super-cohomology theory [6]:

$$\begin{cases} n_1 \in H^1(G_b,\ Z_2), \\ \nu_2 \in C^2(G_b, U_T(1))/B^2(G_b, U_T(1))/\Gamma^2. \end{cases} \tag{16}$$

$$\begin{cases} n_1(gg_0, gg_1) = n_1(g_0, g_1) = n_1(g_0^{-1}g_1), \\ \nu_2(g, ga, gab) = ^g \nu_2(a,b) = \nu_2(a,b)^{1-2s_1(g)} \cdot (-1)^{(\omega_2 \smile n_1)(g,a,b)}. \end{cases} \tag{17}$$

$$\begin{cases} \mathrm{d}n_1 = 0, \\ \mathrm{d}\nu_2 = (-1)^{\omega_2 \smile n_1}. \end{cases} \tag{18}$$

$$\Gamma^2 = \{(-1)^{\omega_2} \in H^2(G_b, U_T(1))\}. \tag{19}$$

By FSLU transformation of 1D FSPT phases with $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$:

$$\Psi\left(\ 0 \overset{c_{(01)}}{\longrightarrow} 1 \overset{c_{(12)}}{\longrightarrow} 2\ \right) = F(g_0, g_1, g_2)\ \ \Psi\left(\ 0 \overset{c_{(02)}}{\longrightarrow} 2\ \right), \tag{20}$$

where the FSLU F operator is defined as

$$F(g_0, g_1, g_2) = \frac{1}{|G_b|^{1/2}} \nu_2(g_0, g_1, g_2) \big(c_{01}^{g_0}\big)^{\dagger n_1(g_0,g_1)} \big(c_{12}^{g_1}\big)^{\dagger n_1(g_1,g_2)} \big(c_{02}^{g_0}\big)^{n_1(g_0,g_2)} \tag{21}$$

we can construct 1D FSPT wavefunction in the same way

$$|\Psi\rangle = \prod_{\mathrm{k}=1}^{N-1} \nu_2(g_0, g_k, g_{k+1}) \Big|\ 0 \overset{c_{(01)}}{\longrightarrow} 1 \overset{c_{(12)}}{\longrightarrow} 2 \quad \cdots \quad \mathrm{N}\ \Big\rangle. \tag{22}$$

As usual, we can choose complex fermion annihilating operators as local observables. However, in this case, due to the onsite symmetry of the system, the complex fermion annihilating operators take on a slightly different form [6], which are $\tilde{c}_{i,i+1} = \sum_{g_i \in G_b} c_{i,i+1}^{g_i}$ and $\tilde{c}_{j,j+1} = \sum_{g_j \in G_b} c_{j,j+1}^{g_j}$. We use their strange correlator to detect the non-triviality of fixed point state $|\Psi\rangle$.

Notice that the 1D FSPT wave function with $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$ takes on the same form as that with $G_f = G_b \times \mathbb{Z}_2^f$, but with the additional constraint that $\nu_2(g_0, g_1, g_2)$ and $n_1(g_0, g_1)$ are subject to the obstruction function $\mathrm{d}\nu_2 = (-1)^{\omega_2 \smile n_1}$. However, although $\nu_2 \in BH^2(G_b, Z_2)$ is constrained by the obstruction function, the obstruction-free $n_1 \in H^1(G_b, Z_2)$ still satisfies the cocycle conditions $\mathrm{d}n_1 = 0$. Hence, we can still partition G_b into two distinct cosets: $Z = \{g | n_1(e, g) = 0\ n_1 \in H^1(G_b, Z_2)\}$ and $O = \sigma Z = \{g | n_1(e, g) = 1\ n_1 \in H^1(G_b, Z_2)\}$.

As mentioned earlier, the term $\Big\langle\ 0 \overset{c_{(01)}}{\longrightarrow} 1 \overset{c_{(12)}}{\longrightarrow} 2 \quad \cdots \quad \mathrm{N}\ \Big| \tilde{c}_{i,i+1}\tilde{c}_{j,j+1} \Big| \cdots \Big\rangle \neq 0$ only if $n_1(g_k, g_{k+1}) = 0$ for all $k \neq i, j$ and $n_1(g_i, g_{i+1}) = n_1(g_j, g_{j+1}) = 1$. This implies that $g_i^{-1}g_{i+1}, g_j^{-1}g_{j+1} \in O$, and $g_k^{-1}g_{k+1} \in Z$ for all $k \neq i, j$. There are two situations to consider. If $g_0 \in Z$, then $g_{i+1}, ..., g_j \in O$ and $g_{j+1}, ..., g_i \in Z$. Conversely, if $g_0 \in O$, then $g_{i+1}, ..., g_j \in Z$ and $g_{j+1}, ..., g_i \in O$.

To calculate the numerator $\langle \Omega | \tilde{c}_{i,i+1}\tilde{c}_{j,j+1} | \Psi \rangle$, we consider periodic boundary conditions with $g_0 = g_N$. The expression can be written as follows:

$$\begin{aligned} &\sum_{g_k} \prod_{\mathrm{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \Big\langle\ 0 \overset{c_{(01)}}{\longrightarrow} 1 \overset{c_{(12)}}{\longrightarrow} 2 \quad \cdots \quad \mathrm{N}\ \Big| \tilde{c}_{i,i+1}\tilde{c}_{j,j+1} \Big| \cdots \Big\rangle \\ &= \sum_{g_{j+1},...,g_i \in Z} \sum_{g_{i+1},...,g_j \in O} \prod_{\mathrm{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) + \sum_{g_{j+1},...,g_i \in O} \sum_{g_{i+1},...,g_j \in Z} \prod_{\mathrm{k}=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \end{aligned} \tag{23}$$

To simplify the calculation, we can use the transfer matrix method again. The numerator can be expressed as:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle = \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{24}$$

Here, T_{OZ}, T_Z, T_{ZO}, T_O are defined as follows: $T_Z(i,j) = \nu_2(e, g_i, g_j)$ for $g_i, g_j \in Z$, $T_O(i,j) = \nu_2(e, g_i, g_j)$ for $g_i, g_j \in O$, $T_{ZO}(i,j) = \nu_2(e, g_i, g_j)$ for $g_i \in Z, g_j \in O$, and $T_{OZ}(i,j) = \nu_2(e, g_i, g_j)$ for $g_i \in O, g_j \in Z$. By using the relationship $T_O = S^{-1}T_Z S$(see Appendix B), where S is a diagonal matrix, we can rewrite the numerator as:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle = \mathrm{tr}(\tilde{T}_{ZO}D^{|i-j|}\tilde{T}_{OZ}D^{N-|i-j|}) + \mathrm{tr}(\tilde{T}_{OZ}D^{|i-j|}\tilde{T}_{ZO}D^{N-|i-j|}) \tag{25}$$

Here, $D = A^{-1}T_Z A$ is the diagonalization of T_Z, and $\tilde{T}_{ZO} = A^{-1}T_{ZO}S^{-1}A$ and $\tilde{T}_{OZ} = A^{-1}ST_{OZ}A$. Furthermore, the denominator can be expressed as:

$$\langle\Omega|\Psi\rangle = \mathrm{tr}(T_Z^N) + \mathrm{tr}(T_O^N) \tag{26}$$

Under the assumption of periodic boundary conditions, we have $\langle\Omega|\tilde{c}_{i,i+1}|\Psi\rangle = 0$, implying that the correlation background is always zero. In Appendix B, it is demonstrated that in the thermodynamic limit ($N \longrightarrow \infty$), the strange correlator will converge to a constant, exhibit power-law decay, or oscillate. These findings are in perfect agreement with the predictions of the strange correlation theory. Moreover, if n_1 is trivial, the problem reverts back to the SPT strange correlation theory. Consequently, we can confidently conclude that the strange correlation theory remains applicable to 1D fermionic symmetry-protected topological (FSPT) states with non-trivial central extensions.

B. Classification of 1D FSPT Phases of Non-trivial Central Extension By Strange Correlator

Having established the validity of the strange correlation theory for 1D FSPT (Fermionic Symmetry Protected Topological) phases with non-trivial central extensions, we can now proceed to further classify the bosonic part of a 1D FSPT state with a non-trivial central extension. To do so, we can choose a non-trivial reference state as described in Equation (27), with non-trivial values assigned to $n_1(g_0, g_1)$ and $\nu_2(g_0, g_1, g_2)$. However, since we have already determined the non-triviality of the fermionic part $n_1(g_0, g_1)$, we can select $n_1(g_0, g_1)$ to be identical to that of the non-trivial state under consideration. This choice allows us to extract and classify the non-trivial bosonic component of our 1D FSPT state.

$$|\Omega\rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu_2'(e, g_k, g_{k+1}) \sum_{\text{all conf.}} |\; \cdot\!-\!\circ\!-\!\cdot\!-\!\circ\!-\!\cdot\!-\!\bullet\!-\!\cdot\!-\!\circ\cdots\bullet\!-\!\cdot\;\rangle \tag{27}$$

and the nontrivial FSPT state under measure is

$$|\Psi\rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu_2''(e, g_k, g_{k+1})|\; \cdot\!-\!\circ\!-\!\cdot\!-\!\circ\!-\!\cdot\!-\!\bullet\!-\!\cdot\!-\!\circ\cdots\bullet\!-\!\cdot\;\rangle. \tag{28}$$

With this peculiar form of the wave function, we can define a transfer matrix T such that $T_{ij} = \nu_2''(e, g_i, g_j)\nu_2'^{*}(e, g_i, g_j) = \nu_2(e, g_i, g_j)$. Therefore, for the strange correlator, the denominator can be expressed as follows:

$$\langle\Omega|\Psi\rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) = \mathrm{tr}(T^N) \tag{29}$$

Similarly, the numerator can be expressed as:

$$\langle\Omega|\hat{O}_i\hat{O}_j|\Psi\rangle = \mathrm{tr}(\hat{O}_i T^{|i-j|}\hat{O}_j T^{N-|i-j|}) \tag{30}$$

Here, $\hat{O}_i$ and $\hat{O}_j$ are operators acting on sites i and j respectively, and $|i-j|$ represents the distance between two sites.

In Appendix C, we demonstrate that if $\nu_2''(e, g_i, g_j) \neq \nu_2'(e, g_i, g_j)\nu_1(g_i, g_j)$, indicating that $\nu_2(e, g_i, g_j)$ is a non-trivial element of group supercohomology $BH^2(G_b, U_T(1))$, the eigenvalues of the corresponding transfer matrix will always exhibit degeneracy due to protected symmetry. On the other hand, if $\nu_2''(e, g_i, g_j)$ and $\nu_2'(e, g_i, g_j)$ represent the same phase, the dominant eigenvalue of the transfer matrix (except for certain extreme cases with accidental degeneracy) will be unique. Consequently, when the detected state is in the same phase as the reference state, the strange correlator will exhibit exponential decay. This behavior arises due to the compatibility of the fermionic and bosonic components in the same phase. However, when the detected state belongs to a different phase than the reference state, the strange correlator may display distinct behaviors such as power-law decay, oscillations, or convergence to a constant value. These diverse behaviors arise from the mismatch between the fermionic and bosonic components in different phases. Therefore, by analyzing the behavior of the strange correlator, we can effectively classify 1D FSPT states with a non-trivial central extension $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$. This classification allows us to distinguish between different phases based on the distinct patterns exhibited by the strange correlator.

IV. SPECIAL CASE OF STRANGE CORRELATOR FOR 1D FSPT STATES

A. $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$

We start with a simple case with $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$. According to Ref.:

$$H^2[Z_2^T, U_T(1)] = Z_2, \tag{31}$$

$$H^1(Z_2^T, Z_2) = Z_2 \tag{32}$$

It is easily checked that:

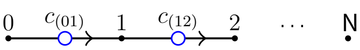
$$n_1(e,e) = n_1(T,T) = 0 \quad n_1(e,T) = n_1(T,e) = 1 \tag{33}$$

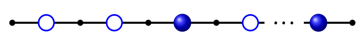
$$\mathrm{d}n_1 = 0 \ (mod 2) \tag{34}$$

$$\nu_2(e,e,e) = \nu_2(T,T,T) = \nu_2(e,T,T) = \nu_2(T,e,e) = \nu_2(e,e,T) = \nu_2(T,T,e) = 1 \tag{35}$$

$$\nu_2(e,T,e) = \nu_2(T,e,T) = -1 \tag{36}$$

$$\mathrm{d}\nu_2 = 1 \tag{37}$$

Suppose periodic boundary condition, there are only two configurations such that
$\Big\langle$ 

$\Big|\, c_{i,i+1}c_{j,j+1} \,\Big|$  $\Big\rangle \neq 0$. They are: (1) $g_{i+1}\dots g_j = T$ and $g_k = e$ for $k \neq i+1\dots j+1$, and (2) $g_{i+1}\dots g_{j+1} = e$ and $g_k = T$ for $k \neq i+1\dots j+1$. In this case, we can choose the total trivial reference state with $n_1 = 0$ and $\nu_2 = 1$, and use the complex fermion annihilation operators $c_{i,i+1}$ and $c_{j,j+1}$ as our local observables. With these choices, we can proceed to compute the strange correlator:

$$C(i,j) = \frac{\langle \Omega | c_{i,i+1}, c_{j,j+1} | \Psi \rangle}{\langle \Omega | \Psi \rangle} = -1 \tag{38}$$

The result obtained manifest the non-triviality of the fermionic part. Next, we aim to prove the non-triviality of the bosonic part. To accomplish this, we select a non-trivial reference state where the 1-cocycle n_1 is the same as that of the state under measurement. We consider the order parameters O_i and O_j as our local observables. With these choices, we can compute the strange correlator as follows:

$$C(i,j) = \frac{\langle \Omega | O_i O_j | \Psi \rangle}{\langle \Omega | \Psi \rangle} = \begin{cases} 1 & (|i-j| \bmod 4 < 2) \\ -1 & (|i-j| \bmod 4 > 1) \end{cases} \tag{39}$$

This result demonstrates oscillation, which is consistent with the predictions of the strange correlation theory. Additionally, if we calculate the eigenvalues of the transfer matrix for the $G_b = Z_2^T$ group, we find that they are $1+i$ and $1-i$, respectively, indicating a two-fold degeneracy. This observation is consistent with the order of $\nu_2 \in H^2(G_b, U_T(1)) = Z_2$, which profoundly explains why the strange correlator of the non-trivial SPT state of $G_b = Z_2^T$ exhibits oscillations.

It is important to note that even when considering the trivial reference state, the computation of the strange correlator using the complex fermion annihilation operators $c_{i,i+1}$ and $c_{j,j+1}$ results in a constant value. Appendix B further reveals that this property can be generalized to more general cases.

B. $G_f = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2^f$

Following the general procedure to construct strange correlator for 1D FSPT state with trivial central extension, however, this time we choose a nontrivial reference state with $n_1(g_0, g_1) = 0$, a trivial cocycle, and $\nu_2(g_0, g_1, g_2)$ the same as that of the nontrivial state under measure. Using the complex fermion annihilating operators $c_{i,i+1}$ and $c_{j,j+1}$ as our local observables, the strange correlator $C(i,j)=\frac{\langle\Omega|c_{i,i+1},c_{j,j+1}|\Psi\rangle}{\langle\Omega|\Psi\rangle}$ is always equal to 1. This result indicates the nontriviality of the fermionic part of our 1D FSPT.

Conversely, to detect the non-triviality of the bosonic part of 1D FSPT state, we could choose a non-trivial reference state with 1-cocycle n_1 same as that of the state under measure and $\nu_2=1$, then it turns into a traditional SPT strange correlator. By previous calculations, we determined that $H^2(Z_4\times Z_4,U_T(1))=Z_4$, indicating the presence of one non-trivial cocycle with order 2 and two other non-trivial cocycles with order 4, representing the non-trivial SPT state. Specifically, we find that $\nu_2(e,a,ab)=(-1)^{a_1b_2}$, where $a=(a_1,a_2)$ with $a_1,a_2\in Z_4$, is a non-trivial cocycle with order 2. Additionally, $\nu_2(e,a,ab)=i^{a_1b_2}$ and $\nu_2(e,a,ab)=(-i)^{a_1b_2}$ correspond to the other two non-trivial cocycles with order 4. Upon verification, we observe that the transfer matrix of the non-trivial cocycle of order 2 possesses two non-zero eigenvalues: $\lambda_1=8$ and $\lambda_2=8$. Similarly, the transfer matrix of the other two non-trivial cocycles of order 4 possesses four non-zero eigenvalues: $\lambda_1=4$, $\lambda_2=4$, $\lambda_3=4$, and $\lambda_4=4$. The degeneracy of the eigenvalues perfectly aligns with the order of the corresponding non-trivial cocycles representing the transfer matrix. Therefore, according to the theorem established in Appendix A, we can confidently conclude that the strange correlator of the bosonic part of the $G_f=\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2^f$ group will either converge to a constant, exhibit power-law decay, or oscillate. Therefore the decaying behaviors of strange correlator with $G_f=\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2^f$ is consistent with our theoretical expectations.

C. $G_f=\mathbb{Z}_{2N_0}^f\times\prod_{i=1}^K\mathbb{Z}_{N_i}$

We now consider the case where the symmetry group G_f is a central extension of the bosonic unitary finite Abelian group $G_b=Z_{N_0}\times\prod_{i=1}^K Z_{N_i}$ [6, 21, 22]. In this case, we have $\omega_2(e,a,ab)=\left\lfloor\frac{a^0+b^0}{N_0}\right\rfloor$, where $\lfloor x\rfloor$ is the floor operator that picks out the largest integer less than or equal to x.

It is well-known that for a given positive integer N, there exists a unique factorization $N=\prod_p p^{n_p}$ where p is a prime number and n_p is a positive integer in $\mathbb{Z}_+$. Furthermore, there is a group isomorphism $\mathbb{Z}_N\cong\prod_p\mathbb{Z}_{p^{n_p}}$. For prime numbers $p>2$, the cohomology group $H^*(\mathbb{Z}_{p^{n_p}},\mathbb{Z}_2)$ is trivial. Therefore, the symmetry group $\prod_{p>2}\mathbb{Z}_{p^{n_p}}$ can only protect bosonic SPT phases and can only affect the FSPT classifications by adding some BSPT phases. To understand genuine FSPT, we can assume $N_i=2^{k_i}$ for $1\leqslant i\leqslant K$, where $k_i\in\mathbb{Z}+$, in the bosonic symmetry group G_b . Without loss of generality, we can also reorder the Abelian groups such that $N_i\leqslant N_{i+1}$ for $1\leqslant i\leqslant K-1$.

We can assume without loss of generality that $N_0=2^{k_0}$ is the k_0-th power of 2 ($k_0\geqslant 1$). Otherwise, $Z_{2N_0}^f$ is isomorphic to $Z_{2^{k_0+1}}^f\times Z_{N_0/2^{k_0}}$, and we can absorb the latter subgroup into Z_{N_i} with $i>0$. It is worth noting that $n_2^{(0)}$ is the nontrivial Z_2-valued 2-cocycle of Z_{N_0}, rather than that of Z_{2N_0}. This is different from $n_2^{(i)}$ for $1\leqslant i\leqslant K$.

Using the ingredients above, we can calculate the obstruction equation $\mathrm{d}\nu_2=(-1)^{\omega_2\smile n_1}$. Assuming $n_1=n_1^i$, we can prove that the right-hand side is obstruction-free if and only if $N_i>N_0$ or equivalently $k_i>k_0$. After a laborious and intricate calculation, we obtain the explicit form of $\nu_2(g_0,g_1,g_2)$:

$$\nu_2(g_0,g_1,g_2)=e^{i\frac{\pi}{N_0}n_1^0(g_0,g_1)n_1^i(g_1,g_2)},\tag{40}$$

where $n_1^j(e,g)=g^j\ mod\ 2$ and $g=(g^0,g^1,g^2,\ldots,g^K)$.

As mentioned previously, for G_f as a central extension of the bosonic unitary finite Abelian group G_b with $\omega_2\neq 0$, we select the complex fermion annihilating operators $\tilde{c}_{i,i+1}=\sum_{g_i\in G_b}c_{i,i+1}^{g_i}$ and $\tilde{c}_{j,j+1}=\sum_{g_j\in G_b}c_{j,j+1}^{g_j}$ as local observables in the system and use their strange correlator to detect the non-triviality of the system. After performing a detailed and intricate calculation, we obtain the following expression for the strange correlator:

$$C(i,j)=\frac{\langle\Omega|\tilde{c}_{i,i+1},\tilde{c}_{j,j+1}|\Psi\rangle}{\langle\Omega|\Psi\rangle}=\frac{\sum_{m=0}^{2}\binom{2}{m}e^{i(m\pi/N_0)}}{2^2}=Const,\tag{41}$$

where $\binom{N}{m}$ denotes the combination coefficient. This result confirms that when using complex fermion annihilation operators as detectors, the strange correlator of the 1D FSPT state with symmetry group $G_f=\mathbb{Z}_{2N_0}^f\times\prod_{i=1}^K\mathbb{Z}_{N_i}$ is a constant. This finding is in perfect agreement with the theory of strange correlations, as it supports the notion that the strange correlator remains constant when the detected state and the reference state are in the same phase.

V. CONCLUSION AND DISCUSSION

In this study, we focused on examining the scaling behavior of the strange correlator in 1D fermionic symmetry-protected topological (FSPT) states which possess symmetries characterized by $G_f = G_b \times \mathbb{Z}_2^f$ and $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$, where G_f represents the fermionic symmetry and G_b represents the bosonic symmetry.Through rigorous analysis, we established that the properties of the strange correlator remain valid within 1D FSPT phases. Furthermore, we provided specific examples that demonstrate the existence of both constant and oscillating strange correlators under certain conditions. However our study only focused on the 1D case, and the availability of the FSPT strange correlator still require further investigation, especially in higher-dimensional systems which possess more degrees of freedom, such as the Kitaev Majorana chain and $p + ip$ superconductor decorations.

Despite the limitations of our study, it holds significant importance as it provides a rigorous and systematic proof of the applicability of the strange correlation theory in both 1D symmetry-protected topological (SPT) states and 1D fractionalized symmetry-protected topological (FSPT) states. This work not only reaffirms the stability of the strange correlation theory but also validates the general SPT model and the subsequent FSPT model, which serves as a crucial link between the two theories for studying symmetry-protected topological states. Additionally, we have found a special property that the eigenvectors of transfer matrix of 1D SPT states carry the projective representation of the local symmetry group G_b, which are very likely to be promoted to research on higher dimensional SPT state.

In our future work, we will primarily focus on investigating the stability of the strange correlator in 2D FSPT states. This presents new challenges, as the availability of complex fermion annihilation operators might not be as straightforward, and the presence of a Majorana chain decoration [6, 23–30] introduces an additional layer of complexity. By addressing these complexities, we aim to deepen our understanding of the behavior and properties of the strange correlator in higher-dimensional FSPT states.

In conclusion, we hope that our work inspires further research in this field and brings us closer to a complete understanding of the topological properties of quantum many-body systems.

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Appendix A: Proof of Correctness of 1D SPT Strange Correlation Theory

1. Proof of Strange Correlation theory under Transfer Matrix Degeneracy

In this section, we present a rigorous proof of the availability of the 1D symmetry-protected topological (SPT) strange correlator using the transfer matrix method.

For any finite symmetry group G_b, we can label the group elements as a sequence $g_1, g_2, ..., g_M$, with g_1 representing the unit element of the group G_b. The transfer matrix can then be expressed as $T_{ij} = \nu_2(e, g_i, g_j)$, where ν_2 denotes the 2nd cohomology group. We assume that the transfer matrix can be diagonalized as $A^{-1}TA = D$, where D is the diagonalized form of T. To compute the numerator of the strange correlator $\langle\Omega|\hat{O}_i\hat{O}_j|\Psi\rangle$, we transform it into $\langle\Omega|\hat{O}_i\hat{O}_j|\Psi\rangle = \mathrm{tr}(\hat{O}'_i D^{|i-j|}\hat{O}'_j D^{N-|i-j|})$, where $\hat{O}'_i = A^{-1}\hat{O}_i A$. The matrix $Y = \hat{O}'_i D^{|i-j|}\hat{O}'_j D^{N-|i-j|}$ can be expressed as:

$$Y = \begin{pmatrix} O'_{11}\lambda_1^L & O'_{12}\lambda_2^L & ... & O'_{1M}\lambda_M^L \\ O'_{21}\lambda_1^L & O'_{22}\lambda_2^L & ... & O'_{2M}\lambda_M^L \\ ... \\ O'_{M1}\lambda_1^L & O'_{M2}\lambda_2^L & ... & O'_{MM}\lambda_M^L \end{pmatrix} \begin{pmatrix} O'_{11}\lambda_1^{N-L} & O'_{12}\lambda_2^{N-L} & ... & O'_{1M}\lambda_M^{N-L} \\ O'_{21}\lambda_1^{N-L} & O'_{22}\lambda_2^{N-L} & ... & O'_{2M}\lambda_M^{N-L} \\ ... \\ O'_{M1}\lambda_1^{N-L} & O'_{M2}\lambda_2^{N-L} & ... & O'_{MM}\lambda_M^{N-L} \end{pmatrix} \tag{A1}$$

Here, we assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant |\lambda_M|$.

Now, we consider two situations. First, if $|\lambda_1| > |\lambda_2|$, then under the thermodynamic limit ($N \to \infty$), the strange correlator can be expressed as ($L = |i - j|$):

$$C(i,j) = \frac{\mathrm{tr}(Y)}{\mathrm{tr}(T^N)} = O'^2_{11} + O'_{12}O'_{21}\left(\frac{\lambda_2}{\lambda_1}\right)^L + \ldots + O'_{1M}O'_{M1}\left(\frac{\lambda_M}{\lambda_1}\right)^L \tag{A2}$$

Additionally, we need to account for the background, which is expressed as:

$$\langle \hat{O}_i \rangle = \frac{\mathrm{tr}(\hat{O}T^N)}{\mathrm{tr}(T^N)} = O'_{11} \tag{A3}$$

Thus, the strange correlator without the background can be expressed as:

$$\langle \hat{O}_i \hat{O}_j \rangle - \langle \hat{O}_i \rangle \langle \hat{O}_j \rangle = O'_{12} O'_{21} \left(\frac{\lambda_2}{\lambda_1} \right)^L + \ldots + O'_{1M} O'_{M1} \left(\frac{\lambda_M}{\lambda_1} \right)^L \tag{A4}$$

Clearly, if $|\lambda_1| > |\lambda_2|$, the strange correlator will always exhibit exponential decay.

Secondly, if the eigenvalues exhibit a p-fold degeneracy such that $|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_p|$ $(1 < p \leqslant M)$, and we let $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$ $(1 \leqslant i < p)$, then the strange correlator can be expressed as:

$$\langle \hat{O}_i \hat{O}_j \rangle - \langle \hat{O}_i \rangle \langle \hat{O}_j \rangle = \frac{\sum_{i,j=1}^{p} O'_{ij} O'_{ji} e^{i\theta_i (N-L)} e^{i\theta_j L}}{1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \ldots + e^{i\theta_p N}} - \left(\frac{O'_{11} + O'_{22} e^{i\theta_2 N} + \ldots + O'_{pp} e^{i\theta_p N}}{1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \ldots + e^{i\theta_p N}} \right)^2 \tag{A5}$$

The behavior of this expression depends on the values of N, L, and the phases θ_i. In general, it converges to a constant, presents power law decay or oscillate. The resulting behavior depends on the specific values of θ_i and θ_j. In summary, when the eigenvalues exhibit a p-fold degeneracy, the strange correlator can converge to a constant, exhibit power-law decay, or oscillate, depending on the values of N, L and the phases θ_i.

2. Degeneracy of Transfer Matrix

Thereafter, our objective is to establish that when a non-trivial 2-cocycle ν_2, which labels the non-trivial SPT phase, the corresponding transfer matrix T, defined as $T_{ij} = \nu_2(e, g_i, g_j)$, exhibits p-fold degeneracy, in which p strongly correlated with the order of ν_2 in the cohomology group $H^2(G_b, U_T(1))$.

a. *Local Symmetry Group G_b without Time Reversal Symmetry*

Firstly, suppose the local symmetry does not include time reversal symmetry. In this case, for any group element $g \in G_b$, the following relation based on the cocycle equation holds:

$$\nu_2(g, g_i, g_j) = \nu_2(e, g_i, g_j) \frac{\nu_2(e, g, g_i)}{\nu_2(e, g, g_j)} \tag{A6}$$

Let's introduce the notations $T^g_{ij} = \nu_2(g, g_i, g_j)$ and $S(g)_{ij} = \delta_{ij} \nu_2(e, g, g_j)$. We can then express their relationship as:

$$T^g = S(g) T S(g)^{-1} \tag{A7}$$

Here, T^g represents the matrix of 2-cocycle values for the group element g, while $S(g)$ is a matrix encoding the 2-cocycle values of the identity element e combined with g. The equation shows how T^g is related to $S(g)$ through matrix multiplication and inversion.

By the property of the 2-cochain, we can write $\nu_2(g, g_i, g_j) = \nu_2(e, g^{-1} g_i, g^{-1} g_j)$. As a result, we observe that the transferred matrix T^g is a rearrangement of the rows and columns of the original matrix T. Consequently, there always exists an invertible matrix $R(g)$ such that $R(g) T^g R(g^{-1}) = T$. Essentially this matrix $R(g)$ is the regular representation of local symmetry group G_b, and we can express the relationships as follows:

$$R(g) T R(g^{-1}) = S(g) T S(g)^{-1} \tag{A8}$$

$$S(g)^{-1} R(g) T = T S(g)^{-1} R(g) \tag{A9}$$

If ν_2 is a coboundary, it satisfies the property $\nu_2(e, g_i, g_j) = \nu_1(g_i, g_j) \frac{\nu_1(e, g_i)}{\nu_1(e, g_j)}$. In this case, we have the relationship:

$$T = S' T' S'^{-1} \tag{A10}$$

where $T'_{ij} = \nu_1(g_i, g_j)$ and $S'_{ij} = \delta_{ij} \nu_1(e, g_i)$. Here, ν_1 is a 1-cochain that satisfies $\nu_1(g g_i, g g_j) = \nu_1(g_i, g_j)$.

Now, let's consider two situations. First, if the local symmetry group G_b is an Abelian group, it only has irreducible representations of order 1. Therefore, $R(g)$ can be reduced to the direct sum of one-dimensional irreducible representations. According to the Ref [31], since T' and $R(g)$ are commutative and both diagonalizable matrices, they can be simultaneously diagonalized. Consequently, the eigenvalues of matrix T' are generally non-degenerate, except for any accidental degeneracy due to intrinsic symmetries. Second, if the local symmetry group G_b is non-Abelian, the regular representation can only be block diagonalized. However, the first block always representing the trivial representation of G_b is always one-dimensional. Similarly, since T' and $R(g)$ commute and T' is diagonalizable, they can be simultaneously block diagonalized. The degeneracy of the eigenvalues of T' corresponds to the size of the diagonalized blocks. However, it is important to note that every group has the irreducible trivial representation of order 1, which means there is always a non-degenerate eigenvalue of the transfer matrix T, which is typically the dominant one. Additionally, if the largest eigenvalue of T' exhibits any accidental degeneracy, it can usually be attributed to the presence of intrinsic symmetries, which are generally unavoidable. Therefore, we can conclude that the largest eigenvalue of matrix T' is generally non-degenerate, which is consistent with the expectation that trivial SPT phases do not possess long-range degeneracy.

If ν_2 is non-trivial, we claim that $S(g)^{-1}R(g)$ represents a projective representation of the local symmetry group G_b. Let's prove this claim:

$$S(g)^{-1}R(g)S(h)^{-1}R(h) \tag{A11}$$
$$= \delta_{ij}\nu_2(e,g,g_i)^{-1}\nu_2(e,h,g^{-1}g_i)^{-1}R(gh) \tag{A12}$$
$$= \delta_{ij}\nu_2(e,g,g_i)^{-1}\nu_2(g,gh,g_i)^{-1}R(gh) \tag{A13}$$
$$= \delta_{ij}\nu_2(e,g,gh)^{-1}\nu_2(e,gh,g_i)^{-1}R(gh) \tag{A14}$$
$$= \nu_2(e,g,g_i)^{-1}S(gh)^{-1}R(gh) \tag{A15}$$

Therefore, $S(g)^{-1}R(g)S(h)^{-1}R(h) = \nu_2(e,g,g_i)^{-1}S(gh)^{-1}R(gh)$. This implies that $S(g)^{-1}R(g)$ represents a projective representation of the group G_b.

On the other hand, projective representations do not have one-dimensional irreducible representations. Therefore, the projective representation $S(g)^{-1}R(g)$ could only be block diagonalized, with each block having a size greater than one. According to Ref [31], since we assumed that the transfer matrix T is diagonalizable and T commutes with $S(g)^{-1}R(g)$, we can conclude that T and the projective representation $S(g)^{-1}R(g)$ can be simultaneously block diagonalized with each block siize bigger than one. Consequently, the eigenvalues of the transfer matrix T generally possess degeneracy.

b. *Local Symmetry Group G_b with Time Reversal Symmetry*

Now let's consider the scenario where time reversal symmetry exists. In this case, we can separate the local symmetry group into two cosets: G_b^0 and G_b^T, where G_b^0 does not contain any time-reversal elements, and G_b^T consists of the time-reversal elements. It can be easily verified that G_b^0 is a subgroup of the local symmetry group G_b.

In the scenario where time reversal symmetry exists, we still maintain the relation $S(g^0)^{-1}R(g^0)A = AS(g^0)^{-1}R(g^0)$, where g^0 is an arbitrary element of G_b^0, and $R(g^0) = \text{diag}[r(g^0), r(g^0)]$. Here, $r(g^0)$ represents the regular representation of the subgroup G_b^0. As a result, the eigenvectors of the transfer matrix A exhibit a projective representation of the subgroup G_b^0. In the subsequent analysis, we will explore the degeneracy of the transfer matrix A based on the specific 2-cocycle ν_2 chosen for our analysis.

First, if ν_2 is a coboundary, it is known from previous analysis that our transfer matrix A is similar to a modified transfer matrix A', where $A'_{ij} = \nu_1(g_i, g_j)$. To simplify the notation, let's define the elements of the subgroup as $g_i^0 \in G_b^0$. Based on this, we can establish the following relations:

$$A'R(g^0) = R(g^0)A' \tag{A16}$$
$$A'^*R(g^0T) = R(g^0T)A' \tag{A17}$$

Additionally, it can be verified that:

$$R(T) = \begin{pmatrix} O & I \\ I & O \end{pmatrix} \tag{A18}$$
$$R(g^0) = \begin{pmatrix} r(g^0) & O \\ O & r(g^0) \end{pmatrix} \tag{A19}$$

As a result, by combining the aforementioned relationships, we can partition our transfer matrix into four block matrices with the following property:

$$A' = \begin{pmatrix} A_1' & A_2' \\ A_2'^* & A_1'^* \end{pmatrix} \tag{A20}$$

Based on our previous analysis, where the extended regular representation $R(g^0)$ can be block diagonalized with more than two blocks of size 1, and assuming the transfer matrix is diagonalizable, we can deduce that the eigenvalues of the transfer matrix may exhibit degeneracy or not. However, except in some extreme cases where accidental degeneracy may occur, there is typically only one dominant eigenvalue. This dominant eigenvalue corresponds to the trivial irreducible representation of the subgroup G_b^0.

Secondly, if ν_2 is a non-trivial 2-cocycle, we can prove that the eigenvalues of the transfer matrix A always exhibit degeneracy. This means that for any eigenvalue λ of A, there will always exist another eigenvalue λ' such that $|\lambda| = |\lambda'|$. We will now provide a proof for this statement under different situations.

Here, it is important to note that while ν_2 is a non-trivial cocycle under the entire local symmetry group G_b, it may not necessarily be a non-trivial 2-cocycle when confined to its subgroup G_b^0.

If ν_2 remains a non-trivial 2-cocycle under the subgroup G_b^0, then based on the previous section, we know that the projective representation of the subgroup G_b^0, $S(g^0)^{-1}R(g^0)$, can only be block diagonalized with each block having a size greater than one. Since $S(g^0)^{-1}R(g^0)$ and the transfer matrix A commute with each other, and we have assumed that A is diagonalizable, according to Ref [31], A and $S(g^0)^{-1}R(g^0)$ can be simultaneously block diagonalized with each block having a size greater than one. Therefore, the eigenvalues of the transfer matrix A will always exhibit degeneracy.

Now, let's consider the situation when ν_2 is a coboundary under the subgroup G_b^0, and G_b^0 is an Abelian group. As usual, we can partition the transfer matrix A into four blocks, and the following relation holds:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \tag{A21}$$

$$\tilde{A} = \begin{pmatrix} I & O \\ O & S_0(T) \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} I & O \\ O & S_0(T)^{-1} \end{pmatrix} = \begin{pmatrix} A_1 & \hat{A}_2 \\ \hat{A}_2^* & A_1^* \end{pmatrix} \tag{A22}$$

Here, $A_1(i,j) = \nu_2(e, g_i^0, g_j^0), A_2(i,j) = \nu_2(e, g_i^0, g_j^0T), A_3(i,j) = \nu_2(e, g_i^0T, g_j^0), A_4(i,j) = \nu_2(e, g_i^0T, g_j^0T)$ and $S_0(T)_{ij} = \delta_{ij}\nu_2(e, T, g_i^0)$ for all $g_i^0 \in G_b^0$. Similarly, we can also partition the projective representation $S(g^0)^{-1}R(g^0)$ into four blocks, and the following relation holds:

$$S(g^0)^{-1}R(g^0) = \begin{pmatrix} S_0(g^0)^{-1}r(g^0) & O \\ O & S_T(g^0)^{-1}r(g^0) \end{pmatrix} \tag{A23}$$

$$\tilde{S}(g^0) = \begin{pmatrix} I & O \\ O & S_0(T) \end{pmatrix} \begin{pmatrix} S_0(g^0)^{-1}r(g^0) & O \\ O & S_T(g^0)^{-1}r(g^0) \end{pmatrix} \begin{pmatrix} I & O \\ O & S_0(T)^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{S}_0(g^0) & O \\ O & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^* \end{pmatrix} \tag{A24}$$

Here, $S_0(g^0)(i,j) = \delta_{ij}\nu_2(e, g^0, g_i^0)$, $S_T(g^0)(i,j) = \delta_{ij}\nu_2(e, g^0, g_i^0T)$ and $\tilde{S}_0(g^0) = S_0(g^0)^{-1}r(g^0)$ for arbitrary $g^0 \in G_b^0$ and all $g_i^0 \in G_b^0$. Given that the transfer matrix A commutes with $S_0(g^0)^{-1}r(g^0)$, we can deduce that $\tilde{A}$ commutes with $\tilde{S}(g^0)$. As we also assume that the subgroup G_b^0 is an Abelian group, meaning it only has 1-dimensional irreducible representations. Consequently, the representation matrix $r(g)$ can be strictly diagonalized, with each diagonal element being unique. Based on the previous proof, we can deduce that $\tilde{S}_0(g^0)$ can also be strictly diagonalized, with each diagonal element being distinct. Now, if ν_2 is a non-trivial 2-cocycle, then the phase factor $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}$ is generally non-trivial. This implies that $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^*$ can also be strictly diagonalized, with each diagonal element being different from those of $\tilde{S}_0(g^0)$. Therefore, we can conclude that $\tilde{S}(g^0)$ can be strictly diagonalized, with each diagonal element being distinct. According to the findings in Ref [31], $\tilde{A}$ and $\tilde{S}(g^0)$ can be diagonalized simultaneously. Now suppose $P^{-1}\tilde{S}(g^0)P$ represents the diagonalization of $\tilde{S}(g^0)$. Then, we have the relation:

$$D = \begin{pmatrix} P^{-1} & O \\ O & P^{-1*} \end{pmatrix} \begin{pmatrix} A_1 & \hat{A}_2 \\ \hat{A}_2^* & A_1^* \end{pmatrix} \begin{pmatrix} P & O \\ O & P^* \end{pmatrix} = \begin{pmatrix} P^{-1}A_1P & O \\ O & P^{-1*}A_1^*P^* \end{pmatrix} \tag{A25}$$

Here, the matrix D represents the diagonalization of $\tilde{A}$. From this expression, it is evident that if λ is an eigenvalue of the transfer matrix $\tilde{A}$, then there exists another eigenvalue λ^* of $\tilde{A}$ (it is important to note that if λ is real, it must be at least two-fold degenerate). Therefore, if ν_2 is a coboundary under subgroup G_b^0, and G_b^0 is an Abelian group, then the transfer matrix A will always exhibit degeneracy with both λ and λ^* as its eigenvalues.

Some may raise doubts regarding whether drawing the same conclusion holds if ν_2 is a coboundary. However, if ν_2 is indeed a coboundary, we can establish the following relationship:

$$S(g^0)^{-1}R(g^0) = \begin{pmatrix} \tilde{S}_0(g^0) & O \\ O & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^* \end{pmatrix} \begin{pmatrix} \nu_1(e,g^0)\hat{S}_0(g^0)r(g^0) & O \\ O & \nu_1(e,g^0)\hat{S}_0(g^0)^*r(g^0)^* \end{pmatrix} \tag{A26}$$

Here, $\hat{S}_0(g^0)(i,j) = \delta_{ij}\frac{\nu_1(e,g_i^0)}{\nu_1(g^0,g_i^0)}$. Consequently, it is evident that the eigenvalues of $\tilde{S}(g^0)$ may not always be distinct. As a result, according to the Ref [31] and equation Eq. (A25), it can be concluded that the eigenvalues of the transfer matrix will typically not exhibit degeneracy. It is worth noting that when ν_2 is a coboundary, the transfer matrix A always possesses a dominant eigenvalue.

Lastly, let's consider the situation where ν_2 is a coboundary under the subgroup G_b^0, and G_b^0 is a non-Abelian group. For non-Abelian group G_b^0, there always exist high-order irreducible representations. Consequently, the matrix $r(g^0)$ can only be block diagonalized, with the presence of both size-1 blocks and higher-order blocks. Based on this, we can block diagonalize $\tilde{S}_0(g^0)$ in the following manner:

$$\begin{pmatrix} P^{-1} & O \\ O & P^{-1*} \end{pmatrix} \begin{pmatrix} \tilde{S}_0(g^0) & O \\ O & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^* \end{pmatrix} \begin{pmatrix} P & O \\ O & P^* \end{pmatrix} = \begin{pmatrix} D & & & \\ & B & & \\ & & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}D^* & \\ & & & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^* \end{pmatrix} \tag{A27}$$

The block D is a strictly diagonalized matrix, while the block B consists of a series of block matrices along the diagonal with each block size larger than one. As we have demonstrated previously, the matrix $\tilde{A}$ can be simultaneously block diagonalized with $\tilde{S}(g^0)$. The eigenvalues corresponding to block B and $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^*$ are always degenerate. Therefore, our focus will be solely on the eigenvalues associated with the diagonalized block D and $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}D^*$. In fact, we can always perform a similar transformation on both $\tilde{A}$ and $\tilde{S}(g^0)$ to align the diagonalized blocks together:

$$X^{-1}\begin{pmatrix} D & & & \\ & B & & \\ & & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}D^* & \\ & & & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^* \end{pmatrix} X = \begin{pmatrix} D & & & \\ & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}D^* & & \\ & & B & \\ & & & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^* \end{pmatrix} \tag{A28}$$

$$X^{-1}\tilde{A}X = \begin{pmatrix} a_1 & \hat{a}_2 & . \\ \hat{a}_2^* & a_1^* & . \\ . & . & . \end{pmatrix} \qquad X^{-1}\tilde{S}(g^0)X = \begin{pmatrix} \tilde{S}_0(g^0) & O & . \\ O & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^* & . \\ . & . & . \end{pmatrix} \tag{A29}$$

$$\tag{A30}$$

From the commutative relation between $\tilde{A}$ and $\tilde{S}(g^0)$, it can be easily verified that:

$$X^{-1}\begin{pmatrix} P^{-1} & O \\ O & P^{-1*} \end{pmatrix} \begin{pmatrix} \tilde{a} & . \\ . & . \end{pmatrix} \begin{pmatrix} P & O \\ O & P^* \end{pmatrix} X = \begin{pmatrix} d_{\tilde{a}} & O \\ O & b_{\tilde{a}} \end{pmatrix} \tag{A31}$$

Here, $d_{\tilde{a}}$ represents strictly diagonalized matrices, and $b_{\tilde{a}}$ is generally not diagonalized or block diagonalized. However, since $b_{\tilde{a}}$ commutes with the block diagonalized part of $\tilde{S}(g^0)$, denoted as diag$[B, \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^]$, and $b_{\tilde{a}}$ is diagonalizable, according to Ref [31], there always exists a similar transformation where $b_{\tilde{a}}$ and diag$[B, \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^]$ can be simultaneously block diagonalized. It should be noted that diag$[B, \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^*]$ can only be block diagonalized with each block having a size larger than one. Therefore, according to the Ref [31], every eigenvalue of $b_{\tilde{a}}$ is degenerate. Additionally, from the commutative relation of $\tilde{A}$ and $\tilde{S}(g^0)$, it can be easily demonstrated that the matrix $\tilde{a}$ and $\tilde{s}(g^0)$ commute with each other. Consequently, based on the previous proof, since $\tilde{s}(g^0)$ can be strictly diagonalized with each diagonal element being distinct, the matrix $\tilde{a}$ can be simultaneously diagonalized with all eigenvalues being degenerate with its conjugate. Thus, we have thoroughly established that if ν_2 is a coboundary under the subgroup G_b^0, and G_b^0 is a non-Abelian group, the eigenvalues of the transfer matrix A always exhibit degeneracy.

In conclusion, we have successfully demonstrated that for non-trivial SPT phases, the eigenvalues of the transfer matrix always possess degeneracy. On the other hand, for trivial SPT phases, apart from the exhibit of intrinsic symmetry, the dominant eigenvalue generally does not exhibit degeneracy. Consequently, in the case of a non-trivial SPT phase, the strange correlator will consistently exhibit long-range correlations, while the trivial SPT phase will always display short-range correlations. This observation aligns perfectly with the theoretical expectations in strange correlation theory.

Appendix B: Proof of Correctness of Strange Correlation Theory for Fermionic Strange Correlator

1. Proof of Correctness of 1D FSPT Strange Correlation Theory with trivial Central Extension

In this subsection, we present a detailed proof of the availability of the 1D FSPT strange correlator based on transfer matrix method.

As mentioned previously, the 1-cocycle labeling complex fermions results in the division of the local symmetry group G_b into two cosets: Z and $O = \sigma Z$. Consequently, we can label the elements in G_b as $g_1, g_2, \ldots, g_M, g'_1, g'_2, \ldots, g'_M$, where g_1 corresponds to the unit element and $g'_i = \sigma g_i$. In the following discussion, we will consider two scenarios based on whether σ includes time reversal symmetry or not.

a. *σ without time reversal*

According to the obstruction function $\mathrm{d}\nu_2(g_0, g_1, g_2, g_3) = (-1)^{\omega_2 \smile n_1}$, if we set $g_2, g_3 \in Z$, then the obstruction function becomes a traditional cocycle equation. Similarly, if we set $g_2, g_3 \in O$, then the obstruction function also becomes a traditional cocycle equation. Now, let's set $g_2, g_3 \in Z$. In this case, the obstruction function becomes:

$$\nu_2(\sigma^{-1}, g_2, g_3) = \frac{\nu_2(\sigma^{-1}, g_1, g_3)\nu_2(g_1, g_2, g_3)}{\nu_2(\sigma^{-1}, g_1, g_2)} \tag{B1}$$

Let $g_1 = e$, then we find the relation:

$$\nu_2(e, g'_2, g'_3) = \frac{\nu_2(e, \sigma, g'_3)}{\nu_2(e, \sigma, g'_2)}\nu_2(e, g_2, g_3) \tag{B2}$$

Now, let's come back to our transfer matrix. According to our definition, $T_Z = \nu_2(e, g_i, g_j)$ for $g_i, g_j \in Z$, and $T_O = \nu_2(e, g'_i, g'_j)$ for $g'_i, g'_j \in O$. Therefore, according to Equation (B32), we have $T_O = S^{-1}T_Z S$, where $S_{ij} = \delta_{ij}\nu_2(e, \sigma, g'_i)$. Since T_O and T_Z are similar matrices, they always possess the same eigenvalues. For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle = \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{B3}$$

Assuming that T_Z is diagonalizable, i.e., $D = A^{-1}T_Z A$ is a diagonalization, we have:

$$\begin{aligned}
&\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle && \text{(B4)}\\
&= \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) && \text{(B5)}\\
&= \mathrm{tr}(A^{-1}ST_{OZ}AD^{|i-j|}A^{-1}T_{ZO}S^{-1}AD^{N-|i-j|}) + \mathrm{tr}(A^{-1}T_{ZO}S^{-1}AD^{|i-j|}A^{-1}ST_{OZ}AD^{N-|i-j|}) && \text{(B6)}\\
&= 2\nu_2(e, \sigma, e)\mathrm{tr}(\tilde{T}D^{|i-j|}\tilde{T}D^{N-|i-j|}) && \text{(B7)}
\end{aligned}$$

Here, $A^{-1}T_{ZO}S^{-1}A = \nu_2(e, \sigma, e)^* A^{-1}ST_{OZ}A = \tilde{T}$. We assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant |\lambda_M|$ (as we can always do) are the eigenvalues of T_Z. Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a trivial cocycle), with $|\lambda_1| > |\lambda_2|$, then in the thermodynamic limit ($N \to \infty$), the strange correlator can be expressed as:

$$C(i,j) = \nu_2(e, \sigma, e)[\tilde{T}(1,1)^2 + \sum_{i\neq 1}\tilde{T}(1,i)\tilde{T}(i,1)\left(\frac{\lambda_i}{\lambda_1}\right)^L] \tag{B8}$$

Clearly, if $|\lambda_1| > |\lambda_2|$, the strange correlator will always converge to a constant.

Secondly, if the eigenvalues exhibit a p-fold degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a non-trivial cocycle) such that $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_p$, then under thernodynamic limit, the strange correlator can be expressed as:

$$\frac{\nu_2(e,\sigma,e)\sum_{i,j=1}^{p}\tilde{T}(j,i)\tilde{T}(i,j)}{p} \tag{B9}$$

Apparently, under this circumstances, the strange correlator will also converge to a constant.

Even in the presence of accidental degeneracy, where $|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_p|$ $(1 < p \leqslant M)$, and we let $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$ $(1 \leqslant i < p)$, the strange correlator can still be expressed as:

$$\frac{\nu_2(e,\sigma,e)\sum_{i,j=1}^{p}[\tilde{T}(j,i)\tilde{T}(i,j)]e^{i\theta_i(N-L)}e^{i\theta_j L}}{1+e^{i\theta_2 N}+e^{i\theta_3 N}+\ldots+e^{i\theta_p N}} \tag{B10}$$

In this situation, the behavior of the strange correlator is determined by the values of N and the phases θ_i. In general, there are two possible cases: power-law decay and oscillation. The specific behavior depends on the values of θ_i and θ_j, as well as the ratio of N and L.

b. σ *contains time reversal*

Similar to the previous section, let's set $g_2, g_3 \in Z$, and the obstruction function becomes:

$$\nu_2(\sigma^{-1},g_2,g_3) = \frac{\nu_2(\sigma^{-1},g_1,g_3)\nu_2(g_1,g_2,g_3)}{\nu_2(\sigma^{-1},g_1,g_2)} \tag{B11}$$

Let $g_1 = e$, this time however, the relation becomes:

$$\nu_2(e,g_2',g_3')^* = \frac{\nu_2(e,\sigma,g_3')^*}{\nu_2(e,\sigma,g_2')^*}\nu_2(e,g_2,g_3) \tag{B12}$$

Now, let's come back to our transfer matrix. According to our definition, $T_Z = \nu_2(e,g_i,g_j)$ for $g_i, g_j \in Z$, and $T_O = \nu_2(e,g_i',g_j')$ for $g_i', g_j' \in O$. Therefore, according to Equation (B32), we have $T_O = S^{-1}T_Z^*S$, where $S_{ij} = \delta_{ij}\nu_2(e,\sigma,g_i')$. Since T_O and T_Z^* are similar matrices, the eigenvalues of T_O the conjugate of those of T_Z. For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle = \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{B13}$$

Assuming that T_Z is diagonalizable, i.e., $D = A^{-1}T_ZA$ is a diagonalization, we have:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle \tag{B14}$$

$$= \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{B15}$$

$$= \mathrm{tr}(A^{-1*}ST_{OZ}AD^{|i-j|}A^{-1}T_{ZO}S^{-1}A^*D^{N-|i-j|*}) + \mathrm{tr}(A^{-1}T_{ZO}S^{-1}A^*D^{|i-j|*}A^{-1*}ST_{OZ}AD^{N-|i-j|}) \tag{B16}$$

$$= \nu_2(e,\sigma,e)[\mathrm{tr}(\tilde{T}D^{|i-j|}\tilde{T}^*D^{N-|i-j|*}) + \mathrm{tr}(\tilde{T}^*D^{|i-j|*}\tilde{T}D^{N-|i-j|})] \tag{B17}$$

Here, $A^{-1}T_{ZO}S^{-1}A^* = \nu_2(e,\sigma,e)[A^{-1*}ST_{OZ}A]^* = \tilde{T}$. We assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant |\lambda_M|$ (as we can always do) are the eigenvalues of T_Z. Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a trivial cocycle), with $|\lambda_1| > |\lambda_2|$, then in the thermodynamic limit $(N \to \infty)$, the strange correlator can be expressed as:

$$C(i,j) = \frac{2|\tilde{T}|^2(1,1)\mathrm{Re}\left[\left(\frac{\lambda_1^*}{\lambda_1}\right)^L\right] + \sum_{i\neq 1}\left[\tilde{T}(1,i)\tilde{T}^*(i,1)\left(\frac{\lambda_i^*}{\lambda_1}\right)^L + \tilde{T}^*(1,i)\tilde{T}(i,1)\left(\frac{\lambda_i}{\lambda_1^*}\right)^L\left(\frac{\lambda_1^*}{\lambda_1}\right)^N\right]}{\nu_2(e,\sigma,e)[1+\left(\frac{\lambda_1^*}{\lambda_1}\right)^N]} \tag{B18}$$

Clearly, if $|\lambda_1| > |\lambda_2|$, the strange correlator will generally present oscillation.

Secondly, if the eigenvalues exhibit a p-fold degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a non-trivial cocycle) such that $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_p$, then under thernodynamic limit, the strange correlator can be expressed as:

$$\frac{\sum_{i,j=1}^{p}\left[\tilde{T}(j,i)\tilde{T}^*(i,j)\left(\frac{\lambda_j^*}{\lambda_j}\right)^L + \tilde{T}^*(j,i)\tilde{T}(i,j)\left(\frac{\lambda_i^*}{\lambda_i}\right)^{N-L}\right]}{p\nu_2(e,\sigma,e)[1+\left(\frac{\lambda_1^*}{\lambda_1}\right)^N]} \tag{B19}$$

Apparently, under this circumstances, the strange correlator will also exhibit oscillatory behavior in general.

Even in the presence of accidental degeneracy, where $|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_p|$ $(1 < p \leqslant M)$, and we let $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$ $(1 \leqslant i < p)$, the strange correlator can still be expressed as:

$$\frac{\sum_{i,j=1}^{p}\nu_2(e,\sigma,e)^*\left[\tilde{T}(j,i)\tilde{T}^*(i,j)\left(\frac{\lambda_j^*}{\lambda_j}\right)^L + \tilde{T}^*(j,i)\tilde{T}(i,j)\left(\frac{\lambda_i^*}{\lambda_i}\right)^{N-L}\right]e^{i\theta_i(N-L)}e^{i\theta_j L}}{[1+e^{i\theta_2 N}+e^{i\theta_3 N}+\ldots+e^{i\theta_p N}]+\left(\frac{\lambda_1^*}{\lambda_1}\right)^N[1+e^{-i\theta_2 N}+e^{-i\theta_3 N}+\ldots+e^{-i\theta_p N}]} \tag{B20}$$

The behavior of the strange correlator expression depends on the values of N and the phases θ_i. In general, there are two possible cases: power-law decay and oscillation. The specific behavior is determined by the values of θ_i and θ_j, as well as the ratio of N and L.

Subsequently, in no situations, the strange correlator would exhibit exponential decay, manifesting a mode of long-range correlation. Therefore, we conclude that if we choose the complex fermion annihilation operator as our observables, the 1D FSPT strange correlator with a non-trivial central extension would always demonstrate long-range correlation.

2. Proof of Correctness of 1D FSPT Strange Correlatioin Theory with Non-trivial Central Extension

In this subsection, we present a detailed proof of the availability of the 1D FSPT strange correlator based on transfer matrix method.

As mentioned previously, the 1-cocycle labeling complex fermions results in the division of the local symmetry group G_b into two cosets: Z and $O = \sigma Z$. Consequently, we can label the elements in G_b as $g_1, g_2, \ldots, g_M, g'_1, g'_2, \ldots, g'_M$, where g_1 corresponds to the unit element and $g'_i = \sigma g_i$. In the following discussion, we will consider two scenarios based on whether σ includes time reversal symmetry or not.

a. *σ without time reversal*

According to the obstruction function $\mathrm{d}\nu_2(g_0, g_1, g_2, g_3) = (-1)^{\omega_2 \smile n_1}$, if we set $g_2, g_3 \in Z$, then the obstruction function becomes a traditional cocycle equation. Similarly, if we set $g_2, g_3 \in O$, then the obstruction function also becomes a traditional cocycle equation. Now, let's set $g_2, g_3 \in Z$. In this case, the obstruction function becomes:

$$\nu_2(\sigma^{-1}, g_2, g_3) = \frac{\nu_2(\sigma^{-1}, g_1, g_3)\nu_2(g_1, g_2, g_3)}{\nu_2(\sigma^{-1}, g_1, g_2)} \tag{B21}$$

Let $g_1 = e$, then we find the relation:

$$\nu_2(e, g'_2, g'_3) = \frac{\nu_2(e,\sigma,g'_3)}{\nu_2(e,\sigma,g'_2)}\nu_2(e, g_2, g_3) \tag{B22}$$

Now, let's come back to our transfer matrix. According to our definition, $T_Z = \nu_2(e, g_i, g_j)$ for $g_i, g_j \in Z$, and $T_O = \nu_2(e, g'_i, g'_j)$ for $g'_i, g'_j \in O$. Therefore, according to Equation (B32), we have $T_O = S^{-1}T_Z S$, where $S_{ij} = \delta_{ij}\nu_2(e, \sigma, g'_i)$. Since T_O and T_Z are similar matrices, they always possess the same eigenvalues. For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle = \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{B23}$$

Assuming that T_Z is diagonalizable, i.e., $D = A^{-1}T_Z A$ is a diagonalization, we have:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle \tag{B24}$$

$$= \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{B25}$$

$$= \mathrm{tr}(A^{-1}ST_{OZ}AD^{|i-j|}A^{-1}T_{ZO}S^{-1}AD^{N-|i-j|}) + \mathrm{tr}(A^{-1}T_{ZO}S^{-1}AD^{|i-j|}A^{-1}ST_{OZ}AD^{N-|i-j|}) \tag{B26}$$

$$= \mathrm{tr}(\tilde{T}_{ZO}D^{|i-j|}\tilde{T}_{OZ}D^{N-|i-j|}) + \mathrm{tr}(\tilde{T}_{ZO}D^{|i-j|}\tilde{T}_{OZ}D^{N-|i-j|}) \tag{B27}$$

Here, $\tilde{T}_{ZO} = A^{-1}T_{ZO}S^{-1}A$ and $\tilde{T}_{OZ} = A^{-1}ST_{OZ}A$. We assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant |\lambda_M|$ (as we can always do) are the eigenvalues of T_Z. Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a trivial cocycle), with $|\lambda_1| > |\lambda_2|$, then in the thermodynamic limit ($N \to \infty$), the strange correlator can be expressed as:

$$C(i,j) = \tilde{T}_{ZO}(1,1)\tilde{T}_{OZ}(1,1) + \frac{\sum_{i\neq 1}[\tilde{T}_{ZO}(1,i)\tilde{T}_{OZ}(i,1) + \tilde{T}_{OZ}(1,i)\tilde{T}_{ZO}(i,1)]\left(\frac{\lambda_i}{\lambda_1}\right)^L}{2} \tag{B28}$$

Clearly, if $|\lambda_1| > |\lambda_2|$, the strange correlator will always converge to a constant.

Secondly, if the eigenvalues exhibit a p-fold degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a non-trivial cocycle) such that $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_p$, then under thernodynamic limit, the strange correlator can be expressed as:

$$\frac{\sum_{i,j=1}^{p}[\tilde{T}_{ZO}(j,i)\tilde{T}_{ZO}(i,j) + \tilde{T}_{OZ}(j,i)\tilde{T}_{ZO}(i,j)]}{2p} \tag{B29}$$

Apparently, under this circumstances, the strange correlator will also converge to a constant.

Even in the presence of accidental degeneracy, where $|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_p|$ $(1 < p \leqslant M)$, and we let $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$ $(1 \leqslant i < p)$, the strange correlator can still be expressed as:

$$\frac{\sum_{i,j=1}^{p}[\tilde{T}_{ZO}(j,i)\tilde{T}_{ZO}(i,j) + \tilde{T}_{OZ}(j,i)\tilde{T}_{ZO}(i,j)]e^{i\theta_i(N-L)}e^{i\theta_j L}}{2[1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \ldots + e^{i\theta_p N}]} \tag{B30}$$

In this situation, the behavior of the strange correlator is determined by the values of N and the phases θ_i. In general, there are two possible cases: power-law decay and oscillation. The specific behavior depends on the values of θ_i and θ_j, as well as the ratio of N and L.

b. σ contains time reversal

Similar to the previous section, let's set $g_2, g_3 \in Z$, and the obstruction function becomes:

$$\nu_2(\sigma^{-1}, g_2, g_3) = \frac{\nu_2(\sigma^{-1}, g_1, g_3)\nu_2(g_1, g_2, g_3)}{\nu_2(\sigma^{-1}, g_1, g_2)} \tag{B31}$$

Let $g_1 = e$, this time however, the relation becomes:

$$\nu_2(e, g_2', g_3')^* = \frac{\nu_2(e, \sigma, g_3')^*}{\nu_2(e, \sigma, g_2')^*}\nu_2(e, g_2, g_3) \tag{B32}$$

Now, let's come back to our transfer matrix. According to our definition, $T_Z = \nu_2(e, g_i, g_j)$ for $g_i, g_j \in Z$, and $T_O = \nu_2(e, g_i', g_j')$ for $g_i', g_j' \in O$. Therefore, according to Equation (B32), we have $T_O = S^{-1}T_Z^*S$, where $S_{ij} = \delta_{ij}\nu_2(e, \sigma, g_i')$. Since T_O and T_Z^* are similar matrices, the eigenvalues of T_O the conjugate of those of T_Z. For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle = \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{B33}$$

Assuming that T_Z is diagonalizable, i.e., $D = A^{-1}T_Z A$ is a diagonalization, we have:

$$\langle\Omega|\tilde{c}_{i,i+1}\tilde{c}_{j,j+1}|\Psi\rangle \tag{B34}$$

$$= \mathrm{tr}(T_{OZ}T_Z^{|i-j|}T_{ZO}T_O^{N-|i-j|}) + \mathrm{tr}(T_{ZO}T_O^{|i-j|}T_{OZ}T_Z^{N-|i-j|}) \tag{B35}$$

$$= \mathrm{tr}(A^{-1*}ST_{OZ}AD^{|i-j|}A^{-1}T_{ZO}S^{-1}A^*D^{N-|i-j|*}) + \mathrm{tr}(A^{-1}T_{ZO}S^{-1}A^*D^{|i-j|*}A^{-1*}ST_{OZ}AD^{N-|i-j|}) \tag{B36}$$

$$= \mathrm{tr}(\tilde{T}_{OZ}D^{|i-j|}\tilde{T}_{ZO}D^{N-|i-j|*}) + \mathrm{tr}(\tilde{T}_{ZO}D^{|i-j|*}\tilde{T}_{OZ}D^{N-|i-j|}) \tag{B37}$$

Here, $\tilde{T}_{ZO} = A^{-1}T_{ZO}S^{-1}A^*$ and $\tilde{T}_{OZ} = A^{-1*}ST_{OZ}A$. We assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant |\lambda_M|$ (as we can always do) are the eigenvalues of T_Z. Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a trivial cocycle), with $|\lambda_1| > |\lambda_2|$, then in the thermodynamic limit ($N \to \infty$), the strange correlator can be expressed as:

$$C(i,j) = \frac{2\tilde{T}_{ZO}(1,1)\tilde{T}_{OZ}(1,1)\mathrm{Re}\left[\left(\frac{\lambda_1^*}{\lambda_1}\right)^L\right] + \sum_{i\neq 1}\left[\tilde{T}_{ZO}(1,i)\tilde{T}_{OZ}(i,1)\left(\frac{\lambda_i^*}{\lambda_1}\right)^L + \tilde{T}_{OZ}(1,i)\tilde{T}_{ZO}(i,1)\left(\frac{\lambda_i}{\lambda_1^*}\right)^L\left(\frac{\lambda_1^*}{\lambda_1}\right)^N\right]}{1+\left(\frac{\lambda_1^*}{\lambda_1}\right)^N} \tag{B38}$$

Clearly, if $|\lambda_1| > |\lambda_2|$, the strange correlator will generally present oscillation.

Secondly, if the eigenvalues exhibit a p-fold degeneracy (that is, when the 2-cocycle $\nu_2 \in H^2(Z, U_T(1))$ is a non-trivial cocycle) such that $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_p$, then under thernodynamic limit, the strange correlator can be expressed as:

$$\frac{\sum_{i,j=1}^{p}\left[\tilde{T}_{ZO}(j,i)\tilde{T}_{OZ}(i,j)\left(\frac{\lambda_j^*}{\lambda_j}\right)^L + \tilde{T}_{OZ}(j,i)\tilde{T}_{ZO}(i,j)\left(\frac{\lambda_i^*}{\lambda_i}\right)^{N-L}\right]}{p[1+\left(\frac{\lambda_1^*}{\lambda_1}\right)^N]} \tag{B39}$$

Apparently, under this circumstances, the strange correlator will also exhibit oscillatory behavior in general.

Even in the presence of accidental degeneracy, where $|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_p|$ ($1 < p \leqslant M$), and we let $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$ ($1 \leqslant i < p$), the strange correlator can still be expressed as:

$$\frac{\sum_{i,j=1}^{p}\left[\tilde{T}_{ZO}(j,i)\tilde{T}_{OZ}(i,j)\left(\frac{\lambda_j^*}{\lambda_j}\right)^L + \tilde{T}_{OZ}(j,i)\tilde{T}_{ZO}(i,j)\left(\frac{\lambda_i^*}{\lambda_i}\right)^{N-L}\right]e^{i\theta_i(N-L)}e^{i\theta_j L}}{[1+e^{i\theta_2 N}+e^{i\theta_3 N}+\ldots+e^{i\theta_p N}]+\left(\frac{\lambda_1^*}{\lambda_1}\right)^N[1+e^{-i\theta_2 N}+e^{-i\theta_3 N}+\ldots+e^{-i\theta_p N}]} \tag{B40}$$

The behavior of the strange correlator expression depends on the values of N and the phases θ_i. In general, there are two possible cases: power-law decay and oscillation. The specific behavior is determined by the values of θ_i and θ_j, as well as the ratio of N and L.

Subsequently, in no situations, the strange correlator would exhibit exponential decay, manifesting a mode of long-range correlation. Therefore, we conclude that if we choose the complex fermion annihilation operator as our observables, the 1D FSPT strange correlator with a non-trivial central extension would always demonstrate long-range correlation.

Appendix C: Classification of 1D FSPT States with Non-trivial Central Extension by Strange Correlator

Firstly, suppose the local symmetry does not include time reversal symmetry. In this case, for any group element $g \in G_b$, the following relation based on group super-cohomology holds:

$$\nu_2(g, g_i, g_j) = \nu_2(e, g_i, g_j)\frac{\nu_2(e, g, g_i)}{\nu_2(e, g, g_j)}(-1)^{(\omega_2 \smile n_1)(e,g,g_i,g_j)} \tag{C1}$$

According to Equation (18), we have the relation $\nu_2(g, g_i, g_j) = \nu_2(e, g^{-1}g_i, g^{-1}g_j)(-1)^{(\omega_2 \smile n_1)(e,g,g_i,g_j)}$. Using this expression, we can derive the following relationships:

$$R(g)TR(g^{-1}) = S(g)TS(g)^{-1} \;\; S(g)^{-1}R(g)T = TS(g)^{-1}R(g) \tag{C2}$$

Here, $R(g)$ represents the regular representation of the local symmetry group G_b, while $S(g)_{ij} = \delta_{ij}\nu_2(e, g, g_j)$.

According to Appendix A, if ν_2 is a coboundary, we can conclude that the largest eigenvalue of the matrix T is generally non-degenerate. This result is consistent with the expectation that the strange correlator between states belonging to the same phase does not exhibit long-range degeneracy.

If ν_2 is non-trivial, we claim that the expression $S(g)^{-1}R(g)$ represents a projective representation of the local symmetry group G_b. Now, let's proceed to prove this claim:

$$S(g)^{-1}R(g)S(h)^{-1}R(h) \tag{C3}$$

$$= \delta_{ij}\nu_2(e, g, g_i)^{-1}\nu_2(e, h, g^{-1}g_i)^{-1}R(gh) \tag{C4}$$

$$= \delta_{ij}\nu_2(e, g, g_i)^{-1}\nu_2(g, gh, g_i)^{-1}R(gh)(-1)^{(\omega_2 \smile n_1)(e,g,gh,g_i)} \tag{C5}$$

$$= \delta_{ij}\nu_2(e, g, gh)^{-1}\nu_2(e, gh, g_i)^{-1}R(gh) \tag{C6}$$

$$= \nu_2(e, g, g_i)^{-1}S(gh)^{-1}R(gh) \tag{C7}$$

Hence, we find that $S(g)^{-1}R(g)S(h)^{-1}R(h) = \nu_2(e, g, g_i)^{-1}S(gh)^{-1}R(gh)$, which leads us to conclude that $S(g)^{-1}R(g)$ indeed represents a projective representation of the group G_b.

As a result, according to Appendix A, we can establish that the eigenvalues of the transfer matrix T are typically degenerate. This is because the block size of its diagonalization must be equal to or greater than 2.

In a similar manner, when considering the presence of time reversal symmetry, we can partition the local symmetry group into two distinct sets: G_b^0 and G_b^T. Here, G_b^0 consists of elements that do not possess any time-reversal properties, while G_b^T comprises elements that do exhibit time-reversal characteristics. It can be easily verified that G_b^0 forms a subgroup of the local symmetry group G_b. We can still observe the relation $S(g^0)^{-1}R^T(g^0)A = AS(g^0)^{-1}R^T(g^0)$, where $g^0 \in G_b^0$, and $R^T(g^0) = \mathrm{diag}[r(g^0), r(g^0)]$. As a result, the eigenvectors of the transfer matrix T carry a projective representation of the group G_b^0. It is important to note that if σ is not the time-reversal element, then under the subgroup G_b^0, ν_2 still belongs to the super-cohomology group $BH(G_b^0, U_T(1))$. Therefore, according to the previous analysis, the eigenvalues of the transfer matrix A always exhibit degeneracy. However, in cases where σ represents the time-reversal element, indicating that ν_2 in the subgroup G_b^0 functions as a 2-cocycle, there are instances where ν_2 may become a coboundary within the subgroup G_b^0. In such situations, we can still verify the availability of Eq. (A21) and Eq. (A23), albeit this time $\nu_2 \in BH(G_b^0, U_T(1))$ is obstructed by Eq. (18). Consequently, by following the same procedure outlined in Appendix A, we can establish that the degeneracy of the transfer matrix A remains intact.

Hence, based on the findings presented in Appendix A, we can conclude that the strange correlator between states belonging to the same phase does not display long-range degeneracy. In contrast, the strange correlator between states belonging to different phases consistently exhibits long-range degeneracy. This observation provides a classification method for 1D FSPT (Fractional Symmetry-Protected Topological) phases that possess a non-trivial central extension.

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