

# Strange Correlator for 1D Fermionic Symmetry-Protected Topological Phases

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## I. INTRODUCTION

Symmetry-protected topological (SPT) phases, including the topological insulators and topological superconductors, are short-range-entangled topological phases beyond Landau's paradigm. An important feature of SPT phases is the bulk-boundary correspondence: when the bulk is topologically nontrivial, its boundary will exhibit anomalous behaviors, including symmetry-protected gaplessness. (The boundary can also exhibit symmetry breaking or realize a gapped topological order with an anomalous symmetry enrichment in 2+1 dimensions.) Therefore, a gapless surface is usually used as a signature to detect a topologically nontrivial bulk. However, this signature is sometimes inconvenient or impossible to use: In numerical simulations such as exact diagonalization and quantum Monte Carlo, the extra computational cost of simulating a boundary (instead of the common practice of using periodic boundary conditions) is often unacceptable; it is impossible to design fully symmetric boundaries for SPT phases protected by crystalline or average symmetries. Therefore, it is desirable to have bulk-detection tools for nontrivial SPT phases.

Unlike free-fermion topological states, which can be detected using topological invariant (such as the Chern number and the Fu-Kane invariant) computed from the bulk band structure, the bulk detection of interacting SPT states is more challenging. In principle, SPT phases can be detected by evaluating the partition function on nontrivial space-time manifold with symmetry fluxes. However, this tool could be inconvenient to use in practice, due to the difficulties of generalizing a lattice model to curved manifolds, and evaluating the partition function for a realistic model. In one dimension, SPT states can be detected using symmetry-transformation properties of the Schmidt eigenstates of the entanglement Hamiltonian, but this approach is difficult to generalize to higher dimensions. Another useful tool for bulk detection is the strange correlator first proposed by You *et al.* [1]. Defined as a correlation function evaluated between two different SPT wave functions, it has proven to be a powerful bulk diagnostic tool for identifying non-trivial topological properties of symmetry protected topological phases, because it uses the ground state wave function which is easily obtained in numerical simulations, and it can be generalized to higher dimensions. Hence, the strange correlator will be the main focus of this work.

The main property of the strange correlator is that [1], for short-range entangled topological phases of matter there exist some on-site observables  $\hat{O}(r), \hat{O}(r')$ , such that it displays a power law decay (or saturates to a finite value) as a function of distance when the state under detection  $|\Psi\rangle$  is a non-trivial topological state. Conversely, if both the detected state  $|\Psi\rangle$  and the reference state  $|\Omega\rangle$  are trivial, all the strange correlators decay exponentially in  $|r - r'|$ .

Indeed, previous research has primarily focused on the strange correlator of bosonic symmetry-protected topological (SPT) phases and its ability to detect such states [1–5]. The properties of the strange correlator in detecting SPT states have been extensively studied. However, there have been no investigation into the effectiveness of the strange correlator method in detecting fermionic symmetry-protected topological (FSPT) states. As a result, the availability and applicability of the strange correlator method in characterizing FSPT states remain unclear and require further exploration. Furthermore, it is worth noting that previous works on the strange correlator have often relied on specific examples to illustrate its properties in symmetry-protected topological states. However, a general proof or comprehensive theoretical framework for the strange correlator method is still lacking. While the strange correlator has shown promise in detecting certain features of symmetry-protected topological states, a more rigorous and systematic understanding of its underlying theory is needed. This would involve establishing a general framework and providing a theoretical foundation for the strange correlator method in characterizing a broader range of symmetry-protected topological states.

In this paper, we extensively investigate the strange correlator in 1D FSPT phases [6]. Our main contribution is a rigorous proof of its applicability to 1D FSPT phases using the transfer matrix method. We also propose a general method for constructing the FSPT strange correlator, considering the fermionic symmetry group's relationship to the bosonic symmetry group. Our work further supports the stability and validity of general SPT and FSPT models [6–20], highlighting the role of the strange correlator in detecting and characterizing FSPT states and enhancing our understanding of these theoretical frameworks.

## II. STRANGE CORRELATOR FOR 1D FSPT PHASES WITH $G_f = G_b \times \mathbb{Z}_2^f$

### A. Constructing Fixed-Point Wavefunction for FSPT Phases in 1D

In this section, we briefly review the fixed-point wave function for a 1D FSPT state, which will be used later to compute the strange correlators. The general form of such wave function is given as [6]:

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi \left( \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \right) | \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \rangle. \quad (1)$$

In the 1D lattice, each locally ordered vertex  $i$  (see the black dot in the figure) has a bosonic degree of freedom labeled by a group element  $g_i \in G_b$ . Recall that the fermionic symmetry-protected topological (FSPT) phases have a total symmetry  $G_f = G_b \times \mathbb{Z}_2^f$ . A spinless complex fermion  $c_{ij}$  is located at the center of each link  $\langle i, j \rangle$  (see the blue ball in the figure), and the fermion occupation number  $n_1(g_i, g_j)$  is either 0 or 1.

Now that the basis is orthonormal, we turn to settle down the coefficient :

$$\Psi \left( \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \right) \quad (2)$$

To obtain a fixed-point wavefunction for Eq. (1), we need to understand the changes of the wavefunction under renormalization. In 1D, renormalization can be understood as removing some bosonic or fermionic degrees of freedom by reducing the number of vertices. The basic renormalization process is known as (2-1) Pachner move of triangulation of 1D manifold [6]. Since we have a bosonic degree of freedom at each vertex and a fermionic degree of freedom at each link, the (2-1) move effectively reduces the Hilbert space of one bosonic mode and one fermionic mode.

To be more precise, the (2-1) move is a FSLU transformation between the fermionic Fock spaces on two different triangulations:

$$\Psi \left( \text{---} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \text{---} \right) = F(g_0, g_1, g_2) \Psi \left( \text{---} 0 \xrightarrow{c_{(02)}} 2 \text{---} \right), \quad (3)$$

where the  $F$  operator is defined as

$$F(g_0, g_1, g_2) = \frac{1}{|G_b|^{1/2}} \nu_2(g_0, g_1, g_2) c_{(12)}^{\dagger n_1(g_1, g_2)} c_{(01)}^{\dagger n_1(g_0, g_1)} c_{(02)}^{n_1(g_0, g_2)}. \quad (4)$$

Here  $|G_b|$  is the order of the group  $G_b$ , and we introduce the normalization factor  $1/|G_b|^{1/2}$  in the above expression due to the change of vertex number.  $\nu_2(g_0, g_1, g_2)$  is a  $U_T(1)$ -valued function with variables  $g_i \in G_b$  [11] and  $c_{(ij)}^\dagger$  is the creation operator for complex fermions at link  $\langle i, j \rangle$ , etc.  $n_1(g_i, g_j) \in \{0, 1\}$  is a  $\mathbb{Z}_2$ -valued function indicating whether there is a complex fermion at link  $\langle i, j \rangle$  or not. Since we are constructing symmetric state, both  $\nu_2$  and  $n_1$  should be symmetric under the action of  $G_b$ :  $\nu_2(gg_0, gg_1, gg_2) = g \bullet \nu_2(g_0, g_1, g_2)$ ,  $n_1(gg_0, gg_1) = g \bullet n_1(g_0, g_1)$ . So they are  $U_T(1)$ -valued 2-cochain and  $\mathbb{Z}_2$ -valued 1-cochain respectively. Because the renormalization process (2-1) Pachner move should preserve the fermion parity, we have  $dn_1(g_0, g_1, g_2) = n_1(g_1, g_2) + n_1(g_0, g_2) + n_1(g_0, g_1) = 0 \pmod{2}$ . Therefore  $n_1$  is in fact a  $\mathbb{Z}_2$ -valued 1-cocycle.

Since we are constructing fixed-point wavefunction, Eq. (1) should be invariant under renormalization. For instance, we can use two different sequences of  $F$  moves Eq. (4) to connect a fixed initial state and a fixed final state. Different approaches should give rise to the same wavefunction. These constraints give us the consistent equations for  $\nu_2$

$$d\nu_2(g_0, g_1, g_2, g_3) = \frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)} = 1. \quad (5)$$

That means  $\nu_2$  should be a  $U_T(1)$ -valued 2-cocycle, provided that the wavefunction Eq. (2) is a fix-point wavefunction. This  $\nu_2$  data is the same as those for construction of bosonic 1D SPT states [11].

With the consistent equation for  $\nu_2$ , we can deduce the complete form of the coefficient  $\Psi$  for 1D FSPT state :

$$\begin{aligned} \Psi \left( \text{---} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \text{---} \dots \text{---} N \text{---} \right) & \left| \text{---} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \text{---} \dots \text{---} N \text{---} \right\rangle, \\ & = \prod_{i=1}^{N-1} F(g_0, g_i, g_{i+1}) \Psi \left( \text{---} 0 \xrightarrow{c_{(0N)}} N \text{---} \right) \left| \text{---} 0 \xrightarrow{c_{(0N)}} N \text{---} \right\rangle. \end{aligned} \quad (6)$$

Substituting the expression of  $F$  move Eq. (4) into this equation, we can find the explicit expression of the coefficient  $\Psi$ :

$$\Psi \left( \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \quad \dots \quad N \end{array} \right) = \frac{1}{|G_b|^{(N-1)/2}} \prod_{i=1}^{N-1} \nu_2(g_0, g_i, g_{i+1}) \Psi \left( \begin{array}{c} 0 \xrightarrow{c_{(0N)}} N \end{array} \right). \quad (7)$$

Now for simplicity, we fix the coefficient for reference state :

$$\frac{1}{|G_b|^{(N-1)/2}} \Psi \left( \begin{array}{c} 0 \xrightarrow{c_{(0N)}} N \end{array} \right) = 1. \quad (8)$$

Finally we have the wavefunction of 1D FSPT chain with  $N+1$  sites:

$$\begin{aligned} \Psi \left( \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \quad \dots \quad N \end{array} \right) &= \prod_{i=1}^{N-1} \nu_2(g_0, g_i, g_{i+1}) \left| \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \quad \dots \quad N \end{array} \right\rangle, \\ &= \prod_{i=1}^{N-1} \nu_2(g_0, g_i, g_{i+1}) \left| \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \quad \dots \quad N \end{array} \right\rangle. \end{aligned} \quad (9)$$

As expected, the coefficient of the 1D fermionic symmetry-protected topological (FSPT) state is the same as that of the 1D bosonic symmetry-protected topological (SPT) state with the same bosonic group  $G_b$  [11]. This is because in the 1D FSPT state with  $G_f = G_b \times \mathbb{Z}_2^f$ , the bosonic and fermionic parts of the state decouple.

### B. Strange Correlator for 1D FSPT Phases with $G_f = G_b \times \mathbb{Z}_2^f$

Now that we have obtained the basic form of the 1D FSPT state, we can construct the strange correlator. The strange correlator is defined for a given wavefunction  $|\Psi\rangle$  as  $C(r, r') = \frac{\langle \Omega | \hat{O}(r) \hat{O}(r') | \Psi \rangle}{\langle \Omega | \Psi \rangle}$ , where  $|\Omega\rangle$  is a symmetric trivial state serving as a reference state, and  $\hat{O}$ 's are local observables in the system. It has been demonstrated that, given a nontrivial SPT wavefunction, the strange correlator is generally long-range or power-law correlated in the long-distance limit. One of the advantages of the strange correlator is that we can robustly diagnose nontrivial SPT phases without restricting to the boundaries, which is particularly useful for SPTs whose protecting symmetry is broken by the existence of boundaries [1–3].

Assuming periodic boundary conditions (In the rest of this passage we will always assume periodic boundary condition) and choosing a reference vertex labeled as  $g^* = e$ , the general wave function for the system can be obtained as:

$$|\Psi\rangle = \sum_{g_k \in G_b} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \left| \begin{array}{c} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \end{array} \right\rangle. \quad (10)$$

In the context of FSPT phases with trivial central extension, the topological properties of the bosonic and fermionic sectors can be considered as effectively decoupled. This implies that the non-trivialities associated with  $\nu_2(e, g_k, g_{k+1})$  characterizing the bosonic sector and  $n_1(g_k, g_{k+1})$  characterizing the fermionic sector are independent of each other [6]. Consequently, to study FSPT phases and uncover their non-trivial nature, it is vital to choose fermionic and bosonic local observables that can detect each sector respectively.

To investigate the non-triviality of the fermionic part FSPT phase, we employ complex fermion annihilating operators denoted as  $c_{i,i+1}$  and  $c_{j,j+1}$  as our local observables.

By utilizing the peculiar strange correlator  $C(i, j) = \frac{\langle \Omega | c_{i,i+1} c_{j,j+1} | \Psi \rangle}{\langle \Omega | \Psi \rangle}$ , where  $|\Omega\rangle$  denotes the trivial reference state, we can probe the non-triviality of the fermionic part of the FSPT state  $|\Psi\rangle$ .

Note that  $n_1(g_k, g_{k+1})$  is a  $\mathbb{Z}_2$ -valued cochain with the property  $n_1(g_k, g_{k+1}) = g \bullet n_1(g_k, g_{k+1}) = n_1(gg_k, gg_{k+1})$  [11]. Therefore, we can divide  $G_b$  into two non-intersecting sets  $Z = \{g | n_1(e, g) = 0\}$  and  $O = \{g | n_1(e, g) = 1\}$ . Since  $n_1(g_k, g_{k+1})$  is also a  $\mathbb{Z}_2$ -valued cocycle with the property  $dn_1(g_0, g_1, g_2) = n_1(g_1, g_2) + n_1(g_0, g_2) + n_1(g_0, g_1) = 0 \pmod{2}$ , we can set  $g_0 = e$  and obtain  $dn_1(e, g_1, g_2) = n_1(g_1, g_2) + n_1(e, g_2) + n_1(e, g_1) = 0 \pmod{2}$ . If  $g_1$  and  $g_2$  belong to subgroup  $Z$ , it follows that  $g_1^{-1}g_2 \in Z$ ,  $g_2^{-1}g_1 \in Z$ ,  $g_1^{-1}g_1 = e \in Z$ ,  $g_1^{-1}e = g_1^{-1} \in Z$ , and  $g_1g_2 \in Z$ . These properties establish that  $Z$  is a subgroup of  $G_b$ . Conversely, if  $g_1$  and  $g_2$  are elements of  $O$ , we observe that  $g_1^{-1}g_2 \in Z$  and  $n_1(g_1, g_2) = 0$ . Furthermore, by examining the case where  $g_1 \in O$  and  $g_2 \in Z$ , we find that  $g_1g_2 \in O$ , implying that the cardinality of  $Z$  matches that of  $O$ , denoted as  $|Z| = |O|$ . Consequently, the sets  $Z$  and  $O$  partition  $G_b$  into



### C. Proof of Correctness of Strange Correlation Theory for 1D SPT Phases

As always, we assume periodic boundary conditions. The 1D symmetry-protected topological (SPT) state under an on-site symmetry  $G_b$  can be regarded as a 1D fermionic symmetry-protected topological (FSPT) state with  $n_1(g_0, g_1) = 0$ . Physically, this implies that there are no complex fermion decorations on each link  $\langle i, j \rangle$ . We can express the 1D SPT fixed-point state as follows:

$$|\Psi\rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \left| \begin{array}{c} \xrightarrow{g^0} \xrightarrow{g^1} \xrightarrow{g^2} \dots \xrightarrow{g^N} \end{array} \right\rangle \quad (17)$$

With this peculiar form of the wave function, we can define a transfer matrix  $T$  such that  $T_{ij} = \nu_2(e, g_i, g_j)$ . This matrix is of size  $|G_b| \times |G_b|$  and represents the peculiar SPT phase. Consequently, for the strange correlator, the denominator can be expressed as follows:

$$\langle \Omega | \Psi \rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) = \text{tr}(T^N) \quad (18)$$

Similarly, the numerator can be expressed as:

$$\langle \Omega | \hat{O}_i \hat{O}_j | \Psi \rangle = \text{tr}(\hat{O}_i T^{|i-j|} \hat{O}_j T^{N-|i-j|}) \quad (19)$$

Here,  $\hat{O}_i$  and  $\hat{O}_j$  are operators acting on sites  $i$  and  $j$  respectively, and  $|i-j|$  represents the distance between two sites.

In Appendix A, we will demonstrate that for any local observable  $\hat{O}_i$ , if the transfer matrix can be diagonalized, the strange correlator exhibits exponential decay only if there is a single dominant eigenvalue (where dominance is determined by the magnitude  $|\lambda|$ ). Otherwise, the strange correlator will display power-law decay, oscillations, or gradually converge to a constant value.

On the other hand, we can prove that for a non-trivial symmetry-protected topological (SPT) state with a non-trivial 2-cocycle  $\nu_2 \in H^2(G_b, U_T(1))$ , the eigenvalues of the corresponding transfer matrix always exhibit degeneracy due to protected symmetry. Conversely, for a trivial SPT state with a trivial 2-cocycle  $\nu_2 \in H^2(G_b, U_T(1))$ , the dominant eigenvalue of the transfer matrix (except for certain extreme cases with accidental degeneracy) is always unique. Consequently, the strange correlator for a trivial SPT state will consistently display exponential decay, while for a non-trivial SPT state, the strange correlator may exhibit power-law decay, oscillations, or converge to a constant value. This observation perfectly aligns with the predictions of the strange correlation theory. Therefore, we have rigorously proven the correctness of the strange correlation theory for 1D FSPT states with  $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$ .

### III. STRANGE CORRELATOR FOR 1D FSPT PHASES WITH NON-TRIVIAL CENTRAL EXTENSION: $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$

#### A. Proof of Correctness of Strange Correlation Theory for 1D FSPT Phases with Non-trivial Central Extension

Now we consider the general case of 1D FSPT state with non-trivial central extension:  $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$ . According to Ref [6], 1D FSPT state  $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$  is classified by the group super-cohomology theory [6]:

$$\begin{cases} n_1 \in H^1(G_b, \mathbb{Z}_2), \\ \nu_2 \in C^2(G_b, U_T(1))/B^2(G_b, U_T(1))/\Gamma^2. \end{cases} \quad (20)$$

$$\begin{cases} n_1(gg_0, gg_1) = n_1(g_0, g_1) = n_1(g_0^{-1}g_1), \\ \nu_2(g, ga, gab) =^g \nu_2(a, b) = \nu_2(a, b)^{1-2s_1(g)} \cdot (-1)^{(\omega_2 \smile n_1)(g, a, b)}. \end{cases} \quad (21)$$

$$\begin{cases} dn_1 = 0, \\ d\nu_2 = (-1)^{\omega_2 \smile n_1}. \end{cases} \quad (22)$$

$$\Gamma^2 = \{(-1)^{\omega_2} \in H^2(G_b, U_T(1))\}. \quad (23)$$

By FSLU transformation of 1D FSPT phases with  $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$ :

$$\Psi \left( \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \end{array} \right) = F(g_0, g_1, g_2) \Psi \left( \begin{array}{c} 0 \xrightarrow{c_{(02)}} 2 \end{array} \right), \quad (24)$$

where the FSLU  $F$  operator is defined as

$$F(g_0, g_1, g_2) = \frac{1}{|G_b|^{1/2}} \nu_2(g_0, g_1, g_2) (c_{01}^{g_0})^{\dagger n_1(g_0, g_1)} (c_{12}^{g_1})^{\dagger n_1(g_1, g_2)} (c_{02}^{g_0})^{n_1(g_0, g_2)} \quad (25)$$

we can construct 1D FSPT wavefunction in the same way

$$|\Psi\rangle = \prod_{k=1}^{N-1} \nu_2(g_0, g_k, g_{k+1}) \left| \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \quad \dots \quad \mathbf{N} \end{array} \right\rangle. \quad (26)$$

As usual, we can choose complex fermion annihilating operators as local observables. However, in this case, due to the onsite symmetry of the system, the complex fermion annihilating operators take on a slightly different form [6], which are  $\tilde{c}_{i,i+1} = \sum_{g_i \in G_b} c_{i,i+1}^{g_i}$  and  $\tilde{c}_{j,j+1} = \sum_{g_j \in G_b} c_{j,j+1}^{g_j}$ . We use their strange correlator to detect the non-triviality of fixed point state  $|\Psi\rangle$ .

Notice that the 1D FSPT wave function with  $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$  takes on the same form as that with  $G_f = G_b \times \mathbb{Z}_2^f$ , but with the additional constraint that  $\nu_2(g_0, g_1, g_2)$  and  $n_1(g_0, g_1)$  are subject to the obstruction function  $d\nu_2 = (-1)^{\omega_2 \sim n_1}$ . However, although  $\nu_2 \in BH^2(G_b, \mathbb{Z}_2)$  is constrained by the obstruction function, the obstruction-free  $n_1 \in H^1(G_b, \mathbb{Z}_2)$  still satisfies the cocycle conditions  $dn_1 = 0$ . Hence, we can still partition  $G_b$  into two distinct cosets:  $Z = \{g|n_1(e, g) = 0 \mid n_1 \in H^1(G_b, \mathbb{Z}_2)\}$  and  $O = \sigma Z = \{g|n_1(e, g) = 1 \mid n_1 \in H^1(G_b, \mathbb{Z}_2)\}$ .

As mentioned earlier, the term

$\left\langle \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \quad \dots \quad \mathbf{N} \end{array} \left| \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} \right| \begin{array}{c} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \end{array} \right\rangle \neq 0$  only if  $n_1(g_k, g_{k+1}) = 0$  for all  $k \neq i, j$  and  $n_1(g_i, g_{i+1}) = n_1(g_j, g_{j+1}) = 1$ . This implies that  $g_i^{-1} g_{i+1}, g_j^{-1} g_{j+1} \in O$ , and  $g_k^{-1} g_{k+1} \in Z$  for all  $k \neq i, j$ . There are two situations to consider. If  $g_0 \in Z$ , then  $g_{i+1}, \dots, g_j \in O$  and  $g_{j+1}, \dots, g_i \in Z$ . Conversely, if  $g_0 \in O$ , then  $g_{i+1}, \dots, g_j \in Z$  and  $g_{j+1}, \dots, g_i \in O$ .

To calculate the numerator  $\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle$ , we consider periodic boundary conditions with  $g_0 = g_N$ . The expression can be written as follows:

$$\begin{aligned} & \sum_{g_k} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \left\langle \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \quad \dots \quad \mathbf{N} \end{array} \left| \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} \right| \begin{array}{c} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \end{array} \right\rangle \\ &= \sum_{g_{j+1}, \dots, g_i \in Z} \sum_{g_{i+1}, \dots, g_j \in O} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) + \sum_{g_{j+1}, \dots, g_i \in O} \sum_{g_{i+1}, \dots, g_j \in Z} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) \end{aligned} \quad (27)$$

To simplify the calculation, we can use the transfer matrix method again. The numerator can be expressed as:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle = \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (28)$$

Here,  $T_{OZ}, T_Z, T_{ZO}, T_O$  are defined as follows:  $T_Z(i, j) = \nu_2(e, g_i, g_j)$  for  $g_i, g_j \in Z$ ,  $T_O(i, j) = \nu_2(e, g_i, g_j)$  for  $g_i, g_j \in O$ ,  $T_{ZO}(i, j) = \nu_2(e, g_i, g_j)$  for  $g_i \in Z, g_j \in O$ , and  $T_{OZ}(i, j) = \nu_2(e, g_i, g_j)$  for  $g_i \in O, g_j \in Z$ . By using the relationship  $T_O = S^{-1} T_Z S$  (see Appendix B), where  $S$  is a diagonal matrix, we can rewrite the numerator as:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle = \text{tr}(\tilde{T}_{ZO} D^{|i-j|} \tilde{T}_{OZ} D^{N-|i-j|}) + \text{tr}(\tilde{T}_{OZ} D^{|i-j|} \tilde{T}_{ZO} D^{N-|i-j|}) \quad (29)$$

Here,  $D = A^{-1} T_Z A$  is the diagonalization of  $T_Z$ , and  $\tilde{T}_{ZO} = A^{-1} T_{ZO} S^{-1} A$  and  $\tilde{T}_{OZ} = A^{-1} S T_{OZ} A$ . Furthermore, the denominator can be expressed as:

$$\langle \Omega | \Psi \rangle = \text{tr}(T_Z^N) + \text{tr}(T_O^N) \quad (30)$$

Under the assumption of periodic boundary conditions, we have  $\langle \Omega | \tilde{c}_{i,i+1} | \Psi \rangle = 0$ , implying that the correlation background is always zero. In Appendix B, it is demonstrated that in the thermodynamic limit ( $N \rightarrow \infty$ ), the strange correlator will converge to a constant, exhibit power-law decay, or oscillate. These findings are in perfect agreement with the predictions of the strange correlation theory. Moreover, if  $n_1$  is trivial, the problem reverts back to the SPT strange correlation theory. Consequently, we can confidently conclude that the strange correlation theory remains applicable to 1D fermionic symmetry-protected topological (FSPT) states with non-trivial central extensions.



## B. Classification of 1D FSPT Phases of Non-trivial Central Extension By Strange Correlator

Having established the validity of the strange correlation theory for 1D FSPT (Fermionic Symmetry Protected Topological) phases with non-trivial central extensions, we can now proceed to further classify the bosonic part of a 1D FSPT state with a non-trivial central extension. To do so, we can choose a non-trivial reference state as described in Equation (31), with non-trivial values assigned to  $n_1(g_0, g_1)$  and  $\nu_2(g_0, g_1, g_2)$ . However, since we have already determined the non-triviality of the fermionic part  $n_1(g_0, g_1)$ , we can select  $n_1(g_0, g_1)$  to be identical to that of the non-trivial state under consideration. This choice allows us to extract and classify the non-trivial bosonic component of our 1D FSPT state.

$$|\Omega\rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu'_2(e, g_k, g_{k+1}) \sum_{\text{all conf.}} | \bullet \circ \bullet \bullet \bullet \circ \cdots \bullet \bullet \rangle \quad (31)$$

and the nontrivial FSPT state under measure is

$$|\Psi\rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu''_2(e, g_k, g_{k+1}) | \bullet \circ \bullet \bullet \bullet \circ \cdots \bullet \bullet \rangle. \quad (32)$$

With this peculiar form of the wave function, we can define a transfer matrix  $T$  such that  $T_{ij} = \nu''_2(e, g_i, g_j) \nu'^*_2(e, g_i, g_j) = \nu_2(e, g_i, g_j)$ . Therefore, for the strange correlator, the denominator can be expressed as follows:

$$\langle \Omega | \Psi \rangle = \sum_{\text{all conf.}} \prod_{k=0}^{N-1} \nu_2(e, g_k, g_{k+1}) = \text{tr}(T^N) \quad (33)$$

Similarly, the numerator can be expressed as:

$$\langle \Omega | \hat{O}_i \hat{O}_j | \Psi \rangle = \text{tr}(\hat{O}_i T^{|i-j|} \hat{O}_j T^{N-|i-j|}) \quad (34)$$

Here,  $\hat{O}_i$  and  $\hat{O}_j$  are operators acting on sites  $i$  and  $j$  respectively, and  $|i-j|$  represents the distance between two sites.

In Appendix C, we demonstrate that if  $\nu''_2(e, g_i, g_j) \neq \nu'_2(e, g_i, g_j) \nu_1(g_i, g_j)$ , indicating that  $\nu_2(e, g_i, g_j)$  is a non-trivial element of group supercohomology  $BH^2(G_b, U_T(1))$ , the eigenvalues of the corresponding transfer matrix will always exhibit degeneracy due to protected symmetry. On the other hand, if  $\nu''_2(e, g_i, g_j)$  and  $\nu'_2(e, g_i, g_j)$  represent the same phase, the dominant eigenvalue of the transfer matrix (except for certain extreme cases with accidental degeneracy) will be unique. Consequently, when the detected state is in the same phase as the reference state, the strange correlator will exhibit exponential decay. This behavior arises due to the compatibility of the fermionic and bosonic components in the same phase. However, when the detected state belongs to a different phase than the reference state, the strange correlator may display distinct behaviors such as power-law decay, oscillations, or convergence to a constant value. These diverse behaviors arise from the mismatch between the fermionic and bosonic components in different phases. Therefore, by analyzing the behavior of the strange correlator, we can effectively classify 1D FSPT states with a non-trivial central extension  $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$ . This classification allows us to distinguish between different phases based on the distinct patterns exhibited by the strange correlator.

## IV. SPECIAL CASE OF STRANGE CORRELATOR FOR 1D FSPT STATES

$$\text{A. } G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$$

We start with a simple case with  $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$ . According to Ref.:

$$H^2[\mathbb{Z}_2^T, U_T(1)] = \mathbb{Z}_2, \quad (35)$$

$$H^1(\mathbb{Z}_2^T, \mathbb{Z}_2) = \mathbb{Z}_2 \quad (36)$$

It is easily checked that:

$$n_1(e, e) = n_1(T, T) = 0 \quad n_1(e, T) = n_1(T, e) = 1 \quad (37)$$

$$dn_1 = 0 \pmod{2} \quad (38)$$

$$\nu_2(e, e, e) = \nu_2(T, T, T) = \nu_2(e, T, T) = \nu_2(T, e, e) = \nu_2(e, e, T) = \nu_2(T, T, e) = 1 \quad (39)$$

$$\nu_2(e, T, e) = \nu_2(T, e, T) = -1 \quad (40)$$

$$d\nu_2 = 1 \quad (41)$$

Suppose periodic boundary condition, there are only two configurations such that

$\left\langle \begin{array}{c} 0 \xrightarrow{c_{(01)}} 1 \xrightarrow{c_{(12)}} 2 \cdots N \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \middle| c_{i,i+1} c_{j,j+1} \right\rangle \neq 0$ . They are: (1)  $g_{i+1} \dots g_j = T$  and  $g_k = e$  for  $k \neq i+1 \dots j+1$ , and (2)  $g_{i+1} \dots g_{j+1} = e$  and  $g_k = T$  for  $k \neq i+1 \dots j+1$ . In this case, we can choose the total trivial reference state with  $n_1 = 0$  and  $\nu_2 = 1$ , and use the complex fermion annihilation operators  $c_{i,i+1}$  and  $c_{j,j+1}$  as our local observables. With these choices, we can proceed to compute the strange correlator:

$$C(i, j) = \frac{\langle \Omega | c_{i,i+1} c_{j,j+1} | \Psi \rangle}{\langle \Omega | \Psi \rangle} = -1 \quad (42)$$

The result obtained manifest the non-triviality of the fermionic part. Next, we aim to prove the non-triviality of the bosonic part. To accomplish this, we select a non-trivial reference state where the 1-cocycle  $n_1$  is the same as that of the state under measurement. We consider the order parameters  $O_i$  and  $O_j$  as our local observables. With these choices, we can compute the strange correlator as follows:

$$C(i, j) = \frac{\langle \Omega | O_i O_j | \Psi \rangle}{\langle \Omega | \Psi \rangle} = \begin{cases} 1 & (|i - j| \bmod 4 < 2) \\ -1 & (|i - j| \bmod 4 > 1) \end{cases} \quad (43)$$

This result demonstrates oscillation, which is consistent with the predictions of the strange correlation theory. Additionally, if we calculate the eigenvalues of the transfer matrix for the  $G_b = Z_2^T$  group, we find that they are  $1 + i$  and  $1 - i$ , respectively, indicating a two-fold degeneracy. This observation is consistent with the order of  $\nu_2 \in H^2(G_b, U_T(1)) = Z_2$ , which profoundly explains why the strange correlator of the non-trivial SPT state of  $G_b = Z_2^T$  exhibits oscillations.

It is important to note that even when considering the trivial reference state, the computation of the strange correlator using the complex fermion annihilation operators  $c_{i,i+1}$  and  $c_{j,j+1}$  results in a constant value. Appendix B further reveals that this property can be generalized to more general cases.

$$\text{B. } G_f = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2^f$$

Following the general procedure to construct strange correlator for 1D FSPT state with trivial central extension, however, this time we choose a nontrivial reference state with  $n_1(g_0, g_1) = 0$ , a trivial cocycle, and  $\nu_2(g_0, g_1, g_2)$  the same as that of the nontrivial state under measure. Using the complex fermion annihilating operators  $c_{i,i+1}$  and  $c_{j,j+1}$  as our local observables, the strange correlator  $C(i, j) = \frac{\langle \Omega | c_{i,i+1} c_{j,j+1} | \Psi \rangle}{\langle \Omega | \Psi \rangle}$  is always equal to 1. This result indicates the nontriviality of the fermionic part of our 1D FSPT.

Conversely, to detect the non-triviality of the bosonic part of 1D FSPT state, we could choose a non-trivial reference state with 1-cocycle  $n_1$  same as that of the state under measure and  $\nu_2 = 1$ , then it turns into a traditional SPT strange correlator. By previous calculations, we determined that  $H^2(Z_4 \times Z_4, U_T(1)) = Z_4$ , indicating the presence of one non-trivial cocycle with order 2 and two other non-trivial cocycles with order 4, representing the non-trivial SPT state. Specifically, we find that  $\nu_2(e, a, ab) = (-1)^{a_1 b_2}$ , where  $a = (a_1, a_2)$  with  $a_1, a_2 \in Z_4$ , is a non-trivial cocycle with order 2. Additionally,  $\nu_2(e, a, ab) = i^{a_1 b_2}$  and  $\nu_2(e, a, ab) = (-i)^{a_1 b_2}$  correspond to the other two non-trivial cocycles with order 4. Upon verification, we observe that the transfer matrix of the non-trivial cocycle of order 2 possesses two non-zero eigenvalues:  $\lambda_1 = 8$  and  $\lambda_2 = 8$ . Similarly, the transfer matrix of the other two non-trivial cocycles of order 4 possesses four non-zero eigenvalues:  $\lambda_1 = 4$ ,  $\lambda_2 = 4$ ,  $\lambda_3 = 4$ , and  $\lambda_4 = 4$ . The degeneracy of the eigenvalues perfectly aligns with the order of the corresponding non-trivial cocycles representing the transfer matrix. Therefore, according to the theorem established in Appendix A, we can confidently conclude that the strange correlator of the bosonic part of the  $G_f = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2^f$  group will either converge to a constant, exhibit power-law decay, or oscillate. Therefore the decaying behaviors of strange correlator with  $G_f = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2^f$  is consistent with our theoretical expectations.

$$\text{C. } G_f = \mathbb{Z}_{2N_0}^f \times \prod_{i=1}^K \mathbb{Z}_{N_i}$$

We now consider the case where the symmetry group  $G_f$  is a central extension of the bosonic unitary finite Abelian group  $G_b = Z_{N_0} \times \prod_{i=1}^K Z_{N_i}$  [6, 21, 22]. In this case, we have  $\omega_2(e, a, ab) = \left\lfloor \frac{a^0 + b^0}{N_0} \right\rfloor$ , where  $\lfloor x \rfloor$  is the floor operator that picks out the largest integer less than or equal to  $x$ .

It is well-known that for a given positive integer  $N$ , there exists a unique factorization  $N = \prod_p p^{n_p}$  where  $p$  is a prime number and  $n_p$  is a positive integer in  $\mathbb{Z}_+$ . Furthermore, there is a group isomorphism  $\mathbb{Z}_N \cong \prod_p \mathbb{Z}_{p^{n_p}}$ . For



prime numbers  $p > 2$ , the cohomology group  $H^*(\mathbb{Z}_{p^{n_p}}, \mathbb{Z}_2)$  is trivial. Therefore, the symmetry group  $\prod_{p>2} \mathbb{Z}_{p^{n_p}}$  can only protect bosonic SPT phases and can only affect the FSPT classifications by adding some BSPT phases. To understand genuine FSPT, we can assume  $N_i = 2^{k_i}$  for  $1 \leq i \leq K$ , where  $k_i \in \mathbb{Z}^+$ , in the bosonic symmetry group  $G_b$ . Without loss of generality, we can also reorder the Abelian groups such that  $N_i \leq N_{i+1}$  for  $1 \leq i \leq K-1$ .

We can assume without loss of generality that  $N_0 = 2^{k_0}$  is the  $k_0$ -th power of 2 ( $k_0 \geq 1$ ). Otherwise,  $Z_{2N_0}^f$  is isomorphic to  $Z_{2^{k_0+1}}^f \times Z_{N_0/2^{k_0}}$ , and we can absorb the latter subgroup into  $Z_{N_i}$  with  $i > 0$ . It is worth noting that  $n_2^{(0)}$  is the nontrivial  $\mathbb{Z}_2$ -valued 2-cocycle of  $Z_{N_0}$ , rather than that of  $Z_{2N_0}$ . This is different from  $n_2^{(i)}$  for  $1 \leq i \leq K$ .

Using the ingredients above, we can calculate the obstruction equation  $d\nu_2 = (-1)^{\omega_2 \sim n_1}$ . Assuming  $n_1 = n_1^i$ , we can prove that the right-hand side is obstruction-free if and only if  $N_i > N_0$  or equivalently  $k_i > k_0$ . After a laborious and intricate calculation, we obtain the explicit form of  $\nu_2(g_0, g_1, g_2)$ :

$$\nu_2(g_0, g_1, g_2) = e^{i \frac{\pi}{N_0} n_1^0(g_0, g_1) n_1^i(g_1, g_2)}, \quad (44)$$

where  $n_1^j(e, g) = g^j \bmod 2$  and  $g = (g^0, g^1, g^2, \dots, g^K)$ .

As mentioned previously, for  $G_f$  as a central extension of the bosonic unitary finite Abelian group  $G_b$  with  $\omega_2 \neq 0$ , we select the complex fermion annihilating operators  $\tilde{c}_{i,i+1} = \sum_{g_i \in G_b} c_{i,i+1}^{g_i}$  and  $\tilde{c}_{j,j+1} = \sum_{g_j \in G_b} c_{j,j+1}^{g_j}$  as local observables in the system and use their strange correlator to detect the non-triviality of the system. After performing a detailed and intricate calculation, we obtain the following expression for the strange correlator:

$$C(i, j) = \frac{\langle \Omega | \tilde{c}_{i,i+1}, \tilde{c}_{j,j+1} | \Psi \rangle}{\langle \Omega | \Psi \rangle} = \frac{\sum_{m=0}^2 \binom{2}{m} e^{i(m\pi/N_0)}}{2^2} = \text{Const}, \quad (45)$$

where  $\binom{N}{m}$  denotes the combination coefficient. This result confirms that when using complex fermion annihilation operators as detectors, the strange correlator of the 1D FSPT state with symmetry group  $G_f = \mathbb{Z}_{2N_0}^f \times \prod_{i=1}^K \mathbb{Z}_{N_i}$  is a constant. This finding is in perfect agreement with the theory of strange correlations, as it supports the notion that the strange correlator remains constant when the detected state and the reference state are in the same phase.

## V. CONCLUSION AND DISCUSSION

In this study, we focused on examining the scaling behavior of the strange correlator in 1D fermionic symmetry-protected topological (FSPT) states which possess symmetries characterized by  $G_f = G_b \times \mathbb{Z}_2^f$  and  $G_f = G_b \times_{\omega_2} \mathbb{Z}_2^f$ , where  $G_f$  represents the fermionic symmetry and  $G_b$  represents the bosonic symmetry. Through rigorous analysis, we established that the properties of the strange correlator remain valid within 1D FSPT phases. Furthermore, we provided specific examples that demonstrate the existence of both constant and oscillating strange correlators under certain conditions. However our study only focused on the 1D case, and the availability of the FSPT strange correlator still require further investigation, especially in higher-dimensional systems which possess more degrees of freedom, such as the Kitaev Majorana chain and  $p + ip$  superconductor decorations.

Despite the limitations of our study, it holds significant importance as it provides a rigorous and systematic proof of the applicability of the strange correlation theory in both 1D symmetry-protected topological (SPT) states and 1D fractionalized symmetry-protected topological (FSPT) states. This work not only reaffirms the stability of the strange correlation theory but also validates the general SPT model and the subsequent FSPT model, which serves as a crucial link between the two theories for studying symmetry-protected topological states. Additionally, we have found a special property that the eigenvectors of transfer matrix of 1D SPT states carry the projective representation of the local symmetry group  $G_b$ , which are very likely to be promoted to research on higher dimensional SPT state.

In our future work, we will primarily focus on investigating the stability of the strange correlator in 2D FSPT states. This presents new challenges, as the availability of complex fermion annihilation operators might not be as straightforward, and the presence of a Majorana chain decoration [6, 23–30] introduces an additional layer of complexity. By addressing these complexities, we aim to deepen our understanding of the behavior and properties of the strange correlator in higher-dimensional FSPT states.

In conclusion, we hope that our work inspires further research in this field and brings us closer to a complete understanding of the topological properties of quantum many-body systems.

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## Appendix A: Proof of Correctness of 1D SPT Strange Correlation Theory

### 1. Proof of Strange Correlation theory under Transfer Matrix Degeneracy

In this section, we present a rigorous proof of the availability of the 1D symmetry-protected topological (SPT) strange correlator using the transfer matrix method.

For any finite symmetry group  $G_b$ , we can label the group elements as a sequence  $g_1, g_2, \dots, g_M$ , with  $g_1$  representing the unit element of the group  $G_b$ . The transfer matrix can then be expressed as  $T_{ij} = \nu_2(e, g_i, g_j)$ , where  $\nu_2$  denotes the 2nd cohomology group. We assume that the transfer matrix can be diagonalized as  $A^{-1}TA = D$ , where  $D$  is the diagonalized form of  $T$ . To compute the numerator of the strange correlator  $\langle \Omega | \hat{O}_i \hat{O}_j | \Psi \rangle$ , we transform it into  $\langle \Omega | \hat{O}_i \hat{O}_j | \Psi \rangle = \text{tr}(\hat{O}'_i D^{|i-j|} \hat{O}'_j D^{N-|i-j|})$ , where  $\hat{O}'_i = A^{-1} \hat{O}_i A$ . The matrix  $Y = \hat{O}'_i D^{|i-j|} \hat{O}'_j D^{N-|i-j|}$  can be expressed as:

$$Y = \begin{pmatrix} O'_{11}\lambda_1^L & O'_{12}\lambda_2^L & \dots & O'_{1M}\lambda_M^L \\ O'_{21}\lambda_1^L & O'_{22}\lambda_2^L & \dots & O'_{2M}\lambda_M^L \\ \dots & \dots & \dots & \dots \\ O'_{M1}\lambda_1^L & O'_{M2}\lambda_2^L & \dots & O'_{MM}\lambda_M^L \end{pmatrix} \begin{pmatrix} O'_{11}\lambda_1^{N-L} & O'_{12}\lambda_2^{N-L} & \dots & O'_{1M}\lambda_M^{N-L} \\ O'_{21}\lambda_1^{N-L} & O'_{22}\lambda_2^{N-L} & \dots & O'_{2M}\lambda_M^{N-L} \\ \dots & \dots & \dots & \dots \\ O'_{M1}\lambda_1^{N-L} & O'_{M2}\lambda_2^{N-L} & \dots & O'_{MM}\lambda_M^{N-L} \end{pmatrix} \quad (\text{A1})$$

Here, we assume that  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_M|$ .

Now, we consider two situations. First, if  $|\lambda_1| > |\lambda_2|$ , then under the thermodynamic limit ( $N \rightarrow \infty$ ), the strange correlator can be expressed as ( $L = |i-j|$ ):

$$C(i, j) = \frac{\text{tr}(Y)}{\text{tr}(T^N)} = O'_{11}{}^2 + O'_{12}O'_{21} \left( \frac{\lambda_2}{\lambda_1} \right)^L + \dots + O'_{1M}O'_{M1} \left( \frac{\lambda_M}{\lambda_1} \right)^L \quad (\text{A2})$$

Additionally, we need to account for the background, which is expressed as:

$$\langle \hat{O}_i \rangle = \frac{\text{tr}(\hat{O}T^N)}{\text{tr}(T^N)} = O'_{11} \quad (\text{A3})$$

Thus, the strange correlator without the background can be expressed as:

$$\langle \hat{O}_i \hat{O}_j \rangle - \langle \hat{O}_i \rangle \langle \hat{O}_j \rangle = O'_{12}O'_{21} \left( \frac{\lambda_2}{\lambda_1} \right)^L + \dots + O'_{1M}O'_{M1} \left( \frac{\lambda_M}{\lambda_1} \right)^L \quad (\text{A4})$$

Clearly, if  $|\lambda_1| > |\lambda_2|$ , the strange correlator will always exhibit exponential decay.

Secondly, if the eigenvalues exhibit a  $p$ -fold degeneracy such that  $|\lambda_1| = |\lambda_2| = |\lambda_3| = \dots = |\lambda_p|$  ( $1 < p \leq M$ ), and we let  $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$  ( $1 \leq i < p$ ), then the strange correlator can be expressed as:

$$\langle \hat{O}_i \hat{O}_j \rangle - \langle \hat{O}_i \rangle \langle \hat{O}_j \rangle = \frac{\sum_{i,j=1}^p O'_{ij} O'_{ji} e^{i\theta_i(N-L)} e^{i\theta_j L}}{1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \dots + e^{i\theta_p N}} - \left( \frac{O'_{11} + O'_{22}e^{i\theta_2 N} + \dots + O'_{pp}e^{i\theta_p N}}{1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \dots + e^{i\theta_p N}} \right)^2 \quad (\text{A5})$$

The behavior of this expression depends on the values of  $N$ ,  $L$ , and the phases  $\theta_i$ . In general, it converges to a constant, presents power law decay or oscillate. The resulting behavior depends on the specific values of  $\theta_i$  and  $\theta_j$ . In summary, when the eigenvalues exhibit a  $p$ -fold degeneracy, the strange correlator can converge to a constant, exhibit power-law decay, or oscillate, depending on the values of  $N$ ,  $L$  and the phases  $\theta_i$ .

### 2. Degeneracy of Transfer Matrix

Thereafter, our objective is to establish that when a non-trivial 2-cocycle  $\nu_2$ , which labels the non-trivial SPT phase, the corresponding transfer matrix  $T$ , defined as  $T_{ij} = \nu_2(e, g_i, g_j)$ , exhibits  $p$ -fold degeneracy, in which  $p$  strongly correlated with the order of  $\nu_2$  in the cohomology group  $H^2(G_b, U_T(1))$ .

*a. Local Symmetry Group  $G_b$  without Time Reversal Symmetry*

Firstly, suppose the local symmetry does not include time reversal symmetry. In this case, for any group element  $g \in G_b$ , the following relation based on the cocycle equation holds:

$$\nu_2(g, g_i, g_j) = \nu_2(e, g_i, g_j) \frac{\nu_2(e, g, g_i)}{\nu_2(e, g, g_j)} \quad (\text{A6})$$

Let's introduce the notations  $T_{ij}^g = \nu_2(g, g_i, g_j)$  and  $S(g)_{ij} = \delta_{ij} \nu_2(e, g, g_j)$ . We can then express their relationship as:

$$T^g = S(g)TS(g)^{-1} \quad (\text{A7})$$

Here,  $T^g$  represents the matrix of 2-cocycle values for the group element  $g$ , while  $S(g)$  is a matrix encoding the 2-cocycle values of the identity element  $e$  combined with  $g$ . The equation shows how  $T^g$  is related to  $S(g)$  through matrix multiplication and inversion.

By the property of the 2-cochain, we can write  $\nu_2(g, g_i, g_j) = \nu_2(e, g^{-1}g_i, g^{-1}g_j)$ . As a result, we observe that the transferred matrix  $T^g$  is a rearrangement of the rows and columns of the original matrix  $T$ . Consequently, there always exists an invertible matrix  $R(g)$  such that  $R(g)T^gR(g^{-1}) = T$ . Essentially this matrix  $R(g)$  is the regular representation of local symmetry group  $G_b$ , and we can express the relationships as follows:

$$R(g)TR(g^{-1}) = S(g)TS(g)^{-1} \quad (\text{A8})$$

$$S(g)^{-1}R(g)T = TS(g)^{-1}R(g) \quad (\text{A9})$$

If  $\nu_2$  is a coboundary, it satisfies the property  $\nu_2(e, g_i, g_j) = \nu_1(g_i, g_j) \frac{\nu_1(e, g_i)}{\nu_1(e, g_j)}$ . In this case, we have the relationship:

$$T = S'T'S'^{-1} \quad (\text{A10})$$

where  $T'_{ij} = \nu_1(g_i, g_j)$  and  $S'_{ij} = \delta_{ij} \nu_1(e, g_i)$ . Here,  $\nu_1$  is a 1-cochain that satisfies  $\nu_1(gg_i, gg_j) = \nu_1(g_i, g_j)$ .

Now, let's consider two situations. First, if the local symmetry group  $G_b$  is an Abelian group, it only has irreducible representations of order 1. Therefore,  $R(g)$  can be reduced to the direct sum of one-dimensional irreducible representations. According to the Ref [31], since  $T'$  and  $R(g)$  are commutative and both diagonalizable matrices, they can be simultaneously diagonalized. Consequently, the eigenvalues of matrix  $T'$  are generally non-degenerate, except for any accidental degeneracy due to intrinsic symmetries. Second, if the local symmetry group  $G_b$  is non-Abelian, the regular representation can only be block diagonalized. However, the first block always representing the trivial representation of  $G_b$  is always one-dimensional. Similarly, since  $T'$  and  $R(g)$  commute and  $T'$  is diagonalizable, they can be simultaneously block diagonalized. The degeneracy of the eigenvalues of  $T'$  corresponds to the size of the diagonalized blocks. However, it is important to note that every group has the irreducible trivial representation of order 1, which means there is always a non-degenerate eigenvalue of the transfer matrix  $T$ , which is typically the dominant one. Additionally, if the largest eigenvalue of  $T'$  exhibits any accidental degeneracy, it can usually be attributed to the presence of intrinsic symmetries, which are generally unavoidable. Therefore, we can conclude that the largest eigenvalue of matrix  $T'$  is generally non-degenerate, which is consistent with the expectation that trivial SPT phases do not possess long-range degeneracy.

If  $\nu_2$  is non-trivial, we claim that  $S(g)^{-1}R(g)$  represents a projective representation of the local symmetry group  $G_b$ . Let's prove this claim:

$$S(g)^{-1}R(g)S(h)^{-1}R(h) \quad (\text{A11})$$

$$= \delta_{ij} \nu_2(e, g, g_i)^{-1} \nu_2(e, h, g^{-1}g_i)^{-1} R(gh) \quad (\text{A12})$$

$$= \delta_{ij} \nu_2(e, g, g_i)^{-1} \nu_2(g, gh, g_i)^{-1} R(gh) \quad (\text{A13})$$

$$= \delta_{ij} \nu_2(e, g, gh)^{-1} \nu_2(e, gh, g_i)^{-1} R(gh) \quad (\text{A14})$$

$$= \nu_2(e, g, g_i)^{-1} S(gh)^{-1} R(gh) \quad (\text{A15})$$

Therefore,  $S(g)^{-1}R(g)S(h)^{-1}R(h) = \nu_2(e, g, g_i)^{-1} S(gh)^{-1} R(gh)$ . This implies that  $S(g)^{-1}R(g)$  represents a projective representation of the group  $G_b$ .

On the other hand, projective representations do not have one-dimensional irreducible representations. Therefore, the projective representation  $S(g)^{-1}R(g)$  could only be block diagonalized, with each block having a size greater than one. According to Ref [31], since we assumed that the transfer matrix  $T$  is diagonalizable and  $T$  commutes with  $S(g)^{-1}R(g)$ , we can conclude that  $T$  and the projective representation  $S(g)^{-1}R(g)$  can be simultaneously block diagonalized with each block size bigger than one. Consequently, the eigenvalues of the transfer matrix  $T$  generally possess degeneracy.

*b. Local Symmetry Group  $G_b$  with Time Reversal Symmetry*

Now let's consider the scenario where time reversal symmetry exists. In this case, we can separate the local symmetry group into two cosets:  $G_b^0$  and  $G_b^T$ , where  $G_b^0$  does not contain any time-reversal elements, and  $G_b^T$  consists of the time-reversal elements. It can be easily verified that  $G_b^0$  is a subgroup of the local symmetry group  $G_b$ .

In the scenario where time reversal symmetry exists, we still maintain the relation  $S(g^0)^{-1}R(g^0)A = AS(g^0)^{-1}R(g^0)$ , where  $g^0$  is an arbitrary element of  $G_b^0$ , and  $R(g^0) = \text{diag}[r(g^0), r(g^0)]$ . Here,  $r(g^0)$  represents the regular representation of the subgroup  $G_b^0$ . As a result, the eigenvectors of the transfer matrix  $A$  exhibit a projective representation of the subgroup  $G_b^0$ . In the subsequent analysis, we will explore the degeneracy of the transfer matrix  $A$  based on the specific 2-cocycle  $\nu_2$  chosen for our analysis.

First, if  $\nu_2$  is a coboundary, it is known from previous analysis that our transfer matrix  $A$  is similar to a modified transfer matrix  $A'$ , where  $A'_{ij} = \nu_1(g_i, g_j)$ . To simplify the notation, let's define the elements of the subgroup as  $g_i^0 \in G_b^0$ . Based on this, we can establish the following relations:

$$A'R(g^0) = R(g^0)A' \quad (\text{A16})$$

$$A'^*R(g^0T) = R(g^0T)A' \quad (\text{A17})$$

Additionally, it can be verified that:

$$R(T) = \begin{pmatrix} O & I \\ I & O \end{pmatrix} \quad (\text{A18})$$

$$R(g^0) = \begin{pmatrix} r(g^0) & O \\ O & r(g^0) \end{pmatrix} \quad (\text{A19})$$

As a result, by combining the aforementioned relationships, we can partition our transfer matrix into four block matrices with the following property:

$$A' = \begin{pmatrix} A'_1 & A'_2 \\ A'_2^* & A'_1 \end{pmatrix} \quad (\text{A20})$$

Based on our previous analysis, where the extended regular representation  $R(g^0)$  can be block diagonalized with more than two blocks of size 1, and assuming the transfer matrix is diagonalizable, we can deduce that the eigenvalues of the transfer matrix may exhibit degeneracy or not. However, except in some extreme cases where accidental degeneracy may occur, there is typically only one dominant eigenvalue. This dominant eigenvalue corresponds to the trivial irreducible representation of the subgroup  $G_b^0$ .

Secondly, if  $\nu_2$  is a non-trivial 2-cocycle, we can prove that the eigenvalues of the transfer matrix  $A$  always exhibit degeneracy. This means that for any eigenvalue  $\lambda$  of  $A$ , there will always exist another eigenvalue  $\lambda'$  such that  $|\lambda| = |\lambda'|$ . We will now provide a proof for this statement under different situations.

Here, it is important to note that while  $\nu_2$  is a non-trivial cocycle under the entire local symmetry group  $G_b$ , it may not necessarily be a non-trivial 2-cocycle when confined to its subgroup  $G_b^0$ .

If  $\nu_2$  remains a non-trivial 2-cocycle under the subgroup  $G_b^0$ , then based on the previous section, we know that the projective representation of the subgroup  $G_b^0$ ,  $S(g^0)^{-1}R(g^0)$ , can only be block diagonalized with each block having a size greater than one. Since  $S(g^0)^{-1}R(g^0)$  and the transfer matrix  $A$  commute with each other, and we have assumed that  $A$  is diagonalizable, according to Ref [31],  $A$  and  $S(g^0)^{-1}R(g^0)$  can be simultaneously block diagonalized with each block having a size greater than one. Therefore, the eigenvalues of the transfer matrix  $A$  will always exhibit degeneracy.

Now, let's consider the situation when  $\nu_2$  is a coboundary under the subgroup  $G_b^0$ , and  $G_b^0$  is an Abelian group. As usual, we can partition the transfer matrix  $A$  into four blocks, and the following relation holds:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad (\text{A21})$$

$$\tilde{A} = \begin{pmatrix} I & O \\ O & S_0(T) \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} I & O \\ O & S_0(T)^{-1} \end{pmatrix} = \begin{pmatrix} A_1 & \hat{A}_2 \\ \hat{A}_2^* & A_1^* \end{pmatrix} \quad (\text{A22})$$

Here,  $A_1(i, j) = \nu_2(e, g_i^0, g_j^0)$ ,  $A_2(i, j) = \nu_2(e, g_i^0, g_j^0T)$ ,  $A_3(i, j) = \nu_2(e, g_i^0T, g_j^0)$ ,  $A_4(i, j) = \nu_2(e, g_i^0T, g_j^0T)$  and  $S_0(T)_{ij} = \delta_{ij}\nu_2(e, T, g_i^0)$  for all  $g_i^0 \in G_b^0$ . Similarly, we can also partition the projective representation  $S(g^0)^{-1}R(g^0)$

into four blocks, and the following relation holds:

$$S(g^0)^{-1}R(g^0) = \begin{pmatrix} S_0(g^0)^{-1}r(g^0) & O \\ O & S_T(g^0)^{-1}r(g^0) \end{pmatrix} \quad (\text{A23})$$

$$\tilde{S}(g^0) = \begin{pmatrix} I & O \\ O & S_0(T) \end{pmatrix} \begin{pmatrix} S_0(g^0)^{-1}r(g^0) & O \\ O & S_T(g^0)^{-1}r(g^0) \end{pmatrix} \begin{pmatrix} I & O \\ O & S_0(T)^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{S}_0(g^0) & O \\ O & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^* \end{pmatrix} \quad (\text{A24})$$

Here,  $S_0(g^0)(i,j) = \delta_{ij}\nu_2(e,g^0,g_i^0)$ ,  $S_T(g^0)(i,j) = \delta_{ij}\nu_2(e,g^0,g_i^0T)$  and  $\tilde{S}_0(g^0) = S_0(g^0)^{-1}r(g^0)$  for arbitrary  $g^0 \in G_b^0$  and all  $g_i^0 \in G_b^0$ . Given that the transfer matrix  $A$  commutes with  $S_0(g^0)^{-1}r(g^0)$ , we can deduce that  $\tilde{A}$  commutes with  $\tilde{S}(g^0)$ . As we also assume that the subgroup  $G_b^0$  is an Abelian group, meaning it only has 1-dimensional irreducible representations. Consequently, the representation matrix  $r(g)$  can be strictly diagonalized, with each diagonal element being unique. Based on the previous proof, we can deduce that  $\tilde{S}_0(g^0)$  can also be strictly diagonalized, with each diagonal element being distinct. Now, if  $\nu_2$  is a non-trivial 2-cocycle, then the phase factor  $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}$  is generally non-trivial. This implies that  $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^*$  can also be strictly diagonalized, with each diagonal element being different from those of  $\tilde{S}_0(g^0)$ . Therefore, we can conclude that  $\tilde{S}(g^0)$  can be strictly diagonalized, with each diagonal element being distinct. According to the findings in Ref [31],  $\tilde{A}$  and  $\tilde{S}(g^0)$  can be diagonalized simultaneously. Now suppose  $P^{-1}\tilde{S}(g^0)P$  represents the diagonalization of  $\tilde{S}(g^0)$ . Then, we have the relation:

$$D = \begin{pmatrix} P^{-1} & O \\ O & P^{-1*} \end{pmatrix} \begin{pmatrix} A_1 & \hat{A}_2 \\ \hat{A}_2^* & A_1^* \end{pmatrix} \begin{pmatrix} P & O \\ O & P^* \end{pmatrix} = \begin{pmatrix} P^{-1}A_1P & O \\ O & P^{-1*}A_1^*P^* \end{pmatrix} \quad (\text{A25})$$

Here, the matrix  $D$  represents the diagonalization of  $\tilde{A}$ . From this expression, it is evident that if  $\lambda$  is an eigenvalue of the transfer matrix  $\tilde{A}$ , then there exists another eigenvalue  $\lambda^*$  of  $\tilde{A}$  (it is important to note that if  $\lambda$  is real, it must be at least two-fold degenerate). Therefore, if  $\nu_2$  is a coboundary under subgroup  $G_b^0$ , and  $G_b^0$  is an Abelian group, then the transfer matrix  $A$  will always exhibit degeneracy with both  $\lambda$  and  $\lambda^*$  as its eigenvalues.

Some may raise doubts regarding whether drawing the same conclusion holds if  $\nu_2$  is a coboundary. However, if  $\nu_2$  is indeed a coboundary, we can establish the following relationship:

$$S(g^0)^{-1}R(g^0) = \begin{pmatrix} \tilde{S}_0(g^0) & O \\ O & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^* \end{pmatrix} \begin{pmatrix} \nu_1(e,g^0)\hat{S}_0(g^0)r(g^0) & O \\ O & \nu_1(e,g^0)\hat{S}_0(g^0)^*r(g^0)^* \end{pmatrix} \quad (\text{A26})$$

Here,  $\hat{S}_0(g^0)(i,j) = \delta_{ij}\frac{\nu_1(e,g_i^0)}{\nu_1(g^0,g_i^0)}$ . Consequently, it is evident that the eigenvalues of  $\tilde{S}(g^0)$  may not always be distinct. As a result, according to the Ref [31] and equation Eq. (A25), it can be concluded that the eigenvalues of the transfer matrix will typically not exhibit degeneracy. It is worth noting that when  $\nu_2$  is a coboundary, the transfer matrix  $A$  always possesses a dominant eigenvalue.

Lastly, let's consider the situation where  $\nu_2$  is a coboundary under the subgroup  $G_b^0$ , and  $G_b^0$  is a non-Abelian group. For non-Abelian group  $G_b^0$ , there always exist high-order irreducible representations. Consequently, the matrix  $r(g^0)$  can only be block diagonalized, with the presence of both size-1 blocks and higher-order blocks. Based on this, we can block diagonalize  $\tilde{S}_0(g^0)$  in the following manner:

$$\begin{pmatrix} P^{-1} & O \\ O & P^{-1*} \end{pmatrix} \begin{pmatrix} \tilde{S}_0(g^0) & O \\ O & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}\tilde{S}_0(g^0)^* \end{pmatrix} \begin{pmatrix} P & O \\ O & P^* \end{pmatrix} = \begin{pmatrix} D & \\ & B \\ & & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}D^* \\ & & & \frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^* \end{pmatrix} \quad (\text{A27})$$

The block  $D$  is a strictly diagonalized matrix, while the block  $B$  consists of a series of block matrices along the diagonal with each block size larger than one. As we have demonstrated previously, the matrix  $\tilde{A}$  can be simultaneously block diagonalized with  $\tilde{S}(g^0)$ . The eigenvalues corresponding to block  $B$  and  $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}B^*$  are always degenerate. Therefore, our focus will be solely on the eigenvalues associated with the diagonalized block  $D$  and  $\frac{\nu_2(e,T,g^0T)}{\nu_2(e,g^0,g^0T)}D^*$ . In

fact, we can always perform a similar transformation on both  $\tilde{A}$  and  $\tilde{S}(g^0)$  to align the diagonalized blocks together:

$$X^{-1} \begin{pmatrix} D & & \\ & B & \\ & \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} D^* & \\ & & \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} B^* \end{pmatrix} X = \begin{pmatrix} D & & \\ & \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} D^* & \\ & & B \\ & & & \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} B^* \end{pmatrix} \quad (\text{A28})$$

$$X^{-1} \tilde{A} X = \begin{pmatrix} a_1 & \hat{a}_2 & \cdot \\ \hat{a}_2^* & a_1^* & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad X^{-1} \tilde{S}(g^0) X = \begin{pmatrix} \tilde{S}_0(g^0) & O & \cdot \\ O & \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} \tilde{S}_0(g^0)^* & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad (\text{A29})$$

$$(\text{A30})$$

From the commutative relation between  $\tilde{A}$  and  $\tilde{S}(g^0)$ , it can be easily verified that:

$$X^{-1} \begin{pmatrix} P^{-1} & O \\ O & P^{-1*} \end{pmatrix} \begin{pmatrix} \tilde{a} & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} P & O \\ O & P^* \end{pmatrix} X = \begin{pmatrix} d_{\tilde{a}} & O \\ O & b_{\tilde{a}} \end{pmatrix} \quad (\text{A31})$$

Here,  $d_{\tilde{a}}$  represents strictly diagonalized matrices, and  $b_{\tilde{a}}$  is generally not diagonalized or block diagonalized. However, since  $b_{\tilde{a}}$  commutes with the block diagonalized part of  $\tilde{S}(g^0)$ , denoted as  $\text{diag}[B, \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} B]$ , and  $b_{\tilde{a}}$  is diagonalizable, according to Ref [31], there always exists a similar transformation where  $b_{\tilde{a}}$  and  $\text{diag}[B, \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} B]$  can be simultaneously block diagonalized. It should be noted that  $\text{diag}[B, \frac{\nu_2(e, T, g^0 T)}{\nu_2(e, g^0, g^0 T)} B^*]$  can only be block diagonalized with each block having a size larger than one. Therefore, according to the Ref [31], every eigenvalue of  $b_{\tilde{a}}$  is degenerate. Additionally, from the commutative relation of  $\tilde{A}$  and  $\tilde{S}(g^0)$ , it can be easily demonstrated that the matrix  $\tilde{a}$  and  $\tilde{s}(g^0)$  commute with each other. Consequently, based on the previous proof, since  $\tilde{s}(g^0)$  can be strictly diagonalized with each diagonal element being distinct, the matrix  $\tilde{a}$  can be simultaneously diagonalized with all eigenvalues being degenerate with its conjugate. Thus, we have thoroughly established that if  $\nu_2$  is a coboundary under the subgroup  $G_b^0$ , and  $G_b^0$  is a non-Abelian group, the eigenvalues of the transfer matrix  $A$  always exhibit degeneracy.

In conclusion, we have successfully demonstrated that for non-trivial SPT phases, the eigenvalues of the transfer matrix always possess degeneracy. On the other hand, for trivial SPT phases, apart from the exhibit of intrinsic symmetry, the dominant eigenvalue generally does not exhibit degeneracy. Consequently, in the case of a non-trivial SPT phase, the strange correlator will consistently exhibit long-range correlations, while the trivial SPT phase will always display short-range correlations. This observation aligns perfectly with the theoretical expectations in strange correlation theory.

## Appendix B: Proof of Correctness of Strange Correlation Theory for Fermionic Strange Correlator

### 1. Proof of Correctness of 1D FSPT Strange Correlation Theory with trivial Central Extension

In this subsection, we present a detailed proof of the availability of the 1D FSPT strange correlator based on transfer matrix method.

As mentioned previously, the 1-cocycle labeling complex fermions results in the division of the local symmetry group  $G_b$  into two cosets:  $Z$  and  $O = \sigma Z$ . Consequently, we can label the elements in  $G_b$  as  $g_1, g_2, \dots, g_M, g'_1, g'_2, \dots, g'_M$ , where  $g_1$  corresponds to the unit element and  $g'_i = \sigma g_i$ . In the following discussion, we will consider two scenarios based on whether  $\sigma$  includes time reversal symmetry or not.

#### a. $\sigma$ without time reversal

According to the obstruction function  $\text{d}\nu_2(g_0, g_1, g_2, g_3) = (-1)^{\omega_2 \sim n_1}$ , if we set  $g_2, g_3 \in Z$ , then the obstruction function becomes a traditional cocycle equation. Similarly, if we set  $g_2, g_3 \in O$ , then the obstruction function also becomes a traditional cocycle equation. Now, let's set  $g_2, g_3 \in Z$ . In this case, the obstruction function becomes:

$$\nu_2(\sigma^{-1}, g_2, g_3) = \frac{\nu_2(\sigma^{-1}, g_1, g_3) \nu_2(g_1, g_2, g_3)}{\nu_2(\sigma^{-1}, g_1, g_2)} \quad (\text{B1})$$



Let  $g_1 = e$ , then we find the relation:

$$\nu_2(e, g'_2, g'_3) = \frac{\nu_2(e, \sigma, g'_3)}{\nu_2(e, \sigma, g'_2)} \nu_2(e, g_2, g_3) \quad (\text{B2})$$

Now, let's come back to our transfer matrix. According to our definition,  $T_Z = \nu_2(e, g_i, g_j)$  for  $g_i, g_j \in Z$ , and  $T_O = \nu_2(e, g'_i, g'_j)$  for  $g'_i, g'_j \in O$ . Therefore, according to Equation (B32), we have  $T_O = S^{-1}T_ZS$ , where  $S_{ij} = \delta_{ij}\nu_2(e, \sigma, g'_i)$ . Since  $T_O$  and  $T_Z$  are similar matrices, they always possess the same eigenvalues. For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle = \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B3})$$

Assuming that  $T_Z$  is diagonalizable, i.e.,  $D = A^{-1}T_ZA$  is a diagonalization, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle \quad (\text{B4})$$

$$= \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B5})$$

$$= \text{tr}(A^{-1}ST_{OZ}AD^{|i-j|}A^{-1}T_{ZO}S^{-1}AD^{N-|i-j|}) + \text{tr}(A^{-1}T_{ZO}S^{-1}AD^{|i-j|}A^{-1}ST_{OZ}AD^{N-|i-j|}) \quad (\text{B6})$$

$$= 2\nu_2(e, \sigma, e) \text{tr}(\tilde{T}D^{|i-j|}\tilde{T}D^{N-|i-j|}) \quad (\text{B7})$$

Here,  $A^{-1}T_{ZO}S^{-1}A = \nu_2(e, \sigma, e)^* A^{-1}ST_{OZ}A = \tilde{T}$ . We assume that  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_M|$  (as we can always do) are the eigenvalues of  $T_Z$ . Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a trivial cocycle), with  $|\lambda_1| > |\lambda_2|$ , then in the thermodynamic limit ( $N \rightarrow \infty$ ), the strange correlator can be expressed as:

$$C(i, j) = \nu_2(e, \sigma, e) [\tilde{T}(1, 1)^2 + \sum_{i \neq 1} \tilde{T}(1, i) \tilde{T}(i, 1) \left( \frac{\lambda_i}{\lambda_1} \right)^L] \quad (\text{B8})$$

Clearly, if  $|\lambda_1| > |\lambda_2|$ , the strange correlator will always converge to a constant.

Secondly, if the eigenvalues exhibit a  $p$ -fold degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a non-trivial cocycle) such that  $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_p$ , then under thermodynamic limit, the strange correlator can be expressed as:

$$\frac{\nu_2(e, \sigma, e) \sum_{i,j=1}^p \tilde{T}(j, i) \tilde{T}(i, j)}{p} \quad (\text{B9})$$

Apparently, under this circumstances, the strange correlator will also converge to a constant.

Even in the presence of accidental degeneracy, where  $|\lambda_1| = |\lambda_2| = |\lambda_3| = \dots = |\lambda_p|$  ( $1 < p \leq M$ ), and we let  $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$  ( $1 \leq i < p$ ), the strange correlator can still be expressed as:

$$\frac{\nu_2(e, \sigma, e) \sum_{i,j=1}^p [\tilde{T}(j, i) \tilde{T}(i, j)] e^{i\theta_i(N-L)} e^{i\theta_j L}}{1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \dots + e^{i\theta_p N}} \quad (\text{B10})$$

In this situation, the behavior of the strange correlator is determined by the values of  $N$  and the phases  $\theta_i$ . In general, there are two possible cases: power-law decay and oscillation. The specific behavior depends on the values of  $\theta_i$  and  $\theta_j$ , as well as the ratio of  $N$  and  $L$ .

#### b. $\sigma$ contains time reversal

Similar to the previous section, let's set  $g_2, g_3 \in Z$ , and the obstruction function becomes:

$$\nu_2(\sigma^{-1}, g_2, g_3) = \frac{\nu_2(\sigma^{-1}, g_1, g_3) \nu_2(g_1, g_2, g_3)}{\nu_2(\sigma^{-1}, g_1, g_2)} \quad (\text{B11})$$

Let  $g_1 = e$ , this time however, the relation becomes:

$$\nu_2(e, g'_2, g'_3)^* = \frac{\nu_2(e, \sigma, g'_3)^*}{\nu_2(e, \sigma, g'_2)^*} \nu_2(e, g_2, g_3) \quad (\text{B12})$$

Now, let's come back to our transfer matrix. According to our definition,  $T_Z = \nu_2(e, g_i, g_j)$  for  $g_i, g_j \in Z$ , and  $T_O = \nu_2(e, g'_i, g'_j)$  for  $g'_i, g'_j \in O$ . Therefore, according to Equation (B32), we have  $T_O = S^{-1}T_Z^*S$ , where  $S_{ij} = \delta_{ij}\nu_2(e, \sigma, g'_i)$ . Since  $T_O$  and  $T_Z^*$  are similar matrices, the eigenvalues of  $T_O$  the conjugate of those of  $T_Z$ . For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle = \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B13})$$

Assuming that  $T_Z$  is diagonalizable, i.e.,  $D = A^{-1}T_Z A$  is a diagonalization, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle \quad (\text{B14})$$

$$= \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B15})$$

$$= \text{tr}(A^{-1*} S T_{OZ} A D^{|i-j|} A^{-1} T_{ZO} S^{-1} A^* D^{N-|i-j|*}) + \text{tr}(A^{-1} T_{ZO} S^{-1} A^* D^{|i-j|*} A^{-1*} S T_{OZ} A D^{N-|i-j|}) \quad (\text{B16})$$

$$= \nu_2(e, \sigma, e) [\text{tr}(\tilde{T} D^{|i-j|} \tilde{T}^* D^{N-|i-j|*}) + \text{tr}(\tilde{T}^* D^{|i-j|*} \tilde{T} D^{N-|i-j|})] \quad (\text{B17})$$

Here,  $A^{-1}T_{ZO}S^{-1}A^* = \nu_2(e, \sigma, e)[A^{-1*}ST_{OZ}A]^* = \tilde{T}$ . We assume that  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_M|$  (as we can always do) are the eigenvalues of  $T_Z$ . Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a trivial cocycle), with  $|\lambda_1| > |\lambda_2|$ , then in the thermodynamic limit ( $N \rightarrow \infty$ ), the strange correlator can be expressed as:

$$C(i, j) = \frac{2|\tilde{T}|^2(1, 1) \text{Re} \left[ \left( \frac{\lambda_1^*}{\lambda_1} \right)^L \right] + \sum_{i \neq 1} \left[ \tilde{T}(1, i) \tilde{T}^*(i, 1) \left( \frac{\lambda_i^*}{\lambda_1} \right)^L + \tilde{T}^*(1, i) \tilde{T}(i, 1) \left( \frac{\lambda_i}{\lambda_1^*} \right)^L \left( \frac{\lambda_1^*}{\lambda_1} \right)^N \right]}{\nu_2(e, \sigma, e) \left[ 1 + \left( \frac{\lambda_1^*}{\lambda_1} \right)^N \right]} \quad (\text{B18})$$

Clearly, if  $|\lambda_1| > |\lambda_2|$ , the strange correlator will generally present oscillation.

Secondly, if the eigenvalues exhibit a  $p$ -fold degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a non-trivial cocycle) such that  $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_p$ , then under thermodynamic limit, the strange correlator can be expressed as:

$$\frac{\sum_{i,j=1}^p \left[ \tilde{T}(j, i) \tilde{T}^*(i, j) \left( \frac{\lambda_j^*}{\lambda_j} \right)^L + \tilde{T}^*(j, i) \tilde{T}(i, j) \left( \frac{\lambda_j}{\lambda_j^*} \right)^{N-L} \right]}{p \nu_2(e, \sigma, e) \left[ 1 + \left( \frac{\lambda_1^*}{\lambda_1} \right)^N \right]} \quad (\text{B19})$$

Apparently, under this circumstances, the strange correlator will also exhibit oscillatory behavior in general.

Even in the presence of accidental degeneracy, where  $|\lambda_1| = |\lambda_2| = |\lambda_3| = \dots = |\lambda_p|$  ( $1 < p \leq M$ ), and we let  $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$  ( $1 \leq i < p$ ), the strange correlator can still be expressed as:

$$\frac{\sum_{i,j=1}^p \nu_2(e, \sigma, e)^* \left[ \tilde{T}(j, i) \tilde{T}^*(i, j) \left( \frac{\lambda_j^*}{\lambda_j} \right)^L + \tilde{T}^*(j, i) \tilde{T}(i, j) \left( \frac{\lambda_j}{\lambda_j^*} \right)^{N-L} \right] e^{i\theta_i(N-L)} e^{i\theta_j L}}{[1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \dots + e^{i\theta_p N}] + \left( \frac{\lambda_1^*}{\lambda_1} \right)^N [1 + e^{-i\theta_2 N} + e^{-i\theta_3 N} + \dots + e^{-i\theta_p N}]} \quad (\text{B20})$$

The behavior of the strange correlator expression depends on the values of  $N$  and the phases  $\theta_i$ . In general, there are two possible cases: power-law decay and oscillation. The specific behavior is determined by the values of  $\theta_i$  and  $\theta_j$ , as well as the ratio of  $N$  and  $L$ .

Subsequently, in no situations, the strange correlator would exhibit exponential decay, manifesting a mode of long-range correlation. Therefore, we conclude that if we choose the complex fermion annihilation operator as our observables, the 1D FSPT strange correlator with a non-trivial central extension would always demonstrate long-range correlation.

## 2. Proof of Correctness of 1D FSPT Strange Correlation Theory with Non-trivial Central Extension

In this subsection, we present a detailed proof of the availability of the 1D FSPT strange correlator based on transfer matrix method.

As mentioned previously, the 1-cocycle labeling complex fermions results in the division of the local symmetry group  $G_b$  into two cosets:  $Z$  and  $O = \sigma Z$ . Consequently, we can label the elements in  $G_b$  as  $g_1, g_2, \dots, g_M, g'_1, g'_2, \dots, g'_M$ , where  $g_1$  corresponds to the unit element and  $g'_i = \sigma g_i$ . In the following discussion, we will consider two scenarios based on whether  $\sigma$  includes time reversal symmetry or not.

a.  $\sigma$  without time reversal

According to the obstruction function  $d\nu_2(g_0, g_1, g_2, g_3) = (-1)^{\omega_2 \sim n_1}$ , if we set  $g_2, g_3 \in Z$ , then the obstruction function becomes a traditional cocycle equation. Similarly, if we set  $g_2, g_3 \in O$ , then the obstruction function also becomes a traditional cocycle equation. Now, let's set  $g_2, g_3 \in Z$ . In this case, the obstruction function becomes:

$$\nu_2(\sigma^{-1}, g_2, g_3) = \frac{\nu_2(\sigma^{-1}, g_1, g_3)\nu_2(g_1, g_2, g_3)}{\nu_2(\sigma^{-1}, g_1, g_2)} \quad (\text{B21})$$

Let  $g_1 = e$ , then we find the relation:

$$\nu_2(e, g'_2, g'_3) = \frac{\nu_2(e, \sigma, g'_3)}{\nu_2(e, \sigma, g'_2)} \nu_2(e, g_2, g_3) \quad (\text{B22})$$

Now, let's come back to our transfer matrix. According to our definition,  $T_Z = \nu_2(e, g_i, g_j)$  for  $g_i, g_j \in Z$ , and  $T_O = \nu_2(e, g'_i, g'_j)$  for  $g'_i, g'_j \in O$ . Therefore, according to Equation (B32), we have  $T_O = S^{-1}T_ZS$ , where  $S_{ij} = \delta_{ij}\nu_2(e, \sigma, g'_i)$ . Since  $T_O$  and  $T_Z$  are similar matrices, they always possess the same eigenvalues. For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle = \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B23})$$

Assuming that  $T_Z$  is diagonalizable, i.e.,  $D = A^{-1}T_ZA$  is a diagonalization, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle \quad (\text{B24})$$

$$= \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B25})$$

$$= \text{tr}(A^{-1}ST_{OZ}AD^{|i-j|}A^{-1}T_{ZO}S^{-1}AD^{N-|i-j|}) + \text{tr}(A^{-1}T_{ZO}S^{-1}AD^{|i-j|}A^{-1}ST_{OZ}AD^{N-|i-j|}) \quad (\text{B26})$$

$$= \text{tr}(\tilde{T}_{ZO}D^{|i-j|}\tilde{T}_{OZ}D^{N-|i-j|}) + \text{tr}(\tilde{T}_{ZO}D^{|i-j|}\tilde{T}_{OZ}D^{N-|i-j|}) \quad (\text{B27})$$

Here,  $\tilde{T}_{ZO} = A^{-1}T_{ZO}S^{-1}A$  and  $\tilde{T}_{OZ} = A^{-1}ST_{OZ}A$ . We assume that  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_M|$  (as we can always do) are the eigenvalues of  $T_Z$ . Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a trivial cocycle), with  $|\lambda_1| > |\lambda_2|$ , then in the thermodynamic limit ( $N \rightarrow \infty$ ), the strange correlator can be expressed as:

$$C(i, j) = \tilde{T}_{ZO}(1, 1)\tilde{T}_{OZ}(1, 1) + \frac{\sum_{i \neq 1} [\tilde{T}_{ZO}(1, i)\tilde{T}_{OZ}(i, 1) + \tilde{T}_{OZ}(1, i)\tilde{T}_{ZO}(i, 1)] \left(\frac{\lambda_i}{\lambda_1}\right)^L}{2} \quad (\text{B28})$$

Clearly, if  $|\lambda_1| > |\lambda_2|$ , the strange correlator will always converge to a constant.

Secondly, if the eigenvalues exhibit a  $p$ -fold degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a non-trivial cocycle) such that  $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_p$ , then under thermodynamic limit, the strange correlator can be expressed as:

$$\frac{\sum_{i,j=1}^p [\tilde{T}_{ZO}(j, i)\tilde{T}_{ZO}(i, j) + \tilde{T}_{OZ}(j, i)\tilde{T}_{ZO}(i, j)]}{2p} \quad (\text{B29})$$

Apparently, under this circumstances, the strange correlator will also converge to a constant.

Even in the presence of accidental degeneracy, where  $|\lambda_1| = |\lambda_2| = |\lambda_3| = \dots = |\lambda_p|$  ( $1 < p \leq M$ ), and we let  $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$  ( $1 \leq i < p$ ), the strange correlator can still be expressed as:

$$\frac{\sum_{i,j=1}^p [\tilde{T}_{ZO}(j, i)\tilde{T}_{ZO}(i, j) + \tilde{T}_{OZ}(j, i)\tilde{T}_{ZO}(i, j)] e^{i\theta_i(N-L)} e^{i\theta_j L}}{2[1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \dots + e^{i\theta_p N}]} \quad (\text{B30})$$

In this situation, the behavior of the strange correlator is determined by the values of  $N$  and the phases  $\theta_i$ . In general, there are two possible cases: power-law decay and oscillation. The specific behavior depends on the values of  $\theta_i$  and  $\theta_j$ , as well as the ratio of  $N$  and  $L$ .

b.  $\sigma$  contains time reversal

Similar to the previous section, let's set  $g_2, g_3 \in Z$ , and the obstruction function becomes:

$$\nu_2(\sigma^{-1}, g_2, g_3) = \frac{\nu_2(\sigma^{-1}, g_1, g_3)\nu_2(g_1, g_2, g_3)}{\nu_2(\sigma^{-1}, g_1, g_2)} \quad (\text{B31})$$

Let  $g_1 = e$ , this time however, the relation becomes:

$$\nu_2(e, g'_2, g'_3)^* = \frac{\nu_2(e, \sigma, g'_3)^*}{\nu_2(e, \sigma, g'_2)^*} \nu_2(e, g_2, g_3) \quad (\text{B32})$$

Now, let's come back to our transfer matrix. According to our definition,  $T_Z = \nu_2(e, g_i, g_j)$  for  $g_i, g_j \in Z$ , and  $T_O = \nu_2(e, g'_i, g'_j)$  for  $g'_i, g'_j \in O$ . Therefore, according to Equation (B32), we have  $T_O = S^{-1}T_Z^*S$ , where  $S_{ij} = \delta_{ij}\nu_2(e, \sigma, g'_i)$ . Since  $T_O$  and  $T_Z^*$  are similar matrices, the eigenvalues of  $T_O$  are the conjugate of those of  $T_Z$ . For the 1D FSPT strange correlator with a non-trivial central extension, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle = \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B33})$$

Assuming that  $T_Z$  is diagonalizable, i.e.,  $D = A^{-1}T_Z A$  is a diagonalization, we have:

$$\langle \Omega | \tilde{c}_{i,i+1} \tilde{c}_{j,j+1} | \Psi \rangle \quad (\text{B34})$$

$$= \text{tr}(T_{OZ} T_Z^{|i-j|} T_{ZO} T_O^{N-|i-j|}) + \text{tr}(T_{ZO} T_O^{|i-j|} T_{OZ} T_Z^{N-|i-j|}) \quad (\text{B35})$$

$$= \text{tr}(A^{-1*} S T_{OZ} A D^{|i-j|} A^{-1} T_{ZO} S^{-1} A^* D^{N-|i-j|*}) + \text{tr}(A^{-1} T_{ZO} S^{-1} A^* D^{|i-j|*} A^{-1*} S T_{OZ} A D^{N-|i-j|}) \quad (\text{B36})$$

$$= \text{tr}(\tilde{T}_{OZ} D^{|i-j|} \tilde{T}_{ZO} D^{N-|i-j|*}) + \text{tr}(\tilde{T}_{ZO} D^{|i-j|*} \tilde{T}_{OZ} D^{N-|i-j|}) \quad (\text{B37})$$

Here,  $\tilde{T}_{ZO} = A^{-1}T_{ZO}S^{-1}A^*$  and  $\tilde{T}_{OZ} = A^{-1*}ST_{OZ}A$ . We assume that  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_M|$  (as we can always do) are the eigenvalues of  $T_Z$ . Under the constraint of periodic boundary conditions, this time, there is no background required to cancel, and we can still discuss the scaling behavior in two situations.

First, if there is no degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a trivial cocycle), with  $|\lambda_1| > |\lambda_2|$ , then in the thermodynamic limit ( $N \rightarrow \infty$ ), the strange correlator can be expressed as:

$$C(i, j) = \frac{2\tilde{T}_{ZO}(1, 1)\tilde{T}_{OZ}(1, 1)\text{Re} \left[ \left( \frac{\lambda_1^*}{\lambda_1} \right)^L \right] + \sum_{i \neq 1} \left[ \tilde{T}_{ZO}(1, i)\tilde{T}_{OZ}(i, 1) \left( \frac{\lambda_i^*}{\lambda_1} \right)^L + \tilde{T}_{OZ}(1, i)\tilde{T}_{ZO}(i, 1) \left( \frac{\lambda_i}{\lambda_1^*} \right)^L \left( \frac{\lambda_1^*}{\lambda_1} \right)^N \right]}{1 + \left( \frac{\lambda_1^*}{\lambda_1} \right)^N} \quad (\text{B38})$$

Clearly, if  $|\lambda_1| > |\lambda_2|$ , the strange correlator will generally present oscillation.

Secondly, if the eigenvalues exhibit a  $p$ -fold degeneracy (that is, when the 2-cocycle  $\nu_2 \in H^2(Z, U_T(1))$  is a non-trivial cocycle) such that  $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_p$ , then under thermodynamic limit, the strange correlator can be expressed as:

$$\frac{\sum_{i,j=1}^p \left[ \tilde{T}_{ZO}(j, i)\tilde{T}_{OZ}(i, j) \left( \frac{\lambda_j^*}{\lambda_j} \right)^L + \tilde{T}_{OZ}(j, i)\tilde{T}_{ZO}(i, j) \left( \frac{\lambda_i}{\lambda_i^*} \right)^{N-L} \right]}{p \left[ 1 + \left( \frac{\lambda_1^*}{\lambda_1} \right)^N \right]} \quad (\text{B39})$$

Apparently, under this circumstances, the strange correlator will also exhibit oscillatory behavior in general.

Even in the presence of accidental degeneracy, where  $|\lambda_1| = |\lambda_2| = |\lambda_3| = \dots = |\lambda_p|$  ( $1 < p \leq M$ ), and we let  $\frac{\lambda_i}{\lambda_1} = e^{i\theta_i}$  ( $1 \leq i < p$ ), the strange correlator can still be expressed as:

$$\frac{\sum_{i,j=1}^p \left[ \tilde{T}_{ZO}(j, i)\tilde{T}_{OZ}(i, j) \left( \frac{\lambda_j^*}{\lambda_j} \right)^L + \tilde{T}_{OZ}(j, i)\tilde{T}_{ZO}(i, j) \left( \frac{\lambda_i}{\lambda_i^*} \right)^{N-L} \right] e^{i\theta_i(N-L)} e^{i\theta_j L}}{[1 + e^{i\theta_2 N} + e^{i\theta_3 N} + \dots + e^{i\theta_p N}] + \left( \frac{\lambda_1^*}{\lambda_1} \right)^N [1 + e^{-i\theta_2 N} + e^{-i\theta_3 N} + \dots + e^{-i\theta_p N}]} \quad (\text{B40})$$

The behavior of the strange correlator expression depends on the values of  $N$  and the phases  $\theta_i$ . In general, there are two possible cases: power-law decay and oscillation. The specific behavior is determined by the values of  $\theta_i$  and  $\theta_j$ , as well as the ratio of  $N$  and  $L$ .

Subsequently, in no situations, the strange correlator would exhibit exponential decay, manifesting a mode of long-range correlation. Therefore, we conclude that if we choose the complex fermion annihilation operator as our observables, the 1D FSPT strange correlator with a non-trivial central extension would always demonstrate long-range correlation.

### Appendix C: Classification of 1D FSPT States with Non-trivial Central Extension by Strange Correlator

Firstly, suppose the local symmetry does not include time reversal symmetry. In this case, for any group element  $g \in G_b$ , the following relation based on group super-cohomology holds:

$$\nu_2(g, g_i, g_j) = \nu_2(e, g_i, g_j) \frac{\nu_2(e, g, g_i)}{\nu_2(e, g, g_j)} (-1)^{(\omega_2 \smile n_1)(e, g, g_i, g_j)} \quad (C1)$$

According to Equation (22), we have the relation  $\nu_2(g, g_i, g_j) = \nu_2(e, g^{-1}g_i, g^{-1}g_j) (-1)^{(\omega_2 \smile n_1)(e, g, g_i, g_j)}$ . Using this expression, we can derive the following relationships:

$$R(g)TR(g^{-1}) = S(g)TS(g)^{-1} S(g)^{-1}R(g)T = TS(g)^{-1}R(g) \quad (C2)$$

Here,  $R(g)$  represents the regular representation of the local symmetry group  $G_b$ , while  $S(g)_{ij} = \delta_{ij}\nu_2(e, g, g_j)$ .

According to Appendix A, if  $\nu_2$  is a coboundary, we can conclude that the largest eigenvalue of the matrix  $T$  is generally non-degenerate. This result is consistent with the expectation that the strange correlator between states belonging to the same phase does not exhibit long-range degeneracy.

If  $\nu_2$  is non-trivial, we claim that the expression  $S(g)^{-1}R(g)$  represents a projective representation of the local symmetry group  $G_b$ . Now, let's proceed to prove this claim:

$$S(g)^{-1}R(g)S(h)^{-1}R(h) \quad (C3)$$

$$= \delta_{ij}\nu_2(e, g, g_i)^{-1}\nu_2(e, h, g^{-1}g_i)^{-1}R(gh) \quad (C4)$$

$$= \delta_{ij}\nu_2(e, g, g_i)^{-1}\nu_2(g, gh, g_i)^{-1}R(gh)(-1)^{(\omega_2 \smile n_1)(e, g, gh, g_i)} \quad (C5)$$

$$= \delta_{ij}\nu_2(e, g, gh)^{-1}\nu_2(e, gh, g_i)^{-1}R(gh) \quad (C6)$$

$$= \nu_2(e, g, g_i)^{-1}S(gh)^{-1}R(gh) \quad (C7)$$

Hence, we find that  $S(g)^{-1}R(g)S(h)^{-1}R(h) = \nu_2(e, g, g_i)^{-1}S(gh)^{-1}R(gh)$ , which leads us to conclude that  $S(g)^{-1}R(g)$  indeed represents a projective representation of the group  $G_b$ .

As a result, according to Appendix A, we can establish that the eigenvalues of the transfer matrix  $T$  are typically degenerate. This is because the block size of its diagonalization must be equal to or greater than 2.

In a similar manner, when considering the presence of time reversal symmetry, we can partition the local symmetry group into two distinct sets:  $G_b^0$  and  $G_b^T$ . Here,  $G_b^0$  consists of elements that do not possess any time-reversal properties, while  $G_b^T$  comprises elements that do exhibit time-reversal characteristics. It can be easily verified that  $G_b^0$  forms a subgroup of the local symmetry group  $G_b$ . We can still observe the relation  $S(g^0)^{-1}R^T(g^0)A = AS(g^0)^{-1}R^T(g^0)$ , where  $g^0 \in G_b^0$ , and  $R^T(g^0) = \text{diag}[r(g^0), r(g^0)]$ . As a result, the eigenvectors of the transfer matrix  $T$  carry a projective representation of the group  $G_b^0$ . It is important to note that if  $\sigma$  is not the time-reversal element, then under the subgroup  $G_b^0$ ,  $\nu_2$  still belongs to the super-cohomology group  $BH(G_b^0, U_T(1))$ . Therefore, according to the previous analysis, the eigenvalues of the transfer matrix  $A$  always exhibit degeneracy. However, in cases where  $\sigma$  represents the time-reversal element, indicating that  $\nu_2$  in the subgroup  $G_b^0$  functions as a 2-cocycle, there are instances where  $\nu_2$  may become a coboundary within the subgroup  $G_b^0$ . In such situations, we can still verify the availability of Eq. (A21) and Eq. (A23), albeit this time  $\nu_2 \in BH(G_b^0, U_T(1))$  is obstructed by Eq. (22). Consequently, by following the same procedure outlined in Appendix A, we can establish that the degeneracy of the transfer matrix  $A$  remains intact.

Hence, based on the findings presented in Appendix A, we can conclude that the strange correlator between states belonging to the same phase does not display long-range degeneracy. In contrast, the strange correlator between states

belonging to different phases consistently exhibits long-range degeneracy. This observation provides a classification method for 1D FSPT (Fractional Symmetry-Protected Topological) phases that possess a non-trivial central extension.

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