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PRUF—a meaning representation language for natural languages†

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PRUF—an acronym for Possibilistic Relational Universal Fuzzy—is a meaning representation language for natural languages which departs from the conventional approaches to the theory of meaning in several important respects.

First, a basic assumption underlying PRUF is that the imprecision that is intrinsic in natural languages is, for the most part, possibilistic rather than probabilistic in nature. Thus, a proposition such as "Richard is tall" translates in PRUF into a possibility distribution of the variable Height (Richard), which associates with each value of the variable a number in the interval [0,1] representing the possibility that Height (Richard) could assume the value in question. More generally, a proposition, p, translates into a procedure, P, which returns a possibility distribution, Π^p, with P and Π^p representing, respectively, the *meaning* of p and the *information* conveyed by p. In this sense, the concept of a possibility distribution replaces that of truth as a foundation for the representation of meaning in natural languages.

Second, the logic underlying PRUF is not a two-valued or multivalued logic, but a fuzzy logic, FL, in which the truth-values are linguistic, that is, are of the form *true*, *not true*, *very true*, *more or less true*, *not very true*, etc., with each such truth-value representing a fuzzy subset of the unit interval. The truth-value of a proposition is defined as its compatibility with a reference proposition, so that given two propositions p and r, one can compute the truth of p *relative to* r.

Third, the quantifiers in PRUF—like the truth-values—are allowed to be linguistic, i.e. may be expressed as *most*, *many*, *few*, *some*, *not very many*, *almost all*, etc. Based on the concept of the cardinality of a fuzzy set, such quantifiers are given a concrete interpretation which makes it possible to translate into PRUF propositions exemplified by "Many tall men are much taller than most men," "All tall women are blonde is not very true," etc.

The translation rules in PRUF are of four basic types: Type I—pertaining to modification; Type II—pertaining to composition; Type III—pertaining to quantification; and Type IV—pertaining to qualification and, in particular, to truth qualification, probability qualification and possibility qualification.

The concepts of semantic equivalence and semantic entailment in PRUF provide a basis for question-answering and inference from fuzzy premises. In addition to serving as a foundation for approximate reasoning, PRUF may be employed as a language for the representation of imprecise knowledge and as a means of precisiation of fuzzy propositions expressed in a natural language.

1. Introduction

In a decade or so from now—when the performance of natural language understanding and question-answering systems will certainly be much more impressive than it is today—it may well be hard to comprehend why linguists, philosophers, logicians and

†To Professor I. M. Gel'fand, who had suggested—a decade ago—the application of the theory of fuzzy sets to natural languages.

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cognitive scientists have been so reluctant to come to grips with the reality of the pervasive imprecision of natural languages and have persisted so long in trying to fit their theories of syntax, semantics and knowledge representation into the rigid conceptual mold of two-valued logic.†

A fact that puts this issue into a sharper perspective is that almost any sentence drawn at random from a text in a natural language is likely to contain one or more words that have a fuzzy‡ denotation—that is, are labels of classes in which the transition from membership to non-membership is gradual rather than abrupt. This is true, for example, of the italicized words in the simple propositions "John is *tall*," "May has *dark* hair," and "May is *much younger* than John," as well as in the somewhat more complex and yet commonplace propositions exemplified by: "*Most* Frenchmen are not *blond*," "It is *very true* that *many* Swedes are *tall*," "It is *quite possible* than *many wealthy* Americans have *high* blood pressure," and "It is *probably quite true* that *most* X's are *much larger* than *most* Y's."

The numerous meaning representation, knowledge representation and query representation languages which have been described in the literature§—prominent among which are semantic networks, predicate calculi, relation algebra, Montague grammar, conceptual dependency graphs, logical networks, AIMDS, ALPHA, CONVERSE, DEDUCE, DILOS, HAM-RPM, HANSA, ILL, KRL, KRS, LIFER, LSP, LUNAR, MAPL, MEANINGEX, MERLIN, OWL, PHILQAI, PLANES, QUEL, REL, REQUEST, SAM, SCHOLAR, SEQUEL, SQUARE and TORUS—are not oriented toward the representation of fuzzy propositions, that is, propositions containing labels of fuzzy sets and hence have no facilities for semantic—as opposed to syntactic—inference from fuzzy premises.‖ However, facilities for the representation and execution of fuzzy instructions are available in the programming languages FUZZY (LeFaivre, 1974), FLOU and FSTDS (Noguchi, Umano, Mizumoto & Tanaka, 1976, 1977) and in the system modelling language of Fellinger (1974).

To clarify this remark, it should be noted that, although a fuzzy proposition such as "Herb is tall," may be—and frequently is—represented in predicate notation as Tall (Herb), such a representation presupposes that Tall is a predicate which partitions a collection of individuals, U, into two disjoint classes: those for which Tall(Herb) is true and those for which Tall(Herb) is false. One could, of course, interpret Tall as a predicate in a multivalued logic—in which case the extension of Tall would be a fuzzy subset of U—but even such more general representations cannot cope with quantified or qualified propositions of the form "Most tall men are fat," "It is very true that X is much larger than Y," "It is quite possible that if X is large then it is very likely that Y is small," etc.

In earlier papers (Zadeh, 1973, 1975*a*, *b*, *c*, 1976*a*, *b*; Bellman & Zadeh, 1976), we

†An incisive discussion of this and related issues may be found in Gaines (1976*b*).

‡Although the terms *fuzzy* and *vague* are frequently used interchangeably in the literature, there is, in fact, a significant difference between them. Specifically, a proposition, p, is *fuzzy* if it contains words which are labels of fuzzy sets; and p is *vague* if it is both fuzzy and insufficiently specific for a particular purpose. For example, "Bob will be back in a few minutes" is fuzzy, while "Bob will be back sometime" is vague if it is insufficiently informative as a basis for a decision. Thus, the vagueness of a proposition is a decision-dependent characteristic whereas its fuzziness is not.

§A list of representative papers and books dealing with the subject of meaning representation languages and related issues is presented in the appended bibliography.

‖Semantic inference differs from syntactic inference in that it involves the meaning of premises while syntactic inference involves only their surface structure.

have argued that traditional logical systems are intrinsically unsuited for the manipulation of fuzzy knowledge—which is the type of knowledge that underlies natural languages as well as most of human reasoning—and have proposed a fuzzy logic, FL, as a model for approximate reasoning. In this logic, the truth-values are *linguistic*, i.e. of the form *true, not true, very true, not very true, more or less true, not very true and not very false* etc., with each truth-value representing a fuzzy subset of the unit interval. In effect, the fuzziness of the truth-values of FL provides a mechanism for the association of imprecise truth-values with imprecise propositions expressed in a natural language, and thereby endows FL with a capability for modeling the type of qualitative reasoning which humans employ in uncertain and/or fuzzy environments.

More recently, the introduction of the concept of a possibility distribution (Zadeh, 1977*a*, *b*) has clarified the role of the concept of a fuzzy restriction† in approximate reasoning, and has provided a basis for the development of a meaning representation language named PRUF (an acronym for *P*ossibilistic‡ *R*elational *U*niversal *F*uzzy) in which—in a significant departure from tradition—it is the concept of a possibility distribution, as opposed to truth, that plays the primary role.

The conceptual structure of PRUF is based on the premise that, in sharp contrast to formal and programming languages, natural languages are intrinsically incapable of precise characterization on either the syntactic or semantic level. In the first place, the pressure for brevity of discourse tends to make natural languages *maximally ambiguous* in the sense that the level of ambiguity in human communication is usually near the limit of what is disambiguable through the use of an external body of knowledge which is shared by the parties in discourse.

Second, a significant fraction of sentences in a natural language cannot be characterized as strictly grammatical or ungrammatical. As is well known, the problem of partial grammaticality is accentuated in the case of sentences which are partially nonsensical in the real world but not necessarily in an imaginary world. Thus, a realistic grammar for a natural language should associate with each sentence its degree of grammaticality—rather than merely generate the sentences which are completely grammatical. The issue of partial grammaticality has the effect of greatly complicating the problem of automatic translation from a natural language into a meaning representation language—which is an important aspect of Montague-type grammars (Montague, 1974; Partee, 1976*b*).

Third, as was alluded to already, a word in a natural language is usually a summary of a complex, multifaceted concept which is incapable of precise characterization. For this reason, the denotation of a word is generally a fuzzy—rather than non-fuzzy—subset of a universe of discourse. For example, if U is a collection of individuals, the denotation of the term *young man* in U is a fuzzy subset of U which is characterized by a membership function $\mu_{\text{young man}}: U \to [0,1]$, which associates with each individual u in U the degree—on the scale from 0 to 1—to which u is a young man. When necessary or expedient, this degree or, equivalently, the grade of membership, $\mu_{\text{young}}(u)$, may be expressed in linguistic

†A *fuzzy restriction* is a fuzzy set which serves as an elastic constraint on the values that may be assigned to a variable. A variable which is associated with a fuzzy restriction or, equivalently, with a possibility distribution, is a *fuzzy variable*.

‡The term "possibilistic" was coined by Gaines & Kohout (1975). The concept of a possibility distribution is distinct from that of possibility in modal logic and related areas (Hughes & Cresswell, 1968; N. Rescher, 1975).

terms such as *high, not high, very high, not very high, low, more or less low*, etc., with each such term representing a fuzzy subset of the unit interval. In this case, the denotation of *young man* is a fuzzy set of Type 2, i.e. a fuzzy set with a fuzzy membership function.†

In essence, PRUF bears the same relation to FL that predicate calculus does to two-valued logic. Thus, it serves to translate a set of premises expressed in a natural language into expressions in PRUF to which the rules of inference in FL (or PRUF) may be applied, yielding other expressions in PRUF which upon retranslation become the conclusions inferred from the original premises. More generally, PRUF may be used as a basis for question-answering systems in which the knowledge-base contains imprecise data, i.e. propositions expressed in a natural or synthetic language which translate into a collection of possibility and/or probability distributions of a set of variables.

Typically, a simple proposition such as "John is young," translates in PRUF into what will be referred to as a *possibility assignment equation* of the form

$$\Pi_{\text{Age(John)}} = \text{YOUNG} \tag{1.1}$$

in which YOUNG—the denotation of young—is a fuzzy subset of the interval [0,100], and $\Pi_{\text{Age (John)}}$ is the possibility distribution of the variable Age(John). What (1.1) implies is that, if on the scale from 0 to 1, the degree to which a numerical age, say 30, is compatible with YOUNG is 0·7, then the possibility that John's age is 30 is also equal to 0·7. Equivalently, (1.1) may be expressed as

$$\text{JOHN}[\Pi_{\text{Age}} = \text{YOUNG}] \tag{1.2}$$

in which JOHN is the name of a relation which characterizes John and Age is an attribute of John which is particularized by the assignment of the fuzzy set YOUNG to its possibility distribution.

In general, an expression in PRUF may be viewed as a procedure which acts on a set of possibly fuzzy relations in a database and computes the possibility distribution of a set of variables. Thus, if p is a proposition in a natural language which translates into an expression P in PRUF, and Π^p is the possibility distribution returned by P, then P may be interpreted as the *meaning of p* while Π^p is the *information conveyed* by *p*.‡ The significance of these notions will be discussed in greater detail in section 3.

The main constituents of PRUF are (a) a collection of translation rules, and (b) a set of rules of inference.§ For the present, at least, the translation rules in PRUF are human-use oriented in that they do not provide a system for an automatic translation from a natural language into PRUF. However, by subordinating the objective of automatic translation to that of achieving a greater power of expressiveness, PRUF provides a system for the translation of a far larger subset of a natural language than is possible with the systems based on two-valued logic. Eventually, it may be possible to achieve the goal of machine translation into PRUF of a fairly wide variety of expressions in a natural language. It is not likely, however, that this goal could be attained through

†Expositions of the relevant aspects of the theory of fuzzy sets may be found in the books and papers noted in the bibliography, especially Kaufmann (1975), Negoita & Ralescu (1975), and Zadeh, Fu, Tanaka & Shimura (1975).

‡In effect, P and Π^p are the counterparts of the concepts of *intension* and *extension* in language theories based on two-valued logic (Cresswell, 1973; Linsky, 1971; Miller & Johnson-Laird, 1976).

§The rules of inference in PRUF and their application to approximate reasoning are described in a companion paper (Zadeh, 1977*b*).

the employment of algorithms of the conventional type in translation programs. Rather, it is probable that recourse would have to be made to the use of fuzzy logic for the representation of imprecise contextual knowledge as well as for the characterization and execution of fuzzy instructions in translation algorithms.

At present, PRUF is still in its initial stages of development and hence our exposition of it in the present paper is informal in nature, with no pretense at definiteness or completeness. Thus, our limited aim in what follows is to explain the principal ideas underlying PRUF; to describe a set of basic translation rules which can serve as a point of departure for the development of other, more specialized, rules; and to illustrate the use of translation rules by relatively simple examples. We shall not consider the translation of imperative propositions nor the issues relating to the implementation of interactive connectives, reserving these and other important topics for subsequent papers.

In the following sections, our exposition of PRUF begins with an outline of some of the basic properties of the concept of a possibility distribution and its role in the representation of the meaning of fuzzy propositions. In section 3, we consider a number of basic concepts underlying PRUF, among them those of possibility assignment equation, fuzzy set descriptor, proposition, question, database, meaning, information, semantic equivalence, semantic entailment and definition.

Section 4 is devoted to the formalization of translation rules of Type I (modification), Type II (composition) and Type III (quantification). In addition, a translation rule for relations is derived as a corollary of rules of Type II, and a rule for forming the negation of a fuzzy proposition is formulated.

The concept of truth is defined in section 5 as a measure of the compatibility of two fuzzy propositions, one of which acts as a reference proposition for the other. Based on this conception of truth, a translation rule for truth-qualified propositions is developed in section 6. In addition, translation rules for probability-qualified and possibility-qualified propositions are established, and the concept of semantic equivalence is employed to derive several meaning-perserving transformations of fuzzy propositions. Finally, in section 7 a number of examples illustrating the application of various translation rules—both singly and in combination—are presented.

2. The concept of a possibility distribution and its role in PRUF

A basic assumption underlying PRUF is that the imprecision that is intrinsic in natural languages is, in the main, possibilistic rather than probabilistic in nature.

As will be seen presently, the rationale for this assumption rests on the fact that most of the words in a natural language have fuzzy rather than non-fuzzy denotation. A conspicuous exception to this assertion are the terms in mathematics. Even in mathematics, however, there are concepts that are fuzzy, e.g. the concept of a sparse matrix, stiff differential equation, approximate equality, etc. More significantly, almost all mathematical concepts become fuzzy as soon as one leaves the idealized universe of mathematical constructs and comes in contact with the reality of pervasive ill-definedness, irreducible uncertainty and finiteness of computational resources.

To understand the relation between fuzziness and possibility,† it is convenient to consider initially a simple non-fuzzy proposition such as‡

†A more detailed account of this and other issues related to the concept of a possibility distribution may be found in Zadeh (1977*a*).

‡The symbol $\triangleq$ stands for "is defined to be" or "denotes."

$$p \triangleq \text{X is an integer in the interval } [0,5].$$

Clearly, what this proposition asserts is that (a) it is possible for any integer in the interval [0,5] to be a value of X, and (b) it is not possible for any integer outside of this interval to be a value of X.

For our purposes, it is expedient to reword this assertion in a form that admits of extension to fuzzy propositions. More specifically, in the absence of any information regarding X other than that conveyed by p, we shall assert that: p induces a *possibility distribution* Π_x which associates with each integer u in [0,5] the possibility that u could be a value of X. Thus,

$$\text{Poss}\{X=u\}=1 \text{ for } 0\leq u\leq 5$$

and

$$\text{Poss}\{X=u\}=0 \text{ for } u<0 \text{ or } u>5$$

where Poss$\{X=u\}$ is an abbreviation for "The possibility that X may assume the value u." For the proposition in question the possibility distribution Π_X is *uniform* in the sense that the possibility-values are equal to unity for u in [0,5] and zero elsewhere.

Next, let us consider a proposition q which may be viewed as a fuzzified version of p, namely,

$$q \triangleq \text{X is a small integer}$$

where "small integer" is the label of a fuzzy set defined by, say,†

$$\text{SMALL INTEGER}=1/0+1/1+0{\cdot}8/2+0{\cdot}6/3+0{\cdot}4/4+0{\cdot}2/5 \tag{2.1}$$

in which + denotes the union rather than the arithmetic sum and a fuzzy singleton of the form 0·6/3 signifies that the grade of membership of the integer 3 in the fuzzy set SMALL INTEGER—or, equivalently, the *compatibility* of 3 with SMALL INTEGER—is 0·6.

At this juncture, we can make use of the simple idea behind our interpretation of p to formulate what might be called the *possibility postulate*—a postulate which may be used as a basis for a possibilistic intepretation of fuzzy propositions. In application to q, it may be stated as follows.

Possibility postulate. In the absence of any information regarding X other than that conveyed by the proposition $q \triangleq$ X is a small integer, q induces a possibility distribution Π_X which equates the possibility of X taking a value u to the grade of membership of u in the fuzzy set SMALL INTEGER. Thus

$$\begin{aligned}
&\text{Poss}\{X=0\}=\text{Poss}\{X=1\}=1\\
&\text{Poss}\{X=2\}=0{\cdot}8\\
&\text{Poss}\{X=3\}=0{\cdot}6\\
&\text{Poss}\{X=4\}=0{\cdot}4\\
&\text{Poss}\{X=5\}=0{\cdot}2
\end{aligned}$$

†To differentiate between a label and its denotation, we express the latter in upper case symbols. To simplify the notation, this convention will not be adhered to strictly where the distinction can be inferred from the context.

and

$$\text{Poss}\{X=u\}=0 \text{ for } u<0 \text{ or } u>5.$$

More generally, the postulate asserts that if X is a variable which takes values in U and F is a fuzzy subset of U, then the proposition

$$q \triangleq X \text{ is } F \tag{2.2}$$

induces a possibility distribution Π_X *which is equal to* F, i.e.

$$\Pi_X=F \tag{2.3}$$

implying that

$$\text{Poss}\{X=u\}=\mu_F(u), \quad u \in U \tag{2.4}$$

where μ_F: $U \to [0,1]$ is the membership function of F, and $\mu_F(u)$ is the grade of membership of u in F.

In essence, then, the possibility distribution of X is a fuzzy set which serves to define the possibility that X could assume any specified value u in U. The function π_X: $U \to [0,1]$ which is equal to μ_F and which associates with each $u \in U$ the possibility that X could take u as its value is called the *possibility distribution function* associated with X. In this connection, it is important to note that the possibility distribution defined by (2.3) depends on the definition of F and hence is purely subjective in nature.

We shall refer to (2.3) as the *possibility assignment equation* because it signifies that the proposition "X is F" translates into the assignment of a fuzzy set F to the possibility distribution of X. More generally, the possibility assignment equation corresponding to a proposition of the form "N is F," where F is a fuzzy subset of a universe of discourse U, and N is the name of (a) a variable, (b) a fuzzy set, (c) a proposition, or (d) an object, may be expressed as

$$\Pi_{X(N)}=F \tag{2.5}$$

or, more simply,

$$\Pi_X=F \tag{2.6}$$

where X is either N itself (when N is a variable) or a variable that is explicit or implicit in N, with X taking values in U. For example, in the case of the proposition "Nora is young," $N \triangleq \text{Nora}$, $X=\text{Age(Nora)}$, $U=[0,100]$ and

$$\text{Nora is young} \to \Pi_{\text{Age(Nora)}}=\text{YOUNG} \tag{2.7}$$

where the symbol $\to$ stands for "translates into."

Since the concept of a possibility distribution is closely related to that of a fuzzy set,† possibility distributions may be manipulated by the rules applying to such sets. In what follows, we shall discuss briefly some of the basic rules of this kind, focusing our attention only on those aspects of possibility distributions which are of direct relevance to PRUF.

†Strictly speaking, the concept of a possibility distribution is coextensive with that of a fuzzy restriction rather than a fuzzy set (Zadeh, 1973, 1975*b*).

POSSIBILITY *VS.* PROBABILITY

Intuitively, possibility relates to our perception of the degree of feasibility or ease of attainment, whereas probability is associated with the degree of likelihood, belief, frequency or proportion. All possibilities are subjective, as are most probabilities.† In general, probabilistic information is not as readily available as possibilistic information and is more difficult to manipulate.

Mathematically, the distinction between probability and possibility manifests itself in the different rules which govern their combinations, especially under the union. Thus, if A is a non-fuzzy subset of U, and Π_X is the possibility distribution induced by the proposition "N is F," then the *possibility measure*,‡ $\Pi(A)$, of A is defined as the supremum of μ_F over A, i.e.

$$\Pi(A) \triangleq \mathrm{Poss}\{X \in A\} = \mathrm{Sup}_{u \in A}\mu_F(u) \tag{2.8}$$

and, more generally, if A is a fuzzy subset of U,

$$\Pi(A) = \mathrm{Poss}\{X \text{ is } A\} = \mathrm{Sup}_u(\mu_A(u) \wedge \mu_F(u)) \tag{2.9}$$

where μ_A is the membership function of A and $\wedge \triangleq \min$.

From the definition of $\Pi(A)$, it follows at once that the possibility measure of the union of two arbitrary subsets of U is given by

$$\Pi(A \cup B) = \Pi(A)\Pi \vee (B) \tag{2.10}$$

where $\vee \triangleq \max$. Thus, possibility measure does not have the basic additivity property of probability measure, namely,

$$P(A \cup B) = P(A) + P(B) \text{ if A and B are disjoint}$$

where P(A) and P(B) are the probability measures of A and B, respectively, and + is the arithmetic sum.

An essential aspect of the concept of possibility is that it does not involve the notion of repeated or replicated experimentation and hence is nonstatistical in nature. Indeed, the importance of the concept of possibility stems from the fact that much—perhaps most—of human decision-making is based on information that is possibilistic rather than probabilistic in nature.§

POSSIBILITY DISTRIBUTIONS *VS.* FUZZY SETS

Although there is a close connection between the concept of a possibility distribution

†There are eminent authorities in probability theory (DeFinetti, 1974) who maintain that all probabilities are subjective.

‡The possibility measure defined by (2.8) is a special case of the more general concept of a fuzzy measure defined by Sugeno (1974) and Terano & Sugeno (1975).

§In many realistic decision processes it is impracticable or impossible to obtain objective probabilistic information in the quantitative form that is needed for the application of statistical decision theory. Thus, the probabilities that are actually used in much of human decision-making are (a) subjective, and (b) linguistic (in the sense defined in Zadeh, 1975). Characterization of linguistic probabilities is related to the issue of probability qualification, which is discussed in section 6. A more detailed discussion of linguistic probabilities may be found in Nguyen (1976*a*).

and that of a fuzzy set, there is also a significant difference between the two that must be clearly understood.

To illustrate the point by a simple example which involves a non-fuzzy set and a uniform possibility distribution, consider a variable labeled Sister(Dedre) to which we assign a set, as in

$$\text{Sister(Dedre)} = \text{Sue} + \text{Jane} + \text{Lorraine} \tag{2.11}$$

or a possibility distribution, as in

$$\Pi_{\text{Sister(Dedre)}} = \text{Sue} + \text{Jane} + \text{Lorraine} \tag{2.12}$$

where + denotes the union. Now, the meaning of (2.11) is that Sue, Jane and Lorraine are sisters of Dedre. By contrast, the meaning of (2.12) is that the sister of Dedre is Sue *or* Jane *or* Lorraine, where *or* is the exclusive or. In effect, (2.12) signifies that there is uncertainty in our knowledge of who is the sister of Dedre, with the possibility that it is Sue being unity, and likewise for Jane and Lorraine. In the case of (2.11), on the other hand, we are certain that Sue, Jane and Lorraine are all sisters of Dedre. Thus, the set {Sue, Jane, Lorraine} plays the role of a possibility distribution in (2.12) but not in (2.11).

Usually, it is clear from the context whether or not a fuzzy (or non-fuzzy) set should be interpreted as a possibility distribution. A difficulty arises, however, when a relation contains a possibility distribution, as is exemplified by the relation RESIDENT whose tableau is shown in Table 2.1.

TABLE 2.1

Resident	Subject	Location
	Jack	New Rochelle
	Jack	White Plains
	- - - - - - -	- - - - - - - -
	Ralph	New Rochelle
	Ralph	Tarrytown

In this case, the rows above the dotted line represent a relation in the sense that Jack resides both in New Rochelle and in White Plains. On the other hand, the rows below the dotted line represent a possibility distribution associated with the location of residence of Ralph, meaning that Ralph resides either in New Rochelle *or* in Tarrytown, but not both. It should be noted parenthetically that there is no provision for dealing with this kind of ambiguity in the conventional representations of relational models of data because the concept of a possibility distribution and the related issue of data uncertainty have not been an object of concern in the analysis of database management systems.

REPRESENTATION BY STANDARD FUNCTIONS

In the manipulation of possibility distributions, it is convenient to be able to express the membership function of a fuzzy subset of the real line as a standard function whose parameters may be adjusted to fit a given membership function in an approximate fashion. A standard function of this type is the S-function, which is a piecewise quadratic function defined by the equations:

$$
\begin{aligned}
S(u;\alpha,\beta,\gamma) &= 0 && \text{for } u \leq \alpha \\
&= 2\left(\frac{u-\alpha}{\gamma-\alpha}\right)^2 && \alpha \leq u \leq \beta \\
&= 1-2\left(\frac{u-\gamma}{\gamma-\alpha}\right)^2 && \text{for } \beta \leq u \leq \gamma \\
&= 1 && \text{for } u \geq \gamma
\end{aligned}
\tag{2.13}
$$

in which the parameter $\beta \triangleq (\alpha+\gamma)/2$ is the *crossover* point, i.e. the value of u at which $S(u;\alpha,\beta,\gamma)=0{\cdot}5$. Other types of standard functions which are advantageous when the arithmetic operations of addition, multiplication and division have to be performed on fuzzy numbers, are (i) piecewise linear (triangular) functions, and (ii) exponential (bell-shaped) functions. A discussion of these functions and their applications may be found in Nahmias (1976) and Mizumoto & Tanaka (1976).

There are two special types of possibility distributions which will be encountered in later sections. One is the *unity* possibility distribution, which is denoted by I and is defined by

$$\pi_I(u)=1 \text{ for } u \in U \tag{2.14}$$

where π_I is the possibility distribution function of I. The other, which is defined on the unit interval, is the *unitary* possibility distribution (or the *unitary fuzzy set* or the *unitor*, for short), which is denoted by $\perp$ and is defined by

$$\pi_{\perp}(v)=v \text{ for } v \in [0,1]. \tag{2.15}$$

In the particular case where a truth-value in FL is the unitary fuzzy set, it will be referred to as the *unitary* truth-value. On denoting this truth-value by *u*-true, we have

$$\mu_{u\text{-true}}(v) \triangleq v,\ v \in [0,1]. \tag{2.16}$$

PROJECTION AND MARGINAL POSSIBILITY DISTRIBUTIONS

The possibility distributions with which we shall be concerned in the following sections are, in general, n-ary distributions denoted by $\Pi_{(X_1,\ldots,X_n)}$, where $X_1,\ldots,X_n$ are variables—or, equivalently, *attributes*—taking values in their respective universes of discourse $U_1,\ldots,U_n$.† As a simple example in which $n=2$, consider the proposition "John is a big man," in which BIG MAN is a fuzzy relation F defined by Table 2.2, with the variables Height and Weight expressed in centimeters and kilograms, respectively.

The relation in question may also be expressed as a linear form

$$\text{BIG MAN}=0{\cdot}5/(165,60)+0{\cdot}6/(170,60)+\ldots+1/(180,80)+\ldots \tag{2.17}$$

†When it is necessary to place in evidence that X takes values in U (i.e. the domain of X is U), we shall express the domain of X as U(X) or, where no confusion can arise, as X (see (2.23)).

TABLE 2.2

BIG MAN	Height	Weight	μ
	165	60	0·5
	170	60	0·6
	175	60	0·7
	170	65	0·75
	—	—	—
	180	70	0·9
	175	75	0·9
	180	75	0·95
	180	80	1
	185	75	1

in which a term such as 0·6/(170,60) signifies that the grade of membership of the pair (170,60) in the relation BIG MAN—or, equivalently, its compatibility with the relation BIG MAN—is 0·6.

The possibility postulate implies that the proposition "John is a big man" induces a binary possibility distribution $\Pi_{(\text{Weight(John)},\ \text{Height(John)})}$ whose tableau is identical with Table 2.2 except that the label of the last column is changed from μ to π in order to signify that the compatibility-values in that column assume the role of possibility-values. What this means is that, by inducing the possibility distribution $\Pi_{(\text{Height(John)},\text{Weight(John)})}$, the proposition "John is a big man" implies that the possibility that John's height and weight are, say, 170 cm and 60 kg, respectively, is 0·6.

It should be noted that, in general, the entries in a relation F need not be numbers, as they are in Table 2.2. Thus, the entries may be pointers to—or identifiers of—physical or abstract objects. For example, the *u*'s in the relation CUP shown in Table 2.3:

TABLE 2.3

CUP	Identifier	μ
	u_1	0·8
	u_2	0·9
	u_3	1·0
	u_4	0·2

may be pictures of cups of various forms. In this case, given the relation CUP, the proposition "X is a cup" induces a possibility distribution Π_X such that Poss$\{X=u_1\}=$ 0·8 and likewise for other rows in the table.

In the translation of expressions in a natural language into PRUF, there are two operations on possibility distributions (or fuzzy relations) that play a particularly important role: *projection* and *particularization*.

Specifically, let $X \triangleq (X_1, \ldots, X_n)$ be a fuzzy variable which is associated with a possibility distribution $\Pi_{(X_1, \ldots, X_n)}$ or, more simply, Π_X, with the understanding that Π_X is an n-ary fuzzy relation in the Cartesian product, $U = U_1 \times \ldots \times U_n$, of the universes of discourse associated with $X_1, \ldots, X_n$. We assume that Π_X is characterized by its possibility distribution function—or, equivalently, membership function—$\pi_{(X_1, \ldots, X_n)}$ (or π_X, for short).

A variable of the form

$$X_{(s)} \triangleq (X_{i_1}, \ldots, X_{i_k}), \tag{2.18}$$

where $s \triangleq (i_1, \ldots, i_k)$ is a subsequence of the index sequence $(1, \ldots, n)$, constitutes a *subvariable* of $X \triangleq (X_1, \ldots, X_n)$. By analogy with the concept of a marginal probability distribution, the *marginal possibility distribution* associated with $X_{(s)}$ is defined by

$$\Pi_{X_{(s)}} = \mathrm{Proj}_{U_{(s)}} \Pi_{(X_1, \ldots, X_n)}, \tag{2.19}$$

where $U_{(s)} \triangleq U_{i_1} \times \ldots \times U_{i_k}$, and the operation of projection is defined—in terms of possibility distribution functions—by

$$\pi_{X_{(s)}}(u_{(s)}) = \mathrm{Sup}_{u_{(s')}}\ \pi_X(u_1, \ldots, u_n), \tag{2.20}$$

where $u_{(s)} \triangleq (u_{i_1}, \ldots, u_{i_k})$ and $u_{(s')} \triangleq (u_{j_1}, \ldots, u_{j_l})$, with s' denoting the index sequence complementary to s (e.g. if $n=5$ and $s=(2,3)$, then $(s')=(1,4,5)$). For example, for $n=2$ and $s=(2)$, (2.20) yields

$$\pi_{X_2}(u_2) = \mathrm{Sup}_{u_1}\ \pi_{(X_1, X_2)}(u_1, u_2) \tag{2.21}$$

as the expression for the marginal possibility distribution function of X_2.

The operation of projection is very easy to perform when Π_X is expressed as a linear form. As an illustration, assume that $U_1 = U_2 = a + b$, or, more conventionally, $\{a,b\}$, and

$$\Pi_{(X_1, X_2)} = 0{\cdot}8aa + 0{\cdot}6ab + 0{\cdot}4ba + 0{\cdot}2bb \tag{2.22}$$

in which a term of the form $0{\cdot}6ab$ signifies that

$$\mathrm{Poss}\{X_1 = a,\ X_2 = b\} = 0{\cdot}6.$$

To obtain the projection of Π_X on, say, U_2 it is sufficient to replace the value of X_1 in each term in (2.22) by the null string Λ. Thus†

$$\begin{aligned} \mathrm{Proj}_{U_2}\ \Pi_{(X_1, X_2)} &= 0{\cdot}8a + 0{\cdot}6b + 0{\cdot}4a + 0{\cdot}2b \\ &= 0{\cdot}8a + 0{\cdot}6b. \end{aligned}$$

To simplify the notation, it is convenient—as is done in SQUARE (Boyce *et al.*, 1974)—to omit the word Proj in (2.19) and interpret $U_{(s)}$ as $X_{i_1} \times \ldots \times X_{i_k}$ (see footnote on p. 404). Thus,

$$\mathrm{Proj}_{U_{(s)}} \Pi_{(X_1, \ldots, X_n)} \triangleq {}_{U_{(s)}}\Pi_{(X_1, \ldots, X_n)} \triangleq {}_{X_{i_1} \times \cdots \times X_{i_k}}\Pi_{(X_1, \ldots, X_n)}. \tag{2.23}$$

†If r and s are two tuples and α and β are their respective possibilities, then $\alpha r + \beta r = (\alpha \vee \beta) r$. Additional details may be found in Zadeh (1977*a*).

In some applications, it is convenient to have at one's disposal not only the operation of projection, as defined by (2.20), but also its *dual, conjunctive projection*,† which is defined by (2.20) with Sup replaced by Inf. It is easy to verify that the latter can be expressed in terms of the former as

$$\overline{\mathrm{Proj}}_{U_{(s)}}\Pi_{(X_1,\ldots,X_n)}=(\mathrm{Proj}_{U_{(s)}}\Pi'_{(X_1,\ldots,X_n)})' \tag{2.24}$$

in which $\overline{\mathrm{Proj}}$ stands for conjunctive projection and $'$ denotes the complement, where the complement of a fuzzy set F in U is a fuzzy set F′ defined by

$$\mu_{F'}(u)=1-\mu_F(u),\ u\in U. \tag{2.25}$$

PARTICULARIZATION

Informally, by the *particularization* of a fuzzy relation or a possibility distribution which is associated with a variable $X\triangleq(X_1,\ldots,X_n)$, is meant the effect of specification of the possibility distributions of one or more subvariables of X. In the theory of non-fuzzy relations, the resulting relation is commonly referred to as a *restriction* of the original relation; and, in the particular case where the values of some of the constituent variables are specified, the degenerate restriction becomes a *section* of the original relation.

Particularization in PRUF may be viewed as the result of forming the conjunction of a proposition of the form "X is F," where X is an n-ary variable, $X\triangleq(X_1,\ldots,X_n)$, with particularizing propositions of the form "$X_{(s)}$ is G," where $X_{(s)}$ is a subvariable of X, and F and G are fuzzy subsets of $U\triangleq U_1\times\ldots\times U_n$ and $U_{(s)}=U_{i_1}\times\ldots\times U_{i_k}$, respectively.

More specifically, let $\Pi_X\triangleq\Pi_{(X_1,\ldots,X_n)}=F$ and $\Pi_{X_{(s)}}\triangleq\Pi_{(X_{i_1},\ldots,X_{i_k})}=G$ be the possibility distributions induced by the propositions "X is F" and "$X_{(s)}$ is G," respectively. By definition, the *particularization of* Π_X *by* $X_{(s)}=G$ (or, equivalently, of F by G) is denoted by $\Pi_X[\Pi_{X_{(s)}}=G]$ (or $F[\Pi_{X_{(s)}}=G]$) and is defined as the intersection‡ of F and G, i.e.

$$\Pi_X[\Pi_{X_{(s)}}=G]=F\cap\overline{G}, \tag{2.26}$$

where $\overline{G}$ is the cylindrical extension of G, i.e. the cylindrical fuzzy set in $U_1\times\ldots\times U_n$ whose projection on $U_{(s)}$ is G and whose membership function is expressed by

$$\mu_{\overline{G}}(u_1,\ldots,u_n)\triangleq\mu_G(u_{i_1},\ldots,u_{i_k}),\quad (u_1,\ldots,u_n)\in U_1\times\ldots\times U_n. \tag{2.27}$$

As a simple illustration, assume that $U_1=U_2=U_3=a+b$,

$$\Pi_{(X_1,X_2,X_3)}=0{\cdot}8aab+0{\cdot}6baa+0{\cdot}1bab+1bbb \tag{2.28}$$

and

$$\Pi_{(X_1,X_2)}=G=0{\cdot}5aa+0{\cdot}2ba+0{\cdot}3bb.$$

†A more detailed discussion of conjunctive projections may be found in Zadeh (1966). It should be noted that the concept of a conjunctive projection is related to that of a conjunctive mapping in SQUARE (Boyce *et al.*, 1974) and to universal quantification in multivalued logic (Rescher, 1969).

‡If A and B are fuzzy subsets of U, their *intersection* is defined by $\mu_{A\cap B}(u)=\mu_A(u)\wedge\mu_B(u)$, $u\in U$. Thus, $\mu_{F\cap\overline{G}}(u_1,\ldots,u_n)=\mu_F(u_1,\ldots,u_n)\wedge\mu_G(u_{i_1},\ldots,u_{i_k})$. Dually, the *union* of A and B is denoted as A+B (or $A\cup B$) and is defined by $\mu_{A+B}(u)\triangleq\mu_A(u)\vee\mu_B(u)$. ($\vee\triangleq$max and $\wedge\triangleq$min.)

In this case

$$\overline{G}=0{\cdot}5aaa+0{\cdot}5aab+0{\cdot}2baa+0{\cdot}2bab+0{\cdot}3bba+0{\cdot}3bbb$$
$$F\cap\overline{G}=0{\cdot}5aab+0{\cdot}2baa+0{\cdot}1bab+0{\cdot}3bbb$$

and hence

$$\Pi_{(X_1,X_2,X_3)}[\Pi_{(X_1,X_3)}=G]=0{\cdot}5aab+0{\cdot}2baa+0{\cdot}1bab+0{\cdot}3bbb.$$

As will be seen in section 4, the right-hand member of (2.26) represents the possibility distribution induced by the conjunction of "X is F" and "$X_{(s)}$ is G," that is, the proposition "X is F and $X_{(s)}$ is G". It is for this reason that the particularized possibility distribution $\Pi_X[\Pi_{X(s)}=G]$ may be viewed as the possibility distribution induced by the proposition "X is F and $X_{(s)}$ is G".

In cases in which more than one subvariable is particularized, e.g. the particularizing propositions are "$X_{(s)}$ is G," and "$X_{(r)}$ is H," the particularized possibility distribution will be expressed as

$$\Pi_X[\Pi_{X(s)}=G;\ \Pi_{X(r)}=H]. \tag{2.29}$$

Furthermore, particularization may be *nested*, as in

$$\Pi_X[\Pi_{X(s)}=G[\Pi_{Y(t)}=J]] \tag{2.30}$$

where the particularizing relation G is, in turn, particularized by the proposition "$Y_{(t)}$ is J," where $Y_{(t)}$ is a subvariable of the variable associated with G.

It is of interest to observe that, as its name implies, particularization involves an imposition of a restriction on the values that may be assumed by a variable. However, by dualizing the definition of particularization as expressed by (2.26), that is, by replacing the intersection with the union the opposite effect is achieved, with the resulting possibility distribution corresponding to the disjunction of "X is F" and "$X_{(s)}$ is G". We shall not make an explicit use of the dual of particularization in the present paper.

As a simple illustration of particularization, consider the proposition $p \triangleq$ John is big, where BIG is defined by Table 2.2, and assume that the particularizing proposition is $q \triangleq$ John is tall, in which TALL is defined by Table 2.4.

The assertion "John is big" may be expressed equivalently as "Size(John) is big," which is of the form "X is F," with X $\triangleq$ Size(John) and F $\triangleq$ BIG. Similarly, "John is tall" may be expressed as "Height(John) is tall," or, equivalently, Y is G, where Y $\triangleq$ Height (John) and G $\triangleq$ TALL.

TABLE 2.4

TALL	Height	μ
	165	0·6
	170	0·7
	175	0·8
	180	0·9
	185	1

TABLE 2.5

$BIG(\Pi_{Height}=TALL)$	Height	Weight	μ
	165	60	0·5
	170	60	0·6
	175	60	0·7
	170	65	0·7
	—	—	—
	180	70	0·9
	175	75	0·8
	180	75	0·9
	180	80	0·9
	185	75	1

Using (2.26), the tableau of the particularized relation $BIG[\Pi_{Height}=TALL]$ is readily found to be given by Table 2.5.

The value of μ for a typical row in this table, say for (Height$=$180, Weight$=$75), is obtained by computing the minimum of the values of μ for the corresponding rows in BIG and TALL (i.e. (180,75) in BIG and (180) in TALL). As is pointed out in section 4, this mode of combination of μ's corresponds to *non-interactive* conjunction, which is assumed to be a standard default definition of conjunction in PRUF. However, PRUF allows any definition of conjunction which is specified by the user to be employed in place of the standard definition.

As an additional example, consider the particularized possibility distribution (see (2.23))

$$\begin{aligned}&PROFESSOR[Name=Simon;\ Sex=Male;\\ &\qquad \Pi_{Age}={}_{\mu\times Age2}APPROXIMATELY[Age1=45]]\end{aligned} \tag{2.31}$$

which describes a subset of a set of professors whose name is Simon, who are male and who are approximately 45 years old. In this case, the possibility distribution of the variable Age is a particularized relation APPROXIMATELY in which the first variable, Age1, is set equal to 45, and which is projected on the Cartesian product of $U(\mu)$ and U(Age2), yielding the fuzzy set of values of Age which are approximately equal to 45.

It should be noted that some of the attributes in (2.31) (e.g. Name) are assigned single values, while others—whose values are uncertain—are associated with possibility distributions. As will be seen in the following sections, this is typical of the particularized possibility distributions arising in the translation of expressions in a natural language into PRUF.

Expressions of the form (2.31) are similar in appearance to the commonly employed semantic network, query language and predicate calculus representations of propositions in a natural language. An essential difference, however, lies in the use of possibility distributions in (2.31) for the characterization of the values of fuzzy variables and in the concrete specification of the manner in which possibility distributions and fuzzy relations are modified by particularization and other operations which will be described in sections 4 and 6.

3. Basic concepts underlying translation into PRUF

The concept of a possibility distribution provides a natural point of departure for the formalization of many other concepts which underlie the translation of expressions in a natural language into PRUF. We shall present a brief exposition of several such concepts in this section, without aiming at the construction of an embracing formal framework.

In speaking somewhat vaguely of expressions in a natural language, what we have in mind is a variety of syntactic, semantic and pragmatic forms exemplified by sentences, propositions, phrases, clauses, questions, commands, exclamations, etc. In what follows, we shall restrict our attention to expressions which are (a) fuzzy propositions (or assertions); (b) fuzzy questions; and (c) what will be referred to as *fuzzy set descriptors* or simply *descriptors*.

PROPOSITIONS

Basically, a fuzzy proposition may be regarded as an expression which translates into a possibility assignment equation in PRUF. This is analogous to characterizing a non-fuzzy proposition as an expression which translates into a well-formed formula (or, equivalently, a closed sentence) in predicate calculus.

The types of fuzzy propositions to which our analysis will apply are exemplified by the following. (Italics place in evidence the words that have fuzzy denotation.)

Ronald is *more or less young*	(3.1)
Miriam was *very rich*	(3.2)
Harry *loves* Ann	(3.3)
X is *much smaller* than Y	(3.4)
X and Y are *approximately equal*	(3.5)
If X is *large* then Y is *small*	(3.6)
Most Swedes are *blond*	(3.7)
Many men are *much taller* than *most men*	(3.8)
Most Swedes are *tall* is *not very true*	(3.9)
The man in the *dark* suit is *walking slowly toward* the door	(3.10)
Susanna gave *several expensive* presents to each of her *close friends*	(3.11)
If X is *much greater* than Y then (Z is *small* is *very probable*)	(3.12)
If X is *much greater* than Y then (Z is *small* is *quite possible*)	(3.13)

In these examples, propositions (3.9), (3.12) and (3.13) are, respectively, truth qualified, probability qualified and possibility qualified; propositions (3.7), (3.8), (3.9) and (3.11) contain fuzzy quantifiers; and proposition (3.10) contains a fuzzy relative clause.

FUZZY SET DESCRIPTORS

Informally, a fuzzy set descriptor or simply a descriptor is an expression which is a label of a fuzzy set or a characterization of a fuzzy set in terms of other fuzzy sets. Simple examples of fuzzy set descriptors in English are

Very tall man	(3.14)
Tall man wearing a brown hat	(3.15)
The dishes on the table	(3.16)

Small integer (3.17)
Numbers which are much larger than 10 (3.18)
Most (3.19)
All (3.20)
Several (3.21)
Many tall women (3.22)
Above the table (3.23)
Much taller than (3.24)

A descriptor differs from a proposition in that it translates, in general, into a fuzzy relation rather than a possibility distribution or a possibility assignment equation. In this connection, it should be noted that a non-fuzzy descriptor (i.e. a description of a non-fuzzy set) would, in general, translate into an open sentence (i.e. a formula with free variables) in predicate calculus. However, while the distinction between open and closed sentences is sharply drawn in predicate calculus, the distinction between fuzzy propositions and fuzzy set descriptors is somewhat blurred in PRUF.

QUESTIONS

For the purposes of translation into PRUF, a question will be assumed to be expressed in the form B is ?A, where B is the body of the question —e.g., How tall is Vera—and A indicates the form of an admissible answer, which might be (a) a possibility distribution or, as a special case, an element of a universe of discourse; (b) a truth-value; (c) a probability-value; and (d) a possibility-value. To differentiate between these cases, A will be expressed as Π in (a) and, more particularly, as α when a numerical value of an attribute is desired; as τ in (b); as λ in (c); and as ω in (d).

To simplify the treatment of questions, we shall employ the artifice of translating into PRUF not the question itself but rather the answer to it, which, in general, would have the form of a fuzzy proposition. As an illustration,

How tall is Tom ?$\Pi \rightarrow$Tom is ?Π (3.25)
How tall is Tom ?$\alpha \rightarrow$Tom is ?α tall (3.26)
Where does Tom live$\rightarrow$Tom lives in ?α (3.27)
Is it true that Fran is blonde$\rightarrow$Fran is blonde is ?τ (3.28)
Is it likely that X is small$\rightarrow$X is small is ?λ (3.29)
Is it possible that (Jan is tall is false)$\rightarrow$(Jan is tall is false) is ?ω (3.30)

In this way, the translation of questions stated in a natural language may be carried out by the application of translation rules for fuzzy propositions, thus making it unnecessary to have separate rules for questions.

POSSIBILITY ASSIGNMENT EQUATIONS

The concept of a possibility assignment equation and its role in the translation of propositions in a natural language into PRUF have been discussed briefly in section 2. In what follows, we shall focus our attention on several additional aspects of this concept which relate to the translation rules which will be formulated in sections 4 and 6.

As was stated earlier, a proposition of the form $p \triangleq$ N is F in which N is the name of (a) a variable, (b) a fuzzy set, (c) a proposition, or (d) an object, and F is a fuzzy subset of a universe of discourse U, translates, in general, into a possibility assignment equation of the form

$$\Pi_{X(N)} = F \tag{3.31}$$

or, more simply,

$$\Pi_X = F \tag{3.32}$$

where X is a variable taking values in U, with X being either N itself (when N is a variable) or a variable that is explicit or implicit in N.

To place in evidence that (3.32) is a translation of "N is F," we write

$$p \triangleq \text{N is F} \rightarrow \Pi_X = F \tag{3.33}$$

and, conversely,

$$p \triangleq \text{N is F} \leftarrow \Pi_X = F \tag{3.34}$$

with the left-hand member of (3.34) referred to as a *retranslation* of its right-hand member.

In general, the variable X is an n-ary variable which may be expressed as $X \triangleq (X_1, \ldots, X_n)$, with $X_1, \ldots, X_n$ varying over $U_1, \ldots, U_n$, respectively. In some instances, the identification of the X_i and F is quite straightforward; in others, it may be a highly non-trivial task requiring a great deal of contextual knowledge.† For this reason, the identification of the X_i is difficult to formulate as a mechanical process. However, as is usually the case in translation processes, the problem can be greatly simplified by a decomposition of p into simpler constituent expressions, translating each expression separately, and then combining the results. The translation rules formulated in sections 4 and 6 are intended to serve this purpose.

In general, a constituent variable, X_i, has a nested structure of the form

$$X_i = \text{Attribute name(Part name(Part name} \ldots \text{(N)))} \tag{3.35}$$

which is similar to the structure of selectors in the Vienna Definition Language (Lucas *et al.*, 1968; Wegner, 1972). As a simple illustration,

$$\text{Myrna is blonde} \rightarrow \Pi_{\text{Color(Hair(Myrna))}} = \text{BLONDE} \tag{3.36}$$

where Color(Hair(Myrna)) is a nested variable of the form (3.35) and BLONDE is the fuzzy denotation of blonde in the universe of discourse which is associated with the proposition in question.

A problem that arises in some cases relates to the lack of an appropriate attribute name. For example, to express the translation of "Manuel is kind," in the form (3.33), we need a designation in English for the attribute which takes "kind" as a value. When such a name is not available in a language, it will be denoted by the symbol A, with a subscript if necessary, to indicate that "kind" is a value of A. However, what is really needed in cases like this is a possibly algorithmic definition of the concept represented

†In one form or another, this problem arises in all meaning representation languages. However, it is a much more difficult problem in machine-oriented languages than in PRUF, because in PRUF the task of identifying the X_i is assumed to be performed by a human.

by A which decomposes it into simpler concepts for which appropriate names are available.

In the foregoing examples, N represents the name of an object, e.g. the name of a person. More generally, N may be a descriptor, which is usually expressed as a relative clause, as in

$$\text{The man standing near the door is tall.} \tag{3.37}$$

N may also be a proposition, as in

$$\text{Lucia is tall is false.} \tag{3.38}$$

In (3.37), N $\triangleq$ The man standing in the door, while in (3.38), N $\triangleq$ Lucia is tall and X(N) is the truth-value of the proposition "Lucia is tall".

An important point concerning propositions of the form "N is F" which can be clarified at this juncture, is that "N is F" should be regarded not as a restricted class of propositions, but as a canonical form for all propositions which admit of translation into a possibility assignment equation†. Thus, if p is any proposition such that

$$p \to \Pi_X = F \tag{3.39}$$

then upon retranslation it may be expressed as "X is F," which is of the form "N is F".

As an illustration, the proposition "Paul was rich," may be translated as

$$\text{Paul was rich} \to \Pi_{(\text{Wealth}(\text{Paul}),\text{Time})} = \text{RICH} \times \text{PAST} \tag{3.40}$$

where (Wealth(Paul),Time) is a binary variable whose first component is the wealth of Paul (expressed as net worth) and the second component is the time at which net worth is assessed; RICH is a fuzzy subset of U(Wealth); PAST is a fuzzy subset of the time-interval extending from the present into the past; and RICH $\times$ PAST is the Cartesian product‡ of RICH and PAST.

Similarly, the proposition "X and Y are approximately equal," where X and Y are real numbers, may be translated as

$$\text{X and Y are approximately equal} \to \Pi_{(X,Y)} = \text{APPROXIMATELY EQUAL} \tag{3.41}$$

where APPROXIMATELY EQUAL is a fuzzy relation in R^2. Upon retranslation, (3.41) yields the equivalent proposition

$$\text{(X,Y) is approximately equal} \tag{3.42}$$

which, though ungrammatical, is in canonical form.

†This is equivalent to saying that "N is F" is a canonical form for all propositions which can be expressed in the form "N is F" through the application of a meaning-preserving transformation. Such transformations will be defined later in this section in connection with the concept of semantic equivalence.

‡If A and B are fuzzy subsets of U and V, respectively, their *Cartesian product* is defined by $\mu_{A\times B}(u,v) \triangleq \mu_A(u) \wedge \mu_B(v)$, $u \in U$, $v \in V$.

A related issue which concerns the form of possibility assignment equations is that, in general, such equations may be expressed equivalently in the form of possibility distributions. More specifically, if we have

$$\text{N is F} \to \Pi_X = \text{F} \tag{3.43}$$

then the possibility assignment equation in (3.43) may be expressed as a possibility distribution (labeled N) of the variable X(N), with the tableau of N having the form:

TABLE 3.1

N	X(N)	π
	u_1	π_1
	u_2	π_2
	.	.
	u_n	π_n

where the π_i are the possibility-values of the u_i.

As a simple illustration, in the translation

$$\text{Brian is tall} \to \Pi_{\text{Height(Brian)}} = \text{TALL} \tag{3.44}$$

where TALL is a fuzzy set defined by, say,

$$\text{TALL} = 0{\cdot}5/160 + 0{\cdot}6/165 + 0{\cdot}7/170 + 0{\cdot}8/175 + 0{\cdot}9/180 + 1/185 \tag{3.45}$$

the possibility assignment equation may be replaced by the possibility distribution

BRIAN	Height	π
	160	0·5
	165	0·6
	170	0·7
	175	0·8
	180	0·9
	185	1·0

which in turn may be expressed as the particularized possibility distribution

$$\text{BRIAN}[\Pi_{\text{Height}} = \text{TALL}] \tag{3.46}$$

on the understanding that, initially,

$$\Pi_{\text{BRIAN}} = \text{I}; \tag{3.47}$$

that is, BRIAN is a unity possibility distribution with

$$\pi_{\mathrm{BRIAN}}(u)=1 \text{ for } u \in \mathrm{U}. \tag{3.48}$$

It is this equivalence between (3.44) and (3.46) that forms the basis for the statement made in section 1 regarding the equivalence of (1.1) and (1.2).

DEFINITION

All natural languages provide a mechanism for defining a concept in terms of other concepts and, more particularly, for designating a complex descriptor by a single label. Consequently, it is essential to have a facility for this purpose in every meaning representation language, including PRUF.†

A somewhat subtle issue that arises in this connection in PRUF relates to the need for normalizing‡ the translation of the definiens into PRUF. As an illustration of this point, suppose that the descriptor *middle-aged* is defined as

$$\text{middle-aged} \triangleq \text{not young and not old}. \tag{3.49}$$

Now, as will be seen in section 5, the translation of the right-hand member of (3.49) is expressed by

$$\text{not young and not old} \to \mathrm{YOUNG}' \cap \mathrm{OLD}' \tag{3.50}$$

where YOUNG and OLD are the translations of young and old, respectively, and ′ denotes the complement. Consequently, for some definitions of YOUNG and OLD the definition of *middle-aged* by (3.49) would result in a subnormal fuzzy set, which would imply that there does not exist any individual who is middle-aged to the degree 1.

While this may be in accord with one's intuition in some cases, it may be counter-intuitive in others. Thus, to clarify the intent of the definition, it is necessary to indicate whether or not the definiens is to be normalized.§ For this purpose, the notation

$$\text{definiendum} \triangleq \text{Norm(definiens)} \tag{3.51}$$

e.g.

$$\text{middle-aged} \triangleq \text{Norm(not young and not old)} \tag{3.52}$$

may be employed to indicate that the translation of the definiens ought to be normalized.

EXPRESSIONS IN PRUF

Expressions in PRUF are not rigidly defined, as they are in formal, programming and machine-oriented meaning representation languages. Typically, an expression in PRUF may assume the following forms.

†Concept definition plays a particularly important role in conceptual dependency graphs (Schank, 1973), in which a small number of primitive concepts are used as basic building blocks for more complex concepts.

‡A fuzzy set F is *normal* if and only if $\mathrm{Sup}_u\, \mu_F(u)=1$. If F is *subnormal*, it may be normalized by dividing μ_F by $\mathrm{Sup}_u\, \mu_F(u)$. Thus, the membership function of normalized F, Norm(F), is given by

$$\mu_{\mathrm{Norm}(F)}(u) \triangleq \mu_F(u)/\mathrm{Sup}_u\, \mu_F(u).$$

§The need for normalization was suggested by some examples brought to the author's attention by P. Kay (U.C., Berkeley) and W. Kempton (U.T., San Antonio). (See Kay 1975).)

(a) A label of a fuzzy relation or a possibility distribution. Examples: CUP, BIG MAN, APPROXIMATELY EQUAL.

(b) A particularized fuzzy relation or a possibility distribution. Examples:

$$\mathrm{CUP}[\Pi_{\mathrm{Color}} = \mathrm{RED};\ \mathrm{Weight} = 35\ \mathrm{gr}] \tag{3.53}$$

$$\begin{aligned}&\mathrm{CAR}[\mathrm{Make} = \mathrm{Ford};\ \Pi_{\mathrm{Size(Trunk)}} = \mathrm{BIG};\\ &\qquad \Pi_{\mathrm{Weight}} = {}_{\mu \times \mathrm{Weight2}}\mathrm{APPROXIMATELY}[\mathrm{Weight1} = 1500\ \mathrm{kg}]].\end{aligned}$$

(c) A possibility assignment equation. Examples:

$$\Pi_{\mathrm{Height(Valentina)}} = \mathrm{TALL} \tag{3.54}$$

$$\Pi_X = \mathrm{CUP}[\Pi_{\mathrm{Color}} = \mathrm{RED};\ \mathrm{Weight} = 35\ \mathrm{gr}].$$

(d) A definition. Examples:

$$F \triangleq H + G[\Pi_{X_{(s)}} = K] \tag{3.55}$$

where + denotes the union, H is a fuzzy relation and $G[\Pi_{X_{(s)}} = K]$ is a particularized fuzzy relation

$$F \triangleq \mathrm{HOUSE}[\Pi_{\mathrm{Color}} = \mathrm{GREY};\ \Pi_{\mathrm{Price}} = \mathrm{HIGH}] \tag{3.56}$$

which defines a fuzzy set of houses which are grey in color and high-priced.

(e) A procedure—expressed in a natural, algorithmic or programming language—for computing a fuzzy relation or a possibility distribution. Examples: examples (t), (u) and (v) in section 7.

In general, a fuzzy set descriptor will translate into an expression of the form (a), (b) or (d), while a fuzzy proposition will usually translate into (b), (c) or (d). In all these cases, an expression in PRUF may be viewed as a procedure which—given a set of relations in a database—returns a fuzzy relation, a possibility distribution or a possibility assignment equation.†

DATABASE, MEANING AND INFORMATION

By a *relational database* or, simply, a *database* in the context of PRUF is meant a collection, $\mathscr{D}$, of fuzzy, time-varying relations which may be characterized in various ways, e.g. by tables, predicates, recognition algorithms, generation algorithms, etc. A simple self-explanatory example of a database, $\mathscr{D}$, consisting of fixed (i.e. time-invariant) relations POPULATION, YOUNG and RESEMBLANCE is shown in Table 3.2. What is implicit in this representation is that each of the variables (i.e. attributes) which appear as column headings, is associated with a specified universe of discourse (i.e. a domain). For example, the universe of discourse associated with the variable Name in POPULATION is given by

$$U(\mathrm{Name}) = \mathrm{Codd} + \mathrm{King} + \mathrm{Chen} + \mathrm{Chang}. \tag{3.57}$$

†It should be noted that an expression in PRUF may also be interpreted as a probability—rather than possibility—manipulating procedure. Because of the need for normalization, operations on probability distributions are, in general, more complex than the corresponding operations on possibility distributions.

In general, two variables which have the same name but appear in different tables may be associated with different universes of discourse.

The relations YOUNG and RESEMBLANCE in Table 3.2 are *purely extensional*† in the sense that YOUNG and RESEMBLANCE are defined directly

TABLE 3.2

POPULATION	Name
	Codd
	King
	Chen
	Chang

RESEMBLANCE	Name1	Name2	μ
	Codd	King	0·8
	Codd	Chen	0·6
	Codd	Chang	0·6
	King	Chen	0·5
	...	...	...
	Chang	Chen	0·8

YOUNG	Name	μ
	Codd	0·7
	King	0·9
	Chen	0·8
	Chang	0·9

as fuzzy subsets of POPULATION and not through a procedure which would allow the computation of YOUNG and RESEMBLANCE for any given POPULATION. To illustrate the point, if POPULATION and YOUNG were defined as shown in Table 3.3, then it would be possible to compute the fuzzy subset YOUNG of any given POPULATION by employing the procedure expressed by

$$\text{YOUNG} =_{\mu \times \text{Name}} \text{POPULATION}[\Pi_{\text{Age}} = \text{YOUNG}] \tag{3.58}$$

TABLE 3.3

POPULATION	Name	Age
	Codd	45
	King	31
	Chen	42
	Chang	33

YOUNG	Age	μ
	30	0·8
	31	0·75
	32	0·70
	33	0·60
	...	...
	42	0·4
	45	0·3

†In the theories of language based on two-valued logic (Linsky, 1971; Quine, 1970*a*; Cresswell, 1973) the dividing line between *extensional* and *intensional* is sharply drawn. This is not the case in PRUF—in which there are levels of intensionality (or, equivalently, levels of procedural generality), with pure extensionality constituting one extreme. This issue will be discussed in greater detail in a forthcoming paper.

where YOUNG in the right-hand member is a fuzzy subset of U(Age), and μ is implicit in POPULATION.

Since an expression in PRUF is a procedure, it involves, in general, not the relations in the database but only their *frames*.† In addition, an expression in PRUF may involve the names of universes of discourse and/or their Cartesian products; the names of some of the relation elements; and possibly the values of some attributes of the relations in the database (e.g. the number of rows).

As an illustration, the *frame of the database* shown in Table 3.2 (i.e. the collection of frames of its constituent relations) is comprised of:

POPULATION	Name

,

YOUNG	Name	μ

RESEMBLANCE	Name1	Name2	μ

Correspondingly, an expression in PRUF such as

$$_{\mu \times \mathrm{Name1}}\ \mathrm{RESEMBLANCE}[\mathrm{Name2} = \mathrm{King}] \tag{3.59}$$

represents a procedure which returns the fuzzy subset of POPULATION comprising names of individuals who resemble King.

Ultimately, each of the symbols or names in a database is assumed to be defined ostensively (Lyons, 1968) or, equivalently, by exemplification; that is, by pointing or otherwise focussing on a real or abstract object and indicating the degree—on the scale from 0 to 1—to which it is compatible with the symbol in question. In this sense, then, a database may viewed as an interface with an external world which might be real or abstract or a combination of the two.‡

In general, the correspondence between a database and an external world is difficult to formalize because the universe of discourse associated with an external world comprises not just a model of that world, say M, but also the set of fuzzy subsets of M, the set of fuzzy subsets of fuzzy subsets of M, etc. To illustrate this point, it is relatively easy to define by exemplification the denotation of *red*, which is a fuzzy subset of M; much more difficult to define the concept of *color*, which is a subset of $\mathscr{P}$(M), the set of fuzzy subsets of M; and much much more difficult to define the concept of *attribute*, which is a subset of $\mathscr{P}(\mathscr{P}$(M)) (Zadeh, 1971*b*).

Viewed in this perspective, the issues related to the correspondence between a database and an external world are similar to those which arise in pattern recognition and are even harder to formulate and resolve within a formal framework. As a direct consequence of this difficulty, a complete formalization of the concept of *meaning* does not appear to be an attainable goal in the foreseeable future.

In the context of PRUF, the concept of meaning is defined in a somewhat restricted way, as follows:

†By the *frame* of a relation is meant its name and column headings (i.e. the names of variables or, equivalently, attributes). The rest of the relation (i.e. the table without column headings) is its *body*.

‡In this sense, the concept of a database is related to that of a possible world in possible world semantics and modal logic (Kripke, 1963; Hughes & Cresswell, 1968; Partee, 1976*a*).

Let e be an expression in a natural language and let E be its translation into PRUF, i.e.

$$e \rightarrow \mathrm{E} \tag{3.60}$$

and, more particularly,

$$p \rightarrow \mathrm{P} \tag{3.61}$$

if e is a proposition; and

$$d \rightarrow \mathrm{D} \tag{3.62}$$

if e is a descriptor. To illustrate:

$$\text{cup} \rightarrow \text{CUP} \tag{3.63}$$

$$\text{red cup} \rightarrow \text{CUP}[\Pi_{\text{Color}} = \text{RED}] \tag{3.64}$$

$$\text{George is young} \rightarrow \Pi_{\text{Age(George)}} = \text{YOUNG} \tag{3.65}$$

or, equivalently,

$$\text{George is young} \rightarrow \text{GEORGE}[\Pi_{\text{Age}} = \text{YOUNG}]. \tag{3.66}$$

Stated informally, the procedure, E, may be viewed as the *meaning* of e in the sense that, if $e \triangleq d$, then for any given database $\mathscr{D}$ on which D is defined, D computes (or returns) a fuzzy relation F^d which is a fuzzy denotation (or extension) of d in its universe of discourse (which may be different from $\mathscr{D}$). Similarly, if $e \triangleq p$, then P is a procedure which, for any given database $\mathscr{D}$ on which P is defined, computes a possibility distribution Π^p. This distribution, then, may be regarded as the *information* conveyed by p.† In particular, if Π^p is the possibility distribution of a variable X and $X_{(s)}$ is a subvariable of X, then the *information conveyed by p about* $X_{(s)}$ is given by the projection of Π^p on $U_{(s)}$. When it is necessary to indicate that Π^p is the result of acting with P on a particular database $\mathscr{D}$, Π^p will be referred to as the possibility distribution induced by p (or the information conveyed by p) *in application to* $\mathscr{D}$.

As an illustration, consider the proposition

$$p \triangleq \text{Mike recently lived near Boston} \tag{3.67}$$

which in PRUF translates into

$$\begin{aligned}\text{RESIDENCE}[&\text{Subject} = \text{Mike};\ \Pi_{\text{Time}} = \text{RECENT PAST};\\ &\Pi_{\text{Location}} = {}_{\mu\times\text{City1}}\ \text{NEAR}[\text{City2} = \text{Boston}]]\end{aligned} \tag{3.68}$$

where NEAR is a fuzzy relation with the frame

NEAR	City1	City2	μ

,

RECENT PAST is a fuzzy relation with the frame

RECENT PAST	Time	μ

†It should be noted that a non-probabilistic measure of information was introduced by Kampe de Feriet & Forte (1967, 1977). In the present paper, however, our concern is with the information itself, which is represented by a possibility distribution, rather than with its measure, which is a real number.

(in which Time is expressed in years counting from the present to the past), and ${}_{\mu\times\mathrm{City1}}\mathrm{NEAR}[\mathrm{City2}=\mathrm{Boston}]$ is the fuzzy set of cities which are near Boston. Given a database, $\mathscr{D}$, (3.68) would return a possibility distribution such as shown (in a partially tabulated form) in Table 3.4, in which

TABLE 3.4

RESIDENCE	Subject	Location	Time	π
	Mike	Cambridge	1	1
	Mike	Cambridge	2	0·8
	Mike	Cambridge	3	0·6
	Mike	Wayland	1	0·9
	Mike	Wayland	2	0·8
	...	...	...	...

the third row, for example, signifies that the possibility that Mike lived in Cambridge 3 years ago is 0·6. In this example, (3.68) constitutes the meaning of p, while the possibility distribution whose tableau is given by Table 3.4 is the information conveyed by p.

In addition to representing the meaning of an expression, e, in a natural language, the corresponding expression, E, in PRUF may be viewed as its *deep structure*—not in the technical sense employed in the literature of linguistics (Chomsky, 1965, 1971)—but in the sense of being dependent not on the surface structure of e but on its meaning. This implies that the form of E is independent of the natural language in which e is expressed, thus providing the basis for referring to PRUF as a universal language. The same can be said, of course, of most of the meaning representation languages that have been described in the literature.

Another characteristic of PRUF that is worthy of mention is that it is an *intentional*† language in the sense that an expression in PRUF is supposed to convey the intended rather than the literal meaning of the corresponding expression in a natural language. For example, if the proposition $p \triangleq$ John is no genius is intended to mean that $q \triangleq$ John is dumb, then the translation of p into PRUF would be that of q rather than p itself. As an example illustrating a somewhat different point, consider the proposition

$$p \triangleq \text{Alla has red hair.} \tag{3.69}$$

In PRUF, its translation could be expressed in one of two ways:

(a) $$\text{Alla has red hair} \to \Pi_{\mathrm{Color(Hair(Alla))}} = \varphi \tag{3.70}$$

where φ is an identifier of the color that is commonly referred to as *red* in the case of hair; or

(b) $$\text{Alla has red hair} \to \Pi_{\mathrm{Color(Hair(Alla))}} = \mathrm{RED}^{f} \tag{3.71}$$

†A thorough discussion of the concept of intentionality may be found in Grice (1968) and Searle (1971).

in which the superscript f (standing for *footnote*) points to a non-standard definition of RED which must be used in (3.71). The same convention is employed, more generally, whenever a non-standard definition of any entity in an expression in PRUF must be employed.

SEMANTIC EQUIVALENCE AND SEMANTIC ENTAILMENT

The concepts of semantic equivalence and semantic entailment are two closely related concepts in PRUF which play an important role in fuzzy logic and approximate reasoning.

Informally, let p and q be a pair of expressions in a natural language and let Π^p and Π^q be the possibility distributions (or the fuzzy relations) induced by p and q in application to a database $\mathscr{D}$. Then, we shall say that p and q are *semantically equivalent*, expressed as

$$p \leftrightarrow q, \tag{3.72}$$

if and only if $\Pi^p = \Pi^q$. Furthermore, if (3.72) holds for all databases,† the semantic equivalence between p and q is said to be *strong*.‡ Thus, the definition of strong semantic equivalence implies that p and q have the same meaning if and only if they are strongly semantically equivalent. In this sense, then, any transformation which maps p into q is *meaning-preserving*.

To illustrate, as will be seen in section 6, the propositions

$$p \triangleq \text{Jeanne is tall is true} \tag{3.73}$$

and

$$q \triangleq \text{Jeanne is not tall is false} \tag{3.74}$$

in which *false* is the antonym of *true*, i.e.

$$\mu_{\mathrm{FALSE}}(v) = \mu_{\mathrm{TRUE}}(1-v), \quad v \in [0,1] \tag{3.75}$$

are semantically equivalent no matter how TALL and TRUE are defined. Consequently, p and q are strongly semantically equivalent and hence have the same meaning. On the other hand, the propositions

$$p \triangleq \text{Jeanne is tall is very true} \tag{3.76}$$

and

$$q \triangleq \text{Jeanne is very tall} \tag{3.77}$$

can be shown to be semantically equivalent when TRUE is the unitary fuzzy set (see (2.15)), that is

$$\mu_{\mathrm{TRUE}}(v) = v, \quad v \in [0,1]$$

but not when TRUE is an arbitrary fuzzy subset of [0,1]. Consequently, p and q are not strongly semantically equivalent.

†Generally, "all databases" should be interpreted as all databases which are related in a specified way to a reference database. This is analogous to the role of the alternativeness relation in possible world semantics (Hughes & Cresswell, 1968).

‡The concept of strong semantic equivalence as defined here reduces to that of semantic equivalence in predicate logic (see Lyndon, 1966) when p and q are non-fuzzy propositions.

Usually, it is clear from the context whether a semantic equivalence is or is not strong. When it is necessary to place in evidence that a semantic equivalence is strong, it will be denoted by $s\leftrightarrow$. Correspondingly, if the equality between Π^p and Π^q is approximate in nature, the approximate semantic equivalence between p and q will be expressed as $p \; a\leftrightarrow q$.

While the concept of semantic equivalence relates to the equality of possibility distributions (or fuzzy relations), that of *semantic entailment* relates to inclusion.† More specifically, on denoting the assertion "p semantically entails q (or q is semantically entailed by p)," by $p \mapsto q$, we have

$$p \mapsto q \text{ iff } \Pi^p \subset \Pi^q \tag{3.78}$$

where Π^p and Π^q are the possibility distributions induced by the propositions p and q, respectively.

As in the case of semantic equivalence, semantic entailment is *strong* if the relation $\mapsto$ holds for all databases. For example, as will be seen in section 4, the possibility distribution induced by the proposition "Gary is very tall" is contained in that induced by "Gary is tall" no matter how TALL is defined. Consequently, we can assert that

$$\text{Gary is very tall } s\mapsto \text{Gary is tall} \tag{3.79}$$

where $s\mapsto$ denotes strong semantic entailment. On the other hand, the validity of the semantic entailment

$$\text{Gary is very tall} \mapsto \text{Gary is not short} \tag{3.80}$$

depends on the definitions of *tall* and *short*, and hence (3.80) does not represent strong semantic entailment.

As was stated earlier, the concepts of semantic equivalence and semantic entailment play an important role in fuzzy logic and approximate reasoning (Zadeh, 1977*b*). In the present paper, we shall make use of the concept of semantic equivalence in sections 4 and 6 to derive several useful meaning-preserving transformations.

4. Translation rules of Types I, II and III

To facilitate the translation of expressions in a natural language into PRUF, it is desirable to have a stock of translation rules which may be applied singly or in combination to yield an expression, E, in PRUF, which is a translation of a given expression, e, in a natural language.

The translation rules which apply to descriptors may readily be deduced from the corresponding rules for propositions. Consequently, we shall restrict our attention in the sequel to the translation of propositions.

The translation rules for propositions may be divided into several basic categories, the more important of which are the following:

Type I. Rules pertaining to modification.
Type II. Rules pertaining to composition.

†If A and B are fuzzy subsets of U, then $A \subset B$ iff $\mu_A(u) \leq \mu_B(u)$, $u \in U$.

Type III. Rules pertaining to quantification.
Type IV. Rules pertaining to qualification.

Simple examples of propositions to which the rules in question apply are the following:

Type I.
X is very small (4.1)
X is much larger than Y (4.2)
Eleanor was very upset (4.3)
The man with the blond hair is very tall (4.4)

Type II.
X is small and Y is large (conjunctive composition) (4.5)
X is small or Y is large (disjunctive composition) (4.6)
If X is small then Y is large (conditional composition) (4.7)
If X is small then Y is large else Y is very large (conditional and conjunctive composition) (4.8)

Type III.
Most Swedes are tall (4.9)
Many men are much taller than most men (4.10)
Most tall men are very intelligent (4.11)

Type IV.
Abe is young is not very true (truth qualification) (4.12)
Abe is young is quite probable (probability qualification) (4.13)
Abe is young is almost impossible (possibility qualification) (4.14)

Rules of Types I, II and III will be discussed in this section. Rules of Type IV will be discussed in section 6, following an exposition of the concepts of consistency, compatibility and truth in section 5.

Translation rules in PRUF are generally expressed in a conditional format exemplified by

$$\begin{array}{ll} \text{If} & p \to \mathrm{P} \\ \text{then} & p^+ \to \mathrm{P}^+ \end{array} \tag{4.15}$$

where p^+ and P^+ are modifications of p and P, respectively. In effect, a rule expressed in this form states that if in a certain context p translates into P, then in the *same context* a specified modification of p, p^+, translates into a specified modification of P, P^+. In this way, the rule makes it explicit that the translation of a modified proposition, p^+, depends on the translation of p. The simpler notation employed in (4.28) conveys the same information, but does so less explicitly.

RULES OF TYPE I

A basic rule of Type I is the *modifier rule*, which may be stated as follows.

If the proposition

$$p \triangleq \mathrm{N} \text{ is } \mathrm{F} \tag{4.16}$$

translates into the possibility assignment equation (see (3.31))

$$\Pi_{(X_1,\ldots,X_n)}=F \tag{4.17}$$

then the translation of the modified proposition

$$p^{+}\triangleq \mathrm{N} \text{ is } mF \tag{4.18}$$

where m is a modifier such as *not*, *very*, *more or less*, *quite*, *extremely*, etc., is given by

$$\mathrm{N} \text{ is } m\mathrm{F}\rightarrow\Pi_{(X_1,\ldots,X_n)}=\mathrm{F}^{+} \tag{4.19}$$

where F^{+} is a modification of F induced by m. In particular:

(a) if $m\triangleq$not, then $F^{+}=F'\triangleq$complement of F; (4.20)
(b) if $m\triangleq$very, then $F^{+}=F^{2}$, where† (4.21)

$$F^{2}=\int_{U}\mu_F^{2}(u)/u; \tag{4.22}$$

(c) if $m\triangleq$more or less, then $F^{+}=\sqrt{F}$ where (4.23)

$$\sqrt{F}=\int_{U}\sqrt{\mu_F(u)}/u; \tag{4.24}$$

or, alternatively,

$$F^{+}=\int_{U}\mu_F(u)K(u) \tag{4.25}$$

where K(u) is the kernel of *more or less*.‡

As a simple illustration of (4.21), let p be the proposition "Lisa is young," where YOUNG is a fuzzy subset of the interval [0,100] whose membership function is expressed in terms of the S-function (2.13) as (omitting the arguments of μ and S):

$$\mu_{YOUNG}=1-S(25,35,45). \tag{4.26}$$

Then, the translation of "Lisa is very young" is given by

$$\text{Lisa is very young}\rightarrow\Pi_{Age(Lisa)}=\mathrm{YOUNG}^{2} \tag{4.27}$$

where

$$\mu_{YOUNG^2}=(1-S(25,35,45))^{2}.$$

†The "integral" representation of a fuzzy set in the form $F=\int_U\mu_F(u)/u$ signifies that F is a union of the fuzzy singletons $\mu_F(u)/u$, $u\in U$, where μ_F is the membership function of F. Thus, (4.22) means that the membership function of F^2 is the square of that of F.

‡More detailed discussions of various types of modifiers may be found in Zadeh (1972*a*, 1975*c*), Lakoff (1973*a*,*b*), Wenstop (1975, 1976), Mizumoto *et al.* (1977), Hersh & Caramazza (1976), and other papers listed in the bibliography. It is important to note that (4.21) and (4.23) should be regarded merely as standardized default definitions which may be replaced, if necessary, by the user-supplied definitions.

Note that we can bypass the conditional format of the translation rule (4.16) and assert directly that

$$\text{Lisa is very young} \rightarrow \Pi_{\text{Age(Lisa)}} = \text{YOUNG}^2 \tag{4.28}$$

on the understanding that YOUNG is the denotation of *young* in the context in which the proposition "Lisa is very young" is asserted. As was stated earlier, the conditional format serves the purpose of making this understanding more explicit.

In some cases, a modifier such as *very* may be implicit rather than explicit in a proposition. Consider, for example, the proposition

$$p \triangleq \text{Vera and Pat are close friends.} \tag{4.29}$$

As an approximation, p may be assumed to be semantically equivalent to

$$q = \text{Vera and Pat are friends}^2 \tag{4.30}$$

so that (using (4.22)) the translation of p may be expressed as (see (7.21))

$$\pi(\text{FRIENDS}) = \mu\text{FRIENDS}^2[\text{Name1} = \text{Vera}; \text{Name2} = \text{Pat}] \tag{4.31}$$

where π(FRIENDS) is the possibility of the relation FRIENDS in $\mathscr{D}$. Thus, what (4.31) implies is that the relation FRIENDS in $\mathscr{D}$ is such that

$$\Pi_X = \perp^2 \tag{4.32}$$

where

$$X \triangleq {}_{\mu}\text{FRIENDS}[\text{Name1} = \text{Vera}; \text{Name2} = \text{Pat}] \tag{4.33}$$

and $\perp$ is the unitor defined by (2.15).

RULES OF TYPE II

Translation rules of Type II pertain to the translation of propositions of the form

$$p = q * r \tag{4.34}$$

where $*$ denotes an operation of composition, e.g. conjunction (and), disjunction (or), implication (if . . . then), etc.

Under the assumption that the operation of composition is non-interactive (Bellman & Zadeh, 1976),† the rules in question may be stated as follows.

If

$$q \triangleq \text{M is F} \rightarrow \Pi_{(X_1, \ldots, X_m)} = \text{F}$$

and

$$r \triangleq \text{N is G} \rightarrow \Pi_{(Y_1, \ldots, Y_n)} = \text{G} \tag{4.35}$$

†Informally, a binary operation $*$ on real numbers u, v is *non-interactive* if an increase in the value of u (or v) cannot be compensated by a decrease in the value of v (or u). It should be understood that the non-interactive definitions of *and* and *or* in (4.36) and (4.37) may be replaced, if necessary, by user-supplied interactive definitions.

then

(a) M is F and N is $G \rightarrow \Pi_{(X_1, \ldots, X_m, Y_1, \ldots, Y_n)} = \overline{F} \cap \overline{G}$
$= F \times G$ (4.36)

(b) M is F or N is $G \rightarrow \Pi_{(X_1, \ldots, X_m, Y_1, \ldots, Y_n)} = \overline{F} + \overline{G}$ (4.37)

and

(c_1) If M is F then N is $G \rightarrow \Pi_{(X_1, \ldots, X_m, Y_1, \ldots, Y_n)} = \overline{F}' \oplus \overline{G}$ (4.38)

or

(c_2) If M is F then N is $G \rightarrow \Pi_{(X_1, \ldots, X_m, Y_1, \ldots, Y_n)} = F \times G + F' \times V$ (4.39)

where F and G are fuzzy subsets of $U \triangleq U_1 \times \ldots \times U_n$ and $V = V_1 \times \ldots \times V_n$, respectively; $\overline{F}'$ and $\overline{G}$ are the cylindrical extensions of F′ and G, i.e.

$$\overline{F}' = F' \times V \tag{4.40}$$

$$\overline{G} = U \times G; \tag{4.41}$$

$F \times G$ is the Cartesian product of F and G which may be expressed as $\overline{F} \cap \overline{G}$ and is defined by

$$\mu_{F \times G}(u,v) = \mu_F(u) \wedge \mu_G(v), \quad u \in U, \quad v \in V; \tag{4.42}$$

$+$ is the union and $\oplus$ is the bounded-sum, i.e.

$$\mu_{\overline{F}' \oplus \overline{G}}(u,v) = 1 \wedge (1 - \mu_F(u) + \mu_G(v)) \tag{4.43}$$

where $u \triangleq (u_1, \ldots, u_n)$, $v \triangleq (v_1, \ldots, v_n)$, $\wedge \triangleq$ min, $+ \triangleq$ arithmetic sum and $- \triangleq$ arithmetic difference.† Note that there are two distinct rules for the conditional composition, (c_1) and (c_2). Of these, (c_1) is consistent with the definition of implication in Łukasiewicz's $Ł_{\text{Aleph}_1}$ logic (Rescher, 1969), while (c_2)—in consequence of (4.53)—corresponds to the relation expressed by the table:

TABLE 4.1

M	N
F	G
F′	V

As a very simple illustration, assume, as in Zadeh (1977*b*), that $U = V = 1 + 2 + 3$, $M \triangleq X$, $N \triangleq Y$,

$$F \triangleq \text{SMALL} \triangleq 1/1 + 0{\cdot}6/2 + 0{\cdot}1/3 \tag{4.44}$$

and

$$G \triangleq \text{LARGE} \triangleq 0{\cdot}1/1 + 0{\cdot}6/2 + 1/3. \tag{4.45}$$

†If the variables $X \triangleq (X_1, \ldots, X_n)$ and $Y \triangleq (Y_1, \ldots, Y_n)$ have a subvariable, say Z, in common, i.e. $X \triangleq (S,Z)$ and $Y \triangleq (T,Z)$, then F and G should be interpreted as cylindrical extensions of F and G in $U(S) \times U(T) \times U(Z)$ rather than in $U(X) \times U(Y)$, where U(S), U(T) and U(Z) denote, respectively, the universes in which S, T and Z take their values. Additionally, the possibility distributions in (7.38) and (7.39) should be interpreted, in a strict sense, as conditional distributions.

Then (4.36), (4.37), (4.38) and (4.39) yield

$$\text{X is small and Y is large} \rightarrow \tag{4.46}$$
$$\Pi_{(X,Y)} = 0{\cdot}1/(1,1) + 0{\cdot}6/(1,2) + 1/(1,3) + 0{\cdot}1/(2,1) + 0{\cdot}6/(2,2) + 0{\cdot}6/(2,3) + 0{\cdot}1/(3,1) + 0{\cdot}1/(3,2) + 0{\cdot}1/(3,3)$$

$$\text{X is small or Y is large} \rightarrow \tag{4.47}$$
$$\Pi_{(X,Y)} = 1/(1,1) + 1/(1,2) + 1/(1,3) + 0{\cdot}6/(2,1) + 0{\cdot}6/(2,2) + 1/(2,3) + 0{\cdot}1/(3,1) + 0{\cdot}6/(3,2) + 1/(3,3)$$

$$\text{If X is small then Y is large} \rightarrow \tag{4.48}$$
$$\Pi_{(X,Y)} = 0{\cdot}1/(1,1) + 0{\cdot}6/(1,2) + 1/(1,3) + 0{\cdot}5/(2,1) + 1/(2,2) + 1/(2,3) + 1/(3,1) + 1/(3,2) + 1/(3,3)$$

$$\text{If X is small then Y is large} \rightarrow \tag{4.49}$$
$$\Pi_{(X,Y)} = 0{\cdot}1/(1,1) + 0{\cdot}6/(1,2) + 1/(1,3) + 0{\cdot}4/(2,1) + 0{\cdot}6/(2,2) + 0{\cdot}6/(2,3) + 0{\cdot}9/(3,1) + 0{\cdot}9/(3,2) + 0{\cdot}9/(3,3)$$

The rules stated above may be employed in combination, yielding a variety of corollary rules which are of use in the translation of more complex forms of composite propositions and descriptors. Among the basic rules of this type are the following.

(d) If M is F then N is G else N is H (4.50)

$$\rightarrow \Pi_{(X_1,\ldots,X_m,Y_1,\ldots,Y_n)} = (\bar{F}' \oplus \bar{G}) \cap (\bar{F} \oplus \bar{H})$$

where $F \subset U \triangleq U_1 \times \ldots \times U_m$ and $G, H \subset V \triangleq V_1 \times \ldots \times V_n$. This rule follows from the semantic equivalence:

$$\text{If M is F then N is G else N is H} \tag{4.51}$$
$$\leftrightarrow \text{(If M is F then N is G) and (If M is not F then N is H)}$$

and the application of (a) and (c_1).

(e) *Translation rule for relations*

Consider a relation, R, whose tableau is of the form shown in Table 4.2.

TABLE 4.2

R	X_1	X_2	...	X_n
	F_{11}	F_{12}	...	F_{1n}
	...	...	...	...
	F_{m1}	...	...	F_{mn}

in which the F_{ij} are fuzzy subsets of the U_j, respectively. On interpreting R as

$$\begin{aligned} R = {} & X_1 \text{ is } F_{11} \text{ and } X_2 \text{ is } F_{12} \text{ and} \ldots \text{and } X_n \text{ is } F_{1n} \text{ or} \\ & X_1 \text{ is } F_{21} \text{ and } X_2 \text{ is } F_{22} \text{ and} \ldots \text{and } X_n \text{ is } F_{2n} \text{ or} \ldots \text{or} \\ & X_1 \text{ is } F_{m1} \text{ and } X_2 \text{ is } F_{m2} \text{ and} \ldots \text{and } X_n \text{ is } F_{mn} \end{aligned} \tag{4.52}$$

it follows from (a) and (b) that

$$R \rightarrow F_{11} \times \ldots \times F_{1n} + \ldots + F_{m1} \times \ldots \times F_{mn} \tag{4.53}$$

which will be referred to as the *tableau rule*. This rule plays an important role in applications to pattern recognition, decision analysis, medical diagnosis and related areas, in which binary relations are employed to describe the features of a class of objects (Zadeh, 1976*a,b*).

As a simple illustration, consider the relation defined by Table 4.3

TABLE 4.3

X	Y
small	large
very small	not very large
not small	very small

in which X and Y are real-valued variables and

$$\begin{aligned} \text{small} & \rightarrow \text{SMALL} \\ \text{large} & \rightarrow \text{LARGE} \end{aligned}$$

where SMALL and LARGE are specified fuzzy subsets of the real line.

First, by the application of (4.20) and (4.21), we have

$$\text{very small} \rightarrow \text{SMALL}^2 \tag{4.54}$$

$$\text{not small} \rightarrow \text{SMALL}' \tag{4.55}$$

$$\text{not very large} \rightarrow (\text{LARGE}^2)'. \tag{4.56}$$

Then, on applying (4.53), we obtain

$$R \rightarrow \text{SMALL} \times \text{LARGE} + (\text{SMALL}^2) \times (\text{LARGE}^2)' + \text{SMALL}' \times \text{SMALL}^2$$

which is the desired translation of the relation in question.

LINGUISTIC VARIABLES

The modifier rule in combination with the translation rules for conjunctive and disjunctive compositions provides a simple method for the translation of linguistic values of so-called *linguistic variables* (Zadeh, 1973, 1975*c*).

Informally, a *linguistic variable* is a variable whose *linguistic values* are words or sentences in a natural or synthetic language, with each such value being a label of a fuzzy subset of a universe of discourse. For example, a variable such as Age may be viewed both as a numerical variable ranging over, say, the interval [0,150], and as a linguistic variable which can take the values *young*, *not young*, *very young*, *not very young*, *quite young*, *old*, *not very young and not very old*, etc. Each of these values may be interpreted as a label of a fuzzy subset of the universe of discourse U = [0,150], whose base variable, *u*, is the generic numerical value of Age.

Typically, the values of a linguistic variable such as Age are built up of one or more *primary terms* (which are the labels of *primary fuzzy sets*†), together with a collection of modifiers which allow a composite linguistic value to be generated from the primary terms through the use of conjunctions and disjunctions. Usually the number of primary terms is two, with one being an antonym of the other. For example, in the case of Age, the primary terms are *young* and *old*, with *old* being the antonym of *young*.

Using the translation rules (4.20), (4.21), (4.36) and (4.37) in combination, the linguistic values of a linguistic variable such as Age may be translated by inspection. To illustrate, suppose that the primary terms *young* and *old* are defined by

$$\mu_{YOUNG} = 1 - S(20,30,40) \tag{4.57}$$

and

$$\mu_{OLD} = S(40,55,70). \tag{4.58}$$

Then

$$\text{not very young} \to (YOUNG^2)' \tag{4.59}$$

and

$$\text{not very young and not very old} \to (YOUNG^2)' \cap (OLD^2)' \tag{4.60}$$

and thus

$$\text{John is not very young} \to \Pi_{Age(John)} = (YOUNG^2)' \tag{4.61}$$

where

$$\mu_{(YOUNG^2)'} = 1 - (1 - S(20,30,40))^2. \tag{4.62}$$

The problem of finding a linguistic value of Age whose meaning approximates to a given fuzzy subset of U is an instance of the problem of *linguistic approximation* (Zadeh, 1975*c*; Wenstop, 1975; Procyk, 1976). We shall not discuss in the present paper the ways in which this non-trivial problem can be approached, but will assume that linguistic approximation is implicit in the retranslation of a possibility distribution into a proposition expressed in a natural language.

RULES OF TYPE III

Translation rules of Type III pertain to the translation of propositions of the general form

$$p \triangleq QN \text{ are } F \tag{4.63}$$

where N is the descriptor of a possibly fuzzy set, Q is a fuzzy quantifier (e.g. *most*, *many*, *few*, *some*, *almost all*, etc.) and F is a fuzzy subset of U. Simple examples of (4.63) are:

$$\text{Most Swedes are tall} \tag{4.64}$$

†Such sets play a role which is somewhat analogous to that of physical units.

$$\text{Many tall men are fat} \tag{4.65}$$

$$\text{Some men are much taller than most men.} \tag{4.66}$$

In general, a fuzzy quantifier is a fuzzy subset of the set of integers, the unit interval or the real line. For example, we may have

$$\text{SEVERAL} \triangleq 0{\cdot}2/3 + 0{\cdot}6/4 + 1/5 + 1/6 + 0{\cdot}6/7 + 0{\cdot}2/8 \tag{4.67}$$

$$\text{MOST} \triangleq \int_0^1 S(u; 0{\cdot}5, 0{\cdot}7, 0{\cdot}9)/u \tag{4.68}$$

(which means that MOST is a fuzzy subset of the unit interval whose membership function is given by S(0·5,0·7,0·9)) and

$$\text{LARGE NUMBER} \triangleq \int_0^\infty (1 + (\tfrac{u}{100})^{-2})^{-1}/u. \tag{4.69}$$

In order to be able to translate propositions of the form (4.63), it is necessary to define the *cardinality* of a fuzzy set, i.e. the number (or the proportion) of elements of U which are in F. Strictly speaking, the cardinality of a fuzzy set should be a fuzzy number, which could be defined as in Zadeh (1977*b*). It is simpler, however, to deal with the *power* of a fuzzy set (DeLuca & Termini, 1972), which in the case of a fuzzy set with a finite support† is defined by‡

$$|F| \triangleq \sum_i \mu_F(u_i), \quad u_i \in \text{Support of F} \tag{4.70}$$

where $\mu_F(u_i)$, $i = 1, \ldots, N$, is the grade of membership of u_i in F and $\sum$ denotes the arithmetic sum. For example, for the fuzzy set SMALL defined by

$$\text{SMALL} \triangleq 1/0 + 1/1 + 0{\cdot}8/2 + 0{\cdot}6/3 + 0{\cdot}4/4 + 0{\cdot}2/5 \tag{4.71}$$

we have

$$|F| = 1 + 1 + 0{\cdot}8 + 0{\cdot}6 + 0{\cdot}4 + 0{\cdot}2 = 4.$$

In the sequel, we shall usually employ the more explicit notation Count(F) to represent the power of F, with the understanding that F should be treated as a bag§ rather than a set. Furthermore, the notation Prop(F/G) will be used to represent the "proportion" of F in G, i.e.

$$\text{Prop}\{F/G\} \triangleq \frac{\text{Count}(F \cap G)}{\text{Count}(G)} \tag{4.72}$$

†The *support* of a fuzzy subset F of U is the set of all points in U at which $\mu_F(u) > 0$.

‡For some applications, it is necessary to eliminate from the count those elements of F whose grade of membership falls below a specified threshold. This is equivalent to replacing F in (4.70) with $F \cap \Gamma$, where Γ is a fuzzy or non-fuzzy set which induces the desired threshold.

§The elements of a bag need not be distinct. For example, the collection of integers {2,3,5,3,5} is a bag if {2,3,5,3,5} $\neq$ {2,3,5}.

and more explicitly

$$\mathrm{Prop}\{F/G\} = \frac{\sum_i (\mu_F(u_i) \wedge \mu_G(u_i))}{\sum_j \mu_G(u_j)} \tag{4.73}$$

where the summation ranges over the values of i for which $u_i \in$ Support of F $\cap$ Support of G. In particular, if G $\triangleq$ U $\triangleq$ finite non-fuzzy set, then (4.73) becomes

$$\mathrm{Prop}\{F/U\} = \frac{1}{N} \sum_{i=1}^{N} \mu_F(u_i) \tag{4.74}$$

where N is the cardinality of U. For convenience, the number Prop{F/U} will be referred to as the *relative cardinality* of F expressed as

$$\mathrm{Prop}(F) \triangleq \mathrm{Prop}\{F/U\} = \frac{1}{N} \sum_{i=1}^{N} \mu_F(u_i). \tag{4.75}$$

As N increases and U becomes a continuum, the expression for the power of F tends in the limit to that of the *additive measure* of F (Zadeh, 1968; Sugeno, 1974), which may be regarded as a continuous analog of the proportion of the elements of U which are "in" F. More specifically, if $\rho(u)$ is a density function defined on U, the measure in question is defined by†

$$\mathrm{Prop}(F) = \int_U \rho(u)\mu_F(u)\mathrm{d}u. \tag{4.76}$$

For example, if $\rho(u)\mathrm{d}u$ is the proportion of Swedes whose height lies in the interval $[u, u+\mathrm{d}u]$, then the proportion of tall Swedes is given by

$$\mathrm{Prop}(\text{tall Swedes}) = \int_0^{200} \rho(u)\mu_{\mathrm{TALL}}(u)\mathrm{d}u \tag{4.77}$$

where μ_{TALL} is the membership function of *tall* and height is assumed to be measured in centimeters.

In a similar fashion, the expression for Prop{F/G} tends in the limit to that of the *relative measure* of F in G, which is defined by

$$\mathrm{Prop}(F/G) \triangleq \frac{\int_{U \times V} \rho(u,v)(\mu_F(u) \wedge \mu_G(v))\mathrm{d}u\mathrm{d}v}{\int_V \rho(v)\mu_G(v)\mathrm{d}v} \tag{4.78}$$

where $\rho(u,v)$ is a density function defined on U $\times$ V and

$$\rho(v) = \int_U \rho(u,v)\mathrm{d}u. \tag{4.79}$$

†We employ the notation Prop(F) even in the continuous case to make clearer the intuitive meaning of measure.

For example, if F$\triangleq$TALL MEN and G$\triangleq$FAT MEN, (4.78) becomes

$$\text{Prop}\{\text{TALL MEN/FAT MEN}\} = \frac{\int_{[0,200]\times[0,100]} \rho(u,v)\mu_{\text{TALL}}(u)\wedge\mu_{\text{FAT}}(v)\mathrm{d}u\mathrm{d}v}{\int_{[0,100]} \rho(v)\mu_{\text{FAT}}(v)\mathrm{d}v} \tag{4.80}$$

where $\rho(u,v)\mathrm{d}u\mathrm{d}v$ is the proportion of men whose height lies in the interval $[u,u+\mathrm{d}u]$ and whose weight lies in the interval $[v,v+\mathrm{d}v]$.

The above definitions provide the basis for the *quantifier rule* for the translation of propositions of the form "QN are F". More specifically, assuming for simplicity that N is a descriptor of a non-fuzzy set, the rule in question may be stated as follows.

If $\text{U}=\{u_1,\ldots,u_N\}$ and

$$N \text{ is } \text{F} \rightarrow \Pi_{\text{X}} = \text{F} \tag{4.81}$$

then

$$\text{Q}N \text{ are } \text{F} \rightarrow \Pi_{\text{Count(F)}} = \text{Q} \tag{4.82}$$

and, if U is a continuum,

$$\text{Q}N \text{ are } \text{F} \rightarrow \Pi_{\text{Prop(F)}} = \text{Q} \tag{4.83}$$

which implies the more explicit rule

$$\text{Q}N \text{ are } \text{F} \rightarrow \pi(\rho) = \mu_{\text{Q}}\left(\int_{\text{U}} \rho(u)\mu_{\text{F}}(u)\mathrm{d}u\right) \tag{4.84}$$

where $\rho(u)\mathrm{d}u$ is the proportion of X's whose value lies in the interval $[u,u+\mathrm{d}u]$, $\pi(\rho)$ is the possibility of ρ, and μ_{Q} and μ_{F} are the membership functions of Q and F, respectively.

As a simple illustration, if MOST and TALL are defined by (4.68) and $\mu_{\text{TALL}}=$ S(160,170,180), respectively, then

$$\text{Most men are tall} \rightarrow \pi(\rho) = \text{S}\left(\int_0^{200} \rho(u)\text{S}(u;160,1970,180)\mathrm{d}u; 0{\cdot}5, 0{\cdot}7, 0{\cdot}9\right) \tag{4.85}$$

where $\rho(u)\mathrm{d}u$ is the proportion of men whose height (in cm) is in the interval $[u,u+\mathrm{d}u]$. Thus, the proposition "Most men are tall" induces a possibility distribution of the height density function ρ which is expressed by the right-hand member of (4.85).

MODIFIER RULE FOR PROPOSITIONS

The modifier rule which was stated earlier in this section (4.16) provides a basis for the formulation of a more general modifier rule which applies to propositions and which leads to a rule for transforming the negation of a proposition into a semantically equivalent form in which the negation has a smaller scope.

The *modifier rule for propositions* may be stated as follows.

If a proposition p translates into a procedure P, i.e.

$$p \rightarrow \text{P} \tag{4.86}$$

and P returns a possibility distribution Π^p in application to a database $\mathscr{D}$, then mp, where m is a modifier, is semantically equivalent to a retranslation of mP, i.e.

$$mp \leftrightarrow q \tag{4.87}$$

where

$$q \leftarrow m\mathrm{P}. \tag{4.88}$$

In (4.88), mP is understood to be a procedure which returns (in application to $\mathscr{D}$):

$$(\Pi^p)' \text{ if } m \triangleq \text{not} \tag{4.89}$$

$$(\Pi^p)^2 \text{ if } m \triangleq \text{very} \tag{4.90}$$

and

$$(\Pi^p)^{0\cdot 5} \text{ if } m \triangleq \text{more or less.} \tag{4.91}$$

For simplicity, the possibility distribution defined by (4.89), (4.90) and (4.91) will be denoted as $m\Pi^p$.

On applying this rule to a proposition of the form $p \triangleq N$ is F and making use of the translation rules (4.20), (4.21), (4.23), (4.36), (4.37) and (4.87), we obtain the following general forms of (strong) semantic equivalence:

(a)
$$m(\text{N is F}) \leftrightarrow \text{N is } m\text{F} \tag{4.92}$$

and, in particular,

$$\text{not(N is F)} \leftrightarrow \text{N is not F} \tag{4.93}$$

$$\text{very(N is F)} \leftrightarrow \text{N is very F} \tag{4.94}$$

$$\text{more or less(N is F)} \leftrightarrow \text{N is more or less F.} \tag{4.95}$$

(b)
$$m(\text{M is F and N is G}) \leftrightarrow (\text{X,Y}) \text{ is } m(\text{F} \times \text{G}) \tag{4.96}$$

and, in particular (in virtue of (4.20), (4.36) and 4.37)),

$$\text{not(M is F and N is G)} \leftrightarrow (\text{X,Y}) \text{ is } (\text{F} \times \text{G})' \tag{4.97}$$
$$\leftrightarrow (\text{X,Y}) \text{ is } \overline{\text{F}}' + \overline{\text{G}}' \tag{4.98}$$
$$\leftrightarrow \text{M is not F or N is not G} \tag{4.99}$$
$$\text{very(M is F and N is G)} \leftrightarrow \text{M is very F and N is very G} \tag{4.100}$$
$$\text{more or less(M is F and N is G)} \leftrightarrow \text{M is more or less F and N is more or less G} \tag{4.101}$$

and dually for disjunctive composition.

(c)
$$m(\text{QN are F}) \leftrightarrow (m\text{Q})\text{N are F} \tag{4.102}$$

and, in particular,

$$\text{not(QN are F)} \leftrightarrow \text{(not Q)N are F} \tag{4.103}$$

which may be regarded as a generalization of the standard negation rules in predicate calculus, viz.

$$\neg (\forall x)F(x) \leftrightarrow (\exists x)\neg F(x), \tag{4.104}$$

$$\neg (\exists x)F(x) \leftrightarrow (\forall x)\neg F(x). \tag{4.105}$$

To see the connection between (4.104), say, and (4.102), we first note that, in consequence of (4.84), we can assert the semantic equivalence

$$\text{QN are F} \leftrightarrow \text{ant Q are not F} \tag{4.106}$$

where ant Q, the antonym of Q, is defined by

$$\mu_{\text{ant Q}}(v) = \mu_Q(1-v), \quad v \in [0,1]. \tag{4.107}$$

Thus, on combining (4.103) and (4.106), we have

$$\text{not(QN are F)} \leftrightarrow \text{(ant(not Q))N are not F} \tag{4.108}$$

which for $Q \triangleq \text{all}$ gives

$$\text{not(all N are F)} \leftrightarrow \text{(ant(not all))N are not F.} \tag{4.109}$$

Then, the right-hand member of (4.109) may be expressed as

$$\text{not(all N are F)} \leftrightarrow \text{some N are not F} \tag{4.110}$$

if we assume that

$$\text{some} \triangleq \text{ant(not all).} \tag{4.111}$$

In a similar fashion, the modifier rule for propositions may be employed to derive the negation rules for qualified propositions of the form $q \triangleq p$ is γ, where γ is a truth-value, a probability-value, or a possibility-value. Rules of this type will be formulated in section 5.

5. Consistency, compatibility and truth

Our aim in this section is to lay the groundwork for the translation of truth-qualified propositions of the form "p is τ," where τ is a linguistic truth-value. To this end, we shall have to introduce two related concepts—consistency and compatibility—in terms of which the *relative* truth of a proposition p with respect to a reference proposition r may be defined.

The concept of truth has traditionally been accorded a central place in logic and philosophy of language. In recent years, it has also come to play a primary role in the theory of meaning—especially in Montague grammar and possible world semantics.

By contrast, it is the concept of a possibility distribution rather than truth that serves as a basis for the definition of meaning as well as other primary concepts in fuzzy logic and PRUF. Thus, as we shall see in the sequel, the concept of truth in PRUF serves in the main as a mechanism for assessing the consistency or compatibility of a pair of propositions rather than—as in classical logic—as an indicator of the correspondence between a proposition and "reality".

CONSISTENCY AND COMPATIBILITY

Let p and q be two propositions of the form $p \triangleq N$ is F and $q \triangleq N$ is G, which translate, respectively, into

$$p \triangleq N \text{ is F} \rightarrow \Pi^{p}_{(X_1, \ldots, X_n)} = \text{F} \tag{5.1}$$

and

$$q \triangleq N \text{ is G} \rightarrow \Pi^{q}_{(X_1, \ldots, X_n)} = \text{G} \tag{5.2}$$

where $(X_1, \ldots, X_n)$ takes values in U. Intuitive considerations suggest that the *consistency* of p with q (or vice versa) be defined as the possibility that "N is F" given that "N is G" (or vice versa). Thus, making use of (2.9), we have

$$\begin{aligned}\text{Cons}\{N \text{ is F}, N \text{ is G}\} &\triangleq \text{Poss}\{N \text{ is F} \mid N \text{ is G}\}\\ &= \operatorname*{Sup}_{u \in \text{U}} (\mu_{\text{F}}(u) \wedge \mu_{\text{G}}(u))\end{aligned} \tag{5.3}$$

where $u \triangleq (u_1, \ldots, u_n)$ denotes the generic value of $(X_1, \ldots, X_n)$, and μ_{F} and μ_{G} are the membership functions of F and G, respectively.

As a simple illustration, assume that

$$p \triangleq N \text{ is a small integer} \tag{5.4}$$

$$q \triangleq N \text{ is not a small integer} \tag{5.5}$$

where

$$\text{SMALL INTEGER} \triangleq 1/0 + 1/1 + 0{\cdot}8/2 + 0{\cdot}6/3 + 0{\cdot}4/4 + 0{\cdot}2/5. \tag{5.6}$$

In this case, $\text{Cons}\{p,q\} = 0{\cdot}4$.

As a less simple example, consider the propositions

$$p \triangleq \text{Most men are tall} \tag{5.7}$$

and

$$q \triangleq \text{Most men are short} \tag{5.8}$$

which translate into (see (4.84))

$$\pi^{p}(\rho) = \mu_{\text{MOST}}\left(\int_0^{200} \rho(u)\mu_{\text{TALL}}(u)du\right) \tag{5.9}$$

and

$$\pi^{q}(\rho) = \mu_{\text{MOST}}\left(\int_0^{200} \rho(u)\mu_{\text{SHORT}}(u)du\right). \tag{5.10}$$

In this case, assuming that μ_{MOST} is a monotone function, we have

$$\text{Cons}\{p,q\} = \mu_{\text{MOST}}\left(\text{Sup}_{\rho}\left(\left(\int_0^{200} \rho(u)\mu_{\text{TALL}}(u)du\right) \wedge \left(\int_0^{200} \rho(u)\mu_{\text{SHORT}}(u)du\right)\right)\right). \tag{5.11}$$

If q is assumed to be a *reference* proposition, which we shall denote by r, then the *truth of p relative to r* could be defined as the consistency of p with r. It appears to be more appropriate, however, to define the truth of p relative to r through the concept of *compatibility* rather than consistency. More specifically, assume that the reference proposition r is of the form

$$r \triangleq N \text{ is } u \tag{5.12}$$

where u is an element of U. Then, by definition,

$$\text{Comp}\{N \text{ is } u/N \text{ is F}\} \triangleq \mu_F(u) \tag{5.13}$$

which coincides with the definition of Poss{X is $u|N$ is F} (see (2.4)) as well as with the definition of the consistency of "N is u" with "N is F". However, when the reference proposition is of the form $r \triangleq N$ is G, the definitions of compatibility and consistency cease to coincide. More specifically, by employing the extension principle,† (5.13) becomes

$$\begin{aligned} \text{Comp}\{N \text{ is G}/N \text{ is F}\} &= \mu_F(G) \\ &= \int_{[0,1]} \mu_G(u)/\mu_F(u) \end{aligned} \tag{5.14}$$

in which the right-hand member is the union over the unit interval of the fuzzy singletons $\mu_G(u)/\mu_F(u)$. Thus, (5.14) signifies that the compatibility of "N is G" with "N is F" is a fuzzy subset of [0,1] defined by (5.14).

The concept of compatibility as defined by (5.14) provides the basis for the following definition of Truth.

Truth. Let p be a proposition of the form "N is F," and let r be a reference proposition, $r \triangleq N$ is G, where F and G are subsets of U. Then, the *truth*, τ, of *p relative to r* is defined as the compatibility of r with p, i.e.

$$\begin{aligned} \tau \triangleq \text{Tr}\{N \text{ is F}/N \text{ is G}\} &\triangleq \text{Comp}\{N \text{ is G}/N \text{ is F}\} \\ &\triangleq \mu_F(G) \\ &\triangleq \int_{[0,1]} \mu_G(u)/\mu_F(u). \end{aligned} \tag{5.15}$$

It should be noted that τ, as defined by (5.15), is a fuzzy subset of the unit interval, implying that a linguistic truth-value may be regarded as a linguistic approximation to the subset defined by (5.15).

A more explicit expression for τ which follows at once from (5.15) is:

$$\mu_\tau(v) = \text{Max}_u \mu_G(u), \quad v \in [0,1] \tag{5.16}$$

subject to

$$\mu_F(u) = v.$$

†The extension principle (Zadeh, 1975*c*) serves to extend the definition of a mapping f: U→V to the set of fuzzy subsets of U. Thus, $f(F) \triangleq \int_U \mu_F(u)/f(u)$, where $f(F)$ and $f(u)$ are, respectively, the images of F and u in V.

Thus, if μ_F is 1–1, then the membership function of τ may be expressed in terms of those of F and G as

$$\mu_\tau(v)=\mu_G(\mu_F^{-1}(v)). \tag{5.17}$$

Another immediate consequence of (5.15) is that the truth-value of p relative to itself is given by

$$\mu_\tau(v)=v$$

rather than unity. Thus, in virtue of (2.15), we have

$$\begin{aligned}\mathrm{Tr}\{N \text{ is F}/N \text{ is F}\} &= \perp \\ &= u\text{-true}.\end{aligned} \tag{5.18}$$

As an illustration of (5.15), assume that

$$p \triangleq N \text{ is not small} \tag{5.19}$$

and

$$r \triangleq N \text{ is small} \tag{5.20}$$

where SMALL is defined by (5.6). Then, (5.15) yields

$$\tau=1/0+0{\cdot}8/0{\cdot}2+0{\cdot}6/0{\cdot}4+0{\cdot}4/0{\cdot}6+0{\cdot}2/0{\cdot}8 \tag{5.21}$$

which may be regarded as a discretized version of the antonym of u-true (see (4.107)). Thus,

$$\mathrm{Tr}\{N \text{ is not small}/N \text{ is small}\}=\text{ant } u\text{-true} \tag{5.22}$$

which, as will be seen later, is a special case of the strong semantic equivalence

$$\mathrm{Tr}\{N \text{ is F}/N \text{ is not F}\}=\text{ant } u\text{-true}. \tag{5.23}$$

As can be seen from the foregoing discussion, in our definition of the truth-value of a proposition p, τ serves as a measure of the compatibility of p with a reference proposition r. To use this definition as a basis for the translation of truth-qualified propositions, we adopt the following postulate.

Postulate. A truth-qualified proposition of the form "p is τ" is semantically equivalent to the reference proposition, r, relative to which

$$\mathrm{Tr}\{p/r\}=\tau. \tag{5.24}$$

We shall use this postulate in the following section to establish translation rules for truth-qualified propositions.

6. Translation rules of Type IV

Our concern in this section is with the translation of qualified propositions of the form $q \triangleq p$ is γ, where γ might be a truth-value, a probability-value, a possibility-value or, more generally, the value of some specified propositional function, i.e. a function from the space of propositions (or n-tuples of propositions) to the set of fuzzy subsets of the unit interval.

Typically, a translation rule of Type IV may be viewed as an answer to the following question: Suppose that a proposition p induces a possibility distribution Π^p. What, then, is the possibility distribution induced by the qualified proposition $q \triangleq p$ is γ, where γ is a specified truth-value, probability-value or possibility-value?

In what follows, we shall state the translation rules pertaining to (a) truth qualification; (b) probability qualification; and (c) possibility qualification. These are the principal modes of qualification which are of more or less universal use in natural languages.

RULE FOR TRUTH QUALIFICATION

Let p be a proposition of the form

$$p \triangleq N \text{ is F} \tag{6.1}$$

and let q be a truth-qualified version of p expressed as

$$q \triangleq N \text{ is F is } \tau \tag{6.2}$$

where τ is a linguistic truth-value. As was stated in section 5, q is semantically equivalent to the reference proposition r, i.e.

$$N \text{ is F is } \tau \leftrightarrow N \text{ is G} \tag{6.3}$$

where F, G and τ are related by

$$\tau = \mu_F(G). \tag{6.4}$$

Equation (6.4) states that τ is the image of G under the mapping μ_F: $U \to [0,1]$. Consequently (Zadeh, 1965), the expression for the membership function of G in terms of those of τ and F is given by

$$\mu_G(u) = \mu_\tau(\mu_F(u)). \tag{6.5}$$

Using this result, the rule for truth qualification may be stated as follows.

If

$$N \text{ is F} \to \Pi_X = F \tag{6.6}$$

then

$$N \text{ is F is } \tau \to \Pi_X = F^+ \tag{6.7}$$

where

$$\mu_{F^+}(u) = \mu_\tau(\mu_F(u)). \tag{6.8}$$

In particular, if τ is the unitary truth-value, that is,

$$\tau = u\text{-true} \tag{6.9}$$

where

$$\mu_{u\text{-true}}(v) = v, \quad v \in [0,1] \tag{6.10}$$

then

$$N \text{ is F is } u\text{-true} \to N \text{ is F}. \tag{6.11}$$

As an illustration of (6.5), if

$$q \triangleq N \text{ is small is very true} \tag{6.12}$$

where

$$\mu_{\text{SMALL}} = 1 - S(5,10,15), \quad u \in [0,\infty) \tag{6.13}$$

and

$$\mu_{\text{TRUE}} = S(0{\cdot}6, 0{\cdot}8, 1{\cdot}0) \tag{6.14}$$

then

$$q \to \pi_X(u) = S^2(1 - S(u;5,10,15); 0{\cdot}6, 0{\cdot}8, 1{\cdot}0). \tag{6.15}$$

RULE FOR PROBABILITY QUALIFICATION

Let p be a proposition of the form (6.1) and let q be a probability-qualified version of p expressed as

$$q \triangleq N \text{ is F is } \lambda \tag{6.16}$$

where λ is a linguistic probability-value such as probable, very probable, not very probable, or, equivalently, likely, very likely, not very likely, etc.

We shall assume that q is semantically equivalent to the proposition

$$\text{Prob}\{N \text{ is F}\} \text{ is } \lambda \tag{6.17}$$

in which $p \triangleq N$ is F is interpreted as a fuzzy event (Zadeh, 1968). More specifically, let $p(u)\mathrm{d}u$ be the probability that $X \in [u, u+\mathrm{d}u]$, where $X \triangleq X(N)$. Then

$$\text{Prob}\{N \text{ is F}\} = \int_U p(u)\mu_F(u)\mathrm{d}u \tag{6.18}$$

and hence (6.17) implies that

$$\Pi_{\int_U p(u)\mu_F(u)\mathrm{d}u} = \lambda. \tag{6.19}$$

Equation (6.19) provides the basis for the following statement of the rule for probability qualification.

If

$$N \text{ is F} \to \Pi_X = F \tag{6.20}$$

then

$$N \text{ is F is } \lambda \to \Pi_{\int_U p(u)\mu_F(u)\mathrm{d}u} = \lambda \tag{6.21}$$

or more explicitly

$$\pi(p(\cdot)) = \mu_\lambda\left(\int_U p(u)\mu_F(u)\mathrm{d}u\right) \tag{6.22}$$

where $\pi(p(\cdot))$ is the possibility of the probability density function $p(\cdot)$.

As an illustration of (6.22), assume that

$$q \triangleq N \text{ is small is likely} \tag{6.23}$$

where LIKELY is defined by

$$\mu_{LIKELY}=(0{\cdot}7,0{\cdot}8,0{\cdot}9) \tag{6.24}$$

and SMALL is given by (6.13). Then

$$N \text{ is small is likely} \rightarrow \pi(p(\cdot)) = \int_0^\infty p(u)(1-S(u;5,10,15))du. \tag{6.25}$$

Note that in this case the proposition in question induces a possibility distribution of the probability density of $X \triangleq N$.

RULE FOR POSSIBILITY QUALIFICATION

Our concern here is with the translation of possibility-qualified propositions of the form

$$q \triangleq N \text{ is F is } \omega \tag{6.26}$$

where ω is a linguistic possibility-value such as *quite possible*, *very possible*, *almost impossible*, etc., with each such value representing a fuzzy subset of the unit interval.

By analogy with our interpretation of probability-qualified propositions, q may be interpreted as

$$N \text{ is F is } \omega \leftrightarrow \text{Poss}\{\text{X is F}\} \text{ is } \omega \tag{6.27}$$

which implies that

$$\Pi_{\text{Poss}\{\text{X is F}\}} = \omega. \tag{6.28}$$

Now suppose that we wish to find a fuzzy set G such that

$$N \text{ is F is } \omega \leftrightarrow N \text{ is G}. \tag{6.29}$$

Then, from the definition of possibility measure (2.9), we have

$$\text{Poss}\{N \text{ is F} \mid N \text{ is G}\} = \operatorname{Sup}_u(\mu_F(u) \wedge \mu_G(u)) \tag{6.30}$$

and hence

$$N \text{ is F is } \omega \rightarrow \pi(\mu_G(\cdot)) = \mu_\omega\left(\operatorname{Sup}_u(\mu_F(u) \wedge \mu_G(u))\right) \tag{6.31}$$

where μ_ω is the membership function of ω. Note that (6.31) is analogous to the translation rule for probability-qualified propositions (6.22).†

Although the interpretation expressed by (6.31) is consistent with (6.22), it is of interest to consider alternative interpretations which are not in the spirit of (6.28). One such interpretation which may be employed as a basis for possibility qualification is the following.

Assume that $\omega \triangleq$ 1-possible (i.e. $\mu_\omega(v)=1$ for $v=1$ and $\mu_\omega(v)=0$ for $v \in [0,1)$), and let

$$p \triangleq N \text{ is F} \rightarrow \Pi_X = \text{F}. \tag{6.32}$$

†A more detailed discussion of this issue may be found in Zadeh (1977*a*).

Then

$$q \triangleq N \text{ is F is 1-possible} \rightarrow \Pi_X = G \tag{6.33}$$

where G is a fuzzy set of Type 2† which has an interval-valued membership function defined by

$$\mu_G(u) = [\mu_F(u), 1], \quad u \in U \tag{6.34}$$

with the understanding that (6.34) implies that Poss$\{X = u\}$ may be any number in the interval $[\mu_F(u), 1]$.

More generally, if $\omega \triangleq \alpha$-possible (i.e. $\mu_\omega(v) = \alpha$ for $v = 1$ and $\mu_\omega(v) = 0$ for $v \in [0,1)$), then

$$\text{N is F is } \alpha\text{-possible} \rightarrow \Pi_X = G \tag{6.35}$$

where G is a fuzzy set of Type 2 defined by

$$\mu_G(u) = [\alpha \wedge \mu_G(u), \alpha \oplus (1 - \mu_F(u))], \quad u \in U \tag{6.36}$$

and $\oplus$ denotes the bounded sum (see (4.43)). The rules expressed by (6.33) and (6.35) should be regarded as provisional in nature, since further experience in the use of possibility distributions may suggest other, more appropriate, interpretations of the concept of possibility qualification.

MODIFIER RULES FOR QUALIFIED PROPOSITIONS

As in the case of translation rules of Types I, II and III, the modifier rule for propositions may be applied to translation rules of Type IV to yield, among others, the negation rule for qualified propositions. In what follows, we shall restrict our attention to the application of this rule to truth-qualified propositions.

Specifically, on applying the modifier rule for propositions to (6.7), we obtain the following general form of strong semantic equivalence

$$m(N \text{ is F is } \tau) \leftrightarrow N \text{ is F is } m\tau \tag{6.37}$$

which implies that

$$\text{not}(N \text{ is F is } \tau) \leftrightarrow N \text{ is F is not } \tau \tag{6.38}$$

$$\text{very}(N \text{ is F is } \tau) \leftrightarrow N \text{ is F is very } \tau \tag{6.39}$$

and

$$\text{more or less}(N \text{ is F is } \tau) \leftrightarrow N \text{ is F is more or less } \tau. \tag{6.40}$$

On the other hand, from (6.7) it also follows that

$$N \text{ is not F is } \tau \leftrightarrow N \text{ is F is ant } \tau \tag{6.41}$$

where ant τ is the antonym of τ. Thus, for example,

$$\text{false} \triangleq \text{ant true} \tag{6.42}$$

†A fuzzy set F is of Type 2 if, for each $u \in U$, $\mu_F(u)$ is a fuzzy subset of Type 1, i.e. $\mu_{\mu_F(u)} : [0,1] \rightarrow [0,1]$.

i.e.

$$\mu_{\mathrm{FALSE}}(v)=\mu_{\mathrm{TRUE}}(1-v), \quad v\in[0,1] \tag{6.43}$$

where FALSE and TRUE are the fuzzy denotations of false and true, respectively. Similarly, from (6.7) it follows that

$$N \text{ is very F is } \tau \leftrightarrow N \text{ is F is } {}^{0\cdot 5}\tau \tag{6.44}$$

where the "left square-root" of τ is defined by

$$\mu_{0\cdot 5_\tau}(v) \triangleq \mu_\tau(v^2), \quad v\in[0,1] \tag{6.45}$$

and, more generally, for a "left-exponent" α,

$$\mu_{\alpha_\tau}(v) \triangleq u_\tau(v^{1/\alpha}), \quad v\in[0,1]. \tag{6.46}$$

On applying these rules in combination to a proposition such as "Barbara is not very rich," we are led to the following chain of semantically equivalent propositions:

Barbara is not very rich (6.47)
Barbara is not very rich is *u*-true (6.48)
Barbara is very rich is ant *u*-true (6.49)
Barbara is rich is ${}^{0\cdot 5}$(ant *u*-true) (6.50)

where

$$\mu_{0\cdot 5_{(\mathrm{ant}\ u\text{-}\mathrm{true})}}(v)=1-v^2. \tag{6.51}$$

If *true* is assumed to be approximately semantically equivalent to *u-true*, the last proposition in the chain may be approximated by

Barbara is rich is not very true. (6.52)

Thus, if we know that "Barbara is not very rich," then by using the chain of reasoning represented by (6.48), (6.49), (6.50) and (6.52), we can assert that an approximate answer to the question "Is Barbara rich?τ" is "not very true".

This example provides a very simple illustration of a combined use of the concepts of semantic equivalence and truth qualification for the purpose of deduction of an approximate answer to a given question, given a knowledge base consisting of a collection of fuzzy propositions. Additional illustrations relating to the application of PRUF to approximate reasoning may be found in Zadeh (1977*b*).

7. Examples of translation into PRUF

As was stated earlier, the translation rules formulated in the preceding sections are intended to serve as an aid to a human user in the translation of propositions (or descriptors) expressed in a natural language into PRUF. The use of the rules in question is illustrated by the following examples, with the understanding that, in general, in the translation of an expression, *e*, in a natural language into an expression, E, in PRUF, E is a procedure whose form depends on the frame of the database and hence is not unique.

For convenience of the reader, the notation employed in the examples is summarized below.

In a translation $e \to$E, if w is a word in e then its correspondent, W, in E is the name of a relation in $\mathscr{D}$ (the database).

$F \triangleq$ fuzzy relation with membership function μ_F

$\Pi_X \triangleq$ possibility distribution of the variable X

$\pi_X \triangleq$ possibility distribution function of Π_X (or X) ((2.2 *et seq.*)

$F[\Pi_{X_{(s)}} = G] \triangleq$ fuzzy relation F which is particularized by the proposition "$X_{(s)}$ is G," where $X_{(s)}$ is a subvariable of the variable, X, associated with F (2.26)

$_{x_{i_1} \times \cdots \times x_{i_k}} F \triangleq$ Proj F on $U_{i_1} \times \ldots \times U_{i_k}$, $U_{i_k} \triangleq U_{i_k}(X_{i_k})$ (2.23)

$F^2 \triangleq$ square of F (4.21)

$\sqrt{F} \triangleq$ square root of F (4.24)

$+ \triangleq$ union or arithmetic sum

$\vee \triangleq$ max

$\wedge \triangleq$ min

$' \triangleq$ complement (2.25)

$\cap \triangleq$ intersection (footnote on p. 407)

$\times \triangleq$ Cartesian product (footnote on p. 413)

$\oplus \triangleq$ bounded sum (4.43)

$\perp \triangleq$ unitor (2.15)

Count(F) $\triangleq$ Cardinality (power) of F (4.70)

Prop(F) $\triangleq$ Count(F)/Cardinality of universe of discourse (4.75)

Prop(F/G) $\triangleq$ Count(F $\cap$G)/Count(G) (4.73)

$\text{Name}_i \triangleq$ Name of ith object in a population

Support(F) $\triangleq$ set of all points u in U for which $\mu_F(u) > 0$

U(X) $\triangleq$ universe of discourse associated with X

Example (*a*)

$$\textit{Ed is 30 years old} \to \text{Age(Ed)} = 30 \tag{7.1}$$

$$\textit{Ed is young} \to \Pi_{\text{Age(Ed)}} = \text{YOUNG} \tag{7.2}$$

$$\textit{Ed is not very young} \to \Pi_{\text{Age(Ed)}} = (\text{YOUNG}^2)', \tag{7.3}$$

where the frame of YOUNG is

YOUNG	Age	μ

.

Alternatively,

$$\textit{Ed is young} \to \text{ED}[\Pi_{\text{Age}} = \text{YOUNG}]. \tag{7.4}$$

Example (*b*)

$$\textit{Sally is very intelligent} \to \Pi_X = \perp^2, \tag{7.5}$$

where

$$X \triangleq_{\mu} \text{INTELLIGENT[Name = Sally]} \tag{7.6}$$

(that is, X is the degree of intelligence of Sally in the table

INTELLIGENT	Name	μ

).

Note that (7.5) implies that

$$\pi(X) = X^2, \quad X \in [0,1]. \tag{7.7}$$

Example (*c*)

$$\textit{Edith is tall and blonde} \rightarrow \Pi_{(\text{Height(Edith)}, \text{Color(Hair(Edith))})} = \text{TALL} \times \text{BLONDE}. \tag{7.8}$$

Alternatively,

$$\textit{Edith is tall and blonde} \rightarrow \text{EDITH}[\Pi_{\text{Height}} = \text{TALL}; \Pi_{\text{Color(Hair)}} = \text{BLONDE}]. \tag{7.9}$$

Example (*d*)

$$\textit{A man is tall} \rightarrow \Pi_{\text{Height(X)}} = \text{TALL} \tag{7.10}$$

where X is the name of the tallest man in the relation

POPULATION	Name	Height

.

Equivalently,

$$\begin{aligned} \textit{A man is tall} \rightarrow\ & \Pi_{\text{Height(Name}_1)} = \text{TALL} \\ \text{or } & \Pi_{\text{Height(Name}_2)} = \text{TALL} \\ & \cdots\cdots\cdots\cdots \\ \text{or } & \Pi_{\text{Height(Name}_N)} = \text{TALL} \end{aligned} \tag{7.11}$$

Example (*e*)

$$\textit{All men are tall} \rightarrow \Pi_{\text{Height(X)}} = \text{TALL} \tag{7.12}$$

where X is the name of the shortest man in the relation

POPULATION	Name	Height

.

Equivalently,

$$\begin{aligned} \textit{All men are tall} \rightarrow\ & \Pi_{\text{Height(Name}_1)} = \text{TALL} \\ & \cdots\cdots\cdots\cdots \\ & \Pi_{\text{Height(Name}_N)} = \text{TALL} \end{aligned} \tag{7.13}$$

Example (*f*)

$$\textit{Most men are tall.} \tag{7.14}$$

Case 1. The frame of $\mathscr{D}$ is comprised of

POPULATION	Name	μ

MOST	ρ	μ

where μ_i in POPULATION is the degree to which Name_i is TALL, and μ_j in MOST is the degree to which ρ_j is compatible with MOST. Then

$$\textit{Most men are tall} \rightarrow \Pi_{\text{Prop(TALL)}} = \text{MOST} \tag{7.15}$$

where

$$\text{Prop(TALL)} = \frac{\sum_{i} {}_{\mu}\text{POPULATION[Name} = \text{Name}_i]}{\text{Count(POPULATION)}}. \tag{7.16}$$

Case 2. The frame of $\mathscr{D}$ is comprised of

POPULATION	Name	Height

TALL	Height	μ

MOST	ρ	μ

.

In this case, the translation is still expressed by (7.15), but with Prop(TALL) given by

$$\text{Prop(TALL)} = \frac{\sum_{i} {}_{\mu}\text{TALL[Height} = {}_{\text{Height}}\text{POPULATION[Name} = \text{Name}_i]]}{\text{Count(POPULATION)}}. \tag{7.17}$$

Example (*g*)

$$\begin{aligned} \textit{Three tall men} \rightarrow \mu(X) &= \text{Min}_i\ \mu_i \text{ for } \text{Name}_i \in \text{Support}(X) \text{ and} \\ & \qquad \text{Count(Support}(X)) = 3 \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{7.18}$$

where X is a fuzzy subset of

POPULATION	Name	μ

and μ_i is the degree to which Name_i is tall. The left-hand member of (7.18) is a descriptor, while the right-hand member defines the membership function of a fuzzy subset of the fuzzy power set of ${}_{\text{Name}}\text{POPULATION}$ (i.e. the set of all fuzzy subsets of the names of individuals in POPULATION).

More generally,

$$\textit{Several tall men} \rightarrow \mu(X) = \text{Min}_i \mu_i \wedge \mu_{\text{SEVERAL}}(\text{Count(Support}(X)) \tag{7.19}$$

where, as in (7.18), Min_i is taken over all i such that $\text{Name}_i \in \text{Support}(X)$.

Example (*h*)

$$\begin{aligned} &\textit{Expensive red car with big trunk} \rightarrow \\ &\qquad \text{CAR}[\Pi_{\text{Price}} = \text{EXPENSIVE};\ \Pi_{\text{Color}} = \text{RED};\ \Pi_{\text{Size(Trunk)}} = \text{BIG}]. \end{aligned} \tag{7.20}$$

Example (*i*)

$$\textit{John loves Pat} \to \Pi_{X} = \perp \tag{7.21}$$

where

$$X \triangleq_{\mu} \text{LOVES(Name1} = \text{John; Name2} = \text{Pat)}, \tag{7.22}$$

with the right-hand member of (7.21) implying that

$$\pi(X) = X. \tag{7.23}$$

It should be noted that in the special case where LOVES is a non-fuzzy relation, (7.21) reduces to the conventional predicate representation LOVES(John,Pat).

Example (*j*)

$$\textit{John loves someone} \to \Pi_{X} = \perp \tag{7.24}$$

where

$$\begin{aligned} \mu_i &\triangleq_{\mu} \text{LOVES[Name1} = \text{John; Name2} = \text{Name}_i] \\ &\triangleq \text{degree to which John loves Name}_i \end{aligned} \tag{7.25}$$

and

$$X \triangleq \text{Max}_i\ \mu_i. \tag{7.26}$$

Note that when LOVES is a non-fuzzy relation, (7.24) reduces to $(\exists y)$LOVES(John,y).

Example (*k*)

$$\textit{John loves everyone} \to \Pi_{X} = \perp \tag{7.27}$$

where

$$X \triangleq \text{Min}_i\ \mu_i \tag{7.28}$$

and μ_i is expressed by (7.25).

Example (*l*)

$$\textit{Someone loves someone} \to \Pi_{X} = \perp \tag{7.29}$$

where

$$X = \text{Max}_{i,j}\ \mu_{ij} \tag{7.30}$$

and μ_{ij} is expressed by

$$\mu_{ij} \triangleq_{\mu} \text{LOVES[Name1} = \text{Name}_i\text{; Name2} = \text{Name}_j]. \tag{7.31}$$

Example (*m*)

$$\textit{Someone loves everyone} \to \Pi_{X} = \perp \tag{7.32}$$

where

$$X \triangleq \text{Max}_i\ \text{Min}_j\ \mu_{ij} \tag{7.33}$$

and μ_{ij} is given by (7.31).

Example (*n*)

$$\textit{Jill has many friends} \to \Pi_{X} = \text{MANY} \tag{7.34}$$

where

$$X \triangleq \text{Count}({}_{\mu \times \text{Name2}}\text{FRIENDS(Name1} = \text{Jill)}). \tag{7.35}$$

Note that the argument of Count is the fuzzy set of friends of Jill.

Example (*o*)

$$\textit{The man near the door is young} \rightarrow \Pi_{\text{Age}(N)} = \text{YOUNG} \tag{7.36}$$

where

$$N = \text{MAN} \cap_{\mu \times \text{Object1}} \text{NEAR}[\text{Object2} = \text{DOOR}]. \tag{7.37}$$

Implicit in (7.37) is the assumption that the descriptor "The man near the door" identifies a man uniquely. The frame of MAN is

MAN	Name

.

Example (*p*)

$$\textit{Kent was walking slowly toward the door} \rightarrow \text{WALKING}[\text{Name} = \text{Kent}; \tag{7.38}$$
$$\Pi_{\text{Speed}} = \text{SLOW}; \Pi_{\text{Time}} = \text{PAST}; \Pi_{\text{Direction}} = \text{TOWARD}(\text{Object} = \text{DOOR})].$$

Example (*q*)

$$\textit{Herta is not very tall is very true} \rightarrow \tag{7.39}$$
$$\pi_{\text{Height(Herta)}}(u) = \mu^2_{\text{TRUE}}(1 - \mu^2_{\text{TALL}}(u)), \quad u \in [0,200]$$

where the frames are

TRUE	v	μ

TALL	Height	μ

$v \in [0,1]$.

Example (*r*)

$$\textit{Carole is very intelligent is very likely.} \tag{7.40}$$

Let

$$v \triangleq_{\mu} \text{INTELLIGENT}[\text{Name} = \text{Carole}] \tag{7.41}$$

i.e. v is the degree to which Carole is intelligent, and the frame of INTELLIGENT is assumed to be

INTELLIGENT	Name	μ

.

Then (see (7.5))

$$\textit{Carole is very intelligent} \rightarrow \Pi_v = \perp^2 \tag{7.42}$$

in which the right-hand member is equivalent to

$$\pi(v) = v^2. \tag{7.43}$$

Next, let

$$\text{X} = \int_0^1 p(v) v^2 \mathrm{d}v \tag{7.44}$$

where $p(v)\mathrm{d}v$ is the probability that Carole's degree of intelligence falls in the interval $[v, v + \mathrm{d}v]$. Then, using the translation rule for probability qualification, we obtain

$$\text{(Carole is very intelligent) is very likely} \rightarrow \Pi_X = \text{LIKELY}^2 \tag{7.45}$$

in which the right-hand member is equivalent to

$$\pi(p(\cdot)) = \mu^2_{\text{LIKELY}}\left(\int_0^1 p(v)v^2 dv\right) \tag{7.46}$$

and the frame of LIKELY is

LIKELY	p	μ

$p \in [0,1]$. Expressed in this form, the translation defines a possibility distribution of the probability density function $p(\cdot)$.

Example (*s*)

X is small is very true is likely$\rightarrow$

$$\pi(p(\cdot)) = \mu_{\text{LIKELY}}\left(\int_0^\infty p(u)\mu^2_{\text{TRUE}}(\mu_{\text{SMALL}}(u))du\right) \tag{7.47}$$

where $U \triangleq [0,\infty)$ and $p(u)du \triangleq \text{Prob}\{X \in [u,u+du]\}$. As in the previous example, (7.47) defines a possibility distribution of the probability density function of X.

Example (*t*)

$$\text{Men who are much taller than most men} \rightarrow F \tag{7.48}$$

where the fuzzy subset F of POPULATION is computed by the following procedure. (For simplicity, the procedure is stated in plain English.)

Assume that the frame of $\mathscr{D}$ is comprised of:

POPULATION	Name

MUCH TALLER	Name1	Name2	μ

MOST	ρ	μ

.

1. Compute
   $F_i \triangleq {}_{\mu \times \text{Name2}}\text{MUCH TALLER}[\text{Name1} = \text{Name}_i]$
   $\triangleq$ fuzzy set of men in relation to whom Name_i is much taller.

2. Compute the relative cardinality of F_i, i.e.

$$\text{Prop}(F_i) = \frac{\text{Count}(F_i)}{\text{Count(POPULATION)}}. \tag{7.49}$$

3. Compute
$\delta_i \triangleq \mu_{MOST}(\mathrm{Prop}(F_i))$ (7.50)
$\triangleq$ degree to which Name_i is much taller than most men.

4. The fuzzy set of men who are much taller than most men is given by
$$F = \delta_1/\mathrm{Name}_1 + \ldots + \delta_N/\mathrm{Name}_N \tag{7.51}$$

where + denotes the union and $\mathrm{Name}_1, \ldots, \mathrm{Name}_N$ are the elements of U(Name) in POPULATION. Alternatively, assume that the frame of $\mathscr{D}$ is comprised of:

POPULATION	Name	Height

MUCH TALLER	Height1	Height2	μ

MOST	ρ	μ

In this case, the procedure assumes the following form.
1. Compute
$h_i \triangleq \mathrm{Height}(\mathrm{Name}_i) = {}_{\mathrm{Height}}\mathrm{POPULATION}[\mathrm{Name} = \mathrm{Name}_i]$. (7.52)
2. Compute
$\gamma_{ij} = \mathrm{MUCH\ TALLER}[\mathrm{Height1} = h_i;\ \mathrm{Height2} = h_j]$ (7.53)
$\triangleq$ degree to which Name_i is much taller than Name_j.
3. Compute the fuzzy set
$F_i = \gamma_{i1}/\mathrm{Name}_1 + \ldots + \gamma_{iN}/\mathrm{Name}_N$ (7.54)
$\triangleq$ fuzzy set of men in relation to whom Name_i is much taller.
4. Same as Step 3 in previous procedure.
5. Same as Step 4 in previous procedure.

Example (*u*)

Many men are much taller than most men $\rightarrow \pi(\mathrm{POPULATION}) = \mu_G$ (7.55)

where μ_G is computed by the following procedure.
Assume that the frame of $\mathscr{D}$ is comprised of

POPULATION	Name

MUCH TALLER	Name1	Name2	μ

MOST	ρ	μ

MANY	ρ	μ

1. Compute F as in Example (t).
2. Compute
   $\gamma = \mathrm{Prop}(F)$
   $\triangleq$ Proportion of men who are much taller than most men.
3. The possibility of the relation POPULATION is given by

$$\pi(\mathrm{POPULATION}) = {}_{\mu}\mathrm{MANY}[\rho = \gamma] \tag{7.56}$$

in which the right-hand member defines μ_G.

Example (v)

Beth gave several big apples to each of her close friends $\to \pi(\mathrm{GAVE}) = \mu_G$. (7.57)

The following procedure computes μ_G on the assumption that the frame of $\mathscr{D}$ is comprised of:

GAVE	Giver	Receiver	Object

BIG	Object	μ

FRIEND	Name1	Name2	μ

SEVERAL	ρ	μ

.

1. Compute
   $G_i \triangleq {}_{\mathrm{Object}}\mathrm{GAVE}[\mathrm{Giver} = \mathrm{Beth};\ \mathrm{Receiver} = \mathrm{Name}_i]$ (7.58)
   $\triangleq$ Set of objects received from Beth by Name_i.
2. Compute
   $H = \mathrm{BIG}[\mathrm{Object} = \mathrm{APPLE}]$ (7.59)
   $\triangleq$ fuzzy set of big apples.
3. Compute
   $K = G_i \cap H$ (7.60)
   $\triangleq$ fuzzy set of big apples received from Beth by Name_i.
4. Compute
   $\gamma_i = {}_{\mu}\mathrm{SEVERAL}[\rho = \mathrm{Count}(K)]$ (7.61)
   $\triangleq$ degree to which Name_i received several big apples from Beth.
5. Compute
   $\delta_i = {}_{\mu}\mathrm{FRIEND}^2[\mathrm{Name1} = \mathrm{Beth};\ \mathrm{Name2} = \mathrm{Name}_i]$ (7.62)
   $\triangle$ degree to which Name_i is a close friend of Beth.
6. Compute
   $\sigma_i \triangleq 1 \wedge (1 - \delta_i + \gamma_i)$ (7.63)
   $\triangleq$ degree to which (If Name_i is a close friend of Beth then Name_i received several big apples from Beth).

7. Compute

$$\pi(\text{GAVE}) \triangleq \text{Min}_i\, \sigma_i \tag{7.64}$$

$\triangleq$ degree to which all close friends of Beth received from her several big apples.

It should be noted that when the translation of a proposition, p, into PRUF requires the execution of a procedure, P, which cannot be expressed as a relatively simple expression in PRUF—as is true of Examples (t), (u) and (v)—the relationship between p and P ceases to be transparent. A higher degree of transparency in cases of this type may be achieved through the introduction into PRUF of higher-level constructions relating to quantification, qualification, particularization and definition. This and other issues concerning the translation of more complex propositions than those considered here will be treated in subsequent papers.

8. Concluding remarks

In essence, PRUF may be regarded as a relation-manipulating language which serves the purposes of (a) precisiation of expressions in a natural language; (b) exhibiting their logical structure; and (c) providing a system for the characterization of the meaning of a proposition by a procedure which acts on a collection of fuzzy relations in a database and returns a possibility distribution.

By serving these purposes, PRUF provides a basis for a formalization of approximate reasoning. More specifically, through the use of PRUF, a set of imprecise premises expressed in a natural or synthetic language may be translated into possibility distributions to which the rules of inference in FL (or PRUF) may be applied, yielding other possibility distributions which upon retranslation lead to approximate consequents of the original premises. In this respect, PRUF plays the same role in relation to fuzzy premises and fuzzy conclusions that predicate calculus does in relation to non-fuzzy premises and non-fuzzy conclusions.

An important aspect of PRUF is a concomitant of its break with the long-standing tradition in logic, linguistics and philosophy of language—the tradition of employing the concept of truth as a foundation for theories of meaning. By adopting instead the concept of a possibility distribution as its point of departure, PRUF permits a uniform treatment of truth-qualification, probability-qualification and possibility-qualification of fuzzy propositions and thereby clarifies the roles played by the concepts of truth, probability and possibility not only in logic and language theory, but also in information analysis, decision analysis and related application areas.

As was stated in the Introduction, our exposition of PRUF in the preceding sections is neither definitive nor complete. There are many issues that remain to be explored, the most complex of which is that of automatic translation from a natural language into PRUF. However, to view this issue in a proper perspective, it must be recognized that the existing systems for automatic translation from a small subset of a natural language into a meaning representation language (and especially, a query language) have very narrow versatility since they are limited in their use to highly restricted domains of semantic discourse and human concept comprehension.

Although PRUF is still in its initial stages of development, its somewhat unconventional conceptual framework puts into a different perspective many of the long-standing issues in language theory and knowledge representation, expecially those pertaining to vagueness, uncertainty and inference from fuzzy propositions. By so doing, PRUF points a way toward the conception of question-answering systems having the capability to act on imprecise, incomplete or unreliable information which is resident in a database. To implement such systems, however, we shall need (a) a better system of linguistic modifiers than those that are available in natural languages, and (b) special-purpose hardware that is oriented toward the storage and manipulation of fuzzy rather than non-fuzzy data.

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