INFO 510 Bayesian Modelling and Inference

Lecture 5 – Conjugate priors and uninformative priors

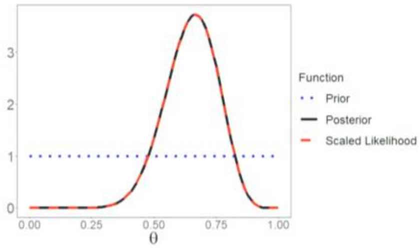
Dr. Kunal Arekar
College of Information
University of Arizona, Tucson

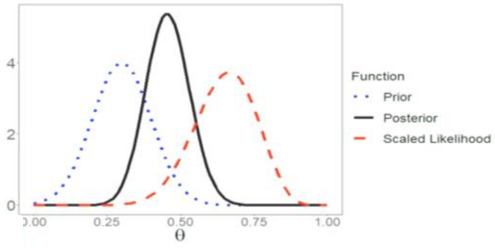
September 10, 2025

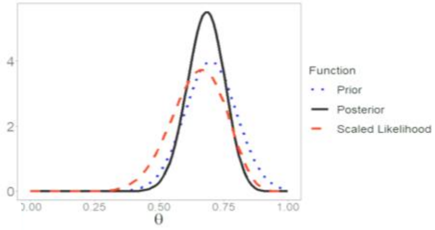
Outline

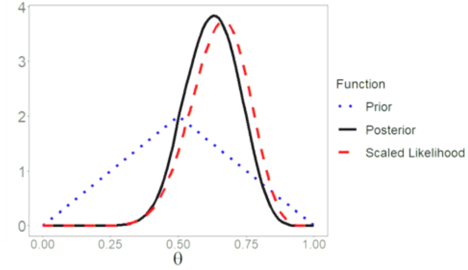
- Conjugacy
  - Conjugate priors
  - Mixture of conjugate prior
- Uninformative priors
  - Jeffreys prior

Different priors good data









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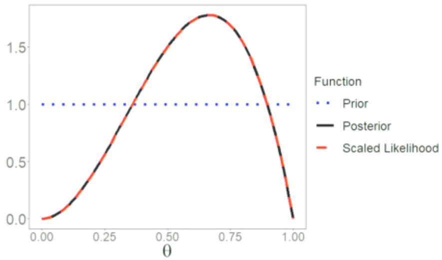

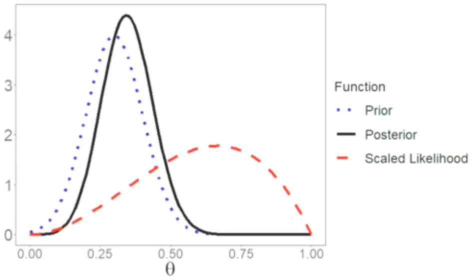

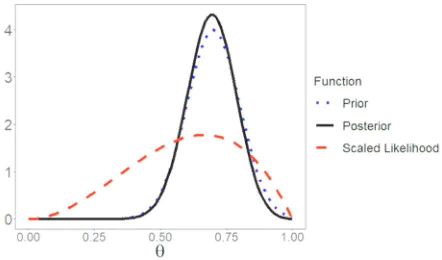

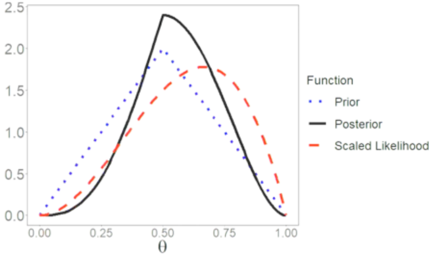

Conjugacy

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- All members of the exponential family have conjugate priors.

Brief List of Conjugate Models

Likelihood	Prior	Posterior
Binomial	Beta	Beta
Negative Binomial	Beta	Beta
Poisson	Gamma	Gamma
Geometric	Beta	Beta
Exponential	Gamma	Gamma
Normal (mean unknown)	Normal	Normal
Normal (variance unknown)	Inverse Gamma	Inverse Gamma
Normal (mean and variance unknown)	Normal/Gamma	Normal/Gamma
Multinomial	Dirichlet	Dirichlet

The Conjugate Beta Prior

- The Beta distribution is conjugate to the Binomial distribution.

$$p(\theta|x) = p(x|\theta)p(\theta) = \text{Binomial}(n,\theta) * \text{Beta}(a,b) =$$

$$\binom{n}{x}\theta^x(1-\theta)^{n-x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{(a-1)}(1-\theta)^{b-1} \tag{9}$$

$$\propto \theta^x(1-\theta)^{n-x}\theta^{(a-1)}(1-\theta)^{b-1}$$

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- The posterior distribution is simply a Beta$(x + a, n - x + b)$ distribution.
- Effectively, our prior is just adding $a-1$ successes and $b-1$ failures to the dataset.

Coin Flipping Example: Model

- Use a Bernoulli likelihood for coin X landing 'heads',

$$p(X = \text{`}H\text{'} | \theta) = \theta, \quad p(X = \text{`}T\text{'} | \theta) = 1 - \theta,$$

$$p(X|\theta) = \theta^{I(X = \text{`}H\text{'})}(1 - \theta)^{I(X = \text{`}T\text{'})}.$$

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- Use a Beta prior for probability θ of 'heads', $\theta \sim \text{Beta}(a, b)$,

$$p(\theta|a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a, b)} \propto \theta^{a-1}(1-\theta)^{b-1}$$

with $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

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- Remember that probabilities sum to one so we have

$$1 = \int_0^1 p(\theta|a, b)d\theta = \int_0^1 \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a, b)}d\theta = \frac{1}{B(a, b)}\int_0^1 \theta^{a-1}(1-\theta)^{b-1}d\theta$$

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which helps us compute integrals since we have

$$\int_0^1 \theta^{a-1}(1-\theta)^{b-1}d\theta = B(a, b).$$

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$$\begin{aligned}
p(\theta|HHH) &= \frac{p(HHH|\theta)p(\theta)}{p(HHH)} && \text{(Bayes' rule)} \\
&\propto p(HHH|\theta)p(\theta) && (p(HHH) \text{ is constant}) \\
&= \theta^3(1-\theta)^0 p(\theta) && \text{(likelihood def'n)} \\
&= \theta^3(1-\theta)^0 \theta^{a-1}(1-\theta)^{b-1} && \text{(prior def'n)} \\
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- Which we've written in the form of a Beta distribution,

$$\theta|HHH \sim \mathrm{Beta}(3+a, b),$$

which let's us skip computing the integral $p(HHH)$.

Coin Flipping Example: Estimates

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- Posterior predictive,

$$\begin{aligned}
p(H|HHH) &= \int_0^1 p(H|\theta)p(\theta|HHH)d\theta \\
&= \int_0^1 \text{Ber}(H|\theta)\text{Beta}\,(\theta|3+a, b)d\theta \\
&= \int_0^1 \theta\text{Beta}(\theta|3+a, b)d\theta = \mathbb{E}[\theta] \\
&= \frac{(3+a)}{(3+a)+b} = \frac{4}{5} = 0.8.
\end{aligned}$$

Coin Flipping Example: Effect of Prior

- We assume all θ equally likely and saw HHH,
  - ML predict it will always land heads.
  - Bayes predict probability of landing heads is only 80%.
    - Takes into account other ways that HHH could happen.

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- Beta$(3, 3)$ prior is like seeing 3 heads and 3 tails (stronger uniform prior),
  - Posterior predictive would be $\frac{3+3}{3+3+3} = 0.667$.
- Beta$(100, 1)$ prior is like seeing 100 heads and 1 tail (biased),
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- Beta$(0.01, 0.01)$ biases towards having unfair coin (head or tail),
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- Dependence on (a, b) is where people get uncomfortable:
  - But basically the same as choosing regularization parameter λ.
  - If your prior knowledge isn't misleading, you will not overfit.

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- If θ comes from the first distribution, the coin is fair, but if it comes from the second it is biased towards heads.

Mixtures of Conjugate Priors (Cont.)

- The prior has the form

$$p(\theta) = \sum_k p(z = k) p(\theta | z = k) \tag{12}$$

where $z = k$ means that θ comes from mixture component k, $p(z = k)$ are called the prior mixing weights, and each $p(\theta | z = k)$ is conjugate.

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- Posterior can also be written as a mixture of conjugate distributions as follows:

$$p(\theta | X) = \sum_k p(z = k | X) p(\theta | X, z = k) \tag{13}$$

where $p(z = k | X)$ are the posterior mixing weights given by

$$p(z = k | X) = \frac{p(z = k) p(X | z = k)}{\sum_{k'} p(z = k' | X) p(\theta | X, z = k')} \tag{14}$$

Uninformative Priors

- If we don't have strong beliefs about what θ should be, it is common to use an uninformative or non-informative prior, and to let the data speak for itself.
- Designing uninformative priors is tricky.

Uninformative Prior for the Bernoulli

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An **uninformative prior** should:

Influence the outcome as little as possible

Reflect minimal knowledge about θ

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- By decreasing the magnitude of the pseudo counts, we can lessen the impact of the prior. By this argument, the most uninformative prior is Beta(0, 0), which is called Haldane prior.

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- By decreasing the magnitude of the pseudo counts, we can lessen the impact of the prior. By this argument, the most uninformative prior is Beta$(0, 0)$, which is called Haldane prior.
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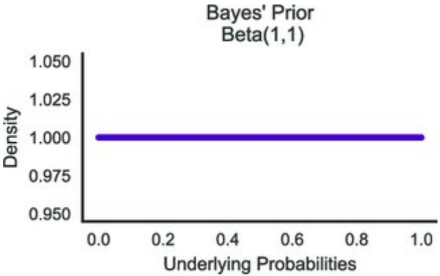
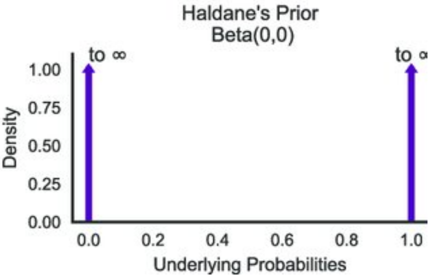
  - Haldane prior results in the posterior Beta$(x, n - x)$ which will be proper as long as $n - x \neq 0$ and $x \neq 0$.





Zhu et al., 2020 Psychological Review 127(5)

Haldane Prior: Beta(0,0)

$\theta|D \sim \text{Beta}(N1,N0)$

$E[\theta|D]=N1/N1+N0$

which is the MLE estimate

if $N_1 = 0$ or $N_0 = 0$, then the posterior is:
$\text{Beta}(0, N_0)$ or $\text{Beta}(N_1, 0)$ $\Rightarrow$ not normalizable!
Posterior mean is undefined.

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- We will see that the "right" uninformative prior is Beta$(\frac{1}{2}, \frac{1}{2})$.

Jeffreys Prior

- Jeffrey argued that a uninformative prior should be invariant to the parametrization used. The key observation is that if $p(\theta)$ is uninformative, then any reparametrization of the prior, such as $\theta = h(\varphi)$ for some function h, should also be uninformative.

Imagine you're describing a shape, like a circle. Whether you describe it using its radius (r) or its diameter (d=2r), your lack of knowledge about the circle's size should remain the same

Similarly, if we change the parameter representation of θ to ϕ=h(θ) using a transformation h, our prior p(θ) should adjust so that it remains "uninformative" in both representations

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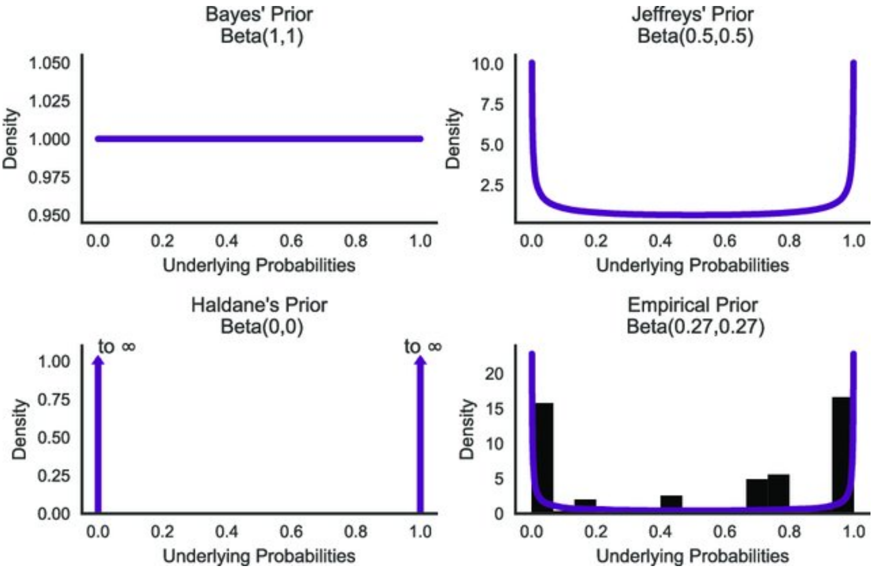

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- The Fisher information is a way of measuring the amount of information that an observable random variable X carries about an unknown parameter θ which the probability of X depends.

$$I(\theta) = E[(\frac{\partial}{\partial\theta} \log f(X;\theta))^2 |\theta] \tag{16}$$

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$$I(\theta) = E[(\frac{\partial}{\partial\theta} \log f(X;\theta))^2 |\theta] \tag{16}$$

- If $\log f(X;\theta)$ is twice differentiable with respect to θ, then

$$I(\theta) = -E[\frac{\partial^2}{\partial\theta^2} \log f(X;\theta)|\theta] \tag{17}$$

Reparametrization for Jeffreys Prior: One parameter case

- For an alternative parametrization ϕ we can derive $p_(\phi) = \sqrt{I(\phi)}$ from $p(\theta) = \sqrt{I(\theta)}$, using the change of variables theorem and the definition of Fisher information:

$$\begin{aligned} p(\phi) = p(\theta)\left|\frac{d\theta}{d\phi}\right| \propto \sqrt{I(\theta)(\frac{d\theta}{d\phi})^2} = \sqrt{E[(\frac{d\ln L}{d\theta})^2](\frac{d\theta}{d\phi})^2} \\ = \sqrt{E[(\frac{d\ln L}{d\theta}\frac{d\theta}{d\phi})^2]} = \sqrt{E[(\frac{d\ln L}{d\phi})^2]} = \sqrt{I(\phi)} \end{aligned} \tag{18}$$

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- The Fisher information is the expected information

$$I(\theta) = E[J(\theta|X)|X \sim \theta] = \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2} = \frac{1}{\theta(1-\theta)} \tag{22}$$

- Hence Jeffreys prior is

$$p(\theta) \propto \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}} \propto \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right). \tag{23}$$

What NOT to do when considering priors

Do NOT choose a prior that assigns 0 probability/density to possible values of the parameter

Do NOT base the prior on the observed data

Do NOT feel like you have to find that one, perfect prior

Do NOT worry too much about the prior!