# Difference of Squares and the Freshman's Dream

Nathalie Luna

MATH 1314

## 1 What's so important about the difference of squares?

In class, we talked about the difference of squares. I advised you to keep an eye out for any difference of squares, which look like this:

$$a^2 - b^2,$$

where $a$ is usually a variable. It's useful to recognize this "pattern" because it can be factorized like this:

$$a^2 - b^2 = (a + b)(a - b).$$

**Why is this true?** Let's start with the right hand side, apply FOIL and confirm that the product (the factorization) is indeed equal to the difference of squares.

$$\begin{aligned}(a + b)(a - b) &= a \cdot a + a \cdot (-b) + b \cdot a + b \cdot (-b),\\ &= a^2 - ab + ab - b^2,\\ &= a^2 + 0 - b^2,\\ &= a^2 - b^2.\end{aligned}$$

So, because all the expressions in the chain of equalities above are the same, then we can factorize the difference of squares $a^2 - b^2$ as $(a - b)(a + b)$.

## 2 Is the difference of squares the same as the square of a difference?

I noticed many people factorized the square of a difference the same way as the difference of squares. So, this begs the question: *Is the difference of squares the same as the square of a difference?* This is not a play on words, although it looks like it.

No student who did this showed any work, but this is how I suspect the train of (erroneous) thought went:

$$\begin{aligned}(a - b)^2 &= a^2 - b^2, && \text{starting with the square of a difference, they "distribute" the square,}\\ &= (a + b)(a - b), && \text{once they had a difference of squares, they could factorize.}\end{aligned}$$

The problem with this train of thought is the first step. **Exponents just don't behave that way.** We can distribute a coefficient into a sum or a difference:

$$c(a + b) = c \cdot a + c \cdot b, \qquad c(a - b) = c \cdot a - c \cdot b,$$

but we can't distribute an exponent this same way.

This misconception is so common that mathematicians have a name for it: The freshman's dream.

## 3 The Freshman's Dream

Distributing a square (or any exponent) into a binomial expression (the sum or difference of two terms) is intuitive, but mathematically incorrect. In algebra, we learn that $c(a-b) = c \cdot a - c \cdot b$. So, when we encounter the expression $(a-b)^2$, it seems natural that

$$(a-b)^2 = a^2 - b^2.$$

The equation above states that the square of a difference (left-hand side) is the same as the difference of squares (right-hand side). However, I called this a misconception and said that mathematicians call it **the freshman's dream**. To verify that the equation above is incorrect, let's take a closer look at each side.

The **left-hand side** is

$$\begin{aligned}(a-b)^2 &= (a-b)\cdot(a-b), \text{ the exponent means we multiply the base, } (a-b), \text{ by itself 2 times,}\\ &= a\cdot a + a\cdot(-b) + (-b)\cdot a + (-b)\cdot(-b), \text{ by applying FOIL,}\\ &= a^2 - ab - ab + b^2,\\ &= a^2 - 2ab + b^2.\end{aligned}$$

The expression we found for the left-hand side, $a^2 - 2ab + b^2$, is decidedly not the same as the **right-hand side**, $a^2 - b^2$. Hopefully, by now, I've convinced you that it's incorrect to equal the square of a difference to the difference of squares.

Once that is cleared up, we are still at a loss as to what to do with the square of a difference.

## 4 So what to do about the Square of a Difference?

If we are in the process of factorizing, and we encounter the square of a difference,

$$(a-b)^2,$$

we are done with the process! As we saw in the chain of equalities of the left-hand side, the square of a difference is already a product. In particular, it's the product of the difference, $a-b$, by itself.