BLOW–UP SOLUTIONS OF WAVE MAP EQUATIONS WITH PERIODIC IN TIME SPEED OF PROPAGATION

A Thesis

by

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ABSTRACT

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We study the initial value problem for the wave map equation with time-dependent speed of propagation. In particular, for arbitrary, small, and smooth initial data we construct blow–up solutions of the wave map with coefficients that are periodic in time. For the proof we use Lyapunov-Floquet theory and Borg's theorem.

DEDICATION

To mom and dad, for always telling me "you can".

ACKNOWLEDGMENTS

Many thanks to my adviser, for his time and patience.

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CHAPTER I

INTRODUCTION

We start with a brief description from the physical point of view for a field theory that will make the subject of our mathematical investigation. In the core of this description is the concept of nonlinear sigma model.

In physical terms a nonlinear sigma model is a scalar field describing maps

$$\phi : L \longrightarrow M,$$

also called field configuration, from a spacetime (L, g), a domain manifold, to a complete Riemannian manifold (M, h), the target manifold, which are formal critical points of the action

$$S = \int_L \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j h_{ab}(\phi)\, dg$$

We consider the nonlinear sigma model for which the spacetime is Lorentzian manifold. This is called the *wave map* being the hyperbolic analogy of what is known as the *harmonic map problem* which is the nonlinear sigma model for the case when domain is Riemannian manifold.

Thus, the basic object of study is a wave map

$$\phi : (L, g_{\mu\nu}) \longrightarrow (M, h_{ab})$$

where L is an $n+1$-dimensional Lorentzian manifold (that is the signature of the metric $g_{\mu\nu}$ is $(n, 1)$) and the target is a k-dimensional Riemannian manifold. The map ϕ is a *wave map* if it is a stationary point for the Lagrangian functional

$$\mathcal{L}[\phi] = \int_L \frac{1}{2} g^{\mu\nu}(x) h_{ab}(\phi) \nabla_\mu \phi^a \nabla_\nu \phi^b \, d\mu_g$$

The Lagrangian is written in local coordinates on the target, for which the notation $\phi^a = \phi^a(x^\mu)$ is used. We denote by $d\mu_g$ the measure with respect to the metric $g^{\mu\nu}$ on the spacetime, and it is given by

$$d\mu_g := \sqrt{|\det(g_{\mu\nu}(x))|} \prod_{\mu=0}^{n} dx^\mu.$$

Here the convention to write $g^{\mu\nu}(x) = (g_{\mu\nu}(x))^{-1}$ and $h^{ab}(\phi) = (h_{ab}(\phi))^{-1}$ for the inverse of two metric tensors is used. These tensors are used also in raising indexes. Moreover we use also the notation conventions

$$\begin{aligned}
\langle \nabla\phi^a, \nabla\phi^b \rangle &:= g^{\mu\nu}(x) \nabla_\mu \phi^a \nabla_\nu \phi^b, \\
\langle \nabla_\mu\phi, \nabla_\nu\phi \rangle &:= h_{ab}(\phi) \nabla_\mu \phi^a \nabla_\nu \phi^b, \\
\langle \nabla\phi, \nabla\phi \rangle &:= g^{\mu\nu}(x) h_{ab}(\phi) \nabla_\mu \phi^a \nabla_\nu \phi^b.
\end{aligned}$$

The Euler-Lagrange equations can be derived via variation argument as follows. Consider small arbitrary variation $\delta\phi$ of the target coordinates, which is vanishing on the boundary of M, and evaluate the change in value of the Lagrangian functional up to first order in $\delta\phi$. Integration by parts leads to

$$\mathcal{L}[\phi + \delta\phi] - \mathcal{L}[\phi] = \int_L \frac{\delta\mathcal{L}}{\delta\phi^a} \delta\phi^a \, d\mu_g$$

where

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta\phi^a} &= h_{ab}(\phi) \Box \phi^b - \frac{\partial h_{ab}(\phi)}{\partial \phi^c} \langle \nabla\phi^b, \nabla\phi^c \rangle + \frac{1}{2} \frac{\partial h_{bc}(\phi)}{\partial \phi^a} \langle \nabla\phi^b, \nabla\phi^c \rangle \\
&= h_{ab}(\phi) \left(\Box \phi^b - \Gamma^b_{cd}(\phi) \langle \nabla\phi^c, \nabla\phi^d \rangle \right)
\end{aligned}$$

In the last formula $\Box$ is the d'Alabertian (or wave) operator

$$\Box := -\nabla_\mu \nabla^\mu$$

and Γ^b_{cd} are the Christoffel symbols on the target manifold

$$\Gamma_{ab;c} := \frac{1}{2}\left(\partial_b h_{ac} + \partial_a h_{bc} - \partial_c h_{ab}\right).$$

A stationary point for the Lagrangian functional is defined by equation

$$\frac{\delta \mathcal{L}}{\delta \phi^a} = 0$$

which implies the following system of equations

$$\Box \phi^b - \Gamma^b_{cd}(\phi)\langle \nabla \phi^c, \nabla \phi^d \rangle = 0$$

This is the *wave map system.*

It is easy to see that if the target manifold is flat, then the wave map system reduced to the uncopled linear wave equations

$$\Box \phi^b = 0$$

In general, **wave maps** from the Minkowski spacetime $\mathbb{R}^{1+n}$ to a Riemanian manifold M are defined as critical points of the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\int_{\mathbb{R}^{1+n}} \left(-\langle \dot{u}, \dot{u}\rangle + \langle \nabla u, \nabla u \rangle\right) dx\, dt$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemanian metric on M, $\dot{u}$ denotes the partial derivative with respect to time, and ∇ denotes the gradient. In local coordinates, the wave map satisfies the system of equations

$$\Box u^i + \sum_{j,k} \Gamma^i_{j,k}(u) \left(\dot{u}^j \dot{u}^k - \nabla u^j \cdot \nabla u^k \right) = 0 \quad i = 1, \dots m, \tag{1.1}$$

where $\Box = \partial^2/\partial t^2 - \Delta$, Δ is the Laplacian,and $\Gamma^i_{j,k}$ is the Christoffel symbol of M, defined as:

$$\Gamma^i_{j,k}(u) := \frac{1}{2} \sum_{m=1}^{n} h^{im} \left(\frac{\partial}{\partial u^j} h_{km} + \frac{\partial}{\partial u^k} h_{jm} - \frac{\partial}{\partial u^m} h_{kj} \right)$$

and h^{ik} is a metric tensor in M.

Let $u \in L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a multi–index. Then the distributional derivative of u is defined as

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

We say that $\partial^\alpha u \in L^2(\Omega)$ if there exists $v \in L^2(\Omega)$ such that

$$\langle \partial^\alpha u, \varphi \rangle := (-1)^{|\alpha|} \int_\Omega u(x) \partial^\alpha \varphi(x)\, dx = \int_\Omega v(x) \varphi(x)\, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Finally, we denote the **Sobolev space** as $W^s(\Omega)$ and we define it as the linear space

$$W^s(\Omega) := \{ u \in L^2(\Omega) \;\big|\; \partial^\alpha u \in L^2(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| \le s \},$$

with the norm

$$\|u\|_{W^s(\Omega)} := \sum_{|\alpha| \le s} \|\partial^\alpha u\|_{L^2(\Omega)}.$$

In the case that $\Omega = \mathbb{R}^n$, the Sobolev space (denoted simply as H^s) is a space of functions whose distributional derivatives (up to order s) exist in an L^2–space:

$$H^s := \{u \in L^2(\mathbb{R}^n) \;\Big|\; \partial^\alpha u \in L^2(\mathbb{R}^n) \text{ for all } \alpha \text{ such that } |\alpha| \leq s\}.$$

We denote by $\mathcal{B}^k$ the Banach space of all functions $u \in C^k(\mathbb{R}^n)$ such that $\partial^\alpha u$, for $|\alpha| \leq k$, is bounded on $\mathbb{R}^n$. That is,

$$\mathcal{B}^k := \{u \in C^k(\mathbb{R}^n) \;\Big|\; \text{ for every } \alpha, |\alpha| \leq k, \exists C_\alpha \text{ such that } |\partial^\alpha u(x)| \leq C_\alpha, \forall x \in \mathbb{R}^n\}.$$

Theorem 1 (Sobolev Embedding Theorem)**.** *If $s > \frac{n}{2} + k$, then the space H^s is continuously embedded in $\mathcal{B}^k$, that is $H^s \subset \mathcal{B}^k$. More precisely, for every α such that $|\alpha| \leq k$, the function $\partial^\alpha u(x)$ is continuous and*

$$\sup_{\mathbb{R}^n} |\partial^\alpha u(x)| \leq C(n,k,s) \|u\|_{H^s}.$$

Theorem 2 (Local Existence Theorem)**.** *Consider the Cauchy problem*

$$\begin{aligned}
&\Box u^i + \sum_{j,k=1}^{m} \Gamma^i_{j,k}(u^1, \ldots, u^m) \left(\dot{u}^j \dot{u}^k - \nabla u^j \cdot \nabla u^k\right) = 0 \\
&u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x), \quad i = 1, \ldots m, \quad x \in \mathbb{R}^n,
\end{aligned}$$

where $\Gamma^i_{j,k}(u)$ are C^∞ functions and $u = (u^1, \ldots, u^m)$. If $u^i_0(x) \in H^{s+1}(\mathbb{R}^n)$ and $u^i_1(x) \in H^s(\mathbb{R}^n)$ for some integer $s > (n+2)/2$ then the Cauchy problem has for some $T > 0$ a solution

$$u \in C^2([0,T] \times \mathbb{R}^n).$$

Proof. Follows from the local existence theorem: Theorem 6.4.11 [4] (Hörmander 1997) $\square$

Theorem 3 (Small Data Global Existence Theorem)**.** *The Cauchy problem*

$$\Box u^i + \sum_{j,k} \Gamma^i_{j,k}(u^1,\ldots,u^m)\left(\dot{u}^j\dot{u}^k - \nabla u^j \cdot \nabla u^k\right) = 0$$
$$u^i(0,x) = \varepsilon u^i_0(x), \quad u^i_t(0,x) = \varepsilon u^i_1(x)\,, \quad i = 1,\ldots m, \quad x \in \mathbb{R}^n\,,$$

with $\Gamma^i_{j,k}(u) \in C^\infty$ functions and $u^i_0(x) \in C^\infty(\mathbb{R}^n)$ and $u^i_1(x) \in C^\infty(\mathbb{R}^n)$ has a C^∞ solution for all $t \geq 0$

$$u \in C^\infty([0,\infty) \times \mathbb{R}^n)\,.$$

if $n \geq 4$ and ε is sufficiently small.

Proof. Follows from Theorem 6.5.2 [4](Hörmander 1997) existence theorem. □

Klainerman and Machedon [5] (1995) proved that the Cauchy problem for (1.1) is time locally (in time) well-posed in the Sobolev space $H^s(\mathbb{R}^{1+n})$ for any $s > n/2$ if $\Gamma^i_{j,k}(u)$ are analytic and $n = 3$. Klainerman and Selberg (1997) extended this result to $n \geq 2$.

Nakanishi and Ohta [8] studied the Cauchy problem for the nonlinear wave equation

$$\begin{cases} \Box u + f(u)\left(\dot{u}^2 - |\nabla u|^2\right) = 0 \quad (t,x) \in \mathbb{R}^{1+n} \\ u(0,x) = u_0(x), \quad \dot{u}(0,x) = u_1(x) \quad x \in \mathbb{R}^n \end{cases} \tag{1.2}$$

where $u = u(t,x)$ is a scalar real-valued unknown function, f is a real valued smooth function.

Equation (1.2) is a model and special case for wave maps.

Theorem 4 (Nakanishi-Ohta)**.** *Let $f \in C^\infty(\mathbb{R})$. Then, if and only if f satisfies*

$$\int_0^\infty \exp\left(\int_0^s f(r)dr\right) ds = \infty \textit{ and } \int_{-\infty}^0 \exp\left(\int_0^s f(r)dr\right) ds = \infty, \tag{1.3}$$

the problem (1.2) has a global classical solution $u \in C^\infty(\mathbb{R}^{1+n})$ for any $u_0 \in C^\infty(\mathbb{R}^n)$ and $u_1 \in C^\infty(\mathbb{R}^n)$.

Note here, that in theorem (4), the initial data u_0, u_1 are not assumed small.

CHAPTER II

MAIN THEOREM

We consider the system of equations

$$\begin{aligned}&\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u^i \\ &+\sum_{j,k}\Gamma^i_{j,k}(u^1,\ldots,u^m)\left(\dot{u}^j\dot{u}^k - b^2(t)\nabla u^j\cdot\nabla u^k\right) = 0,\ \ i = 1,\ldots m,\end{aligned} \tag{2.1}$$

where $b = b(t)$ is a smooth positive periodic function.

We are concerned with what happens to the Cauchy problem for equation (1.1) if we have small initial data. That is, if the initial value problem is globally solvable for small initial data if speed of propagation depends on time, for instance, is "almost" constant if ε is small and

$$\begin{aligned}&\left(\partial^2/\partial t^2 - (1+\varepsilon\sin(t))\Delta\right) u^i \\ &+\sum_{j,k}\Gamma^i_{j,k}(u^1,\ldots,u^m)\left(\dot{u}^j\dot{u}^k - (1+\varepsilon\sin(t))\nabla u^j\cdot\nabla u^k\right) = 0, \\ &i = 1,\ldots,m.\end{aligned}$$

Theorem 5 (Main Theorem)**.** *Let $b = b(t)$ be a defined on R, a periodic, non-constant, smooth, and positive function. Assume that*

$$\sum_{j,k=1}^{m}\Gamma^1_{j,k}(\xi,\ldots,\xi) = \sum_{j,k=1}^{m}\Gamma^2_{j,k}(\xi,\ldots,\xi) = \cdots = \sum_{j,k=1}^{m}\Gamma^m_{j,k}(\xi,\ldots,\xi) = f(\xi), \tag{2.2}$$

and that (1.3) does not hold. Then for every n, s, and for every positive δ there are initial data

$u_0^i, u_1^i \in C_0^\infty(\mathbb{R}^n)$ such that

$$\sum_{i=1}^{m} \|u_0^i\|_{(s+1)} + \|u_1^i\|_{(s)} \le \delta \tag{2.3}$$

but a global solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ to the problem with data

$$u^i(0,x) = u_0^i(x), \quad u_t^i(0,x) = u_1^i(x)\,, \quad i = 1,\dots,m, \quad x \in \mathbb{R}^n\,, \tag{2.4}$$

does not exist.

Proof will be given in several steps.

Example 1: Consider the system (2.1) with $m = 2$:

$$\begin{aligned}
\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u^1 + \sum_{j,k=1}^{2} \Gamma^1_{j,k}(u^1,u^2)\left(\dot u^j \dot u^k - b^2(t)\nabla u^j \cdot \nabla u^k\right) &= 0,\\
\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u^2 + \sum_{j,k=1}^{2} \Gamma^2_{j,k}(u^1,u^2)\left(\dot u^j \dot u^k - b^2(t)\nabla u^j \cdot \nabla u^k\right) &= 0.
\end{aligned} \tag{2.5}$$

We define $h_{ik}(u^1,u^2) := f(u^1,u^2)\delta_{ik}$ and

$$\frac{\partial h}{\partial u^k}(u^1,u^2) = \frac{\partial h}{\partial u^l}(u^1,u^2) \quad \text{if} \quad u^1 = u^2 = u^n \quad \text{for} \quad k,l = 1,2.$$

Then, the Christoffel symbols are:

$$\begin{aligned}
\Gamma^i_{j,k} &= \frac{1}{2}\sum_{m=1}^{2} h^{im}\left(\frac{\partial}{\partial u^j}h_{km} + \frac{\partial}{\partial u^k}h_{jm} - \frac{\partial}{\partial u^m}h_{kj}\right),\\
&= \frac{1}{2}\sum_{m=1}^{2} \frac{1}{f(u^1,u^2)}\delta_{im}\left(\frac{\partial}{\partial u^j}f(u^1,u^2)\delta_{km} + \frac{\partial}{\partial u^k}f(u^1,u^2)\delta_{jm} - \frac{\partial}{\partial u^m}f(u^1,u^2)\delta_{kj}\right),\\
&= \frac{1}{2f(u^1,u^2)}\delta_{ii}\left(\frac{\partial}{\partial u^j}f(u^1,u^2)\delta_{ki} + \frac{\partial}{\partial u^k}f(u^1,u^2)\delta_{ji} - \frac{\partial}{\partial u^i}f(u^1,u^2)\delta_{kj}\right),\\
&= \frac{1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^j}f(u^1,u^2)\delta_{ki} + \frac{\partial}{\partial u^k}f(u^1,u^2)\delta_{ji} - \frac{\partial}{\partial u^i}f(u^1,u^2)\delta_{kj}\right).
\end{aligned}$$

Now, we calculate each Christoffel symbol:

$$
\begin{aligned}
\Gamma^1_{1,1} &= \frac{1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^1}f(u^1,u^2)\right),\\
\Gamma^1_{1,2} &= \frac{1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^2}f(u^1,u^2)\right),\\
\Gamma^1_{2,1} &= \frac{1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^2}f(u^1,u^2)\right),\\
\Gamma^1_{2,2} &= \frac{-1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^1}f(u^1,u^2)\right),\\
\Gamma^2_{2,2} &= \frac{1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^2}f(u^1,u^2)\right),\\
\Gamma^2_{1,2} &= \frac{1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^1}f(u^1,u^2)\right),\\
\Gamma^2_{2,1} &= \frac{1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^1}f(u^1,u^2)\right),\\
\Gamma^2_{1,1} &= \frac{-1}{2f(u_1,u_2)}\left(\frac{\partial}{\partial u^2}f(u^1,u^2)\right).
\end{aligned}
$$

If we let $u^1 = u^2 = \xi \in \mathbb{R}$, we apply the assumptions on h_{ik} to obtain

$$
\begin{aligned}
\Gamma^1_{1,1} &= \frac{1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right),\\
\Gamma^1_{1,2} &= \frac{1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right),\\
\Gamma^1_{2,1} &= \frac{1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right),\\
\Gamma^1_{2,2} &= \frac{-1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right),\\
\Gamma^2_{2,2} &= \frac{1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right),\\
\Gamma^2_{1,2} &= \frac{1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right),\\
\Gamma^2_{2,1} &= \frac{1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right),\\
\Gamma^2_{1,1} &= \frac{-1}{2f(\xi,\xi)}\left(\frac{\partial}{\partial u^1}f(\xi,\xi)\right).
\end{aligned}
$$

Therefore,

$$\Gamma^i_{j,k}(\xi,\xi)=\begin{cases}\frac{-1}{2f}\left(\frac{\partial}{\partial u^1}f\right), & \text{if } i\neq j=k,\\ \frac{1}{2f}\left(\frac{\partial}{\partial u^1}f\right), & \text{otherwise.}\end{cases}$$

Finally, note that:

$$\begin{aligned}\sum_{j,k=1}^{2}\Gamma^1_{jk}&=\Gamma^1_{11}+\Gamma^1_{12}+\Gamma^1_{21}+\Gamma^1_{22},\\ &=\frac{1}{2f}\left(\frac{\partial}{\partial u^1}f\right)+\frac{1}{2f}\left(\frac{\partial}{\partial u^1}f\right)+\frac{1}{2f}\left(\frac{\partial}{\partial u^1}f\right)+\frac{-1}{2f}\left(\frac{\partial}{\partial u^1}f\right),\\ &=\frac{1}{f}\left(\frac{\partial}{\partial u^1}f\right),\\ \sum_{j,k=1}^{2}\Gamma^2_{jk}&=\Gamma^2_{11}+\Gamma^2_{12}+\Gamma^2_{21}+\Gamma^2_{22},\\ &=\frac{-1}{2f}\left(\frac{\partial}{\partial u^1}f\right)+\frac{1}{2f}\left(\frac{\partial}{\partial u^1}f\right)+\frac{1}{2f}\left(\frac{\partial}{\partial u^1}f\right)+\frac{1}{2f}\left(\frac{\partial}{\partial u^1}f\right),\\ &=\frac{1}{f}\left(\frac{\partial}{\partial u^1}f\right).\end{aligned}$$

Then, we have that

$$\hat{f}(\xi):=\frac{1}{f(\xi)}\left(\frac{\partial}{\partial\xi}f(\xi)\right)=\sum_{j,k=1}^{2}\Gamma^1_{jk}=\sum_{j,k=1}^{2}\Gamma^2_{jk}.$$

Now, if we have the following initial data for problem (2.5):

$$u^1(0,x)=u^2(0,x)=u_0(x),\quad u^1_t(0,x)=u^2_t(0,x)=u_1(x),$$

then we will have that $u:=u_1=u_2=\xi$, by uniqueness in Sobolov spaces. Then, the Cauchy problem for system (2.5) is simplified to the following equation:

$$\left(\partial^2/\partial t^2-b^2(t)\Delta\right)\xi+\hat{f}(\xi)\left(\dot{\xi}^2-b^2(t)\nabla\xi\cdot\nabla\xi\right)=0,$$

Example 2: Let $h_{ik}(u^1,\ldots,u^n) = h(u^1,\ldots,u^n)\delta_{ik} + H((u^\alpha - u^\beta)^2)$, where $\alpha, \beta \in \{1,\cdots,n\}$. Assume that $h(u^1,\ldots,u^n) > 0$, that $H(0)$ is small and that

$$\frac{\partial}{\partial u^k} h(u^1,\ldots,u^n) = \frac{\partial}{\partial u^l} h(u^1,\ldots,u^n) \text{ if } u^1 = \cdots = u^n, \forall k, l.$$

First, we calculate the Christoffel symbols for $h_{ik}(u^1,\ldots,u^n)$:

$$\begin{aligned}
\Gamma^i_{jk}(u) &= \frac{1}{2}\sum_{m=1}^{n} h^{im}\left(\frac{\partial}{\partial u^j}h_{km} + \frac{\partial}{\partial u^k}h_{jm} - \frac{\partial}{\partial u^m}h_{jk}\right), \\
&= \frac{1}{2}\sum_{m=1}^{n}\left(\frac{1}{h}\delta_{im} + H((u^\alpha - u^\beta)^2)\right)\left(\frac{\partial}{\partial u^j}\left[h\delta_{km} + H((u^\alpha - u^\beta)^2)\right] + \frac{\partial}{\partial u^k}\left[h\delta_{jm} + H((u^\alpha - u^\beta)^2)\right]\right. \\
&\qquad\qquad \left. - \frac{\partial}{\partial u^m}\left[h\delta_{jk} + H((u^\alpha - u^\beta)^2)\right]\right), \\
&= \frac{1}{2}\left(\frac{1}{h} + H((u^\alpha - u^\beta)^2)\right)\left(\frac{\partial}{\partial u^j}\left[h\delta_{ki} + H((u^\alpha - u^\beta)^2)\right] + \frac{\partial}{\partial u^k}\left[h\delta_{ji} + H((u^\alpha - u^\beta)^2)\right]\right. \\
&\qquad\qquad \left. - \frac{\partial}{\partial u^i}\left[h\delta_{jk} + H((u^\alpha - u^\beta)^2)\right]\right).
\end{aligned}$$

Now, if we define the initial data

$$u^i(0,x) = u_0(x) \text{ and } \frac{\partial}{\partial t}u^i(0,x) = u_1(x), \forall i \in \{1,\ldots,n\},$$

then by uniqueness in Sobolev spaces, we have that $u := u^1 = \cdots = u^n$. Also, then the Christoffel symbols become:

$$\begin{aligned}
\Gamma^i_{jk}(u) &= \frac{1}{2}\left(\frac{1}{h(u)} + H((u-u)^2)\right)\left(\frac{\partial}{\partial u}\left(h\delta_{ki} + H((u-u)^2)\right) + \frac{\partial}{\partial u}\left(h\delta_{ji} + H((u-u)^2)\right)\right. \\
&\qquad\qquad \left. - \frac{\partial}{\partial u}\left(h\delta_{jk} + H((u-u)^2)\right)\right), \\
&= \frac{1}{2}\left(\frac{1}{h(u)} + H(0)\right)\left(\frac{\partial}{\partial u}\left(h\delta_{ki} + H(0)\right) + \frac{\partial}{\partial u}\left(h\delta_{ji} + H(0)\right) - \frac{\partial}{\partial u}\left(h\delta_{jk} + H(0)\right)\right), \\
&= \frac{1}{2}\left(\frac{1}{h(u)} + H(0)\right)\left(\frac{\partial}{\partial u}h\delta_{ki} + \frac{\partial}{\partial u}h\delta_{ji} - \frac{\partial}{\partial u}h\delta_{jk}\right).
\end{aligned}$$

Now, if we fix $i \in \{1, \ldots, n\}$, we get that

$$\begin{aligned}
\sum_{j,k=1}^{n} \Gamma^{i}_{jk}(u) &= \sum_{j,k=1}^{n} \frac{1}{2}\left(\frac{1}{h(u)} + H(0)\right)\left(\frac{\partial}{\partial u} h\delta_{ki} + \frac{\partial}{\partial u} h\delta_{ji} - \frac{\partial}{\partial u} h\delta_{jk}\right), \\
&= \frac{1}{2}\left(\frac{1}{h(u)} + H(0)\right)\left(n\frac{\partial}{\partial u} h + n\frac{\partial}{\partial u} h - n\frac{\partial}{\partial u} h\right), \\
&= \frac{1}{2}\left(\frac{1}{h(u)} + H(0)\right)\left(n\frac{\partial}{\partial u} h\right), \\
&= \frac{n}{2}\left(\frac{1}{h(u)} + H(0)\right)\frac{\partial}{\partial u} h,
\end{aligned}$$

which does not depend on our choice of i. Therefore, if we define

$$f(u) := \frac{n}{2}\left(\frac{1}{h(u)} + H(0)\right)\frac{\partial}{\partial u} h,$$

then we have that

$$\sum_{j,k=1}^{n} \Gamma^{1}_{jk}(u) = \sum_{j,k=1}^{n} \Gamma^{2}_{jk}(u) = \cdots = \sum_{j,k=1}^{n} \Gamma^{n}_{jk}(u) = f(u).$$

Example 3: Let $b(t) = 1 + \varepsilon \sin(t)$ be a defined on R, a periodic, non-constant, smooth, and positive function. Assume that $m = 2$

$$\left(\partial^2/\partial t^2 - (1 + \varepsilon \sin(t))\Delta\right) u + |\dot{v}|^2 - (1 + \varepsilon \sin(t))|\nabla v|^2 = 0$$
$$\left(\partial^2/\partial t^2 - (1 + \varepsilon \sin(t))\Delta\right) v + |\dot{u}|^2 - (1 + \varepsilon \sin(t))|\nabla u|^2 = 0$$

Then for every n, s, and for every positive δ there are data $u_0, v_0, u_1, v_1 \in C_0^\infty(R^n)$ such that

$$\|u_0\|_{(s+1)} + \|u_1\|_{(s)} + \|v_0\|_{(s+1)} + \|v_1\|_{(s)} \leq \delta$$

but a solution $u, v \in C^2(R_+ \times R^n)$ to the problem with data

$$u(0,x) = u_0(x),\, u_t(0,x) = u_1(x), \quad x \in R^n\,,$$
$$v(0,x) = v_0(x), v_t(0,x) = v_1(x)\,, \quad x \in R^n\,,$$

does not exist. For the same data if $\varepsilon = 0$ then a small data solution exists globally.

CHAPTER III

PROOF OF MAIN THEOREM

3.1 (Step 1) Uniqueness

Theorem 6 (Uniqueness Theorem)**.** *Let $u \in C^2$ be a solution of the equation*

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(t,x) u_{x_j} + a(t,x) u_t + c(t,x) u = 0, \tag{3.1}$$
$$(t,x) \in [0,T] \times \mathbb{R}^n$$

where the positive functions $a_{ij}(t,x)$ and their first derivative (with respect to time) are bounded by α, the functions $b_j(t,x)$ are bounded by β,the function $a(t,x)$ is bounded by A, and bounded by B and $b'(t)$ bounded by B_0, the function $c(t,x)$ is bounded by C, and for every $t > 0$ $u(t,x) = 0$ if $|x|$ is large. We also assume that

$$0 < \sigma_0 |\xi|^2 \le a_{ij}(t,x) \xi_i \xi_j \le \sigma_1 |\xi|^2, \quad \forall \xi \neq 0.$$

Then any other solution $v \in C^2$ of (3.1) with same property and

$$\partial_t^k u(0,x) = \partial_t^k v(0,x) \quad and \quad k = 0,1$$

is equal to u for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Proof. Let

$$F(x,u,u',u'') := u_{tt} - b^2(t) \sum_{i,j=1}^{n} a_{ij}(t,x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(t,x) u_{x_j} + a(t,x) u_t + c(t,x) u$$

with $x \in \mathbb{R}^{1+n}$. Now let $u \in C^2$ be a solution of $F(x,u,u',u'') = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Note that we have the following values of $F_{jk} = \frac{\partial F}{\partial u_j \partial u_k}$:

$$F_{00} = 1,$$
$$F_{ij} = -b^2(t) a_{ij}(t,x), \text{ where } 1 \leq i,j \leq n.$$

Then:

$$(\xi_0, \dots, \xi_n) \begin{pmatrix} 1 & -b^2(t)a_{01} & \cdots & -b^2(t)a_{0n} \\ -b^2(t)a_{10} & -b^2(t)a_{11} & & \\ \vdots & & \ddots & \vdots \\ -b^2(t)a_{n0} & & \cdots & -b^2(t)a_{nn} \end{pmatrix} \begin{pmatrix} \xi_0 \\ \\ \vdots \\ \xi_n \end{pmatrix} = 0,$$

$$\xi_0^2 - b^2(t) \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j = 0,$$
$$\xi_0^2 = b^2(t) \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j,$$
$$\xi_0^2 \leq b^2(t) \sigma_1 |\xi|^2,$$
$$\xi_0 = \pm \sqrt{\sigma_1} b(t) |\xi|.$$

Since $b(t)$ is different from 0, then we have that $F(x,u,u',u'') = 0$ is hyperbolic in the direction of time when $\xi = (\xi^1, \dots, \xi^n) \neq 0$.

Suppose that $v \in C^2$ be another solution to (3.1), such that

$$\partial_t^k u(0,x) = \partial_t^k v(0,x) \quad \textit{and} \quad k = 0,1.$$

This implies that $(u-v)$ is a solution to the equation of the form:

$$F(x,u,u',u'') - F(x,v,v',v'') = \sum_{|\alpha|\leq 2} a_\alpha(x,u,u',u'',v,v',v'')\partial^\alpha(u-v) = 0.$$

which we have shown to be hyperbolic in the direction of time, $t = \xi_0$.

Since $w := (u-v)$ is a solution to the equation

$$w_{tt} - b^2(t)\sum_{i,j=1}^{n} a_{ij}(t,x)w_{x_ix_j} + \sum_{j=1}^{n} b_j(t,x)w_{x_j} + a(t,x)w_t + c(t,x)w = 0,$$

with initial data $w(0,x) = w_t(0,x) = 0$, $\forall x \in \mathbb{R}^n$.

3.1.1 Energy Estimates and Uniqueness for Scalar Linear Equation

Consider the solution w to the equation

$$w_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x)w_{x_ix_j} + \sum_{j=1}^{n} b_j(t,x)w_{x_j} + a(t,x)w_t + c(t,x)w = f, \tag{3.2}$$

with initial data $w(0,x) = \varphi_0(x)$, $w_t(0,x) = \varphi_1(x)$, for all $x \in \mathbb{R}^n$.

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^n}\left[w_t^2 + \sum_{i,j=1}^{n} a_{ij}(t,x)w_{x_i}w_{x_j}\right]dx \geq 0.$$

We are going to prove the so-called energy estimate.

Theorem 7 (Energy Estimate for scalar linear equation)**.** *Assume that $w \in C^2([0,T]\times\mathbb{R}^n)$ is solution of (3.2) such that $w(t,x) = 0$ for all $t \in [0,T]$ if $x \notin B$ for some ball $B \subseteq \mathbb{R}^n$ of finite radius. Then*

$$E(t) \leq cE(0) + c\int_0^t\int_{\mathbb{R}^n}|f(s,x)|^2\,dxds \qquad \text{for all} \quad t \in [0,T] \tag{3.3}$$

Proof. From the equation above, we have

$$
\begin{aligned}
f &= w_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_i x_j} + \sum_{j=1}^{n} b_j(t,x) w_{x_j} + a(t,x) w_t + c(t,x) w, \\
f w_t &= w_t \left[w_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_i x_j} + \sum_{j=1}^{n} b_j(t,x) w_{x_j} + a(t,x) w_t + c(t,x) w \right], \\
f w_t &= w_t w_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_i x_j} w_t + \sum_{j=1}^{n} b_j(t,x) w_t w_{x_j} + a(t,x) w_t^2 + c(t,x) w_t w.
\end{aligned}
$$

We integrate this identity

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}^n} f w_t \, dxds \\
= \; & \int_0^t \int_{\mathbb{R}^n} \left[w_t w_{tt} - \sum_{i,j=1}^{n} a_{ij}(s,x) w_{x_i x_j} w_t + \sum_{j=1}^{n} b_j(s,x) w_t w_{x_j} + a(s,x) w_t^2 + c(s,x) w_t w \right] dxds, \\
& \int_0^t \int_{\mathbb{R}^n} f w_t \, dxds \\
= \; & \int_0^t \int_{\mathbb{R}^n} w_t w_{tt} \, dxds - \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_i x_j} w_t \, dxds \\
& + \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} b_j(s,x) w_t w_{x_j} \, dxds + \int_0^t \int_{\mathbb{R}^n} a(s,x) w_t^2 \, dxds + \int_0^t \int_{\mathbb{R}^n} c(s,x) w_t w \, dxds,
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}^n} w_t w_{tt} \, dxds - \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_i x_j} w_t \, dxds \\
= \; & \int_0^t \int_{\mathbb{R}^n} f w_t \, dxds - \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} b_j(s,x) w_t w_{x_j} \, dxds \\
& - \int_0^t \int_{\mathbb{R}^n} a(s,x) w_t^2 \, dxds - \int_0^t \int_{\mathbb{R}^n} c(s,x) w_t w \, dxds, \qquad (3.4)
\end{aligned}
$$

Now we will individually look at each term of equation (3.4) considering the definition of **energy**:

$$
E(t) := \frac{1}{2} \int_{\mathbb{R}^n} \left[w_t^2 + \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_i} w_{x_j} \right] dx \geq 0.
$$

The *first term* of equation (3.4) can be written in the following way:

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^n} w_t(s,x) w_{tt}(s,x)\, dx\, ds &= \frac{1}{2}\int_0^t \int_{\mathbb{R}^n} \frac{\partial}{\partial t} w_t^2(s,x)\, dx\, ds, \\
&= \frac{1}{2}\int_0^t \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} w_t^2(s,x)\, dx\right) ds, \\
&= \frac{1}{2}\int_{\mathbb{R}^n} w_t^2(t,x)\, dx - \frac{1}{2}\int_{\mathbb{R}^n} w_t^2(0,x)\, dx \qquad (3.5)
\end{aligned}$$

The *second term* of equation (3.4) can be written in the following way:

$$\begin{aligned}
& -\sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_i x_j} w_t \, dxds \\
= \; & \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} a_{ij}(s,x) w_t(s,x)\right) w_{x_i}\, dxds \\
= \; & \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} a_{ij}(t,x) w_{x_j t}(t,x) w_{x_i}\, dxds + \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} a_{ij}(s,x)\right) w_t(t,x) w_{x_i}\, dxds \\
= \; & \frac{1}{2}\frac{\partial}{\partial t} \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_j} w_{x_i}\, dxds \\
& -\frac{1}{2}\sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} a_{ij}(s,x)\right) w_{x_j} w_{x_i}\, dxds + \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} a_{ij}(s,x)\right) w_t w_{x_i}\, dxds \\
= \; & \frac{1}{2}\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} a_{ij}(t,x) w_{x_j}(s,x) w_{x_i}(t,x)\, dxds - \frac{1}{2}\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} a_{ij}(0,x) w_{x_j}(0,x) w_{x_i}(0,x)\, dxds \\
& -\frac{1}{2}\sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} a_{ij}(s,x)\right) w_{x_j} w_{x_i}\, dxds + \sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} a_{ij}(s,x)\right) w_t w_{x_i}\, dxds
\end{aligned}$$

Here

$$\left|\frac{1}{2}\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} a_{ij}(0,x) w_{x_j}(0,x) w_{x_i}(0,x)\, dxds\right| \leq \frac{1}{2}E(0) \qquad (3.6)$$

$$\left|\frac{1}{2}\sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} a_{ij}(s,x)\right) w_{x_j} w_{x_i}\, dxds\right| \leq C_\alpha \int_0^t E(s)\, ds \qquad (3.7)$$

$$\left|\sum_{i,j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} a_{ij}(s,x)\right) w_t w_{x_i}\, dxds\right| \leq C_\beta \int_0^t E(s)\, ds \qquad (3.8)$$

Hence

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} w_t w_{tt}\, dxds - \sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_i x_j} w_t\, dxds \\
= \;& \frac{1}{2}\int_{\mathbb{R}^n} w_t^2\, dx + \frac{1}{2}\sum_{i,j=1}^n \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_j}(s,x) w_{x_i}(s,x)\, dxds + T(t) \\
= \;& E(t) + T(t) \qquad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
T(t) \;:=\; & -\frac{1}{2}\int_{\mathbb{R}^n} w_t^2(0,x)\, dx - \frac{1}{2}\sum_{i,j=1}^n \int_{\mathbb{R}^n} a_{ij}(0,x) w_{x_j}(0,x) w_{x_i}(0,x)\, dxds \\
& + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} a_{ij}(t,x)\right) w_{x_j} w_{x_i}\, dxdt \\
& + \sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} a_{ij}(t,x)\right) w_t w_{x_i}\, dxdt \\
|T(t)| \;\leq\; & c_1 E(0) + C_1 \int_0^t E(s)\, ds \qquad (3.10)
\end{aligned}$$

The *first term* of the right hand side of equation (3.4) is bounded the following way:

$$\begin{aligned}
& \left| \int_0^t \int_{\mathbb{R}^n} f(s,x) w_t(s,x)\, dxds \right| \\
\leq \;& \int_0^t \int_{\mathbb{R}^n} |f(s,x)||w_t(s,x)|\, dxds \\
\leq \;& \int_0^t \int_{\mathbb{R}^n} \frac{1}{2}\left(|f(s,x)|^2 + |w_t(s,x)|^2\right) dxds \\
\leq \;& \int_0^t \int_{\mathbb{R}^n} \frac{1}{2}|f(s,x)|^2\, dxds + \frac{1}{2}\int_0^t E(s)\, ds \qquad (3.11)
\end{aligned}$$

The *second term* of the right hand side of equation (3.4) is bounded the following way:

$$
\begin{aligned}
& \left| \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} b_j(s,x) w_t w_{x_j} \, dxds \right| \\
\leq \; & \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} |b_j(s,x)||w_t||w_{x_j}| \, dxds \\
\leq \; & \max_{t,x} |b_j(t,x)| \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} |w_t||w_{x_j}| \, dxds \\
\leq \; & C \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} \left(\frac{1}{2}|w_t|^2 + \frac{1}{2}|w_{x_j}|^2 \right) dxds \\
\leq \; & C \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} \left(\frac{1}{2}|w_t|^2 + \frac{1}{2}|w_{x_j}|^2 \right) dxds \\
\leq \; & \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^{n} \left(\frac{1}{2}|w_t|^2 + \frac{C}{2\sigma_0} a_{ij}(s,x) w_{x_i} w_{x_j} \right) dxds \\
\leq \; & C_2 \int_0^t E(s) \, ds
\end{aligned}
\tag{3.12}
$$

The *third term* of the right hand side of equation (3.4) is bounded the following way:

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} a(s,x) w_t^2 \, dxds \\
&\leq \left| \int_0^t \int_{\mathbb{R}^n} a(s,x) w_t^2 \, dxds \right|, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \left| a(s,x) w_t^2 \right| \, dxds, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \underbrace{|a(s,x)|}_{\leq A} \left| w_t^2 \right| \, dxds, \\
&\leq A \int_0^t \underbrace{\int_{\mathbb{R}^n} w_t^2 dx}_{\leq E(s)} ds, \\
&\leq A \int_0^t E(s) \, ds.
\end{aligned} \tag{3.13}
$$

The *fourth term* of the right hand side of equation (3.4) is bounded the following way:

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^n} c(s,x) w_t w \, dx \, ds \right| \\
&\leq \left| \int_0^t \int_{\mathbb{R}^n} c(s,x) w w_t \, dxds \right| \\
&\leq \int_0^t \int_{\mathbb{R}^n} |c(s,x) w w_t| \, dxds, \\
&= \int_0^t \int_{\mathbb{R}^n} \underbrace{|c(s,x)|}_{\leq C} \quad \underbrace{|w|\,|w_t|}_{\leq \frac{1}{2}\left(w^2 + w_t^2\right)} \quad dxds, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \frac{C}{2} \left(w^2 + w_t^2 \right) \, dxds, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \left(\frac{C}{2} w_t^2 + \frac{C}{2} w^2 \right) \, dxds, \\
&\leq C \int_0^t \underbrace{\frac{1}{2} \int_{\mathbb{R}^n} w_t^2 dx}_{\leq E(s)} ds + \frac{C}{2} \int_0^t \int_{\mathbb{R}^n} w^2 dxds
\end{aligned} \tag{3.14}
$$

To estimate the second term of equation (3.14), we use

$$\begin{aligned}
|w(t,x)|^2 &= |w(0,x)+\int_0^t w_t(s,x)\,ds||w(0,x)+\int_0^t w_t(s_1,x)\,ds_1| \\
&\leq 2|w(0,x)|^2+2\int_0^t |w_t(s,x)|\,ds\int_0^t |w_t(s_1,x)|\,ds_1 \\
&= 2|w(0,x)|^2+2\int_0^t\int_0^t ds\,ds_1|w_t(s,x)||w_t(s_1,x)| \\
&\leq 2|w(0,x)|^2+2\int_0^t\int_0^t ds\,ds_1\frac{1}{2}\left(|w_t(s,x)|^2+|w_t(s_1,x)|^2\right) \\
&\leq 2|w(0,x)|^2+\int_0^t\int_0^t ds\,ds_1|w_t(s,x)|^2 \\
&\leq 2|w(0,x)|^2+t\int_0^t |w_t(s,x)|^2\,ds \\
&\leq 2|w(0,x)|^2+C\int_0^t |w_t(s,x)|^2\,ds
\end{aligned}$$

Since

$$\int_{\mathbb{R}^n}|w(0,x)|^2dx \leq c\int_{\mathbb{R}^n} a_{ij}(0,x)w_{x_j}(0,x)w_{x_i}(0,x)\,dx$$

it follows

$$\left|\int_0^t\int_{\mathbb{R}^n} c(s,x)w_t(s,x)w(s,x)\ dx\ dt\right| \leq c_4E(0)+C_4\int_0^t E(s)\ ds. \tag{3.15}$$

Finally, we substitute expressions (3.5) through (3.14) into equation (3.4) to obtain the following inequality:

$$E(t)\leq cE(0)+\int_0^t\int_{\mathbb{R}^n}\frac{1}{2}|f(s,x)|^2\,dxds+C\int_0^t E(s)\ ds \text{ for all } t\in[0,T]. \tag{3.16}$$

For the last step, we can use **Gronwall's Lemma**:

Lemma 6.3.6 [Gronwall's Lemma]

If φ, κ, and f are non–negative, f is increasing and

$$\varphi(t) \le f(t) + \int_0^t \varphi(\tau)\kappa(\tau)d\tau, \quad 0 \le t \le \tau, \tag{3.17}$$

then

$$\varphi(t) \le f(t) + \int_0^t \kappa(\tau) f(\tau) \exp\left(\int_\tau^t \kappa(s)ds\right)d\tau, \quad 0 \le t \le T.$$

Proof. Assume the hypothesis and the following inequality:

$$\varphi(t) \le f(t) + \int_0^t \varphi(\tau)\kappa(\tau)d\tau, \quad 0 \le t \le \tau.$$

Now, we define $Z(t) := \int_0^t \varphi(\tau)\kappa(\tau)d\tau$. And we have that $Z'(t) = \kappa(t)\varphi(t)$ while $Z(0) = 0$. Next, we multiply both sides of the inequality above, by the non–negative function $\kappa(t)$ to obtain:

$$\kappa(t)\varphi(t) \le \kappa(t)f(t) + \kappa(t)\int_0^t \varphi(\tau)\kappa(\tau)d\tau,$$

$$Z'(t) \le \kappa(t)f(t) + \kappa(t)Z(t),$$

$$Z'(t) - \kappa(t)Z(t) \le \kappa(t)f(t).$$

Next, we multiply both sides by $\exp\left(-\int_0^t \kappa(s)ds\right)$ to obtain:

$$
\begin{aligned}
\left(Z'(t)-\kappa(t)Z(t)\right)\exp\left(-\int_0^t \kappa(s)ds\right) &\leq \kappa(t)f(t)\exp\left(-\int_0^t \kappa(s)ds\right),\\
\frac{d}{dt}\left(Z(t)\exp\left(-\int_0^t \kappa(s)ds\right)\right) &\leq \kappa(t)f(t)\exp\left(-\int_0^t \kappa(s)ds\right),\\
\int_0^t \frac{d}{d\tau}\left(Z(\tau)\exp\left(-\int_0^\tau \kappa(s)ds\right)\right)d\tau &\leq \int_0^t \kappa(\tau)f(\tau)\exp\left(-\int_0^\tau \kappa(s)ds\right)d\tau,\\
Z(t)\exp\left(-\int_0^t \kappa(s)ds\right)-Z(0) &\leq \int_0^t \kappa(\tau)f(\tau)\exp\left(-\int_0^\tau \kappa(s)ds\right)d\tau,\\
Z(t)\exp\left(-\int_0^t \kappa(s)ds\right) &\leq Z(0)+\int_0^t \kappa(\tau)f(\tau)\exp\left(-\int_0^\tau \kappa(s)ds\right)d\tau.
\end{aligned}
$$

Finally, we multiply both sides by $\exp\left(\int_0^t \kappa(s)ds\right)$ to obtain:

$$
Z(t) \leq \exp\left(\int_0^t \kappa(s)ds\right)\left[Z(0)+\int_0^t \kappa(\tau)f(\tau)\exp\left(-\int_0^\tau \kappa(s)ds\right)d\tau\right],
$$

Because $Z(t) := \int_0^t \varphi(\tau)\kappa(\tau)d\tau$ and $Z(0) := \int_0^0 \varphi(\tau)\kappa(\tau)d\tau = 0$, then

$$
\begin{aligned}
\int_0^t \varphi(\tau)\kappa(\tau)d\tau &\leq \exp\left(\int_0^t \kappa(s)ds\right)\int_0^t \kappa(\tau)f(\tau)\exp\left(-\int_0^\tau \kappa(s)ds\right)d\tau,\\
&\leq \exp\left(\int_0^t \kappa(s)ds\right)\int_0^t \kappa(\tau)f(\tau)\exp\left(\int_\tau^0 \kappa(s)ds\right)d\tau,\\
&\leq \int_0^t \kappa(\tau)f(\tau)\exp\left(\int_\tau^0 \kappa(s)ds+\int_0^t \kappa(s)ds\right)d\tau,\\
&\leq \int_0^t \kappa(\tau)f(\tau)\exp\left(\int_\tau^t \kappa(s)ds\right)d\tau.
\end{aligned}
$$

Finally, we substitute the inequality above into (3.17) to obtain the desired inequality:

$$
\begin{aligned}
&\varphi(t) \leq f(t)+\int_0^t \varphi(\tau)\kappa(\tau)d\tau,\\
&\varphi(t) \leq f(t)+\int_0^t \kappa(\tau)f(\tau)\exp\left(\int_\tau^t \kappa(s)ds\right)d\tau.
\end{aligned}
$$

□

Since $e^{-ct} \geq 0$, $\forall t > 0$, then $\int_0^t E(s)ds \leq 0$. But we know that $E(t)$ is non–negative, so $\int_0^t E(s)ds \geq 0$. Therefore, $\int_0^t E(s)ds = 0$, which implies that $E(t) \equiv 0$. Then, we have that:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} \left[w_t^2 + b^2(t)|\nabla w|^2\right] dx = 0, \quad \forall t \in [0,T].$$

Also, since we know that $b(t)$ is positive and the squares of the first derivatives of w are non–negative, then the equation above implies that all of the first derivatives of w are identically zero. That is

$$w_t(t,x) \equiv w_{x_1}(t,x) \equiv \cdots \equiv w_{x_n}(t,x) \equiv 0, \quad t \in [0,T], \quad x \in \mathbb{R}^n.$$

Therefore, $w(t,x) = c \in \mathbb{R}$. Since the initial data is $w(0,x) = 0$, then $w(t,x) \equiv 0$. Therefore,

$$u(t,x) = v(t,x).$$

□

Theorem 8 (Local Uniqueness Theorem for Linear Equations)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and $0 \in \Omega$. Consider in $\Omega \times [-T,T]$ the solution w to the equation*

$$w_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)w_{x_i})_{x_j} + \sum_{j=1}^{n} b_j(t,x)w_{x_j} + c(t,x)w_t + d(t,x)w = 0, \tag{3.18}$$

where $a_{ij} \in C^1(\Omega \times [-T,T])$, $b_j(t,x) \in C(\Omega \times [-T,T])$, $a(t,x) \in C(\Omega \times [-T,T])$, $c(t,x) \in C(\Omega \times [-T,T])$.

Then there exist a number $T_0 > 0$ and a domain $\Omega_0 \subseteq \Omega$ with $0 \in \Omega_0$, such that any solution $w \in C^2(\Omega \times [-T,T])$ of (3.18) with the property supp $w \subseteq \Omega \times [0,T]$ vanishes in $\Omega_0 \times [0,T_0]$.

Proof. We consider the Holmgren's transform

$$y = x, \quad \tau = t + |x|^2.$$

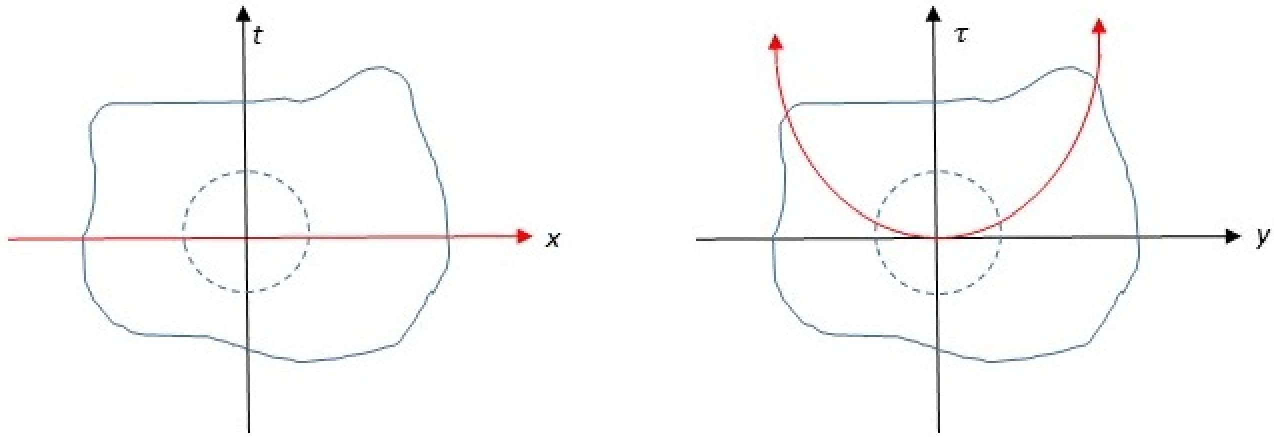

Figure 3.1: Holmgren's Transform

First, we calculate the partial derivatives.

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{w}(\tau,y) &= \sum_{j=1}^{n}\underbrace{\frac{\partial y_j}{\partial t}}_{=0}\frac{\partial}{\partial y_j}\tilde{w}+\underbrace{\frac{\partial \tau}{\partial t}}_{=1}\frac{\partial}{\partial \tau}\tilde{w}=\frac{\partial}{\partial \tau}\tilde{w}=\tilde{w}_{\tau},\\
\frac{\partial^2}{\partial t^2}\tilde{w}(\tau,y) &= \frac{\partial}{\partial t}\tilde{w}_{\tau}\sum_{j=1}^{n}\underbrace{\frac{\partial y_j}{\partial t}}_{=0}\frac{\partial}{\partial y_j}\tilde{w}_{\tau}+\underbrace{\frac{\partial \tau}{\partial t}}_{=1}\frac{\partial}{\partial \tau}\tilde{w}_{\tau}=\frac{\partial}{\partial \tau}\tilde{w}_{\tau}=\tilde{w}_{\tau\tau},\\
\frac{\partial}{\partial x_i}\tilde{w}(\tau,y) &= \sum_{j=1}^{n}\underbrace{\frac{\partial y_j}{\partial x_i}}_{=\sigma_{ij}}\frac{\partial}{\partial y_j}\tilde{w}+\underbrace{\frac{\partial \tau}{\partial x_i}}_{=2x_i}\frac{\partial}{\partial \tau}\tilde{w}=\underbrace{\frac{\partial y_i}{\partial x_i}}_{=1}\frac{\partial}{\partial y_i}\tilde{w}+2x_i\frac{\partial}{\partial \tau}\tilde{w},\\
&= \tilde{w}_{y_i}+2y_i\tilde{w}_{\tau},\\
\frac{\partial^2}{\partial x_j x_i}\tilde{w}(\tau,y) &= \frac{\partial}{\partial x_j}\tilde{w}_{x_i}=\sum_{l=1}^{n}\underbrace{\frac{\partial y_l}{\partial x_j}}_{=\sigma_{jl}}\frac{\partial}{\partial y_l}\tilde{w}_{x_i}+\underbrace{\frac{\partial \tau}{\partial x_j}}_{=2x_j}\frac{\partial}{\partial \tau}\tilde{w}_{x_i}=\underbrace{\frac{\partial y_j}{\partial x_j}}_{=1}\frac{\partial}{\partial y_j}\tilde{w}_{x_i}+2x_j\frac{\partial}{\partial \tau}\tilde{w}_{x_i},\\
&= \frac{\partial}{\partial y_j}\left[\tilde{w}_{y_i}+2y_i\tilde{w}_{\tau}\right]+2x_j\frac{\partial}{\partial \tau}\left[\tilde{w}_{y_i}+2y_i\tilde{w}_{\tau}\right],\\
&= \tilde{w}_{y_iy_j}+4y_jy_i\tilde{w}_{\tau\tau}+2y_i\tilde{w}_{\tau y_j}+2y_j\tilde{w}_{\tau y_i}.
\end{aligned}
$$

Therefore, $\tilde{w}(\tau,y)$ is a solution for the equation:

$$\begin{aligned}\tilde{w}_{\tau\tau} - &\sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)\left[\tilde{w}_{y_i y_j} + 4y_j y_i \tilde{w}_{\tau\tau} + 2y_i \tilde{w}_{\tau y_j} + 2y_j \tilde{w}_{\tau y_i}\right] \\ &+ \sum_{j=1}^{n} \hat{b}_j(\tau,y)\left[\tilde{w}_{y_j} + 2y_j \tilde{w}_\tau\right] + \hat{c}(\tau,y)\tilde{w}_\tau + \hat{d}(\tau,y)\tilde{w}(\tau,y) = 0,\end{aligned}$$

where

$$\begin{aligned}\hat{a}_{ij}(\tau,y) &:= a_{ij}(t,x),\\ \hat{b}_j(\tau,y) &:= b_j(t,x),\\ \hat{c}(\tau,y) &:= c(t,x),\\ \hat{d}(\tau,y) &:= d(t,x).\end{aligned}$$

Note that we can re–arrange the PDE above as:

$$\begin{aligned}&\left[1 - \sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)4y_j y_i\right]\tilde{w}_{\tau\tau} - \sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)\tilde{w}_{y_i y_j} + \sum_{j=1}^{n} \hat{b}_j(\tau,y)\tilde{w}_{y_j} \\ &\quad + \left[\hat{c}(\tau,y) + 2\sum_{j=1}^{n} \hat{b}_j(\tau,y)y_j\right]\tilde{w}_\tau + \hat{d}(\tau,y)\tilde{w}(\tau,y) + 4\sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)y_i \tilde{w}_{\tau y_j} = 0.\end{aligned}$$

Now, we define the following functions:

$$\begin{aligned}\tilde{a}_{ij}(\tau,y) &:= \frac{\hat{a}_{ij}(\tau,y)}{1 - \sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)4y_j y_i},\\ \tilde{b}_j(\tau,y) &:= \frac{\hat{b}_j(\tau,y)}{1 - \sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)4y_j y_i},\\ \tilde{c}(\tau,y) &:= \frac{\hat{c}(\tau,y) + 2\sum_{j=1}^{n} \hat{b}_j(\tau,y)y_j}{1 - \sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)4y_j y_i},\\ \tilde{d}(\tau,y) &:= \frac{\hat{d}(\tau,y)}{1 - \sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)4y_j y_i},\\ \tilde{f}_j(\tau,y) &:= \frac{4\sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)y_i}{1 - \sum_{i,j=1}^{n} \hat{a}_{ij}(\tau,y)4y_j y_i}.\end{aligned}$$

Here, $1-\sum_{i,j=1}^{n}\hat{a}_{ij}(\tau,y)4y_jy_i$ is different from zero if $|y|$ is small. That is, $1-\sum_{i,j=1}^{n}\hat{a}_{ij}(\tau,y)4y_jy_i \neq 0$ in some neighborhood of the origin. Therefore, $\tilde{w}(\tau,y)$ is a solution to the equation:

$$\tilde{w}_{\tau\tau}-\sum_{i,j=1}^{n}\tilde{a}_{ij}(\tau,y)\tilde{w}_{y_iy_j}+\sum_{j=1}^{n}\tilde{b}_j(\tau,y)\tilde{w}_{y_j}+\tilde{c}(\tau,y)\tilde{w}_\tau$$
$$+\tilde{d}(\tau,y)\tilde{w}(\tau,y)+\tilde{f}_j(\tau,y)\tilde{w}_{\tau y_j}=0, \tag{3.19}$$

where $\tilde{a}_{ij}(\tau,y),\tilde{b}_j(\tau,y),\tilde{c}(\tau,y),\tilde{d}(\tau,y),\tilde{f}_j(\tau,y)$ are smooth functions in some neighborhood of the origin.

Now, we apply a second transformation to reduce to canonical form:

$$t=\tau,$$
$$z_i=f^i(y,\tau).$$

First, we define the (time) partial derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial \tau} &= \underbrace{\frac{\partial t}{\partial \tau}}_{=1}\frac{\partial}{\partial t} + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial}{\partial z_k} = \frac{\partial}{\partial t} + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial}{\partial z_k},\\
\frac{\partial^2}{\partial \tau^2} &= \left(\frac{\partial}{\partial t} + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial}{\partial z_k}\right)\left(\frac{\partial}{\partial t} + \sum_{l=1}^{n} \frac{\partial z_l}{\partial \tau}\frac{\partial}{\partial z_l}\right),\\
&= \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\left(\sum_{l=1}^{n} \frac{\partial z_l}{\partial \tau}\frac{\partial}{\partial z_l}\right) + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial}{\partial z_k}\frac{\partial}{\partial t} + \sum_{k,j=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial}{\partial z_k}\left(\frac{\partial z_l}{\partial \tau}\frac{\partial}{\partial z_l}\right),\\
&= \frac{\partial^2}{\partial t^2} + \sum_{l=1}^{n}\left(\frac{\partial}{\partial t}\frac{\partial z_l}{\partial \tau}\frac{\partial}{\partial z_l} + \frac{\partial z_l}{\partial \tau}\frac{\partial^2}{\partial t \partial z_l}\right) + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial^2}{\partial z_k \partial t} + \sum_{k,j=1}^{n} \frac{\partial z_k}{\partial \tau}\left(\frac{\partial^2 z_l}{\partial z_k \partial \tau}\frac{\partial}{\partial z_l} + \frac{\partial z_l}{\partial \tau}\frac{\partial^2}{\partial z_k \partial z_l}\right),\\
&= \frac{\partial^2}{\partial t^2} + \sum_{l=1}^{n}\left(\frac{\partial \tau}{\partial t}\frac{\partial^2 z_l}{\partial \tau^2}\frac{\partial}{\partial z_l} + \sum_{i=1}^{n}\frac{\partial y_i}{\partial t}\frac{\partial^2 z_l}{\partial y_i \partial \tau}\frac{\partial}{\partial z_l} + \frac{\partial z_l}{\partial \tau}\frac{\partial^2}{\partial t \partial z_l}\right) + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial^2}{\partial z_k \partial t}\\
&\quad + \sum_{k,j=1}^{n} \frac{\partial z_k}{\partial \tau}\left(\frac{\partial^2 z_l}{\partial z_k \partial \tau}\frac{\partial}{\partial z_l} + \frac{\partial z_l}{\partial \tau}\frac{\partial^2}{\partial z_k \partial z_l}\right),\\
&= \frac{\partial^2}{\partial t^2} + \sum_{k,l=1}^{n} \frac{\partial z_k}{\partial \tau}\frac{\partial z_l}{\partial \tau}\frac{\partial^2}{\partial z_k \partial z_l} + \sum_{l=1}^{n}\left(\frac{\partial^2 z_l}{\partial \tau^2} + \sum_{k=1}^{n}\frac{\partial z_k}{\partial \tau}\frac{\partial^2 z_l}{\partial z_k \partial \tau} + \sum_{i=1}^{n}\frac{\partial^2 z_l}{\partial y_i \partial \tau}\right)\frac{\partial}{\partial z_l}\\
&\quad + 2\sum_{l=1}^{n} \frac{\partial z_l}{\partial \tau}\frac{\partial^2}{\partial z_l \partial t},
\end{aligned}
$$

Now, we define the (direction) derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial y_i} &= \underbrace{\frac{\partial t}{\partial y_i}}_{=0}\frac{\partial}{\partial t} + \sum_{l=1}^{n} \frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial z_l} = \sum_{l=1}^{n} \frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial z_l},\\
\frac{\partial^2}{\partial y_j \partial y_i} &= \frac{\partial}{\partial y_j}\left[\sum_{l=1}^{n} \frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial z_l}\right] = \sum_{l=1}^{n}\left(\frac{\partial}{\partial y_j}\frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial z_l} + \frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial y_j}\frac{\partial}{\partial z_l}\right),\\
&= \sum_{l=1}^{n} \frac{\partial^2 z_l}{\partial y_j \partial y_i}\frac{\partial}{\partial z_l} + \sum_{l=1}^{n} \frac{\partial z_l}{\partial y_i}\left(\underbrace{\frac{\partial t}{\partial y_j}}_{=0}\frac{\partial^2}{\partial t \partial z_l} + \sum_{k=1}^{n}\frac{\partial z_k}{\partial y_j}\frac{\partial^2}{\partial z_k \partial z_l}\right),\\
&= \sum_{l=1}^{n} \frac{\partial^2 z_l}{\partial y_j \partial y_i}\frac{\partial}{\partial z_l} + \sum_{l,k=1}^{n} \frac{\partial z_l}{\partial y_i}\frac{\partial z_k}{\partial y_j}\frac{\partial^2}{\partial z_k \partial z_l}.
\end{aligned}
$$

Finally, we define the mixed partial derivative:

$$\begin{aligned}
\frac{\partial^2}{\partial\tau\partial y_i} &= \frac{\partial}{\partial\tau}\left[\sum_{l=1}^{n}\frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial z_l}\right],\\
&= \sum_{l=1}^{n}\left(\frac{\partial}{\partial\tau}\frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial z_l}+\frac{\partial z_l}{\partial y_i}\frac{\partial}{\partial\tau}\frac{\partial}{\partial z_l}\right),\\
&= \sum_{l=1}^{n}\frac{\partial^2 z_l}{\partial\tau\partial y_i}\frac{\partial}{\partial z_l}+\sum_{l=1}^{n}\frac{\partial z_l}{\partial y_i}\left(\frac{\partial t}{\partial\tau}\frac{\partial^2}{\partial t\partial z_l}+\sum_{k=1}^{n}\frac{\partial z_l}{\partial z_k\partial\tau}\frac{\partial^2}{\partial z_k\partial z_l}\right),\\
&= \sum_{l=1}^{n}\frac{\partial^2 z_l}{\partial\tau\partial y_i}\frac{\partial}{\partial z_l}+\sum_{l=1}^{n}\frac{\partial z_l}{\partial y_i}\left(\underbrace{\frac{\partial t}{\partial\tau}}_{=1}\frac{\partial^2}{\partial t\partial z_l}+\sum_{k=1}^{n}\frac{\partial z_l}{\partial z_k\partial\tau}\frac{\partial^2}{\partial z_k\partial z_l}\right),\\
&= \sum_{l=1}^{n}\frac{\partial^2 z_l}{\partial\tau\partial y_i}\frac{\partial}{\partial z_l}+\sum_{l=1}^{n}\frac{\partial z_l}{\partial y_i}\left(\frac{\partial^2}{\partial t\partial z_l}+\sum_{k=1}^{n}\frac{\partial z_l}{\partial z_k\partial\tau}\frac{\partial^2}{\partial z_k\partial z_l}\right),\\
&= \sum_{l=1}^{n}\frac{\partial^2 z_l}{\partial\tau\partial y_i}\frac{\partial}{\partial z_l}+\sum_{l=1}^{n}\frac{\partial z_l}{\partial y_i}\frac{\partial^2}{\partial t\partial z_l}+\sum_{l,k=1}^{n}\frac{\partial z_l}{\partial y_i}\frac{\partial z_l}{\partial z_k\partial\tau}\frac{\partial^2}{\partial z_k\partial z_l}.
\end{aligned}$$

Then, the new partial derivatives are:

$$\begin{aligned}
w_{\tau\tau} &= w_{tt}+\sum_{k,l=1}^{n}\left(\frac{\partial z_k}{\partial\tau}\frac{\partial z_l}{\partial\tau}\right)w_{z_k z_l}+\sum_{l=1}^{n}\left(\frac{\partial^2 z_l}{\partial\tau^2}+\sum_{k=1}^{n}\frac{\partial z_k}{\partial\tau}\frac{\partial^2 z_l}{\partial z_k\partial\tau}+\sum_{i=1}^{n}\frac{\partial^2 z_l}{\partial y_i\partial\tau}\right)w_{z_l}\\
&\quad+\sum_{l=1}^{n}\left(2\frac{\partial z_l}{\partial\tau}\right)w_{z_l t},\\
w_{y_i y_j} &= \sum_{l=1}^{n}\left(\frac{\partial^2 z_l}{\partial y_j\partial y_i}\right)w_{z_l}+\sum_{l,k=1}^{n}\left(\frac{\partial z_l}{\partial y_i}\frac{\partial z_k}{\partial y_j}\right)w_{z_k z_l},\\
w_{y_j} &= \sum_{l=1}^{n}\left(\frac{\partial z_l}{\partial y_j}\right)w_{z_l},\\
w_\tau &= w_t+\sum_{k=1}^{n}\left(\frac{\partial z_k}{\partial\tau}\right)w_{z_k},\\
w_{\tau y_j} &= \sum_{l=1}^{n}\left(\frac{\partial^2 z_l}{\partial\tau\partial y_j}\right)w_{z_l}+\sum_{l=1}^{n}\left(\frac{\partial z_l}{\partial y_j}\right)w_{t z_l}+\sum_{l,k=1}^{n}\left(\frac{\partial z_l}{\partial y_j}\frac{\partial z_l}{\partial z_k\partial\tau}\right)w_{z_k z_l}.
\end{aligned}$$

Substituting these derivatives into equation (3.19), we obtain:

$$
\begin{aligned}
& w_{tt} + \sum_{k,l=1}^{n} \left(\frac{\partial z_k}{\partial \tau} \frac{\partial z_l}{\partial \tau} \right) w_{z_k z_l} + \sum_{l=1}^{n} \left(\frac{\partial^2 z_l}{\partial \tau^2} + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau} \frac{\partial^2 z_l}{\partial z_k \partial \tau} + \sum_{i=1}^{n} \frac{\partial^2 z_l}{\partial y_i \partial \tau} \right) w_{z_l} \\
& + \sum_{l=1}^{n} \left(2 \frac{\partial z_l}{\partial \tau} \right) w_{z_l t} - \sum_{i,j=1}^{n} a_{ij}(t,z) \left(\sum_{l=1}^{n} \left(\frac{\partial^2 z_l}{\partial y_j \partial y_i} \right) w_{z_l} + \sum_{l,k=1}^{n} \left(\frac{\partial z_l}{\partial y_i} \frac{\partial z_k}{\partial y_j} \right) w_{z_k z_l} \right) \\
& + \sum_{j=1}^{n} b_j(t,z) \sum_{l=1}^{n} \left(\frac{\partial z_l}{\partial y_j} \right) w_{z_l} + c(t,z) \left(w_t + \sum_{k=1}^{n} \left(\frac{\partial z_k}{\partial \tau} \right) w_{z_k} \right) + d(t,z) w \\
& + \sum_{j=1}^{n} f_j(t,z) \left(\sum_{l=1}^{n} \left(\frac{\partial^2 z_l}{\partial \tau \partial y_j} \right) w_{z_l} + \sum_{l=1}^{n} \left(\frac{\partial z_l}{\partial y_j} \right) w_{t z_l} + \sum_{l,k=1}^{n} \left(\frac{\partial z_l}{\partial y_j} \frac{\partial z_l}{\partial z_k \partial \tau} \right) w_{z_k z_l} \right) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
a_{ij}(t,z) &:= \tilde{a}_{ij}(\tau,y), \\
b_j(t,z) &:= \tilde{b}_j(\tau,y), \\
c(t,z) &:= \tilde{c}(\tau,y), \\
d(t,z) &:= \tilde{d}(\tau,y), \\
f_j(t,z) &:= \tilde{f}_j(\tau,y)
\end{aligned}
$$

Once we group similar terms, we obtain the following equation:

$$
\begin{aligned}
& w_{tt} + \sum_{l,k=1}^{n} \left[\frac{\partial z_k}{\partial \tau} \frac{\partial z_l}{\partial \tau} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial z_l}{\partial y_i} \frac{\partial z_k}{\partial y_j} + \sum_{j=1}^{n} f_j \frac{\partial z_l}{\partial y_j} \frac{\partial z_l}{\partial z_k \partial \tau} \right] w_{z_l z_k} \\
& + \sum_{l=1}^{n} \left[\frac{\partial^2 z_l}{\partial \tau^2} + \sum_{k=1}^{n} \frac{\partial z_k}{\partial \tau} \frac{\partial^2 z_l}{\partial z_k \partial \tau} + \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 z_l}{\partial \tau \partial y_i} + \sum_{j=1}^{n} b_j \frac{\partial z_l}{\partial \tau} + c \frac{\partial z_l}{\partial \tau} + \sum_{j=1}^{n} f_j \frac{\partial^2 z_l}{\partial \tau \partial y_j} \right] w_{z_l} \\
& + c w_t + d w + \sum_{l=1}^{n} \left[2 \frac{\partial z_l}{\partial \tau} + \sum_{j=1}^{n} f_j \frac{\partial z_l}{\partial y_j} \right] w_{t z_l} = 0.
\end{aligned} \tag{3.20}
$$

Now, we take a look at the coefficient for the mixed partial derivative. That is,

$$
2 \frac{\partial z_l}{\partial \tau} + \sum_{j=1}^{n} f_j \frac{\partial z_l}{\partial y_j}, \text{ where } l = 1, \ldots, n.
$$

Consider the system of linear partial differential equations:

$$2\frac{\partial z_l}{\partial \tau} + \sum_{j=1}^{n} f_j \frac{\partial z_l}{\partial y_j} = 0, \text{ for } l = 1, \ldots, n,$$

with the initial data

$$z_l(0, y) = y_l, \text{ for } l = 1, \ldots, n.$$

Because we know that the solution to this system exists, we can pick the functions $z_l(\tau, y)$ to be a solution to this initial value problem. Then, the equation (3.20) becomes

$$w_{tt} + \sum_{l,k=1}^{n} A_{lk}(t,z) w_{z_l z_k} + \sum_{l=1}^{n} B_j(t,z) w_{z_l} + C(t,z) w_t + D(t,z) w = 0. \tag{3.21}$$

Because the functions $A_{ij}(t,z), B_j(t,z), C(t,z)$, and $D(t,z)$ satisfy the hypothesis from Theorem 7 and the transformed equation (3.21) is hyperbolic with respect to time, we can apply the energy estimates from this theorem.

With this, the proof is complete.

□

3.1.2 Energy Estimates and Uniqueness for System of Linear Equations

Consider the following system of hyperbolic equations of second order for the vector function $W = {}^{tr}(w^1, w^2, \ldots, w^m)$

$$w^k_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x) w^k_{x_i x_j} + \sum_{l=1}^{m} \sum_{j=1}^{n} b_{klj}(t,x) w^l_{x_j} + \sum_{l=1}^{m} c_{kl}(t,x) w^l_t + \sum_{l=1}^{m} d_{kl}(t,x) w^l = f^k, \quad k = 1, 2 \ldots, m$$

with initial data $w^k(0,x) = \varphi_0^k(x), w_t^k(0,x) = \varphi_1^k(x)$ for all $x \in \mathbb{R}^n$ and $k = 1,2\ldots,m$. This system can be also written in the matrix form

$$W_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x)W_{x_ix_j} + \sum_{j=1}^{n} B_j(t,x)W_{x_j} + C(t,x)W_t + D(t,x)W = F,$$

We define the energy as follows

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^n} \sum_{l=1}^{m} \left[(w_t^l(t,x))^2 + \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_i}^l(t,x) w_{x_j}^l(t,x) \right] dx \geq 0.$$

Theorem 9 (Energy Estimates for System of Linear Equations)**.** *Assume that the coefficient functions $a_{ij}(t,x)$ are such that*

$$0 < \sigma_0|\xi|^2 \leq a_{ij}(t,x)\xi_i\xi_j \leq \sigma_1|\xi|^2, \quad \forall \xi \neq 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

and their first derivatives are bounded, the matrix-valued functions $B_j(t,x)$, $C(t,x)$, and $D(t,x)$ are bounded on $[0,T] \times \mathbb{R}^n$. Suppose that $W(t,x) \equiv 0$ for all $t \in [0,T]$ if $x \notin B$, for some ball $B \subset \mathbb{R}^n$ of finite radius.

Then for the energy $E(t)$ the following estimate holds

$$E(t) \leq cE(0) + c\int_0^t \int_{\mathbb{R}^n} |F(s,x)|^2 \, dxds \qquad \text{for all} \quad t \in [0,T] \tag{3.22}$$

Proof. From the system of equations above, we have the following equation for $k = 1,\ldots,m$

$$\begin{aligned}
f^k &= w_{tt}^k - \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_ix_j}^k + \sum_{l=1}^{n}\sum_{j=1}^{n} b_{lj}^k(t,x) w_{x_j}^l + \sum_{l=1}^{n} c_l^k(t,x) w_t^l + \sum_{l=1}^{n} d_l^k(t,x) w^l, \\
f^k w_t^k &= w_t^k \left[w_{tt}^k - \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_ix_j}^k + \sum_{l=1}^{n}\sum_{j=1}^{n} b_{lj}^k(t,x) w_{x_j}^l + \sum_{l=1}^{n} c_l^k(t,x) w_t^l + \sum_{l=1}^{n} d_l^k(t,x) w^l \right], \\
f^k w_t^k &= w_t^k w_{tt}^k - \sum_{i,j=1}^{n} a_{ij}(t,x) w_{x_ix_j}^k w_t^k + \sum_{l=1}^{n}\sum_{j=1}^{n} b_{lj}^k(t,x) w_{x_j}^l w_t^k + \sum_{l=1}^{n} c_l^k(t,x) w_t^l w_t^k + \sum_{l=1}^{n} d_l^k(t,x) w^l w_t^k.
\end{aligned}$$

We integrate this identity to obtain:

$$\int_0^t \int_{\mathbb{R}^n} f^k w_t^k dxds = \int_0^t \int_{\mathbb{R}^n} \Bigg[w_t^k w_{tt}^k - \sum_{i,j=1}^n a_{ij}(s,x) w_{x_i x_j}^k w_t^k + \sum_{l=1}^n \sum_{j=1}^n b_{lj}^k(s,x) w_{x_j}^l w_t^k$$
$$+ \sum_{l=1}^n c_l^k(s,x) w_t^l w_t^k + \sum_{l=1}^n d_l^k(s,x) w^l w_t^k \Bigg] dxds,$$
$$\int_0^t \int_{\mathbb{R}^n} f^k w_t^k \, dxds = \int_0^t \int_{\mathbb{R}^n} w_t^k w_{tt}^k \, dxds - \sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_i x_j}^k w_t^k \, dxds$$
$$+ \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \sum_{j=1}^n b_{lj}^k(s,x) w_{x_j}^l w_t^k dxds + \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n c_l^k(s,x) w_t^l w_t^k dxds$$
$$+ \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n d_l^k(s,x) w^l w_t^k dxds,$$

Thus we have

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^n} w_t^k w_{tt}^k \, dxds - \sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^n} a_{ij}(s,x) w_{x_i x_j}^k w_t^k \, dxds \\ &\quad = \int_0^t \int_{\mathbb{R}^n} f^k w_t^k \, dxds - \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \sum_{j=1}^n b_{lj}^k(s,x) w_{x_j}^l w_t^k dxds \\ &\quad - \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n c_l^k(s,x) w_t^l w_t^k dxds + \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n d_l^k(s,x) w^l w_t^k dxds. \end{aligned} \tag{3.23}$$

Now we will individually look at each term of equation (3.23) considering the definition of **energy**:

$$E(t) := \frac{1}{2} \sum_{k=1}^m \int_{\mathbb{R}^n} \left[(w^k)_t^2 + \sum_{i,j=1}^n a_{ij}(t,x)(w^k)_{x_i}(w^k)_{x_j} \right] dx \geq 0.$$

The *first term* of equation (3.23) can be written in the following way:

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^n} w_t^k(s,x) w_{tt}^k(s,x) \, dx \, ds &= \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} \frac{\partial}{\partial t} (w_t^k)^2(s,x) \, dx \, ds, \\ &= \frac{1}{2} \int_0^t \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^n} (w_t^k)^2(s,x) \, dx \right) \, ds, \\ &= \frac{1}{2} \int_{\mathbb{R}^n} (w_t^k)^2(t,x) \, dx - \frac{1}{2} \int_{\mathbb{R}^n} (w_t^k)^2(0,x) \, dx \end{aligned} \tag{3.24}$$

The *second term* of equation (3.23) can be written in the following way:

$$
\begin{aligned}
&-\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n} a_{ij}(s,x)w^k_{x_ix_j}w^k_t\,dxds\\
&=\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_j}a_{ij}(s,x)w^k_t(s,x)\right)w^k_{x_i}\,dxds\\
&=\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n} a_{ij}(s,x)w^k_{x_jt}(s,x)w^k_{x_i}\,dxds+\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_j}a_{ij}(s,x)\right)w^k_t(s,x)w^k_{x_i}\,dxds\\
&=\frac{1}{2}\frac{\partial}{\partial t}\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n} a_{ij}(s,x)w^k_{x_j}w^k_{x_i}\,dxds\\
&-\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial t}a_{ij}(s,x)\right)w^k_{x_j}w^k_{x_i}\,dxds+\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_j}a_{ij}(s,x)\right)w^k_tw^k_{x_i}\,dxds\\
&=\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n} a_{ij}(s,x)w^k_{x_j}(s,x)w^k_{x_i}(s,x)\,dxds-\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n} a_{ij}(0,x)w^k_{x_j}(0,x)w^k_{x_i}(0,x)\,dx\\
&-\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial t}a_{ij}(s,x)\right)w^k_{x_j}w^k_{x_i}\,dxds+\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_j}a_{ij}(s,x)\right)w^k_tw^k_{x_i}\,dxds \qquad (3.25)
\end{aligned}
$$

The *first term* of the right hand side of equation (3.23) is bounded the following way:

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^n} f^k(s,x)w^k_t(s,x)\,dxds\right|\\
&\le\int_0^t\int_{\mathbb{R}^n}|f^k(s,x)||w^k_t(s,x)|\,dxds\\
&\le\int_0^t\int_{\mathbb{R}^n}\frac{1}{2}\left(|f^k(s,x)|^2+|w^k_t(s,x)|^2\right)dxds\\
&\le\int_0^t\int_{\mathbb{R}^n}\frac{1}{2}|f^k(s,x)|^2\,dxds+\frac{1}{2}\int_0^t E(s)ds. \qquad (3.26)
\end{aligned}
$$

The *second term* of the right hand side of equation (3.23) is bounded the following way:

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \sum_{j=1}^n b_{lj}^k(s,x) w_{x_j}^l w_t^k dxds \right| \\
&\leq \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \sum_{j=1}^n |b_{lj}^k(s,x)||w_t^k||w_{x_j}^l| dxds \\
&\leq \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \sum_{j=1}^n \max_{t,x} |b_{lj}^k(t,x)||w_t^k||w_{x_j}^l| dxds \\
&\leq C \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \sum_{j=1}^n \left(\frac{1}{2}|w_t^k|^2 + \frac{1}{2}|w_{x_j}^l|^2 \right) dxds \\
&\leq C \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^n \left(\frac{1}{2}|w_t^k|^2 + \sum_{l=1}^n \frac{1}{2}|w_{x_j}^l|^2 \right) dxds \\
&\leq \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^n \left(\frac{1}{2}|w_t^k|^2 + \sum_{l=1}^n \frac{C}{2\sigma_0} a_{ij}(s,x) w_{x_i}^l w_{x_j}^l \right) dxds \\
&\leq C_2 \int_0^t E(s)ds.
\end{aligned}
\tag{3.27}
$$

The *third term* of the right hand side of equation (3.23) is bounded the following way:

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n c_l^k(s,x)(w_t^k)(w_t^l)\ dxds \\
&\leq \left| \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n c_l^k(s,x)(w_t^k)(w_t^l)\ dxds \right|, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \left| c_l^k(s,x) \right| \left| w_t^k \right| \left| w_t^l \right|\ dxds, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \max_{t,x} \left| c_l^k(s,x) \right| \left(\frac{1}{2}(w_t^k)^2 + \frac{1}{2}(w_t^l)^2 \right)\ dxds, \\
&\leq C \int_0^t \int_{\mathbb{R}^n} \left(\frac{1}{2}(w_t^k)^2 + \sum_{l=1}^n \frac{1}{2}(w_t^l)^2 \right)\ dxds, \\
&\leq C \int_0^t \underbrace{\frac{1}{2} \int_{\mathbb{R}^n} (w_t^k)^2 dx}_{\leq E(s)} ds + C \int_0^t \underbrace{\frac{1}{2} \int_{\mathbb{R}^n} \sum_{l=1}^n (w_t^l)^2\ dx}_{\leq E(s)} ds, \\
&\leq C_3 \int_0^t E(s)\ ds.
\end{aligned}
\tag{3.28}
$$

The *fourth term* of the right hand side of equation (3.23) is bounded the following way:

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n d_l^k(s,x) w_t^k w^l \; dx \; ds\right| \\
&\leq \left|\int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n d_l^k(s,x) w^l w_t^k \; dxds\right| \\
&\leq \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \left|d_l^k(s,x) w^l w_t^k\right| dxds, \\
&= \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \underbrace{\left|d_l^k(s,x)\right|}_{\leq D} \quad \underbrace{\left|w^l\right|\left|w_t^k\right|}_{\leq \frac{1}{2}\left((w^l)^2+(w_t^k)^2\right)} \quad dxds, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{D}{2}\left((w^l)^2+(w_t^k)^2\right) \; dxds, \\
&\leq \int_0^t \int_{\mathbb{R}^n} \left(\frac{D}{2}(w_t^k)^2 + \frac{D}{2}\sum_{l=1}^n (w^l)^2\right) \; dxds, \\
&\leq D\int_0^t \frac{1}{2}\int_{\mathbb{R}^n} (w_t^k)^2 dxds + \frac{D}{2}\sum_{l=1}^n \int_0^t \int_{\mathbb{R}^n} (w^l)^2 dxds
\end{aligned} \tag{3.29}$$

Now, note that the first term of the equation above can be estimated the following way:

$$D\int_0^t \frac{1}{2}\int_{\mathbb{R}^n} (w_t^k)^2 dxds \leq D\int_0^t E(s)ds$$

To estimate the last term of equation (3.29), we use

$$
\begin{aligned}
|w^l(t,x)|^2 &= |w^l(0,x)+\int_0^t w_t^l(s,x)\,ds||w^l(0,x)+\int_0^t w_t^l(s_1,x)\,ds_1| \\
&\le 2|w^l(0,x)|^2+2\int_0^t |w_t^l(s,x)|\,ds\int_0^t |w_t^l(s_1,x)|\,ds_1 \\
&= 2|w^l(0,x)|^2+2\int_0^t\int_0^t ds\,ds_1|w_t^l(s,x)||w_t^l(s_1,x)| \\
&\le 2|w^l(0,x)|^2+2\int_0^t\int_0^t ds\,ds_1\frac{1}{2}\left(|w_t^l(s,x)|^2+|w_t^l(s_1,x)|^2\right) \\
&\le 2|w^l(0,x)|^2+\int_0^t\int_0^t ds\,ds_1|w_t^l(s,x)|^2 \\
&\le 2|w^l(0,x)|^2+t\int_0^t |w_t^l(s,x)|^2\,ds \\
&\le 2|w^l(0,x)|^2+D\int_0^t |w_t^l(s,x)|^2\,ds
\end{aligned}
$$

Therefore, this is how we estimate the second term of equation (3.29)

$$
\begin{aligned}
\left|\frac{D}{2}\int_0^t\int_{\mathbb{R}^n}(w^k)^2dxds\right| &\le \frac{D}{2}\int_0^t\int_{\mathbb{R}^n}|w^k|^2dxds, \\
&\le \frac{D}{2}\int_0^t\int_{\mathbb{R}^n}\left(2|w(0,x)|^2+D\int_0^t|w_t(s,x)|^2\,ds\right)dxds, \\
&= \frac{D}{2}\int_0^t\int_{\mathbb{R}^n}2|w(0,x)|^2dxds+\frac{D}{2}\int_0^t\int_{\mathbb{R}^n}D\int_0^t|w_t(s,x)|^2\,dsdxds \\
&= D\int_0^t c\int_{\mathbb{R}^n}a_{ij}(0,x)w_{x_j}(0,x)w_{x_i}(0,x)\,dxds+\frac{D}{2}\int_0^t E(s)ds \\
&= c_4E(0)+C_4\int_0^t E(s)ds. \qquad (3.30)
\end{aligned}
$$

Finally, we return to equation (3.23):

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^n} w_t^k w_{tt}^k\,dxds-\sum_{i,j=1}^n\int_0^t\int_{\mathbb{R}^n}a_{ij}(s,x)w_{x_ix_j}^k w_t^k\,dxds \\
&\quad= \int_0^t\int_{\mathbb{R}^n} f^k w_t^k\,dxds-\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^n\sum_{j=1}^n b_{lj}^k(s,x)w_{x_j}^l w_t^k dxds \\
&\quad-\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^n c_l^k(s,x)w_t^l w_t^k dxds+\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^n d_l^k(s,x)w^l w_t^k dxds.
\end{aligned}
$$

And now, we substitute expressions (3.24) and (3.25) into the equation above:

$$\underbrace{\frac{1}{2}\int_{\mathbb{R}^n}(w_t^k)^2(t,x)\;dx-\frac{1}{2}\int_{\mathbb{R}^n}(w_t^k)^2(0,x)\;dx}_{\text{estimate (3.24)}}+\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}a_{ij}(s,x)w_{x_j}^k(s,x)w_{x_i}^k(s,x)\,dxds$$
$$\underbrace{-\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}a_{ij}(0,x)w_{x_j}^k(0,x)w_{x_i}^k(0,x)\,dx-\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial t}a_{ij}(s,x)\right)w_{x_j}^k w_{x_i}^k\,dxds+\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_j}a_{ij}(s,x)\right)w_t^k w_{x_i}^k\,dxds}_{\text{estimate (3.25)}}$$
$$=\int_0^t\int_{\mathbb{R}^n}f^k w_t^k\,dxds-\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n}\sum_{j=1}^{n}b_{lj}^k(s,x)w_{x_j}^l w_t^k dxds$$
$$-\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n}c_l^k(s,x)w_t^l w_t^k dxds+\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n}d_l^k(s,x)w^l w_t^k dxds.$$

We now rearrange some terms to obtain:

$$\begin{aligned}E(t)=T(t)&+\int_0^t\int_{\mathbb{R}^n}f^k w_t^k\,dxds-\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n}\sum_{j=1}^{n}b_{lj}^k(s,x)w_{x_j}^l w_t^k dxds\\&-\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n}c_l^k(s,x)w_t^l w_t^k dxds+\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n}d_l^k(s,x)w^l w_t^k dxds.\end{aligned}\tag{3.31}$$

where

$$\begin{aligned}T(t):=&\frac{1}{2}\int_{\mathbb{R}^n}(w_t^k)^2(0,x)\;dx+\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}a_{ij}(0,x)w_{x_j}^k(0,x)w_{x_i}^k(0,x)\,dx\\&+\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial t}a_{ij}(s,x)\right)w_{x_j}^k w_{x_i}^k\,dxds\\&+\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_j}a_{ij}(s,x)\right)w_t^k w_{x_i}^k dxds.\end{aligned}$$

Note that, because

$$\left| -\frac{1}{2}\int_{\mathbb{R}^n} (w_t^k)^2(0,x)\ dx \right| \le \frac{1}{2}E(0),$$
$$\left| \frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n} a_{ij}(0,x)w_{x_j}^k(0,x)w_{x_i}^k(0,x)\,dxds \right| \le \frac{1}{2}E(0),$$
$$\left| \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t}a_{ij}(s,x)\right) w_{x_j}w_{x_i}\,dxds \right| \le C_\alpha \int_0^t E(s),ds$$
$$\left| \sum_{i,j=1}^{n}\int_0^t\int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j}a_{ij}(s,x)\right) w_t w_{x_i}\,dxds \right| \le C_\beta \int_0^t E(s),ds$$

we have the following estimate for $T(t)$:

$$|T(t)| \le c_1E(0) + C_1\int_0^t E(s)ds.$$

Now, we apply absolute value to equation (3.31) to obtain the following inequality:

$$E(t) \le |T(t)| + \left|\int_0^t\int_{\mathbb{R}^n} f^k w_t^k\,dxds\right| + \left|\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n}\sum_{j=1}^{n} b_{lj}^k(s,x)w_{x_j}^l w_t^k dxds\right|$$
$$+ \left|\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n} c_l^k(s,x)w_t^l w_t^k dxds\right| + \left|\int_0^t\int_{\mathbb{R}^n}\sum_{l=1}^{n} d_l^k(s,x)w^l w_t^k dxds\right|.$$

Next, we apply the estimates (3.26) through (3.30) and the estimate for $T(t)$ to obtain:

$$E(t) \le c_1E(0) + C_1\int_0^t E(s)ds + \int_0^t\int_{\mathbb{R}^n}\frac{1}{2}|f^k(s,x)|^2\,dxds + \frac{1}{2}\int_0^t E(s)ds + C_2\int_0^t E(s)ds$$
$$+ C_3\int_0^t E(s)\ ds + c_4E(0) + C_4\int_0^t E(s)ds$$

Therefore,

$$E(t) \le cE(0) + \int_0^t\int_{\mathbb{R}^n}\frac{1}{2}|f^k(s,x)|^2\,dxds + C\int_0^t E(s)\ ds \text{ for all } t\in[0,T].$$

Finally, we apply **Gronwall's Lemma** with $\phi(t) = E(t)$, $f(t) = cE(0) + \int_0^t\int_{\mathbb{R}^n}\frac{1}{2}|f^k(s,x)|^2dxds$, and $\kappa(t) = C$ to obtain:

$$E(t) \leq \left(cE(0) + \int_0^t \int_{\mathbb{R}^n} \frac{1}{2} |f^k(s,x)|^2 dx ds \right) \exp\left(\int_0^t C ds \right),$$
$$E(t) \leq \left(cE(0) + \int_0^t \int_{\mathbb{R}^n} \frac{1}{2} |f^k(s,x)|^2 dx ds \right) \exp(Ct)$$

□

Corollary 1 (Uniqueness in Sobolev Spaces)**.** *Assume that the coefficient functions $a_{ij}(t,x)$ are such that*

$$0 < \sigma_0 |\xi|^2 \leq a_{ij}(t,x)\xi_i\xi_j \leq \sigma_1 |\xi|^2, \quad \forall \xi \neq 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

and their first derivatives are bounded, the matrix-valued functions $B_j(t,x)$, $C(t,x)$, and $D(t,x)$ are bounded on $[0,T] \times \mathbb{R}^n$.

If $w^k(t,x) \in C^2([0,T] \times \mathbb{R}^n)$ and $w^k(0,x) = 0$, $w_t^k(0,x) = 0$ for all $x \in \mathbb{R}^n$ and $k = 1,2\dots,m$, then $w^k(t,x) = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Corollary 2 (Local Uniqueness for System of Linear Equations)**.** *Let $w(t,x) = (w^1(t,x),\dots,w^m(t,x)) \in C^2([0,T] \times \mathbb{R}^n)$ be a solution to the system of equations*

$$w_{tt}^k - \sum_{i,j=1}^n a_{ij}(t,x) w_{x_i x_j}^k + \sum_{l=1}^m \sum_{j=1}^n b_{klj}(t,x) w_{x_j}^l + \sum_{l=1}^m c_{kl}(t,x) w_t^l + \sum_{l=1}^m d_{kl}(t,x) w^l = f^k, \quad k = 1,2\dots,m$$

with initial data $w^k(0,x) = \varphi_0^k(x), w_t^k(0,x) = \varphi_1^k(x)$ for all $x \in \mathbb{R}^n$ and $k = 1,2\dots,m$.

3.1.3 Uniqueness Theorem for System of Non–Linear Equations

Theorem 10 (Uniqueness Theorem for System of Non–Linear Equations)**.** *Let $u = (u^1,\dots,u^m)$ be a solution on $[0,T] \times \mathbb{R}^n$ for the system of equations:*

$$\begin{aligned}
&\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u^i \\
&+ \sum_{j,k} \Gamma^i_{j,k}(u^1,\dots,u^m) \left(\dot{u}^j \dot{u}^k - b^2(t) \nabla u^j \cdot \nabla u^k\right) = 0, \ i = 1,\dots m,
\end{aligned}$$

which satisfies the conditions of the main theorem. Then any other solution $v = (v^1, \ldots, v^m) \in C^2$ on $[0,T] \times \mathbb{R}^n$ of the system such that

$$\partial_t^k u(0,x) = \partial_t^k v(0,x) \quad \textit{and} \quad k = 0,1$$

is equal to u for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Proof. We will begin by defining the following m functions:

$$F^i(u,u',u'') := \left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u^i + G^i(u,u'), \text{ where } 1 \leq i \leq m.$$

Note that for $1 \leq i \leq m$, we have that:

$$\begin{aligned}
F^i(u,u',u'') - F^i(v,v',v'') &= \int_{(v,v',v'')}^{(u,u',u'')} \frac{dF^i(s)}{dn} ds, \\
&= \int_{(v,v',v'')}^{(u,u',u'')} \left(n_u \frac{\partial F^i(s)}{\partial u} + n_{u'} \frac{\partial F^i(s)}{\partial u'} + n_{u''} \frac{\partial F^i(s)}{\partial u''} \right) d|s|, \\
&= \int_{(v,v',v'')}^{(u,u',u'')} \left(n_u \frac{\partial F^i(s)}{\partial u} + n_{u'} \frac{\partial F^i(s)}{\partial u'} + n_{u''} \frac{\partial F^i(s)}{\partial u''} \right) d|s|, \\
&= \int_0^1 \frac{\partial F^i(s)}{\partial u} \cdot (u-v) dt + \int_0^1 \frac{\partial F^i(s)}{\partial u'} \cdot (u'-v') dt + \int_0^1 \frac{\partial F^i(s)}{\partial u''} \cdot (u''-v'') dt, \\
&= \underbrace{\int_0^1 \frac{\partial F^i(s)}{\partial u} dt}_{=a^i_{(1,0,0)}} \cdot (u-v) + \underbrace{\int_0^1 \frac{\partial F^i(s)}{\partial u'} dt}_{=a^i_{(0,1,0)}} \cdot (u'-v') + \underbrace{\int_0^1 \frac{\partial F^i(s)}{\partial u''} dt}_{=a^i_{(0,0,1)}} \cdot (u''-v''), \\
&= a^i_{(1,0,0)}(u,u',v,v') \cdot (u-v) + a^i_{(0,1,0)}(u,u',v,v') \cdot (u'-v') \\
&\quad + a^i_{(0,0,1)}(u,u',v,v') \cdot (u''-v''),
\end{aligned}$$

where $a^i_{(1,0,0)}, a^i_{(0,1,0)}, a^i_{(0,0,1)} \in C^1$.

Considering that $x \in \mathbb{R}^{1+n}$, there will be n^2+3n+3 coefficients a^i_α, such that the multi–variable indexes $|\alpha| \leq 2$. There is one coefficient for $\alpha = 0$, which corresponds to the term $(u-v)$. Since there are $n+1$ variables, then there are $n+1$ first derivatives with its corresponding coefficients. Finally, there are $(n+1)(n+1) = n^2+2n+1$ second derivatives with corresponding coefficients.

This implies that $(u^i - v^i)$ is a solution to the equation of the form:

$$F^i(x,u,u',u'') - F^i(x,v,v',v'') = \sum_{j=1}^{m} \sum_{|\alpha|\le 2} a_\alpha^{ij}(x,u,u',v,v')\partial^\alpha (u^j - v^j) = 0.$$

which is hyperbolic in the direction of time, $t = \xi_0$.

Now, we can define $w = (w^1, \ldots, w^m) = (u^1 - v^1, \ldots, u^m - v^m)$ and it is a solution to the system of equations of the form:

$$\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) w^i + G^i(w,w') = 0, \text{ where } 1 \le i \le m.$$

We can combine all the equations in the system to obtain:

$$\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) w + \sum_{j=1}^{m} \sum_{|\alpha|\le 1} a_\alpha^{ij}(x,w,w')\partial^\alpha w = 0.$$

Finally, because the functions $a_\alpha^{ij} \in C^1$, then we can apply the theorem 8 to the equation above and then the uniqueness proof for the non–linear equations is complete. □

3.2 (Step 2) Local existence

Theorem 11. *Consider the following equation*

$$\partial_t^m u = \sum_{j=0}^{m-1} A_j(t,x,D^{m-1}u,D_x)\partial_t^j u + C(t,x,D^{m-1}u)$$

with initial conditions:

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \ldots, \partial_t^{m-1} u(0,x) = u_{m-1}(x).$$

If the equation is strictly hyperbolic, and the initial data $u_j \in H^{s+m-1-j}(M)$ with $s > \frac{n}{2}+1$, then there is a unique local solution

$$u \in C(I, H^{s+m-j}(M)) \cap C^{m-1}(I, H^s(M)),$$

which persists as long as for some $r > 0$,

$$\|u(t)\|_{C^{m+r}} + \cdots + \|\partial_t^{m-1} u(t)\|_{C^{1+r}}$$

is bounded.

This theorem is proved in [10] (Taylor, 2011).

Theorem 12 (Local Existence Theorem)**.** *Consider the system*

$$\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u^i + \sum_{j,k} \Gamma^i_{j,k}(u) \left(\dot{u}^j \dot{u}^k - b^2(t) \nabla u^j \cdot \nabla u^k\right) = 0, \quad i = 1, \ldots m,$$

where $\Gamma^i_{j,k}(u)$, $b(t)$ are C^∞ functions. If $u^i_0(x) \in H^{s+1}(\mathbb{R}^n)$ and $u^i_1(x) \in H^s(\mathbb{R}^n)$ for some integer $s > (n+2)/2$ then the Cauchy problem with initial data

$$u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x), \quad i = 1, \ldots m, \quad x \in \mathbb{R}^n,$$

has for some $T > 0$ a solution $u \in C^2([0,T] \times \mathbb{R}^n)$.

Proof. This is a special case of theorem (11), when $m = 2$

□

3.3 (Step 3) Cone of dependence

Consider the following initial value problem:

$$u_{tt} = c^2 \Delta u, \text{ for } x \in \mathbb{R}^n, t > 0 \tag{3.32}$$

$$u(x,0) = g(x). \tag{3.33}$$

Theorem 13 (Dependence domain)**.** *Let $u(x,t)$ be a solution of* (3.32)*,* (3.33)*. Let $x_0 \in \mathbb{R}^n$ and $t_0 > 0$. If $g(x) = h(x) = 0$ for every x in $\overline{B_0} = \{x \in \mathbb{R}^n : \ |x - x_0| \le ct_0\}$, then $u(x_0,t_0) = 0$.*

Proof. Assume that $u \in C^2(\mathbb{R}^n \times (0,\infty))$ is a solution of (3.32), (3.33), and let $x_0 \in \mathbb{R}^n$ and $t_0 > 0$.

Let $\tau \in [0,t_0]$, and we define $\overline{B_\tau} = \{x \in \mathbb{R}^n | \ |x - x_0| \le c(t_0 - \tau)\}$.

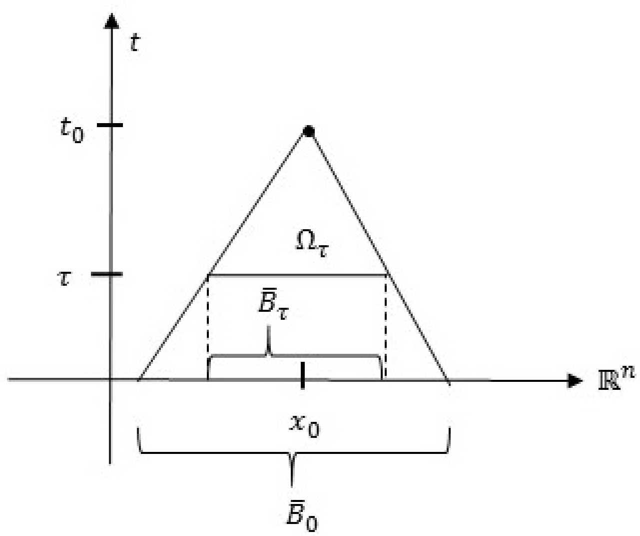

Figure 3.2: Triangle with defined sets

Consider the local energy function: $\mathcal{E}_{x_0,t_0}(\tau) = \frac{1}{2}\int_{B_\tau} \left(u_t^2 + c^2|\nabla u|^2\right)|_{t=\tau}\, dx$, for $0 \le \tau \le t_0$. We will prove that $\mathcal{E}_{x_0,t_0}$ is a non-increasing function. That is,

$$\mathcal{E}_{x_0,t_0}(\tau) \le \mathcal{E}_{x_0,t_0}(0), \text{ for } 0 \le \tau \le t_0.$$

We begin by defining the following sets:

$$\begin{aligned} \Omega_\tau &= \{(x,t) : |x - x_0| < c(t_0 - t), 0 < t < \tau\} \\ C_\tau &= \{(x,t) : |x - x_0| = c(t_0 - t), 0 < t < \tau\} \end{aligned}$$

Then the boundary of Ω_τ is the following disjoint union:

$$\delta\Omega_\tau = C_\tau \cup (\overline{B_0} \times \{0\}) \cup (\overline{B_\tau} \times \{\tau\}).$$

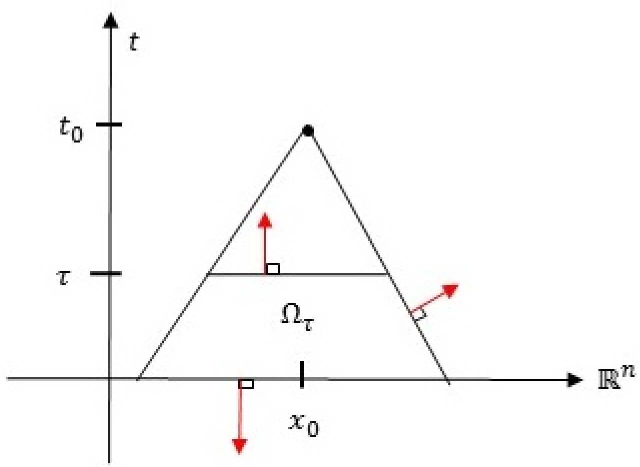

Figure 3.3: Boundary of Ω_τ and representative external unit vector

Note that the external unit vector ν on $\delta\Omega$ is given by:

$$\begin{aligned} \text{On } \overline{B_\tau} \times \{\tau\}, &\qquad \nu = \langle 0, \dots, 0, 1\rangle; \\ \text{On } \overline{B_0} \times \{0\}, &\qquad \nu = \langle 0, \dots, 0, -1\rangle; \\ \text{On } C_\tau, &\qquad \nu \text{ satisfies the following equations} \\ &\qquad \begin{cases} (\nu_1^2 + \dots + \nu_n^2)c^2 = \nu_{n+1}^2, \\ \nu_1^2 + \dots + \nu_{n+1}^2 = 1. \end{cases} \end{aligned}$$

From the first equation, we get that

$$v_1^2 + \cdots + v_n^2 = \frac{v_{n+1}^2}{c^2}. \tag{3.34}$$

And from the second equation, we get

$$\begin{aligned}
\frac{1}{1+c^2} &= \frac{v_1^2 + \cdots + v_{n+1}^2}{1+c^2}, \\
&= (v_1^2 + \cdots + v_{n+1}^2)\frac{1}{1+c^2}, \\
&= (v_1^2 + \cdots + v_{n+1}^2)\frac{1}{1+\frac{v_{n+1}^2}{v_1^2+\cdots+v_n^2}}, \text{ by the first equation,} \\
&= (v_1^2 + \cdots + v_{n+1}^2)\frac{1}{\frac{(v_1^2+\cdots+v_n^2)+v_{n+1}^2}{v_1^2+\cdots+v_n^2}}, \\
&= (v_1^2 + \cdots + v_{n+1}^2)\frac{v_1^2+\cdots+v_n^2}{v_1^2+\cdots+v_{n+1}^2}, \\
&= v_1^2 + \cdots + v_n^2, \\
&= \frac{v_{n+1}^2}{c^2}, \text{ by the first equation.}
\end{aligned}$$

Then, we have that

$$\frac{1}{1+c^2} = \frac{v_n^2}{c^2}. \tag{3.35}$$

Given a solution $u(x,t) \in C^2(\mathbb{R}^n \times (0,\infty))$ of equation (3.32), we define the following vector field:

$$\vec{V} = \langle 2c^2 u_t u_{x_1}, \ldots, 2c^2 u_t u_{x_n}, -(c^2|\nabla u|^2 + u_t^2)\rangle.$$

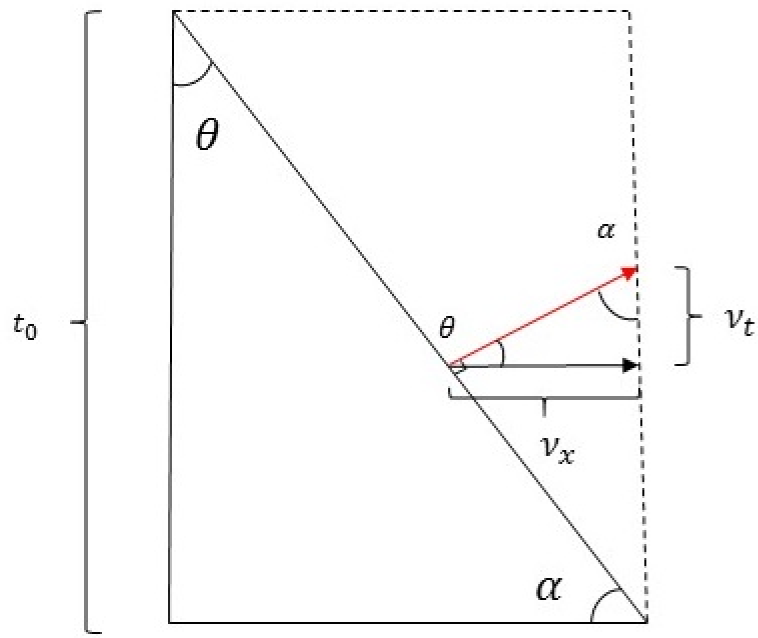

Figure 3.4: Justification for first equation

Now, we calculate the divergence of $\vec{V}$:

$$
\begin{aligned}
\operatorname{div}\vec{V} &= \frac{\partial}{\partial x_1} 2c^2 u_t u_{x_1} + \cdots + \frac{\partial}{\partial x_n} 2c^2 u_t u_{x_n} + \frac{\partial}{\partial t}[-(c^2|\nabla u|^2 + u_t^2)], \\
&= 2c^2 u_{tx_1} u_{x_1} + 2c^2 u_t u_{x_1x_1} + \cdots + 2c^2 u_{tx_n} u_{x_n} + 2c^2 u_t u_{x_nx_n} - c^2 \frac{\partial}{\partial t}|\nabla u|^2 - 2u_t u_{tt}, \\
&= 2c^2[u_{tx_1} u_{x_1} + u_t u_{x_1x_1} + \cdots + u_{tx_n} u_{x_n} + u_t u_{x_nx_n}] - c^2 \frac{\partial}{\partial t}[u_{x_1}^2 + \cdots + u_{x_n}^2] - 2u_t u_{tt}, \\
&= 2c^2[u_{tx_1} u_{x_1} + \cdots + u_{tx_n} u_{x_n}] + 2c^2[u_t u_{x_1x_1} + \cdots + u_t u_{x_nx_n}] - c^2 \frac{\partial}{\partial t}[u_{x_1}^2 + \cdots + u_{x_n}^2] - 2u_t u_{tt}, \\
&= 2c^2[u_{tx_1} u_{x_1} + \cdots + u_{tx_n} u_{x_n}] + 2c^2[u_t u_{x_1x_1} + \cdots + u_t u_{x_nx_n}] - c^2[2u_{x_1} u_{x_1t} + \cdots + 2u_{x_n} u_{x_nt}] - 2u_t u_{tt}, \\
&= 2c^2[u_{tx_1} u_{x_1} + \cdots + u_{tx_n} u_{x_n}] + 2c^2[u_t u_{x_1x_1} + \cdots + u_t u_{x_nx_n}] - 2c^2[u_{x_1} u_{x_1t} + \cdots + u_{x_n} u_{x_nt}] - 2u_t u_{tt}, \\
&= 2c^2[u_{tx_1} u_{x_1} + \cdots + u_{tx_n} u_{x_n}] + 2c^2[u_t u_{x_1x_1} + \cdots + u_t u_{x_nx_n}] - 2c^2[u_{x_1} u_{x_1t} + \cdots + u_{x_n} u_{x_nt}] - 2u_t u_{tt}, \\
&= 2c^2[u_t u_{x_1x_1} + \cdots + u_t u_{x_nx_n}] - 2u_t u_{tt}, \\
&= 2u_t[c^2(u_{x_1x_1} + \cdots + u_{x_nx_n}) - u_{tt}], \\
&= 2u_t[\underbrace{c^2\Delta u - u_{tt}}_{=0,\text{ because } u \text{ is a solution of (3.32)}}] = 0.
\end{aligned}
$$

Since $\operatorname{div}\vec{V} = 0$, then by the divergence theorem, we have that

$$\int_{\partial\Omega_\tau} \vec{V} \cdot \nu \, dS = 0. \tag{3.36}$$

We will now show that the following inequality holds on C_τ:

$$2u_t(u_{x_1}\nu_1 + \cdots + u_{x_n}\nu_n) - \frac{c}{\sqrt{1+c^2}}|\nabla u|^2 - \frac{1}{c\sqrt{1+c^2}}u_t^2 \leq 0. \tag{3.37}$$

Note that

$$\begin{aligned}
2u_t(u_{x_1}\nu_1 + \cdots + u_{x_n}\nu_n) &= 2(u_t)(\nabla u \cdot \nu_x), \\
&\leq 2|u_t||\nabla u \cdot \nu_x|, \\
&\leq 2|u_t||\nabla u||\nu_x|, \\
&\leq (c\sqrt{1+c^2})|\nabla u|^2|\nu_x| + \frac{1}{c\sqrt{1+c^2}}|u_t|, \\
&= (c\sqrt{1+c^2})|\nabla u|^2\frac{1}{1+c^2} + \frac{1}{c\sqrt{1+c^2}}|u_t|, \text{ by equations (3.34) and (3.35)}, \\
&= \frac{c}{\sqrt{1+c^2}}|\nabla u|^2 + \frac{1}{c\sqrt{1+c^2}}|u_t|.
\end{aligned}$$

Thus, we have our desired inequality.

Note then, that we have on C_τ:

$$\begin{aligned}
\int_{C_\tau} \vec{V}\cdot\nu \, dS &= \int_{C_\tau} \langle 2c^2 u_t u_{x_1},\ldots,2c^2 u_t u_{x_n}, -(c^2|\nabla u|^2+u_t^2)\rangle \cdot \langle \nu_1,\ldots,\nu_{n+1}\rangle \, dS,\\
&= \int_{C_\tau} \sum_{1=i}^{n} 2c^2 u_t u_{x_i}\nu_i - (c^2|\nabla u|^2 - u_t^2)\nu_{n+1} \, dS,\\
&= \int_{C_\tau} 2c^2 u_t \sum_{1=i}^{n} u_{x_i}\nu_i - c^2|\nabla u|^2\nu_{n+1} - u_t^2\nu_{n+1} \, dS,\\
&= \int_{C_\tau} 2c^2 u_t \sum_{1=i}^{n} u_{x_i}\nu_i - c^2|\nabla u|^2 \frac{c}{\sqrt{1+c^2}} - u_t^2\frac{c}{\sqrt{1+c^2}} \, dS, \text{ by equation (3.35)}\\
&= c^2 \int_{C_\tau} \underbrace{2u_t \sum_{1=i}^{n} u_{x_i}\nu_i - \frac{c}{\sqrt{1+c^2}}|\nabla u|^2 - \frac{1}{c\sqrt{1+c^2}}u_t^2}_{\leq 0, \text{ by inequality (3.37)}} \, dS \leq 0.
\end{aligned}$$

Therefore, we have that

$$\int_{C_\tau} \vec{V}\cdot\nu \, dS \leq 0. \tag{3.38}$$

Recalling equation (3.36), we have that

$$\begin{aligned}
\int_{\partial\Omega_\tau} \vec{V}\cdot\nu \, dS &= 0,\\
\int_{C_\tau} \vec{V}\cdot\nu \, dS + \int_{\overline{B_0}\times\{0\}} \vec{V}\cdot\nu \, dS + \int_{\overline{B_\tau}\times\{\tau\}} \vec{V}\cdot\nu \, dS &= 0,\\
\int_{\overline{B_0}\times\{0\}} \vec{V}\cdot\nu \, dS + \int_{\overline{B_\tau}\times\{\tau\}} \vec{V}\cdot\nu \, dS &\geq 0, \text{ by inequality (3.38)},\\
\int_{\overline{B_0}\times\{0\}} \vec{V}\cdot\langle 0,\ldots,0,-1\rangle \, dS + \int_{\overline{B_\tau}\times\{\tau\}} \vec{V}\cdot\langle 0,\ldots,0,1\rangle \, dS &\geq 0,\\
\int_{\overline{B_0}\times\{0\}} (-[c^2|\nabla u|^2+u_t^2])(-1) \, dS + \int_{\overline{B_\tau}\times\{\tau\}} (-[c^2|\nabla u|^2+u_t^2])(1) \, dS &\geq 0,\\
\int_{\overline{B_0}\times\{0\}} c^2|\nabla u|^2+u_t^2 \, dS - \int_{\overline{B_\tau}\times\{\tau\}} [c^2|\nabla u|^2+u_t^2] \, dS &\geq 0,\\
\int_{\overline{B_0}} \left(c^2|\nabla u|^2+u_t^2\right)|_{t=0} \, dS - \int_{\overline{B_\tau}} (c^2|\nabla u|^2+u_t^2)|_{t=\tau} \, dS &\geq 0,\\
\mathcal{E}_{x_0,t_0}(0) - \mathcal{E}_{x_0,t_0}(\tau) &\geq 0,\\
\mathcal{E}_{x_0,t_0}(0) &\geq \mathcal{E}_{x_0,t_0}(\tau).
\end{aligned}$$

Thus, we have that the local energy function is non-increasing. Therefore, since we have that

$u(x,t)=0$ at $\overline{B_0}$ by hypothesis, this implies that $u(x_0,t_0)=0$. Finally, since x_0,t_0 is arbitrary in Ω_τ, then $u(x,t)=0$ in this region.

Before we state the next theorem, we need the following definitions.

Definition 1. *A covariant vector $V=(V_0,V')$ at (t_0,x_0) is* ***timelike*** *with respect to L if the equation*

$$L_m(t_0,x_0,X+\tau V)=0$$

has m distinct real roots $\tau_1,\dots,\tau_m$ when X is not proportional to V.

Definition 2. *A hypersurface S is* ***spacelike*** *with respect to L if its normals are all timelike with respect to L.*

Remark. The condition of strict hyperbolicity of L is equivalent to saying that each hyperplane $\{t=\text{constant}\}$ is spacelike.

Theorem 14 (Cone of Dependence Theorem)**.** *Consider the following equation*

$$Lu=f, \tag{3.39}$$

with Cauchy data

$$u\Big|_S=g_1,\quad \frac{\partial}{\partial\nu}u\Big|_S=g_2,\dots,\quad \frac{\partial^{m-1}}{\partial\nu^{m-1}}u\Big|_S=g_m, \tag{3.40}$$

where $L=L(t,x,D_{t,x})$ is strictly hyperbolic and S is a timelike hyperspace in $\mathbb{R}^{n+1}$. Let G be a bounded domain in $\mathbb{R}^{n+1}$ whose boundary consists of two parts, $\partial G=S_0\cup I$, where $I=\partial G\cap\{t=0\}$ and S_0 is part of a one–parameter family of spacelike surfaces S_σ foliating a neighborhood of $\overline{G}$ (See figure 3.5).

Suppose that S_0 is spacelike, that $u\in C(\mathbb{R},H^s)$ is a solution of (3.39), (3.40)*, that the Cauchy data vanishes on I, and that $Lu\Big|_G=0$. Then, $u=0$ on G.*

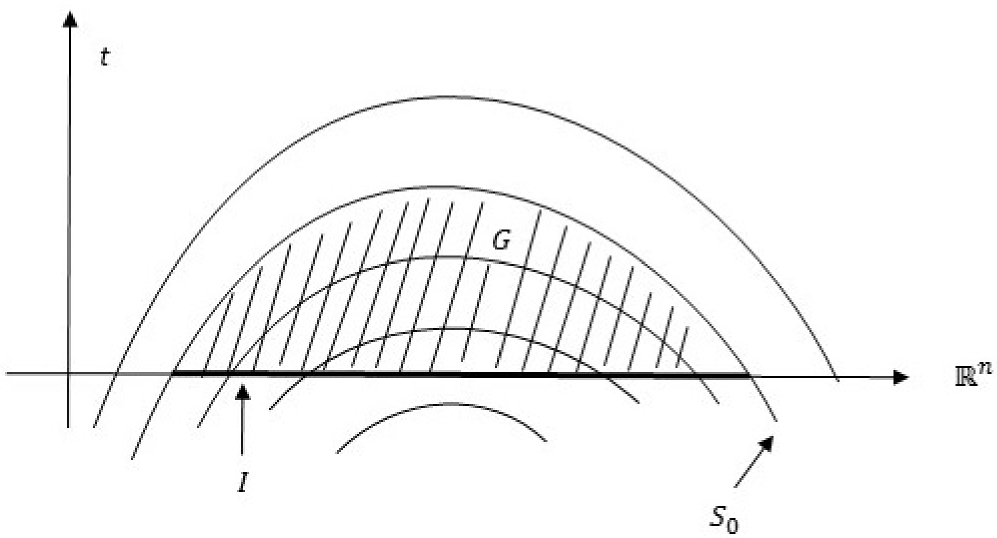

Figure 3.5: Definition of G and its boundary

Proof. Our approach will follow Taylor [10] and it will be to show that $\langle u, \varphi \rangle = 0$ for all $\varphi \in C^\infty(G)$.

Because the principle part of L and L^* must be equal, up to sign, then the surfaces S_σ are also spacelike for L^*. Let Φ be a solution to the Cauchy problem:

$$L^*\Phi = \varphi, \quad \frac{\partial^j}{\partial \nu^j}\Phi\Big|_{S_0} = 0, \quad j = 0, \ldots, m-1$$

in a neighborhood of $\overline{G}$. We can suppose that Φ is zero on the opposite side of G from S_0. Therefore, we can conclude that

$$\langle \varphi, u \rangle = \langle L^*\Phi, u \rangle = \langle \Phi, Lu \rangle = 0.$$

This concludes the proof. □

We can apply this theorem to the following scalar equation

$$\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u + \hat{f}(u)\left(|\dot{u}|^2 - b^2(t)|\nabla u|^2\right) = 0,$$

where $b(t) \in C^\infty$. Let $u_0^i(x) \in H^{s+1}(\mathbb{R}^n)$ and $u_1^i(x) \in H^s(\mathbb{R}^n)$ for some integer $s > (n+2)/2$ and

$x_0 \in \mathbb{R}^n$ and $t_0 > 0$. If

$$u^i(0,x) = 0, \quad u_t^i(0,x) = 0\,, \quad i = 1,\dots m, \quad x \in \overline{B_0}\,, \tag{3.41}$$

where $c = \max b(t)$ and $\overline{B_0} = \{x \in \mathbb{R}^n : \ |x - x_0| \leq ct_0\}$, then we conclude that the solution $u(t_0, x_0) = 0$.

3.4 (Step 4) Reduction to the scalar equation

Theorem 15. *Consider the system of equation*

$$\begin{aligned}&\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u^i \\ &+ \sum_{j,k} \Gamma^i_{j,k}(u^1,\dots,u^m)\left(\dot{u}^j\dot{u}^k - b^2(t)\nabla u^j \cdot \nabla u^k\right) = 0, \quad i = 1,\dots m,\end{aligned}$$

where $\Gamma^i_{j,k}(u)$, $b(t)$ are C^∞ functions satisfying hypothesis of theorem (II). *If*

$$u^i(0,x) = u_0(x), \quad u_t^i(0,x) = u_1(x)\,, \qquad i = 1,\dots m, \quad x \in \mathbb{R}^n\,,$$

then

$$u^1(t,x) = u^2(t,x) = \ldots = u^m(t,x) \quad \forall x \in \mathbb{R}^n, \quad t \geq 0\,.$$

Proof. If we assume that the initial data for $u^1(t,x),\dots,u^m(t,x)$ is the same, then by uniqueness in Sobolev spaces, we have that $u(t,x) := u^1(t,x) = \cdots = u^m(t,x)$. Therefore,

$$\sum_{j,k=1}^n \Gamma^i_{j,k}(u^1,\dots,u^m) = \sum_{j,k=1}^n \Gamma^i_{j,k}(u,\dots,u) =: f(u).$$

And the system is reduced to the following scalar equation

$$\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u + \hat{f}(u)\left(|\dot{u}|^2 - b^2(t)|\nabla u|^2\right) = 0.$$

□

3.5 (Step 5) The scalar equation

Consider the scalar equation

$$\left(\partial^2/\partial t^2 - b^2(t)\Delta\right) u + f(u)\left(|\dot{u}|^2 - b^2(t)|\nabla u|^2\right) = 0, \tag{3.42}$$

where $f(u)$, $b(t)$ are C^∞ functions and $f(u)$ is from condition (2.2) and does not satisfy (1.3). The initial conditions are

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n.$$

For this Cauchy problem we find arbitrarily small initial data and prove that the solution blows up in finite time. This implies that solution to the problem (2.1)-(2.4) blows up and this proves the Main Theorem.

3.6 (Step 6) Hill's Equations

Consider the following second-order differential equation:

$$\omega_{tt} + \lambda b^2(t)\omega = 0, \tag{3.43}$$

where λ is a real constant, ω and b are real-valued functions, and $b(t)$ is a periodic function.

Then, equation 3.43 is equivalent to the following system:

$$\frac{d}{dt}x(t) = A(t)x(t). \tag{3.44}$$

We define the following Cauchy problem:

$$\frac{d}{dt}X(t,t) = A(t)X(t,t), X(t_0,t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.45}$$

3.7 (Step 7) The Monodromy Matrix (and some results from Floquet Theory)

3.7.1 Definitions and Properties

The following 2×2 matrix is called the **matrizant**:

$$X(t,t_0) = I_2 + \sum_{k=1}^{\infty}\int_{t_0}^{t} A(t_1)dt_1 \int_{t_0}^{t_1} A(t_2)dt_2 \cdots \int_{t_0}^{t_k-1} A(t_k)dt_k.$$

A **fundamental set of solutions** to a differential equation is one consisting of solutions whose Wronskian is different than 0. Since the columns of the matrizant are linearly independent solutions of equation 3.43, then it is a fundamental solution to the equivalent equation 3.44.

It is also the solution to the Cauchy problem 3.45 since it satisfies the equation and the initial condition $X(0,0) = I_2$:

$$\begin{aligned} X(0,0) &= I_2 + \sum_{k=1}^{\infty}\int_{0}^{0} A(t_1)dt_1 \int_{0}^{0} A(t_2)dt_2 \cdots \int_{0}^{0} A(t_k)dt_k, \\ X(0,0) &= I_2, \end{aligned}$$

Property 1: $X(t,t_0) = X(t,t_1)X(t_1,t_0)$.

Proof. Note that: $X(t,t_0)$ is a solution to the equation 3.44, and $X(t,t_1)X(t_1,t_0)$ is also a solution to the equation because

$$\begin{aligned} \frac{d}{dt}\left[X(t,t_1)X(t_1,t_0)\right] &= \frac{d}{dt}X(t,t_1)X(t_1,t_0) + X(t,t_1)\underbrace{\frac{d}{dt}X(t_1,t_0)}_{=0} \\ &= \frac{d}{dt}X(t,t_1)X(t_1,t_0) \\ &= A(t)X(t,t_1)X(t_1,t_0) \end{aligned}$$

Then, these equations vary by a constant, $C \in M_2(\mathbb{R})$. This means

$$X(t,t_0) = X(t,t_1)X(t_1,t_0) + C, \ \ \forall t \in \mathbb{R}.$$

If we let $t = t_1$, then $X(t_1,t_0) = X(t_1,t_1)X(t_1,t_0) + C$. But

$$\begin{aligned} X(t_1,t_1) &= I_2 + \sum_{k=1}^{\infty} \int_{t_1}^{t} A(t_1)dt_1 \int_{t_1}^{t_1} A(t_2)dt_2 \cdots \int_{t_1}^{t_k-1} A(t_k)dt_k, \\ &= I_2 + \sum_{k=1}^{\infty} \int_{t_1}^{t} A(t_1)dt_1(0) \cdots \int_{t_1}^{t_k-1} A(t_k)dt_k, \\ &= I_2. \end{aligned}$$

Therefore, $X(t_1,t_0) = X(t_1,t_0) + C$, which implies that $C = 0 \in M_2(\mathbb{R})$ and

$$X(t,t_0) = X(t,t_1)X(t_1,t_0).$$

□

Property 2: $[X(t,t_0)]^{-1} = X(t_0,t)$

Proof. First, we show that the matrizant is invertible by using the following theorem:

Theorem 16 (The Liouville Formula)**.** *Let $\frac{d}{dt}x(t) = A(t)x(t)$ be a two-dimensional, first-order homogeneous linear differential equation, and let $X(t)$ be a matrix-valued solution to the equation. If $tr(A(t))$ is a continuous function on its domain, then*

$$\det X(t) = \det X(t_0) \exp\left(\int_{t_0}^{t} tr\, A(\varepsilon)d\varepsilon\right).$$

The matrizant satisfies the Liouville hypothesis since $X(t,t_0)$ is a solution to the system 3.44, and

$$tr(A(t,t_0)) = 0.$$

Then, we can apply the Liouville formula to get:

$$\begin{aligned} W(t) := \det X(t,t_0) &= \det X(t_0,t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr} A(\varepsilon,t_0) d\varepsilon\right), \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \exp\left(\int_{t_0}^{t} 0 d\varepsilon\right) = 1 \cdot e^0 = 1. \end{aligned}$$

Since $\det X(t,t_0) \neq 0$, then $(X(t,t_0))^{-1}$ exists. Also, note that

$$\begin{aligned} X(t_0,t)X(t,t_0) &= X(t_0,t_0), \text{by property 1}, \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{by definition of the Cauchy problem 3.45}, \\ &= I_2, \text{the identity in } M_2(\mathbb{R}). \end{aligned}$$

Since $X(t_0,t)X(t,t_0) = I_2$, then $[X(t,t_0)]^{-1} = X(t_0,t)$. □

Property 3: If $A(t)$ is a 1-periodic matrix, then $X(t+t_0,t_0)$ is independent of $t_0 \in \mathbb{N}$.

Proof. We begin by defining the matrix $Y(t) := X(t+t_0,t_0)$, where $t \in \mathbb{N}$; and the Cauchy problem

$$\frac{d}{dt}Y(t) = A(t)Y(t), \;\; Y(0) = I_2$$

Then $Y(t)$ is a solution of the equation. Since $Y(0) = X(t_0,t_0) = I_2$, then Y also satisfies the initial condition.

Now, we define $Y_0(t) := X(t,0)$. Note that it is also a solution of the equation, and since $Y_0(0) := X(0,0) = I_2$, then it also satisfies the initial condition.

Then, by uniqueness in the Cauchy problem, both solutions are identical. That is, $Y(t) = Y_0(t)$. Then, $X(t+t_0,t_0) = X(t,0)$ by definition, which implies that $X(t+t_0,t_0)$ is independent of t_0.

In particular,

If $t_0 = 1$, then

$$X(t+1,1) = X(t,0), \ \ \forall t \in \mathbb{R}.$$

If $t = n \in \mathbb{N}$, then

$$X(n+1,n) = X(1,0), \ \ \forall n \in \mathbb{N}.$$

$\square$

Property 4: If $A(t)$ is a 1-periodic matrix, then $X(t+n,0) = X(t,0)\,[C(0)]^n, \ \ \forall t \in \mathbb{R}, n \in \mathbb{Z}$, where $C(t,t_0) := X(t_0,t)X(t+1,t_0)$

Proof. The derivative of $C(t,t_0)$ with respect to t is

$$\begin{aligned}
\frac{d}{dt}C(t,t_0) &= \frac{d}{dt}\left[X(t_0,t)X(t+1,t_0)\right] \\
\frac{d}{dt}C(t,t_0) &= \frac{d}{dt}\left[X(t_0,t)\right]X(t+1,t_0) + X(t_0,t)\underbrace{\frac{d}{dt}\left[X(t+1,t_0)\right]} \\
\frac{d}{dt}C(t,t_0) &= \frac{d}{dt}\left[X(t_0,t)\right]X(t+1,t_0) + X(t_0,t)\underbrace{A(t+1)}X(t+1,t_0) \\
\frac{d}{dt}C(t,t_0) &= \underbrace{\frac{d}{dt}\left[X(t_0,t)\right]}X(t+1,t_0) + X(t_0,t)A(t)X(t+1,t_0) \quad (*)
\end{aligned}$$

We will work on simplifying the expression $\frac{d}{dt}\left[X(t_0,t)\right]$:

$$\begin{aligned}
\left[X(t,t_0)\right]^{-1}X(t,t_0) &= I_2,\\
\frac{d}{dt}\left[X(t,t_0)^{-1}X(t,t_0)\right] &= \frac{d}{dt}\left[I_2\right],\\
\frac{d}{dt}\left[X(t,t_0)^{-1}\right]X(t,t_0)+X(t,t_0)^{-1}\frac{d}{dt}\left[X(t,t_0)\right] &= 0,\\
\frac{d}{dt}\left[X(t_0,t)\right]X(t,t_0)+X(t_0,t)\frac{d}{dt}\left[X(t,t_0)\right] &= 0.
\end{aligned}$$

We re-arrange $\frac{d}{dt}\left[X(t_0,t)\right]X(t,t_0)+X(t_0,t)\frac{d}{dt}\left[X(t,t_0)\right]=0$ as follows:

$$\begin{aligned}
\frac{d}{dt}\left[X(t_0,t)\right]X(t,t_0) &= -X(t_0,t)\frac{d}{dt}\left[X(t,t_0)\right],\\
\frac{d}{dt}\left[X(t_0,t)\right] &= -X(t_0,t)\frac{d}{dt}\left[X(t,t_0)\right]X(t,t_0)^{-1},\\
\frac{d}{dt}\left[X(t_0,t)\right] &= -X(t_0,t)A(t)X(t,t_0)X(t,t_0)^{-1},\\
\frac{d}{dt}\left[X(t_0,t)\right] &= -X(t_0,t)A(t).
\end{aligned}$$

Now we have an expression for $\frac{d}{dt}\left[X(t_0,t)\right]$.

Substitute $\frac{d}{dt}\left[X(t_0,t)\right]=-X(t_0,t)A(t)$ equation $(*)$:

$$\begin{aligned}
\frac{d}{dt}C(t,t_0) &= \frac{d}{dt}\left[X(t_0,t)\right]X(t+1,t_0)+X(t_0,t)A(t)X(t+1,t_0),\\
\frac{d}{dt}C(t,t_0) &= -X(t_0,t)A(t)X(t+1,t_0)+X(t_0,t)A(t)X(t+1,t_0),\\
\frac{d}{dt}C(t,t_0) &= 0.
\end{aligned}$$

This implies that $C(t,t_0)$ is independent of t. Therefore,

$$C(t_0)=C(t_0,t_0)=X(t_0,t_0)X(t_0+1,t_0)=X(t_0+1,t_0).$$

Since we have shown that $C(t_0) = X(t_0+1,t_0)$, then

$$C := C(0) = X(1,0).$$

We begin by showing the property when $n = 1$. We start with the definition of $C(t,t_0) = C(t_0)$:

$$\begin{aligned}
X(t_0,t)X(t+1,t_0) &= C(t_0), \\
X(t+1,t_0) &= X(t_0,t)^{-1}C(t_0), \\
X(t+1,t_0) &= X(t,t_0)C(t_0), \\
X(t+1,0) &= X(t,0)C.
\end{aligned}$$

We finish the proof of this property by induction. Assume that

$$X(t+k,0) = X(t,0)C^k, \text{ for some } k \in \mathbb{N}.$$

Then,

$$\begin{aligned}
X(t+(k+1),0) &= X(t+k+1,1)X(1,0), \text{ by property 1,} \\
&= X(t+k,0)X(1,0), \text{ since } A \text{ is 1-periodic,} \\
&= X(t,0)C^kX(1,0), \text{ by the inductive hypothesis,} \\
&= X(t,0)C^kC, \text{ by definition of } C, \\
&= X(t,0)C^{k+1}.
\end{aligned}$$

□

Property 5. The matrizant of any 1-periodic system satisfies the identity

$$X(t+1,t_0+1) = X(t,t_0),\ \forall t,t_0 \in \mathbb{R}.$$

Proof. We apply some of the properties we have proven before. Note that

$$\begin{aligned}
X(t+1,t_0+1) &= X(t+1,t_0)X(t_0,t_0+1), \text{ by property 1,} \\
&= X(t+1,t_0)X(t_0+1,t_0)^{-1}, \text{ by property 2,} \\
&= X(t,t_0)C(t_0)X(t_0+1,t_0)^{-1}, \text{ by definition of } C, \\
&= X(t,t_0)C(t_0)C(t_0)^{-1}, \text{ by definition of } C, \\
&= X(t,t_0).
\end{aligned}$$

□

Before we state Property 6, we will need a few more definitions.

Definition. The following 2×2 matrix is called the **monodromy matrix**:

$$X(1,0) = I_2 + \sum_{k=1}^{\infty} \int_0^1 A(t_1)dt_1 \int_0^{t_1} A(t_2)dt_2 \cdots \int_0^{t_k-1} A(t_k)dt_k$$

Definition. Its eigenvalues are called **multipliers** of the Cauchy problem 3.45.

Remark. If μ is a multiplier, then it will satisfy the characteristic equation:

$$\det[X(1,0) - \mu I_2] = 0. \tag{3.46}$$

Property 6. If μ is a multiplier and $x(t)$ a solution of equation 3.44 with initial value $x(0) = a$, then $x(t+1) = \mu x(t),\ \forall t \in \mathbb{R}$.

Let $a \in M_{2,1}(\mathbb{R})$ be an eigenvector of the monodromy matrix associated with the multiplier μ, and let $x(t)$ some solution of the system 3.44. Then we can express $x(t)$ in terms of the matrizant the following way: $x(t) = X(t,0)a$ because it satisfies the equation:

$$\frac{d}{dt}x(t) = \frac{d}{dt}X(t,0)a = A(t)X(t,0)a = A(t)x(t),$$

and the initial condition:

$$x(0) = X(0,0)a = I_2 a = a.$$

Finally, note that

$$\begin{aligned}
x(t+1) &= X(t+1,0)a, \text{ by definition of } x(t), \\
&= X(t+1,1)X(1,0)a, \text{ by property 1}, \\
&= X(t,0)X(1,0)a, \text{ by property 3}, \\
&= X(t,0)\mu a, \text{ since } a \text{ is the eigenvector of } \mu, \\
&= \mu X(t,0)a, \text{ by properties of matrix multiplication}, \\
&= \mu x(t), \text{ by definition of } x(t).
\end{aligned}$$

Proposition 1. *If μ is a multiplier for the Hill's equation (3.43), then $\frac{1}{\mu}$ is also a multiplier.*

Proof. A multiplier for the Hill's equation is an eigenvalue of the monodromy matrix, by definition. Therefore, both multipliers, μ_1 and μ_2 will satisfy the following equation:

$$\det[X(1,0) - \mu I] = 0.$$

And, we can re-write it as follows:

$$\begin{aligned}
\begin{vmatrix} b_{11}-\mu & b_{12} \\ b_{21} & b_{22}-\mu \end{vmatrix} &= 0, \\
\mu^2 - (b_{11}+b_{22})\mu + b_{11}b_{22} - b_{21}b_{12} &= 0, \\
(\mu-\mu_1)(\mu-\mu_2) &= 0.
\end{aligned}$$

Then, we have that

$$\mu_1\mu_2 = b_{11}b_{22} - b_{21}b_{12} = \det X(1,0).$$

Since we have shown that the determinant of the matrizant is $W(t) := \det X(t,0) = 1$, then

$$\mu_1\mu_2 = \det X(1,0) = W(1) = 1.$$

Therefore,

$$\mu_1 = \frac{1}{\mu_2}.$$

□

3.7.2 Lyapunov-Floquet Theorem

Theorem 17 (Lyapunov-Floquet Theorem)**.** *In the system 3.45, let $A(t)$ be a continuous matrix with period P. Then the matrizant $X(t,0)$ of 3.45 has a representation of the form*

$$X(t,0) = Z(t)e^{tB}, \textit{ where } Z \textit{ is } P\textit{-periodic.}$$

Proof. Recall that $X(t+P,0) = X(t,0)C$, where $C = X(1,0)$. Since $\det C \neq 0$, then we can express C the following way: $C = e^{PB}$, where $B \in M_2(\mathbb{R})$ such that $\det B \neq 0$.

We define $Z(t) := X(t,0)e^{-tB}$. Now, we note that

$$X(t,0) = X(t,0)e^{-tB}e^{tB} = Z(t)e^{tB}.$$

And now we show that $Z(t)$ is P-periodic:

$$\begin{aligned}
Z(t+P) &= X(t+P,0)e^{-(t+P)B} \\
&= X(t+P,0)e^{-PB}e^{-tB} \\
&= X(t,0)Ce^{-PB}e^{-tB} \\
&= X(t,0)e^{PB}e^{-PB}e^{-tB} \\
&= X(t,0)e^{-tB} \\
&= Z(t)
\end{aligned}$$

$\square$

3.7.3 Lemma

Before we state the Lemma, we will introduce new notation and definitions.

Let $y_1(t)$ and $y_2(t)$ be two solutions of equation 3.43, which take the initial data:

$$y_1(0) = 0,\;\; y_1'(0) = 1,\;\; y_2(0) = 1,\;\; y_2'(0) = 0.$$

Since the matrizant can be expressed as $X_\lambda(t,0) = \begin{pmatrix} y_1'(t) & y_2'(t) \\ y_1(t) & y_2(t) \end{pmatrix}$ then the **monodromy matrix** can be expressed as $X_\lambda(1,0) = \begin{pmatrix} y_1'(1) & y_2'(1) \\ y_1(1) & y_2(1) \end{pmatrix}$.

The **characteristic equation** (3.46) can be re-written as:

$$
\begin{aligned}
\det\left[X(1,0)-\mu I_2\right] &= 0 \\
\begin{vmatrix} y_1'(1)-\mu & y_2'(1) \\ y_1(1) & y_2(1)-\mu \end{vmatrix} &= 0 \\
\mu^2 - \underbrace{\left[y_1'(1)+y_2(1)\right]}_{\text{trace of } X(1,0)}\mu + \underbrace{y_1'(1)y_2(1)-y_1(1)y_2'(1)}_{\text{determinant of } X(1,0)} &= 0 \\
\mu^2 - \left[y_1'(1)+y_2(1)\right]\mu + 1 &= 0.
\end{aligned}
$$

We define the *discriminant* of Hill's equation as

$$\Delta(\lambda) := y_1'(1)+y_2(1) = \text{trace } X_\lambda(1,0).$$

Consider the family of equations

$$\omega_{tt} + \lambda^2\alpha(t)\omega = 0, \tag{3.47}$$

where λ is a real parameter and α is non-negative, smooth, π-periodic function.

Lemma 1. *The discriminant* $\Delta(\lambda) := y_1'(\lambda,\pi)+y_2(\lambda,\pi) =$ *trace* $X_\lambda(\pi,0)$ *is an entire function such that* $\Delta(0) = 2$ *and*

$$(\partial_\lambda^2\Delta)(0) = -2\pi\int_0^\pi \alpha(s)ds. \tag{3.48}$$

Proof. Let $y_1(\lambda,t)$ and $y_2(\lambda,t)$ be two solutions of equation 3.47, which take the initial data:

$$y_1(\lambda,0)=0,\ \ y_1'(\lambda,0)=1,\ \ y_2(\lambda,0)=1,\ \ y_2'(\lambda,0)=0.$$

Note that these functions are also differentiable in λ.

If we let $\lambda = 0$, then equation 3.47 becomes $\omega_{tt} = 0$. So, $y_1(0,t) = t$, $y_2(0,t) = 1$ are solutions and

$y_1'(0,t) = 1$, and $y_2'(0,t) = 0$. By definition of $\Delta(\lambda)$, we have that

$$\Delta(0) = \underbrace{y_1'(0,\pi)}_{=1} + \underbrace{y_2(0,\pi)}_{=1} = 2.$$

Now, to find an expression for $(\partial_\lambda^2 \Delta)(0)$ we derive $\Delta(\lambda)$ with respect to λ to get:

$$\begin{aligned}
\partial_\lambda \Delta(\lambda) &= \partial_\lambda y_1'(\lambda,\pi) + \partial_\lambda y_2(\lambda,\pi) \\
\partial_\lambda^2 \Delta(\lambda) &= \underbrace{\partial_\lambda^2 y_1'(\lambda,\pi)}_{\textit{Part i}} + \underbrace{\partial_\lambda^2 y_2(\lambda,\pi)}_{\textit{Part ii}}
\end{aligned}$$

Part *i*)

Since y_1 is a solution to equation 3.47, then

$$\begin{aligned}
y_{1\ tt} + \lambda^2 \alpha(t) y_1 &= 0, \\
\partial_\lambda y_{1\ tt} + \partial_\lambda \left[\lambda^2 \alpha(t) y_1\right] &= \partial_\lambda(0), \\
\partial_\lambda y_{1\ tt} + \lambda^2 \alpha(t) \partial_\lambda y_1 + 2\lambda \alpha(t) y_1 &= 0, \\
\partial_\lambda y_{1\ tt} + \lambda^2 \alpha(t) \partial_\lambda y_1 + 2\lambda \alpha(t) y_1 &= 0, \\
\underbrace{\int_0^t \partial_\lambda y_{1\ tt}(\lambda,s) ds}_{\text{Watch me integrate!}} + \int_0^t \lambda^2 \alpha(s) \partial_\lambda y_1(\lambda,s) ds + 2\int_0^t \lambda \alpha(s) y_1(\lambda,s) ds &= 0, \\
\partial_\lambda y_{1\ t}(\lambda,t) - \partial_\lambda \underbrace{y_{1\ t}(\lambda,0)}_{=0} + \int_0^t \lambda^2 \alpha(s) \partial_\lambda y_1(\lambda,s) ds + 2\int_0^t \lambda \alpha(s) y_1(\lambda,s) ds &= 0, \\
\partial_\lambda y_{1\ t}(\lambda,t) + \lambda^2 \int_0^t \alpha(s) \partial_\lambda y_1(\lambda,s) ds + 2\lambda \int_0^t \alpha(s) y_1(\lambda,s) ds &= 0
\end{aligned}$$

We derive a second time, to get:

$$\partial_\lambda^2 y_{1\ t}(\lambda,t) + \partial_\lambda \left[\lambda^2 \int_0^t \alpha(s) \partial_\lambda y_1(\lambda,s) ds\right] + \partial_\lambda \left[2\lambda \int_0^t \alpha(s) y_1(\lambda,s) ds\right] = 0,$$

$$\partial_\lambda^2 y_{1\ t}(\lambda,t) = -\partial_\lambda\left[\lambda^2\int_0^t \alpha(s)\partial_\lambda y_1(\lambda,s)ds\right] - \partial_\lambda\left[2\lambda\int_0^t \alpha(s)y_1(\lambda,s)ds\right].$$

When $\lambda = 0$, we have that

$$(\partial_\lambda^2 y_{1t})(0,\pi) = -2\int_0^\pi \alpha(s)y_1(0,s)ds$$

$$(\partial_\lambda^2 y_{1t})(0,\pi) = -2\int_0^\pi \alpha(s)s\,ds. \quad (i)$$

Part *ii*)

Similarly, since y_2 satisfies equation 3.47, then

$$\begin{aligned}
y_{2tt} + \lambda^2\alpha(t)y_2 &= 0,\\
\partial_\lambda[y_{2tt} + \underbrace{\lambda^2\alpha(t)y_2}_{\text{Product rule}}] &= \partial_\lambda[0],\\
\partial_\lambda y_{2tt} + 2\lambda\alpha(t)y_2 + \lambda^2\alpha(t)\partial_\lambda y_2 &= 0,\\
\partial_\lambda y_{2tt} + \lambda^2\alpha(t)\partial_\lambda y_2 &= -2\lambda\alpha(t)y_2.
\end{aligned}$$

As in part *i*, we integrate as follows:

$$\begin{aligned}
\int_0^t [\partial_\lambda y_{2tt}(\lambda,s) + \lambda^2\alpha(s)\partial_\lambda y_2(\lambda,s)]ds &= \int_0^t [-2\lambda\alpha(s)y_2(\lambda,s)],\\
\partial_\lambda y_{2t}(\lambda,t) - \partial_\lambda \underbrace{y_{2t}(\lambda,0)}_{=0} + \lambda^2\int_0^t \alpha(s)\partial_\lambda y_2(\lambda,s)ds &= -2\lambda\int_0^t \alpha(s)y_2(\lambda,s)ds.
\end{aligned}$$

Now we derive again the previous equation to get:

$$\partial_\lambda y_2(\lambda,t) + \lambda^2\int_0^t d\tau\int_0^\tau \alpha(s)\partial_\lambda y_2(\lambda,s)ds =$$

$$-2\lambda\int_0^t d\tau\int_0^\tau \alpha(s)y_2(\lambda,s)ds,$$

$$\partial_\lambda[\partial_\lambda y_2(\lambda,t) + \lambda^2\int_0^t d\tau\int_0^\tau \alpha(s)\partial_\lambda y_2(\lambda,s)ds] =$$

$$\partial_\lambda[-2\lambda\int_0^t d\tau\int_0^\tau \alpha(s)y_2(\lambda,s)ds],$$

$$\partial_\lambda^2 y_2(\lambda,t)+\partial_\lambda\lambda^2\int_0^t d\tau\int_0^\tau \alpha(s)\partial_\lambda y_2(\lambda,s)ds=$$

$$-2\lambda\partial_\lambda^2\int_0^t d\tau\int_0^\tau \alpha(s)y_2(\lambda,s)ds]-2\int_0^t d\tau\int_0^\tau \alpha(s)y_2(\lambda,s)ds.$$

When we plug in $\lambda=0$ in the previous equation, only the last term survives, and we get:

$$\begin{aligned}
(\partial_\lambda^2 y_2)(0,\pi) &= -2\int_0^\pi d\tau\int_0^\tau \alpha(s)y_2(0,s)ds,\\
&= -2\int_0^\pi d\tau\int_0^\tau \alpha(s)ds,\\
&= -2\int_0^\pi \alpha(s)ds\int_s^\pi d\tau,\\
&= -2\int_0^\pi \alpha(s)(\pi-s)ds,\\
&= -2\pi\int_0^\pi \alpha(s)ds+2\int_0^\pi \alpha(s)sds. \quad (ii)
\end{aligned}$$

Returning to our original expression for $\partial_\lambda\Delta$, we have that

$$\begin{aligned}
\partial_\lambda^2\Delta(\lambda) &= \partial_\lambda^2 y_1'(\lambda,\pi)+\partial_\lambda^2 y_2(\lambda,\pi),\\
\partial_\lambda^2\Delta(\lambda) &= -2\int_0^\pi \alpha(s)s\,ds+-2\pi\int_0^\pi \alpha(s)ds+2\int_0^\pi \alpha(s)sds,\\
\partial_\lambda^2\Delta(\lambda) &= -2\pi\int_0^\pi \alpha(s)ds.
\end{aligned}$$

The Lemma is proven. □

3.7.4 Floquet Theory

Theorem 18 (Floquet's Theorem)**.** *(a) Assume that $\Delta^2>4$ for some λ. Then there exists a real number γ and two continuous functions $p_1(t)$ and $p_2(t)$, which are π-periodic or π-semiperiodic functions (depending on whether $\Delta>2$ or $\Delta<-2$), such that any solution w to equation (3.43) can be written as follows:*

$$w(t)=C_1p_1(t)e^{\gamma t}+C_2p_2(t)e^{-\gamma t},\ \textit{where } C_1,C_2\in\mathbb{C}.$$

(b) Assume that $\Delta^2 = 4$. Then equation (3.43) admits at least one 2π-periodic solution. Namely, it has a π-periodic solution when $\Delta = 2$ and π-semiperiodic solution when $\Delta = -2$.

Proof. (a) Assume that $\Delta > 2$.

Since the multipliers μ_1 and μ_2 of the monodromy matrix $X_\lambda(\pi, 0)$, satisfy the equation $\mu_1\mu_2 = 1$, then they must have the same sign. And, since $\mu_1 + \mu_2 = \Delta > 2$, then both multipliers must be positive.

By proposition 5, there are non-trivial solutions $y_1(t)$ and $y_2(t)$ such that

$$y_1(t+\pi) = \mu_1 y_1(t) = e^{\ln\mu_1} y_1(t), \quad y_2(t+\pi) = \mu_2 y_2(t) = e^{\ln\mu_2} y_2(t)$$

for all $t \in \mathbb{R}$. Then we define the functions

$$p_1(t) := e^{-t(\ln\mu_1)/\pi} y_1(t) \text{ and } p_2(t) := e^{-t(\ln\mu_2)/\pi} y_2(t).$$

Not that for $k = 1, 2$, we have that

$$\begin{aligned}
p_k(t+\pi) &= e^{-(t+\pi)(\ln\mu_k)/\pi} y_k(t+\pi), \\
&= e^{-(t+\pi)(\ln\mu_k)/\pi} e^{\ln\mu_k} y_k(t), \\
&= e^{-t(\ln\mu_k)/\pi} y_k(t), \\
&= p_k(t).
\end{aligned}$$

Then, $p_1(t)$ and $p_2(t)$ are π -periodic.

All we have left to prove is that the functions $y_1(t) = p_1(t)e^{t(\ln\mu_1)/\pi}$ and $y_2(t) = p_2(t)e^{t(\ln\mu_2)/\pi}$ form fundamental system of solutions.

Let $w(t)$ be a solution of 3.47. Then there exist $C_1, C_2 \in \mathbb{C}$ such that

$$\begin{aligned} w(t) &= C_1 y_1(t) + C_2 y_2, \\ &= C_1 e^{-t\frac{\ln\mu_1}{\pi}} y_1(t) e^{t\frac{\ln\mu_1}{\pi}} + C_2 e^{-t\frac{\ln\mu_2}{\pi}} y_2(t) e^{t\frac{\ln\mu_2}{\pi}}, \\ &= C_1 p_1(t) e^{t\frac{\ln\mu_1}{\pi}} + C_2 p_2(t) e^{t\frac{\ln\mu_2}{\pi}}. \end{aligned}$$

Now, we define the constant $\gamma = \frac{\ln\mu_1}{\pi} = \frac{\ln(1/\mu_2)}{\pi} = \frac{-\ln\mu_2}{\pi}$. Then

$$w(t) = C_1 p_1(t) e^{\gamma t} + C_2 p_2(t) e^{-\gamma t}.$$

Assume now that $\Delta < -2$. By an argument similar to the first case, we can conclude that μ_1 and μ_2 are negative. Then

$$\ln|\mu_k| = \ln|\mu_k| + i\pi, \text{ for } k = 1, 2.$$

Let $y_1(t)$ and $y_2(t)$ be solutions to the equation 3.47. And now we define the functions:

$$\begin{aligned} p_1(t) &:= e^{-t(\ln|\mu_1|)/\pi} y_1(t), \\ p_2(t) &:= e^{-t(\ln|\mu_2|)/\pi} y_2(t). \end{aligned}$$

These functions are not π-periodic.

We show the following property for $k = 1, 2$ to use in this proof:

$$\begin{aligned} p_k(t) &= e^{-t(\ln|\mu_k|)/\pi} y_k(t), \qquad (3.49) \\ p_k(t) e^{t(\ln|\mu_k|)/\pi} &= y_k(t), \\ p_k(t) e^{t(i\pi + \ln|\mu_k|)/\pi} &= y_k(t), \\ p_k(t) e^{it} e^{t(\ln|\mu_k|)/\pi} &= y_k(t), \\ p_k(t) e^{it} &= y_k(t) e^{-t(\ln|\mu_k|)/\pi}. \end{aligned}$$

Now, we show that the functions $p_k(t)e^{it}$, $k=1,2$, are π-semiperiodic. Indeed,

$$\begin{aligned} p_k(t+\pi)e^{i(t+\pi)} &= y_k(t+\pi)e^{-(t+\pi)(\ln|\mu_k|)/\pi} \\ &= \mu_k y_k(t)e^{-(t+\pi)(\ln|\mu_k|)/\pi} \\ &= e^{\ln|\mu_k|}y_k(t)e^{-(t+\pi)(\ln|\mu_k|)/\pi} \\ &= e^{\ln|\mu_k|+i\pi}y_k(t)e^{-(t+\pi)(\ln|\mu_k|)/\pi} \\ &= e^{i\pi}y_k(t)e^{-t(\ln|\mu_k|)/\pi} \\ &= -p_k(t)e^{it}, \quad k=1,2. \end{aligned}$$

Let $w(t)$ be a solution of equation 3.47. Then, there exist $C_1, C_2 \in \mathbb{C}$ such that

$$\begin{aligned} w(t) &= C_1y_1(t)+C_2y_2(t), \\ &= C_1 \underbrace{y_1(t)e^{-t\frac{\ln|\mu_1|}{\pi}}}_{\text{apply property (3.49)}} e^{t\frac{\ln|\mu_1|}{\pi}} + C_2 \underbrace{y_2(t)e^{-t\frac{\ln|\mu_1|}{\pi}}}_{\text{apply property (3.49)}} e^{t\frac{\ln|\mu_1 1|}{\pi}}, \\ &= C_1p_1(t)e^{it}e^{t\frac{\ln|\mu_1|}{\pi}} + C_2p_2(t)e^{it}e^{t\frac{\ln|\mu_2|}{\pi}}, \\ &= C_1p_1(t)e^{t\frac{\ln|\mu_1|+i\pi}{\pi}} + C_2p_2(t)e^{t\frac{\ln|\mu_2|+i\pi}{\pi}}, \\ &= C_1p_1(t)e^{t\frac{\ln|\mu_1|}{\pi}} + C_2p_2(t)e^{t\frac{\ln|\mu_2|}{\pi}}. \end{aligned}$$

We define $\gamma := \frac{\ln|\mu_1|}{\pi} = \frac{\ln|1/\mu_2|}{\pi} = \frac{-\ln|\mu_2|}{\pi}$. Then, we have that

$$w(t) = C_1p_1(t)e^{t\gamma} + C_2p_2(t)e^{-t\gamma}.$$

Proof. (b) Assume $\Delta^2 = 4$. If $\Delta = 2$, then characteristic equation 3.46 becomes

$$\mu^2 - 2\mu + 1 = 0, (\mu-1)^2 = 0.$$

And so, it has the root $\mu = 1$ with multiplicity of 2. Then the function $X(t,0)c$, where c is eigenvector of C, is π-periodic solution.

If $\Delta = -2$, then characteristic equation becomes $(\mu+1)^2 = 0$ and has root $\mu = -1$ with multiplicity of 2. Then the function $X(t,0)c$, defined as in the previous case, is a π-semiperiodic solution. □

3.8 (Step 8) Borg's Theorem

Theorem 19 (Borg's Theorem)**.** *Assume that α is non-constant, positive, and π-periodic function. Then there exists an open interval $\Lambda \subset (0,\infty)$ such that for every given $\lambda \in \Lambda$ any solution of ((3.43)) can be written in the form*

$$w(t) = C_1 p_1(t) e^{\gamma t} + C_2 p_2(t) e^{-\gamma t}, \qquad C_1, C_2 \in \mathbb{C},$$

where γ is a positive number and both functions $p_1(t)$ and $p_2(t)$ are π-periodic or π-semiperiodic functions.

We will work this proof in four parts:

- Step 1. The function $\Delta^2(\lambda) - 4$ has only real zeros.
- Step 2. The entire function $\Delta(\lambda)$ has exponential growth with type not greater than $\int_0^\pi \sqrt{\alpha(s)}ds$, that is for every positive δ there is $C(\delta)$ such that

$$|\Delta(\lambda)| \leq C(\delta) \exp\left(\left\{\int_0^\pi \sqrt{\alpha(s)}ds + \delta\right\}|\lambda|\right) \quad \forall\, \lambda \in \mathbb{C}.$$

- Step 3. If $\Delta^2(\lambda) \leq 4$ for any real λ, then there exists a complex number B such that

$$\Delta(\lambda) = 2\cos(B\lambda).$$

- Step 4. Conclusion.

3.8.1 Step 1 (Borg's Theorem)

Proof. Assume that $\Delta^2(\lambda_0) = 4$. Then by *Floquet's theorem*, there exists a nontrivial solution $w_0(t)$ to equation $w_{tt} + \lambda_0^2 \alpha(t) w = 0$, which is π-periodic or π-semiperiodic.

Multiplying by $\overline{w_0(t)}$ and integrating we derive

$$\int_0^\pi w_{0tt}(s)\overline{w_0(s)}ds + \lambda_0^2 \int_0^\pi \alpha(s) w_0(s) \overline{w_0(s)} ds = 0.$$

It follows

$$w_{0t}(\pi)\overline{w_0(\pi)} - w_{0t}(0)\overline{w_0(0)} - \int_0^\pi w_{0t}(s)\overline{w_{0t}(s)}ds + \lambda_0^2 \int_0^\pi \alpha(s)|w_0(s)|^2 ds = 0.$$

Hence

$$-\int_0^\pi |w_{0t}(s)|^2 ds + \lambda_0^2 \int_0^\pi \alpha(s)|w_0(s)|^2 ds = 0.$$

Thus λ_0 is real. $\square$

3.8.2 Step 2 (Borg's Theorem)

The entire function $\Delta(\lambda)$ has exponential growth with type not greater than $\int_0^\pi \sqrt{\alpha(s)}ds$, that is for every positive δ there is $C(\delta)$ such that

$$|\Delta(\lambda)| \leq C(\delta) \exp\left(\left\{\int_0^\pi \sqrt{\alpha(s)}ds + \delta\right\}|\lambda|\right) \quad \forall\, \lambda \in \mathbb{C}.$$

Proof. For given $\lambda \in \mathbb{C}$ we define the energy $E(t;\lambda)$ of the solution w of Hill's equation by

$$E(t;\lambda) = |\lambda|^2 \alpha(t)|w(t)|^2 + |w'(t)|^2.$$

Then we derive $E(t;\lambda)$ with respect to t to get:

$$\begin{aligned}
\frac{d}{dt}E(t;\lambda) &= |\lambda|^2\alpha'(t)|w(t)|^2 + |\lambda|^2\alpha(t)2\Re(\overline{w(t)}w'(t)) + 2\Re(\overline{w'(t)}w''(t)) \\
&= |\lambda|^2\alpha'(t)|w(t)|^2 + |\lambda|^2\alpha(t)2\Re(\overline{w(t)}w'(t)) - 2\Re(\overline{w'(t)}\lambda^2\alpha(t)w(t)) \\
&\leq \frac{|\alpha'(t)|}{\alpha(t)}E(t;\lambda) + 2|\lambda|\sqrt{a(t)}(|\lambda|\sqrt{\alpha(t)}|w(t)|)|w'(t)| \\
&\quad + 2|\lambda|\sqrt{\alpha(t)}(|\lambda|\sqrt{\alpha(t)}|w(t)|)|w'(t)| \\
&\leq \left(\frac{|\alpha'(t)|}{a(t)} + 2|\lambda|\sqrt{\alpha(t)}\right)E(t;\lambda).
\end{aligned}$$

It follows

$$E(t;\lambda) \leq E(0;\lambda)\exp\left(\int_0^t \left\{\frac{|\alpha'(s)|}{\alpha(s)} + 2|\lambda|\sqrt{\alpha(s)}\right\} ds\right)$$

This inequality holds for any solution $w(t)$. If we set $w = y_1$ and $w = y_2$, then we get for the energies $E_1(\pi;\lambda)$ and $E_2(\pi;\lambda)$ of that solutions

$$\begin{aligned}
E_1(\pi;\lambda) &\leq E_1(0;\lambda)\exp\left(\int_0^\pi \left\{\frac{|\alpha'(s)|}{\alpha(s)} + 2|\lambda|\sqrt{\alpha(s)}\right\} ds\right), \\
&= \exp\left(\int_0^\pi \left\{\frac{|\alpha'(s)|}{\alpha(s)} + 2|\lambda|\sqrt{\alpha(s)}\right\} ds\right), \\
E_2(\pi;\lambda) &\leq E_2(0;\lambda)\exp\left(\int_0^\pi \left\{\frac{|\alpha'(s)|}{\alpha(s)} + 2|\lambda|\sqrt{\alpha(s)}\right\} ds\right) \\
&= |\lambda|^2\alpha(0)\exp\left(\int_0^\pi \left\{\frac{|\alpha'(s)|}{\alpha(s)} + 2|\lambda|\sqrt{\alpha(s)}\right\} ds\right).
\end{aligned}$$

respectively, since $E_1(0;\lambda) = 1$ and $E_2(0;\lambda) = |\lambda|^2\alpha(0)$. Hence for the discriminant we have

$$\begin{aligned}
|\Delta(\lambda)| &= |y_1'(\lambda,\pi)+y_2(\lambda,\pi)| \le |y_1'(\lambda,\pi)|+|y_2(\lambda,\pi)|, \\
&\le \sqrt{E_1(\pi;\lambda)}+\sqrt{\frac{E_2(\pi;\lambda)}{|\lambda|^2\alpha(\pi)}}, \\
&\le \left(1+\sqrt{\alpha(0)/\alpha(\pi)}\right)\exp\left(\int_0^\pi \left\{\frac{1}{2}\frac{|\alpha'(s)|}{\alpha(s)}+|\lambda|\sqrt{\alpha(s)}\right\}ds\right).
\end{aligned}$$

Then for $\delta = 0$ we obtain

$$|\Delta(\lambda)| \le C(\delta)\exp\left(\{\int_0^\pi \sqrt{\alpha(s)}ds+\delta\}|\lambda|\right),$$

which proves the claimed statement. □

3.8.3 Step 3 (Borg's Theorem)

If $\Delta^2(\lambda) \le 4$ for any real λ, then there exists a complex number B such that

$$\Delta(\lambda) = 2\cos(B\lambda).$$

Proof. Indeed, the function $4-\Delta^2(\lambda)$ does not vanish (Step 1) for non-real λ.

Assume that it is non-negative on $\mathbb{R}$. Then all zeros of $4-\Delta^2(\lambda)$ are of even multiplicity. Hence $\sqrt{4-\Delta^2(\lambda)}$ is an entire function, too. According to Step 2 function $\sqrt{4-\Delta^2(\lambda)}$ has exponential growth.

Recall that $\Delta(0) = 2$. Denote

$$f(\lambda) := \sqrt{1-\Delta^2(\lambda)/4} \quad \text{and } g(\lambda) := \Delta(\lambda)/2, \text{ such that } g(0) = 1.$$

The proof of this step will be completed if we apply the following theorem from complex analysis:

Let f and g be entire functions with exponential growth. If $f(z)^2+g(z)^2=1$ and $g(0)=1$ then there is a complex number $B\in\mathbb{C}$ such that $f(z)=\sin(Bz)$ and $g(z)=\cos(Bz)$.

To prove the last theorem we consider $g+if$ and $g-if$, then $(g+if)(g-if)\equiv 1$, so that they do not vanish anywhere on $\mathbb{C}$.

Therefore $g+if=\exp(A+Bz)$ and $g-if=\exp(-A-Bz)$ while $g(0)=1$ implies $A=0$. $\square$

3.8.4 Step 4 (Borg's Theorem)

Conclusion

If the statement of the theorem is not true, then, by Floquet's theorem, $\Delta^2(\lambda)\leq 4$ on $[0,\infty)$. But $\Delta(\lambda)$ is even function, so that $\Delta^2(\lambda)\leq 4$ on $\mathbb{R}$. By Step 3, this implies $\Delta(\lambda)=2\cos(B\lambda)$ for some $B\in\mathbb{C}$. It follows that

$$B^2=-\frac{1}{2}(\partial_\lambda^2\Delta)(0).$$

Moreover, by Lemma 1, we have

$$B^2=-\frac{1}{2}(\partial_\lambda^2\Delta)(0)=\pi\int_0^\pi\alpha(s)ds.$$

By Step 2, the entire function $\Delta(\lambda)$ has exponential growth with type not greater than $\int_0^\pi\sqrt{\alpha(s)}ds$. Hence,

$$|B|\leq\int_0^\pi\sqrt{\alpha(s)}ds\quad\Longrightarrow\quad\pi\int_0^\pi\alpha(s)ds\leq\left(\int_0^\pi\sqrt{\alpha(s)}ds\right)^2.$$

At the same time, by the Schwarz-Hölder inequality

$$\left(\int_0^\pi \sqrt{\alpha(s)}ds\right)^2 \leq \left(\int_0^\pi \alpha(s)ds\right)\left(\int_0^\pi 1ds\right) = \pi\int_0^\pi \alpha(s)ds.$$

This implies that

$$\left(\int_0^\pi \sqrt{\alpha(s)}ds\right)^2 \leq \left(\int_0^\pi \alpha(s)ds\right)\left(\int_0^\pi 1ds\right) = \pi\int_0^\pi \alpha(s)ds.$$

Therefore,

$$\pi\int_0^\pi \alpha(s)ds = \left(\int_0^\pi \sqrt{\alpha(s)}ds\right)^2,$$

which only holds when the function $\alpha(t)$ is constant. Thus, the theorem is proved.

3.9 (Step 9) Construction of an exponentially increasing solution to Hill's equation

3.9.1 Yagdjian's First Lemma

Lemma 2. *Let $b(t)$ defined on $\mathbb{R}$ be a non-constant, positive, smooth 1-periodic function. Then, there exists an open subset $\Lambda^o \subset \Lambda$, such that $b_{21} \neq 0$, for all $\lambda \in \Lambda^o$.*

Proof. Assume the hypothesis and that $b_{21} = 0$. Let $\omega(t) := x_{21}(t,0)$, such that

$$X_\lambda(t,0) = \begin{pmatrix} x_{11}(t,0) & x_{12}(t,0) \\ x_{21}(t,0) & x_{22}(t,0) \end{pmatrix}.$$

Then $\omega(t)$ solves Hill's equation, by the way the matrizant is defined. And $\omega(t)$ takes boundary values $\omega(0) = \omega(1) = 0$ because

$$X_\lambda(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11}(0,0) & x_{12}(0,0) \\ x_{21}(0,0) & x_{22}(0,0) \end{pmatrix}.$$

3.9.2 A useful identity

Let $b := b(t)$ and let μ_0, μ_0^{-1} be the eigenvalues of $X_\lambda(1,0) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

We claim that the following equality holds:

$$1 - \frac{b_{21}}{\mu_0^{-1} - b_{22}} \frac{b_{12}}{\mu_0 - b_{11}} = \left(\mu_0 - \mu_0^{-1}\right) \frac{1}{b_{22} - \mu_0^{-1}} \neq 0.$$

The function $\Delta(\lambda)$ has been defined as the trace of X_λ, that is $\Delta(\lambda) = b_{11} + b_{22}$. And, since we know the eigenvalues of X_λ, then we have that

$$b_{11} + b_{22} = \mu_0 + \mu_0^{-1}. \tag{3.50}$$

From the previous equation, we obtain:

$$\begin{aligned}
b_{11} + b_{22} &= \mu_0 + \mu_0^{-1}, \\
b_{11} + b_{22} - 2\mu_0 &= \mu_0 + \mu_0^{-1} - 2\mu_0, \\
(b_{11} - \mu_0) + (b_{22} - \mu_0) &= -\mu_0 + \mu_0^{-1}, \\
|(b_{11} - \mu_0) + (b_{22} - \mu_0)| &= |-\mu_0 + \mu_0^{-1}|, \\
|b_{11} - \mu_0| + |b_{22} - \mu_0| &\geq |\mu_0 - \mu_0^{-1}|.
\end{aligned}$$

Therefore, since $|\mu_0 - \mu_0^{-1}| > 0$, we have that

$$\max\{|b_{11} - \mu_0|, |b_{22} - \mu_0|\} \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|. \tag{3.51}$$

Note that, also from equation (3.50), we have that

$$
\begin{aligned}
b_{11} + b_{22} &= \mu_0 + \mu_0^{-1}, \\
b_{11} + b_{22} - \mu_0 - b_{22} &= \mu_0 + \mu_0^{-1} - \mu_0 - b_{22}, \\
b_{11} - \mu_0 &= \mu_0^{-1} - b_{22}. \qquad\qquad (*) \\
|b_{11} - \mu_0| &= |b_{22} - \mu_0|
\end{aligned}
$$

Therefore, from equation (3.51), we can say without loss of generality:

$$|b_{11} - \mu_0| = |b_{22} - \mu_0| > \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0.$$

Furthermore, we have that

$$
\begin{aligned}
1 - \frac{b_{21}}{\mu_0^{-1} - b_{22}} \frac{b_{12}}{\mu_0 - b_{11}} &= \frac{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11}) - b_{21}b_{12}}{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11})}, \\
&= \frac{\mu_0\mu_0^{-1} - b_{11}\mu_0^{-1} - b_{22}\mu_0 + b_{11}b_{22} - b_{21}b_{12}}{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11})}, \\
&= \frac{\mu_0\mu_0^{-1} - b_{11}\mu_0^{-1} - b_{22}\mu_0 + 1}{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11})}, \text{ since } \det X_\lambda(1,0) = 1, \\
&= \frac{\mu_0\mu_0^{-1} - b_{22}\mu_0 - b_{11}\mu_0^{-1} + \mu_0\mu_0^{-1}}{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11})}, \\
&= \frac{\mu_0(\mu_0^{-1} - b_{22}) - \mu_0^{-1}(b_{11} - \mu_0)}{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11})}, \\
&= \frac{\mu_0(\mu_0^{-1} - b_{22}) - \mu_0^{-1}(\mu_0^{-1} - b_{22})}{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11})}, \text{ by equation } (*), \\
&= \frac{(\mu_0 - \mu_0^{-1})(\mu_0^{-1} - b_{22})}{(\mu_0^{-1} - b_{22})(\mu_0 - b_{11})}, \\
&= \left(\mu_0 - \mu_0^{-1}\right) \frac{1}{b_{22} - \mu_0^{-1}} \neq 0.
\end{aligned}
$$

Finally, we have our claimed identity:

$$1 - \frac{b_{21}}{\mu_0^{-1} - b_{22}} \frac{b_{12}}{\mu_0 - b_{11}} = \left(\mu_0 - \mu_0^{-1}\right) \frac{1}{b_{22} - \mu_0^{-1}} \neq 0.$$

3.9.3 Finding the diagonalizer B of the Monodromy matrix $X(1,0)$

Let $B := \begin{pmatrix} \frac{b_{12}}{\mu_0 - b_{11}} & 1 \\ 1 & \frac{b_{21}}{\mu_0^{-1} - b_{22}} \end{pmatrix}$. Then the determinant is

$$\det B = \frac{b_{12}}{\mu_0 - b_{11}} \frac{b_{21}}{\mu_0^{-1} - b_{22}} - 1.$$

By the previous section, we have shown that $\det B \neq 0$. Then we can determine the inverse of the diagonalizer:

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} \frac{b_{21}}{\mu_0^{-1} - b_{22}} & -1 \\ -1 & \frac{b_{12}}{\mu_0 - b_{11}} \end{pmatrix}.$$

3.9.4 Yagdjian's Second Lemma

Lemma 3. *Let $W := W(t)$ be a solution to Hill's equation with parameter λ such that $b_{21} \neq 0$. If W takes initial data*

$$W(0) = 0, \quad W_t(0) = 1,$$

then, for every $M \in \mathbb{N}$, we have that

$$W(M) = \frac{b_{21}}{\mu_0 - \mu_0^{-1}} \left(\mu_0^M - \mu_0^{-M}\right).$$

Proof. Let $w := w(t)$ and $z := z(t)$ be solutions to Hill's equation with initial data

$$w(0) = 1, \quad w_t(0) = \frac{b_{12}}{\mu_0 - b_{11}}, \quad \text{and,} \quad z(0) = b_{21}(\mu_0^{-1} - b_{22}), \quad z_t(0) = 1.$$

Part 1: Find value of function $w(t)$ at integer values $M \in \mathbb{N}$.

Note that

$$
\begin{aligned}
\begin{pmatrix} \frac{d}{dt}w(M) \\ w(M) \end{pmatrix} &= X(M,0)\begin{pmatrix} \frac{d}{dt}w(0) \\ w(0) \end{pmatrix}, \\
&= \underbrace{X(M,M-1)X(M-1,M-2)\cdots X(1,0)}_{M\text{-multipliers}}\begin{pmatrix} \frac{d}{dt}w(0) \\ w(0) \end{pmatrix}, \text{ by property 1 of the matrizant,} \\
&= \underbrace{X(1,0)X(1,0)\cdots X(1,0)}_{M\text{-multipliers}}\begin{pmatrix} \frac{d}{dt}w(0) \\ w(0) \end{pmatrix}, \text{ by property 5 of the matrizant,} \\
&= BB^{-1}X(1,0)B\cdots B^{-1}X(1,0)BB^{-1}\begin{pmatrix} \frac{d}{dt}w(0) \\ w(0) \end{pmatrix}, \\
&= B(B^{-1}X(1,0)B)\cdots(B^{-1}X(1,0)B)B^{-1}\begin{pmatrix} \frac{d}{dt}w(0) \\ w(0) \end{pmatrix}, \\
&= B\underbrace{\mathcal{M}\cdots\mathcal{M}}_{M\text{-multipliers}}B^{-1}\begin{pmatrix} \frac{d}{dt}w(0) \\ w(0) \end{pmatrix}, \text{ where } \mathcal{M} = \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix}\left[\left(\frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}}-1\right)^{-1}\right]\begin{pmatrix} \frac{b_{21}}{\mu_0^{-1}-b_{22}} & -1 \\ -1 & \frac{b_{12}}{\mu_0-b_{11}} \end{pmatrix}\begin{pmatrix} \frac{b_{12}}{\mu_0-b_{11}} \\ 1 \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix}\left[\left(\frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}}-1\right)^{-1}\right]\begin{pmatrix} \frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}}-1 \\ \frac{-b_{21}}{\mu_0^{-1}-b_{22}}+\frac{b_{21}}{\mu_0^{-1}-b_{22}} \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix}\left[\left(\frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}}-1\right)^{-1}\right]\begin{pmatrix} \frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}}-1 \\ 0 \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
&= \begin{pmatrix} \frac{b_{12}}{\mu_0-b_{11}} & 1 \\ 1 & \frac{b_{21}}{\mu_0^{-1}-b_{22}} \end{pmatrix}\begin{pmatrix} \mu_0^M \\ 0 \end{pmatrix}, \\
&= \begin{pmatrix} \frac{\mu_0^M b_{12}}{\mu_0-b_{11}} \\ \mu_0^M \end{pmatrix}.
\end{aligned}
$$

Therefore, we have

$$\frac{d}{dt}w(M) = \frac{\mu_0^M b_{12}}{\mu_0 - b_{11}}, \quad w(M) = \mu_0^M.$$

Part 2: Find value of function $z(t)$ at integer values $M \in \mathbb{N}$.

Similarly to the function $w(t)$, we have that:

$$
\begin{aligned}
\begin{pmatrix} \frac{d}{dt}z(M) \\ z(M) \end{pmatrix} &= X(M,0)\begin{pmatrix} \frac{d}{dt}z(0) \\ z(0) \end{pmatrix}, \\
&= B\mathcal{M}^M B^{-1}\begin{pmatrix} \frac{d}{dt}z(0) \\ z(0) \end{pmatrix}, \text{ where } \mathcal{M} = \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix} B^{-1}\begin{pmatrix} \frac{d}{dt}z(0) \\ z(0) \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix}\left(\frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}}-1\right)^{-1}\begin{pmatrix} \frac{b_{21}}{\mu_0^{-1}-b_{22}} & -1 \\ -1\frac{b_{12}}{\mu_0-b_{11}} & \end{pmatrix}\begin{pmatrix} 1 \\ \frac{b_{21}}{\mu_0^{-1}-b_{22}} \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix}\left(\frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}}-1\right)^{-1}\begin{pmatrix} \frac{b_{21}}{\mu_0^{-1}-b_{22}} - \frac{b_{12}}{\mu_0-b_{11}} \\ -1+\frac{b_{21}}{\mu_0^{-1}-b_{22}}\frac{b_{12}}{\mu_0-b_{11}} \end{pmatrix}, \\
&= B\begin{pmatrix} \mu_0^M & 0 \\ 0 & \mu_0^{-M} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
&= B\begin{pmatrix} 0 \\ \mu_0^{-M} \end{pmatrix}, \\
&= \begin{pmatrix} \frac{b_{12}}{\mu_0-b_{11}} & 1 \\ 1 & \frac{b_{21}}{\mu_0^{-1}-b_{22}} \end{pmatrix}\begin{pmatrix} 0 \\ \mu_0^{-M} \end{pmatrix}, \\
&= \begin{pmatrix} \mu_0^{-M} \\ \frac{\mu_0^{-M}b_{21}}{\mu_0^{-1}-b_{22}} \end{pmatrix}.
\end{aligned}
$$

Therefore, we have

$$
\frac{d}{dt}z(M) = \mu_0^{-M}, \quad z(M) = \frac{\mu_0^{-M}b_{21}}{\mu_0^{-1}-b_{22}}.
$$

Part 3: Building the function $W(t)$ from part 1 and part 2.

Since $w(t)$ and $z(t)$ are linearly independent solutions of Hill's equation, then they form a fundamental set of solutions. By hypothesis, $W(t)$ is also a solution of Hill's equation. Therefore, there exist scalars c_w and c_z such that:

$$W(t) = c_w w(t) + c_z z(t).$$

Finally, to determine the values of c_w and c_z, we will use the initial data from the hypothesis:

$$\begin{aligned} W(0) = 0, &\qquad W_t(0) = 1, \\ c_w w(0) + c_z z(0) = 0, &\qquad c_w w_t(0) + c_z z_t(0) = 1 \end{aligned}$$

Then, from the first equation, we get that

$$c_w = \frac{-c_z z(0)}{w(0)},$$

which we plug into the second equation to get:

$$\begin{aligned} \left(\frac{-c_z z(0)}{w(0)}\right) w_t(0) + c_z z_t(0) &= 1, \\ -c_z \left(\frac{z(0) w_t(0)}{w(0)} - z_t(0)\right) &= 1, \\ -c_z \left(\frac{z(0) w_t(0) - z_t(0) w(0)}{w(0)}\right) &= 1, \\ -c_z &= \left(\frac{z(0) w_t(0) - z_t(0) w(0)}{w(0)}\right)^{-1}, \\ -c_z &= \frac{w(0)}{z(0) w_t(0) - z_t(0) w(0)}, \\ c_z &= -\frac{w(0)}{z(0) w_t(0) - z_t(0) w(0)}. \end{aligned}$$

Now, since we know the initial data for $z(t)$ and $w(t)$, we can express c_z as:

$$c_z \quad = \quad -\frac{1}{b_{21}(\mu_0^{-1} - b_{22})\frac{b_{12}}{\mu_0 - b_{11}} - (1)(1)}.$$

3.10 (Step 10) Construction of blow–up solution to the scalar PDE

We can transform the following equation:

$$u_{tt} - b^2 \Delta u + f(u)\left(u_t^2 - b^2(t)|\nabla u|^2\right) = 0,$$

into the linear wave equation:

$$v_{tt} - b^2(t)\Delta v = 0, \tag{3.52}$$

by using the following substitution:

$$v := G(u) = \int_0^u \exp\left(\int_0^s f(r)\, dr\right)\, ds.$$

We will work through the proof using the case $f(u) = 1$.

Suppose that $f(u) = 1$. Then equation (3.10) becomes:

$$u_{tt} - b^2 \Delta u + u_t^2 - b^2(t)|\nabla u|^2 = 0.$$

Assume that $u(t,x)$ is a solution to (3.10) with the smooth initial data:

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{3.53}$$

In this case, $G(u)$ becomes:

$$v := G(u) = \int_0^u \exp\left(\int_0^s dr\right)\, ds = \int_0^u \exp(s-0)\, ds = \int_0^u e^s\, ds = e^u - 1.$$

That is, $v+e^u-1$, or $u=\ln(v+1)$.

Note that

$$\begin{gathered}
\int_0^\infty \exp\left(\int_0^s dr\right)\,ds = \int_0^\infty \exp(s-0)\,ds = \int_0^\infty e^s\,ds = \infty, \\
\int_{-\infty}^0 \exp\left(\int_0^s dr\right)\,ds = \int_{-\infty}^0 \exp(s-0)\,ds = \int_{-\infty}^0 e^s\,ds = 0-e^0 = -1 < \infty, \\
a := \lim_{u\to-\infty} G(u) = \lim_{u\to-\infty} e^u - 1 = -1, \text{ and } b := \lim_{u\to-\infty} G(u) = \lim_{u\to-\infty} e^u - 1 = \infty.
\end{gathered}$$

Therefore, $f(u)=1$ does not satisfy equation (1.3) from the hypothesis of the Nakanishi-Ohta Theorem. Then, the problem (3.10),(3.53) does not have a global classical solution for all initial data u_0, u_1. However, we will show that no matter how small the initial data is, the problem will not have a global solution. Also, note that $G'(u)>0$. This implies that for all $t\in[0,\infty)$ and for all $x\in\mathbb{R}^n$, we have that $G(u)>a$, which is equivalent to $v(t,x)>-1$.

Now, we calculate the partial derivatives of u, using the expression $u=\ln(v+1)$. That is,

$$\begin{aligned}
u_t &= \frac{1}{v+1}\frac{\partial}{\partial t}v = \frac{v_t}{v+1}, \\
u_{tt} &= \frac{\partial}{\partial t}\frac{v_t}{v+1} = \frac{v_{tt}(v+1)-v_t v_t}{(v+1)^2} = \frac{v_{tt}}{v+1} - \frac{v_t^2}{(v+1)^2}, \\
u_{x_i} &= \frac{\partial}{\partial x_i}\ln(v+1) = \frac{v_{x_i}}{v+1}, \\
|\nabla u|^2 &= \sum_{i=1}^n \left(\frac{v_{x_i}}{v+1}\right)^2 = sum_{i=1}^n \frac{v_{x_i}^2}{(v+1)^2}, \\
u_{x_i x_i} &= \frac{\partial}{\partial x_i}\frac{v_{x_i}}{v+1} = \frac{v_{x_i x_i}(v+1)-v_{x_i}v_{x_i}}{(v+1)^2} = \frac{v_{x_i x_i}}{v+1} - \frac{v_{x_i}^2}{(v+1)^2}, \\
\Delta u &= \sum_{i=1}^n u_{x_i x_i} = \sum_{i=1}^n \left(\frac{v_{x_i x_i}}{v+1} - \frac{v_{x_i}^2}{(v+1)^2}\right).
\end{aligned}$$

Now, we can plug these expressions into the left-hand side of equation (3.10) to get:

$$\begin{aligned}
u_{tt} - b^2(t)\Delta u + u_t^2 - b^2(t)|\nabla u|^2 &= 0, \\
\left(\frac{v_{tt}}{v+1} - \frac{v_t^2}{(v+1)^2}\right) - b^2(t)\sum_{i=1}^{n}\left(\frac{v_{x_i x_i}}{v+1} - \frac{v_{x_i}^2}{(v+1)^2}\right) + \left(\frac{v_t}{v+1}\right)^2 - b^2(t)\sum_{i=1}^{n}\frac{v_{x_i}^2}{(v+1)^2} &= 0, \\
\frac{v_{tt}}{v+1} - b^2(t)\sum_{i=1}^{n}\frac{v_{x_i x_i}}{v+1} &= 0, \\
v_{tt} - b^2(t)\sum_{i=1}^{n} v_{x_i x_i} &= 0, \\
v_{tt} - b^2(t)\Delta v &= 0.
\end{aligned}$$

Since we assumed that $u(t,x)$ is a solution of (3.10), (3.53). Then $v(t,x)$ solves equation (3.52) and takes initial data:

$$\begin{cases} v(0,x) := e^{u(0,x)} - 1 = e^{u_0} - 1; \\ v_t(0,x) := u_t(0,x)e^{u(0,x)} = u_1 e^{u_0}. \end{cases}$$

Now, we pick initial data for the Cauchy problem (3.10), (3.53):

$$\begin{cases} u_0(x) := \frac{1}{M^S}\chi\left(\frac{x}{M^2}\right), \\ u_1(x) := \frac{A}{M^S}\chi\left(\frac{x}{M^2}\right) e^{-\int_0^{u_0} f(r)dr}\cos(x\cdot y) = \frac{A}{M^S}\chi\left(\frac{x}{M^2}\right) e^{-u_0}\cos(x\cdot y). \end{cases}$$

where M is a large natural number; $s > 2n$; $y \in \mathbb{R}^n$ such that $|y|^2 = \lambda$ and this value of λ satisfies the hypothesis from Lemma (5.8); $A = \pm 1$; and χ is a cut-off function such that $\chi\left(\frac{x}{M^2}\right) = 1$ when $|x| \leq M^2$, and $\chi\left(\frac{x}{M^2}\right) = 0$ otherwise.

Now that we have defined the initial data of the Cauchy problem, we can re-write the initial data for $v(t,x)$:

$$\begin{cases} v(0,x) = e^{u_0} - 1 = e^{\frac{1}{M^S}\chi\left(\frac{x}{M^2}\right)} - 1, \\ v_t(0,x) = u_1 e^{u_0} = \frac{A}{M^S}\chi\left(\frac{x}{M^2}\right) e^{-u_0}\cos(x\cdot y) e^{\frac{1}{M^S}\chi\left(\frac{x}{M^2}\right)}. \end{cases}$$

Now, we claim that the function

$$V(t,x) := \int_0^{\frac{1}{M^S}} e^{\int_0^s f(r)dr} + W(t)\frac{A}{M^S}\cos(x\cdot y) = e^{\frac{1}{M^S}} - 1 + W(t)\frac{A}{M^S}\cos(x\cdot y),$$

(where $W(t)$ is a solution of (3.43), $W(0)$, and $W_t(0) = 1$) is a solution for (3.52).

To show this, we first calculate the partial derivatives of $V(t,x)$:

$$\begin{aligned}
V_t &= W_t(t)\frac{A}{M^S}\cos(x\cdot y),\\
V_{tt} &= W_{tt}(t)\frac{A}{M^S}\cos(x\cdot y),\\
V_{x_i} &= -W(t)\frac{Ay_i}{M^S}\sin(x\cdot y),\\
V_{x_ix_i} &= -W(t)\frac{Ay_i^2}{M^S}\cos(x\cdot y).
\end{aligned}$$

Then, we plug in these expressions into the left-hand side of equation (3.52):

$$\begin{aligned}
V_{tt} - b^2(t)\Delta V &= V_{tt} - b^2(t)\sum_{i=1}^n V_{x_ix_i},\\
&= W_{tt}(t)\frac{A}{M^S}\cos(x\cdot y) - b^2(t)\sum_{i=1}^n\left(-W(t)\frac{Ay_i^2}{M^S}\cos(x\cdot y)\right),\\
&= \frac{A}{M^S}\cos(x\cdot y)\left[W_{tt}(t) - b^2(t)\sum_{i=1}^n\left(-W(t)y_i^2\right)\right],\\
&= \frac{A}{M^S}\cos(x\cdot y)[W_{tt}(t) + b^2(t)W(t)\underbrace{\sum_{i=1}^n y_i^2}_{=|y|^2=\lambda}],\\
&= \frac{A}{M^S}\cos(x\cdot y)\left(\underbrace{W_{tt}(t) + \lambda b^2(t)W(t)}_{=0}\right),\\
&= 0.
\end{aligned}$$

Therefore, $V(t,x)$ satisfies equation (3.52).

Now, by definition of V, we have the following initial data defined on $\mathbb{R}^n$:

$$\begin{aligned} V(0,x) &= e^{\frac{1}{M^S}} - 1 + \underbrace{W(0)}_{=0}\frac{A}{M^S}\cos(x\cdot y) = e^{\frac{1}{M^S}} - 1, \\ V_t(0,x) &= \underbrace{W_t(0)}_{=1}\frac{A}{M^S}\cos(x\cdot y) = \frac{A}{M^S}\cos(x\cdot y). \end{aligned}$$

On the other hand, by definition of initial data for $v(t,x)$ and of the cut-off function, we have that if $|x| \leq M^2$, then:

$$v(0,x) = e^{\frac{1}{M^S}} - 1, \quad \text{and} \quad v_t(0,x) = \frac{A}{M^S}\cos(x\cdot y).$$

Then, by finite propagation speed in the Cauchy problem, we have that

$$V(t,x) = v(t,x) \text{ in } \Pi_M := [0,M]\times\{x\in\mathbb{R}^n; |x| \leq M^{\frac{2}{3}}, \text{ for large } M$$

Then, the solution $v(t,x)$ of (3.52) is $v(t,x) = e^{\frac{1}{M^S}} - 1 + W(t)\frac{A}{M^S}cos(x\cdot y)$ in Π_M. In particular, when $x=0$, we have $v(t,0) = e^{\frac{1}{M^S}} - 1 + W(t)\frac{A}{M^S}$. Finally, by Lemma (5.8) for any large positive integer $M\in\mathbb{N}$, we have

$$v(M,0) = e^{\frac{1}{M^S}} - 1 + \frac{A}{M^S}\frac{b_{21}}{\mu_0 - \mu_0^{-1}}(\mu_0^M - \mu_0^{-M}).$$

If we pick $A=-1$, then $v(M,0)$ tends to $-\infty$ as M grows. Therefore, we can pick a natural number M_0 such that $v(M_0) < -1$. Since $v(t,x)$ is a continuous function, and we had previously shown that $v(t,x) > -1$ then there exist $t_0 \in [0,\infty), x_0 \in \mathbb{R}^n$ such that $v(t_0,x_0) = -1$. Then, at this point, the value of u is

$$u(t_0,x_0) = \ln(v(t_0,x_0)+1) = \ln(-1+1) \to \infty.$$

Therefore, the solution has a blow-up at time t_0.

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BIOGRAPHICAL SKETCH

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