

Assignment8

CS20Btech11035 -NYALAPOGULA MANASWINI

Download python code from

<https://github.com/N-Manaswini23/Assignment8/tree/main/python%20codes>

Download latex code from

<https://github.com/N-Manaswini23/Assignment8/blob/main/assignment8.tex>

CSIR UGC JUNE 2013 QUESTION 70

Let X and Y be independent random variables each following a uniform distribution on $(0, 1)$. Let $W = XI_{\{Y \leq X^2\}}$, where I_A denotes the indicator function of set A . Then which of the following statements are true?

- 1) The cumulative distribution function of W is given by

$$F_W(t) = t^2 I_{\{0 \leq t \leq 1\}} + I_{\{t > 1\}} \quad (0.0.1)$$

- 2) $P[W > 0] = \frac{1}{3}$
- 3) The cumulative distribution function of W is continuous
- 4) The cumulative distribution function of W is given by

$$F_W(t) = \left(\frac{2+t^3}{3}\right) I_{\{0 \leq t \leq 1\}} + I_{\{t > 1\}} \quad (0.0.2)$$

SOLUTION

Given X and Y be two independent random variables.

Given $W = XI_{\{Y \leq X^2\}}$

$X \in (0, 1)$, $Y \in (0, 1)$, $W \in [0, 1]$

The PDF for X is

$$p_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.3)$$

The CDF for X is

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & \text{otherwise} \end{cases} \quad (0.0.4)$$

The PDF for Y is

$$p_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.5)$$

The CDF for Y is

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & \text{otherwise} \end{cases} \quad (0.0.6)$$

$I_{\{Y \leq X^2\}}$ is defined as follows

$$I_{\{Y \leq X^2\}} = \begin{cases} 1 & y \leq x^2 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.7)$$

W is defined as follows

$$W = \begin{cases} x & y \leq x^2 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.8)$$

From (0.0.8)

$$p_W(W = 0) = \Pr(I_{\{Y \leq X^2\}} = 0) \quad (0.0.9)$$

$$= \Pr(x^2 < y) \quad (0.0.10)$$

Let $Z = X^2 - Y$ be a random variable where $Z \in (-1, 1)$

$$F_{X^2}(u) = \Pr(X^2 \leq u) \quad (0.0.11)$$

$$= \Pr(X \leq \sqrt{u}) \quad (0.0.12)$$

$$= F_X(\sqrt{u}) \quad (0.0.13)$$

From (0.0.4), The CDF for X^2 is

$$F_{X^2}(u) = \begin{cases} 0 & u \leq 0 \\ \sqrt{u} & 0 < u < 1 \\ 1 & \text{otherwise} \end{cases} \quad (0.0.14)$$

The PDF for X^2 is

$$p_{X^2}(u) = \begin{cases} \frac{1}{2\sqrt{u}} & 0 < u < 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.15)$$

$$F_{\{-Y\}}(v) = \Pr(-Y \leq v) \quad (0.0.16)$$

$$= \Pr(Y \geq -v) \quad (0.0.17)$$

$$= 1 - F_Y(-v) \quad (0.0.18)$$

From (0.0.6), The CDF for $(-Y)$ is

$$F_{\{-Y\}}(v) = \begin{cases} 0 & v \leq -1 \\ 1 + v & -1 < v < 0 \\ 1 & \text{otherwise} \end{cases} \quad (0.0.19)$$

The PDF for $(-Y)$ is

$$p_{\{-Y\}}(v) = \begin{cases} 1 & -1 < v < 0 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.20)$$

$$Z = X^2 - Y \implies z = u + v$$

Using convolution

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X^2}(z - v) p_{\{-Y\}}(v) dv \quad (0.0.21)$$

Solving (0.0.21) using (0.0.20), (0.0.15) for $z \in (-1, 1)$, we get PDF of Z as follows

$$p_Z(z) = \begin{cases} \sqrt{z+1} & -1 < z \leq 0 \\ 1 - \sqrt{z} & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.22)$$

CDF of Z as follows

$$F_Z(z) = \begin{cases} \frac{2}{3}(z+1)^{\frac{3}{2}} & -1 < z \leq 0 \\ z - \frac{2}{3}z^{\frac{3}{2}} & 0 < z < 1 \\ 1 & \text{otherwise} \end{cases} \quad (0.0.23)$$

using (0.0.23) to find $p_W(W = 0)$

$$p_W(W = 0) = \Pr(x^2 < y) \quad (0.0.24)$$

$$= F_z(0) \quad (0.0.25)$$

$$= \frac{2}{3} \quad (0.0.26)$$

$$W = t \implies X = t \text{ where } t \in (0, 1)$$

$$p_W(t) = \int_{-\infty}^{\infty} p_X(t) I_{\{y \leq t^2\}} dy \quad (0.0.27)$$

$$0 < y < 1 \quad (0.0.28)$$

$$0 < y \leq t^2 \quad (0.0.29)$$

For $0 < t < 1$,

$$p_W(t) = \int_0^{t^2} p_X(t) I_{\{y \leq t^2\}} dy \quad (0.0.30)$$

$$= t^2 \quad (0.0.31)$$

\therefore PDF of W is as follows

$$p_W(t) = \begin{cases} \frac{2}{3} & t = 0 \\ t^2 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.32)$$

The CDF of W is as follows:

$$F_W(t) = \begin{cases} 0 & t < 0 \\ \frac{2+t^3}{3} & 0 \leq t \leq 1 \\ 1 & \text{otherwise} \end{cases} \quad (0.0.33)$$

CDF of W is discontinuous at $W = 0$.

The CDF above can be written as

$$F_W(t) = \left(\frac{2+t^3}{3} \right) I_{\{0 \leq t \leq 1\}} + I_{\{t > 1\}} \quad (0.0.34)$$

$$\Pr(W > 0) = 1 - F_W(0) \quad (0.0.35)$$

$$= \frac{1}{3} \quad (0.0.36)$$

$$\therefore \Pr(W > 0) = \frac{1}{3} \quad (0.0.37)$$

\therefore option 2 and 4 are correct.



