Assignment8

CS20Btech11035 -NYALAPOGULA MANASWINI

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CSIR UGC JUNE 2013 QUESTION 70

Let X and Y be independent random variables each following a uniform distribution on $(0,1)$.Let $W = XI_{\{Y\leq X^2\}}$,where I_A denotes the indicator function of set A.Then which of the following statements are true?

1) The cumulative distribution function of W is given by
$$F_W(t) = t^2 I_{\{0\leq t\leq 1\}} + I_{\{t>1\}} \tag{0.0.1}$$
2) $P[W > 0] = \frac{1}{3}$
3) The cumulative distribution function of W is continuous
4) The cumulative distribution function of W is given by
$$F_W(t) = \left(\frac{2+t^3}{3}\right) I_{\{0\leq t\leq 1\}} + I_{\{t>1\}} \tag{0.0.2}$$

ANSWER

Option 2 and 4 are correct.

SOLUTION

Given X and Y be two independent random variables.
Given $W = XI_{\{Y\leq X^2\}}$
$X \in (0,1)$, $Y \in (0,1)$, $W \in [0,1)$
The PDF for X is

$$p_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{0.0.3}$$

The CDF for X is

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & otherwise \end{cases} \tag{0.0.4}$$

The PDF for Y is

$$p_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & otherwise \end{cases} \tag{0.0.5}$$

The CDF for Y is

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & otherwise \end{cases} \tag{0.0.6}$$

$I_{\{Y\leq X^2\}}$ is defined as follows

$$I_{\{Y\leq X^2\}} = \begin{cases} 1 & y \leq x^2 \\ 0 & otherwise \end{cases} \tag{0.0.7}$$

W is defined as follows

$$W = \begin{cases} x & y \leq x^2 \\ 0 & otherwise \end{cases} \tag{0.0.8}$$

From (0.0.8)

$$p_W(W = 0) = \Pr(I_{\{Y\leq X^2\}} = 0) \tag{0.0.9}$$
$$= \Pr(x^2 < y) \tag{0.0.10}$$

Let $Z = X^2 - Y$ be a random variable where $Z \in (-1,1)$

$$F_{X^2}(u) = \Pr(X^2 \leq u) \tag{0.0.11}$$
$$= \Pr(X \leq \sqrt{u}) \tag{0.0.12}$$
$$= F_X(\sqrt{u}) \tag{0.0.13}$$

From (0.0.4),The CDF for X^2 is

$$F_{X^2}(u) = \begin{cases} 0 & u \leq 0 \\ \sqrt{u} & 0 < u < 1 \\ 1 & otherwise \end{cases} \tag{0.0.14}$$

The PDF for X^2 is

$$p_{X^2}(u) = \begin{cases} \frac{1}{2\sqrt{u}} & 0 < u < 1 \\ 0 & otherwise \end{cases} \quad (0.0.15)$$

$$F_{\{-Y\}}(v) = \Pr(-Y \le v) \quad (0.0.16)$$
$$= \Pr(Y \ge -v) \quad (0.0.17)$$
$$= 1 - F_Y(-v) \quad (0.0.18)$$

From (0.0.6),The CDF for $(-Y)$ is

$$F_{\{-Y\}}(v) = \begin{cases} 0 & v \le -1 \\ 1+v & -1 < v < 0 \\ 1 & otherwise \end{cases} \quad (0.0.19)$$

The PDF for $(-Y)$ is

$$p_{\{-Y\}}(v) = \begin{cases} 1 & -1 < v < 0 \\ 0 & otherwise \end{cases} \quad (0.0.20)$$

$Z = X^2 - Y \implies z = u + v$
Using convolution

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X^2}(z-v)p_{\{-Y\}}(v)\text{dv} \quad (0.0.21)$$

Solving (0.0.21) using (0.0.20),(0.0.15) for $z \in (-1,1)$, we get PDF of Z as follows

$$p_Z(z) = \begin{cases} \sqrt{z+1} & -1 < z \le 0 \\ 1 - \sqrt{z} & 0 < z < 1 \\ 0 & otherwise \end{cases} \quad (0.0.22)$$

CDF of Z as follows

$$F_Z(z) = \begin{cases} \frac{2}{3}(z+1)^{\frac{3}{2}} & -1 < z \le 0 \\ z - \frac{2}{3}z^{\frac{3}{2}} & 0 < z < 1 \\ 1 & otherwise \end{cases} \quad (0.0.23)$$

using (0.0.23) to find $p_W(W = 0)$

$$p_W(W=0) = \Pr(x^2 < y) \quad (0.0.24)$$
$$= F_z(0) \quad (0.0.25)$$
$$= \frac{2}{3} \quad (0.0.26)$$

$W = t \implies X = t$ where $t \in (0,1)$

$$p_W(t) = \int_{-\infty}^{\infty} p_X(t)I_{\{y \le t^2\}}\text{dy} \quad (0.0.27)$$

$$0 < y < 1 \quad (0.0.28)$$
$$0 < y \le t^2 \quad (0.0.29)$$

For $0 < t < 1$,

$$p_W(t) = \int_0^{t^2} p_X(t)I_{\{y \le t^2\}}\text{dy} \quad (0.0.30)$$
$$= t^2 \quad (0.0.31)$$

$\therefore$ PDF of W is as follows

$$p_W(t) = \begin{cases} \frac{2}{3} & t = 0 \\ t^2 & 0 < t < 1 \\ 0 & otherwise \end{cases} \quad (0.0.32)$$

The CDF of W is as follows:

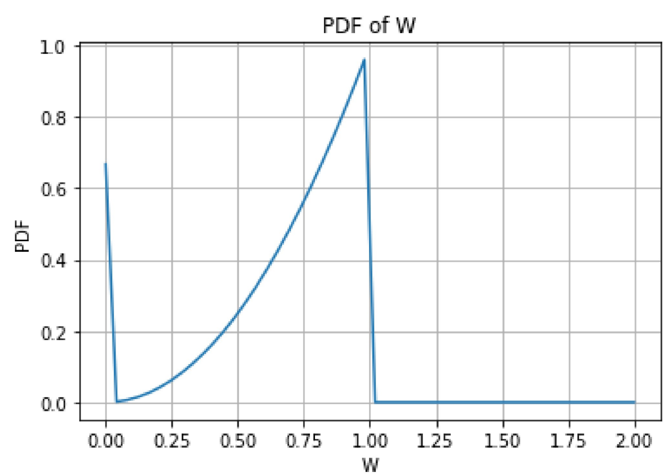
$$F_W(t) = \begin{cases} 0 & t < 0 \\ \frac{2+t^3}{3} & 0 \le t \le 1 \\ 1 & otherwise \end{cases} \quad (0.0.33)$$

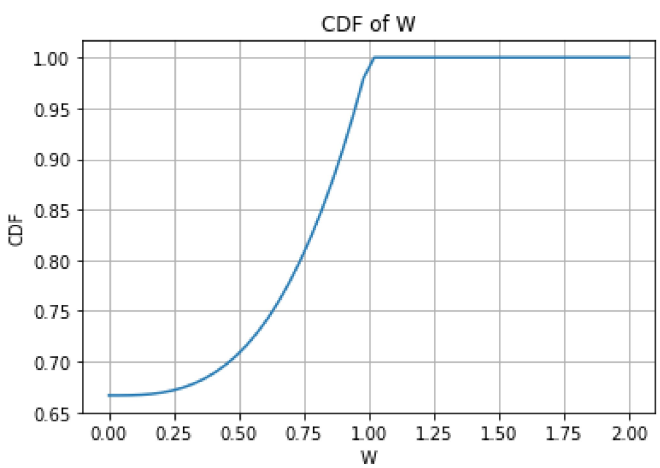
CDF of W is discontinuous at $W = 0$.
The CDF above can be written as

$$F_W(t) = \left(\frac{2+t^3}{3}\right)I_{\{0 \le t \le 1\}} + I_{\{t > 1\}} \quad (0.0.34)$$

$$\Pr(W > 0) = 1 - F_W(0) \quad (0.0.35)$$
$$= \frac{1}{3} \quad (0.0.36)$$
$$\therefore \Pr(W > 0) = \frac{1}{3} \quad (0.0.37)$$

$\therefore$ option 2 and 4 are correct.





CDF of W
CDF
W
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00