Logarithmic regret algorithms for online convex optimization

Report

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May 1, 2023

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Problem Statement

Online convex optimization is a fundamental problem in machine learning, control systems, and finance. The problem involves making sequential decisions in an unknown and dynamic environment where the goal is to minimize the cumulative regret incurred over time. The goal of online convex optimization is to design algorithms that achieve sublinear regret bound of $O(logT)$.

Existing online convex optimization algorithms of Zinkevich achieve sublinear regret bound of $O(\sqrt{T})$. However, these bounds may not be sufficient for practical applications where the regret needs to be minimized as much as possible. Therefore, the aim is to propose and analyze new algorithms for online convex optimization that achieve logarithmic regret bounds, which would represent a significant improvement over existing methods for which we will use the paper, titled "Logarithmic regret algorithms for online convex optimization," which is authored by Elad Hazan, Amit Agarwal, and Satyen Kale. The authors propose and analyze new online convex optimization algorithms that achieve logarithmic regret bounds. The paper focuses on developing algorithms that have low computational and memory requirements while also achieving logarithmic regret bounds.

Introduction

The proposed algorithms are modification of existing algorithms, which are widely used methods in online convex optimization. The authors modifies the algorithms to achieve logarithmic regret bounds. Specifically, they propose three algorithms: one that achieves logarithmic regret bounds for strongly convex functions, and other two that achieves logarithmic regret bounds for general convex functions. The three algorithms are Online Gradient Descent(OGD), Online Newton Step(ONS) and Follow-The-Approximate-Leader(FTAL).

The proposed algorithms have practical applications in fields such as machine learning, control systems, and finance. For example, the algorithms can be used in online portfolio optimization, where the goal is to allocate resources optimally among different assets in a dynamic environment. The proposed algorithms can also be used in online learning, where the goal is to learn a hypothesis that minimizes the expected loss over a sequence of examples.

In conclusion, the paper "Logarithmic regret algorithms for online convex optimization" proposes and analyzes new algorithms for online convex optimization that achieve logarithmic regret bounds. The proposed algorithms achieve logarithmic regret bounds for strongly convex and general convex functions. The authors provide theoretical analysis and numerical experiments that demonstrate the effectiveness of the proposed algorithms in practice. The proposed algorithms have practical applications in fields such as machine learning, control systems, and finance.

Notations and Definitions

3.1 Online Convex optimization

Online convex optimization (OCO) is a subfield of machine learning and optimization that deals with optimizing a convex function over a sequence of data points that arrive in an online fashion. In other words, OCO is concerned with making optimal decisions in an online setting where the objective function is not known in advance, but rather, revealed sequentially over time.
In OCO, the decision maker receives a sequence of inputs and must make a decision for each input, based on the current knowledge of the objective function. The decision maker then receives feedback in the form of a convex cost function, which evaluates the quality of the decision made. The goal is to minimize the cumulative cost over the entire sequence of inputs.

3.2 Regret

Regret in online convex optimization refers to the difference between the total cost incurred by an online algorithm over a sequence of decisions and the cost that would have been incurred by an optimal offline algorithm that has access to the entire sequence of decisions in advance. In other words, regret measures the performance loss of an online algorithm compared to the best possible algorithm that has complete knowledge of the sequence of decisions.

In online convex optimization, the decision maker does not have full knowledge of the problem and has to make decisions based on partial information that is revealed over time. The goal is to minimize the cost incurred by the decisions made while learning from the feedback provided by the environment. Regret is a key performance measure in this setting, as it quantifies the effectiveness of the learning algorithm and its ability to adapt to changing environments. Sometimes, We calculate regret in terms of payoff instead of cost incurred.

In online convex optimization, an online player iteratively chooses a point from a set in Euclidean space denoted $\mathcal{P} \subset \mathbb{R}^n$.Let $\mathcal{A}$ be an algorithm used by an online player.Let us denote number of iterations by T. Let x_t be the point that player chooses in t^{th} iteration. After choosing x_t, $f_t : \mathcal{P} \leftarrow \mathbb{R}$ is revealed. Now regret is defined as follows:

$$x_t = \mathcal{A}(\{f_1, f_2, ..., f_{t-1}\})$$
$$\text{Regret}(\mathcal{A}, \{f_1, f_2, ..., f_t\}) = \mathbb{E}\left[\sum_{t=1}^{T} f_t(x_t)\right] - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x)$$

Worst case guaranteed regret is denoted and defined as follows:

$$\text{Regret}_T(\mathcal{A}) = \sup_{\{f_1, f_2, \ldots\ldots, f_{t-1}\}} \{\text{Regret}(\mathcal{A}, \{f_1, f_2, \ldots\ldots, f_t\})\}$$

3.3 Notations

3.3.1 Diameter of the convex set

Denote by $\mathcal{D}$ the diameter of the underlying convex set $\mathcal{P}$, i.e.

$$\mathcal{D} = \max_{x,y \in \mathcal{P}} \|x - y\|_2$$

3.3.2 Upper bound of gradient

Denoted by G, we say that the cost functions have gradients upper bounded if the following holds:

$$\sup_{x \in \mathcal{P}, t \in [T]} \|\nabla f_t(x)\|_2 \leq G$$

3.3.3 Hessian of a function

The Hessian of a function f at a point x is a matrix $\nabla^2 f(X)$ such that

$$\nabla^2 f(x)[i, j] = \frac{\delta^2}{\delta x_i \delta x_j} f(x)$$

We say that the Hessian of the cost function is lower bounded by $H > 0$ (also called H-strong convex) if

$$\forall x \in \mathcal{P}, t \in [T] : \nabla^2 f_t(x) \succcurlyeq H\mathbf{I}_n$$

3.3.4 α-exp-concavity property

We say that a function f_t satisfies α-exp-concavity property if

$$\forall x \in \mathcal{P}, t \in [T] : \nabla^2[\exp\{-\alpha f_t(x)\}] \preccurlyeq 0$$

3.3.5 *Remark*

There are two classes of class of cost functions. One are those which have bounded gradient and are H-strong convex for some $H > 0$. The second are those that satisfy the α-exp-concavity property.
We require the payoff functions to be concave instead of convex and the payoff functions will be assumed to be (α)-exp-concave.

Online Gradient Descent

Author modified Zinkevich's Online gradient descent a bit to achieve logarithmic regret.
Runtime: $O(n)$ per iteration given gradient + $O(n^4 log\frac{R}{r})$ for projection in each iteration.

4.1 Algorithm

Theorem 1. *Updating step size with $\eta_t = \frac{1}{H_t}$ achieves logarithmic regret.*

$$Regret_t(OGD) \leq \frac{G^2}{2H}(1 + \log(T))$$

Proof. Let f_t be a H-strong convex function

$$Regret_t(OGD) = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*)$$

By using taylor series approximation and H-strong convexity, we get

$$f_t(x^*) \geq f_t(x_t) + \nabla_t^T(x^* - x_t) + \frac{H}{2}\|x^* - x_t\|^2 \tag{4.1}$$

Which is followed by Zenkevich's analysis to upper bound $\nabla_t^\top(x_t - x^*)$, also using the update rule for x_{t+1} and the properties of projection (see lemma 3), we get:

$$\|x_{t+1} - x^*\|^2 = \|\Pi(x_t - \eta_{t+1}\nabla_t) - x^*\|^2 \tag{4.2}$$

$$\leq \|x_t - \eta_{t+1}\nabla_t - x^*\|^2 \tag{4.3}$$

$$5\nabla_t^\top(x_t - x^*) \leq \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{\eta_{t+1}} + \eta_{t+1}G^2 \tag{4.4}$$

Simplifying above equation and substituting $\eta_t = \frac{1}{H_t}$ along with simplified equation in 4.1, we get

$$\begin{aligned} 2\sum_{t=1}^{T} f_t(x_t) - f_t(x^*) &\leq \sum_{t=1}^{T} \|x_t - x^*\|^2 \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} - H\right) + G^2\sum_{t=1}^{T}\eta_{t+1} \\ &= G^2\sum_{t=1}^{T}\frac{1}{H_t} \\ &\leq \frac{G^2}{H}(1 + \log T) \end{aligned}$$

Hence proved. □

4.2 Running Time

This algorithm is pretty straight forward and has a complexity of $O(n)$ per iteration given the gradient. However the projection step may take longer. this has been discussed in Chapter 7

Online Newton Step(ONS)

Online Newton Step is a slight modification of Newton–Raphson method. The difference lies in the step of update where in the Newton-Raphson method uses the vector which is the multiple of Hessian and the gradient whereas here the author uses $A_t^{-1}\nabla_t$ where the matrix A_t is releted to Hessian but not the same.

5.1 Algorithm

Lemma 1. *Let $f : \mathcal{P} \to \mathbb{R}$ be an α-exp concave function such that $\forall x \in \mathcal{P}, \|\nabla f(x)\| \leq G$. D is the diameter of $\mathcal{P}$. Then for $\forall x, y \in \mathcal{P}, \beta \leq \frac{1}{2}min\{\frac{1}{4GD}, \alpha\}$, the following holds:*

$$f(x) \geq f(y) + \nabla^\top(x-y) + \frac{\beta}{2}(x-y)^\top \nabla f(y) \nabla f(y)^\top (x-y) \tag{5.1}$$

Proof. let $h(x) = \exp\{-2\beta f(x)\}$.
Given that f is α-exp concave $\implies$ $\exp\{-2\alpha f(x)\}$ is concave $\implies$ $h(x)$ is concave. (since $2\beta \leq \alpha$).

$$h(x) \leq h(y) + \nabla h(y)^\top (x-y)$$

We know that $\nabla h(y) = -2\beta \exp\{-2\beta f(y)\} \nabla f(y)$. Substituting this in above equation and simplifying it, we get

$$f(x) \geq f(y) - \frac{1}{2\beta} \log[1 - 2\beta \nabla f(y)^\top (x-y)]. \tag{5.2}$$

We know that $\|\nabla f(y)\| \leq G \implies \left\|\nabla f(y)^\top\right\| \leq G$ and $|x-y| \leq D$.
So, $2\beta \nabla f(y)^\top (x-y) \leq \beta G D \leq \frac{1}{4}$.
This implies,

$$-\log[1 - 2\beta \nabla f(y)^\top (x-y)] \geq 2\beta \nabla f(y)^\top (x-y) + \frac{1}{4}(2\beta \nabla f(y)^\top (x-y))^2 \tag{5.3}$$

Substitute equation (5.3) in equation (5.2) to obtain equation (5.1).

□

Theorem 2. *Assume that for all t the loss function $f_t : \mathcal{P} \to \mathbb{R}^n$ is α-exp-concave and has the property that $\forall x \in \mathcal{P}, \|\nabla f(x)\| \leq G$. Then the algorithm ONS has the following regret bound:*

$$Regret_T(ONS) \leq 5\left(\frac{1}{\alpha} + GD\right) n \log T$$

where $\mathcal{P}$ is the convex set, D is the diameter of the convex set $\mathcal{P}$ and G is the upper bound of the gradient of the function f.

Proof. Let $x^* \in argmin_{x\in\mathcal{P}} \sum_{t=1}^{T} f_t(x)$ be the best decision so far. Substituting $y = x_t$ and $x = x^*$ lemma 1, we get

$$f_t(x_t) - f_t(x^*) \leq R_t \triangleq \nabla_t^\top (x_t - x^*) - \frac{\beta}{2}(x^* - x_t)^\top \nabla_t \nabla_t^\top (x^* - x_t) \tag{5.4}$$

where $\beta = \frac{1}{2} min\{\frac{1}{4GD}, \alpha\}$
Lets define $y_{t+1} = x_t - \frac{1}{\beta} A_t^{-1} \nabla_t$ for convenience where, $A_t = \sum_{i=1}^{t} \nabla_i \nabla_i^\top + \epsilon \mathbf{I}_n, \epsilon = \frac{1}{\beta^2 D^2}$. Note that A_t is symmetric.
Now according to the update rule of the algorithm we have $x_{t+1} = \Pi_{s_n}^{A_t}(y_{t+1})$. Now from definition of y_{t+1}:

$$y_{t+1} - x^* = x_t - x^* - \frac{1}{\beta} A_t^{-1} \nabla_t \tag{5.5}$$

$$A_t(y_{t+1} - x^*) = A_t(x_t - x^*) - \frac{1}{\beta}\nabla_t \tag{5.6}$$

Multiplying the transpose of 5.5 and 5.6 we get:

$$\begin{aligned}
&(y_{t+1} - x^*)^\top A_t (y_{t+1} - x^*) \\
&= (x_t - x^* - \frac{1}{\beta} A_t^{-1} \nabla_t)^\top (A_t(x_t - x^*) - \frac{1}{\beta}\nabla_t) \\
&= (x_t - x^*)^\top A_t (x_t - x^*) - \frac{1}{\beta}(x_t - x^*)^\top \nabla_t - \frac{1}{\beta}(A_t^{-1}\nabla_t)^\top A_t(x_t - x^*) + \frac{1}{\beta^2}(A_t^{-1}\nabla_t)^\top \nabla_t
\end{aligned}$$

$$(y_{t+1} - x^*)^\top A_t (y_{t+1} - x^*) = (x_t - x^*)\top A_t (x_t - x^*) - \frac{2}{\beta}\nabla_t^\top (x_t - x^*) + \frac{1}{\beta^2}\nabla_t^\top A_t^{-1} \nabla_t \tag{5.7}$$

Since the projection of the points as opposed to the actual points are considered, we observe that from Lemma 3 (x_{t+1} is the projection of y_{t+1}):

$$(y_{t+1} - x^*)^\top A_t (y_{t+1} - x^*) \geq (x_{t+1} - x^*)^\top A_t (x_{t+1} - x^*)$$

This together with equation 5.7 gives us:

$$\nabla_t^\top (x_{t+1} - x^*) \leq \frac{1}{2\beta}\nabla_t^\top A_t^{-1} \nabla_t + \frac{\beta}{2}(x_t - x^*)^\top A_t (x_t - x^*) - \frac{\beta}{2}(x_{t+1} - x^*)^\top A_t (x_{t+1} - x^*)$$

Now using summation from $t = 1$ to T:

$$\begin{aligned}
\sum_{t=1}^{T} \nabla_t^\top (x_{t+1} - x^*) \leq{} & \frac{1}{2\beta} \sum_{t=1}^{T} \nabla_t^\top A_t^{-1} \nabla_t \\
& + \frac{\beta}{2} \sum_{t=1}^{T} (x_t - x^*)^\top A_t (x_t - x^*) \\
& - \frac{\beta}{2} \sum_{t=1}^{T} (x_{t+1} - x^*)^\top A_t (x_{t+1} - x^*) \\
\\
\leq{} & \frac{1}{2\beta} \sum_{t=1}^{T} \nabla_t^\top A_t^{-1} \nabla_t \\
& + \frac{\beta}{2} (x_1 - x^*)^\top A_1 (x_1 - x^*) \\
& + \frac{\beta}{2} \sum_{t=2}^{T} (x_t - x^*)^\top (A_t - A_{t-1}) (x_t - x^*) \\
& - \frac{\beta}{2} (x_{T+1} - x^*)^\top A_T (x_{T+1} - x^*)
\end{aligned}$$

$$A_t - A_{t-1} = \nabla_t \nabla_t^\top$$

$$\begin{aligned}
\sum_{t=1}^{T} \nabla_t^\top (x_{t+1} - x^*) \leq{} & \frac{1}{2\beta} \sum_{t=1}^{T} \nabla_t^\top A_t^{-1} \nabla_t \\
& + \frac{\beta}{2} (x_1 - x^*)^\top A_1 (x_1 - x^*) \\
& + \frac{\beta}{2} \sum_{t=2}^{T} (x_t - x^*)^\top (\nabla_t \nabla_t^\top) (x_t - x^*) \\
& - \frac{\beta}{2} (x_{T+1} - x^*)^\top A_T (x_{T+1} - x^*)
\end{aligned}$$

Adding and Subtracting $(x_1 - x^*)^\top \nabla_1 \nabla_1^\top (x_1 - x^*)$ on RHS and neglecting $\frac{\beta}{2}(x_{T+1} - x^*)^\top A_T (x_{T+1} - x^*)$ as we consider inequality

$$\begin{aligned}\sum_{t=1}^{T} \nabla_t^\top (x_{t+1} - x^*) \leq \frac{1}{2\beta} &\sum_{t=1}^{T} \nabla_t^\top A_t^{-1} \nabla_t \\ &+ \frac{\beta}{2} \sum_{t=1}^{T} (x_t - x^*)^\top (\nabla_t \nabla_t^\top)(x_t - x^*) \\ &+ \frac{\beta}{2} (x_1 - x^*)^\top (A_1 - \nabla_1 \nabla_1^\top)(x_1 - x^*)\end{aligned}$$

Taking the second term in the RHS to the LHS, it gives the expression for $\sum_{t=1}^{T} R_t$ (refer eq (5.4)), we get:

$$\sum_{t=1}^{T} R_t \leq \frac{1}{2\beta} \sum_{t=1}^{T} \nabla_t^\top A_t^{-1} \nabla_t + \frac{\beta}{2} (x_1 - x^*)^\top (A_1 - \nabla_1 \nabla_1^\top)(x_1 - x^*)$$

Using $A_1 - \nabla_1 \nabla_1^\top = \epsilon I_n$ and $\|x_1 - x^*\|^2 \leq D^2$ and $\epsilon = \frac{1}{\beta^2 D^2}$

$$\begin{aligned}Regret_T(ONS) &\leq \sum_{t=1}^{T} R_t \\ &\leq \frac{1}{2\beta} \sum_{t=1}^{T} \nabla_t^\top A_t^{-1} \nabla_t + \frac{1}{2\beta}\end{aligned}$$

Using Lemma 4, we can put $V_t = A_t$, $u_t = \nabla_t$, $r = G$ and $\epsilon = \frac{1}{\beta^2 D^2}$, we get for the first term,

$$\begin{aligned}\frac{1}{2\beta} \sum_{t=1}^{T} \nabla_t^\top A_t^{-1} \nabla_t &\leq \frac{n}{2\beta} \log\left(\frac{G^2 T}{\epsilon} + 1\right) \\ &\leq \frac{n}{2\beta} \log\left(T G^2 \beta^2 D^2 + 1\right) \\ &\leq \frac{n}{2\beta} \log(T)\end{aligned}$$

Since $\beta = \frac{1}{2} min\{\frac{1}{4GD}, \alpha\}$, we have $\frac{1}{\beta} \leq 8(GD + \frac{1}{\alpha})$. Then we get,

$$\begin{aligned}Regret_T(ONS) &\leq \frac{n}{2} \log(T)(8(GD + \frac{1}{\alpha}) \\ &\leq 4n \left(GD + \frac{1}{\alpha}\right) \log(T)\end{aligned}$$

This here proofs the stated regret bound. $\square$

5.2 Running time

Space complexity of ONLINE NEWTON STEP is $\mathcal{O}(n^2)$ which is required to store the matrix $\mathbf{A}$. In every step, we need to compute A_t^{-1}, current gradient, matrix-vector product, projection. A_t^{-1} can be computed from $\mathbf{A}_{t-1}^{-1}$ in $\mathcal{O}(n^2)$ using the equation below:

$$A_t^{-1} = (\mathbf{A}_{t-1} + \nabla\nabla^T) = \mathbf{A}_{t-1}^{-1} - \frac{\mathbf{A}_{t-1}^{-1}\nabla\nabla^T\mathbf{A}_{t-1}^{-1}}{1 + \nabla^\top\mathbf{A}_{t-1}^{-1}\nabla} \tag{5.8}$$

Projection step $\arg\min_{x\in\mathcal{P}}(y-x)^\top A_t^{-1}(y-x)$ is a convex optimization problem and takes polynomial time. To sum up, space and time complexities of ONLINE NEWTON STEP is $\mathcal{O}(n^2)$.

Follow the Approximate Leader(FTAL)

6.1 Algorithm

Follow the Approximate Leader is a modification of Follow the leader(FTL) and it guarentees logarithmic regret.

Lemma 2. *FTAL is equivalent to FTL.*

Proof. Optimization step of FTL at period T is as follows:

$$\begin{aligned}
x_t &\triangleq \arg\min_{x\in\mathcal{P}} \sum_{\tau=1}^{t-1} \tilde{f}_\tau(x) \\
&= \arg\min_{x\in\mathcal{P}} \sum_{\tau=1}^{t-1} f_\tau(x_\tau) + \nabla_t^\top (x - x_\tau) + \frac{\beta}{2}(x - x_\tau)^\top \nabla\nabla^\top (x - x_\tau) \\
&= \arg\min_{x\in\mathcal{P}} \sum_{\tau=1}^{t-1} f_\tau(x_\tau) + \nabla_t^\top x - \nabla_t^\top x_\tau + \frac{\beta}{2}x^\top \nabla_t\nabla_t^\top x - \frac{\beta}{2}x^\top \nabla_t\nabla_t^\top x_\tau - \frac{\beta}{2}x_\tau^\top \nabla_t\nabla_t^\top x + \frac{\beta}{2}x_\tau^\top \nabla_t\nabla_t^\top x_\tau \\
&= \arg\min_{x\in\mathcal{P}} \sum_{\tau=1}^{t-1} \nabla_t^\top x + \frac{\beta}{2}x^\top \nabla_t\nabla_t^\top x - \frac{\beta}{2}x^\top \nabla_t\nabla_t^\top x_\tau - \frac{\beta}{2}x_\tau^\top \nabla_t\nabla_t^\top x \\
&= \arg\min_{x\in\mathcal{P}} \sum_{\tau=1}^{t-1} -(\beta x_\tau^\top \nabla_t\nabla_t^\top - \nabla^\top)x + \frac{\beta}{2}x^\top \nabla_t\nabla_t^\top x
\end{aligned}$$

Substituting $b_{t-1} = \sum_{\tau=1}^{t-1} \nabla_\tau\nabla_\tau^\top x_\tau - \frac{1}{\beta}\nabla_\tau$ and $\mathbf{A}_{t-1} = \sum_{\tau=1}^{t-1} \nabla_\tau\nabla_\tau^\top$, we get

$$\begin{aligned}
x_t &= \arg\min_{x\in\mathcal{P}} \left\{ \frac{\beta}{2}x^\top \mathbf{A}_{t-1}x - \beta b_{t-1}^\top x \right\} \\
&= \arg\min_{x\in\mathcal{P}} \left\{ x^\top \mathbf{A}_{t-1}x - 2b_{t-1}^\top x \right\} \\
&= \arg\min_{x\in\mathcal{P}} \left\{ (x - \mathbf{A}_{t-1}^{-1}b_{t-1})^\top \mathbf{A}_{t-1}(x - \mathbf{A}_{t-1}^{-1}b_{t-1}) - b_{t-1}^\top \mathbf{A}_{t-1}^{-1}b_{t-1} \right\} \\
&= \prod_{\mathcal{P}}^{\mathbf{A}_{t-1}} (\mathbf{A}_{t-1}^{-1}b_{t-1})
\end{aligned}$$

This is equivalent to FTAL.
Hence proved. □

6.2 Analysis of FTL and FTAL

FTL algorithm analysis is done in standard way as follows:
First, lemma (7) is proved by induction. Next it is shown that leader doesn't change i.e, $x_t \approx x_{t+1}$. This implies low regret.
But the FTAL algorithm analysis does not follow this method. In the FTAL analysis, it is shown that on average $x_t \approx x_{t+1}$.
Lemma (1) proves that α-exp-concave functions can be lower-bounded by a paraboloid $\tilde{f}_t(x) = a + (v^\top x - b)^2$ where $v \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$ is a multiple of $\nabla f_t(x_t)$. The important technical step now is to demonstrate that FOLLOW THE LEADER has regret $O(logT)$ when executed on a certain class of cost functions, which includes the paraboloid functions mentioned above.

Theorem 3. *Assume that $\forall t$, the function $f_t : \mathcal{P} \to \mathbb{R}^n$ can be written as $f_t(x) = g_t(v_t^\top x)$ for a univariate convex function $g_t : \mathbb{R} \to \mathbb{R}$ and some vector $v_t \in \mathbb{R}^n$. Assume that for some $R, a, b > 0$, we have $\|v_t\|_2 \leq R$, and $\forall x \in \mathcal{P}$, we have $\left|g_t(v_t^\top x)\right| \leq b$ and $g_t^{''}(v_t^\top x) \geq a$. Then the FOLLOW THE LEADER algorithm on the functions f_t satisfies the following regret bound:*

$$Regret_T(FTL) \leq \frac{2nb^2}{a}\left[\log\left(\frac{DRaT}{b}\right) + 1\right]$$

Proof. From Lemma 5 we can see that:

$$\sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x) \leq \sum_{t=1}^{T} [f_t(x_t) - f_t(x_{t+1})] \tag{6.1}$$

Here some notations are introduced to increase readability:

$$F_t \triangleq \sum_{\tau=1}^{t-1} f_\tau$$

$$\Delta x_t = x_{t+1} - x_t$$

$$\Delta\nabla F_t(x_t) = (\nabla F_{t+1}(x_{t+1}) - \nabla F_t(x_t))$$

We observe that $\forall \tau$

$$\nabla f_\tau(x) = g_\tau^{'}(v_\tau^\top x)v_\tau$$

$$\nabla_t = \nabla f_t(x_t) = g_t^{'}(v_t^\top x)v_t$$

Since f is a convex function we know that:

$$f_t(x_t) - f_t(x_{t+1}) \leq -\nabla f_t(x_t)^\top (x_{t+1} - x_t) = -\nabla_t \Delta x_t \tag{6.2}$$

Now consider $\nabla F_{t+1}(x_{t+1}) - \nabla F_{t+1}(x_t)$, we have,

$$\nabla F_{t+1}(x_{t+1}) - \nabla F_{t+1}(x_t) = \sum_{\tau=1}^{t} \nabla f_\tau(x_{t+1} - \nabla f_\tau(x_t) \tag{6.3}$$

$$= \sum_{\tau=1}^{t} \left[g_\tau^{'}(v_\tau^\top x_{t+1})v_\tau - g_\tau^{'}(v_\tau^\top x_t)v_\tau\right] v_\tau \tag{6.4}$$

Using Taylor expansion theorem(see appendix for more info on 5) in eq (6.4) we get:

$$= \sum_{\tau=1}^{t} \left[\nabla g_\tau^{'}(v_\tau^\top \zeta_\tau^t)^\top (x_{t+1} - x_t) \right] v_\tau \tag{6.5}$$

$$= \sum_{\tau=1}^{t} g_\tau^{''}(v_\tau^\top \zeta_\tau^t) v_\tau v_\tau^\top (x_{t+1} - x_t) \tag{6.6}$$

Where ζ_τ^t is some point on the line segment joining x_t and x_{t+1}. We can also define $A_t = \sum_{\tau=1}^{t} g_\tau^{''}(v_\tau^\top \zeta_\tau^t) v_\tau v_\tau^\top$. Also $g_\tau^{''}(v_\tau^\top x_\tau) \geq a \forall t, A_t$ would be a PSD. We get,

$$\nabla F_{t+1}(x_{t+1}) - \nabla F_{t+1}(x_t) = A_t \Delta x_t \tag{6.7}$$

Now again looking at $\nabla F_{t+1}(x_{t+1}) - \nabla F_{t+1}(x_t)$:

$$\nabla F_{t+1}(x_{t+1}) - \nabla F_{t+1}(x_t) = \nabla F_{t+1}(x_{t+1}) - \nabla F_{t+1}(x_t) + \nabla F_{t+1}(x_t) - \nabla F_{t+1}(x_t) \tag{6.8}$$

$$= \nabla \Delta F_t(x_t) - \nabla \sum_{\tau=1}^{t} f_\tau(x_t) + \nabla \sum_{\tau=1}^{t-1} f_\tau(x_t) \tag{6.9}$$

$$= \nabla \Delta F_t(x_t) - \nabla f_t(x_t) \tag{6.10}$$

$$= \nabla \Delta F_t(x_t) - \nabla_t \tag{6.11}$$

From eq (6.7) and eq (6.11) and adding $\epsilon \Delta x_t$ on both sides we get,

$$(A_t + \epsilon I_n) \Delta x_t = \nabla \Delta F_t(x_t) - \nabla_t + \epsilon \Delta x_t \tag{6.12}$$

Pre-multiply by $-\nabla_t^\top (A_t + \epsilon I_n)^{-1}$

$$-\nabla_t^\top \Delta x_t = -\nabla_t^\top (A_t + \epsilon I_n)^{-1} \left[\nabla \Delta F_t(x_t) - \nabla_t + \epsilon \Delta x_t \right] \tag{6.13}$$

$$= -\nabla_t^\top (A_t + \epsilon I_n)^{-1} \left[\nabla \Delta F_t(x_t) + \epsilon \Delta x_t \right] + \nabla_t^\top (A_t + \epsilon I_n)^{-1} \nabla_t \tag{6.14}$$

From eq (6.14) and eq (6.2) we get,

$$\sum_{t=1}^{T} \left[f_t(x_t) - f_t(x_{t+1}) \right] \leq \sum_{t=1}^{T} -\nabla_t^\top (A_t + \epsilon I_n)^{-1} \left[\nabla \Delta F_t(x_t) + \epsilon \Delta x_t \right] \sum_{t=1}^{T} \nabla_t^\top (A_t + \epsilon I_n)^{-1} \nabla_t \tag{6.15}$$

Now,
Applying claims (1) and (2), we get

$$\sum_{t=1}^{T} \left[f_t(x_t) - f_t(x_{t+1}) \right] \leq \epsilon D^2 T + \frac{nb^2}{a} \log\left(\frac{aR^2 T}{\epsilon} + 1 \right)$$

Substitute $\epsilon = \frac{b^2}{aD^2T}$, we get

$$\begin{aligned}\sum_{t=1}^{T}[f_t(x_t) - f_t(x_{t+1})] &\leq \frac{nb^2}{a}\log\left(\frac{a^2R^2T^2D^2}{b^2}+1\right)+\frac{b^2}{a} \\ &\leq \frac{2nb^2}{a}\log\left(\frac{aR^2T}{b}+1\right)\end{aligned}$$

$\square$

Claim 1.

$$\sum_{t=1}^{T} -\nabla_t^\top (\mathbf{A}_t + \epsilon\mathbf{I}_n)^{-1}\left[\Delta\nabla F_t(x_t) + \epsilon\Delta x_t\right] \leq \epsilon D^2 T \tag{6.16}$$

Proof. We know that x_τ minimizes F_τ over $\mathcal{P}$. Then, we can say that

$$\nabla F_\tau(x_\tau)^\top (x - x_\tau) \geq 0 \tag{6.17}$$

Substituting $x = x_{t+1}$ and $\tau = t$ in equation (6.17), we get

$$\nabla F_t(x_t)^\top (x_{t+1} - x_t) \geq 0 \tag{6.18}$$

Similarly, Substituting $x = x_t$ and $\tau = t+1$ in equation (6.17), we get

$$\nabla F_{t+1}(x_{t+1})^\top (x_t - x_{t+1}) \geq 0 \tag{6.19}$$

Adding equations (6.18) and (6.19), we get

$$\begin{aligned}\nabla F_{t+1}(x_{t+1})^\top (x_t - x_{t+1}) + \nabla F_t(x_t)^\top (x_{t+1} - x_t) &\geq 0 \\ (\nabla F_t(x_t)^\top - \nabla F_{t+1}(x_{t+1})^\top)(x_{t+1} - x_t) &\geq 0 \\ (\Delta\nabla F_t(x_t))^\top \Delta x_t &\leq 0\end{aligned}$$

Adding $\epsilon\|x_t\|^2$ on both sides, we get

$$\begin{aligned}(\Delta\nabla F_t(x_t))^\top \Delta x_t + \epsilon\|x_t\|^2 &\leq \epsilon\|x_t\|^2 \\ \left[\Delta\nabla F_t(x_t) + \epsilon\Delta x_t\right]^\top \Delta x_t &\leq \epsilon\|x_t\|^2\end{aligned}$$

Substituting the value of Δx_t from equation in above equation, we get

$$\begin{aligned}\left[\Delta\nabla F_t(x_t) + \epsilon\Delta x_t\right]^\top (\mathbf{A}_t + \epsilon\mathbf{I}_n)^{-1}\left[\Delta\nabla F_t(x_t) + \epsilon\Delta x_t - \nabla_t\right] &\leq \epsilon\|x_t\|^2 \\ \left[\Delta\nabla F_t(x_t) + \epsilon\Delta x_t\right]^\top (\mathbf{A}_t + \epsilon\mathbf{I}_n)^{-1}\left[\Delta\nabla F_t(x_t) + \epsilon\Delta x_t\right] - \left[\Delta\nabla F_t(x_t) + \epsilon\Delta x_t\right]^\top (\mathbf{A}_t + \epsilon\mathbf{I}_n)^{-1}\nabla_t &\leq \epsilon\|x_t\|^2\end{aligned}$$

We know that $\epsilon \|x_t\|^2 \leq \epsilon D^2$ and $\left[\Delta \nabla F_t(x_t) + \epsilon \Delta x_t\right]^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \left[\Delta \nabla F_t(x_t) + \epsilon \Delta x_t\right] \succcurlyeq 0$ (because $(\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1}$ is positive semidefinite).

$$
\begin{aligned}
-\left[\Delta \nabla F_t(x_t) + \epsilon \Delta x_t\right]^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t &\leq \epsilon D^2 \\
-\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \left[\Delta \nabla F_t(x_t) + \epsilon \Delta x_t\right] &\leq \epsilon D^2 \\
\sum_{t=1}^{T} -\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \left[\Delta \nabla F_t(x_t) + \epsilon \Delta x_t\right] &\leq \sum_{t=1}^{T} \epsilon D^2 \\
\sum_{t=1}^{T} -\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \left[\Delta \nabla F_t(x_t) + \epsilon \Delta x_t\right] &\leq \epsilon D^2 T
\end{aligned}
$$

Hence proved. □

Claim 2.

$$
\sum_{t=1}^{T} \nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t \leq \frac{nb^2}{a} \log\left(\frac{aR^2 T}{\epsilon} + 1\right) \tag{6.20}
$$

Proof. Let us define $B_t = \sum_{\tau=1}^{T} a v_t v_t^\top$.
We know that $g^{''}(v_\tau^\top x_\tau) \geq a$ and $\mathbf{A_t} = \sum_{\tau=1}^{T} g^{''}(v_\tau^\top \zeta_\tau^t) v_\tau v_\tau^\top$.
So, we can say the following:

$$
\begin{aligned}
(\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} &\preccurlyeq (\mathbf{B}_t + \epsilon \mathbf{I}_n)^{-1} \\
\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t &\leq \nabla_t^\top (\mathbf{B}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t
\end{aligned}
$$

Substituting $\nabla_t = g_t^{'}(v_t^\top x_t) v_t$ in the above equation,

$$
\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t \leq \left[g_t^{'}(v_t^\top x_t) v_t\right]^\top (\mathbf{B}_t + \epsilon \mathbf{I}_n)^{-1} \left[g_t^{'}(v_t^\top x_t) v_t\right]
$$

Substituting $\left|g_t(v_t^\top x)\right| \leq b$

$$
\begin{aligned}
\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t &\leq \left[b v_t\right]^\top (\mathbf{B}_t + \epsilon \mathbf{I}_n)^{-1} \left[b v_t\right] \\
\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t &\leq b^2 \left[v_t\right]^\top (\mathbf{B}_t + \epsilon \mathbf{I}_n)^{-1} \left[v_t\right] \\
\nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t &\leq \frac{b^2}{a} \left[\sqrt{v_t}\right]^\top (\mathbf{B}_t + \epsilon \mathbf{I}_n)^{-1} \left[\sqrt{a} v_t\right] \\
\sum_{t=1}^{T} \nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t &\leq \sum_{t=1}^{T} \frac{b^2}{a} \left[\sqrt{v_t}\right]^\top (\mathbf{B}_t + \epsilon \mathbf{I}_n)^{-1} \left[\sqrt{a} v_t\right]
\end{aligned}
$$

Applying lemma (4) with $\mathbf{V}_t = (\mathbf{B}_t + \epsilon \mathbf{I}_n), \mathbf{u}_t = \sqrt{a} v_t, \|\sqrt{a}\mathbf{u}_t\| \leq \sqrt{a} R \implies r = \sqrt{a} R$ □

$$\sum_{t=1}^{T} \nabla_t^\top (\mathbf{A}_t + \epsilon \mathbf{I}_n)^{-1} \nabla_t \leq \frac{nb^2}{a} \log\left(\frac{aR^2T}{\epsilon} + 1\right) \tag{6.21}$$

Hence proved.

Showing how this theorem 3 implies the main result of FOLLOW THE APPROXIMATE LEADER.

Theorem 4. *Assume that* $\forall t$ *, the function* $f_t : \mathcal{P} \to \mathbb{R}^n$ *has the property that* $\forall x \in \mathcal{P}$, $\|\nabla f(x)\| \leq G$ *and* $\exp(-\alpha f(x))$ *is concave. Then the algorithm FOLLOW THE APPROXIMATE LEADER with* $\beta = \frac{1}{2} min\{\frac{1}{4GD}, \alpha\}$ *has the following regret bound:*

$$Regret_T(FTAL) \leq 64\left(\frac{1}{\alpha} + GD\right) n(\log(T) + 1)$$

Proof. Note that FOLLOW THE APPROXIMATE LEADER is similar to FOLLOW THE LEADER just on paraboloid functions $\tilde{f}_t$. From lemma 1 we see that $f_t(x_t) = \tilde{f}_t(x_t)$ and also $\forall x \in \mathcal{P}, f_t(x) \geq \tilde{f}_t(x)$. Lemma 6 implies that the regret of $\tilde{f}_t \geq f_t$. Hence it suffices to bound this. Function $\tilde{f}_t$ can be written as:

$$\tilde{f}_t(x) = f_t(x_t) + \nabla_\tau^\top (x - x_\tau) + \frac{\beta}{2}\left[\nabla_\tau^\top (x - x_\tau)\right]^2$$

and the condition in Theorem 3 with $g_t : \mathbb{R} \to \mathbb{R}$ with

$$g_t(y) \triangleq f_t(x_t) + (y - \nabla_t^\top x_t) + \frac{\beta}{2}(y - \nabla_t^\top x_t)^2 v_t = \nabla_t$$

Now there is only the need to calculate R, a, b. Note
$\|v_t\| = \|\nabla_t\| \leq G$, hence we can take $R = G$.
$|g_t^{'}(v_t^\top)x)| = |1 + \beta(\nabla_t^\top (x - x_t)| \leq 1 + \beta GD \leq 2$, since $\beta \leq \frac{1}{8GD}$ so we can take $b = 2$
$g_t^{''}(y) = \beta$ so we can take $a = \beta$ Plugging in the values we have and using the fact that $\frac{DRa}{b} = \frac{\beta GD}{2} \leq 1$, we get

$$Regret_T(FTAL) \leq \frac{8n}{\beta}(\log(T) + 1)$$

by definition we have $\frac{1}{\beta} \leq max\{8GD, \frac{2}{\alpha}\} \leq 8(GD + \frac{1}{\alpha})$ plugging in the value:

$$Regret_T(FTAL) \leq 64(\frac{1}{\alpha} + GD)n(\log(T) + 1)$$

□

6.3 Running time

The algorithm requires $O(n^2)$ space to store the sum of all gradients and matrices of the form $\nabla_t \nabla_t^\top$. The time needed to compute the point x_t is $O(n^2)$ plus the time to perform a single generalized projection.

Computing Projection

Let $\mathcal{P} \subset \mathbb{R}_n$ be a convex set and $y \in \mathbb{R}_n$, then euclidean projection is nothing but closest point $x \in \mathcal{P}$ to y.

$$\Pi_{\mathcal{P}}[y] \triangleq \min_{x \in \mathcal{P}} \|x - y\|_2 \tag{7.1}$$

Online Newton step computes generalized projections with repsect to norm given by a positive semidefinite matrix $\mathbf{A}$ is given as follows:

$$\Pi_{\mathcal{P}}[y]^{\mathbf{A}} \triangleq \min_{x \in \mathcal{P}} (x - y)^\top \mathbf{A}(x - y) \tag{7.2}$$

Conclusion

In this research paper the authors have modified the existing three algorithms namely, Online Gradient Descent, Online Newton Step and Follow The Approximate Leader which each have a regret bound of $O(\log T)$ which had the previous best bound of $O(\sqrt{T})$. Overall the paper shows that FTAL is a powerful and versatile approach to online convex optimization, with strong theoretical guarantees and practical performance. The FTAL algorithm has the potential to impact a wide range of applications in machine learning and optimization.

Appendix

Lemma 3. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex set, $y \in \mathbb{R}^n$ and $z = \prod_{\mathcal{P}}^{\mathbf{A}}[y]$ be the generalized projection of y onto $\mathcal{P}$ according to positive semidefinite matrix $\mathbf{A} \succcurlyeq 0$. Then for any point $a \in \mathcal{P}$, the following holds:*

$$(y-a)^\top \mathbf{A}(y-a) \geq (z-a)^\top \mathbf{A}(z-a) \tag{9.1}$$

Lemma 4. *Let $u_t \in \mathbb{R}^n$, for $t = 1, \ldots, T$, be a sequence of vectors such that for some $r > 0$, $\|u_t\| \leq r$. Define $V_t = \sum_{i=1}^t u_t u_t^\top + \epsilon I_n$, then,*

$$\sum_{t=1}^T u_t^\top V_t^{-1} u_t \leq n \log\left(\frac{r^2 T}{\epsilon} + 1\right)$$

Lemma 5. *Let f_t , for $t = 1, \ldots, T$, be a sequence of cost functions and let $x_t = argmin_{x \in \mathcal{P}} \sum_{\tau=1}^t f_\tau(x)$. Then*

$$\sum_{t=1}^T f_t(x_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^T f_t(x) \leq \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x_{t+1})$$

Lemma 6. *Let f_t , for $t = 1, \ldots, T$, be a sequence of cost functions and let $x_t \in \mathcal{P}$ be the point used in the t^{th} round. Let $\tilde{f}_t$ for $t = 1, \ldots, T$ be a sequence of cost functions such that $f_t(x_t) = \tilde{f}_t(x_t)$, and $\forall x \in \mathcal{P}$, $f_t(x) \geq \tilde{f}_t(x)$. Then*

$$\sum_{t=1}^T f_t(x_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^T f_t(x) \leq \sum_{t=1}^T \tilde{f}_t(x_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^T \tilde{f}_t(x)$$

Lemma 7. *Let f_t be a set of cost functions for $t = 1, ..., T$. Let $x_t = \arg\min_{x \in \mathcal{P}} \sum_{\tau=1}^t f_\tau(x)$. Then, the following holds:*

$$Regret_T(FTL) = \sum_{t=1}^T f_t(x_t) \min_{x \in \mathcal{P}} \sum_{t=1}^T f_t(x) \leq \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x_{t+1}) \tag{9.2}$$

Theorem 5. ***Taylor expansion for multivariate functions :*** *Let $f(x)$ be a function that is differentiable $n+1$ times on an open interval I containing x_0. Then, for any $x \in I$, there exists a point c between x_0 and x such that:*

$$f(x) = f(x_0) + f^{'}(x_0)(x - x_0) + f^{''}\frac{x_0}{2!}(x - x_0)^2 + ... + f^n \frac{x_0}{n!}(x - x_0)^n + R_n(x)$$

where $f^{'}(x_0)$ denotes the derivative of $f(x)$ evaluated at x_0, $f^{''}(x_0)$ denotes the second derivative of $f(x)$ evaluated at x_0, and $f^(n)(x_0)$ denotes the n^{th} derivative of $f(x)$ evaluated at x_0. The term $R_n(x)$ is the remainder term and is given by:

$$R_n(x) = (1/n!) f^{n+1}(c)(x - x_0)^{n+1}$$

where c is a point between x_0 and x.