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Logarithmic regret algorithms for online convex optimization

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Title

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Problem Statement

Goal of the paper

Online convex optimization is a fundamental problem in machine learning, control systems, and finance. The aim is to propose and analyze new algorithms for online convex optimization that achieve logarithmic regret bounds, which would represent a significant improvement over existing methods.

Introduction

The authors modifies the existing algorithms to achieve logarithmic regret bounds. The modified algorithms are Online Gradient Descent(OGD), Online Newton Step(ONS) and Follow-The-Approximate-Leader(FTAL). The proposed algorithms have practical applications in fields such as machine learning, control systems, and finance.

Definitions

Regret

Regret in online convex optimization refers to the difference between the total cost incurred by an online algorithm over a sequence of decisions and the cost that would have been incurred by an optimal offline algorithm that has access to the entire sequence of decisions in advance.

$$x_t = \mathcal{A}(\{f_1, f_2, ..., f_{t-1}\})$$

$$\text{Regret}(\mathcal{A}, \{f_1, f_2, ..., f_t\}) = \mathbb{E}\left[\sum_{t=1}^{T} f_t(x_t)\right] - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x)$$

Definitions Contd.

Online Convex Optimization

Online convex optimization (OCO) is a subfield of machine learning and optimization that deals with optimizing a convex function over a sequence of data points that arrive in an online fashion

α-exp concavity

We say that a function f_t satisfies α-exp-concavity property if

$$\forall x \in \mathcal{P}, t \in [T] : \nabla^2[\exp -\alpha f_t(x)] \preccurlyeq 0$$

Zinkevich's Online Gradient Descent

Running time:$\mathcal{O}(n)$ per iteration

Algorithm Online Gradient Descent

Require: convex set $\mathcal{P} \subset \mathbb{R}^n$, step sizes $\eta_1, \eta_2, ... \geq 0$, initial $\mathbf{x_1} \in \mathcal{P}$.

In iteration 1, use point $\mathbf{x_1} \in \mathcal{P}$

for iteration $t > 1$: use point **do**

$\mathbf{x}_t = \Pi_{\mathcal{P}}(\mathbf{x_{t-1}} - \eta_\mathbf{t} \nabla \mathbf{f_{t-1}}(\mathbf{x_{t-1}}))$

Here,

$$\Pi_{\mathcal{P}}[y]^{\mathbf{A}} \triangleq \min_{x \in \mathcal{P}} (x-y)^\top \mathbf{A} (x-y)$$

Modifed Online Gradient Descent

Modification

Assuming the cost functions $f_1, f_2, ...$ are strictly convex, learning rates are modified as $\eta_t = \frac{1}{H_t}$.

Modifed Online Gradient Descent

Theorem

Updating step size with $\eta_t = \frac{1}{H_t}$ achieves logarithmic regret.

$$Regret_t(OGD) \leq \frac{G^2}{2H}(1 + \log(T))$$

Lemma

Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex set, $y \in \mathbb{R}^n$ and $z = \prod_{\mathcal{P}}^{\mathbf{A}}[y]$ be the generalized projection of y onto $\mathcal{P}$ according to positive semidefinite matrix $\mathbf{A} \succcurlyeq 0$. Then for any point $a \in \mathcal{P}$, the following holds:

$$(y - a)^\top \mathbf{A}(y - a) \geq (z - a)^\top \mathbf{A}(z - a) \tag{1}$$

Proof sketch

By using taylor series approximation and H-strong convexity, we get

$$f_t(x^*) \geq f_t(x_t) + \nabla_t^T (x^* - x_t) + \frac{H}{2} \|x^* - x_t\|^2 \tag{2}$$

Using update rule in algorithm and lemma (1), we get

$$5\nabla_t^\top (x_t - x^*) \leq \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{\eta_{t+1}} + \eta_{t+1} G^2 \tag{3}$$

Simplifying above equation and substituting $\eta_t = \frac{1}{H_t}$ along with simplified equation in 2, we get

$$2 \sum_{t=1}^{T} f_t(x_t) - f_t(x^*) \leq \frac{G^2}{H}(1 + \log T)$$

Online Newton Step

Running time: $\mathcal{O}(n^2)$ per iteration

Algorithm Online Newton Step

Require: convex set $\mathcal{P} \subset \mathbb{R}^n$

Select an arbitary point $x_1 \in \mathcal{P}$

for Iteration $t > 1$ use point **do**

$x_t = \Pi_{\mathcal{P}}^{A_{t-1}} \left(x_{t-1} - \frac{1}{\beta} A_{t-1}^{-1} \nabla_{t-1} \right)$

end for Here,

$\beta = \frac{1}{2} min\{\frac{1}{4GD}, \alpha\}$

$\nabla_t = \nabla f_t(x_t)$, $A_t = \sum_{i=1}^{t} \nabla_i \nabla_i^\top + \epsilon \mathbf{I}_n$, $\epsilon = \frac{1}{\beta^2 D^2}$

$\Pi_{\mathcal{P}}^{A_{t-1}}$ is the projection in the norm induced by A_{t-1}:

$$\Pi_{\mathcal{P}}^{A_{t-1}}(y) = \arg\min_{x \in \mathcal{P}} (y - x)^\top A_{t-1} (y - x)$$

Online Newton Step

Theorem

Assume that $\forall t$, the loss function $f_t : \mathcal{P} \to \mathbb{R}^n$ is α-exp-concave and has the property that $\forall x \in \mathcal{P}$, $\|\nabla f(x)\| \leq G$. Then the algorithm ONLINE NEWTON STEP has the following regret bound:

$$Regret_T(ONS) \leq 5 \left(\frac{1}{\alpha} + GD \right) n \log T$$

Lemma

Let $f : \mathcal{P} \to \mathbb{R}$ be an α-exp concave function such that $\forall x \in \mathcal{P}, \|\nabla f(x)\| \leq G$. D is the diameter of $\mathcal{P}$. Then for $\forall x, y \in \mathcal{P}, \beta \leq \frac{1}{2} min\{\frac{1}{4GD}, \alpha\}$, the following holds:

$$f(x) \geq f(y) + \nabla^\top (x-y) + \frac{\beta}{2}(x-y)^\top \nabla f(y) \nabla f(y)^\top (x-y) \quad (4)$$

Proof Sketch

Let $x^* \in argmin_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x)$ be the best decision so far. Using lemma 4, we get

$$f_t(x_t) - f_t(x^*) \leq R_t \triangleq \nabla_t^\top (x_t - x^*) - \frac{\beta}{2}(x^* - x_t)^\top \nabla_t \nabla_t^\top (x^* - x_t) \quad (5)$$

where $\beta = \frac{1}{2} min\{\frac{1}{4GD}, \alpha\}$
Lets define $y_{t+1} = x_t - \frac{1}{\beta} A_t^{-1} \nabla_t$ for convenience where,
$A_t = \sum_{i=1}^{t} \nabla_i \nabla_i^\top + \epsilon \mathbf{I}_n, \epsilon = \frac{1}{\beta^2 D^2}$. Note that A_t is symmetric.

Proof Sketch Contd.

From the update rule from definition we know that $x_{t+1} = \Pi_{S_n}^{A_t}(y_{t+1})$. from the definition of y_{t+1} and subtracting x^*, multiplying with A_t on both sides then and multiplying it with its transpose and observing from the equation 4 we get:

$$\nabla_t^\top (x_{t+1} - x^*) \leq \frac{1}{2\beta} \nabla_t^\top A_t^{-1} \nabla_t + \frac{\beta}{2}(x_t - x^*)^\top A_t (x_t - x^*) - \frac{\beta}{2}(x_{t+1} - x^*)^\top A_t (x_{t+1} - x^*)$$

After summing it from $t = 1$ to T and solving and get the required regret bounds, i.e,

$$Regret_T(ONS) \leq 4n\left(GD + \frac{1}{\alpha}\right)\log(T) \tag{6}$$

Here it ends the proof.

Follow the Approximate Leader

Running time:$\mathcal{O}(n^2)$ per iteration

Algorithm Follow the Approximate Leader

Require: convex set $\mathcal{P} \subset \mathbb{R}^n$, parameter β

function

(I)n iteration 1, use a random point $\mathbf{x_1} \in \mathcal{P}$

for iteration $t > 1$: use point as in the equations below **do**

$\nabla_{t-1} = \nabla f_{t-1}(x_{t-1})$

$\mathbf{A_{t-1}} = \sum_{\tau=1}^{t-1} \nabla_t \nabla_t^\top$

$\mathbf{b}_{t-1} = \sum_{\tau=1}^{t-1} \nabla_t \nabla_t^\top \mathbf{x}_\tau - \frac{1}{\beta} \nabla_t$

$\mathbf{x}_t = \Pi_{\mathcal{P}}^{A_{t-1}}(\mathbf{A_{t-1}^{-1} b_{t-1}})$

end for

Here,

$$\Pi_{\mathcal{P}}^{A_{t-1}}(y) = \arg\min_{x \in \mathcal{P}} (y-x)^\top A_{t-1}(y-x)$$

Follow The Approximate Leader Regret Bound

Lemma

Follow The Approximate Leader is equivalent to Follow The Leader.

Theorem for Follow The Leader

Assume that $\forall t$, the function $f_t : \mathcal{P} \to \mathbb{R}^n$ can be written as $f_t(x) = g_t(v_t^\top x)$ for a univariate convex function $g_t : \mathbb{R} \to \mathbb{R}$ and some vector $v_t \in \mathbb{R}^n$. Assume that for some $R, a, b > 0$, we have $\|v_t\|_2 \leq R$, and $\forall x \in \mathcal{P}$, we have $g_t(v_t^\top x) \leq b$ and $g_t''(v_t^\top x) \geq a$. Then the FOLLOW THE LEADER algorithm on the functions f_t satisfies the following regret bound:

$$Regret_T(FTL) \leq \frac{2nb^2}{a}\left[\log(\frac{DRaT}{b}) + 1\right] \tag{7}$$

Follow the Approximate Leader Regret Bound

Theorem for Follow the Approximate Leader

Assume that $\forall t$, the function $f_t : \mathcal{P} \rightarrow \mathbb{R}^n$ has the property that $\forall x \in \mathcal{P}$, $\|\nabla f(x)\| \leq G$ and $\exp(-\alpha f(x))$ is concave. Then the algorithm FOLLOW THE APPROXIMATE LEADER with $\beta = \frac{1}{2} min\{\frac{1}{4GD}, \alpha\}$ has the following regret bound:

$$Regret_T(FTAL) \leq 64 \left(\frac{1}{\alpha} + GD \right) n(\log(T) + 1)$$

Lemma

Let f_t , for $t = 1, \ldots, T$, be a sequence of cost functions and let $x_t \in \mathcal{P}$ be the point used in the t^{th} round. Let $\tilde{f}_t$ for $t = 1, \ldots, T$ be a sequence of cost functions such that $f_t(x_t) = \tilde{f}_t(x_t)$, and $\forall x \in \mathcal{P}$, $f_t(x) \geq \tilde{f}_t(x)$. Then

$$\sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x) \leq \sum_{t=1}^{T} \tilde{f}_t(x_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} \tilde{f}_t(x) \tag{8}$$

Proof sketch

The proof idea here is to make use of the lemma 4, lemma 8 along with theorem 7. We would then calculate the values of R, a, b. Substituting these and the fact that $\frac{1}{\beta} \leq max\{8GD, \frac{2}{\alpha}\} \leq 8(GD + \frac{1}{\alpha})$ we get the required regret bound, i.e,:

$$Regret_T(FTAL) \leq 64(\frac{1}{\alpha} + GD)n(\log(T) + 1)$$

Conclusion

In this research paper the authors have modified the existing three algorithms to achieve a regret bound of $O(logT)$. Overall the paper shows that FTAL is a powerful and versatile approach to online convex optimization, with strong theoretical guarantees and practical performance. The FTAL algorithm has the potential to impact a wide range of applications in machine learning and optimization.

Thank You