CSIR UGC JUNE 2013 Q-70

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Question

Let X and Y be independent random variables each following a uniform distribution on $(0,1)$.Let $W = XI_{\{Y \leq X^2\}}$,where I_A denotes the indicator function of set A.Then which of the following statements are true?

1. The cumulative distribution function of W is given by

$$F_W(t) = t^2 I_{\{0 \leq t \leq 1\}} + I_{\{t > 1\}} \tag{1}$$

2. $P[W > 0] = \frac{1}{3}$
3. The cumulative distribution function of W is continuous
4. The cumulative distribution function of W is given by

$$F_W(t) = \frac{2 + t^3}{3} I_{\{0 \leq t \leq 1\}} + I_{\{t > 1\}} \tag{2}$$

SOLUTION

Given X is an independent random variable,where $X \in (0,1)$ following a uniform distribution.

$$\int_0^1 p_X(x)\mathrm{dx} = 1 \tag{3}$$

$$\therefore p_X(x) = 1 \tag{4}$$

The PDF for X is

$$p_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{5}$$

By definition of CDF for $X \in (0,1)$

$$F_X(x) = \int_0^x p_X(x)\mathrm{dx} \tag{6}$$

$$= x \tag{7}$$

Solution

The CDF for X is

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & otherwise \end{cases} \tag{8}$$

Solution

Given Y is an independent random variable,where $Y \in (0,1)$ following a uniform distribution.

$$\int_0^1 p_Y(y)\mathrm{dy} = 1 \tag{9}$$

$$\therefore p_Y(y) = 1 \tag{10}$$

The PDF for Y is

$$p_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & otherwise \end{cases} \tag{11}$$

By definition of CDF for $Y \in (0,1)$

$$F_Y(y) = \int_0^y p_Y(y)\mathrm{dy} \tag{12}$$

$$= y \tag{13}$$

Solution

The CDF for Y is

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & otherwise \end{cases} \tag{14}$$

Indicator function

The indicator function of an event E is defined as follows

$$I_E(w) = \begin{cases} 1 & w \in E \\ 0 & otherwise \end{cases} \tag{15}$$

So,indicator function given in question is defined as follows where $X, Y \in (0, 1)$ are independent random variables

$$I_{\{Y \leq X^2\}} = \begin{cases} 1 & y \leq x^2 \\ 0 & otherwise \end{cases} \tag{16}$$

Solution

$W = XI_{\{X^2 \leq Y\}}$ is defined as follows

$$W = \begin{cases} x & y \leq x^2 \\ 0 & otherwise \end{cases} \tag{17}$$

From above definition

$$p_W(W = 0) = \Pr(I_{\{Y \leq X^2\}} = 0) \tag{18}$$
$$= \Pr(x^2 < y) \tag{19}$$

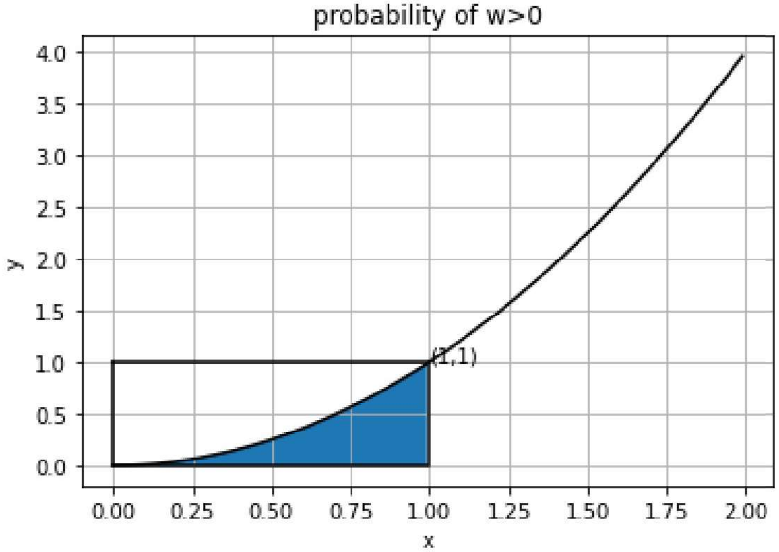
Solution

$\Pr(x^2 < y)$

This can be found in two ways

1. graphical method
2. convolution

Solution(graphical method)



From the graph

$$\Pr(x^2 \geq y) = \frac{\text{area of shaded part}}{\text{Area of rectangle}} \tag{20}$$

$$= \int_0^1 x^2 \mathrm{dx} \tag{21}$$

$$\therefore P[W > 0] = \frac{1}{3} \tag{22}$$

$$P[W = 0] = 1 - \frac{1}{3} = \frac{2}{3} \tag{23}$$

Solution(convolution for $\Pr(x^2 < y)$)

Let $Z = X^2 - Y$ be a random variable where $Z \in (-1, 1)$

$$\begin{aligned} F_{X^2}(u) &= \Pr(X^2 \leq u) && (24) \\ &= \Pr(X \leq \sqrt{u}) && (25) \\ &= F_X(\sqrt{u}) && (26) \end{aligned}$$

From (8),The CDF for X^2 is

$$F_{X^2}(u) = \begin{cases} 0 & u \leq 0 \\ \sqrt{u} & 0 < u < 1 \\ 1 & \textit{otherwise} \end{cases} \tag{27}$$

Solution(convolution for $\Pr(x^2 < y)$)

By definition of PDF for $X^2 \in (0,1)$

$$p_{X^2}(u) = \frac{\mathrm{d}F_{X^2}(u)}{\mathrm{du}} \tag{28}$$

$$= \frac{\mathrm{d}\sqrt{u}}{\mathrm{du}} \tag{29}$$

$$= \frac{1}{2\sqrt{u}} \tag{30}$$

The PDF for X^2 is

$$p_{X^2}(u) = \begin{cases} \frac{1}{2\sqrt{u}} & 0 < u < 1 \\ 0 & otherwise \end{cases} \tag{31}$$

Solution(convolution for $\Pr(x^2 < y)$)

$$F_{\{-Y\}}(v) = \Pr(-Y \leq v) \tag{32}$$
$$= \Pr(Y \geq -v) \tag{33}$$
$$= 1 - F_Y(-v) \tag{34}$$

From (14),The CDF for $(-Y)$ is

$$F_{\{-Y\}}(v) = \begin{cases} 1-0 & -v \geq 0 \\ 1-(-v) & 0 < -v < 1 \\ 1-1 & -v \leq 1 \end{cases} \tag{35}$$

Solving this we get

$$F_{\{-Y\}}(v) = \begin{cases} 0 & v \leq -1 \\ 1+v & -1 < v < 0 \\ 1 & otherwise \end{cases} \tag{36}$$

Solution(convolution for $\Pr(x^2 < y)$)

By definition of PDF for $(-Y) \in (-1, 0)$

$$p_{\{-Y\}}(v) = \frac{\mathrm{d}F_{\{-Y\}}(v)}{\mathrm{dv}} \tag{37}$$

$$= \frac{\mathrm{d}(1+v)}{\mathrm{dv}} \tag{38}$$

$$= 1 \tag{39}$$

The PDF for $(-Y)$ is

$$p_{\{-Y\}}(v) = \begin{cases} 1 & -1 < v < 0 \\ 0 & otherwise \end{cases} \tag{40}$$

Solution(convolution for $\Pr(x^2 < y)$)

$Z = X^2 - Y \implies z = u + v$

Using convolution

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X^2}(z-v)p_{\{-Y\}}(v)\mathrm{dv} \tag{41}$$

$$-1 < v < 0 \tag{42}$$

$$0 < z - v < 1 \implies z - 1 < v < z \tag{43}$$

For $z \in (-1, 0]$

$$p_Z(z) = \int_{max(-1,z-1)}^{min(0,z)} p_{X^2}(z-v)p_{\{-Y\}}(v)\mathrm{dv} \tag{44}$$

$$= \int_{-1}^{z} \frac{1}{2\sqrt{z-v}}\mathrm{dv} \tag{45}$$

$$= \sqrt{z+1} \tag{46}$$

Solution(convolution for $\Pr(x^2 < y)$)

For $z \in (0,1)$

$$p_Z(z) = \int_{max(-1,z-1)}^{min(0,z)} p_{X^2}(z-v)p_{\{-Y\}}(v)\mathrm{dv} \tag{47}$$

$$= \int_{z-1}^{0} \frac{1}{2\sqrt{z-v}}\mathrm{dv} \tag{48}$$

$$= 1 - \sqrt{z} \tag{49}$$

PDF of Z as follows

$$p_Z(z) = \begin{cases} \sqrt{z+1} & -1 < z \leq 0 \\ 1-\sqrt{z} & 0 < z < 1 \\ 0 & otherwise \end{cases} \tag{50}$$

Solution(convolution for $\Pr(x^2 < y)$)

By definition of CDF

$$F_Z(z) = \int_{-\infty}^{\infty} p_Z(z)\mathrm{dz} \tag{51}$$

$$\tag{52}$$

for $z \in (-1, 0]$

$$F_Z(z) = \int_{-1}^{z} p_Z(z)\mathrm{dz} \tag{53}$$

$$= \int_{-1}^{z} \sqrt{z+1}\mathrm{dz} \tag{54}$$

$$= \frac{2}{3}(z+1)^{\frac{3}{2}} \tag{55}$$

Solution(convolution for $\Pr(x^2 < y)$)

for $z \in (0,1)$

$$F_Z(z) = \int_0^z p_Z(z)\mathrm{dz} \tag{56}$$

$$= \int_0^z (1-\sqrt{z})\mathrm{dz} \tag{57}$$

$$= z - \frac{2}{3}z^{\frac{3}{2}} \tag{58}$$

CDF of Z as follows

$$F_Z(z) = \begin{cases} \frac{2}{3}(z+1)^{\frac{3}{2}} & -1 < z \leq 0 \\ z - \frac{2}{3}z^{\frac{3}{2}} & 0 < z < 1 \\ 1 & otherwise \end{cases} \tag{59}$$

Solution

using (59) to find $p_W(W=0)$

$$p_W(W=0) = \Pr(x^2 < y) \quad (60)$$
$$= F_z(0) \quad (61)$$
$$= \frac{2}{3}(0+1)^{\frac{3}{2}} \quad (62)$$
$$= \frac{2}{3} \quad (63)$$

We need to find PDF of W

$W = t \implies X = t$ where $t \in (0,1)$

$$p_W(t) = \int_{-\infty}^{\infty} p_X(t) I_{\{y \le t^2\}} \mathrm{dy} \quad (64)$$

$$0 < y < 1 \quad (65)$$
$$0 < y \le t^2 \quad (66)$$

Solution

For $0 < t < 1$,

$$p_W(t) = \int_0^{min(1,t^2)} p_X(t) I_{\{y \le t^2\}} \mathrm{dy} \tag{67}$$

$$= \int_0^{t^2} p_X(t) I_{\{y \le t^2\}} \mathrm{dy} \tag{68}$$

$$= t^2 \tag{69}$$

$\therefore$ PDF of W is as follows

$$p_W(t) = \begin{cases} \frac{2}{3} & t = 0 \\ t^2 & 0 < t < 1 \\ 0 & otherwise \end{cases} \tag{70}$$

By definition of CDF

$$F_W(t) = \int_{-\infty}^{\infty} p_W(t) \mathrm{dt} \tag{71}$$

$$\tag{72}$$

Solution

for $t \in [0, 1]$

$$F_W(t) = \int_0^t p_W(t)\mathrm{dt} \tag{73}$$

$$= \frac{2}{3} + \int_0^t t^2\mathrm{dt} \tag{74}$$

$$= \frac{2+t^3}{3} \tag{75}$$

The CDF of W is as follows:

$$F_W(t) = \begin{cases} 0 & t < 0 \\ \frac{2+t^3}{3} & 0 \leq t \leq 1 \\ 1 & otherwise \end{cases} \tag{76}$$

Solution

We need to find $P[W > 0]$

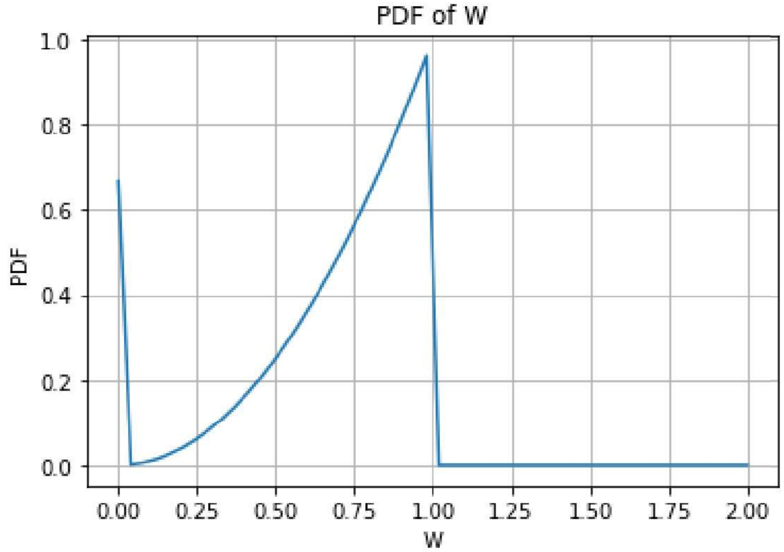
$$\Pr(W > 0) = 1 - F_W(0) \tag{77}$$
$$= \frac{1}{3} \tag{78}$$
$$\therefore \Pr(W > 0) = \frac{1}{3} \tag{79}$$

CDF of W is discontinuous at $W = 0$.
$\therefore$ option 3 is incorrect.
The CDF in (76) can be written as

$$F_W(t) = \frac{2 + t^3}{3} I_{\{0 \le t \le 1\}} + I_{\{t > 1\}} \tag{80}$$

$\therefore$ option 2 and 4 are correct.

graphs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0.4
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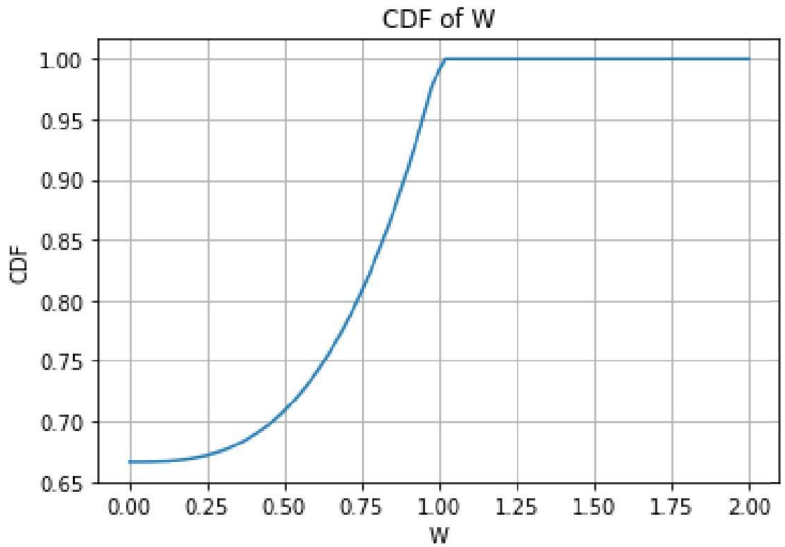
W


CDF of W
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1.75
2.00
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