

Problem: 9.10.6.8

Nikam Pratik Balasaheb (EE21BTECH11037)

1 PROBLEM

Bisector of angles A,B and C of a triangle ABC intersect its circumcircle at D,E,and F respectively. Prove that the angles of triangle DEF are $90^\circ - \frac{A}{2}$, $90^\circ - \frac{B}{2}$ and $90^\circ - \frac{C}{2}$.

2 SOLUTION

1) Let the vertices of the triangle ABC be:

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{A} = c \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (2.0.3)$$

For ease of calculation let's assume $\theta = 60^\circ, a = c = 5$.

2) side length b is given by:

$$b = \|\mathbf{A} - \mathbf{C}\| \quad (2.0.4)$$

$$= \left\| \begin{pmatrix} c \cos \theta - a \\ c \sin \theta \end{pmatrix} \right\| \quad (2.0.5)$$

$$= \sqrt{a^2 + c^2 - 2ac \cos \theta} \quad (2.0.6)$$

$$(2.0.7)$$

3) Circumcenter of triangle ABC:

Circumcenter of triangle can be found by calculating point of intersection of perpendicular bisectors of two sides.

a) perpendicular bisector of side BC:

$$(\mathbf{C} - \mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{C} + \mathbf{B}}{2} \right) = 0 \quad (2.0.8)$$

$$(\mathbf{C} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{C}\|^2 - \|\mathbf{B}\|^2}{2} \quad (2.0.9)$$

$$\mathbf{C}^\top \mathbf{x} = \frac{a^2}{2} \quad (2.0.10)$$

b) perpendicular bisector of side AB:

$$\mathbf{A}^\top \mathbf{x} = \frac{c^2}{2} \quad (2.0.11)$$

Therefore, the circumcenter is given by:

$$\begin{pmatrix} a & 0 \\ c \cos \theta & c \sin \theta \end{pmatrix} \mathbf{O} = \begin{pmatrix} \frac{a^2}{2} \\ \frac{c^2}{2} \end{pmatrix} \quad (2.0.12)$$

4) Circumradius of triangle ABC,

$$R = \|\mathbf{A} - \mathbf{O}\| \quad (2.0.13)$$

5) For Circumcircle of triangle ABC, Writing equation of circle in form $\mathbf{x}^\top \mathbf{V} \mathbf{x} + \mathbf{u}^\top \mathbf{x} + f = 0$,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.14)$$

$$\mathbf{u} = -\mathbf{O} \quad (2.0.15)$$

$$f = \|\mathbf{O}\|^2 - R^2 \quad (2.0.16)$$

$$= 2\mathbf{O}^\top \mathbf{A} - \|\mathbf{A}\|^2 \quad (2.0.17)$$

6) angular bisectors: Expressing the line equations in form $\mathbf{x} = \mathbf{h} + \mu \mathbf{m}$,

a) Angular Bisector of angle A:

$$\mathbf{m} = (\mathbf{C} - \mathbf{A}) + (\mathbf{B} - \mathbf{A}) \quad (2.0.18)$$

$$= \mathbf{B} + \mathbf{C} - 2\mathbf{A} \quad (2.0.19)$$

$$\mathbf{h} = \mathbf{A} \quad (2.0.20)$$

b) Angular Bisector of angle B:

$$\mathbf{m} = (\mathbf{C} - \mathbf{B}) + (\mathbf{A} - \mathbf{B}) \quad (2.0.21)$$

$$= \mathbf{A} + \mathbf{C} - 2\mathbf{B} \quad (2.0.22)$$

$$\mathbf{h} = \mathbf{B} \quad (2.0.23)$$

c) Angular Bisector of angle C:

$$\mathbf{m} = (\mathbf{A} - \mathbf{C}) + (\mathbf{B} - \mathbf{C}) \quad (2.0.24)$$

$$= \mathbf{A} + \mathbf{B} - 2\mathbf{C} \quad (2.0.25)$$

$$\mathbf{h} = \mathbf{C} \quad (2.0.26)$$

7) The point of intersection of any line $\mathbf{x} = \mathbf{h} + \mu \mathbf{m}$ and conic $\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$:

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V} \mathbf{h} + \mathbf{u}) \pm \sqrt{(\mathbf{m}^\top (\mathbf{V} \mathbf{h} + \mathbf{u}))^2 - g(\mathbf{h}) (\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (2.0.27)$$

where,

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f \quad (2.0.28)$$

- 8) The points of intersection D, E and F are found using above method. Refer codes/9.10.6.8.py. The resulting **D, E, F** are given in table 8.

D	$\begin{pmatrix} 2.5 \\ -1.44 \end{pmatrix}$
E	$\begin{pmatrix} 5 \\ 2.886 \end{pmatrix}$
F	$\begin{pmatrix} 0 \\ 2.886 \end{pmatrix}$

TABLE 8: Table

- 9) The desired angle i.e $\angle DEF$ is found by using,

$$\cos(\angle DEF) = \frac{(\mathbf{D} - \mathbf{E})^\top (\mathbf{F} - \mathbf{E})}{\|\mathbf{D} - \mathbf{E}\| \|\mathbf{F} - \mathbf{E}\|} \quad (2.0.29)$$

- 10) In codes/9.10.6.8.py, the problem is solved for $a = c = 5$ and $\theta = 60^\circ$. It is seen that the angle $\angle DEF$ is 60° which satisfies the statement to be proved.

Parameter	Value	Description
a	5	length of side opposite to Vertex B
c	5	length of side opposite to vertex A
θ	60°	$\angle ABC$

TABLE 10: Table

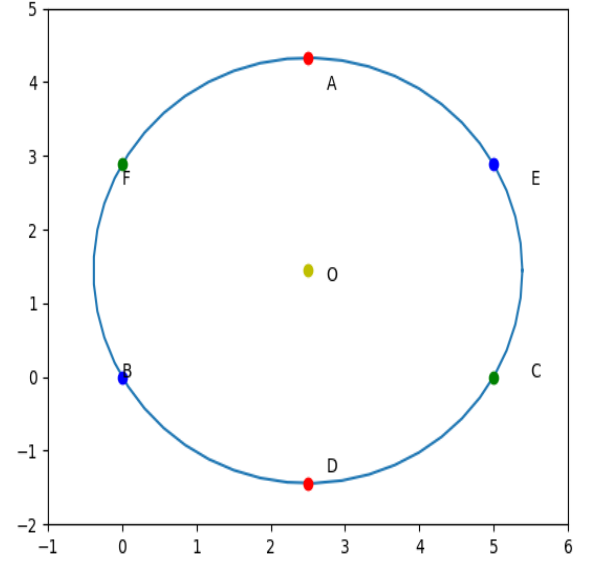


Fig. 10: Figure