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1 Problem

In triangle ABC with vertices $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, FInd the equation and length of altitude from vertex $\mathbf{A}$

2 Solution

1) Direction vector of side BC

$$\mathbf{m} = \mathbf{B} - \mathbf{C} \tag{2.0.1}$$

$$= \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{2.0.2}$$

Direction vector of side BC is normal of altitude from $\mathbf{A}$

2) Equation of the altitude

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.0.3}$$

$$\begin{pmatrix} 3 & -3 \end{pmatrix} \mathbf{x} = -3 \tag{2.0.4}$$

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -1 \tag{2.0.5}$$

3) Equation of line BC

$$\mathbf{x} = \mathbf{B} + \mu \mathbf{m} \tag{2.0.6}$$

$$\mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{2.0.7}$$

$$i.e. \quad \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{2.0.8}$$

4) Optimization problem
The length of altitide can be expressed as a optimization problem,

$$\min \|\mathbf{A} - \mathbf{x}\|^2 \tag{2.0.9}$$

such that

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.0.10}$$

where,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.11}$$

$$c = 3 \tag{2.0.12}$$

5) The line equation can be expressed as

$$\mathbf{x} = \mathbf{B} + \mu \mathbf{m} \tag{2.0.13}$$

where,

$$\mathbf{m} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \tag{2.0.15}$$

Substituting in the optimization problem,

$$\min \|\mathbf{A} - (\mathbf{B} + \mu \mathbf{m})\|^2 \tag{2.0.16}$$

$$\min f(\mu) = \mu^2 \|\mathbf{m}\|^2 - 2\mu \mathbf{m}^\top (\mathbf{A} - \mathbf{B}) + \|\mathbf{A} - \mathbf{B}\|^2 \tag{2.0.17}$$

coefficient of μ^2 is positive, the function is convex.

$$f''(\mu) = 2 \|\mathbf{m}\|^2 > 0 \tag{2.0.18}$$

$$\therefore f'(\mu_{min}) = 0 \tag{2.0.19}$$

$$\frac{\partial f}{\partial \mu} = -2\mathbf{m}^\top (\mathbf{A} - \mathbf{B}) + 2\mu \|\mathbf{m}\|^2 = 0 \tag{2.0.20}$$

$$\mu_{min} = \frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{B})}{\|\mathbf{m}\|^2} \tag{2.0.21}$$

$$= \frac{\begin{pmatrix} 3 & -3 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}}{18} \tag{2.0.22}$$

$$= \frac{-18}{18} \tag{2.0.23}$$

$$= -1 \tag{2.0.24}$$

Substituting in the line equation,

$$\mathbf{x}_{min} = \mathbf{B} + \mu_{min} \mathbf{m} \tag{2.0.25}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.26}$$

6) Length of altitude,

$$l = \|\mathbf{A} - \mathbf{x}_{min}\| \tag{2.0.27}$$
$$= \sqrt{2} \tag{2.0.28}$$

Parameter	Value	Desription
A	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	Vertex 'A' of the triangle
B	$\begin{pmatrix} 4 \\ -1 \end{pmatrix}$	Vertex 'B' of triangle
C	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	Vertex 'C' of triangle

TABLE 6: Table

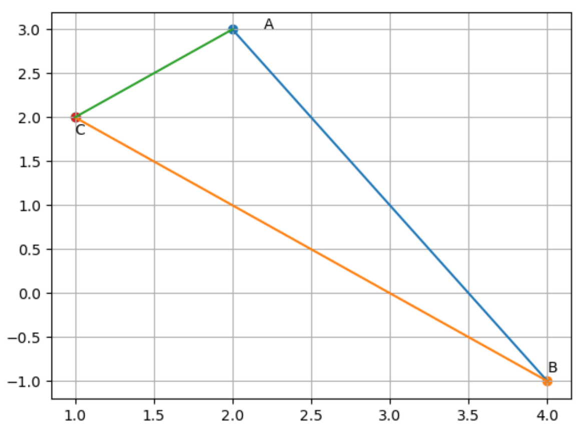

Fig. 6: Figure

Note that $\mathbf{x}_{min}$ i.e. the foot of the altitude is coincident with the vertex **C**. Therefore, the altitude is side AC itself.