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1 Problem

Bisector of angles A,B and C of a triangle ABC intersect its circumcircle at D,E,and F respectively. Prove that the angles of triangle DEF are $90° - \frac{A}{2}$, $90° - \frac{B}{2}$ and $90° - \frac{C}{2}$.

2 Solution

1) Let the vertices of the triangle ABC be:

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A} = c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.0.3}$$

For ease of calculation let's assume a = 5, c = 3, $\theta = 60°$.

2) side length b is given by:

$$b = \|A - C\| \tag{2.0.4}$$

$$= \left\| \begin{pmatrix} c\cos\theta - a \\ c\sin\theta \end{pmatrix} \right\| \tag{2.0.5}$$

$$= \sqrt{a^2 + c^2 - 2ac\cos\theta} \tag{2.0.6}$$

$$= \sqrt{19} \tag{2.0.7}$$

3) Circumcenter of triangle ABC:
   Circumcenter of triangle can be found by calculating point of intersection of perpedicular bisectors of two sides.
   a) perpendicular bisector of side BC:

$$(\mathbf{C} - \mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{C} + \mathbf{B}}{2} \right) = 0 \tag{2.0.8}$$

$$(\mathbf{C} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{C}\|^2 - \|\mathbf{B}\|^2}{2} \tag{2.0.9}$$

$$\mathbf{C}^\top \mathbf{x} = \frac{a^2}{2} \tag{2.0.10}$$

   b) perpendicular bisector of side AB:

$$\mathbf{A}^\top \mathbf{x} = \frac{c^2}{2} \tag{2.0.11}$$

Therefore, the circumcenter is given by:

$$\begin{pmatrix} a & 0 \\ c\cos\theta & c\sin\theta \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{a^2}{2} \\ \frac{c^2}{2} \end{pmatrix} \tag{2.0.12}$$

$$\mathbf{x} = \begin{pmatrix} \frac{5}{2} \\ -\frac{2}{\sqrt{3}} \end{pmatrix} \tag{2.0.13}$$

4) Incenter of triangle ABC:

$$\mathbf{I} = \frac{a\mathbf{A} + b\mathbf{B} + c\mathbf{C}}{a + b + c} \tag{2.0.14}$$

$$= \frac{15\sqrt{3}}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{2.0.15}$$

5) angular bisectors: