Problem: 11.11.3.9

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1 Problem

Find the co-ordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of latus rectum of the ellipse $4x^2 + 9y^2 = 36$.

2 Solution

1) Given ellipse equation:

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.1}$$

$$here, \quad \mathbf{V} = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.2}$$

$$f = -4 \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.4}$$

2) points of intersection of a line $\mathbf{x} = \mathbf{A} + \mu\mathbf{h}$ with ellipse are given by:

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-m^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}))^2 - g(\mathbf{h})(\mathbf{m}^\top \mathbf{V}\mathbf{m})} \right) \tag{2.0.5}$$

where,

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f \tag{2.0.6}$$

3) Center of the ellipse,

$$\mathbf{C} = -\mathbf{V}^{-1} u \tag{2.0.7}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.8}$$

4) Major axis

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.9}$$

$$i.e., \quad \mathbf{x} = \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.10}$$

5) Vertices
Vertices lie on major axis, therefore let

$$\mathbf{v} = \mu_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.11}$$

$$\mu_i = \pm 3 \tag{2.0.12}$$

Vertices are $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

6) Length of major axis

length of major axis = distance between vertices (2.0.13)

$$= \|\mathbf{v_1} - \mathbf{v_2}\| \tag{2.0.14}$$

$$= 6 \tag{2.0.15}$$

7) minor axis

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.16}$$

$$i.e., \quad \mathbf{x} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.17}$$

Points of intersection of minor axis with ellipse be $\mu_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
On solving,

$$\mu_i = \pm 2 \tag{2.0.18}$$

Points of intersection of minor axis with ellipse $= \pm \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ (2.0.19)

8) Length of minor axis

length of minor axis = distance between Points of inters (2.0.20)

$$= 4 \tag{2.0.21}$$

Normal to directrix,

$$\mathbf{n} = \text{direction vector of major axis} \tag{2.0.22}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.23}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.24}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.25}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 = -4 \tag{2.0.26}$$

9) Eccentricity

$$e = \frac{\sqrt{5}}{3} \tag{2.0.27}$$

$$\mathbf{F} = \frac{5c}{9} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.28}$$

$$c = \pm \frac{9}{\sqrt{5}} \tag{2.0.29}$$

Parameter	Value	Description
$\mathbf{V}$	$\begin{pmatrix} \frac{4}{9} & 0 \\ 0 & 1 \end{pmatrix}$	matrix $\mathbf{V}$from ellipse equation
$\mathbf{u}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	vector $\mathbf{u}$ from ellipse equation
f	-4	constant f from ellipse equation

TABLE 12: Table 1

10) Foci

$$\mathbf{F} = \begin{pmatrix} \pm\sqrt{5} \\ 0 \end{pmatrix} \tag{2.0.30}$$

11) equation of Latus Recta

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{2.0.31}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 3 \tag{2.0.32}$$

$$i.e., \mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.33}$$

Let points of intersection of latus rectum and curve be,

$$\mathbf{x} = \mathbf{F} + \mu_i \mathbf{m} \tag{2.0.34}$$

here,

$$\mu_i = \pm \frac{4}{3} \tag{2.0.35}$$

12) Length of latus recta

length of latus recta = distance between points o

(2.0.36)

$$= \frac{8}{3} \tag{2.0.37}$$

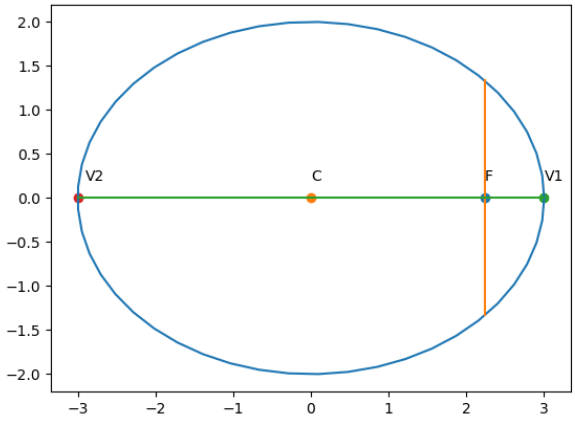

Fig. 12: Figure 1