Question : 11.10.4.23

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1 Problem

Prove that the products of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2-b^2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{a^2-b^2} \\ 0 \end{pmatrix}$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

2 Solution

$$\mathbf{P} = \begin{pmatrix} \sqrt{a^2-b^2} \\ 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{Q} = \begin{pmatrix} -\sqrt{a^2-b^2} \\ 0 \end{pmatrix} \tag{2.0.2}$$

Given line is,

$$\mathbf{x} = \begin{pmatrix} \frac{a}{\cos\theta} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sin\theta}{b} \\ \frac{\cos\theta}{a} \end{pmatrix} \tag{2.0.3}$$

A point on the line,

$$\mathbf{x}_0 = \begin{pmatrix} \frac{a}{\cos\theta} \\ 0 \end{pmatrix} \tag{2.0.4}$$

Normal vector,

$$\mathbf{n} = \begin{pmatrix} \frac{\cos\theta}{a} \\ \frac{-\sin\theta}{b} \end{pmatrix} \tag{2.0.5}$$

The line equation,

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{x}_0) = 0 \tag{2.0.6}$$

$$\begin{pmatrix} \frac{\cos\theta}{a} \\ \frac{-\sin\theta}{b} \end{pmatrix} \mathbf{x} = 1 \tag{2.0.7}$$

Comparing with $\mathbf{n}^\top \mathbf{x} = c$, $c = 1$.
Distance from point $\mathbf{P}$ to the line,

$$d_1 = \frac{\left| n^\top \mathbf{P} - c \right|}{\|\mathbf{n}\|} \tag{2.0.8}$$

Distance from point $\mathbf{Q}$ to the line,

$$d_2 = \frac{\left| n^\top \mathbf{Q} - c \right|}{\|\mathbf{n}\|} \tag{2.0.9}$$

$$= \frac{\left| -n^\top \mathbf{P} - c \right|}{\|\mathbf{n}\|} \tag{2.0.10}$$

$$= \frac{\left| n^\top \mathbf{P} + c \right|}{\|\mathbf{n}\|} \tag{2.0.11}$$

$$d_1 d_2 = \frac{\left| (n^\top \mathbf{P})^2 - c^2 \right|}{\|\mathbf{n}\|} \tag{2.0.12}$$

$$= \frac{\left| \frac{\cos^2\theta (a^2-b^2)}{a^2} - 1 \right|}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}} \tag{2.0.13}$$

$$= \frac{\left(b^2\cos^2\theta + a^2\sin^2\theta\right) a^2 b^2}{\left(b^2\cos^2\theta + a^2\sin^2\theta\right) a^2} \tag{2.0.14}$$

$$= b^2 \tag{2.0.15}$$

Hence Proved.

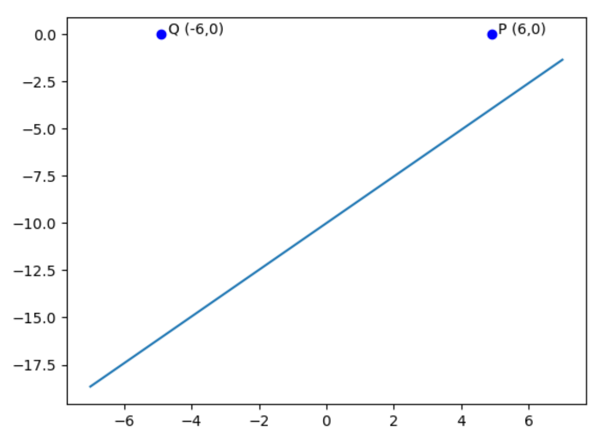

Fig. 0: Figure 1