Question: 12.11.2.15

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1 Problem

Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

2 Solution

The lines l_1 and l_2 in vector form can be written as

$$\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{x_1} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{2.0.3}$$

We first check whether the given lines are skew. The lines

$$\mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1}, \ \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2.0.4}$$

intersect if

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{2.0.5}$$

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{2.0.6}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{2.0.7}$$

$$\tag{2.0.8}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \tag{2.0.9}$$

$$\mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{2.0.10}$$

We check whether the equation (**??**) has a solution

$$\begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{2.0.11}$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ -6 & -2 & 6 \\ 1 & 1 & 8 \end{array}\right) \tag{2.0.12}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{7}R_1]{R_2 \leftarrow R_2 + \frac{6}{7}R_1} \tag{2.0.13}$$

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ 0 & -\frac{8}{7} & \frac{66}{7} \\ 0 & \frac{6}{7} & -\frac{52}{7} \end{array}\right) \tag{2.0.14}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + \frac{3}{4}R_2} \tag{2.0.15}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{5}{14} \end{array}\right) \tag{2.0.16}$$

The rank of the matrix is 3. So the given lines are skew. The closest points on two skew lines defined by (**??**) are given by

$$\mathbf{M}^\top \mathbf{M} \boldsymbol{\lambda} = \mathbf{M}^\top \left(\mathbf{x_2} - \mathbf{x_1}\right) \tag{2.0.17}$$

$$\begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{2.0.18}$$

$$\begin{pmatrix} 86 & 20 \\ 20 & 6 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

The augmented matrix of the above equation (**??**) is given by,

$$\left(\begin{array}{cc|c} 86 & 20 & 0 \\ 20 & 6 & 0 \end{array}\right) \tag{2.0.20}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{10}{43}R_1} \tag{2.0.21}$$

$$\left(\begin{array}{cc|c} 86 & 20 & 0 \\ 0 & \frac{58}{43} & 0 \end{array}\right) \tag{2.0.22}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{43}{58} R_2]{R_1 \leftarrow \frac{1}{86}\left(R_1 - \frac{430}{29} R_2\right)} \tag{2.0.23}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \tag{2.0.24}$$

So, we get

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.25}$$

The closest points $\mathbf{A}$ on line l_1 and $\mathbf{B}$ on line l_2 are given by,

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{2.0.26}$$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + 0\mathbf{m_1} \tag{2.0.27}$$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{2.0.28}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2.0.29}$$

$$= \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + 0\mathbf{m_2} \tag{2.0.30}$$

$$= \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \tag{2.0.31}$$

The minimum distance between the lines is given by,

$$\|\mathbf{B} - \mathbf{A}\| = \left\|\begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}\right\| \tag{2.0.32}$$

$$= 2\sqrt{29} \tag{2.0.33}$$

The shortest distance between the given lines is $2\sqrt{29}$ units.