Que: 11.11.4.9

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1 Problem

Find the equations of hyperbola having Vertices $\begin{pmatrix}0\\\pm3\end{pmatrix}$ and Foci $\begin{pmatrix}0\\\pm5\end{pmatrix}$

2 Solution

$\mathbf{V}_1$ and $\mathbf{F}_1$ be vertex and focus on a side of center,and $\mathbf{V}_2$ and $\mathbf{F}_2$ on other

$$\mathbf{V}_1 = \begin{pmatrix}0\\3\end{pmatrix} \tag{2.0.1}$$

$$\mathbf{V}_2 = \begin{pmatrix}0\\-3\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{F}_1 = \begin{pmatrix}0\\5\end{pmatrix} \tag{2.0.3}$$

$$\mathbf{F}_2 = \begin{pmatrix}0\\-5\end{pmatrix} \tag{2.0.4}$$

Transverse axis: Line joining two foci

$$\mathbf{m} = \mathbf{F}_1 - \mathbf{F}_2 \tag{2.0.5}$$

$$= \begin{pmatrix}0\\10\end{pmatrix} \tag{2.0.6}$$

$$\mathbf{n} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.0.7}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}(\mathbf{x} - \mathbf{F}_1) = 0 \tag{2.0.8}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.0.9}$$

Center of hyperbola, $\mathbf{O}$ is given by:

$$\mathbf{O} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} \tag{2.0.10}$$

$$\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.11}$$

Eccentricity,

$$e = \frac{\|\mathbf{F_1} - \mathbf{O}\|}{\|\mathbf{V_1} - \mathbf{O}\|} \tag{2.0.12}$$

$$= \frac{5}{3} \tag{2.0.13}$$

The distance between center and directrix,

$$\frac{\|\mathbf{V_1} - \mathbf{O}\|}{e} = \frac{9}{5} \tag{2.0.14}$$

Also, the directrix is perpendicular to the transverse axis.
Hence, the equation of directrix is,

$$\begin{pmatrix}0 & 1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}0\\\frac{9}{5}\end{pmatrix}\right) = 0 \tag{2.0.15}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \frac{9}{5} \tag{2.0.16}$$

$$\mathbf{V} = \|\mathbf{n}\|^2\,\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \tag{2.0.17}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \frac{25}{9}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.0.18}$$

$$= \begin{pmatrix}1 & 0\\0 & -\frac{16}{9}\end{pmatrix} \tag{2.0.19}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2\,\mathbf{F} \tag{2.0.20}$$

$$= \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.21}$$

$$f = \|\mathbf{n}\|^2\,\|\mathbf{F}\|^2 - c^2e^2 \tag{2.0.22}$$

$$= 16 \tag{2.0.23}$$

Equation of the hyperbola:

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2.0.24}$$

$$\mathbf{x}^\top\begin{pmatrix}1 & 0\\0 & -\frac{16}{9}\end{pmatrix}\mathbf{x} + 16 = 0 \tag{2.0.25}$$

$$\tag{2.0.26}$$

$\mathbf{F}_1$	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	Focus
$\mathbf{F}_2$	$\begin{pmatrix} 0 \\ -5 \end{pmatrix}$	Focus
$\mathbf{V}_1$	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	Vertex
$\mathbf{V}_2$	$\begin{pmatrix} 0 \\ -3 \end{pmatrix}$	Vertex

TABLE 0: Table1

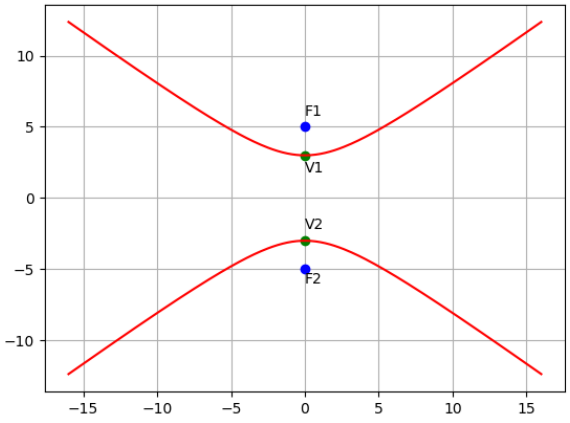

Fig. 0: Figure 1